

Mitsuo Kono
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Nonlinear Physics of Plasmas

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Nonlinear Physics of Plasmas

With 180 Figures



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*In Memory of Dik ter Haar, a Teacher,
Mentor and a Friend*

Preface

Plasma physics has been developed in struggles of taming rich behaviors of plasmas. High temperature plasmas observed in a laboratory, space, and astrophysical objects are often subjected to various kinds of instabilities because of excess free energy stored in inhomogeneity and anisotropy inherent to magnetically confined plasmas. In the processes of relaxation from non-equilibrium to equilibrium states, nonlinear interactions are playing important roles. This situation has given plasma physics a special position in the field of physics by providing new concepts and theories and leading to the development of nonlinear sciences.

Soliton was discovered as a robust nonlinear localized wave behaving like a particle by Zabusky and Kruskal, leading to the remarkable exact solution method of the inverse scattering transform by Gardner, Greene, Kruskal, and Miura for the K-dV equation and by Zakharov and Shabat for the nonlinear Schrödinger equation. The method has been extended and applied to nonlinear wave equations such as the derivative nonlinear Schrödinger equation for Alfvén solitons and the Petviashvili equation for temperature gradient drift waves. The success of the soliton theory encourages efforts to find nonlinear stationary solutions for higher dimensional equations associated with vortices and secondary flows induced by turbulence which are supposed to control transport in plasmas.

Vortices in nature are often involved in violent transport. They store energy in their localized structures in the form of rotational motions and carry it over a long distance. Vortices have been shown inherent to magnetized plasmas in which $\mathbf{E} \times \mathbf{B}$ drifts play a crucial role. The Hasegawa and Mima equation derived to explain the high level of density fluctuations and the broad frequency spectrum observed in a tokamak has been shown to have a dipole vortex solution which is fairly stable against collisions. This basic equation and several model equations derived thereafter are anticipated to link strong turbulence to self-organized vortical motions which are supposed to be one of the main factors in anomalous transport in magnetized plasmas.

Chaos was intensively studied in plasma physics under the name of stochastic heating associated with an auxiliary heating methods in fusion plasmas, long before the term “chaos” was introduced. Chirikov showed that particle dynamics under the wave heating was described by the Standard map which has become a central basis of chaos theory in Hamiltonian systems. Magnetic field lines in Tokamaks are also

shown to be described by modified Standard maps which reveal the existence of the transport barriers for improving the plasma confinement.

Complexity in plasma physics comes from wave–wave and wave–particle self-consistent interactions which force us to treat nonlinearity fully. Although computer simulations are powerful and inevitable and have been successful in providing new findings and notions, we have not spared many pages on computational aspects and methods, as they are beyond the scope of our text, and also keeping in mind that computational modelings are guided by our knowledge of fundamental theories. At present, in plasma physics, there are well established nonlinear notions and theories which are necessary not only to understand present status of the plasma phenomena but also to make a success in solving many remaining unsolved problems and in developing new methods for emerging future problems.

This book aims at making a bridge between the fundamental linear and nonlinear theories of plasmas and nonlinear phenomena involving structures, chaos, and turbulence in plasmas. It is directed mainly to graduate students, advanced undergraduates with some background in plasma physics, and to researchers seeking to advance in the field of nonlinear plasmas. Certainly, we had to omit a number of topics in developing fields, such as, for example, zonal flows in magnetized plasma turbulence, although the formulation and derivations of the basic equations are given. The book also omits basics of fusion plasmas as they are well covered in a number of good textbooks and monographs.

The manuscript consists of four parts. Part I provides basic concepts and fundamental theories necessary to understand contemporary plasma physics. Part II gives ideas and methodologies for exploring nonlinear phenomena associated with coherent structures, turbulence, and chaos. Soliton-like excitations, vortical motions, Langmuir turbulence, and wave collapse are mainly discussed since they are well established concepts and pedagogical to graduate students and physicists in other fields. Part III is devoted to spatiotemporal structures in strong plasma turbulence. Departure from exact one-dimensional soliton solutions in two- and three-dimensional models introduces notions of soliton stability and collapse, self-similarity, scaling, and direct numerical simulations. Complexity in plasmas is outlined in the nonlinear wave–wave interaction paradigm, which, together with fundamentals of relativistic plasma interactions is relevant to current research on laser plasmas. Part IV, for interested advanced readers, is on new, emerging topics on characterization of turbulence and integrated modeling in multiscale plasma interactions. Chapters 1–8 are written by M. Kono and Chaps. 9–15 by M.M. Škorić; the complete text is compiled and fully integrated. Moreover, the choice of presented topics from the vast field of nonlinear plasma phenomena reflects the biases and the original results of the authors, some stemming from a joint work at the University of Oxford, three decades ago.

The text was partially delivered in a series of lectures to graduate students at Chuo University, Tokyo (M.K) and to PhD students at the Graduate University of Advanced Studies, National Institute for Fusion Science, Japan (M.M.Š), and at two Plasma Colleges at ICTP, Trieste. The work on this manuscript was commenced during a sabbatical leave of M.K., who thanks Chuo University for hospitality.

Our pleasure and privilege is to express special gratitude to many colleagues and friends for fruitful collaboration over the years, professors: B.V. Stanić, S. Ishiguro, R. Horiuchi, E.A. Kuznetsov, H. Pécseli, M. Y. Tanaka, H. Sugama, J. Vranješ, Lj. Hadžievski, A. Maluckov, M. Rajković, Dr Lj. Nikolić, and Dr LiBaiwen. We are indebted to Prof. A.N. Kaufman, Prof. R.O. Dendy and Prof. I.G. Kevrekidis for motivating discussions on new research avenues to be explored. Our unique tribute is due to friends and longtime collaborators, Prof. D. ter Haar and Prof. M. Jovanović who are no longer with us. Finally, continuing support and encouragements by Prof. Y.H. Ichikawa, Prof. K. Mima, Prof. O. Motojima, Prof. A. Komori, and Prof. T. Sato are gratefully acknowledged. In conclusion, our sincere gratitude is due to Dr. Habil. Claus E. Ascheron, the Springer Executive Editor in Physics, whose unflagging effort and patience made a completion of this book possible.

Tokyo and Gifu
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Mitsuo Kono
Miloš M. Škorić

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Part I

Fundamentals of Plasma Physics

Chapter 1

Basic Properties of High Temperature Plasmas

As temperature is increased, the state of matter changes from solid to liquid, and then to gas. With further increase in temperature, some or all of the atoms of neutral gas are ionized. A plasma is a partially or fully ionized gas where ionization and recombination are balanced to keep the ionized gas to be charge neutral as a whole. Once the charge neutrality is broken, a large electric field is locally created to drive the plasma unstable. In this sense, the charge neutrality (rigorously quasi-charge neutrality) is essentially important. This implies that the potential in the plasma is usually determined through not Poisson's equation but equations of motion for the electrons and ions under the charge neutrality.

Particles in a plasma interact through the Coulomb force which reaches over a long distance. Therefore, the motion of one particle affects those of all the other particles and the reaction of the other particles also influences all the particles. This complicated mutual influence among the particles is organized into collective motions extended over the whole plasma depending on the species of particles in the plasmas and the existence of external magnetic fields.

The first observation of the collective motion was made by Tonks and Langmuir [1] in a gas discharge, showing that the gas of charged particles behaves rather differently compared with ordinary gases.

The long range part of the Coulomb potential is masked by the collective behavior of the particles and the particles are interacting with the screened Coulomb potential. Thus, the particle participates in the collective motion in the long wavelength and at the same time carries an individual motion in the short wavelength through the screened Coulomb potential. The boundary between the collective motion and the individual motion is given by the Debye length. When the wavelength of the collective motion becomes short compared to the Debye length, the collective motion is certainly destroyed.

Because of the inertial effect of the finite mass of particles, the collective motions propagate as waves. The waves are stable and long lived when the collision frequency is small compared to the wave frequencies, which is usually the case in high temperature plasmas.

Plasmas are rich in collective motions. The collective motion is the coherent response of the particles to the potential produced self-consistently by the particles.

Self-consistency brings nonlinearity which is responsible for complexity of physical phenomena in plasmas.

When the temperature of a plasma is finite, the particles are agitated and not completely respond coherently to the wave potential. The particle agitation, which is so called the thermal motion, affects the propagation properties and the wave profile by bringing the dispersion. There are some other causes of the dispersion such as the breakdown of the charge neutrality.

In this chapter as an introduction of the whole book, we deal with basic concepts and properties of plasmas which are bases of the nonlinear plasma physics.

There have been already published monographs on plasma physics [2–37]. Some are restricted to basic physics and some are for advanced topics. We try to make a bridge between basic plasma physics and advanced contemporary plasma physics.

1.1 Ionization Equilibrium

Plasmas are found everywhere in the universe except for on the Earth where they are seen in special occasions under the natural environment or produced in laboratories. This is because the room temperature is not high enough to create the plasma state.

The number of particles with energy level E_i under temperature T is given by the Boltzmann distribution as

$$n_i = g_i e^{-E_i/\kappa T}, \quad (1.1)$$

where g_i is the degeneracy of states for the ion with the level i . The ratio of the number of the same species of particles with the energy level E_i and in the ground state E_0 is given by

$$\frac{n_i}{n_0} = \frac{g_i}{g_0} e^{-E_{i0}/\kappa T}, \quad E_{i0} = E_i - E_0. \quad (1.2)$$

The energy difference of two states is expressed in terms of the ionization potential $V_i[eV]$ and kinetic energy of the electron as

$$E_{i0} = eV_i + \frac{p^2}{2m_e}, \quad (1.3)$$

where m_e is the electron mass. The degree of freedom of the system at the level i is given by the product of the degree of freedom of the ion g_i and that of the electron which is equivalent to the degree of freedom of a single electron in the phase space volume element $d\mathbf{p}$. The degree of freedom of one electron with the momentum between $\mathbf{p}$ and $\mathbf{p} + d\mathbf{p}$ is given by the degree of freedom in the phase space as $2d\mathbf{p}/h^3 n_e$ with Planck's constant h and the spin degree of freedom 2. Then, the ratio of the densities of the charged and neutral particles in a weakly ionized gas is

given by the Saha equation [38]

$$\begin{aligned}\frac{n_i}{n_0} &= \frac{g_i}{g_0} \frac{2}{h^3 n_e} \int_{-\infty}^{\infty} e^{-(eV_i + p^2/2m_e)/\kappa T} d\mathbf{p} \\ &= \frac{2g_i}{g_0} \frac{1}{n_e} \left(\frac{2\pi m_e \kappa T}{h^2} \right)^{3/2} e^{-eV_i/\kappa T} \simeq 2.4 \times 10^{21} \frac{T^{3/2}}{n_e} e^{-eV_i/\kappa T},\end{aligned}\quad (1.4)$$

where $2g_i/g_0 \sim 1$ has been used and n_i and n_e are the ion and electron densities, respectively. Since an ion remains charged until it meets an electron to recombine, the ion density n_i is proportional to $1/n_e$ since in equilibrium the recombination rate is proportional to the electron density.

Defining the ionization ratio r_i and the degree of ionization α as

$$r_i = \frac{n_i}{n_0} \simeq \frac{n_e}{n_0}, \quad \alpha = \frac{n_i}{n_0 + n_i} = \frac{r_i}{1 + r_i}, \quad (1.5)$$

we have the degree of the ionization with $r_i \ll 1$

$$\alpha^2 \simeq r_i^2 = \frac{2.4}{n_0} \times 10^{21} T^{3/2} e^{-eV_i/\kappa T}. \quad (1.6)$$

From this, the degree of ionization increases with temperature and decreases with neutral density and ionization potential.

For a hydrogen gas at the room temperature with

$$n_0 \sim 3 \times 10^{25} \text{ m}^{-3}, \quad T \sim 300 \text{ K}, \quad \frac{eV_i}{\kappa T} = \frac{13.6}{300/11,600} \sim 526,$$

we have by putting $n_i \sim n_e$

$$\frac{n_i}{n_0} \sim 10^{-114},$$

which shows that the plasma state is not naturally realized.

When the temperature is increased, Coulomb interaction takes place to lead high ionization and n_i/n_0 abruptly increases after κT becomes a few times smaller than eV_i , leading to a fully ionized state with $n_i \gg n_0$.

1.2 Plasma Temperature

The most probable particle velocity distribution of a gas at thermal equilibrium is given by

$$f(\mathbf{v}) = \left(\frac{m}{2\pi\kappa T} \right)^{3/2} n \exp\left(-\frac{m\mathbf{v}^2}{2\kappa T} \right). \quad (1.7)$$

The average kinetic energy then is estimated by

$$E = \frac{1}{n} \int_{-\infty}^{\infty} \frac{1}{2} m v^2 f(v) dv = \frac{3}{2} \kappa T, \quad (1.8)$$

and is expressed by temperature. Thus in plasma physics, temperatures are given in units of energy. The conversion factor is obtained in the following. Putting

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J} = \kappa T,$$

we have

$$T = \frac{1.6 \times 10^{-19}}{1.38 \times 10^{-23}} = 11,600 \text{ K}. \quad (1.9)$$

The average kinetic energy of a plasma with temperature 1 eV is $E = (3/2) \text{ eV}$. In the following, temperature is expressed in energy unit, that is $\kappa T \rightarrow T$.

Since the temperature, density, and volume of plasmas used in laboratory experiments for basic studies of plasma physics are 100 eV, $10^{18} - 10^{19} \text{ m}^{-3}$, and $1 - 3 \text{ m}^3$, respectively, the heat load on the wall is not serious. However, large machines with higher temperature plasmas like fusion devices are subject to large heat load.

1.3 Collision Frequency

Collision frequencies are determined by solving the Boltzmann equation. However, they are estimated qualitatively as follows. An impact parameter b is given by setting kinetic energy equal to potential energy as

$$\frac{e^2}{b} \sim \frac{T}{2}. \quad (1.10)$$

The collision cross section is then given by

$$\sigma \sim \pi b^2 \sim 4\pi \left(\frac{e^2}{T} \right)^2. \quad (1.11)$$

Therefore, the collision time τ is estimated by

$$\frac{1}{\tau} \sim n \sigma v \sim n \sigma \sqrt{T m} \sim \frac{1}{4\pi} \frac{\omega_p}{n \lambda_D^3}, \quad (1.12)$$

where $\lambda_D = \sqrt{T/4\pi n e^2}$, the Debye length and $\omega_p = \sqrt{4\pi n e^2/m}$, the plasma frequency. As is shown later for high temperature plasmas, we have

$$n \lambda_D^3 \gg 1, \quad (1.13)$$

which indicates that the high temperature plasmas are supposed collisionless as to Coulomb collisions, that is,

$$\omega_p \tau \gg 1. \quad (1.14)$$

A plasma is relaxed to the equilibrium due to collisions. In the processes of approaching the equilibrium, there could appear different temperatures at the same time because of different relaxation times. Collision frequency $1/\tau_{\alpha\beta}$ of a charged particle of species α against a particle of β is approximately given by

$$\frac{1}{\tau_{ei}} \sim \frac{\omega_{pe}}{n \lambda_D^3}, \quad (1.15)$$

$$\frac{1}{\tau_{ee}} \sim \frac{1}{\tau_{ei}}, \quad (1.16)$$

$$\frac{1}{\tau_{ii}} \sim \sqrt{\frac{m_e}{m_i}} \frac{1}{\tau_{ee}}, \quad (1.17)$$

$$\frac{1}{\tau_{ie}} \sim \frac{m_e}{m_i} \frac{1}{\tau_{ei}}. \quad (1.18)$$

Since the ratio of the magnitude of the collision frequencies is given by

$$\frac{1}{\tau_{ee}} : \frac{1}{\tau_{ii}} : \frac{1}{\tau_{ie}} = 1 : \sqrt{\frac{m_e}{m_i}} : \frac{m_e}{m_i}, \quad (1.19)$$

the plasma relaxes to the thermal equilibrium in a way that the thermal equilibrium among the electrons is realized first, then that between the ions follows, and finally the thermal equilibrium between the electrons and the ions is established. Thus, two-temperature state lasts long before the thermal equilibrium as a whole is reached. Thus, a plasma can have several temperatures at the same time, that is, the electrons and the ions have separately the Boltzmann distributions with different temperatures.

When there is a magnetic field, even a single species plasma can have two temperatures. Motions of a charged particle are different in the directions perpendicular to and parallel to the magnetic field, resulting in two temperatures.

1.4 Conditions for High Temperature Plasmas

A condition for a plasma to exist is that the average electron kinetic energy is much larger than the Coulomb potential, that is,

$$T \gg \frac{e^2}{\langle r \rangle}, \quad (1.20)$$

where $\langle r \rangle$ is the average particle distance and is given from

Table 1.1 Plasma parameters for various plasmas

	$n(\text{cm}^{-3})$	$T(\text{eV})$	$n\lambda_D^3$
ionosphere	10^5	0.05	1.5×10^4
glow discharge	10^9	2	3.7×10^4
flame	10^8	0.1	1.3×10^3
interstellar space	1	0.01	4×10^5
magnetic fusion plasma	10^{15}	10^4	1.3×10^7
inertial fusion plasma	10^{26}	100	1

$$\frac{4}{3}\pi\langle r \rangle^3 n = 1.$$

A high temperature plasma is characterized by smallness of the ratio of the average potential energy to the electron kinetic energy:

$$\Gamma = \frac{e^2/\langle r \rangle}{T} = \frac{1}{3} \left(\frac{4\pi}{3} n \lambda_D^3 \right)^{-2/3} \ll 1. \quad (1.21)$$

If we introduce the plasma parameter g as

$$g = n \lambda_D^3, \quad (1.22)$$

the condition for the high temperature plasma is equivalent to $g \gg 1$. Since the right hand side of (1.22) is the number of particles in the Debye sphere, $g \gg 1$ means that the number of particles in the Debye sphere is large enough for shielding to be effective. Plasma parameters are given in Table 1.1 for various plasmas.

The density for the plasma parameter to be nearly equal to 1 like inertial fusion plasmas is given by

$$n \sim 1.68 \times 10^{17} T^3 (\text{cm}^{-3}), \quad [T] = \text{eV},$$

for which the plasma is a degenerate Fermi liquid and quantum effect should be taken into account.

An equation of state for plasmas whose plasma parameters are large is approximated by that of an ideal gas:

$$p = C n^\gamma, \quad \gamma = \frac{c_p}{c_v}, \quad (1.23)$$

or

$$\nabla \ln p = \gamma \nabla \ln n. \quad (1.24)$$

For the isothermal process $\gamma = 1$, ($\nabla p = \nabla(nT) = T \nabla n$), and for the adiabatic process $\gamma = (2 + d)/d$, where d is the degree of freedom.

1.5 Debye Screening

A plasma has a remarkable property to protect itself against an electric potential that is applied to it. This is known as the Debye shielding. When a test charge Q is introduced into a plasma, the plasma particles with charges of the same sign as that of the test charge are repelled from the test charge and those of the opposite sign are attracted to the test charge. As a result, the polarization field is excited to shield the test charge Q . Thus, the spatial extent of the potential produced by the test charge is effectively reduced compared to that of the Coulomb potential. This effectively reduced extent of the test charge force is called the Debye length.

The electrons respond to the test charge immediately after it is introduced into the plasma while the ions are immobile because of their heavy mass. Then, the response of the electrons to the test charge causes a local charge separation to produce a potential and the electrons distribute according to the Boltzmann distribution determined by the potential.

$$n_e = n_0 e^{e\phi/T} \simeq n_0 \left(1 + \frac{e\phi}{T} + \dots \right), \quad (1.25)$$

where n_0 is the background density. The ion density is constant, $n_i = n_0$. Now, the Poisson equation is written as

$$\nabla^2 \phi = 4\pi e(n_e - n_i) + 4\pi Q\delta(\mathbf{r}) \simeq \frac{\phi}{\lambda_D^2} + 4\pi Q\delta(\mathbf{r}), \quad (1.26)$$

whose solution is given by

$$\phi(\mathbf{r}) = \frac{Q}{r} e^{-r/\lambda_D}. \quad (1.27)$$

Thus, the Coulomb potential of the test charge Q/r is reduced by e^{-r/λ_D} . The total charge necessary to shield the electrostatic potential produced by the test charge is compensated by the particles in the Debye sphere.

The Debye length is given by

$$\lambda_D \simeq 7.43 \times 10^{-2} \sqrt{\frac{T}{n}}. \quad (1.28)$$

For fusion plasmas ($T \sim 1 \text{ KeV}$, $n \sim 10^{14} \text{ cm}^{-3}$) $\lambda_D \sim 2 \times 10^{-3} \text{ cm}$, and for space plasmas ($T \sim 100 \text{ eV}$, $n \sim 1 \text{ cm}^{-3}$) $\lambda_D \sim 7 \times 10^3 \text{ cm}$.

The Debye length is increased with temperature T because electrons near the edge of the electron cloud can escape from the test charge and the shielding becomes incomplete. The Debye length is decreased with density n because the higher the density the more particles participate the shielding.

1.6 Plasma Diamagnetism

The equation of motion for a single particle of species α in a homogeneous magnetic field is given by

$$m_\alpha \frac{d\mathbf{v}}{dt} = \frac{e_\alpha}{c} \mathbf{v} \times \mathbf{B}, \quad (\mathbf{e}, \mathbf{B} \text{ in cgs}). \quad (1.29)$$

Taking $\mathbf{B} = (0, 0, B)$, we have

$$\begin{aligned} \frac{dv_x}{dt} &= \Omega_\alpha v_y, \\ \frac{dv_y}{dt} &= -\Omega_\alpha v_x, \\ \frac{dv_z}{dt} &= 0, \end{aligned}$$

where Ω_α is the cyclotron frequency defined as $\Omega_\alpha = e_\alpha B / m_\alpha c$, and

$$\frac{|\Omega_e|}{2\pi} = 2.8B \text{ MHz}, \quad [B] = \text{Gauss}. \quad (1.30)$$

Introducing a new variable by $V = v_x + iv_y$, the equation of motion is reduced to

$$\frac{dV}{dt} = -i\Omega_\alpha V, \quad (1.31)$$

which is readily integrated to give

$$V = v_x + iv_y = v_\perp e^{-i\Omega_\alpha t}, \quad (1.32)$$

and

$$x + iy = x_0 + iy_0 + i \frac{v_\perp}{\Omega_\alpha} e^{-i\Omega_\alpha t}. \quad (1.33)$$

From this, the electron and the ion rotate around the magnetic field line in the opposite direction to each other, and the direction of the rotation is the direction of the diamagnetism, that is, the magnetic field produced by the charge particles is opposite to the external magnetic field and therefore the plasma particles reduce the external magnetic field. This is seen as follows.

Suppose an electron with the velocity $v_\perp$ is rotating around a magnetic field line with the Larmor radius $\rho_L = v_\perp / |\Omega_e|$. The density of the single electron on a loop around the magnetic field line is $1/2\pi\rho_L$. The magnetic moment produced by the current loop of the electron gyration motion is given by

$$\mu = \frac{1}{c} JS \frac{\mathbf{B}}{B}, \quad (1.34)$$

where the current is

$$J = -nev = -\frac{1}{2\pi\rho_L}e\rho_L|\Omega_e| = -\frac{e|\Omega_e|}{2\pi},$$

and the area surrounded by the loop is

$$S = \pi\rho_L^2 = \pi\left(\frac{v_\perp}{\Omega_e}\right)^2.$$

Thus, the magnetic moment is shown to be given

$$\mu = -\frac{e}{2c}\frac{v_\perp^2}{|\Omega_e|}\frac{\mathbf{B}}{B} = -\frac{W_\perp}{B}\frac{\mathbf{B}}{B}. \quad (1.35)$$

where $W_\perp = mv_\perp^2/2$.

With the particle density n and the magnetic moment μ , the magnetization is simply given by $\mathbf{M} = n\mu$. Then, the external magnetic field $\mathbf{H}$ is expressed in terms of the plasma magnetic field $\mathbf{B}$ by

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = \left(1 + 4\pi\frac{nW_\perp}{B^2}\right)\mathbf{B}, \quad (1.36)$$

Therefore, the magnetic field in the plasma is smaller than the external magnetic field as

$$\mathbf{B} = \frac{1}{1 + 4\pi nW_\perp/B^2}\mathbf{H}. \quad (1.37)$$

The plasma diamagnetization is a crucial property for the confinement.

The classical Boltzmann statistics tells that magnetization in thermal equilibrium is zero (van Leeuwen's theorem [39,40]) since the density of the charged particles is homogeneous and the angular momentum associated with particles due to the cyclotron oscillations cancels each other. This is seen in a different way. Since a stationary magnetic field does not give energy to the charged particles, the free energy of the charged particle system does not change either. From this, the magnetization is also said to be zero. However, confined plasmas we have in laboratories are no longer homogeneous and the cancellation of the angular momentum is not complete.

1.7 Single Particle Motions

In plasmas under a strong external magnetic field such as magnetic fusion plasmas, the self-generated magnetic field by the motion of the plasma particles does not affect the external magnetic field. Therefore, knowledge of a single particle motion in an external magnetic field often helps understanding some of plasma behaviors as a whole.

1.7.1 Guiding Center Drifts

The equation of motion is given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = \frac{e}{m} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right), \quad (1.38)$$

where $\mathbf{E}$ and $\mathbf{B}$ are assumed to be slowly varying in space and time, that is, the temporal and spatial scales are assumed much larger than the cyclotron oscillation period and the Larmor radius ρ_L . A unit vector along the magnetic field is taken as $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ and the cyclotron frequency is defined by $\Omega = eB/mc$. Then, we may express the velocity as

$$\mathbf{v} = v_{\parallel} \mathbf{b} + \mathbf{v}_{\perp}. \quad (1.39)$$

Substituting (1.39) into (1.38) and taking vector and scalar product of $\mathbf{b}$ with the resultant equation, we have

$$\mathbf{b} \times \frac{d\mathbf{v}}{dt} = \frac{e}{m} \mathbf{b} \times \mathbf{E} + \Omega \mathbf{v}_{\perp} - v_{\parallel} \mathbf{b} \times \frac{d\mathbf{b}}{dt}, \quad (1.40)$$

$$\frac{dv_{\parallel}}{dt} = \frac{e}{m} \mathbf{b} \cdot \mathbf{E} - v_{\parallel} \mathbf{b} \cdot \frac{d\mathbf{b}}{dt}. \quad (1.41)$$

We introduce guiding center coordinates $(\mathbf{x}_G, v_G)$ through

$$\mathbf{v}_{\perp} = \mathbf{v}_G + \tilde{\mathbf{v}}_{\perp}, \quad \tilde{\mathbf{v}}_{\perp} = -v_{\perp} (\mathbf{e}_1 \sin \Omega_G t + \mathbf{e}_2 \cos \Omega_G t), \quad (1.42)$$

and

$$\mathbf{x} = x_{\parallel} \mathbf{b} + \mathbf{x}_G + \tilde{\boldsymbol{\rho}}, \quad \tilde{\boldsymbol{\rho}} = \frac{v_{\perp}}{\Omega_G} (\mathbf{e}_1 \cos \Omega_G t - \mathbf{e}_2 \sin \Omega_G t), \quad (1.43)$$

where Ω_G is the cyclotron frequency estimated at the guiding center and

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{b}.$$

Note $\tilde{\mathbf{v}}_{\perp}$ is a solution for

$$\mathbf{b} \times \frac{d\mathbf{v}_{\perp}}{dt} = \Omega_G \mathbf{v}_{\perp}.$$

Since the Larmor radius is assumed small compared with the inhomogeneity of the electric and magnetic field, we may expand $\mathbf{E}$, $\mathbf{B}$, $\mathbf{b}$, and Ω with respect to $\boldsymbol{\rho}$ like

$$E(\mathbf{x}, t) \simeq \left\{ 1 + \boldsymbol{\rho} \cdot \nabla_{\perp} + \frac{1}{2} (\boldsymbol{\rho} \cdot \nabla_{\perp})^2 + \cdots \right\} E(x_{\parallel}, \mathbf{x}_G, t).$$

Averaging (1.40) over the cyclotron oscillation period we obtain the guiding center drift as

$$\begin{aligned}
\mathbf{v}_G &= c \left\langle \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right\rangle + \left\langle \frac{\mathbf{B}}{\Omega B} \times \frac{d\mathbf{v}_G}{dt} \right\rangle \\
&\quad - \left\langle \frac{\Omega - \Omega_G}{\Omega} \tilde{\mathbf{v}}_\perp \right\rangle + \frac{v_\parallel}{\Omega} \left\langle \mathbf{b} \times \frac{d\mathbf{b}}{dt} \right\rangle \\
&= c \left(1 + \frac{1}{4} \rho_L^2 \nabla_\perp^2 \right) \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{c}{\Omega B} \frac{d}{dt} \mathbf{E}_\perp \\
&\quad + \frac{v_\perp^2}{2\Omega} \frac{\mathbf{B} \times \nabla B}{B^2} + \frac{v_\parallel^2}{\Omega} \nabla \times \mathbf{b},
\end{aligned} \tag{1.44}$$

where $\langle \dots \rangle$ is an average over the cyclotron oscillation period and the right hand side is estimated at the guiding center. In deriving (1.44), we have used

$$(\mathbf{b} \cdot \nabla) \mathbf{b} = -\mathbf{b} \times (\nabla \times \mathbf{b}),$$

and

$$\begin{aligned}
\left\langle \mathbf{b} \times \frac{d\mathbf{b}}{dt} \right\rangle &= \langle \mathbf{b} \times (\mathbf{v} \cdot \nabla) \mathbf{b} \rangle = v_\parallel \mathbf{b} \times (\mathbf{b} \cdot \nabla) \mathbf{b} \\
&= -v_\parallel \mathbf{b} \times \{ \mathbf{b} \times (\nabla \times \mathbf{b}) \} = v_\parallel \nabla \times \mathbf{b}.
\end{aligned}$$

The first term of the right hand side of (1.44) is the $\mathbf{E} \times \mathbf{B}$ drift with a finite Larmor radius effect, the second is the polarization drift, the third is the grad- B drift and the last is the curvature drift. The $\mathbf{E} \times \mathbf{B}$ drift is independent on charge, driving the electrons and ions in the same direction, while the grad- B and curvature drifts and the polarization drift depend on charge to drive the electrons and ions in the opposite directions, resulting in the charge separation to build up a potential. Since the polarization drift is an inertial effect causing a retarded response to the oscillating electric field, it is negligible for the electrons.

In a simple torus configuration of magnetic fields with a constant radius of curvature $\mathbf{R}$ where

$$\nabla \times \mathbf{b} = \frac{\mathbf{R} \times \mathbf{B}}{R^2 B}, \quad \frac{\nabla B}{B} = -\frac{\mathbf{R}}{R^2}, \tag{1.45}$$

the grad- B drift and curvature drift are strengthened each other as is given by

$$\frac{v_\perp^2}{2\Omega} \frac{\mathbf{B} \times \nabla B}{B^2} + \frac{v_\parallel^2}{\Omega} \nabla \times \mathbf{b} = \frac{mc}{e} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right) \frac{\mathbf{R} \times \mathbf{B}}{R^2 B^2}, \tag{1.46}$$

and drive a charge separation. The resultant electric field carries the plasma to the wall by the $\mathbf{E} \times \mathbf{B}$ drift. Thus, a simple torus cannot confine plasmas, and magnetic field lines have to be toroidally wound so as to cancel the charge separation, which is called the rotational transform.

1.7.2 Gravitational Drift

When some other forces are applied to plasmas, the expression of the $E \times B$ drift is used by replacing $e\mathbf{E}$ with the force $\mathbf{F}$ to give a drift $\mathbf{v}_F$ as

$$\frac{c}{e} \frac{\mathbf{F} \times \mathbf{B}}{B^2}. \quad (1.47)$$

For the gravitational force, we have a gravitational drift

$$\mathbf{v}_G = \frac{mc}{e} \frac{\mathbf{g} \times \mathbf{B}}{B^2}. \quad (1.48)$$

The curvature drift is a drift due to the centrifugal force acting particles moving along a curved magnetic field line.

Under the gravitational force, electrons and ions drift in the opposite directions to create a charge separation which is a cause of instability. For a plasma stratified in a way that a high density plasma is on a low density plasma, a gravitational force drives the Rayleigh–Taylor instability. In the inertial fusion when the plasma is radiated by laser beams, at an ablation surface, a high density plasma pushes a low density plasma to trigger the Rayleigh–Taylor instability which makes a fuel surface undulated, resulting in a poor compression yield.

1.7.3 Magnetic Mirrors

1.7.3.1 Magnetic Moment

Suppose that q and p are variables of canonical conjugates in a periodic dynamical system. Then, an action integral J taken over one period

$$J = \oint p dq \quad (1.49)$$

is known to be an adiabatic invariant. Applying a slowly varying field to the system, the action integral is conserved although the motion becomes aperiodic. Since the action integral is related to a magnetic moment μ through

$$J_{\perp} = \oint p dq = \oint m v_{\perp} \rho_L d\theta = 2\pi m \rho_L v_{\perp} = 4\pi \frac{m}{|e|} \mu, \quad (1.50)$$

the magnetic moment is also conserved.

1.7.3.2 Trapping in Magnetic Mirror

For an axi-symmetric magnetic field we have

$$\frac{1}{r} \frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0, \quad (1.51)$$

which gives

$$B_r = -\frac{1}{2} r \frac{\partial B_z}{\partial z}. \quad (1.52)$$

Using the Lorentz force given by

$$F_{\parallel} = -\frac{e}{c} v_{\theta} B_r = \frac{e}{2c} v_{\theta} r \frac{\partial B_z}{\partial z} = -\mu \frac{\partial B_z}{\partial z}, \quad (1.53)$$

the equation of motion is written as

$$m \frac{dv_{\parallel}}{dt} = -\mu \frac{\partial B}{\partial z}. \quad (1.54)$$

Multiplying $v_{\parallel}$ to the both sides, we have

$$\frac{d}{dt} \left\{ \frac{m}{2} v_{\parallel}^2 + \mu B \right\} = \frac{d\mu}{dt} B. \quad (1.55)$$

Combining with the energy conservation

$$\frac{d}{dt} \left\{ \frac{m}{2} (v_{\parallel}^2 + v_{\perp}^2) \right\} = \frac{d}{dt} \left(\frac{m}{2} v_{\parallel}^2 + \mu B \right) = 0, \quad (1.56)$$

μ is shown as an invariant:

$$\frac{d\mu}{dt} = 0. \quad (1.57)$$

From this, when a particle moves from a place with the weak magnetic field to a place with the strong magnetic field, $v_{\perp}$ is increased since the magnetic moment is conserved and $v_{\parallel}$ is decreased from the energy conservation. When the magnetic field at the center is strong enough for $v_{\parallel}$ near the edge of the mirror field to be zero, then the particle is reflected. Thus, a particle is trapped in a magnetic mirror.

1.7.3.3 Loss Cone

From the conserved quantities

$$\mu = \frac{mv_{\perp}^2/2}{B}, \quad v^2 = v_{\parallel}^2 + \frac{2}{m} \mu B, \quad (1.58)$$

we have

$$v_{\parallel} = \pm \sqrt{v^2 - \frac{2}{m}\mu B}. \quad (1.59)$$

Putting the maximum value of the magnetic field to be B_M , the particles are reflected whose velocities satisfy

$$v^2 < \frac{2}{m}\mu B_M = \frac{v_{\perp}^2}{B} B_M. \quad (1.60)$$

Therefore, a trapping condition is given by

$$\frac{v_{\perp}^2}{v^2} > \frac{B}{B_M} = \sin^2 \theta_M. \quad (1.61)$$

Here, putting $v_{\perp} = v \sin \theta$, a particle satisfying $\theta < \theta_M$ is not trapped and escapes from the magnetic mirror. Particles in the region of a cone specified by θ_M in phase space are not trapped. This cone is called a loss cone. As a result, a mirror-confined plasma is not isotropic.

1.7.3.4 Magnetic Pumping

Since a magnetic moment is conserved in a magnetic mirror, if the magnetic field is slowly increased, the particle energy is also gradually increased through $W_{\perp} \propto B$. This is called a magnetic pumping.

1.7.3.5 Adiabatic Invariant $J_{\parallel}$

A particle trapped by a mirror field executes a periodic motion along the magnetic field. Therefore, $J_{\parallel} = m \int v_{\parallel} d\ell$ is the second invariant.

Denoting the length between the mirrors as ℓ and the average velocity of the particles as $\langle v_{\parallel} \rangle$, we have

$$J_{\parallel} = 2m \langle v_{\parallel} \rangle \ell. \quad (1.62)$$

From this if ℓ is shorten slowly, then $\langle v_{\parallel} \rangle$ is increased. This is proposed by Fermi to explain the acceleration of cosmic ray (Fermi acceleration).

1.8 Collective and Individual Motions

Equations of motion for plasma particles interacting with the Coulomb force are given by

$$\frac{d\mathbf{x}_{\alpha i}}{dt} = \mathbf{v}_{\alpha i}, \quad (1.63)$$

$$\frac{d\mathbf{v}_{\alpha i}}{dt} = -\frac{e_{\alpha}}{m_{\alpha}} \sum_{\beta} \sum_{j \neq i}^n \frac{\partial}{\partial \mathbf{x}_{\alpha i}} \frac{e_{\beta}}{|\mathbf{x}_{\alpha i} - \mathbf{x}_{\beta j}|}. \quad (1.64)$$

Defining the density by

$$\rho_\alpha(\mathbf{x}, t) = \sum_{i=1}^n \delta(\mathbf{x} - \mathbf{x}_{\alpha i}(t)) = \sum_j \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_{\alpha i}(t))} = \sum_{\mathbf{k}} \rho_\alpha(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (1.65)$$

where

$$\rho_\alpha(\mathbf{k}, t) = \sum_i e^{-i\mathbf{k} \cdot \mathbf{x}_{\alpha i}(t)}, \quad (1.66)$$

the equation of motion (1.64) is rewritten as

$$\begin{aligned} \frac{d\mathbf{v}_{\alpha i}}{dt} &= -\frac{e_\alpha}{m_\alpha} \sum_\beta e_\beta \sum_{\mathbf{q}} 4\pi \frac{i\mathbf{q}}{q^2} e^{i\mathbf{q} \cdot \mathbf{x}_{\alpha i}} \rho_\beta(\mathbf{q}, t) \\ &= \frac{e_\alpha}{m_\alpha} \sum_{\mathbf{q}} (-i\mathbf{q}) \phi(\mathbf{q}, t) e^{i\mathbf{q} \cdot \mathbf{x}_{\alpha i}}, \end{aligned} \quad (1.67)$$

where

$$\phi(\mathbf{q}, t) = \sum_\alpha \frac{4\pi e_\alpha}{q^2} \rho_\alpha(\mathbf{q}, t). \quad (1.68)$$

Equation (1.68) is nothing but the Poisson equation. With the help of (1.67), an equation for the density is written as

$$\begin{aligned} \frac{d^2}{dt^2} \rho_\alpha(\mathbf{k}, t) &= - \sum_j (\mathbf{k} \cdot \mathbf{v}_{\alpha j})^2 e^{-i\mathbf{k} \cdot \mathbf{x}_{\alpha j}(t)} - i \sum_j \mathbf{k} \cdot \frac{d\mathbf{v}_{\alpha j}}{dt} e^{-i\mathbf{k} \cdot \mathbf{x}_{\alpha j}(t)} \\ &= - \sum_j (\mathbf{k} \cdot \mathbf{v}_{\alpha j})^2 e^{-i\mathbf{k} \cdot \mathbf{x}_{\alpha j}(t)} \\ &\quad - \frac{e_\alpha}{m_\alpha} \left\{ n_\alpha k^2 \phi(\mathbf{k}, t) + \sum_{\mathbf{q} \neq \mathbf{k}} \mathbf{k} \cdot \mathbf{q} \rho_\alpha(\mathbf{k} - \mathbf{q}, t) \phi(\mathbf{q}, t) \right\}. \end{aligned} \quad (1.69)$$

For a high frequency collective motion to which the ions cannot respond, the potential is approximated from (1.68)

$$\phi(\mathbf{k}, t) \simeq -\frac{4\pi e}{k^2} \rho_e(\mathbf{k}, t), \quad (1.70)$$

by which (1.69) is reduced to

$$\begin{aligned} \left(\frac{d^2}{dt^2} + \omega_{pe}^2 \right) \rho_e(\mathbf{k}, t) &= - \sum_j (\mathbf{k} \cdot \mathbf{v}_{\alpha j})^2 e^{-i\mathbf{k} \cdot \mathbf{x}_{\alpha j}(t)} \\ &\quad + \frac{\omega_{pe}^2}{n_{e0}} \sum_{\mathbf{q} \neq \mathbf{k}} \frac{\mathbf{k} \cdot \mathbf{q}}{q^2} \rho_e(\mathbf{k} - \mathbf{q}, t) \rho_e(\mathbf{q}, t). \end{aligned} \quad (1.71)$$

where $\omega_{pe}^2 = 4\pi n e^2 / m_e$ is the electron plasma frequency.

The second term of the left hand side of (1.71) represents the collective motion, indicating that the electrons execute an organized motion (oscillation) characterized by a single frequency ω_{pe} . The first term of the right hand side of (1.71) includes the electron velocities which are averaged over the Boltzmann distribution to give

$$\sum_j (\mathbf{k} \cdot \mathbf{v}_j)^2 e^{-i\mathbf{k} \cdot \mathbf{r}_j} \sim \sum_j e^{-i\mathbf{k} \cdot \mathbf{r}_j} \int d\mathbf{v} (\mathbf{k} \cdot \mathbf{v})^2 F_{e0}(\mathbf{v}) = k^2 v_{Te}^2 \rho(\mathbf{k}, t), \quad (1.72)$$

where v_{Te} is the thermal velocity defined by $v_{Te}^2 = T_e/m_e$. This is the effect of the electron thermal motion and gives the dispersion of the wave as the collective motion.

The $\rho(\mathbf{k}, t)$ is a sum of vectors directing from the origin to the unit circle over the number of particles. The phase of each vector is given by $-\mathbf{k} \cdot \mathbf{r}_j$ and distributes randomly for a homogeneous plasma as long as $\mathbf{k} \neq 0$. Therefore, we may have $\langle \rho(\mathbf{k}, t) \rangle = 0$ where $\langle \dots \rangle$ is an average over the phases. The second term of the right hand side of (1.71) is neglected only when the amplitudes of fluctuations are small and $\rho(\mathbf{q}, t)$ and $\rho(\mathbf{k} - \mathbf{q}, t)$ are not correlated. The approximation

$$\langle \rho(\mathbf{q}, t) \rho(\mathbf{k} - \mathbf{q}, t) \rangle = 0, \quad (1.73)$$

is called the random phase approximation.

Thus, in the linear approximation, the equation for the density becomes

$$\frac{d^2}{dt^2} \rho(\mathbf{k}, t) + (\omega_{pe}^2 + k^2 v_{Te}^2) \rho(\mathbf{k}, t) = 0. \quad (1.74)$$

If we put $\rho(\mathbf{k}, t) \propto e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$, then we have for the dispersion relation

$$\omega^2 = \omega_{pe}^2 + k^2 v_{Te}^2. \quad (1.75)$$

In (1.75), ω_{pe}^2 denotes the coherent response of the electrons to the density fluctuation and $(k v_{Te})^2$ is the average contribution of the thermal motions to the frequency of the density fluctuation. The Debye length is a length by which the coherent electron motion (the phase velocity) and the thermal motion (the thermal velocity) are compatible:

$$k = \frac{\omega_{pe}}{v_{Te}} = \lambda_D^{-1} = k_D. \quad (1.76)$$

Thus, the collective motion is well defined when

$$k^2 < k_D^2. \quad (1.77)$$

For the short wave length $k^2 > k_D^2$, the collective motion is destructed by the thermal motion and the individual motions are the main feature of the plasma.

The collective motion derived above is called the electron plasma wave and is also called the Langmuir wave. The Langmuir wave is characterized by the finite frequency ω_{pe} in the limit $k \rightarrow 0$. When a dispersion relation is given by

$$\omega^2 = \omega_0^2 + \sum_p a_p k^p, \quad (1.78)$$

the linear wave amplitude in the long wave limit has $A(x, t)e^{i\omega_0 t}$. Therefore, the equation for the amplitude is given by

$$i2\omega_0 \frac{\partial}{\partial t} A(x, t) + \sum_p a_p \left(i \frac{\partial}{\partial x} \right)^p A(x, t) = 0. \quad (1.79)$$

For (1.79) to be a wave equation, p has to be even and is equal to or larger than 2. In this case, the dispersion is said strong. In fact the difference between the phase velocity $v_{ph} = \omega/k$ and the group velocity $v_g = d\omega/dk$ is large in the long wave limit. Waves with this property are called strongly dispersive waves.

The electron plasma frequency is given by

$$\omega_{pe} \sim 5.64 \times 10^4 \sqrt{n_0} \text{ [rad/sec]}, \quad f_{pe} = \frac{\omega_{pe}}{2\pi} = 9 \times 10^3 \sqrt{n_0}. \quad (1.80)$$

For a plasma with $n_0 \sim 10^{12} \text{ cm}^{-3}$, the electron plasma frequency is in a range of microwave frequencies, $f_{pe} \sim 9 \times 10^9 \text{ [1/sec]} = 9 \text{ [GHz]}$.

For a low frequency, collective motion to which the electrons respond statically, that is, from (1.70) and (1.74) with $d^2\rho/dt^2$ neglected $\rho_e(\mathbf{k}, t) \simeq e\phi(\mathbf{k}, t)/T_e$, then (1.68) is rewritten as

$$\phi(\mathbf{k}, t) = \frac{4\pi e}{k^2 + k_D^2} \rho_i(\mathbf{k}, t), \quad (1.81)$$

which is substituted into (1.69) to give

$$\left(\frac{d^2}{dt^2} + \frac{k^2 \omega_{pi}^2}{k^2 + k_D^2} + k^2 v_{Ti}^2 \right) \rho_i(\mathbf{k}, t) = 0, \quad (1.82)$$

where the nonlinear term is neglected. Noting for $T_i/T_e \ll 1$

$$\frac{k^2 \omega_{pi}^2}{k^2 + k_D^2} + k^2 v_{Ti}^2 = k^2 C_s^2 \left(\frac{1}{1 + k^2 \lambda_D^2} + \frac{T_i}{T_e} \right) \simeq \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2},$$

where $C_s = \sqrt{T_e/m_i} = \omega_{pi}/\lambda_D$, then (1.82) is reduced to

$$\left(\frac{d^2}{dt^2} + \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2} \right) \rho_i(\mathbf{k}, t) = 0. \quad (1.83)$$

The dispersion relation is given by

$$\omega^2 = \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2}. \quad (1.84)$$

This low frequency collective oscillation is called the ion acoustic wave. This is an ion plasma oscillation in which the ions are interacting with the screened Coulomb potential. This is seen by replacing the Fourier transform of the Coulomb potential energy e^2/k^2 with the screened Coulomb potential energy $e^2/(k^2 + k_D^2)$ as

$$\omega_{pi}^2 = \frac{4\pi n_0 e^2}{m_i} \rightarrow \frac{4\pi n_0}{m_i} \frac{e^2}{1 + k_D^2/k^2} = \frac{\omega_{pi}^2}{1 + k_D^2/k^2}. \quad (1.85)$$

When $k \sim 1/\lambda_D$, the ion acoustic wave tends to the ion plasma frequency for which the electron shielding is no longer effective.

The reason why C_s appears instead of the ion thermal velocity v_{Ti} in (1.84) is that the ions are driven by the potential for which the electrons are adiabatic, that is, the potential is balanced with the electron pressure.

For the ion acoustic wave, the dispersion is caused by the breaking of the charge neutrality. From (1.84), the frequency diminishes in the limit $k \rightarrow 0$ while the phase velocity remains constant C_s and the group velocity reduces to the phase velocity in the same limit. When a dispersion relation is given by

$$\omega = k \left(C + \sum_p a_p k^p \right), \quad (1.86)$$

the linear wave amplitude $A(x, t)$ obeys the equation

$$\left(\frac{\partial}{\partial \tau} + C \frac{\partial}{\partial x} \right) A(x, t) + \sum_p a_p \left(i \frac{\partial}{\partial \xi} \right)^{p+1} A(x, t) = 0. \quad (1.87)$$

For (1.87) to be a wave equation, p should be even and is equal to or larger than 2. In this case, the dispersion is said to be weak. In fact, the difference between the phase velocity and the group velocity is of the order of $(p+1)a_p k^p$ which is small in the long wave limit. Waves with this dispersion are called weakly dispersive waves.

The Langmuir wave and the ion acoustic wave are the counterparts of optical modes and acoustic modes in solids. Oscillations of one dimensional crystal with two kinds of atoms which are aligned by turn are described by

$$m \frac{d^2}{dt^2} u_{2n} = \beta(u_{2n+1} - 2u_{2n} + u_{2n-1}), \quad (1.88)$$

$$M \frac{d^2}{dt^2} u_{2n+1} = \beta(u_{2n+2} - 2u_{2n+1} + u_{2n}). \quad (1.89)$$

Putting the displacements from the equilibrium positions as

$$u_{2n} = \xi e^{i(\omega t + 2nka)}, \quad (1.90)$$

$$u_{2n+1} = \eta e^{i(\omega t + (2n+1)ka)}, \quad (1.91)$$

and substituting them into the above equations of motion, we have

$$-m\omega^2\xi = i2\beta\eta \sin ka - 2\beta\xi, \quad (1.92)$$

$$-M\omega^2\eta = i2\beta\xi \sin ka - 2\beta\eta, \quad (1.93)$$

which give a dispersion relation

$$\omega^2 = \beta \left(\frac{1}{m} + \frac{1}{M} \right)^2 \pm \beta \left\{ \left(\frac{1}{m} + \frac{1}{M} \right)^2 - \frac{4 \sin^2 ka}{mM} \right\}^{1/2}. \quad (1.94)$$

For $ka \ll 1$, we have two branches of modes

$$\omega^2 = 2\beta \left(\frac{1}{m} + \frac{1}{M} \right), \quad (1.95)$$

$$\omega^2 = \frac{2\beta}{M+m} k^2 a^2, \quad (1.96)$$

one is the optical mode in which two kinds of atoms oscillate out of phase and the other is the acoustic mode for which two kinds of atoms oscillate in phase. The Langmuir oscillation corresponds to the optical mode and the ion acoustic wave corresponds to the acoustic mode.

1.9 Wave-Particle Resonant Interaction

1.9.1 Landau Damping

Plasma particles are divided into two categories; one is adiabatic particles which oscillate many times within one oscillation period of a wave in the plasma and do not exchange energy with the wave in average. Therefore, the adiabatic particles support the wave itself. The other is resonant particles whose velocities are near to the phase velocity and exchange energy with the wave. The resonant particles with velocities little less than the phase velocity gain energy and those with velocities little faster than the phase velocity lose energy. In an equilibrium particle, velocities obey the Boltzmann distribution, and the number of particles whose velocities are little less than the phase velocity is larger than the number of particles whose velocities are little faster. Therefore, the net flow of energy from the wave to the particles is driven

to lead the wave damping. This is called the Landau damping and the damping rate is proportional to the derivative of the velocity distribution function at the phase velocity:

$$\gamma \propto \left(\frac{df}{dv} \right)_{v=\omega/k}. \quad (1.97)$$

When the main body of the particles resonates with the wave, in other words, the thermal velocity is equal to the phase velocity, the wave number equals to the Debye wave number. Therefore, the collective motion is absorbed by the thermal motion and the wave cannot be excited.

1.9.2 Kinetic Instability

Since a way of exchanging energy between particles and waves depends on the derivative of the particle velocity distribution function at the phase velocity, waves can grow for velocity distributions with a positive slope at the phase velocity. This is called the inverse Landau damping or the kinetic instability. There are several types of velocity distribution functions preferable for the kinetic instability. A bump on tail distribution is produced by launching a beam with a velocity near to the phase velocity. A shifted Maxwellian is realized in a current carrying plasma. An anisotropic velocity distribution is associated with the loss cone in magnetic mirror devices.

1.10 Dispersion and Wave Spreading

If there are no thermal motions, all the particles in a plasma oscillate coherently and support waves. The thermal motions give disturbances to the particle coherent response to the wave and as a result the group velocity depends on the wave number. This is related to the fact that the thermal particles are agitated randomly and therefore strongly affect short waves but not long waves.

When the group velocity depends on the wave number, the group velocity is different from the phase velocity and the wave packet starts to spread.

Suppose a packet of waves whose dispersion relation is given by

$$\omega = \omega(k), \quad (1.98)$$

and the wavenumbers are distributed around k_0 with a small spread:

$$\psi(x, t) = \sum_k \psi(k) e^{-i(kx - \omega(k)t)}, \quad \psi(k) = \psi_0 e^{-(k-k_0)^2/\delta^2}. \quad (1.99)$$

Putting the group velocity as $v_g = d\omega(k)/dk$, and expanding the frequency around k_0 , we have

$$\omega(k) = \omega(k_0) + v_g(k_0)(k - k_0) + \frac{1}{2} \left(\frac{dv_g}{dk} \right)_{k_0} (k - k_0)^2 + \dots \quad (1.100)$$

Here, changing summation over k into integral with $K = k - k_0$, we obtain

$$\begin{aligned} \psi(x, t) &= \psi_0 e^{-i(k_0 x - \omega_0 t)} \int \frac{dK}{2\pi} e^{-iK(x - v_g(k_0)t) + i \left(\frac{dv_g}{dk} \right)_{k_0} K^2 t / 2 - K^2 / \delta^2} \\ &= \frac{\psi_0}{2\sqrt{\pi}} \frac{e^{-i(k_0 x - \omega_0 t)}}{\left\{ \frac{1}{\delta^4} + \frac{t^2}{4} \left(\frac{dv_g}{dk} \right)_{k_0}^2 \right\}^{1/4}} \exp \left\{ - \frac{(x - v_g(k_0)t)^2}{\sqrt{\frac{1}{\delta^4} + \frac{t^2}{4} \left(\frac{dv_g}{dk} \right)_{k_0}^2}} \right\}, \end{aligned} \quad (1.101)$$

which shows that the wave packet propagates with a group velocity and spreads in time. The wave propagating with the phase velocity in the wave packet is a carrier wave.

1.11 Nonlinearity and Wave Steepening

Suppose an equation describing one dimensional nonlinear wave propagation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \quad (1.102)$$

Since the solutions are constant on the characteristics of the equation

$$\frac{d\xi}{dt} = u(\xi, t), \quad (1.103)$$

we may write the solution as

$$u(x, t) = u(x - \xi(t)) = u \left(x - \int u(\xi(t), t) dt \right). \quad (1.104)$$

This shows that the velocity of the wave is large where the amplitude is large and small where the amplitude is small. Therefore, the part with the large amplitude takes over the part with the small amplitude, that is, wave steepening occurs. This leads to a multiple valued profile. However, in reality, short waves excited by wave steepening are thermalized to damp by the dissipation.

The wave steepening is understood in a different way. The nonlinear term in (1.102) constantly creates the higher harmonics of the initially given wave and steepens the wave profile. This is a normal cascade of wavenumber spectrum.

There is an inverse cascade as well. For special types of dispersion relations which allow resonance couplings among waves, long waves are excited through nonlinearity to lead to structure formation.

1.12 Complexity in Plasmas

Complexity in nature is originated from nonlinearity. A linear system is characterized by a harmonic motion (an eigen state) of each degree of freedom of the system and is based on the principle of superposition. On the other hand, nonlinearity diversifies physical phenomena by integrating/mixing degrees of freedom to excite secondary flows, solitons, vortical motions, self-organized structures, chaos, turbulence, explosion, collapse, and so on.

Certainly, nonlinearity plays roles when a system deviates far from its equilibrium or a stationary state. Inhomogeneous or anisotropic states in space or velocity tend to homogeneous or isotropic states by releasing excess free energy which supports the inhomogeneity or anisotropy. Fluctuations are excited in these relaxation processes. Since there are many kinds of instabilities in plasmas, nonlinearity is a main feature and provides a variety of phenomena as is noted above. This is the reason why plasma physics could be a pioneer in the course of development of nonlinear physics.

Nonlinear problems are mostly not tractable analytically and solved numerically. Numerical simulations have revealed many notions such as solitons, chaos, and fractals which provide ideas to develop various kinds of nonlinear theories. Computer simulation is one of the most powerful tools to attack nonlinear problems not only in plasma but also other field of natural sciences.

Chapter 2

The Kinetic Theory of Plasmas

Since particles in a plasma interact with each other through the Coulomb force, their behaviors can be described by a set of Newton's equation of motion. Formally, dynamics of N particles is described by a trajectory in $6N$ dimensional phase space. We can define the distribution function in the phase space and derive the equation for the distribution function from the set of Newton's equation of motion. This is called the Klimontovich equation [7]. The Klimontovich distribution function is not smooth since the particle discreteness is kept. The Klimontovich equation is exact but not practical to be solved.

Instead of keeping all the dynamical information in the system, we average the Klimontovich equation over the discreteness to derive a continuous flow in 6 dimensional phase space. The flow in the phase space is described by a single particle distribution function which is continuous and no longer singular. The distribution function is determined not only by the self-consistent electromagnetic field which is produced by the distribution function itself but also the two point particle correlation function which represents collisional processes. In high temperature plasmas, collective motions are usually dominant and the collisional processes are often negligible. Such cases are described by the Vlasov equation [41] which simply neglect the collisional effects. Although the particle correlations are neglected, the Vlasov equation still includes wave-wave and wave-particle interactions which are origins of a variety of physical phenomena. The mathematical structure of the Vlasov equation is the same as that of the Klimontovich equation though the underlying notions are very different.

The kinetic theory is simple for deriving dispersion equations of waves in plasmas except for the cases that the magnetic field configuration is too complicated. At the same time, there appear many singularities related to various types of wave-particle resonances. Results depend on how the singularities are dealt. The Landau damping [42] is a good example. When the Landau damping was introduced, it was not accepted and regarded as a mathematical artifact. It took long time for the Landau damping to be recognized as a real process. van Kampen [43] developed a theory on an undamped stationary linear solution of the Vlasov equation and provided another view of the Landau damping. The notion introduced by van

Kampen was developed by introducing trapped particles to be applicable to large amplitude stationary waves by Bernstein, Greene, and Kruskal [44] and used by Dawson [45] to give an intuitive physical picture of the Landau damping. An experiment on the plasma echo by Malmberg and Wharton [46] was a direct proof of the Landau damping and van Kampen modes.

The important aspect brought by the Landau damping is kinetic instabilities. When resonant particles exist, kinetic instabilities are expected for velocity distribution functions, which have a positive slope at the phase velocities of modes concerned. The quasi-linear theory describes the development of a kinetic instability up to the saturation by taking account of the feedback effect of the wave growth on the origin of the instability. The feedback processes are sometime called quasi-linear effects and used in not only kinetic theories but also fluid theories.

In this chapter first, the Klimontovich equation and the Vlasov equation are derived. Then, the propagation properties of linear waves are studied with and without a magnetic field. After the Landau damping and plasma echo are explained, the van Kampen modes are discussed. The quasi-linear theory is formulated in the next section. Finally, the drift kinetic and gyrokinetic equations are derived.

2.1 Klimontovich Equation and Vlasov Equation

Dynamical equations for N charged particles are given by

$$\frac{d}{dt}\mathbf{x}_i = \mathbf{v}_i, \quad (2.1)$$

$$\frac{d}{dt}\mathbf{v}_i = -\frac{e_\alpha}{m_\alpha} \sum_{\beta} \sum_{j=1}^N \frac{\partial}{\partial \mathbf{x}_i} \frac{e_\beta}{|\mathbf{x}_i - \mathbf{x}_j|}, \quad (2.2)$$

where α denotes the species of the particles. Introducing a distribution function as

$$f_\alpha(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^N \delta(\mathbf{x} - \mathbf{x}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)), \quad (2.3)$$

the electric field produced in this particle system is given by

$$\hat{E}(\mathbf{x}, t) = -\nabla \sum_{\beta} \int \frac{e_\beta}{|\mathbf{x} - \mathbf{x}'|} f_\beta(\mathbf{x}', \mathbf{v}', t) d\mathbf{x}' d\mathbf{v}'. \quad (2.4)$$

The equation for the distribution function is given as follows:

$$\frac{\partial f_\alpha}{\partial t} = - \sum_{i=1}^N \left\{ \frac{d\mathbf{x}_i}{dt} \cdot \nabla + \frac{d\mathbf{v}_i}{dt} \cdot \frac{\partial}{\partial \mathbf{v}} \right\} \delta(\mathbf{x} - \mathbf{x}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t))$$

$$\begin{aligned}
&= - \sum_{i=1}^N \left\{ \mathbf{v}_i \cdot \nabla - \frac{e_\alpha}{m_\alpha} \sum_{\beta} \sum_{j=1}^N \left(\frac{\partial}{\partial \mathbf{x}_i} \frac{e_\beta}{|\mathbf{x}_i - \mathbf{x}_j|} \right) \cdot \frac{\partial}{\partial \mathbf{v}} \right\} \\
&\quad \times \delta(\mathbf{x} - \mathbf{x}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)) \\
&= -\mathbf{v} \cdot \nabla f_\alpha + \frac{e_\alpha}{m_\alpha} \nabla \sum_{\beta} \int \frac{e_\beta}{|\mathbf{x} - \mathbf{x}'|} f_\beta(\mathbf{x}', \mathbf{v}', t) d\mathbf{x}' d\mathbf{v}' \cdot \frac{\partial}{\partial \mathbf{v}} f_\alpha, \quad (2.5)
\end{aligned}$$

which leads to

$$\frac{\partial}{\partial t} f_\alpha + \mathbf{v} \cdot \nabla f_\alpha + \frac{e_\alpha}{m_\alpha} \hat{\mathbf{E}} \cdot \frac{\partial}{\partial \mathbf{v}} f_\alpha = 0. \quad (2.6)$$

Since this distribution function depends on the $6N$ initial values of the N particles, the equation for the distribution function is equivalent to $6N$ sets of Newton's equations of motion. It is noted that the electric field is also microscopic electric field depending on the initial conditions of the N particles. This is called the Klimontovich equation.

A macroscopic distribution of representatives in phase space is given by averaging the distribution function (2.3) over the probability distribution of the initial values. Putting the probability distribution function of the initial values as

$$P(\{\mathbf{x}_{i0}\}, \{\mathbf{v}_{i0}\}),$$

we define the macroscopic distribution function as

$$F_\alpha(\mathbf{x}, \mathbf{v}, t) = \int f_\alpha(\mathbf{x}, \mathbf{v}, t; \{\mathbf{x}_{i0}\}, \{\mathbf{v}_{i0}\}) P(\{\mathbf{x}_{i0}\}, \{\mathbf{v}_{i0}\}) \prod_{i=1}^N d\mathbf{x}_{i0} d\mathbf{v}_{i0}. \quad (2.7)$$

The distribution function F_α is now a smoothed function by smearing out the discreteness stemmed from the δ function related to the individual particle motions. The macroscopic electric field is defined by

$$\mathbf{E}(\mathbf{x}, t) = -\frac{\partial}{\partial \mathbf{x}} \sum_{\beta} \int \frac{e_\beta}{|\mathbf{x} - \mathbf{x}'|} F_\beta(\mathbf{x}', \mathbf{v}', t) d\mathbf{x}' d\mathbf{v}'. \quad (2.8)$$

Fluctuations from particle discreteness are given by

$$\delta f_\alpha = f_\alpha - F_\alpha, \quad (2.9)$$

$$\delta \mathbf{E} = \hat{\mathbf{E}} - \mathbf{E}. \quad (2.10)$$

Multiplying the Klimontovich equation with the probability distribution of the initial values and integrating over the initial values, we have

$$\frac{\partial}{\partial t} F_\alpha + \mathbf{v} \cdot \nabla F_\alpha + \frac{e_\alpha}{m_\alpha} \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}} F_\alpha = -\frac{e_\alpha}{m_\alpha} \left\langle \delta \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}} \delta f_\alpha \right\rangle, \quad (2.11)$$

where the right hand side describes physical processes resulting from the particle discreteness such as collisions, radiation, and absorption. In high temperature plasmas, the physical processes associated with the discreteness are order of $O(1/n\lambda_D^3)$ and can be neglected to give the Vlasov equation:

$$\frac{\partial}{\partial t} F_\alpha + \mathbf{v} \cdot \nabla F_\alpha + \frac{e_\alpha}{m_\alpha} \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}} F_\alpha = 0, \quad (2.12)$$

$$\mathbf{E}(\mathbf{x}, t) = -\frac{\partial}{\partial \mathbf{x}} \sum_\beta \int \frac{e_\beta}{|\mathbf{x} - \mathbf{x}'|} F_\beta(\mathbf{x}', \mathbf{v}', t) d\mathbf{x}' d\mathbf{v}'. \quad (2.13)$$

The Vlasov equation is characterized by collisionlessness and nonlinearity. The Vlasov equation does not include particle correlations but does include wave correlations. Thus, the Klimontovich equation and the Vlasov equation are mathematically expressed in the same forms but physically different.

In the above, the electrostatic field has been considered. An extension to the cases with electromagnetic fields is straightforward. The Vlasov equation obtained by the following replacement $\mathbf{E} \rightarrow \mathbf{E} + (\mathbf{v} \times \mathbf{B})/c$,

$$\frac{\partial}{\partial t} F_\alpha + \mathbf{v} \cdot \nabla F_\alpha + \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}} F_\alpha = 0, \quad (2.14)$$

has to be solved at the same time with the induction equation,

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (2.15)$$

Ampere's equation,

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \sum_\alpha e_\alpha \int \mathbf{v} F_\alpha(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \quad (2.16)$$

Poisson's equation

$$\nabla \cdot \mathbf{E} = 4\pi \sum_\alpha e_\alpha \int F_\alpha d\mathbf{v}, \quad (2.17)$$

and the equation for the divergence-free magnetic field

$$\nabla \cdot \mathbf{B} = 0. \quad (2.18)$$

The Vlasov equation has the following properties.

1. Invariant with respect to time reversal ($t \rightarrow -t$, $\mathbf{x} \rightarrow \mathbf{x}$, $\mathbf{v} \rightarrow -\mathbf{v}$)
2. Constant H function

Defining an H function for F_α obeying the Vlasov equation as

$$H = \int F_\alpha \log F_\alpha d\mathbf{x} d\mathbf{v},$$

we have

$$\frac{dH}{dt} = 0, \quad (2.19)$$

under the boundary condition $F \rightarrow 0$ for $\mathbf{x} \rightarrow \pm\infty$, $\mathbf{v} \rightarrow \pm\infty$. This means that there is no production and dissipation in the entropy of the Vlasov plasma.

2.2 Linear Electrostatic Wave Without External Magnetic Field

When a small amplitude fluctuation is excited in an unmagnetized plasma, the deviation from the thermal equilibrium is small and the distribution function is the sum of the equilibrium distribution $F_{\alpha 0}$ and the fluctuating part δF_α as

$$F_\alpha(\mathbf{x}, \mathbf{v}, t) = n_{\alpha 0} \{ F_{\alpha 0}(\mathbf{v}) + \delta F_\alpha(\mathbf{x}, \mathbf{v}, t) \}, \quad (2.20)$$

where $F_{\alpha 0}$ is given by

$$F_{\alpha 0}(\mathbf{v}) = \left(\frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} e^{-m_\alpha v^2 / 2T_\alpha}. \quad (2.21)$$

For the Vlasov equation in equilibrium without electromagnetic fields, any distribution function can formally be a solution. However, equilibrium is physically realized through collisions and usually the Maxwellian is taken as an equilibrium distribution function.

Substituting (2.20) into the Vlasov equation (2.12) without a magnetic field and linearizing the resultant equation, we have

$$\frac{\partial}{\partial t} \delta F_\alpha + \mathbf{v} \cdot \nabla \delta F_\alpha + \frac{e_\alpha}{m_\alpha} \mathbf{E} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} = 0, \quad (2.22)$$

$$\nabla \cdot \mathbf{E} = - \sum_\alpha 4\pi n_{\alpha 0} e_\alpha \int \delta F_\alpha d\mathbf{v}. \quad (2.23)$$

Putting $\delta F \propto e^{i(\mathbf{k}\mathbf{x} - \omega t)}$ and using the potential ϕ instead of the electric field $\mathbf{E}$, we have

$$\varepsilon(\mathbf{k}, \omega) \phi(\mathbf{k}, \omega) = 0, \quad (2.24)$$

where a dielectric function $\varepsilon(\mathbf{k}, \omega)$ is defined by

$$\varepsilon(\mathbf{k}, \omega) = 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \int \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} d\mathbf{v}. \quad (2.25)$$

The potential fluctuation is excited as a wave with the wavenumber $\mathbf{k}$ and the frequency ω which are related to each other through the equation

$$\varepsilon(\mathbf{k}, \omega) = 0. \quad (2.26)$$

Equation (2.26) is called a dispersion equation. Since the dielectric function has a singularity at $\omega = \mathbf{k} \cdot \mathbf{v}$ in the integrand, we have to specify a method to eliminate the divergence. Vlasov restricted the integral to the principal value. For $\mathbf{k} \cdot \mathbf{v} / \omega \ll 1$, we have

$$\begin{aligned} \varepsilon(\mathbf{k}, \omega) &= 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \int \frac{\wp}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} d\mathbf{v} \\ &= 1 - \sum_{\alpha} \omega_{p\alpha}^2 \int \frac{F_{\alpha 0}}{(\omega - \mathbf{k} \cdot \mathbf{v})^2} d\mathbf{v} \\ &\simeq 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2} \int \left(1 + 2 \frac{\mathbf{k} \cdot \mathbf{v}}{\omega} + 3 \left(\frac{\mathbf{k} \cdot \mathbf{v}}{\omega} \right)^2 + \dots \right) F_{\alpha 0} d\mathbf{v} \\ &\simeq 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2} \left(1 + 3 \frac{k^2 v_{T\alpha}^2}{\omega^2} \right). \end{aligned} \quad (2.27)$$

Thus, the high frequency normal mode is obtained as

$$\omega^2 = \omega_{pe}^2 (1 + 3k^2 \lambda_D^2). \quad (2.28)$$

For $v_{Ti} < \omega/k < v_{Te}$ we have

$$\begin{aligned} \varepsilon(\mathbf{k}, \omega) &= 1 - \frac{\omega_{pe}^2}{k^2} \int \frac{1}{\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{e0}}{\partial \mathbf{v}} d\mathbf{v} + \omega_{pi}^2 \int \frac{\wp}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{i0}}{\partial \mathbf{v}} d\mathbf{v} \\ &= 1 + \frac{1}{k^2 \lambda_D^2} - \frac{\omega_{pi}^2}{\omega^2}. \end{aligned} \quad (2.29)$$

The low frequency normal mode is given by

$$\omega^2 = \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2}. \quad (2.30)$$

2.3 Landau Damping

The Landau damping is a collisionless mechanism of wave damping appeared under the condition that the particles resonate with the wave. Landau thought that since the resonant particles satisfying $v = \omega/k$ exist though the number is small, instead of taking the principal value integral, an initial value problem should be solved to discuss what kind of waves an initial fluctuation develops into. He assumed that there is no fluctuation for $t < 0$ and an action $g(v)\delta(t)$ is added to produce a fluctuation at $t = 0$. Introducing the Laplace transformation

$$F_p(\mathbf{k}, v) = \int_0^\infty e^{-pt} F(\mathbf{k}, v, t) dt, \quad \Re p > 0, \quad (2.31)$$

and rewriting the Vlasov equation as

$$F_p(\mathbf{k}, v) = \frac{ig(v)}{ip - \mathbf{k} \cdot \mathbf{v}} + \sum_\alpha \frac{e_\alpha}{m_\alpha} \frac{1}{ip - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} \phi_p, \quad (2.32)$$

we have

$$\phi_p(\mathbf{k}) = \frac{4\pi n_0 e}{k^2 \varepsilon(\mathbf{k}, p)} \int \frac{g(v)}{ip - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}. \quad (2.33)$$

From this, the potential is obtained as

$$\begin{aligned} \phi(\mathbf{k}, t) &= \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} e^{pt} \phi_p(\mathbf{k}) dp \\ &= \frac{2n_0 e}{k^2} \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{e^{-i\omega t}}{\varepsilon(\mathbf{k}, \omega)} \left\{ \int \frac{g(v)}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v} \right\} d\omega. \end{aligned} \quad (2.34)$$

This describes how an initial fluctuation develops into a potential. In this expression, $\mathbf{k}$ and ω are not related in a definite way and for a given $\mathbf{k}$ an integration over all ω is taken. However, if there is no singularity in $g(v)$, an asymptotic form of the integral at large t is determined by a zero of $\varepsilon(\mathbf{k}, \omega)$. This is because

$$\int_{-\infty}^{\infty} e^{-ik \cdot vt} g(v) dv \rightarrow 0 \quad \text{for } t \rightarrow \infty. \quad (2.35)$$

From this putting

$$\varepsilon(\mathbf{k}, \omega(\mathbf{k})) = 0, \quad (2.36)$$

we have

$$\phi(\mathbf{k}, t) \propto e^{-i\omega(\mathbf{k})t} \quad \text{for } t \rightarrow \infty. \quad (2.37)$$

Thus, for large t a mode appears with an eigen frequency $\omega(\mathbf{k})$ determined by $\varepsilon(\mathbf{k}, \omega(\mathbf{k})) = 0$. In calculating $\varepsilon(\mathbf{k}, \omega)$, noting $\Im \omega > 0$ we have to put

$$\frac{1}{\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma} = \frac{\wp}{\omega - \mathbf{k} \cdot \mathbf{v}} - i\pi\delta(\omega - \mathbf{k} \cdot \mathbf{v}), \quad (2.38)$$

which makes the dielectric function complex

$$\varepsilon(\mathbf{k}, \omega) = 1 + \frac{\omega_p^2}{k^2} \int \frac{\wp}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_0}{\partial \mathbf{v}} d\mathbf{v} - i \frac{\omega_p^2}{k^2} \int \delta(\omega - \mathbf{k} \cdot \mathbf{v}) \mathbf{k} \cdot \frac{\partial F_0}{\partial \mathbf{v}} d\mathbf{v}. \quad (2.39)$$

The third term on the right hand side of (2.39) represents absorption of waves by the resonant particles. Using (2.21) and taking z axis of the velocity space along the wavenumber $\mathbf{k}$ to integrate over v_x and v_y , we have

$$\begin{aligned} \varepsilon(\mathbf{k}, \omega) &= 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \sqrt{\frac{m_{\alpha}}{2\pi T_{\alpha}}} \int \frac{kv_z}{kv_z - \omega + i\sigma} e^{-m_{\alpha}v_z^2/2T_{\alpha}} dv_z \\ &= 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2 v_{T\alpha}^2} W\left(\frac{\omega}{kv_{T\alpha}}\right), \end{aligned} \quad (2.40)$$

where $W(z)$ is the plasma dispersion function [47] and expressed by

$$\begin{aligned} W(z) &= \frac{1}{\sqrt{2\pi}} \int_C \frac{xe^{-x^2/2}}{x-z} dx \\ &= 1 - ze^{-z^2/2} \int_0^z e^{x^2/2} dx + i\sqrt{\frac{\pi}{2}} ze^{-z^2/2} \\ &= \begin{cases} i\sqrt{\frac{\pi}{2}} ze^{-z^2/2} + 1 - z^2 + \frac{z^4}{3} - \dots, & |z| < 1, \\ i\sqrt{\frac{\pi}{2}} ze^{-z^2/2} - \frac{1}{z^2} - \frac{3}{z^4} - \dots, & |z| \gg 1, \end{cases} \end{aligned} \quad (2.41)$$

For $kv_{T\alpha}/\omega \ll 1$, we have

$$\varepsilon(\mathbf{k}, \omega) = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2} \left(1 + 3 \frac{k^2 v_{T\alpha}^2}{\omega^2}\right) + i\sqrt{\frac{\pi}{2}} \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2 v_{T\alpha}^2} \frac{\omega}{kv_{T\alpha}} e^{-\omega^2/2k^2 v_{T\alpha}^2}. \quad (2.42)$$

Putting $\omega = \omega_k + i\gamma_k$ and assuming $|\gamma_k/\omega_k| \ll 1$, we approximate the dielectric function as

$$\varepsilon(\mathbf{k}, \omega) = \varepsilon(\mathbf{k}, \omega_k) + i \frac{\partial \varepsilon(\mathbf{k}, \omega_k)}{\partial \omega} \gamma_k. \quad (2.43)$$

Then, ω_k and γ_k are determined by

$$\Re \varepsilon(\mathbf{k}, \omega_k) = 0, \quad \gamma_k = - \left(\frac{\partial \varepsilon(\mathbf{k}, \omega_k)}{\partial \omega} \right)^{-1} \Im \varepsilon(\mathbf{k}, \omega_k). \quad (2.44)$$

Thus, we get

$$\omega_k^2 = \sum_{\alpha} \omega_{p\alpha}^2 \left(1 + 3 \frac{k^2 v_{T\alpha}^2}{\omega_k^2} \right) \simeq \omega_{pe}^2 \left(1 + 3 \frac{k^2 v_{Te}^2}{\omega_{pe}^2} \right), \quad (2.45)$$

and

$$\gamma = -\sqrt{\frac{\pi}{8}} \frac{\omega_k^4}{k^3 v_{Te}^3} e^{-\omega_k^2/2(kv_{Te})^2} \simeq -\sqrt{\frac{\pi}{8}} \frac{\omega_{pe}}{(k\lambda_D)^3} e^{-1/2(k\lambda_D)^2}. \quad (2.46)$$

For $kv_{Ti} \ll \omega \ll kv_{Te}$, we have

$$\varepsilon(\mathbf{k}, \omega) = 1 + \frac{1}{k^2 \lambda_D^2} - \frac{\omega_{pi}^2}{\omega^2} + i \sqrt{\frac{\pi}{2}} \frac{\omega_{pi}^2}{k^2 v_{Ti}^2} \frac{\omega}{kv_{Ti}} e^{-\omega^2/2k^2 v_{Ti}^2}, \quad (2.47)$$

yielding

$$\omega_k^2 = \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2}. \quad (2.48)$$

and

$$\gamma_k = -\sqrt{\frac{\pi}{8}} \frac{\omega_k^4}{k^3 v_{Ti}^3} e^{-\omega_k^2/2(kv_{Ti})^2} \simeq -\sqrt{\frac{\pi}{8}} \left(\frac{T_e}{T_i} \right)^{3/2} e^{-T_e/2T_i}. \quad (2.49)$$

2.4 Linear Electrostatic Wave with External Magnetic Field

Assuming a uniform magnetic field the Vlasov equation reads

$$(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}} F_{\alpha 0} = 0, \quad (2.50)$$

$$\frac{\partial}{\partial t} \delta F_{\alpha} + \mathbf{v} \cdot \nabla \delta F_{\alpha} + \frac{e_{\alpha}}{m_{\alpha} c} (\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}} \delta F_{\alpha} + \frac{e_{\alpha}}{m_{\alpha}} \mathbf{E} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} = 0, \quad (2.51)$$

$$\nabla \cdot \mathbf{E} = \sum_{\alpha} 4\pi e_{\alpha} n_{\alpha 0} \int \delta F_{\alpha} d\mathbf{v}. \quad (2.52)$$

From (2.50), the background distribution is a function of the perpendicular velocity $v_{\perp}$ and the parallel velocity $v_{\parallel}$, i.e.,

$$F_{\alpha 0}(\mathbf{v}) = F_{\alpha 0}(v_{\perp}, v_{\parallel}). \quad (2.53)$$

Using the Fourier transform of the fluctuation

$$\delta F_\alpha(\mathbf{x}, \mathbf{v}, t) = \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \delta F_\alpha(\mathbf{k}, \mathbf{v}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad (2.54)$$

putting the cylindrical coordinates as

$$\mathbf{v} = (v_\perp \cos \varphi, v_\perp \sin \varphi, v_\parallel), \quad \mathbf{k} = (k_\perp \cos \theta, k_\perp \sin \theta, k_\parallel),$$

equation (2.51) is rewritten as

$$\begin{aligned} & \left\{ i\Omega_\alpha \frac{\partial}{\partial \varphi} - \omega + k_\parallel v_\parallel + k_\perp v_\perp \cos(\varphi - \theta) \right\} \delta F_\alpha(\mathbf{k}, \mathbf{v}, \omega) \\ &= \frac{e_\alpha}{m_\alpha} \phi(\mathbf{k}, \omega) \left\{ k_\perp \cos(\varphi - \theta) \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right\} F_{\alpha 0}, \end{aligned} \quad (2.55)$$

where $\Omega_\alpha = e_\alpha B_0 / m_\alpha c$.

Introducing a function $H_m(\varphi)$ defined by

$$\left\{ i\Omega_\alpha \frac{\partial}{\partial \varphi} + k_\perp v_\perp \cos(\varphi - \theta) \right\} H_m(\varphi) = m\Omega_\alpha H_m(\varphi), \quad (2.56)$$

then we have

$$H_m(\varphi) = e^{-im\varphi + iZ_\alpha \sin(\varphi - \theta)} = \sum_{\ell} J_\ell(Z_\alpha) e^{-i(m-\ell)\varphi - i\ell\theta}, \quad (2.57)$$

where $Z_\alpha = k_\perp v_\perp / \Omega_\alpha$. $H_m(\varphi)$ forms a complete set:

$$\int H_m^*(\varphi) H_n(\varphi) d\varphi = \delta_{mn}, \quad (2.58)$$

for which the following identities have been used.

$$\sum_{\ell} J_{\ell+k}(z) J_{\ell-k}(z) = \delta_{k0}, \quad \sum_{\ell} J_{\ell+k+1}(z) J_{\ell-k}(z) = 0, \quad \sum_{\ell} J_{\ell}^2(z) = 1. \quad (2.59)$$

Therefore, $\delta F_\alpha(\mathbf{k}, \mathbf{v}, \omega)$ can be expanded in terms of $H_m(\varphi)$,

$$\delta F_\alpha(\mathbf{k}, \mathbf{v}, \omega) = \sum_m g_m(\mathbf{k}, \mathbf{v}, \omega) H_m(\varphi), \quad (2.60)$$

then (2.55) gives

$$g_m(\mathbf{k}, \mathbf{v}, \omega) = \phi(\mathbf{k}, \omega) \Pi_m(\mathbf{k}, \omega), \quad (2.61)$$

where

$$\Pi_m(\mathbf{k}, \omega) = \frac{e_\alpha}{m_\alpha} \frac{e^{im\theta}}{m\Omega_\alpha - (\omega - k_\parallel v_\parallel)} J_m(Z_\alpha) \left\{ \frac{m\Omega_\alpha}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right\} F_{\alpha 0}. \quad (2.62)$$

Substituting (2.60) into (2.52) with (2.61) and (2.62), we have

$$\varepsilon(\mathbf{k}, \omega) \phi(\mathbf{k}, \omega) = \left\{ 1 + \sum_\alpha \chi_\alpha(\mathbf{k}, \omega) \right\} \phi(\mathbf{k}, \omega) = 0, \quad (2.63)$$

where the susceptibility $\chi_\alpha(\mathbf{k}, \omega)$ is defined by

$$\chi_\alpha(\mathbf{k}, \omega) = \frac{\omega_\alpha^2}{2\pi k^2} \sum_m \int \frac{J_m^2(Z_\alpha)}{m\Omega_\alpha - (\omega - k_\parallel v_\parallel)} \left\{ \frac{m\Omega_\alpha}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right\} F_{\alpha 0} dv. \quad (2.64)$$

From (2.64), the resonance condition is given by

$$v_\parallel = \frac{\omega - m\Omega_\alpha}{k_\parallel}. \quad (2.65)$$

For $m = 0$, the resonance gives the Landau damping. For $m \neq 0$, the resonance condition is rewritten as $(v_\parallel - \omega/k_\parallel)/|\Omega_\alpha| = m/k_\parallel$ to show that the particle moves along the magnetic field during one cyclotron period to come back to the same phase of the wave, leading to the wave damping. This is called the cyclotron damping.

2.5 Plasma Wave Echo

Wave damping in a collisionless plasma is related to resonant particles and is not contradicted with the reversibility of the Vlasov equation. A typical example of this situation is a plasma wave echo.

A wave with the wave number $k = k_1$ launched at $t = t_1$ damps out by the Landau damping while the distribution function keeps the information of the wave launched at $t = t_1$:

$$F = F_0 + F(k_1, v, t_1) e^{-ik_1 v(t-t_1)} + \frac{e}{m} \int_{t_1}^t e^{-ik_1 v(t-t')} E(k_1, t') \frac{\partial F_0}{\partial v} dt' + \text{cc}. \quad (2.66)$$

When the second wave is added at $t = t_2$ after the first wave is damped, then the distribution function is given by

$$\begin{aligned}
F = F_0 + F(k_2, v, t_2) e^{-ik_2 v(t-t_2)} + \frac{e}{m} \int_{t_2}^t e^{-ik_2 v(t-t')} E(k_2, t') \frac{\partial F_0}{\partial v} dt' \\
+ F(k_1, v, t_1) \{e^{ik_1 v(t_2-t_1)} + e^{-ik_1 v(t_2-t_1)}\} \\
\times F(k_2, v, t_2) \{e^{i(k_2-k_1)v(t-t_2)} + e^{-i(k_2-k_1)v(t-t_2)}\} + \text{cc}, \quad (2.67)
\end{aligned}$$

which shows that even after the second wave is Landau damped the charge distribution given by

$$\rho = \int e^{ik_1 v(t-t_1)} e^{-ik_2 v(t-t_2)} F(k_1, v, t_1) F(k_2, v, t_2) dv, \quad (2.68)$$

which gives an echo of the electric field at the time

$$t = t_3 = \frac{k_2 t_2 - k_1 t_1}{k_2 - k_1}, \quad (2.69)$$

for which the phase of the charge distribution becomes zero. The plasma wave echo is regarded as a proof of the Landau damping. The wave damps by the phase mixing, but the information is preserved in the distribution function. The reversibility of the Vlasov equation is not broken.

2.6 van Kampen Mode

van Kampen [43] showed that the linearized Vlasov system has a stationary plane wave solution which consists of a modulated beam whose velocity is equal to the phase velocity of the wave and a polarization cloud produced by the interaction of the beam with the plasma particles.

We start with the linearized Vlasov equation for a plasma with the immobile ion background

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla f(\mathbf{x}, \mathbf{v}, t) + \frac{e}{m} \mathbf{E}(\mathbf{x}, t) \frac{\partial F_0(v)}{\partial v} = 0, \quad (2.70)$$

$$\nabla \cdot \mathbf{E}(\mathbf{x}, t) = 4\pi n_0 e \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \quad (2.71)$$

where $F_0(v)$ is the unperturbed velocity distribution function and $f(\mathbf{x}, \mathbf{v}, t)$ is the fluctuation of the velocity distribution function. In the following, we look for a solution of (2.70) in the form of

$$f(\mathbf{x}, \mathbf{v}, t) = \int d\mathbf{k} \int d\omega g_{\mathbf{k}, \omega}(\mathbf{v}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad (2.72)$$

and substitute it into (2.70) to obtain

$$-i(\omega - \mathbf{k} \cdot \mathbf{v})g_{k,\omega}(\mathbf{v}) = i\frac{\omega_{pe}^2}{k^2}\mathbf{k} \cdot \frac{\partial F_0(\mathbf{v})}{\partial \mathbf{v}} \int g_{k,\omega}(\mathbf{v}')d\mathbf{v}'. \quad (2.73)$$

Taking $v_z = (\mathbf{k} \cdot \mathbf{v})/k$ along the wavenumber vector $\mathbf{k}$ which is in the z direction and integrating (2.73) over $\mathbf{v}_\perp$ ($\mathbf{k} \cdot \mathbf{v}_\perp = 0$) under the isotropic distribution $F_0(\mathbf{v}) = F_0(|\mathbf{v}|)$, we have

$$(\omega - kv_z)g_{k,\omega}(v_z) = 2\pi\omega_{pe}^2 \frac{v_z}{k} F_0(v_z) \int_{-\infty}^{\infty} g_{k,\omega}(v'_z)dv'_z, \quad (2.74)$$

where we have used

$$\int_{-\infty}^{\infty} g_{k,\omega}(\mathbf{v})d\mathbf{v}_\perp = g_{k,\omega}(v_z),$$

and

$$\int_{-\infty}^{\infty} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} F_0(|\mathbf{v}|)d\mathbf{v}_\perp = kv_z \int_{v_z}^{\infty} \int_0^{2\pi} \frac{1}{v} \frac{dF_0(v)}{dv} v dv d\theta = -2\pi kv_z F_0(v_z).$$

Since $g(\mathbf{v})e^{i(kz-\omega t)}$ is the solution we look for, the integral $\int g(v_z)dv_z$ must not be zero so that it may be normalized to 1.

$$\int_{-\infty}^{\infty} g_{k,\omega}(\mathbf{v})d\mathbf{v}_\perp dv_z = \int_{-\infty}^{\infty} g_{k,\omega}(v_z)dv_z = 1. \quad (2.75)$$

The solution of (2.74) is given by

$$g_{k,\omega}(v_z) = 2\pi\omega_{pe}^2 \frac{v_z}{k} F_0(v_z) \left\{ \wp \frac{1}{\omega - kv_z} + \lambda(k, \omega) \delta(\omega - kv_z) \right\}, \quad (2.76)$$

where λ is an arbitrary function of k and ω . Equation (2.76) shows that the solution is expressed by a beam traveling with the phase velocity ω/k and the polarization cloud. The λ denotes the number of the resonant particles. Substituting (2.76) into the normalization condition, (2.75) determines λ through

$$2\pi \frac{\omega_{pe}^2}{k} \left\{ \wp \int \frac{v_z}{\omega - kv_z} F_0(v_z)dv_z + \frac{\lambda(k, \omega/k)}{|k|} \frac{\omega}{k} F_0\left(\frac{\omega}{k}\right) \right\} = 1. \quad (2.77)$$

Although (2.77) looks like the dispersion equation which determines ω in terms of k , since (2.77) includes two unknown variables ω and λ , the relation between k and ω is not uniquely determined. This implies that for a given k , ω is arbitrarily chosen, that is, ω is continuous. Thus, for an arbitrary ω , the density of the resonant particle λ can be chosen so as for the solution (2.76) to represent an undamped stationary wave.

Using the following replacements $\omega/k = u$ and $2\pi v_z F_0(v_z) = F(v_z)$, (2.76) and (2.77) are rewritten as

$$g_{k,u}(v_z) = \frac{\omega_{pe}^2}{k^2} F(v_z) \left\{ \wp \frac{1}{u - v_z} + \lambda(k, u) \delta(u - v_z) \right\}, \quad (2.78)$$

and with the normalization condition (2.75)

$$\wp \int_{-\infty}^{\infty} \frac{F(v_z)}{u - v_z} dv_z + \lambda(k, u) F(u) = \frac{k^2}{\omega_{pe}^2}. \quad (2.79)$$

Here, we decompose $F(u)$ into a positive frequency part $F_+(u)$ and a negative frequency part $F_-(u)$ by

$$F_{\pm}(u) = \frac{1}{2} F(u) \pm \frac{i}{2\pi} \wp \int \frac{F(v)}{u - v} dv \quad (2.80)$$

where $F_+(u)$ has an analytic continuation without singularities in the upper half of the complex u plane and $F_-(u)$ in the lower half plane. Note $F_-(v_z) = F_+^*(v_z)$. If we introduce

$$\delta_{\pm}(x) = \delta(x) \pm \frac{1}{\pi i} \wp \frac{1}{x}, \quad (2.81)$$

$F_{\pm}(v_z)$ is expressed as

$$F_{\pm}(v_z) = \frac{1}{2} \int \delta_{\pm}(v_z - u) F(u) du. \quad (2.82)$$

Note

$$F_+(u) + F_-(u) = F(u), \quad F_+(u) - F_-(u) = \frac{i}{\pi} \wp \int \frac{F(v)}{u - v} dv = F_*(u). \quad (2.83)$$

Then, (2.79) is rewritten as

$$-i\pi F_*(u) + \lambda(k, u) F(u) = \frac{k^2}{\omega_{pe}^2}. \quad (2.84)$$

Since $g_{k,u}(v_z)$ is complete as is shown by

$$\int g_{k,u}(v) g_{k,u}(v') du = \delta(v - v'), \quad (2.85)$$

the plane wave solution can be expanded with $g_{k,u}(v_z)$ by

$$f(z, v_z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(k, u) g_{k,u}(v_z) e^{ik(z-ut)} dk du, \quad (2.86)$$

where $C(k, u)$ is an expansion coefficient and is determined for a given initial condition $f(z, v_z, 0)$.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(k, u) g_{k,u}(v_z) e^{ikz} dk du = f(z, v_z, 0), \quad (2.87)$$

which is Fourier transformed to give

$$\int_{-\infty}^{\infty} C(k, u) g_{k,u}(v_z) du = \tilde{f}(k, v_z). \quad (2.88)$$

Substituting (2.78) into (2.88) to give with (2.83)

$$i\pi C_*(k, u) + \lambda(k, u)C(k, v_z) = \frac{k^2}{\omega_{pe}^2} \frac{\tilde{f}(k, v_z)}{F(v_z)}. \quad (2.89)$$

Eliminating $\lambda(k, u)$ in (2.84) and (2.89), we have

$$\tilde{f}(k, u) = Z(k, u)C_+(k, u) + Z^*(k, u)C_-(k, u), \quad (2.90)$$

where

$$Z(k, u) = 1 + i2\pi \frac{\omega_{pe}^2}{k^2} F_+(u). \quad (2.91)$$

Since the first term is analytic in the upper half plane and the second in the lower half plane, we may put

$$C_+(k, u) = \frac{\tilde{f}_+}{Z(k, u)}, \quad C_-(k, u) = \frac{\tilde{f}_-}{Z^*(k, u)}, \quad (2.92)$$

$$\tilde{f}_{\pm}(k, u) = \int dv_z f(k, v_z, 0) \delta_{\pm}(u - v_z). \quad (2.93)$$

Now, the solution for an initial condition $f(z, v_z, 0) = \int dz e^{ikz} \tilde{f}(k, v_z)$ is given by

$$\begin{aligned} f(z, v_z, t) &= \int dk \int du e^{ik(z-ut)} C(k, u) g_{k,u}(v_z) \\ &= \int dk \int du e^{ik(z-ut)} \left\{ \frac{\tilde{f}_+(k, u)}{Z(k, u)} + \frac{\tilde{f}_-(k, u)}{Z^*(k, u)} \right\} \\ &\quad \times \frac{\omega_{pe}^2}{k^2} F(v_z) \left\{ \wp \frac{1}{u - v_z} + \lambda(k, u) \delta(u - v_z) \right\} \\ &= \int dk \int du e^{ik(z-ut)} \left\{ \frac{\tilde{f}_+(k, u)}{Z(k, u)} + \frac{\tilde{f}_-(k, u)}{Z^*(k, u)} \right\} \end{aligned}$$

$$\begin{aligned}
& \times \frac{1}{2} \{ Z(k, v_z) \delta_+(u - v_z) + Z^*(k, v_z) \delta_-(u - v_z) \} \\
& = \int dk \int du e^{ik(z-ut)} \int dv'_z \tilde{f}(k, v'_z) \frac{1}{2} \left\{ \frac{\delta_+(u - v'_z)}{Z(k, u)} + \frac{\delta_-(u - v'_z)}{Z^*(k, u)} \right\} \\
& \quad \times \{ Z(k, v_z) \delta_+(u - v_z) + Z^*(k, v_z) \delta_-(u - v_z) \}, \tag{2.94}
\end{aligned}$$

where we have used (2.81).

Then, the density fluctuation is given by

$$\begin{aligned}
n(z, t) &= \int_{-\infty}^{\infty} f(z, v_z, t) dv_z \\
&= \int dk \int du e^{ik(z-ut)} \int dv'_z \tilde{f}(k, v'_z) \left\{ \frac{\delta_+(u - v'_z)}{Z(k, u)} + \frac{\delta_-(u - v'_z)}{Z^*(k, u)} \right\} \\
&= \int dk \int du e^{ik(z-ut)} \left\{ \frac{\tilde{f}_+(k, u)}{Z(k, u)} + \frac{\tilde{f}_-(k, u)}{Z^*(k, u)} \right\} \\
&= \int dk \int du e^{ik(z-ut)} \frac{\tilde{f}_+(k, u)}{Z(k, u)}. \tag{2.95}
\end{aligned}$$

where we have remained the first term because the second term is analytic in the lower half plane and damps for $t > 0$. When the initial perturbation is monochromatic, $\tilde{f}(k, u) = \tilde{f}(u) \delta(k - k_0)$, (2.95) becomes

$$n(z, t) = e^{ik_0 z} \int e^{-ik_0 u t} \frac{\tilde{f}_+(u)}{Z(k_0, u)} du. \tag{2.96}$$

Thus, the density fluctuation is represented by a superposition of waves with the same wave number but different phase velocities. When $\tilde{f}(u)$ is a mono-energetic beam $\delta(u - v_0)$, that is,

$$\tilde{f}_+(u) = \frac{1}{2} \int \delta_+(u - s) \tilde{f}(s) ds = \frac{1}{2} \delta(u - v_0) + \frac{i}{2\pi} \wp \frac{1}{u - v_0}, \tag{2.97}$$

we have

$$\begin{aligned}
n(z, t) &= e^{ik_0 z} \int_{-\infty}^{\infty} \frac{e^{-ik_0 u t}}{Z(k_0, u)} \frac{1}{2} \left\{ \delta(u - v_0) + \frac{i}{\pi} \wp \frac{1}{u - v_0} \right\} du \\
&= \frac{e^{ik_0(z-v_0 t)}}{2Z(k_0, v_0)} - \wp \int_{-\infty}^{\infty} \frac{1}{2\pi i} \frac{e^{ik_0(z-ut)}}{Z(k_0, u)(u - v_0)} du. \tag{2.98}
\end{aligned}$$

Here, the first term represents the fluctuation carried by the beam and the second term is the polarization cloud having the phase velocity near the beam velocity.

Thus, waves excited in a plasma can be expressed in terms of a superposition of van Kampen modes which are phase-mixed in time to the wave damping. This is

another view of the Landau damping. The damping is a result of the fact that ω is continuous.

A finite amplitude stationary solution of (2.70) is obtained by Bernstein, Green, and Kruskal [44]. In the stationary state, the general solution F_α of (2.70) is given by a function of the energy $E_\alpha = m_\alpha v^2/2 + e_\alpha \phi$ through

$$F_\alpha = F_\alpha(E_\alpha). \quad (2.99)$$

Once the wave potential is given, the particles are uniquely divided into trapped and untrapped particles depending on the energy. The untrapped particle distributions are determined by the boundary condition at $x \rightarrow \pm\infty$. The distribution functions of the trapped particles which are localized in space remain arbitrary. Thus, if the trapped particle distributions are specified, then the potential is determined by the Poisson equation. The BGK modes are continuously reduced to the van Kampen mode in the limit of the small amplitude for which only the resonance particles are trapped.

The BGK modes are essential to have the trapped particles which oscillate back and forth in the potential well (or trough). Therefore, if a perturbation applied to the BGK modes resonates with the trapped particles, they are driven detrapped and the BGK modes will be destroyed. In higher dimension systems, the motion of trapped particles is not easy to figure out and the BGK modes seem not likely stable in those systems.

2.7 Kinetic Instability and Quasi-Linear Theory

In a plasma whose velocity distribution function has a hump around a certain velocity, waves with phase velocities v_{ph} which are in the range of $\partial F_0(v_{ph})/\partial v > 0$ grow and then scatter the particles to deform the velocity distribution function, resulting in the suppression of the instability. This saturation mechanism is called the quasi-linear effect which is formulated in the following.

We start with the Vlasov equation

$$\frac{\partial F_\alpha}{\partial t} + \mathbf{v} \cdot \nabla F_\alpha + \frac{e_\alpha}{m_\alpha} \mathbf{E} \cdot \frac{\partial F_\alpha}{\partial \mathbf{v}} = 0, \quad (2.100)$$

$$\nabla \cdot \mathbf{E} = \sum_\alpha 4\pi n_0 e_\alpha \int F_\alpha d\mathbf{v}. \quad (2.101)$$

Splitting the distribution function and the electric field into the slowly changing part $F_{\alpha 0}$ and $\mathbf{E}_0$ and the fluctuating part $\tilde{f}_\alpha$ and $\tilde{\mathbf{E}}$ as

$$F_\alpha(\mathbf{x}, \mathbf{v}, t) = F_{\alpha 0}(\mathbf{v}) + \tilde{f}_\alpha(\mathbf{x}, \mathbf{v}, t), \quad \langle \tilde{f}_\alpha \rangle = 0, \quad (2.102)$$

$$\mathbf{E}(\mathbf{x}, t) = \tilde{\mathbf{E}}(\mathbf{x}, t), \quad \langle \tilde{\mathbf{E}} \rangle = 0, \quad (2.103)$$

where $\langle \dots \rangle$ is an average over time longer than the fluctuation oscillation period. Taking time averaging (2.100) and subtracting the resultant equation from (2.100), we get

$$\frac{\partial F_{\alpha 0}}{\partial t} + \mathbf{v} \cdot \nabla F_{\alpha 0} = -\frac{e_{\alpha}}{m_{\alpha}} \left\langle \tilde{\mathbf{E}} \cdot \frac{\partial \tilde{f}_{\alpha}}{\partial \mathbf{v}} \right\rangle, \quad (2.104)$$

$$\frac{\partial \tilde{f}_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f}_{\alpha} + \frac{e_{\alpha}}{m_{\alpha}} \tilde{\mathbf{E}} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} = -\frac{e_{\alpha}}{m_{\alpha}} \left\{ \mathbf{E} \cdot \frac{\partial F_{\alpha}}{\partial \mathbf{v}} - \left\langle \tilde{\mathbf{E}} \cdot \frac{\partial \tilde{f}_{\alpha}}{\partial \mathbf{v}} \right\rangle \right\}, \quad (2.105)$$

$$\nabla \cdot \tilde{\mathbf{E}} = \sum_{\alpha} 4\pi n_0 e_{\alpha} \int \tilde{f}_{\alpha} d\mathbf{v}. \quad (2.106)$$

The right hand side of (2.105) represents the correlation of the fluctuations which is supposed to be higher order in magnitude compared with the fluctuation itself. Therefore, we may put the right hand side of (2.105) as neglected. Substituting the Fourier transform of the fluctuation

$$\tilde{f}_{\alpha}(\mathbf{x}, \mathbf{v}, t) = \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} f_{\alpha}(\mathbf{k}, \mathbf{v}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (2.107)$$

into (2.105), we have

$$f_{\alpha}(\mathbf{k}, \mathbf{v}, \omega) = -\frac{e_{\alpha}}{m_{\alpha}} \frac{\phi(\mathbf{k}, \omega)}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}}. \quad (2.108)$$

Substituting (2.108) into the Poisson equation (2.101), we have

$$\varepsilon(\mathbf{k}, \omega) \phi(\mathbf{k}, \omega) = \left\{ 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \int \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} d\mathbf{v} \right\} \phi(\mathbf{k}, \omega) = 0. \quad (2.109)$$

Defining the real frequency of the wave by

$$\Re \varepsilon(\mathbf{k}, \omega_k) = 0, \quad (2.110)$$

equation (2.109) is rewritten as

$$\varepsilon(\mathbf{k}, \omega) \phi(\mathbf{k}, \omega) = \left\{ i \Im \varepsilon(\mathbf{k}, \omega_k) + \frac{\partial \varepsilon(\mathbf{k}, \omega_k)}{\partial \omega} (\omega - \omega_k) \right\} \phi(\mathbf{k}, \omega) = 0. \quad (2.111)$$

Transforming (2.111) back into time representation, we have

$$\frac{\partial \phi(\mathbf{k}, t)}{\partial t} = (-i\omega_k + \gamma_k) \phi(\mathbf{k}, t), \quad (2.112)$$

$$\gamma_k = -\frac{\Im \varepsilon(\mathbf{k}, \omega_k)}{\partial \varepsilon(\mathbf{k}, \omega_k) / \partial \omega} = \frac{\pi \omega_k^3}{2 k^2} \frac{k}{|k|} \left(\frac{\partial \bar{F}_{e0}}{\partial v} \right)_{v=\omega_k/k}, \quad (2.113)$$

where $\bar{F}_{e0}(v) = \int F_{e0}(\mathbf{v}) d\mathbf{v}_\perp$, $\mathbf{k} \cdot \mathbf{v}_\perp = 0$ and we have retained only the electron contribution. Multiplying $\phi^*(\mathbf{k}, t)$ to (2.112) and adding the conjugate of the resultant equation we obtain

$$\frac{\partial}{\partial t} |\phi(\mathbf{k}, t)|^2 = 2\gamma_k |\phi(\mathbf{k}, t)|^2. \quad (2.114)$$

Now, we consider the evolution of $F_{e0}(\mathbf{v}, t)$. Substituting (2.108) into (2.104) gives

$$\frac{\partial F_{e0}}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}(\mathbf{v}) \cdot \frac{\partial F_{e0}}{\partial \mathbf{v}}, \quad (2.115)$$

where the diffusion tensor D_{ij} is given by

$$D_{ij}(\mathbf{v}) = \left(\frac{e}{m_e} \right)^2 \sum_k k_i k_j |\phi(\mathbf{k}, t)|^2 \left\{ \wp \frac{\gamma_k}{(\omega_k - \mathbf{k} \cdot \mathbf{v})^2 + \gamma_k^2} + \pi \delta(\omega_k - \mathbf{k} \cdot \mathbf{v}) \right\}. \quad (2.116)$$

A set of (2.113), (2.114), and (2.115) describe the development of the instability to the saturation in a closed form. When the instability develops, the amplitude $|\phi(\mathbf{k}, t)|^2$ grows by (2.114) to scatter the resonant particles to flatten the hump in the velocity distribution by (2.115). The wave ceases to grow when a plateau is formed in the vicinity of the resonant region.

In (2.115), the delta function part indicates that the resonant particles are engaged in diffusion while the principal part denotes that the adiabatic particles are exchanging energy with the wave. If $\omega_k \gg \mathbf{k} \cdot \mathbf{v}$, (2.114) with the nonresonant part of (2.116) becomes

$$\begin{aligned} \frac{\partial F_{e0}}{\partial t} &= \left(\frac{e}{m_e} \right)^2 \sum_k \frac{\gamma_k |\phi(\mathbf{k}, t)|^2}{\omega_k^2} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \right)^2 F_{e0} \\ &= \left(\frac{e}{m_e} \right)^2 \sum_k \frac{\partial |\phi(\mathbf{k}, t)|^2}{\partial t} \frac{1}{2\omega_k^2} \left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \right)^2 F_{e0}, \end{aligned} \quad (2.117)$$

where we have used (2.114). Multiplying $m_e v^2$ to (2.117) and integrate over the velocity space

$$\frac{n_0 m_e}{2} \frac{\partial}{\partial t} \int v^2 F_{e0}(\mathbf{v}) d\mathbf{v} = \frac{\partial}{\partial t} \sum_k \frac{\omega_{pe}^2}{\omega_k^2} \frac{k^2 |\phi(\mathbf{k}, t)|^2}{8\pi}. \quad (2.118)$$

As is discussed in Chap. 4.4, the wave energy W is given by

$$W = \sum_{\mathbf{k}} \frac{\partial}{\partial \omega} (\omega \varepsilon(\mathbf{k}, \omega)) \frac{|\mathbf{E}(\mathbf{k}, \omega_{\mathbf{k}})|^2}{8\pi}, \quad (2.119)$$

which is a sum of the field energy and the average particle kinetic energy. This is because the adiabatic particles oscillate coherently with the wave to support the wave oscillation. Since for the electron plasma wave we have

$$\frac{\partial}{\partial \omega} (\omega \varepsilon(\mathbf{k}, \omega)) = 1 + \frac{\omega_{pe}^2}{\omega_{\mathbf{k}}^2}, \quad (2.120)$$

equation (2.118) is rewritten to confirm the above statement as

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \frac{n_0 m_e}{2} \int v^2 F_{e0}(v) dv + \sum_{\mathbf{k}} \frac{k^2 |\phi(\mathbf{k}, t)|^2}{8\pi} \right\} \\ = \frac{\partial}{\partial t} \sum_{\mathbf{k}} \frac{\partial}{\partial \omega} (\omega \varepsilon(\mathbf{k}, \omega)) \frac{k^2 |\phi(\mathbf{k}, t)|^2}{8\pi}. \end{aligned} \quad (2.121)$$

2.8 Drift Kinetic Theory

The Vlasov equation under an electromagnetic field is complicated to be solved. One way to overcome this complicacy is to use the fact that the magnetic moment is an adiabatic invariant.

In a strong magnetic field plasma particles are subject to the Larmor motion whose spatial extent (the Larmor radius ρ_L) may be often assumed much smaller than the characteristic scale length L of the phenomena we are concerned,

$$\varepsilon_D = \frac{\rho_L}{L} \ll 1. \quad (2.122)$$

In this case, the plasma motion is localized in the plane perpendicular to the magnetic field and is well described with the guiding center drift. If, at the same time, the time scale $1/\omega$ of the phenomena is much longer than the gyroperiod $1/\Omega$, which is the case for wide range of phenomena:

$$\frac{\omega}{\Omega} \sim \varepsilon_D \ll 1, \quad (2.123)$$

the distribution function and the electromagnetic field are supposed almost constant. Even when these conditions are fulfilled we may include the magnetic field inhomogeneity and curvature whose effects are sometimes more important than the

finite Larmor radius effects which are of the order of ρ_L/L . This is called the drift approximation which is based on the fact that the magnetic moment is conserved.

We start with a single particle Lagrangian in terms of the canonical variables $(\mathbf{q}, \mathbf{p})$

$$L(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}) = \mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t), \quad \dot{\mathbf{q}} = \frac{d\mathbf{q}}{dt}, \quad (2.124)$$

where the Hamiltonian is given by

$$H(\mathbf{q}, \mathbf{p}, t) = \frac{1}{2m}(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q}, t))^2 + e\phi(\mathbf{q}, t), \quad (2.125)$$

and $\mathbf{A}(\mathbf{q}, t)$ and $\phi(\mathbf{q}, t)$ are the vector and the electrostatic potentials, respectively. The Poisson brackets of the canonical variables are given by

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}.$$

With an arbitrary coordinate system $\mathbf{Z} = \mathbf{Z}(\mathbf{q}, \mathbf{p}, t)$, the Lagrangian is transformed into

$$L = \mathbf{p} \cdot \left(\sum_j \frac{\partial \mathbf{q}}{\partial \mathbf{Z}_j} \dot{\mathbf{Z}}_j + \frac{\partial \mathbf{q}}{\partial t} \right) - H = \mathbf{p} \cdot \sum_j \frac{\partial \mathbf{q}}{\partial \mathbf{Z}_j} \dot{\mathbf{Z}}_j - \left(H - \mathbf{p} \cdot \frac{\partial \mathbf{q}}{\partial t} \right). \quad (2.126)$$

The canonical variables $(\mathbf{q}, \mathbf{p})$ are transformed into the guiding center variables $(\mathbf{x}, \mathbf{v})$ through

$$\mathbf{q} = \mathbf{x} + \boldsymbol{\rho}, \quad \mathbf{p} = m\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t), \quad (2.127)$$

where $\mathbf{v}$ is the physical velocity and with a unit vector $\mathbf{b}$ parallel to the equilibrium magnetic field line

$$\boldsymbol{\rho} = \frac{w}{\Omega}\mathbf{a}, \quad \Omega = \frac{eB}{mc}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (2.128)$$

$$\mathbf{v} = u\mathbf{b} + w\mathbf{c}, \quad (2.129)$$

$$\mathbf{a} = \mathbf{e}_1 \cos \theta - \mathbf{e}_2 \sin \theta, \quad (2.130)$$

$$\mathbf{c} = -\mathbf{e}_1 \sin \theta - \mathbf{e}_2 \cos \theta, \quad (2.131)$$

$$\mathbf{a} = \mathbf{b} \times \mathbf{c}, \quad \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{b}. \quad (2.132)$$

The u and w are velocities parallel and perpendicular to the equilibrium magnetic field and note that $\mathbf{a}$ and $\mathbf{c}$ are oscillatory in time through the gyrophase (θ) dependence. Then, the Lagrangian is rewritten as

$$\begin{aligned} L = & \left\{ \frac{e}{c}\mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t) + m\mathbf{v} \right\} \cdot \dot{\mathbf{x}} + \frac{e}{c} \left(\dot{\mathbf{x}} \cdot \nabla \frac{w}{\Omega} \mathbf{a} \right) \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t) \\ & + \frac{e}{c} \frac{1}{\Omega} \mathbf{a} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t) \dot{w} + \left\{ \frac{mw^2}{\Omega} + \frac{e}{c} \frac{w}{\Omega} \mathbf{c} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t) \right\} \dot{\theta} \end{aligned}$$

$$- \left\{ \frac{m}{2}(u^2 + w^2) + e\phi(\mathbf{x} + \boldsymbol{\rho}, t) \right\}. \quad (2.133)$$

Then, the equations of motion are invariant under substitution $L \rightarrow L + dS/dt = L_D$ where S is a generating function of the transformation. From (2.133), the transformed Lagrangian becomes

$$\begin{aligned} L_D = & \left\{ \frac{e}{c} \mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t) + m\mathbf{v} + \frac{\partial S}{\partial \mathbf{x}} \right\} \cdot \dot{\mathbf{x}} + \frac{e}{c} \left(\dot{\mathbf{x}} \cdot \nabla \frac{w}{\Omega} \mathbf{a} \right) \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t) \\ & + \frac{\partial S}{\partial u} \dot{u} + \left\{ \frac{e}{c} \frac{1}{\Omega} \mathbf{a} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t) + \frac{\partial S}{\partial w} \right\} \dot{w} \\ & + \left\{ \frac{e}{c} \frac{w}{\Omega} \mathbf{c} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\rho}, t) + \frac{mw^2}{\Omega} + \frac{\partial S}{\partial \theta} \right\} \dot{\theta} \\ & - \left\{ \frac{m}{2}(u^2 + w^2) + e\phi(\mathbf{x} + \boldsymbol{\rho}, t) \right\} + \frac{\partial S}{\partial t}. \end{aligned}$$

If the generating function is chosen as

$$\frac{\partial S}{\partial w} + \frac{e}{c} \frac{1}{\Omega} \mathbf{a} \cdot \mathbf{A}(\mathbf{x} + \boldsymbol{\rho}) = 0, \quad (2.134)$$

the new Lagrangian L_D averaged over the gyrophase θ is obtained up to the first order of ε_D

$$\bar{L}_D = \frac{e}{c} \mathbf{A}^* \cdot \dot{\mathbf{x}} + \frac{mc}{e} \mu \dot{\theta} - \bar{H}_D, \quad (2.135)$$

where μ is the magnetic moment defined by $\mu = mw^2/2B$ and

$$\mathbf{A}^* = \mathbf{A}(\mathbf{x}, t) + \frac{mc}{e} u \mathbf{b}(\mathbf{x}, t), \quad (2.136)$$

$$\bar{H}_D = \frac{m}{2} u^2 + \mu B + e\phi. \quad (2.137)$$

The equations of motion are derived from

$$\frac{d}{dt} \left(\frac{\partial \bar{L}_D}{\partial \dot{Z}_i} \right) - \frac{\partial \bar{L}_D}{\partial Z_i} = 0, \quad (2.138)$$

where $\mathbf{Z} = \{\mathbf{x}, u, w, \theta\}$. For $Z_i = u, w, \theta$, we have $u = \mathbf{b} \cdot \dot{\mathbf{x}}$, $\dot{\theta} = \Omega$, and $d(mw^2/2\Omega)/dt = 0$ showing the magnetic moment is invariant; $d\mu/dt = 0$. For $\mathbf{x}$, we have

$$m \frac{d\mathbf{u}}{dt} \mathbf{b} + \mu \nabla B = e \left(\mathbf{E}^* + \frac{1}{c} \dot{\mathbf{x}} \times \mathbf{B}^* \right), \quad (2.139)$$

where

$$\begin{aligned}\mathbf{B}^* &= \nabla \times \mathbf{A}^* = \mathbf{B} + \frac{mc}{e} u \nabla \times \mathbf{b}, \\ \mathbf{E}^* &= -\nabla \phi - \frac{\partial \mathbf{A}^*}{\partial t} = \mathbf{E} - \frac{mc}{e} u \frac{\partial \mathbf{b}}{\partial t}.\end{aligned}$$

Equation (2.139) is solved to give with $B_{\parallel}^* = \mathbf{b} \cdot \mathbf{B}^*$

$$\dot{\mathbf{x}} = \frac{1}{B_{\parallel}^*} \left\{ u \mathbf{B}^* + \frac{c}{e} \mathbf{b} \times (\mu \nabla B - e \mathbf{E}^*) \right\}, \quad (2.140)$$

$$\dot{u} = -\frac{1}{m B_{\parallel}^*} \mathbf{B}^* \cdot (\mu \nabla B - e \mathbf{E}^*). \quad (2.141)$$

Equations (2.140) and (2.141) are the drift equations in which the grad-B and curvature drifts are included.

The Poisson brackets of the variables $\mathbf{Z}$ are given by

$$\begin{aligned}\{\mathbf{x}, \mathbf{x}\} &= \frac{c}{e B_{\parallel}^*} \mathbf{b} \times \mathbf{I}, \quad \{\mathbf{x}, u\} = \frac{\mathbf{B}^*}{m B_{\parallel}^*}, \quad \{\mathbf{x}, w\} = 0, \quad \{\mathbf{x}, \theta\} = 0, \\ \{u, w\} &= 0, \quad \{u, \theta\} = 0, \quad \{w, \theta\} = -\frac{\Omega}{mw}.\end{aligned} \quad (2.142)$$

The drift kinetic equation is derived from the Liouville theorem of the volume conservation. Under the transformation of the canonical coordinates $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ to the guiding center coordinates $\mathbf{Z} = (\mathbf{x}, u, \mu, \theta)$ with Jacobian $J(\mathbf{Z})$, we have

$$\int d\mathbf{z} F(\mathbf{z}, t) = \int d\mathbf{Z} J(\mathbf{Z}) f(\mathbf{Z}, t) = N, \quad (2.143)$$

where N is the total number of particles in the system. For the canonical variables, we have

$$\frac{\partial}{\partial \mathbf{z}} \cdot \{H, \mathbf{z}\} = 0,$$

which is the direct consequence of Hamilton's equation:

$$\dot{\mathbf{q}} = \{\mathbf{q}, H\} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = \{\mathbf{p}, H\} = -\frac{\partial H}{\partial \mathbf{q}}.$$

The corresponding equation for $\mathbf{Z}$ is given by

$$\frac{\partial}{\partial \mathbf{Z}} \cdot (J(\mathbf{Z}) \{\bar{H}_D, \mathbf{Z}\}) = 0. \quad (2.144)$$

The conservation of the phase space volume is written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, \bar{H}_D\} = \frac{\partial f}{\partial t} + \{\mathbf{Z}, \bar{H}_D\} \cdot \frac{\partial f}{\partial \mathbf{Z}} = 0. \quad (2.145)$$

Since $\{\dot{\mathbf{x}}, \dot{u}, \dot{\theta}\}$ do not depend on the gyrophase θ and $\dot{\mu} = 0$, we may assume $f(\mathbf{x}, u, \mu, \theta) = f(\mathbf{x}, u)$. Thus, the drift kinetic Vlasov equation is obtained as

$$\frac{\partial f}{\partial t} + \dot{u} \frac{\partial f}{\partial u} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0. \quad (2.146)$$

If we combine (2.144) with (2.146), we have

$$\frac{\partial}{\partial t} (J(\mathbf{Z}) f) = \frac{\partial}{\partial \mathbf{Z}} (J(\mathbf{Z}) \{\bar{H}_D, \mathbf{Z}\} f), \quad (2.147)$$

which is explicitly written as

$$\frac{\partial}{\partial t} (B_{\parallel}^* f) + \frac{\partial}{\partial u} (\dot{u} B_{\parallel}^* f) + \frac{\partial}{\partial \mathbf{x}} \cdot (\dot{\mathbf{x}} B_{\parallel}^* f) = 0, \quad (2.148)$$

where $J(\mathbf{Z}) = B_{\parallel}^*/m$ has been used.

2.9 Gyrokinetic Theory

When the condition (2.122) is relaxed, that is, an electromagnetic fluctuation with wavenumber comparable to ρ_L is excited, the magnetic moment is no longer constant under the guiding center approximation. Therefore, the guiding center variables are transformed into the gyro-center variables in order for the magnetic moment to be conserved, which is called the gyro-center approximation [48–51].

The electromagnetic fluctuation is assumed to be described by the vector potential $\mathbf{A}_1(\mathbf{q}, t)$ and the electrostatic potential $\phi(\mathbf{q}, t) = \phi_1(\mathbf{q}, t)$ whose amplitude is assumed of the order ε_G where

$$\frac{e\phi_1}{mv^2} \sim \frac{e|\mathbf{A}_1|}{mcv} \sim \varepsilon_G \ll 1. \quad (2.149)$$

The ratio of the frequency of the electromagnetic field ω to the cyclotron frequency ω/Ω is again assumed to be of the order of ε_D . In the following, we use $\varepsilon_G \sim \varepsilon_D \sim \varepsilon$.

The equilibrium magnetic field is given by $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ and the vector potential is expressed by

$$\mathbf{A}(\mathbf{q}, t) = \mathbf{A}_0(\mathbf{q}) + \mathbf{A}_1(\mathbf{q}, t).$$

A unit vector along the equilibrium magnetic field is taken as $\mathbf{b}$.

We again start with a single particle Lagrangian

$$L(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, t) = \mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t), \quad (2.150)$$

where the Hamiltonian is given by

$$H(\mathbf{q}, \mathbf{p}, t) = \frac{1}{2m} \left(\mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{q}, t) \right)^2 + e\phi(\mathbf{q}, t), \quad (2.151)$$

The canonical variables $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ are transformed into the guiding center variables $\mathbf{Z} = (\mathbf{x}, \mathbf{v}_0)$ through

$$\mathbf{q} = \mathbf{x} + \boldsymbol{\rho}, \quad \mathbf{p} = m\mathbf{v}_0 + \frac{e}{c} \mathbf{A}_0(\mathbf{x} + \boldsymbol{\rho}), \quad (2.152)$$

where $\mathbf{v}_0$ is related to the physical velocity $\mathbf{v}$ by

$$\mathbf{v}_0 = \mathbf{v} + \frac{e}{mc} \mathbf{A}_1(\mathbf{q}, t),$$

and $\boldsymbol{\rho}, \mathbf{a}, \mathbf{c}, \mathbf{e}_1, \mathbf{e}_2$ are given by (2.128) and (2.130)–(2.132). Instead of $\mathbf{v}$ defined by (2.129), we define $\mathbf{v}_0$ as

$$\mathbf{v}_0 = u\mathbf{b} + w\mathbf{c}. \quad (2.153)$$

The reason for using $\mathbf{v}_0$ instead of the physical velocity $\mathbf{v}$ is to separate the electromagnetic field from the equilibrium magnetic field and to stuff it into the Hamiltonian. The Lagrangian is rewritten as

$$\begin{aligned} L = & \left\{ \frac{e}{c} \mathbf{A}_0(\mathbf{x} + \boldsymbol{\rho}) + m\mathbf{v}_0 + \frac{e}{c} \nabla \frac{w}{\Omega} \mathbf{a} \cdot \mathbf{A}_0(\mathbf{x} + \boldsymbol{\rho}) \right\} \cdot \dot{\mathbf{x}} \\ & + \frac{e}{c} \frac{1}{\Omega} \mathbf{a} \cdot \mathbf{A}_0(\mathbf{x} + \boldsymbol{\rho}) \dot{w} + \left\{ \frac{mw^2}{\Omega} + \frac{e}{c} \frac{w}{\Omega} \mathbf{c} \cdot \mathbf{A}_0(\mathbf{x} + \boldsymbol{\rho}) \right\} \dot{\theta} \\ & - \left\{ \frac{m}{2} (u^2 + w^2) + e\phi_1(\mathbf{x} + \boldsymbol{\rho}, t) - \frac{e}{c} \mathbf{v}_0 \cdot \mathbf{A}_1(\mathbf{x} + \boldsymbol{\rho}, t) \right. \\ & \left. + \frac{e^2}{2mc^2} |\mathbf{A}_1(\mathbf{x} + \boldsymbol{\rho}, t)|^2 \right\}. \end{aligned} \quad (2.154)$$

The Lagrangian is transformed under the generating function (2.134) into

$$L_D = \frac{e}{c} \mathbf{A}_0^* \cdot \dot{\mathbf{x}} + \frac{mc}{e} \mu \dot{\theta} - H_D, \quad (2.155)$$

where the Hamiltonian is

$$H_D = \frac{m}{2} u^2 + \mu B_0 + e\psi(\mathbf{x} + \boldsymbol{\rho}, t), \quad (2.156)$$

and the effective potential ψ is given by

$$\psi(\mathbf{x}, t) = \phi_1(\mathbf{x}, t) - \frac{1}{c}(\mathbf{u}\mathbf{b} + \mathbf{w}\mathbf{c}) \cdot \mathbf{A}_1(\mathbf{x}, t) + \frac{e}{2mc^2} |\mathbf{A}_1(\mathbf{x}, t)|^2. \quad (2.157)$$

Now, the fluctuation is packed into the Hamiltonian and because of the fluctuation with the gyrophase, the magnetic moment is not conserved. However, as is shown by Cary [52], the Lie transformation provides systematically an invariant magnetic moment in every order of ε .

We consider a transformation from the guiding center coordinates to the gyro-center coordinates $\mathbf{Z} \rightarrow \bar{\mathbf{Z}}(\mathbf{Z}, t, \varepsilon)$ and an operator T which evaluates any function g at the transformed point $\bar{\mathbf{Z}}$ and creates a new function G at the original point $\mathbf{Z}$ as

$$G(\mathbf{Z}, t) = T(t, \varepsilon)g(\mathbf{Z}) = g(\bar{\mathbf{Z}}(\mathbf{Z}, t, \varepsilon)). \quad (2.158)$$

If g is the identity function $g(\mathbf{Z}) = \mathbf{Z}$, then we have

$$\bar{\mathbf{Z}} = T\mathbf{Z}. \quad (2.159)$$

On the other hand, we may consider a function $W(\bar{\mathbf{Z}}, t, \varepsilon)$ which generates $\bar{\mathbf{Z}}$ through

$$\frac{d\bar{\mathbf{Z}}}{d\varepsilon} = \{\bar{\mathbf{Z}}, W(\bar{\mathbf{Z}}(\mathbf{Z}, t, \varepsilon), t, \varepsilon)\}, \quad (2.160)$$

where $\{\ , \}$ is the Poisson bracket. Then, the operator T is shown to be expressed in terms of the generating function W . If we introduce the Lie operator L

$$L_W = \{W, \ \}, \quad (2.161)$$

and differentiate (2.159) with respect to ε , we have

$$\frac{d\bar{\mathbf{Z}}}{d\varepsilon} = -TL_W, \quad (2.162)$$

giving a formal solution

$$T = \exp\left[-\int L_W(\varepsilon)d\varepsilon\right]. \quad (2.163)$$

When the evolution of $\mathbf{Z}$ is determined by a Hamiltonian $H(\mathbf{Z}, t, \varepsilon)$, the evolution of the transformed variables $\bar{\mathbf{Z}}$ is determined by a new Hamiltonian $K(\bar{\mathbf{Z}}, t, \varepsilon)$ as

$$\frac{d\bar{\mathbf{Z}}_i}{dt} = \{\bar{\mathbf{Z}}_i, K(\bar{\mathbf{Z}}, t, \varepsilon)\} = \frac{\partial \bar{\mathbf{Z}}_i}{\partial t} + \{\bar{\mathbf{Z}}_i, T^{-1}H(\bar{\mathbf{Z}}, t, \varepsilon)\}, \quad (2.164)$$

where $H(\mathbf{Z}, t, \varepsilon) = TH(\mathbf{z}, t, \varepsilon)$ has been used. If we put the new Hamiltonian as

$$K = T^{-1}H + R(\bar{\mathbf{Z}}, t, \varepsilon), \quad (2.165)$$

we have

$$\frac{\partial \bar{Z}_i}{\partial t} = \{\bar{Z}_i, R(\bar{Z}, t, \varepsilon)\}, \quad (2.166)$$

which is written in an operator form as is done for (2.162)

$$\frac{\partial T}{\partial t} = -TL_R, \quad L_R = \{R, \quad \}. \quad (2.167)$$

From (2.162) and (2.167), we have

$$\begin{aligned} \frac{\partial^2 T}{\partial t \partial \varepsilon} &= -\frac{\partial T}{\partial t} \{W, \quad \} - T \left\{ \frac{\partial W}{\partial t}, \quad \right\} = -T \{R, \{W, \quad \}\} - T \left\{ \frac{\partial W}{\partial t}, \quad \right\}, \\ \frac{\partial^2 T}{\partial \varepsilon \partial t} &= -\frac{\partial T}{\partial \varepsilon} \{R, \quad \} - T \left\{ \frac{\partial R}{\partial \varepsilon}, \quad \right\} = T \{W, \{R, \quad \}\} - T \left\{ \frac{\partial R}{\partial \varepsilon}, \quad \right\}. \end{aligned}$$

Equating $\partial^2 T / \partial t \partial \varepsilon = \partial^2 T / \partial \varepsilon \partial t$ we obtain

$$T \left\{ \frac{\partial R}{\partial \varepsilon} - \frac{\partial W}{\partial t} + \{R, W\}, \quad \right\} = 0,$$

which gives

$$\frac{\partial R}{\partial \varepsilon} + \{R, W\} = \frac{\partial W}{\partial t}. \quad (2.168)$$

A formal solution is

$$R = T^{-1} \int d\varepsilon T(\varepsilon) \frac{\partial W(\varepsilon)}{\partial t}.$$

The new Hamiltonian is given by

$$K(\bar{Z}, t, \varepsilon) = T^{-1} H(\bar{Z}, t, \varepsilon) + T^{-1} \int d\varepsilon T(\varepsilon) \frac{\partial W(\bar{Z}, t, \varepsilon)}{\partial t}. \quad (2.169)$$

We come to develop a perturbation theory. We expand W , L_W , T , H , and K as

$$\begin{pmatrix} W \\ L_W \\ T \\ H \\ K \end{pmatrix} = \sum_{n=0}^{\infty} \varepsilon^n \begin{pmatrix} W_{n+1} \\ L_{W_{n+1}} \\ T_n \\ H_n \\ K_n \end{pmatrix}, \quad (2.170)$$

with $T_0 = 1$. The reason that W is order less by one than the others is the integration over ε in (2.169). Substituting T and L_W into (2.162) gives

$$T_n = -\frac{1}{n} \sum_{m=0}^{n-1} T_m L_{W_{n-m}}. \quad (2.171)$$

In a similar way, we have with the help of the identity $TT^{-1} = 1$

$$T_n^{-1} = \frac{1}{n} \sum_{m=0}^{n-1} L_{W_{n-m}} T_m^{-1}, \quad T_0^{-1} = 1. \quad (2.172)$$

Differentiating (2.169) with respect to ε gives

$$\frac{\partial T}{\partial \varepsilon} K + T \frac{\partial K}{\partial \varepsilon} = \frac{\partial H}{\partial \varepsilon} + T \frac{\partial W}{\partial t},$$

leading with (2.162) to

$$\frac{\partial W}{\partial t} = \frac{\partial K}{\partial \varepsilon} - L_W K - T^{-1} \frac{\partial H}{\partial \varepsilon}. \quad (2.173)$$

Substituting (2.170) into (2.173) we obtain

$$\frac{\partial W_n}{\partial t} = nK_n - \sum_{m=0}^{n-1} L_{W_{n-m}} K_m - \sum_{m=1}^n m T_{n-m}^{-1} H_m, \quad W_0 = 0. \quad (2.174)$$

Noting

$$L_{W_n} K_0 = L_{W_n} H_0 = \{W_n, H_0\}, \quad T_0^{-1} H_n = H_n,$$

we get

$$D_0 W_n = n(K_n - H_n) - \sum_{m=1}^{n-1} (L_{W_{n-m}} K_m + m T_{n-m}^{-1} H_m), \quad (2.175)$$

where

$$D_0 W_n = \frac{\partial W_n}{\partial t} + \{W_n, H_0\}.$$

Up to the third order we have

$$D_0 W_1 = K_1 - H_1, \quad (2.176)$$

$$D_0 W_2 = 2(K_2 - H_2) - L_{W_1}(K_1 + H_1), \quad (2.177)$$

$$D_0 W_3 = 3(K_3 - H_3) - L_{W_1}(K_2 + 2H_2) - L_{W_2} \left(K_1 + \frac{1}{2} H_1 \right) - \frac{1}{2} L_{W_1}^2 H_1. \quad (2.178)$$

Now, we go back to (2.155) and (2.156) and transform from the guiding center coordinates $\mathbf{Z} = (x, u, \mu, \theta)$ to the gyro-center coordinates $\bar{\mathbf{Z}} = (\bar{X}, \bar{U}, \bar{\mu}, \bar{\theta})$ for which the magnetic moment is conserved under the electromagnetic field. In the following, we consider up to the first order of ε . We rewrite (2.156) as

$$H_D = H_0 + H_1, \quad \psi = \sum_{n=1} \varepsilon^n \psi_n,$$

$$H_0 = \frac{1}{2} m \bar{U}^2 + \mu B_0(\bar{X}), \quad H_1 = e \psi_1(\bar{\mathbf{Z}}, t),$$

and

$$H_1 = \langle H_1 \rangle + \tilde{H}_1, \quad \langle H_1 \rangle = \int d\bar{\theta} e \psi(\bar{\mathbf{Z}}, t), \quad \tilde{\psi}_1 = \psi_1 - \langle \psi_1 \rangle.$$

The generating function W_1 is obtained from (2.176) by putting $K_1 = \langle H_1 \rangle$, that is,

$$D_0 W_1 = \frac{\partial W_1}{\partial t} + \frac{\partial W_1}{\partial \bar{\mathbf{Z}}} \cdot \{\bar{\mathbf{Z}}, H_0\} \simeq \Omega \frac{\partial W_1}{\partial \bar{\theta}} = K_1 - H_1 = -e \tilde{\psi}_1, \quad (2.179)$$

which is integrated to give

$$W_1 = -\frac{1}{\Omega} \int d\bar{\theta} e \tilde{\psi}(\bar{\mathbf{Z}}). \quad (2.180)$$

The new Hamiltonian is given up to ε by

$$K = K_0 + K_1 = H_0 + e \langle \psi_1 \rangle. \quad (2.181)$$

The gyro-center variables and guiding center variables are related through (2.160) which is integrated to give

$$\mathbf{Z} = \bar{\mathbf{Z}} - \varepsilon \{\bar{\mathbf{Z}}, W_1(\bar{\mathbf{Z}})\} = \bar{\mathbf{Z}} - \varepsilon \sum_i \{\bar{\mathbf{Z}}, \bar{Z}_i\} \frac{\partial W_1}{\partial \bar{Z}_i}. \quad (2.182)$$

The gyro-center Lagrangian up to the first order of ε is given by

$$L_G = \frac{e}{c} \mathbf{A}(\bar{\mathbf{X}})^* \cdot \dot{\bar{\mathbf{X}}} + \bar{\mu} \dot{\bar{\theta}} - \left\{ \frac{1}{2} m \bar{U}^2 + \bar{\mu} B_0 + e \langle \psi_1 \rangle \right\}, \quad (2.183)$$

which is independent of the gyrophase $\bar{\theta}$, indicating $\bar{\mu}$ is constant. The Poisson brackets are obtained as

$$\{\bar{\mathbf{X}}, \bar{\mathbf{X}}\} = \frac{c}{e B_{0\parallel}^* (\bar{\mathbf{X}})} \mathbf{b}(\bar{\mathbf{X}}) \times \mathbf{I}, \quad \{\bar{\mathbf{X}}, \bar{U}\} = \frac{\mathbf{B}_0^*}{m B_{0\parallel}^*}, \quad \{\bar{\theta}, \bar{\mu}\} = \frac{e}{mc}, \quad (2.184)$$

where

$$\mathbf{B}_0^* = \nabla \times \mathbf{A}_0^*, \quad B_{0\parallel}^* = \mathbf{b} \cdot \mathbf{B}_0^*.$$

The equations of motion are derived from

$$\frac{d}{dt} \left(\frac{\partial L_G}{\partial \dot{\bar{Z}}_i} \right) - \frac{\partial L_G}{\partial \bar{Z}_i} = 0, \quad (2.185)$$

which gives

$$\dot{\bar{\mathbf{X}}} = \frac{1}{B_{0\parallel}^*} \left\{ \left(\bar{U} + \frac{e}{m} \frac{\partial}{\partial \bar{U}} \langle \psi_1 \rangle \right) \mathbf{B}_0^* + \frac{c}{e} \mathbf{b} \times (\bar{\mu} \nabla B_0 + e \nabla \langle \psi_1 \rangle) \right\}, \quad (2.186)$$

$$\dot{\bar{U}} = -\frac{1}{m B_{0\parallel}^*} \mathbf{B}_0^* \cdot (\bar{\mu} \nabla B_0 + e \nabla \langle \psi_1 \rangle), \quad (2.187)$$

$$\dot{\bar{\mu}} = 0, \quad (2.188)$$

$$\dot{\bar{\theta}} = \Omega + \frac{e^2}{mc} \frac{\partial}{\partial \bar{\mu}} \langle \psi_1 \rangle. \quad (2.189)$$

The explicit expression of the transformation from the guiding center coordinates to the gyro-center coordinates is obtained from (2.182) and (2.184)

$$\begin{aligned} \mathbf{x} &= \bar{\mathbf{X}} - \varepsilon \frac{c}{e B_{0\parallel}^*} \left[\mathbf{b} \times \nabla + \Omega \frac{\mathbf{B}_0^*}{B_0} \frac{\partial}{\partial \bar{U}} \right] W_1, \\ u &= \bar{U} + \varepsilon \frac{1}{m B_{0\parallel}^*} \mathbf{B}_0^* \cdot \nabla W_1, \\ \mu &= \bar{\mu} + \varepsilon \frac{e}{mc} \frac{\partial}{\partial \bar{\theta}} W_1, \\ \theta &= \bar{\theta} - \varepsilon \frac{e}{mc} \frac{\partial}{\partial \bar{\mu}} W_1. \end{aligned}$$

The higher order W_n is systematically obtained following the above procedure. The ponderomotive Hamiltonian is obtained from (2.177). The condition that the right hand side of (2.177) is independent of the gyrophase gives

$$K_2 = \langle H_2 \rangle + \frac{1}{2} \langle L_{W_1} (K_1 + H_1) \rangle = \langle H_2 \rangle + \frac{1}{2} \langle \{W_1, H_1\} \rangle. \quad (2.190)$$

The second term of the right hand side of (2.190) is the ponderomotive Hamiltonian.

The gyrokinetic equation is derived in the same way as the drift kinetic equation:

$$\frac{\partial g}{\partial t} + \dot{\bar{U}} \frac{\partial g}{\partial \bar{U}} + \dot{\bar{\mathbf{X}}} \cdot \frac{\partial g}{\partial \bar{\mathbf{X}}} = 0, \quad (2.191)$$

where the gyro-center distribution $g(\bar{\mathbf{Z}})$ is related to the guiding center distribution $f(\mathbf{Z})$ through

$$g(\bar{\mathbf{Z}}) = T^{-1} f(\mathbf{Z}). \quad (2.192)$$

Equation (2.191) is also written as

$$\frac{\partial}{\partial t}(B_{0\parallel}^* g) + \frac{\partial}{\partial \bar{U}}(\dot{\bar{U}} B_{0\parallel}^* g) + \frac{\partial}{\partial \bar{X}}(\dot{\bar{X}} B_{0\parallel}^* g) = 0. \quad (2.193)$$

The gyrokinetic equation is to be supplemented with the Poisson equation and the Amperè law. The poisson equation in canonical coordinates is expressed as

$$\nabla^2 \phi_1(\mathbf{q}, t) = 4\pi \sum_{\alpha} e_{\alpha} \int F_{\alpha}(\mathbf{q}', \mathbf{p}', t) \delta(\mathbf{q} - \mathbf{q}') d\mathbf{z}', \quad (2.194)$$

where α denotes particle species. Using (2.143), the Poisson equation is given in terms of the guiding center distribution function as

$$\nabla^2 \phi_1(\mathbf{q}, t) = 4\pi \sum_{\alpha} e_{\alpha} \int f_{\alpha}(\mathbf{Z}, t) \delta(\mathbf{X} - \mathbf{q} + \boldsymbol{\rho}_{\alpha}(\mathbf{Z})) J_{\alpha}(\mathbf{Z}) d\mathbf{Z}, \quad (2.195)$$

or is expressed by integration with respect to the gyro-center coordinates as

$$\nabla^2 \phi_1(\mathbf{q}, t) = 4\pi \sum_{\alpha} e_{\alpha} \int \{T g_{\alpha}(\bar{\mathbf{Z}}, t)\} \delta(\bar{\mathbf{X}} - \mathbf{q} + \bar{\boldsymbol{\rho}}_{\alpha}(\bar{\mathbf{Z}})) J_{\alpha}(\bar{\mathbf{Z}}) d\bar{\mathbf{Z}}, \quad (2.196)$$

where from (2.192) the guiding center distribution is explicitly given up to ε by

$$f(\mathbf{Z}) = T g(\bar{\mathbf{Z}}) = g(\bar{\mathbf{Z}}) - \varepsilon \{g(\bar{\mathbf{Z}}), W_1(\bar{\mathbf{Z}})\}.$$

The Amperè law is given up to ε in a similar way as

$$\begin{aligned} \nabla^2 A_1(\mathbf{q}, t) = & -\frac{4\pi}{c} \sum_{\alpha} e_{\alpha} \int \{v_{\alpha 0}(\mathbf{Z}) f_{\alpha}(\mathbf{Z}) - f_{\alpha 0}(\mathbf{Z}) \frac{e_{\alpha}}{m_{\alpha} c} A_1(\mathbf{q}, t)\} \\ & \times \delta(\mathbf{X} - \mathbf{q} + \boldsymbol{\rho}_{\alpha}(\mathbf{Z})) J_{\alpha}(\mathbf{Z}) d\mathbf{Z}, \end{aligned} \quad (2.197)$$

where $f_{\alpha 0}$ is a local Maxwellian distribution defined at each flux surface and assumed to satisfy $|f_{\alpha} - f_{\alpha 0}|/f_{\alpha 0} \sim \varepsilon$. The current responsible for the electromagnetic field is shown to be expressed in terms of the sum of the polarization and magnetization currents by Sugama [53, 54].

As is understood from the above derivation, the gyrokinetic equation describes low frequency turbulence by eliminating high frequency motions and fluctuations. Although the above derivation is restricted to order of ε , the higher order gyrokinetics is developed by Dubin [55], Sugama [53] and Brizard [56] who also showed the energy conservation in the higher order. Conservation laws are important for numerical simulations. The derivation of the gyrokinetic equation is standardized and is reviewed in detail by Brizard and Hahm [57] in which references are almost covered. The gyrokinetic equation has been intensively studied and successfully applied to investigate various kinds of phenomena in plasma turbulence and associated transport.

Chapter 3

The Fluid Theory of Plasmas

When the phase velocity of a wave excited in a plasma is larger than the thermal velocity, the main body of the plasma particles are out of resonance with the wave and the number of the resonant particles which exchange the energy with the wave is small. In this case, the plasma can be treated as a fluid. In the fluid plasma, the electron and ion fluids are supposed to be inter-penetrating each other. They interact through the electromagnetic field and exchange the momentum and energy through the collisions. Although transport coefficients based on collisional processes are given by the kinetic theory, transport coefficients associated with the waves excited in the plasma are determined in the frame work of the fluid theory.

In this chapter, we derive the fluid equations for plasmas by taking moments of the kinetic equation. With the help of the Langevin equation, we obtain collisional transport coefficients which are shown to be the direct results of the fluctuation–dissipation theorem. Then, we discuss some types of drifts which play important roles in the nonlinear physics of the plasmas. Finally, one fluid model to describe slow scale dynamical behavior of plasmas is introduced.

3.1 The Fluid Equations for Plasmas

The kinetic equation with contributions from particle discreteness is given by

$$\frac{\partial}{\partial t} F_{\alpha} + \mathbf{v} \cdot \nabla F_{\alpha} + \frac{e_{\alpha}}{m_{\alpha}} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial}{\partial \mathbf{v}} F_{\alpha} = C_{\alpha}, \quad (3.1)$$

where the collision term $C_{\alpha} = \sum_{\beta} C_{\alpha\beta}(F_{\alpha}, F_{\beta})$ satisfies the following relations if the ionization and dissociation are neglected.

$$\int C_{\alpha\beta} d\mathbf{v} = 0, \quad (3.2)$$

$$\int m_{\alpha} \mathbf{v} C_{\alpha\alpha} d\mathbf{v} = 0, \quad \int m_{\alpha} v^2 C_{\alpha\alpha} d\mathbf{v} = 0, \quad (3.3)$$

$$\int \{m_\alpha C_{\alpha\beta} + m_\beta C_{\beta\alpha}\} \mathbf{v} d\mathbf{v} = 0, \quad (3.4)$$

$$\int \{m_\alpha C_{\alpha\beta} + m_\beta C_{\beta\alpha}\} v^2 d\mathbf{v} = 0. \quad (3.5)$$

Introducing the density, the velocity, the pressure tensor, and the internal energy through

$$n_\alpha(\mathbf{x}, t) = \int F_\alpha(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \quad (3.6)$$

$$\mathbf{v}_\alpha(\mathbf{x}, t) = \frac{1}{n_\alpha(\mathbf{x}, t)} \int \mathbf{v} F_\alpha(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \quad (3.7)$$

$$\mathbf{P}_\alpha(\mathbf{x}, t)_{ij} = m_\alpha \int (\mathbf{v} - \mathbf{v}_\alpha)_i (\mathbf{v} - \mathbf{v}_\alpha)_j F_\alpha(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \quad (3.8)$$

$$U_\alpha(\mathbf{x}, t) = \frac{1}{2} m_\alpha \int (\mathbf{v} - \mathbf{v}_\alpha)^2 F_\alpha(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}, \quad (3.9)$$

we have for the equation of continuity with the help of (3.2)

$$\frac{\partial}{\partial t} n_\alpha + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0. \quad (3.10)$$

For the equation of motion we have

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{v}_\alpha + (\mathbf{v}_\alpha \cdot \nabla) \mathbf{v}_\alpha &= \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{1}{m_\alpha n_\alpha} \nabla \cdot \mathbf{P}_\alpha \\ &+ \frac{1}{n_\alpha} \int \mathbf{v} C_\alpha d\mathbf{v}. \end{aligned} \quad (3.11)$$

The stress tensor is given by the pressure p and the viscosity stress tensor $\boldsymbol{\pi}$ as

$$\mathbf{P}_{ij} = p \delta_{ij} - \pi_{ij}, \quad (3.12)$$

The viscosity appears when the neighboring parts of the fluid have different velocities, that is, when the fluid has shear in the velocity. Therefore the momentum transfer due to the viscosity is proportional to the first derivative of the velocity. For unmagnetized plasmas the viscosity stress tensor π_{ij} is the off-diagonal elements of the stress tensor and appears when the velocity distribution function deviates from the spherical symmetry. Note that $\pi_{ij} = \pi_{ji}$. Thus the viscosity tensor is written as

$$\pi_{ij} = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \left(\zeta - \frac{2}{3} \eta \right) \delta_{ij} \nabla \cdot \mathbf{v},$$

where the viscosity coefficients η and ζ are functions of thermodynamic quantities and in general are not constants. However, they do not change over the plasma and

can be regarded as constant. The η is estimated to be $\eta = nT\tau$ where τ is the collision time and ζ is zero for monatomic gases such as plasmas. Then, we have

$$\nabla \cdot \boldsymbol{\pi} = \eta \nabla^2 \mathbf{v} + \frac{1}{3} \eta \nabla \nabla \cdot \mathbf{v}. \quad (3.13)$$

The kinematic viscosity is defined by $\nu_\alpha = \eta_\alpha / (n_\alpha m_\alpha)$. For magnetized plasmas, the isotropy is violated and the viscosity term is given by Braginskii [58]. When the z axis is taken parallel to the magnetic field $\hat{\mathbf{z}}$, corresponding to (3.13), we have for the case of $\Omega\tau \gg 1$

$$\nabla_\perp \cdot \boldsymbol{\pi}_\perp = -\eta_m \nabla^2 \hat{\mathbf{z}} \times \mathbf{v}, \quad (3.14)$$

where the magnetized viscosity is given by $\eta_m = \eta / (\Omega\tau)$ with the cyclotron frequency Ω .

The friction term is given by the Krook approximation

$$\frac{1}{n_\alpha} \int \mathbf{v} C_\alpha d\mathbf{v} = \sum_\beta \frac{\mathbf{v}_\beta - \mathbf{v}_\alpha}{\tau_{\alpha\beta}}, \quad (3.15)$$

where $\tau_{\alpha\beta}$ is the collision time when the incident particle α collides with the target particle β and satisfies in order for (3.4) to hold

$$\frac{m_\alpha n_\alpha}{\tau_{\alpha\beta}} = \frac{m_\beta n_\beta}{\tau_{\beta\alpha}}. \quad (3.16)$$

Thus, the equation of motion is given by

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{v}_\alpha + (\mathbf{v}_\alpha \cdot \nabla) \mathbf{v}_\alpha &= \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{1}{m_\alpha n_\alpha} \nabla p_\alpha \\ &+ \nabla \cdot \boldsymbol{\pi} + \sum_\beta \frac{\mathbf{v}_\beta - \mathbf{v}_\alpha}{\tau_{\alpha\beta}}. \end{aligned} \quad (3.17)$$

For the energy equation, we have

$$\begin{aligned} \frac{\partial}{\partial t} \left(U_\alpha + \frac{1}{2} m_\alpha n_\alpha v_\alpha^2 \right) + \nabla \cdot \left\{ \left(U_\alpha + \frac{1}{2} m_\alpha n_\alpha v_\alpha^2 + p_\alpha \right) \mathbf{v}_\alpha \right. \\ \left. - \boldsymbol{\pi}_\alpha \cdot \mathbf{v}_\alpha + \mathbf{q}_\alpha \right\} - e_\alpha n_\alpha \mathbf{E} \cdot \mathbf{v}_\alpha &= Q_\alpha, \end{aligned} \quad (3.18)$$

where the heat flux $\mathbf{q}_\alpha$ and the heat generation Q_α are given by

$$\mathbf{q}_\alpha = \frac{1}{2} m_\alpha \int (\mathbf{v} - \mathbf{v}_\alpha)^2 (\mathbf{v} - \mathbf{v}_\alpha) F_\alpha d\mathbf{v}, \quad (3.19)$$

and

$$Q_\alpha = \frac{1}{2} m_\alpha \sum_\beta \int (\mathbf{v} - \mathbf{v}_\alpha)^2 C_{\alpha\beta} d\mathbf{v}. \quad (3.20)$$

Equation (3.18) is rewritten by using the equation of motion (3.17) and the equation of continuity (3.10) as

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) U_\alpha + (U_\alpha + p_\alpha)(\nabla \cdot \mathbf{v}_\alpha) + \nabla \cdot \mathbf{q}_\alpha \\ &= - \sum_i \sum_j \pi_{ij} \frac{\partial v_i}{\partial x_j} - \frac{m_\alpha n_\alpha}{\tau_{\alpha\beta}} \sum_\beta \mathbf{v}_\alpha \cdot (\mathbf{v}_\beta - \mathbf{v}_\alpha) + Q_\alpha. \end{aligned} \quad (3.21)$$

Here, we assume that the distribution function F_α deviates a little from the Maxwellian with the average velocity $\mathbf{v}_\alpha$ in d -dimensional velocity space which is expressed by

$$F_\alpha(\mathbf{x}, t) = n_\alpha(\mathbf{x}, t) \left(\frac{m_\alpha}{2\pi T_\alpha(\mathbf{x}, t)} \right)^{d/2} e^{-m_\alpha(\mathbf{v} - \mathbf{v}_\alpha(\mathbf{x}, t))^2 / 2T_\alpha(\mathbf{x}, t)}, \quad (3.22)$$

where n_α , T_α , and $\mathbf{v}_\alpha$ are spatially and temporally dependent. Then, we may neglect the stress tensor, friction terms, heat flux, and the heat generation in (3.21). Noting

$$U_\alpha = \frac{d}{2} n_\alpha T_\alpha, \quad p_\alpha = n_\alpha T_\alpha, \quad (3.23)$$

where d is the number of degrees of freedom, (3.21) becomes

$$\frac{d}{2} \left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) (n_\alpha T_\alpha) - \frac{d+2}{2} T_\alpha \left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) n_\alpha = 0, \quad (3.24)$$

or in a compact form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) \ln(p_\alpha n_\alpha^{-\gamma}) = 0, \quad (3.25)$$

which is called the adiabatic law with the specific heat ratio

$$\gamma = (d+2)/d. \quad (3.26)$$

This is valid when energy equipartition is established in d directions in a short time. In magnetic fields, the energy exchange between the perpendicular directions and the parallel direction is not adiabatic and we take $d = 2$, leading to $\gamma = 2$.

In summary, a plasma in which the several fluids of different species $\{\alpha\}$ are mutually interpenetrating is described by the following set of equations:

$$\frac{\partial}{\partial t} n_\alpha + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0, \quad (3.27)$$

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{v}_\alpha + (\mathbf{v}_\alpha \cdot \nabla) \mathbf{v}_\alpha &= \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{1}{m_\alpha n_\alpha} \nabla p_\alpha \\ &+ \nabla \cdot \boldsymbol{\pi} + \sum_{\beta} \frac{\mathbf{v}_\beta - \mathbf{v}_\alpha}{\tau_{\alpha\beta}}, \end{aligned} \quad (3.28)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) \ln(p_\alpha n_\alpha^{-\gamma}) = 0, \quad (3.29)$$

$$\nabla \cdot \mathbf{E} = 4\pi \sum_{\alpha} e_\alpha n_\alpha, \quad (3.30)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (3.31)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.32)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \sum_{\alpha} e_\alpha n_\alpha \mathbf{v}_\alpha + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}. \quad (3.33)$$

3.2 Collisional Transport in Plasmas

3.2.1 Collision Integral

Transport coefficients due to collisional processes are determined by the kinetic theory by taking account of collision terms. The complication arises from the long range nature of the Coulomb interaction for which the Boltzmann collision integral diverges for large distance, that is, small angle scattering is important in a plasma. Then, the differential cross-section for small angle scattering under the Coulomb force is given by Rutherford formula which gives a logarithmically divergent collision integral. The divergence is avoided by taking account of the screening effects. Landau [59] introduced the lower limit of the integral with respect to the scattering angle by physical intuition to give the Landau collision integral

$$\begin{aligned} C_\alpha &= \sum_{\beta} \frac{2\pi e_\alpha^2 e_\beta^2 \log \Lambda_c}{m_\alpha} \frac{\partial}{\partial \mathbf{v}} \cdot \int \frac{1}{|\mathbf{v} - \mathbf{v}'|} \left(\mathbf{I} - \frac{(\mathbf{v} - \mathbf{v}') : (\mathbf{v} - \mathbf{v}')}{(\mathbf{v} - \mathbf{v}')^2} \right) \\ &\times \left(\frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{v}'} \right) F_\alpha(\mathbf{v}) F_\beta(\mathbf{v}') d\mathbf{v}', \end{aligned} \quad (3.34)$$

where $\log \Lambda_c$ is the Coulomb logarithm.

By taking into account the collective effects responsible for the screening effects, Balescu [60], Lenard [61], and Guernsey [62] derived a collision integral separately which is called the Balescu–Lenard–Guernsey collision integral

$$C_\alpha = \sum_\beta \frac{2\pi e_\alpha^2 e_\beta^2}{m_\alpha} \frac{\partial}{\partial \mathbf{v}} \cdot \int \mathcal{Q}_{\alpha\beta}(\mathbf{v}, \mathbf{v}') \cdot \left(\frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}} - \frac{1}{m_\beta} \frac{\partial}{\partial \mathbf{v}'} \right) F_\alpha(\mathbf{v}) F_\beta(\mathbf{v}') d\mathbf{v}', \quad (3.35)$$

$$\mathcal{Q}_{\alpha\beta}(\mathbf{v}, \mathbf{v}') = \int \frac{\mathbf{k} : \mathbf{k}}{k^4} \frac{\delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'))}{|\varepsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} d\mathbf{k}. \quad (3.36)$$

The collision integrals have a form similar to that in the Fokker–Planck equation, dynamical friction term (the second term in (3.34) and (3.35)) and the diffusion term in velocity space (the first term in (3.34) and (3.35)).

Transport coefficients can be calculated by taking moments of the collision integral. However, calculations involved in determining the transport coefficients based on the collision integral are lengthy and complicated. An alternative way of looking for the transport coefficients is to utilize the Langevin equation.

3.2.2 The Langevin Equation

Particles in a plasma change their momentum by collisions. The collisions are stochastic processes and the momentum change due to the collisions is regarded equivalent to the picture that the particles are subject to random force. Therefore, the equation of motion is written as

$$m \frac{d\mathbf{v}}{dt} = e \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) + \mathbf{F}(t), \quad (3.37)$$

where $\mathbf{F}(t)$ is a stochastic variable representing a randomly fluctuating force. The collisions happen not continuously but discretely. If the time scale of physical processes concerned is longer than the one collision time, we may study the physical effects of the collisions by averaging the collision processes. Denoting the average over the time longer than the collision time by $\langle \cdots \rangle$ and introducing new variables by

$$\langle \mathbf{v} \rangle = \mathbf{u}, \quad \frac{m}{2} \langle \mathbf{v}^2 \rangle = U, \quad (3.38)$$

we obtain

$$m \frac{d\mathbf{u}}{dt} = e \left(\mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B} \right) + \langle \mathbf{F} \rangle, \quad (3.39)$$

$$\frac{dU}{dt} = e \mathbf{u} \cdot \mathbf{E} + \langle \mathbf{v} \cdot \mathbf{F} \rangle. \quad (3.40)$$

Using the collision time τ , the momentum change and the energy dissipation are expressed by

$$\langle \mathbf{F} \rangle = -\frac{m\mathbf{u}}{\tau}, \quad \langle \mathbf{v} \cdot \mathbf{F} \rangle = -\frac{U - U_0}{\tau}. \quad (3.41)$$

Therefore, we may put

$$\mathbf{F} = \langle \mathbf{F} \rangle + \delta \mathbf{F}, \quad \langle \delta \mathbf{F} \rangle = 0, \quad (3.42)$$

where $\langle \mathbf{F} \rangle$ is the friction force. Equation (3.37) is rewritten as

$$m \frac{d\mathbf{v}}{dt} = e \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) - \frac{m\mathbf{v}}{\tau} + \delta \mathbf{F}(t). \quad (3.43)$$

This is called the Langevin equation.

3.2.3 The Fluctuation–Dissipation Theorem

Suppose $\mathbf{E}$ and $\mathbf{B}$ are constant and $\mathbf{B}$ is in the z direction. Equation (3.43) is rewritten for the velocity components as

$$m \frac{dv_x}{dt} = eE_x + \frac{B}{c} v_y - \frac{mv_x}{\tau} + \delta F_x(t), \quad (3.44)$$

$$m \frac{dv_y}{dt} = eE_y - \frac{B}{c} v_x - \frac{mv_y}{\tau} + \delta F_y(t), \quad (3.45)$$

$$m \frac{dv_z}{dt} = eE_z - \frac{mv_z}{\tau} + \delta F_z(t). \quad (3.46)$$

Introducing a new variable by $V = v_x + iv_y$, $E = E_x + iE_y$, $\delta F = \delta F_x + i\delta F_y$, (3.44) and (3.45) now read

$$\frac{dV}{dt} = \frac{e}{m} E - \left(i\Omega + \frac{1}{\tau} \right) V + \frac{1}{m} \delta F, \quad (3.47)$$

where $\Omega = eB/mc$. Furthermore, if we replace

$$V = \frac{eE}{m} \left(i\Omega + \frac{1}{\tau} \right)^{-1} + \delta V, \quad (3.48)$$

then (3.47) is reduced to

$$\frac{d\delta V}{dt} = - \left(i\Omega + \frac{1}{\tau} \right) \delta V + \frac{1}{m} \delta F(t), \quad (3.49)$$

which is integrated to give

$$\delta V(t) = \delta V_0 e^{-(i\Omega + 1/\tau)t} + \frac{1}{m} \int_0^t dt' e^{-(i\Omega + 1/\tau)(t-t')} \delta F(t'). \quad (3.50)$$

Then, the energy is given by

$$\begin{aligned} \frac{m}{2} |\delta V(t)|^2 &= \frac{m}{2} |\delta V_0|^2 e^{-2t/\tau} + e^{-2t/\tau} \left\{ \int_0^t e^{(i\Omega + 1/\tau)t'} \delta V_0 \delta F^*(t') + \text{cc} \right\} \\ &+ \frac{1}{2m} e^{-2t/\tau} \int_0^t dt' \int_0^t dt'' e^{i\Omega(t'-t'') + (t'+t'')/\tau} \delta F(t') \delta F^*(t''), \end{aligned} \quad (3.51)$$

which is averaged over the time longer than the collision time to give

$$\begin{aligned} \frac{m}{2} \langle |\delta V(t)|^2 \rangle &= \frac{m}{2} |\delta V_0|^2 e^{-2t/\tau} \\ &+ \frac{1}{2m} e^{-2t/\tau} \int \int dt' dt'' e^{i\Omega(t'-t'') + (t'+t'')/\tau} \langle \delta F(t') \delta F^*(t'') \rangle. \end{aligned} \quad (3.52)$$

Assuming that the system is in a stationary state and the correlation time of the fluctuating force is zero, we may take

$$\langle \delta F_i(t') \delta F_j^*(t'') \rangle = A \delta_{ij} \delta(t' - t''), \quad (3.53)$$

and obtain

$$U = \frac{m}{2} |\delta V_0|^2 e^{-2t/\tau} + \frac{A\tau}{4m} (1 - e^{-2t/\tau})(1 - e^{-2t/\tau}), \quad (3.54)$$

which gives in the limit $t \rightarrow \infty$

$$A = \frac{4mU}{\tau}. \quad (3.55)$$

If the system is in the thermal equilibrium, since $U = T$ for two degrees of freedom, we have $A = 4mT/\tau$. Thus, we obtain the relation connecting the fluctuation and the dissipation

$$\sum_i \langle \delta F_i(t') \delta F_i^*(t'') \rangle = \frac{4mT}{\tau} \delta(t' - t''). \quad (3.56)$$

This is called the fluctuation–dissipation theorem. Since the thermal fluctuations are originally caused by the collisions, the correlation of the thermal fluctuations is naturally related to the dissipation. Certainly, we have

$$\langle \delta F_z(t') \delta F_z^*(t'') \rangle = \frac{2mT}{\tau} \delta(t' - t''). \quad (3.57)$$

3.2.4 Diffusion and Mobility

A diffusion coefficient is obtained in the following. Integrating (3.50) with respect to time gives the spatial displacement,

$$\delta R(t) = \int \delta V(t) dt = \frac{1}{m} \int_0^t dt' \int_0^t dt'' e^{-(i\Omega + 1/\tau)(t'-t'')} \delta F(t''),$$

where we have put $\delta V_0 = 0$. The spatial displacement correlation is estimated as

$$\begin{aligned} \langle \delta R_i(t) \delta R_j^*(t) \rangle &= \frac{1}{m^2} \int_0^t dt_1 \int_0^t dt_2 \int_0^{t_1} dt'_1 \int_0^{t_2} dt'_2 e^{-i(\Omega + 1/\tau)(t_1 - t'_1)} \\ &\quad \times e^{-(-i\Omega + 1/\tau)(t_2 - t'_2)} \langle \delta F_i(t'_1) \delta F_j^*(t'_2) \rangle. \end{aligned} \quad (3.58)$$

Here, using the correlation for the fluctuations

$$\langle \delta F_i(t'_1) \delta F_j^*(t'_2) \rangle = \frac{2mT}{\tau} \delta_{ij} \delta(t'_1 - t'_2),$$

and rearranging the integral as

$$\begin{aligned} &\int_0^t dt_1 \int_0^t dt_2 e^{-i\Omega(t_1 - t_2) - (t_1 + t_2)/\tau} \int_0^{t_1} dt'_1 \int_0^{t_2} dt'_2 e^{(t'_1 + t'_2)/\tau} \delta(t'_1 - t'_2) \\ &= \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-i\Omega(t_1 - t_2) - (t_1 + t_2)/\tau} \frac{e^{2t_2/\tau} - 1}{2/\tau} \\ &\quad + \int_0^t dt_1 \int_{t_1}^t dt_2 e^{-i\Omega(t_1 - t_2) - (t_1 + t_2)/\tau} \frac{e^{2t_1/\tau} - 1}{2/\tau} \\ &= \int_0^t dt_1 \int_0^{t_1} dt_2 \{ e^{-i\Omega(t_1 - t_2)} + e^{i\Omega(t_1 - t_2)} \} e^{-(t_1 + t_2)/\tau} \frac{e^{2t_2/\tau} - 1}{2/\tau}, \end{aligned}$$

we obtain

$$\begin{aligned} \langle \delta R_i(t) \delta R_j^*(t) \rangle &= \frac{2T}{m} \frac{\tau}{1 + \Omega^2 \tau^2} \delta_{ij} \left\{ t + \frac{\tau}{2} (1 - e^{-2t/\tau}) \right. \\ &\quad \left. - \frac{2\tau}{1 + \Omega^2 \tau^2} (1 - \tau e^{-t/\tau} (\Omega \sin \Omega t - \frac{1}{\tau} \cos \Omega t)) \right\} \end{aligned}$$

which gives the fluctuation–dissipation theorem in the limit $t \rightarrow \infty$ as

$$\frac{\langle \delta R_i \delta R_j^* \rangle}{2t} = \frac{T\tau}{m} \frac{1}{1 + \Omega^2 \tau^2} \delta_{ij}. \quad (3.59)$$

In a similar way, if we put

$$v_z = \frac{eE_z \tau}{m} + \delta v_z, \quad (3.60)$$

and $\delta z = \int \delta v_z dt$, we obtain

$$\frac{\langle (\delta z)^2 \rangle}{t} = \frac{T\tau}{m} = D, \quad (3.61)$$

which is equivalent to (3.59) with $B = 0$. Since the excursion length δz during the collision time is roughly given by $\delta z = v_T \tau$, the diffusion coefficient D is obtained as

$$D = \frac{(\delta z)^2}{\tau} = \frac{(v_T \tau)^2}{\tau} = \frac{T \tau}{m}, \quad (3.62)$$

to recover (3.61). The cross magnetic field diffusion coefficient is modified as

$$D_{\perp} = \frac{D}{1 + \Omega^2 \tau^2}. \quad (3.63)$$

In the small collision frequency case $\Omega \gg 1/\tau$, the diffusion is certainly affected by the magnetic field as is seen by

$$D_{\perp} \simeq \frac{D}{\Omega^2 \tau^2} = \frac{(v_T/\Omega)^2}{\tau} = \frac{r_L^2}{\tau}, \quad (3.64)$$

implying that the excursion length during the collision time is the Larmor radius rather than the mean free path. Since the ratio of the electron and the ion perpendicular diffusion coefficient is given by

$$\frac{D_{e\perp}}{D_{i\perp}} \sim \sqrt{\frac{m_e}{m_i}} \frac{T_e}{T_i} < 1, \quad (3.65)$$

implying the electrons are left behind the ions across the magnetic field. On the other hand along the magnetic field, the electrons diffuse fast and the ions lag behind the electrons.

$$\frac{D_e}{D_i} \sim \sqrt{\frac{m_i}{m_e}} \frac{T_e}{T_i} > 1. \quad (3.66)$$

The particle response to the electric field is given by (3.48) and (3.60)

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{e}{m} \frac{\tau^2}{\Omega^2 \tau^2} \begin{pmatrix} 1/\tau & \Omega \\ -\Omega & 1/\tau \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad v_z = \frac{e E_z \tau}{m}, \quad (3.67)$$

The diagonal elements of the matrix coefficient of the electric field in the right hand side of (3.67) are the mobility and the off-diagonal elements are the $E \times B$ drift. Thus, the mobility is given by

$$\mu_z = \frac{e \tau}{m}, \quad \mu_{\perp} = \frac{e}{m} \frac{\tau}{1 + \Omega^2 \tau^2}. \quad (3.68)$$

3.2.5 The Einstein Relation

In a plasma in a stationary state without a magnetic field, if the velocity is sufficiently small so that the term $\mathbf{v} \cdot \nabla \mathbf{v}$ in the equation of motion is neglected, (3.28)

reads

$$\mathbf{v}_\alpha = \frac{\tau_\alpha}{m_\alpha n_\alpha} \{e_\alpha n_\alpha \mathbf{E} - T_\alpha \nabla n_\alpha\} = \frac{e_\alpha}{|e_\alpha|} \mu_\alpha \mathbf{E} - D_\alpha \nabla \ln n_\alpha, \quad (3.69)$$

where the mobility μ_α and the diffusion coefficient D_α are defined as

$$\mu_\alpha = \frac{|e_\alpha| \tau_\alpha}{m_\alpha}, \quad D_\alpha = \frac{T_\alpha \tau_\alpha}{m_\alpha}. \quad (3.70)$$

Then, the flux is expressed by these coefficients

$$\mathbf{\Gamma}_\alpha = n_\alpha \mathbf{v}_\alpha = \frac{e_\alpha}{|e_\alpha|} \mu_\alpha n_\alpha \mathbf{E} - D_\alpha \nabla n_\alpha. \quad (3.71)$$

The relation between the mobility and the diffusion coefficient is given from (3.70)

$$\mu_\alpha = \frac{|e_\alpha| D_\alpha}{T_\alpha}, \quad (3.72)$$

which is called the Einstein relation. The mobility μ_α is the response of the plasma to the electric field and describes the effect of the dissipation, while the diffusion coefficient is to describe the average effect of the thermal fluctuations on the motion of the fluid elements. The thermal fluctuations are caused by collisions, which bring the friction force. Thus, the mobility and the diffusion coefficient are different aspects of the same physical process. This physical situation is formulated by the fluctuation–dissipation theorem which gives the relation directly in three dimension:

$$D_{ij} = \frac{\langle r_i r_j \rangle}{\tau} = \frac{\langle (v_i \tau)(v_j \tau) \rangle}{\tau} = \frac{v^2 \tau}{3} \delta_{ij} = \frac{T \tau}{m} \delta_{ij} = \frac{T}{e} \mu_{ij}. \quad (3.73)$$

In systems at thermal equilibrium, the net flux $\mathbf{J}$ is given by

$$J_i = \sum_j \left(n \mu_{ij} E_j - D_{ij} \frac{\partial}{\partial x_j} n \right). \quad (3.74)$$

Since the density and the electric field are given by the potential

$$n = n_0 e^{e\phi/T}, \quad E_j = -\frac{\partial}{\partial x_j} \phi, \quad (3.75)$$

the flux is rewritten as

$$J_i = \sum_j n \left(\mu_{ij} - \frac{e}{T} D_{ij} \right) E_j. \quad (3.76)$$

At equilibrium, we may put the flux equal to zero and obtain the Einstein relation,

$$\mu_{ij} = \frac{e}{T} D_{ij}. \quad (3.77)$$

Thus, the Einstein relation is the general relation in systems at thermal equilibrium.

3.2.6 Ambipolar Diffusion

Suppose a collisional plasma in which the collision dominates over the other processes. The plasma slowly diffuses toward the edge region with the flux (3.71). If the electrons and ions diffuse at different rates, the charge balance is locally broken to build up the potential which adjusts both species to diffuse at the same rate. Since the electrons are lighter, they leave the plasma first and the ions are left behind. Then, the local polarization is to set up the electric field to draw the electrons back and push the ions forward. The electric field is obtained by setting $\Gamma_e = \Gamma_i = \Gamma$, we have

$$\mathbf{E} = \frac{D_i - D_e}{\mu_i + \mu_e} \nabla \ln n, \quad (3.78)$$

where we have assumed the charge neutrality which is the case when the plasma extends over larger than the Debye length. Then, the flux is given by

$$\Gamma = -\frac{\mu_e D_i + \mu_i D_e}{\mu_e + \mu_i} \nabla n = -D_a \nabla n. \quad (3.79)$$

The coefficient D_a is called the ambipolar diffusion coefficient.

$$D_a = \frac{\mu_e D_i + \mu_i D_e}{\mu_e + \mu_i}. \quad (3.80)$$

3.3 Fluid Drifts

3.3.1 Cross Magnetic Field Drifts

If stress tensor and friction force are omitted, the equation of motion (3.28) is rewritten as

$$\frac{\partial}{\partial t} \mathbf{v}_\alpha + (\mathbf{v}_\alpha \cdot \nabla) \mathbf{v}_\alpha = \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{1}{n_\alpha m_\alpha} \nabla p_\alpha. \quad (3.81)$$

Taking the vector product of $\hat{\mathbf{z}} = \mathbf{B}/|\mathbf{B}|$ and the equation of motion (3.81), we have

$$\begin{aligned}
\mathbf{v}_{\alpha\perp} &= \mathbf{v}_{\alpha E} + \mathbf{v}_{\alpha D} + \frac{1}{\Omega_\alpha} \hat{\mathbf{z}} \times \left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) \mathbf{v}_{\alpha\perp} \\
&\simeq \mathbf{v}_{\alpha E} + \mathbf{v}_{\alpha D} + \frac{1}{\Omega_\alpha} \hat{\mathbf{z}} \times \left(\frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \right) \mathbf{u}_\alpha
\end{aligned} \tag{3.82}$$

where

$$\mathbf{u}_\alpha = \mathbf{v}_{\alpha E} + \mathbf{v}_{\alpha D},$$

and $\Omega_\alpha = e_\alpha |\mathbf{B}| / m_\alpha c$. $\mathbf{v}_{\alpha E}$ is the $\mathbf{E} \times \mathbf{B}$ drift defined by

$$\mathbf{v}_{\alpha E} = -\frac{e_\alpha}{m_\alpha \Omega_\alpha} \hat{\mathbf{z}} \times \mathbf{E} = c \frac{\mathbf{E} \times \mathbf{B}}{B^2} = \mathbf{v}_E, \tag{3.83}$$

and $\mathbf{v}_{\alpha D}$ is the diamagnetic drift defined by

$$\mathbf{v}_{\alpha D} = \frac{1}{n_\alpha m_\alpha \Omega_\alpha} \hat{\mathbf{z}} \times \nabla p_\alpha = \frac{v_{T_\alpha}^2}{\Omega_\alpha} \hat{\mathbf{z}} \times \nabla \ln n_\alpha = \frac{c T_\alpha}{e_\alpha} \frac{\mathbf{B} \times \nabla_\perp \ln n_\alpha}{B^2}. \tag{3.84}$$

The diamagnetic drift is due to the imbalance of the plasma particles coming from a region of higher density and from a region of lower density. The third term of the right hand side of (3.82) is the polarization drift $\mathbf{v}_{\alpha p}$ defined by

$$\mathbf{v}_{\alpha p} = \hat{\mathbf{z}} \times \frac{1}{\Omega_\alpha} \left(\frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \right) \mathbf{u}_\alpha. \tag{3.85}$$

The $\mathbf{E} \times \mathbf{B}$ drift is independent on charge, driving the electrons and the ions in the same direction, while both the diamagnetic drift and the polarization drift depend on charge to drive the electrons and the ions in the opposite directions, producing the current.

3.3.2 Collisional Cross Magnetic Field Drifts

Now, we see the effects from the viscosity term and the collision term whose importance depends on physical situations. Here for simplicity, we consider the case that the collision term is approximated as follows

$$\sum_\beta \frac{\mathbf{v}_\alpha - \mathbf{v}_\beta}{\tau_{\alpha\beta}} \sim \frac{\mathbf{v}_\alpha}{\tau_\alpha}, \quad \frac{1}{\tau_\alpha} = \sum_\beta \frac{1}{\tau_{\alpha\beta}}.$$

Then, (3.28) reads

$$\frac{d\mathbf{v}_\alpha}{dt} = \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{1}{n_\alpha m_\alpha} (\nabla p_\alpha - \nabla \cdot \boldsymbol{\pi}_\alpha) - \frac{\mathbf{v}_\alpha}{\tau_\alpha}. \tag{3.86}$$

In a similar way as before, we have

$$\mathbf{v}_{\alpha\perp} = \mathbf{u}_\alpha + \hat{\mathbf{v}}_{\alpha p} + \frac{1}{\Omega_\alpha \tau_\alpha} \hat{\mathbf{z}} \times \mathbf{v}_{\alpha\perp}, \quad (3.87)$$

where

$$\hat{\mathbf{v}}_{\alpha p} = \mathbf{v}_{\alpha p} - \frac{1}{n_\alpha m_\alpha \Omega_\alpha} \hat{\mathbf{z}} \times \nabla \cdot \boldsymbol{\pi}_\alpha. \quad (3.88)$$

According to Braginskii [58], we have

$$(\nabla \cdot \boldsymbol{\pi})_\perp = -\eta_1 \nabla^2 \mathbf{u}_\perp + \eta_3 \nabla^2 \hat{\mathbf{z}} \times \mathbf{u}_\perp + \mathbf{U} \cdot \nabla \eta_1 + \mathbf{W} \cdot \nabla \eta_3, \quad (3.89)$$

where we have used $\nabla_\perp \cdot \mathbf{u}_\perp = 0$ and

$$\eta_1 = n_i T_i \tau_i, \quad \eta_3 = \frac{n_i T_i}{\Omega_i},$$

$$\begin{aligned} U_{xx} = -U_{yy} &= \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}, & U_{xy} = U_{yx} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \\ W_{xx} = -W_{yy} &= U_{xy}, & W_{xy} = W_{yx} &= -U_{xx}. \end{aligned}$$

From (3.87), we have

$$\hat{\mathbf{z}} \times \mathbf{v}_{\alpha\perp} = \frac{1}{\Omega_\alpha} \left\{ \frac{e_\alpha}{m_\alpha} \mathbf{E}_\perp - \frac{1}{n_\alpha m_\alpha} \nabla_\perp p_\alpha \right\} + \hat{\mathbf{z}} \times \mathbf{v}_{\alpha p} - \frac{1}{\Omega_\alpha \tau_\alpha} \mathbf{v}_{\alpha\perp}, \quad (3.90)$$

which is substituted into (3.87) to give

$$\begin{aligned} \mathbf{v}_{\alpha\perp} &= \frac{\Omega_\alpha^2 \tau_\alpha^2}{1 + \Omega_\alpha^2 \tau_\alpha^2} \left\{ \mathbf{v}_E + \mathbf{v}_{\alpha D} + \left(1 + \frac{1}{\Omega_\alpha \tau_\alpha} \right) \hat{\mathbf{v}}_{\alpha p} \right\} \\ &\quad + \frac{e_\alpha}{|e_\alpha|} \mu_{\alpha\perp} \mathbf{E}_\perp - \frac{D_{\alpha\perp}}{n_\alpha T_\alpha} \nabla_\perp p_\alpha. \end{aligned} \quad (3.91)$$

where the perpendicular mobility (3.68) and diffusion coefficient (3.63) have been used.

It is worthwhile to note that the diamagnetic convective term in the ion polarization is canceled with a part of the ion viscosity term for strong magnetized plasmas as is first pointed out by Weiland [32], which is seen as follows. The convection term of the polarization drift and the ion viscosity term are rewritten as

$$\nabla \cdot \{n_i \hat{\mathbf{z}} \times (\mathbf{u} \cdot \nabla) \mathbf{u}\} = -n_i (\mathbf{u} \cdot \nabla) \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{u}_\perp) + \nabla n_i \cdot (\hat{\mathbf{z}} \times (\mathbf{u} \cdot \nabla) \mathbf{u}_\perp),$$

$$\nabla \cdot \hat{\mathbf{z}} \times (\nabla \cdot \boldsymbol{\pi}) = -\eta_1 \nabla^2 \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{u}) - 2\hat{\mathbf{z}} \cdot (\nabla \eta_1 \times \nabla^2 \mathbf{u}) + 2\nabla \eta_3 \cdot \nabla^2 \mathbf{u}_\perp.$$

The contribution from the diamagnetic drift and the viscosity term η_3 are combined to give

$$-n_i(\mathbf{u}_{iD} \cdot \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{u}_\perp) - \frac{2}{m_i}(\nabla \eta_3) \cdot \nabla^2 \mathbf{u}_\perp = n_i \hat{\mathbf{z}} \cdot (\mathbf{v}_{iD} \times \nabla^2 \mathbf{u}_\perp),$$

where we have used $\nabla_\perp \cdot \mathbf{u}_\perp = 0$ and $\nabla \eta_3 / m_i = n_0 \mathbf{v}_{iD}$. Thus, we have

$$\nabla \cdot (n_i \hat{\mathbf{v}}_{ip}) = -\frac{n_i C_s^2}{\Omega_i^2} \left\{ \frac{\partial}{\partial t} - \eta_i \nabla^2 + \frac{C_s^2}{\Omega_i} \left(\hat{\mathbf{z}} \times \nabla \frac{e\phi}{T_e} \right) \cdot \nabla \right\} \nabla_\perp^2 \frac{e\phi}{T_e}, \quad (3.92)$$

where $\eta_i = \eta_1 / (n_i m_i)$ and we have assumed the scale length of the fluctuation is shorter than that of the background density inhomogeneity.

3.4 Magnetohydrodynamics

The fluid motion of a plasma as a whole is slow compared with the thermal motions of the particles. The macroscopic motion under the condition of charge neutrality is mainly controlled by the ion mass, resulting in low frequency motions. Introducing the mass density, charge density, and average flow velocity as

$$\rho = \sum_\alpha m_\alpha n_\alpha, \quad \rho_e = \sum_\alpha e_\alpha n_\alpha, \quad \mathbf{u} = \frac{1}{\rho} \sum_\alpha m_\alpha n_\alpha \mathbf{v}_\alpha, \quad (3.93)$$

we have from (3.27) to (3.29)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (3.94)$$

$$\rho \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right\} = \rho_e \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}, \quad (3.95)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) (p \rho^{-\gamma}) = 0, \quad (3.96)$$

where the pressure tensor is given by (3.12) and (3.13). In the right hand side of (3.95), we may omit the term $\rho_e \mathbf{E}$ in the plasma with high conductivity, since a small charge separation produces a large current, the $\mathbf{J} \times \mathbf{B}$ is more important than $\rho_e \mathbf{E}$. This is seen as follows. Putting the spatial size of the charge density as L , then we estimate the electric field as $E \sim \rho_e L$, the current as $J \sim \sigma \rho_e L$, and the magnetic field as $B \sim jL/c \sim \sigma \rho_e L^2/c$. Therefore,

$$\frac{jB/c}{\rho_e E} \sim \left(\frac{\sigma L}{c} \right)^2 \gg 1.$$

Thus, (3.95) is reduced to

$$\rho \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right\} = \frac{1}{c} \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}. \quad (3.97)$$

In deriving (3.96), we consider the case that the time scale concerned is shorter than the diffusion time due to the collision and assume that the dissipation and heat generation are neglected so that the energy equation is converted to the adiabatic law.

In order to close the set of equations, we use Ohm's law and the Maxwell equations as well

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B} \right). \quad (3.98)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (3.99)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (3.100)$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho_e, \quad (3.101)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.102)$$

where σ is the conductivity. In (3.100), we have omitted the displacement current which is the response of the plasma and is small compared with $\mathbf{J}$ in a conducting medium ($\partial/\partial t \ll \sigma$).

The magnetohydrodynamics (MHD) equations are valid in the time scale that the distribution functions relax to the Maxwellian. The cause of the relaxation to the equilibrium is not necessarily the collision. The mixing of the collective motions leads to the relaxation of the distribution function to the Maxwellian.

3.5 Frozen-in and Diffusion of Magnetic Field Line

Eliminating $\mathbf{E}$ in (3.98) and (3.99) and using (3.100), we have

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{c}{\sigma} \nabla \times \mathbf{J} \\ &= \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad \eta = \frac{c^2}{4\pi\sigma}. \end{aligned} \quad (3.103)$$

3.5.1 Frozen-in Field Line

For perfect conducting plasmas ($\sigma = \infty$ or $\eta = 0$), (3.103) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}). \quad (3.104)$$

The meaning of this equation is understood as follows. The time change of a magnetic flux through a surface S , following the plasma motion defined by

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}, \quad (3.105)$$

is a change in $\mathbf{B}$ itself and a change in the boundary C of the surface S moving with the plasma flow

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_C \mathbf{B} \cdot (\mathbf{u} \times d\boldsymbol{\ell}) \\ &= \int_S \left\{ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) \right\} \cdot d\mathbf{S}, \end{aligned} \quad (3.106)$$

which is zero because of (3.104). Thus, the magnetic flux is constant and the magnetic field lines are frozen-in the plasma. In perfect conducting plasmas, relative motions between magnetic field lines and the plasmas are not allowed. This is because if the relative motion between the magnetic field lines and the plasma exists, the effective field $\mathbf{u} \times \mathbf{B}/c$ is driven and an infinitely large current is to flow.

This is analogous to the vortex filament in a barotropic flow which is given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla w. \quad (3.107)$$

Taking the rotation of (3.107) and introducing the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = 0, \quad (3.108)$$

which has the same structure as (3.104) and indicates that the vorticity filaments move attached with the fluid.

3.5.2 Diffusion of Magnetic Field Lines

For simplicity, we consider the case of $\mathbf{u} = 0$ and have a diffusion equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}. \quad (3.109)$$

With a boundary condition $\mathbf{B}(\infty) = 0$, we have

$$\frac{d}{dt} \int B^2 d\mathbf{x} = -\eta \int \sum_{i,j} \left(\frac{\partial B_i}{\partial x_j} \right)^2 < 0, \quad (3.110)$$

which implies that the magnetic field lines are separating from each other. Introducing the magnetic Reynolds number through

$$R_m = \frac{[\nabla \times (\mathbf{u} \times \mathbf{B})]}{\eta \nabla^2 \mathbf{B}} = \frac{U/L}{\eta/L^2} = \frac{UL}{\eta}, \quad (3.111)$$

the “Frozen-in” is realized for $R_m \gg 1$ and the picture of “Frozen-in” does not hold for $R_m \sim 1$.

3.6 MHD Equilibrium

The equilibrium of the MHD equation is given by

$$\nabla p = \frac{1}{c} \mathbf{J} \times \mathbf{B}, \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (3.112)$$

with a subsidiary condition

$$\nabla \cdot \mathbf{B} = 0. \quad (3.113)$$

Equation (3.112) yields $\mathbf{B} \cdot \nabla p = 0$ and $\mathbf{J} \cdot \nabla p = 0$. That is, the pressure is constant along the magnetic field line and the current flow, and the plasma extends freely along $\mathbf{B}$ and $\mathbf{J}$, or the magnetic field lines and the current flow are on the isobaric surface.

3.6.1 Equilibrium for Plasma Column

For a cylindrically symmetric plasma column (3.112) reads

$$\frac{d}{dr} \left(p + \frac{B_\theta^2 + B_z^2}{8\pi} \right) + \frac{B_\theta^2}{4\pi r} = 0. \quad (3.114)$$

There are three dependent variables ($p(r)$, $B_\theta(r)$, $B_z(r)$) for one equation.

3.6.1.1 θ Pinch

When the plasma is confined by the toroidal field produced by the θ -component of the plasma current, we have

$$\frac{d}{dr} \left(p + \frac{B_z^2}{8\pi} \right) = 0,$$

leading to

$$p + \frac{B_z^2}{8\pi} = \frac{B_{z0}^2}{8\pi}. \quad (3.115)$$

The magnetic field inside the plasma is smaller than that of outside the plasma, which is the plasma diamagnetism. The magnetic field B_z produced by the θ component of the plasma current is to prevent the plasma from bending.

3.6.1.2 z Pinch

When the plasma is confined by the poloidal field produced by the z component plasma current, we have

$$\frac{dp}{dr} + \frac{1}{8\pi r^2} \frac{d}{dr} (r B_\theta)^2 = 0, \quad (3.116)$$

where

$$B_\theta = \frac{4\pi}{c} \frac{1}{r} \int_0^r dr' r' j_z(r') = \frac{2I(r)}{cr}.$$

Integrating (3.116) from an arbitrary point $r = r$ inside the plasma to the plasma edge $r = R$ where we may assume $p(R) = 0$, we have

$$p(r) = \frac{1}{2\pi c^2} \int_r^R dr' \frac{1}{r'^2} \frac{dI^2(r')}{dr'}.$$

Integration $p(r)$ again over the cross section of the plasma column gives

$$\begin{aligned} 2\pi \int_0^R dr' r' p(r') &= \frac{1}{c^2} \int_0^R dr' r' \int_{r'}^R dr'' \frac{1}{r''^2} \frac{dI^2(r'')}{dr''} \\ &= \frac{1}{c^2} \int_0^R dr'' \frac{1}{r''^2} \frac{dI^2(r'')}{dr''} \int_0^{r''} dr' r' \\ &= \frac{1}{2c^2} (I^2(R) - I^2(0)). \end{aligned} \quad (3.117)$$

This implies that there is an upper limit of the total current above which the pressure cannot resist the compression by the B_θ .

3.6.1.3 Sausage Instability

Suppose a plasma column of radius a sustained by a magnetic field B_z . When a uniform current I_z is applied to this plasma in the z direction, the poloidal magnetic field B_θ produced by I_z compresses the plasma. If the plasma radius changes from

a to $a + \delta a$ by this compression, from the conservation of the magnetic flux

$$\delta(B_z a^2) = a^2 \delta B_z + 2a B_z \delta a = 0,$$

we obtain

$$\delta B_z = -2B_z \frac{\delta a}{a}.$$

On the other hand, the conservation of the current

$$\delta(B_\theta a) = a \delta B_\theta + B_\theta \delta a = 0,$$

gives

$$\delta B_\theta = -B_\theta \frac{\delta a}{a}.$$

For the pinch ($\delta a < 0$), $\delta B_z > 0$ strengthens the magnetic field inside the plasma and resists against the compression by B_θ for the stabilization, while $\delta B_\theta > 0$ strengthens the compression for further destabilization. The difference of the magnetic pressure is given by

$$\delta P = \delta \left(\frac{B_z^2}{8\pi} - \frac{B_\theta^2}{8\pi} \right) = \frac{1}{4\pi} (B_\theta^2 - 2B_z^2) \frac{\delta a}{a}.$$

If $\delta P > 0$ ($2B_z^2 > B_\theta^2$) for $\delta a < 0$, that is, two times of the toroidal magnetic field as a resistant to the compression is larger than the poloidal field as an enhancer of the pinch, the system is stable. However, if I_z becomes larger so as to $\delta P < 0$, the plasma is unstable against the pinch.

3.6.1.4 Kink Instability

When the plasma column is deformed bent in the z direction, the toroidal magnetic field B_z resists against the deformation. The tension of B_z is given by

$$\frac{B_z^2}{8\pi} \pi a^2 \frac{\lambda}{R}, \quad (3.118)$$

where R is the radius of curvature and λ is the length of the kink deformation. The pressure of the deformed poloidal magnetic field is outward and is given by

$$\frac{\lambda}{2R} \int dS \frac{B_\theta^2(r)}{8\pi} \simeq \frac{B_\theta^2(a)}{8\pi} \pi a^2 \frac{\lambda}{R} \int_a^\lambda \frac{1}{r} = \frac{B_\theta^2(a)}{8\pi} \pi a^2 \frac{\lambda}{R} \ln \left(\frac{\lambda}{a} \right). \quad (3.119)$$

The stability condition is given from (3.118) and (3.119)

$$\frac{B_z^2}{B_\theta^2(a)} > \ln \frac{\lambda}{a}. \quad (3.120)$$

From the equilibrium condition $p + B_z^2/8\pi = B_\theta^2/8\pi$, we have always $B_z \leq B_\theta$. Therefore, the plasma is always unstable against the deformation disturbance with large λ/a .

3.6.1.5 Force Free Configuration

When a plasma current is parallel to a magnetic field everywhere in the plasma, that is, $\mathbf{J} = (4\pi/c)\nabla \times \mathbf{B} \parallel \mathbf{B}$, the pressure distribution is constant since $\nabla p = \mathbf{J} \times \mathbf{B}/c = 0$. Therefore, this is called the force free configuration. Then, we have for a scalar α

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \quad (3.121)$$

which gives

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times (\alpha \mathbf{B}) = \alpha^2 \mathbf{B}.$$

In rewriting the above equation, we have

$$\nabla^2 \mathbf{B} + \alpha^2 \mathbf{B} = 0. \quad (3.122)$$

In a cylindrical coordinates, (3.122) is expressed as

$$\nabla^2 B_z + \alpha^2 B_z = 0, \quad (3.123)$$

and

$$\nabla^2 B - i \frac{2}{r^2} \frac{\partial B}{\partial \theta} - \frac{B}{r^2} + \alpha^2 B = 0, \quad (3.124)$$

where $B = B_\theta + iB_r$. Putting $(B_z, B) \propto e^{i(m\theta + kz)}$, we have

$$B_z(r, \theta, z) = J_m(\sqrt{\alpha^2 - k^2}r)e^{i(m\theta + kz)},$$

and

$$B(r, \theta, z) = J_{m+1}(\sqrt{\alpha^2 - k^2}r)e^{i(m\theta + kz)}.$$

The z component of the magnetic field decreases as r increases and changes its sign at the first zero of the Bessel function.

The force free field is realized in the reversed field pinch as a result of energy relaxation and is formulated by J. B. Taylor [63] who derived (3.121) through a variational problem as follows.

In a plasma with high conductivity, since a magnetic field is assumed to be frozen in the plasma, we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}).$$

Introducing a vector potential $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} + \nabla \xi,$$

which leads to

$$\frac{\partial}{\partial t} \int \mathbf{A} \cdot \mathbf{B} d\mathbf{r} = 0.$$

The integral

$$K = \int \mathbf{A} \cdot \mathbf{B} d\mathbf{r} = \int \mathbf{A} \cdot (\nabla \times \mathbf{A}) d\mathbf{r}$$

is called the helicity and an index to represent the degree of connectedness of the magnetic field lines.

For the case of finite conductivity, the reconnection of the magnetic field lines occur and the helicity is no longer invariant. However, if the change in the magnetic field in the process of the reconnection of the magnetic field lines is small, the helicity is approximately conserved. Thus, minimizing the energy of the magnetic field with the helicity K fixed is associated with a variational problem

$$\delta \int \left\{ \frac{B^2}{8\pi} + \alpha \mathbf{A} \cdot \mathbf{B} \right\} d\mathbf{r} = 0,$$

we have (3.121). The force free configuration is supposed to be realized when the magnetic field energy is converted to the energy of the turbulence.

3.6.2 Simple Torus

It is worthwhile to point out that a simple torus expressed by a magnetic field $\mathbf{B} = (0, B_\theta(r, z), 0)$ cannot sustain the MHD equilibrium. The force balance is given by

$$\frac{\partial p}{\partial r} = -\frac{1}{2r^2} \frac{\partial}{\partial r} (r B_\theta)^2, \quad \frac{\partial p}{\partial z} = -\frac{1}{8\pi} \frac{\partial B_\theta^2}{\partial z}.$$

On the other hand, we have from (3.112)

$$\nabla \times \nabla p = \frac{1}{4\pi} \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] = -\frac{1}{4\pi r} \frac{\partial B_\theta^2}{\partial z} \boldsymbol{\theta} = 0,$$

showing that B_θ is independent of z and the force balance in the z direction is violated. However, B_θ must depend on z to confine a plasma, indicating that an additional poloidal magnetic field is necessary. This corresponds to the fact that the

charge separation induced by the grad-B and curvature drift is never canceled in the simple torus and the electrons and ions are driven by the resultant $E \times B$ drift to the wall.

3.6.3 Magnetic Surface

The magnetic field defined by $\mathbf{B} = \nabla\psi \times \nabla\varphi$, where ψ and φ are scalar functions, $\nabla \cdot \mathbf{B} = 0$ is automatically satisfied. If p is a function of ψ only, $\mathbf{B} \cdot \nabla p = 0$ is also satisfied. If the magnetic surface ψ spanned by the magnetic field lines is defined by

$$\nabla\psi \cdot \mathbf{B} = 0, \quad (3.125)$$

the isobaric surface coincides with the magnetic surface and the current flow on the magnetic surface. Thus at the equilibrium, the plasma is surrounded by the magnetic surface. The configuration of the magnetic surface is limited to a cylinder extended to the infinity or a torus. The function φ is to describe the degree of freedom inside the magnetic surface.

3.6.4 Grad–Shafranov Equation

The equation for the MHD equilibrium is reduced to an equation for a single dependent variable in a system with axial symmetry or helical symmetry.

3.6.4.1 Axisymmetric Plasma

From (3.125) rewritten in an explicit form

$$B_r \frac{\partial\psi}{\partial r} + B_z \frac{\partial\psi}{\partial z} = 0,$$

we may put

$$rB_r = -\frac{\partial\psi}{\partial z}, \quad rB_z = \frac{\partial\psi}{\partial r}, \quad (3.126)$$

which guarantees $\nabla \cdot \mathbf{B} = 0$. Then, $\mathbf{B} \cdot \nabla p = 0$ is rewritten as

$$-\frac{\partial p}{\partial r} \frac{\partial\psi}{\partial z} + \frac{\partial p}{\partial z} \frac{\partial\psi}{\partial r} = 0,$$

giving

$$p = p(\psi). \quad (3.127)$$

On the other hand, $\nabla \times \mathbf{B} \cdot \nabla p = 0$ gives

$$-\frac{\partial p}{\partial r} \frac{\partial}{\partial z}(rB_\theta) + \frac{\partial p}{\partial z} \frac{\partial}{\partial r}(rB_\theta) = 0,$$

from which we have

$$rB_\theta = I(\psi). \quad (3.128)$$

Then finally, $\mathbf{j} \times \mathbf{B} = 4\pi \nabla p$ is rewritten as

$$\frac{\partial p}{\partial r} = -\frac{1}{4\pi} \frac{1}{r^2} \frac{\partial \psi}{\partial r} \left\{ \Delta^* \psi + I \frac{\partial I}{\partial \psi} \right\},$$

$$\frac{\partial p}{\partial z} = -\frac{1}{4\pi} \frac{1}{r^2} \frac{\partial \psi}{\partial z} \left\{ \Delta^* \psi + I \frac{\partial I}{\partial \psi} \right\},$$

where

$$\Delta^* \psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}.$$

Since p is a function of only ψ , we obtain

$$\Delta^* \psi + 4\pi r^2 \frac{dp(\psi)}{d\psi} + I(\psi) \frac{dI(\psi)}{d\psi} = 0. \quad (3.129)$$

Equation (3.129) is called the Grad–Shafranov equation for a system with an axial symmetry. The magnetic surface ψ is solved for a given $p(\psi)$ and $I(\psi) = rB_\theta$ as a boundary value problem.

For $I(\psi)$ independent of ψ and

$$p(\psi) = \begin{cases} \alpha \psi & \text{for } r^2 + z^2 \leq R^2, \\ 0 & \text{for } r^2 + z^2 > R^2, \end{cases}$$

(3.129) is reduced to

$$\Delta^* \psi = \begin{cases} 4\pi \alpha r^2 \psi, & \text{for } r^2 + z^2 \leq R^2, \\ 0, & \text{for } r^2 + z^2 > R^2, \end{cases}$$

which is equivalent to the equation for Hill's spherical vortex. The solution is given by

$$\psi = \begin{cases} \frac{2\pi}{5} \alpha r^2 \{R^2 - (r^2 + z^2)\}, & \text{for } r^2 + z^2 \leq R^2, \\ -\frac{4\pi}{15} R^2 \alpha r^2 \left(1 - \frac{R^3}{(r^2 + z^2)^{3/2}}\right), & \text{for } r^2 + z^2 > R^2. \end{cases}$$

The equilibrium is called the field reversal configuration.

3.6.4.2 Helically Symmetric Plasma

The helical symmetry is described by $(r, \phi = \theta - \alpha z)$. In a similar way as before, we have

$$B_r \frac{\partial \psi}{\partial r} + \left(\frac{B_\theta}{r} - \alpha B_z \right) \frac{\partial \psi}{\partial \phi} = 0,$$

giving

$$r B_r = \frac{\partial \psi}{\partial \phi}, \quad B_\theta - \alpha r B_z = -\frac{\partial \psi}{\partial r}. \quad (3.130)$$

Again from $\mathbf{B} \cdot \nabla p = 0$, we have

$$p = p(\psi). \quad (3.131)$$

Then, since $\mathbf{j} \cdot \nabla p = 0$ gives

$$B^*(r, \phi) = B_z + \alpha r B_\theta = B^*(\psi),$$

we have with (3.130)

$$B_r = \frac{1}{r} \frac{\partial \psi}{\partial \phi}, \quad B_\theta = \frac{-\frac{\partial \psi}{\partial r} + \alpha r B^*}{1 + (\alpha r)^2}, \quad B_z = \frac{\alpha r \frac{\partial \psi}{\partial r} + B^*}{1 + (\alpha r)^2}. \quad (3.132)$$

Thus, we have

$$\begin{aligned} \frac{4\pi}{c} j_r &= \frac{1}{r} \frac{dB^*}{d\psi} \frac{\partial \psi}{\partial \phi}, \\ \frac{4\pi}{c} j_\theta &= -\alpha r L(\psi) - \frac{\frac{dB^*}{d\psi} \frac{\partial \psi}{\partial r}}{1 + (\alpha r)^2} + \frac{2\alpha^2 r B^*}{(1 + (\alpha r)^2)^2}, \\ \frac{4\pi}{c} j_z &= -L(\psi) + \frac{\alpha r \frac{dB^*}{d\psi} \frac{\partial \psi}{\partial r}}{1 + (\alpha r)^2} + \frac{2\alpha B^*}{(1 + (\alpha r)^2)^2}, \end{aligned}$$

where

$$L(\psi) = \left(\frac{1}{r} \frac{\partial}{\partial r} \frac{r}{1 + (\alpha r)^2} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) \psi.$$

Thus, we obtain the Grad–Shafranov equation for the helical symmetry,

$$L(\psi) + \frac{B^* \frac{dB^*}{d\psi}}{1 + (\alpha r)^2} - \frac{2\alpha B^*}{1 + (\alpha r)^2} = -4\pi \frac{dp}{d\psi}. \quad (3.133)$$

3.7 Reduced MHD Equations

We first consider a strongly magnetized plasma with an external constant magnetic field B_0 in the z direction as well as a poloidal field produced by the plasma motion. The total magnetic field is expressed by

$$\mathbf{B} = B_0 \hat{\mathbf{z}} + \nabla \Psi \times \hat{\mathbf{z}}, \quad (3.134)$$

where Ψ is a flux function and satisfied $\mathbf{B} \cdot \nabla \Psi = 0$, implying that B is tangential to the surface $\Psi(x, y) = \text{const}$. From (3.100), we have

$$\hat{\mathbf{z}} \cdot \mathbf{J} = \frac{c}{4\pi} \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{B}) = -\nabla^2 \Psi. \quad (3.135)$$

Substituting (3.134) into Faraday's induction law (3.99) yields

$$\nabla \times \left(\frac{\partial}{\partial t} \Psi \hat{\mathbf{z}} + c \mathbf{E} \right) = 0, \quad (3.136)$$

which gives with Ohm's law (3.98)

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \Psi = \frac{c^2}{4\pi\sigma} \nabla^2 \Psi + E, \quad (3.137)$$

where E is an external electric field satisfying $\nabla \times (E \hat{\mathbf{z}}) = 0$. The equation of motion is given by (3.97). Assuming the mass density ρ is constant, taking a rotation of (3.97) and denoting $\omega = \hat{\mathbf{z}} \cdot (\nabla \times \mathbf{u})$, we obtain

$$\rho \left\{ \frac{\partial \omega}{\partial t} - \hat{\mathbf{z}} \cdot (\nabla \times (\mathbf{u} \times \omega)) \right\} = \hat{\mathbf{z}} \cdot (\nabla \times (\mathbf{J} \times \mathbf{B})). \quad (3.138)$$

Since the pressure may be assumed to be z -independent, the flow is almost two dimensional $|u_\perp| \gg |u_\parallel|$. We introduce a stream function through

$$\mathbf{u} = \hat{\mathbf{z}} \times \nabla \psi, \quad (3.139)$$

then (3.138) is reduced to

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \nabla^2 \psi = \frac{c}{4\pi} \hat{\mathbf{z}} \times \nabla \Psi \cdot \nabla (\nabla^2 \Psi). \quad (3.140)$$

Equations (3.137), (3.139), and (3.140) give a set of equations for two scalar functions Ψ and ψ . A set of (3.137) and (3.140) is called the reduced MHD equations which are used to analyze linear stability/instability of MHD modes.

3.7.1 Toroidal Plasma

Now, we take into account the effect of plasma diamagnetism on the magnetic field as well as the curvature effect of the magnetic field line. At an equilibrium, we have

$$\nabla p = \mathbf{J} \times \mathbf{B}, \quad (3.141)$$

which is combined with (3.100) to give

$$\nabla \left(p + \frac{B^2}{8\pi} \right) = \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B} = -\frac{B^2}{4\pi} \frac{\mathbf{R}}{R}, \quad (3.142)$$

where the curvature of the magnetic field line is expressed in terms of the local curvature radius $\mathbf{R}$ as

$$\frac{1}{B^2} \mathbf{B} \cdot \nabla \mathbf{B} = -\frac{\mathbf{R}}{R}. \quad (3.143)$$

For a toroidal system, the magnetic field is given up to the order of r/R_0 (R_0 is the major radius) by

$$B^2 = B_0^2 \left\{ 1 - 2 \left(\frac{p}{B_0^2/4\pi} + \frac{x}{R_0} \right) \right\}, \quad (3.144)$$

Then, (3.134) is now modified as

$$\mathbf{B} = B \hat{\mathbf{z}} + \nabla \Psi \times \hat{\mathbf{z}}. \quad (3.145)$$

In a similar way as deriving (3.137) from the Faraday's induction law, we have

$$\frac{\partial \Psi}{\partial t} = \mathbf{B} \cdot \nabla_{\perp} \psi + \frac{c^2}{4\pi\sigma} \nabla_{\perp}^2 \Psi + E, \quad (3.146)$$

where ψ is a stream function and E is an external electric field satisfying $\nabla \times (E \hat{\mathbf{z}}) = 0$. In order to take into account the plasma diamagnetism, we start with

$$\nabla \cdot \mathbf{J} = \nabla_{\perp} \cdot \mathbf{J}_{\perp} + \nabla_{\parallel} J_{\parallel} = 0. \quad (3.147)$$

We have from the equation of motion (3.97)

$$\mathbf{J}_{\perp} = \frac{c}{B^2} \left\{ \rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} + c \nabla p \right\} \times \mathbf{B}, \quad (3.148)$$

and from (3.141)

$$J_{\parallel} = -\frac{c}{4\pi} \nabla_{\perp}^2 \Psi. \quad (3.149)$$

Substituting (3.148) and (3.149) into (3.147) and using the following approximation

$$\begin{aligned}\nabla \cdot \left(\frac{\rho}{B^2} \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \times \mathbf{B} \right) &\sim \mathbf{B} \cdot \left(\nabla \times \frac{\rho}{B^2} \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} \right) \sim \frac{\rho}{B} \frac{d}{dt} \nabla_{\perp}^2 \psi, \\ \nabla \cdot \left(\nabla p \times \frac{\mathbf{B}}{B^2} \right) &\sim \left(\nabla \frac{1}{B^2} \times \nabla p \right) \cdot \mathbf{B},\end{aligned}$$

we have

$$\rho \frac{d}{dt} \nabla_{\perp}^2 \psi = \frac{1}{4\pi} \mathbf{B} \cdot \nabla \nabla_{\perp}^2 \psi + \frac{2c}{B^2} (\nabla B \times \nabla p) \cdot \mathbf{B}. \quad (3.150)$$

A set of equations is closed by the heat balance (3.96) which now reads

$$\frac{\partial p}{\partial t} + (\hat{\mathbf{z}} \times \nabla \psi) \cdot \nabla p = 0, \quad (3.151)$$

where we have used the continuity equation and the incompressibility condition $\nabla \cdot \mathbf{u} = 0$.

A set of (3.145), (3.146), (3.150), and (3.151) is the reduced MHD equations for plasmas in the toroidal geometry.

3.7.2 Flute Instability

When magnetic field lines are convex to a plasma, the plasma particles traveling along the magnetic field lines suffer from the outward centrifugal force which drives gravitational drifts to excite the charge separation. The resultant $E \times B$ drift is added to the effective gravitational drift to enhance the fluctuations. The fluctuation grows to form a flute-like structure.

On the contrary when the magnetic field lines are concave to the plasma, the centrifugal force is inward and the resultant $E \times B$ drift is opposite to the gravitational drift to reduce the fluctuations. This is called the min-B configuration.

3.7.3 Tearing Mode Instability

When magnetic field lines in the opposite direction come close to each other, a current sheet is formed. With a finite resistivity in the current sheet, the reconnection of the neighboring magnetic field lines occurs due to the pinch of the current. In this process, the current sheet is teared and the mode excited in the process is called the tearing mode.

A plasma is initially stationary with $\mathbf{u}_0 = 0$ and $\mathbf{B}_0 = (0, B_y(x), 0)$, where the magnetic field B_y is given by

$$B_y(x) \propto \begin{cases} x & \text{for } x \sim 0, \\ \pm B_{y0} & \text{for } x \rightarrow \pm\infty. \end{cases} \quad (3.152)$$

We start from (3.137) and (3.140) and linearize them with respect to the two dimensional perturbations $(\delta\Psi(x, y), \delta\psi(x, y))$ to obtain

$$\frac{\partial \delta\Psi}{\partial t} + (\hat{z} \times \nabla \delta\psi) \cdot \nabla \Psi_0 = \eta \nabla^2 \delta\Psi, \quad \eta = \frac{c^2}{4\pi\sigma}, \quad (3.153)$$

$$\rho_0 \frac{\partial}{\partial t} \nabla^2 \delta\psi = \frac{c}{4\pi} \{ (\hat{z} \times \nabla \Psi_0) \cdot (\nabla^2 \delta\Psi) + \hat{z} \times \nabla \delta\Psi \cdot (\nabla^2 \Psi_0) \}. \quad (3.154)$$

Putting

$$B_y(x) = -\frac{d\Psi_0(x)}{dx}, \quad \begin{pmatrix} \delta\Psi(x, y) \\ \delta\psi(x, y) \end{pmatrix} = \begin{pmatrix} \Psi_1(x) \\ \psi_1(x) \end{pmatrix} e^{\gamma t - iky},$$

we have

$$\eta \left(\frac{d^2}{dx^2} - k^2 - \frac{\gamma}{\eta} \right) \Psi_1 = ik B_y(x) \psi_1, \quad (3.155)$$

$$4\pi\rho_0\gamma \left(\frac{d^2}{dx^2} - k^2 \right) \psi_1 = ik B_y(x) \left(\frac{d^2}{dx^2} - k^2 - \frac{1}{B_y(x)} \frac{d^2 B_y(x)}{dx^2} \right) \Psi_1. \quad (3.156)$$

Near the neutral sheet ($x \sim 0$) $B_y(x)$ can be neglected and (3.155) and (3.156) reduce to

$$\frac{d^2 \Psi_1}{dx^2} - \left(k^2 + \frac{\gamma}{\eta} \right) \Psi_1 = 0, \quad (3.157)$$

$$\frac{d^2 \Psi_1}{dx^2} - k^2 \Psi_1 = 0, \quad (3.158)$$

from which we have for Ψ_1

$$\Psi_1(x) = \Psi_1^{(-)} \cosh[\sqrt{k^2 + (4\pi\sigma\gamma/c^2)}x], \quad (3.159)$$

γ is determined by connecting this solution with a far field solution. Far from the neutral sheet, the magnetic field lines are frozen in the plasma, we may put $\eta \rightarrow 0$ and obtain

$$\psi_1 = 0 \quad (3.160)$$

$$\frac{d^2 \Psi_1}{dx^2} - \left(k^2 + \frac{1}{B_y(x)} \frac{d^2 B_y(x)}{dx^2} \right) \Psi_1 = 0. \quad (3.161)$$

Noting that the magnetic field behaves as is given by (3.152), we may assume

$$B_y(x) = B_{y0} \tanh\left(\frac{x}{L}\right),$$

which is an equilibrium solution of the magnetic neutral sheet. Then, (3.161) reduces to

$$\frac{d^2\psi_1}{dx^2} + \left(\frac{1}{L^2} - k^2\right)\psi_1 = 0.$$

Thus, we get

$$\psi_1 = \psi_1^{(+)} \sin(\sqrt{1/L^2 - k^2}x). \quad (3.162)$$

Assuming the scale length of the magnetic neutral sheet in the x direction ε and connecting (3.159) and (3.162) smoothly at $x = \varepsilon$, we have the growth rate as

$$\gamma = \frac{\eta}{\varepsilon^2} = \frac{c^2}{4\pi\sigma\varepsilon^2}.$$

3.7.4 Ballooning Mode Instability

In a torus plasma, the pressure gradient ∇p is positive in the inner region and negative in the outer region while the centrifugal force due to the magnetic curvature is positive everywhere. As a result, the inner region of the plasma is stable (good curvature), while the outer region is unstable (bad curvature). Thus, the instability is localized in the outer region, and a mode grows like a balloon. The instability is called the ballooning mode instability.

The equilibrium solution is given by $(\rho_0, \mathbf{B}_0, \psi_0, p_0)$. In the toroidal coordinates (x, y, z) , x is in the major radius direction, y is the direction parallel to the axis of the torus, and z is the toroidal direction. For the magnetic field, the toroidal field dominates over the poloidal field and therefore the magnetic field is assumed along the z direction. The pressure gradient is in the x direction and the scale length of the gradient is assumed much larger than the wavenumber of the perturbation.

We start with (3.146), (3.150), and (3.151) which are linearized with respect to the perturbation as

$$\frac{\partial \delta\psi}{\partial t} = (\mathbf{B}_0 \cdot \nabla) \delta\psi + \frac{c^2}{4\pi\sigma} \nabla_\perp^2 \delta\psi, \quad (3.163)$$

$$\rho_0 \frac{\partial}{\partial t} \nabla_\perp^2 \delta\psi = \frac{1}{4\pi} \mathbf{B}_0 \cdot \nabla \nabla_\perp^2 \delta\psi + \frac{1}{4\pi} \delta\mathbf{B} \cdot \nabla \nabla_\perp^2 \psi_0 - \left(\nabla \frac{2x}{R_0} \times \nabla \delta p \right) \cdot \mathbf{z}, \quad (3.164)$$

$$\frac{\partial \delta p}{\partial t} + (\mathbf{z} \times \nabla \delta\psi) \cdot \nabla p_0 = 0, \quad (3.165)$$

where (3.144) has been used. Putting the perturbation in a form $\delta A \propto \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$, (3.163)–(3.165) are reduced to

$$\begin{aligned} -\omega \delta \Psi &= B_0 k_z \delta \psi, \\ \rho_0 \omega k_\perp^2 \delta \psi &= -B_0 k_z k_\perp^2 \delta \Psi - \frac{2c}{R} k_y \delta p, \\ -\omega \delta p &= k_y \delta \psi \frac{dp_0}{dx}, \end{aligned}$$

where the resistivity and parallel current $J_\parallel \propto \nabla_\perp^2 \Psi_0$ are neglected. The dispersion relation is given by

$$\omega^2 = \frac{k_z^2 B_0^2}{\rho_0} + \frac{2c}{\rho_0 R_0} \frac{k_y^2}{k_\perp^2} \frac{dp_0}{dx}. \quad (3.166)$$

In the outer region where $dp_0/dx < 0$, ω becomes imaginary for $|dp_0/dx| > (R_0/2c)(1 + (k_x/k_y)^2)k_z^2 B_0^2$ and gives an instability.

Chapter 4

Waves in Plasmas

Since a plasma consists of light electrons and heavy ions, waves in plasmas are roughly divided into two : high frequency waves for which the electrons are carriers of the mass motion and low frequency waves whose dynamical motion is carried by the ions. The high frequency waves are characterized by a finite frequency when the wave number goes to zero and often called hard modes. On the other hand, the low frequency modes are called soft modes since the frequency goes to zero when the wave number goes to zero. Waves are collective motions of plasma particles and are excited by a variety of instabilities to release the excess free energy stored in the plasmas in order to relax to the equilibrium.

In this chapter, electrostatic and electromagnetic waves are studied. Then, the wave energy is shown to be equal to the sum of the field energy and the particle kinetic energy. The wave energy is shown to be not always positive. The wave energy can be negative when there are relative motions such as beams and flows. Three types of wave instabilities are discussed in the last section.

4.1 Electrostatic Waves

4.1.1 *Waves Without an External Magnetic Field*

Basic equations are given by

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0, \quad (4.1)$$

$$\frac{\partial \mathbf{v}_\alpha}{\partial t} + (\mathbf{v}_\alpha \cdot \nabla) \mathbf{v}_\alpha = \frac{e_\alpha}{m_\alpha} \mathbf{E} - \frac{T_\alpha}{m_\alpha n_\alpha} \nabla n_\alpha, \quad (4.2)$$

$$\nabla \cdot \mathbf{E} = 4\pi \sum_\alpha e_\alpha n_\alpha, \quad (4.3)$$

where the equilibrium solutions are taken as

$$n_0 = n_{\alpha 0} = \text{const}, \quad v_\alpha = \mathbf{E} = 0.$$

Introducing a fluctuation through

$$A = A_0 + \tilde{A} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)},$$

and linearizing the basic equations, we have

$$-i\omega \tilde{n}_\alpha + in_{\alpha 0} \mathbf{k} \cdot \tilde{\mathbf{v}}_\alpha = 0, \quad (4.4)$$

$$-i\omega \tilde{\mathbf{v}}_\alpha = -i \frac{e_\alpha}{m_\alpha} \mathbf{k} \tilde{\phi} - i \frac{T_\alpha}{m_\alpha n_0} \mathbf{k} \tilde{n}_\alpha, \quad (4.5)$$

$$-k^2 \tilde{\phi} = 4\pi \sum e_\alpha \tilde{n}_\alpha, \quad (4.6)$$

where we have used $\tilde{\mathbf{E}} = -\nabla \tilde{\phi}$. Equations (4.4)–(4.6) are combined to give

$$\tilde{n}_\alpha = \frac{n_0 e_\alpha}{m_\alpha} \frac{k^2}{\omega^2 - k^2 v_{T\alpha}^2} \tilde{\phi}, \quad (4.7)$$

$$\tilde{\mathbf{v}}_\alpha = \frac{e_\alpha}{m_\alpha} \frac{\omega \mathbf{k}}{\omega^2 - k^2 v_{T\alpha}^2} \tilde{\phi}, \quad (4.8)$$

where $v_{T\alpha}^2 = T_\alpha/m_\alpha$ is the thermal velocity. Substituting the above equations into the Poisson equation, we obtain

$$\left(1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - k^2 v_{T\alpha}^2}\right) \tilde{\phi} = 0, \quad \omega_{p\alpha}^2 = \frac{4\pi n_0 e_\alpha^2}{m_\alpha}. \quad (4.9)$$

Defining a dielectric function by

$$\varepsilon(\mathbf{k}, \omega) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - k^2 v_{T\alpha}^2}, \quad (4.10)$$

the dispersion relation of an electrostatic wave is given by $\varepsilon(\mathbf{k}, \omega) = 0$ which is explicitly written as

$$1 = \frac{\omega_{pe}^2}{\omega^2 - k^2 v_{Te}^2} + \frac{\omega_{pi}^2}{\omega^2 - k^2 v_{Ti}^2}. \quad (4.11)$$

There are two kinds of modes: the high frequency and low frequency modes. For the high frequency mode, (4.11) is rewritten as

$$1 - \frac{1}{(\omega/\omega_{pe})^2 - k^2 \lambda_D^2} = \frac{m_e}{m_i} \frac{1}{(\omega/\omega_{pe})^2 - (T_i/T_e)(m_i/m_e)k^2 \lambda_D^2},$$

which gives the Langmuir wave

$$\omega^2 = \omega_{pe}^2 (1 + k^2 \lambda_D^2). \quad (4.12)$$

For the low frequency mode, we may assume $k^2 v_{Ti}^2 < \omega^2 < k^2 v_{Te}^2$ and rewrite (4.11) as

$$1 \simeq -\frac{1}{k^2 \lambda_D^2} + \frac{\omega_{pe}^2}{\omega^2 - k v_{Ti}^2},$$

which gives an ion acoustic mode

$$\omega^2 = \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2} + \frac{T_i}{T_e} k^2 C_s^2, \quad C_s^2 = \frac{T_e}{m_i}. \quad (4.13)$$

The dispersion of the Langmuir wave is caused by electron thermal motions and that of the ion acoustic wave is by breaking of quasi-charge neutrality. The dispersion relations (4.12) and (4.13) are given in Fig. 4.1.

4.1.2 Waves With an External Magnetic Field B_0

First, high frequency waves are considered. Then, ions are just a background to satisfy charge neutrality. The linearized basic equations are given for the magnetic field $\mathbf{B}_0 = (0, 0, B_0)$ by

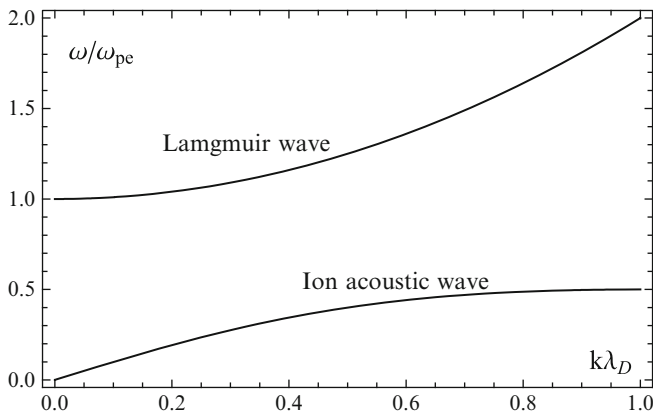


Fig. 4.1 Dispersion relations of the Langmuir wave and an ion acoustic wave whose frequency is multiplied by the mass ratio

$$\frac{\partial \tilde{n}_e}{\partial t} + \nabla \cdot (n_0 \tilde{\mathbf{v}}_e) = 0, \quad (4.14)$$

$$\frac{\partial \tilde{\mathbf{v}}_e}{\partial t} = -\frac{e}{m_e} \left(\tilde{\mathbf{E}} + \frac{1}{c} \tilde{\mathbf{v}}_e \times \mathbf{B}_0 \right) - \frac{T_e}{m_e n_0} \nabla \tilde{n}_e, \quad (4.15)$$

$$\nabla \cdot \tilde{\mathbf{E}} = -4\pi e \tilde{n}_e. \quad (4.16)$$

which give

$$\tilde{v}_{ex} = -\frac{\omega k_x - i|\Omega_e|k_y}{\omega^2 - \Omega_e^2} v_{Te}^2 \left(\frac{e\tilde{\phi}}{T_e} - \frac{\tilde{n}_e}{n_0} \right), \quad (4.17)$$

$$\tilde{v}_{ey} = -\frac{\omega k_y + i|\Omega_e|k_x}{\omega^2 - \Omega_e^2} v_{Te}^2 \left(\frac{e\tilde{\phi}}{T_e} - \frac{\tilde{n}_e}{n_0} \right), \quad (4.18)$$

$$\tilde{v}_{ez} = -\frac{k_z}{\omega} v_{Te}^2 \left(\frac{e\tilde{\phi}}{T_e} - \frac{\tilde{n}_e}{n_0} \right). \quad (4.19)$$

The dispersion equation is given by

$$\left(1 - \frac{\omega_{pe}^2 + k_{\perp}^2 v_{Te}^2}{\omega^2 - \Omega_e^2} \right) k_{\perp}^2 + \left(1 - \frac{\omega_{pe}^2 + k_z^2 v_{Te}^2}{\omega^2} \right) k_z^2 = 0. \quad (4.20)$$

For $\mathbf{k} \perp \mathbf{B}_0$, we have

$$\omega^2 = \Omega_e^2 + \omega_{pe}^2 + k_{\perp}^2 v_{Te}^2, \quad (4.21)$$

which is called the Upper Hybrid wave. For $\mathbf{k} \parallel \mathbf{B}_0$, it certainly reduces to the Langmuir wave.

$$\omega^2 = \omega_{pe}^2 + k_z^2 v_{Te}^2.$$

For a weak magnetic field ($\Omega_e \ll \omega_{pe}$), the dispersion relation is given by

$$\omega^2 \simeq \omega_{pe}^2 \left(1 + \frac{\Omega_e^2 k_{\perp}^2}{\omega_{pe}^2 k^2} \right) (1 + k^2 \lambda_D^2). \quad (4.22)$$

For low frequency waves, the electron density fluctuation is given from (4.17) to (4.19) and (4.16) as

$$\left(1 - \frac{k_{\perp}^2 v_{Te}^2}{\omega^2 - \Omega_e^2} - \frac{k_z^2 v_{Te}^2}{\omega^2} \right) \frac{\tilde{n}_e}{n_0} = - \left(\frac{k_{\perp}^2 v_{Te}^2}{\omega^2 - \Omega_e^2} + \frac{k_z^2 v_{Te}^2}{\omega^2} \right) \frac{e\tilde{\phi}}{T_e}. \quad (4.23)$$

In a similar way, the ion density fluctuation is obtained as

$$\left(1 - \frac{k_{\perp}^2 v_{Ti}^2}{\omega^2 - \Omega_i^2} - \frac{k_z^2 v_{Ti}^2}{\omega^2} \right) \frac{\tilde{n}_i}{n_0} = \left(\frac{k_{\perp}^2 C_s^2}{\omega^2 - \Omega_i^2} + \frac{k_z^2 C_s^2}{\omega^2} \right) \frac{e\tilde{\phi}}{T_e}. \quad (4.24)$$

The dispersion equation is given from the charge neutrality condition $\tilde{n}_e = \tilde{n}_i$,

$$\frac{\frac{k_{\perp}^2 C_s^2}{\omega^2 - \Omega_i^2} + \frac{k_z^2 C_s^2}{\omega^2}}{1 - \frac{k_{\perp}^2 v_{Ti}^2}{\omega^2 - \Omega_i^2} - \frac{k_z^2 v_{Ti}^2}{\omega^2}} + \frac{\frac{k_{\perp}^2 v_{Te}^2}{\omega^2 - \Omega_e^2} + \frac{k_z^2 v_{Te}^2}{\omega^2}}{1 - \frac{k_{\perp}^2 v_{Te}^2}{\omega^2 - \Omega_e^2} - \frac{k_z^2 v_{Te}^2}{\omega^2}} = 0. \quad (4.25)$$

Noting that $|\Omega_e| \gg \omega$, (4.25) is reduced to

$$1 - \left(\frac{k_{\perp}^2 C_s^2}{\omega^2 - \Omega_i^2} + \frac{k_z^2 C_s^2}{\omega^2} \right) \left(1 + \frac{T_i}{T_e} \right) = 0. \quad (4.26)$$

This approximation corresponds to the fact that the electrons move freely along the magnetic field lines and cancel charge separation and therefore the electron density follows the Boltzmann distribution

$$n_e = n_0 e^{e\tilde{\phi}/T} \simeq n_i.$$

Solutions of (4.26) is readily obtained as

$$\omega^2 = \begin{cases} \Omega_i^2 + k^2 C_s^2 \left(1 + \frac{T_i}{T_e} \right), \\ k_z^2 C_s^2 \left(1 + \frac{T_i}{T_e} \right) \end{cases} \quad (4.27)$$

where $\Omega_i^2 > k^2 C_s^2$ has been used. The upper branch is called the electrostatic ion cyclotron wave and the lower branch is simply the ion acoustic wave.

For $\mathbf{k}$ to be nearly perpendicular to $\mathbf{B}_0$, the electrons are not driven along the magnetic field, we have a wave in a frequency range $\Omega_i < \omega < |\Omega_e|$. Equation (4.25) is now rewritten as

$$\left(1 - \frac{k_{\perp}^2 v_{Ti}^2}{\omega^2} \right) \left(-\frac{k_{\perp}^2 v_{Te}^2}{\Omega_e^2} + \frac{k_z^2 v_{Te}^2}{\omega^2} \right) + \left(1 + \frac{k_{\perp}^2 v_{Te}^2}{\Omega_e^2} \right) \frac{k^2 C_s^2}{\omega^2} = 0, \quad (4.28)$$

which gives

$$\omega^2 \simeq |\Omega_e| \Omega_i \left\{ 1 + k_{\perp}^2 \rho_{Le}^2 \left(1 + \frac{T_i}{T_e} \right) + \frac{k_z^2}{k_{\perp}^2} \frac{m_i}{m_e} \right\}, \quad (4.29)$$

where ρ_{Le} is the electron Larmor radius and we have assumed $k_z/k_{\perp} < \sqrt{m_e/m_i}$. This is called the Lower Hybrid wave.

If the charge neutrality is broken, (4.25) has an additional term $k^2 \lambda_{De}^2$ on the right hand side and the $|\Omega_e| \Omega_i$ in (4.29) is replaced by $|\Omega_e| \Omega_i / (1 + \Omega_e^2 / \omega_{pe}^2)$.

4.2 Electromagnetic Waves

Since only electrons can respond to high frequency electromagnetic waves, ions are treated as an immobile background.

4.2.1 Waves Without an External Magnetic Field

Basic equations are given by

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_0 \mathbf{v}_e) = 0, \quad (4.30)$$

$$\frac{\partial \mathbf{v}_e}{\partial t} = -\frac{e}{m} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_e \times \mathbf{B} \right) - \frac{T_e}{mn_0} \nabla n_e, \quad (4.31)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (4.32)$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} (-n_0 e \mathbf{v}_e), \quad (4.33)$$

$$\nabla \cdot \mathbf{E} = -4\pi e n_e, \quad (4.34)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (4.35)$$

Differentiating the Biot Savart equation once with respect to time to eliminate a magnetic field, we have

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = \frac{4\pi n_0 e}{c^2} \frac{\partial \mathbf{v}_e}{\partial t} = -\frac{\Omega_e^2}{c^2} \mathbf{E} + \frac{v_{Te}^2}{c^2} \nabla(\nabla \cdot \mathbf{E}),$$

which reads

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + \omega_{pe}^2 \mathbf{E} - c^2 \nabla^2 \mathbf{E} + (c^2 - v_{Te}^2) \nabla(\nabla \cdot \mathbf{E}) = 0. \quad (4.36)$$

An electromagnetic wave in a plasma is a mixture of longitudinal and transverse waves which is readily understood by comparing the electromagnetic wave equation in a vacuum. Putting

$$\mathbf{E} \propto e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)},$$

the dispersion equation reads

$$(\omega^2 - \omega_{pe}^2 - k^2 c^2) \mathbf{E} + (c^2 - v_{Te}^2) \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) = 0. \quad (4.37)$$

The dispersion relation is given for $\mathbf{k} \parallel \mathbf{E}$ (a longitudinal wave)

$$\omega^2 = \Omega_e^2 + k^2 v_T^2,$$

and for $\mathbf{k} \perp \mathbf{E}$ (a transverse wave)

$$\omega^2 = \Omega_e^2 + k^2 c^2.$$

An electromagnetic wave with $\omega \leq \Omega_e$ cannot propagate in a plasma. This is known as the reflection of an electromagnetic wave at the Van Allen Belt.

4.2.2 Waves With an External Magnetic Field $\mathbf{B}_0$

4.2.2.1 Ordinary Wave ($\mathbf{E} \parallel \mathbf{B}_0$)

Since the electric field is parallel to the magnetic field, the electron motion is not affected by the magnetic field. Therefore, the dispersion relation is the same as the case without magnetic field $\mathbf{B}_0 = 0$:

$$\omega^2 = \Omega_e^2 + k^2 c^2.$$

4.2.2.2 Extraordinary Wave ($\mathbf{E} \perp \mathbf{B}_0$)

The dispersion equation is given by

$$n^2 = \frac{k^2 c^2}{\omega^2} = 1 - \frac{\Omega_e^2}{\omega^2} \frac{\omega^2 - \Omega_e^2}{\omega^2 - (\Omega_e^2 + \Omega_e^2)}, \quad (4.38)$$

where n is a reflection index. Since the wave is reflected when $n^2 = 0$, its frequency is called the cut-off frequency. On the other hand, the wave is absorbed at $n^2 = \infty$, which is called the resonance.

4.2.2.3 L-wave and R-wave

When $\mathbf{k} \parallel \mathbf{B}_0$, the wave is circularly polarized.

$$\text{L-wave } \varepsilon_L = E_x + iE_y : \quad n^2 = 1 - \frac{\Omega_e^2}{\omega(\omega + \Omega_e)}$$

$$\text{R-wave } \varepsilon_R = E_x - iE_y : \quad n^2 = 1 - \frac{\Omega_e^2}{\omega(\omega - \Omega_e)}.$$

The R-wave rotates in the direction of the electron rotation and the L-wave rotates in the direction of the ion rotation. The low frequency wave of the R-waves is called the Whistler wave.

4.3 MHD Wave

A set of the MHD equations are given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (4.39)$$

$$\rho \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right\} = \rho_e \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}, \quad (4.40)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) (p \rho^{-\gamma}) = 0, \quad (4.41)$$

where the pressure tensor is given by (3.12) and (3.13). Equations (4.39)–(4.41) are supplemented with Ohm's law and the Maxwell equations

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B} \right). \quad (4.42)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (4.43)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (4.44)$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho_e, \quad (4.45)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (4.46)$$

A stationary state is specified by

$$\rho = \rho_0, \quad p = p_0, \quad \mathbf{u} = 0, \quad \mathbf{B} = \mathbf{B}_0. \quad (4.47)$$

4.3.1 Alfvén Wave

Suppose that a plasma is incompressible and in the frozen-in state. Linearizing the basic equations with respect to the fluctuations around the stationary state we have

$$\nabla \cdot \delta \mathbf{u} = 0, \quad (4.48)$$

$$\frac{\partial}{\partial t} \delta \mathbf{u} = \frac{1}{4\pi n_0 m_i} \mathbf{B}_0 \times (\nabla \times \delta \mathbf{B}), \quad (4.49)$$

$$\frac{\partial}{\partial t} \delta \mathbf{B} = \nabla \times (\delta \mathbf{u} \times \mathbf{B}_0), \quad (4.50)$$

$$\nabla \cdot \delta \mathbf{B} = 0. \quad (4.51)$$

If we put $\delta A \propto e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$, the dispersion relation is obtained by

$$\omega^2 = \frac{(\mathbf{k} \cdot \mathbf{B}_0)^2}{4\pi n_0 m_i} = (\mathbf{k} \cdot \mathbf{u}_A)^2, \quad (4.52)$$

where the Alfven velocity is defined by

$$u_A = \frac{|\mathbf{B}_0|}{\sqrt{4\pi n_0 m_i}}. \quad (4.53)$$

In an ordinary fluid, there is no transverse wave because there is no stress. However, in a MHD plasma, a transverse wave can propagate because of the tension of the magnetic field line. When a plasma is subject to displacement, the restoration force due to magnetic field lines frozen in the plasma acts to drive the wave.

4.3.2 Magnetosonic Wave

A sound wave propagating across a magnetic field has an extra restoration force in a form of the magnetic pressure. A set of the linearized equations is given by

$$\frac{\partial}{\partial t} \delta \rho = -\rho_0 \nabla \cdot \delta \mathbf{u}, \quad (4.54)$$

$$\frac{\partial}{\partial t} \delta \mathbf{u} = -\frac{1}{\rho_0} \frac{pd(\rho_0)}{d\rho_0} \nabla \delta \rho + \frac{1}{4\pi n_0 m_i} \mathbf{B}_0 \times (\nabla \times \delta \mathbf{B}), \quad (4.55)$$

$$\frac{\partial}{\partial t} \delta \mathbf{B} = \nabla \times (\delta \mathbf{u} \times \mathbf{B}_0), \quad (4.56)$$

$$\nabla \cdot \delta \mathbf{B} = 0. \quad (4.57)$$

The dispersion relation is given by

$$\omega^2 = \frac{1}{2} \left\{ k^2 (u_s^2 + u_A^2) \pm k^2 \sqrt{(u_s^2 + u_A^2) - 4(\mathbf{k} \cdot \mathbf{u}_A)^2 / k^2} \right\}, \quad (4.58)$$

where $u_s = \sqrt{(1/\rho_0)(dp(\rho_0)/d\rho_0)}$. For $\mathbf{k} \perp \mathbf{B}_0$, we have the dispersion relation for the magnetosonic wave

$$\omega^2 = k^2 (u_s^2 + u_A^2). \quad (4.59)$$

For $\mathbf{k} \parallel \mathbf{B}_0$, the dispersion relation reduces to $\omega^2 = k^2 u_s^2$ as a longitudinal wave and $\omega^2 = k^2 u_A^2$ as a transverse wave.

4.4 Wave Energy

Equations for electromagnetic fields in a plasma is given by

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j}, \quad (4.60)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (4.61)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4.62)$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (4.63)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (4.64)$$

Introducing an electric displacement field $\mathbf{D}$ through

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{j}, \quad (4.65)$$

we have

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla \cdot \mathbf{D} = 0, \quad (4.66)$$

which gives

$$\frac{1}{4\pi} \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) + \frac{c}{4\pi} \nabla \cdot (\mathbf{E} \times \mathbf{B}) = 0. \quad (4.67)$$

The first term is interpreted as change in the electromagnetic field energy W ,

$$\frac{\partial W}{\partial t} = \frac{1}{4\pi} \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right). \quad (4.68)$$

The second term is the Poynting vector which represents the transport of electromagnetic field energy. The electric displacement field is a physical quantity to describe the plasma response to the electric field and is represented by

$$\mathbf{D}(\mathbf{k}, \omega) = \varepsilon(\mathbf{k}, \omega) \mathbf{E}(\mathbf{k}, \omega). \quad (4.69)$$

For the case that the field amplitude varies slowly in time, we have

$$\frac{\partial \mathbf{D}}{\partial t} = -i\omega \varepsilon(\mathbf{k}, \omega(\mathbf{k})) \mathbf{E} + \left(\frac{\partial}{\partial \omega} (\omega \varepsilon) \right)_{\omega=\omega(\mathbf{k})} \frac{\partial \mathbf{E}}{\partial t} = \left(\frac{\partial}{\partial \omega} (\omega \varepsilon) \right)_{\omega=\omega(\mathbf{k})} \frac{\partial \mathbf{E}}{\partial t},$$

where $\varepsilon(\mathbf{k}, \omega(\mathbf{k})) = 0$ has been used. Thus, the energy associated with the mode $\mathbf{k}$ is obtained by

$$W_k = \frac{1}{8\pi} (\langle E_k^2 \rangle \frac{\partial}{\partial \omega} (\omega \varepsilon)|_{\omega=\omega(\mathbf{k})} + \langle B_k^2 \rangle). \quad (4.70)$$

Therefore, the wave energy consists of the electric field energy in vacuum and the energy stored in the plasma, that is, the kinetic energy of the plasma particles moving coherently with the wave. In fact, for the plasma oscillation whose dielectric function is given by

$$\varepsilon(\mathbf{k}, \omega) = 1 - \frac{\omega_p^2 + k^2 v_T^2}{\omega^2},$$

we have

$$\frac{\partial}{\partial \omega}(\omega \varepsilon)|_{\omega=\omega(k)} = 1 + \frac{\omega_p^2 + k^2 v_T^2}{\omega^2(k)} = 2,$$

which is exactly two times of the electric field energy in a vacuum.

Introducing the number of waves by N_k , the wave energy and the wave momentum are expressed by

$$W_k = \omega(k) N_k, \quad P_k = k N_k = \frac{k}{\omega(k)} W_k. \quad (4.71)$$

Since the ratio of the wave energy to the wave momentum is equal to the phase velocity, the wave momentum is larger for smaller phase velocity with fixed wave energy.

When the medium moves with a constant velocity V , the wave energy W_k in a moving frame and the wave energy W_k^0 in the rest frame are related through

$$W_k^0 = \omega^0(k) N_k = (\omega(k) + \mathbf{k} \cdot \mathbf{V}) N_k, \quad (4.72)$$

where the Doppler shift is taken into account based on the fact that the number of the waves does not depend on reference frames.

4.5 Negative Energy Wave

Although in a medium at equilibrium the wave energy is positive definite, in a medium at non-equilibrium, the wave energy is not always positive. If wave energy is negative, energy has to be extracted from the medium in order to excite waves.

The energy of a plane wave with a frequency $\omega = \omega(k)$ propagating in the x direction is given by

$$W_k = \omega(k) N_k.$$

When the medium moves with a constant velocity $V = -v_0$ in a direction opposite to the wave propagation direction, the wave energy in the rest frame W_k^0 is given by

$$W_k^0 = (\omega(k) - k v_0) N_k.$$

If the velocity of the medium is increased, the wave in the rest frame does not move when $v_0 = \omega/k$ and the wave energy becomes zero. Further increase in the velocity of the medium makes the wave energy negative. Thus, when the velocity of the medium is larger than the phase velocity of the wave, the wave is carried by the medium in the negative x direction. Certainly, the wave propagates in the positive x direction relative to the medium but the energy of the wave left behind the medium is negative. The sign of the wave energy is given from (4.70) by

$$\frac{\partial}{\partial \omega}(\omega \varepsilon(\mathbf{k}, \omega))|_{\omega=\omega(\mathbf{k})}. \quad (4.73)$$

An example of negative energy waves is given by an ion beam-plasma system. The dispersion relation of the wave excited in such a system is obtained in the following. Since a low frequency wave is excited in this system, the electron obeys the Boltzmann distribution:

$$\delta n_e = n_0(e^{\delta \phi / T_e} - 1) \simeq n_0 \frac{e \delta \phi}{T_e}.$$

The equation of continuity and the equation of motion for the ion and the ion-beam are given by

$$\frac{\partial n_{i,b}}{\partial t} + \nabla \cdot (n_{i,b} \mathbf{v}_{i,b}) = 0, \quad (4.74)$$

$$\frac{\partial \mathbf{v}_{i,b}}{\partial t} + (\mathbf{v}_{i,b} \cdot \nabla) \mathbf{v}_{i,b} = -\frac{e}{m_i} \nabla \phi, \quad (4.75)$$

where the ion density is denoted by n_i , the ion beam density by n_b , and the beam velocity by v_b . The Poisson equation is

$$\nabla^2 \phi = -\sum_{\alpha} 4\pi e_{\alpha} n_{\alpha}.$$

Putting the equilibrium and the fluctuation as

$$\begin{pmatrix} n_{i,b} \\ \mathbf{v}_{i,b} \\ \phi \end{pmatrix} = \begin{pmatrix} n_{0,b} \\ \mathbf{v}_b \\ 0 \end{pmatrix} + \begin{pmatrix} \delta n_i \\ \delta \mathbf{v}_{i,b} \\ \delta \phi \end{pmatrix} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (4.76)$$

and linearizing the basic equations, we have

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}_b) \delta n_{i,b} + i n_0 \mathbf{k} \cdot \delta \mathbf{v}_{i,b} = 0,$$

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}_b) \delta \mathbf{v}_{i,b} = -i \frac{e}{m_i} \mathbf{k} \delta \phi,$$

$$-k^2 \delta \phi = -\sum_{\alpha} 4\pi e_{\alpha} \delta n_{\alpha},$$

leading to the dispersion relation

$$\varepsilon(\mathbf{k}, \omega) = 1 + \frac{1}{k^2 \lambda_D^2} - \frac{\omega_{pi}^2}{\omega^2} - b \frac{\omega_{pi}^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_b)^2}, \quad (4.77)$$

where

$$b = \frac{n_b}{n_0}.$$

Rewriting the dispersion relation

$$(\omega^2 - \omega^2(k))(\omega - \mathbf{k} \cdot \mathbf{v}_b)^2 = b\omega^2\omega^2(k),$$

where

$$\omega^2(k) = \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2},$$

we have for the ion acoustic mode

$$\omega^2 \simeq \omega^2(k) + b \frac{\omega^4(k)}{(\omega(k) - \mathbf{k} \cdot \mathbf{v}_b)^2}, \quad (4.78)$$

and for the fast and slow beam modes under the condition $(\mathbf{k} \cdot \mathbf{v}_b)^2 > \omega(k)^2$

$$\omega \simeq \mathbf{k} \cdot \mathbf{v}_b \pm \sqrt{b \frac{(\mathbf{k} \cdot \mathbf{v}_b)^2 \omega^2(k)}{(\mathbf{k} \cdot \mathbf{v}_b)^2 - \omega^2(k)}}. \quad (4.79)$$

The sign of the wave energy is given by

$$\frac{\partial}{\partial \omega}(\omega \mathcal{E}) = 2 \frac{\omega_{pi}^2}{\omega^2} \left\{ 1 + b \left(\frac{\omega}{\omega - \mathbf{k} \cdot \mathbf{v}_b} \right)^3 \right\},$$

which becomes negative for

$$\frac{\mathbf{k} \cdot \mathbf{v}_b}{1 + b^{1/3}} < \omega < \mathbf{k} \cdot \mathbf{v}_b.$$

This condition is satisfied by the slow beam mode as long as

$$(\mathbf{k} \cdot \mathbf{v}_b)^2 > \left\{ 1 + \frac{(1 + b^{1/3})^2}{b^{1/3}} \right\} \omega(k)^2.$$

Thus, the energy is transferred from the slow beam mode to the fast beam mode and the ion acoustic mode and as a result all three waves grow.

4.6 Instabilities in Plasmas

A spatially confined plasma is inevitably inhomogeneous in the density and temperature so that a variety of motions are excited in the course of the relaxation to the equilibrium. Instabilities in plasmas are to convert the free energy stored in the inhomogeneity into waves and macroscopic motions. There are many ways to store free energy in flows, inhomogeneity and anisotropy in thermodynamic quantities

and velocity distributions, leading to various kinds of instabilities. These instabilities bring a large amplitude coherent wave or wave turbulence depending on the structure of the wave number spectrum of the instability growth rates and on the types of nonlinear interactions to saturate the instabilities.

4.6.1 Streaming Instability

A relative motion of constituent fluids in a plasma drives an instability to relax to the equilibrium by exciting waves. For simplicity, we assume that there is no magnetic field and a warm electron fluid drifts relative to a cold ion fluid ($T_i = 0$). The physical quantities are written as

$$\begin{pmatrix} n_e \\ n_i \\ v_e \\ v_i \\ \phi \end{pmatrix} = \begin{pmatrix} n_0 \\ n_0 \\ v_{e0} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} n_0 \delta n_e \\ n_0 \delta n_i \\ \delta v_e \\ \delta v_i \\ \delta \phi \end{pmatrix} e^{i(k \cdot x - \omega t)}. \quad (4.80)$$

From the equation of continuity and the equation of motion, we have

$$\delta n_e = \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_{e0}} \mathbf{k} \cdot \delta \mathbf{v}_e, \quad (4.81)$$

$$\delta n_i = \frac{1}{\omega} \mathbf{k} \cdot \delta \mathbf{v}_i, \quad (4.82)$$

$$\delta \mathbf{v}_e = -\frac{\mathbf{k} v_{Te}^2}{\omega - \mathbf{k} \cdot \mathbf{v}_{e0}} \left(\frac{e}{T_e} \delta \phi - \delta n_e \right) \quad (4.83)$$

$$\delta \mathbf{v}_i = C_s^2 \frac{\mathbf{k}}{\omega} \frac{e}{T_e} \delta \phi \quad (4.84)$$

$$\frac{e}{T_e} \delta \phi = -\frac{1}{k^2 \lambda_D^2} (\delta n_i - \delta n_e), \quad (4.85)$$

which give the dispersion equation

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_{e0})^2 - k^2 v_{Te}^2} = 0. \quad (4.86)$$

Equation (4.86) is the 4-th order algebraic equation in ω . Therefore, the instability is expected when the solutions become complex. When $v_{e0} < v_{Te}$, the solutions are all real and the instability does not occur. In this case, the drift is not distinguishable from the thermal motion and the Landau damping dominates. Therefore in the following, we consider the case of $v_{e0} > v_{Te}$ under which we may neglect v_{Te} for

simplicity and use the following dispersion equation instead of (4.86)

$$\varepsilon(\mathbf{k}, \omega) = 1 - \frac{\omega_{\text{pi}}^2}{\omega^2} - \frac{\omega_{\text{pe}}^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_{\text{e0}})^2} = 0. \quad (4.87)$$

The complex solutions of (4.87) are realized under the condition

$$\varepsilon(\mathbf{k}, \omega_0) < 0, \quad (4.88)$$

where ω_0 is determined by

$$\frac{\partial \varepsilon(\mathbf{k}, \omega_0)}{\partial \omega} = 0, \quad (4.89)$$

giving

$$\omega_0 = \frac{\mathbf{k} \cdot \mathbf{v}_{\text{e0}}}{1 + (m_i/m_e)^{1/3}}. \quad (4.90)$$

The instability condition posed by (4.88) is given by

$$\varepsilon(\mathbf{k}, \omega_0) = 1 - \left(\frac{\mathbf{k}_0 \cdot \mathbf{v}_{\text{e0}}}{\mathbf{k} \cdot \mathbf{v}_{\text{e0}}} \right)^2 < 0, \quad \mathbf{k}_0 \cdot \mathbf{v}_{\text{e0}} = \omega_{\text{pi}} \left(1 + \left(\frac{m_i}{m_e} \right)^{1/3} \right). \quad (4.91)$$

Thus, the wave number of the unstable waves is less than the critical wave number $\mathbf{k}_0$ fixed for the electron streaming velocity. The critical instability growth rate is estimated by expanding the dielectric function with respect to ω around ω_0 and using (4.89)

$$\varepsilon(\mathbf{k}, \omega) = \varepsilon(\mathbf{k}, \omega_0) + \frac{1}{2} \frac{\partial^2 \varepsilon(\mathbf{k}, \omega_0)}{\partial \omega^2} (\omega - \omega_0)^2 = 0, \quad (4.92)$$

which gives

$$\omega = \omega_0 + \left\{ -\varepsilon(\mathbf{k}, \omega_0) / \left(\frac{1}{2} \frac{\partial^2 \varepsilon(\mathbf{k}, \omega_0)}{\partial \omega^2} \right) \right\}^{1/2}. \quad (4.93)$$

Here

$$\frac{\partial^2 \varepsilon(\mathbf{k}, \omega_0)}{\partial \omega^2} = -6 \frac{\omega_{\text{pi}}^2}{\omega_0^4} \left(1 + \left(\frac{m_e}{m_i} \right)^{1/3} \right) \simeq -6 \frac{\omega_{\text{pi}}^2}{\omega_0^4}. \quad (4.94)$$

Therefore, the growth rate near the critical instability point is given by

$$\gamma = \sqrt{\frac{1}{3} \left\{ \left(\frac{\mathbf{k}_0 \cdot \mathbf{v}_{\text{e0}}}{\mathbf{k} \cdot \mathbf{v}_{\text{e0}}} \right)^2 - 1 \right\} \frac{\omega_0^2}{\omega_{\text{pi}}}}. \quad (4.95)$$

Near the critical point, the frequency ω_0 is proportional to $\mathbf{k} \cdot \mathbf{v}_{\text{e0}}$, indicating that the wave is an oscillation of the electron stream.

When the oscillation $\mathbf{k} \cdot \mathbf{v}_{e0}$ of the electron stream resonates with the electron plasma oscillation ω_{pe} , the large growth rate is anticipated because of the feedback process of the self-interaction of the electron fluid. We rewrite (4.87) as

$$\frac{(\omega - \mathbf{k} \cdot \mathbf{v}_{e0} - \omega_{pe})(\omega - \mathbf{k} \cdot \mathbf{v}_{e0} + \omega_{pe})}{(\omega - \mathbf{k} \cdot \mathbf{v}_{e0})^2} = \frac{\omega_{pi}^2}{\omega^2}, \quad (4.96)$$

and use the resonant condition $\mathbf{k} \cdot \mathbf{v}_{e0} = \omega_{pe}$ and $\omega < \mathbf{k} \cdot \mathbf{v}_{e0}$ to obtain

$$-2 \frac{\omega}{\omega_{pe}} = \frac{\omega_{pi}^2}{\omega^2}, \quad (4.97)$$

giving

$$\Re \omega = -\frac{1}{2} \left(\frac{m_e}{2m_i} \right)^{1/3} \omega_{pe}, \quad \Im \omega = \frac{\sqrt{3}}{2} \left(\frac{m_e}{2m_i} \right)^{1/3} \omega_{pe}. \quad (4.98)$$

The growth rate in the resonance is so large to be compatible with the real frequency. This is because the energy stored in the electron stream is released to be converted into the wave. When a small fluctuation in the potential acts on the electrons to accelerate at the crest and decelerate in the trough and on the ions in the reverse way, resulting in the bunching of the electrons at the trough and of the ions at the crest, which in turn builds the potential further. This instability is called the Buneman instability. The growth rate is certainly maximum at the resonance. The dispersion relation obtained by solving (4.87) is depicted in Fig. 4.2 in which two branches of the Doppler shifted electron plasma oscillation mode ($\omega_{pe} + \mathbf{k} \cdot \mathbf{v}_{e0}$) and the ion plasma oscillation mode (ω_{pi}) are depicted. The other modes $\omega_{pe} + \mathbf{k} \cdot \mathbf{v}_{e0}$ and $-\omega_{pi}$ are outside the figure region. A dotted line is the growth rate which is maximum at the resonance $\mathbf{k} \cdot \mathbf{v}_{e0} = \omega_{pe}$. In the unstable region of the wave number, the Doppler

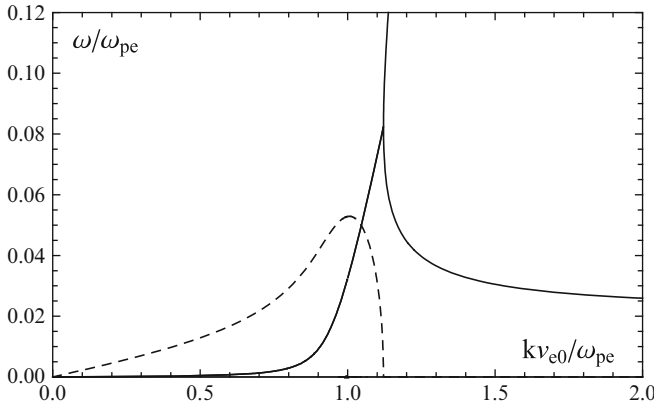


Fig. 4.2 Dispersion relations of the Buneman instability obtained by solving (4.87) for $m_e/m_i = 1/2000$. A dotted line is the growth rate

shifted electron plasma oscillation mode and the ion plasma oscillation mode are degenerate.

Far from the resonance, we approximate $\omega - \mathbf{k} \cdot \mathbf{v}_{e0} \sim -\mathbf{k} \cdot \mathbf{v}_{e0}$ in the last term of (4.87) to get

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\mathbf{k} \cdot \mathbf{v}_{e0})^2} = 0, \quad (4.99)$$

giving for $(\mathbf{k} \cdot \mathbf{v}_{e0})^2 / \omega_{pe}^2 < 1$

$$\gamma = \frac{\omega_{pi}}{\sqrt{\frac{\omega_{pe}^2}{(\mathbf{k} \cdot \mathbf{v}_{e0})^2} - 1}}. \quad (4.100)$$

The wave energy is shown in the previous section to be proportional to $\partial \omega \varepsilon(\mathbf{k}, \omega) / \partial \omega$. From (4.87), we have

$$\frac{\partial}{\partial \omega} (\omega \varepsilon(\mathbf{k}, \omega)) = 1 + \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2 (\omega + \mathbf{k} \cdot \mathbf{v}_{e0})}{(\omega - \mathbf{k} \cdot \mathbf{v}_{e0})^3}. \quad (4.101)$$

From this, the ion oscillation branch (the second term of (4.101)) has a positive contribution and the electron stream oscillation branch (the third term of (4.101) with $\omega < \mathbf{k} \cdot \mathbf{v}_{e0}$) has a negative contribution to the wave energy. In the course of the development of the instability, the energy is transferred from the electron stream oscillation to the ion plasma oscillation, resulting in the growth of both.

When a beam is launched into a plasma, a similar analysis gives again an instability whose growth rate depends on the ratio of the beam density to the plasma density. In this case in addition to the eigen modes in the plasma, there appear modes related to the beam. Therefore, beam-plasma systems are rich in nonlinear wave phenomena.

4.6.2 Gradient Instability

Plasmas confined in experimental devices are inevitably inhomogeneous in the densities. In a plasma whose density is inhomogeneous, the density gradient causes the diamagnetic drift in a magnetic field. A perturbed electric field perpendicular to the magnetic field drives an $\mathbf{E} \times \mathbf{B}$ drift which carries the plasma from the high density side to the low density side and produces a density perturbation. From the equation of continuity, the density perturbation propagates with the diamagnetic drift velocity in the direction perpendicular to the density gradient when the wavelength of the perturbation along the magnetic field is sufficiently small. Thus, the phase difference between the density and potential fluctuations is real and 90° . The electrons can flow along the magnetic field to establish a thermodynamic equilibrium, leading to the Boltzmann distribution. Thus, the drift wave is oscillatory and the potential does not

build up if the plasma is collisionless. Now, suppose that the plasma is collisional. Then, the electrons lag behind the potential, resulting in a complex phase difference between the density and potential fluctuations. The complex phase difference drives the plasma to the region where the plasma density has already been high and vice versa. Thus, the drift wave is excited. Although there are some other causes to drive the drift wave unstable, here we only consider a collisional effect on the stability of the drift wave.

Suppose that the plasma is confined in a cylindrical device. The magnetic field is in the z direction and the density gradient is in the radial direction $n_0(r)$. The basic equations are given by

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0, \quad (4.102)$$

$$\begin{aligned} \frac{\partial \mathbf{v}_\alpha}{\partial t} + (\mathbf{v}_\alpha \cdot \nabla) \mathbf{v}_\alpha &= \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{T_\alpha}{m_\alpha} \nabla \ln n_\alpha \\ &+ \sum_\beta \frac{1}{\tau_{\alpha\beta}} (\mathbf{v}_\beta - \mathbf{v}_\alpha), \end{aligned} \quad (4.103)$$

where $\tau_{\alpha\beta}$ is a collision time of the particle α colliding to the particle β and $m_\alpha n_\alpha / \tau_{\alpha\beta} = m_\beta n_\beta / \tau_{\beta\alpha}$. In the following, we consider that the plasma is fully ionized and the collisions are between the electrons and the ions. The cross field drift is given by

$$\mathbf{v}_{\alpha\perp} = \mathbf{v}_E + \mathbf{v}_{\alpha D} + \mathbf{v}_{\alpha P} + \mathbf{v}_{\alpha C}, \quad (4.104)$$

where $\mathbf{v}_E$ is the $\mathbf{E} \times \mathbf{B}$ drift, $\mathbf{v}_{\alpha D}$ is the diamagnetic drift, $\mathbf{v}_{\alpha P}$ is the polarization drift, and $\mathbf{v}_{\alpha C}$ is the collisional drift, respectively, defined by

$$\mathbf{v}_E = c \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \quad (4.105)$$

$$\mathbf{v}_{\alpha D} = -\frac{c T_\alpha}{e_\alpha B^2} \nabla \ln n_\alpha \times \mathbf{B}, \quad (4.106)$$

$$\mathbf{v}_{\alpha P} = \frac{c}{\Omega_\alpha B^2} \frac{\partial}{\partial t} \left\{ \mathbf{E} - \frac{T_\alpha}{e_\alpha} \nabla \ln n_\alpha \right\} \times \mathbf{B}, \quad (4.107)$$

$$\begin{aligned} \mathbf{v}_{\alpha C} &= \sum_\beta \frac{1}{\Omega_\alpha \tau_{\alpha\beta}} \tilde{\mathbf{z}} \times (\mathbf{v}_{\alpha\perp} - \mathbf{v}_{\beta\perp}) \\ &= -\sum_\beta \frac{c}{\Omega_\alpha \tau_{\alpha\beta} B} \left\{ \frac{T_\alpha}{e_\alpha} \nabla_\perp \ln n_\alpha - \frac{T_\beta}{e_\beta} \nabla_\perp \ln n_\beta \right\}. \end{aligned} \quad (4.108)$$

Note under the quasi neutrality condition and $|\Omega_e| \tau_{ei} = \Omega_i \tau_{ie}$ that

$$\nabla_\perp \cdot \mathbf{v}_E = 0, \quad \nabla_\perp \cdot (n_\alpha \mathbf{v}_{\alpha D}) = 0, \quad \mathbf{v}_{eC} = \mathbf{v}_{iC}. \quad (4.109)$$

The parallel drift is determined by

$$\frac{\partial v_{\alpha\parallel}}{\partial t} = -\frac{e_{\alpha}}{m_{\alpha}} \nabla_{\parallel} \phi - v_{T\alpha}^2 \nabla_{\parallel} \ln n_{\alpha} + \frac{1}{\tau_{\alpha\beta}} (v_{\beta\parallel} - v_{\alpha\parallel}). \quad (4.110)$$

Expressing the fluctuations by $\{n_{\alpha}, v_{\alpha}, \phi\} \propto e^{i(\mathbf{k}_{\perp} \cdot \mathbf{r}_{\perp} + k_{\parallel} z - \omega t)}$, assuming that $\omega \tau_{ei} < 1$ and neglecting the nonlinear term, we have

$$v_{e\parallel} = ik_{\parallel} \tau_{ei} v_{Te}^2 \left(\frac{e\phi}{T_e} - \frac{\tilde{n}_e}{n_0} \right) + v_{i\parallel}, \quad (4.111)$$

$$\begin{aligned} v_{i\parallel} &= \frac{1}{-i\omega} \left\{ -ik_{\parallel} C_s^2 \left(\frac{e\phi}{T_e} - \frac{T_i}{T_e} \frac{n_i}{n_0} \right) + \frac{1}{\tau_{ie}} (v_{e\parallel} - v_{i\parallel}) \right\} \\ &= C_s^2 \left(1 - \frac{T_i}{T_e} \right) \frac{k_{\parallel}}{\omega} \frac{n_e}{n_0}. \end{aligned} \quad (4.112)$$

From (4.111) and (4.112), we have $|v_{e\parallel}| \gg |v_{i\parallel}|$. The equation of continuity (4.102) together with (4.105)–(4.109) reads

$$\frac{\partial n_{\alpha}}{\partial t} = -\mathbf{v}_E \cdot \nabla_{\perp} n_{\alpha} - \nabla_{\perp} \cdot \{n_{\alpha} (\mathbf{v}_{\alpha p} + \mathbf{v}_{\alpha c})\} - \nabla_{\parallel} (n_{\alpha} v_{\alpha\parallel}). \quad (4.113)$$

For the electron density, we have in the lowest order

$$n_e = \frac{c}{\omega e B^2} \mathbf{B} \cdot (\mathbf{k}_{\perp} \times \nabla_{\perp} n_0) \phi, \quad (4.114)$$

which is substituted into (4.111) to give

$$v_{e\parallel} = ik_{\parallel} \tau_{ei} v_{Te}^2 \left\{ 1 - \frac{c T_e}{\omega e B^2} \mathbf{B} \cdot (\mathbf{k}_{\perp} \times \nabla_{\perp} \ln n_0) \right\} \frac{e\phi}{T_e}. \quad (4.115)$$

where $|v_{e\parallel}| \gg |v_{i\parallel}|$ has been used. Equation (4.113) for the electron and ion fluid with the quasi neutrality condition gives

$$\begin{aligned} \nabla \cdot n_0 (\mathbf{v}_e - \mathbf{v}_i) &= \nabla_{\perp} \cdot n_0 (\mathbf{v}_{ep} - \mathbf{v}_{ip}) + \nabla_{\parallel} n_0 (v_{e\parallel} - v_{i\parallel}) \\ &\simeq -n_0 (\nabla_{\perp} \cdot \mathbf{v}_{ip} - \nabla_{\parallel} v_{e\parallel}) = 0, \end{aligned} \quad (4.116)$$

where we have used that $|v_{ip}| \gg |v_{ep}|$ and remained only linear terms with respect to the fluctuations. Substituting (4.107) and (4.115) into (4.116), we obtain the dispersion relation

$$\omega - \omega_* - i\omega^2 \tau_* = 0, \quad (4.117)$$

where

$$\omega_* = \frac{c T_e}{e B^2} \mathbf{B} \cdot (\mathbf{k}_{\perp} \times \nabla_{\perp} \ln n_0), \quad \tau_* = \left(\frac{k_{\perp}}{k_{\parallel}} \right)^2 \frac{1}{\Omega_i |\Omega_e| \tau_{ei}}. \quad (4.118)$$

Since the condition for the Landau damping to be ignored is given by

$$v_{Ti} \ll \frac{\omega}{k_{\parallel}} \ll v_{Te}, \quad (4.119)$$

we may assume

$$\omega_* \tau_* \simeq \frac{k_{\perp}^2 v_{Te}^2}{\Omega_e^2} \frac{1}{\Omega_i \tau_{ei}} \left(\frac{k_{\perp}}{k_{\parallel}} \right)^2 \ll 1, \quad (4.120)$$

and replace ω by ω_* in the last term of (4.117) to obtain

$$\omega = \omega_* + i\omega_*^2 \tau_*. \quad (4.121)$$

A good way to see the cause of the instability is to check the response of the electron density fluctuation to the potential fluctuation. The electron density fluctuation is estimated from (4.113) as

$$\omega \frac{n_e}{n_0} + \frac{c T_e}{e B^2} \frac{e \phi}{T_e} (\mathbf{k}_{\perp} \times \mathbf{B}) \cdot \nabla_{\perp} \ln n_0 - k_{\parallel} v_{e\parallel} = 0, \quad (4.122)$$

which leads with (4.115) to

$$\frac{n_e}{n_0} = \left\{ \frac{\omega_*}{\omega} + i \frac{\tau_{ei}}{\omega} k_{\parallel}^2 v_{Te}^2 \left(1 - \frac{\omega_*}{\omega} \right) \right\} \frac{e \phi}{T_e}. \quad (4.123)$$

Equation (4.123) shows a finite phase difference between the electron density fluctuation and the potential, implying that the electron responds to the potential not statically but with a lag. The phase difference comes from the collision frequency which is responsible for the instability.

If the plasma is weakly ionized and the collisions with neutral particles are dominant, we have instead of (4.117)

$$\omega \left(1 + \frac{\tau_*}{\tau_{in}} \right) - \omega_* - i\omega_*^2 \tau_* = 0, \quad (4.124)$$

where τ_{ei} in the definition of τ_* is replaced by $\tau_e = \tau_{ei} \tau_{en} / (\tau_{ei} + \tau_{en})$ and τ_{an} is the collision time with the neutral particles. Therefore as long as $\tau_* < \tau_{in}$, the result does not much change.

4.6.3 Gravitational Instability

Gravitational unstable configurations often appear in a plasma such that the higher density part pushes the less density part or the less density part supports the higher density part. For example, in a cylindrical magnetized plasma the density is usually

higher at the center and decreases toward the edge. When an electric field drives the plasma in the azimuthal direction by the $E \times B$ drift, the plasma is subject to the centrifugal force. In that situation, the inner higher density plasma pushes the outer less density plasma to bring the plasma to a dynamically unstable state. Plasmas in a curved magnetic field are also in a similar situation.

Now, suppose that there is a cold nonuniform plasma whose density gradient is in the direction as that for a gravitational field $\mathbf{g}$ and a uniform magnetic field $\mathbf{B}$ is in the z direction. In the equilibrium, the equation of motion for the ions is given by

$$\frac{e}{c} \mathbf{v}_{i0} \times \mathbf{B} + m_i \mathbf{g} = 0, \quad (4.125)$$

where we have assumed that $\mathbf{g}$ is constant and therefore $\mathbf{v}_{i0}$ is also. Then, we have a usual gravitational drift

$$\mathbf{v}_{i0} = -\frac{1}{\Omega_i} \hat{\mathbf{z}} \times \mathbf{g}. \quad (4.126)$$

The electrons drift in the opposite direction to the ions, but the electron drift velocity $\mathbf{v}_{e0}$ is small by m_e/m_i compared with the ion drift velocity.

When a perturbation is applied to the plasma, the ion equation of motion is linearized around the equilibrium to give

$$\frac{\partial}{\partial t} \tilde{\mathbf{v}}_i + (\mathbf{v}_{i0} \cdot \nabla) \tilde{\mathbf{v}}_i = \frac{e}{m_i} \left(-\nabla \tilde{\phi} + \frac{1}{c} \tilde{\mathbf{v}}_i \times \mathbf{B} \right) + \frac{\tilde{n}_i}{n_0} (\Omega_i \mathbf{v}_{i0} \times \hat{\mathbf{z}} + \mathbf{g}), \quad (4.127)$$

where the last term on the right hand side vanishes because of (4.125). For the perturbation in the form of $e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$, (4.127) becomes

$$(\omega - \mathbf{k} \cdot \mathbf{v}_{i0}) \tilde{\mathbf{v}}_i = \frac{e}{m_i} \mathbf{k} \tilde{\phi} - i \Omega_i \hat{\mathbf{z}} \times \tilde{\mathbf{v}}_i, \quad (4.128)$$

which gives

$$\tilde{\mathbf{v}}_{i\perp} = \frac{e}{m_i} \frac{\mathbf{k}_\perp (\omega - \mathbf{k} \cdot \mathbf{v}_{i0}) - i \Omega_i \hat{\mathbf{z}} \times \mathbf{k}}{(\omega - \mathbf{k} \cdot \mathbf{v}_{i0})^2 - \Omega_i^2} \tilde{\phi}, \quad (4.129)$$

and

$$\tilde{v}_{i\parallel} = \frac{e}{m_i} \frac{k_\parallel}{(\omega - \mathbf{k} \cdot \mathbf{v}_{i0})} \tilde{\phi}. \quad (4.130)$$

For the electron equation of motion, we have

$$\frac{\partial}{\partial t} \tilde{\mathbf{v}}_e = -\frac{e}{m_e} \left(-\nabla \tilde{\phi} + \frac{1}{c} \tilde{\mathbf{v}}_e \times \mathbf{B} \right), \quad (4.131)$$

which gives for $|\Omega_e| \gg \partial/\partial t$

$$\tilde{\mathbf{v}}_{e\perp} = i \frac{e}{m_i \Omega_i} \hat{\mathbf{z}} \times \mathbf{k} \tilde{\phi}, \quad \tilde{v}_{e\parallel} = \frac{e k_\parallel}{m_e \omega} \tilde{\phi}. \quad (4.132)$$

The equation of continuity for the ions reads

$$\frac{\partial \tilde{n}_i}{\partial t} + \nabla \cdot (n_0 \tilde{\mathbf{v}}_i + \tilde{n}_i \mathbf{v}_{i0}) = 0, \quad (4.133)$$

leading to

$$\begin{aligned} \frac{\tilde{n}_i}{n_0} &= \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_{i0}} (\mathbf{k} - i \nabla \ln n_0) \cdot \tilde{\mathbf{v}}_i \\ &= \frac{e}{m_i} \tilde{\phi} \left\{ \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v}_{i0})^2 - \Omega_i^2} (\mathbf{k}_\perp - i \nabla_\perp \ln n_0) \cdot \left(\mathbf{k}_\perp - \frac{i \Omega_i}{\omega - \mathbf{k} \cdot \mathbf{v}_{i0}} \hat{\mathbf{z}} \times \mathbf{k} \right) \right. \\ &\quad \left. + \frac{k_\parallel^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_{i0})^2} \right\}. \end{aligned} \quad (4.134)$$

For the electron equation of continuity, we have

$$\frac{\tilde{n}_e}{n_0} = \frac{1}{\omega} (\mathbf{k} - i \nabla \ln n_0) \cdot \tilde{\mathbf{v}}_e = \frac{e}{m_e \omega |\Omega_e|} \tilde{\phi} \left\{ \hat{\mathbf{z}} \times \mathbf{k} \cdot \nabla \ln n_0 + \frac{|\Omega_e|}{\omega} k_\parallel^2 \right\}. \quad (4.135)$$

The quasi-neutrality condition $\tilde{n}_i = \tilde{n}_e$ gives the following dispersion equation under the conditions $k_\parallel \simeq 0$ and $(\omega - \mathbf{k} \cdot \mathbf{v}_{i0})^2 \ll \Omega_i^2$

$$\begin{aligned} \omega(\omega - \mathbf{k} \cdot \mathbf{v}_{i0}) &= \frac{\Omega_i (\mathbf{k} \cdot \mathbf{v}_{i0}) (\hat{\mathbf{z}} \times \mathbf{k} \cdot \nabla \ln n_0)}{k_\perp^2} \\ &= \frac{\mathbf{k} \cdot (\hat{\mathbf{z}} \times \mathbf{g}) \mathbf{k} \cdot (\hat{\mathbf{z}} \times \nabla \ln n_0)}{k_\perp^2}, \end{aligned} \quad (4.136)$$

whose solution is expressed by

$$\omega = \frac{1}{2} \left\{ \mathbf{k} \cdot \mathbf{v}_{i0} \pm \sqrt{(\mathbf{k} \cdot \mathbf{v}_{i0})^2 + 4 \mathbf{k} \cdot (\hat{\mathbf{z}} \times \mathbf{g}) \mathbf{k} \cdot (\hat{\mathbf{z}} \times \nabla \ln n_0)} \right\}. \quad (4.137)$$

An instability occurs when

$$(\mathbf{k} \cdot \mathbf{v}_{i0})^2 + 4 \mathbf{k} \cdot (\hat{\mathbf{z}} \times \mathbf{g}) \mathbf{k} \cdot (\hat{\mathbf{z}} \times \nabla \ln n_0) < 0, \quad (4.138)$$

from which the instability requires that the density gradient $\nabla \ln n_0$ is opposite to $\mathbf{g}$. This implies that the less dense plasma supports the higher density plasma, which is obviously a dynamically unstable configuration. This instability is called the interchange instability.

Part II

Nonlinear Theory of Plasmas

Chapter 5

Nonlinear Interactions in Plasmas

There are two types of nonlinear interactions, one is a resonant interaction which is a real process to engage in energy transport and the other is an adiabatic process which is usually renormalized into physical parameters such as temperature, mass, and charge. The resonant interaction consists of wave–wave interactions and wave–particle interactions. The lowest order nonlinear wave–wave interaction is a scattering process of three waves which need satisfy the resonance conditions

$$\sum_{i=1}^3 \mathbf{k}_i = 0, \quad \sum_{i=1}^3 \omega_i = 0,$$

where $\mathbf{k}_i$ and ω_i are the wavenumber and the frequency of the wave, related to each other through the dispersion relation $\varepsilon(\mathbf{k}_i, \omega_i) = 0$. The three wave interaction provides a variety of physics such as the decay and explosive interactions, formation of solitons and onset of chaos.

The resonance interaction among waves from the same family is rather limited since the three waves need to satisfy the resonance condition which is not always the case. In this case, four wave interactions have to be considered. However, the self-interaction gives a frequency shift by which the waves sometime become in resonance or the resonance among the linear waves is broken.

When many waves are excited and the phases are not strongly correlated, which is the case in most turbulence, we need a kinetic theory of the waves. However, the structure of the kinetic equation of turbulence is similar to the nonlinear wave equation of coherent waves since the fundamental processes of wave–wave interactions are the same for the case of coherent waves.

Two waves of $(\mathbf{k}_1, \omega_1)$ and $(\mathbf{k}_2, \omega_2)$ interact to produce a beat wave $(\mathbf{k}_1 \pm \mathbf{k}_2, \omega_1 \pm \omega_2)$ which can be in resonance with the particle of velocity satisfying the following resonant condition

$$(\mathbf{k}_1 \pm \mathbf{k}_2) \cdot \mathbf{v} = \omega_1 \pm \omega_2.$$

This corresponds to the three wave resonance in which one wave is replaced by a particle. This process is a scattering of the waves by the particle and is called the nonlinear Landau damping or stimulated scattering. Since the phase velocity of the wave is much larger than the particle thermal velocity, the Landau damping is often ignored. However, the number of particles interacting with a beat wave can be large. Thus, the nonlinear Landau damping is the lowest order process in nonlinear energy transport in which particles are involved.

Beside the resonant interaction between waves and particles, particles are quivered by high frequency waves and increase their average kinetic energy which reflects to the temperature. When the wave amplitude is modulated, then the quivered particles are driven in the direction of the gradient of the averaged wave potential. The force acting on the particles in this process is known as the ponderomotive force. The ponderomotive force mainly acts on the electrons to repel them from the high amplitude region and dig a density depletion to trap the wave. In nonlinear wave propagation problems, the ponderomotive force plays crucial roles.

In this chapter, we provide basic theoretical frames to treat fundamental processes of nonlinear interactions in plasmas such as resonant wave–wave and wave–particle interactions and adiabatic wave–particle interactions, and then formulate the weak turbulence theory and wave kinetic theory under the eikonal approximation for wave turbulence [4, 18, 64].

5.1 Nonlinear Wave Equation

Starting from the Vlasov equation

$$\frac{\partial F_\alpha}{\partial t} + \mathbf{v} \cdot \nabla F_\alpha + \frac{e_\alpha}{m_\alpha} \mathbf{E} \cdot \frac{\partial F_\alpha}{\partial \mathbf{v}} = 0, \quad (5.1)$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \sum_\alpha 4\pi e_\alpha \int d\mathbf{v} F_\alpha(\mathbf{r}, \mathbf{v}, t), \quad (5.2)$$

and substituting the Fourier transform of the distribution function and the electric field into the Vlasov equation,

$$F_\alpha(\mathbf{r}, \mathbf{v}, t) = n_0 F_{\alpha 0}(\mathbf{v}) + \sum_k \int \frac{d\omega}{2\pi} f_\alpha(\mathbf{k}, \mathbf{v}, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (5.3)$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla \phi(\mathbf{r}, t) = -i \sum_k \int \frac{d\omega}{2\pi} \mathbf{k} \phi(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (5.4)$$

the fluctuation part of the distribution function is expressed by

$$\begin{aligned}
f_\alpha(\mathbf{k}, \mathbf{v}, \omega) &= -\frac{n_{\alpha 0} e_\alpha}{m_\alpha} \frac{\phi(\mathbf{k}, \omega)}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} F_{\alpha 0} \\
&\quad - \frac{e_\alpha}{m_\alpha} \sum_{\mathbf{k}'} \int \frac{d\omega'}{2\pi} \frac{\phi(\mathbf{k}', \omega')}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} f_\alpha(\mathbf{k} - \mathbf{k}', \mathbf{v}, \omega - \omega') \\
&= -\frac{n_{\alpha 0} e_\alpha}{m_\alpha} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} F_{\alpha 0} \phi(\mathbf{k}, \omega) \\
&\quad + n_{\alpha 0} \left(\frac{e_\alpha}{m_\alpha} \right)^2 \sum_{\mathbf{k}'} \int \frac{d\omega'}{2\pi} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega - \omega' - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}} \\
&\quad \times (\mathbf{k} - \mathbf{k}') \cdot \frac{\partial F_\alpha}{\partial \mathbf{v}} \phi(\mathbf{k}', \omega') \phi(\mathbf{k} - \mathbf{k}', \omega - \omega') + \dots, \tag{5.5}
\end{aligned}$$

which is substituted into the Poisson equation

$$\phi(\mathbf{k}, \omega) = \sum_\alpha \frac{4\pi n_0 e_\alpha}{k^2} \int d\mathbf{v} f_\alpha(\mathbf{k}, \mathbf{v}, \omega), \tag{5.6}$$

to give a nonlinear wave coupling equation.

$$\begin{aligned}
\varepsilon(y)\phi(y) &= \int dy' v(y, y') \phi(y') \phi(y - y') \\
&\quad + \int dy' \int dy'' w(y, y', y'') \phi(y') \phi(y'') \phi(y - y' - y'') \\
&\quad + \dots, \tag{5.7}
\end{aligned}$$

where we have put

$$y = (\mathbf{k}, \omega), \quad \int dy = \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi}, \quad g_\alpha(y) = -\frac{e_\alpha}{m_\alpha} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}},$$

and

$$\varepsilon(y) = 1 - \sum_\alpha \frac{4\pi n_{\alpha 0} e_\alpha}{k^2} \int d\mathbf{v} g_\alpha(y) \mathbf{k} \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}}, \tag{5.8}$$

$$\begin{aligned}
v(y, y') &= \sum_\alpha \frac{4\pi n_{\alpha 0} e_\alpha}{k^2} \int d\mathbf{v} g_\alpha(y) \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} g_\alpha(y - y') \\
&\quad \times (\mathbf{k} - \mathbf{k}') \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}}, \tag{5.9}
\end{aligned}$$

$$\begin{aligned}
w(y, y', y'') = \sum_{\alpha} \frac{4\pi n_{\alpha 0} e_{\alpha}}{k^2} \int d\mathbf{v} g_{\alpha}(y) \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} g_{\alpha}(y - y') \\
\times \mathbf{k}'' \cdot \frac{\partial}{\partial \mathbf{v}} g_{\alpha}(y - y' - y'') (\mathbf{k} - \mathbf{k}' - \mathbf{k}'') \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}}. \quad (5.10)
\end{aligned}$$

In (5.7), the first and the second terms of the right hand side represent the three wave interactions and four wave interactions, respectively. The three wave interactions include the resonance decay interactions and the self-interactions in which the zero and the higher harmonics are involved. In the following, we consider up to the four wave interactions. In order to take into account the self-interaction through the zero and the higher harmonics, we have to iterate once in the first term of (5.7) which is converted to

$$\begin{aligned}
\varepsilon(y)\phi(y) = \int dy' v(y, y') \phi(y') \phi(y - y') \\
+ \int dy' \int dy'' \left\{ \frac{v(y, y') v(y - y', y'')}{\varepsilon(y - y')} + w(y, y', y'') \right\} \\
\times \phi(y') \phi(y'') \phi(y - y' - y''). \quad (5.11)
\end{aligned}$$

5.2 Resonant Three Wave Interaction

The lowest order wave interaction is given by three-wave interaction. For waves specified by the wavenumbers and the frequencies $\{(\mathbf{k}_j, \omega_j); j = 1, 2, 3\}$ the interaction is well defined when the waves are in resonance

$$\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 = 0, \quad \omega_1 \pm \omega_2 \pm \omega_3 = 0, \quad (5.12)$$

which are not realized at the same time. Therefore, one type of the resonance is enough to consider. In the following, we assume the resonance in the form of

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3, \quad \omega_1 = \omega_2 + \omega_3. \quad (5.13)$$

However, the resonance interaction does not always work. A Langmuir wave cannot decay into two other Langmuir waves. The resonance condition is satisfied, for example, among two Langmuir waves and one ion acoustic wave and among three different modes in an ion beam-plasma system. We may allow a small mismatch

$$\delta \mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, \quad \delta \omega = \omega_1 - \omega_2 - \omega_3. \quad (5.14)$$

In addition, we include the self-interactions which give detuning of the resonance condition. The amplitude of the each wave would be changing in space and time because of the interactions among the waves. We may put

$$\begin{aligned}
\phi(\mathbf{x}, t) &= \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \phi(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\
&= \sum_j \{ \phi_{k_j}(\mathbf{x}, t) e^{i(\mathbf{k}_j \cdot \mathbf{x} - \omega(k_j)t)} + \phi_{k_j}^*(\mathbf{x}, t) e^{-i(\mathbf{k}_j \cdot \mathbf{x} - \omega(k_j)t)} \} \\
&= \sum_j \sum_{\mathbf{K}} \int \frac{d\Omega}{2\pi} \{ \phi_j(\mathbf{K}, \Omega) e^{i(\mathbf{k}_j + \mathbf{K}) \cdot \mathbf{x} - i(\omega_j + \Omega)t} \\
&\quad + \phi_j^*(\mathbf{K}, \Omega) e^{-i(\mathbf{k}_j - \mathbf{K}) \cdot \mathbf{x} + i(\omega_j - \Omega)t} \}, \tag{5.15}
\end{aligned}$$

where $(\mathbf{K}, \Omega)$ are slow variables to describe the modulation of the amplitude, and $\phi_j(\mathbf{K}, \Omega) = \phi_{k_j}(\mathbf{K}, \Omega)$ and $\omega_j = \omega(\mathbf{k}_j)$. Here, $\omega(\mathbf{k})$ is determined by $\Re \varepsilon(\mathbf{k}, \omega(\mathbf{k})) = 0$. From (5.15), we have

$$\begin{aligned}
\phi(\mathbf{k}, \omega) &= \sum_j \sum_{\mathbf{K}} \int \frac{d\Omega}{2\pi} \{ \phi_j(\mathbf{K}, \Omega) \delta(\mathbf{k} - \mathbf{k}_j - \mathbf{K}) \delta(\omega - \omega_j - \Omega) \\
&\quad + \phi_j^*(\mathbf{K}, \Omega) \delta(\mathbf{k} + \mathbf{k}_j - \mathbf{K}) \delta(\omega + \omega_j - \Omega) \} \\
&= \sum_j \int dY \{ \phi_j(Y) \delta(y - y_j - Y) + \phi_j^*(Y) \delta(y + y_j - Y) \}. \tag{5.16}
\end{aligned}$$

where $Y = (\mathbf{K}, \Omega)$ is a slow scale variable and $y_j = (\mathbf{k}_j, \omega_j)$. Then, (5.11) is reduced to

$$\begin{aligned}
&\int dY \delta(y - y_1 - Y) [\varepsilon(y_1 + Y) \phi_1(Y) - \frac{1}{2} \int dY_2 \int dY_3 \\
&\quad V(y, y_2 + Y_2) \phi_2(Y_2) \phi_3(Y_3) \delta(Y - Y_2 - Y_3 + \delta y) \\
&\quad - \int dY_1 \int dY_2 \int dY_3 \left\{ \frac{V(y_1, 2y_1) V(2y_1, y_1)}{4\varepsilon(2y_1)} \phi_1(Y_2) \phi_1(Y_1 - Y_2) \phi_1^*(Y_3) \right. \\
&\quad + \frac{V(y_1 + Y, Y_1) V(Y_1, y_1 + Y_2)}{2\varepsilon(Y_1)} \phi_1(Y_2) \phi_1^*(Y_1 - Y_2) \phi_1(Y_3) \\
&\quad + W(y_1 + Y, y_1 + Y_1, y_1 + Y_2) \phi_1(Y_1) \phi_1(Y_2) \phi_1^*(Y_3) \\
&\quad \left. + w(y_1, -y_1, y_1) \phi_1^*(Y_1) \phi_1(Y_2) \phi_1(Y_3) \right\} \delta(Y - Y_1 - Y_2 - Y_3)] = 0, \tag{5.17}
\end{aligned}$$

where δy is the wavenumber and frequency mismatch defined by $\delta y = (\delta \mathbf{k}, \delta \omega)$ and

$$V(y, y') = v(y, y') + v(y, y - y'), \tag{5.18}$$

$$W(y, y', y'') = w(y, y', y'') + w(y, y', y - y' - y''). \tag{5.19}$$

The coefficients of (5.17) are dependent on the waves involved. For $\Omega/K \ll v_{Te} \ll \omega_1/k_1$ they are estimated as

$$\begin{aligned}
V(y_1, 2y_1) &= 4V(2y_1, y_1) \simeq -3 \frac{\omega_{pe}^2}{\omega_1^2} \frac{k_1^2 v_{Te}^2}{\omega_1^2} \frac{e}{T_e}, \quad \varepsilon(2y_0) \simeq 1 - \frac{1}{4} \frac{\omega_{pe}^2}{\omega_1^2}, \\
V(y_1 + Y, Y_1) &= \frac{K_1^2}{k_1^2} V(Y_1, y_1 + Y_2) \simeq \frac{\omega_{pe}^2}{\omega_1^2} \frac{e}{T_e}, \\
W(y_1, y_1 + Y_1, y_1 + Y_2) &\simeq -\frac{\omega_{pe}^2}{\omega_1^2} \frac{k_1^2 v_{Te}^2}{\omega_1^2} \left(\frac{e}{T_e} \right)^2, \\
w(y_1, -y_1, y_1) &\simeq -\frac{15}{2} \frac{\omega_{pe}^2}{\omega_1^2} \frac{k_1^4 v_{Te}^4}{\omega_1^4} \left(\frac{e}{T_e} \right)^2, \tag{5.20}
\end{aligned}$$

while for $v_{Ti} \ll \omega_1/k_1 \ll v_{Te}$ and $v_{Ti} \ll \Omega/K \sim C_s$

$$\begin{aligned}
V(y_1, 2y_1) &= 4V(2y_1, y_1) \simeq -\frac{\omega_{pe}^2}{k_1^2 v_{Te}^2} \frac{e}{T_e}, \quad \varepsilon(2y_0) \simeq 1 + \frac{1}{4} \left(\frac{1}{k_1^2 \lambda_D^2} - \frac{\omega_{pi}^2}{\omega_1^2} \right), \\
V(y_1 + Y, Y_1) &= \frac{K_1^2}{k_1^2} V(Y_1, y_1 + Y_2) \simeq -\frac{\omega_{pe}^2}{k_1^2 v_{Te}^2} \frac{e}{T_e} \left\{ 1 + \frac{k_1^2 C_s^2}{\omega_1^2} \right\}, \\
W(y_1, y_1 + Y_1, y_1 + Y_2) &\simeq -\frac{\omega_{pe}^2}{k_1^2 v_{Te}^2} \left(\frac{e}{T_e} \right)^2 \left\{ 1 + \left(\frac{k_1 C_s}{\omega_1} \right)^4 \right\}, \\
w(y_1, -y_1, y_1) &\simeq -\frac{1}{2} \frac{\omega_{pe}^2}{k_1^2 v_{Te}^2} \left(\frac{e}{T_e} \right)^2 \left(1 + 15 \frac{k_1^6 C_s^6}{\omega_1^6} \right), \tag{5.21}
\end{aligned}$$

and for both cases

$$\varepsilon(Y_1) = 1 + \frac{\omega_{pe}^2}{K_1^2 v_{Te}^2} + \chi_i(K_1, \Omega_1). \tag{5.22}$$

Multiplying $e^{i(k \cdot x - \omega t)}$ to both sides of the (5.17), expanding $\varepsilon(y_1 + Y)$ with respect to Y on the left hand side and integrating with respect to y , we get

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g1} \cdot \nabla - \gamma_1 \right) \frac{\partial \varepsilon(y_1)}{\partial \omega_1} \phi_1 = \frac{V(y_1, y_2)}{2} \phi_2 \phi_3 e^{-i\delta\theta(x,t)} + \mu_1 |\phi_1|^2 \phi_1, \tag{5.23}$$

where

$$\delta\theta(\mathbf{x}, t) = \delta\mathbf{k} \cdot \mathbf{x} - \delta\omega t, \quad \mathbf{v}_{g1} = -\frac{\frac{\partial \varepsilon(y_1)}{\partial \mathbf{k}_1}}{\frac{\partial \varepsilon(y_1)}{\partial \omega_1}}, \quad \gamma_1 = -\frac{\Im \varepsilon(y_1)}{\frac{\partial \varepsilon(y_1)}{\partial \omega_1}}.$$

and μ_1 is the coefficient of the frequency shift which is determined after the wave is specified by using (5.20) and (5.21). If the waves involved are assumed to be well defined and the contribution to $V(y_1, y_2)$ and μ_1 from the resonant particles is neglected, $V(y_1, y_2)$ and μ_1 are real.

In a similar way, other two equations are obtained

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g2} \cdot \nabla - \gamma_2 \right) \frac{\partial \varepsilon(y_2)}{\partial \omega_2} \phi_2 = \frac{V(y_2, y_1)}{2} \phi_1 \phi_3^* e^{i\delta\theta(x,t)} + \mu_2 |\phi_2|^2 \phi_2, \quad (5.24)$$

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g3} \cdot \nabla - \gamma_3 \right) \frac{\partial \varepsilon(y_3)}{\partial \omega_3} \phi_3 = \frac{V(y_3, y_1)}{2} \phi_1 \phi_2^* e^{i\delta\theta(x,t)} + \mu_3 |\phi_3|^2 \phi_3. \quad (5.25)$$

Defining

$$V_1 = \frac{T_e}{2e} \frac{k_1}{k_2 k_3} \frac{V(y_1, y_2)}{C}, \quad V_2 = \frac{T_e}{2e} \frac{k_2}{k_3 k_1} \frac{V(y_2, y_1)}{C}, \quad V_3 = \frac{T_e}{2e} \frac{k_3}{k_1 k_2} \frac{V(y_3, y_1)}{C},$$

$$C = |s_1 s_2 s_3|^{1/2}, \quad s_i = \frac{\partial \varepsilon(y_i)}{\partial \omega_i}, \quad V^2 = V_2 V_3, \quad \hat{\mu}_i = \left(\frac{T_e}{ek_i} \right)^2 \frac{V_1}{V_i} \mu_i,$$

and normalizing the amplitude by

$$A_1 = |s_1|^{1/2} k_1 \frac{e\phi_1}{T_e}, \quad \left\{ A_j = \sqrt{\frac{V_j}{V_1}} |s_j|^{1/2} k_j \frac{e\phi_j}{T_e}, \quad j = 2, 3 \right\} \quad (5.26)$$

the following set of equations is obtained.

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g1} \cdot \nabla - \gamma_1 \right) A_1 = \hat{s}_1 V A_2 A_3 e^{-i\delta\theta(x,t)} + \hat{s}_1 \hat{\mu}_1 |A_1|^2 A_1, \quad (5.27)$$

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g2} \cdot \nabla - \gamma_2 \right) A_2 = \hat{s}_2 V A_3^* A_1 e^{i\delta\theta(x,t)} + \hat{s}_2 \hat{\mu}_2 |A_2|^2 A_2, \quad (5.28)$$

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g3} \cdot \nabla - \gamma_3 \right) A_3 = \hat{s}_3 V A_1 A_2^* e^{i\delta\theta(x,t)} + \hat{s}_3 \hat{\mu}_3 |A_3|^2 A_3, \quad (5.29)$$

where V is real and $\hat{s}_i = \text{sgn}(s_i)$ denotes the sign of the wave energy of the wave i .

When the three waves are in resonance and the self-interactions are neglected, (5.27)–(5.29) are reduced to

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g1} \cdot \nabla - \gamma_1 \right) A_1 = \hat{s}_1 V A_2 A_3, \quad (5.30)$$

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g2} \cdot \nabla - \gamma_2 \right) A_2 = \hat{s}_2 V A_1 A_3^*, \quad (5.31)$$

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g3} \cdot \nabla - \gamma_3 \right) A_3 = \hat{s}_3 V A_1 A_2^*. \quad (5.32)$$

The numbers of the wave $|A_i|^2$ are related to each other through

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_{g1} \cdot \nabla - 2\gamma_1\right) \hat{s}_1 |A_1|^2 + \left(\frac{\partial}{\partial t} + \mathbf{v}_{g2} \cdot \nabla - 2\gamma_2\right) \hat{s}_2 |A_2|^2 = 0, \quad (5.33)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_{g1} \cdot \nabla - 2\gamma_1\right) \hat{s}_1 |A_1|^2 + \left(\frac{\partial}{\partial t} + \mathbf{v}_{g3} \cdot \nabla - 2\gamma_3\right) \hat{s}_3 |A_3|^2 = 0, \quad (5.34)$$

$$\sum_{j=1}^3 \left(\frac{\partial}{\partial t} + \mathbf{v}_{gj} \cdot \nabla - 2\gamma_j\right) \left(\frac{\mathbf{k}_j}{\omega_j}\right) \hat{s}_j |A_j|^2 = 0. \quad (5.35)$$

This shows that the sign of the wave energy plays an important role for the resonant interaction.

When the waves are spatially homogeneous and the damping/growth is neglected, we see the direction of the cascade on the wave energy. Defining the number of the wave by $N_j = |A_j|^2$ and the change in the number due to interaction by ΔN_j , we obtain the Manley–Rowe relation [65] from (5.35)

$$\left(\frac{\mathbf{k}_1}{\omega_1}\right) \hat{s}_1 \Delta N_1 + \left(\frac{\mathbf{k}_2}{\omega_2}\right) \hat{s}_2 \Delta N_2 + \left(\frac{\mathbf{k}_3}{\omega_3}\right) \hat{s}_3 \Delta N_3 = 0, \quad (5.36)$$

which gives

$$\hat{s}_1 \Delta N(\mathbf{k}_1) = -\hat{s}_2 \Delta N(\mathbf{k}_2) = -\hat{s}_3 \Delta N(\mathbf{k}_3). \quad (5.37)$$

Therefore, if all the waves are positive, the energy is transferred from the wave 1 to the waves 2 and 3 or from the waves 2 and 3 to the wave 1. This is the resonant decay interaction. However, the wave 1 is a negative energy wave and the others are positive energy waves, $\Delta N_j > 0$ or $\Delta N_j < 0$ holds for all j , implying the wave 1 grows by giving the energy to the waves 2 and 3 or damps by being given the energy from the waves 2 and 3. This is the resonant explosive interaction.

5.2.1 Parametric Instability

Suppose that three waves are all positive energy waves and the wave 1 is unstable ($\gamma_1 > 0$) and the waves 2 and 3 are stable ($\gamma_2 < 0$ and $\gamma_3 < 0$). In addition, the amplitudes are spatially independent. The wave 1 initially at a noise level grows up to a certain amplitude and decays into the waves 2 and 3. This process is analysed by assuming that the amplitude of the wave 1 is constant $A_1(0) = A_{10}$. Then, linearizing (5.31) and (5.32) with respect to A_2 and A_3

$$i \left(\frac{\partial}{\partial t} - \gamma_2\right) A_2 = V A_3^* A_{10}, \quad (5.38)$$

$$i \left(\frac{\partial}{\partial t} - \gamma_3\right) A_3 = V A_{10} A_2^*, \quad (5.39)$$

and putting $A_{2,3} \propto e^{\lambda t}$, we obtain

$$\lambda = \frac{1}{2} \left\{ \gamma_2 + \gamma_3 \pm \sqrt{(\gamma_2 - \gamma_3)^2 + 4V^2|A_{10}|^2} \right\}. \quad (5.40)$$

Thus, the instability occurs when the following condition is satisfied

$$\gamma_2\gamma_3 < V^2|A_{10}|^2. \quad (5.41)$$

This instability is called the parametric instability. As a result of the instability, the waves 2 and 3 grow to reach saturation states

$$|A_1|^2 = \frac{\gamma_2\gamma_3}{V^2}, \quad |A_2|^2 = \frac{\gamma_1|\gamma_3|}{V^2}, \quad |A_3|^2 = \frac{\gamma_1|\gamma_2|}{V^2}. \quad (5.42)$$

5.2.2 Resonant Decay Interaction

When $\hat{s}_i > 0$ and $\gamma_i = 0$ for $i = 1, 2, 3$ and again the amplitudes are independent of space, there appears a closed cycle of the energy transfer among three waves, that is, the amplitude oscillation. Starting with

$$\frac{dA_1}{dt} = -iVA_2A_3, \quad (5.43)$$

$$\frac{dA_2}{dt} = -iVA_1A_3^*, \quad (5.44)$$

$$\frac{dA_3}{dt} = -iVA_1A_2^*, \quad (5.45)$$

and putting

$$A_i = a_i e^{i\theta_i}, \quad (5.46)$$

we have

$$\frac{da_1}{dt} = -Va_2a_3 \sin \theta, \quad (5.47)$$

$$\frac{da_2}{dt} = Va_1a_3 \sin \theta, \quad (5.48)$$

$$\frac{da_3}{dt} = Va_1a_2 \sin \theta, \quad (5.49)$$

$$\frac{d}{dt} \ln(a_1a_2a_3 \cos \theta) = 0, \quad (5.50)$$

where

$$\theta = \theta_1 - \theta_2 - \theta_3. \quad (5.51)$$

There are three integrals

$$a_1 a_2 a_3 \cos \theta = C_1, \quad a_1^2 + a_2^2 = C_2, \quad a_1^2 + a_3^2 = C_3, \quad (5.52)$$

which guarantee the finiteness of the amplitudes. Using (5.52), (5.47) is reduced to

$$\frac{da_1^2}{dt} = -2V \sqrt{a_1^2(C_2 - a_1^2)(C_3 - a_1^2) - C_1^2}. \quad (5.53)$$

The solution is expressed by the Jacobian elliptic function sn. Putting the three roots of the equation $a_1^2(C_2 - a_1^2)(C_3 - a_1^2) - C_1^2 = 0$ with respect to a_1^2 as $\alpha_1 > \alpha_2 > \alpha_3$, (5.53) is rewritten into

$$\int \frac{da_1^2}{\sqrt{(a_1^2 - \alpha_1)(a_1^2 - \alpha_2)(a_1^2 - \alpha_3)}} = \frac{2}{\sqrt{\alpha_1 - \alpha_3}} \text{sn}^{-1}(\sin \psi, k), = -2Vt,$$

where $\sin \psi = \sqrt{(\alpha_1 - a_1^2)/(\alpha_1 - \alpha_2)}$, $k^2 = (\alpha_1 - \alpha_2)/(\alpha_1 - \alpha_3)$ and α_i is expressed in terms of C_i through

$$C_1^2 = \alpha_1 \alpha_2 \alpha_3, \quad C_2 + C_3 = \alpha_1 + \alpha_2 + \alpha_3, \quad C_2 C_3 = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1. \quad (5.54)$$

Thus, the solution is given by

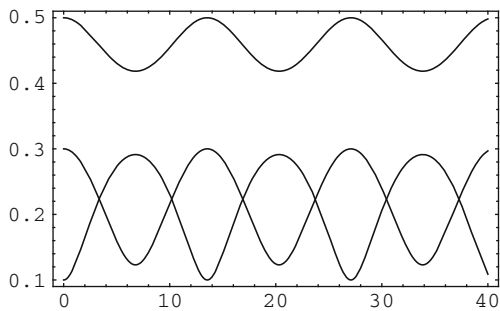
$$a_1^2(t) = \alpha_1 - (\alpha_1 - \alpha_2) \text{sn}^2(\sqrt{\alpha_1 - \alpha_3} V t, k). \quad (5.55)$$

A solution of (5.43)–(5.45) is shown in Fig. 5.1 for $V = 0.5$, with $A_1(0) = 0.1$, $A_2(0) = 0.3$, $A_3(0) = 0.5$.

For the special case that $\theta = \pi/2$ and $C_2 = C_3$ (5.53) is reduced to

$$\frac{da_1}{dt} = -2V(C_2 - a_1^2), \quad (5.56)$$

Fig. 5.1 Temporal evolution of the solutions for (5.43)–(5.45) for $V = 0.5$ with the initial condition $A_1(0) = 0.1$, $A_2(0) = 0.3$, $A_3(0) = 0.5$



which is integrated to give

$$a_1(t) = \sqrt{C_2} \tanh(\sqrt{C_2} Vt + \delta), \quad a_{2,3}(t) = \sqrt{C_2} \operatorname{sech}(\sqrt{C_2} Vt + \delta), \quad (5.57)$$

where $\delta = \ln |(a_1(0) - \sqrt{C_2}) / (a_1(0) + \sqrt{C_2})|$.

5.2.3 Resonant Explosive Interaction

When one of the three resonantly interacting waves is a negative energy wave and the others are positive energy waves, energy is transferred from a negative energy wave to positive energy waves and all the three waves grow, resulting in explosion. Suppose that the wave 1 is a negative energy wave and the waves 2 and 3 are positive energy waves. Then, the basic equations are given by

$$\frac{dA_1}{dt} = iVA_2A_3, \quad (5.58)$$

$$\frac{dA_2}{dt} = -iVA_1A_3^*, \quad (5.59)$$

$$\frac{dA_3}{dt} = -iVA_1A_2^*. \quad (5.60)$$

Putting

$$A_i = a_i e^{i\theta_i}, \quad (5.61)$$

the basic equations are converted into

$$\frac{da_1}{dt} = Va_2a_3 \sin \theta, \quad (5.62)$$

$$\frac{da_2}{dt} = Va_1a_3 \sin \theta, \quad (5.63)$$

$$\frac{da_3}{dt} = Va_1a_2 \sin \theta, \quad (5.64)$$

$$\frac{d}{dt} \ln(a_1a_2a_3 \cos \theta) = 0, \quad (5.65)$$

where

$$\theta = \theta_1 - \theta_2 - \theta_3. \quad (5.66)$$

There are three integrals

$$a_1a_2a_3 \cos \theta = C_1, \quad a_1^2 - a_2^2 = C_2, \quad a_1^2 - a_3^2 = C_3, \quad (5.67)$$

which show there is no limit on the amplitude. Using (5.67), (5.62) is rewritten as

$$\frac{da_1^2}{dt} = 2V \sqrt{a_1^2(a_1^2 - C_2)(a_1^2 - C_3) - C_1^2}, \quad (5.68)$$

which is readily integrated to give

$$a_1^2 = \frac{\alpha_1 - \alpha_2 \operatorname{sn}^2(\sqrt{\alpha_1 - \alpha_2} Vt, k)}{1 - \operatorname{sn}^2(\sqrt{\alpha_1 - \alpha_2} Vt, k)}, \quad (5.69)$$

where $\{\alpha_i; i = 1, 2, 3\}$ are the three solutions of the equation $a_1^2(a_1^2 - C_2)(a_1^2 - C_3) - C_1^2 = 0$ with $\alpha_1 > \alpha_2 > \alpha_3$, and $k^2 = (\alpha_2 - \alpha_3)/(\alpha_1 - \alpha_3)$. The solution diverges at the time t_*

$$t_* = \frac{K(k)}{\sqrt{\alpha_1 - \alpha_2} V}, \quad (5.70)$$

which is determined by

$$\operatorname{sn}^2(\sqrt{\alpha_1 - \alpha_2} Vt_*, k) = 1. \quad (5.71)$$

and $K(k)$ is the complete elliptic integral of the first kind.

5.2.4 *Chaos in Resonant Three-Wave Interaction with Dissipation and Frequency Mismatch*

In three-wave interactions without damping/ growth, the energy is periodically transferred among the three waves. Introduction of the growth/damping to the resonant three-wave interaction brings the system into a stationary state. Here, we see the effect of the frequency mismatch. We assume spatial homogeneity and retain the frequency mismatch $|\delta\omega| \ll |\omega_i|$ in (5.27)–(5.29) to have

$$i \left(\frac{d}{dt} - \gamma_1 \right) A_1 = \hat{s}_1 V A_2 A_3 e^{i\delta\omega t}, \quad (5.72)$$

$$i \left(\frac{d}{dt} - \gamma_2 \right) A_2 = \hat{s}_2 V A_1 A_3^* e^{-i\delta\omega t}, \quad (5.73)$$

$$i \left(\frac{d}{dt} - \gamma_3 \right) A_3 = \hat{s}_3 V A_1 A_2^* e^{-i\delta\omega t}. \quad (5.74)$$

If $\gamma_2 = \gamma_3 < 0$ and $\hat{s}_2 = \hat{s}_3$, we have from (5.73) and (5.74)

$$|A_2|^2 - |A_3|^2 \propto e^{-2|\gamma_2|t} \rightarrow 0 \quad \text{for } t \rightarrow \infty. \quad (5.75)$$

In the following, we assume that $\gamma_1 > 0$, $\gamma_2 = \gamma_3 < 0$, $\hat{s}_1 = \hat{s}_2 = \hat{s}_3$ and $A_2 = A_3$. Then, our starting equations are rewritten as

$$i \left(\frac{d}{dt} - \gamma_1 \right) A_1 = V A_2^2 e^{i\delta\omega t}, \quad (5.76)$$

$$i \left(\frac{d}{dt} - \gamma_2 \right) A_2 = V A_1 A_2^* e^{-i\delta\omega t}. \quad (5.77)$$

Putting

$$A_i = a_i e^{i\theta_i}, \quad (5.78)$$

with replacements

$$x_1 = V a_1 \sin \theta, \quad x_2 = V a_1 \cos \theta, \quad x_3 = V^2 a_2^2, \quad \theta = \theta_1 - 2\theta_2 - \delta\omega t, \quad (5.79)$$

we obtain the following dynamical system [66], [67], [68]

$$\frac{dx_1}{dt} = \gamma_1 x_1 - \delta\omega x_2 - x_3 + 2x_2^2, \quad (5.80)$$

$$\frac{dx_2}{dt} = \delta\omega x_1 + \gamma_1 x_2 - 2x_1 x_2, \quad (5.81)$$

$$\frac{dx_3}{dt} = 2(\gamma_2 + x_1)x_3. \quad (5.82)$$

Since the volume of the phase space is given by

$$\sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{dx_i}{dt} = 2(\gamma_1 + \gamma_2), \quad (5.83)$$

we need $\gamma_1 + \gamma_2 < 0$ for this system to be a dissipative dynamical system. There are two fixed points

$$(x_{1*}, x_{2*}, x_{3*}) = (0, 0, 0), \quad \left(-\gamma_2, \frac{\gamma_2 \delta\omega}{\gamma_1 + 2\gamma_2}, -\gamma_1 \gamma_2 \left\{ 1 + \left(\frac{\delta\omega}{\gamma_1 + 2\gamma_2} \right)^2 \right\} \right). \quad (5.84)$$

The origin is obviously unstable because $\gamma_1 > 0$. The stability of the other fixed point is examined by linearizing (5.80)–(5.82) around the fixed point and assuming the deviation from the fixed point is proportional to $e^{\lambda t}$. The equation for λ is given by

$$\begin{aligned} & \lambda^3 - 2(\gamma_1 + \gamma_2)\lambda^2 + \left\{ \gamma_1^2 + \frac{\gamma_1(\gamma_1 - 4\gamma_2)}{(\gamma_1 + 2\gamma_2)^2} \delta\omega^2 \right\} \lambda \\ & = -2\gamma_1 \gamma_2 (\gamma_1 + 2\gamma_2) \left\{ 1 + \frac{\delta\omega^2}{(\gamma_1 + 2\gamma_2)^2} \right\}. \end{aligned} \quad (5.85)$$

Since $\gamma_1 + \gamma_2 < 0$ and the right hand side of (5.85) is negative definite, when there are three real roots, one of them is positive and the fixed point is unstable. When there is only one real root, it is negative and the real parts of the complex roots should be negative for the fixed point to be stable. The stability criterion is obtained by making the sum and difference of (5.85) and its complex conjugate and putting $\lambda + \lambda^* = 0$. Thus, we get the critical value of the magnitude of the mismatch $\delta\omega_c$ for the onset of the instability

$$\delta\omega_c^2 = (\gamma_1 + 2\gamma_2)^2 \frac{2\gamma_2^2 + 2\gamma_1\gamma_2 + \gamma_1^2}{2\gamma_2^2 + 2\gamma_1\gamma_2 - \gamma_1^2}, \quad (5.86)$$

for which we need another condition

$$|\gamma_2| > \frac{1}{2}(1 + \sqrt{3})\gamma_1. \quad (5.87)$$

Thus, the fixed point is stable for $\delta\omega^2 > \delta\omega_c^2$. When $\delta\omega^2 < \delta\omega_c^2$, the fixed point becomes unstable and the unstable orbit shows a transition from periodic oscillation to chaos through the period doubling bifurcation with changing γ_2 for fixed γ_1 and $\delta\omega$. One example of the numerical solution is shown in Fig. 5.2 for $\gamma_1 = 0.1$, $\gamma_2 = -1.5$, and $\delta\omega = 0.3$.

The frequency mismatch brings the three wave resonance system into a chaotic state. This implies that the self-interaction giving the frequency shift is anticipated to have a similar detuning effect on the resonance coupling.

In (5.27)–(5.29), we neglect the spatial modulation to have

$$i \left(\frac{d}{dt} - \gamma_1 \right) A_1 = \hat{s}_1 V A_2 A_3 + (\delta\omega + \hat{s}_1 \hat{\mu}_1 |A_1|^2) A_1, \quad (5.88)$$

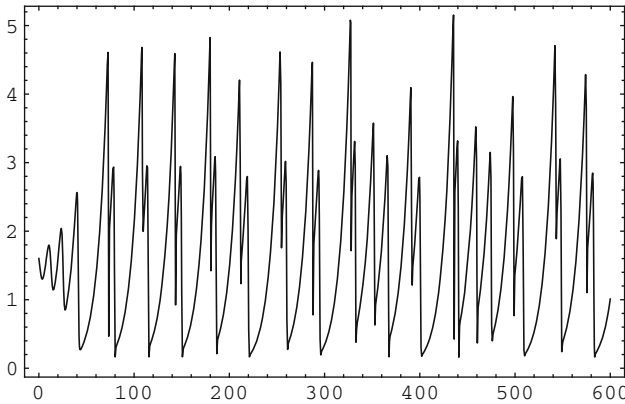


Fig. 5.2 Temporal evolution of $x_1(t)$ for (5.80)–(5.82) for $\gamma_1 = 0.1$, $\gamma_2 = -1.5$, and $\delta\omega = 0.3$ with the initial condition $x_1(0) = x_{1*} + 0.1$, $x_2(0) = x_{2*} + 0.1$ and $x_3(0) = x_{3*} + 1$

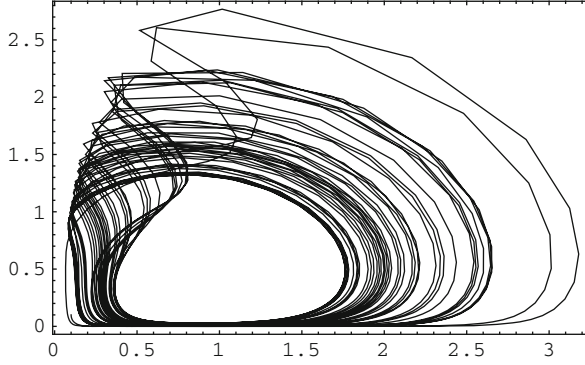


Fig. 5.3 The projection of the trajectory on $(|A_1(t)|, |A_3(t)|)$ plane in the phase space spanned by (5.88)–(5.90) for $\delta\omega = 0.0$ with $\gamma_1 = 0.1, \gamma_2 = -1.5, \gamma_3 = -0.5, m_1 = 0.2, m_2 = m_3 = 0$, and the initial condition $A_1(0) = 0.1, A_2(0) = 0.1, A_3(0) = 0.1$

$$i \left(\frac{d}{dt} - \gamma_2 \right) A_2 = \hat{s}_2 V A_1 A_3^* + \hat{s}_2 \hat{\mu}_2 |A_2|^2 A_2, \quad (5.89)$$

$$i \left(\frac{d}{dt} - \gamma_3 \right) A_3 = \hat{s}_3 V A_1 A_2^* + \hat{s}_3 \hat{\mu}_3 |A_3|^2 A_3, \quad (5.90)$$

where we have replaced A_1 by $A_1 e^{-i\delta\omega t}$. The self-interaction is almost equivalent to the frequency mismatch. The solutions of (5.88)–(5.90) with $\delta\omega = 0$ exhibit a similar transition route from stationary to chaotic states through the period doubling bifurcations in the case of $\hat{s}_1 = \hat{s}_2 = \hat{s}_3$. In Fig. 5.3, the projection of trajectory on $(|A_1(t)|, |A_3(t)|)$ plane is depicted when the frequency mismatch is set to zero ($\delta\omega = 0$) while the frequency shifts due to the self-interactions are kept as $\mu_1 = 0.2, \mu_2 = \mu_3 = 0$.

5.2.5 Coupled Solitons in Resonant Three-Wave Interaction

Here, we consider wave packets instead of monochromatic waves. We start with (5.27)–(5.29) without damping/growth terms,

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g1} \cdot \nabla \right) A_1 = \hat{s}_1 V A_2 A_3, \quad (5.91)$$

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g2} \cdot \nabla \right) A_2 = \hat{s}_2 V A_3^* A_1, \quad (5.92)$$

$$i \left(\frac{\partial}{\partial t} + \mathbf{v}_{g3} \cdot \nabla \right) A_3 = \hat{s}_3 V A_1 A_2^*. \quad (5.93)$$

For simplicity, if all the waves are assumed to propagate in the same direction, we may consider one dimensional propagation, say, in the x direction and look

for solutions of the form $A_i(x, t) = A_i(\xi)$, $\xi = x - \lambda t$. Then, (5.91)–(5.93) are reduced to

$$\frac{dA_1}{d\xi} = -i\hat{s}_1 \frac{V}{v_{g1} - \lambda} A_2 A_3, \quad (5.94)$$

$$\frac{dA_2}{d\xi} = -i\hat{s}_2 \frac{V}{v_{g2} - \lambda} A_3^* A_1, \quad (5.95)$$

$$\frac{dA_3}{d\xi} = -i\hat{s}_3 \frac{V}{v_{g3} - \lambda} A_1 A_2^*. \quad (5.96)$$

Introducing the following replacements

$$A_i = \frac{\tilde{A}_i}{\sqrt{|v_{gi} - \lambda|}}, \quad \hat{V} = \frac{V}{\sqrt{\prod_{i=1}^3 |v_{gi} - \lambda|}}, \quad \beta_i = \hat{s}_i \operatorname{sgn}(v_{gi} - \lambda),$$

we have

$$\frac{d\tilde{A}_1}{d\xi} = -i\beta_1 \hat{V} \tilde{A}_2 \tilde{A}_3, \quad (5.97)$$

$$\frac{d\tilde{A}_2}{d\xi} = -i\beta_2 \hat{V} \tilde{A}_3^* \tilde{A}_1, \quad (5.98)$$

$$\frac{d\tilde{A}_3}{d\xi} = -i\beta_3 \hat{V} \tilde{A}_1 \tilde{A}_2^*. \quad (5.99)$$

There are conserved quantities

$$\beta_1 |\tilde{A}_1|^2 + \beta_2 |\tilde{A}_2|^2 = C_2, \quad \beta_1 |\tilde{A}_1|^2 + \beta_3 |\tilde{A}_3|^2 = C_3. \quad (5.100)$$

Since (5.97)–(5.99) are isomorphic to (5.43)–(5.45) or (5.58)–(5.60), the solutions are given in a similar way. If we put

$$\tilde{A}_i = a_i e^{i\theta_i}, \quad (5.101)$$

we have the following conserved quantities.

$$a_1 a_2 a_3 \cos \theta = C_1, \quad \beta_1 a_1^2 + \beta_2 a_2^2 = C_2, \quad \beta_1 a_1^2 + \beta_3 a_3^2 = C_3, \quad (5.102)$$

where $\theta = \theta_1 - \theta_2 - \theta_3$. Thus, if $\theta = \pi/2$, $C_2 = C_3$ and $\beta_i = 1$, then we have coupled soliton solutions (a kink solution).

$$a_1(\xi) = \sqrt{C_2} \tanh(\sqrt{C_2} V \xi + \delta_0), \quad a_{2,3}(\xi) = \sqrt{C_2} \operatorname{sech}(\sqrt{C_2} V \xi + \delta_0), \quad (5.103)$$

where $\delta_0 = \ln |(a_1(0) - \sqrt{C_2}) / (a_1(0) + \sqrt{C_2})|$.

Equations (5.97)–(5.99) are shown to be reduced to the Sine–Gordon equation [69] when two of the group velocities coincide, implying there is an N -soliton solution for the resonant three-wave interaction system if the two of the group velocities are equal. It is also shown numerically that coupled soliton solutions (5.103) are stable against collision [70].

5.2.6 Spatio-Temporal Evolution in Three Wave Resonant Interaction

Equations (5.27)–(5.29) have chaotic solutions when the space dependence is neglected and soliton solutions for the case of $\gamma = \delta\omega = 0$. Thus, (5.27)–(5.29) are anticipated to have two faces of chaos and solitons. However, the number of the parameters are too large to study. In the following, we omit the self-interaction and growth/damping. Then, (5.27)–(5.29) are rewritten as

$$\left(\frac{\partial}{\partial t} + v_{g1} \cdot \nabla\right) A_1 = -\hat{s}_1 V A_2 A_3 - i\delta\omega A_1, \quad (5.104)$$

$$\left(\frac{\partial}{\partial t} + v_{g2} \cdot \nabla\right) A_2 = \hat{s}_2 V A_3^* A_1, \quad (5.105)$$

$$\left(\frac{\partial}{\partial t} + v_{g3} \cdot \nabla\right) A_3 = \hat{s}_3 V A_1 A_2^*, \quad (5.106)$$

where we have replaced A_i by $A_i e^{i\delta_i}$ and $\delta_1 - \delta_2 - \delta_3 = \pi/2$. In the following, we set $v_{g1} = 0.3$, $v_{g2} = -0.254$, $v_{g3} = 0.023$, $\lambda = -0.27$ and $\beta_i = 1$. When a set of stationary coupled soliton solutions (5.103) are launched at the initial moment, the propagation depends on the relative magnitude of L/v_g (L is the soliton width) and $2\pi/\delta\omega$. The coupled solitons are disturbed by the resonance mismatch and have a tendency to propagate with their own group velocities v_{gi} instead of the common phase velocity λ . For $\delta\omega = 0.01$, they propagate with λ for some time and then only A_1 and A_3 emit linear waves. A_1 and A_3 change their phase by π when they interact with linear waves. A_2 propagates without changing its form since the group velocity is chosen close to λ . This case is shown in Fig. 5.4. When $\delta\omega$ increases, A_2 and A_3 are locked in phase and this time A_1 and A_3 (therefore A_2) emit a baby coupled soliton which grows until the amplitude equal to the initially launched amplitude and then again emit a baby coupled soliton as is shown in Fig. 5.5. For a further increase in $\delta\omega$, each wave has a tendency to propagate with its own group velocity without interaction with other waves.

When sinusoidal waves are given at the initial moment for the case of $\delta\omega = 0$ such as $A_1 = -a_1 \cos(k(x - L/2))$, $A_2 = a_2 \sin(k(x - L/2))$, $A_3 = (a_3/2)\{1 - \cos(2k(x - L/2))\}$, spatio-temporal chaotic evolution is observed as is shown in Fig. 5.6.

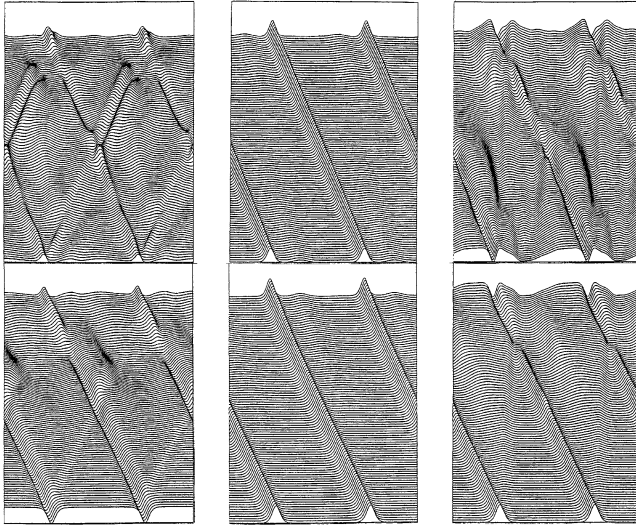
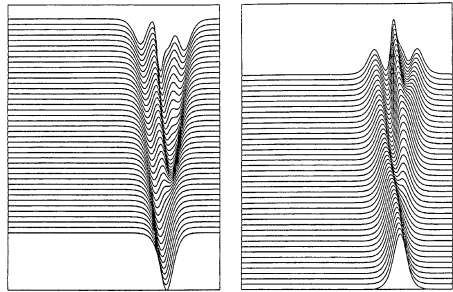


Fig. 5.4 Evolution of a coupled soliton solution with $\delta\omega = 0.01$. The left column is A_1 , the middle is A_2 , and the right is A_3 . The bottom box is $t = 0 - 1200$ and the top is $t = 1200 - 2400$

Fig. 5.5 Propagation of coupled soliton solutions with $\delta\omega = 0.05$. The left is A_1 and the right is A_3 . The time is $t = 0 - 1200$



5.3 Self-Interaction and Modulation Instability

Beside physics involved in three-wave interactions in which mode couplings (decay or explosive interactions) are important, simple harmonic generation is also an important mechanism of nonlinearity. In this section, we see the effect of the self-interaction on the propagation properties of the wave concerned.

We start with (5.27) without the linear growth/damping and the decay/explosive interaction and add a dispersion term, that is,

$$i \left(\frac{\partial}{\partial t} + v_g(\mathbf{k}) \cdot \nabla \right) A + \frac{d^2 \omega(\mathbf{k})}{d\mathbf{k} d\mathbf{k}} : \nabla \nabla A + \mu |A|^2 A = 0. \quad (5.107)$$

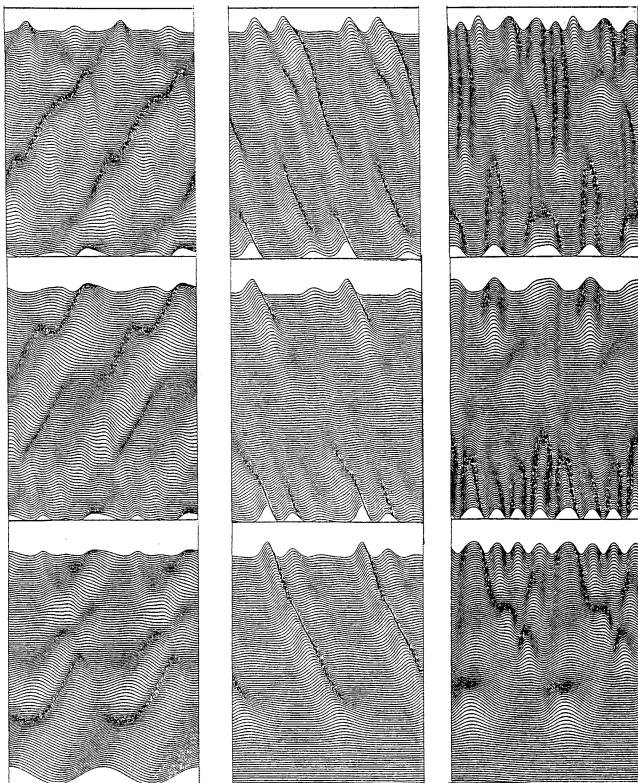


Fig. 5.6 Spatio-temporal chaos described by (5.104)–(5.106). The left column for A_1 , the middle for A_2 , and the right for A_3 with $a_1 = 0.1$, $a_2 = a_3 = 0.01$. The bottom box is $t = 0 - 1200$, the middle box is $t = 1200 - 2400$, and the top box is $t = 2400 - 3600$

This equation is called the nonlinear Schrödinger equation describing the time evolution of the amplitude of a strongly dispersive wave $\phi(x, t) = A(x, t)e^{i(k \cdot x - \omega(k)t)}$. The Langmuir wave is a typical strongly dispersive wave and an ion acoustic wave is weakly dispersive in the long wave limit but strongly dispersive in the short wave limit. Therefore, the nonlinear Schrödinger equation is widely used for waves propagating in nonlinear dispersive media.

Equation (5.107) has a solution of a finite amplitude wave,

$$A = A_0 e^{i(K \cdot x - \Omega t)}, \quad \Omega = K \cdot v_g(k) + \left(K \cdot \frac{d}{dk} \right)^2 \omega(k) - \mu |A_0|^2. \quad (5.108)$$

Applying a modulation to the solution (5.108)

$$A = e^{i(K \cdot x - \Omega t)} \{ A_0 + \delta A_1 e^{i(q \cdot x - \omega t)} + \delta A_2 e^{-i(q \cdot x - \omega t)} \}, \quad (5.109)$$

we have the following coupled equations

$$\begin{aligned} \{\omega - \mathbf{q} \cdot \mathbf{v}_g(\mathbf{k}) - b_k(\mathbf{q}, \mathbf{K}) - a_k(\mathbf{q}) - \mu|A_0|^2\} \delta A_1 &= -\mu A_0^2 \delta A_2^*, \\ \{\omega - \mathbf{q} \cdot \mathbf{v}_g(\mathbf{k}) - b_k(\mathbf{q}, \mathbf{K}) + a_k(\mathbf{q}) - \mu|A_0|^2\} \delta A_2^* &= -\mu A_0^{*2} \delta A_1, \end{aligned}$$

where

$$\begin{aligned} a_k(\mathbf{q}) &= \left(\mathbf{q} \cdot \frac{d}{d\mathbf{k}} \right)^2 \omega(k) = \left(1 - \frac{(\mathbf{q} \cdot \mathbf{k})^2}{q^2} \right) \frac{q^2}{k} \frac{d\omega(k)}{dk} + \frac{(\mathbf{q} \cdot \mathbf{k})^2}{k^2} \frac{d^2\omega(k)}{dk^2}, \\ b_k(\mathbf{q}, \mathbf{K}) &= \mathbf{K} \cdot \frac{d}{d\mathbf{k}} \mathbf{q} \cdot \frac{d}{d\mathbf{k}} \omega(k) \\ &= \left(\frac{\mathbf{q} \cdot \mathbf{K}}{k} - \frac{(\mathbf{K} \cdot \mathbf{k})(\mathbf{q} \cdot \mathbf{k})}{k^3} \right) \frac{d\omega(k)}{dk} + \frac{(\mathbf{q} \cdot \mathbf{k})(\mathbf{K} \cdot \mathbf{k})}{k^2} \frac{d^2\omega(k)}{dk^2}. \end{aligned}$$

The dispersion relation is obtained as

$$\omega = \mathbf{q} \cdot \mathbf{v}_g(\mathbf{k}) + 2b_k(\mathbf{q}, \mathbf{K}) \pm |a_k(\mathbf{q})| \sqrt{1 - 2\mu|A_0|^2/a_k(\mathbf{q})}. \quad (5.110)$$

The instability occurs when

$$1 - 2\mu|A_0|^2/a_k(\mathbf{q}) < 0. \quad (5.111)$$

Since the frequency shift due to the finiteness of the amplitude is given by $\delta\Omega = -\mu|A_0|^2$ from (5.108) and $a_k(\mathbf{q}) \propto d^2\omega(k)/dk^2$, the necessary condition for the instability is given by

$$\delta\Omega \frac{d^2\omega(k)}{dk^2} < 0, \quad (5.112)$$

which is known as the Lighthill criterion [71] for the modulation instability.

The physical mechanism of the modulation instability is the following. Suppose $d^2\omega(k)/dk^2 = dv_g/dk > 0$ and $\mu > 0$. Then, at the crest of the modulation amplitude, the frequency decreases and accordingly the phase velocity decreases too. On the upstream side of the crest, the number of nodes increases and the wavenumber increases, leading to a larger group velocity, while on the downstream side the situation is opposite. On the other hand in the trough of the modulation amplitude, the carrier wave on the upstream side propagates slower and the wave on the downstream side does faster. Thus, the crest of the modulation becomes higher and the trough becomes flatter.

For waves with $d^2\omega(k)/dk^2 = dv_g/dk > 0$, $\mu < 0$ means that the wave propagates under the repulsive potential, while the wave propagates under the attractive potential for $\mu > 0$. The modulation instability occurs when the potential energy is larger than the wave kinetic energy which is denoted by (5.111).

5.4 Nonlinear Wave-Particle Interaction

There are two types of interaction between waves and particles. One is a resonant interaction in which the velocity of the particle is in resonance with the phase velocity or the group velocity of the wave. The former is the Landau damping which is the absorption or emission of the waves by the resonant particles while the latter is a scattering of the waves by the particles.

The other is an adiabatic interaction in which the quiver motions of the particles remaining after averaged over time have contributions of exciting secondary flows and increasing temperature, charge, and mass effectively by renormalizing their quiver motions.

In this section, we consider the nonlinear Landau damping as a resonant wave particle interaction and the ponderomotive force as an adiabatic wave particle interaction.

5.4.1 Nonlinear Landau Damping

Two waves of $(\mathbf{k}_1, \omega_1)$ and $(\mathbf{k}_2, \omega_2)$ interact to produce a beat wave $(\mathbf{k}_1 \pm \mathbf{k}_2, \omega_1 \pm \omega_2)$ which resonates with the particle of velocity satisfying the following resonant condition

$$(\mathbf{k}_1 \pm \mathbf{k}_2) \cdot \mathbf{v} = \omega_1 \pm \omega_2.$$

This process is wave scattering by the particle and is called the nonlinear Landau damping or stimulated scattering by the analogy with the Landau damping describing emission and absorption of waves by particles.

A plasma eigen-mode is meaningful when the damping due to thermal particles is ignored, that is, when the phase velocity of the wave is much larger than the thermal velocity which is a condition for fluid approximation to hold. However, since the number of particles interacting with a beat wave can be large, the nonlinear Landau damping is the lowest order process in nonlinear energy transport in which particles are involved.

Here, we consider a coherent Langmuir wave of $(\mathbf{k}_0, \omega_0)$ where $\varepsilon(\mathbf{k}_0, \omega_0) = 0$. Through the nonlinearity, the fundamental wave excites second harmonic wave and zero frequency wave which in turn couple with the fundamental wave to give a frequency shift and the nonlinear Landau damping. The amplitude is modulated by this self-interaction. The decay interaction is not possible among the Langmuir waves. We may put the potential as

$$\begin{aligned} \phi(\mathbf{x}, t) &= \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \phi(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\ &= \sum_{\ell} \{ \phi_{\ell}(\mathbf{x}, t) e^{i\ell(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + \phi_{\ell}^*(\mathbf{x}, t) e^{-i\ell(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} \} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\ell} \sum_{\mathbf{K}} \int \frac{d\Omega}{2\pi} \{ \phi_{\ell}(\mathbf{K}, \Omega) e^{i(\ell \mathbf{k}_0 + \mathbf{K}) \cdot \mathbf{x} - i(\ell \omega_0 + \Omega)t} \\
&\quad + \phi_{\ell}^*(\mathbf{K}, \Omega) e^{-i(\ell \mathbf{k}_0 - \mathbf{K}) \cdot \mathbf{x} + i(\ell \omega_0 - \Omega)t} \}, \tag{5.113}
\end{aligned}$$

where $(\mathbf{K}, \Omega)$ are slow variables to describe the modulation of the amplitude, and $\phi_{-\ell}(\mathbf{K}, \Omega) = \phi_{\ell}^*(\mathbf{K}, \Omega)$. Therefore in the following, we use the relation

$$\begin{aligned}
\phi(\mathbf{k}, \omega) &= \sum_{\ell} \sum_{\mathbf{K}} \int \frac{d\Omega}{2\pi} \{ \phi_{\ell}(\mathbf{K}, \Omega) \delta(\mathbf{k} - \ell \mathbf{k}_0 - \mathbf{K}) \delta(\omega - \ell \omega_0 - \Omega) \\
&\quad + \phi_{\ell}^*(\mathbf{K}, \Omega) \delta(\mathbf{k} + \ell \mathbf{k}_0 - \mathbf{K}) \delta(\omega + \ell \omega_0 - \Omega) \} \\
&= \sum_{\ell} \int dY \{ \phi_{\ell}(Y) \delta(y - \ell y_0 - Y) + \phi_{\ell}^*(Y) \delta(y + \ell y_0 - Y) \}, \tag{5.114}
\end{aligned}$$

where $Y = (\mathbf{K}, \Omega)$, $y_0 = (\mathbf{k}_0, \omega_0)$ and $\int dY = \sum_{\mathbf{K}} \int d\Omega / (2\pi)$.

We start from (5.11) which is rewritten as

$$\begin{aligned}
&\int dY \delta(y - y_0 - Y) [\varepsilon(y_0 + Y) \phi_1(Y) \\
&\quad - \int dy' dy'' dy''' \left\{ \frac{V(y_0, 2y_0) V(2y_0, y_0)}{4\varepsilon(2y_0)} \phi_1(Y''') \phi_1(Y' - Y''') \phi_1^*(Y'') \right. \\
&\quad \left. + \frac{V(y_0 + Y, Y') V(Y', y_0 + Y''')}{2\varepsilon(Y')} \phi_1(Y'') \phi_1(Y''') \phi_1^*(Y' - Y''') \right\} \\
&\quad \times \delta(Y - Y' - Y'') \\
&\quad - \{ W(y, y_0 + Y', -y_0 + Y'') \phi_1(Y') \phi_1^*(Y'') \phi_1(Y''') \\
&\quad + w(y, -y_0 + Y', y_0 + Y'') \phi_1^*(Y') \phi_1(Y'') \phi_1(Y''') \} \\
&\quad \times \delta(Y - Y' - Y'' - Y''')] = 0. \tag{5.115}
\end{aligned}$$

The coefficients are given in (5.20). The nonlinear Landau damping comes from $\varepsilon(Y')$ which is given by

$$\varepsilon(Y') \simeq \frac{1}{K'^2 \lambda_D^2} \left(1 - i\pi \frac{K'}{|K'|} A \right), \tag{5.116}$$

where $v_{\parallel} = \mathbf{K}' \cdot \mathbf{v} / |\mathbf{K}'|$, $v_{g\parallel} = \mathbf{K}' \cdot \mathbf{v}_g / |\mathbf{K}'|$ and we have used the following approximation

$$\frac{\Omega'}{K'} \sim v_{g\parallel}, \quad A = C_s^2 \int dv_{\parallel} \delta(v_{\parallel} - v_{g\parallel}) \frac{\partial F_{i0}}{\partial v_{\parallel}}. \tag{5.117}$$

Expanding the dielectric function with respect to Y as

$$\varepsilon(y_0 + Y) \simeq \frac{\partial \varepsilon(y_0)}{\partial \omega} \Omega + \frac{\partial \varepsilon(y_0)}{\partial \mathbf{k}} \cdot \mathbf{K} + \frac{1}{2} \left(\mathbf{K} \cdot \frac{\partial}{\partial \mathbf{k}} \right)^2 \varepsilon(y_0), \quad (5.118)$$

multiplying $e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$ to both side of (5.115) and integrating with respect to y , we obtain

$$\begin{aligned} i \left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla \right) \hat{\phi}_1(\mathbf{x}, t) + \frac{3}{2} \omega_0 \lambda_D^2 \nabla^2 \hat{\phi}_1(\mathbf{x}, t) \\ = -\frac{1}{4} \omega_0 (k_0 \lambda_D)^2 \left\{ |\hat{\phi}_1(\mathbf{x}, t)|^2 + \frac{A}{2} \oint \frac{|\hat{\phi}_1(x'_\parallel, \mathbf{x}_\perp)|^2}{x_\parallel - x'_\parallel} dx'_\parallel \right\} \hat{\phi}_1(\mathbf{x}, t). \end{aligned} \quad (5.119)$$

where $\hat{\phi} = e\phi/T_e$ and $x_\parallel = \mathbf{K} \cdot \mathbf{x}/|\mathbf{K}|$. The second term of the right hand side of (5.119) is the nonlinear Landau damping which is expressed in form of nonlocal integral.

The number of waves are conserved in scattering processes of waves by particles. When waves lose energy due to the nonlinear Landau damping of the particles which obey the Boltzmann distribution, the energy is transferred to the waves with smaller frequency. Thus, the energy cascades from a short wave region to a long wave region through the nonlinear Landau damping. If there is no sink in the long wave region, the energy is accumulated in the long wave region and a large amplitude monochromatic wave grows. However, the large amplitude monochromatic wave is unstable against modulation and breaks into solitons with a finite wave length, transferring the energy back to the short wave region.

5.4.2 Ponderomotive Potential Force and Magnetization

5.4.2.1 Ponderomotive Potential Force

When the wave amplitude is modulated, because of the gradient of the potential amplitude, the quiver motion of the particle is not symmetric in back and forth and, as a result, the particle is driven in the direction that the amplitude decreases. The effective force by the gradient of the squared high frequency electric field is called the ponderomotive force.

Since the quiver motion is effective for electrons, we consider first the electron response to a high frequency wave, say the Langmuir wave. We start from the Vlasov equation without an external magnetic field

$$\frac{\partial}{\partial t} F_e + \mathbf{v} \cdot \nabla F_e + \frac{e}{m_e} \nabla \phi \cdot \frac{\partial}{\partial \mathbf{v}} F_e = 0. \quad (5.120)$$

We divide the distribution function into a slowly varying part F_{e0} and a rapidly varying part f_e as

$$F_e(\mathbf{x}, \mathbf{v}, t) = F_{e0}(\mathbf{x}, \mathbf{v}, t) + f_e(\mathbf{x}, \mathbf{v}, t), \quad (5.121)$$

and average the Vlasov equation over the time longer than the inverse of the wave frequency to obtain

$$\begin{aligned} \frac{\partial}{\partial t} F_{e0} + \mathbf{v} \cdot \nabla F_{e0} + \frac{e}{m_e} \nabla \Phi \cdot \frac{\partial}{\partial \mathbf{v}} F_{e0} = -i \frac{e}{m_e} \int dY e^{i(\mathbf{K} \cdot \mathbf{x} - \Omega t)} \\ \times \int d\mathbf{y} \phi(Y - \mathbf{y})(\mathbf{K} - \mathbf{k}) \cdot \frac{\partial f_e(\mathbf{y})}{\partial \mathbf{v}}, \end{aligned} \quad (5.122)$$

where $Y = (\mathbf{K}, \Omega)$ and $\mathbf{y} = (\mathbf{k}, \omega)$ are slow and fast Fourier scale variables, and $\int dY = \sum_{\mathbf{K}} \int d\Omega / (2\pi)$, $\int d\mathbf{y} = \sum_{\mathbf{k}} \int d\omega / (2\pi)$. In (5.122), we have retained the low frequency potential Φ through which the ion dynamics is related to the averaged electron quiver motion. The rapidly varying part is obtained by subtracting (5.122) from (5.120)

$$f_e(\mathbf{y}) = \frac{e}{m_e} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{e0}}{\partial \mathbf{v}} \phi(\mathbf{y}), \quad (5.123)$$

which is substituted into (5.122) to give

$$\begin{aligned} \frac{\partial}{\partial t} F_{e0} + \mathbf{v} \cdot \nabla F_{e0} + \frac{e}{m_e} \nabla \Phi \cdot \frac{\partial}{\partial \mathbf{v}} F_{e0} = -i \left(\frac{e}{m_e} \right)^2 \int dY e^{i(\mathbf{K} \cdot \mathbf{x} - \Omega t)} \\ \times \int d\mathbf{y} \phi(Y - \mathbf{y}) \phi(\mathbf{y})(\mathbf{K} - \mathbf{k}) \cdot \frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{e0}}{\partial \mathbf{v}}, \end{aligned} \quad (5.124)$$

From (5.114), we have

$$\phi(Y - \mathbf{y}) \phi(\mathbf{y}) = 2 \int dY' \phi_{y_0}(Y') \phi_{y_0}^*(Y - Y'), \quad (5.125)$$

where $y_0 = (\mathbf{k}, \omega(\mathbf{k}))$ and $\omega(\mathbf{k})$ is determined by the dispersion equation

$$\varepsilon(\mathbf{k}, \omega(\mathbf{k})) = 1 + \chi_e(\mathbf{k}, \omega(\mathbf{k})) = 0. \quad (5.126)$$

Multiplying $\mathbf{v}$ to (5.124) and integrating the resultant equation with respect to $\mathbf{v}$, we obtain with the definitions $n_e \mathbf{u}_e = \int \mathbf{v} F_{e0} d\mathbf{v}$ and $p_e \mathbf{I} + n_e \mathbf{u}_e : \mathbf{u}_e = \int \mathbf{v} : \mathbf{v} F_{e0} d\mathbf{v}$,

$$\begin{aligned} \frac{\partial}{\partial t} n_e \mathbf{u}_e + \nabla \cdot (n_e \mathbf{u}_e : \mathbf{u}_e) = -\nabla \frac{p_e}{m_e} + \frac{n_e e}{m_e} \nabla \Phi \\ - \nabla \frac{2e^2}{m_e^2} \sum_{\mathbf{k}} \frac{k^2}{\omega_{pe}^2} \Re \chi_e(\mathbf{k}, \omega(\mathbf{k})) |\phi_{\mathbf{k}}|^2, \end{aligned} \quad (5.127)$$

where we have used $\Re \chi(-\mathbf{k}, \omega(-\mathbf{k})) = \Re \chi(\mathbf{k}, \omega(\mathbf{k}))$. The right hand side is called the ponderomotive force. Integrating (5.120) over $\mathbf{v}$, we have the continuity equation

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (5.128)$$

which is combined with (5.127) to give

$$\frac{\partial^2 n_e}{\partial t^2} - \frac{1}{m_e} \nabla \cdot (\nabla p_e - n_e e \nabla \Phi) = \nabla \cdot \frac{2e^2}{m_e^2} \sum_{\mathbf{k}} \frac{k^2}{\omega_{pe}^2} \Re \chi_e(\mathbf{k}, \omega(\mathbf{k})) |\phi_{\mathbf{k}}|^2, \quad (5.129)$$

where the convection term is neglected. In a similar way, we have the equation for the ion density

$$\frac{\partial^2 n_i}{\partial t^2} - \frac{1}{m_i} \nabla \cdot (\nabla p_i + n_i e \nabla \Phi) = 0. \quad (5.130)$$

Assuming the charge neutrality $n_e = n_i = n$ and eliminating Φ from (5.129) and (5.130), we obtain

$$\frac{\partial^2 n}{\partial t^2} - C_s^2 \nabla^2 n = \nabla \cdot \frac{2e^2}{m_e m_i} \sum_{\mathbf{k}} \frac{k^2}{\omega_{pe}^2} \Re \chi_e(\mathbf{k}, \omega(\mathbf{k})) |\phi_{\mathbf{k}}|^2, \quad (5.131)$$

where we have neglected terms with m_e/m_i . Thus, the ponderomotive force drives the electrons which leave the ions behind and then the ambipolar potential is built to pull the ions toward the electrons, resulting in the density depletion. The high frequency wave is trapped in this density cavity and digs the density more, leading to the collapse of the wave.

5.4.2.2 Ponderomotive Magnetization

The ponderomotive force discussed above is a potential force. There is another effect of high frequency waves exerting the low frequency plasma motion, that is, ponderomotive magnetization [72] which is obtained when the high frequency wave amplitude is spatially inhomogeneous/anisotropic. Here, we derive the ponderomotive magnetization as a fluid response to high frequency waves and show that the ponderomotive potential force and magnetization are complementary and uniquely related to the ponderomotive potential simply expressed in terms of the linear susceptibility of a plasma [73].

Starting from a fluid equation,

$$\frac{\partial \mathbf{v}_\alpha}{\partial t} + (\mathbf{v}_\alpha \cdot \nabla) \mathbf{v}_\alpha = \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right), \quad (5.132)$$

and taking time average of (5.132) over the high frequency wave period, we have an equation for the low frequency dynamics up to the second order with respect to the high frequency oscillations:

$$\frac{\partial \mathbf{v}_{\alpha 0}}{\partial t} + (\mathbf{v}_{\alpha 0} \cdot \nabla) \mathbf{v}_{\alpha 0} = \frac{e_{\alpha}}{m_{\alpha}} \left(\mathbf{E}_0 + \frac{1}{c} \mathbf{v}_{\alpha 0} \times \mathbf{B}_0 \right) - \left\langle (\tilde{\mathbf{v}}_{\alpha} \cdot \nabla) \tilde{\mathbf{v}}_{\alpha} - \frac{e_{\alpha}}{m_{\alpha} c} \tilde{\mathbf{v}}_{\alpha} \times \tilde{\mathbf{B}} \right\rangle, \quad (5.133)$$

where all the physical quantities have been split into low frequency and high frequency parts as $\mathbf{A} = \mathbf{A}_0 + \tilde{\mathbf{A}}$ and $\langle \cdots \rangle$ indicates the time average: $\langle \tilde{\mathbf{A}} \rangle = 0$. For the nonlinear terms associated with the high frequency motion in (5.133), we need only the linear response to the electric field:

$$\frac{\partial \tilde{\mathbf{v}}_{\alpha}}{\partial t} = \frac{e_{\alpha}}{m_{\alpha}} \left(\tilde{\mathbf{E}} + \frac{1}{c} \tilde{\mathbf{v}}_{\alpha} \times \mathbf{B}_0 \right). \quad (5.134)$$

Here, we introduce a vector potential $\tilde{\mathbf{A}}$ and a displacement of the fluid through

$$\tilde{\mathbf{E}} = -\frac{1}{c} \frac{\partial \tilde{\mathbf{A}}}{\partial t}, \quad \tilde{\mathbf{v}}_{\alpha} = \frac{\partial \tilde{\mathbf{r}}_{\alpha}}{\partial t}, \quad (5.135)$$

where we have chosen the radiation gauge. Equation (5.134) is integrated:

$$\tilde{\mathbf{v}}_{\alpha} = \frac{e_{\alpha}}{m_{\alpha} c} (-\tilde{\mathbf{A}} + \tilde{\mathbf{r}}_{\alpha} \times \mathbf{B}_0). \quad (5.136)$$

Taking the rotation of (5.136) leads to

$$\nabla \times \tilde{\mathbf{v}}_{\alpha} + \frac{e_{\alpha}}{m_{\alpha} c} \tilde{\mathbf{B}} = \nabla \times (\tilde{\mathbf{r}}_{\alpha} \times \boldsymbol{\Omega}_{\alpha}), \quad (5.137)$$

where $\boldsymbol{\Omega}_{\alpha} = e_{\alpha} \mathbf{B}_0 / m_{\alpha} c$. Using (5.137), the second term of the right hand side of (5.133) reads

$$\begin{aligned} & \left\langle (\tilde{\mathbf{v}}_{\alpha} \cdot \nabla) \tilde{\mathbf{v}}_{\alpha} - \frac{e_{\alpha}}{m_{\alpha} c} \tilde{\mathbf{v}}_{\alpha} \times \tilde{\mathbf{B}} \right\rangle \\ &= \frac{1}{2} \nabla \langle \tilde{v}_{\alpha}^2 \rangle - \langle \tilde{\mathbf{v}}_{\alpha} \times (\nabla \times \tilde{\mathbf{v}}_{\alpha} + \frac{e_{\alpha}}{m_{\alpha} c} \tilde{\mathbf{B}}) \rangle \\ &= \frac{1}{2} \nabla \langle \tilde{v}_{\alpha}^2 \rangle - \boldsymbol{\Omega}_{\alpha} \times \langle (\tilde{\mathbf{v}}_{\alpha} \cdot \nabla) \tilde{\mathbf{r}}_{\alpha} \rangle - \frac{1}{2} \nabla \langle \tilde{\mathbf{v}}_{\alpha} \cdot (\tilde{\mathbf{r}}_{\alpha} \times \boldsymbol{\Omega}_{\alpha}) \rangle \\ &= \frac{1}{2} \nabla \langle \tilde{v}_{\alpha}^2 \rangle - \frac{1}{2 N_{\alpha}} \boldsymbol{\Omega}_{\alpha} \times \nabla \times \langle (N_{\alpha} \tilde{\mathbf{v}}_{\alpha} \times \tilde{\mathbf{r}}_{\alpha}) \rangle \\ &\quad - \frac{1}{2} \nabla (\boldsymbol{\Omega}_{\alpha} \cdot \langle \tilde{\mathbf{v}}_{\alpha} \times \tilde{\mathbf{v}}_{\alpha} \rangle) + \frac{1}{N_{\alpha}} \boldsymbol{\Omega}_{\alpha} \times \langle \tilde{n}_{\alpha} \tilde{v}_{\alpha} \rangle, \end{aligned}$$

where we have introduced $\tilde{n}_\alpha = -\nabla \cdot (N_\alpha \tilde{\mathbf{r}}_\alpha)$. From (5.133) and (5.138), we have for the slow part of the equation of motion

$$\begin{aligned} \frac{\partial \mathbf{v}_{\alpha 0}}{\partial t} + (\mathbf{v}_{\alpha 0} \cdot \nabla) \mathbf{v}_{\alpha 0} &= \frac{e_\alpha}{m_\alpha} \mathbf{E}_0 + \left(\mathbf{v}_{\alpha 0} + \frac{1}{N_\alpha} \langle \tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha \rangle \right) \times \boldsymbol{\Omega}_\alpha \\ &+ \nabla F_P + \mathbf{F}_M, \end{aligned} \quad (5.138)$$

where the ponderomotive potential force is given by

$$\mathbf{F}_P = -\frac{1}{2} \nabla \cdot (\langle \tilde{\mathbf{v}}_\alpha^2 \rangle + \boldsymbol{\Omega}_\alpha \cdot \langle \tilde{\mathbf{r}}_\alpha \times \tilde{\mathbf{v}}_\alpha \rangle), \quad (5.139)$$

and the force involving the induced magnetization current

$$\mathbf{F}_M = \frac{1}{N_\alpha} \boldsymbol{\Omega} \times \nabla \times \mathbf{M}_\alpha, \quad \mathbf{M}_\alpha = \frac{N_\alpha}{2} \langle \tilde{\mathbf{r}}_\alpha \times \tilde{\mathbf{v}}_\alpha \rangle. \quad (5.140)$$

The ponderomotive force and the magnetization are determined by the ponderomotive potential defined as

$$\Phi_\alpha = \frac{1}{2} (\langle \tilde{\mathbf{v}}_\alpha^2 \rangle + \boldsymbol{\Omega}_\alpha \cdot \langle \tilde{\mathbf{r}}_\alpha \times \tilde{\mathbf{v}}_\alpha \rangle), \quad (5.141)$$

through the following equations

$$\mathbf{F}_P = -\nabla \Phi_\alpha, \quad \mathbf{M}_\alpha = N_\alpha \frac{\partial}{\partial \boldsymbol{\Omega}_\alpha} \Phi_\alpha. \quad (5.142)$$

The magnetization gives a solenoidal current

$$\mathbf{j}_\alpha = \nabla \times \mathbf{M}_\alpha, \quad (5.143)$$

which is equivalent to the stationary current, as is clearly seen from the force free condition in (5.138):

$$N_\alpha \mathbf{v}_0 + \langle \tilde{n}_\alpha \tilde{\mathbf{v}}_\alpha \rangle = \nabla \times \mathbf{M}_\alpha. \quad (5.144)$$

We show the $K - \chi$ theorem on the basis of the fluid theory. The $K - \chi$ theorem formulated by Cary and Kaufman [74] is to show that the ponderomotive Hamiltonian K is related to the susceptibility χ . Multiplying (5.136) by $\tilde{\mathbf{v}}_\alpha$ and taking time average, we obtain

$$\langle \tilde{\mathbf{v}}_\alpha^2 \rangle + \boldsymbol{\Omega}_\alpha \cdot \langle \tilde{\mathbf{r}}_\alpha \times \tilde{\mathbf{v}}_\alpha \rangle = -\frac{e_\alpha}{m_\alpha c} \langle \tilde{\mathbf{v}}_\alpha \cdot \tilde{\mathbf{A}} \rangle, \quad (5.145)$$

which relates the ponderomotive potential Φ and the linear response $\tilde{\mathbf{v}}$. If we put

$$\tilde{\mathbf{A}}(\mathbf{x}, t) = \sum_j \{ \tilde{\mathbf{A}}(\mathbf{k}_j, \omega_j; \mathbf{x}, t) e^{i\theta_j} + \text{c.c.} \}, \quad (5.146)$$

where $\omega_j = -\partial\theta_j/\partial t$ and $\mathbf{k}_j = \nabla\theta_j$ are related to each other through the dispersion relation. Then, we obtain from (5.134) and (5.135)

$$\tilde{A} = -i \frac{c}{\omega_j} \left(1 - \frac{i}{\omega_j} \frac{\partial}{\partial t} \right) \tilde{E}, \quad (5.147)$$

$$\tilde{v}_p = V_{pq}(\mathbf{k}_j, \omega_j) \tilde{E}_q + i \frac{\partial V_{pq}}{\partial \omega_j} \frac{\partial \tilde{E}_q}{\partial t} - \frac{m_\alpha}{e_\alpha} V_{pr} \frac{\partial V_{rq}}{\partial t} \tilde{E}_q, \quad (5.148)$$

where

$$V_{pq}(\mathbf{k}_j, \omega_j) = i \frac{e_\alpha}{m_\alpha} \frac{\omega_j}{\omega_j^2 - \Omega_\alpha^2} \left\{ \delta_{pq} + \frac{i}{\omega_j} \varepsilon_{pqr} \Omega_{\alpha r} - \frac{1}{\omega_j^2} \Omega_{\alpha p} \Omega_{\alpha q} \right\}, \quad (5.149)$$

and ε_{pqr} is the totally antisymmetric third rank unit tensor. Substituting (5.147) and (5.148) into (5.145), we obtain the ponderomotive potential in terms of the linear susceptibility if we neglect the inhomogeneity

$$\Phi_\alpha = \sum_j \frac{1}{8\pi N_\alpha m_\alpha} \chi_{pq}(\mathbf{k}_j, \omega_j) \tilde{E}_p(\mathbf{k}_j, \omega_j) \tilde{E}_q^*(\mathbf{k}_j, \omega_j), \quad (5.150)$$

where we have used

$$\chi_{pq}(\mathbf{k}_j, \omega_j) = i \frac{4\pi}{\omega_j} e_\alpha N_\alpha V_{pq}(\mathbf{k}_j, \omega_j). \quad (5.151)$$

Equation (5.150) shows a relation between the ponderomotive potential and the susceptibility, indicating the $K - \chi$ theorem.

5.5 Weak Turbulence Theory

When many waves are excited and their phases are random, then the wave field is a stochastic variable and is described by the ensemble average over the wave phases. The amplitude of each wave would be changing in space and time because of the interactions among the waves. We may use the expression (5.15) and (5.16) for the wave amplitudes.

Suppose the phases of the excited waves are random and the average over the phases is denoted as $\langle \cdots \rangle$. Then, we use the random phase approximation for the homogeneous turbulence

$$\langle \phi_j(Y) \rangle = 0, \quad (5.152)$$

$$\langle \phi_i(Y') \phi_j^*(Y'') \rangle = \int dY I_i(Y) \delta(y_i - y_j) \delta(Y' - Y'') \delta(Y' + Y'' - Y), \quad (5.153)$$

which give

$$\begin{aligned} \langle \phi(y) \phi(z) \rangle &= \sum_i \int dY I_i(Y) \\ &\times \left\{ \delta\left(y - y_i - \frac{Y}{2}\right) \delta\left(z + y_i - \frac{Y}{2}\right) + \delta\left(y + y_i - \frac{Y}{2}\right) \delta\left(z - y_i - \frac{Y}{2}\right) \right\}. \end{aligned} \quad (5.154)$$

In order to construct the wave kinetic theory, we multiply $\phi(z)$ to (5.7) and average over the phases. The three body correlation thus appears in the resultant equation is converted into the four body correlation by iteration of (5.7)

$$\begin{aligned} &\langle \phi(y') \phi(y - y') \phi(z) \rangle \\ &= \frac{1}{2} \int dy'' \left\{ \frac{V(y - y', y'')}{\varepsilon(y - y')} \langle \phi(y'') \phi(y - y' - y'') \phi(y') \phi(z) \rangle \right. \\ &\quad + \frac{V(y', y'')}{\varepsilon(y')} \langle \phi(y'') \phi(y' - y'') \phi(y - y') \phi(z) \rangle \\ &\quad \left. + \frac{V(z, y'')}{\varepsilon(z)} \langle \phi(y'') \phi(z - y'') \phi(y') \phi(y - y') \rangle \right\}. \end{aligned} \quad (5.155)$$

Furthermore, we approximate the four body correlation by the product of the two body correlation

$$\begin{aligned} \langle \phi(y) \phi(y') \phi(y'') \phi(y''') \rangle &= \langle \phi(y) \phi(y') \rangle \langle \phi(y'') \phi(y''') \rangle \\ &\quad + \langle \phi(y) \phi(y'') \rangle \langle \phi(y') \phi(y''') \rangle \\ &\quad + \langle \phi(y) \phi(y''') \rangle \langle \phi(y') \phi(y'') \rangle. \end{aligned} \quad (5.156)$$

Thus, we have

$$\begin{aligned} \varepsilon(y) \langle \phi(y) \phi(z) \rangle &= \\ &\int dy' \int dy'' \left[\frac{1}{2} \frac{V(y, y') V(z, y'')}{\varepsilon(z)} \langle \phi(y') \phi(y'') \rangle \langle \phi(y - y') \phi(z - y'') \rangle \right. \\ &\quad + \left\{ \frac{V(y, y') V(y - y', y'')}{\varepsilon(y - y')} + W(y, y', y'') \right\} \\ &\quad \times \langle \phi(y') \phi(y'') \rangle \langle \phi(y - y' - y'') \phi(z) \rangle \\ &\quad \left. + w(y, y', y'') \langle \phi(y') \phi(z) \rangle \langle \phi(y'') \phi(y - y' - y'') \rangle \right], \end{aligned} \quad (5.157)$$

which is rewritten as

$$\begin{aligned}
& \sum_i \int dY \delta\left(y - y_i - \frac{Y}{2}\right) \delta\left(z + y_i - \frac{Y}{2}\right) \left\{ \varepsilon\left(y_i + \frac{Y}{2}\right) I_i(Y) \right. \\
& - \sum_j \sum_\ell \int dy' \int dy'' I_j(Y') I_\ell(Y'') \delta(Y - Y' - Y'') \\
& \left[\frac{|V(y_i, y_j)|^2}{2\varepsilon^*(y_i)} \delta(y_i - y_j - y_\ell) + \frac{V(y_i, -y_j)V(y_i, y_j)}{2\varepsilon(y_i)} \delta(y_i - y_j + y_\ell) \right. \\
& + \left\{ \frac{V(y_i, y_j)V(y_i - y_j, -y_j)}{\varepsilon(y_i - y_i)} + W(y_i, y_j, -y_j) \right\} \delta(y_\ell - y_i) \\
& \left. \left. + w(y_i, y_j, y_\ell) \delta(y_j + y_i) \right] \right\} = 0. \tag{5.158}
\end{aligned}$$

If the three wave decay interaction is assigned for $y_i = y_j + y_\ell$, (5.158) is simplified to

$$\begin{aligned}
\varepsilon(y_i + Y/2) I_i(Y) = \sum_j \int dY' \{ & t(y_i, y_j) I_j(Y') I_{i-j}(Y - Y') \\
& + m(y_i, y_j) I_j(Y') I_i(Y - Y') \}, \tag{5.159}
\end{aligned}$$

where

$$t(y_i, y_j) = \frac{1}{2} \frac{|V(y_i, y_j)|^2}{\varepsilon^*(y_i)}, \tag{5.160}$$

$$\begin{aligned}
m(y_i, y_j) = & \frac{V(y_i, y_j)V(y_i - y_j, y_j)}{\varepsilon(y_i - y_j)} + W(y_i, y_j, -y_j) \\
& + w(y_i, -y_i, -y_i). \tag{5.161}
\end{aligned}$$

Now we expand the left hand side of (5.159) to give

$$\begin{aligned}
\{\Omega - \mathbf{v}_g(\mathbf{k}_i) \cdot \mathbf{K} - i2\gamma_L(\mathbf{k}_i)\} \frac{\partial \Re \varepsilon(y_i)}{\partial \omega} I_i(Y) = 2 \sum_j \int dY' \\
\{t(y_i, y_j) I_j(Y') I_{i-j}(Y - Y') + m(y_i, y_j) I_j(Y') I_i(Y - Y')\}. \tag{5.162}
\end{aligned}$$

where

$$\gamma_L(\mathbf{k}_i) = -\frac{\Im \varepsilon(y_i)}{\frac{\partial \Re \varepsilon(y_i)}{\partial \omega}}, \quad \mathbf{v}_g(\mathbf{k}_i) = -\frac{\frac{\partial \Re \varepsilon(y_i)}{\partial \mathbf{k}_i}}{\frac{\partial \Re \varepsilon(y_i)}{\partial \omega}}.$$

Introducing the number of the waves defined by

$$N_i = \frac{\partial \Re \omega \varepsilon(y_i)}{\partial \omega} \frac{k_i^2}{\omega_i} \frac{e^2 I_i}{8\pi T_e^2}, \quad (5.163)$$

we have the wave kinetic equation for the weak turbulence,

$$\begin{aligned} \frac{\partial}{\partial t} N_i + \mathbf{v}_g(\mathbf{k}_i) \cdot \nabla N_i &= 2(\gamma(\mathbf{k}_i) + \gamma_{\text{NL}}(\mathbf{k}_i)) N_i \\ &+ \sum_j \sum_\ell T(y_i, y_j, y_\ell) N_j N_\ell \delta(y_i - y_j - y_\ell), \end{aligned} \quad (5.164)$$

where

$$\begin{aligned} \gamma_{\text{NL}}(\mathbf{k}_i) &= \sum_j M(y_i, y_j) N_j, \quad M(y_i, y_j) = \frac{8\pi T_e^2}{e^2} \frac{1}{k_j^2} \frac{\Im m(y_i, y_j)}{\frac{\partial \Re \omega \varepsilon(y_i)}{\partial \omega} \frac{\partial \Re \omega \varepsilon(y_j)}{\partial \omega}}, \\ T(y_i, y_j, y_\ell) &= 2 \frac{8\pi T_e^2}{e^2} \frac{k_i^2}{k_j^2 k_\ell^2} \frac{\Im t(y_i, y_j)}{\frac{\partial \Re \omega \varepsilon(y_j)}{\partial \omega} \frac{\partial \Re \omega \varepsilon(y_\ell)}{\partial \omega}}. \end{aligned}$$

The second term of the right hand side of (5.164) is the decay process in which the waves $(\mathbf{k}_j, \omega_j)$ and $(\mathbf{k}_\ell, \omega_\ell)$ are merged into the wave $(\mathbf{k}_i, \omega_i)$. The decay process needs the conservation law of energy and momentum

$$\omega_i = \omega_j + \omega_\ell, \quad \mathbf{k}_i = \mathbf{k}_j + \mathbf{k}_\ell, \quad (5.165)$$

which are not always satisfied and depend on the structure of the dispersion relations.

Here, we consider a cascade of the wave spectrum in the wavenumber space for the case of the Langmuir wave turbulence. Since the decay interaction is not allowed for the Langmuir turbulence and if the turbulence is spatially homogeneous, the linearly unstable system with $\gamma_L(\mathbf{k}_i) > 0$ is brought to a stationary state whose wave spectrum is determined by

$$\gamma_L(\mathbf{k}_i) + \gamma_{\text{NL}}(\mathbf{k}_i) = 0. \quad (5.166)$$

When the linear Landau damping is neglected, the number of the waves is conserved. This is seen from (5.5)

$$\gamma_{\text{NL}}(\mathbf{k}_i) \propto \sum_j \frac{(\mathbf{k}_i \cdot \mathbf{k}_j)^2}{k_i^2 k_j^2} (\mathbf{k}_i - \mathbf{k}_j)^2 \Im \varepsilon^*(\mathbf{k}_i - \mathbf{k}_j, \omega_i - \omega_j) N_j, \quad (5.167)$$

from which we have

$$\sum_i \gamma_{\text{NL}}(\mathbf{k}_i) N_i = 0, \quad (5.168)$$

that is

$$\frac{d}{dt} \sum_i N_i = 0. \quad (5.169)$$

The total momentum is conserved in a similar way as above. The equation of the background distribution function is given by

$$\frac{\partial}{\partial t} F_{\alpha 0} + \mathbf{v} \cdot \nabla F_{\alpha 0} = \frac{e_\alpha}{m_\alpha} \left\langle \nabla \phi \cdot \frac{\partial}{\partial \mathbf{v}} f_\alpha(\mathbf{x}, \mathbf{v}, t) \right\rangle, \quad (5.170)$$

whose right hand side is Fourier transformed with the use of (5.5) into

$$\begin{aligned} \frac{\partial}{\partial t} F_{\alpha 0} + \mathbf{v} \cdot \nabla F_{\alpha 0} &= i \frac{e_\alpha}{m_\alpha} \int dY e^{i(\mathbf{K} \cdot \mathbf{x} - \Omega t)} \int d\mathbf{y}' \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \\ &\times \left\{ g_\alpha(Y - y')(\mathbf{K} - \mathbf{k}') \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} \langle \phi(y') \phi(Y - y') \rangle \right. \\ &+ \int d\mathbf{y}'' g_\alpha(Y - y') \mathbf{k}'' \cdot \frac{\partial}{\partial \mathbf{v}} g_\alpha(Y - y' - y'')(\mathbf{K} - \mathbf{k}' - \mathbf{k}'') \cdot \frac{\partial F_{\alpha 0}}{\partial \mathbf{v}} \\ &\times \langle \phi(y') \phi(y'') \phi(Y - y' - y'') \rangle + \dots \left. \right\}. \end{aligned} \quad (5.171)$$

Again assuming the homogeneity and neglecting the decay interactions, we have

$$\begin{aligned} \frac{\partial}{\partial t} \int m_i \mathbf{v} F_{i0}(\mathbf{v}) d\mathbf{v} &= -\frac{i}{4\pi n_0} \int dY e^{i(\mathbf{K} \cdot \mathbf{x} - \Omega t)} \int d\mathbf{y}' \mathbf{k}' (\mathbf{K} - \mathbf{k}')^2 \\ &\times \{ \chi_{\text{ion}}(Y - y) \langle \phi(y') \phi(Y - y') \rangle \\ &+ v_{\text{ion}}(Y - y', y'') \langle \phi(y') \phi(y'') \phi(Y - y' - y'') \rangle \} \\ &= -2 \sum_i \left\{ \Im \chi_{\text{ion}} + \sum_j M(y_i, y_j) N_j \right\} \mathbf{k}_i N_i = -\frac{\partial}{\partial t} \sum_i \mathbf{k}_i N_i, \end{aligned} \quad (5.172)$$

where χ_{ion} is the ion susceptibility and v_{ion} is the ion contribution to (5.9)

Since the momentum is transferred from the wave to the particles, we have

$$\frac{d}{dt} \sum_i \mathbf{k}_i N_i < 0. \quad (5.173)$$

From (5.169) and (5.173), the wave energy cascades toward the smaller wave number region.

Thus, the nonlinear Landau damping transfers the wave energy to the small wave number region where a long wave is piled up since there is no effective dissipation

mechanism in the long wave region. However, the coherent long wave is shown to be unstable against modulation. In one dimension, the modulation instability is stabilized by forming solitons, while in the higher dimension, Langmuir solitons are no longer stable and collapse in a finite time. In those cases, wave–particle nonlinear interactions such as trapping and particle reflection have to be taken into account.

5.6 Kinetic Theory for Waves as Quasi-Particles

For the wave turbulence, whose wave numbers are well defined, we may regard the waves as quasi-particles. Then, the wave is expressed instead of (5.152) by

$$\phi(\mathbf{x}, t) = \sum_j \phi_j(\mathbf{x}, t) e^{i\theta_j(\mathbf{x}, t)}, \quad (5.174)$$

where the amplitude $\phi_j(\mathbf{x}, t)$ is modulated in space and time. The phase is determined by the wave number and the frequency through

$$\nabla \theta_j(\mathbf{x}, t) = -\mathbf{k}_j, \quad \frac{\partial}{\partial t} \theta_j = \omega_j, \quad (5.175)$$

where $\mathbf{k}_j$ and ω_j are related by the dispersion relation

$$\varepsilon(\mathbf{k}_j, \omega_j) = 0. \quad (5.176)$$

In addition, $\mathbf{k}_j$ and ω_j satisfy another relation directly derived from (5.175)

$$\frac{\partial \mathbf{k}_j}{\partial t} + \nabla \omega_j = 0. \quad (5.177)$$

From the dispersion relation, we have

$$\nabla \omega_j = \nabla \left(\frac{\partial \omega_j}{\partial \mathbf{k}_j} \cdot \mathbf{k}_j \right) = \nabla (\mathbf{v}_g \cdot \mathbf{k}_j). \quad (5.178)$$

The equation for the number of the waves is given by (5.164)

$$\begin{aligned} \frac{\partial}{\partial t} N_i + \mathbf{v}_g(\mathbf{k}_i) \cdot \nabla N_i &= 2(\gamma(\mathbf{k}_i) + \gamma_{\text{NL}}(\mathbf{k}_i)) N_i \\ &+ \sum_j \sum_\ell T(y_i, y_j, y_\ell) N_j N_\ell \delta(y_i - y_j - y_\ell), \end{aligned} \quad (5.179)$$

where γ_{NL} and $T(y_i, y_j, y_\ell)$ are given by (5.5) and (5.5). Introducing a distribution function of the quasi-particles

$$I_{\mathbf{k}}(\mathbf{x}, t) = \sum_j N_j \delta(\mathbf{k} - \mathbf{k}_j(\mathbf{x}, t)), \quad (5.180)$$

the kinetic equation for the quasi-particles is obtained by

$$\begin{aligned} & \frac{\partial}{\partial t} I_{\mathbf{k}}(\mathbf{x}, t) + \mathbf{v}_g \cdot \nabla I_{\mathbf{k}}(\mathbf{x}, t) - \nabla \omega \cdot \frac{\partial}{\partial \mathbf{k}} I_{\mathbf{k}}(\mathbf{x}, t) \\ &= 2(\gamma(\mathbf{k}) + \gamma_{\text{NL}}(\mathbf{k})) I_{\mathbf{k}}(\mathbf{x}, t) \\ &+ \int d\mathbf{k}' \int d\mathbf{k}'' T(\mathbf{k}, \mathbf{k}', \mathbf{k}'') I_{\mathbf{k}'}(\mathbf{x}, t) I_{\mathbf{k}''}(\mathbf{x}, t) \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}''). \end{aligned} \quad (5.181)$$

If there is a flow V in the medium, then we need to make the following replacement

$$\omega_j \rightarrow \omega_j + \mathbf{k}_j \cdot V, \quad (5.182)$$

and (5.181) is rewritten as

$$\begin{aligned} & \frac{\partial}{\partial t} I_{\mathbf{k}}(\mathbf{x}, t) + \left(\frac{d\omega}{d\mathbf{k}} + V \right) \cdot \nabla I_{\mathbf{k}}(\mathbf{x}, t) - \nabla(\omega + \mathbf{k} \cdot V) \cdot \frac{\partial}{\partial \mathbf{k}} I_{\mathbf{k}}(\mathbf{x}, t) \\ &= 2(\gamma(\mathbf{k}) + \gamma_{\text{NL}}(\mathbf{k})) I_{\mathbf{k}}(\mathbf{x}, t) \\ &+ \int d\mathbf{k}' \int d\mathbf{k}'' T(\mathbf{k}, \mathbf{k}', \mathbf{k}'') I_{\mathbf{k}'}(\mathbf{x}, t) I_{\mathbf{k}''}(\mathbf{x}, t) \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}''). \end{aligned} \quad (5.183)$$

5.7 Modulation Instability of Plasmon Gas

Since the dispersion relation of a Langmuir wave depends on the plasma density, the propagation properties of the Langmuir wave is affected by change in the plasma density. Dynamics of a Langmuir wave turbulence in a plasma without flow is described by

$$\frac{\partial}{\partial t} I_{\mathbf{k}}(\mathbf{x}, t) + \frac{d\omega(\mathbf{k})}{d\mathbf{k}} \cdot \nabla I_{\mathbf{k}}(\mathbf{x}, t) - \nabla \omega(\mathbf{k}) \cdot \frac{\partial}{\partial \mathbf{k}} I_{\mathbf{k}}(\mathbf{x}, t) = 0, \quad (5.184)$$

$$\nabla \omega(\mathbf{k}) = \frac{\partial \omega(\mathbf{k})}{\partial n} \nabla n \simeq \frac{\omega_{\text{pe}}}{2n} \nabla n, \quad (5.185)$$

where n is the plasma density which is affected by the Langmuir wave through the ponderomotive force. A slowly varying part of the electron equation of motion is given by

$$\langle \tilde{\mathbf{v}}_e \cdot \nabla \tilde{\mathbf{v}}_e \rangle = v_{\text{Te}}^2 \nabla \left(\frac{e\phi}{T_e} - \ln n \right). \quad (5.186)$$

Using the rapidly varying part of the electron equation of motion, we have

$$\tilde{v}_e \sim -\frac{k v_{Te}^2}{\omega} \frac{e\tilde{\phi}}{T_e}, \quad (5.187)$$

which is substituted into (5.186) to give the slowly varying part of the potential

$$\frac{e\phi}{T_e} = \ln n + \sum_k \left(\frac{\omega_{pe}}{\omega(k)} \right)^2 I_k \simeq \ln n + \sum_k I_k. \quad (5.188)$$

From the ion equation of continuity and equation of motion

$$\frac{\partial n}{\partial t} + \nabla(n\mathbf{v}) = 0, \quad (5.189)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -C_s^2 \nabla \frac{e\phi}{T_e}, \quad (5.190)$$

an equation for the low frequency part of the density is obtained by

$$\frac{\partial^2}{\partial t^2} n - C_s^2 \nabla^2 n = n C_s^2 \nabla^2 \sum_k I_k, \quad (5.191)$$

where the convection term is neglected. Thus, the basic equations for a coupled system of the plasmon gas and the low frequency density fluctuation are given by Vedenov and Rudakov [75]

$$\frac{\partial}{\partial t} I_k(\mathbf{x}, t) + \frac{d\omega}{d\mathbf{k}} \cdot \nabla I_k(\mathbf{x}, t) - \frac{\omega_{pe}}{2n} (\nabla n) \cdot \frac{\partial}{\partial \mathbf{k}} I_k(\mathbf{x}, t) = 0, \quad (5.192)$$

$$\frac{\partial^2}{\partial t^2} n - C_s^2 \nabla^2 n = n C_s^2 \nabla^2 \sum_k I_k. \quad (5.193)$$

The physical meaning of (5.192) is easily seen as follows. The characteristic equations of (5.192) are given by

$$\frac{d\mathbf{x}}{dt} = \frac{d\omega(\mathbf{k})}{d\mathbf{k}} = \mathbf{v}_g, \quad \frac{d\mathbf{k}}{dt} = -\frac{\omega_{pe}}{2} \nabla \ln n. \quad (5.194)$$

The second equation of (5.194) is rewritten as

$$\frac{d\mathbf{v}_g}{dt} = -\frac{\omega_{pe}}{2} \nabla \ln n \cdot \frac{d^2 \omega(\mathbf{k})}{d\mathbf{k} d\mathbf{k}}. \quad (5.195)$$

Since $(d^2 \omega(\mathbf{k})/d\mathbf{k} d\mathbf{k})^{-1}$ is an effective mass of the plasmon, the first equation of (5.194) and (5.195) describe the dynamics of the plasmons in the phase space

spanned by $(\mathbf{x}, \mathbf{v}_g)$ under the potential n which is determined by the pressure of the plasmons through (5.193). When density fluctuations are introduced, the plasmons are trapped in the well of the density which in turn is deepened and attracts more plasmons. The positive feedback processes lead to the modulation instability described by (5.192) and (5.193). Thus, a homogeneous plasmon gas is broken into blobs. A criterion for the modulation instability is given by the sign of the product of the effective mass and the potential.

Suppose that a perturbation is applied to the homogeneous solution $I_k^0 = \text{const}$ and $n_0 = \text{const}$ of (5.192) and (5.193) as

$$I_k = I_k^0 + \delta I_k e^{i(\mathbf{q} \cdot \mathbf{x} - \Omega t)}, \quad n = n_0 + \delta n e^{i(\mathbf{q} \cdot \mathbf{x} - \Omega t)}. \quad (5.196)$$

Substituting (5.196) into (5.192) and (5.193) and linearizing the resultant equations, we have a dispersion relation

$$\Omega^2 - q^2 C_s^2 = -\frac{1}{2} q^2 C_s^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\omega_{pe}}{\Omega - \mathbf{q} \cdot \frac{d\omega(\mathbf{k})}{d\mathbf{k}}} \mathbf{q} \cdot \frac{\partial I_k^0}{\partial \mathbf{k}}, \quad (5.197)$$

where $\sum_k$ is replaced by $\int d\mathbf{k}/(2\pi)^3$.

A solution of the dispersion relation is given

$$\Omega = q C_s + i\gamma, \quad (5.198)$$

$$\gamma = \frac{\omega_{pe}}{4} q C_s \int \frac{d\mathbf{k}}{(2\pi)^3} \mathbf{q} \cdot \frac{\partial I_k^0}{\partial \mathbf{k}} \delta \left(q C_s - \mathbf{q} \cdot \frac{d\omega(\mathbf{k})}{d\mathbf{k}} \right). \quad (5.199)$$

The resonance between the group velocity of the plasmon and the phase velocity of the modulation is analogous to the nonlinear Landau damping and gives rise to an asymmetric deformation of the plasmon distribution.

When the plasmons are not in resonance with the low frequency density fluctuation, (5.197) becomes

$$\begin{aligned} \Omega^2 - q^2 C_s^2 &= \frac{\omega_{pe}}{2} q^2 C_s^2 \int \frac{(\mathbf{q} \cdot \frac{d}{d\mathbf{k}})^2 \omega(\mathbf{k})}{\left(\Omega - \mathbf{q} \cdot \frac{d\omega(\mathbf{k})}{d\mathbf{k}} \right)^2} I_k^0 \frac{d\mathbf{k}}{(2\pi)^3} \\ &= \frac{\omega_{pe}}{2} q^4 C_s^2 \int \frac{\frac{dv_g}{dk_{\parallel}}}{(\Omega - q v_g)^2} I^0(k_{\parallel}) \frac{dk_{\parallel}}{2\pi}, \end{aligned} \quad (5.200)$$

where $k_{\parallel} = \mathbf{q} \cdot \mathbf{k}/|\mathbf{q}|$, $v_g(k_{\parallel}) = d\omega(k_{\parallel})/dk_{\parallel}$ and $I_{k_{\parallel}}^0 = \int I_k^0 d\mathbf{k}_{\perp}/(2\pi)^2$. For simplicity, we assume that the plasmon distribution is narrow and peaked at $k_{\parallel} = k_0$ and may put

$$I_{k_{\parallel}}^0 = 2\pi I_0 \delta(k_{\parallel} - k_0). \quad (5.201)$$

Then, (5.200) is reduced to

$$f(\Omega) = (\Omega^2 - q^2 C_s^2)(\Omega - qv_g(k_0))^2 = \frac{\omega_{pe}}{2} q^4 C_s^2 \frac{dv_g}{dk_0} I_0. \quad (5.202)$$

A condition for the modulation instability is obtained by

$$v_g(k_0) > C_s, \quad f(\Omega_m) < \frac{\omega_{pe}}{2} q^4 C_s^2 \frac{dv_g}{dk_0} I_0 = \frac{1}{2} \frac{m_i}{m_e} (q C_s)^4 I_0, \quad (5.203)$$

where Ω_m is determined by $f'(\Omega_m) = 0$ and is given by

$$\Omega_m = \frac{1}{4} q v_g(k_0) \left\{ 1 + \sqrt{1 + 8 \frac{m_e}{m_i}} \right\} \simeq \frac{1}{2} q v_g(k_0),$$

$$f(\Omega_m) = \frac{q^4 v_g(k_0)^4}{16} \left(1 - \frac{4}{k_0^2 \lambda_D^2} \frac{m_e}{m_i} \right).$$

Then, (5.203) is rewritten as

$$k_0 \lambda_D > \sqrt{\frac{m_e}{m_i}}, \quad I_0 > \frac{(k_0 \lambda_D)^4}{8} \frac{m_i}{m_e}. \quad (5.204)$$

The growth rate of the modulation instability is estimated by putting $\Omega = qv_g(k_0)/2 + \delta\Omega$ in (5.202)

$$\delta\Omega^2 = \frac{1}{4} q^2 v_g(k_0)^2 \left\{ 1 - \sqrt{8 \frac{m_e}{m_i} \frac{I_0}{(k_0 \lambda_D)^4}} \right\} \simeq -q^2 v_g(k_0)^2 \sqrt{\frac{1}{2} \frac{m_e}{m_i} \frac{I_0}{(k_0 \lambda_D)^4}},$$

which gives

$$\gamma = q C_s \left\{ \frac{1}{2} \frac{m_i}{m_e} \frac{I_0}{(k_0 \lambda_D)^4} \right\}^{1/4}. \quad (5.205)$$

The modulation instability thus develops to bring a homogeneous turbulence into blobs.

Chapter 6

Solitons in Plasmas

A large amplitude coherent wave has been shown to evolve into solitary waves which behave like particles and are named solitons. A solitary wave was first discovered to propagate in a long distance without changing the shape and the speed in a shallow water in 1844 [76] and then Korteweg and de Vries [77] derived an equation which describes the propagation of solitary waves in 1895. Fermi, Pasta, and Ulam [78, 79] used the first computer to study the relaxation to thermal equilibrium in one dimensional lattice system interacting under a nonharmonic potential and found the recurrent phenomena instead of equipartition of the energy. Zabusky and Kruskal [80] showed that the discrete dynamical equation for the FPU problem is reduced to the KdV equation in a continuum limit and solved it numerically to discover particle-like behaviors of solitary waves which are responsible for the recurrent phenomena found by FPU. Then, a remarkable progress was made by Gardner, Green, Kruskal, and Miura [81] who formulated the inverse scattering method as an exact solution method to solve an initial value problem for the K-dV equation. Another remarkable development was made by Zakharov and Shabat [82] who extended the inverse scattering method to solve the nonlinear Schrödinger equation. Then Ablowitz, Kaup, Newell, and Segur [83] generalized the Zakharov–Shabat method to find nonlinear evolution equations solvable with the inverse scattering method formulated in terms of a set of 2×2 matrix forms of the linear eigenvalue problem. Hirota [84] also developed a bilinear transformation method to obtain a N-soliton solution.

The K-dV equation is re-discovered in plasma physics by Gardner and Morikawa [85] for a magnetosonic wave and also by Washimi and Taniuti [86] for an ion acoustic wave. Taniuti [87] formulated the reductive perturbation theory which is applied to waves in not only plasmas but also other fields of physics to derive various types of nonlinear equations.

Thus, solitons had been a subject of intensive studies and the notion of soliton is widely established. However, the studies had been restricted to solitons in linearly stable systems in which only one linear eigen mode exists. A question arises whether solitons are still meaningful in unstable plasmas or in multi-mode systems such as an ion beam-plasma system and play key roles in nonlinear processes.

This chapter deals with basic soliton theories in the first part and then studies soliton physics in a realistic plasmas with multiple modes and a source of instability

where a family of ion acoustic waves (low frequency weakly dispersive waves) are mainly involved.

The theory of solitons is fully developed. There are excellent books on the solution methods and applications [30, 88]

6.1 Ion Acoustic Waves and K-dV Equation

Since the dispersion relation of an ion acoustic wave is given by

$$\omega = \frac{kC_s}{\sqrt{1 + k^2\lambda_D^2}} \simeq kC_s \left(1 - \frac{1}{2}k^2\lambda_D^2\right), \quad (6.1)$$

the time scale of the change in the wave amplitude in the frame moving with the ion sound wave velocity is determined by the dispersion

$$\Omega = \omega - kC_s = -\frac{1}{2}\omega_{pi}k^3\lambda_D^3. \quad (6.2)$$

Therefore, the equation for the wave potential ϕ normalized by T_e/e in the Fourier space is written as

$$\left(\Omega + \frac{1}{2}\omega_{pi}K^3\right)\phi(\Omega, K) = 0, \quad K = k\lambda_D, \quad (6.3)$$

which is transformed into the coordinate space $\xi = (x - C_s t)/\lambda_D$ to give

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\omega_{pi}\frac{\partial^3\phi}{\partial\xi^3} = 0. \quad (6.4)$$

Now, let us consider the nonlinearity which balances with the dispersion. In one dimension, the equations for the ion fluid are given by

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \quad (6.5)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + C_s^2\frac{\partial\phi}{\partial x} = 0, \quad (6.6)$$

$$\lambda_D^2\frac{\partial^2\phi}{\partial x^2} = e\phi - \frac{n}{n_0}. \quad (6.7)$$

Equations (6.5) and (6.6) are combined to give

$$\frac{\partial^2}{\partial t^2}n - C_s^2\frac{\partial}{\partial x}\left(n\frac{\partial\phi}{\partial x}\right) - \frac{\partial^2}{\partial x^2}(nv^2) = 0. \quad (6.8)$$

In the moving frame $\xi = (x - C_s t)/\lambda_D$, we consider a wave propagation in one direction and then may approximate the second order time derivative and the velocity as

$$\frac{\partial^2}{\partial t^2} = \left(\frac{\partial}{\partial t} - \omega_{pi} \frac{\partial}{\partial \xi} \right)^2 \simeq \omega_{pi} \frac{\partial}{\partial \xi} \left(\omega_{pi} \frac{\partial}{\partial \xi} - 2 \frac{\partial}{\partial t} \right), \quad (6.9)$$

$$v \simeq C_s, \quad (6.10)$$

and the Poisson equation is rewritten as

$$n \simeq n_0 \left\{ 1 + \phi + \frac{1}{2} \phi^2 - \lambda_D^2 \frac{\partial^2}{\partial \xi^2} \phi \right\}. \quad (6.11)$$

Equations (6.9)–(6.11) are substituted into (6.8) to give

$$-2\omega_{pi} \frac{\partial}{\partial \xi} \left\{ \frac{1}{\omega_{pi}} \frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial^2 \phi}{\partial \xi^3} + \frac{1}{2} \frac{\partial \phi^2}{\partial \xi} \right\} = 0, \quad (6.12)$$

which is integrated under the boundary condition $\phi(\pm\infty) = 0$ to give with a new time scale $\omega_{pi} t = \tau$

$$\frac{\partial \phi}{\partial \tau} + \frac{1}{2} \frac{\partial^3 \phi}{\partial \xi^3} + \phi \frac{\partial \phi}{\partial \xi} = 0. \quad (6.13)$$

This equation is originally derived by Korteweg and de Vries in 1895 to describe a shallow water wave and is called the K-dV equation [77]. Long later, Gardner and Morikawa [85] rediscovered the K-dV equation for nonlinear hydromagnetic waves in a cold magnetized plasma. They derived an equation for the dispersive wave spreading to be balanced with the nonlinear wave steepening. Then, Washimi and Taniuti demonstrated that a large amplitude ion acoustic wave is described by the K-dV equation. Taniuti developed a kind of multi-scale perturbation method called the reductive perturbation theory which applied to derive model equations to describe nonlinear behavior of various modes in not only plasmas but also other fields of physics and was successful to extract essential features of nonlinear wave propagation.

6.1.1 Reductive Perturbation Theory and K-dV Equation

The reductive perturbation theory is to utilize the balance between wave spreading due to dispersion and wave steepening due to nonlinearity explicitly. In a weakly dispersive wave such as an ion acoustic wave the wave number is taken small and of the order of the smallness parameter $\varepsilon^{1/2}$ and then the dispersion term is estimated as $\varepsilon^{3/2}$. Thus, the space and time coordinates are anticipated as

$$\xi = \varepsilon^{1/2}(x - C_s t)/\lambda_D, \quad \tau = \varepsilon^{3/2}t. \quad (6.14)$$

Then, (6.5)–(6.7) are rewritten as

$$\varepsilon \frac{\partial n}{\partial \tau} - \frac{\partial n}{\partial \xi} + \frac{\partial}{\partial \xi}(nv) = 0, \quad (6.15)$$

$$\varepsilon \frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial \xi} + v \frac{\partial v}{\partial \xi} + \frac{\partial \phi}{\partial \xi} = 0, \quad (6.16)$$

$$\varepsilon \frac{\partial^2 \phi}{\partial \xi^2} = e^\phi - \frac{n}{n_0}. \quad (6.17)$$

Now, we expand the physical variables as

$$\begin{pmatrix} n \\ v \\ \phi \end{pmatrix} = \begin{pmatrix} n_0 \\ 0 \\ 0 \end{pmatrix} + \sum_{i=1}^{\infty} \varepsilon^i \begin{pmatrix} n^{(i)} \\ v^{(i)} \\ \phi^{(i)} \end{pmatrix}, \quad (6.18)$$

and substitute (6.18) into (6.15)–(6.17) to give the relations among the first order quantities:

$$\frac{n^{(1)}(\xi, \tau)}{n_0} = v^{(1)}(\xi, \tau) = \phi^{(1)}(\xi, \tau), \quad (6.19)$$

where we have assumed $n^{(1)}(\pm\infty, \tau) \rightarrow 0$.

In the second order of ε , we have

$$\frac{\partial n^{(1)}}{\partial \tau} - \frac{\partial n^{(2)}}{\partial \xi} + n_0 \frac{\partial v^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}(n^{(1)}v^{(1)}) = 0, \quad (6.20)$$

$$\frac{\partial v^{(1)}}{\partial \tau} - \frac{\partial v^{(2)}}{\partial \xi} + v^{(1)} \frac{\partial v^{(1)}}{\partial \xi} + \frac{\partial \phi^{(2)}}{\partial \xi} = 0, \quad (6.21)$$

$$\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = \phi^{(2)} - \frac{n^{(2)}}{n_0} + \frac{1}{2}(\phi^{(1)})^2, \quad (6.22)$$

which are combined to give the K-dV equation:

$$\frac{\partial \phi^{(1)}}{\partial \tau} + \frac{1}{2} \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} = 0. \quad (6.23)$$

The higher order terms are successively obtained to give corrections to the first order terms.

6.1.2 Kinetic Theory Derivation of K-dV Equation

The K-dV equation is certainly derived from the kinetic equation as well for which the effect of the resonant particle interaction can be evaluated. From the kinetic equation, we have

$$\varepsilon(\mathbf{k}, \omega)\phi(\mathbf{k}, \omega) = \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} V_{\mathbf{k}_1, \mathbf{k}_2} \phi(\mathbf{k}_1, \omega_1) \phi(\mathbf{k}_2, \omega_2) \delta(\omega - \omega_1 - \omega_2), \quad (6.24)$$

where

$$\varepsilon(\mathbf{k}, \omega) = 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \int d\mathbf{v} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial F_{\alpha}}{\partial \mathbf{v}}, \quad (6.25)$$

$$V_{\mathbf{k}_1, \mathbf{k}_2} = \sum_{\alpha} \frac{e_{\alpha}}{m_{\alpha}} \frac{\omega_{p\alpha}^2}{k^2} \int d\mathbf{v} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k}_1 \cdot \frac{\partial}{\partial \mathbf{v}} \frac{1}{\omega_2 - \mathbf{k}_2 \cdot \mathbf{v}} \mathbf{k}_2 \cdot \frac{\partial F_{\alpha}}{\partial \mathbf{v}}. \quad (6.26)$$

In the following, one dimensional case is considered. Introducing the slow time scale $\omega - kC_s = \Omega$ which balances the dispersion, $\varepsilon(\mathbf{k}, \omega)$ is given by

$$\begin{aligned} \varepsilon(k, \omega) &= 1 + \frac{1}{k^2 \lambda_D^2} - \frac{\omega_{pi}^2}{\omega^2} + i \text{Im} \varepsilon(k, \omega) \\ &= \frac{2\omega_{pi}^2}{k^3 C_s^3} \left(\Omega + \frac{1}{2} \frac{k^3 C_s^3}{\omega_{pi}^2} \right) + i\pi \frac{\omega_{pi}^2}{k^2} \frac{k}{|k|} \int d\mathbf{v} \delta(C_s - v) \frac{\partial F_i}{\partial v}. \end{aligned} \quad (6.27)$$

Taking the ion contribution and using the relation $\omega \sim kC_s \gg kv_{Ti}$ after partial integration, $V_{\mathbf{k}_1, \mathbf{k}-\mathbf{k}_1}$ reduces to

$$V_{\mathbf{k}_1, \mathbf{k}-\mathbf{k}_1} \simeq \frac{e}{m_i} \frac{\omega_{pi}^2}{k^2} \left\{ \frac{2k^2 k_1 (k - k_1)}{\omega^3 (\omega - \omega_1)} + \frac{kk_1 (k - k_1)^2}{\omega^2 (\omega - \omega_1)^2} \right\} = \frac{3e}{m_i} \frac{\omega_{pi}^2}{C_s^4} \frac{k_1}{k^3}. \quad (6.28)$$

Thus, we have for the potential ϕ normalized by T_e/e

$$\begin{aligned} &\left\{ \Omega + \frac{1}{2} \omega_{pi} k^3 \lambda_D^3 + A(C_s) \omega_{pi} \lambda_D \frac{k^2}{|k|} \right\} \phi(k, \Omega) \\ &= \frac{3}{2} C_s \sum_{k_1} \int \frac{d\omega_1}{2\pi} k_1 \phi(k_1, \Omega_1) \phi(k - k_1, \Omega - \Omega_1), \end{aligned} \quad (6.29)$$

where

$$A(C_s) = 2C_s^2 \int d\mathbf{v} \delta(C_s - v) \frac{\partial F_i(v)}{\partial v}. \quad (6.30)$$

Equation (6.29) is inverse-Fourier transformed with $\tau = \omega_{pi} t$ and $\xi = (x - C_s t)/\lambda_D$ into

$$\frac{\partial \phi}{\partial \tau} + \frac{1}{2} \frac{\partial^3 \phi}{\partial \xi^3} + \frac{3}{4} \frac{\partial \phi^2}{\partial \xi} - A(C_s) \int \frac{\wp}{\xi - \xi'} \frac{\partial}{\partial \xi'} \phi(\xi', \tau) d\xi' = 0, \quad (6.31)$$

where $\wp$ denotes the principal part of the integral. The last term represents the Landau damping which distorts the wave form.

6.1.3 Stationary Solutions of K-dV Equation

Since the K-dV equation is rescaled by the transformation $\phi \rightarrow \alpha\phi$, here we consider the following form of the K-dV equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + \phi \frac{\partial \phi}{\partial x} = 0. \quad (6.32)$$

Suppose to look for a solution moving with λ . Then, the K-dV equation is readily integrated into

$$\left(\frac{d\phi}{d\xi} \right)^2 = \lambda \phi^2 - \frac{1}{3} \phi^3 + C\phi + D, \quad (6.33)$$

where C and D are integration constants. A periodic solution exists for finite C and D and is expressed by the Jacobean elliptic cnoidal function. For both C and D to be zero, a solution diminishes at infinity together with its derivative and is characterized by single hump which travels at a constant speed. The solitary wave solution is given by

$$\phi(\xi, \tau) = 12K^2 \text{sech}^2[K(\xi - 4K^2\tau)], \quad (\lambda = 4K^2), \quad (6.34)$$

which indicates that the larger the amplitude, the faster it travels and the narrower the width becomes.

6.1.4 Sagdeev Potential

We obtain a stationary solution of a nonlinear ion acoustic wave described by (6.5)–(6.7). The stationary solution is expressed by $A(x, t) = A(x - \lambda t)$ where λ is the phase velocity. The electrons are assumed to obey the Boltzmann distribution

$$n_e = n_0 e^\phi, \quad (6.35)$$

where ϕ is the potential normalized by T_e/e . The equation of motion for the ion fluid (6.6) is integrated to give

$$(u - \lambda)^2 + 2\phi = (u_0 - \lambda)^2 + 2\phi_0, \quad (6.36)$$

where u and λ are normalized by the ion acoustic velocity C_s , and u_0 and ϕ_0 are integration constants. From the ion equation of continuity (6.5), we have

$$n_i = \frac{n_0}{\sqrt{1 + \frac{2(\phi_0 - \phi)}{(u_0 - \lambda)^2}}}. \quad (6.37)$$

The electron and ion density are substituted into Poisson's equation to give

$$\frac{d^2\phi}{dx^2} = e^\phi - \frac{1}{\sqrt{1 + \frac{2(\phi_0 - \phi)}{(u_0 - \lambda)^2}}} = -\frac{dV(\phi)}{d\phi}, \quad (6.38)$$

where x is normalized by the Debye length. This is integrated to give

$$\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 = -V(\phi) = (e^\phi - 1) + (u_0 - \lambda)^2 \left(\sqrt{1 - \frac{2\phi}{(u_0 - \lambda)^2}} - 1 \right), \quad (6.39)$$

where we have put $\phi_0 = 0$. The right hand side is called the Sagdeev potential named after Sagdeev [89]. For solutions to exist in (6.39), we require that the Sagdeev potential should be concave at $\phi = 0$ and positive at the potential $\phi_c = (u_0 - \lambda)^2/2$ above which the ion reflection occurs:

$$\frac{d^2V(\phi)}{d\phi^2} \Big|_{\phi=0} = \frac{1}{(u_0 - \lambda)^2} - 1 < 0, \quad (6.40)$$

$$V \left(\frac{(u_0 - \lambda)^2}{2} \right) = 1 - \exp \left(\frac{(u_0 - \lambda)^2}{2} \right) + (u_0 - \lambda)^2 > 0. \quad (6.41)$$

For the nonlinear solution to exist, the phase velocity has to satisfy the following inequality

$$1 < |u_0 - \lambda| < 1.5852. \quad (6.42)$$

This indicates that the nonlinear ion acoustic wave is compressional and travels with a super sonic speed for the case of $u_0 = 0$.

Now, we see under what condition the K-dV approximation can be used. The potential associated with the K-dV approximation (6.13) is given by

$$V_{KdV}(\phi) = -\lambda_{KdV}\phi^2 + \frac{1}{3}\phi^3, \quad (6.43)$$

where λ_{KdV} is the phase velocity in the frame moving with the sound velocity. Therefore, we have

$$\lambda = 1 + \lambda_{KdV}. \quad (6.44)$$

Expanding the Sagdeev potential with respect to ϕ and using the relation (6.44), we have

$$\begin{aligned}
V(\phi) &= \frac{1-\lambda^2}{2\lambda^2}\phi^2 + \frac{3-\lambda^4}{6\lambda^4}\phi^3 + \dots \\
&= -\frac{\lambda_{\text{KdV}}(1+\lambda_{\text{KdV}}/2)}{(1+\lambda_{\text{KdV}})^2}\phi^2 \\
&\quad + \frac{2-\lambda_{\text{KdV}}(2+\lambda_{\text{KdV}})(2+2\lambda_{\text{KdV}}+\lambda_{\text{KdV}}^2)}{6(1+\lambda_{\text{KdV}})^4}\phi^3 + \dots, \quad (6.45)
\end{aligned}$$

which is compared with (6.43). Thus, the K-dV approximation is valid when λ_{KdV} is positive and small compared with unity.

6.1.5 Fermi–Pasta–Ulam Problem

When an electronic computer became available, Fermi, Pasta, and Ulam thought that the ergodic problem is appropriate to be solved with the computer. Consider a dynamical system of N identical particles of mass m on a line with fixed end points with forces acting between nearest neighbors. Denoting the displacement from equilibrium of the n th particle by y_n , and the interaction potential by ϕ , the equation of motion for the lattice oscillation is given by

$$m \frac{d^2 y_n}{dt^2} = \phi'(y_{n+1} - y_n) - \phi'(y_n - y_{n-1}). \quad (6.46)$$

If the interaction is harmonic, the solution is expressed by the superposition of normal modes, and there is no energy transfer among the modes. For the linear spring which obeys Hook's law, the potential is given by

$$\phi(y) = \frac{1}{2}\kappa y^2. \quad (6.47)$$

If the linear solution is assumed in a form of

$$y_n = e^{i(kn - \omega t)}, \quad (6.48)$$

the dispersion relation is given by

$$\omega_\ell = 2\sqrt{\frac{\kappa}{m}} \sin \frac{k_\ell}{2}, \quad k_\ell = \frac{\pi \ell}{N+1}. \quad (6.49)$$

Therefore, the general solution is simply expressed by superposing the linear eigen modes as

$$y_n = \sum_{\ell} A_n^{\ell} \sin\left(\frac{\pi \ell}{N+1}n\right) \cos(\omega_\ell t + \delta_\ell). \quad (6.50)$$

FPU [79] expected that a nonlinear interaction potential would distribute the energy among the modes, leading to thermal equilibrium. The potentials used are

$$\phi(r) = \kappa \left(\frac{r^2}{2} + \alpha \frac{r^3}{3} \right), \quad (6.51)$$

and

$$\phi(r) = \kappa \left(\frac{r^2}{2} + \alpha \frac{r^4}{4} \right). \quad (6.52)$$

The number of particles is 32 and 64 and the initial condition is given by

$$y_n(0) = A \sin \frac{\pi n}{N+1}, \quad (\ell = 1), \quad (6.53)$$

which implies that the energy is given to the fundamental mode ($\ell = 1$). In the course of time evolution, the energy is transferred to several neighboring modes as expected. However, in the long run, the energy does not spread throughout all the normal modes and the recurrence to the initial state is observed. It was a big surprise that nonlinearity did not drive the system into thermal equilibrium and the recurrence phenomenon is observed. After FPU, many works were done in the two different directions. One is to understand the recurrence phenomena in terms of solitons and the other is to obtain thermal equilibrium by launching energy to higher modes at the initial moment.

6.1.6 *K-dV Equation in the Continuum Limit of FPU Problem and Solitons*

Noting that FPU launched a long wave as an initial value, Zabusky and Kruskal showed that the discrete dynamical equation of the FPU lattice system is transformed into the K-dV equation in the continuum limit [80]. Setting the lattice distance as h and approximating the displacement of the particle at the $(n + 1)$ th position as

$$y_{n+1} = r + \frac{\partial r}{\partial x} h + \frac{1}{2} \frac{\partial^2 r}{\partial x^2} h^2 + \cdots, \quad (6.54)$$

the equation of motion for the discrete FPU problem is approximated as

$$\left(\frac{\partial}{\partial t} - \sqrt{\frac{\kappa h^2}{m}} \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + \sqrt{\frac{\kappa h^2}{m}} \frac{\partial}{\partial x} \right) r = \alpha \frac{\partial^4 r}{\partial x^4} + \beta \frac{\partial r}{\partial x} \frac{\partial^2 r}{\partial x^2}. \quad (6.55)$$

Taking into account the wave propagating in one direction

$$r(x, t) = r \left(x - \sqrt{\kappa h^2 / m} t, t \right) = r(\xi, t),$$

and putting $u = \partial r / \partial \xi$, they derived

$$\frac{\partial u}{\partial t} + \delta \frac{\partial^3 u}{\partial \xi^3} + u \frac{\partial u}{\partial \xi} = 0. \quad (6.56)$$

Zabusky and Kruskal solved the K-dV equation numerically to show

1. A sinusoidal wave launched at the initial moment evolves into a train of solitary waves. Because of the periodic boundary condition, the solitary waves initially lined up in order according to the magnitude of the amplitude start overtaking.
2. The solitary waves preserve their identities at collision and suffer small phase shifts.
3. The solitary waves repeat overtaking in a course of time evolution and at certain time all the solitary waves merge at one point, returning to the initial state. Since each solitary wave has a definite velocity depending on the magnitude of the amplitude, the recurrence is observed around the least common multiple of the times the solitary waves traverse the system. The recurrence time is not exactly the least common multiple of the traverse times because of the phase shift at the collisions.

Thus, they explained the recurrence phenomena observed in the FPU problem and named a solitary wave a soliton since it behaves like a particle. The above mentioned properties are seen from Figs. 6.1 and 6.2.

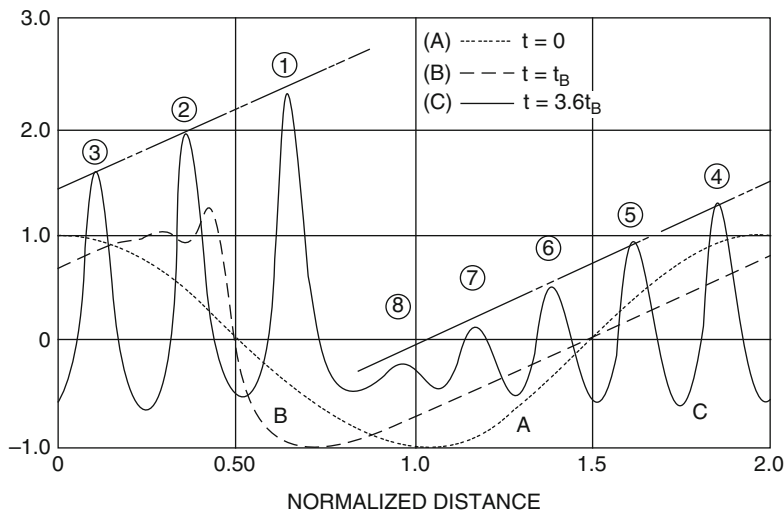


Fig. 6.1 The temporal development of the wave form (from Zabusky and Kruskal [80])

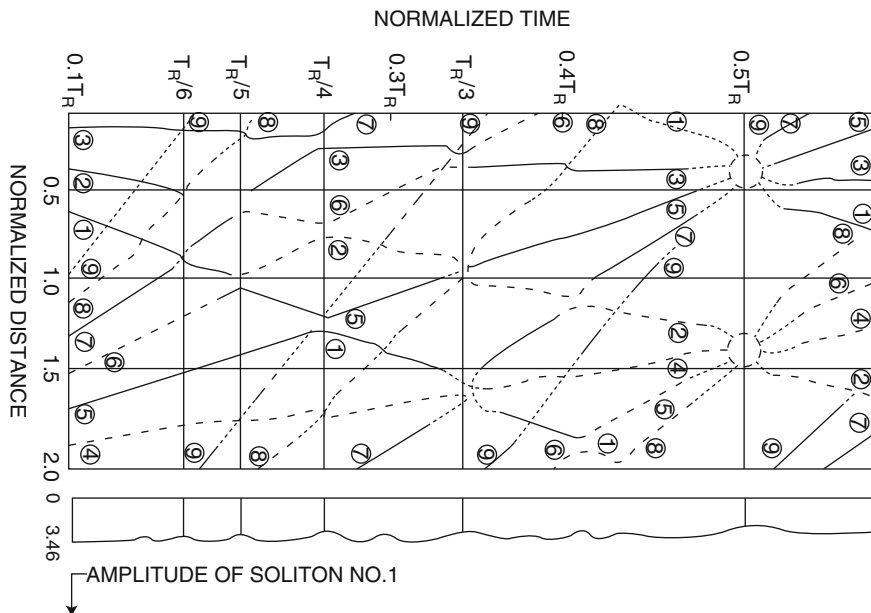


Fig. 6.2 Soliton trajectories on the space-time diagram (from Zabusky and Kruskal [80])

6.2 Langmuir Waves and Envelope Solitons

6.2.1 Langmuir Waves and Nonlinear Schrödinger Equation

Since the dispersion relation of a Langmuir wave is given by

$$\omega = \omega_{pe} \left(1 + \frac{1}{2} k^2 \lambda_D^2 \right), \quad (6.57)$$

the linear wave satisfies the following relation

$$\left\{ \omega - \omega_{pe} \left(1 + \frac{1}{2} k^2 \lambda_D^2 \right) \right\} \phi(\omega, k) = 0, \quad (6.58)$$

which is transformed into

$$\left(i \frac{\partial}{\partial t} - \omega_{pe} \right) \phi + \frac{d^2 \omega}{dk^2} \frac{\partial^2 \phi}{\partial x^2} = 0. \quad (6.59)$$

Noting that the zero-th order solution is given by $\phi \propto e^{i\omega_{pe}t}$, the nonlinear term is in the odd order of the potential and the lowest order of the nonlinear term is proportional to $|\phi|^2 \phi$. Thus, the nonlinear equation for the amplitude is given by

$$i \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} + \alpha |\phi|^2 \phi = 0. \quad (6.60)$$

This is called the nonlinear Schrödinger equation. The stationary solution is given by

$$\phi(x, t) = A e^{i\theta} \text{sech}[a(x - bt)], \quad A = \sqrt{3}, \quad \theta = \frac{b}{2}x - \left(\frac{b^2}{4} - a^2\right)t. \quad (6.61)$$

Since the amplitude is in the shape of soliton, this is also called as an envelope soliton. The K-dV equation is an evolution equation for real amplitude and the solution is a one parameter family, while the nonlinear Schrödinger equation is an evolution equation for a complex amplitude and is a two parameter family.

An N soliton solutions of the nonlinear Schrödinger equation is obtained by Zakharov–Shabat using the inverse scattering method [82].

6.2.2 Modulation Instability of Finite Amplitude Langmuir Wave

A finite amplitude Langmuir wave is described by the nonlinear Schrödinger equation. Here, let us consider the stability of a plane wave. Apparently, a solution is given by

$$\phi = \phi_0 e^{i(Kx - \Omega t)}, \quad \Omega = K^2 - \alpha |\phi_0|^2. \quad (6.62)$$

Applying a modulation to this solution

$$\phi = e^{i(Kx - \Omega t)} [\phi_0 + \delta \phi_1 e^{i(qx - \omega t)} + \delta \phi_2 e^{-i(qx - \omega t)}], \quad (6.63)$$

the dispersion relation is obtained as

$$\omega = 2Kq \pm q \sqrt{q^2 - 2\alpha |\phi_0|^2}, \quad (6.64)$$

which gives

$$\begin{cases} \text{Stable} & \text{for } \alpha < 0 \quad (\text{repulsive}), \\ \text{Unstable} & \text{for } \alpha > 0 \quad (\text{attractive}) \quad \text{and} \quad |\phi_0|^2 > \frac{q^2}{2\alpha}. \end{cases} \quad (6.65)$$

Thus, a finite amplitude wave becomes unstable for $\alpha > 0$ and $|\phi_0|^2 > q^2/(2\alpha)$ and grows to develop into an envelop soliton. The condition of the modulation instability represents that the potential energy is so large compared with the kinetic energy of the quasi-particle (modulated wave) that the quasi-particle is trapped in the potential.

6.3 Solitons and Inverse Scattering Method

6.3.1 *K-dV Equation*

An initial value problem for the K-dV equation was solved by Gardner, Green, Kruskal, and Miura [81]. When $u(x, t)$ obeys the K-dV equation given by

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad (6.66)$$

if we require that the following scattering problem with $u(x, t)$ as a potential holds at an arbitrary time

$$-\frac{\partial^2 \phi}{\partial x^2} + u\phi = \lambda\phi, \quad (6.67)$$

the eigen value of the scattering problem is shown time-independent under the boundary condition $\phi(x = \pm\infty) = 0$

$$\frac{d\lambda}{dt} = 0. \quad (6.68)$$

This is shown in the following. From (6.67), the potential is expressed as

$$u = \lambda + \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2}, \quad (6.69)$$

which is substituted into (6.66) to give

$$\frac{d\lambda}{dt} \phi^2 + \frac{\partial}{\partial x} \left\{ \phi \frac{\partial R}{\partial x} - \frac{\partial \phi}{\partial x} R \right\} = 0, \quad (6.70)$$

where

$$R = \frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} - 3(u + \lambda) \frac{\partial \phi}{\partial x}. \quad (6.71)$$

Integrating (6.70) over the space, we have

$$\frac{d\lambda}{dt} \int \phi^2 dx + [\phi \frac{\partial R}{\partial x} - \frac{\partial \phi}{\partial x} R]_{-\infty}^{\infty} = 0, \quad (6.72)$$

which reduces under the boundary condition to

$$\frac{d\lambda}{dt} = 0. \quad (6.73)$$

Therefore, we have from (6.70)

$$\phi \frac{\partial R}{\partial x} - \frac{\partial \phi}{\partial x} R = W(R, \phi) = \text{const}, \quad (6.74)$$

where $W(R, \phi)$ is the Wronskian. Using (6.74), we have

$$\frac{\partial}{\partial x} \frac{R}{\phi} = \frac{1}{\phi^2} \left(\phi \frac{\partial R}{\partial x} - R \frac{\partial \phi}{\partial x} \right) = \frac{\text{const}}{\phi^2} = \frac{D}{\phi^2}, \quad (6.75)$$

which is integrated to give

$$R = C\phi + D\phi \int \frac{1}{\phi^2} dx. \quad (6.76)$$

Then, (6.71) becomes

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} - 3(u + \lambda) \frac{\partial \phi}{\partial x} = C\phi + D\phi \int \frac{1}{\phi^2} dx, \quad (6.77)$$

which describes the time evolution of the scattering data. From above, an initial value problem for the K-dV equation can be solved in the following.

1. Solve (6.67) with the initial value $u(x, 0)$.
2. Solve (6.77) for the scattering data.
3. Solve the inverse problem with the scattering data.

This procedure is to transform a nonlinear problem to a linear problem of the scattering data and is applied to many nonlinear partial differential equations which have soliton solutions.

Suppose that $u(\pm\infty, t) \rightarrow 0$ at an arbitrary time. An eigenvalue problem (6.67) is assumed to be solved for the initial value $u(x, 0)$ to give discrete eigenvalues $\lambda = -\kappa_n^2$, $n = 1, 2, \dots, N$ for the negative energy state, and continuous spectrum of eigenvalues $\lambda = k^2$ for the positive energy state. The eigenfunction is given for the discrete eigenvalues

$$\phi^{(n)} \rightarrow c_n(t) e^{-\kappa_n x} \quad \text{for } x \rightarrow \infty, \quad (6.78)$$

and for the continuous spectrum

$$\phi \sim e^{-ikx} + b_R(k, t) e^{ikx} \quad \text{for } x \rightarrow \infty, \quad (6.79)$$

$$\sim a_T(k, t) e^{-ikx} \quad \text{for } x \rightarrow -\infty, \quad (6.80)$$

where

$$\int_{-\infty}^{\infty} \phi^{(n)2} dx = 1, \quad |a_T(k, t)|^2 + |b_R(k, t)|^2 = 1. \quad (6.81)$$

The time evolution of the scattering data $a_T(k, t)$, $b_R(k, t)$, $c_n(t)$ can be evaluated using the property that $u \rightarrow 0$ at $x \rightarrow \infty$.

For the discrete eigen states since

$$\phi \int \frac{1}{\phi^2} dx \rightarrow \infty \quad \text{for } x \rightarrow \infty, \quad (6.82)$$

we put $D = 0$ in (6.77). Eliminating u in (6.77) gives

$$\frac{\partial}{\partial t} \frac{\phi^2}{2} + \frac{\partial}{\partial x} \left(\phi \frac{\partial^2 \phi}{\partial x^2} - 2 \left(\frac{\partial \phi}{\partial x} \right)^2 - 3\lambda \phi^2 \right) = C \phi^2. \quad (6.83)$$

Then, the condition $\int \phi^2 dx = 1$ leads to $C = 0$. Thus, we obtain

$$\frac{d}{dt} c_n = 4\kappa_n^3 c_n, \quad (6.84)$$

which is solved to give

$$c_n(t) = c_n(0)e^{4\kappa_n^3 t}. \quad (6.85)$$

For the continuous spectrum, we have from (6.77)

$$\frac{d}{dt} a_T + i4k^3 a_T = C a_T + D a_T \int \frac{1}{\phi^2} dx \quad \text{for } x \rightarrow -\infty. \quad (6.86)$$

Since a_T is independent of space, we put $D = 0$. Furthermore, we have

$$\left(\frac{d}{dt} b_R - i8k^3 b_R \right) e^{ikx} + (C - i4k^3) e^{-ikx} = 0, \quad \text{for } x \rightarrow \infty. \quad (6.87)$$

From (6.86) and (6.87), we obtain

$$b_R(k, t) = b_R(k, 0)e^{i8k^3 t}, \quad C = i4k^3, \quad a_T(k, t) = a_T(k, 0). \quad (6.88)$$

The inverse problem is formulated in the following way. The potential for the scattering problem is given by

$$u(x, t) = -2 \frac{d}{dx} K(x, x; t), \quad (6.89)$$

where $K(x, y; t)$ is given by the Gel'fand–Levitan [90] equation

$$K(x, y; t) + B(x + y; t) + \int_x^\infty B(y + z; t) K(x, z; t) dz = 0, \quad y > x, \quad (6.90)$$

and $B(x + y; t)$ is defined in terms of the scattering data by

$$B(x+y; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} b_R(k, t) e^{ik(x+y)} dk + \sum_{n=1}^N c_n^2(t) e^{-\kappa_n(x+y)}. \quad (6.91)$$

6.3.1.1 One-Soliton Solution

Suppose that there is only one bound state for $u(x, 0)$ and no reflection $b_R(k, t) = 0$. Then, the G-L equation becomes

$$K(x, y; t) + c^2(0) e^{8\kappa^3 t - \kappa(x+y)} + c^2(0) e^{8\kappa^3 t} \int_x^{\infty} e^{-\kappa(y+z)} K(x, z; t) dz = 0, \quad y > x. \quad (6.92)$$

Differentiating (6.92) with respect to y , we have

$$\frac{\partial}{\partial y} K(x, y; t) = -\kappa K(x, y; t), \quad (6.93)$$

which gives

$$K(x, y; t) = e^{-\kappa y} h(x, t). \quad (6.94)$$

Substituting (6.94) into the G-L equation (6.92) to obtain

$$K(x, y; t) = -\frac{c(0)^2 e^{8\kappa^3 t - \kappa(x+y)}}{1 + c(0)^2 (2\kappa)^{-1} e^{8\kappa^3 t - 2\kappa x}}. \quad (6.95)$$

Thus, one soliton solution of the K-dV equation is given by

$$u(x, t) = -2 \frac{d}{dx} K(x, x; t) = -2\kappa^2 \operatorname{sech}^2[\kappa(x - 4\kappa^2 t) - \delta], \quad (6.96)$$

where

$$\delta = \frac{1}{2} \log \left[\frac{c^2(0)}{2\kappa} \right]. \quad (6.97)$$

A soliton is a solution for the initial condition to have only one bound state and no continuous spectrum for the scattering problem.

6.3.1.2 Two-Soliton Solution

Again we consider the case that there are two bound states for $u(x, 0)$ and no reflection $b_R(k, t) = 0$. In a similar way, we solve the G-L equation to obtain

$$K(x, x; t) = 2e^{(\theta_1 + \theta_2 + \delta)/2} \left\{ \cosh \left(\frac{\theta_1 + \theta_2 + \delta}{2} \right) + \alpha \cosh \left(\frac{\theta_1 - \theta_2 + \eta}{2} \right) \right\}, \quad (6.98)$$

where $\theta_i = 8\kappa_i^3 t - 2\kappa_i x$, $\alpha = |(\kappa_1 + \kappa_2)/(\kappa_1 - \kappa_2)|$, $\delta = -\log[4\alpha^2 \kappa_1 \kappa_2]$ and $\eta = \log(\kappa_2/\kappa_1)$. From (6.98) and (6.89), we have

$$u(x, t) = -4 \frac{(\kappa_1 + \kappa_2)^2 + \alpha(\kappa_2^2 \cosh(\theta_1 + \phi_1) + \kappa_1^2 \cosh(\theta_2 + \phi_2))}{\{\cosh[(\theta_1 + \theta_2 + \delta)/2] + \alpha \cosh[(\theta_1 - \theta_2 + \eta)/2]\}^2}, \quad (6.99)$$

where $\phi_1 = -\log(2\alpha\kappa_1)$ and $\phi_2 = -\log(2\alpha\kappa_2)$.

An N soliton solution is also obtained.

6.3.2 Nonlinear Shrödinger Equation

Following Zakharov–Shabat [82], we start with the one dimensional nonlinear Schrödinger equation given by

$$i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \kappa |u|^2 u = 0, \quad (6.100)$$

where κ is assumed to be positive since the soliton is formed as a result of the modulational instability. Equation (6.100) is rewritten in an operator form as

$$\frac{\partial L}{\partial t} = i[L, A], \quad (6.101)$$

where $[L, A] = LA - AL$ and

$$L = i \begin{pmatrix} 1 + \rho & 0 \\ 0 & 1 - \rho \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \quad (6.102)$$

$$A = -\rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} |u|^2/(1 + \rho) & i(\partial u^*/\partial x) \\ -i(\partial u/\partial x) & -|u|^2/(1 - \rho) \end{pmatrix}, \quad (6.103)$$

with $\rho^2 = (\kappa - 2)/\kappa$. We may set $\kappa > 2$ without loss of generality. When $u(x, t)$ obeys (6.100), the eigenvalue of L is shown time independent. If we define the eigenvalue problem of L as

$$L\varphi = \lambda\varphi, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad (6.104)$$

we have

$$\frac{d\lambda}{dt}\varphi = (L - \lambda) \left(\frac{\partial \varphi}{\partial t} + iA\varphi \right).$$

Taking the inner product with φ and integrating over x , we have

$$\begin{aligned} \frac{d\lambda}{dt} \int (\varphi, \varphi) dx &= \int \left(\varphi, (L - \lambda) \left(\frac{\partial \varphi}{\partial t} + iA\varphi \right) \right) dx \\ &= \int \left((L - \lambda)\varphi, \frac{\partial \varphi}{\partial t} + iA\varphi \right) dx = 0. \end{aligned}$$

Therefore, the eigenvalue is time-independent and we may choose for some $C(t)$

$$i \frac{\partial \varphi}{\partial t} = A\varphi + C(t)\varphi. \quad (6.105)$$

First, we solve a direct problem of scattering. Suppose that $u(x, t)$ decreases to zero rapidly when $|x| \rightarrow \infty$. Equation (6.104) is transformed into

$$\frac{\partial v_1}{\partial x} + i\zeta v_1 = q(x)v_2, \quad (6.106)$$

$$\frac{\partial v_2}{\partial x} - i\zeta v_2 = -q^*(x)v_1, \quad (6.107)$$

where

$$\varphi = \begin{pmatrix} 0 & \sqrt{1-\rho} \\ \sqrt{1+\rho} & 0 \end{pmatrix} e^{-i\zeta x/\rho} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (6.108)$$

$$q(x) = \frac{i u(x, t)}{\sqrt{1-\rho^2}}, \quad \zeta = \frac{\lambda \rho}{1-\rho^2}. \quad (6.109)$$

Suppose that ϕ and ψ are the solutions of (6.106) and (6.107) for $\zeta = \zeta_1$ and ζ_2 . Then, we obtain

$$\frac{d}{dx}(\phi_1 \psi_2 - \phi_2 \psi_1) + i(\zeta_1 - \zeta_2)(\phi_1 \psi_2 + \phi_2 \psi_1) = 0. \quad (6.110)$$

Furthermore, if v is a solution of (6.106) and (6.107) for $\zeta_1 = \xi_1 + i\eta_1$,

$$\bar{v} = \begin{pmatrix} v_2^* \\ -v_1^* \end{pmatrix}$$

is a solution of (6.106) and (6.107) for $\zeta_2 = \xi_1 - i\eta_1 = \zeta_1^*$.

Now, we determine scattering data. We consider solutions ϕ and ψ of (6.106) and (6.107) which have the following asymptotic forms for the real eigenvalue $\zeta = \xi$:

$$\phi \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}, \quad \bar{\phi} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x} \quad \text{for } x \rightarrow -\infty, \quad (6.111)$$

$$\psi \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x}, \quad \bar{\psi} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \quad \text{for } x \rightarrow +\infty, \quad (6.112)$$

Since ψ and $\bar{\psi}$ form a complete set, we may express ϕ in terms of ψ and $\bar{\psi}$ as

$$\phi(x, \xi) = a(\xi)\bar{\psi}(x, \xi) + b(\xi)\psi(x, \xi), \quad (6.113)$$

$$\bar{\phi}(x, \xi) = -a^*(\xi)\psi(x, \xi) + b^*(\xi)\bar{\psi}(x, \xi). \quad (6.114)$$

Substituting ϕ and $\bar{\phi}$ into (6.110) leads to

$$|a(\xi)|^2 + |b(\xi)|^2 = 1. \quad (6.115)$$

The ϕ and ψ are analytically continued in the upper half plane $\Im \zeta > 0$. As from (6.110) we have

$$a(\xi) = \phi_1 \psi_2 - \phi_2 \psi_1, \quad (6.116)$$

$a(\xi)$ is also analytically continued to the upper half plane $\Im \zeta > 0$. Note

$$a(\zeta) \rightarrow 1 \quad \text{for } \Im \zeta > 0, \quad |\zeta| \rightarrow \infty. \quad (6.117)$$

For the bound states, from (6.113) rewritten as

$$\frac{\phi(x, \zeta)}{a(\zeta)} = \bar{\psi}(x, \zeta) + \frac{b(\zeta)}{a(\zeta)} \psi(x, \zeta), \quad (6.118)$$

the eigenfunctions are expressed by

$$\phi(x, \zeta_j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta_j x}, \quad x \rightarrow -\infty, \quad (6.119)$$

$$\phi(x, \zeta_j) = \frac{b(\zeta_j)}{a'(\zeta_j)} \psi(x, \zeta_j) = c_j \psi(x, \zeta_j), \quad x \rightarrow \infty, \quad (6.120)$$

where eigenvalues $\zeta = \zeta_j, j = 1, \dots, N$ are determined by $a(\zeta) = 0$ in the upper plane $\Im \zeta > 0$.

The eigenvalues ζ_j are not generally pure imaginary. However, noting for $q = q^*$

$$\phi(x, -\xi) = \phi^*(x, \xi), \quad \psi(x, -\xi) = \psi^*(x, \xi),$$

we have

$$a(\xi) = a^*(-\xi),$$

which may be analytically continued to the upper plane to give

$$a(\zeta) = a^*(-\zeta).$$

This implies that the zeros of $a(\zeta)$ locate on the imaginary axis.

Next, we consider time development of the scattering data. The eigenfunction obeys (6.105) which simply reduces for $|x| \rightarrow \infty$ to

$$i \frac{\partial \varphi}{\partial t} = -\rho \frac{\partial^2 \varphi}{\partial x^2} + C(t)\varphi. \quad (6.121)$$

Using the transformation (6.108) to (6.121) leads to

$$i \frac{\partial v}{\partial t} = \frac{\xi^2}{\rho} v + 2i\xi \frac{\partial v}{\partial x} - \rho \frac{\partial^2 v}{\partial x^2} + C(t)v, \quad (6.122)$$

which holds at both $x \rightarrow -\infty$ and $x \rightarrow \infty$. Substituting (6.111) for $x \rightarrow -\infty$ and (6.113) for $x \rightarrow \infty$ into (6.122) gives

$$\begin{aligned} 0 &= \frac{(1+\rho)^2}{\rho} \xi^2 + C(t), \\ i \frac{da(\xi, t)}{dt} &= \left\{ \frac{(1+\rho)^2}{\rho} \xi^2 + C(t) \right\} a(\xi, t), \\ i \frac{db(\xi, t)}{dt} &= \left\{ \frac{(1-\rho)^2}{\rho} \xi^2 + C(t) \right\} b(\xi, t). \end{aligned}$$

Thus, we get

$$a(\xi, t) = a(\xi, 0), \quad b(\xi, t) = b(\xi, 0)e^{i4\xi^2 t}. \quad (6.123)$$

For the bound state in a similar way, we substitute (6.119) and (6.120) into (6.122) to have

$$c_j(t) = c_j(0)e^{i4\xi_j^2 t}. \quad (6.124)$$

The inverse problem is solved with the help of the Gel'fand–Levitan equation. Noting ψ is shown to be expressed with a kernel K by

$$\psi(x, \zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} + \int_x^\infty K(x, z) e^{i\xi z} dz, \quad \Im \zeta > 0, \quad (6.125)$$

where $K(x, z)$ satisfies the Gel'fand–Levitan equation

$$K_1(x, y) = F^*(x+y) + \int_x^\infty K_2^*(x, z) F^*(z+y) dz, \quad (6.126)$$

$$K_2^*(x, y) = - \int_x^\infty K_1(x, z) F(z+y) dz, \quad (6.127)$$

with

$$F(x) = \frac{1}{2\pi} \int \frac{b(\xi)}{a(\xi)} e^{i\xi x} d\xi - i \sum_j \frac{b(\zeta_j)}{a'(\zeta_j)} e^{i\zeta_j x}. \quad (6.128)$$

Detailed derivation is given in Zakharov and Shabat [91]. The potential $q(x)$ is given by

$$q(x) = -2K_1(x, x), \quad \int_x^\infty |q(z)|^2 dz = -2K_2^*(x, x). \quad (6.129)$$

One soliton solution is given by

$$u(x, t) = \sqrt{2\kappa} \frac{e^{-i4(\xi^2 - \eta^2)t - i2\xi x + i\theta}}{\cosh[2\eta(x - x_0) + 8\xi\eta t]}. \quad (6.130)$$

The N -soliton solution is also obtained in [91].

The method developed by Zakharov and Shabat is extended by Ablowitz, Kaup, Newell, and Segur [83, 92, 93] as an exact solution method covering wide classes of integrable nonlinear equations including the Nonlinear Schrödinger equation, The Modified KdV equation, The Sine–Gordon equation, and so on.

6.4 Solitons and Bilinear Transformation

6.4.1 K-dV Equation

Suppose the K-dV equation in a form

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 12u \frac{\partial u}{\partial x} = 0. \quad (6.131)$$

The transformation

$$u(x, t) = \frac{\partial^2}{\partial x^2} \log f(x, t) \quad (6.132)$$

reduces (6.131) to a homogeneous equation in $f(x, t)$ as

$$f \frac{\partial^2 f}{\partial t \partial x} - \frac{\partial f}{\partial t} \frac{\partial f}{\partial x} + f \frac{\partial^4 f}{\partial x^4} - 4 \frac{\partial f}{\partial x} \frac{\partial^3 f}{\partial x^3} + 3 \left(\frac{\partial^2 f}{\partial x^2} \right)^2 = 0, \quad (6.133)$$

which is rewritten in a bilinear form

$$(D_t D_x + D_x^4) f \cdot f = 0, \quad (6.134)$$

where

$$D_x f \cdot f = \left\{ \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) f(x) f(x') \right\}_{x=x'}. \quad (6.135)$$

A single soliton solution is obtained by putting

$$f(x, t) = 1 + e^\theta, \quad \theta = ax - a^3 t + \delta, \quad (6.136)$$

for which (6.132) gives

$$u(x, t) = \frac{a^2}{4} \operatorname{sech}^2 \left(\frac{a}{2} (x - a^3 t + \delta) \right). \quad (6.137)$$

For a two-soliton solution, we similarly introduce

$$f(x, t) = 1 + \sum_{i=1}^2 \varepsilon^i f^{(i)}, \quad (6.138)$$

where ε is an expansion parameter. Substituting (6.138) into (6.133), we have at the first order in ε

$$\frac{\partial^2 f^{(1)}}{\partial t \partial x} + \frac{\partial^4 f^{(1)}}{\partial x^4} = 0, \quad (6.139)$$

which allow us to take

$$f^{(1)} = 1 + \sum_{i=1}^2 e^{\theta_i}, \quad \theta_i = a_i x - a_i^3 t + \delta_i. \quad (6.140)$$

This is substituted into the equation at the second order in ε to give

$$\frac{\partial^2 f^{(2)}}{\partial t \partial x} + \frac{\partial^4 f^{(2)}}{\partial x^4} = 3a_1 a_2 (a_1 - a_2)^2. \quad (6.141)$$

This is integrated to give

$$f^{(2)} = \left(\frac{a_1 - a_2}{a_1 + a_2} \right)^2 e^{\theta_1 + \theta_2}. \quad (6.142)$$

A remarkable thing is that we can take $f^{(i)} = 0$ for $i \geq 3$. Thus, a two-soliton solution is given with (6.132) by

$$f = 1 + e^{\theta_1} + e^{\theta_2} + \left(\frac{a_1 - a_2}{a_1 + a_2} \right)^2 e^{\theta_1 + \theta_2}. \quad (6.143)$$

The same process works for an N soliton solution. This bilinear transformation method is introduced by Hirota and is called Hirota's method [84].

6.4.2 Nonlinear Schrödinger Equation

The bilinear transformation is applied to the nonlinear Schrödinger equation as well

$$i \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} + \kappa |\phi|^2 \phi = 0. \quad (6.144)$$

Introducing functions f and g through

$$\phi = \frac{g}{f}, \quad (6.145)$$

and substituting (6.145) into (6.144) gives

$$\frac{1}{f^2} \{-iD_t f \cdot g + D_x^2 f \cdot g\} + \frac{g}{f} \left\{ -2 \frac{\partial^2}{\partial x^2} \ln f + \kappa \frac{|g|^2}{|f|^2} \right\} = 0. \quad (6.146)$$

We may put

$$iD_t f \cdot g = D_x^2 f \cdot g, \quad \frac{|g|^2}{|f|^2} = \frac{2}{\kappa} \frac{\partial^2}{\partial x^2} \ln f. \quad (6.147)$$

A one-parameter family is given by

$$f = 1 + e^{\theta + \theta^*}, \quad g = 2a \sqrt{\frac{2}{\kappa}} e^{\theta} \quad (6.148)$$

where

$$\Re \theta = a(x - bt), \quad \Im \theta = \frac{1}{2}bx + \left(a^2 - \frac{1}{4}b^2\right)t. \quad (6.149)$$

From (6.145) and (6.148), we get a one-soliton solution

$$\phi(x, t) = a \sqrt{\frac{2}{\kappa}} e^{i[bx/2 + (a^2 - b^2/4)t]} \operatorname{sech}[a(x - bt)]. \quad (6.150)$$

6.5 Soliton-Like Excitations in Plasmas With Multiple Modes

An interesting concern with soliton phenomena is to study whether the notion of soliton is effective even for non-integrable systems, in other words, if solitons play a similar role in real systems as they do in integrable systems. Here, we consider soliton-like excitations in a multi-mode system which has several linear eigen modes interacting each other. In a single mode system, a soliton is formed through the self-interaction which gives steepening. In a multi-mode system, there are mode couplings among eigen modes as well as the self-interaction of each eigen mode. Thus, multi-mode systems are appropriate to check if a soliton can be regarded as a realistic physical entity. Here, we take an ion beam-plasma system as an example of multi-mode systems.

6.5.1 Basic Equations for Nonlinear Wave Propagation in An Ion Beam-Plasma System

An ion beam-plasma system is a plasma with an ion beam in which low frequency modes are excited because its dynamics is carried by the ions and the electrons are supposed to respond to the potential statically. The ion dynamics is driven by the potential which is balanced with the electron pressure, the ion temperature may be neglected under the condition $T_e \gg T_i$ which is the case for laboratory plasmas. The ion beam is a drifting stream of ions introduced to the plasma and is characterized by a uni-directional energy, implying that the temperature of the ion beam is also neglected. Therefore, the basic equations of an ion beam-plasma system are given for the ion (ion beam) density $n_i(n_b)$ and the ion (ion beam) velocity $v_i(v_b)$ by

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i v_i) = 0, \quad (6.151)$$

$$\frac{\partial v_i}{\partial t} + (v_i \cdot \nabla) v_i + C_s^2 \nabla \phi = 0, \quad (6.152)$$

$$\frac{\partial n_b}{\partial t} + \nabla \cdot (n_b v_b) = 0, \quad (6.153)$$

$$\frac{\partial v_b}{\partial t} + (v_b \cdot \nabla) v_b + C_s^2 \nabla \phi = 0, \quad (6.154)$$

$$\nabla^2 \phi = n_{e0} e^\phi - n_i - n_b, \quad (6.155)$$

where ϕ is the potential normalized by T_e/e and n_{e0} is the electron density at the equilibrium. Suppose that the ion beam is in the x direction and the average beam velocity is v_{b0} . Introducing a frame moving with v_{b0} , that is, $\xi = x - v_{b0}t$ and $\tau = t$, (6.151)–(6.155) are rewritten as

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi} \right) n + \nabla_\perp \cdot [(1+n)v] = 0, \quad (6.156)$$

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi} \right) v + (v_\perp \cdot \nabla_\perp) v + \mu^{-2} \nabla \phi = 0, \quad (6.157)$$

$$\frac{\partial n_b}{\partial \tau} + \nabla \cdot [(1+n_b)v_b] = 0, \quad (6.158)$$

$$\frac{\partial v_b}{\partial \tau} + (v_b \cdot \nabla) v_b + \mu^{-2} \nabla \phi = 0, \quad (6.159)$$

$$\lambda^2 \nabla^2 \phi = e^\phi - (1-b)(1+n) - b(1+n_b), \quad (6.160)$$

where the ion density and the ion beam density are expressed by $n_{i0}(1+n)$ and $n_{b0}(1+n_b)$ with the charge neutrality at the equilibrium $n_{i0} + n_{b0} = n_{e0} = n_0$ and $b = n_{b0}/n_0$. The ion velocity and the ion beam velocity fluctuations are normalized by v_{b0} as $v_{b0}v_i$ and $v_{b0}v_b$ with $\mu = v_{b0}/C_s$. The space and time coordinates are

normalized by the characteristic scale length L and L/v_{b0} . The λ denotes λ_D/L where λ_D is the Debye length.

Since the fluctuations vary slowly in time in the beam frame, if the variation in the direction perpendicular to the beam is assumed small, we may have in the zero-th order approximation for the case of $\mu \simeq 1$

$$n \sim v_{ix} \sim \phi, \quad \frac{\partial}{\partial \xi} v_{\perp} \sim \frac{\partial \phi}{\partial \mathbf{r}_{\perp}}. \quad (6.161)$$

Equations (6.156), (6.157), and (6.160) are now combined to give [94]

$$\frac{\partial}{\partial \xi} \left[\frac{\partial \phi}{\partial \tau} + \frac{\mu^2 - 1 - b}{2} \frac{\partial \phi}{\partial \xi} + \frac{\lambda^2}{2} \frac{\partial^3 \phi}{\partial \xi^3} + \phi \frac{\partial \phi}{\partial \xi} + \frac{b}{2} \frac{\partial n_b}{\partial \xi} \right] + \frac{1}{2} \nabla_{\perp} \phi = 0, \quad (6.162)$$

which is to take into account the ion acoustic wave propagating in the same direction of the beam because the interaction between the ion acoustic wave and the beam is strong when the relative velocity is small. Since the effect of the beam on the propagation of the ion acoustic wave is proportional to the small parameter b in (6.162), the linear approximation is enough for the beam dynamics. From (6.159) and (6.160), we have

$$\frac{\partial^2 n_b}{\partial \tau^2} - \mu^2 \frac{\partial^2 \phi}{\partial \xi^2} = 0. \quad (6.163)$$

Equations (6.162) and (6.163) are reduced to the Kadomtsev–Petviashvili equation if the beam modulation is neglected. When the perpendicular variation is neglected, (6.162) becomes one dimensional K-dV equation with the effect of the beam modulation.

$$\frac{\partial \phi}{\partial \tau} + \frac{\mu^2 - 1 - b}{2} \frac{\partial \phi}{\partial \xi} + \frac{\lambda^2}{2} \frac{\partial^3 \phi}{\partial \xi^3} + \phi \frac{\partial \phi}{\partial \xi} + \frac{b}{2} \frac{\partial n_b}{\partial \xi} = 0. \quad (6.164)$$

Replacing the variables as $u = \sqrt{2}/(\lambda v_0^{3/2})\phi$, $\rho = \sqrt{2}/(\lambda v_0^{5/2})n_b$, $t = (\lambda v_0^{3/2}/\sqrt{2})\tau$, $x = (\lambda \sqrt{v_0/2})\xi$ where $v_0 = 2/(1 + b - \mu^{-2})$, we have

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} + \frac{\beta}{2} \frac{\partial \rho}{\partial x} = 0, \quad (6.165)$$

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{v_0}{\mu^2} \frac{\partial^2 u}{\partial x^2} = 0, \quad (6.166)$$

where $\beta = b v_0^2$.

The linear dispersion equation is obtained by putting physical quantities proportional to $e^{i(kx - \omega t)}$ as

$$\frac{\omega^3}{k^3} + (1 + k^2) \frac{\omega^2}{k^2} - \frac{a^3}{2} = 0, \quad a^3 = \frac{\beta v_0}{\mu^2}, \quad (6.167)$$

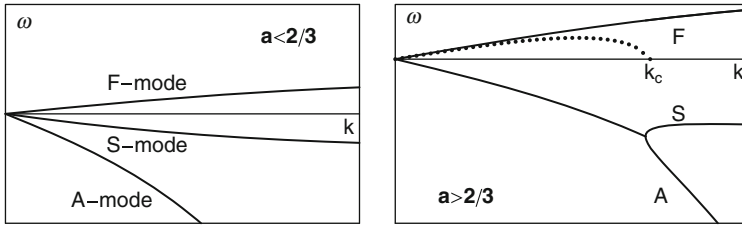


Fig. 6.3 The linear dispersion relation in the beam frame for stable case (*left*) and unstable case (*right*)

Three solutions of (6.167) correspond to the fast beam mode (referred to as F-mode), slow beam mode (S-mode) and ion acoustic mode (A-mode). The stability condition for the linear wave is given as

$$k^2 > \frac{3}{2}a - 1. \quad (6.168)$$

The dispersion relation is depicted in Fig. 6.3.

Under the linearly stable condition $a < 2/3$, the nonlinear evolution of the linear eigen modes can be derived from (6.165) and (6.166) which are rewritten as

$$\frac{\partial W}{\partial t} + A(W) \frac{\partial W}{\partial x} + D \frac{\partial^3 W}{\partial x^3} = 0, \quad (6.169)$$

$$W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad A(W) = \begin{pmatrix} w_1 - 1 & a^3/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (6.170)$$

where $w_1 = (a/b^{1/3})u$ and $w_2 = (b^{1/3}/a)\rho$. Here, denoting the right eigenvector of $A(W=0)$ as R_j and the eigenvalue as μ_j and the corresponding left eigenvector as L_i belonging to the eigenvalue μ_i , we have

$$A(W=0)R_j = \mu_j R_j, \quad L_i A(W=0) = \mu_i L_i, \quad (6.171)$$

and

$$L_i R_j = \frac{3\mu_i + 2}{\mu_i} \delta_{ij}. \quad (6.172)$$

The eigenvalues $\{\mu_j\}$ give the phase velocities of the linear modes whose dispersion is neglected and are real and distinct. Here, expressing W in terms of the linear eigenmodes $\{f_j; j = A, F, S\}$ as

$$W = \sum_j f_j R_j, \quad (6.173)$$

and multiplying the left eigenvector L_i to (6.169) from the left, we obtain [95]

$$\begin{aligned} & \frac{\partial f_j}{\partial t} + \mu_j \frac{\partial f_j}{\partial x} + \alpha_j \frac{\partial^3 f_j}{\partial x^3} + \alpha_j \left(f_j + \sum_{i \neq j} f_i \right) \frac{\partial f_j}{\partial x} \\ & - \alpha_j \sum_{i \neq j} \frac{\partial^3 f_i}{\partial x^3} - \alpha_j \left(\sum_{\ell} f_{\ell} \right) \frac{\partial}{\partial x} \left(\sum_{i \neq j} f_i \right), \quad \alpha_j = \frac{\mu_j}{3\mu_j + 2}, \end{aligned} \quad (6.174)$$

and $\{f_j\}$ are related to $\{w_j\}$ through

$$f_j = \frac{\mu_j^2}{(\mu_j - \mu_i)(\mu_j - \mu_k)} \{w_1 + \mu_i \mu_k w_2 - (\mu_i + \mu_k) w_3\}. \quad (6.175)$$

It turns out that (6.174) is characterized by two types of nonlinear interactions. One is the self-interaction which leads to the soliton formation through wave steepening and the other is the resonance mode coupling among three inherent eigen modes which yields the explosive interaction since one of the modes in this system is a negative energy wave.

6.5.2 Characteristic Times for Onset of The Explosive Instability and Soliton Formation

In an ion beam-plasma system, the explosive instability is unavoidable. Therefore for the soliton-like excitations to be relevant, the time scale of the soliton formation is shorter than the time for the onset of the explosive instability. Here, we compare the characteristic times for the onset of the explosive instability and the soliton formation.

First, we examine the explosive instability under the resonant condition $k_S = k_F + k_A$, and $\omega_S = \omega_F + \omega_A$. Here, since the resonance condition is affected by the dispersion, we may recalculate the phase velocities $\{\hat{\mu}_j\}$ of the eigen modes with the eigenvalues of

$$\begin{pmatrix} -1 - k^2 a^3/2 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

instead of those of $A(0)$. Then, the equations for the eigen modes are given by (6.174) in which μ_j should be replaced by $\hat{\mu}_j$.

Putting f_j into

$$f_j(x, t) = g_j(t) e^{ik_j(x - \hat{\mu}_j t)}, \quad (6.176)$$

and substituting it into (6.174), we have

$$\frac{dg_j}{dt} + \frac{\alpha_j}{2} i k_j \sum_i \sum_\ell g_k g_\ell e^{i(k_i + k_\ell - k_j)x - i(k_i \hat{\mu}_i + k_\ell \hat{\mu}_\ell - k_j \hat{\mu}_j)t} = 0. \quad (6.177)$$

Expressing g_j in terms of real functions $\Phi_j(t)$ and $\theta_j(t)$ as

$$g_j(t) = (k_j |\alpha_j|)^{1/2} \Phi_j(t) e^{i\theta_j(t)}, \quad (6.178)$$

we get for the explosive instability

$$\frac{d}{dt} \Phi_F = \beta \Phi_S \Phi_A \sin \theta, \quad (6.179)$$

$$\frac{d}{dt} \Phi_S = \beta \Phi_F \Phi_A \sin \theta, \quad (6.180)$$

$$\frac{d}{dt} \Phi_A = \beta \Phi_S \Phi_F \sin \theta, \quad (6.181)$$

$$\frac{d}{dt} \ln(\Phi_F \Phi_S \Phi_A \cos \theta) = 0, \quad (6.182)$$

where $\theta = \theta_S - \theta_A - \theta_F$ and $\beta = |k_F k_S k_A \hat{\alpha}_F \hat{\alpha}_S \hat{\alpha}_A|^{1/2}$ where $\hat{\alpha}_j = \hat{\mu}_j / (3\hat{\mu}_j + 2)$. The solution is given for $|\Phi_i(0)|, |\Phi_j(0)| > |\Phi_k(0)|$ and $\theta(0) \ll 1$

$$\Phi_i(t) = \frac{\Phi_i(0)}{\text{sn}(K(\eta) - \beta \Phi_i(0)t, \eta)}, \quad (6.183)$$

where sn is the Jacobian elliptic cnoidal function, $K(\eta)$ is the complete elliptic integral of the first kind, and $\eta = (\Phi_i(0)^2 - \Phi_j(0)^2 - \Phi_k(0)^2) / \Phi_i(0)^2$. The explosion time of the mode i is estimated from the zero of the denominator of (6.183):

$$T_{\text{expl}}^{(i)} = \frac{K(\eta)}{2\beta \Phi_i(0)} \simeq \begin{cases} \frac{1}{4\beta \Phi_i(0)} \ln \frac{1}{1-\eta^2} & \text{for } \eta \rightarrow 1, \\ \frac{\pi}{4\beta \Phi_i(0)}, & \eta \ll 1. \end{cases} \quad (6.184)$$

Now, the time for the soliton formation is roughly approximated by the crossing time of the characteristics of the left hand side of (6.174). Then, we have

$$T_{\text{soliton}}^{(i)} = (k a_i |\alpha_i|)^{-1}, \quad (6.185)$$

where k is the wave number of the initially launched linear mode i with the amplitude a_i . When an F-mode is initially given, we have from (6.184) and (6.185) for the ratio of these two time scales with $k = 0.0628$, $a_F = 0.12$ and $a = 0.533$ ($\mu_F = 0.248$, $\mu_A = -0.340$, $\mu_A = -0.912$)

$$\frac{T_{\text{expl}}^{(F)}}{T_{\text{soliton}}^{(F)}} \simeq \frac{k|\alpha_F|}{2\beta} \ln \left| \frac{\Phi_F(0)^2}{\Phi_S(0)^2 + \Phi_A(0)^2} \right| \sim 0.98, \quad (6.186)$$

where we have solved (6.177) for $g_S(0)$ and $g_A(0)$ under the action of the pump wave $\Phi_F(0)$ neglecting the effect of the dispersion to get

$$\Phi_S(0) \sim \alpha_S \frac{\Phi_F(0)^2}{2(\mu_S - \mu_F)}, \quad \Phi_A(0) \sim \alpha_A (\alpha_F - \alpha_A) \frac{\Phi_F(0)^2}{(\mu_F - \mu_A)^2}.$$

In this case, the two time scales are comparable and the soliton formation is almost masked by the explosive instability. However, under the same parameters when an initial amplitude is allocated to an A -mode ($a_A = 0.12$) or S -mode ($a_S = 0.12$), then we have in a similar way

$$\frac{T_{\text{expl}}^{(A)}}{T_{\text{soliton}}^{(A)}} \simeq \frac{k|\alpha_A|}{2\beta} \ln \left| \frac{\Phi_A(0)^2}{\Phi_F(0)^2 + \Phi_S(0)^2} \right| \sim 37, \quad (6.187)$$

$$\frac{T_{\text{expl}}^{(S)}}{T_{\text{soliton}}^{(S)}} \simeq \frac{k|\alpha_S|}{2\beta} \ln \left| \frac{\Phi_S(0)^2}{\Phi_F(0)^2 + \Phi_A(0)^2} \right| \sim 20, \quad (6.188)$$

implying that for these two cases the explosive instability is practically unimportant and the solitons are formed. However, as is seen in the next subsection, there is a critical amplitude above which the A - and S -solitons explode in a finite time. Therefore, provided that the amplitudes of the solitons are less than the critical amplitudes, the solitons are stable entities.

6.5.3 Numerical Solutions

A nonlinear stationary soliton solution is readily obtained as

$$W(x, t) = \left(\frac{1}{(K/\Omega)^2} \right) 12K^2 \text{sech}^2(Kx - \Omega t), \quad (6.189)$$

where Ω is determined by the following soliton dispersion equation

$$\left(\frac{\Omega}{K} \right)^3 + (1 - 4K^2) \left(\frac{\Omega}{K} \right)^2 - \frac{a^3}{2} = 0, \quad a^3 = \frac{\beta v_0}{\mu^2}, \quad (6.190)$$

and is shown in Fig. 6.4. For $a < 2/3$ (linearly stable case), three soliton modes are possible; the fast beam soliton (F-soliton), slow beam soliton (S-soliton), and ion acoustic soliton (A-soliton). In this case, there is a critical wave number K_c beyond which A -soliton and S -soliton cannot exist:

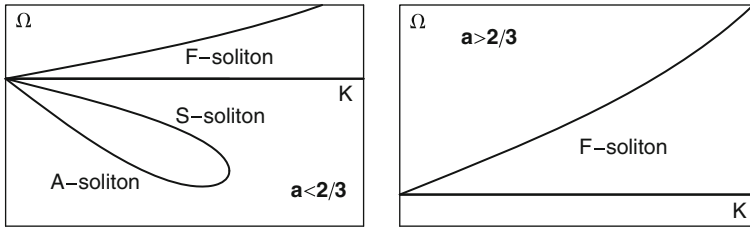


Fig. 6.4 The soliton dispersion relation for stable case (*left*) and unstable (*right*)

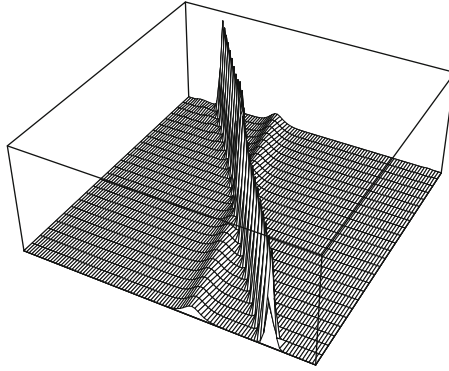


Fig. 6.5 Stable head-on collision between A-soliton ($k_A = 0.212$) and S-soliton ($k_S = 0.071$) for $a = 0.427$ ($K_c = 0.3$) when $k_A + k_S < K_c$

$$K_c = \frac{1}{2} \sqrt{1 - \frac{3}{2}a}. \quad (6.191)$$

On the other hand, for $a > 2/3$ (linearly unstable case) only F-solitons are allowed. Since Ω associated with the A- and S-solitons becomes complex for both $K > K_c$ in the linearly stable case $a < 2/3$ and the linearly unstable case $a > 2/3$, the A- and S-solitons explode in a finite time, which is anticipated from (6.189). The explosion solitons are a nonlinear counter part of the explosive instability of an ion beam-plasma system.

Now (6.165) and (6.166) are numerically solved [94] to study the stability, interactions, and evolution of the solitons. A stable head-on collision between A- and S-solitons is depicted in Fig. 6.5 with $a = 0.427$ ($K_c = 0.3$) for $k_A = 0.212$ and $k_S = 0.071$ ($k_A + k_S < K_c$). An unstable head-on collision is depicted in Fig. 6.6 with $a = 0.427$ ($K_c = 0.3$) for $k_A = 0.212$ and $k_S = 0.141$ ($k_A + k_S > K_c$). The A- and S-solitons behave like ordinary solitons and preserve their identities after collision for $k_A + k_S < K_c$, while they explode for $k_A + k_S = 0.353 > K_c$.

An F-soliton excites A- and S-modes and drives the whole system to an explosion as in Fig. 6.7. The A- and S-solitons are stable up to near the critical amplitudes for which a strong coupling between the A- and S-solitons is anticipated from the dispersion relation.

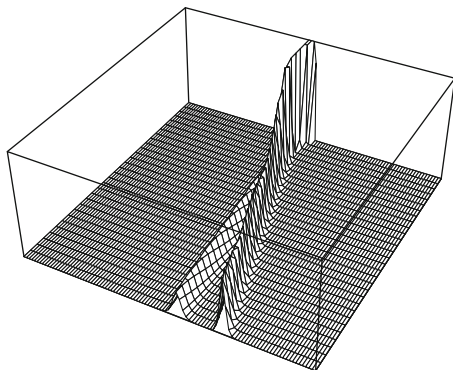


Fig. 6.6 Unstable head-on collision between A-soliton ($k_A = 0.212$) and S-soliton ($k_S = 0.141$) for $a = 0.427$ ($K_c = 0.3$) when $k_A + k_S > K_c$

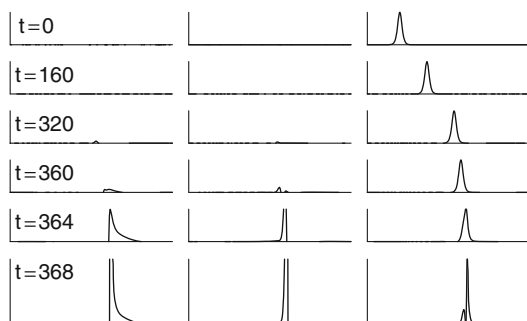


Fig. 6.7 An F-soliton ($k_F = 0.283$) (right) excites A-soliton (left) and S-soliton (middle) to drive the whole system to an explosion for $a = 0.427$ ($K_c = 0.3$)

The collision processes between both like-solitons and unlike-solitons with small amplitudes are very similar to those of the K-dV solitons. When the amplitudes of the solitons are increased, an explosion occurs at the collision even though they are themselves stable.

Thus, nonlinear modes in an ion beam-plasma system are characterized by mutually independent K-dV solitons for the case that not only each amplitude but also the sum of them are smaller than the threshold and by coupled nonlinear explosion modes for the supercritical case.

6.5.4 K-dV Approximation for Small Amplitude Nonlinear Modes

With a smallness parameter δ introducing the coordinate ξ_j attached to each mode and the time scale τ so as to balance between the nonlinearity and the dispersion,

$$\xi_j = \delta(x - \mu_j t + \delta \Psi_j), \quad \tau = \delta^3 t, \quad (6.192)$$

and expanding f_j in (6.174) as

$$f_j = \delta^2 f_j^{(1)}(\xi_j, \tau) + \delta^4 f_j^{(2)}(\xi_j, \tau) + \dots, \quad (6.193)$$

we collect terms of the order of δ^3 in (6.174)

$$\sum_{\ell} (\mu_j - \mu_{\ell}) \frac{\partial f_j^{(1)}}{\partial \xi_{\ell}} = 0, \quad (6.194)$$

leading to

$$f_j^{(1)} = f_j^{(1)}(\xi_j, \tau). \quad (6.195)$$

Of the order of δ^5 , we have

$$\begin{aligned} & \sum_{\ell} (\mu_j - \mu_{\ell}) \frac{\partial f_j^{(2)}}{\partial \xi_{\ell}} + \alpha_j \left(\sum_i f_i^{(1)} \right) \sum_{\ell \neq j} \frac{\partial f_{\ell}^{(1)}}{\partial \xi_{\ell}} + \alpha_j \sum_{\ell \neq j} \frac{\partial^3 f_{\ell}^{(1)}}{\partial \xi_{\ell}^3} \\ & + \frac{\partial f_j^{(1)}}{\partial \tau} + \alpha_j \frac{\partial^3 f_j^{(1)}}{\partial \xi_j^3} + \alpha_j f_j^{(1)} \frac{\partial f_j^{(1)}}{\partial \xi_j} \\ & \sum_{\ell \neq j} \left\{ (\mu_j - \mu_{\ell}) \frac{\partial \Psi_j}{\partial \xi_{\ell}} + \alpha_j \sum_{\ell \neq j} f_{\ell}^{(1)} \right\} \frac{\partial f_j^{(1)}}{\partial \xi_j} = 0. \end{aligned} \quad (6.196)$$

In the above if we set

$$\Psi_j = - \sum_{\ell \neq j} \frac{\alpha_j}{\mu_j - \mu_{\ell}} \int^{\xi_j} f_{\ell}^{(1)}(\xi, \tau), \quad (6.197)$$

a non-secular (boundedness) condition for $f_j^{(2)}$ in (6.196) gives the K-dV equation for $f_j^{(1)}$ as

$$\frac{\partial f_j^{(1)}}{\partial t} + \alpha_j \frac{\partial^3 f_j^{(1)}}{\partial \xi_j^3} + \alpha_j f_j^{(1)} \frac{\partial f_j^{(1)}}{\partial \xi_j} = 0. \quad (6.198)$$

Thus, small amplitude nonlinear waves are shown to be described by the K-dV solitons.

6.5.5 Nonlinear Explosion Modes

The explosion occurs at $K \simeq K_c$, where A- and S-modes are strongly coupled. To analyze the explosive behavior, we restrict ourselves to the vicinity of the instability point. At $a = 2/3$, the eigenvalues of $A(W = 0)$ are $\mu = 1/3$ and $-2/3$, the latter of

which is a degenerate phase velocity of A- and S-modes. When $a \simeq 2/3$, the deviation of μ_A and μ_S from $-2/3$ is so small that we may use a singular perturbation method of multi-time scales to separate the evolution of the coupled system of A- and S-modes from the dynamics of F-mode. We introduce a smallness parameter ε giving a measure of the deviation from the critical point as $a^3 = (8/27)(1 + 3\varepsilon)$ and variables $\xi = x - (1/3)t + \chi$, $\eta = x + (2/3)t + \psi$ and $\tau = t$. Using the multi-time scales

$$\frac{\partial}{\partial t} = -\frac{1}{3} \left(1 + \frac{\partial \chi}{\partial \xi} - 2 \frac{\partial \chi}{\partial \eta} \right) \frac{\partial}{\partial \xi} + \frac{2}{3} \left(1 - \frac{1}{2} \frac{\partial \chi}{\partial \xi} + \frac{\partial \chi}{\partial \eta} \right) \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_2} + \dots,$$

we expand dependent variables with respect to a smallness parameter δ ;

$$w_1 - \frac{1}{9}w_2 = \sum_n \delta^{n+1} G^{(n)}, \quad w_1 - \frac{4}{9}w_2 = \sum_n \delta^{n+1} H^{(n)}, \quad (6.199)$$

where $G^{(n)}$ and $H^{(n)}$ represent the degenerated A- and S-modes and the F-mode, respectively. The ordering is fixed as $\varepsilon \sim \delta^2$, $\partial/\partial \xi \sim \partial/\partial \eta$, $\partial/\partial \tau_1 \sim \delta^2$, $\partial/\partial \tau_2 \sim \delta^3$. Substituting (6.199) into (6.165) and (6.166), we have in the lowest order of δ ,

$$G^{(1)} = g(\eta, \tau), \quad H^{(1)} = h(\xi, \tau), \quad (6.200)$$

where $G^{(1)}$ and $H^{(1)}$ are assumed to be finite at $\xi \rightarrow \pm\infty$. In the next order, we have

$$\frac{\partial h}{\partial \tau_1} = 0, \quad G^{(2)} = g^{(2)}(\eta), \quad \frac{\partial H^{(2)}}{\partial \eta} = 4 \frac{\partial g}{\partial \tau_1}. \quad (6.201)$$

From the nonsecular condition in the order of δ^3 , we have

$$\frac{\partial^2 g}{\partial \tau_1^2} + \frac{4}{9} \left[\left(\varepsilon - \frac{h}{3} \right) \frac{\partial^2 g}{\partial \eta^2} + \frac{\partial^4 g}{\partial \eta^4} + \frac{2}{3} \frac{\partial^2 g^2}{\partial \eta^2} \right] = 0, \quad (6.202)$$

$$\frac{\partial h}{\partial \tau_2} + \frac{4}{9} \varepsilon \frac{\partial h}{\partial \xi} + \frac{1}{9} \frac{\partial^3 h}{\partial \xi^3} - \frac{1}{54} \frac{\partial h^2}{\partial \xi} = 0, \quad (6.203)$$

$$\frac{\partial \psi}{\partial \xi} = -\frac{4}{27} h, \quad \frac{\partial \chi}{\partial \eta} = -\frac{4}{27} g. \quad (6.204)$$

Equations (6.202) and (6.203) show that even for negative ε , an F-soliton (rarefactive pulse) has a tendency to drive coupled A- and S-modes unstable by changing (6.202) from hyperbolic to elliptic, that is, an F-soliton excites A- and S-modes, leading them to explosion.

Under the small amplitude approximation, an F-mode is self-sustained as shown in (6.203) and does not couple to A- and S-modes, implying that F-modes can be neglected in (6.202) if they are initially zero. Then, (6.202) reduces to

$$\frac{\partial^2 g}{\partial \tau^2} + \varepsilon \frac{\partial^2 g}{\partial \eta^2} + 6 \frac{\partial^2 g^2}{\partial \eta^2} + \frac{\partial^4 g}{\partial \eta^4} = 0, \quad (6.205)$$

where $\tau_1 \rightarrow \tau$, and $g \rightarrow 9g$.

For $\varepsilon < 0$, (6.205) is linearly stable and has a soliton solution

$$g = K^2 \text{sech}^2[K(\eta - V\tau)], \quad V = \pm \sqrt{|\varepsilon| - 4K^2}, \quad (6.206)$$

where $\pm$ are allocated to S- and A-solitons.

The soliton solution (6.206) is to explode for $K^2 > |\varepsilon|/4$ and is replaced by

$$g = K^2 \frac{1 - A \cosh(Kx) \sin(\Omega t)}{[\cosh(Kx) - A \sin(\Omega t)]^2}, \quad (6.207)$$

$$\Omega = K \sqrt{K^2 - |\varepsilon|}, \quad A = \sqrt{\frac{4K^2 - |\varepsilon|}{K^2 - |\varepsilon|}}. \quad (6.208)$$

The solution with $A > 1$ explode at the time $t_{\text{blow}} = (1/\Omega) \sin^{-1}(1/A)$. The explosion modes (6.207) are also possible for $\varepsilon > 0$. In this case, Ω and A are given by (6.208) but $|\varepsilon|$ is replaced by $-\varepsilon$. The explosion modes in the linearly unstable region means that (6.169) is lacking in nonlinear saturation mechanisms.

6.5.6 Beam Reflection and Soliton Emission

The unphysical explosion solutions appear since the nonlinearity of the ion beam dynamics is ignored. In fact when the wave amplitude increases, the ion beam start to be reflected by the wave and the fluid description is violated. Here, let us evaluate the critical amplitude above which the ion beam is reflected. The full nonlinear equations for the ion beam-plasma system are given by

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} + \frac{\beta}{2} \frac{\partial \rho}{\partial x} = 0, \quad (6.209)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}[(1 + \rho)w] = 0, \quad (6.210)$$

$$\frac{\partial w}{\partial t} + w \frac{\partial}{\partial x} w + \frac{v_0}{\mu^2} \frac{\partial u}{\partial x} = 0, \quad (6.211)$$

where w is the beam velocity fluctuation normalized by v_{b0}/v_0 . If we look for a stationary solution $f(x - \lambda t)$ which is diminishes at $x \rightarrow \pm\infty$, we have

$$\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{1}{3}u^3 + (\lambda + 1)u^2 + \beta s \left\{ \sqrt{1 - 2\frac{u}{s}} - 1 + \frac{u}{s} \right\}. \quad (6.212)$$

where $s = (\mu\lambda)^2/v_0$. The ion beam is reflected when the potential amplitude is larger than $u_{\text{beam}} = (\mu\lambda)^2/(2v_0)$, while the ion fluid is reflected when the potential amplitude is larger than $u_{\text{fluid}} = v_0\mu^2(1 + \lambda/v_{b0})^2/2$. Since λ is a propagation velocity in the frame moving with the ion beam velocity, usually $u_{\text{fluid}} > u_{\text{beam}}$, that is, the ion fluid is not reflected for the wave amplitude for which the ion beam is reflected.

When the ion beam is reflected, the fluid description is violated and the particle description is used to describe the ion dynamics. However, for the plasma, the fluid description is still valid. Thus, the nonlinear development of the ion beam-plasma instability can be studied numerically based on a hybrid code which consists of a fluid model for the plasma and a particle code for the ion beam [96, 97].

Using the difference scheme developed by Zabusky and Kruskal [80] for the plasma and the PIC code for the ion beam, the following equations are solved numerically.

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} + \frac{\beta}{2} \frac{\partial \rho}{\partial x} = 0, \quad (6.213)$$

$$\rho(x, t) = \int dv f(x, v, t) - 1, \quad (6.214)$$

$$f(x, v, t) = \frac{1}{N} < \sum_j \delta(x - x_j(t)) \delta(v - v_j(t)) >, \quad (6.215)$$

$$\frac{dx_j}{dt} = v_j - v_0, \quad (6.216)$$

$$\frac{dv_j}{dt} = -\frac{v_0}{\mu^2} \frac{\partial u}{\partial x_j}. \quad (6.217)$$

Here, simulations are done for the following two cases. Under the linearly stable conditions, a stationary soliton solution is given as an initial condition to observe the time evolution. Under the linearly unstable conditions, a tiny amplitude sinusoidal wave is launched initially. The periodic boundary condition is used for both cases.

6.5.6.1 Linearly Stable Cases

An initial condition is a stationary soliton solution $u(x, 0) = 12K^2 \text{sech}^2(Kx)$. The course of development is shown in Figs. 6.8 and 6.9.

- A-soliton with $K = 0.18$ and $a = 0.357$.
 1. The soliton starts to grow in a self-similar manner because of the bunching of the ion beam. The beam bunching is followed by a further buildup in the potential whose profile responds statically to the beam bunching. Thus up to the onset of the beam reflection, we have from (6.209) to (6.211)

$$u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \frac{\beta}{2} \frac{\partial \rho}{\partial x} = 0, \quad (6.218)$$

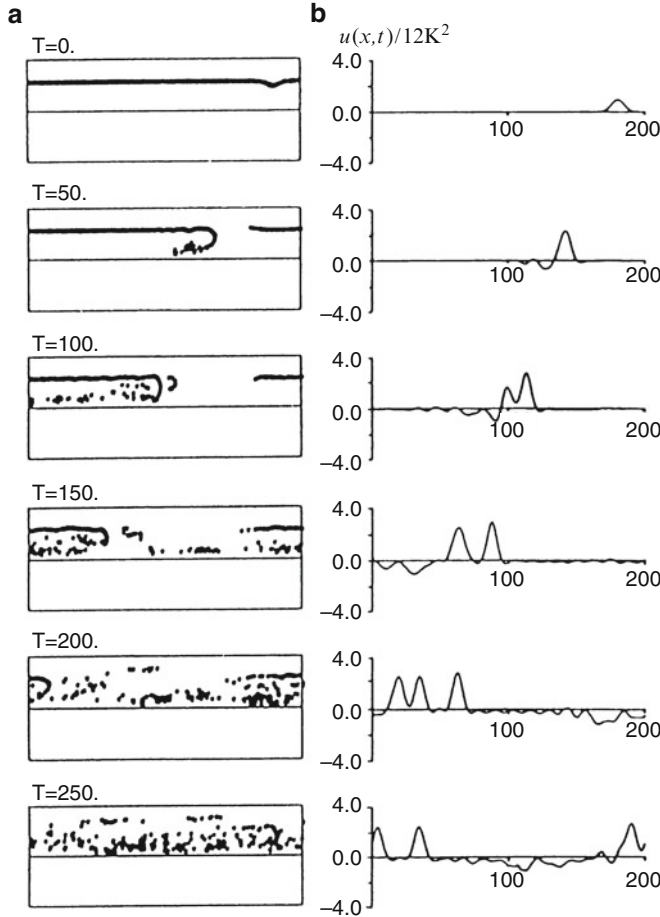


Fig. 6.8 Soliton formation in the development of the ion beam-plasma instability: (a) the phase space and (b) the wave profile

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{v_0}{\mu^2} \frac{\partial}{\partial x} \left(\rho \frac{\partial u}{\partial x} \right) = 0, \quad (6.219)$$

which allow a self-similar solution given by

$$u = \frac{f(\eta)}{T_{\text{expl}} - t}, \quad \rho = \frac{g(\eta)}{(T_{\text{expl}} - t)^2}, \quad \eta = \frac{x - \lambda t}{T_{\text{expl}} - t}. \quad (6.220)$$

2. The soliton reflects the beam and continues to grow with a feed of the energy from the beam. After the onset of the beam reflection, the energy loss of the reflected beam is transferred to the potential. Then, the time evolution of the amplitude is evaluated from (6.209) by

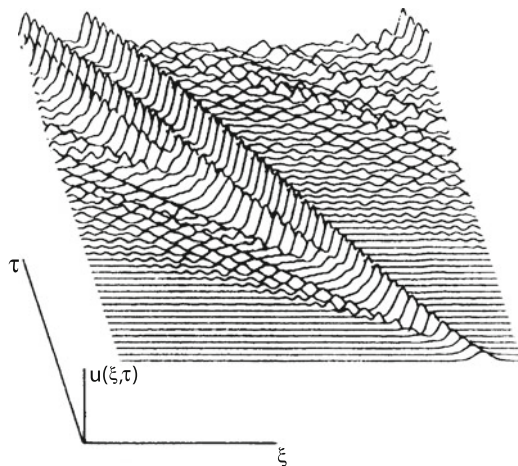


Fig. 6.9 Growth and emission of solitons in the development of the ion beam-plasma instability

$$\frac{\partial}{\partial t} \int dx u^2 = - \int dx u \frac{\partial \rho_R}{\partial x}, \quad (6.221)$$

where ρ_R is approximated by the stationary solution of (6.209)–(6.211):

$$\rho_R = \frac{1}{\sqrt{1 - 2v_0 u / (\mu \lambda)^2}} - 1. \quad (6.222)$$

Noting that the potential u is continuous at the reflection point, we have

$$\frac{\partial}{\partial t} \int dx u^2 = \beta \frac{(\mu \lambda)^2}{2v_0} \frac{\rho_R^2}{1 + \rho_R}, \quad (6.223)$$

where ρ_R is the beam density at the reflection point which is finite. If we set $u = A \operatorname{sech}^2(\sqrt{A/12}x)$, then we obtain

$$A = A_R \left\{ 1 + \frac{\beta}{\sqrt{12A_R}}(t - t_R) \right\}^{2/3}, \quad A_R = \frac{(\mu \lambda)^2}{2v_0}. \quad (6.224)$$

This describes the amplification after the onset of the beam reflection.

3. The soliton grows up to the level for which only one eigenvalue exists under the inverse scattering analysis and then is stabilized by emitting a baby soliton. Since the condition for the soliton to evolve into N solitons is given by

$$\frac{u(t)}{u(0)} > \frac{1}{2}N(N + 1), \quad (6.225)$$

whenever the soliton amplitude grows to be over three times as large as the initial amplitude, it reaches the amplitude to emit a baby soliton.

4. The emitted soliton gains the energy from the beam and grows until it emits a baby soliton.
5. This process is repeated and a train of solitons is formed in a downstream side.

- S-soliton

The same sort of reflection as that for the A-soliton is observed. However in this case since the propagation velocity λ_S is much less than λ_A in the beam frame, the amplitude for the onset of reflection is so small that S-solitons are practically not observed.

- F-soliton

The F-solitons propagate with velocities larger than the ion beam velocity, causing a beam acceleration. At the initial stage, the F-soliton grows because of the accumulation of the beam ions and steepens to form a shock. Then, it accelerates the beam while a train of solitons is emitted behind the shock.

6.5.6.2 Linearly Unstable Cases

An initial condition is given by a linear solution:

$$\begin{aligned} u(x, 0) &= A \sin(Kx), \\ x_j(0) &= \Delta x \cdot (j - 1) + \frac{v_0}{\mu^2} \frac{K^2}{\Omega^2} u(\Delta x \cdot (j - 1), 0), \\ v_j(0) &= v_0 \left\{ 1 + \frac{K}{\Omega^2} u(\Delta x \cdot (j - 1), 0) \right\}. \end{aligned}$$

The parameters are chosen in such a way that the growth rate is nearly at the maximum, its magnitude is strong enough to be comparable with the frequency, and the higher harmonics are all stable ($a = 0.865$, $k_c = 0.545$, $\gamma = 0.107$, $\omega = -0.244$). The development of the instability is shown in Figs. 6.10 and 6.11.

1. The sinusoidal wave grows according to the linear growth rate.
2. The linear growth is followed by the nonlinear growth which is analyzed in the following. Eliminating ρ in (6.165) and (6.166), we have

$$\left(\frac{\partial}{\partial t} + \lambda_F \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + \lambda_S \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + \lambda_A \frac{\partial}{\partial x} \right) u + \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{2} u^2 \right) = 0.$$

For the unstable case, we have λ_F is real and $\lambda_{A,B} = \lambda \pm i\delta$. Transforming to the frame moving with λ , that is, $\xi = x - \lambda t$, the above equation is rewritten as

$$\left\{ \frac{\partial}{\partial t} + (\lambda_F - \lambda) \frac{\partial}{\partial \xi} \right\} \left(\frac{\partial}{\partial t^2} + \delta^2 \frac{\partial^2}{\partial \xi^2} \right) u + \left(\frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial \xi} \right)^2 \frac{\partial}{\partial \xi} \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{2} u^2 \right) = 0,$$

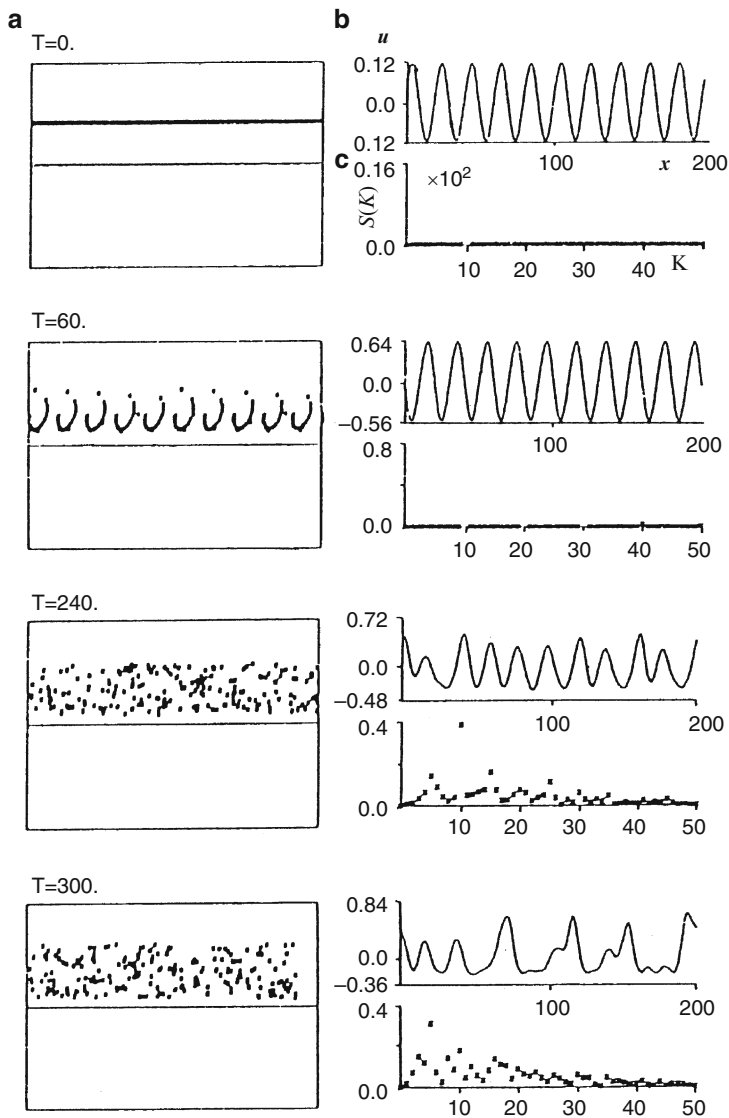


Fig. 6.10 Nonlinear development of the ion beam-plasma instability: (a) the beam particles in the phase space, (b) the spatial profile of the wave potential and (c) the wavenumber spectrum of the wave potential

which reduces for $|\delta/\lambda| \ll 1$ to

$$\frac{\partial^2 u}{\partial t^2} + \delta^2 \frac{\partial^2 u}{\partial \xi^2} + \frac{\lambda^2}{\lambda_F - \lambda} \frac{\partial^2}{\partial \xi^2} \left(\frac{\partial u}{\partial \xi^2} + \frac{1}{2} u^2 \right) = 0. \quad (6.226)$$

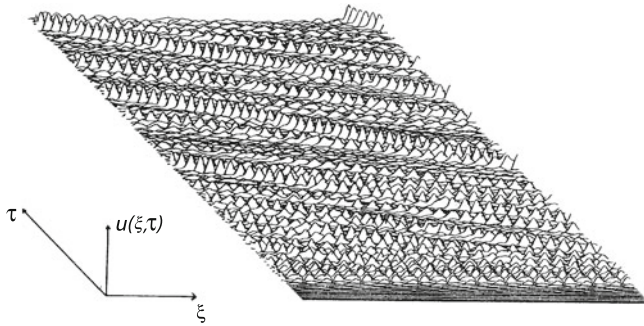


Fig. 6.11 Nonlinear development of the ion beam-plasma instability

An exact solution of (6.226) which is continuously connected to the linearly growing solution $u(\xi, t) = Ae^{\gamma t} \cos(k\xi)$, $\gamma = k\delta\sqrt{1 - k^2/k_c^2}$, $k_c = \delta\sqrt{\lambda_F - \lambda}/\lambda$ is given by

$$u(\xi, t) = 12k^2 \frac{|B| \cosh(\gamma t + \psi) \cos(k\xi - \omega t) - 1}{[|B| \cosh(\gamma t + \psi) - \cos(k\xi - \omega t)]^2}, \quad (6.227)$$

where $\phi = \ln(AB/24k^2)$ and $B = \sqrt{(k_c^2 - 4k^2)/(k_c^2 - k^2)}$.

3. The nonlinear growth is saturated by trapping the beam and the wave is steepened by exciting higher harmonics and is converted to a train of pulses.
4. The trapped beam particles form islands in the phase space and then the particles trapped at the edge of the islands start to wander in the phase space and produce long wave fluctuations in the beam density.
5. The amplitudes of the pulses start to change because of the detrapped particles and the pulses move with different speeds to repeat soliton-like collisions.
6. The deeply trapped beam particles parametrically couple with the initially given wave to excite waves peaked at the wavenumber of the subharmonic, destroying the periodicity the system originally possessed and leading to broadening in the wavenumber spectrum of the wave potential.
7. The deeply trapped beam particles start detrapped and a train of the pulses is destroyed to develop into turbulence which is akin to an ensemble of solitons under the action of random forces.

6.6 Soliton-Like Excitations in Linearly Unstable Plasmas

There is a big question of how a finite amplitude wave can be excited in linearly stable plasmas. In laboratory experiments, a finite amplitude wave can be launched into a stable plasma from an external source to excite solitons. However, a small amplitude wave cannot grow to the level beyond which the nonlinearity starts to

play. Thus, the nonlinear development of linearly unstable plasmas is to be studied in order to see the roles of solitons. Here as an example of linearly unstable plasmas, we take the plasma with an electron drift which leads to the Buneman instability (two stream instability).

6.6.1 Basic Equations for Long Wave Buneman Instability

The Buneman instability is usually referred to as a resonant type of instability whose growth rate is comparable with the real frequency. The growth is so fast that the bunching of the electron is a primary phenomena which leads to explosion. However, in the long wave region where the growth rate is moderate, we may anticipate several types of stationary nonlinear solutions [98].

Suppose that the electron fluid drifts with the velocity v_0 , while the ion fluids is at rest. The basic equations are given by

$$\frac{\partial}{\partial t} n_\alpha + \frac{\partial}{\partial x} (n_\alpha v_\alpha) = 0, \quad (6.228)$$

$$\frac{\partial}{\partial t} v_\alpha + v_\alpha \frac{\partial}{\partial x} v_\alpha = -\frac{e_\alpha}{m_\alpha} \frac{\partial \phi}{\partial x} - \frac{T_\alpha}{m_\alpha n_\alpha} \frac{\partial}{\partial x} n_\alpha, \quad (6.229)$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi \sum_{\alpha} e_\alpha n_\alpha, \quad (6.230)$$

whose dispersion equation is given by

$$\frac{\omega_{pe}^2}{(\omega - kv_0)^2 - k^2 v_{Te}^2} + \frac{\omega_{pi}^2}{\omega^2} = 1, \quad (6.231)$$

where $v_{Te} = \sqrt{T_e/m_e}$ and the ion temperature is neglected. In the long wave limit (6.231) can be rewritten as

$$(\omega - kv_0)^2 - k^2 v_0^2 = \frac{\omega_{pe}^2 \omega^2}{\omega^2 - \omega_{pi}^2} \simeq -\frac{m_i}{m_e} \omega^2, \quad (6.232)$$

from which the growth rate is estimated for $1 - (v_{Te}/v_0)^2 > m_e/m_i$ as

$$\omega \sim \sqrt{\frac{m_e}{m_i}} kv_0 \left(\sqrt{\frac{m_e}{m_i}} + i \sqrt{1 - \frac{v_{Te}^2}{v_0^2}} \right) \sim ikv_0 \sqrt{\frac{m_e}{m_i} \left(1 - \frac{v_{Te}^2}{v_0^2} \right)}, \quad (6.233)$$

Then, we may use the quasi-static approximation for the electrons to obtain

$$\ln \rho_e + \frac{\mu^2}{2} \left(\frac{1}{\rho^2} - 1 \right) = \Psi, \quad (6.234)$$

where

$$\rho_e = \frac{n_e}{n_0}, \quad \Psi = \frac{e\phi}{T_e}, \quad \mu = \frac{v_0}{v_{T_e}} (> 1 \text{ for instability}). \quad (6.235)$$

Combining (6.228) and (6.229) for the ions, we obtain

$$\frac{\partial^2 \rho_i}{\partial t^2} - C_s^2 \frac{\partial^2 \Psi}{\partial x^2} = C_s^2 \frac{\partial^2}{\partial x^2} (\rho_i v_i^2) + C_s^2 \frac{\partial}{\partial x} \left((\rho_i - 1) \frac{\partial \Psi}{\partial x} \right), \quad (6.236)$$

where $\rho_i = n_i/n_0$, $v_i = v_i/C_s$, $C_s = \sqrt{T_e/m_i}$. Using Poisson's equation (6.230), we eliminate ρ_i in (6.236) to obtain

$$\frac{\partial^2}{\partial t^2} \left(\rho_e - \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial x^2} \right) = \frac{\partial^2}{\partial x^2} \left[\left(\rho_e - \frac{\partial^2 \Psi}{\partial x^2} \right) v_i^2 \right] + \frac{\partial}{\partial x} \left[\left(\rho_e - 1 - \frac{\partial^2 \Psi}{\partial x^2} \right) \frac{\partial \Psi}{\partial x} \right], \quad (6.237)$$

where t and x are normalized by ω_{pi}^{-1} and λ_D (the Debye length), respectively. From (6.235), ρ_e can be expanded in terms of Ψ as

$$\rho_e \simeq 1 - \frac{1}{\mu^2 - 1} \Psi + \frac{1}{2} \frac{3\mu^2 - 1}{\mu^2 - 1} \Psi^2 + \dots \quad (6.238)$$

Substituting (6.238) into (6.237), we have up to the second order with respect to Ψ ,

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} \left(\Psi + (\mu^2 - 1) \frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{2} \frac{3\mu^2 - 1}{(\mu^2 - 1)^2} \Psi^2 \right) + (\mu^2 - 1) \frac{\partial^2 \Psi}{\partial x^2} \\ &= -(\mu^2 - 1) \frac{\partial^2}{\partial x^2} v_i^2 + \frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi^2, \end{aligned} \quad (6.239)$$

Since the second order time derivative in the dispersion and nonlinear terms may be approximated by

$$\frac{\partial^2}{\partial t^2} \simeq -(\mu^2 - 1) \frac{\partial^2}{\partial x^2}, \quad (6.240)$$

which comes from the dispersion relation (6.233). Furthermore from the ion equation of motion, we have $v_i^2 \sim -\Psi^2/(\mu^2 - 1)$ based on (6.240). Thus, we finally obtain

$$\frac{\partial^2 \Psi}{\partial t^2} + \frac{\partial^2 \Psi}{\partial \xi^2} - \frac{\partial^4 \Psi}{\partial \xi^4} + \varepsilon \frac{\partial^2 \Psi^2}{\partial \xi^2} = 0, \quad (6.241)$$

where we have used the following transformation $\xi = x/\sqrt{\mu^2 - 1}$ and

$$\varepsilon = \left(\frac{3\mu^2 - 1}{(\mu^2 - 1)^2} - 3 \right) \frac{1}{2(\mu^2 - 1)}. \quad (6.242)$$

Equation (6.241) is different from the Boussinesq equation in the sign of the second derivative with respect to the spatial coordinate. By replacing ξ with $i\xi$, where $i^2 = -1$, (6.241) maps into the Boussinesq equation, which is integrable, suggesting that (6.241) is also integrable. However, the reality condition does not so map. Real solutions of the Boussinesq equation will not map into a real solution of (6.241). Therefore, we need to rework for the solution. Fortunately, (6.241) can be transformed into Hirota's bilinear form. Introducing the dependent variable transformation,

$$\Psi = -6\varepsilon \frac{\partial^2}{\partial \xi^2} \ln f, \quad (6.243)$$

we have

$$(D_\tau^2 + D_\xi^2 - D_\xi^4) f \cdot f = 0, \quad (6.244)$$

where

$$D_x f \cdot f = \left\{ \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) f(x) f(x') \right\}_{x=x'}. \quad (6.245)$$

There are a variety of solutions for (6.241). Since ε is positive for $1 < \mu^2 < (9 + \sqrt{33})/6$ and negative for $\mu^2 > (9 + \sqrt{33})/6$, (6.243) indicates that the solution can be both positive and negative, which reflects the ion bunching and electron bunching, respectively.

6.6.2 Pulsating Solitons

The solution is given by

$$f = 2e^{K(\xi - \xi_0)} \{ \cosh[K(\xi - \xi_0)] - A \sin(\Omega \tau) \}, \quad (6.246)$$

where

$$\Omega = K\sqrt{1 - K^2}, \quad A = \sqrt{(4K^2 - 1)/(1 - K^2)}, \quad \xi_0 = K^{-1} \ln A, \quad (6.247)$$

and

$$\Psi = -6\varepsilon K^2 \frac{1 - A \cosh[K(\xi - \xi_0)] \sin(\Omega \tau)}{\{ \cosh[K(\xi - \xi_0)] - A \sin(\Omega \tau) \}^2}. \quad (6.248)$$

When K is less than $1/2$, then A does not exceed 1 and the denominator of (6.248) does not vanish, implying the solution (6.248) is a pulsating soliton for the case of

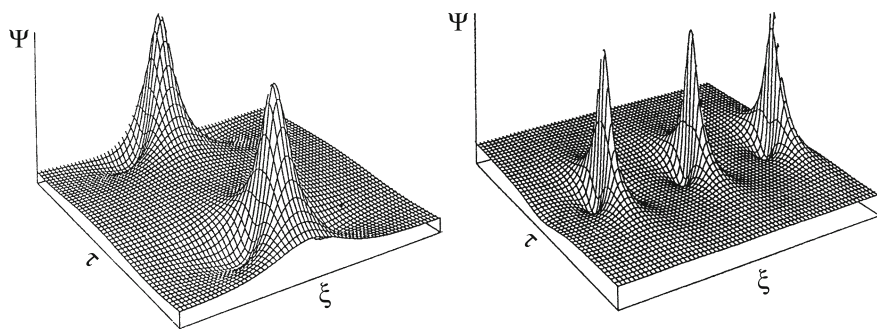


Fig. 6.12 Pulsating soliton (*left*) and spatially periodic soliton (*right*)

$\varepsilon < 0$. The amplitude of this positive pulsating soliton can be small enough that the ion reflection does not occur. The pulsating soliton is depicted in Fig. 6.12.

When K is increased so as to be $1/2 < K < 1$, we obtain, instead of (6.246),

$$f = -2e^{K(\xi - \xi_0)} \{ \sinh[K(\xi - \xi_0)] - B \cos(\Omega\tau + \delta) \}, \quad (6.249)$$

where

$$\Omega = K\sqrt{1 - K^2}, \quad B = \sqrt{(4K^2 - 1)/(1 - K^2)}, \quad \xi_0 = K^{-1} \ln B,$$

and

$$\psi = 6\varepsilon K^2 \frac{1 - B \sinh[K(\xi - \xi_0)] \cos(\Omega\tau + \delta)}{\{ \sinh[K(\xi - \xi_0)] - B \cos(\Omega\tau + \delta) \}^2}. \quad (6.250)$$

The solution (6.250) is singular and explodes along the curve

$$\xi = \xi_0 + \frac{1}{K} \sinh^{-1}[B \cos(\Omega\tau + \delta)]. \quad (6.251)$$

This unphysical explosion solution is, however, not realized since the ion or electron reflection occurs and invalidates the fluid description. Instead, this singular solution may be replaced by the following stable soliton obtained for $K > 1/2$.

6.6.3 Ordinary Solitons

For $K > \frac{1}{2}$, there is a soliton solution which is stable:

$$f = 1 + e^{2(K\xi - \Omega\tau)}, \quad \Omega = K\sqrt{4K^2 - 1}. \quad (6.252)$$

and

$$\Psi = -6\varepsilon K^2 \text{sech}^2(K\xi - \Omega\tau). \quad (6.253)$$

The ion reflection does not occur even when ε is negative provided that $\mu^2 > (27 + \sqrt{465})/6$.

6.6.4 Temporally Localized and Spatially Periodic Solitons

To see how a linearly unstable wave evolves, we look for a stationary solution which is connected to the linear unstable solution at the initial moment

$$\Psi \propto e^{\gamma t} \sin(K\xi), \quad \gamma = K\sqrt{K^2 + 1}. \quad (6.254)$$

Again Hirota's method gives

$$f = 1 + e^{2\gamma(\tau + \tau_0)} + 2e^{\gamma\tau} \sin(K\xi), \quad \tau_0 = \frac{1}{\gamma} \ln C, \quad C = \sqrt{(4K^2 + 1)/(K^2 + 1)}, \quad (6.255)$$

which leads to

$$\Psi = 6\varepsilon K^2 \frac{1 + C \cosh[\gamma(\tau + \tau_0)] \sin(K\xi)}{\{C \cosh[\gamma(\tau + \tau_0)] + \sin(K\xi)\}^2}, \quad (6.256)$$

The solution is symmetric with respect to time reversal and grows at the beginning with $e^{\gamma t}$, reaches the maximum value and then damps according to the linear damping $e^{-\gamma t}$. The tempo-spatio evolution is given in Fig. 6.12.

Chapter 7

Vortical Motions in Plasmas

Vortices in nature are often involved in violent transport. Examples are hurricanes, typhoons, cyclones, tornadoes, and so on. They store energy in their localized structures in a form of rotational motions and carry it over a long distance. Tidal vortices do not move and attract things passing by to swallow.

Flows are generally divided into two categories, flows with and without circulation. The circulation is defined by the integral taken along some closed contour C ,

$$\Gamma = \oint_C \mathbf{v} \cdot d\boldsymbol{\ell}. \quad (7.1)$$

When there is a flow circulating along C , the velocity component along the closed contour has a definite sign to give a finite Γ . However, the circulation defined by (7.1) is dependent on the ways of choosing closed contours. Therefore, we had better found the way to express circulation at each point of the space by the local velocity field $\mathbf{v}$. The integral (7.1) is converted by the Stokes formula to

$$\Gamma = \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S}, \quad (7.2)$$

where S is the closed area whose boundary is C . Equation (7.2) implies that the circulation around a certain point in space exists if the rotation of the velocity field

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \quad (7.3)$$

is not zero. The $\boldsymbol{\omega}$ is called the vorticity. A tube parallel to the vorticity vector $\boldsymbol{\omega}$ is called a vortex tube. Since from (7.3) the divergence of the vorticity is zero,

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{v}) = 0, \quad (7.4)$$

the vortex tube is continuous in the fluid and is terminated at the surface or the wall and otherwise is in the form of a ring.

The velocity field for a given vorticity $\boldsymbol{\omega}(\mathbf{r})$ is obtained in the following. For simplicity, we assume the fluid is incompressible:

$$\nabla \cdot \mathbf{v} = 0, \quad (7.5)$$

which gives

$$\mathbf{v} = \nabla \times \mathbf{A}. \quad (7.6)$$

The velocity field determined by (7.6) is invariant under the gauge transformation

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\psi, \quad (7.7)$$

where ψ is a scalar and we may choose

$$\nabla \cdot \mathbf{A} = 0. \quad (7.8)$$

Then, (7.3) is reduced to

$$\nabla \times \mathbf{v} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A} = \boldsymbol{\omega}, \quad (7.9)$$

which is readily solved to give

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \quad (7.10)$$

and the velocity field

$$\mathbf{v} = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}'. \quad (7.11)$$

The structure is common to the static magnetic field theory: the correspondence is between velocity $\mathbf{v}$ and magnetic field $\mathbf{B}$ and between vorticity $\boldsymbol{\omega}$ and current $\mathbf{j}$.

Corresponding to two types of the flow, there are two types of forces driving these flows: potential forces and solenoidal forces. When the force acting on the fluid is the potential force U , the motion of the fluid is described by

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \boldsymbol{\omega} = -\nabla \left(\frac{1}{2} v^2 + U \right) - \frac{1}{mn} \nabla p + \eta \nabla^2 \mathbf{v}, \quad (7.12)$$

where we have used $\mathbf{v} \cdot \nabla \mathbf{v} = v^2/2 - \mathbf{v} \times \boldsymbol{\omega}$. Taking rotation of (7.12) yields the equation for the vorticity

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = \frac{1}{mn^2} \nabla n \times \nabla p + \eta \nabla^2 \boldsymbol{\omega}, \quad (7.13)$$

which shows that $\nabla n \times \nabla p(T, n)$ is a source of vorticity. If the pressure is a function of the density n only (a barotropic flow) and the viscosity is neglected, the right hand side of (7.13) vanishes and (7.13) is reduced to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = 0, \quad (7.14)$$

which is called the Euler equation and is analogous to the equation for the frozen-in of magnetic field line in a perfect conducting plasma. Equation (7.14) implies that the circulation along an arbitrary closed contour moving with the fluid is conserved:

$$\begin{aligned} \frac{d\Gamma}{dt} &= \frac{d}{dt} \int_S \boldsymbol{\omega} \cdot d\mathbf{S} \\ &= \int_S \frac{\partial \boldsymbol{\omega}}{\partial t} \cdot d\mathbf{S} + \oint_C \boldsymbol{\omega} \cdot (\mathbf{v} \times d\boldsymbol{\ell}) \\ &= \int_S (\nabla \times (\mathbf{v} \times \boldsymbol{\omega})) \cdot d\mathbf{S} - \oint_C (\mathbf{v} \times \boldsymbol{\omega}) \cdot d\boldsymbol{\ell} \\ &= \int_S \{ \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) \} \cdot d\mathbf{S} = 0. \end{aligned}$$

Thus, if the vorticity is initially zero everywhere in the fluid, it is zero in the course of time development.

The vorticity is not always associated with rotating fluids. For example, the flow with a gradient in the velocity such as $\mathbf{v} = (vz/h, 0, 0)$ has the vorticity $\boldsymbol{\omega} = (0, v/h, 0)$. Wherever velocity shear exists, the vorticity is not zero. The shear is generated due to the viscosity in a boundary layer which is a vortex layer to supply vortices continuously.

In plasmas, the convective term driven by the $E \times B$ drift gives a vector non-linearity which excites vortical motions. Therefore, vortical motions are intrinsic in plasmas under a magnetic field.

Vortices are observed over the wide range of physical systems, from quantum to galactic systems, suggesting that studies on vortical motions in plasmas both theoretically and experimentally would provide deep insights utilized in other fields.

7.1 Two Dimensional Vortices

For two dimensional incompressible fluids in which (7.5) holds, the vorticity has only z -component. Assuming the z -component of the vector potential $A_z = -\psi$ (ψ is a stream function), the vorticity and the velocity are given by

$$\omega = \nabla^2 \psi, \quad v_x = -\frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \psi}{\partial x}, \quad (7.15)$$

which is substituted into (7.14) to give the two dimensional Euler equation

$$\frac{\partial}{\partial t} \nabla^2 \psi + \{\psi, \nabla^2 \psi\} = 0, \quad (7.16)$$

where $\{\dots, \dots\}$ is the Poisson bracket,

$$\{A, B\} = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x}. \quad (7.17)$$

Equation (7.16) indicates that ψ is a Hamiltonian and any $\omega = F(\psi)$ expressed as an arbitrary function F of ψ is a stationary solution since

$$\{\psi, \omega\} = \{\psi, F(\psi)\} = 0,$$

implying there are infinite number of invariants. Among them, physical invariants are energy E and enstrophy W

$$E = \frac{1}{2} \int v^2 d\mathbf{x} = \frac{1}{2} \int (\nabla \psi)^2 d\mathbf{x}, \quad W = \frac{1}{2} \int \omega^2 d\mathbf{x} = \frac{1}{2} \int (\nabla^2 \psi)^2 d\mathbf{x}. \quad (7.18)$$

which are shown by multiplying ψ and $\nabla^2 \psi$ to (7.16), respectively, and rewriting as

$$\begin{aligned} \frac{\partial}{\partial t} (\nabla \psi)^2 &= 2 \nabla \cdot \left(\psi \frac{\partial}{\partial t} \nabla \psi \right) - \hat{z} \cdot \nabla \times (\psi^2 \nabla \nabla^2 \psi), \\ \frac{\partial}{\partial t} (\nabla^2 \psi)^2 &= -\hat{z} \cdot \nabla \times ((\nabla^2 \psi) \nabla \psi), \end{aligned}$$

where $\hat{z}$ is a unit vector along the z -axis.

Equation (7.16) is rewritten as

$$\frac{\partial}{\partial t} \nabla^2 \psi + \hat{z} \cdot \{\nabla \psi \times \nabla (\nabla^2 \psi)\} = 0, \quad (7.19)$$

whose second term visualizes the nonlinear interaction that the velocity shear layer entangles itself to drive a vortical motion.

Here, we consider stationary solutions of (7.16) which is given by

$$\nabla^2 \psi = F(\psi), \quad (7.20)$$

where F is an arbitrary function.

A solid-body rotation is realized when ω is constant for which $F(\psi) = \omega$. and (7.20) is simply written as

$$\nabla^2 \psi = \omega. \quad (7.21)$$

The stream function is given by

$$\psi = \frac{\omega}{4} r^2. \quad (7.22)$$

If ω is confined in a finite circular region of radius R and zero outside the region, that is, the flow outside the circular region is a potential flow

$$\nabla^2 \psi = 0, \quad \text{for } R < r, \quad (7.23)$$

then the stream function is given by

$$\psi = a \ln r. \quad (7.24)$$

Since the inner and outer solutions are continuous at the boundary, we have

$$\psi(r) = \begin{cases} \frac{\omega}{4} r^2, & \text{for } r \leq R, \\ \frac{\omega}{4} R^2 \left(1 + 2 \ln \left(\frac{r}{R} \right) \right), & \text{for } R < r. \end{cases} \quad (7.25)$$

For a linear function $F(x)$, we have multipole solutions as well as monopole solutions. For a localized solution, we may choose

$$F(x) = \begin{cases} -\beta^2 x, & \text{for } r \leq R, \\ \gamma^2 x, & \text{for } R < r. \end{cases} \quad (7.26)$$

Then, (7.20) becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = \begin{pmatrix} -\beta^2 \\ \gamma^2 \end{pmatrix} \psi, \quad \text{for } \begin{pmatrix} r \leq R \\ R < r \end{pmatrix} \quad (7.27)$$

The solution is given by

$$\psi(\mathbf{r}) = \begin{cases} \frac{J_\ell(\beta r)}{J_\ell(\beta R)} \cos(\ell\theta), & \text{for } r \leq R, \\ \frac{K_\ell(\gamma r)}{K_\ell(\gamma R)} \cos(\ell\theta), & \text{for } R < r, \end{cases} \quad (7.28)$$

where β and γ are related through

$$\beta \frac{J_{\ell+1}(\beta R)}{J_\ell(\beta R)} = \gamma \frac{K_{\ell+1}(\gamma R)}{K_\ell(\gamma R)}. \quad (7.29)$$

A multipole vortex surrounded by a potential flow is also constructed in a similar way. A vortex is confined in a region $r \leq R$ whose outside is a potential flow so that

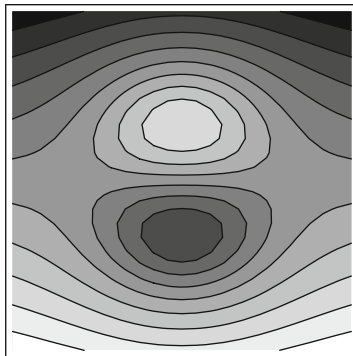


Fig. 7.1 Contour plot of the dipole solution given by (7.32)

$F(x)$ is taken as

$$F(x) = \begin{cases} -\beta^2 x, & \text{for } r \leq R, \\ 0, & \text{for } R < r. \end{cases} \quad (7.30)$$

Then, (7.20) becomes

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = \begin{pmatrix} -\beta^2 \\ 0 \end{pmatrix} \psi, \quad \text{for } \begin{pmatrix} r \leq R \\ R < r \end{pmatrix}. \quad (7.31)$$

The inner and outer solutions are given by

$$\psi(r) = \begin{cases} AJ_1(\beta r) \sin \theta & \text{for } r \leq R, \\ \left(Br + \frac{C}{r} \right) \sin \theta & \text{for } R < r. \end{cases} \quad (7.32)$$

From the conditions for the inner and outer solutions to be continuous at $r = R$, we have

$$J_1(\beta R) = 0, \quad B = \frac{\beta}{2} J_0(\beta R) A, \quad C = -R^2 B, \quad (7.33)$$

where J_0 and J_1 are the Bessel functions of the first kind. The figure of the solution is depicted in Fig. 7.1.

7.2 Point Vortex

When vortical motions are well defined, the vorticity is localized in a small region, suggesting we may introduce a point vortex to approximate a fluid vortex whose vorticity is assumed to be concentrated at the rotation center. One single fluid vortex may correspond to a single point vortex or a collection of many point vortices. The advantage of introducing point vortices is to convert partial differential equations

with an infinite degrees of freedom to a set of ordinary differential equations which is sometime simpler to solve.

As the simplest equation to describe vortical motions, we start with two dimensional Euler equation:

$$\frac{\partial}{\partial t} \nabla^2 \psi + [\psi, \nabla^2 \psi] = 0, \quad (7.34)$$

where

$$\nabla^2 \psi = \omega. \quad (7.35)$$

Now, suppose the vorticity is distributed in space as N point vortices as

$$\omega = \sum_{j=1}^N \gamma_j \delta(\mathbf{r} - \mathbf{r}_j(t)), \quad (7.36)$$

where γ_j is the strength of j -th point vortex and is assumed constant. Then, (7.35) gives an explicit form of the stream function

$$\psi(\mathbf{r}, t) = \sum_{j=1}^N \frac{\gamma_j}{2\pi} \ln |\mathbf{r} - \mathbf{r}_j(t)|. \quad (7.37)$$

Substituting (7.36) and (7.37) into (7.34) to give

$$\sum_{j=1}^N \gamma_j \left\{ \frac{d\mathbf{r}_j}{dt} - \hat{\mathbf{z}} \times \nabla \psi(\mathbf{r}, t) \right\} \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_j(t)) = 0, \quad (7.38)$$

from which, we may have

$$\frac{d}{dt} \mathbf{r}_j = \sum_{\ell \neq j} \gamma_\ell \frac{\mathbf{z} \times (\mathbf{r}_j - \mathbf{r}_\ell)}{|\mathbf{r}_j - \mathbf{r}_\ell|^2}. \quad (7.39)$$

There are three constants of motion:

$$\mathbf{G} = \sum_j \gamma_j \mathbf{r}_j, \quad (7.40)$$

$$L = \sum_j \gamma_j |\mathbf{r}_j|^2, \quad (7.41)$$

$$E = \sum_{j \neq \ell} \gamma_j \gamma_\ell \ln |\mathbf{r}_j - \mathbf{r}_\ell|, \quad (7.42)$$

which are related to the symmetry properties of the system. Because of these conserved quantities, systems with three vortices or less are integrable, while the dynamical behavior of systems with more than three vortices can become chaotic [140, 142].

Equilibrium configurations of point vortices for the two dimensional Euler equation have been discussed by Morikawa and Swenson [99] for the case that N vortices are uniformly distributed on a circle with and without a center vortex and by Campbell and Ziff [100] with patterns consisting of up to 217 vortices. Morikawa and Swenson found that the configuration is locally stable for $2 \leq N \leq 7$ without a center vortex and for $3 \leq N \leq 9$ with a center vortex. Although, as already discussed, the dynamical behavior of systems with more than three vortices can become chaotic, vortex crystallization is observed in electron plasmas whose dynamical behavior is supposed to be described by two dimensional Euler equation. In reality, vortices and fluctuations coexist, and we need a theory of taking into account of relevant degrees of freedom involved in the physical systems.

Statistical properties of a point vortex system are studied by Onsager [101] based that the system is a Hamilton system as is confirmed by

$$\frac{dx_j}{dt} = \frac{1}{\gamma_j} \frac{\partial H}{\partial y_j}, \quad \frac{dy_j}{dt} = -\frac{1}{\gamma_j} \frac{\partial H}{\partial x_j}, \quad (7.43)$$

where

$$H = \sum_{i \neq j} \gamma_i \gamma_j \ln |\mathbf{r}_i - \mathbf{r}_j|. \quad (7.44)$$

Since x and y are canonical conjugate in this system, configuration space is nothing but phase space. This is essentially different from ordinary Hamilton systems where configuration space is a sub-space of phase space. Suppose that the fluid is enclosed by a boundary so that the vortices are confined to an area A . Then, the volume of phase space is given by

$$\int d\Omega = \left(\int dx dy \right)^N = A^N = \text{finite}. \quad (7.45)$$

On the other hand, energy of the system can range from $-\infty$ to ∞ : when two vortices of the same sign come to the same point, the energy becomes ∞ , while when two vortices of the opposite sign come to the same point, the energy becomes $-\infty$. The phase volume which corresponds to energies less than a given value E , $H(x_1, y_1, \dots, x_N, y_N) < E$, is a differentiable function of the energy and is given by

$$\Phi(E) = \int_{H < E} d\Omega = \int_{-\infty}^E \Psi'(E) dE, \quad (7.46)$$

with

$$\Phi(-\infty) = 0, \quad \Phi(\infty) = A^N. \quad (7.47)$$

Since $\Phi(E)$ is a monotonically increasing function of E and levels off for $E \rightarrow \infty$, $\Phi'(E) = \Omega(E)$ cannot increase monotonically and must have a negative slope for a finite E_m , that is, $\Phi'(E)$ is maximum at E_m and $\Omega'(E) < 0$ for $E > E_m$. From the definition of temperature

$$\frac{1}{T} = \frac{\partial S}{\partial E} = k_B \frac{\Omega'(E)}{\Omega(E)} = k_B \frac{\Phi''(E)}{\Phi'(E)}, \quad (7.48)$$

temperature is negative for $E > E_m$. For an equilibrium state characterized by the Boltzmann factor $\exp(-E/k_B T)$, the most realizable state is the state with the highest energy since $T < 0$, which corresponds to the state with the highest order. Therefore, the vortices of the same sign will tend to cluster.

Equation (7.43) is rewritten by using a complex variable $z = x + iy$ as

$$\frac{dz_\alpha}{dt} = \frac{i}{2\pi} \sum_{\beta \neq \alpha} \gamma_\beta \frac{z_\alpha - z_\beta}{|z_\alpha - z_\beta|^2}. \quad (7.49)$$

With the stream function (7.37) in terms of the complex variable

$$\psi(z) = \frac{i}{2\pi} \sum_{\beta} \gamma_\beta \ln |z - z_\beta|, \quad (7.50)$$

(7.49) is expressed by

$$\frac{dz_\alpha}{dt} = \frac{\partial \psi(z_\alpha)}{\partial \bar{z}_\alpha}, \quad (7.51)$$

where $\bar{z}_\alpha$ is the complex conjugate to z_α .

The advantage of using complex variables is that a circle boundary is simply replaced by the mirror images of the vortices inside the boundary [102]. When the boundary C is given by $|z| = R$, the potential is given by

$$\psi_C(z) = \psi(z) + \bar{\psi}\left(\frac{R^2}{z}\right) = \frac{i}{2\pi} \sum_{\beta} \gamma_\beta \left\{ \ln |z - z_\beta| - \ln \left| \frac{R^2}{z} - \bar{z}_\beta \right| \right\}. \quad (7.52)$$

On the circle boundary which is expressed by $\bar{z} = R^2/z$, $\psi_C(z)$ is zero, implying that the boundary is a streamline. If the point z is outside C , the point R^2/z is inside C and vice versa. Thus, $\psi_C(z)$ has the same properties as $\psi(z)$.

7.3 Vortical Motions in Plasmas

7.3.1 Drifts for Driving Vortical Motions

Vortical motions in plasmas are driven by the $\mathbf{E} \times \mathbf{B}$ drift. The $\mathbf{E} \times \mathbf{B}$ drift substituted into the convective term of the continuity equation brings the plasma into a rotation motion. We start from the fluid equations for electrons and ions

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0, \quad (7.53)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla\right) \mathbf{v}_\alpha = \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B}\right) - \frac{1}{n_\alpha m_\alpha} \nabla p_\alpha + \frac{1}{n_\alpha m_\alpha} \nabla \cdot \boldsymbol{\pi} - \sum_\beta \frac{\mathbf{v}_\alpha - \mathbf{v}_\beta}{\tau_{\alpha\beta}}, \quad (7.54)$$

where $\tau_{\alpha\beta}$ is the collision time of particle α against a target particle β and ν_α is the kinematic viscosity. Suppose that the magnetic field is uniform and in the z -direction. The drifts across the magnetic field are given by

$$\mathbf{v}_{e\perp} = \mathbf{v}_E + \mathbf{v}_{eD}, \quad \mathbf{v}_{i\perp} = \mathbf{v}_E + \mathbf{v}_{iD} + \mathbf{v}_p - \frac{1}{n_\alpha m_\alpha \Omega_i} \nabla \cdot \boldsymbol{\pi}, \quad (7.55)$$

where the $\mathbf{E} \times \mathbf{B}$ drift, diamagnetic drift and polarization drift are given by

$$\mathbf{v}_E = \frac{c}{B} \hat{\mathbf{z}} \times \nabla \phi, \quad (7.56)$$

$$\mathbf{v}_{\alpha D} = \frac{v_{T\alpha}^2}{\Omega_\alpha} \hat{\mathbf{z}} \times \nabla \ln n_\alpha, \quad (7.57)$$

$$\mathbf{v}_p = \frac{1}{\Omega_i} \hat{\mathbf{z}} \times \left(\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla\right) \mathbf{u}_\perp, \quad \mathbf{u}_\perp = \mathbf{v}_E + \mathbf{v}_{iD}. \quad (7.58)$$

The polarization drift is an inertia effect and is neglected for the electrons. In a uniform magnetic field, the $\mathbf{E} \times \mathbf{B}$ drift and the diamagnetic flux are divergent free as

$$\nabla \cdot \mathbf{v}_E = 0, \quad \nabla \cdot (n_\alpha \mathbf{v}_{\alpha D}) = 0. \quad (7.59)$$

In an non-uniform or curved magnetic field, the $\mathbf{E} \times \mathbf{B}$ drift and diamagnetic flux are no longer divergent free and are given by

$$\nabla \cdot \mathbf{v}_E = \frac{2c}{B^2} \hat{\mathbf{z}} \cdot (\nabla B \times \nabla \phi), \quad \nabla \cdot (n_\alpha \mathbf{v}_{\alpha D}) = \frac{2c}{e_\alpha B^2} \hat{\mathbf{z}} \cdot (\nabla B \times \nabla p_\alpha). \quad (7.60)$$

Substituting (7.55) into the continuity equation for a homogeneous plasma where $|v_E| \gg |v_{iD}|$, we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla\right) \tilde{n}_e + \frac{\partial}{\partial z} v_{e\parallel} = 0, \quad (7.61)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla\right) (\tilde{n}_i - \rho_s^2 \nabla_\perp^2 \tilde{\phi}) + \rho_s^2 \eta_i \nabla_\perp^4 \tilde{\phi} + \frac{\partial}{\partial z} v_{i\parallel} = 0, \quad (7.62)$$

where $\tilde{n}_\alpha = n_\alpha/n_{\alpha 0} - 1$, $\tilde{\phi} = e\phi/T_e$, and $\rho = C_s/\Omega_i$. Assuming the charge neutrality and subtracting (7.62) from (7.61), we have

$$\left(\frac{\partial}{\partial t} + \frac{C_s^2}{\Omega_i} \hat{\mathbf{z}} \times \nabla \tilde{\phi} \cdot \nabla\right) \nabla_\perp^2 \tilde{\phi} = -\frac{1}{\rho_s^2} \frac{\partial}{\partial z} v_{e\parallel} + \eta_i \nabla^4 \tilde{\phi}, \quad (7.63)$$

where we have assumed that $|v_{e\parallel}| \gg |v_{i\parallel}|$. The parallel velocity $v_{e\parallel}$ is obtained from (7.54) as

$$v_{e\parallel} = v_{Te}^2 \tau_{ei} \frac{\partial}{\partial z} (\tilde{\phi} - \tilde{n}_e), \quad (7.64)$$

Thus (7.61) with (7.64) and (7.63) give a set of equations for vortical motions in a homogeneous plasma. The vector nonlinearity of the left hand side of (7.63) is responsible for exciting vortical motions.

7.3.2 Vortices in Electron Plasmas

Electron plasmas are known to support vortical motions in two dimension. Substituting the electron drift (7.55) into (7.53) with the help of the Poisson equation,

$$\nabla^2 \phi = 4\pi e n, \quad (7.65)$$

we obtain

$$\frac{\partial}{\partial t} \nabla^2 \phi + [\phi, \nabla^2 \phi] = 0, \quad (7.66)$$

where space and time are normalized by a characteristic length λ_D and $|\Omega_e|/\omega_{pe}^2$, and the potential is normalized by T_e/e . This is exactly the same as that for two dimensional Euler equation in fluids. Here, the density is equivalent to the vorticity.

The emergence of self-organized motions has been observed experimentally in electron plasmas by Fine, Case, Flynn, and Driscoll [103] in which an electron plasma is shown to break into electron columns which, depending on the initial conditions, either collapse to a single vortex or self-organize to form a long lived vortex crystal.

Density discretization of an electron plasma is based on the diocotron instability for which the plasma density is required to be hollow [104]–[108]. A fluctuation excited in the electron plasma column which propagates in the azimuthal direction can be enhanced only at the place where the fluctuation resonates with the rotating plasma column. Here, we show that linear eigenfunctions are localized with positive growth rates for hollow density profiles, leading to density discretization which may lead to vortex crystal formation in a nonlinear stage.

We start with (7.66) which describe the motion of two dimensional electron fluid confined in the cylinder of a radius R . Substituting

$$\begin{pmatrix} \phi(r, t) \\ n(r, t) \end{pmatrix} = \begin{pmatrix} \phi_0(r) \\ n_0(r) \end{pmatrix} + \sum_{\ell} \int \frac{d\omega}{2\pi} \begin{pmatrix} \phi_{\ell}(r, \omega) \\ n_{\ell}(r, \omega) \end{pmatrix} e^{i(\ell\theta - \omega t)}, \quad (7.67)$$

into (7.65)–(7.66) and linearizing the resultant equations, an eigenvalue equation now reads

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\ell^2}{r^2} \right) \phi_\ell(r) = \frac{\ell}{r} \frac{dn_0}{dr} \frac{1}{-\omega + \ell\omega_0} \phi_\ell(r), \quad (7.68)$$

where the $\mathbf{E} \times \mathbf{B}$ rotation frequency ω_0 is given by

$$\omega_0 = \frac{1}{r} \frac{d\phi_0}{dr}. \quad (7.69)$$

Multiplying ϕ^* on the both sides of (7.69), integrating with respect to r from the center of the plasma to the edge, and using boundary conditions

$$\phi(0) = \phi(1) = 0, \quad (7.70)$$

then, we have

$$\int_0^1 r \left\{ \left| \frac{d\phi_\ell}{dr} \right|^2 - \left(\frac{\ell^2}{r^2} + \frac{\ell}{r} \frac{dn_0}{dr} \frac{1}{-\omega + \ell\omega_0} \right) |\phi_\ell(r)|^2 \right\} dr = 0. \quad (7.71)$$

For $\omega = \omega_r + i\gamma$, the imaginary part of the above equation is given by

$$\gamma \int_0^1 \frac{dn_0(r)}{dr} \frac{1}{(-\omega + \ell\omega_0(r))^2 + \gamma^2} dr = 0, \quad (7.72)$$

from which the unperturbed density has to have at least one maximum for an instability, indicating that the instability is due to the shear of the azimuthal drift velocity and is called the diocotron instability. Thus, the unperturbed density $n_0(r)$ may be chosen as

$$n_0(r) = A e^{-b(r^2 - a/2b)^2}, \quad (7.73)$$

where A is a normalization factor ($\int_0^1 n_0(r) dr = 1$) given by

$$A = \frac{2}{\pi} \sqrt{\frac{\pi}{b}} \left\{ \operatorname{erf} \left(\sqrt{b} \left(1 - \frac{a}{2b} \right) \right) - \operatorname{erf} \left(-\frac{a}{2\sqrt{b}} \right) \right\},$$

and $\operatorname{erf}(x)$ is an error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

The parameters a and b are chosen so as for the instability to occur. Then, the structure of the unperturbed potential $\phi_0(r)$ is determined from (7.65) as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \phi_0(r) = \hat{A} e^{-b(r^2 - a/2b)^2}, \quad \hat{A} = \frac{\lambda_D^2}{R^2} A, \quad (7.74)$$

which gives

$$\omega_0(r) = \frac{\hat{A}}{4} \sqrt{\frac{\pi}{b}} \frac{1}{r^2} \left[\operatorname{erf} \left(\sqrt{b} \left(r^2 - \frac{a}{2b} \right) \right) - \operatorname{erf} \left(-\frac{a}{\sqrt{b}} \right) \right]. \quad (7.75)$$

The density fluctuation $n_\ell(r)$ is given by

$$n_\ell(r) = \frac{1}{r} \frac{dn_0(r)}{dr} \frac{1}{-\omega + \ell \omega_0(r)} \phi_\ell(r). \quad (7.76)$$

The eigenvalue equation (7.68) is numerically solved by a shooting method. A contour plot of the density fluctuation for the hollow unperturbed density shown in Fig. 7.2 is depicted in Fig. 7.3 for $\ell = 4$ which clearly shows localized structures.

It is not obvious if linear eigenfunctions are observed in laboratory experiments. However, if nonlinear stationary or quasi-stable states are realized, the structures of linear eigenfunctions are somehow anticipated to remain.

7.3.3 Convective Cells

For low frequency fluctuations to which only ions respond, the ion equation of motion is reduced from (7.13) to

$$\frac{\partial}{\partial t} \omega - (\omega \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \omega + \omega (\nabla \cdot \mathbf{v}) = \eta \nabla^2 \omega, \quad (7.77)$$

where the viscosity term is retained and the pressure is assumed to be barotropic. In a two dimensional homogeneous magnetized plasma where the magnetic field is in the z direction, if we consider the $E \times B$ drift for the ion cross-field drift, (7.77) is

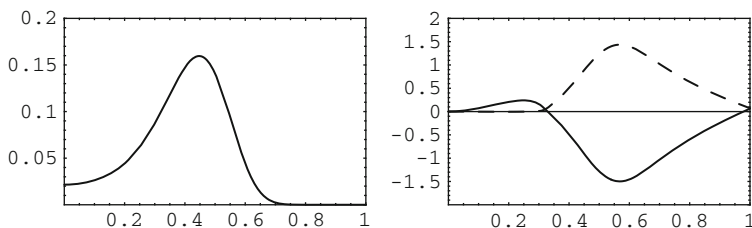


Fig. 7.2 The hollow structure of the unperturbed density ($a = 20$ and $b = 50$) (left) and the eigen function for $\ell = 4$, $\omega_r = 0.113$, and $\gamma = 0.002$ (right)

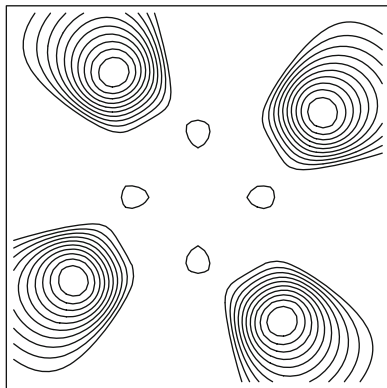


Fig. 7.3 The contour plot of the density fluctuation for $\ell = 4$, $\omega_r = 0.113$, and $\gamma = 0.002$

reduced to

$$\frac{\partial}{\partial t} \nabla_{\perp}^2 \phi + [\phi, \nabla_{\perp}^2 \phi] = \eta \nabla_{\perp}^4 \phi, \quad (7.78)$$

where the potential is normalized by T_e/e and the space and time are normalized by $\rho_s = C_s/\Omega_i$ and $1/\Omega_i$. This is a standard convective cell mode whose linear wave is damped according to the dispersion relation $\omega = -i\eta k_{\perp}^2$. Therefore, the convective cells are a secondary mode which is driven by other linearly unstable modes. When the convective cells are excited the diffusion is estimated by

$$D \propto \frac{\rho_s^2}{1/\Omega_i} = \frac{cT_e}{eB}, \quad (7.79)$$

which is enhanced compared with the classical diffusion r_L^2/τ . Thus, the convective cell modes are supposed to be important for the plasma transport.

7.3.4 Drift Wave Vortices

Nonlinear dynamics of drift waves has been subject to intensive studies since these waves are regarded as responsible for anomalous transport in plasmas. Hasegawa and Mima [109] derived a model equation to explain the high level of density fluctuations and the broad frequency spectrum observed in a tokamak [110]. Since then, many properties of drift wave turbulence have been revealed: the spectrum evolution is characterized by an inverse cascade [124], the wavenumber spectrum obeys the Kolmogorov–Kraichnan law k^{-3} [112, 113], the broad frequency spectrum is demonstrated in several ways based on such a soliton gas model [114, 115] and a truncated mode-coupling model exhibiting chaos [116, 117], the saturation in an unstable system is initiated by the $\mathbf{E} \times \mathbf{B}$ drift [118], etc.

In addition to the studies of the drift wave turbulence, coherent structures have been shown inherent to the Hasegawa–Mima equation. Larichev and Reznik [120] found vortex-pair solutions in the Rossby wave equation which is equivalent to the Hasegawa–Mima equation. These dipole vortices have been shown to be fairly stable against collisions and perturbations [128], although in the strict sense they are not as stable as solitons. Thus, the Hasegawa–Mima equation is anticipated to link strong turbulence to self-organized motions. Similar phenomena of self-organization have been demonstrated in two-dimensional hydrodynamic turbulence [121–125] and in magnetohydrodynamic turbulence [126, 127].

Suppose that the density gradient is in the x -direction with constant $(d/dx) \ln n_0$. Substituting (7.55) into (7.53) for the ions, we obtain

$$\frac{\partial}{\partial t}(1 - \rho_s^2 \nabla_\perp^2)\phi + v_{De} \frac{\partial}{\partial y}\phi - \Omega_i \rho_s^4 (\mathbf{z} \times \nabla\phi) \cdot \nabla \nabla_\perp^2 \phi = 0, \quad (7.80)$$

where the potential is normalized by T_e/e and $\rho_s = C_s/\Omega_i$. In deriving (7.80), we have assumed that the electron density obeys the Boltzmann distribution $n_e = n_0 e^{e\phi/T_e}$ and the charge neutrality $n_i = n_e = n$. If we normalize the space and time by ρ_s and $1/\Omega_i$, (7.80) is reduced to

$$\frac{\partial}{\partial t}(1 - \nabla_\perp^2)\phi + v_* \frac{\partial}{\partial y}\phi - \{\phi, \nabla_\perp^2 \phi\} = 0, \quad (7.81)$$

where $v_* = v_{De}/C_s$. Equation (7.80) is derived by Hasegawa and Mima [109] and is equivalent to the Rossby wave equation.

Here, again the energy E and enstrophy W are conserved:

$$E = \frac{1}{2} \int \{\phi^2 + (\nabla_\perp \phi)^2\} d\mathbf{x}, \quad W = \frac{1}{2} \int \{(\nabla_\perp^2 \phi)^2 + (\nabla_\perp \phi)^2\} d\mathbf{x}. \quad (7.82)$$

which are obtained by multiplying ϕ and $\nabla_\perp^2 \phi$ to (7.81) and rewriting as

$$\frac{\partial}{\partial t} \{\phi^2 + (\nabla_\perp \phi)^2\} = 2\nabla_\perp \cdot (\phi \frac{\partial}{\partial t} \nabla_\perp \phi) - \frac{\partial \phi^2}{\partial y} + \hat{\mathbf{z}} \cdot \nabla_\perp \times (\phi^2 \nabla_\perp \nabla_\perp^2 \phi),$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \{(\nabla_\perp^2 \phi)^2 + (\nabla_\perp \phi)^2\} &= 2\nabla_\perp \cdot \left\{ (\nabla_\perp \phi) \left(\frac{\partial \phi}{\partial t} + v_* \frac{\partial \phi}{\partial y} \right) \right\} - v_* \frac{\partial}{\partial y} (\nabla_\perp \phi)^2 \\ &\quad - \hat{\mathbf{z}} \cdot \nabla_\perp \times ((\nabla_\perp^2 \phi)^2 \nabla_\perp \phi). \end{aligned}$$

7.3.4.1 Dipole Vortex Solutions

Equation (7.81) is reduced to if we look for a stationary solution in a form of $\phi(\mathbf{r}, t) = \phi(\sqrt{x^2 + (y - ct)^2})$,

$$\{\nabla_{\perp}^2 \phi - \phi - v_* x, \phi + cx\} = 0. \quad (7.83)$$

The solution to (7.83) is given by

$$\nabla_{\perp}^2 \phi - \phi - v_* x = F(\phi + cx), \quad (7.84)$$

where F is an arbitrary function. Because of x in (7.84), there is no monopole solution. Therefore, we look for localized dipole solutions so that $\phi \rightarrow 0$ for $r \rightarrow \infty$. The form of F is determined from (7.84) in $x \rightarrow \infty$ as

$$F(x) = -\frac{v_*}{c}x. \quad (7.85)$$

Then, (7.84) becomes

$$\nabla_{\perp}^2 \phi - \left(1 - \frac{v_*}{c}\right)\phi = 0, \quad (7.86)$$

whose solution diminishes at infinity only when $(c/v_*)(c/v_* - 1) > 0$ and is given by

$$\phi(r, \theta) = AK_1(\beta r) \cos \theta, \quad \beta = \sqrt{1 - \frac{v_*}{c}}. \quad (7.87)$$

For the inner solution to be finite at $r = 0$, we may choose F as

$$F(x) = -(1 + \gamma^2)x. \quad (7.88)$$

Then, (7.63) becomes

$$\nabla_{\perp}^2 \phi + \gamma^2 \phi + c(\beta^2 + \gamma^2)x = 0, \quad (7.89)$$

whose solution is given by

$$\phi(r, \theta) = \{Br + CJ_1(\gamma r)\} \cos \theta. \quad (7.90)$$

From the condition that the inner and outer solutions are continuous at $r = R$, we have

$$A = -\frac{cR}{K_1(\beta R)}, \quad B = -c \left(1 + \frac{\beta^2}{\gamma^2}\right), \quad C = \frac{\beta^2}{\gamma^2} \frac{cR}{J_1(\gamma R)},$$

and the parameter γ is determined by β and R by the relation

$$\frac{K_2(\beta R)}{K_1(\beta R)} = -\frac{\beta}{\gamma} \frac{J_2(\gamma R)}{J_1(\gamma R)}. \quad (7.91)$$

Thus, the solution is given by

$$\phi(r, \theta) = \begin{cases} c \left\{ - \left(1 + \frac{\beta^2}{\gamma^2} \right) \frac{r}{R} + \frac{\beta^2}{\gamma^2} \frac{J_1(\gamma r)}{J_1(\gamma R)} \right\} R \cos \theta, & \text{for } r \leq R, \\ -c \frac{K_1(\beta r)}{K_1(\beta R)} R \cos \theta, & \text{for } R < r. \end{cases} \quad (7.92)$$

This solution is first given by Larichev and Reznik [120] for the nonlinear Rossby equation.

For fixed β , there are discrete but an infinite number of γ . The dipole vortex is realized for the smallest γ . This is a ground state. As γ becomes larger, the number of nodes is increased and the dipole vortices dress dresses equal to the number of nodes. These are supposed to be excited states. The dipoles in the ground state and in the second excited state are depicted in Fig. 7.4.

The ground state dipole vortex propagates in the y direction without any change if the symmetry axis is placed on the y axis, while the excited state dipole vortices propagate leaving the dresses behind to relax to a ground state dipole vortex even if their symmetry axes are placed on the y axis which is shown in Fig. 7.5 [121].

If the ground state dipole vortex is placed tilted with a finite angle between the symmetry axis and the y axis, it starts to propagate along an oscillating orbit around the y axis [128]. This is because the vortex is pulled to the y axis by the force proportional to x in (7.84) when it deviates from the y axis. In Fig. 7.6, the propagation of the ground state dipole vortex is shown when (left) the symmetry axis is placed on the y axis and (right) the symmetry axis is tilted. The ground state dipole vortices are stable against collision if the impact parameter is zero. When two ground state dipole vortices are placed initially in a way that their symmetry axes are crossed with a finite angle or parallel, they emit wakes and become coalescent to a monopole like vortex to stay rather long although a monopole is not a stationary solution.

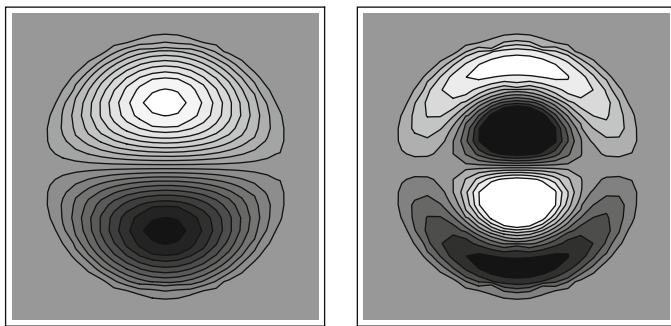


Fig. 7.4 Ground (left: $\gamma = 3.82$) and excited (right: $\gamma = 7$) state dipole solutions of (7.92) for $\beta = 4$. The horizontal axis is y

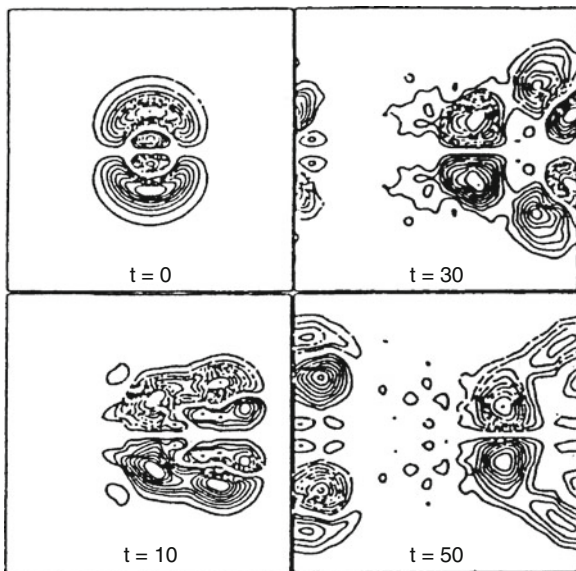


Fig. 7.5 Second excited state dipole is undressed as it propagates

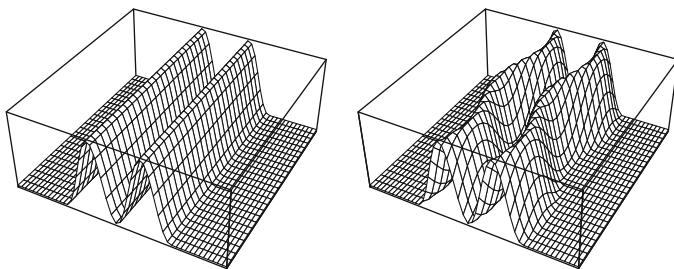


Fig. 7.6 Propagation of the squared dipole vortex amplitude: (left) initially not tilted, (right) initially tilted

7.3.5 Particle Transport Due to Drift Wave Vortices

Vortex excitation is supposed to enhance particle transport. Hasegawa and Wakatani [111] derived a set of equations to describe particle transport under the action of drift wave vortices which develop by the dissipative drift wave instability. The drift wave instability needs collisions which prevent axial electrons from canceling the charge separation.

We start from (7.53) and (7.54) together with (7.55)–(7.59). Assuming the electron parallel velocity is much larger than the ion parallel velocity we may have from (7.54)

$$v_{e\parallel} = v_{Te}^2 \tau_{ei} \frac{\partial}{\partial z} (\phi - \ln n_e), \quad (7.93)$$

where the potential is normalized by T_e/e and the electron viscosity is neglected. Substituting the electron cross field drifts (7.55) and the electron parallel velocity into the electron continuity equation (7.53), we obtain an equation for the particle transport

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \ln n_e = -v_{Te}^2 \tau_{ei} \frac{\partial^2}{\partial z^2} (\phi - \ln n_e). \quad (7.94)$$

In a similar way substituting the ion cross field drift (7.58) into the ion continuity equation, we have

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right\} (\ln n_i - \rho_s^2 \nabla_\perp^2 \phi) = -\rho_s^2 \eta_i \nabla^2 \nabla_\perp^2 \phi. \quad (7.95)$$

Assuming the charge neutrality $n_e = n_i = n$ and subtracting (7.95) from (7.94), we have

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right\} \nabla_\perp^2 \phi = -\frac{v_{Te}^2 \tau_{ei}}{\rho_s^2} \frac{\partial^2}{\partial z^2} (\phi - \ln n) + \eta_i \nabla^2 \nabla_\perp^2 \phi. \quad (7.96)$$

Equations (7.94) and (7.96) are to give a set of equations to describe the particle transport under the collisional drift wave instability and are rewritten with the normalization of space and time by ρ_s and $1/\Omega_i$

$$\left\{ \frac{\partial}{\partial t} + \mathbf{z} \times \nabla \phi \cdot \nabla \right\} \ln n = -\frac{v_{Te}^2 \tau_{ei}}{\Omega_i \rho_s^2} \frac{\partial^2}{\partial z^2} (\phi - \ln n). \quad (7.97)$$

$$\left\{ \frac{\partial}{\partial t} + \mathbf{z} \times \nabla \phi \cdot \nabla \right\} \nabla_\perp^2 \phi = -\frac{v_{Te}^2 \tau_{ei}}{\Omega_i \rho_s^2} \frac{\partial^2}{\partial z^2} (\phi - \ln n) + \frac{\eta_i}{\Omega_i \rho_s^2} \nabla^2 \nabla_\perp^2 \phi. \quad (7.98)$$

Note that using the Boltzmann distribution for the density $n = n_0 e^\phi \simeq n_0 (1 + \phi)$ and ignoring the viscosity term, (7.95) is simply reduced to the Hasegawa–Mima equation.

7.3.6 Ion Heat Transport Due to Drift Wave Vortices

Ion heat transport has been widely accepted to be controlled by ion temperature gradient modes. The ion temperature gradient mode is due to the coupling between the drift mode and ion acoustic mode for which ion pressure parallel to the magnetic field is taken into account. The continuity equation is given by

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \left(\ln n_i - \frac{C_s^2}{\Omega_i^2} \nabla_\perp^2 \phi \right) + \mathbf{v}_{iB} \cdot \left(\nabla \phi + \frac{T_i}{T_e} \nabla \ln p_i \right) \\ & + \eta_i \frac{C_s^2}{\Omega_i^2} \nabla_\perp^4 \phi + \frac{\partial}{\partial z} v_{iz} = 0, \end{aligned} \quad (7.99)$$

where

$$\mathbf{v}_{iB} = \frac{2C_s^2}{\Omega_i} \hat{\mathbf{z}} \times \nabla \ln B.$$

The equation of motion for the parallel velocity is given by

$$\frac{\partial v_{iz}}{\partial t} + \mathbf{v}_E \cdot \nabla v_{iz} = -\frac{e}{m_i} \frac{\partial \phi}{\partial z} - \frac{1}{m_i n_0} \frac{\partial p_i}{\partial z}. \quad (7.100)$$

The ion pressure is given by the heat balance equation

$$\frac{3}{2} \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) p_i - \frac{5}{2} T_i \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) n_i + \nabla \cdot \mathbf{q}_i = 0, \quad (7.101)$$

where $\mathbf{q}_i$ is the cross field ion heat flux given by Braginskii [58] as

$$\mathbf{q}_i = -(\chi_{\parallel} \nabla_{\parallel} + \chi_{\perp} \nabla_{\perp}) T_i + \frac{5}{2} \frac{p_i}{m_i \Omega_i} \hat{\mathbf{z}} \times \nabla T_i. \quad (7.102)$$

$$\nabla \cdot \mathbf{q}_i = -(\chi_{\parallel} \nabla_{\parallel}^2 + \chi_{\perp} \nabla_{\perp}^2) T_i - \frac{5}{2} \frac{p_i}{m_i \Omega_i B} \left(\hat{\mathbf{z}} \times \nabla \ln \frac{p_i}{B^2} \right) \cdot \nabla T_i. \quad (7.103)$$

Equations (7.99)–(7.101) together with the charge neutrality and the adiabatic response of the electron to the potential provide a closed set of equations to describe the ion heat transport driven by the drift wave turbulence.

7.3.7 Zonal Flow

A zonal flow has been intensively studied under the concerns related to a transport barrier which is widely accepted to control confinement. A zonal flow is a secondary shear flow driven by, for example, drift waves. The shear flow deforms and tears the drift wave potential structures into fragments, affecting plasma transport and confinement.

The drift waves are assumed to have a small scale and the secondary flow has a large scale. We start with the Hasegawa–Wakatani equations (7.97) and (7.98) for the drift waves and the equation for the convective cell (7.78) for the secondary flow.

Expressing the potential fluctuations as a sum of the large scale fluctuations ϕ_0 and the small scale fluctuations $\tilde{\phi}$ as

$$\phi = \phi_0 + \tilde{\phi}. \quad (7.104)$$

and substituting it into (7.97) and (7.98), we have for the drift waves

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla \right) (1 - \nabla_{\perp}^2) \tilde{\phi} + \mathbf{v}_* \cdot \nabla \tilde{\phi} - \tilde{\mathbf{v}}_E \cdot \nabla \nabla_{\perp}^2 \phi_0 = 0, \quad (7.105)$$

where $\mathbf{v}_0 = \hat{\mathbf{z}} \times \nabla \phi_0$, $\mathbf{v}_* = -\hat{\mathbf{z}} \times \nabla \ln n_0$, $\tilde{\mathbf{v}}_E = \hat{\mathbf{z}} \times \nabla \tilde{\phi}$ and we have neglected the viscosity term and used $\ln(n_e/n_0) = \phi$. For the secondary flow, we have from (7.78)

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla_{\perp} \right) \nabla_{\perp}^2 \phi_0 = -\langle \tilde{\mathbf{v}}_E \cdot \nabla_{\perp} \nabla_{\perp}^2 \tilde{\phi} \rangle, \quad (7.106)$$

where we have again omitted the viscosity term. Noting $\nabla_{\perp}^2 \tilde{\phi} = \hat{\mathbf{z}} \cdot \nabla \times \tilde{\mathbf{v}}_E = \tilde{\omega}$, (7.106) is rewritten as

$$\hat{\mathbf{z}} \cdot \nabla \times \left\{ \frac{\partial}{\partial t} \mathbf{v}_0 - \langle \tilde{\mathbf{v}}_E \times \tilde{\omega} \hat{\mathbf{z}} \rangle \right\} = 0, \quad (7.107)$$

which gives

$$\frac{\partial}{\partial t} \mathbf{v}_0 = \langle \tilde{\mathbf{v}}_E \times \tilde{\omega} \hat{\mathbf{z}} \rangle + \nabla w. \quad (7.108)$$

In the following, we put $w = 0$.

Fourier transforming $\tilde{\phi}$ in space as

$$\tilde{\phi} = \sum_{\mathbf{k}} \tilde{\phi}(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \phi_0 = \sum_{\mathbf{K}} \phi_0(\mathbf{K}, t) e^{i\mathbf{K} \cdot \mathbf{x}},$$

where $\mathbf{k}$ is for small scale and $\mathbf{K}$ for large scale, (7.105) becomes

$$i \frac{\partial}{\partial t} (1 + k_{\perp}^2) \tilde{\phi}(\mathbf{k}, t) = \sum_{\mathbf{K}} \mathbf{k} \cdot \{ \mathbf{v}_*(\mathbf{K}) - \mathbf{v}_0(\mathbf{K}) (1 + (\mathbf{k} - \mathbf{K})_{\perp}^2 - K_{\perp}^2) \} \tilde{\phi}(\mathbf{k} - \mathbf{K}). \quad (7.109)$$

If we introduce $\psi(\mathbf{k}, t) = (1 + k_{\perp}^2) \tilde{\phi}(\mathbf{k}, t)$, (7.109) is rewritten as

$$\frac{\partial}{\partial t} \psi(\mathbf{k}) = -i(\omega_*(\mathbf{K}) - \mathbf{k} \cdot \mathbf{v}_0(\mathbf{K})) \psi(\mathbf{k} - \mathbf{K}), \quad (7.110)$$

where $\omega_* = \mathbf{k} \cdot \mathbf{v}_*/(1 + k_{\perp}^2)$ and we have used $(\mathbf{k} - \mathbf{K})_{\perp}^2 \gg K_{\perp}^2$. From (7.110), we have

$$\begin{aligned} \frac{\partial}{\partial t} \langle \psi(\mathbf{k}) \psi(\mathbf{k}') \rangle &= -i \sum_{\mathbf{K}} \{ (\omega_*(\mathbf{K}) - \mathbf{k} \cdot \mathbf{v}_0(\mathbf{K})) \langle \psi(\mathbf{k} - \mathbf{K}) \psi(\mathbf{k}') \rangle \\ &\quad + (\omega_*(\mathbf{K}) - \mathbf{k}' \cdot \mathbf{v}_0(\mathbf{K})) \langle \psi(\mathbf{k}' - \mathbf{K}) \psi(\mathbf{k}) \rangle \}. \end{aligned} \quad (7.111)$$

Here, the correlation function is given by (5.154) as

$$\langle \psi(\mathbf{k}, t) \psi(\mathbf{k}', t) \rangle = I \left(\frac{\mathbf{k} - \mathbf{k}'}{2}, \mathbf{k} + \mathbf{k}', t \right) = \sum_{\mathbf{K}} I \left(\mathbf{k} - \frac{\mathbf{K}}{2}, \mathbf{K}, t \right) \delta(\mathbf{k} + \mathbf{k}' - \mathbf{K}). \quad (7.112)$$

Then, (7.111) is inverse Fourier transformed by multiplying $e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}$ and summing over $\mathbf{k}'$ to yield

$$\frac{\partial}{\partial t} I(\mathbf{k}, \mathbf{x}, t) + \left(\frac{d\omega_*}{d\mathbf{k}} + \mathbf{v}_0 \right) \cdot \nabla I(\mathbf{k}, \mathbf{x}, t) - \nabla(\omega_* + \mathbf{k} \cdot \mathbf{v}_0) \cdot \frac{\partial}{\partial \mathbf{k}} I(\mathbf{k}, \mathbf{x}, t) = 0. \quad (7.113)$$

In the similar way from (7.108), we have

$$\frac{\partial}{\partial t} \mathbf{v}_0 = \nabla \cdot \sum_{\mathbf{k}} 2 \frac{\mathbf{k} : \hat{\mathbf{z}} \times \mathbf{k}}{(1 + k^2)^2} I(\mathbf{k}, \mathbf{x}, t). \quad (7.114)$$

Equations (7.113) and (7.114) are a closed set of equations describing the interaction between the zonal flow and the drift wave turbulence. The zonal flow is driven by the drift wave turbulence which in turn is convected by the zonal flow to be fragmented.

The above derivation is the skeleton of the mathematical structure of the zonal flow and the drift wave turbulence within the frame work of the weak turbulence theory and the quasi-linear feedback interaction. A large amount of works have been devoted to studies of the zonal flow in connection with the plasma transport by taking many other effects which are not included above.

7.3.8 Self-Organization of Monopole Vortices in Temperature Inhomogeneity

Monopoles are more prevailing in nature. A monopole described by the Euler equation cannot propagate by itself. For the planetary atmospheric motion, the pressure gradient with the Coriolis force induces geophysical flow which gives a scalar nonlinearity as well as a vector nonlinearity as

$$\frac{\partial}{\partial t} (\nabla^2 - 1)h + (\nabla \ln f) \cdot (\hat{\mathbf{z}} \times \nabla) \left(h + \frac{h^2}{2} \right) + \{h, \nabla^2 h\} = 0, \quad (7.115)$$

where h is the deviation from the free surface and f is the Coriolis parameter. This is called a generalized Charney–Obukov equation [129, 130].

In a magnetized plasma if the temperature is inhomogeneous, then the Hasegawa–Mima equation is reduced to (7.115). The ion cross field drift is substituted into the ion equation of continuity to give

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) (\ln n_i - \rho_{s0}^2 \nabla_{\perp}^2 \phi) = 0, \quad (7.116)$$

where the potential is normalized by T_{e0}/e , T_{e0} is the electron temperature measured, for example, at the center, and $\rho_{s0} = \sqrt{T_{e0}/m_i/\Omega_i}$. Equation (7.116) is the

same as (7.95). Here, invoking the charge neutrality and the electron adiabaticity $n_e(\mathbf{x}, t) = n_0 \exp\{e\phi(\mathbf{x}, t)/T_e(\mathbf{x})\}$, we have

$$\left(\frac{\partial}{\partial t} + \hat{\mathbf{z}} \times \nabla \phi \cdot \nabla\right) (\phi - \nabla^2 \phi) + \mathbf{v}_* \cdot \nabla \phi + \frac{1}{2} \mathbf{v}_{T_e} \cdot \nabla \phi^2 = 0, \quad (7.117)$$

where the space and time are normalized by ρ_{s0} and $1/\Omega_i$, and

$$\mathbf{v}_* = -\hat{\mathbf{z}} \times \nabla \ln n_0, \quad \mathbf{v}_{T_e} = -\hat{\mathbf{z}} \times \nabla \ln T_e = \kappa \mathbf{v}_*, \quad \kappa = \frac{\partial \ln T_e}{\partial \ln n_0}.$$

In deriving (7.117), we have used an approximation $T_e/T_{e0} \sim 1$.

In (7.117), the scalar nonlinearity $\mathbf{v}_{T_e} \cdot \nabla \phi^2/2$ gives a solitary structure of the K-dV type (a monopole vortex) and a vector nonlinearity $[\phi, \nabla^2 \phi]$ describes a vortical structure. If the scale lengths of the structure and the temperature gradient are denoted by ℓ and L_{T_e} , respectively, the order of magnitudes of these two terms are estimated as

$$\mathbf{v}_{T_e} \cdot \nabla \phi^2/2 \sim \phi^2/L_{T_e}\ell, \quad \{\phi, \nabla^2 \phi\} \sim \phi^2/\ell^4.$$

When the following inequality holds,

$$\ell > L_{T_e}^{1/3}, \quad (7.118)$$

the scalar nonlinearity dominates over the vector nonlinearity, indicating the monopole type of vortices are excited.

When the large scale of vortices characterized by (7.118) are dominated, a long wave approximation may be used to (7.117) to give

$$\frac{\partial}{\partial t} \phi + \mathbf{v}_* \cdot \nabla \left(\phi + \nabla^2 \phi + \frac{1}{2} \kappa \phi^2 \right) - \{\phi, \nabla^2 \phi\} = 0. \quad (7.119)$$

The core part of the monopole vortex may be described by a localized function F as

$$\phi(\mathbf{x}, t) = \sum_{\alpha} \gamma_{\alpha} F_{\alpha}(|\mathbf{x} - \mathbf{x}_{\alpha}(t)|), \quad (7.120)$$

which is substituted into (7.119) to give

$$\begin{aligned} \sum_{\alpha} \gamma_{\alpha} \frac{\mathbf{x} - \mathbf{x}_{\alpha}}{|\mathbf{x} - \mathbf{x}_{\alpha}|} F'_{\alpha} \left\{ -\frac{d\mathbf{x}_{\alpha}}{dt} + \sum_{\beta} \gamma_{\beta} \frac{\hat{\mathbf{z}} \times (\mathbf{x} - \mathbf{x}_{\alpha})}{|\mathbf{x} - \mathbf{x}_{\alpha}|} F''_{\beta} \right. \\ \left. + \hat{\mathbf{z}} \times \mathbf{v}_* \left(1 + \frac{F'''_{\alpha}}{F'_{\alpha}} + \kappa \sum_{\beta} \gamma_{\beta} F_{\beta} \right) \right\} = 0. \end{aligned} \quad (7.121)$$

Here, the unknown function F is determined so as for the nonlinearity to balance with the dispersion,

$$(1 - \lambda_\alpha)F'_\alpha + F'''_\alpha + \kappa F_\alpha F'_\alpha = 0, \quad (7.122)$$

which is solved to give

$$F_\alpha(|\mathbf{x} - \mathbf{x}_\alpha|) = A_\alpha \operatorname{sech}^2 \left(\sqrt{\frac{\gamma_\alpha A_\alpha}{12}} |\mathbf{x} - \mathbf{x}_\alpha| \right), \quad \lambda_\alpha = 1 + \frac{\kappa}{3} \gamma_\alpha A_\alpha, \quad (7.123)$$

where λ is a parameter to represent the spatial extent of the localized function of F . Then, the motion of the center of monopole vortex is described by

$$\frac{d\mathbf{x}_\alpha}{dt} = \sum_\beta \gamma_\beta \frac{\hat{\mathbf{z}} \times (\mathbf{x}_\alpha - \mathbf{x}_\beta)}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|} F'''_\beta + \hat{\mathbf{z}} \times \mathbf{v}_* \left(\lambda_\alpha + \kappa \sum_{\beta \neq \alpha} \gamma_\beta F_\beta(|\mathbf{x}_\alpha - \mathbf{x}_\beta|) \right). \quad (7.124)$$

The monopole vortices are subject to chaotic motion as the Euler vortices exhibit chaotic behavior when the number of vortices are larger than 3. In order to integrate the microscopic chaotic motions into the macroscopic ordered motion, we consider a Langevin equation by introducing fluctuations $\tilde{f}(t)$ and dissipation ν

$$\begin{aligned} \frac{d\mathbf{x}_\alpha}{dt} = & \sum_\beta \gamma_\beta \frac{\hat{\mathbf{z}} \times (\mathbf{x}_\alpha - \mathbf{x}_\beta)}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|} F'''_\beta(|\mathbf{x}_\alpha - \mathbf{x}_\beta|) \\ & + \left(\lambda_\alpha + \kappa \sum_{\beta \neq \alpha} \gamma_\beta F_\beta(|\mathbf{x}_\alpha - \mathbf{x}_\beta|) \right) \hat{\mathbf{z}} \times \mathbf{v}_* + \tilde{f}_\alpha(t) - \nu \mathbf{r}_\alpha. \end{aligned} \quad (7.125)$$

Here, the fluctuations $\tilde{f}(t)$ is assumed to obey the Gaussian process, that is,

$$\langle \tilde{f}_\alpha(t) \rangle = 0, \quad (7.126)$$

$$\langle \tilde{f}_\alpha(t) \tilde{f}_\beta(t') \rangle = \Gamma \delta_{\alpha\beta} \delta(t - t'). \quad (7.127)$$

Since the position of the vortices are agitating because of the fluctuations, the vortex position is divided into two part, the fluctuating part and the average part:

$$\mathbf{x}_\alpha = \mathbf{R}_\alpha + \tilde{\mathbf{r}}_\alpha, \quad \langle \tilde{\mathbf{r}}_\alpha \rangle = 0, \quad (7.128)$$

where $\tilde{\mathbf{r}}_\alpha$ is a stochastic variable. The fluctuating part of the orbit is expressed in terms of the Green function $G_{\alpha\beta}(t|t')$ of (7.125),

$$\tilde{\mathbf{r}}_\alpha = \sum_\beta \int G_{\alpha\beta}(t|t') \tilde{f}_\beta(t').$$

The Green function may be approximated by

$$G_{\alpha\beta}(t|t') \simeq \delta_{\alpha\beta} e^{-2\nu(t-t')}.$$

Substituting (7.128) into (7.125) and averaging the resultant equation over the fluctuations, we have

$$\begin{aligned} \frac{d\mathbf{R}_\alpha}{dt} = & \sum_{\beta} \kappa_{\beta} \frac{\hat{\mathbf{z}} \times (\mathbf{R}_\alpha - \mathbf{R}_\beta)}{|\mathbf{R}_\alpha - \mathbf{R}_\beta|} \left(1 + \frac{D_{\alpha\beta}}{|\mathbf{R}_\alpha - \mathbf{R}_\beta|^2} \right) F_{\beta}'''(|\mathbf{R}_\alpha - \mathbf{R}_\beta|) \\ & + \left(\lambda_{\alpha} + \kappa \sum_{\beta \neq \alpha} \kappa_{\beta} F_{\beta}(|\mathbf{R}_\alpha - \mathbf{R}_\beta|) \right) \hat{\mathbf{z}} \times \mathbf{v}_* - \nu \mathbf{R}_\alpha, \end{aligned} \quad (7.129)$$

where

$$D_{\alpha\beta}(t) = \frac{1}{2} \langle (\tilde{\mathbf{r}}_\alpha - \tilde{\mathbf{r}}_\beta)^2 \rangle \simeq \Gamma \frac{1 - e^{-2\nu t}}{2\nu} (1 - \delta_{\alpha\beta}), \quad (7.130)$$

and we have assumed that $\tilde{x}_\alpha$ and $\tilde{y}_\alpha$ are independent. The time evolution of the solution obtained by solving (7.129) in a cylindrical system ($\mathbf{v}_* = \nu_* \mathbf{R}/R$) is shown in Fig. 7.7, where the initial configuration is determined by a random number generator. Vortices move away from each other because of the diffusion caused by the fluctuations and find their equilibrium positions (fixed points) which are the edge of the range of the interaction force. The dissipation guarantees that the fixed points are stable.

7.4 Collisional Drift Wave Instability and Formation of Dipole Vortices

Dipole vortices described by the Hasegawa–Mima equation are shown to be stable against collisions and perturbations almost like solitons. A question arises if the dipole vortices are survived even in an unstable system. There are several causes to

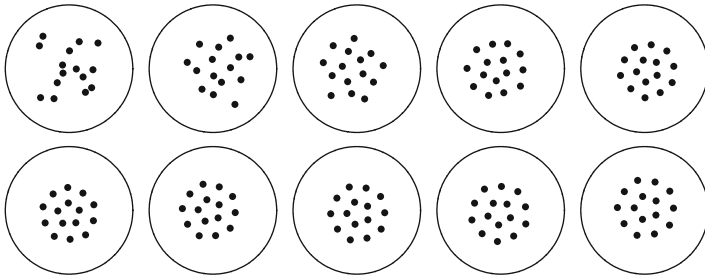


Fig. 7.7 An effective large vortex consisting of point vortices which self-organized themselves and revolve both on the center of mass and around the cylinder axis

destabilize the drift wave. Here, we consider a collision as a source of the instability. Thus, we add a collision term to (7.54) for the instability:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \right) \mathbf{v}_\alpha = \frac{e_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_e \times \mathbf{B} \right) - \frac{1}{n_\alpha m_\alpha} (\nabla p_\alpha - \nabla \cdot \boldsymbol{\pi}_\alpha) - \frac{\mathbf{v}_\alpha - \mathbf{v}_\beta}{\tau_{\alpha\beta}}, \quad (7.131)$$

where the ion viscosity is introduced for the growth rate of the instability to have a finite bandwidth in wavenumber space. The electron viscosity is neglected because of its smallness. The magnetic field is constant and the density gradient is in the x direction and $L_n^{-1} = (d/dx) \ln n_0$ is constant. In the following, we consider the case where both electron and ion temperature are homogeneous and electrostatic waves propagate almost perpendicular to the magnetic field ($k_\perp \gg k_\parallel$). Then, the $\mathbf{E} \times \mathbf{B}$ and diamagnetic drifts are given by (7.56) and (7.57), while the polarization drift is given by

$$\mathbf{v}_p = \frac{1}{\Omega_i} \mathbf{z} \times \left\{ \left(\frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla_\perp \right) \mathbf{u}_i - \frac{1}{n_i m_i} \nabla \pi_i \right\}, \quad \mathbf{u}_i = \mathbf{v}_E + \mathbf{v}_{iD}. \quad (7.132)$$

Note that the ion viscosity in a strongly magnetized plasma is expressed by sum of the convective derivative term with $\mathbf{v}_{iD}$ cancels a part of the ion viscosity term for the strongly magnetized plasma but

$$\nabla \cdot (n_i \mathbf{v}_p) \simeq -\frac{n_i C_s^2}{\Omega_i^2} \left\{ \frac{\partial}{\partial t} - \eta_i \nabla^2 + \frac{C_s^2}{\Omega_i} \left(\hat{\mathbf{z}} \times \nabla \frac{e\phi}{T_e} \right) \cdot \nabla \right\} \nabla_\perp^2 \frac{e\phi}{T_e}, \quad (7.133)$$

where $\eta_i = v_{Ti}^2 \tau_i$ and the scale length of the density inhomogeneity is larger than that of the drift wave.

Since the electron inertia is so small, the electron parallel velocity may be approximated as

$$v_{e\parallel} \simeq D_\parallel \nabla_\parallel \left(\frac{e\phi}{T_e} - \ln n_e \right), \quad (7.134)$$

where $D_\parallel = T_e \tau_{ei} / m_e$. Then, the electron equation of continuity becomes

$$\frac{\partial}{\partial t} n_e + (\mathbf{v}_E \cdot \nabla_\perp) n_0 + n_0 D_\parallel \nabla_\parallel^2 \left(\frac{e\phi}{T_e} - \frac{\tilde{n}_e}{n_0} \right) = 0, \quad (7.135)$$

where the nonlinear term is neglected. We may replace $\nabla_\parallel$ by $ik_\parallel$ in (7.135) to obtain

$$\tilde{n}_e \simeq n_{e0} \left\{ 1 - \frac{1}{k_\parallel^2 D_\parallel} \left(\frac{\partial}{\partial t} + v_{De} \frac{\partial}{\partial y} \right) \right\} \frac{e\phi}{T_e}, \quad (7.136)$$

where an iteration is once used based on the fact that the deviation of the electron density from the Boltzmann distribution is small.

Substituting (7.56), (7.57), (7.132), and (7.133) into the ion equation of continuity, invoking the charge neutrality and retaining the terms up to ρ_s/L_n , $\rho_s = C_s/\Omega_i$, we obtain

$$\begin{aligned} & \left(1 - \frac{1}{k_{\parallel}^2 D_{\parallel}} \frac{\partial}{\partial t}\right) \left(\frac{\partial}{\partial t} + v_{D_e} \frac{\partial}{\partial y}\right) \phi - \frac{C_s^2}{\Omega_i^2} \left(\frac{\partial}{\partial t} - \eta_i \nabla^2\right) \nabla_{\perp}^2 \phi + \frac{\partial}{\partial z} v_{i\parallel} \\ &= \frac{C_s^4}{\Omega_i^3} \{\phi, \nabla_{\perp}^2 \phi\}, \end{aligned} \quad (7.137)$$

where $e\phi/T_e$ is replaced by ϕ .

For the ion motion parallel to the magnetic field, we may neglect the viscosity and have

$$\frac{\partial}{\partial t} v_{i\parallel} + (v_E \cdot \nabla) v_{i\parallel} = -\frac{e}{m_i} \nabla_{\parallel} \phi - \frac{1}{n_0 m_i} \nabla_{\parallel} p_i. \quad (7.138)$$

Using the normalization $x/\rho_s = x$, $\Omega_i t = t$, $v_{i\parallel}/C_s = v_z$, $\mu = C_s/k_{\parallel}^2 D_{\parallel}$, $\eta = \eta_i/\rho_s^2$, $v_* = C_s/v_{D_e}$, we obtain the model equations describing the nonlinear evolution of the collisional drift wave instability:

$$\left\{ \left(1 - \mu \frac{\partial}{\partial t}\right) \left(\frac{\partial}{\partial t} + v_* \frac{\partial}{\partial y}\right) - \left(\frac{\partial}{\partial t} - \eta \nabla^2\right) \nabla^2 \right\} \phi + \frac{\partial}{\partial z} v_z = \{\phi, \nabla_{\perp}^2 \phi\}, \quad (7.139)$$

$$\frac{\partial}{\partial t} v_z + \frac{\partial}{\partial z} \phi = -\{\phi, v_z\}. \quad (7.140)$$

When collisions and viscosity are neglected, the set of equations is shown by Meiss and Horton [131] to have dipole solutions as stationary solutions. This may suggest that the evolution of the collisional drift wave instability governed by (7.139) and (7.140) eventually leads to the formation of dipole vortices. However, three-dimensional dynamics are seemingly quite complicated. In the following, we restrict our attention to evolution in a two-dimensional system for which the ion motion in the z direction is neglected in (7.139) and (7.140). This reduction of the dimensionality physically corresponds to the case where ion acoustic waves are decoupled from the drift waves, and is verified when $k_{\perp} \gg k_{\parallel}$. Furthermore, invoking the result that $\partial/\partial t + v_* \partial/\partial y \sim 0$ for waves with a long wavelength, the collisional term responsible for the instability, $\mu \partial/\partial t$ is replaced by $-\mu v_* \partial/\partial y$. Thus, (7.139) and (7.140) reduce to a single nonlinear equation:

$$\left\{ \left(1 + \mu v_* \frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial t} + v_* \frac{\partial}{\partial y}\right) - \left(\frac{\partial}{\partial t} - \eta \nabla^2\right) \nabla \right\} \phi = (\mathbf{z} \times \nabla_{\perp} \phi) \cdot \nabla_{\perp} \nabla_{\perp}^2 \phi, \quad (7.141)$$

which is equivalent to the Hasegawa–Wakatani equation to describe drift waves in an unstable system. This is shown by putting $\partial^2/\partial z^2 = -k_{\parallel}^2$ in (7.94) and (7.95) to solve n_e in term of the potential and substituting the resultant into (7.95) under the charge neutrality $n_e = n_i$.

The dispersion relation of a linear wave is obtained from (7.141) to give the real frequency as

$$\omega(\mathbf{k}) = \frac{k_y v_*}{(1 + k^2)^2 + \mu^2 k_y^2 v_*^2} \{1 + k^2 + \mu(\mu k_y^2 v_*^2 - \eta k^4)\}, \quad (7.142)$$

and the instability growth rate as

$$\gamma(\mathbf{k}) = \frac{k^2}{(1 + k^2)^2 + \mu^2 k_y^2 v_*^2} \{\mu k_y^2 v_*^2 - \eta k^2(1 + k^2)\}, \quad (7.143)$$

which are certainly comparable with those derived from the Hasegawa–Wakatani equation. In Fig. 7.8, the growth rate $\gamma(\mathbf{k})$ is depicted against k_y for various values of k_x at $v_* = 0.8$, $\mu = 0.3$, and $\eta = 0.03$. The spectral structure of the growth rate is characterized by the finite bandwidth for the instability, which itself manifests a possibility of forming a coherent structure because the wave is cascaded from the instability region to longer and shorter stable regions.

In solving (7.141), we used the algorithm developed by Gourlay and Moris [132]. The number of mesh points is 64×64 and the periodic boundary conditions are imposed. The stability of the numerical scheme is ensured by monitoring changes in conserved quantities of (7.141) without the collision and viscosity, that is, the Hasegawa–Mima equation. For (7.141), we launched many waves at the initial moment and confirmed that the observed growth rate spectra agree well with the theoretical linear growth rate spectra. Simulations with a higher resolution (128×128 grids) performed for several parameters confirmed the results in the 64×64 simulations. This is because, as a result of the viscosity in (7.141), short waves are heavily damped and play no role in the physics. The initial condition is chosen as

$$\phi(x, y, t = 0) = \phi_0 \sin(k_x x + k_y y),$$

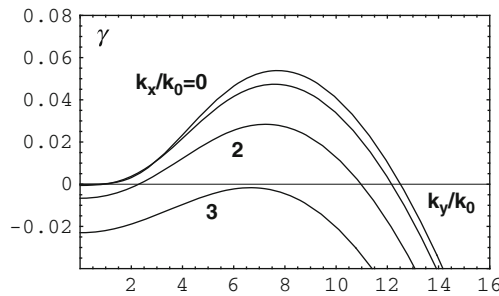


Fig. 7.8 The growth rates against k_y/k_0 for fixed $k_x/k_0 = 0 - 8$ ($k_0 = \pi/16$) in the case of $v_* = 0.8$, $\mu = 0.3$, and $\eta = 0.03$

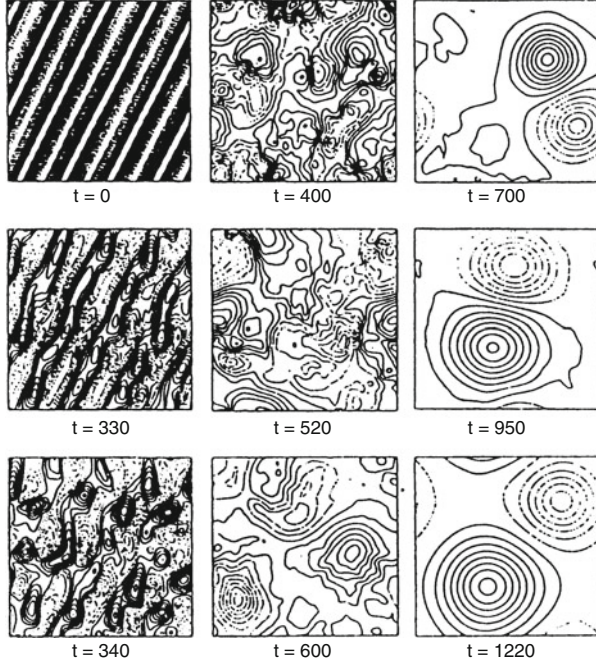


Fig. 7.9 Development of the instability when a sinusoidal wave is given initially as $\phi(x, y, t = 0) = \phi_0 \sin(k_x x + k_y y)$, $(k_x, k_y) = (2k_0, 4k_0)$, $(k_0 = \pi/16)$ for $\phi_0 = 0.001$ with $v_* = 0.8$, $\mu = 0.3$, and $\eta = 0.03$

$$(k_x, k_y) = (2k_0, 4k_0), \quad (k_0 = \pi/16),$$

with the amplitude $\phi_0 = 0.001$ for $v_* = 0.8$, $\mu = 0.3$, and $\eta = 0.03$. The course of development is shown in Figs. 7.9 and 7.10, in which the contour of the potential and the spectrum are depicted, respectively. The maximum amplitudes of the potential and the spectrum at each time step are used for the normalization.

The wave grows at the linear growth rate $\gamma \simeq 0.023$, followed by an excitation of higher harmonics. Around $t = 300$, when the amplitude of the initially launched waves exceeds a certain value of about $\phi_0 e^{\gamma t} \simeq 0.99$, energy transfer to many modes occurs abruptly (from $t = 290$ to $t = 310$ in Fig. 7.9b), leading to the breakup of the wave into a train of small vortices. This secondary instability is identified with the parametric instability as follows. By Fourier transforming (7.141), we obtain

$$\left(\frac{\partial}{\partial t} + i\omega(\mathbf{k}) - \gamma(\mathbf{k}) \right) \phi(\mathbf{k}) = \sum_{\mathbf{k}' + \mathbf{k}'' = \mathbf{k}} \Lambda_k(\mathbf{k}', \mathbf{k}'') \phi(\mathbf{k}') \phi(\mathbf{k}''), \quad (7.144)$$

where $\omega(\mathbf{k})$ and $\gamma(\mathbf{k})$ are given by (7.142) and (7.143), respectively, and

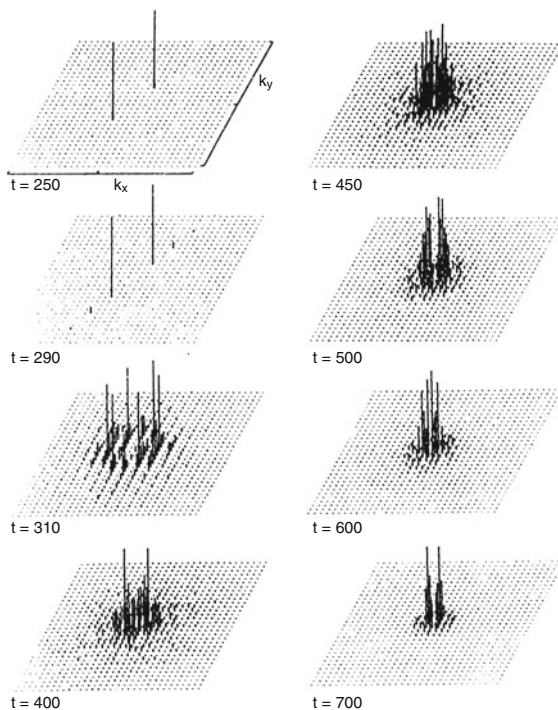


Fig. 7.10 Development of spectrum of the potential corresponding to Fig. 7.9. The amplitude is normalized by the largest at each time step

$$\Lambda_k(\mathbf{k}', \mathbf{k}'') = \frac{1}{2} \frac{(\mathbf{k}' \times \mathbf{k}'') \cdot \mathbf{z}}{1 + k^2 + i\mu k_y v_*} (k''^2 - k'^2). \quad (7.145)$$

Taking the initially launched waves $\mathbf{k} = (2k_0, 4k_0)$ and $(-2k_0, -4k_0)$ as a pump since they already have grown up to the finite amplitude, the conventional analysis for the parametric interaction yields the growth rate Γ and the frequency shift Ω as

$$\Gamma = \frac{1}{2} [\gamma(\mathbf{k}_1) + \gamma(\mathbf{k} - \mathbf{k}_1)] \pm \frac{1}{\sqrt{2}} [(A^2 + B^2)^{1/2} + A]^{1/2}, \quad (7.146)$$

$$\Omega = \pm \frac{1}{\sqrt{2}} [(A^2 + B^2)^{1/2} - A]^{1/2}, \quad (7.147)$$

where

$$A = \frac{1}{4} \{ [\gamma(\mathbf{k}_1) - \gamma(\mathbf{k} - \mathbf{k}_1)]^2 - \Delta^2 \} + |\psi_0|^2 \Re G,$$

$$B = \frac{1}{2} \Delta [\gamma(\mathbf{k}_1) - \gamma(\mathbf{k} - \mathbf{k}_1)] + |\psi_0|^2 \Im G,$$

$$\Delta = \omega(\mathbf{k}) - \omega(\mathbf{k}') - \omega(\mathbf{k} - \mathbf{k}_1),$$

$$G = \frac{[(\mathbf{k}' \times \mathbf{k}'') \cdot \mathbf{z}]^2 (k^2 - k_1^2)}{(1 + k^2 + i\mu k_y v_*) [1 + (\mathbf{k} - \mathbf{k}_1)^2 + i\mu (\mathbf{k} - \mathbf{k}_1)_y v_*]}.$$

Since $\gamma(\mathbf{k})$ is of the order of μ , the threshold for the instability is roughly estimated by

$$|\psi_0|_{th}^2 \simeq \frac{\Delta^2}{e\Re G}, \quad (7.148)$$

which is meaningful only when $G > 0$, that is,

$$\min[k_1^2, (\mathbf{k} - \mathbf{k}_1)^2] < k^2 < \max[k_1^2, (\mathbf{k} - \mathbf{k}_1)^2]. \quad (7.149)$$

A set of the wavenumber $\mathbf{k}_1$ satisfying the relation (7.149) for the $\mathbf{k} = (2k_0, 4k_0)$, is shown by solid points in Fig. 7.11, which, together with the points symmetric with respect to the origin for the pump of $(-2k_0, -4k_0)$, reproduces the observed structure of the spectrum.

The small vortices excited as a results of the parametric instability then interact to fuse into larger ones. The inverse cascade of the wave energy on the spectrum clearly observed in Fig. 7.10 is responsible for the fusion of vortices. Finally, one large dipole vortex is formed and propagates in a rotating manner. In Fig. 7.12, the contour and spectrum of the vortex pair at the final stage are compared with those of the dipole vortex, (7.92) with $R = 14.8$ and $c = 0.56$. The spectrum of the dipole vortex whose symmetry axis is at the angle of $\pi/4$ to the y axis is given as the sum of the spectrum of Fig. 7.12d and that obtained by interchanging k_x and k_y in Fig. 7.12d, which is quite similar to the spectrum in Fig. 7.12b. Thus, the dipole vortex at the final stage may be identified with the theoretical one.

The amplitudes of the dipole vortex in the final stage are determined by assuming a stationary energy flow from the linearly unstable modes to the damped ones.

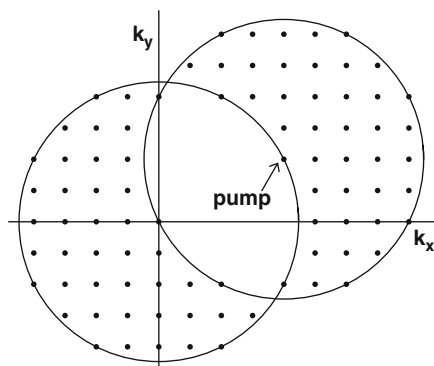


Fig. 7.11 A set of wavenumbers subject to the parametric instability when the pump wave is taken to be $(k_x, k_y) = (2k_0, 4k_0)$

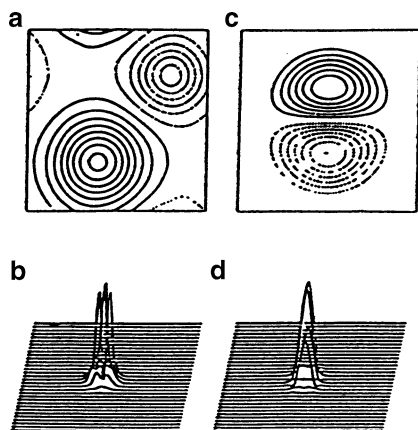


Fig. 7.12 Spatial and spectral structures of the observed vortex pair and the stationary solution. (a) the contour of the potential observed at the final stage of the simulation for $v_* = 0.8$, $\mu = 0.3$, and $\eta = 0.03$, (b) the spectral structure for the case of (a), (c) the contour of the stationary solution for $R = 14.8$ and $c = 0.56$, and (d) the spectrum in the case of (d)

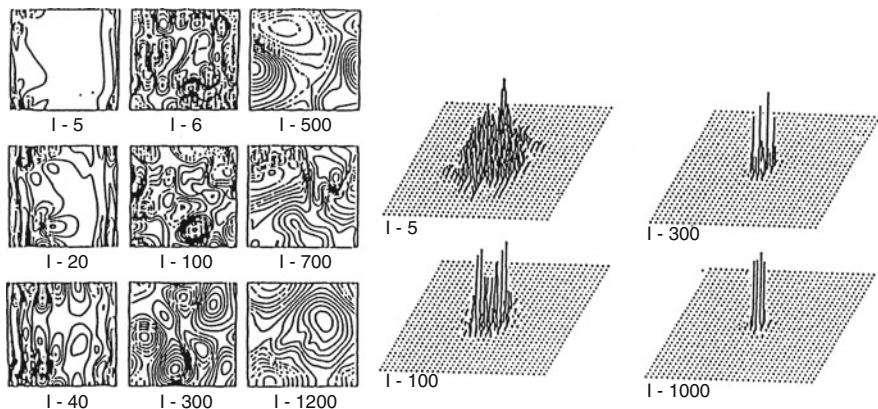


Fig. 7.13 The time evolution of (a) the spatial structure and (b) the spectral structure of the potential for $v_* = 0.8$, $\mu = 0.4$, and $\eta = 0.01$, when many waves are initially launched. The amplitude of the spectrum is normalized by the largest at each time step

Since the energy is mainly contained in $(k_x/k_0, k_y/k_0) = (\pm 1, 0)$, $(0, \pm 1)$, and $(\pm 1, \pm 1)$ as is seen in Fig. 7.13a, we may look for stationary solutions of the truncated equation derived from (7.144):

$$|\psi(1, 0)|^2 \sim \frac{1}{4k_0^6}, \quad (7.150)$$

$$|\psi(0, 1)|^2 \sim \frac{\eta\delta}{4k_0^3}, \quad (7.151)$$

$$|\psi(1, \pm 1)|^2 \sim \frac{\eta \delta}{2k_0^3}. \quad (7.152)$$

The magnitude of the amplitudes given by (7.150)–(7.152) is of the same order as that observed in the simulation. It is comparable to the saturation amplitude obtained by Terry and Horton [133] ($e\phi/T_e \sim 10L_n/\rho_s$), which is claimed too high by Biskamp and Kaifen [117]. The reason for this might be related to the neglect of the coupling with ion acoustic waves [117].

When many waves are initially launched for the parameters $\mu = 0.4$ and $\eta = 0.01$, small scaled vortices are readily formed since the linear instability is dominated in the initial stage of the evolution. The wave number spectrum of the potential resembles that of the linear growth rate. Apparently, the parametric instability cannot be identified. The fusion of smaller vortices into larger ones becomes the main process of the evolution. The time developments of the potential and the wave spectrum are shown in Fig. 7.13a and b.

We have studied the formation of a coherent structure through turbulence in the course of nonlinear development of the collisional drift wave instability. We derive a model equation which is the Hasegawa–Mima equation with the effect of collision and viscosity. The linear instability is followed by a parametric instability to allocate the wave energy to many waves, which, in turn, cause the destruction of the sinusoidal wave pattern into a random ensemble of small vortices. The small vortices are then fused to form larger vortices because of an inverse cascade. The fusion process is repeated to give a single large dipole vortex identified as a stationary solution of the Hasegawa–Mima equation.

Although many efforts have been devoted to establishing a scenario for nonlinear development of the drift wave instability, so far the scenario is terminated by the saturation of the instability and the resultant turbulence. Here, we find that there is another stage beyond the turbulence state, that is, the onset and development of self-organized motions. The Hasegawa–Mima equation seems capable of describing a variety of nonlinear wave dynamics.

7.4.1 Point Vortex Description for Drift Wave Vortices

A crucial difference of the Hasegawa–Mima equation from Euler’s equation is the existence of the drift term giving rise to dispersive waves, which requires that the vorticity attached to each point vortex no longer be constant but to vary in space and time, in contrast with the point vortex for Euler’s equation characterized by constant strength vorticities. This modulated point vortex model was first introduced by Kono and Yamagata [134] based on the fact that the Rossby wave equation (the Hasegawa–Mima equation) conserves the vorticity along the trajectory, and then later by Zabusky and McWilliams [135] who studied the configurations of the vortices corresponding to a stationary solution of the Rossby wave equation and stability of the configurations.

First, we rederive the point vortex equation for the Hasegawa–Mima equation. Then, an exact solution for a vortex pair is shown to recover the dynamical properties of the drift wave vortices revealed by numerical simulations. The collision processes of two vortex pairs are numerically studied. Finally, a statistical theory of a many-vortex system is formulated where the vortex diffusion coefficient is analytically derived to give the empirical formula given by Horton [136].

Starting with the Hasegawa–Mima equation

$$\frac{\partial \pi}{\partial t} + \{\phi, \pi - v_* x\}, \quad \pi = \phi - \nabla^2 \phi, \quad (7.153)$$

where v_* is assumed constant, we introduce the vortices through

$$\pi(\mathbf{r}, t) = \sum_{\alpha} \gamma_{\alpha}(t) V_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}(t)), \quad (7.154)$$

where $V_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}(t))$ is a localized function at $\mathbf{r} = \mathbf{r}_{\alpha}(t)$ and $\mathbf{r}_{\alpha}(t)$ is determined by the characteristics of (7.153);

$$\frac{d\mathbf{r}_{\alpha}(t)}{dt} - \{\phi(\mathbf{r}_{\alpha}, t), \mathbf{r}_{\alpha}\} = 0. \quad (7.155)$$

Substituting (7.154) into (7.153), we obtain

$$\sum_{\alpha} \left\{ \frac{d\gamma_{\alpha}(t)}{dt} - \gamma_{\alpha}(t) \nabla \cdot \left(\frac{d\mathbf{r}_{\alpha}}{dt} - \mathbf{z} \times \nabla \phi(\mathbf{r}, t) \right) \right\} V_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}(t)) = -v_* \frac{\partial}{\partial y} \phi(\mathbf{r}, t). \quad (7.156)$$

Since $V_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}(t))$ is a function localized around $\mathbf{r} = \mathbf{r}_{\alpha}(t)$, we may replace the arguments $\mathbf{r}$ appearing in the coefficients of $V_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha})$ with $\mathbf{r}_{\alpha}(t)$, and then the second term of the left-hand side of (7.156) vanishes according to (7.155). Then, we have

$$\sum_{\alpha} \frac{d\gamma_{\alpha}(t)}{dt} V_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}(t)) = -v_* \frac{\partial}{\partial y} \phi(\mathbf{r}, t). \quad (7.157)$$

Multiplying both sides of (7.156) by $V_{\ell}(\mathbf{r} - \mathbf{r}_{\ell}(t))$ and integrating with respect to $\mathbf{r}$, where we may approximate the overlap integral as follows:

$$\int d\mathbf{r} V_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}(t)) V_{\ell}(\mathbf{r} - \mathbf{r}_{\ell}(t)) \simeq \delta_{\alpha\ell}, \quad (7.158)$$

then we obtain

$$\frac{d\gamma_{\alpha}}{dt} = v_* \frac{dx_{\alpha}}{dt}. \quad (7.159)$$

When the localized function $V_{\alpha}(\mathbf{r} - \mathbf{r}_{\alpha}(t))$ is approximated by a delta function $\delta(\mathbf{r} - \mathbf{r}_{\alpha}(t))$, that is, the vortex is assumed a point vortex, we have from (7.154)

$$\phi(\mathbf{r}, t) = \frac{1}{2\pi} \sum_{\alpha} \gamma_{\alpha}(t) K_0(|\mathbf{r} - \mathbf{r}_{\alpha}(t)|), \quad (7.160)$$

where K_0 is the modified Bessel function of the second kind. In the following, v_* is assumed positive constant,

$$\gamma_{\alpha}(t) = \gamma_{j,0} + v_* x_{\alpha}(t), \quad (7.161)$$

where $\gamma_{j,0}$ is an integration constant. Then, (7.155) becomes

$$\frac{d\mathbf{r}_{\alpha}}{dt} = \frac{1}{2\pi} \sum_{\alpha \neq \beta} (\gamma_{\beta 0} + v_* x_{\beta}) \mathbf{z} \times \nabla_{\beta} K_0(r_{\alpha\beta}), \quad (7.162)$$

where $r_{\alpha\beta} = |\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}|$. Equation has the same form as that introduced by Kono and Yamagata [134] first and then later by Zabusky and McWilliams [135] who introduced the name modulated point vortex for the variation $\gamma_{\alpha} = \gamma_{\alpha 0} + v_* x_{\alpha}$. The simple case of constant γ_{α} valid when $v_* = 0$ is studied by Hasegawa, Maclellan, and Kodama [119].

There are two conserved quantities for the Hasegawa–Mima equation; the energy E and the enstrophy W given by (7.18). In the point vortex description, the energy and enstrophy are expressed as

$$E = \frac{1}{2\pi} \sum_{\alpha \neq \beta} (\gamma_{\alpha 0} + v_* x_{\alpha})(\gamma_{\beta 0} + v_* x_{\beta}) K_0(r_{\alpha\beta}), \quad (7.163)$$

$$W = \frac{1}{8\pi} \sum_{\alpha \neq \beta} (\gamma_{\alpha 0} + v_* x_{\alpha})(\gamma_{\beta 0} + v_* x_{\beta}) [K_0(r_{\alpha\beta}) + K_2(r_{\alpha\beta})], \quad (7.164)$$

where the self-energy has been subtracted. Although (7.163) and (7.164) are not conserved in general since

$$\frac{dE}{dt} = -\frac{1}{2} \sum_{\alpha} v_* \frac{\partial}{\partial y_{\alpha}} \phi^2(\mathbf{r}_{\alpha}, t), \quad (7.165)$$

and

$$\frac{dW}{dt} = -\frac{1}{2} \sum_{\alpha} v_* \frac{\partial}{\partial y_{\alpha}} [\nabla \phi(\mathbf{r}_{\alpha}, t)]^2, \quad (7.166)$$

this does not mean that the point vortex description is invalid. The discrepancy comes from the fact that (7.18) takes into account contributions from both vortices and waves, while (7.163) and (7.164) are based on vortices only. Therefore, for the vortex pairs corresponding to the stationary solution of the Hasegawa–Mima equation, that is, for those propagating straight in the y direction, (7.163) and (7.164) are certainly constants of motion since the waves are never excited. In general, however, changes in energy and enstrophy of the vortex system in the course of time evolution

may occur whenever waves are involved in the fundamental processes of the vortices such as emission or absorption of waves by the vortices, which is observed in numerical experiments [131, 138] based on the Hasegawa–Mima equation.

The Hamiltonian is given by

$$H = \sum_{\alpha} \phi(\mathbf{r}_{\alpha}, t), \quad (7.167)$$

from which we have

$$\frac{dx_{\alpha}}{dt} = -\frac{\partial H}{\partial y_{\alpha}} = \frac{1}{2\pi} \sum_{\beta} (\gamma_{\beta 0} + v_{*} x_{\beta}) \frac{y_{\alpha} - y_{\beta}}{r_{\alpha\beta}} K_1(r_{\alpha\beta}), \quad (7.168)$$

$$\frac{dy_{\alpha}}{dt} = \frac{\partial H}{\partial x_{\alpha}} = -\frac{1}{2\pi} \sum_{\beta} (\gamma_{\beta 0} + v_{*} x_{\beta}) \frac{x_{\alpha} - x_{\beta}}{r_{\alpha\beta}} K_1(r_{\alpha\beta}). \quad (7.169)$$

Since H is a translational invariant along the y axis, the translational momentum in x is conserved:

$$P = \sum_{\alpha} (\gamma_{\alpha 0} + v_{*} x_{\alpha})^2. \quad (7.170)$$

7.4.1.1 Dipole Vortex Solutions

Now, we solve (7.168) and (7.169) for two vortices. In this case, in addition to (7.170), the relative distance of the vortices is a constant of motion,

$$r_{12}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = r_0^2, \quad (7.171)$$

which leads to

$$\frac{d}{dt} \cos \theta = \pm \frac{v_{*} K_1(r_0)}{2\pi} \sqrt{(A + \cos \theta)(B - \cos \theta)(1 - \cos^2 \theta)}, \quad (7.172)$$

where $x_1 - x_2 = r_0 \cos \theta$, $A = [\sqrt{2P} + (\gamma_{20} - \gamma_{10})/v_{*} r_0]$, and $B = [\sqrt{2P} - (\gamma_{20} - \gamma_{10})/v_{*} r_0]$. Equation (7.172) is readily integrated to yield a solution expressed in terms of the Jacobi elliptic function. The solution is given by

$$\cos \theta = \gamma \frac{1 - \beta^2 \operatorname{sn}^2(\omega t, k)}{1 - a^2 \operatorname{sn}^2(\omega t, k)}, \quad (7.173)$$

where for $v_{*} r_0 + |\gamma_{10} - \gamma_{20}| > \sqrt{2P} > |v_{*} r_0 - |\gamma_{10} - \gamma_{20}||$,

$$a^2 = \frac{1 + B}{A + B}, \quad \beta^2 = A a^2, \quad \gamma = -\operatorname{sgn}(\gamma_{10} - \gamma_{20}),$$

$$\omega = \sqrt{2(A + B)} \frac{v_{*} K_1(r_0)}{4\pi}, \quad k = \sqrt{(1 + A)(1 + B)/2(A + B)},$$

and for $\sqrt{2P} < |v_* r_0 - |\gamma_{10} - \gamma_{20}||$,

$$a^2 = \frac{A+B}{1+B}, \quad \beta^2 = a^2/A, \quad \gamma = -A \operatorname{sgn}(\gamma_{10} - \gamma_{20}),$$

$$\omega = \sqrt{(1+A)(1+B)} \frac{v_* K_1(r_0)}{4\pi}, \quad k = \sqrt{2(A+B)/[(1+A)(1+B)]}.$$

Equations for the center of gravity are given by

$$\frac{1}{2}(x_1 + x_2) = -\frac{\pi r_0}{v_*} K_1(r_0) \frac{d\theta}{dt} - \frac{\gamma_{10} + \gamma_{20}}{2v_*}, \quad (7.174)$$

$$\frac{1}{2} \frac{d}{dt}(y_1 + y_2) = \frac{K_1(r_0)}{4\pi} (\gamma_{10} - \gamma_{20} + v_* r_0 \cos \theta) \cos \theta, \quad (7.175)$$

whose explicit solutions are readily obtained by using (7.173).

It must be noted that a vortex pair ($\gamma_{10} + \gamma_{20} = 0$) propagates in the y direction without oscillation in the orbit $\theta_0 = \theta(t=0) = n\pi$ (n :integer) and with oscillation for $\theta_0 \neq n\pi$ as is shown in Fig. 7.14, which has been numerically observed by Makino, Kamimura, and Taniuti [128] based on the Hasegawa–Mima equation. Computations are monitored by keeping P constant within ten effective figures. Nycander and Isichenko [137] derived the equation for the center of gravity of a vortex pair from the momentum equations of the Hasegawa–Mima equation and obtained the frequency of the trajectory which is also well fitted to the results by Makino, Kamimura, and Taniuti. A nonpropagating solution shown in Fig. 7.14c is realized for such initial angles of the symmetry axis of the vortex pair to the coordinate axis that the velocity of the vortex pair given by

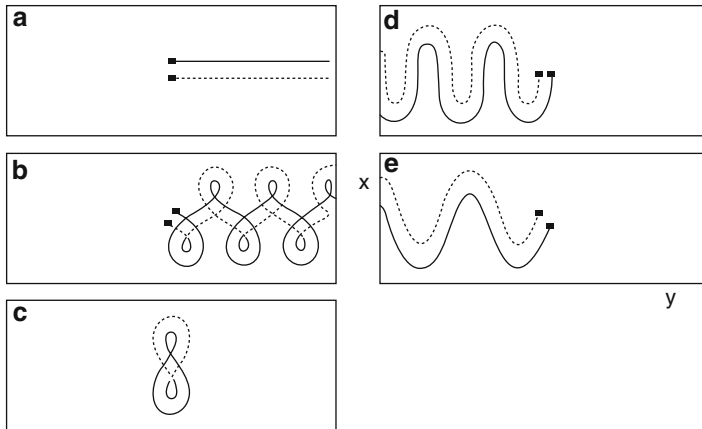


Fig. 7.14 Trajectories of an opposite signed vortex pair of $\gamma/v_* = 2.0$ and $r_0 = 0.4$: *solid lines* and *dotted lines* indicate a positive and negative vortex, respectively. Angles θ_0 between the symmetry axis and the x axis are (a) 0, (b) $\pi/6$, (c) 0.8976, (d) $\pi/2$, and (e) $2\pi/3$

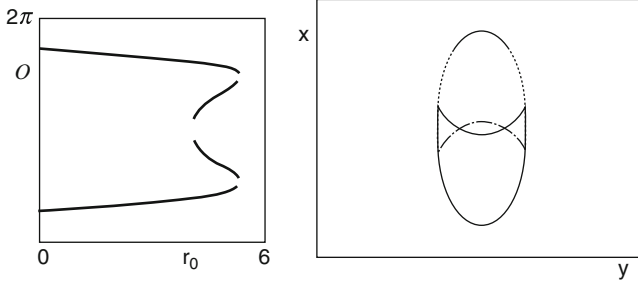


Fig. 7.15 (a) Tilted angle versus the size of the vortex pair for nonpropagating vortex pair of $\gamma/v_* = 2.0$, and (b) another type of non-propagating vortex pair of $\gamma/v_* = 2.0$, $r_0 = 5.0$ and $\theta_0 = 1.6755$

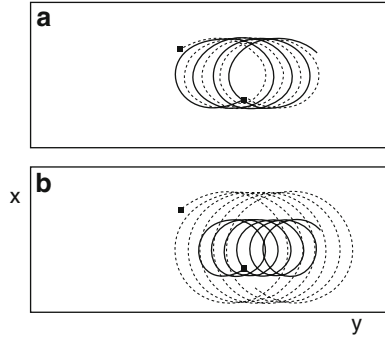


Fig. 7.16 Trajectories of a like-signed vortex pair: (a) equal vorticity strength $(\gamma_1, r_1) = (\gamma_2, r_2) = (20, 2)$ and (b) strong-weak vortices $(\gamma_1, r_1) = (50, 4)$ and $(\gamma_2, r_2) = (20, 2)$

$$\begin{aligned} \frac{1}{2} < \frac{d}{dt}(y_1 + y_2) > &= \frac{1}{8K(k)} \int_0^4 K(k) dt \frac{K_1(r_0)}{2\pi} (\gamma_{10} - \gamma_{20} + v_* r_0 \cos \theta) \cos \theta, \\ &= \sqrt{2P} \left(1 - 2 \frac{E(k)}{K(k)} \right) \frac{K_1(r_0)}{4\pi} = c, \end{aligned} \quad (7.176)$$

is zero where $K(k)$ and $E(k)$ are the complete elliptic integral of the first and second kind, respectively. Figure 7.15a shows the initial angle versus the size of the nonpropagating vortex pairs for the case of $\gamma_0/v_* = 2.0$. Another example of a nonpropagating vortex pair is given by Fig. 7.15b.

Two like-signed vortices ($\gamma_{10} = \gamma_{20}$) are mutually trapped, rotating around the center of gravity which is easily seen from (7.174) and (7.175) (Fig. 7.16). This mutual trapping leads to a coarse graining of the correlation over directions and may be considered a mechanism behind the fusion of vortices in the sense that a group of point vortices positioned sufficiently near one another act at large distances as a single vortex with the sum of the intensity, $\gamma_T \simeq \sum_\alpha \gamma_\alpha$. A coalescence of like-vortices and a long-lived monopole numerically observed by Horton may be interpreted by

this mutual trapping process. The inverse cascade of the energy associated with the conservation of enstrophy is also regarded in the vortex representation as a trapping and as a kind of snowballing process.

7.4.1.2 Collision Processes of Dipole Vortices

Collision processes of two vortex pairs are shown to recover those observed numerically [114, 128, 138] for elastic cases with zero impact parameters (Fig. 7.17) and for inelastic cases with nonzero impact parameters (Fig. 7.18).

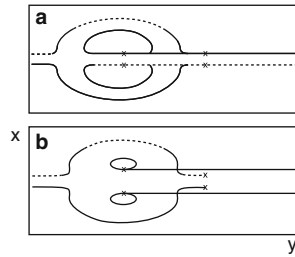


Fig. 7.17 Head-on collision between two opposite signed vortex-pairs with zero impact parameter

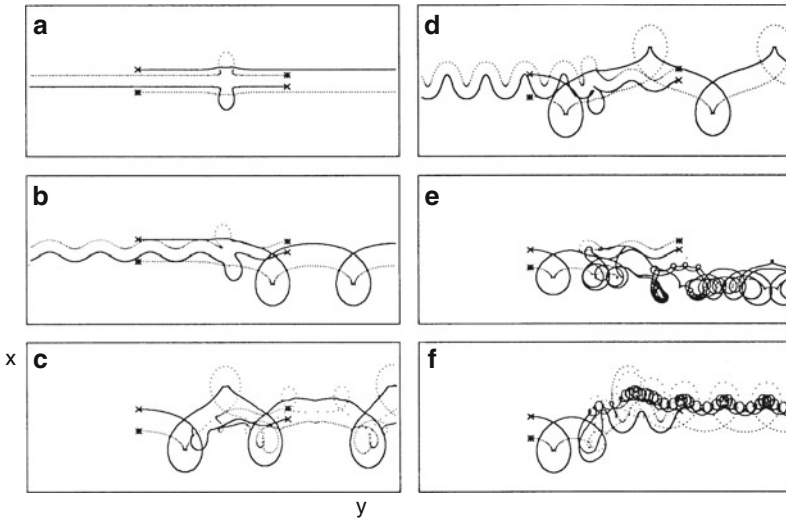


Fig. 7.18 Head-on collision between two opposite signed vortex-pairs of $\gamma_1/v_* = -1.0$, $\gamma_2/v_* = 1.0$, $\gamma_3/v_* = 2.0$, and $\gamma_4/v_* = 2.0$ with the initial positions: $x_1 = x_{10} + \Delta x_1$, $x_2 = -x_{10} + \Delta x_1$, $x_3 = x_{20} + \Delta x_2$, $x_4 = -x_{20} + \Delta x_2$, $x_{10} = 0.25$, $x_{20} = 0.50$, $y_1 = y_2 = -y_3 = -y_4 = 4.0$ with $(\Delta x_1, \Delta x_2) =$ (a) (0.0,0.0), (b) (0.15,0.0), (c) (0.15,0.15), (d) (0.25,0.25), (e) (0.25,0.50), (f) (0.50,0.50)

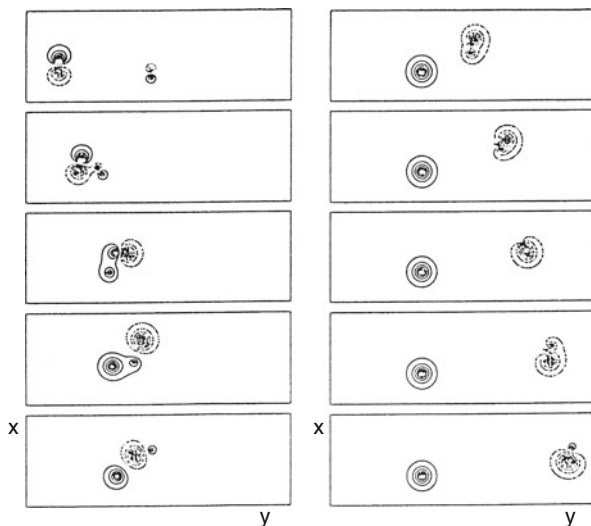


Fig. 7.19 Contours of stream function obtained from (7.160) using the point vortex trajectories in Fig. 7.18e

Since our point vortex model does not take into account the effects of interaction with the wake fields, the inelastic collisions with an emission of wake fields observed by McWilliams and Zabusky [138] cannot be described by the vortex field component in (7.160) alone. However, the position dependence of the vorticity gives the vortex system studied here a variety of complicated behaviors including an exchange scattering and a boomerang scattering, indicating that our point vortex system is likely to become turbulent when many vortices are involved. However, the potential structure constructed from (7.160) is quite orderly as is shown in Fig. 7.19, which corresponds to Fig. 7.18e; mutually trapped vortices behave as a single vortex though the dynamics of constituent point vortices is very complicated. Therefore, the complication of the dynamics of the point vortices is rather analogous to complicated behaviors of constituent particles in an ordinary gas or fluid dynamics and averaged properties may be of primary importance although the dynamical properties of the point vortex system are academically interesting since the chaotic behavior may be characterized by intermittent structures and clusters of vortices in which local order is a preferred state because of the short range interaction force between point vortices. The range of the interaction $\rho_s = c(m_i T_e)^{1/2}/eB$ is given by the parallel electron motion shielding the charge separation in the Euler vortex.

7.4.2 Kinetic Theory of Vortex Diffusion

Here, we turn to a statistical system with N point vortices and derive an equation for vortex diffusion. Introducing a distribution function of vortices of α species by

$$F_\alpha(\mathbf{r}, t) = \sum_j \delta(\mathbf{r} - \mathbf{r}_j^{(\alpha)}(t)), \quad (7.177)$$

we immediately obtain the Klimontovich equation for vortices using (7.168) and (7.169):

$$\frac{\partial}{\partial t} F_\alpha + \mathbf{z} \times \nabla \Phi \cdot \nabla F_\alpha = 0, \quad (7.178)$$

$$\Phi(\mathbf{r}, t) = \frac{1}{2\pi} \sum_\alpha \int d\mathbf{r}' (\gamma_{\alpha 0} + v_* x') K_0(|\mathbf{r} - \mathbf{r}'|) F_\alpha(\mathbf{r}'). \quad (7.179)$$

The average distribution function is defined by the average over the ensemble of initial data of the vortices

$$\langle F_\alpha(\mathbf{r}, t) \rangle = \int d\mathbf{r}_{10} d\mathbf{r}_{20} \cdots d\mathbf{r}_{N0} P_\alpha(\mathbf{r}_{10}, \cdots, \mathbf{r}_{N0}) F_\alpha(\mathbf{r}, t), \quad (7.180)$$

while the fluctuation part of the distribution function is simply given

$$\tilde{f}_\alpha(\mathbf{r}, t) = F_\alpha(\mathbf{r}, t) - \langle F_\alpha(\mathbf{r}, t) \rangle, \quad (7.181)$$

which includes fluctuations due to the interactions of the vortices and the discreteness of the vortices. Taking the ensemble average of (7.178), and subtracting the result from (7.178), we obtain

$$\frac{\partial}{\partial t} \langle F_\alpha \rangle + \mathbf{z} \times \nabla \langle \Phi \rangle \cdot \nabla \langle F_\alpha \rangle = -\langle \mathbf{z} \times \nabla \tilde{\Phi} \cdot \nabla \tilde{F}_\alpha \rangle, \quad (7.182)$$

and

$$\frac{\partial}{\partial t} \tilde{f}_\alpha + \mathbf{z} \times \nabla (\langle \Phi \rangle + \tilde{\Phi}) \cdot \nabla \tilde{f}_\alpha = -\mathbf{z} \times \nabla \tilde{\Phi} \cdot \nabla \langle F_\alpha \rangle. \quad (7.183)$$

Introducing the Green's function

$$\left\{ \frac{\partial}{\partial t} + \mathbf{z} \times \nabla (\langle \Phi \rangle + \tilde{\Phi}) \cdot \nabla \right\} G(\mathbf{r}, t | \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'), \quad (7.184)$$

equation (7.183) is formally solved in terms of G by

$$\begin{aligned} \tilde{f}_\alpha(\mathbf{r}, t) &= \int d\mathbf{r}' dt' G(\mathbf{r}, t | \mathbf{r}', t') \delta f(\mathbf{r}', t') \\ &\quad - \int d\mathbf{r}' dt' G(\mathbf{r}, t | \mathbf{r}', t') \mathbf{z} \times \nabla' \tilde{\Phi} \cdot \nabla' \langle F_\alpha \rangle. \end{aligned} \quad (7.185)$$

The Green's function is given by a solution of the characteristic equation of (7.184) as

$$G(\mathbf{r}, t | \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}(t; \mathbf{r}', t')), \quad (7.186)$$

where

$$\begin{aligned}
\mathbf{r}(t; \mathbf{r}', t') &= \mathbf{r}' + \int_{t'}^t dt'' \mathbf{z} \times \nabla [\langle \Phi(\mathbf{r}(t''), t'') \rangle + \tilde{\Phi}(\mathbf{r}(t''), t'')] \\
&= \mathbf{r}' + \mathbf{U}_E(t - t') + \tilde{\mathbf{r}}(t; \mathbf{r}', t').
\end{aligned} \tag{7.187}$$

Therefore, the average Green's function is expected in terms of cumulants with respect to $\tilde{\mathbf{r}}$:

$$\langle G(\mathbf{r}, t | \mathbf{r}', t') \rangle = \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}' - \mathbf{U}_E(t - t'))} e^{\sum_n C_n(\mathbf{k} \cdot \tilde{\mathbf{r}})}, \tag{7.188}$$

where $C_n(\mathbf{k} \cdot \tilde{\mathbf{r}})$ denotes the cumulants and the first two terms are given by

$$C_1(\mathbf{k} \cdot \tilde{\mathbf{r}}) = -i \langle \mathbf{k} \cdot \tilde{\mathbf{r}} \rangle, \tag{7.189}$$

$$C_2(\mathbf{k} \cdot \tilde{\mathbf{r}}) = -\frac{1}{2} \{ \langle (\mathbf{k} \cdot \tilde{\mathbf{r}})^2 \rangle - C_1(\mathbf{k} \cdot \tilde{\mathbf{r}})^2 \}. \tag{7.190}$$

Since our vortex system is characterized by complicated dynamical behavior of the constituent point vortices, the fluctuations are likely to deviate from the Gaussian, implying that the higher order cumulants do not vanish. However, we may assume, as a model, that the second order cumulant dominates over the others. Then, (7.188) is approximated by

$$\langle G(\mathbf{r}, t | \mathbf{r}', t') \rangle \simeq \sum_{\mathbf{k}} \exp \{ i\mathbf{k} \cdot [\mathbf{r} - \mathbf{r}' - \mathbf{U}_E(t - t')] - \mathbf{k} \cdot \mathbf{D} \cdot \mathbf{k}(t - t') \}, \tag{7.191}$$

where

$$\mathbf{D} = \frac{1}{2} \int_0^\infty d\mathbf{r} \{ \mathbf{z} \times \nabla \tilde{\Phi}(\mathbf{r}(t), t) \mathbf{z} \times \nabla \tilde{\Phi}(\mathbf{r}(t - t'), t - t') \}, \tag{7.192}$$

and we have assumed that the correlation time of the fluctuations is short. Substituting (7.185) and (7.191) into (7.182), we have

$$\frac{\partial}{\partial t} \langle F_\alpha \rangle + \mathbf{z} \times \nabla \langle \Phi \rangle \cdot \nabla \langle F_\alpha \rangle = \nabla \cdot \mathbf{D} \cdot \nabla \langle F_\alpha \rangle + \mathbf{A} \cdot \nabla \langle F_\alpha \rangle, \tag{7.193}$$

where

$$\mathbf{D} = \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \frac{\mathbf{z} \times \mathbf{k} : \mathbf{z} \times \mathbf{k}}{-i(\omega - \mathbf{k} \cdot \mathbf{U}_E) + \mathbf{k} \cdot \mathbf{D} \cdot \mathbf{k}} |\tilde{\Phi}(\mathbf{k}, \omega)|^2, \tag{7.194}$$

$$\begin{aligned}
\mathbf{A} &= -\frac{i}{2\pi} \sum_{\mathbf{k}} \int \frac{d\omega}{2\pi} \frac{\mathbf{z} \times \mathbf{k}}{\varepsilon(\mathbf{k}, \omega)} (\gamma_0 \hat{K}_0(\mathbf{k}) \\
&\quad + i\nu_* \frac{\partial \hat{K}_0(\mathbf{k})}{\partial k_x} \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{U}_E)^2 + (\mathbf{k} \cdot \mathbf{D} \cdot \mathbf{k})^2}),
\end{aligned} \tag{7.195}$$

$$\varepsilon(\mathbf{k}, \omega) = 1 + \frac{i}{2\pi} \sum_{\mathbf{k}} \frac{\gamma_0 \hat{K}_0(\mathbf{k}) + i v_* (\partial \hat{K}_0 / \partial k_x)}{-i(\omega - \mathbf{k} \cdot \mathbf{U}_E) + \mathbf{k} \cdot \mathbf{D} \cdot \mathbf{k}} \mathbf{k} \cdot \mathbf{z} \times \nabla \langle F_\alpha(\mathbf{r}) \rangle, \quad (7.196)$$

and $\hat{K}_0$ is the Fourier transform of $K_0(|\mathbf{r}|)$. The second term on the right-hand side of (7.193) is a drag term due to the emission of wake fields by the vortices because of their discreteness. Since the characteristic frequency of the vortex fluctuation is simply the vortex turnover time, that is, $\omega \sim \mathbf{k} \cdot \mathbf{U}_E$ which is given by $\varepsilon(\mathbf{k}, \omega) = 0$, we may evaluate the vortex diffusion from (7.194) as

$$D \sim \frac{1}{D} \sum_{\mathbf{k}, \omega} |\tilde{\Phi}(\mathbf{k}, \omega)|^2 \sim \sqrt{\sum_{\mathbf{k}, \omega} |\tilde{\Phi}(\mathbf{k}, \omega)|^2}, \quad (7.197)$$

which by (7.179) with $\gamma_0 < v_* r_0$ approximately reduces to

$$D \sim N v_* r_0, \quad (7.198)$$

where r_0 is the average size of the vortices. The idea behind (7.194) is illustrated in the following way. The quasilinear approximation for the vortex diffusion is to take into account the agitation of the orbit of a test vortex under the fluctuations induced by the other vortices that are assumed to be in a free motion, and it is given by

$$D \sim \frac{k^2 |\tilde{\Phi}|^2}{\omega - \mathbf{k} \cdot \mathbf{U}_E}. \quad (7.199)$$

However, the vortices causing the fluctuations are themselves subject to diffusive motions. Therefore, a self-consistent diffusion is given by replacing ω in the above expression by $\omega + ik^2 D$, leading to (7.194) or (7.197).

From numerical experiments on the vortex collision, Horton [136] found that the cross section σ for strong inelastic collisions is peaked at the impact parameter compatible to r_0 where $\sigma_{\max} \simeq 2r_0$. Taking the average vortex speed as $c > v_*$, he estimated the vortex–vortex collision frequency as

$$\nu \sim n_v c \sigma \sim 2 n_v v_* r_0. \quad (7.200)$$

This leads to an effective diffusion D of the vortices

$$D \sim \nu r_0^2 \sim N v_* r_0, \quad (7.201)$$

which is the same result as that obtained in (7.198) by the statistical theory for point vortices.

It is worthwhile to point out that the point vortex system introduced in this section may be subject to a phase transition to form a vortex lattice, since the interaction force between the point vortices is short range and a local order is likely to be formed. We suggest that further studies of the packing fraction $f_p = n_v \pi r_0^2$ and the vortex–vortex correlation function would be used to distinguish among the turbulent

states described as vortex gas [139], a vortex liquid and a density packed system approaching a vortex lattice.

7.5 Vortex Collapse Revisited

7.5.1 Vortex Collapse

When the diamagnetic drift is neglected, the equations for the motion of vortices are given by

$$\frac{d}{dt} \mathbf{r}_\alpha = \frac{1}{2\pi} \sum_\beta \gamma_\beta \frac{\mathbf{z} \times (\mathbf{r}_\alpha - \mathbf{r}_\beta)}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|} K_1(|\mathbf{r}_\alpha - \mathbf{r}_\beta|). \quad (7.202)$$

The short range nature of the interaction between the vortices stems from the shielding effect. This implies that the distance between the vortices is short enough for the shielding to be neglected, the dynamical behavior of vortices is described by the following equation:

$$\frac{d}{dt} \mathbf{r}_\alpha = \frac{1}{2\pi} \sum_\beta \gamma_\beta \frac{\mathbf{z} \times (\mathbf{r}_\alpha - \mathbf{r}_\beta)}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|^2}, \quad (7.203)$$

which are the point vortex equations corresponding to Euler's equation. There are three conserved quantities associated with (7.203), that is, total circulation G , the angular momentum L and the energy E which are given, respectively, by

$$G = \int \omega d\mathbf{r}, \quad (7.204)$$

$$L = -\frac{1}{2} \int \omega r^2 d\mathbf{r}, \quad (7.205)$$

$$E = -\frac{1}{2} \int (\nabla \phi)^2 d\mathbf{r} = -\frac{1}{2} \int \omega \phi d\mathbf{r}. \quad (7.206)$$

Many studies [101, 140, 141] have been reported on solutions of (7.203). Among them are also remarkable works concerned with the vortex collapse for which three vortices converge self-similarly. For the collapse to occur, suitable initial conditions have to be satisfied in addition to the condition that $\gamma_1 \gamma_2 + \gamma_2 \gamma_3 + \gamma_3 \gamma_1 = 0$ [141]. Depending on the sign of vorticities γ_α , $\{\alpha = 1, 2 \text{ and } 3\}$, the vortices can also diverge [144]. This type of “explosive” motion can be considered as an “inversion” of the collapse.

The vortex collapse is algebraically unstable: a small deviation from the condition no longer leads to the collapse. In those cases, three vortices first converge to some extent and then turn to diverge. Once they start to diverge, they keep diverging as long as (7.203) is applicable. This is, however, not the case for (7.202). When

they diverge, two of them are to form a pair to travel together and the third one is left behind, because of the short range nature of the interaction. However, since the vorticities of two vortices traveling together are opposite in sign and have to be different in magnitude, the two vortices perform a circular motion. They eventually return to the place where the third one remains. At the time they come to the position where the two sides of the triangle become equal, the pair is renewed by changing the partner. The new pair travels in a similar way as before until it comes back to the place and again, exchange its partner.

Introducing complex variables $z_\alpha = x_\alpha + iy_\alpha$ representing the vortex position $\mathbf{r}_\alpha = (x_\alpha, y_\alpha)$, (7.202) is rewritten as

$$\frac{d}{dt}z_\alpha = \frac{1}{2\pi} \sum_{\beta} \frac{z_\alpha - z_\beta}{|z_\alpha - z_\beta|} K_1(|z_\alpha - z_\beta|). \quad (7.207)$$

With the aid of the cyclic permutation, the set of equations (7.207) for three vortices reduces to

$$\frac{d\rho}{dt} = -\frac{\gamma_1}{2\pi} \left[\eta \frac{K_1(\ell)}{\ell} - K_1(\eta) \right] \sin \Psi, \quad (7.208)$$

$$\frac{d\eta}{dt} = \frac{\gamma_2}{2\pi} \left[\rho \frac{K_1(\ell)}{\ell} - K_1(\rho) \right] \sin \Psi, \quad (7.209)$$

where

$$z_2 - z_3 = \rho e^{i\theta}, \quad z_3 - z_1 = \eta e^{i\phi}, \quad (7.210)$$

$$\Psi = \theta - \phi, \quad \ell = |z_1 - z_2| = \sqrt{\rho^2 + \eta^2 + 2\rho\eta \cos \Psi}.$$

From (7.204) and (7.210), we have for $\gamma_1 + \gamma_2 + \gamma_3 \neq 0$

$$z_1 = -\frac{1}{\gamma_1 + \gamma_2 + \gamma_3} \{(\gamma_2 + \gamma_3)\eta e^{i\phi} + \gamma_2 \rho e^{i\theta} - G\}, \quad (7.211)$$

$$z_2 = \frac{1}{\gamma_1 + \gamma_2 + \gamma_3} \{\gamma_1 \eta e^{i\phi} + (\gamma_1 + \gamma_3)\rho e^{i\theta} + G\}, \quad (7.212)$$

$$z_3 = \frac{1}{\gamma_1 + \gamma_2 + \gamma_3} \{\gamma_1 \eta e^{i\phi} - \gamma_2 \rho e^{i\theta} + G\}. \quad (7.213)$$

The constants of motion L and E are now expressed by

$$L = \frac{1}{\gamma_1 + \gamma_2 + \gamma_3} \{\gamma_1 \gamma_2 \ell^2 + \gamma_2 \gamma_3 \rho^2 + \gamma_3 \gamma_1 \eta^2 + |G|^2\}, \quad (7.214)$$

$$E = \gamma_1 \gamma_2 K_0(\ell) + \gamma_2 \gamma_3 K_0(\rho) + \gamma_3 \gamma_1 K_0(\eta). \quad (7.215)$$

With the use of (7.214) and (7.215), either (7.208) or (7.209) is enough for obtaining the solution in principle. However, (7.215) is not convenient to use. Therefore, we consider (7.208) and (7.209) with (7.214).

First, we review the collapse for which the modified Bessel function can be approximated as

$$K_0(x) \simeq -0.577 + \ln 2 - \ln x \quad \text{for } x \ll 1, \quad (7.216)$$

and for which the following conditions hold:

$$L = 0 \quad \text{and} \quad \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1 = 0, \quad (7.217)$$

where G is chosen to be zero. The first condition of (7.217) is explicitly written as

$$\frac{\gamma_2}{\gamma_1 + \gamma_2} \rho_0^2 + \frac{\gamma_1}{\gamma_1 + \gamma_2} \eta_0^2 = \ell_0^2, \quad (7.218)$$

where ρ_0, η_0 , and ℓ_0 are their initial values. Equation (7.218) gives the relation among the initial positions of the three vortices for the collapse to occur.

Then, (7.208) and (7.209) become

$$\frac{d\rho}{dt} = -\frac{\gamma_1}{2\pi} \left(\frac{\eta}{\ell^2} - \frac{1}{\eta} \right) \sin \Psi, \quad (7.219)$$

$$\frac{d\eta}{dt} = \frac{\gamma_2}{2\pi} \left(\frac{\rho}{\ell^2} - \frac{1}{\rho} \right) \sin \Psi, \quad (7.220)$$

which are combined to give

$$\frac{d\rho}{d\eta} = \frac{\rho}{\eta}. \quad (7.221)$$

Thus, we have

$$\frac{\rho}{\eta} = \frac{\rho_0}{\eta_0} \equiv \alpha. \quad (7.222)$$

For this case, Ψ is also constant:

$$\cos \Psi = \frac{\gamma_3(\gamma_2 + \alpha^2\gamma_1)}{2\alpha\gamma_1\gamma_2}. \quad (7.223)$$

Then, (7.220) reduces to

$$\frac{d\eta}{dt} = \frac{(\alpha^2 - 1)\gamma_1\gamma_2}{2\pi\alpha(\gamma_1 + \alpha^2\gamma_2)\eta} \sin \Psi \equiv \frac{A}{\eta}, \quad (7.224)$$

which gives

$$\eta = \sqrt{\eta_0^2 + 2At}, \quad \rho = \alpha\eta, \quad \ell = \sqrt{-\frac{\gamma_3}{\gamma_1\gamma_2}(\gamma_1 + \alpha^2\gamma_2)\eta}. \quad (7.225)$$

From the imaginary part of $d(z_2 - z_3)/dt$, we have

$$\begin{aligned} \frac{d\theta}{dt} &= -\frac{1}{2\pi\rho} \left\{ (\gamma_2 + \gamma_3)K_1(\rho) + \gamma_1 \left[\frac{\rho + \eta \cos \Psi}{\ell} K_1(\ell) - K_1(\eta) \cos \Psi \right] \right\} \\ &\simeq -\frac{1}{2\pi\rho} \left\{ \frac{\gamma_2 + \gamma_3}{\rho} + \gamma_1 \left[\frac{\rho + \eta \cos \Psi}{\ell^2} - \frac{\cos \Psi}{\eta} \right] \right\} \\ &\equiv \frac{B}{\eta_0^2 + 2At}, \end{aligned} \quad (7.226)$$

leading to

$$\theta = \frac{B}{2A} \ln \left| \frac{\eta_0^2 + 2At}{\eta_0^2} \right| + \theta_0, \quad (7.227)$$

where

$$B = -\frac{1}{2\pi\alpha} \left\{ \frac{\gamma_1 + \gamma_2}{\alpha} - \gamma_1 \left[\frac{\gamma_1\gamma_2(\alpha + \cos \Psi)}{\gamma_3(\gamma_1 + \alpha^2\gamma_2)} + \cos \Psi \right] \right\}.$$

Equations (7.225)–(7.227) give a self-similar solution which diverges or converges depending on the sign of A , that is, $\alpha^2 - 1$ except for the following two cases:

$$(a) \quad \alpha = \frac{1}{-\gamma_1\gamma_3} [\gamma_1\gamma_2 \pm \sqrt{-\gamma_1\gamma_2\gamma_3(\gamma_1 + \gamma_2 + \gamma_3)}], \quad (7.228)$$

$$(b) \quad \alpha = 1, \quad (7.229)$$

The case (a) comes from $\sin \Psi = 0$. The both cases correspond to a rigid rotation.

7.5.2 Boomerang Interaction of Three Vortices

Now, we relax one of the conditions for the collapse given by (7.217), that is,

$$L \neq 0 \quad \text{and} \quad \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1 = 0. \quad (7.230)$$

First, in order to see the effect of (7.230) on the converging collapse solution, we may use the approximate expression (7.216) for the vortices with short distances. Under the conditions (7.230), the conserved quantities (7.214) and (7.215) are rewritten as

$$\gamma_1\gamma_2\ell^2 + \gamma_2\gamma_3\rho^2 + \gamma_3\gamma_1\eta^2 = \varepsilon, \quad (7.231)$$

$$\left(\frac{\ell}{\ell_0}\right)^{\gamma_1\gamma_2} \left(\frac{\rho}{\rho_0}\right)^{\gamma_2\gamma_3} \left(\frac{\eta}{\eta_0}\right)^{\gamma_3\gamma_1} = 1, \quad (7.232)$$

where $\varepsilon = (\gamma_1 + \gamma_2 + \gamma_3)L$ and G is chosen zero. The ℓ_0 , ρ_0 , and η_0 are their initial values, respectively. For simplicity, in the following, we consider a special case:

$$(\gamma_1, \gamma_2, \gamma_3) = (\gamma, \gamma, -\gamma/2), \quad (7.233)$$

which satisfies one of the conditions given by (7.230). Then, (7.232) simply becomes

$$\frac{\ell^2}{\rho\eta} = \frac{\ell_0^2}{\rho_0\eta_0} = C, \quad (7.234)$$

which is combined with (7.231) and (7.210) to give

$$\rho^2 + \eta^2 - 2C\rho\eta + \frac{2\varepsilon}{\gamma^2} = 0, \quad (7.235)$$

$$\cos \Psi = \frac{\ell^2 - \rho^2 - \eta^2}{2\rho\eta} = -\frac{C}{2} + \frac{\varepsilon}{\gamma^2\rho\eta}. \quad (7.236)$$

Then from (7.218), (7.219), and (7.234), we have

$$\frac{d}{dt}\ell^2 = \pm \frac{\gamma}{2\pi\ell^4} \{(C^2 - 1)(4 - C^2)(\ell^2 - \ell_1^2)(\ell^2 - \ell_2^2)(\ell^2 - \ell_3^2)(\ell^2 - \ell_4^2)\}^{1/2}, \quad (7.237)$$

where

$$\{\ell_1^2, \ell_2^2, \ell_3^2, \ell_4^2\} = \left\{ \frac{C}{C-1} \frac{\varepsilon}{\gamma^2}, \frac{C}{C+1} \frac{\varepsilon}{\gamma^2}, \frac{2C}{2+C} \frac{\varepsilon}{\gamma^2}, \frac{2C}{C-2} \frac{\varepsilon}{\gamma^2} \right\}, \quad (7.238)$$

with

$$\ell_1 > \ell_2 > \ell_3 > \ell_4. \quad (7.239)$$

Equation (7.237) gives the same solution as (7.225) for $\varepsilon = 0$. The solution to (7.237) is given by

$$\ell_{1g}^4 \left(\frac{\beta_1}{\beta} \right)^4 \{V_0 + 2 \frac{\beta^2 - \beta_1^2}{\beta_1^2} V_1 + \frac{(\beta^2 - \beta_1^2)^2}{\beta^4} V_2\} = \pm \frac{\gamma}{2\pi} \sqrt{(C^2 - 1)(4 - C^2)t}, \quad (7.240)$$

where

$$\begin{aligned} V_0 &= F(\phi, k), \quad V_1 = \Pi(\phi, \beta^2, k), \\ V_2 &= \frac{1}{2(\beta^2 - 1)(k^2 - \beta^2)} \{ \beta^2 E(u) + (k^2 - \beta^2)u \\ &\quad + (2\beta^2 k^2 + 2\beta^2 - \beta^4 - 3k^2) \Pi(\phi, \beta^2, k) - \frac{\beta^4 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1 - \beta^2 \operatorname{sn}^2 u} \}, \end{aligned}$$

$$\phi = \sin^{-1} \sqrt{\frac{(\ell_2^2 - \ell_4^2)(\ell^2 - \ell_1^2)}{(\ell_1^2 - \ell_4^2)(\ell^2 - \ell_2^2)}}, \quad \text{sn } u = \sin \phi,$$

$$\beta^2 = \frac{\ell_1^2 - \ell_4^2}{\ell_2^2 - \ell_4^2}, \quad \beta_1^2 = \frac{\ell_2^2}{\ell_1^2} \beta^2,$$

$$k^2 = \frac{\ell_2^2 - \ell_3^2}{\ell_1^2 - \ell_3^2} \beta^2, \quad g = \frac{2}{\sqrt{(\ell_1^2 - \ell_3^2)(\ell_2^2 - \ell_4^2)}}.$$

The solution is singular with respect to ε and does not smoothly continue to the collapse solution. At the point where ℓ^2 becomes minimum, that is,

$$\ell^2 = \ell_1^2 = \frac{C}{C-1} \frac{\varepsilon}{\gamma^2} = C\rho\eta, \quad (7.241)$$

ρ is equal to η , which is easily seen from (7.235). This is the point where the partner of the boomerang journey is exchanged.

The boomerang interaction is shown in Fig. 7.20(left) as well as the vortex collapse in (right). Thus, for unshielded vortices, the divergence or explosion, gives unbounded vortex motion, which becomes bounded by the shielding effect. This is, after all, a result which was to be expected. We emphasize that three unshielded vortices can also perform a bounded or localized motion, given a proper choice of vortices and initial positions, but these parameters will then not give a collapse by inversion. The boomerang interaction is also observed numerically in a fluid model [143]. The vortex collapse is thought to provide a key to understanding of fundamental processes of strong turbulence. As discussed for instance by Novikov [141] and Novikov and Sedov [144], the collapse accounts for energy being systematically transferred to small scales, and for vortex dispersion, or “explosion”, energy cascades into the large scales. The boomerang effect described here, an effect which owes its existence to the shielding effect ($\gamma \neq 0$), consequently corresponds to a “sloshing” of energy between large and small scales.

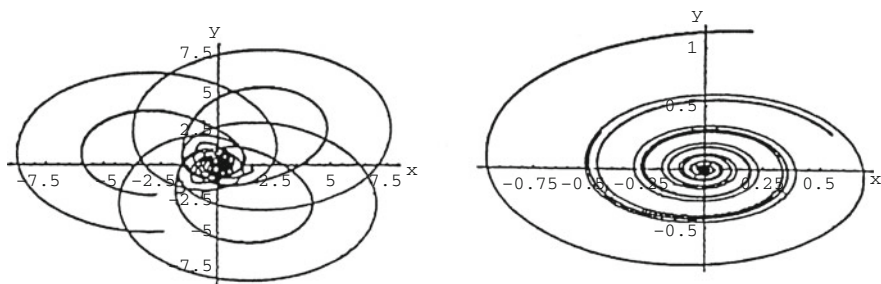


Fig. 7.20 Numerical solutions of the “boomerang interaction” (left) and the vortex collapse (right) along logarithmic spiral trajectories

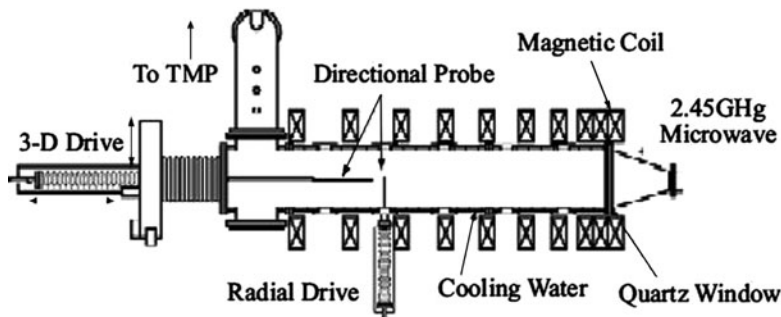


Fig. 7.21 Experimental set-up for the spiral structure

7.6 Spiral Structures in Magnetized Rotating Plasmas

Spiral structures observed in ECR plasmas (Fig. 7.21) [145] and in gun-produced plasmas [146] are also a kind of vortices. The spiral structures observed in ECR plasmas are static under special conditions while those observed in gun-produced plasmas are generally rotating. The characteristic features of the observed spiral structures in the ECR plasmas are (1) the stretching direction of the arm is reversed when the magnetic field is reversed, (2) the stationary structure is observed only in a narrow range of the background pressure, and (3) the spatial extent of the structure in the axial direction is of the order of the plasma radius. In the ECR plasmas, the ratio of the ion collision frequency with neutral particles to the ion cyclotron frequency is small as $\nu_i/\Omega_i \sim 0.05$ and there is another smallness parameter $C_s/r_d\Omega_i$ which is the ratio of nonlinear term to the Lorentz force and is of the same order as ν_i/Ω_i , where C_s and r_d are the ion acoustic velocity and the plasma radius. Furthermore, the ratio of the ion azimuthal drift velocity to the sound velocity v_{i0}/C_s is $0.2 \sim 0.4$. On the other hand in the gun-produced plasmas where ν_i is comparable with Ω_i and the ion azimuthal flow is supersonic, spiral structures are two dimensional and claimed due to the Kelvin–Helmholtz instability in [146].

In this section, we present a theory of the spiral structure observed in the ECR plasmas. The low frequency instabilities such as the collisional drift wave instability, the centrifugal instability and the Kelvin–Helmholtz instability are taken into account, and the linear eigenvalue problem is numerically solved to show the existence of spiral solutions.

Plasmas in a cylindrical vessel are inevitably driven to rotate with the $\mathbf{E} \times \mathbf{B}$ drift due to an ambipolar potential. Then, ions are subject to centrifugal force and their rotation frequency is affected by an effective gravitational drift, while electrons are driven by a diamagnetic drift as well. The difference of those velocities induce charge separation which cannot be fully canceled by electrons whose axial motions are dragged by collision with neutral particles. Thus, fluctuations are excited and particle azimuthal motions are organized in such a way that the core part of the plasmas is almost rotating rigidly while the outer part lags behind the core part

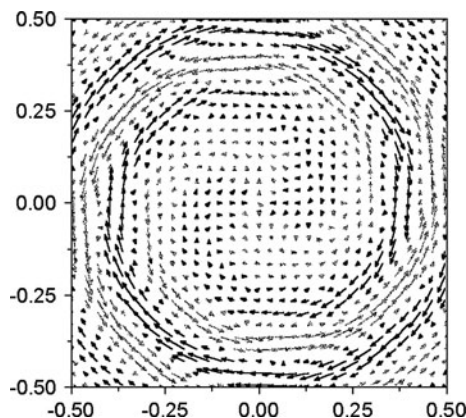


Fig. 7.22 Flow pattern associated with the spiral structure observed in the experiment

because azimuthal velocities do not increase in proportion to the radius, producing a spiral structure. The observed flow pattern is depicted in Fig. 7.22.

Equations for ions and electrons in a magnetized plasma read

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0, \quad (7.242)$$

$$\frac{\partial \mathbf{v}_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \mathbf{v}_\alpha = \frac{e_\alpha}{m_\alpha} \left(-\nabla \phi + \frac{1}{c} \mathbf{v}_\alpha \times \mathbf{B} \right) - \frac{1}{n} \nabla p_\alpha - \nu_\alpha \mathbf{v}_\alpha. \quad (7.243)$$

where n_α , $\mathbf{v}_\alpha$, and ν_α are the density, velocity, and collision frequency of electrons and ions with the neutral particles, respectively, and ϕ is the potential.

In the following, physical quantities are divided into the stationary part and fluctuating part:

$$\begin{pmatrix} n \\ \phi \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} n_0(r, z) \\ \phi_0(r, z) \\ \mathbf{v}_0(r, z) \end{pmatrix} + \sum_{\ell} \begin{pmatrix} n_{\ell}(r, z) \\ \phi_{\ell}(r, z) \\ \mathbf{v}_{\ell}(r, z) \end{pmatrix} e^{i(\ell\theta - \omega t)} + \text{c.c.}$$

For the ion drift, an effective gravitational drift due to a centrifugal force is taken into account in the azimuthal direction, and is neglected in the radial direction since $v_i/\Omega_i \ll 1$. For the electron drift, the diamagnetic drift is dominated over the gravitational drift due to the centrifugal force because of $v_e/\Omega_e \ll 1$.

The rotation frequencies of the ion and electron azimuthal drift now read

$$\omega_{i0} \simeq \omega_E \left(1 - \frac{\omega_E}{\Omega_i} \right), \quad \omega_{e0} \simeq \omega_E + \omega_*, \quad (7.244)$$

where ω_E and ω_* are the frequencies associated with the $\mathbf{E} \times \mathbf{B}$ drift and the diamagnetic drift, respectively, defined by

$$\omega_E = \frac{C_s^2}{\Omega_i} \frac{1}{r} \frac{d\phi_0}{dr}, \quad \omega_* = -\frac{v_T^2}{\Omega_e} \frac{1}{r} \frac{d \ln n_0}{dr},$$

where $C_s^2 = T_e/m_i$, $v_T^2 = T_e/m_e$, $\Omega_i = eB/m_i c$, and $\Omega_e = eB/m_e c$. The $e\phi/T_e$ has been replaced by ϕ as well. The second term in the expression for ω_{i0} is a contribution from the gravitational drift.

The space potential produced by the ion radial transport is short-circuited by the electron axial transport so that we have $\nabla(n_{i0}v_{i0}) = \nabla(n_{e0}v_{e0})$, which determines the structures of the density and potential. Since a solution of this equation is sensitively dependent on the boundary conditions at the end of the field lines unless the plasma is so long that parallel diffusion can be neglected altogether, it is unlikely to obtain a self-consistent solution $n_0(r, z)$ and $\phi_0(r, z)$ to the problem of ambipolar diffusion across magnetic field. Instead in the following, we take a phenomenological approach to assume the density and potential profiles compatible with those of laboratory plasmas. The radial profile of the azimuthal velocity observed in the experiment is shown by a dotted line in Fig. 7.23 which is approximated by a solid line expressed by

$$\frac{v_{i\theta}}{C_s} = \frac{C_s}{\Omega_i} \frac{\partial \phi}{\partial r} = 2.65 r^2 \left(1 - \frac{25}{16} r^2\right), \quad (7.245)$$

which gives

$$\phi = \frac{2.65}{3} \frac{\Omega_i}{C_s} r^3 \left(1 - \frac{15}{16} r^2\right). \quad (7.246)$$

The observed density profile is almost Gaussian giving a constant diamagnetic drift frequency ω_* .

The fluctuating parts of the electron velocities given by (7.242) and (7.243) are

$$u_{e\ell} = \frac{v_T^2}{\Sigma_e(\omega)} \left[-\frac{i\ell(\Omega_e - 2\omega_{e0})}{r} + \Gamma_e(\omega) \frac{\partial}{\partial r} \right] \left(\phi_\ell - \frac{n_{e\ell}}{n_0} \right), \quad (7.247)$$

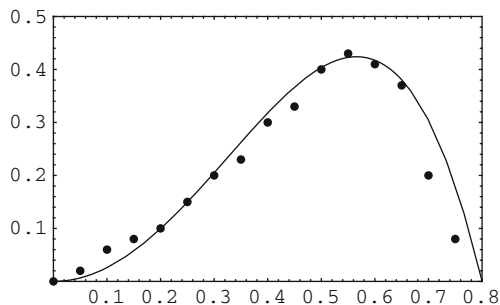


Fig. 7.23 The azimuthal velocity observed in the experiment (*dotted line*) and the *solid line* is used in the calculation

$$v_{e\ell} = \frac{v_T^2}{\Sigma_e(\omega)} \left[\left(\Omega_e - \omega_{e0} - \frac{dv_{e0}}{dr} \right) \frac{\partial}{\partial r} + \frac{i\ell \Gamma_e(\omega)}{r} \right] \left(\phi_\ell - \frac{n_{e\ell}}{n_0} \right), \quad (7.248)$$

$$w_{e\ell} = \frac{v_T^2}{\Gamma_e(\omega)} \frac{\partial}{\partial z} \left(\phi_\ell - \frac{n_{e\ell}}{n_0} \right), \quad (7.249)$$

where

$$\Gamma_e(\omega) = v_e - i(\omega - \ell\omega_{e0}), \quad (7.250)$$

$$\Sigma_e(\omega) = (\Omega_e - 2\omega_{e0}) \left(\Omega_e - \omega_{e0} - \frac{dv_{e0}}{dr} \right) + \Gamma_e(\omega)^2. \quad (7.251)$$

On the other hand, the fluctuating parts of the ion velocities are given by

$$u_{i\ell} = -\frac{C_s^2}{\Sigma_i(\omega)} \left[\frac{i\ell(\Omega_i + 2\omega_{i0})}{r} \phi_\ell + \Gamma_i(\omega) \frac{\partial \phi_\ell}{\partial r} \right], \quad (7.252)$$

$$v_{i\ell} = \frac{C_s^2}{\Sigma_i(\omega)} \left[\left(\Omega_i + \omega_{i0} + \frac{dv_{i0}}{dr} \right) \frac{\partial \phi_\ell}{\partial r} - \frac{i\ell \Gamma_i(\omega)}{r} \phi_\ell \right], \quad (7.253)$$

$$w_{i\ell} = -\frac{C_s^2}{\Gamma_i(\omega)} \frac{\partial \phi_\ell}{\partial z}, \quad (7.254)$$

where

$$\Gamma_i(\omega) = v_i - i(\omega - \ell\omega_{i0}), \quad (7.255)$$

$$\Sigma_i(\omega) = (\Omega_i + 2\omega_{i0}) \left(\Omega_i + \omega_{i0} + \frac{dv_{i0}}{dr} \right) + \Gamma_i(\omega)^2. \quad (7.256)$$

Substituting the above velocities into the electron and ion continuity equations, invoking charge neutrality $n_{e\ell} = n_{i\ell} = n_\ell$, and assuming the axial dependence of the potential with the normalizations $\xi = r/r_d$ and $\eta = z/r_d$ as

$$\phi_\ell(\xi, \eta) = \phi_\ell(\xi) e^{-(\mu - ik)\eta}, \quad (7.257)$$

we have

$$\frac{d^2 \phi_\ell}{d\xi^2} + \left[\frac{1}{\xi} + \frac{d \ln n_0}{d\xi} \right] \frac{d \phi_\ell}{d\xi} + \left[\beta(\xi) - \frac{\ell^2}{\xi^2} \right] \phi_\ell = 0, \quad (7.258)$$

where

$$\begin{aligned} \beta(\xi) = & \frac{(\mu - ik)^2 \Omega_e \Omega_i}{\Gamma_e(\omega) \Gamma_i(\omega)} \left(1 - \frac{\ell \omega_*}{\omega - \ell \omega_E} \right) \\ & - \frac{i\ell}{\xi} \frac{1}{\Gamma_i(\omega)} \left\{ \left(\omega_{i0} + \frac{1}{r_d} \frac{dv_{i0}}{d\xi} + \frac{\ell \omega_E^2}{\omega - \ell \omega_E} \right) \frac{d \ln n_0}{d\xi} + \frac{d\omega_{i0}}{d\xi} + \frac{1}{r_d} \frac{d^2 v_{i0}}{d\xi^2} \right\}. \end{aligned} \quad (7.259)$$

In deriving (7.259), we have neglected terms of the order of or less than $O(\Omega_i/\Omega_e)$ and $O(\omega_{i0}/\Omega_i)$ and used the following approximation derived from the electron or ion continuity equation,

$$\frac{n_\ell}{n_0} \simeq -\frac{v_T^2}{\Omega_e^2} \frac{\ell \Omega_e}{\omega - \ell(\omega_{e0} - \omega_*)} \frac{1}{r} \frac{d \ln n_0}{dr} \phi_\ell = \frac{\ell \omega_*}{\omega - \ell \omega_E} \phi_\ell. \quad (7.260)$$

Equation (7.258) describes low frequency fluctuations excited by both the collisional drift wave instability and the flute mode instability such as the gravitational instability due to the centrifugal force acting on ions and the Kelvin–Helmholtz instability. The difference between (7.258) and the equation derived by Rosenbluth and Simon [147] is that the quasi-charge neutrality is assumed in (7.258) while an ion diamagnetic drift is taken into account instead of an electron diamagnetic drift in [147].

Putting that $\phi_\ell(\xi) = \psi_\ell(\xi)/\sqrt{n_0(\xi)}$, (7.258) is transformed to

$$\frac{d^2 \psi_\ell}{d\xi^2} + \frac{1}{\xi} \frac{d\psi_\ell}{d\xi} + \left[A(\xi) - \frac{\ell^2}{\xi^2} \right] \psi_\ell = 0, \quad (7.261)$$

where

$$A(\xi) = \beta(\xi) - \frac{1}{4} \left(\frac{d \ln n_0}{d\xi} \right)^2 - \frac{1}{2} \left(\frac{d}{d\xi} + \frac{1}{\xi} \right) \frac{d \ln n_0}{d\xi}.$$

The ratio of the contribution from the drift wave to that from the flute mode is estimated to be $(\mu^2 + k^2)(\Omega_e/\nu_e)(r_d \Omega_i/C_s)$. Then, the collisional drift wave instability is dominant when $\mu^2 + k^2 \gg (\nu_e/\Omega_e)(C_s/r_d \Omega_i)$ is satisfied. The quantity $(\nu_e/\Omega_e)(C_s/r_d \Omega_i)$ is of the order of 10^{-4} to 10^{-5} for the laboratory plasmas and thus we only consider the cases of fluctuations due to the collisional drift wave instability:

$$A(\xi) \simeq \frac{(\mu - ik)^2 \Omega_e \Omega_i}{\Gamma_e(\omega) \Gamma_i(\omega)} \frac{\omega - \ell \omega_{e0}}{\omega - \ell \omega_E}$$

Now, the characteristic features of the solutions can be examined as follows. The solution is approximated in the case of weak ξ -dependence of the zero-th order quantities by $\psi_\ell(\xi) \simeq J_\ell(\xi \sqrt{A(\xi)})$, where J_ℓ is the Bessel function of the first kind. The real part of the argument of the Bessel function should be positive to give a convergent behavior while the imaginary part is responsible for a spiral structure which comes from the imaginary part of ω , $\Gamma_e(\omega)$, and $\Gamma_i(\omega)$. Multiplying ψ_ℓ^* to (7.261) and integrating the resultant equation from the center to the edge of the plasma under the boundary condition $\psi_\ell(0) = \psi_\ell(1) = 0$, we have

$$\int_0^1 \xi \left\{ \left| \frac{d\psi_\ell}{d\xi} \right|^2 + \frac{\ell^2}{\xi^2} |\psi_\ell|^2 - A(\xi) |\psi_\ell|^2 \right\} d\xi = 0. \quad (7.262)$$

From the imaginary part of this equation, we have

$$\int \Im A(\xi) \xi |\psi_\ell(\xi)|^2 d\xi = 0,$$

which corresponds to the Rayleigh condition. For a collisional drift mode, we have at the marginal instability ($\gamma \sim 0$) for $\mu \gg k$

$$\Im A(\xi) = \frac{v_e \mu^2 \Omega_e \Omega_i (\omega_r - \ell \omega_{e0}) (\omega_r - \ell \omega_{i0}) (\omega_r - \ell \omega_E)}{|\Gamma_e(\omega)|^2 |\Gamma_i(\omega)|^2 [(\omega_r - \ell \omega_E)^2 + \gamma^2]},$$

from which the Rayleigh condition holds even when $\omega_r = 0$, indicating the formation of stationary spiral structures. The winding direction of spiral arms is reversed when the magnetic fields is changed in sign since the imaginary part of $A(\xi)$ is proportional to the odd power of the magnetic field for $\omega_r = 0$. Certainly, rotating spirals exist as well.

Equation (7.258) is numerically solved under the boundary condition $\psi_\ell(0) = \psi_\ell(1) = 0$ [148]. There are both stationary ($\omega_r = 0$) and rotating ($\omega_r \neq 0$) spiral solutions for a given profile of the density and the potential. The numerical results for the stationary solution are shown in Figs. 7.24 and 7.25, where Fig. 7.24 is the radial potential profile, and Fig. 7.25 is the density perturbation contour calculated by (7.260). The spiral structure is clearly seen in the figure, and its arm radius is a linearly increasing function of azimuthal angle. The contour structure is similar to the observed spiral. The spiral structure is identified with the Archimedes spiral which is seen from an eikonal approximation for the solution of (7.258), that is, $\phi_\ell(\xi) \sim \exp(\pm \int \sqrt{\beta(\xi)} d\xi) \sim \exp(\pm i \Im \{ \sqrt{\beta(\xi)} \} \xi)$ for $\xi \neq 0$.

The vector field plot of ion velocity Fig. 7.22 associated with the spiral structure exhibits the similar spiral structure, which well explained by the $\mathbf{E} \times \mathbf{B}$ drift due to the perturbed potential ϕ_ℓ . It should be noted that the flow field associated with this spiral indicates a circulation of the flow between the core and peripheral regions.

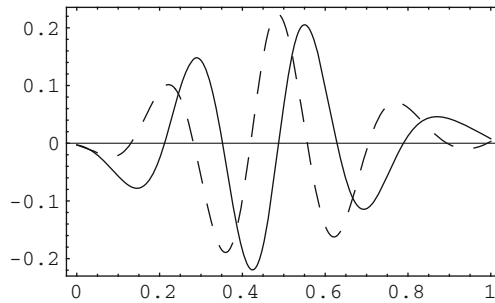


Fig. 7.24 The eigen function of the stationary spiral solution for $M/m = 80,000$, $k = 0.0225$, $\gamma/\Omega_i = 0.024$: the radial profile of the potential: a solid line is the real part, and a dotted line the imaginary part

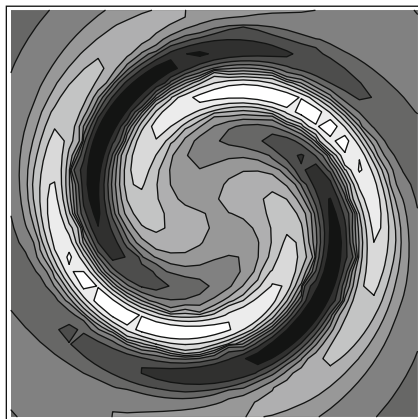


Fig. 7.25 Density perturbation contour plot of the stationary spiral solution for $M/m = 80,000$, $k = 0.0225$, $\gamma/\Omega_i = 0.024$:

Even when the contributions from the centrifugal and the Kelvin–Helmholtz instabilities are neglected in numerics, there is no change in this pattern. Thus, the spiral structure for this choice of parameters is made up only by the collisional drift wave instability. The imaginary part of the eigenvalue γ decreases with an azimuthal mode number m , which corresponds to the fact that observed stationary spiral structures are always with two arms.

In the above, a special profile for the potential (7.246) is used since numerical spiral structures are supposed to be compared with those observed in [145]. However, spiral structures have been checked insensitive to spatial profiles of ω_E .

The axial structure of the spiral is shown to be described by the Ginzburg–Landau equation [149].

Formation of spiral structures is a rather general characteristic in magnetized rotating plasmas since the energy stored in the form of plasma inhomogeneity such as density and velocity shear is released to give an instability which causes a phase difference between the real and imaginary parts of eigen-functions, driving a spiral structure. Instabilities could be the collisional drift wave instability, centrifugal instability, and Kelvin–Helmholtz instability, although the collisional drift wave instability is dominant for ordinary laboratory plasmas. At the marginal instability where the growth rate is zero, however, collision plays the same role as instabilities of producing phase difference between the real and imaginary parts of the eigen-function. Therefore, collision is also essential to the formation of spiral structures. Under the special condition that the potential is such that ω_E becomes zero somewhere in the radial direction within the plasma, the stationary spiral structure becomes similar to those observed in the experiment. For a wide variety of density and potential profiles rotating spiral structures are normally excited.

Chapter 8

Chaos in Plasmas

Chaotic motions of particles in plasmas are intensively studied in connection with an auxiliary heating for fusion plasmas such as wave heating, stochastic heating, and turbulent heating which are all concerned with diffusion in velocity space. Chirikov [150] who had been engaged in stochastic heating introduced the standard mapping which has been a basis of chaos in Hamilton systems.

Studying particle dynamics under waves provides many new notions to understand and tools to explore nonlinear world. In this chapter, we introduce basic concepts of dynamical chaos and then give a review of chaos in plasmas, a web of stochastic layers which brings a global diffusion in phase space even in two dimensional dynamical systems, chaos in a magnetic mirror which is responsible for the generation of high energy electrons, chaos in current-carrying sheath, chaos of magnetic field lines, and a transport barrier produced by the Tokamap. In addition, one example of macroscopic property affected by the onset of chaos in particle dynamics is illustrated.

8.1 Chaos in Conservative Systems

8.1.1 Pendulum

Denoting ψ as an angle between a pendulum of the length ℓ and the vertical axis, the equation of motion for the pendulum is given by

$$v = \ell \frac{d\psi}{dt}, \quad m \frac{dv}{dt} = m\ell \frac{d^2\psi}{dt^2} = -mg \sin \psi, \quad (8.1)$$

which is rewritten as

$$\frac{dp}{dt} = -F \sin \psi, \quad \frac{d\psi}{dt} = Gp, \quad (8.2)$$

where

$$p = mv, \quad F = mg, \quad G = \frac{1}{m\ell}.$$

A Hamiltonian is given by

$$H = \frac{1}{2}Gp^2 - F \cos \psi, \quad (8.3)$$

The motion is classified according to the magnitude of $H = E$ as

$E > F$: rotation ($p \neq 0$, ψ : unbounded motion)

$E < F$: oscillation (motion in a well of the potential ψ : bounded)

$E = F$: separatrix (the boundary between the rotation and the oscillation and infinite period)

There are three fixed points (mod(2π)), $(p_0, \psi_0) = (0, 0), (0, \pm\pi)$. Putting the dynamical variables as

$$p = p_0 + \delta p, \quad \psi = \psi_0 + \delta \psi,$$

linearizing the equation of motion around the fixed points (p_0, ψ_0) , we have

$$\frac{d\delta p}{dt} = F \cos \psi_0 \delta \psi, \quad (8.4)$$

$$\frac{d\delta \psi}{dt} = G \delta p. \quad (8.5)$$

Assuming $\delta p, \delta \psi \propto e^{\lambda t}$, we have

$$\lambda^2 = -FG \cos \psi_0. \quad (8.6)$$

Thus, the fixed point $(p_0, \psi_0) = (0, 0)$ is stable (elliptic fixed point) since λ is complex (pure imaginary), while the fixed point $(p_0, \psi_0) = (0, \pm\pi)$ is unstable (hyperbolic fixed point or saddle point) since λ is real.

The period of the pendulum is given by

$$T = \frac{1}{\sqrt{2G}} \oint \frac{d\psi}{\sqrt{E + F \cos \psi}}. \quad (8.7)$$

Generating function to transform the variables from (p, ψ) to the action and angle variables (J, θ) is given by

$$W = W(J, \psi) = \int^{\psi} p d\psi, \quad (8.8)$$

where

$$\begin{aligned} J(E) &= \frac{2}{\pi} \oint \sqrt{\frac{2}{G}(E + F \cos \psi)} d\psi \\ &= \frac{8}{\pi} \sqrt{\frac{F}{G}} \begin{cases} E(\kappa) - (1 - \kappa^2)K(\kappa) & \text{for } \kappa < 1 \text{ (liberation)} \\ \frac{1}{2}\kappa E(1/\kappa) & \text{for } \kappa > 1 \text{ (rotation)} \end{cases}, \end{aligned} \quad (8.9)$$

$$\begin{aligned}
\theta &= \frac{\partial W}{\partial J} = \frac{dE}{dJ} \frac{\partial W}{\partial E} = \frac{1}{G \frac{dJ}{dE}} \int_0^\psi \frac{d\psi}{\sqrt{2(E + F \cos \psi)/G}} \\
&= \frac{\pi}{2} \begin{cases} E(\eta, \kappa)/K(\kappa) & \text{for } \kappa < 1 \\ 2F(\psi/2, 1/\kappa)/K(1/\kappa) & \text{for } \kappa > 1 \end{cases} \quad (8.10)
\end{aligned}$$

where K and E are the complete elliptic function of the first and second kinds

$$\begin{aligned}
K(\kappa) &= \int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - \kappa^2 \sin^2 \xi}}, \quad E(\kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2 \xi} d\xi, \\
\kappa &= \sqrt{\frac{1}{2} \left(1 + \frac{E}{F} \right)}, \quad \eta = \sin^{-1}(\sin(\psi/2)/\kappa).
\end{aligned}$$

Then, a new Hamiltonian is given by

$$H = \omega J, \quad \omega = \frac{\pi}{2} \omega_0 \begin{cases} 1/K(\kappa) & \text{for } \kappa < 1 \\ 2\kappa/K(1/\kappa) & \text{for } \kappa > 1 \end{cases}, \quad \omega_0 = \sqrt{FG}. \quad (8.11)$$

and the equation of motion is given by

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial J} = \omega, \quad \frac{dJ}{dt} = -\frac{\partial H}{\partial \theta} = 0, \quad (8.12)$$

which yields

$$J = \text{const}, \quad \theta = \omega t + \theta_0, \quad (8.13)$$

The frequency is in general a function of energy and in the limit of $\kappa \rightarrow 0$ becomes independent of energy because it reduces to a harmonic oscillator. In the vicinity of the separatrix, we have

$$\lim_{\kappa \rightarrow 1} \frac{\omega(\kappa)}{\omega_0} = \begin{cases} \frac{\pi}{2 \ln[4/\sqrt{1 - \kappa^2}]} & \text{for } \kappa < 1, \\ \frac{\pi}{\ln[4/\sqrt{\kappa^2 - 1}]} & \text{for } \kappa > 1, \end{cases}$$

which shows that the frequency diverges for $\kappa \rightarrow 1$.

The orbit of the separatrix is obtained by putting $E = F$ for which we have

$$p = \frac{\sqrt{2}\omega_0}{G} \sqrt{1 + \cos \psi} = \pm \frac{2\omega_0}{G \cos \frac{\psi}{2}}. \quad (8.14)$$

Then, the equation of motion becomes

$$\frac{d\psi}{dt} = \pm 2\omega_0 \cos \frac{\psi}{2}, \quad (8.15)$$

which is solved to give

$$\psi = 4 \tan^{-1}(e^{\omega_0 t}) - \pi. \quad (8.16)$$

8.1.2 Resonance

Suppose an unperturbed state is a periodic system of one degree of freedom described by $H_0(J)$ and a perturbed Hamiltonian is given by

$$H_1(J, \theta, t) = \sum_{m,n} H_{mn}(J) e^{i(m\theta + n\Omega t)}. \quad (8.17)$$

The total Hamiltonian is given by

$$H = H_0(J) + H_1(J, \theta, t), \quad (8.18)$$

where

$$\omega = \frac{dH_0}{dJ}.$$

Both the energy and action change by the perturbation and therefore the frequency is also changed. This change is calculated by introducing a new Hamiltonian obtained under the variable transformation from (J, θ) to $(\bar{J}, \bar{\theta})$ and requiring that the new Hamiltonian does not change over the time longer than the time scale of the perturbation. Choosing a generating function of transformation as

$$S(\bar{J}, \theta) = \bar{J}\theta + S_1(\bar{J}, \theta, t), \quad (8.19)$$

the old and new variables are given by

$$J = \frac{\partial S}{\partial \theta}, \quad \bar{\theta} = \frac{\partial S}{\partial \bar{J}}, \quad (8.20)$$

and the new Hamiltonian is expressed by

$$\begin{aligned} \bar{H}(\bar{J}, \bar{\theta}, t) &= H(J, \theta, t) + \frac{\partial S_1}{\partial t} \\ &= H\left(\bar{J} + \frac{\partial S_1}{\partial \theta}, \bar{\theta} - \frac{\partial S_1}{\partial \bar{J}}, t\right) + \frac{\partial S_1}{\partial t} \\ &= H_0(\bar{J}) + \langle H_1 \rangle + \frac{\partial S_1}{\partial t} + \omega \frac{\partial S_1}{\partial \bar{\theta}} + \tilde{H}_1, \end{aligned} \quad (8.21)$$

where the perturbation H_1 is divided into non-oscillating part $\langle H_1 \rangle$ and oscillating part $\tilde{H}_1$. From the requirement that the new Hamiltonian has no oscillating part, we choose the generating function so as to satisfy the following equation

$$\frac{\partial S_1}{\partial t} + \omega \frac{\partial S_1}{\partial \bar{\theta}} + \tilde{H}_1 = 0, \quad (8.22)$$

which gives the generating function as

$$S_1 = i \sum_{m,n \neq 0} \frac{H_{mn}(\bar{J})}{m\omega + n\Omega} e^{i(m\bar{\theta} + n\Omega t)}. \quad (8.23)$$

The oscillation of the unperturbed dynamical system is in resonance with the frequency of the perturbation at the zero of the denominator

$$m_0\omega(J) + n_0\Omega = 0, \quad (8.24)$$

and the amplitude increases linearly in time. This is called a secular term and is seen by calculating the change of action $\Delta J = \bar{J} - J$ as

$$\Delta J = -\frac{\partial S_1}{\partial \theta} \simeq \frac{m_0 H_{m_0 n_0}(\bar{J})}{m_0\omega + n_0\Omega} e^{i(m_0\omega + n_0\Omega)t} - \text{c.c.} \propto t. \quad (8.25)$$

However, since the frequency changes with J , the resonance condition violates and the amplitude does not increase infinitely. Instead, the orbit satisfying the resonance condition becomes a periodic solution and deviates from the linear orbit. From (8.21) and (8.17), we have

$$\langle H_1(\bar{J}, \bar{\theta}) \rangle = \sum_m H_{m0}(\bar{J}) e^{im\bar{\theta}} = H_{00}(\bar{J}) + 2H_{10}(\bar{J}) \cos \bar{\theta}, \quad (8.26)$$

where we have retained terms with $m = 0, \pm 1$. Therefore, (8.21), (8.22), and (8.26) are combined to give

$$\tilde{H}(\bar{J}, \bar{\theta}, t) = H_0(\bar{J}) + H_{00}(\bar{J}) + 2H_{10}(\bar{J}) \cos \bar{\theta}, \quad (8.27)$$

which is a pendulum Hamiltonian. Thus, the resonance changes the orbit described by $H_0(J)$ into that described by (8.27) which is called a resonance island. The perturbed orbit is deformed largely when the resonance occurs. When the perturbation amplitude is large enough for the neighboring resonance islands to overlap, the particle can transfer from one orbit to another successively to move in all the phase space. This is the onset of global chaos.

In the vicinity of the separatrix, the width of the orbits covers both the inside and outside of the separatrix and the solution exhibits rotation and oscillation irregularly. On the other hand, an orbit near the bottom of the potential does not change its oscillation motion provided that the width of the resonance island is small. This is understood by seeing how the frequency responds to the energy.

$$\frac{d\omega}{dE} \propto \begin{cases} \frac{1}{\kappa^2} \rightarrow \infty & \text{for } \kappa \rightarrow 0, \\ \frac{1}{K(\kappa)} \rightarrow 0 & \text{for } \kappa \rightarrow 1. \end{cases} \quad (8.28)$$

When the resonance islands of neighboring orbits overlap, a particle can switch one orbit to another at the crossing of two orbits and move globally in the phase space.

When a perturbation is applied to a periodic system,

- (local chaos) particles near the separatrix are affected and in a small region of the phase space covering the separatrix the particle motion becomes chaotic.
- (global chaos) for a large perturbation, the resonance islands belonging to the orbits overlap. When the resonance islands are connected, the particle motion becomes globally chaotic.

8.1.3 Resonance in Multiple Periodic Systems

A single periodic system has resonance only when an oscillating perturbation is applied. In a multiple periodic system, resonance is intrinsic, which is obvious by considering that a single periodic system together with a perturbation is regarded as a new doubly periodic system.

Suppose a doubly periodic motion of a particle on a torus whose Hamiltonian is given by

$$H_0 = H_0(J_1, J_2). \quad (8.29)$$

In this system, the energy and action are conserved and the action variable parametrizes the conservation surface in the phase space as the radius of the concentric circle. The motion is characterized by the toroidal frequency ω_1 and the poloidal frequency ω_2 . When the aspect ratio

$$\alpha = \frac{\omega_1}{\omega_2}, \quad (8.30)$$

is commensurate ($\alpha = r/s, r, s : \text{integer}$), the motion is periodic after rotating r turns in the toroidal direction and s turns in the poloidal direction, while when the aspect ratio is incommensurate, one orbit covers the surface of a torus densely to be a quasi-periodic motion. If we plot the positions the particle crosses a fixed surface tangential to the toroidal axis, a periodic solution is presented by discrete points distributed on a circle with equal distance and a quasi-periodic solution is a set of points distributed densely on a circle. The cross section is called the Poincaré cross section and the procedure of plotting an orbit on the Poincaré cross section is called the Poincaré map.

In a doubly periodic motion, tori belonging to different action variables are concentric and never overlap. However, in higher dimension systems, since torus can cross each other, a particle repeats changing from one orbit to another, leading to diffusion. This is called the Arnold diffusion.

Suppose a perturbation

$$H_1(\mathbf{J}, \boldsymbol{\theta}) = \sum_{m,n} H_{m,n}(\mathbf{J}) e^{i(m\theta_1 + n\theta_2)}, \quad (8.31)$$

is applied to a doubly periodic system $H_0(\mathbf{J})$. The frequencies are given by

$$\omega_1(\mathbf{J}) = \frac{\partial H_0(\mathbf{J})}{\partial J_1}, \quad \omega_2(\mathbf{J}) = \frac{\partial H_0(\mathbf{J})}{\partial J_2}, \quad \frac{\omega_1}{\omega_2} = \frac{r}{s}. \quad (8.32)$$

For a resonance $r\omega_1 \sim s\omega_2$ which gives a secular term, noting that $|r\omega_1 - s\omega_2|/(\omega_1 \text{ or } \omega_2) \ll 1$, we consider to erase one of the action variables $\{J_1, J_2\}$. A generating function of transformation $(\mathbf{J}, \boldsymbol{\theta}) \rightarrow (\hat{\mathbf{J}}, \hat{\boldsymbol{\theta}})$ is given by

$$F_2 = (r\theta_1 - s\theta_2)\hat{J}_1 + \theta_2\hat{J}_2. \quad (8.33)$$

The new and old variables are related through

$$J_1 = \frac{\partial F_2}{\partial \theta_1} = r\hat{J}_1, \quad (8.34)$$

$$J_2 = \frac{\partial F_2}{\partial \theta_2} = \hat{J}_2 - s\hat{J}_1, \quad (8.35)$$

$$\hat{\theta}_1 = \frac{\partial F_2}{\partial \hat{J}_1} = r\theta_1 - s\theta_2, \quad (8.36)$$

$$\hat{\theta}_2 = \frac{\partial F_2}{\partial \hat{J}_2} = \theta_2. \quad (8.37)$$

This transformation is to describe a deviation from the resonance since

$$\frac{d}{dt} \hat{\theta}_1 = r \frac{d\theta_1}{dt} - s \frac{d\theta_2}{dt}. \quad (8.38)$$

We may assume $|\omega_2| < |\omega_1|$ without loss of generality. The new Hamiltonian is given by

$$\begin{aligned} \hat{H} &= \hat{H}_0(\hat{\mathbf{J}}) + \hat{H}_1(\hat{\mathbf{J}}, \hat{\boldsymbol{\theta}}) \\ &= \hat{H}_0(\hat{\mathbf{J}}) + \sum_{m,n} H_{m,n}(\hat{\mathbf{J}}) e^{(i/r)(m\hat{\theta}_1 + (ms+nr)\hat{\theta}_2)}. \end{aligned}$$

Noting that $\hat{\theta}_2$ is a rapidly varying variable, the Hamiltonian is averaged over $\hat{\theta}_2$ to give

$$\begin{aligned} \bar{H} &= \bar{H}_0(\hat{\mathbf{J}}) + \bar{H}_1(\hat{\mathbf{J}}, \hat{\boldsymbol{\theta}}) \\ &= \bar{H}_0(\hat{\mathbf{J}}) + \sum_p H_{pr, -ps}(\hat{\mathbf{J}}) e^{-ip\hat{\theta}_1}, \quad p = \frac{m}{r}. \end{aligned} \quad (8.39)$$

Since $\bar{H}$ does not depend on $\hat{\theta}_2$, $\hat{J}_2$ is an integral: $\hat{J}_2 = J_2 + (s/r)J_1$. Since $\hat{J}_2$ is constant, the Hamiltonian is a function of $(\hat{J}_1, \hat{\theta}_1)$,

$$\bar{H} = \bar{H}(\hat{J}_1, \hat{\theta}_1),$$

which describes the motion in the system of one degree of freedom. Remaining $p = 0, \pm 1$, the Hamiltonian is reduced to that of a pendulum.

$$\bar{H} = \hat{H}_0 + H_{0,0}(\hat{J}) + 2H_{r,-s}(\hat{J}) \cos \hat{\theta}_1, \quad (8.40)$$

This implies that the motion associated with the resonance is equivalent to the motion of a pendulum and the orbit is presented by the resonance island. This is the basis of the self-similar structure of chaos in Hamilton systems.

8.2 Poincarè Mapping

8.2.1 Integrable System

Points on a Poincarè cross section for the dynamical system specified by a Hamiltonian $H_0(J_1, J_2)$ appear on a circle $J_1 = \text{const.}$ at every $\Delta t = 2\pi/\omega_2$ and the separation distance in the θ_1 direction is $\omega_1 \Delta t = 2\pi\omega_1/\omega_2 = 2\pi\alpha$. In general, $\alpha = \alpha(J_1, J_2)$. However, because of $H(J_1, J_2) = E$ and $J_2 = J_2(J_1, E)$, we have $\alpha = \alpha(J_1)$. Thus, a transition of n -th point to $(n + 1)$ -th point is given by

$$J_{n+1} = J_n \quad (8.41)$$

$$\theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}), \quad (8.42)$$

which is a mapping from a circle to a circle and is called a twist map. If the α is commensurate, there are finite number of points on a circle, and otherwise, the points are densely distributed on a circle.

8.2.2 Non Integrable System

8.2.2.1 Perturbed Twist Map

Since in a non integrable system described by a Hamiltonian

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + \varepsilon H_1(\mathbf{J}, \boldsymbol{\theta}), \quad (8.43)$$

during the motion from the n -th point to the $(n + 1)$ -th point on the Poincarè cross section both the action and angle are changed, the map is expressed by

$$J_{n+1} = J_n + \varepsilon f(J_{n+1}, \theta_n), \quad (8.44)$$

$$\theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}) + \varepsilon g(J_{n+1}, \theta_n). \quad (8.45)$$

This is called a perturbed twist map. Here, the reason why J_{n+1} is used on the right hand side is to guarantee for the map to be conserved up to the order of ε .

$$\begin{aligned} \frac{\partial(J_{n+1}, \theta_{n+1})}{\partial(J_n, \theta_n)} &= \frac{\partial J_{n+1}}{\partial J_n} \frac{\partial \theta_{n+1}}{\partial \theta_n} - \frac{\partial \theta_{n+1}}{\partial J_n} \frac{\partial J_{n+1}}{\partial \theta_n} \\ &= \frac{1}{1 - \varepsilon \frac{\partial f}{\partial J_{n+1}}} \left(1 + \varepsilon \frac{\partial g}{\partial \theta_n} \right) \\ &\simeq 1 + \varepsilon \left(\frac{\partial f}{\partial J_{n+1}} + \frac{\partial g}{\partial \theta_n} \right) + \dots. \end{aligned} \quad (8.46)$$

A generating function for the perturbed twist map is given by

$$F = J_{n+1}\theta_n + 2\pi A(J_{n+1}) + \varepsilon B(J_{n+1}, \theta_n).$$

Putting

$$\alpha = \frac{dA}{dJ_{n+1}}, \quad f = -\frac{\partial B}{\partial \theta_n}, \quad g = \frac{\partial B}{\partial J_{n+1}},$$

we obtain a perturbed twist map.

$$\begin{aligned} J_n &= \frac{\partial F}{\partial \theta_n} = J_{n+1} + \varepsilon \frac{\partial B}{\partial \theta_n} = J_{n+1} - \varepsilon f, \\ \theta_{n+1} &= \frac{\partial F}{\partial J_{n+1}} = \theta_n + 2\pi\alpha + \varepsilon g. \end{aligned}$$

Therefore, we have

$$\frac{\partial f}{\partial J_{n+1}} = -\frac{\partial^2 B}{\partial J_{n+1} \partial \theta_n} = -\frac{\partial g}{\partial \theta_n},$$

from which (8.46) leads to a conservation condition up to ε

$$\frac{(J_{n+1}, \theta_{n+1})}{(J_n, \theta_n)} = 1. \quad (8.47)$$

8.2.2.2 Radial Twist Map

When $g = 0$ holds, we have

$$J_{n+1} = J_n + \varepsilon f(J_{n+1}, \theta_n), \quad (8.48)$$

$$\theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}), \quad (8.49)$$

which is called a radial twist map. The fixed points of the unperturbed system are given by

$$J_{n+1} = J_n = J_0, \quad \text{mod } (2\pi\alpha(J_0)) = 0. \quad (8.50)$$

For a perturbed system if we put $J_n = J_0 + \Delta J_n$, then the map is rewritten as

$$\begin{aligned} \Delta J_{n+1} &= \Delta J_n + \varepsilon f(\theta_n), \\ \theta_{n+1} &= \theta_n + 2\pi\alpha(J_0 + \Delta J_{n+1}) \simeq \theta_n + 2\pi\alpha'(J_0)\Delta J_{n+1}. \end{aligned}$$

If we introduce a new variable through $I_n = 2\pi\alpha'(J_0)\Delta J_n$, the map is expressed by

$$I_{n+1} = I_n + Kf(\theta_n), \quad (8.51)$$

$$\theta_{n+1} = \theta_n + I_{n+1}. \quad (8.52)$$

8.3 Standard Map

A radial twist map with

$$f(\theta_n) = \sin \theta_n, \quad (8.53)$$

is called a standard map.

$$I_{n+1} = I_n + K \sin \theta_n, \quad (8.54)$$

$$\theta_{n+1} = \theta_n + I_{n+1}, \quad (8.55)$$

The name is based on the fact that many dynamical systems are reduced to this map. A standard map is a discrete dynamical system that a particle is kicked every time the orbit returns to the Poincaré cross section. Here, we regard n as time and construct a continuous dynamical system. Rewriting the map as

$$\frac{dI}{dn} = K \sin \theta \sum_{m=-\infty}^{\infty} \delta(m - n), \quad (8.56)$$

$$\frac{d\theta}{dn} = I, \quad (8.57)$$

then we have a Hamiltonian corresponding to the map

$$\begin{aligned} H(I, \theta, n) &= \int^I dI' I' - K \int^{\theta} d\theta' \sin \theta' \sum_{m=-\infty}^{\infty} \delta(m - n) \\ &= \frac{1}{2} I^2 + K \cos \theta \sum_{m=-\infty}^{\infty} \delta(m - n) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}I^2 + K \cos \theta + 2K \cos \theta \sum_{m=1}^{\infty} \cos(2\pi mn) \\
&= \frac{1}{2}I^2 + K \cos \theta + K \sum_{m=-\infty}^{\infty} \cos(\theta - 2\pi mn) \\
&= H_0 + H_1,
\end{aligned} \tag{8.58}$$

where

$$H_0(I, \theta) = \frac{1}{2}I^2 + K \cos \theta, \quad H_1(I, \theta, n) = K \sum_{m=-\infty}^{\infty} \cos(\theta - 2\pi mn). \tag{8.59}$$

Clearly, it has the same structure as a Hamiltonian of a perturbed pendulum.

8.3.1 Chaos in Standard Map

A periodic solution for a standard map is obtained as follows. A solution of period 1 is given by

$$(I_1, \theta_1) \rightarrow (I_2, \theta_2) = (I_1, \theta_1), \pmod{2\pi}, \tag{8.60}$$

whose fixed points are

$$(I, \theta) = (2\pi m, 0), (2\pi m, \pm\pi). \tag{8.61}$$

From this, the solutions of period 1 distribute infinitely with a separation distance 2π along the I axis in the phase space (I, θ) . The width of the island of the separatrix is determined by the unperturbed Hamiltonian

$$H_0 = \frac{1}{2}I^2 + K \cos \theta, \tag{8.62}$$

which gives under the condition $H_0 = K$ and $\theta = \pi$

$$\Delta I = 2\sqrt{2K(1 - \cos \pi)} = 4\sqrt{K}. \tag{8.63}$$

A solution of period 2 is given by

$$(I_1, \theta_1) \rightarrow (I_2, \theta_2) \rightarrow (I_3, \theta_3) = (I_1, \theta_1),$$

that is,

$$I_2 = I_1 + K \sin \theta_1, \tag{8.64}$$

$$\theta_2 = \theta_1 + I_2 - 2m_1\pi, \tag{8.65}$$

$$I_1 = I_2 + K \sin \theta_2, \tag{8.66}$$

$$\theta_1 = \theta_2 + I_1 - 2m_2\pi, \tag{8.67}$$

whose fixed points are given by

$$\sin \theta_1 + \sin \theta_2 = 0, \quad I_1 + I_2 = 2\pi(m_1 + m_2). \quad (8.68)$$

There are two cases (1) $\theta_2 = -\theta_1$ and (2) $\theta_2 = \theta_1 - \pi$. For the case (1), we have

$$I_1 = 2m_2\pi + 2\theta_1, \quad I_2 = 2m_1 - 2\theta_1, \quad 2\pi(m_1 - m_2) - 4\theta_1 = K \sin \theta_1,$$

which gives

$$I_{1,2} = (2m + 1)\pi, \quad \theta_{1,2} = n \pm \frac{\pi}{2}. \quad (8.69)$$

For the case (2), we have

$$I_{1,2} = (2m + 1)\pi, \quad \theta_1 = \pi, \quad \theta_2 = 0. \quad (8.70)$$

The solutions are distributed between the solution of period 1 along the I axis.

The width of the island of the solution of period 2 is obtained perturbatively. The time variation of θ in the perturbation term H_1 of the Hamiltonian

$$H = \frac{1}{2}I^2 + \varepsilon K \sum_m \cos(\theta - 2\pi mn) = H_0 + H_1,$$

gives a contribution to the averaged Hamiltonian. From the equation of motion

$$\frac{\partial I}{\partial n} = -\frac{\partial H}{\partial \theta} = \varepsilon K \sum_m \sin(\theta - 2\pi mn), \quad (8.71)$$

$$\frac{\partial \theta}{\partial n} = \frac{\partial H}{\partial I} = I, \quad (8.72)$$

the solution up to the first order of ε is given by

$$I = I_0 + \sum_m \frac{\varepsilon K}{2\pi m} \cos(\theta - 2\pi mn), \quad (8.73)$$

$$\theta = \theta_0 - \sum_m \frac{\varepsilon K}{(2\pi m)^2} \sin(\theta - 2\pi mn). \quad (8.74)$$

These are substituted into the Hamiltonian to give within the second order of ε

$$\begin{aligned} H &= \frac{1}{2}I^2 + \varepsilon K \sum_m \{\cos(\theta_0 - 2\pi mn) - \sin(\theta_0 - 2\pi mn)(\theta - \theta_0)\} \\ &= \frac{1}{2}I^2 + \varepsilon K \sum_m \cos(\theta_0 - 2\pi mn) \end{aligned}$$

$$\begin{aligned}
& + \sum_{m,m'} \left(\frac{\varepsilon K}{4\pi m'} \right)^2 \{ \cos(2\theta_0 - 2\pi(m + m')n) + \cos(2\pi(m - m')n) \} \\
& = \frac{1}{2} I^2 + \left(\frac{\varepsilon K}{4} \right)^2 \cos(2\hat{\theta}) + \Delta H_0,
\end{aligned}$$

where

$$2\theta_0 - 2\pi(m + m')n \simeq 2n[I_0 - \pi(m + m')] = 2n\pi(2p + 1 - m - m'),$$

and we have used the fact that the contribution is not zero only when $m + m' = 2p + 1$ and also the identity

$$\sum_m \frac{1}{(m - 2p - 1)^2} = \pi^2.$$

The constant term is set to $\Delta H_0 = 0$ and $2\hat{\theta} = 2\theta_0 - 2\pi(2p + 1)n$. Thus, the new Hamiltonian up to the second order of ε is expressed by

$$\bar{H} = \frac{1}{2} \hat{I}^2 + \left(\frac{K}{4} \right)^2 \cos 2\hat{\theta}, \quad (8.75)$$

and the width of the island is given by

$$\Delta I_2 = K. \quad (8.76)$$

8.3.1.1 Primary Resonance Overlap

When the resonance island of the solution of period 1 lined up along the I axis overlap, the solution orbit can move in the phase space without restriction. This is a global chaos. The condition for overlapping to occur is that the width of the resonance islands is larger than the distance between the solutions of period 1.

$$\Delta I > 2\pi,$$

that is

$$K > \left(\frac{\pi}{2} \right)^2 \simeq 2.47. \quad (8.77)$$

This value is much larger than the critical value $K_c = 0.9716$ of the onset of the chaos obtained by computer experiments. This is because before the neighboring solutions of period 1 overlap, overlapping between the solution of period 1 and that of period 2 occurs.

8.3.1.2 Secondary Resonance Overlap

The secondary resonance overlap needs the condition that the sum of the width ΔI of the resonance island of the solution of period 1 and the width ΔI_2 of the solution of period 2 is larger than the separation distance of the solutions of period 1,

$$\Delta I + \Delta I_2 > 2\pi,$$

which is rewritten as

$$2\sqrt{K} + \frac{K}{2} > \pi.$$

The critical value for the onset of global chaos is given

$$K_c = 1.456, \quad (8.78)$$

which is still larger than the experimental value.

If a contribution from the width of the separatrix is taken into account (Chirikov [150]), the critical value is given by

$$K_c = 1.2, \quad (8.79)$$

which is closer to the experimental value but still a bit larger. The Poincaré plot of the standard map is shown in Fig. 8.1.

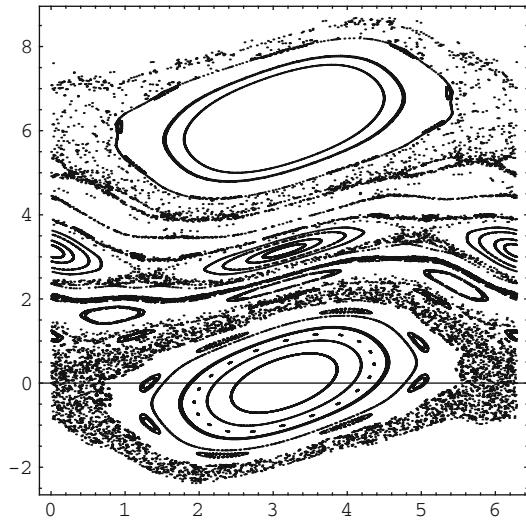


Fig. 8.1 Poincaré plot of the Standard Map for $K = 1$

8.3.2 Global Chaos: Greene's Method

When frequencies $\omega_j = \partial H_0 / \partial I_j$ of a unperturbed system are linearly independent, the integrals except for the vicinity of the rational surface are distorted but not destroyed provided that a perturbation is small enough (KAM theorem). The condition for the KAM tori to exist is

1. When the perturbation is small
2. When the tori are far from the rational surface
3. When the tori are far from the separatrix

Noting that the breakdown of the KAM tori for the incommensurate aspect ratio $\alpha = \omega_1 / \omega_2$ corresponds to the destabilization of the elliptic points of the periodic orbit with a rational number of the aspect ratio $\alpha_n = r/s$ which tends to irrational number as $s \rightarrow \infty$, Greene [151] thought that the KAM torus with a rational number which is the most farthest to irrational numbers is the last KAM torus remained not destructed by the perturbation until the onset of chaos and the global chaos onsets by destabilization of the last KAM torus. He argued that the last KAM torus is destructed when the aspect ratio is expressed by the Golden section and obtained $K = 0.9716$.

The best method to approximate an irrational number by rational numbers is to use a contracted fraction. An irrational number α is expanded by a contracted fraction as

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} = [a_1, a_2, a_3, \cdots], \quad (8.80)$$

where a_n is a positive integer. The irrational number farthest from the rational number is constructed by the smallest number of a_n , that is $a_n = 1$, therefore, we have

$$\alpha = [1, 1, 1, \cdots] = \frac{\sqrt{5} - 1}{2}.$$

8.4 Chaos in Dissipative Systems

8.4.1 Attractors and Strange Attractors

Suppose that an N -dimensional vector obeys a set of dynamical equations described by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) = \prod_i \frac{\partial x_i(\mathbf{x}_0, t)}{\partial x_{i0}} \Delta x_{i0}. \quad (8.81)$$

Then, the time evolution of the system is represented by a trajectory in N dimensional phase space. A dissipative dynamical system is characterized by the contrac-

tion of the volume occupied by the orbit in phase space. The contraction rate $\lambda(\mathbf{x})$ of the volume $\Delta V(\mathbf{x}_0, t) = \prod_i \delta x_i$ around a representative point $\mathbf{x}_0$ in phase space is given by

$$\begin{aligned}\lambda(\mathbf{x}) &= \frac{1}{\Delta V} \frac{\partial \Delta V}{\partial t} = \sum_i \frac{1}{\Delta x_i} \frac{\partial \Delta x_i}{\partial t} \\ &= \sum_i \frac{1}{\Delta x_i} \frac{\partial}{\partial x_{i0}} \frac{\partial x_{i0}}{\partial t} \Delta x_{i0} \\ &= \sum_i \frac{\Delta x_{i0}}{\Delta x_i} \frac{\partial}{\partial x_{i0}} F_i(\mathbf{x}_0, t).\end{aligned}\quad (8.82)$$

In the vicinity of $t = 0$, the rate is estimated as

$$\lambda = \sum_i \frac{\partial F_i}{\partial x_i}, \quad (8.83)$$

and the volume contracts for $\lambda < 0$ and expands for $\lambda > 0$. Dynamical systems for which the following relation satisfies at any point $\mathbf{x}_0$ on the trajectory in phase space are called dissipative dynamical systems.

$$\lambda_0 = \lambda(\mathbf{x}_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left| \frac{\Delta V(\mathbf{x}_0, t)}{\Delta V(\mathbf{x}_0, 0)} \right| < 0. \quad (8.84)$$

Since the volume occupied by the trajectory in phase space contracts, stable stationary states in N dimensional dynamical systems are on the manifolds whose dimension is less than N . This manifold is called attractors.

Chaotic behavior is possible by an orbit instability for which trajectories starting from near-by points diverge in time, while the volume in phase space contracts. This means an attractor in chaos is a bounded region in phase space to which trajectories are attracted, visiting every points on the attractor. The volume in phase space contracts not in all directions but in some direction and expands in the other directions to make the basin of attraction complicated structure. An attractor in a dissipative dynamical system which exhibits chaotic behavior is called a strange attractor [153, 154].

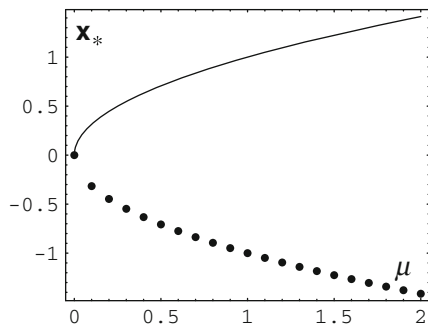
8.4.2 Bifurcation Theory

8.4.2.1 Tangent Bifurcation

A differential equation

$$\frac{dx}{dt} = V(x, \mu) = \mu - x^2 = -\frac{d\psi}{dx}, \quad \Psi(x) = -\mu x + \frac{1}{3}x^3, \quad (8.85)$$

Fig. 8.2 Tangent bifurcation: stable and unstable fixed points are denoted by *solid* and *dotted* lines, respectively



gives different stationary solutions depending on μ which is called a bifurcation parameter. The fixed points are given only for $\mu \geq 0$ by

$$x_* = \pm \sqrt{\mu}. \quad (8.86)$$

The stability of these fixed points is determined by the linearized equation around the fixed points

$$\frac{d}{dt} \delta x = -2x_* \delta x, \quad (8.87)$$

Putting $\delta x \propto \exp(-\lambda t)$, we have

$$\lambda = -2x_*, \quad (8.88)$$

which shows that the fixed point $\sqrt{\mu}$ is stable and $-\sqrt{\mu}$ is unstable. Since there are no real fixed points for $\mu < 0$, stable solutions do not exist. When μ changes its sign from negative to positive, a stable solution appears (Fig. 8.2).

8.4.2.2 Exchange of Stability

Fixed points of the following equation

$$\frac{dx}{dt} = \mu x - x^2 = -\frac{d\psi}{dx}, \quad \psi(x) = -\frac{\mu}{2}x^2 + \frac{1}{3}x^3, \quad (8.89)$$

are given by $x_* = 0$ and μ . Since the exponent of the stability λ is given by

$$\lambda = \mu - 2x_* = \begin{cases} \mu & \text{for } x_* = 0, \\ -\mu & \text{for } x_* = \mu, \end{cases} \quad (8.90)$$

for $\mu < 0$ the fixed point $x_* = \mu$ is unstable and not realized, while another fixed point $x_* = 0$ is stable and realized. When μ turns from negative to positive, the fixed point $x_* = 0$ becomes unstable and is replaced by another fixed point $x_* = \mu$ which becomes stable. Thus, this bifurcation is called an exchange of stability (Fig. 8.3).

Fig. 8.3 Exchange of stability bifurcation: stable and unstable fixed points are denoted by *solid* and *dotted* lines, respectively

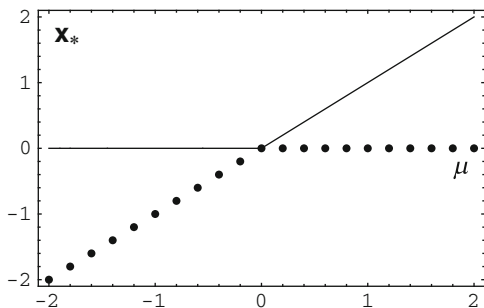
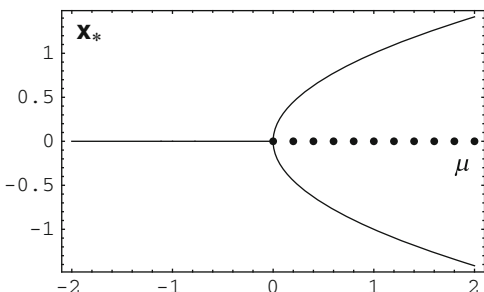


Fig. 8.4 Pitchfork bifurcation: stable and unstable fixed points are denoted by *solid* and *dotted* lines, respectively



8.4.2.3 Pitchfork Bifurcation

For an equation

$$\frac{dx}{dt} = \mu x - x^3, \quad \Psi(x) = -\frac{\mu}{2}x^2 + \frac{1}{4}x^4, \quad (8.91)$$

the fixed points are $x_* = 0, \pm \sqrt{\mu}$. Since the stability exponent is given by

$$\lambda = \mu - 3x_*^2 = \begin{cases} \mu & \text{for } x_* = 0, \\ -2\mu & \text{for } x_* = \pm\sqrt{\mu}, \end{cases} \quad (8.92)$$

as μ turns from negative to positive, $x_* = 0$ becomes unstable and $x_* = \pm\sqrt{\mu}$ is realized (Fig. 8.4).

8.4.2.4 Reversed Pitchfork Bifurcation

For an equation

$$\frac{dx}{dt} = -\mu x + x^3, \quad \Psi(x) = \frac{\mu}{2}x^2 - \frac{1}{4}x^4, \quad (8.93)$$

the fixed points are $x_* = 0, \pm \sqrt{\mu}$. The stability exponent is given by

$$\lambda = -\mu + 3x_*^2 = \begin{cases} -\mu & \text{for } x_* = 0, \\ 2\mu & \text{for } x_* = \pm \sqrt{\mu}, \end{cases} \quad (8.94)$$

Thus, when μ becomes from negative to positive, $x_* = \pm \sqrt{\mu}$ turns unstable and $x_* = 0$ is realized.

8.4.2.5 Hopf Bifurcation

For an equation

$$\frac{dx}{dt} = 2x(\gamma(\mu) + a(\mu)x + b(\mu)x^2 + \dots) = 2\Gamma(x)x, \quad (8.95)$$

one of the fixed points $x_* = 0$ is unstable for $\gamma(\mu) > 0$. The parameter μ_0 such that $\gamma(\mu_0) = 0$ is called a critical point.

(1) Soft Mode (Hopf Bifurcation)

When $a(\mu_0) < 0$, as μ increases to be $\mu > \mu_0$, then $x_* = 0$ becomes unstable and a new fixed point determined by

$$\Gamma = \gamma(\mu_0) + \frac{d\gamma(\mu_0)}{d\mu}(\mu - \mu_0) + a(\mu_0)x = 0, \quad (8.96)$$

appears. The new fixed point is given by

$$x_* = \frac{1}{a} \frac{d\gamma(\mu_0)}{d\mu}(\mu - \mu_0). \quad (8.97)$$

The amplitude of the new solution increases from zero as the super-criticality $\mu - \mu_0$ increases.

(2) Hard Mode (Inverted Hopf Bifurcation)

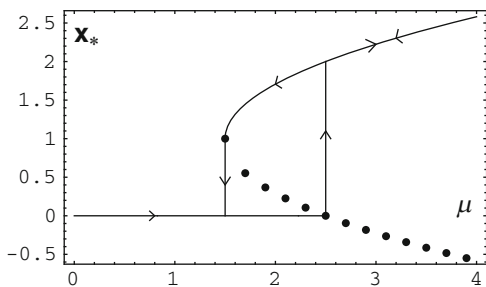
When $a(\mu_0) > 0$ and $b(\mu_0) < 0$, taking $b(\mu)x^2$ into account gives a new fixed point which is determined by

$$\Gamma = \frac{d\gamma(\mu_0)}{d\mu}(\mu - \mu_0) + a(\mu_0)x + b(\mu_0)x^2 = 0, \quad (8.98)$$

to give

$$x_{*\pm} = \frac{1}{2b(\mu_0)} \left\{ -a(\mu_0) \pm \sqrt{a(\mu_0)^2 - 4b(\mu_0) \frac{d\gamma(\mu_0)}{d\mu}(\mu - \mu_0)} \right\}. \quad (8.99)$$

Fig. 8.5 Inverted Hopf bifurcation: stable and unstable fixed points are denoted by *solid* and *dotted* lines, respectively



Thus, when the parameter μ increases, the amplitude jumps from zero to x_{*+} at $\mu = \mu_0$, while when the parameter μ decreases, the fixed point jumps from x_{*-} to zero at

$$\mu = \mu_0 + \frac{a(\mu_0)^2}{4b(\mu_0)} \left(\frac{d\gamma(\mu_0)}{d\mu} \right)^{-1} < \mu_0. \quad (8.100)$$

Thus, the path that the fixed point amplitude increases as μ increases is different from the path that the fixed point amplitude decreases as μ decreases. This hysteresis is a characteristic feature of the inverted Hopf bifurcation (Fig. 8.5).

8.4.3 Period Doubling Route to Chaos: Logistic Map

The Logistic map is a basic map in dissipative dynamical systems. The Logistic map is given by

$$x_{n+1} = f(x_n, C) = Cx_n(1 - x_n). \quad (8.101)$$

8.4.3.1 Fixed Points of f

The fixed point of f is obtained by

$$x_1 = f(x_1) = Cx_1(1 - x_1), \quad (8.102)$$

to give

$$x_{*,0} = 0, \quad x_{*,1} = 1 - \frac{1}{C}. \quad (8.103)$$

The stability of the fixed points is analyzed by linearizing (8.101) around the fixed points x_* . The linearized equation is given by

$$\Delta x_{n+1} = f'(x_*)\Delta x_n = f'(x_*)^{n+1}\Delta x_0, \quad (8.104)$$

from which the fixed point is stable for $|f'(x_*)| = |C(1 - 2x_*)| < 1$. This condition gives the value of the parameter C for the fixed points, respectively:

$$\begin{aligned}
|C| &< \frac{1}{2} \quad \text{for } x_{*,0} = 0, \\
|2 - C| &< 1 \quad \text{for } x_{*,1} = 1 - \frac{1}{C}.
\end{aligned} \tag{8.105}$$

Thus, $x_{*,0}$ is stable and $x_{*,1}$ is unstable for $-1 < C < 1$, that is, $x_{*,0}$ is realized. At $C = 3$ $x_{*,0}$ is destabilized and $x_{*,1}$ is stabilized. The fixed point $x_{*,1}$ becomes unstable for $C > 3$.

8.4.3.2 Fixed Points of f^2

For $C > 3$, the fixed points of f are unstable and the fixed points of f^2 appear. The fixed points of f^2 are given by solving

$$x_2 = f(f(x_2)) = C^2 x_2 (1 - x_2) (1 - C x_2 (1 - x_2)) \equiv f_2(x_2). \tag{8.106}$$

The $f(x)$ has one maximum and $f_2(x) = f^2(x)$ has three maximums. This is seen by noting $f'_2(x) = f'(f(x))f'(x) = 0$ and $f'(1/2) = 0$. The $f_2(x)$ is at maximum for x so as to be $f(x) = 1/2$ and there are two x 's to satisfy $f(x) = 1/2$ for $C > 3$. Another important thing is that the fixed points of f are also fixed points of f^2 since

$$x_1 = f(f(x_1)) = f(x_1). \tag{8.107}$$

This implies if the fixed points of f are unstable, then they are also unstable with respect to f_2 . If the slope of f_2 at $x_{*,0} = 0$ is less than 1, there is only one fixed point of f_2 . Thus, if

$$|f'(x_*)| > 1, \tag{8.108}$$

the following holds

$$f'_2(x_*) = f'(f(x_*))f'(x_*) = f'(x_*)^2 > 1. \tag{8.109}$$

When the fixed points of f become unstable for $C > 3$, the fixed points of f_2 are newly born to be $x_{2,+}$ and $x_{2,-}$. These new fixed points satisfy

$$x_{2,+} = f(x_{2,-}) = f_2(x_{2,+}), \tag{8.110}$$

$$x_{2,-} = f(x_{2,+}) = f_2(x_{2,-}). \tag{8.111}$$

Putting

$$x_{2,n} = x_{2\pm} + \Delta x_{2,n}, \tag{8.112}$$

and linearizing (8.106), we get

$$\Delta x_{2,n} = \lambda_2^n \Delta x_0, \tag{8.113}$$

where

$$\lambda_{2,+} = f'_2(x_{2,+}) = f'(x_{2,+})f'(x_{2,-}) = f_2(x_{2,-}) = \lambda_{2,-}. \quad (8.114)$$

The stability of the two fixed points of f^2 are destabilized at the same value of the bifurcation parameter. When two fixed points are destabilized, new two fixed points appear corresponding to each of the destabilized fixed points. Therefore, when the fixed points of f^2 become unstable, period 2^2 motion appears. That is, whenever the fixed points are destabilized, the period is doubled. An infinite repetition of the period doubling bifurcation leads to chaos.

8.4.3.3 Accumulation of Period Doubling and Renormalization

The fixed points of f^2 are obtained by solving (8.106). Since the fixed points f^2 include the fixed points of f , (8.106) is factorized as

$$x \left(x - 1 + \frac{1}{C} \right) \left\{ x^2 - \left(1 + \frac{1}{C}x \right) + \frac{1}{C} \left(1 + \frac{1}{C} \right) \right\} = 0. \quad (8.115)$$

The solutions are given by

$$x_{2,\pm} = \frac{1}{2} \left\{ 1 + \frac{1}{C} \pm \sqrt{\left(1 + \frac{1}{C} \right) \left(1 - \frac{3}{C} \right)} \right\}, \quad (8.116)$$

implying that the solutions of f^2 exist for $C < -1$ or $C > 3$. The solutions of f become unstable at $C = -1$ or $C = 3$.

The unstable points $C = 3$ and $C = -1$ are mirror symmetry to each other. Putting

$$x = \bar{x} + x_* = \bar{x} + \left(1 - \frac{1}{C} \right), \quad (8.117)$$

the Logistic map is rewritten as

$$\bar{x}_{n+1} = (2 - C)\bar{x}_n \left\{ 1 - \frac{C}{(2 - C)}\bar{x}_n \right\}, \quad (8.118)$$

Here, changing a variable through

$$\bar{x}_n = \left(\frac{2}{C} - 1 \right) y_n, \quad (8.119)$$

we obtain a map

$$y_{n+1} = (2 - C)y_n(1 - y_n), \quad (8.120)$$

which is isomorphic to the Logistic map. Therefore, by replacement

$$C \rightarrow \bar{C} = 2 - C, \quad (8.121)$$

the orbit for $C > 0$ and the orbit for $C < 0$ are mutually interchanged.

Stability of the fixed points of f^2 is studied in the following. Putting

$$x_n = x_{2,+} + \Delta x_n, \quad (8.122)$$

$$x_{n+1} = x_{2,-} + \Delta x_{n+1}, \quad (8.123)$$

we have

$$\Delta x_{n+1} = C(1 - 2x_{2,+})\Delta x_n - C\Delta x_n^2, \quad (8.124)$$

$$\Delta x_{n+1} = C(1 - 2x_{2,-})\Delta x_{n+1} - C\Delta x_{n+1}^2. \quad (8.125)$$

In the following, we consider only for $C < 0$. For $C > 0$ the transition point can be obtained from the result of $C < 0$ by using the mirror symmetry.

In (8.124) and (8.125), we delete the intermediate state Δx_{n+1} and retain up to Δx^2 to obtain

$$\begin{aligned} \Delta x_{n+2} = & C^2(1 - 2x_{2,-})(1 - 2x_{2,+})\Delta x_n \\ & - C^2\{(1 - 2x_{2,-}) + C(1 - 2x_{2,+})^2\}\Delta x_n^2. \end{aligned} \quad (8.126)$$

Introducing a transformation

$$\Delta x_n = \alpha^{-1} y_n, \quad (8.127)$$

we have

$$y_{n+2} = C' y_n (1 - y_n), \quad C' = -C^2 + 2C + 4, \quad (8.128)$$

which is isomorphic to the original map. Since the fixed point of f of the original map is destabilized at $C = -1$, the fixed points of the same type of map is destabilized at $C' = -1$. That is, the fixed points of f^2 are destabilized at

$$C' = -C^2 + 2C + 4 = -1, \quad C_2 = 1 - \sqrt{6} \sim -1.44948, \quad (8.129)$$

at which they transit to the fixed points of f^4 .

The accumulation point of the period doubling is given by Feigenbaum [152]

$$C' = C = C_\infty. \quad (8.130)$$

From this, we obtain

$$-C_\infty^2 + 2C_\infty + 4 = C_\infty, \quad C_\infty = \frac{1 - \sqrt{17}}{2} \sim -1.5615. \quad (8.131)$$

In numerical experiments, the transition to chaos is given by $C_\infty = -1.56994$. From the mirror symmetry, the critical value for $C > 0$ is given by

$$C_\infty^{(+)} = 2 - C_\infty^{(-)} = 3.5615, \quad (8.132)$$

and the numerical experiment gives $C_\infty^{(+)} = 3.5699456$.

The value C_n of the bifurcation parameter at the bifurcation points approaches in the vicinity of the accumulation point

$$C_n \sim C_\infty + \delta^{-n}. \quad (8.133)$$

Therefore, the exponent δ is given by

$$\begin{aligned} \delta &= \lim_{n \rightarrow \infty} \frac{C_n - C_{n-1}}{C_{n+1} - C_n} \\ &= \lim_{n \rightarrow \infty} \frac{(C_n - C_{n+1})(C_n + C_{n+1} - 2)}{C_{n+1} - C_n}, \\ &= \lim_{n \rightarrow \infty} (2 - C_n - C_{n+1}) \\ &= 2 - 2C_\infty = 5.123. \end{aligned} \quad (8.134)$$

Numerical experiments gives $\delta = 4.6692$.

8.5 Fractal Structure

Strange attractors has a self-similar structure which repeats itself in a smaller scale. A self-similar system which does not have a characteristic scale length is specified by a fractal dimension. The fractal dimension is defined by

$$d(S) = \lim_{\varepsilon \rightarrow 0} \frac{\ln M(\varepsilon)}{\ln(1/\varepsilon)}, \quad (8.135)$$

where S is a subspace of the N dimensional space and $M(\varepsilon)$ is a number of the N dimensional unit volume of a side length ε necessary to cover S .

Divide a unit line into three equal pieces and eliminate the middle, then divide the remaining two sections into three equal pieces, respectively, and eliminate the middle of each group. The Cantor set produced by repeating this procedure n times has a self-similar structure and the fractal dimension is given by noting $M = 2^n$ and $\varepsilon = 1/3^n$ as

$$d = \lim_{n \rightarrow \infty} \frac{\ln 2^n}{\ln 3^n} \sim 0.63. \quad (8.136)$$

8.6 Lyapunov Exponents

For a dissipative dynamical system

$$\frac{dx_i}{dt} = F_i(\mathbf{x}), \quad i = 1, 2, \dots, N, \quad (8.137)$$

the distance between an orbit starting from $\mathbf{x}_0$ and another orbit starting from $\mathbf{x}_0 + \Delta\mathbf{x}_0$ at time t is given by

$$d_i(\mathbf{x}_0, t) = ||\Delta x_i(\mathbf{x}_0, t)||, \quad (8.138)$$

$$\frac{d}{dt} \Delta \mathbf{x}(\mathbf{x}_0, t) = \mathbf{F}(\mathbf{x}_0, t) - \mathbf{F}(\mathbf{x}_0 + \Delta \mathbf{x}_0) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}. \quad (8.139)$$

The divergent rate of the two orbits is expressed by

$$\sigma_i(\mathbf{x}_0, \mathbf{F}) = \lim_{t \rightarrow \infty, d_i(0) \rightarrow 0} \frac{1}{t} \ln \frac{d_i(\mathbf{x}_0, t)}{d_i(\mathbf{x}_0, 0)}. \quad (8.140)$$

The σ is called the Lyapunov exponent. Line up N σ s in order of magnitude as $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_N$. The flow becomes chaotic when at least one of the Lyapunov exponents is positive. The average volume shrinking rate is expressed by the sum of the Lyapunov exponents.

$$\lambda_0 = \sum_{i=1}^N \sigma_i < 0. \quad (8.141)$$

Obviously, one dimensional flow cannot be chaotic.

When N dimensional flow is expressed by $(N - 1)$ dimensional map, the Lyapunov exponent is known to be proportional to the Lyapunov exponent of the flow except for the zero exponent. Therefore, two dimensional flow is also unable to show chaos. Since two dimensional flow is expressed by a one dimensional map and $\lambda_0 = \sigma_1$, dissipative systems ($\lambda_0 < 0$) are not compatible with chaos ($\sigma_1 > 0$). Thus, the lowest dynamical systems exhibit chaos is three dimensional flows and two dimensional map which are reduced from the three dimensional flow. Since three dimensional flows are reduced to two dimensional maps,

$$\sigma_1 > 0, \quad \lambda_0 = \sigma_1 + \sigma_2 < 0, \quad (8.142)$$

dissipative systems are compatible with chaos.

Since the Lyapunov exponent is positive, the attractor is stretched in one direction and at the same time folded since the volume is bounded. Through the processes that stretching and folding are repeated, the self-similar structure is created.

8.7 Dimension of Attractor

Suppose a d dimensional dynamical system whose solution is given by

$$\mathbf{x}(t) = \{x_1(t), x_2(t), \dots, x_d(t)\}. \quad (8.143)$$

Then, the time series $\{\mathbf{x}(t_j), j = 1, 2, \dots, n\}$ is expressed as an orbit in d dimensional phase space. Dividing the phase space into cells ℓ^d of the side length ℓ , denoting the number of the orbit point $\{\mathbf{x}(t)\}$ found in the i -th cell ($\{i = 1, 2, \dots, M(\ell)\}$) by N_i , then the probability p_i to find the point of the attractor in the i -th cell is given by

$$p_i = \lim_{N \rightarrow \infty} \frac{N_i}{N}. \quad (8.144)$$

If the dimension is defined in terms of p_i^n as

$$D_n = \lim_{\ell \rightarrow 0} \frac{1}{n-1} \frac{\log(\sum_{i=1}^{M(\ell)} p_i^n)}{\log \ell}, \quad n = 0, 1, 2, \dots, \quad (8.145)$$

then we have

$$D_0 = - \lim_{\ell \rightarrow 0} \frac{\log(\sum 1)}{\log \ell} = - \lim_{\ell \rightarrow 0} \frac{\log M(\ell)}{\log \ell}, \quad (8.146)$$

$$D_1 = - \lim_{\ell \rightarrow 0} \frac{S(\ell)}{\log \ell}, \quad S(\ell) = - \sum_{i=1}^{M(\ell)} p_i \log p_i, \quad (8.147)$$

where D_0 is the Hausdorff dimension and D_1 is the information dimension. In deriving (8.147), we have used

$$\lim_{n \rightarrow 1} \frac{\log(\sum p_i^n)}{n-1} = \frac{d}{dn} \log(\sum p_i^n)|_{n=1} = \frac{\sum p_i \log p_i}{\sum p_i} = \sum p_i \log p_i. \quad (8.148)$$

If an attractor is homogeneous, p_i is constant $p_i = 1/M(\ell)$ everywhere and

$$S(\ell) = - \sum_{i=1}^{M(\ell)} \frac{1}{M(\ell)} \log \frac{1}{M(\ell)} = \log M(\ell). \quad (8.149)$$

Therefore, we have $D_0 = D_1 = D$. The difference between D_0 and D_1 is a measure of the inhomogeneity of the attractor. In general since

$$S(\ell) \leq \log M(\ell), \quad (8.150)$$

we conclude

$$D_1 \leq D_0. \quad (8.151)$$

8.8 Correlation Dimension

Dimension for $n = 2$ in (8.145)

$$D_2 = \lim_{\ell \rightarrow 0} \frac{\log(\sum p_i^2)}{\log \ell}, \quad (8.152)$$

is called the correlation dimension [155]. Define a correlation integral with

$$C(\ell) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i,j} \theta(\ell - |\mathbf{x}_i - \mathbf{x}_j|). \quad (8.153)$$

This is a limit of the number of pairs satisfying $|\mathbf{x}_i - \mathbf{x}_j| < \ell$ divided by the squared total number of pairs and is equal to the probability that the two points on the attractor are separated less than ℓ or the probability that two point on the attractor are in the cell ℓ^d . Therefore, we have

$$C(\ell) = \sum_{i=1}^{M(\ell)} p_i^2, \quad (8.154)$$

that is,

$$D_2 = \lim_{\ell \rightarrow 0} \frac{\log C(\ell)}{\log \ell}. \quad (8.155)$$

Thus, we have

$$C(\ell) \sim \ell^{D_2}. \quad (8.156)$$

8.9 Construction of Attractor with Observed Signal

It is difficult to observe the time series of all the variables of a dynamical system, particularly in experiments. Therefore, we consider if it is possible to construct an attractor from the time series of data of only one variable.

Consider a flow in two dimensional phase space.

$$\frac{d}{dt} \mathbf{x} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} = (x, y). \quad (8.157)$$

Any point $\{x(t + \tau), y(t + \tau)\}$ of the trajectory in the phase space is produced from the point $\{x(t), y(t)\}$ uniquely and they are 1 to 1 correspondent.

Consider a vector

$$\boldsymbol{\xi}(t) = \{x(t), x(t + \tau)\} = \{\xi_1(t), \xi_2(t)\}. \quad (8.158)$$

The vector $\xi(t + \tau)$ created by shifting time by τ is approximated for small τ as

$$\xi_1(t) = x(t), \quad (8.159)$$

$$\begin{aligned} \xi_2(t) &= x(t + \tau) = \int_t^{t+\tau} dt' F_1(x(t'), y(t')) + x(t) \\ &\simeq \tau F_1(x(t), y(t)) + x(t). \end{aligned} \quad (8.160)$$

The information included in the time series data $\{x(t_j), t_j = j\tau\}$ is the same as the information of the data $\{\xi(t_j)\}$ and therefore it gives the same dimension.

According to Taken [156], a d dimensional dynamical system is embedded in n dimensional phase space constructed by an arbitrary one time series data. That is, the attractor of a dynamical system is to match the smoothly deformed attractor of the constructed phase space. The embedded dimension n to be taken is roughly given in terms of the dimension d of the original dynamical system through

$$n = 2d + 1, \quad (8.161)$$

When the dimension of a dynamical system is unknown, the optimum embedded dimension has to be determined by trial and error. The correlation dimension of the Lorenz attractor is known as $D_2 = 2.05$. Since the correlation dimension obtained using the three dimensional orbit of the one variable of the Lorenz equation is shown to be equal to that obtained by the Lorenz equation itself [155]. Thus, a new procedure is empirically ensured.

When the correlation integral is expressed by

$$C(\ell) \propto \ell^\nu, \quad (8.162)$$

ν is called the correlation exponent. First, calculate the correlation exponent of the phase space constructed from the time series data. If the constructed dimension is smaller than the dimension of the real attractor, the correlation exponent is equal to d since the attractor covers the contracted phase space. If the correlation dimension saturates as d is increased, the correlation exponent gives the correlation dimension.

In order to determine a time delay τ decrease τ until the correlation exponent no longer changes. However, trial and error takes long time. The better way is to minimize the error estimation function so as to neglect the linear correlation. A coefficient λ to minimize the error estimation function

$$E = \sum_{t=1}^{N-\tau} \{x(t + \tau) - \bar{x} - \lambda(x(t) - \bar{x})\}^2, \quad \bar{x} = \sum_{t=1}^N x(t), \quad (8.163)$$

is given by

$$\lambda = \frac{\sum_{t=1}^{N-\tau} (x(t) - \bar{x})(x(t + \tau) - \bar{x})}{\sum_{t=1}^{N-\tau} (x(t) - \bar{x})^2}, \quad (8.164)$$

which is nothing but a self-correlation and τ is obtained when the self-correlation is minimized.

When time series data follows random processes, the orbit distributes densely in the phase space and the correlation exponent coincides with the embedded dimension. From this, chaos can be separated from noise. Suppose a strange attractor embedded in d dimensional space with an external white noise of amplitude ℓ_0 . Since the noise occupies the d dimensional space, the correlation exponent gives an embedded dimension for $\ell \ll \ell_0$, while the correlation exponent gives an attractor dimension for $\ell \gg \ell_0$. Thus, chaos can be separated from noise.

The methods of using the correlation integral are applied to many problems. However, since the total number of data is defined by the limit of $N \rightarrow \infty$, the number of the data has a lower bound. When the correlation integral $C(\ell)$ is depicted against ℓ , if a linear part is obtained for $\ell_1 < \ell < \ell_2$, the distance $\ell_2 - \ell_1$ is to be long so that

$$\frac{\ell_2}{\ell_1} \geq 10. \quad (8.165)$$

Since the slope of the linear line is the correlation dimension

$$D_2 = \frac{\log_{10} C(\ell_2) - \log_{10} C(\ell_1)}{\log_{10} \ell_2 - \log_{10} \ell_1}, \quad (8.166)$$

the lower and upper bound of the correlation integral is estimated by

$$C(\ell_1) > \frac{1}{N^2}, \quad C(\ell_2) < \frac{N(N-1)}{N^2} < 1. \quad (8.167)$$

Thus, we obtain

$$\log_{10} C(\ell_2) - \log_{10} C(\ell_1) < \log_{10} N^2, \quad (8.168)$$

which gives

$$D_2 = \log_{10} N^2. \quad (8.169)$$

Therefore, the number of data necessary for the correlation dimension to be reliable has to satisfy

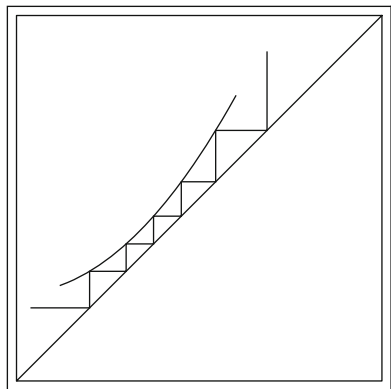
$$N > 10^{D_2/2}. \quad (8.170)$$

If the correlation dimension of an attractor is around 8, then the number of the data should be $N \sim 10^4$.

8.10 Intermittent Chaos

Among time series of data irregular changes lasting for short period appear randomly in a long lasting regular variation. This is called an intermittent chaos which was first found in the Lorenz model by Manneville and Pomeau [157, 158] and later is widely discovered. There are three types of intermittent chaos whose maps

Fig. 8.6 The regular channel embedded in the chaos sea



$x_{n+1} = f(x_n)$ have a narrow region close to a line $x_{n+1} = x_n$, where a regular motion is observed as is shown in Fig. 8.6.

8.10.1 Type I Intermittency

A map

$$x_{n+1} = f(x_n) = \varepsilon + x_n(1 + \alpha x_n), \quad (\alpha > 0) \quad (8.171)$$

has a fixed point

$$x_* = \pm \sqrt{\frac{-\varepsilon}{\alpha}}.$$

Since the eigenvalue is given by

$$\lambda = \frac{df(x_*)}{dx} = 1 + 2\alpha x_* = \begin{cases} 1 \pm 2\sqrt{-\varepsilon\alpha} & \text{for } \varepsilon < 0, \\ 1 - i2\sqrt{-\varepsilon\alpha} & \text{for } \varepsilon > 0, \end{cases} \quad (8.172)$$

one of the eigenvalue is stable for $\varepsilon < 0$ and unstable for $\varepsilon > 0$. When ε changes from negative to positive, the real part of the eigenvalue moves from inside a convergent circle to outside. The amplitude of the regular oscillation increases monotonically and enters into sea of chaos (Fig. 8.7).

8.10.2 Type II Intermittency

A map

$$x_{n+1} = f(x_n) = (1 + \varepsilon)x_n + \alpha x_n^3 \quad (8.173)$$

has a fixed point

$$x_* = 0, \quad \pm \sqrt{\frac{-\varepsilon}{\alpha}}.$$

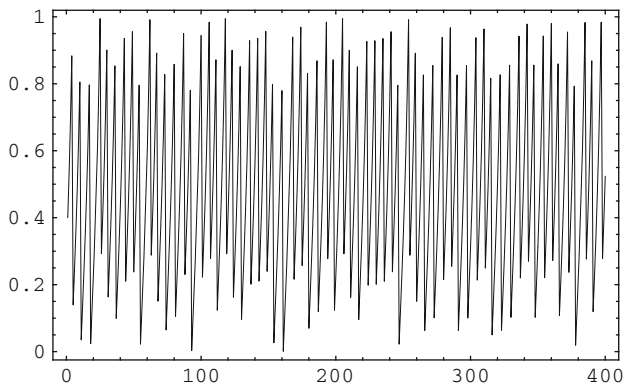


Fig. 8.7 The time series data of the type I intermittent chaos

The eigenvalue is given for $\varepsilon < 0$

$$\lambda = \frac{df(x_*)}{dx} = 1 + \varepsilon + 3\alpha x_*^2 = \begin{cases} 1 + \varepsilon & \text{for } x_* = 0, \\ 1 - 2\varepsilon & \text{for } x_* = \pm \sqrt{-\frac{\varepsilon}{\alpha}}, \end{cases} \quad (8.174)$$

and for $\varepsilon > 0$

$$\lambda = \frac{df(x_*)}{dx} = 1 + \varepsilon. \quad (8.175)$$

The conjugate solutions cross the convergent circle simultaneously (Fig. 8.8).

8.10.3 Type III Intermittency

A map

$$x_{n+1} = -(1 + \varepsilon)x_n - \alpha x_n^3 \quad (8.176)$$

has fixed points

$$x_* = 0, \quad \pm \sqrt{-\frac{\varepsilon}{\alpha}}.$$

The eigenvalue is given by

$$\lambda = \frac{df(x_*)}{dx} = -(1 + \varepsilon) - 3\alpha x_*^2. \quad (8.177)$$

As ε changes from negative to positive, the motion of the system leads to chaos through regular oscillation (Fig. 8.9).

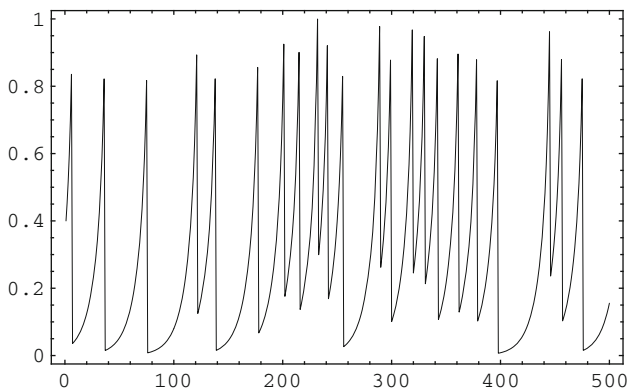


Fig. 8.8 The time series data of the type II intermittent chaos

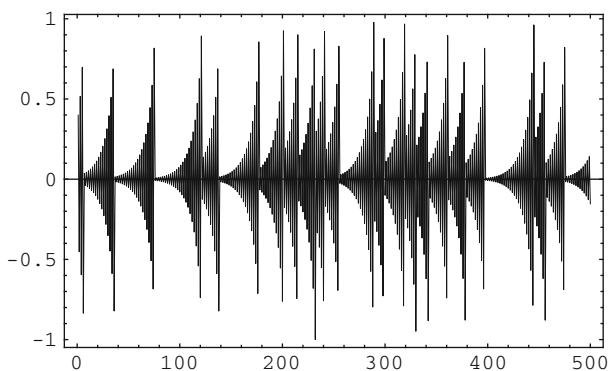


Fig. 8.9 The time series data of the type III intermittent chaos

8.11 Chaos in Plasmas

8.11.1 Stochastic Web

When a perturbation with the frequency ω is applied to a periodic dynamical system characterized by Ω , the particle gets kicks from the perturbation repetitively at the resonance. Even if the perturbation amplitude is small, the separatrix is destroyed and a stochastic layer is formed. The repetitive resonances connect the stochastic layers to develop a web of the stochastic layers [159]. Since the kick depends on integers r and s where $\omega/\Omega = r/s = \text{rational}$, the map produced from the flow is twisted, the web structure reflects the symmetry of the resonance.

An equation of motion for a particle in a homogeneous magnetic field is given by

$$\frac{d}{dt} \mathbf{x} = \mathbf{v}, \quad (8.178)$$

$$\frac{d}{dt} \mathbf{v} = \frac{e}{m} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}_0 \right). \quad (8.179)$$

The magnetic field and the electrostatic field are assumed to be

$$\mathbf{B}_0 = (0, 0, B_0), \quad \mathbf{E} = \left(\sum_k E_k \sin(kx - \omega_k t), 0, 0 \right),$$

the equation of motion is rewritten as

$$\frac{d}{dt} \left\{ \frac{d^2 x}{dt^2} + \Omega^2 x - \sum_k E_k \sin(kx - \omega_k t) \right\} = 0, \quad \Omega = \frac{eB_0}{mc},$$

which is reduced under a proper initial condition

$$\frac{d^2 x}{dt^2} + \Omega^2 x = \sum_k E_k \sin(kx - \omega_k t). \quad (8.180)$$

This is a harmonic oscillator with perturbations and describes a web independent of the amplitude when $\Omega/\omega_k = \text{integer}$ and the separatrix is destroyed by the perturbation.

Now, taking a wave packet as the perturbation

$$E(x, t) = E_0 \sum_{n=-\infty}^{\infty} \sin(k_0 - \omega_0 t - n\Delta\omega t) \quad (8.181)$$

and using an identity with $T = 2\pi/\Delta\omega$

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} \cos(n\Delta\omega t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad (8.182)$$

the perturbation is expressed by

$$E(x, t) = E_0 T \sin(k_0 x - \omega_0 t) \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad (8.183)$$

implying that the system is equivalent to an oscillator subject to kicks at discrete time. Between the two consecutive kicks, the particle obeys (8.180) without the fluctuation force

$$\frac{d^2 x}{dt^2} + \Omega^2 x = 0, \quad (8.184)$$

which is integrated to

$$x(t) = A \cos(\Omega t) + B \sin(\Omega t), \quad (8.185)$$

$$v_x(t) = \Omega \{A \cos(\Omega t) - B \sin(\Omega t)\}. \quad (8.186)$$

The particle kicked at $t_n = nT$ satisfies the boundary condition

$$x(t_n + 0) = x(t_n - 0), \quad v(t_n + 0) = v(t_n - 0) - E_0 T \sin(k_0 x(t_n) - \omega_0 t_n),$$

which determine A and B . Then, by putting $v(t = nT - 0) = v(n)$, the equation is discretized to

$$v_x(n+1) = v_y(n) \sin(\Omega T) + \left\{ v_x(n) + TE_0 \sin\left(n\omega_0 T - \frac{k_0 v_y(n)}{\Omega}\right) \right\} \cos(\Omega T), \quad (8.187)$$

$$v_y(n+1) = v_y(n) \cos(\Omega T) - \left\{ v_x(n) + TE_0 \sin\left(n\omega_0 T - \frac{k_0 v_y(n)}{\Omega}\right) \right\} \sin(\Omega T), \quad (8.188)$$

where $x(n) = v_y(n)/\Omega$ has been used. Introducing new variables through $u = k_0 v_x/\Omega$, $v = k_0 v_y/\Omega$, $\alpha = \Omega T$ and ignoring the acceleration mode $\omega_0 = 0$, we have

$$u(n+1) = (u(n) + K \sin v(n)) \cos \alpha + v(n) \sin \alpha, \quad (8.189)$$

$$v(n+1) = -(u(n) + K \sin v(n)) \sin \alpha + v(n) \cos \alpha, \quad (8.190)$$

where α is a transformation angle and satisfies for resonance

$$\alpha_{p,q} = \frac{2\pi p}{q}. \quad (8.191)$$

For $p = 1$, $q = 4$, and $\alpha_{1,4} = \pi/2$, (8.189) and (8.190) read

$$u(n+1) = v(n), \quad (8.192)$$

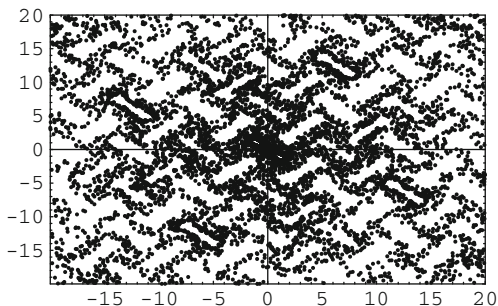
$$v(n+1) = -u(n) - K \sin v(n). \quad (8.193)$$

Figure 8.10 shows the stochastic web obtained by solving (8.192) and (8.193) for $q = 4$ and $K = 2.5$.

When $K \ll 1$ the twisting in (8.192) and (8.193) is disentangled as follows. Noting

$$\begin{aligned} u(n+2) &= v(n+1) = -u(n) - K \sin v(n), \\ v(n+2) &= -u(n+1) - K \sin v(n+1) = -v(n) - K \sin(-u(n) - K \sin v(n)) \\ &\sim -v(n) + K \sin u(n) + O(K^2), \end{aligned}$$

Fig. 8.10 Stochastic web given by (8.189) and (8.190) for $q = 4$ and $K = 2.5$



$$\begin{aligned} u(n+3) &= -u(n+1) - K \sin v(n+1) \sim -v(n) + K \sin u(n) + O(K^2), \\ v(n+3) &= -v(n+1) + K \sin u(n+1) \sim u(n) + 2K \sin v(n), \end{aligned}$$

we have up to $O(K)$

$$\begin{pmatrix} u(n+4) \\ v(n+4) \end{pmatrix} = \begin{pmatrix} u(n) + 2K \sin v(n) \\ v(n) - 2K \sin u(n) \end{pmatrix}. \quad (8.194)$$

A corresponding Hamiltonian is given by

$$\begin{aligned} H_4 &= -\frac{K}{2T} \cos v - 2K \cos \sum_{n=-\infty}^{\infty} \delta(t - 4nT), \\ &= -\frac{K}{2T} (\cos v + \sin u) - \frac{K}{2T} \cos u \sum_{n \neq 0} \cos \left(\frac{n\Delta t}{4} \right). \end{aligned} \quad (8.195)$$

Putting $\Omega_4 = K/2T$ and assuming $\Omega_4/\Delta\omega \ll 1$, then time averaged Hamiltonian is given by

$$\bar{H}_4 = -\Omega_4 (\cos v + \sin u). \quad (8.196)$$

The separatrix is obtained by $\bar{H}_4 = 0$ as

$$v = \pm(u + \pi) + 2\pi n, \quad (8.197)$$

which gives a web. Although the width of the web channel is exponentially small, it covers all the phase space and the particle may diffuse to a high energy region.

8.11.2 Chaos of Particle Motion in a Magnetic Mirror Field

Stochastic heating had long been a topic associated with auxiliary heating in fusion plasmas and studied both experimentally and theoretically [160–168]. However, a difficulty in studying particle acceleration by an electromagnetic wave is that the

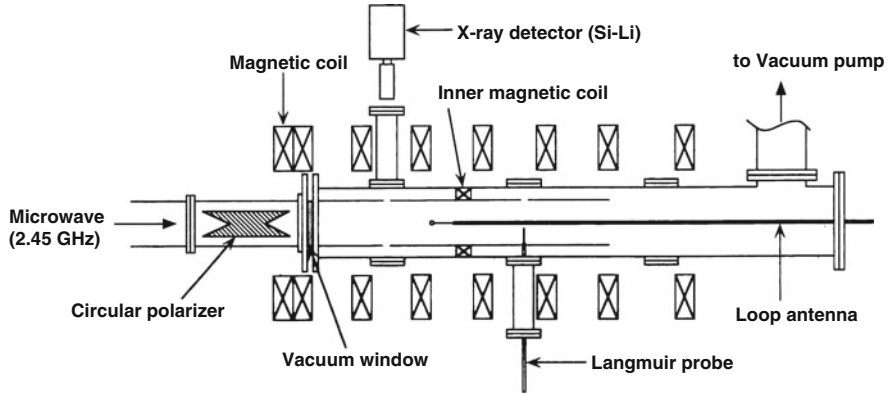
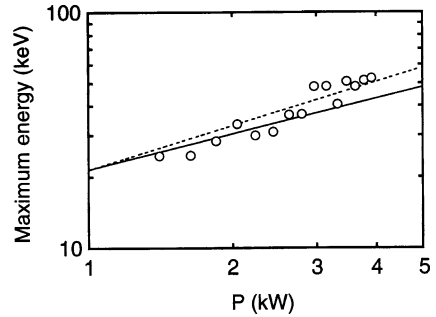


Fig. 8.11 The experimental set up

Fig. 8.12 The maximum electron energy as a function of input microwave power: observation (circle), numerical simulation (dotted line) and theory (solid line)



wave damping was so weak to cause the multiple reflection by the chamber wall. Then, an experiment was done to overcome the difficulty by an electron cyclotron wave which strongly interacts with electrons trapped in a weak mirror field and fully damps even in low density plasmas [169]. The experimental device is shown in Fig. 8.11. The particle acceleration was confirmed by detecting X-rays. The X-ray emission was observed with the field intensity close to the electron cyclotron resonance. The X-ray energy spectra for different input powers are studied to give the scaling with respect to the input power, which is shown in Fig. 8.12.

Suppose a monochromatic wave is applied to the system. The equation of motion is given by

$$\frac{dv_x}{dt} = -|\Omega_e|v_y - \frac{eE_0}{m_e} \sin(kz - \omega t), \quad (8.198)$$

$$\frac{dv_y}{dt} = |\Omega_e|v_x - \frac{eE_0}{m_e} \cos(kz - \omega t), \quad (8.199)$$

$$\frac{dv_z}{dt} = -\frac{v_x^2 + v_y^2}{2B_0} \frac{\partial B_z}{\partial z}, \quad (8.200)$$

$$\frac{dz}{dt} = v_z, \quad (8.201)$$

where Ω_e is the electron cyclotron frequency. In this system, there are three invariants, magnetic moment, action of the oscillation motion in axial direction and action of the drift motion in the azimuthal direction. Introducing a variable transformation as

$$\begin{pmatrix} U \\ V \\ Z \end{pmatrix} = \begin{pmatrix} \cos(kz - \omega t) & -\sin(kz - \omega t) & 0 \\ \sin(kz - \omega t) & \cos(kz - \omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ z \end{pmatrix}, \quad (8.202)$$

the equation of motion reads

$$\frac{dU}{dt} = \Omega V, \quad (8.203)$$

$$\frac{dV}{dt} = -\Omega U - \frac{eE_0}{m_e}, \quad (8.204)$$

$$\frac{dW}{dt} = -\frac{U^2 + V^2}{2B_0} \frac{\partial B_z}{\partial Z}, \quad (8.205)$$

$$\frac{dZ}{dt} = W, \quad (8.206)$$

where $W = v_z A \Omega = \omega - |\Omega_e| - kW$. This is a conserved system seen by

$$\frac{\partial \dot{U}}{\partial U} + \frac{\partial \dot{V}}{\partial V} + \frac{\partial \dot{W}}{\partial W} + \frac{\partial \dot{Z}}{\partial Z} = 0, \quad (8.207)$$

We define U_n and V_n as the perpendicular velocities of the electron going to the negative z -direction ($W < 0$) and crossing the $Z = 0$ plane, in the n -th bouncing motion. Since the acceleration by the ECW is perpendicular to the magnetic field, the parallel velocity $W_n(Z = 0)$ does not change in the subsequent bouncing motions, and, therefore, is equal to its initial velocity $W_n = -W_0$. Integrating (8.203)–(8.206) by one bouncing period with the initial condition $\{U_n, V_n, -W_0, Z = 0\}$, a simple map is obtained as

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} \cos \alpha_n & -\sin \alpha_n \\ \sin \alpha_n & \cos \alpha_n \end{pmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix} + K \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (8.208)$$

where

$$\alpha_n = 2\pi ak W_0 / \sqrt{U_n^2 + V_n^2 + 2kL}, \quad K = \frac{eE_0}{m_e} \frac{L}{W_0},$$

and L is the mirror length. Since the constant phase $2kL$ in α_n does not make substantial change in the phase space structure, we set $2kL = 0$ in the following, without loss of generality.

Figure 8.13 shows the Poincaré surface of section plots; (a) is obtained by numerical integrating (8.203)–(8.206), and (b) by mapping (8.208) 300 times, in which $ak = 255.5$, $K/W_0 = 2.4$ and 31 particles at different initial velocities are used in the calculation. The vertical axis is the absolute value of the normalized

perpendicular velocity $\sqrt{U_n^2 + V_n^2}/W_0$, and the horizontal axis is the angle $\theta = \tan^{-1}(V_n/U_n)$. There is no significant difference between (a) and (b) of Fig. 8.13 in the phase space structure, and the map correctly represents the phase space structure of the original system. There exist islands as $\sqrt{U^2 + V^2}/W_0 = 128, 85, 64, \dots$, and these islands accumulate in the low energy region. There also exists a stochastic ‘sea’ in the low energy region and its boundary to the regular motion expands to eat the regular region as the ‘kick’ K is increased. The electrons located in the low energy region can diffuse to reach the edge of the stochastic sea through the multiple interaction with the ECW, but never enter into the regular region. Hence, this boundary is the upper limit of the acceleration by the ECW.

The structure of the Poincarè map is similar to that of the Fermi map [170], which is seen in the following. By introducing a transformation

$$U_n + iV_n = I_n e^{i\theta_n}, \quad (8.209)$$

we rewrite (8.208) as

$$\begin{aligned} I_{n+1} e^{i\theta_{n+1}} &= I_n e^{i(\alpha_n + \theta_n)} - iK \\ &= I_n e^{i(\alpha_n + \theta_n + \chi_n)} \sqrt{1 + \frac{2K}{I_n} \sin(\alpha_n + \theta_n) + \left(\frac{K}{I_n}\right)^2}, \end{aligned} \quad (8.210)$$

where $\tan \chi_n = -(K/I_n) \cos(\theta_n + \alpha_n) / [1 - (K/I_n) \sin(\alpha_n + \theta_n)]$. When $K/I_n \ll 1$, we neglect the small angle χ_n and the following equation is obtained within the first order of K/I_n :

$$I_{n+1} e^{i\theta_{n+1}} = I_n \left[1 + \frac{K}{I_n} \sin(\alpha_n + \theta_n) \right] e^{i(\alpha_n + \theta_n)}.$$

Then, we have the standard form of the Fermi map

$$I_{n+1} = I_n + K \sin \theta_{n+1}, \quad (8.211)$$

$$\theta_{n+1} = \theta_n + \frac{\pi a k}{I_n}. \quad (8.212)$$

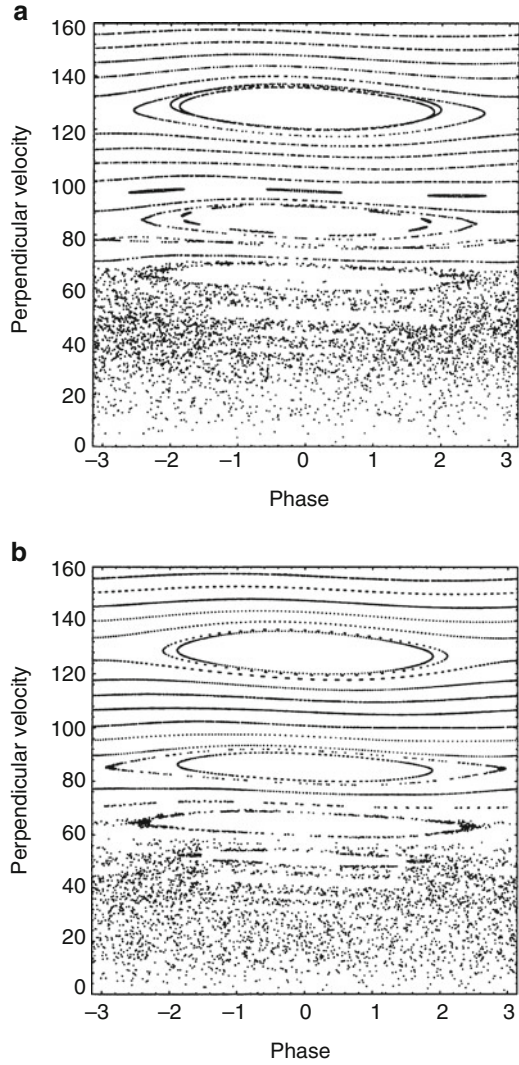
The m -th island (period 1 fixed point) is defined by

$$\frac{2\pi a k W_0}{I^{(m)}} = 2m\pi, \quad m = 1, 2, 3, \dots \quad (8.213)$$

where $I^{(m)}$ is the position of the island in the phase space spanned by (θ, I) . Equation (8.213) predicts the islands at $I^{(m)}/W_0 = 255.5/m$ ($m = 1, 2, 3, \dots$) for the present case, which well agree with the numerical results shown in Fig. 8.13.

The boundary between the regular and the stochastic regions is given by $I_{\max}$ which is calculated from the Fermi map (8.211) and (8.212). For a certain I_n value,

Fig. 8.13 Poincarè plot obtained by solving (a) (8.203)–(8.206), (b) discrete map (8.208)



the total phase change after one bouncing motion is given by substituting (8.211) into (8.212)

$$\theta_{n+1} = \theta_n + \frac{2\pi ak W_0}{I_{n-1} + K \sin \theta_n} = \theta_n + \frac{2\pi ak W_0}{I_{n-1}} \left(1 - \frac{K}{I_{n-1}} \sin \theta_n \right), \quad (8.214)$$

where $K/I_n \ll 1$ has been used. Equation (8.214) gives a stochasticity threshold as

$$\frac{2\pi ak W_0 K}{I_{\max}^2} \simeq 1, \quad (8.215)$$

where $I_{\max}$ is the boundary value between the regular and the stochastic regions, that is, the upper limit of acceleration, and hence gives the maximum energy scaling with respect to input power as $K \propto I_{\max}^2$. Noting $K \propto E$ and $I_{\max}^2 \propto E_{\max}$ (maximum energy), we have

$$E_{\max} \propto E \propto \sqrt{P}, \quad (8.216)$$

which is compared with Fig. 8.12.

8.11.3 Chaos in a Current-Carrying Ion Sheath

The notion of chaos motivated studies in various fields of plasma physics. In a current carrying plasma, there also reported several works on chaos [171]–[172] which did not show a route to chaos although observation of chaos was claimed. After a coherent instability was found to be related to a set of cascading bifurcations in a current-carrying ion sheath by Ohno [174], Komori [175] made an experiment showing a clear cut onset of chaos in a current-carrying ion sheath by applying an external periodic oscillation to a fine-meshed grid which divides the plasma produced by dc discharge. The experimental set-up is illustrated in Fig. 8.14. The period doubling bifurcation to chaos is observed when the frequency of the external field is increased as is shown in Fig. 8.15 where the period doubling $\omega/2^n$ is observed up to $n = 2$. Further period doublings for $n > 2$ hardly measured, possibly because of the rapid convergence to chaos. Increasing the frequency brings the spectrum continuous. Further increase in the frequency causes period tripling.

In the experimental set-up, the potential at the grid is negatively biased to the extent that the electrons cannot penetrate up to the grid and, therefore, the resultant sheath is an ion sheath which is detached from the plasma. The ion sheath on both

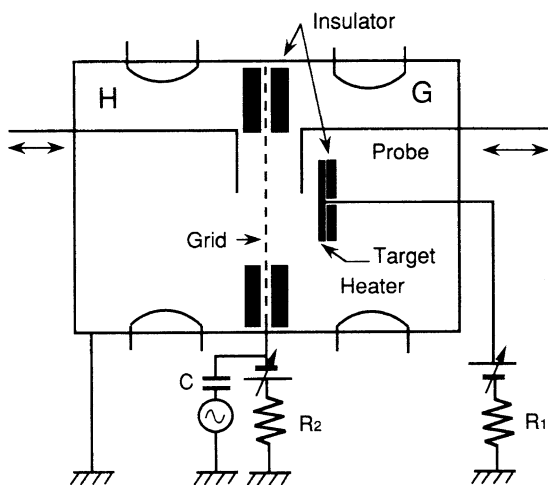
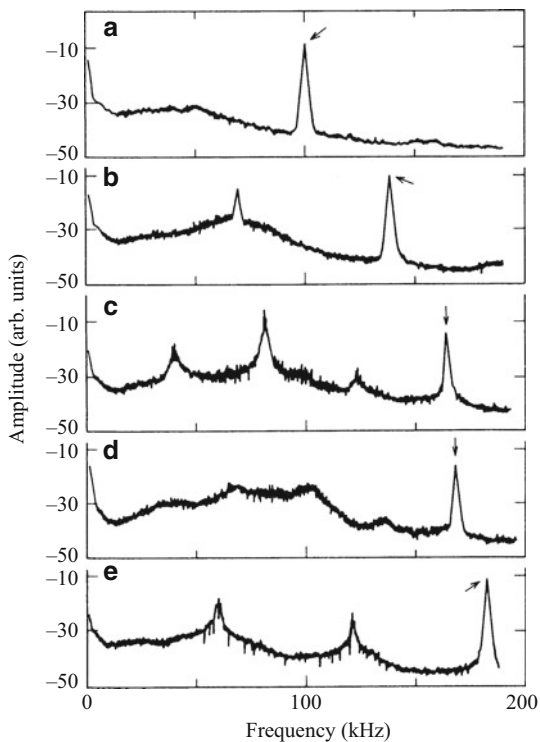


Fig. 8.14 The experimental set up

Fig. 8.15 The observed period doubling bifurcation to chaos. The *arrows* indicate the driving frequency



sides of the grid forms a potential well in which the ions oscillate to induce a primary motion responding to the external oscillation.

The structure of the ion sheath is studied based on one-dimensional ion fluid:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \quad (8.217)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{e}{m_i} E, \quad (8.218)$$

$$\frac{\partial E}{\partial x} = 4\pi en. \quad (8.219)$$

For the stationary sheath, the flux is constant:

$$nv = n_0 v_0 = \frac{J_0}{4\pi e}. \quad (8.220)$$

From (8.218) and (8.219), we have

$$\frac{d}{dx} \left\{ \frac{J_0^2}{n} - \frac{E^2}{8\pi m_i} \right\} = 0, \quad (8.221)$$

which gives

$$n = \frac{n_0}{1 + (A/2)(E^2/E_0^2 - 1)}, \quad (8.222)$$

where $A = (\omega_{\text{pi}} E_0 / J_0)^2$. Substituting (8.222) into the Poisson equation (8.219), we have an implicit expression for E with respect to x as

$$x - x_0 = \frac{e E_0}{m_i \omega_{\text{pi}}^2} \left\{ \left(1 - \frac{A}{2}\right) \left(\frac{E}{E_0} - 1\right) + \frac{A}{6} \left(\frac{E^3}{E_0^3} - 1\right) \right\}. \quad (8.223)$$

The potential is expressed in term of E as

$$\phi - \phi_0 = -\frac{1}{2} \frac{e E_0^2}{m_i \omega_{\text{pi}}^2} \left(\frac{E^2}{E_0^2} - 1\right) \left\{ 1 + \frac{A}{4} \left(\frac{E^2}{E_0^2} - 1\right) \right\}. \quad (8.224)$$

For a large electric field, we have from (8.223) and (8.224),

$$\frac{E}{E_0} = \left(\frac{6}{A} \frac{x - x_0}{e E_0 / m_i \omega_{\text{pi}}^2} \right)^{1/3}, \quad \phi - \phi_0 = -\frac{A}{8} \frac{e E_0^2}{m_i \omega_{\text{pi}}^2} \left(\frac{E}{E_0}\right)^4, \quad (8.225)$$

which are combined to give an explicit expression of the Child–Langmuir law of space-charge limited current in a plane diode:

$$\frac{\phi_0 - \phi}{e E_0^2 / m_i \omega_{\text{pi}}^2} \simeq \frac{6^{4/3}}{8} \left(\frac{x - x_0}{e E_0 / m_i \omega_{\text{pi}}^2} \right)^{4/3} \left(\frac{J_0}{\omega_{\text{pi}} E_0} \right)^{2/3}. \quad (8.226)$$

Now, consider the ion dynamics in the sheath potential, which is described by the equation of motion

$$\frac{dx}{dt} = v, \quad (8.227)$$

$$\frac{dv}{dt} = \frac{e}{m_i} (E_0 - E). \quad (8.228)$$

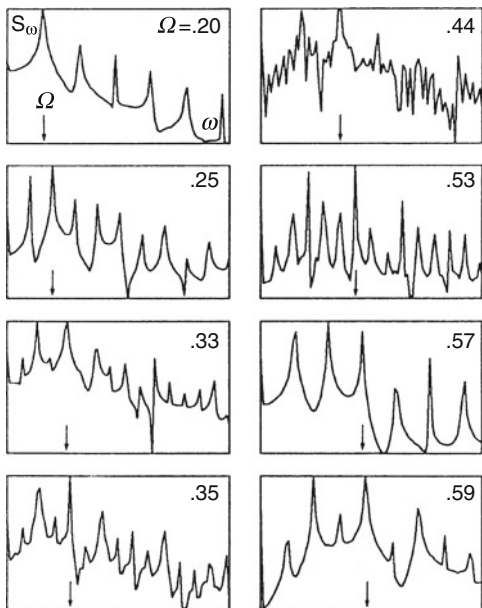
In order to see the response to an oscillating external field, (8.228) is replaced as

$$\frac{dv}{dt} = \frac{e}{m_i} (E_0 - E) - \nu v + E_{\text{ext}} \sin(\Omega t), \quad (8.229)$$

where the damping term is introduced since the experiments are done in the plasma with dissipation. From (8.223), we may approximately express $E_0 - E$ in terms of $(x - x_0)/(e E_0 / m_i \omega_{\text{pi}}^2) \equiv x$ as

$$\frac{E_0 - E}{E_0} \simeq -\frac{x}{1 + A(x + (1 - 3A)x^2/3)}. \quad (8.230)$$

Fig. 8.16 The numerical solution of (8.231) to exhibit the period doubling bifurcation to chaos. The phase space trajectories and the Fourier spectrum of $J(t)$ with $A = 0.2$, $\nu = 0.18$, and $E_{\text{ext}} = 2.2$ for various values of Ω



Combining (8.227), (8.229), and (8.230) and replacing E_0 by $-|E_0|$ since E_0 is chosen as negative, we finally

$$\frac{d^2x}{dt^2} + \nu \frac{dx}{dt} + \frac{x}{1 + A[x + (1 - 3A)x^2/3]} + E_{\text{ext}} \sin(\Omega t) = 0, \quad (8.231)$$

where E_0 is taken negative and the following normalizations are used:

$$\omega_{\text{pi}} t \rightarrow t, \quad \frac{\nu}{\omega_{\text{pi}}} \rightarrow \nu, \quad \frac{\Omega}{\omega_{\text{pi}}} \rightarrow \Omega, \quad \frac{I}{\omega_{\text{pi}}|E_0|} \rightarrow I, \quad \frac{E_{\text{ext}}}{|E_0|} \rightarrow E.$$

The numerical solution of (8.231) is given in Fig. 8.16 for the case of period doubling bifurcation to chaos, which shows a good agreement with the experimental data Fig. 8.15.

Equation (8.231) is a dynamical system of $3/2$ degrees of freedom (three independent variables). Therefore, the dimension of the chaos of this system is supposed to be less than three. We can check this by estimating the correlation dimension defined by

$$D = \lim_{\ell \rightarrow \infty} \frac{\log C(\ell)}{\ell}, \quad C(\ell) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i,j} \theta(\ell - |\mathbf{x}_i - \mathbf{x}_j|),$$

where a vector $\mathbf{x}_i$ is constructed by the time series of $\mathbf{x}(t)$ as

$$\mathbf{x}_i(t) = \{x(t), x(t + \tau), \dots, x(t + (d - 1)\tau)\}.$$

A numerically estimated correlation dimension is given by

$$D = 1.54 \pm 0.04, \quad (8.232)$$

which agrees well with the value $D_{\text{exp}} = 1.54 \pm 0.22$ obtained from the experimental data.

8.11.4 Chaos of Magnetic Field Lines

A magnetic surface is a vessel to confine a plasma. Chaos of magnetic field line is to destruct the magnetic surface, resulting in an anomalous transport. In this sense, controlling chaos of magnetic field lines is practically important for plasma confinement.

An equation for magnetic field line in the cylindrical coordinate is given by

$$\frac{dr}{dz} = \frac{B_r}{B_z}, \quad \frac{d\theta}{dt} = \frac{1}{r} \frac{B_\theta}{B_z}, \quad (8.233)$$

From the divergent-free condition

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \frac{1}{r} \frac{\partial}{\partial r}(rB_r) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0, \quad (8.234)$$

we have for a tokamak

$$B_z = B_0 = \text{const}, \quad B_r = -\frac{1}{r} \frac{\partial}{\partial \theta} \int dr B_\theta.$$

Changing a variable from (r, z) to (I, ϕ) through

$$I = \frac{1}{R} \int_0^r dr r B_z = \frac{B_0}{2R} r^2, \quad z = R\phi, \quad (8.235)$$

and defining

$$H = \int_0^r dr B_\theta(r, \theta, z) = H(I, \theta, \phi), \quad (8.236)$$

the equation for the magnetic field line is written as

$$\frac{dr}{dz} = \frac{R}{B_0 r} \frac{dI}{R d\phi} = \frac{B_r}{B_0}, \quad (8.237)$$

and

$$\frac{d\theta}{dz} = \frac{1}{R} \frac{d\theta}{d\phi} = \frac{1}{r} \frac{B_\theta}{B_0}, \quad (8.238)$$

which give

$$\frac{dI}{d\phi} = rB_r = -\frac{\partial}{\partial\theta} \int dr B_\theta = -\frac{\partial H}{\partial\theta}. \quad (8.239)$$

and

$$\frac{d\theta}{d\phi} = \frac{RB_\theta}{rB_0} = \frac{R}{rB_0} \frac{\partial H}{\partial r} = \frac{\partial H}{\partial I}. \quad (8.240)$$

Suppose an equilibrium state (unperturbed state) with a safety factor

$$H = H_0(I), \quad q(r) = \frac{B_0}{B_\theta} \frac{r}{R} = \left(\frac{\partial H_0}{\partial I} \right)^{-1}. \quad (8.241)$$

A magnetic surface for which the safety factor is commensurate is a rational surface.

$$q(I_r) = \frac{m_0}{n_0}. \quad (8.242)$$

When a magnetic perturbation

$$H_1(I, \theta, \phi) = \sum V_{mn}(I) e^{i(m\theta - n\phi)} + \text{c.c.} \quad (8.243)$$

is applied, the Hamiltonian in the vicinity of the rational surface is approximated by

$$\begin{aligned} H = & \frac{1}{2} \frac{\partial^2 H_0(I_r)}{\partial I^2} (I - I_r)^2 + V_{m_0 n_0}(I_r) \cos(m_0 \theta - n_0 \phi) \\ & + \sum V_{mn}(I_r) \cos(m \theta - n \phi). \end{aligned} \quad (8.244)$$

Putting

$$I - I_r = m_0 p, \quad \xi = m_0 \theta - n_0 \phi, \quad (8.245)$$

the Hamiltonian is rewritten as

$$\begin{aligned} H = & \frac{1}{2} \frac{\partial^2 H_0(I_r)}{\partial I^2} (m_0 p)^2 + V_{m_0 n_0}(I_r) \cos \xi \\ & + \sum V_{mn}(I_r) \cos \left[\frac{m}{m_0} (\xi + n_0 \phi) - n \phi \right], \end{aligned} \quad (8.246)$$

which is nothing but a Hamiltonian for the standard map.

8.11.5 Anomalous Transport in Tokamak and Tokamap

The characteristics of the tokamak discharge are summarized as [22]

- The current distribution $J(r)$ is a monotonically decreasing function of r
- Rotational transform $q(r) = B_0 r / B_\theta R (B_\theta \propto J(r))$ is a monotonically increasing function of r
- Magnetic shear $s = dq(r)/dr \geq 0$

8.11.5.1 Toroidal Coupling and Transport Barrier

Fluctuations in a tokamak are localized at rational surfaces since the fluctuations expressed by

$$\phi(r, \theta, \psi, t) = \sum_{m,n} \phi_{m,n}(r) e^{i(m\theta - n\psi) - i\omega t},$$

resonate with the pitch of the magnetic field line. Since the width of the localized eigenfunction is proportional to $1/\sqrt{s}$ and the distance between the resonance is proportional to $1/s$, when the magnetic shear is large, the eigenfunctions overlap (toroidal coupling) to enhance the diffusion and worsen the plasma confinement. On the contrary, when the magnetic shear becomes small, overlapping of eigenfunctions becomes also small (breakdown of the toroidal coupling) to suppress the diffusion.

When neutral beams are injected to the center of the plasma to heat and increase the electric conductivity, the current penetration to the center is delayed due to the skin effect and the current distribution becomes transiently convex and the magnetic shear is zero at the minimum, negative in the inner region and positive in the outer region. Thus, the toroidal coupling is suppressed and the transport is limited in the region where the magnetic shear is reversed. This reversed magnetic shear is called the transport barrier.

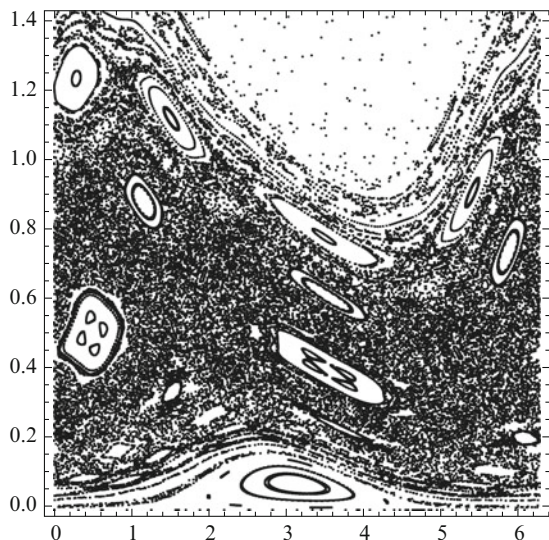
8.11.5.2 Tokamap

The aspect ratio of the standard map is a monotonically increasing function of current I ($\omega(I_{n+1}) = I_{n+1}$), while the aspect ratio in a tokamak is a monotonically decreasing function of current $I (\propto r^2)$. The map which takes into account the correct dependence of the aspect ratio on the current in a tokamak is given by Balescu [177] and is called the Tokamap,

$$I_{n+1} = I_n - K \frac{I_{n+1}}{1 + I_{n+1}} \sin \theta_n \quad (8.247)$$

$$\theta_{n+1} = \theta_n + 2\pi\omega(I_{n+1}) - K \frac{1}{(1 + I_{n+1})^2} \cos \theta_n, \quad (8.248)$$

Fig. 8.17 Poincarè plot of Tokamap for the normal discharge (8.249) with $K = 4.8/2\pi$ and $a = 1$



where the aspect ratios of the normal discharge and the discharge with the reversed magnetic shear are given by

- Normal discharge

$$\omega(I) = \frac{a}{4}(2 - I)[2 - 2I + I^2], \quad (8.249)$$

- Discharge with the reversed magnetic shear

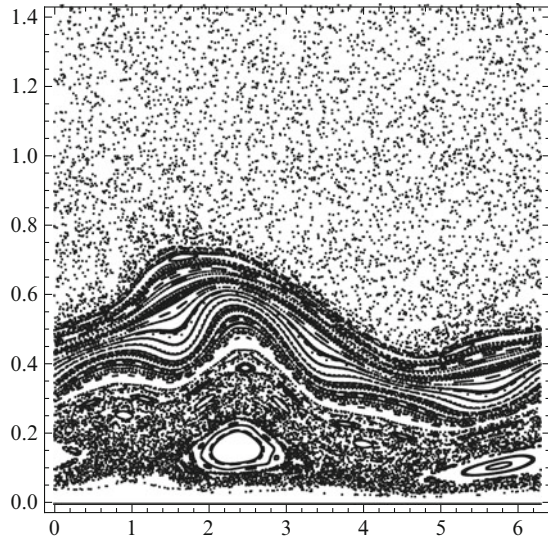
$$\omega(I) = a[1 - b(cI - 1)^2]. \quad (8.250)$$

The Tokamap shows that for the discharge with the reversed magnetic shear there exist integrals in the core part of the plasma and chaos is limited in the region outside the core. The integral separating the chaos region is the transport barrier and the particle transport is suppressed by this integral. The Poincarè plot for the Tokamap is shown in Figs. 8.17 and 8.18 where the transport barrier is clearly observed.

8.11.6 Ponderomotive Force at the Onset of Chaos

The ponderomotive force is based on adiabatic interactions of particles with waves, that is, a weak field approximation of particle dynamics in which the motion of an oscillation center is well separated from a quiver motion. When the amplitudes of waves become so large as to trap particles, the bounce motion of the trapped particles can be resonated with the wave, leading to stochastic motion of the particles. Thus, the oscillation center approximation becomes invalidated and the ponderomotive force loses its footing. Schmidt [178] studied the surface of a section mapping

Fig. 8.18 Poincarè plot of Tokamap for the reversed magnetic shear (8.250) with $K = 4.5/2\pi$, $a = 0.67$, $b = 0.5025$, and $c = 2.2226$



of the motion of a particle in a standing wave and demonstrated that a fixed point corresponding to the bottom of the ponderomotive potential well is destabilized to be swallowed in chaos. A similar problem arises for the ponderomotive force in a magnetized plasma at cyclotron resonance. A singular behavior of the ponderomotive force in an adiabatic approximation at resonance has been shown to disappear experimentally and numerically by Dimonte, Lamb, and Morales [179, 180]. Kono and Sanuki [72] showed that this nonsingular behavior at resonance closely related to the onset of particle orbit instability and obtained an analytical expression for the ponderomotive force, which diminishes at resonance and recovers the adiabatic expression far off the resonance.

Here, we derive an analytical expression of the ponderomotive force in the case considered by Schmidt. The equation of motion in dimensionless units is given as

$$\frac{d^2x}{dt^2} = K \cos(2\pi x) \sin(2\pi t), \quad (8.251)$$

with the single parameter

$$K = \frac{2\pi eEk}{m\omega^2} = 2\pi \left(\frac{\omega_b}{\omega} \right)^2. \quad (8.252)$$

where k , ω , and E are the wavenumber, frequency, and amplitude of the standing wave, respectively. It should be noted that the parameter K is given as a ratio of the bounce frequency ω_b to the wave frequency, or a ratio of the quiver velocity to the phase velocity. Now, we split the particle motion into its slowly (s) and rapidly (f) varying parts

$$x = x_s + x_f, \quad \langle x_f \rangle = 0, \quad (8.253)$$

where $\langle \dots \rangle = \int_{-\pi/2}^{\pi/2} \dots dt$. This split is possible only when the motion of an oscillation center is well separated on the time scale from a quiver motion. In the following, we restrict ourselves to the case up to the onset of chaos. From (8.251), we obtain the equations for the slow motion

$$\frac{d^2 x_s}{dt^2} = K \langle \cos[2\pi(x_s + x_f)] \sin(2\pi t) \rangle, \quad (8.254)$$

and for the fast motion

$$\frac{d^2 x_f}{dt^2} = K \cos[2\pi(x_s + x_f)] \sin(2\pi t) - \frac{d^2 x_s}{dt^2}, \quad (8.255)$$

The crucial point is to take into account the effects of the finite amplitude wave on the particle trajectory. The lowest order approximation is to retain the effects of the successive scattering by the wave in the course of time development

$$\frac{d^2 x_f}{dt^2} + 2\pi K \sin(2\pi x_s) [x_f \sin(2\pi t) - \langle x_f \sin(2\pi t) \rangle] = K \cos(2\pi x_f) \sin(2\pi t). \quad (8.256)$$

Equation (8.256) is an inhomogeneous Mathieu equation, now rewritten in the Fourier space as

$$g_n + a_n(g_{n+1} - g_{n-1}) = b_n, \quad n = 0, \pm 1, \dots, \quad (8.257)$$

where

$$g_n = x(\omega + 2n\pi), \quad a_n = i\pi K \sum (2\pi x_s)/(\omega + 2n\pi)^2, \quad (8.258)$$

and

$$b_n = -\frac{K}{2} \frac{\cos(2\pi x_s)}{(\omega + 2n\pi)^2} \left(\frac{1}{\omega + 2\pi} - \frac{1}{\omega - 2\pi} \right). \quad (8.259)$$

By induction, we obtain

$$\begin{aligned} g_{\pm(N-\ell)} = & \pm \frac{a_{\pm(N-\ell)}}{\Sigma^{(\pm)}(N-\ell)} g_{\pm(N-\ell-1)} \pm \frac{\prod_{m=1}^{\ell+1} a_{\pm(N+1-m)}}{\prod_{m=1}^{\ell} \Sigma^{(\pm)}(N-m)} g_{\pm(N+1)} \\ & + \frac{\beta^{(\pm)}(N-\ell)}{\Sigma^{(\pm)}(N-\ell)}, \quad (\ell = 1, 2, \dots, N-1), \end{aligned} \quad (8.260)$$

where

$$\Sigma^{(\pm)}(N-\ell) = 1 + \frac{a_{\pm(N-\ell)} a_{\pm(N+1-\ell)}}{\Sigma^{(\pm)}(N+1-\ell)}, \quad (8.261)$$

and

$$\beta^{(\pm)}(N-\ell) = b_{\pm(N-\ell)} - \frac{a_{\pm(N-\ell)} \beta^{(\pm)}(N+1-\ell)}{\Sigma^{(\pm)}(N+1-\ell)}. \quad (8.262)$$

Putting $\ell = N - 1$ in (8.260), $g_{\pm}$ can be expressed in terms of g_0 and $g_{\pm(N+1)}$ with an arbitrary large N . From the n dependence of a_n and b_n , we may consider a rapid convergence $g_{\pm N} \rightarrow 0$ for $N \rightarrow \infty$. Thus from (8.257), we obtain

$$g_0 = \frac{1}{\Sigma(\omega)} \left[b_0 - a_0 \left(\frac{\beta^{(+)}(1)}{\Sigma^{(+)}(1)} + \frac{\beta^{(-)}(1)}{\Sigma^{(-)}(1)} \right) \right], \quad (8.263)$$

where

$$\Sigma(\omega) = 1 + \frac{a_0 a_1}{\Sigma^{(+)}(1)} + \frac{a_0 a_{-1}}{\Sigma^{(-)}(1)}, \quad \Sigma(-\omega) = \Sigma(\omega). \quad (8.264)$$

Furthermore, neglecting higher order resonances, for the second term of the right-hand side of (8.263), we obtain

$$x_f(\omega) \simeq -\frac{K \cos(2\pi x_s)}{2 \omega^2 \Sigma(\omega)} \left(\frac{i}{\omega + 2\pi} - \frac{i}{\omega - 2\pi} \right). \quad (8.265)$$

Thus, the evolution of $x_f(t)$ is determined by the poles of (8.265). Here, the zeros of $\Sigma(\omega)$ are estimated at the fixed point of the oscillation center $x_s = 1/4$. Although the number of the zeros of $\Sigma(\omega)$ is infinite for fixed K , frequencies larger than 2π do not contribute to (8.254). As K increases, the two smallest frequencies ($\omega = 0$ and 2π at $K = 0$) approach each other and merge at $\text{Re } \omega = \pi$ to become complex when $K = 2.8528$, where the periodic solution is destabilized. For K near K_c , the orbit is represented as

$$x_f(t) = A_0 K \sin(2\pi x_s) \sin(\pi t), \quad (8.266)$$

where the residue A_0 corresponding to the pole $\omega = \pi$ is represented as

$$A_0 = -\frac{4}{3\pi^4} \left(\frac{\partial \Sigma}{\partial \omega} \right)_{K_c}^{-1} \propto (K_c - K)^{-1/2}. \quad (8.267)$$

Now, we are ready to estimate the ponderomotive force, defined by the right-hand side of (8.254), which is rewritten as

$$\begin{aligned} \frac{d^2 x_s}{dt^2} &= -i \frac{K}{4} (e^{i2\pi x_s} < e^{i2\pi(x_f+t)} - e^{i2\pi(x_f-t)} > -\text{c.c.}) \\ &= K \sum_{\ell} \sum_m J_{\ell}(\pi K |A_0|) J_m(\pi K |A_0|) \sin[2\pi(\ell - m + 1)x_s] \\ &\quad \times \left(\frac{\sin[(\ell + m)\pi/2 + \pi i]}{(\ell + m)\pi + 2\pi} - \frac{\sin[(\ell + m)\pi/2 - \pi i]}{(\ell + m)\pi - \pi} \right). \end{aligned} \quad (8.268)$$

Near the onset point of the stochastic instability, we just retain main contributions from $\ell + m = \pm 1$. Equation (8.268) is reduced to

$$\frac{d^2 x_s}{dt^2} = \sum_{\ell > 0} F_\ell(K) \sin(4\pi \ell x_s), \quad (8.269)$$

where

$$F_\ell(K) = (-1)^{\ell+1} \frac{4\ell}{\pi^2 A_0} J_\ell^2(\pi K |A_0|) \propto K_c - K. \quad (8.270)$$

Thus, the ponderomotive force decreases to zero as K approaches K_c .

On the other hand, for small K , the poles $\omega = \pm 2\pi$ give main contributions as

$$\frac{d^2 x_s}{dt^2} = \sum_{\ell > 0} G_\ell(K) \sin(4\pi \ell x_s), \quad (8.271)$$

where

$$G_\ell(K) = (-1)^\ell \frac{1}{2\pi^3 \Sigma(2\pi) J_\ell^2(K/8\pi |\Sigma(2\pi)|)}. \quad (8.272)$$

Thus for small K , G_ℓ is proportional to K^2 with the smallest $\ell = 1$, which is nothing but the conventional ponderomotive force.

The scaling obtained above is confirmed by numerical experiments. We make the Poincaré map for the slow motion by regarding the average of the time sequences of the solution of (8.251) within the unit time interval as $(x_s, dx_s/dt)$. In this way, the ponderomotive potential well can be visualized around the fixed point $x_s = 1/4$. The Poincaré map is depicted for various K in Fig. 8.19, which shows how the stable periodic oscillation is destabilized as K increases. The destabilization of the elliptic fixed point gives a hyperbolic fixed point near which outgoing and incoming trajectories intersect, leading to an infinite number of homoclinic points and chaotic motion. A further increase in K leads to a cascade of period doubling. We may then anticipate that the strength of the ponderomotive force is proportional to the square of the shortest distance from the separatrix to the fixed point $(x_s, dx_s/dt) = (1/4, 0)$. Figure 8.20 shows the K dependence of the measured ponderomotive force, indicating excellent agreement with the theory.

So far, we have considered the ponderomotive force based on a single particle model for which each particle executes simple harmonic motion about its equilibrium position independent of its amplitude and independent of what the rest of the particles are doing, provided the ordering of particles is maintained. When the amplitude of the wave gets larger, stochastic instability results in a chaotic motion of the particle, which affects the wave dynamics. In the terminology of a fluid description, overtaking of the fluid elements occurs to yield wave breaking. This happens when the quiver velocity becomes equal to the phase velocity [182–186]. For oscillations with such large amplitudes, there will be fine-scale mixing of the various parts of the oscillation, which, in turn, destroys the coherent oscillation to drive the system into a turbulent state. Thus, two distinct time scales underlying the ponderomotive force are mixed up and diffusion becomes the characteristic feature. However, mode–mode couplings then eventually cause modulations in space and

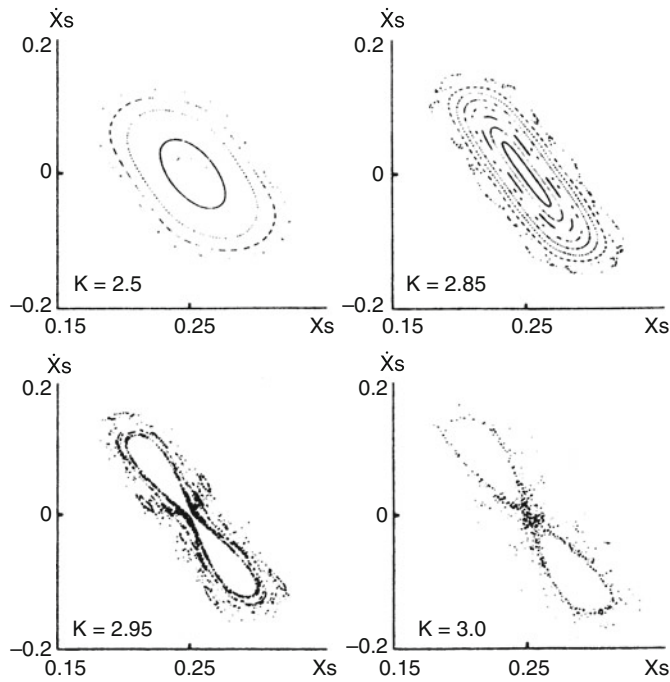
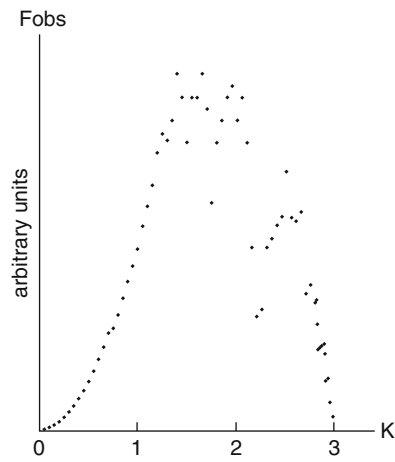


Fig. 8.19 The Poincarè map of the slow motion $(x_s, dx_s/dt)$ for various K

Fig. 8.20 The measured ponderomotive force versus K



time. This slow scale modulation is clearly distinguished on the time scale from the turbulent motion, and again provides a basis to define the ponderomotive force acting on the medium after the turbulent motions are averaged out. In that way, adiabaticity might survive in a new stage.

Chapter 9

Ponderomotive Potential and Magnetization

High-intensity electromagnetic waves interacting with a plasma may cause a variety of ponderomotive effects, where the term ponderomotive usually refers to nonlinear slow timescale (low-frequency) phenomena induced by the fast timescale oscillating (high-frequency) fields. It appears as a generic multi-scale plasma paradigm, represented by two, nonlinearly interlinked, slow and fast timescale plasma dynamics. These ponderomotive effects play an important role in many physical–laboratory and fusion–and astrophysical situations. In many treatments of ponderomotive effects, single-particle dynamics has been employed [9, 187]; that is, the effective force was taken the particle number density times the single particle force. This would mean that collective effects were neglected. Moreover, although the need for a more exact treatment was often mentioned, the only term taken into account in the ponderomotive force was the one involving the gradient of the time-averaged high-frequency electromagnetic radiation field energy density. Derivations of expressions for the ponderomotive force have used a variety of approaches, such as, fluid equations [188, 189], kinetic approaches [190–195], Lie transform technique [196] and phenomenological methods [197]. Typically, these derivations are all relatively complicated [198], and its is our aim to present a simple and rigorous Hamiltonian treatment of ponderomotive interactions in a Vlasov plasma following Škorić and ter Haar [199].

9.1 Hamiltonian Formulation of Ponderomotive Interactions in a Vlasov Plasma

A Hamiltonian formulation of ponderomotive interactions in a Vlasov plasma is presented. Transformation from the rest frame to the oscillation center frame (OCF) is achieved, and a Liouville equation is derived from the low-frequency OCF distribution function; the Hamiltonian in this equation includes the ponderomotive terms. The equation is solved in the adiabatic approximation, the corresponding charge and current densities evaluated, and the first-order nonadiabatic correction to the current density is derived.

Here, we shall consider a hot, collisionless, nonrelativistic electron plasma [199] and neglect ion motions which serve as a neutralizing background. In a Vlasov plasma, the Liouville equation reduces to the Vlasov equation for the single electron distribution function $f(\mathbf{r}, \mathbf{v}, t)$ as

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla)f - \frac{e}{m} \left(\mathbf{E} + (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} \right) = 0, \quad (9.1)$$

where $\mathbf{E}$ and $\mathbf{B}$ are (self-consistent) electric and magnetic fields from the Maxwell equations ($c = 1$) and where $-e$ and m are the electron charge and mass, respectively. We shall restrict ourselves to the case where there are no external electric and magnetic fields. Consider now the situation where a (strong) high-frequency (transverse or longitudinal) electromagnetic wave, with electric and magnetic fields $\mathbf{E}_H$ and $\mathbf{B}_H$, is excited in the plasma. We follow [196, 199], and introduce, instead of the rest-frame distribution function f , the so-called OCF distribution function $F(\mathbf{r}, \mathbf{v}, t)$ where

$$\mathbf{u} = \mathbf{v} - \mathbf{v}_q, \quad (9.2)$$

with $\mathbf{v}_q$ the linear electron quiver velocity in the high-frequency field, and

$$\frac{d\mathbf{v}_q}{dt} = -\frac{e}{m} \mathbf{E}_H. \quad (9.3)$$

The transformation from $f(\mathbf{r}, \mathbf{v}, t)$ to $F(\mathbf{r}, \mathbf{v}, t)$ reflects the fact that once the high-frequency transient effects have died out, the electrons respond linearly to a high-frequency field by relaxing toward so-called the oscillating (Maxwell) distribution function; that is, to a distribution function f which is the initial (equilibrium) distribution function f_0 with the velocity argument shifted by the linear quiver velocity, thus

$$F = f_0(\mathbf{v} - \mathbf{v}_q). \quad (9.4)$$

Put mathematically, the transformation from $f \rightarrow F$, when made in (9.1) enables us to remove the lowest-order linear response, thus easing the perturbative calculation of higher-order nonlinear terms. It also turns out that the transformed (9.1) is suitable for compact Hamiltonian formalism. The equation for F is the (Hamiltonian) Liouville equation

$$\frac{\partial F}{\partial t} + [F, H] = 0, \quad (9.5)$$

where $[\dots]$ is a Poisson bracket, in terms of $\mathbf{r}$ and $\mathbf{p} = m\mathbf{u}$, and H is the OCF Hamiltonian.

$$H = \frac{1}{2}m(\mathbf{u} + \mathbf{v}_q)^2. \quad (9.6)$$

As a standard, we split the various quantities into two— fast and slow timescales; i.e., a high frequency part and a low-frequency (time averaged, indicated by angular brackets $\langle \dots \rangle$, indicated, respectively, by the indices H and L . Equation (9.5) then

reduces to the following pair of coupled equations:

$$\frac{\partial F_L}{\partial t} + [F_L, H_L] + \langle [F_H, H_H] \rangle = 0, \quad (9.7)$$

$$\frac{\partial F_H}{\partial t} + [F_H, H_L] + [F_L, H_H] = 0, \quad (9.8)$$

where we neglect terms corresponding to second and higher-order harmonics, an approximation as discussed elsewhere [198]; that is, concentrate on the high-frequency dynamics around the fundamental frequency Ω of the high-frequency wave. The slow-timescale (L) and fast-timescale (H) Hamiltonians are given by the equations

$$H_L = \frac{1}{2}(u^2 + \langle v_q^2 \rangle), H_H = m(\mathbf{u} \cdot \mathbf{v}_q). \quad (9.9)$$

The distribution functions F_H and F_L determine the moments, electron density n and current density $\mathbf{j}$.. in the rest frame, as follows:

$$\begin{aligned} \mathbf{n} &= \int f d^3 \mathbf{v} = \int f d^3 \mathbf{u}, \\ \mathbf{j} &= -e \int f \mathbf{v} d^3 \mathbf{v} = -en \mathbf{v}_q - \int \mathbf{u} f d^3 \mathbf{u}, \end{aligned} \quad (9.10)$$

or, split into low-frequency and high-frequency components

$$\mathbf{n}_L = \int F_L d^3 \mathbf{u}, \mathbf{n}_H = \int F_H d^3 \mathbf{u}, \quad (9.11)$$

$$\mathbf{j}_L = -e \langle n_H \mathbf{v}_q \rangle - \int \mathbf{u} F_L d^3 \mathbf{u}, \quad (9.12)$$

$$\mathbf{j}_H = -en_L \mathbf{v}_q - \int \mathbf{u} F_H d^3 \mathbf{u}.$$

We need, therefore to solve (9.7) and (9.8), and to do this we shall use the perturbation treatment This has been done by other authors [188, 190–192]. but we shall present a simpler, albeit rigorous, derivation [199, 202] of the final results. We introduce a small parameter δ_T which characterizes the thermal dispersion and which is the ratio of the electron thermal velocity v_T to the phase velocity of the high-frequency waves [199, 200].

$$\delta_T = (v_T k_H / \omega) \ll 1, \quad (9.13)$$

where k_H is a typical high-frequency wavenumber. Assuming the validity of the inequality (9.13), we are at the same time ensured of the absence of resonant kinetic high-frequency wave-electron effects. We now solve (9.8) for F_H in the linear approximation, keeping only terms of first order in δ_T . This leads to

$$\frac{\partial F_H}{\partial t} + [F_L, H_H] = 0, \quad (9.14)$$

or

$$F_H = \left[F_L, \int H_H dt \right] = -[F_L, m(\mathbf{u} \cdot \mathbf{r}_q)], \quad (9.15)$$

where the integration is on the high-frequency scale while $\mathbf{r}_q = \int \mathbf{v}_q dt$ is the linear quiver displacement. One can check by inspection that the term $[F_H, H_L]$, neglected in writing down (9.14) which gives nonlinear contribution, is small whenever $\bar{W} \ll 1$, where $\bar{W}$ is the so-called plasma turbulence parameter $\bar{W} = v_q^2/v_T^2$. Substituting (9.15) into (9.7) we find [199]

$$\frac{\partial F_L}{\partial t} + [F_L, H_L] + \langle [H_H, [F_L, m(\mathbf{u} \cdot \mathbf{r}_q)]] \rangle = 0. \quad (9.16)$$

If the plasma is not very nonuniform so that typical wavenumbers corresponding to the spatial variation of F_L are much smaller than k_H and since, as can be verified a posteriori, the velocity gradient of F_L is sufficiently small, we can neglect all second derivatives of F_L . (We note that the terms involving $\partial^2 F_L / \partial u_i \partial u_j$ lead to a diffusion in velocity space [18]. More precisely, in the presence of a growing oscillation, this effect could result in an increase in the time-averaged momentum and energy of nonresonant electrons. The so-called fake diffusion has been studied by some authors [196, 201]. In such a case, the last term on the left-hand-side of (9.16) can be transformed as follows,

$$\begin{aligned} & \langle [H_H, [F_L, m(\mathbf{u} \cdot \mathbf{r}_q)]] \rangle \\ &= \left\langle \nabla(\mathbf{u} \cdot \mathbf{v}_q) \cdot \frac{\partial}{\partial \mathbf{u}} - (\mathbf{v}_q \cdot \nabla) \cdot (\mathbf{r}_q \cdot \nabla) - \nabla(\mathbf{u} \cdot \mathbf{r}_q) \cdot \frac{\partial}{\partial \mathbf{u}} F_L \right\rangle \\ &= \langle [\nabla F_L \cdot (\mathbf{r}_q \cdot \nabla) \mathbf{v}_q] \rangle - \left\langle \nabla(\mathbf{r}_q \cdot \nabla)(\mathbf{u} \cdot \mathbf{v}_q) \cdot \frac{\partial}{\partial \mathbf{u}} F_L \right\rangle \\ &\quad - \langle (\mathbf{v}_q \cdot \nabla)(\mathbf{r}_q \cdot \nabla) + (\mathbf{r}_q \cdot \nabla)(\mathbf{v}_q \cdot \nabla) \rangle F_L \\ &\quad + u_n \langle [(\mathbf{r}_q \cdot \nabla) \nabla_m v_{qn}] + [(\mathbf{v}_q \cdot \nabla) \nabla_m r_{qn}] \rangle \frac{\partial}{\partial u_m} F_L, \end{aligned} \quad (9.17)$$

where summation over repeated indices is implied. The last two terms on the right-hand-side of (9.17) are both time-averages of time derivatives and therefore vanish for slowly modulated waves. The first two terms on the right-hand-side of (9.17) can be combined into a single Poisson bracket so that we end up with the equation

$$\frac{\partial F_L}{\partial t} + [F_L, H_P] = 0, \quad (9.18)$$

where

$$H_P = H_L + \langle (\mathbf{r}_q \cdot \nabla) H_H \rangle, \quad (9.19)$$

is the effective ponderomotive Hamiltonian, which, together with any low-frequency fields, determines the slow-timescale dynamics of electrons in high-frequency waves. The approximation $H_P = H_L$ is the one mentioned in the introduction, while the extra term corresponds to what Cary and Kaufman [196] called the ponderomotive Hamiltonian. We can rewrite (9.19) as follows,

$$H_P = \frac{1}{2} m [(\mathbf{u} - \mathbf{u}_0)^2 + \langle v_q^2 \rangle - u_0^2], \quad (9.20)$$

where we have introduced an ansatz for slow-timescale induced drift velocity $\mathbf{u}_0$ by the equation

$$\mathbf{u}_0 = -\langle (\mathbf{r}_q \cdot \nabla) \mathbf{v}_q \rangle \quad (9.21)$$

and (9.18) can be written in the form

$$\frac{\partial F_L}{\partial t} + [(\mathbf{u} - \mathbf{u}_0) \cdot \nabla F_L] + \nabla \left[-\frac{1}{2} \langle v_q^2 \rangle + (\mathbf{u} \cdot \mathbf{u}_0) \right] \cdot \frac{\partial F_L}{\partial \mathbf{u}} = 0, \quad (9.22)$$

a result also obtained via a different procedure [191]. We see that apart from the well-known gradient ponderomotive force $\nabla \frac{1}{2} \langle v_q^2 \rangle$, there is an additional term in the effective ponderomotive force which derives from the deformation of F_L due to the slow-timescale flow induced by ponderomotive interactions (see (9.21)). In the case of a warm plasma, when the phase velocity ω/k_L of the low-frequency motions is small compared with v_T ,

$$\omega/k_L \ll v_T, \quad (9.23)$$

we can look for a stationary solution of (9.18). In steady state, this equation has the general adiabatic solution,

$$F_L = F_0(H_P), \quad (9.24)$$

where F_0 is an arbitrary distribution function. Before discussing this solution, a few remarks are appropriate. First, the rest frame distribution is obtained just by replacing $\mathbf{u}$ with $\mathbf{u} - \mathbf{u}_0$. Second, if one wants to take into account the low-frequency charge separation field, one should simply replace H_P with $H_P - e\Phi$ in, (9.18) and (9.24), where Φ is the appropriate scalar potential. Third, in the absence of high-frequency waves, the solution (9.24) should reduce to the initial velocity distribution which we shall assume to be Maxwellian. Fourth, in the OCF, the solution for F_L is the initial distribution shifted by $\mathbf{u}_0$ as a result of nonlinear ponderomotive interactions. This leads to a linear, first-order term that should affect the current density (9.12). Finally, we briefly examine the validity of adiabatic approximation. This approximation characterizes most of earlier works [192], while Cary and Kaufman [196] also work essentially in this approximation when obtaining their result for the low-frequency ponderomotive response. One might feel that as far as the

nonadiabatic resonant type of correction to the high-frequency electron dynamics is concerned, the smallness of δ_T justifies the adiabatic procedure. However, the situation is different in the case of the slow-timescale low-frequency response. Condition (9.23) implies that a considerable number of electrons are in resonance with the low-frequency motion. It would therefore be useful to find the quantitative measure of such a nonadiabatic effect. To do this, we have assumed that $F_L = F_0 + \delta F$, where $|\delta F| \ll F_0$ and, substituting this expression into (9.18), linearizing and Fourier transforming and further assuming that F_0 is Maxwellian, we find in the leading term of the Fourier transform $\delta j_L^{*,\omega}$ of the (nonadiabatic) resonant contribution to the nonlinear low-frequency current density, the expression

$$\delta j_L^{*,\omega} = -i(\pi/2)^{1/2}(\Omega/kv_T)[n_L \mathbf{u}_0]_{\mathbf{k},\omega}^\perp, \quad (9.25)$$

where the superscript $\perp$ indicates the component at the right angle to $\mathbf{k}$. We see that this correction gives a nonlinear Landau damping mechanism for high-frequency waves, which could be small as long as the condition (9.23) holds. Let us now return to (9.24). Assuming that F_0 is Maxwellian, and using the fact that $u_0^2/\langle v_q^2 \rangle \sim \overline{W}\delta_T^2 \ll 1$ to neglect u_0^2 in H_p in the exponent, we get from (9.11) and (9.12)

$$n_L = n_0 \exp(-<v_q^2>/2v_T^2), \quad (9.26)$$

a well-known result for the electron density, modified because of ponderomotive interactions [203]. We now turn to the low-frequency ponderomotive current density $\mathbf{j}_L$. To evaluate $n_H v_H$, we can use either (9.11) and (9.15) or the high-frequency equation of continuity. Either procedure gives us

$$n_H = -\nabla \cdot (n_L \mathbf{r}_q) \quad (9.27)$$

The second contribution to $\mathbf{j}_L$ becomes $n_L \mathbf{u}_0$ if we use for F_L the shifted Maxwellian distribution. Using (9.27) and (9.21) and the vanishing of the time average of expressions such as $(r_{qi} v_{qj} + v_{qi} r_{qj})$ (cf. discussion following (9.17)), we find

$$\mathbf{j}_L = e \left((\nabla \cdot (n_L \mathbf{r}_q)) \mathbf{v}_q - e n_L (\mathbf{v}_q \cdot \nabla) \mathbf{r}_q \right) = \frac{1}{2} e [\nabla \times (\mathbf{v}_q \times \mathbf{r}_q)], \quad (9.28)$$

which is important, and hence a less-known result for the nonlinear ponderomotive electron current density. We note that this equation is consistent with the requirement of slow-timescale quasineutrality ($\nabla \cdot \mathbf{j}_L = 0$). The result given by (9.28) was obtained four decades ago in the search for a possible collisionless mechanism for the nonlinear excitation of quasistatic magnetic fields by Langmuir waves [190, 204]. However, it should be stressed that it is just a general manifestation of ponderomotive interactions as the more widely exploited density modification formula (9.26). Naturally, the basic generation equation for the low-frequency magnetic field $\delta \mathbf{B}$ due to the ponderomotive current density (9.28) follows simply from the appropriate Maxwell equation in which we can neglect the displacement current and the ion contribution, therefore

$$\nabla^2 \delta \mathbf{B} = -(4\pi e/c)(\nabla \times \mathbf{j}_L),$$

$$\nabla^2 \delta \mathbf{B} = -(2\pi e/c)[\nabla \times (\nabla \times \langle (\mathbf{v}_q \times \mathbf{r}_q) \rangle)], \quad (9.29)$$

which for longitudinal $\mathbf{E}_H$ simplifies to the formula

$$\delta \mathbf{B} = (2\pi e/c)n_L \langle (\mathbf{v}_q \times \mathbf{r}_q) \rangle. \quad (9.30)$$

If one goes beyond the adiabatic approximation, an integral term to the left-hand side of (9.29) would appear which represents the anomalous skin damping of the magnetic field, characteristic of the propagation of quasi-static (electro)-magnetic wave in an overdense Vlasov plasma $\Omega \ll \omega_{pe}$ so that the condition (9.23) is satisfied[11]. In conclusion, we note that above results indicate that slow-scale magnetic fields together with well-known density modification are both expected as general features of ponderomotive effects in turbulent plasmas [206–209].

9.2 The Hydrodynamics of Ponderomotive Interactions in a Collisionless Plasma

Ponderomotive effects, that is, nonlinear low-frequency phenomena induced by high-intensity electromagnetic waves, have been the subject of many studies [187]. Ponderomotive interactions have been invoked to explain a number of important phenomena in laboratory, fusion, space, and astrophysical plasmas in particular, such as in a mechanism of nonlinear generation of ponderomotively driven large quasistatic magnetic fields in laser plasmas. As discussed above, it has been shown [190, 198] that in a collisionless plasma, magnetic fields can be excited by a ponderomotive solenoidal electron current $\mathbf{j}_L$, given by the expression

$$\mathbf{j}_L = -e \langle n \mathbf{v} \rangle = \frac{1}{2} e \nabla \times (n_L \langle \mathbf{v}_q \times \mathbf{r}_q \rangle),$$

where $-e, m$, and $\mathbf{v}$ are, respectively, the electron charge, number density, and velocity, while the pointed brackets, as before, indicate time averaging over the period of high-frequency waves. The quantities $\mathbf{v}_q$ and $\mathbf{r}_q$ are, respectively, the electron linear quiver velocity and displacement in the high-frequency electromagnetic (transverse or longitudinal) field $(\mathbf{E}_H, \mathbf{B}_H)$, which satisfy the relations $d^2 \mathbf{r}_q / dt^2 \equiv d \mathbf{v}_q / dt = -e \mathbf{E}_H / m$, where m is the nonrelativistic electron mass, and indices L and H denote the high-frequency (fast timescale) and the low-frequency (slow timescale) parts of the relevant quantities, respectively. In earlier papers, e.g., in [190, 198] (see above), the above equation for $\mathbf{j}_L$ was obtained by using the adiabatic kinetic theory in a warm collisionless plasma assuming that mobile electrons are in equilibrium with the slow timescale motions. In that way, the equation, as expected, is consistent with the requirement for the slow timescale quasineutrality, $(\nabla \cdot \mathbf{j}_L) = 0$. Later kinetic studies [188, 191, 192, 196, 204, 205] have examined

nonlinear collisionless mechanism in more detail confirming the original result. Subsequent study using a Hamiltonian approach, given in a previous section, gave a simple formalism and physical insight into the mechanism [196, 199, 200, 206–208]. It has been often claimed that only a kinetic theory can derive the correct expression for $\mathbf{j}_L$ in a collisionless plasma and that all attempts to use hydrodynamics without “a correction for the particle stress perturbed by ponderomotive terms” are doomed to failure [190–192]. This statement was further reinforced by the fact that most of the earlier hydrodynamic attempts failed to obtain complete results [189, 204, 210, 211]. It is the purpose of the following to present a simple rigorous hydrodynamic derivation of the correct result, while showing the basic physical approximations involved [72, 207–209].

As usual, we split all physical quantities into their slow and fast timescale parts, the total slow-timescale ponderomotive electron current density will consist of two terms, a fast-timescale beat contribution and a slow-timescale part:

$$\mathbf{j}_L = -e \langle n\mathbf{v} \rangle = -e(\langle n_H \mathbf{v}_H \rangle + n_L \mathbf{v}_L). \quad (9.31)$$

As we are interested only in the contributions to $\mathbf{j}_L$, which are quadratic in the high-frequency field amplitude, it is sufficient to evaluate the first “beat” term in (9.31) by solving the linear evolution for each n_H and $\mathbf{v}_H$, or more precisely, by using the linearized equation of continuity for n_H and the equation of electron motion for $\mathbf{v}_H$, respectively. The thermal correction which is of the second-order in the dispersion parameter is neglected, cf. (13) in [199]. Thus, we simply have

$$-e \langle n_H \mathbf{v}_H \rangle = e \langle \nabla \cdot (n_L \mathbf{r}_q) \mathbf{v}_q \rangle \quad (9.32)$$

The slow-timescale velocity in (9.31) is more difficult to evaluate, since as $n_L (\approx n_0)$ is the zero-order quantity, we have to calculate $\mathbf{v}_L$ to the second order in the high-frequency field amplitude. We now introduce a hydrodynamic (fluid) description of the nonrelativistic electron motion:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \times \left(\nabla \times \mathbf{v} - \frac{e}{mc} \mathbf{B} \right) + \frac{e}{m} \mathbf{E} + \nabla v^2 + \frac{1}{nm} \nabla p_e = 0 \quad (9.33)$$

where p_e is the electron pressure term, which we shall take into account in its standard, isotropic form. In that connection, we should note an important result of the adiabatic kinetic theory (see above); the only effect of the collisionless ponderomotive interactions on the steady-state electron distribution is a shift by mean high- and low-frequency velocities; this means that if the electron distribution was initially Maxwellian, the electron pressure term remains isotropic [191, 199, 207]. We now look for an expression for $\mathbf{v}_L$. Since $(\nabla \cdot \mathbf{j}_L) = 0$, and together with (9.31) and (9.33) it already defines the potential part of $\mathbf{v}_L$, it will be sufficient to determine the remaining solenoidal (transverse) component. Therefore, we take the **curl** of (9.33) to arrive at a general (Helmholtz) equation [in a nonuniform plasma $\nabla \times (\nabla p_e / nm)$ can give rise to a (non-ponderomotive) thermoelectric magnetization

current proportional to $(\nabla n_L \times \nabla T)$ (e.g., see [191, 192]):

$$\frac{\partial \omega}{\partial t} - \nabla \times (\mathbf{v} \times \omega) = 0, \quad (9.34)$$

describing the vortex dynamics of an electron fluid; the electron vorticity is defined by the equation $\omega = \nabla \times \mathbf{v} - e\mathbf{B}/mc$. As usual, splitting the physical quantities into low-frequency and high-frequency parts and neglecting second- (and higher- order) harmonics [190], we get from (9.34)

$$\frac{\partial \omega_H}{\partial t} - \nabla \times (\mathbf{v}_L \times \omega_H) - \nabla \times (\mathbf{v}_H \times \omega_L) = \mathbf{0}, \quad (9.35)$$

$$\frac{\partial \omega_L}{\partial t} - \nabla \times (\mathbf{v}_L \times \omega_L) - \nabla \times (\mathbf{v}_H \times \omega_H) = \mathbf{0}; \quad (9.36)$$

if we assume that electrons are in a steady state as far as the slow-timescale motion is concerned, we can drop the time derivative in (9.36). As the linear part ω_q of the high-frequency vorticity ω_H vanishes, we solve (9.35) for the nonlinear (third order in high-frequency amplitude) part of ω_H . The zero value of ω_q follows directly from the equation of motion for $\mathbf{v}_q$, as

$$\frac{\partial}{\partial t} \nabla \times \mathbf{v}_q = -\frac{\nabla \times \mathbf{E}_H}{m} = \frac{e}{mc} \frac{\partial \mathbf{B}_H}{\partial t}.$$

From the above, it follows that the introduction of the vorticity equation (9.34) removes the lowest order linear response which appears to be analogous to the transformation to the oscillating center frame in the kinetic approach (see above [199]). We can now solve (9.35) for the nonlinear ω_H , by using the linear high-frequency quantities for $\mathbf{v}_q$ and ω_q , hence we have

$$\omega_H = \nabla \times (\mathbf{r}_q \times \omega_L). \quad (9.37)$$

Substituting (9.37) into (9.36), we readily get

$$\frac{\partial \omega_L}{\partial t} - \nabla \times [(\mathbf{v}_L \times \omega_L) + \langle \mathbf{v}_q \times \nabla \times (\mathbf{r}_q \times \omega_L) \rangle] = 0, \quad (9.38)$$

where in the steady state, we can put the time-derivative term equal to zero.

After some tedious manipulations (shown below), we find that the last term in the (9.38) can be transformed into

$$\nabla \times \langle \mathbf{v}_q \times \nabla \times (\mathbf{r}_q \times \omega_L) \rangle = \nabla \times [\omega_L \times \langle (\mathbf{v}_q \cdot \nabla) \mathbf{r}_q \rangle], \quad (9.39)$$

so that in the steady state equation (9.38) reduces to

$$\nabla \times [\omega_L \times (\mathbf{v}_L - \langle (\mathbf{v}_q \cdot \nabla) \mathbf{r}_q \rangle)] = 0, \quad (9.40)$$

which directly yields a trivial solution

$$\mathbf{v}_L = \langle (\mathbf{v}_q \cdot \nabla) \mathbf{r}_q \rangle, \quad (9.41)$$

which fully agrees with the kinetic result [190–192], contrary to some earlier statements. At this point, we stress that, indeed, (9.40) is the fourth order in the amplitude, which is necessary in order to derive the correct formulae for slow-timescale ponderomotive velocity term.

By combining (9.41) and (9.42) with (9.31) and (9.32), we readily get the required equation for the ponderomotive electron current

$$\mathbf{j}_L = e \langle \mathbf{v}_q (\nabla \cdot n_L \mathbf{r}_q) - n_L (\mathbf{v}_q \cdot \nabla) \mathbf{r}_q \rangle = \frac{1}{2} e \nabla \times (n_L \langle \mathbf{v}_q \times \mathbf{r}_q \rangle). \quad (9.42)$$

The basic equation for the generation of slow-timescale magnetic fields by the ponderomotive magnetization current (9.42) follows simply from Ampère's law with the displacement current and the ion contribution neglected, as was pointed out earlier. Let us briefly comment on some of earlier work on this subject. Some of hydrodynamic derivation in a cold electron fluid limit [189, 210] typically recovered only the first term in (9.36) and thus obtained $(\partial/\partial t)[\nabla \times \mathbf{v}_L - (e\mathbf{B}_L/mc)] = 0$. As was pointed out in [72, 191, 196] such results are applicable just for the short-time response, before electrons have time to equilibrate with the slow timescale ponderomotive dynamics. Moreover, it was claimed that in most experimental situations in laser plasmas, the actual scaling does not follow the cold plasma approximation, but it supports the steady state solution given by (9.42). Moreover, we note, the above result for the ponderomotive magnetization current is typically not recovered in the static and collisionless limit of many results on nonstationary ponderomotive interactions [212] (and references therein).

Appendix: Derivation of Equation (9.39)

To prove (9.39) we shall use tensor notation involving the totally antisymmetric third rank unit tensor ϵ_{ijk} , which satisfies the relations

$$\epsilon_{ijk} = -\epsilon_{ikj} = -\epsilon_{jik},$$

and

$$\sum_k \epsilon_{ijk} \epsilon_{klm} = \delta_{li} \delta_{mj} - \delta_{mi} \delta_{lj},$$

where δ_{ij} is the Kronecker symbol.

We also use the fact that $\mathbf{v}_q$ and $\mathbf{r}_q$ are high-frequency quantities while ω_L is a low-frequency quantity, so that on averaging over the high-frequency motions we have relations such as (compare the discussion in the above section)

$$\langle v_i \partial_j r_k \omega_l + r_i \partial_j v_k \omega_l \rangle = 0, \quad \langle r_i v_i \partial_j \omega_l \rangle = 0, \quad (9.43)$$

where v_i , ∂_i , r_i and ω_i denote, respectively, the components of $\mathbf{v}_q$, $\mathbf{r}_q$, ∇ , and ω_L . We finally remind that from the definition of vorticity, $(\nabla \cdot \omega_L) = 0$. We now write the i th component of the key vector expression in (9.39)

$$\begin{aligned}
 \langle \mathbf{v}_q \times \nabla \times (\mathbf{r}_q \times \mathbf{L}) \rangle_i &= \sum_{\substack{j,k,l \\ m,p,s}} \langle \epsilon_{ijk} \epsilon_{klm} \epsilon_{mps} v_j \partial_l r_p \omega_s \rangle \\
 &= \sum_{j,p,s} \langle \epsilon_{jps} v_j \partial_i r_p \omega_s + \epsilon_{ips} v_j \partial_j r_s \omega_p \rangle \\
 &= \sum_{j,p,s} \langle \epsilon_{ips} \omega_p v_j \partial_j r_s + \epsilon_{ips} r_s v_j \partial_j \omega_p + \epsilon_{jps} v_j \partial_j r_p \omega_s \rangle \\
 &= \omega_L \times \langle (\mathbf{v}_q \cdot \nabla) \mathbf{r}_q \rangle_i + \frac{1}{2} \left\langle \sum_{j,p,s} (\epsilon_{jps} (\partial_i (v_j r_p \omega_s) \right. \\
 &\quad \left. + v_j r_p \partial_i \omega_s) + \epsilon_{ips} (r_s v_j \partial_j \omega_p + v_p r_j \partial_j \omega_s)) \right\rangle \\
 &= \langle \mathbf{v}_q \times \nabla \times (\mathbf{r}_q \times \mathbf{L}) \rangle_i + \frac{1}{2} \langle \nabla (\mathbf{v}_q \cdot \mathbf{r}_q \times \omega_L) \rangle_i \\
 &\quad + \frac{1}{2} \sum_{j,p,s} \langle \epsilon_{jps} v_j r_p \partial_i \omega_s + \epsilon_{ips} (r_s v_j \partial_j \omega_p + v_p r_j \partial_j \omega_s) \rangle,
 \end{aligned}$$

where we have used (9.43) and the properties of ϵ_{ijk} . We, now in the first sum involving ϵ_{jps} , take into account that one out of j, p, s must necessarily equal i so that we can write

$$\sum_{j,p,s} \epsilon_{jps} r_p v_j \partial_i \omega_s = \sum_{p,s} \epsilon_{ips} (r_p v_i \partial_i \omega_s + r_s v_p \partial_i \omega_i + r_i v_s \partial_i \omega_p), \quad (9.44)$$

where we have indulged in some renaming of dummy indices. In the sums involving ϵ_{ips} , we use the fact that j must, for given p and s , take on the values i, p , and s so that those sums can be written in the form

$$\begin{aligned}
 \sum_{j,p,s} \epsilon_{ips} (r_s v_j \partial_j \omega_p + v_p r_j \partial_j \omega_s) &= \sum_{p,s} \epsilon_{ips} (r_s v_i \partial_i \omega_p + r_i v_p \partial_i \omega_s + r_s v_p \partial_p \omega_p \\
 &\quad + r_p v_p \partial_p \omega_s + r_s v_s \partial_s \omega_p + r_s v_p \partial_s \omega_s). \quad (9.45)
 \end{aligned}$$

Combining (9.44) and (9.45) and using the properties of ϵ_{ips} , we find the expression

$$\begin{aligned}
 &\left\langle \sum_{p,s} \epsilon_{ips} [r_s v_p (\partial_i \omega_i + \partial_p \omega_p + \partial_s \omega_s) + r_p v_p \partial_p \omega_s + r_s v_s \partial_s \omega_p] \right\rangle \\
 &= \langle \mathbf{v}_q \times \mathbf{r}_q (\nabla \cdot \omega_L)_i \rangle
 \end{aligned}$$

where we have used (9.43). Hence, we finally get

$$\begin{aligned} \langle \mathbf{v}_q \times \nabla \times (\mathbf{r}_q \times \omega_L) \rangle &= \omega_L \times \langle (\mathbf{v}_q \cdot \nabla) \mathbf{r}_q \rangle + \frac{1}{2} \langle \nabla (\mathbf{v}_q \cdot \mathbf{r}_q \times \omega_L) \\ &\quad + \frac{1}{2} \langle \mathbf{v}_q \times \mathbf{r}_q (\nabla \cdot \omega_L) \rangle, \end{aligned}$$

from which (9.39) follows, if we use that $(\nabla \cdot \omega_L) = 0$.

9.3 Spontaneous Generation of Magnetostatic Fields

In a collisionless plasma regime, large amplitude magnetostatic fields can be driven by the ponderomotive electron magnetization current. Accordingly, under an action of transverse pump field, generally two types of parametric instabilities, magneto-modulational instability and stimulated magnetostatic scattering instability, were found [208, 209]. Parametric growth rates indicate possible importance in high-intensity laser plasma interaction studies. Under an action of strong laser light (pump), various plasma oscillation modes can become coupled and can grow in time and space before getting saturated at large amplitudes and dissipated in a plasma. A linear parametric theory of laser light instabilities, including a decay, oscillating-two-stream (OTS), stimulated Brillouin (SBS), stimulated Raman (SRS), and stimulated Compton scattering and filamentation, is well developed [213, 314, 321]. In particular, much of the ongoing effort has been put into studies of scattering instabilities which have a potential to substantially reduce laser light absorption in a target. In this section, we wish to discuss less known competing parametric processes due to excitation of magnetostatic field instabilities. In a collisionless nonrelativistic regime, the basic mechanism-driving magnetostatic (slow timescale) fields has appeared to be due to the ponderomotive electron magnetization current, derived above (9.42), written in the form

$$\mathbf{j}_M = -\frac{ie\omega_{pe}^2}{16\pi m\omega_0^3} \nabla \times (\mathbf{E} \times \mathbf{E}^*), \quad (9.46)$$

where, $\mathbf{E}$ is the complex amplitude of the high-frequency (ω_0) laser field.

The basic magnetostatic field generation equation is readily obtained from the Amperes law, following the procedure indicated above (for details, see Kono et al. [188]), written as

$$\nabla^2 \delta \mathbf{B} + \mathbf{S}_{\text{kin}} = \frac{ie\omega_{pe}^2}{4mc\omega_0^3} \nabla \times \nabla \times (\mathbf{E} \times \mathbf{E}^*), \quad (9.47)$$

with the displacement current neglected, and where $\mathbf{S}_{\text{kin}}$ represents the skin (anomalous) damping term which characterizes propagation of a magnetostatic mode ($\omega \ll \omega_{pe}$) through a warm overcritical density plasma.

9.3.1 Coupled Mode Equations

We shall focus on a nonlinear state of a nonisothermal ($T_e \gg T_i$) collisionless plasma subjected to an action of a finite amplitude laser pump wave $\tilde{\mathbf{E}}(\omega_0, \mathbf{k}_0)$. We generalize the standard derivation of parametrically coupled modes [213, 314, 321], by taking into account, apart from the usual low-frequency density perturbation, the additional ponderomotive effect in the form of ponderomotively driven magnetostatic fluctuations. Leaving out details of the straightforward procedure (see [208, 209]), the basic set of coupled equations is written as

$$\frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2} + \omega_{pe}^2 \tilde{\mathbf{E}} + \mathbf{c}^2 \nabla \times (\nabla \times \tilde{\mathbf{E}}) - 3v_{Te}^2 \nabla (\nabla \cdot \tilde{\mathbf{E}}) = \frac{\delta n}{n_0} \omega_{pe}^2 \tilde{\mathbf{E}} + \frac{e}{mc} \frac{\partial \tilde{\mathbf{E}}}{\partial t} \times \delta \mathbf{B}, \quad (9.48)$$

$$\frac{\partial^2 \delta n}{\partial t^2} - \nabla^2 \delta n = \frac{1}{16\pi M} \nabla^2 |\tilde{\mathbf{E}}|^2, \quad (9.49)$$

$$\nabla^2 \delta \mathbf{B} = \frac{ie\omega_{pe}^2}{4mc\omega_0^3} \nabla \times \nabla \times (\mathbf{E} \times \mathbf{E}^*) - \mathbf{S}_{kin}, \quad (9.50)$$

where

$$\mathbf{S}_{kin} = -\frac{\omega_{pe}^2}{(2\pi)^{3/2} c^2 v_{Te}} \frac{\partial}{\partial t} \int \frac{\delta \mathbf{B}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \quad (9.51)$$

where $\tilde{\mathbf{E}}$ is the high-frequency laser field with a complex amplitude $\mathbf{E}$ and δn and $\delta \mathbf{B}$ stand for the low-frequency density and magnetostatic field perturbation, respectively. An integral term (9.51) represents the anomalous skin effect (Landau damping). From (9.48) to (9.51), it appears that the parametric excitation of magnetostatic fields [208, 209] coexists with the standard type of laser-plasma instabilities, such as, e.g., OTS and stimulated Brillouin and Raman-scattering instabilities.

9.3.2 Parametric Instability Analysis

Applying a linear parametric theory method, we investigate the instability growth rate against an excitation of magnetostatic fields $\delta \mathbf{B}(\Omega, \mathbf{q})$. One found that generally taken, the parametric coupling involves a pump (t), a magnetostatic wave (m) and high-frequency side-bands. Depending on the nature of these side-bands, two types of magnetostatic instabilities were found by Škorić, [208, 209].

I MMI- magneto-modulational instability involving the Langmuir (l) side-bands;

II MSI- magnetostatic stimulated scattering instability which corresponds to a fully electromagnetic (t) instability.

In order to further illustrate, we put some typical values for calculated instability growth rates, which depend on the laser pump intensity v_0/c ($v_0 \equiv eE_0/m\omega_0$), the frequency mismatch $\Delta\omega = \omega_0 - \omega_{pe}$, and the electron temperature v_{Te} .

(a) Magneto-modulational Instability (*MMI*). ($t \rightarrow l + m$)

This instability is of a purely growing type (4-wave) involving both resonant Langmuir sidebands. An interesting case is when $\mathbf{k}_0 \ll 2(\mathbf{k}_0 \perp \mathbf{q})$ with a dispersion relation

$$(i\xi - 1)(\Omega^2 - \Delta\omega_0^2) + \frac{1}{4} \frac{\omega_{pe}^2}{\omega_0} \frac{q^2}{q^2 + k_0^2} \Delta\omega_0 \left(\frac{v_0}{c}\right)^2 = 0, \quad (9.52)$$

where $\Delta\omega_0 = (c^2 k_0^2 - 3v_{Te}^2 q^2) / 2\omega_0$, $\omega_0^2 = \omega_{pe}^2 + c^2 k_0^2$, and the Landau damping term gives $\xi = (\pi/2) (\omega_{pe}^2 \Omega / c^2 v_{Te} q^3)$. In the case of weak damping, i.e., $\xi \ll 1$, one gets for a maximum growth rate

$$\gamma = \frac{\omega_{pe}^2}{8\omega_0} \left(\frac{v_0}{c}\right)^2, \quad (9.53)$$

for $q_{\max}^2 = \left[c^2 k_0^2 + \left(\omega_{pe}^2 / 4 \right) \left(\frac{v_0}{c} \right)^2 / 3v_{Te}^2 \right]$, which is a slow instability, of the order of weakly relativistic correction (relativistic mass increase). For a nonzero damping, $\xi \sim \sigma(1)$, one normally expects an instability growth rate smaller than (9.53). In a dipole pump limit, the above results agree with the other results, being similar to the magneto-modulational instability studied in detail for the Langmuir pump by Kono et al. [188, 204].

(b) Stimulated magnetostatic scattering instability (*MSI*) ($t \rightarrow t' + m$)

This instability, proposed by Škorić [208, 209], looks particularly interesting, as a 3-wave process (resonant Stokes sideband) with a dispersion relation in the form

$$(\Omega + i\omega_*) (\Omega - \Delta\omega_0) - i\omega_* \frac{\omega_{pe}^2}{\omega_0} \left(\frac{v_0}{c}\right) \sin^2 \theta = 0, \quad (9.54)$$

where $\omega_* = \Omega / \xi$, (ξ is given above) comes from the Landau damping term ($\mathbf{S}_{\text{kin}}$), $\Delta\omega_0 = c^2 q (2k_0 \cos \theta - q) / 2\omega_0$, and angle $\theta \angle (\mathbf{k}_0, \mathbf{q})$. Following (9.54), we get for a finite k_0 , a maximum growth rate $\gamma \equiv \text{Im}(\Omega)$

$$\gamma \simeq 0.3\omega_0 \left(\frac{v_{Te}}{c}\right)^{\frac{1}{2}} \left(\frac{v_0}{c}\right) \left(\frac{ck_0}{\omega_0}\right)^{\frac{3}{2}}, \quad (9.55)$$

for

$$\theta \sim 40^\circ, \text{ if } \frac{k_0 c}{\omega_0} \gg \frac{v_{Te}}{c} \left(\frac{\omega_0}{\omega_{pe}}\right)^2 \left(\frac{v_0}{c}\right)^2,$$

which can become pronounced in a high-intensity laser-driven hot plasma regimes. For the resonance regime, in a limit of dipole pump, both sidebands are driven resonant (4-wave), resulting in a growth rate lower than (9.55), given as

$$\gamma \simeq \frac{\omega_{pe}}{2\sqrt{2}} \frac{ck_0}{\omega_0} \frac{v_0}{c}, \quad (9.56)$$

for

$$\theta = \pi/2, \quad \text{if } 1 \ll (q/k_0)^2 < \left(1/\sqrt{2}\right) (\omega_{pe}/ck_0) (v_0/c).$$

Generally taken, MSI which maximizes for perturbations transverse to the pump field is pronounced at high (laser) intensities in low-density hot plasmas; coexisting and possibly competing with electromagnetic stimulated scattering instabilities, such as, in SBS, SRS and Compton; as well as with the 4-wave filamentation type of instability. Magnetostatic field excitation, appears as a general feature of parametrically unstable warm plasmas. For a more complete and consistent description of electromagnetic parametric instabilities by laser light, inclusion of (weak) relativistic effects would be appropriate [214].

Part III

Structures in Strong Plasma Turbulence

Chapter 10

Strong Langmuir Turbulence

In a consistent and tutorial way, a set of generalized Zakharov equations for non-linear Langmuir waves is derived, which is valid for both electrostatic and electromagnetic, that is, potential and transverse, perturbations which include corrections due to higher electron nonlinearities and allow for a breakdown of slow-time scale quasi-neutrality. We demonstrate a paradigmatic Langmuir collapse and show how correction terms may affect dynamics in two and three dimensions.

10.1 Introduction

An important ongoing question in nonlinear plasma physics is the problem of strong Langmuir turbulence [4, 215–219]. It has been early well established both through numerical simulations [218, 220, 221] and through experimental data [218, 222] that if sufficient energy is put into a plasma, Langmuir solitons will be formed, that is, localized structures which are both density depressions and electric field maxima. The numerical early work is often concerned with one-dimensional models, although Nicholson and Goldman [223] already deal with a two-dimensional plasma, and it is normally based on the so-called Zakharov equations which were derived by Zakharov [215] to describe the development of the modulational instability in an unmagnetized plasma. From these equations it follows that in two or three dimensions collapse will occur, and much numerical work has been devoted to a study of the existence and dynamics of the so-called Langmuir collapse (for a review, see, e.g., Rudakov and Tsytovich [216], Thornhill and ter Haar [217], Goldman [218], Zakharov and Kuznetsov [219]), it is necessary to use numerical methods as the only known analytical solution of the two- or three-dimensional Zakharov equations is the planar Langmuir soliton. Unfortunately, the numerical procedure can only be applied to the early stages of the collapse, since the Zakharov equations lose their validity when the field intensities become too large. It is the aim of this section to consider a generalization of the Zakharov equations which will be valid at higher amplitudes in order to show consistent derivation and possibly answer the question of whether the collapse will continue until the size of the Langmuir cavitons become of the order of the Debye length so that Landau damping

can play a role. In this context, it is interesting to note that the experiments by Antipov and collaborators [222] indicated structures that are smaller than the Langmuir solitons, but large enough for Landau damping to still be negligible. There has been relatively little work on the limitation of the Zakharov equations. Khakimov and Tsytovich [224] (see also Tsytovich [225]) used their nonlinear dielectric formalism to investigate the limit of applicability of the Zakharov equations. They claim to have taken all nonlinearities up to the fifth order in the electric field into account and they derived a generalized set of equations. Using an approach similar to the original approach of Zakharov's and to the approach to be used here [226], Kuznetsov [227] examined the effect of higher electrostatic nonlinearities and came to the conclusion that they become important at large field amplitudes. It is our purpose to clarify the effect of electron nonlinearities upon the collapse, including both electrostatic and electromagnetic perturbations. It is important to include both potential and nonpotential modes, as the frequently used electrostatic (potential) approximation is normally violated in the case of developed (strong) Langmuir turbulence of hot plasmas (see, e.g., Thornhill and ter Haar [217] ; Nicholson and Goldman [223]). We shall follow original work of Zakharov [215] by separating "fast" and "slow" time scales and applying a hydrodynamic approach, but in addition we shall include higher electron nonlinearities describing the scattering by stimulated fluctuations in the low frequency electron velocity and in the $2\omega_p$ -components of the electron density and velocity (ω_p - electron plasma frequency). We thus allow the possibility of the breakdown of the quasi-neutrality for slow motions (compare also [228]). We also shall apply the adiabatic scaling approach [226, 229] to examine the collapse and the possibility of the existence of quasi-stationary structures in two or three dimensions. Although we find that in the spherically symmetric case the correction terms introduced by us halt the collapse, that is, prevent the appearance of a mathematical singularity, we have not yet determined whether the collapse is halted before Landau damping becomes effective, that is, before the caviton reaches a size of the order of the Debye length.

10.2 Derivation of the Generalized Zakharov Equations

We consider an isotropic, unmagnetized plasma in which the electron temperature T_e , is much higher than the ion temperature T_i and we restrict ourselves to considering long-wavelength electrostatic and electromagnetic modes so that

$$(k^l r_{De})^2 \ll 1 \quad (k^t c / \omega_p)^2 \ll 1, \quad (10.1)$$

where k^l and k^t are, respectively, the wavenumber of the electrostatic and electromagnetic waves, c is the velocity of light, and r_{De} is the electron Debye radius.

We consider Langmuir turbulence when the dominant plasma mode is that of the Langmuir waves with frequencies close to the electron plasmas frequency (ω_p). We

follow the original idea of well separated “slow” (ion) and “fast” electron time scales [215] and split the electron density n_e , electron velocity $\mathbf{v}_e$, electric field strength $\mathbf{E}$, and current density $\mathbf{j}$ in terms corresponding to different time scales [226] :

$$n_e = n_0 + n_s + n_1 + n_2 + \dots, \quad (10.2)$$

$$\mathbf{v}_e = \mathbf{v}_s + \mathbf{v}_1 + \mathbf{v}_2 + \dots, \quad (10.3)$$

$$\mathbf{E} = \mathbf{E}_s + \mathbf{E}_1 + \mathbf{E}_2 + \dots, \quad (10.4)$$

$$\mathbf{j}_e = \mathbf{j}_s + \mathbf{j}_1 + \mathbf{j}_2 + \dots \quad (10.5)$$

Here, n_0 is the initial uniform electron density corresponding to the situation where there are no waves present, the quantities with index s vary on the slow time scale, those with index 1 on the fast $(2\pi/\omega_p)$ time scale, those with index 2 on the fast $2(2\pi/\omega_p)$ time scale (2nd harmonic), and so on. It is convenient to write

$$n_l = \tilde{n}_l \exp(-i\omega_p t) + \tilde{n}_l^* \exp(i\omega_p t), n_2 = \tilde{n}_2 \exp(-2i\omega_p t) + \text{c.c.}, \quad (10.6)$$

$$\mathbf{v}_l = \tilde{\mathbf{v}}_l \exp(-i\omega_p t) + \tilde{\mathbf{v}}_l^* \exp(i\omega_p t), \mathbf{v}_2 = \tilde{\mathbf{v}}_2 \exp(-2i\omega_p t) + \text{c.c.}, \quad (10.7)$$

and so on, where the quantities with tildes are slowly varying and where the asterisk sign indicates the complex conjugate quantity.

We assume that the density perturbation is not too strong, that is,

$$n_0 \gg n_s, n_1, n_2, \quad (10.8)$$

where, however, n_s may sometimes be taken to be comparable with, though still well below, n_0 , while the basic assumption of the predominance of the Langmuir mode leads to the inequalities

$$|\mathbf{v}_1| \gg |\mathbf{v}_s|, |\mathbf{v}_2| \dots, \quad |\mathbf{E}_1| \gg |\mathbf{E}_s|, |\mathbf{E}_2| \dots, \quad (10.9)$$

Our basic equations are the Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \nabla \times \mathbf{H} = \frac{1}{c} \left(4\pi\mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \right), \quad (10.10)$$

where ρ is the charge density; the quantities ρ and $\mathbf{j}$ satisfy the relations

$$\rho = -e(n_e - n_i), \mathbf{j} = -e(n_e \mathbf{v}_e - n_i \mathbf{v}_i), \quad (10.11)$$

where n_i and $\mathbf{v}_i$ are the ion density and the ion velocity and $-e$ is the electron charge.

We shall split the ion density as follows:

$$n_i = n_0 + \delta n_i, \quad (10.12)$$

where δn_i changes on the slow time scale and where δn_i is not necessarily equal to n_s : we are not, as in Zakharov's original treatment [215], a priori assuming the quasi-neutrality on the slow time scale.

The Maxwell equations (10.10) lead to the wave equation,

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + c^2 \nabla \times \nabla \times \mathbf{E} = -4\pi \frac{\partial \mathbf{j}}{\partial t}, \quad (10.13)$$

which is linear in both, $\mathbf{E}$ and $\mathbf{j}$ and convenient for further calculations. We can readily split up the above equation corresponding to various time scales. The equation corresponding to the ω_p -time scale is

$$\frac{\partial^2 \mathbf{E}_1}{\partial t^2} + c^2 \nabla \times \nabla \times \mathbf{E}_1 = -4\pi \frac{\partial \mathbf{j}_1}{\partial t}. \quad (10.14)$$

We must draw attention to the fact that the representation (10.7), (10.8) in the case of $\mathbf{E}$ differs by a factor 2 from the one normally used. The quantity $\mathbf{j}_1$ follows from (10.11) and satisfies the relation

$$\mathbf{j}_1 = -e (n_0 \tilde{\mathbf{v}}_1 + n_s \tilde{\mathbf{v}}_1 + \tilde{n}_1 \mathbf{v}_s + \tilde{n}_1^* \tilde{\mathbf{v}}_2 + \tilde{n}_1 \tilde{\mathbf{v}}_2^*). \quad (10.15)$$

Substituting expression (10.15) into (10.14) and neglecting, as usual (i.e., slowly varying envelope), $\partial^2 \tilde{\mathbf{E}}_1 / \partial t^2$ as compared to $\omega_p \partial \tilde{\mathbf{E}}_1 / \partial t$, we find

$$-2i\omega_p \frac{\partial \tilde{\mathbf{E}}_1}{\partial t} - \omega_p^2 \tilde{\mathbf{E}}_1 + c^2 \nabla \times \nabla \times \tilde{\mathbf{E}}_1 = 4\pi e [n_0 \tilde{\mathbf{v}}_1 + n_s \tilde{\mathbf{v}}_1 + \tilde{n}_1 \mathbf{v}_s + \tilde{n}_1^* \tilde{\mathbf{v}}_2 + \tilde{n}_1 \tilde{\mathbf{v}}_2^*]. \quad (10.16)$$

To close the equation we need expressions for n_s , $\mathbf{v}_s$, $\tilde{n}_1$, $\tilde{\mathbf{v}}_1$, $\tilde{n}_2$, and $\tilde{\mathbf{v}}_2$. It is obvious that the dominant term on the right-hand side of (10.16) is the linear one involving $n_0 \tilde{\mathbf{v}}_1$, while the other four terms are much smaller and may be regarded to be nonlinear corrections.

Our basic assumptions are (10.1), (10.8), and (10.9) and they allow us to introduce a hydrodynamic description for the electron fluid, that is, to use the equation of continuity and equation of motion,

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0, \quad (10.17)$$

$$\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e = -\frac{e}{m} \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_e \times \mathbf{B}] \right) - 3v_{Te}^2 \nabla \left(\frac{n_e}{n_0} \right). \quad (10.18)$$

Using inequalities (10.8) and (10.9) to linearize (10.18) in $\tilde{n}_1$ and $\tilde{\mathbf{v}}_1$ and assuming that the plasma is nonrelativistic so that we can drop the term in the Lorentz force involving high-frequency magnetic field $\mathbf{B}$, we get

$$\begin{aligned} \frac{\partial \tilde{\mathbf{v}}_1}{\partial t} = & - \left[(\tilde{\mathbf{v}}_1 \cdot \nabla) \mathbf{v}_s + (\tilde{\mathbf{v}}_1^* \cdot \nabla) \tilde{\mathbf{v}}_2 + (\mathbf{v}_s \cdot \nabla) \tilde{\mathbf{v}}_1 \right. \\ & \left. \times (\tilde{\mathbf{v}}_2 \cdot \nabla) \tilde{\mathbf{v}}_1^* \right] - \frac{e}{m} \tilde{\mathbf{E}}_1 - 3v_{Te}^2 \nabla \left(\frac{\tilde{n}_1}{n_0} \right). \end{aligned} \quad (10.19)$$

To the zeroth order we get from (10.19)

$$\tilde{\mathbf{v}}_1^{(0)} = -\frac{ie}{m\omega_p} \tilde{\mathbf{E}}_1, \quad (10.20)$$

and using the Poisson equation in the form

$$\text{div } \tilde{\mathbf{E}}_1 = -4\pi e \tilde{n}_1, \quad (10.21)$$

we get to the first order

$$\begin{aligned} \frac{\partial \tilde{\mathbf{v}}_1^{(1)}}{\partial t} = & \frac{ie}{m\omega_p} \left[(\tilde{\mathbf{E}}_1 \cdot \nabla) \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla) \tilde{\mathbf{E}}_1 - (\tilde{\mathbf{E}}_1^* \cdot \nabla) \tilde{\mathbf{v}}_2 \right. \\ & \left. - (\tilde{\mathbf{v}}_2 \cdot \nabla) \tilde{\mathbf{E}}_1^* \right] - \frac{e}{m} \tilde{\mathbf{E}}_1 + \frac{3e}{m} r_{De}^2 \nabla (\nabla \cdot \tilde{\mathbf{E}}_1). \end{aligned} \quad (10.22)$$

We can now use (10.20) and (10.22) to obtain from (10.16) the relation

$$\begin{aligned} i \frac{\partial \tilde{\mathbf{E}}_1}{\partial t} - \frac{c^2}{2\omega_p} \nabla \times \nabla \times \tilde{\mathbf{E}}_1 + \frac{3}{2} \omega_p r_{De}^2 \nabla \cdot \nabla \cdot \tilde{\mathbf{E}}_1 - \frac{\omega_p}{2n_0} n_s \tilde{\mathbf{E}}_1 \\ = -\frac{1}{2} i \left[(\tilde{\mathbf{E}}_1 \cdot \nabla) \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla) \tilde{\mathbf{E}}_1 + \mathbf{v}_s \nabla \cdot \tilde{\mathbf{E}}_1 \right] - \frac{\omega_p}{2n_0} \tilde{n}_2 \tilde{\mathbf{E}}_1^* \\ + \frac{1}{2} i \left[(\tilde{\mathbf{E}}_1^* \cdot \nabla) \tilde{\mathbf{v}}_2 + (\tilde{\mathbf{v}}_2 \cdot \nabla) \tilde{\mathbf{E}}_1^* - \tilde{\mathbf{v}}_2 \nabla \cdot \tilde{\mathbf{E}}_1^* \right]. \end{aligned} \quad (10.23)$$

If the right-hand side of (10.23) were zero, this equation would be one of the Zakharov equations which describes both electrostatic and electromagnetic perturbations [217, 226, 227]. The terms on the right-hand side of (10.23) correspond to higher electron nonlinearities describing both electrostatic (Langmuir) and electromagnetic perturbations. The equation is a generalization of Kuznetsov's result [227], in which only potential perturbations were considered. Indeed, one readily recovers his results by putting $\tilde{\mathbf{E}}_1 = \nabla \psi$.

To close the set of equations we still need relations for n_s , $\mathbf{v}_s$, $\tilde{n}_2$, and $\tilde{\mathbf{v}}_2$. The electron motions at frequencies close to $2\omega_p$ can be described in the hydrodynamic framework as the phase velocity is much larger than the thermal velocity. Linearizing (10.17) and (10.18) with respect to n_2 and $\mathbf{v}_2$, we find

$$\frac{\partial n_2}{\partial t} + n_0 \nabla \cdot \mathbf{v}_2 + \nabla \cdot (n_1 \mathbf{v}_1) = 0, \quad (10.24)$$

$$\frac{\partial \mathbf{v}_2}{\partial t} + (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1 + \frac{e}{mc} [\mathbf{v}_1 \times \mathbf{B}_1] = -\frac{e}{m} \mathbf{E}_2, \quad (10.25)$$

where $\mathbf{E}_2$, which represents the electric field component due to charge separation at frequencies close to $2\omega_p$, satisfies the continuity relation

$$\nabla \cdot \mathbf{E}_2 = -4\pi e n_2. \quad (10.26)$$

Using (10.20) for $\mathbf{v}_1$ and the appropriate Maxwell equation (10.10) for $\mathbf{B}_1$ we get from (10.25) the following equation for $\mathbf{v}_2$, where we have retained the terms with the correct frequency dependence:

$$m \frac{\partial \mathbf{v}_2}{\partial t} = -e \mathbf{E}_2 + \mathbf{F}_2, \quad (10.27)$$

where $\mathbf{F}_2$ is the potential (ponderomotive or Miller, [230]) force at the second harmonic frequency $2\omega_p$. If we write

$$\mathbf{F}_2 = \tilde{\mathbf{F}}_2 \exp(-2i\omega_p t) + \tilde{\mathbf{F}}_2^* \exp(2i\omega_p t), \quad (10.28)$$

we have

$$\tilde{\mathbf{F}}_2 = -m \left((\tilde{\mathbf{v}}_1 \cdot \nabla) \tilde{\mathbf{v}}_1 + \frac{e}{mc} [\tilde{\mathbf{v}}_1 \times \tilde{\mathbf{B}}_1] \right) = \frac{-1}{8\pi n_0} \nabla (\tilde{\mathbf{E}}_1 \tilde{\mathbf{E}}_1). \quad (10.29)$$

We note from (10.27) that at the frequency $2\omega_p$ the driving force derives both from charge separation and from the ponderomotive force. To find n_2 we first combine (10.24), (10.26), (10.20), and (10.21) to find $\nabla \cdot \mathbf{v}_2$ in the form

$$\nabla \cdot \mathbf{v}_2 = \frac{i}{12\pi n_0 \omega_p} \nabla \cdot (\tilde{\mathbf{E}}_1 \nabla \cdot \tilde{\mathbf{E}}_1 + \nabla (\tilde{\mathbf{E}}_1 \tilde{\mathbf{E}}_1)), \quad (10.30)$$

and hence

$$\tilde{n}_2 = \frac{1}{6\pi m \omega_p^2} \nabla \cdot \left(\tilde{\mathbf{E}}_1 \nabla \cdot \tilde{\mathbf{E}}_1 + \frac{1}{4} \nabla (\tilde{\mathbf{E}}_1 \tilde{\mathbf{E}}_1) \right). \quad (10.31)$$

Equations (10.30) and (10.31) are identical with Kuznetsov's equations [227] which were derived assuming that there were only electrostatic perturbations.

Next we must derive the "slow timescale" electron equations. As the slow-timescale phase velocity v_{ph}^s will be of the order of the ion-sound speed which is small compared to the electron thermal velocity, we may assume that the averaged electron density, defined by the relation

$$\langle n_e \rangle = \langle n_0 + n_s + n_1 + n_2 + \dots \rangle = n_0 + n_s, \quad (10.32)$$

will be given by a Boltzmann distribution in the field of an effective potential U_{eff} [215, 217]. This effective potential consists of a slow-timescale charge separation

potential $-e\phi_s$ and a ponderomotive potential U_{pond} obtained, like $\langle n_e \rangle$, by averaging over the fast-timescale motion:

$$U_{\text{eff}} = -e\phi_s + U_{\text{pond}}, \quad (10.33)$$

where

$$U_{\text{pond}} = \frac{e^2}{m\omega_p^2} (\tilde{\mathbf{E}}_1 \tilde{\mathbf{E}}_1^*). \quad (10.34)$$

For $\langle n_e \rangle$ we have thus

$$\langle n_e \rangle = n_0 \exp \frac{-e\phi_s + U_{\text{pond}}}{T_e}, \quad (10.35)$$

whence

$$n_s = n_0 [\exp \{ (e\phi_s - U_{\text{pond}}) / T_e \} - 1]. \quad (10.36)$$

If the electron distribution is stationary, we get, by linearizing the equation of continuity (10.17) with respect to n_s and $\mathbf{v}_s$ and retaining only the slow-timescale terms,

$$\nabla \cdot \mathbf{v}_s = -\nabla \cdot (\tilde{n}_1 \tilde{\mathbf{v}}_1^* + \tilde{n}_1^* \tilde{\mathbf{v}}_1) / n_0, \quad (10.37)$$

and hence, using (10.20) and (10.21)

$$\nabla \cdot \mathbf{v}_s = \frac{i}{4\pi m n_0 \omega_p} \nabla \cdot [\tilde{\mathbf{E}}_1^* \nabla \cdot \tilde{\mathbf{E}}_1 - \tilde{\mathbf{E}}_1 \nabla \cdot \tilde{\mathbf{E}}_1^*]. \quad (10.38)$$

To eliminate $e\phi_s$ we shall need an equation for the ion motions. As the ion temperature is lower than T_e , v_{ph}^s will be larger than the ion thermal velocity v_{Ti} and we can apply a hydrodynamic description for the ion motion. (If $v_{\text{ph}}^s \ll v_{Ti}$, one can use the so-called “static” approximation [215].)

For the moment, we shall assume that we may linearize the ion equations (see the discussion at the end of this section). We then have

$$\frac{\partial \delta n_i}{\partial t} + n_0 \nabla \cdot \mathbf{v}_i = 0, \quad (10.39)$$

$$\frac{\partial \mathbf{v}_i}{\partial t} = -\frac{e}{M} \nabla \phi_s, \quad (10.40)$$

where we have, in the equation of motion, omitted the pressure term, since $T_e \gg T_i$ by assumption, and neglecting the direct ion ponderomotive force (as $m/M \ll 1$) in comparison to the ambipolar “electron” force.

We restrict ourselves to density perturbations for which $n_s \ll n_0$ – which means that we can expand and linearize (10.36) to obtain

$$\frac{n_s T_e}{n_0} + U_{\text{pond}} = e\phi_s, \quad (10.41)$$

while Poisson's equation becomes

$$\nabla^2 \phi_s = 4\pi e (n_s - \delta n_i). \quad (10.42)$$

Combining (10.41) and (10.42) to eliminate ϕ_s , we have

$$n_s = \delta n_i + \nabla^2 U_{\text{pond}} / 4\pi e^2 + r_{\text{De}}^2 \nabla^2 n_s. \quad (10.43)$$

If we further assume that the spatial variations of n_s are on length scales much larger than the Debye radius, we can drop the last term on the right-hand side of (10.43) to get the departure from the quasi-neutrality, by

$$n_s = \delta n_i + \nabla^2 U_{\text{pond}} / 4\pi e^2. \quad (10.44)$$

We can now eliminate n_s and $\mathbf{v}_i$ from (10.39), (10.40), and (10.44) and using (10.34) we find the familiar, ponderomotively driven ion-sound equation

$$\frac{\partial^2 \delta n_i}{\partial t^2} - c_s^2 \nabla^2 \delta n_i = \frac{1}{4\pi M} \nabla^2 (\tilde{\mathbf{E}}_1 \tilde{\mathbf{E}}_1^*). \quad (10.45)$$

Let us briefly remind ourselves of a fundamental requirement for the validity of our results [226, 230]. This is the requirement that the distance traveled by an electron during a time ω_p^{-1} is small compared to the characteristic fast timescale length scale k^{-1} , that is,

$$v_s \omega_p^{-1} \ll k^{-1} \quad \text{and} \quad v_1 \omega_p^{-1} \ll k^{-1}, \quad (10.46)$$

we note that we shall allow for v_1 to be more important than v_s ; however, by using (10.20) and (10.38) this leads to the condition

$$\frac{W}{n_0 T_e} \ll (k r_{\text{De}})^{-2}, \quad (10.47)$$

where W is the energy density in the Langmuir wave.

The condition (10.47) also ensures that one can neglect the effect of electrons trapped in the finite-amplitude, fast timescale electric field of the wave [13]. The trapping is a special case of a strong Landau-like resonant interaction involving strong nonlinear electron-orbit modifications. Trapping in spatially localized wavepackets is also negligible for sufficiently broad wavepacket spectra when condition (10.47) is satisfied. We must add that, anyway, trapping is inherently a kinetic effect so that we cannot consider it in the hydrodynamic description used here.

Under additional simplifications, our equations will readily reduce to the celebrated Zakharov equations. Namely, by restricting ourselves to the potential electric field, i.e., $\tilde{\mathbf{E}}_1 = -\nabla\varphi$, dropping higher electron nonlinearities on the right-hand side of (10.23) and assuming the quasi-neutrality by putting $n_s = \delta n_i$, we obtain a

system of two Zakharov equations describing the nonlinear Langmuir waves coupled to the ion sound. These equations have been studied into great depth, bringing the notion of Langmuir solitons in strong Langmuir turbulence models into focus in numerous analytical, numerical, and experimental investigations. By introducing dimensionless units

$$\begin{aligned} r &= \frac{3}{2} r_{\text{De}} \sqrt{M/m} r', \quad t = \frac{3}{2} (M/m) t', \\ \delta n/n_0 &= \frac{4}{3} (m/M) n, \quad \varphi = (T_e/e) \sqrt{12} \varphi', \end{aligned} \quad (10.48)$$

we obtain Zakharov equations in the original form

$$\begin{aligned} \nabla^2 (\varphi_t + \nabla^2 \varphi) - \nabla \cdot (n \nabla \varphi) &= 0, \\ (\partial^2 / \partial t^2 - \nabla^2) \varphi &= \nabla^2 |\varphi|^2. \end{aligned} \quad (10.49)$$

In the one-dimensional case, above equations admit the four-parameter family of soliton solutions, given by

$$\varphi_x = \sqrt{2(1 - \beta^2)} \lambda \operatorname{sech}[\lambda(x - \beta t - x_0)] \exp i[(\lambda^2 - \beta^2/4)t + \beta x/2 + \alpha_0]. \quad (10.50)$$

The type of these solutions is strongly influenced by the soliton velocity β which in physical units is equal to $3v_{Te}k_0r_{\text{De}}$, where k_0 is a wavenumber at the maximum in the Langmuir wavepacket spectrum (see Kuznetsov et al. [262]). For a soliton at rest, $k_0r_{\text{De}} \ll W/n_0T_e$, the electric field varies monotonically. However, when $(k_0r_{\text{De}})^2 \gg W/n_0T_e$, this is an envelope soliton with a quasi-monochromatic carrier frequency. In such a case, one can make a simplification with the slow time variation of the envelope to arrive at the above set of Zakharov equations. In the static limit ($W/n_0T_e \ll m/M$), the Zakharov system reduces to one equation in a form of the famous nonlinear Schroedinger equation (NLS), [270],

$$\nabla^2 (\varphi_t + \nabla^2 \varphi) + \nabla \cdot (|\nabla \varphi|^2 \nabla \varphi) = 0. \quad (10.51)$$

10.3 Adiabatic Scaling and Spherical Collapse

In this part, we investigate the effect of the nonlinear correction terms on the stability of spherical or circular configurations in the three- and two-dimensional cases, respectively. At the end, we shall briefly discuss less symmetrical structures, while more general discussion on wave collapse will be given in the next chapter. The spherical collapse was already discussed in Zakharov's original paper [215], but the solutions he considered were criticized by Litvak et al. [231] and by Degtyarev et al.

[232]. For the spherically symmetric case, (23) becomes

$$iE_t + \frac{\partial}{\partial r} \left[\frac{1}{r^{d-1}} \frac{\partial}{\partial r} (r^{d-1} E) \right] - nE - \frac{\beta |E|^2 E}{r^2} = 0, \quad (10.52)$$

where we have introduced dimensionless variables by the substitutions given in (10.48). The quantity d is the dimensionality ($d = 1, 2$ and 3) of the system, and β is the dimensionality parameter, given by

$$\beta(d = 1) = 0, \quad \beta(d = 2, 3) = (2/3)^3 m/M. \quad (10.53)$$

The term with β comes from higher electron nonlinearities which thus do not contribute in the one-dimensional case, as was independently confirmed by Kuznetsov [227] and by Khakimov and Tsytovich [224] and Škorić and ter Haar [226] (see also [233]). In the derivation of (10.52), we have eliminated n_s , $\mathbf{v}_s$, $\tilde{n}_2$, and $\tilde{\mathbf{v}}_2$ from (10.23) by using (10.30), after integrating over r , (10.31), (10.37), and (10.38).

We shall follow the approach used in an earlier paper [226, 229] and, for the moment, for stationary states represent the density perturbation n as an as yet unspecified function $-Q(|E|^2)$ of the plasmon density $|E|^2$

$$n = -Q(|E|^2). \quad (10.54)$$

Of course, n satisfies the equation

$$n_{tt} - \nabla^2 n = \nabla^2 |E|^2. \quad (10.55)$$

In the static case, where the time-derivative can be neglected, we have clearly

$$Q(x) = x. \quad (10.56)$$

Using (10.54), we get (10.52) simply rewritten in the form

$$iE_t + \frac{\partial}{\partial r} \left[\frac{1}{r^{d-1}} \frac{\partial}{\partial r} (r^{d-1} E) \right] + Q(|E|^2) E - \beta \frac{|E|^2 E}{r^2} = 0. \quad (10.57)$$

Equation (10.57) has the constants of motion

$$N = \int |E|^2 d^d r, \quad (10.58)$$

the plasmon number, and

$$H = \int \left\{ |\nabla E|^2 - R(|E|^2) + \beta \frac{|E|^4}{2r^2} \right\} d^d r, \quad (10.59)$$

the Hamiltonian. In (10.59), we have $R(|E|^2)$ which is defined by the equation

$$Q(x) = \frac{dR(x)}{dx}. \quad (10.60)$$

We now consider a scaling factor $\lambda(t)$ such that

$$r \rightarrow r/\lambda, \quad E \rightarrow \lambda^{\frac{d}{2}} E; \quad (10.61)$$

this scaling leaves the plasmon number- N (action) invariant and may thus be called to be an adiabatic scaling. The Hamiltonian scales as follows

$$H \rightarrow C \int r^{d-1} dr \left\{ \lambda^2 \left| \frac{1}{r^{d-1}} \frac{\partial}{\partial r} (r^{d-1} E) \right|^2 - \lambda^{-d} R(r^d |E|^2) + \lambda^{d+2} \beta \frac{|E|^4}{2r^2} \right\}, \quad (10.62)$$

where C is a numerical factor depending on the value of d

$$C(d=1) = 1, \quad C(d=2) = 2\pi, \quad C(d=3) = 4\pi. \quad (10.63)$$

If now we assume that

$$\lim_{x \rightarrow \infty} Q(x) \propto x^\nu, \quad (10.64)$$

stability against collapse, that is, as $\lambda \rightarrow \infty$, is guaranteed, if

$$\nu < \sigma + \frac{2}{d} = \nu_{\text{crit}}, \quad (10.65)$$

where σ is a constant depending on d

$$\sigma(d=1) = 0, \quad \sigma(d=2, 3) = 1. \quad (10.66)$$

Condition (10.65) changes into condition $\nu < \frac{2}{d}$ for the Zakharov equations (see, e.g. [229]) when $\beta = 0$. We also see that in the case (10.56), when $\nu = 1$, the one-dimensional case leads to stationary solutions – which are, of course, the Langmuir solitons – even without the electron nonlinearities. Finally, we note that, if the electron nonlinearities are taken into account, condition (10.65) is satisfied even for two- and three-dimensional plasmas in the case of the ponderomotive force nonlinearity (10.56). In the spherical approximation, the corrections due to the higher-order electron nonlinearities thus lead to the possibility of quasi-stationary solutions also in the two- or three-dimensional case. We still need to consider, however, whether the absence of a mathematical singularity also implies the absence of a physical collapse to dimensions of the size of the Debye length, as seems to be the case judging

from the experimental results of e.g., Antipov et al. [222] who formed apparently stable structures of the size of several Debye lengths.

In order to study this question, we shall simply estimate the magnitude of the extra correction term in (10.48). As long as $W/n_0 T_e \ll m/M$, we can neglect the n_{tt} term in (10.55) (see [217]) so that (10.56) holds and the self-focusing term in (10.50) is simply $|E|^2 E$. If we require, say, that the correction term is an order of magnitude smaller than the self-focusing term, we have

$$\beta \frac{|E|^2 E}{r^2} \sim \frac{1}{10} |E|^2 E, \text{ or } r^2 \sim 10\beta, \quad (10.67)$$

whence, restoring physical dimensions, we get $r \sim 3r_{De}$.

This means that the stabilizing action of the higher-order electron nonlinearities does not start to be fully effective until the spherical collapse has already proceeded quite far.

10.4 Qualitative Discussion of the Collapse

Before discussing the collapse of a caviton structure, that is, a localized Langmuir wave accompanied by a density depression, centered at the origin, we must draw attention that the second term in (10.50) can be written in the form $\partial^2 E / \partial r^2 + (d-1)r^{-1} \partial E / \partial r - (d-1)E/r^2$, so that the origin is a singular point of that equation, already for $\beta = 0$. One should therefore for, the spherical case, impose the boundary condition $E(0, t) = 0$. This may have been the reason why Degtyarev et al. [232] considered the spherical collapse of a spherical layer of radius R and thickness δ , with $\delta \ll R$, with a soliton-like field structure in the r -direction. As $\delta \gg r_{De}$, so that a fortiori $R \gg r_{De}$, the evolution of that model will not be affected by our correction term. By the same token, however, it is only possible in this model to study the very early stages of the collapse, as the condition on δ and R will soon be violated.

In the more general case (see also [231] and [234]), we could expect that there will be two regions with different dynamical properties. There will be a “core” region with a radius of the order of a few times r_{De} which is stabilized through the effect of the higher-order electron nonlinearities, and there will be a “shell” region which collapses toward the center (originally in [226] preceding a notion of “nucleated” collapse by [239]). As the collapse proceeds, the field amplitude increases and the condition necessary for the static approximation to hold will be violated. This means that one should consider the hydrodynamic approximation [217], but that would mean using the full equation (10.55) when $W/n_0 T_e$ approaches the value of m/M , but is still below it so that the collapse is subsonic. The self-focusing term will become increasingly important and it will drive the collapse toward the sonic, or even supersonic ($W/n_0 T_e > m/M$) regime. The transition to the supersonic regime was subject to a certain amount of controversy (see [235]) as (10.55) is

no longer valid for near-sonic motion. However, numerical experiments [218, 236] seem to indicate, at least for nondissipative cases, the existence of a supersonic regime. This part of the collapse was a subject of further studies, especially the question whether such a supersonic stage could actually occur in real plasmas. If (10.50) remains valid during the collapse, we may perhaps expect in the later stages of the collapse a hydrodynamic stabilization so that stable spherical cavitons with spatial dimensions of the order of the “core” radius may be formed and these may be the structure seen by Antipov and coworkers. Strictly speaking, as soon as the value of $(W/n_0 T_e)(kr_{De})^2 M/m$ becomes of the order of unity, corrections to the ion motion will have to be considered ([260] and vide infra) and linearized (10.39) and (10.40) must be replaced by the proper nonlinear equations [228]. Let us finally briefly consider the general, nonspherically-symmetric case. As compared to the original Zakharov equations, we note that in (10.16) we have three extra terms. These terms are of the relative order $(kr_{De})^2$ in the static regime, but they may become important when $W/n_0 T_e$ becomes larger. For example, in the simple model of a self-similar supersonic collapse [254], we have $n_s/n_0 = \chi(W/n_0 T_e)^{2/3}$, where $\chi \sim 0.1$ to 1 and for the relative magnitude of the correction terms, we get $(W/n_0 T_e)^{1/3}(kr_{De})^2 \chi^{-1}$ which shows the possible importance of the correction terms in the hydrodynamic regime.

Chapter 11

Wave Collapse in Plasmas

11.1 Langmuir Soliton Stability and Collapse

A detailed study of the linear stability and nonlinear wave collapse of Langmuir solitons in a weakly magnetized plasma is performed [215,237]. An analytical investigation of the linear soliton instability with respect to long-wavelength transverse perturbation versus magnetic field effect is presented. For a more complete understanding of the growth-rate structure, a numerical solution of the eigenvalue problem that corresponds to the model equations is obtained and compared with analytical predictions. Comparison with other results is given. Furthermore, numerical results obtained by a direct simulation method in two dimensions are also presented. In a linear regime, detailed agreement with the results of the corresponding eigenvalue problem is found. In the nonlinear regime of the soliton instability, all considered cases exhibit a collapse dynamics. Moreover, in the developed, highly nonlinear stage of the soliton collapse, self-similar behavior consistent with a “weak” collapse regime is found [264].

11.1.1 Introduction

One-dimensional stationary localized soliton-like wave structures were expected to be the basic elements of strong plasma turbulence [218, 253–255]. In one-dimensional systems, solitons are often stable, evolving rapidly from an arbitrary initial plasma state, and therefore determining the basic features of the emerging plasma turbulence. However, in real plasmas, as a rule, solitons appear to be unstable with respect to transverse perturbations [238]. In a nonlinear stage of evolution, this instability often leads to a soliton collapse, a unique nonlinear wave phenomenon of the formation of a singularity in a finite time. Accordingly, the appearance of the collapsing nonstationary wave structures (cavities) that exhibit a rapid field growth followed by an intensive spatial localization (self-focusing) results in a qualitative change of the turbulence character [239]. The dispersion relation describing linear Langmuir waves in a weak magnetic field has a form

$$\omega_k = \omega_{pe} \left[1 + \frac{3}{2} k^2 r_{De}^2 + \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}^2} \frac{k_{\perp}^2}{k^2} \right], \quad (11.1)$$

where ω_{ce} and ω_{pe} ($\omega_{ce} \ll \omega_{pe}$) are the electron cyclotron and the electron plasma frequency, respectively, r_{De} is the Debye radius, and $k_{\perp}$ is the wave number component transverse to the magnetic field. It is evident that the transverse perturbation increases the wave frequency and is therefore energetically unfavorable. Accordingly, unstable modes should only correspond to long-wavelength transverse perturbations, with a frequency increase on the order of the instability growth rate. Moreover, a magnetic field increase results in an increase of the frequency of transverse oscillations and appears as a stabilizing factor.

Soliton instability in a weak magnetic field was already studied in [240]. It was shown that for moving solitons with a velocity $V/v_{te} > \omega_{ce}/\omega_{pe}$ (v_{te} is the electron thermal velocity) the magnetic field produces no changes in the soliton stability apart from increasing the transverse instability length scales, according to

$$l_{\perp} \sim l_0 \frac{\omega_{ce}^2}{\omega_{pe}^2} \frac{l_0}{r_{De}},$$

where $l_0 \sim (8\pi nT/E_0^2)^{\frac{1}{2}}$, is the soliton characteristic length.

In the opposite limit, in the case of a standing soliton ($V = 0$) in the long-wavelength region for the instability growth rate (γ), the following analytical solution of the corresponding eigenvalue problem [239] was obtained

$$\gamma = 2\omega_{pe} \left[\frac{E_0^2}{8\pi nT} [12 - 7\zeta(3)] - \frac{21}{4} \zeta(3) \frac{\omega_{ce}^2}{\omega_{pe}^2} \right] k_{\perp}^2 r_{De}^2, \quad (11.2)$$

where

$$\zeta(x) = \sum_n n^{-x},$$

is Riemann's zeta function.

At first sight, for sufficiently large values ($\omega_{ce}^2/\omega_{pe}^2 > 0.43E_0^2/8\pi nT$), it seems that the magnetic field stabilizes the linear instability. However, instability may reappear if one takes into account the next terms in the expansion in transverse wave number $k_{\perp}$. The dispersion relation (11.2) in the limit $k_{\perp} \rightarrow 0$, turns into a marginally stable mode corresponding to a small variation of the soliton amplitude. On the other hand, the expression (11.2) by itself does not appear to be sufficiently exact. In the treatment [238, 240], only the first term in the expansion $\gamma(k)$ was calculated. Moreover, the existence and solvability of the perturbation scheme, typically has not been proved, nor was the convergence of the series expansion. Numerical results obtained by Rowland [241] indicate that the magnetic field is unable to stabilize the soliton instability. Yet, these results were based on a few values of the parameters, so it is unclear if other regions of soliton stability can exist.

The influence of the magnetic field on the growth rate structure and on the transverse instability length scales remains a very important issue. These characteristics would give us an opportunity to estimate the parameters of the emerging collapsing Langmuir wave packets. The magnetic field effect on the collapse of Langmuir wave packets for different model equations has been a subject of many studies [242–245].

In this section, we present a detailed study of the influence of the magnetic field on the soliton stability [264]. First, we formulate the basic model equation and discuss the physical background of the problem. On the basis of the variational principle, we present an analytical investigation of the growth rate structure in a linear regime for finite values of the transverse perturbation wave number. Then, we give results of the numerical solution of the corresponding eigenvalue problem. We have obtained a complete spectral structure of the growth rate and corresponding eigenfunctions. Somewhat unexpectedly, the calculated form of $\gamma(k)$ does not agree with the analytically obtained (1.2). We discuss the possible reason for this discrepancy, mainly due to the inadequate accuracy of the perturbation treatment. Finally, the last part is devoted to the nonlinear stage of the soliton instability which exhibits a soliton collapse. In order to study the magnetic field effect on the nonlinear stage of the soliton instability, we show direct numerical simulations in two dimensions (2D). In the linear regime, we find detailed agreement with the results of the corresponding eigenvalue problem. In the nonlinear regime, all considered cases exhibit collapse dynamics. Moreover, in the developed highly nonlinear stage of the soliton collapse, self-similar behavior consistent with a “weak” collapse regime is found. Present analysis [264] differs from most of the previous studies [242–245] on Langmuir collapse in terms of the model equation, initial conditions (soliton), level of nonlinearity, and observed phenomena. The topic of Langmuir soliton stability and collapse important on its own merit, also have appeared in many applications involving observational data from space, astro- and ionospheric plasmas.

11.1.2 Basic Equations

Nonlinear evolution of Langmuir waves in a weak magnetic field is conveniently described by a time-averaged dynamical equation for the envelope of the high-frequency Langmuir wave potential ψ , which in dimensionless units [264]

$$t \rightarrow \frac{3}{2} \frac{M}{m} \omega_{pe}^{-1} t, \quad r \rightarrow \frac{3}{2} \left[\frac{M}{m} \right]^{1/2} r_{De} r, \quad \psi \rightarrow \frac{T}{e} (12)^{1/2} \psi,$$

reads

$$\Delta(i\psi_t + \Delta\psi) - \sigma\Delta_\perp\psi + \nabla(|\nabla\psi|^2\nabla\psi) = 0, \quad (11.3)$$

where $\sigma = \frac{3}{4}\omega_{ce}^2/\omega_{pe}^2 (M/m)$, M and m are as usual the ion and the electron mass, respectively. The external magnetic field $\mathbf{B}$ ($\omega_{ce} \equiv eB/mc$) is in the x -direction while the dimensionless (11.3) is valid for $\sigma \ll \frac{3}{4}M/m$. The linear part of (2.1)

corresponds to the dispersion relation (1.1) while the nonlinear term is described through a static plasma response to the ponderomotive force action. We assume that the characteristic nonlinear timescales are slower than the ion-sound motions. The above is justified in a small amplitude region $E_0^2/8\pi nT < m/M$, i.e., in the subsonic regime.

For a more complete insight into the soliton stability problem and corresponding growth rate structure, the inclusion of the ion inertia is essential (see [242–245, 271]). As will be seen below, the magnetic field increase results in a growth of transverse perturbation length scales. Under the assumption that the characteristic transverse length scales are sufficiently larger than longitudinal ones, the (11.3) substantially simplifies to [271]

$$\frac{\partial^2}{\partial x^2} (i\psi_t + \psi_{xx}) - \sigma \Delta_\perp \psi + \frac{\partial}{\partial x} (|\psi_x|^2 \psi_x) = 0. \quad (11.4)$$

As mentioned above, we shall investigate stability of a planar standing soliton, given with the analytical solution of (11.1) and (11.2)

$$\psi_0 = \sqrt{2} \arctan [\sinh (\lambda x)] \exp (i \lambda^2 t), \quad (11.5)$$

where λ is the soliton strength parameter.

We study the stability of (11.3) with respect to small transverse perturbations with a potential $\psi_p = f + ig$, in a form

$$f, g \sim \exp (i \lambda^2 t + \gamma \lambda^2 t + i k \lambda y), \quad k \equiv k_\perp / \lambda.$$

Linearizing (11.1) on the background of the soliton (11.3) and taking scaling transformation $x \rightarrow x\lambda$, and $\sigma \rightarrow \sigma/\lambda^2$, we obtain

$$\begin{aligned} \gamma \left[\frac{d^2}{dx^2} - k^2 \right] f + \left[\frac{d^4}{dx^4} - (1 + 2k^2) \frac{d^2}{dx^2} + k^2 (1 + \sigma + k^2) \right. \\ \left. - \frac{2k^2}{\cosh^2 x} + \frac{d}{dx} \frac{2}{\cosh^2 x} \frac{d}{dx} \right] g = 0, \end{aligned} \quad (11.6)$$

$$\begin{aligned} -\gamma \left[\frac{d^2}{dx^2} - k^2 \right] g + \left[\frac{d^4}{dx^4} - (1 + 2k^2) \frac{d^2}{dx^2} + k^2 (1 + \sigma + k^2) \right. \\ \left. - \frac{2k^2}{\cosh^2 x} + \frac{d}{dx} \frac{6}{\cosh^2 x} \frac{d}{dx} \right] f = 0. \end{aligned}$$

The increment γ (growth rate) of an instability is given by the eigenvalues of the (11.4) corresponding to the spatially localized eigenfunctions.

In the literature, there exist some standard methods of solving for $\gamma(k)$, in the long-wavelength limit, based on the local proximity of the eigen-functions (2.4) for neutrally stable perturbations [238]. It is evident that odd and even (with respect to x) solutions of (11.4) can be treated independently. Odd modes (antisymmetric) correspond to marginally stable soliton deformations in the long-wavelength region. However, for even (symmetric) modes in [240], the following analytical solution was obtained:

$$\gamma^2 = 2k^2 [12 - 7\zeta(3) - 7\sigma\zeta(3)]. \quad (11.7)$$

Due to instability, the soliton is split into a number of wave packets that each exhibit a local growth of the amplitude. From the expression (2.5), it seems that for sufficiently large values of σ ,

$$\sigma > \frac{12 - 7\zeta(3)}{7\sigma\zeta(3)},$$

the magnetic field can stabilize the instability. However, as already mentioned, in a real situation this might not occur. Namely, the instability may reappear in the calculations if one takes into account the next terms in the expansion in k . It seems that standard analytical methods do not appear to be successful in that case. Therefore, we shall try to investigate the structure of the growth rate $\gamma(k)$ for finite values of k by applying an approximative variational method.

11.1.3 Variational Treatment of Soliton Stability

A basic idea of an approximative “brute force” treatment of a soliton instability in its nonlinear stage (see Trubnikov et al. [246] and references therein) is as follows. Equation (11.1) can be obtained by the variational principle [264]

$$\Delta S = 0, \quad (11.8)$$

where S is the action defined by

$$S = \int \int \left[\frac{i}{2} (\nabla \psi \nabla \psi_t^* - \nabla \psi^* \nabla \psi_t) + |\Delta \psi|^2 + \sigma |\nabla_{\perp} \psi|^2 - \frac{1}{2} |\nabla \psi|^4 \right] \mathbf{dr} dt. \quad (11.9)$$

Let us substitute ψ in (11.9) as a set of trial functions with varying parameters. In our case, we chose ψ in the following form:

$$\psi_0 = \sqrt{2} \arctan \{ \sinh [\lambda(y, t)x] \} \exp [-i\varphi(y, t)], \quad |\lambda_{t,y}| \ll |\varphi_{t,y}|. \quad (11.10)$$

Accordingly, (11.9) reduces to a much simpler system of differential equations for λ and φ , which can be treated by standard analytical methods. However, the success of the above procedure essentially depends on our choice of trial functions. As already

mentioned, an unstable mode appears as a local modulation of the soliton amplitude and phase. Our chosen trial function corresponds to such type of perturbation and should in a long-wavelength limit recover expression (11.7).

We substitute the trial function (11.10) into the action S , where, after a straightforward procedure, we obtain

$$\begin{aligned}
 S = \int \int & \left[4\lambda\varphi_t - \frac{4}{3}\lambda^3 + 24\varphi_y^2\lambda - (\lambda_y\varphi_{yt}\lambda^{-2} - \varphi_y\lambda_{yt}\lambda^{-2} \right. \\
 & - 2\varphi_y^2\varphi_t\lambda^{-1} - 2\varphi_{yy}^2\lambda^{-1} - 2\sigma\varphi_y^2\lambda^{-1} - 2\lambda_{yy}\varphi_y^2\lambda^{-2} \\
 & \left. + 4\varphi_y\varphi_{yy}\lambda_y\lambda^{-2} - 2\varphi^4\lambda^{-1} \right) I_0 + 4\lambda_{yy}^2\lambda^{-2} I_2 \Big] dy dt, \quad (11.11)
 \end{aligned}$$

where

$$I_0 = \int_0^\infty [\arctan(\sinh x) - \pi/2]^2 dx, \quad I_2 = \int_0^\infty \frac{x^2}{\cosh^2 x} dx.$$

By varying the functional S over σ and λ , it is possible to derive a system of two equations (if the initial function was appropriately chosen). We limit ourselves to check if in a long-wavelength limit ($k \rightarrow 0$) of the linearized version of equations $\Delta S/\Delta\varphi = 0$, and $\Delta S/\Delta\lambda = 0$, one can recover the results of [240]. By linearizing the equations $\Delta S/\Delta\varphi = 0$, and $\Delta S/\Delta\lambda = 0$, on the background of the stationary solutions $\varphi_0 = t\lambda^2$, $\lambda_0 = \text{const}$, $\varphi = \varphi_0 + \Delta\varphi$, $\lambda = \lambda_0 + \Delta\lambda$, we get

$$\begin{aligned}
 -4\Delta\lambda_t - 48\lambda_0\Delta\varphi_{yy} - (2\lambda_0^{-2}\Delta\lambda_{yyt} + 4\lambda_0\Delta\varphi_{yy} + 4\lambda_0^{-2}\Delta\varphi_{yyy}) \\
 - 4\sigma\lambda_0^{-1}\Delta\varphi_{yy})I_0 = 0, \quad (11.12)
 \end{aligned}$$

$$4\Delta\varphi_t - 8\lambda_0\Delta\lambda + 2\lambda_0^{-2}I_0\Delta\varphi_{yyt} + 8\lambda_0^{-3}I_2\Delta\lambda_{yyyy} = 0. \quad (11.13)$$

For perturbations in a form $\Delta\lambda, \Delta\varphi = \exp(\gamma t + ik y)$, after simple calculations we obtain the formula for $\gamma(k)$ as

$$\gamma^2(k) = 2 \frac{[12\lambda_0^2 - 7(\lambda_0^2 + \sigma)\zeta(3)]k^2 + 7\zeta(3)k^4}{[1 + \frac{7}{2}\zeta(3)k^2\lambda_0^{-2}]^2}. \quad (11.14)$$

It is evident that in a long-wavelength limit, our result (11.14) agrees with (11.7). However, it is also obvious that the instability may reappear if we take into account the higher order terms in k . The maximum growth rate depends only weakly on σ . Accordingly, generally taken, in the framework of (11.3), the magnetic field does not stabilize the soliton stability. However, based on (11.14) for sufficiently strong magnetic fields, islands (regions) of stability around

$$\sigma = \frac{12 - 7\zeta(3)}{7\zeta(3)}\lambda_0^2,$$

should exist.

We have already emphasized that the accuracy of the above-noted variational treatment critically depends on our choice of trial functions. This procedure seems convenient to predict the rough qualitative features of the instability increment, however, hardly adequate enough to describe the fine structure of the eigenmode corresponding to (11.7). Therefore, to single out the detailed structure of the instability increment, we shall have to numerically solve the eigenvalue problem (11.6).

11.1.4 Numerical Treatment

In order to find a detailed structure of the instability increment $\gamma(k)$, it is necessary to calculate a set of eigenfunctions which vanish at infinity with the corresponding eigenvalues for γ , for different values of the perturbation wave number k and the magnetic field σ . As mentioned above, unstable modes, at least in the long-wavelength limit, correspond to a symmetric (even) type of the electric field perturbations. Accordingly, the electric potential perturbations are antisymmetric. This situation enables us to solve the system (11.6) in the interval $[0, \infty)$ with the boundary conditions [264]

$$\left. \frac{d^4 \Delta\psi}{dx^4} \right|_{x=0} = \left. \frac{d^2 \Delta\psi}{dx^2} \right|_{x=0} = 0, \quad \Delta\psi = f + ig. \quad (11.15)$$

Further, based on (11.6) the following condition is automatically satisfied:

$$\Delta\psi|_{x=0} = 0.$$

In order to find spatially localized (vanishing at infinity) solutions which satisfy (11.15) we have adopted the following method. On the right-hand side (r.h.s.) of the interval $[0, R]$, $R \gg 1$, for given k , we assume the following asymptotic solution of (11.6):

$$\Delta\psi = f + ig = C_1 \exp(k_1 x) + C_2 \exp(k_2 x), \quad (11.16)$$

where C_1 and C_2 are the arbitrary complex constants. By making use of (11.16) as boundary conditions for (11.6), it is possible to solve the system (11.6) as a Cauchy problem and to calculate the eigenfunction $\Delta\psi$ and its derivatives at the l.h.s. of the interval (at $x = 0$). As a next step, we shall define an auxiliary function $F(\gamma, C_1, C_2)$ in the following way:

$$F(\gamma, C_1, C_2) = [f^2(0) + f_{xx}^2(0) + f_{xxxx}^2(0) + g^2(0) + g_{xx}^2(0) + g_{xxxx}^2(0)]^{1/2}.$$

If F revolves at a zero point, then the obtained functions $f(x)$ and $g(x)$ appear to be the eigenfunctions and $\gamma(k)$ the corresponding eigenvalue of the system (11.6), with boundary conditions (11.15) and (11.16). The function F depends on five independent parameters. Based on a linearity of (11.6), it is possible to fix one of

them, e.g., $ImC_1 = 1$. By varying the remaining parameters, we look for a minimum of the function F . The minimization is performed by the method of steepest descent, starting with arbitrary values of γ_0 , C_1^0 and C_2^0 . Typical maximum values for f and g were equal or larger than unity. The procedure was terminated when the value of F became smaller than 10^{-3} . In order to perform calculations in the small k region, with the slowly decaying asymptotic solutions, we have chosen a substantially larger interval of calculations $R = 8$.

It has been proved that for all considered initial states, a single minimum of F exists being independent on the initial conditions. Therefore, in Fig. 11.1, we plot the calculated spectrum $\gamma(k)$ for different values of σ . It is evident in (Fig. 11.1) that the magnetic field increase results in a continuous change of $\gamma(k)$; its maximum shifted to the long-wavelength region while the maximum value weakly increases with σ . Accordingly, the magnetic field increase leads to a growth of the transverse perturbation scale length. For larger values of $l_\perp/l_\parallel$, i.e., for larger magnetic fields σ , which corresponds to (11.4) the spectral dependence $\gamma(k)$ on the magnetic field strength appears to be universal:

$$\gamma(k, \sigma) = \gamma(k\sqrt{\sigma}). \quad (11.17)$$

Our calculations indicate that the transition to this universal behavior (11.17) already appears at $\sigma \sim 10$. It is a very important fact that the numerically calculated

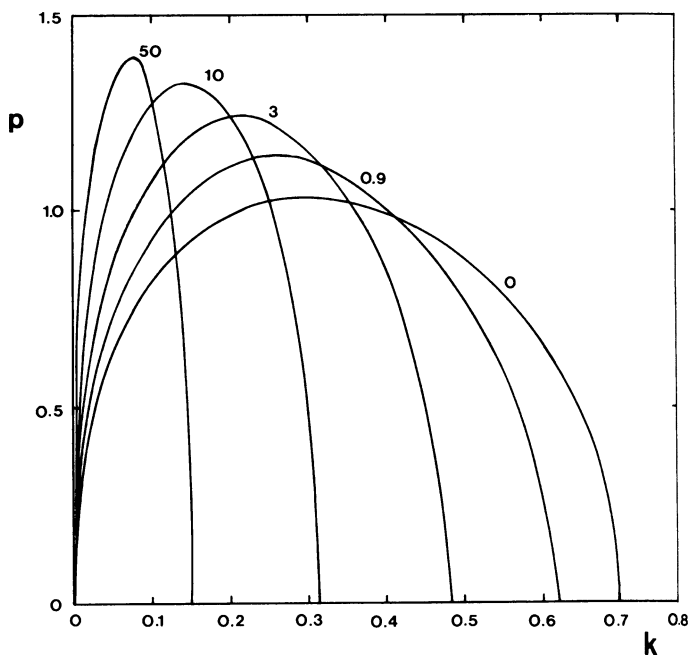


Fig. 11.1 Linear growth rate (p) versus transverse perturbation wavenumber (k) for different magnetic field strengths (σ): 0, 0.9, 3, 10, and 50

spectral dependence differs qualitatively from the analytical formulas obtained above. Namely, numerical results show that the magnetic field increase does not produce an island of stability near $k = 0$; i.e., instability exists in the entire interval between $k = 0$ and the cutoff (critical) value $k = k_c$. On the other hand, the spectral structure of the growth rate in the small k region considerably differs from the analytically predicted dependence $\gamma(k) \sim k$, exhibiting a nonlinear behavior according to $\gamma(k) \sim k^{1/3}$ (see Fig. 11.2).

The above situation convinces us that the formulation of the perturbation theory for the plasma soliton stability proposed in [238] and [240] does not appear to be sufficiently accurate. As an additional check of this problem we have performed further investigations in the region $k \rightarrow 0$. It seems obvious that for $k = 0$ and $\gamma = 0$, neutrally stable perturbations correspond to infinitely small variations of the soliton parameters. Accordingly, the eigenfunctions f and g of a symmetric type turn out to be

$$f = 0, \quad g = \sqrt{2} \arctan(\sinh x). \quad (11.18)$$

Our numerically calculated solutions, as $k \rightarrow 0$, continuously assume the form of (11.18). Furthermore, we study the spectral behavior near the cutoff value k_c . Solutions of the eigenvalue problem (11.6) for $k \neq 0$ and $\gamma = 0$ were obtained in an independent way. It appears that two types of solutions exist. The first one, with the critical (cutoff) wave number $k_c = 1 - \sigma$, if $k \neq k_c$, turns into a stable mode ($\gamma^2 < 0$). The second type corresponds to an unstable branch (for $\sigma = 0$ and $k_c \simeq 0.7$) with a continuous transition for $k \neq k_c$, to a solution of the complete system (11.6).

In order to check the accuracy of our numerical method, we have investigated the soliton instability in the framework of the nonlinear Schrodinger (NLS) equation. The calculated spectral form of $\gamma(k)$ coincides with the one found in [247]. In the case of the NLS equation, the soliton stability problem for a small k is solvable with the perturbation theory [238] to any order of expansion, therefore $\gamma \sim k$ for $k \rightarrow 0$, as was confirmed in our calculations. Moreover, we have attempted to construct a

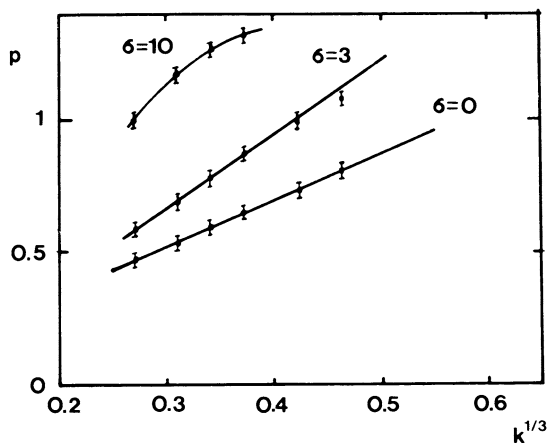


Fig. 11.2 Linear growth rate as a function of transverse perturbation wavenumber ($k^{1/3}$) for magnetic field strength: 0, 3, and 10

novel analytical perturbation scheme which for $k \rightarrow 0$ could correctly recover the results of the above numerical calculations. We look for a solution of (11.6) in a form of series expansion for a small k values [264],

$$\gamma = Ak^{1/3} + \dots, \quad (11.19)$$

$$g = g_0 + k^{2/3}g_{2/3} + k^{4/3}g_{4/3} + \dots \equiv g'_0 + g'_{2/3} + \dots,$$

$$f = k^{1/3} \left(f_{1/3} + k^{2/3}f_{3/3} + k^{4/3}f_{5/3} + \dots \right) \equiv f'_{1/3} + f'_{3/3} + f'_{5/3} + \dots$$

In the first order of the perturbation theory, we get

$$\frac{d}{dx}H_+ \frac{d}{dx}f'_{1/3} = \gamma \frac{d^2}{dx^2}g'_0. \quad (11.20)$$

Further, to successive orders one obtains

$$\begin{aligned} \frac{d}{dx}H_- \frac{d}{dx}g'_{2/3} &= -\gamma \frac{d^2}{dx^2}f'_{1/3}, \\ \frac{d}{dx}H_+ \frac{d}{dx}f'_{3/3} &= \gamma \frac{d^2}{dx^2}g'_{2/3}, \\ \frac{d}{dx}H_- \frac{d}{dx}g'_{4/3} &= -\gamma \frac{d^2}{dx^2}f'_{3/3}, \\ \frac{d}{dx}H_+ \frac{d}{dx}f'_{5/3} &= \gamma \frac{d^2}{dx^2}g'_{4/3}, \end{aligned} \quad (11.21)$$

where

$$\begin{aligned} H_+ &= \frac{d^2}{dx^2} - 1 + \frac{6}{\cosh^2 x}, \\ H_- &= \frac{d^2}{dx^2} - 1 + \frac{2}{\cosh^2 x} \end{aligned}$$

Finally, at the sixth order, we shall come across the terms proportional to k^2 .

$$\frac{d}{dx}H_- \frac{d}{dx}g'_{6/3} + k^2 \left[-\frac{d^2}{dx^2} + \sigma - 1 + \frac{2}{\cosh^2 x} \right] g_0 + \gamma \frac{d^2}{dx^2}f'_{5/3} = 0. \quad (11.22)$$

In that way, so far as $f'_{5/3} \sim k^5$, we have constructed the perturbation scheme for which $\gamma \sim k^{1/3}$. Although operators H_+ and H_- and their inverse counterparts are well defined, a fact that, in principle, should allow the derivation of $f'_{5/3}(x)$, the proof of the convergence of the above perturbation scheme is rather complex. A particular point lies in the fact that the expansion for a small k is justified only in the region $|x| \ll 1/k$. This is connected with a slow decay of the solution of the complete system (11.6) $f, g \sim \exp(-kx)$. Naturally, this was a reason why we try

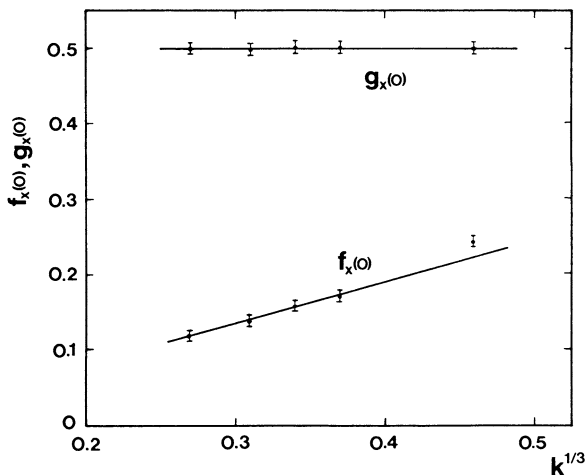


Fig. 11.3 Spatial derivatives of eigenfunctions (f, g) at $x = 0$ as a function of transverse perturbation wavenumber ($k^{1/3}$)

to check the proposed scheme upon our numerical calculations. It is evident from Fig. 11.2, that the dependence $\gamma \sim k^{1/3}$ is obeyed with a high degree of accuracy for small k values. The deviation from the above dependence, for large σ , comes from the fact that in this case, as $\gamma = f(k\sqrt{\sigma})$, the expansion is justified for substantially smaller values of k . In Fig. 11.3, we plot the dependence of $dg/dx|_{x=0}$ and $df/dx|_{x=0}$ on $k^{1/3}$. As expected from the expansion (11.19), $g_x(0) = \text{const.}$ and $f_x(0) = Bk^{1/3}$. In this way, our numerical results give strong support for the proposed perturbation scheme.

11.1.5 Nonlinear Stage of Soliton Instability

Further, we discuss a nonlinear stage of the soliton instability. In an isotropic plasma, in a linear regime, this instability results in a transverse modulation of the Langmuir soliton amplitude. The nonlinear growth of the perturbation can lead to a soliton breakup into a number of collapsing wave packets (Langmuir caviton) [248, 249]. Therefore, in our problem, it is expected that the magnetic field can substantially affect this nonlinear stage of the soliton instability (see [242–245, 271]).

In Fig. 11.4, we plot the eigenfunctions of the system (2.4) calculated above that correspond to the maximum linear growth rate for different values of the magnetic field. It is evident that the increase of σ does not bring a qualitative change in a structure of the growing perturbations. Therefore, it seems reasonable to expect that the magnetic field growth just increases the transverse length scales of the wave packet leaving all basic qualitative features of the collapse process preserved.

In order to investigate the soliton instability, in particular, in its highly nonlinear stage, we have further performed a direct numerical simulation of (11.3) and (11.4)

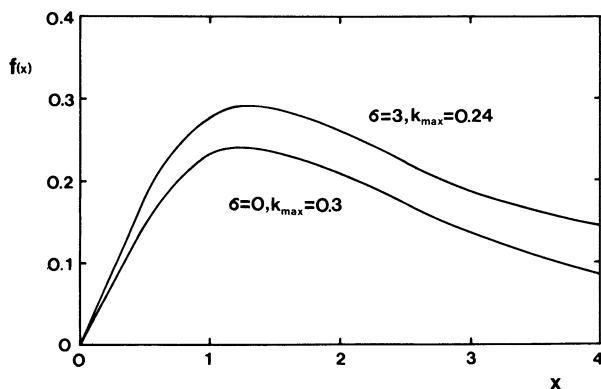


Fig. 11.4 The eigenfunctions (f, g) for $\sigma = 0$ and $\sigma = 3$

in two dimensions ($2D$). We have used the spectral Fourier method with respect to the space coordinates with an explicit time integration scheme. The instability development was studied by imposing initial conditions in a form of the standing planar soliton (2.3) periodically perturbed in a transverse direction. For sufficiently large values of σ , computations based on (2.1) and (2.2) produce very close results. Moreover, direct simulation results are checked upon those obtained in the previous section. In the initial stage, when instability exhibits an exponential growth of the perturbation, we have chosen sufficiently small initial perturbation levels. We have used periodic boundary conditions (L_x, L_y) , a numerical grid of 64×64 (checked upon 128×128 points, a time step of 0.001 and the perturbation level $\epsilon = 0.01$, with regularly checking the conserved integrals of (11.3) and (11.4), i.e., the plasmon number (N) and the Hamiltonian (H). In contrast to, e.g., a work of Pereira et al. [248], our perturbation level ϵ was decreased [264] until the growth rate γ has become independent on ϵ . We have performed runs with different values for k and σ . However, in the range of small perturbation wave numbers ($k < 0.2$), a care must be taken, since the grid resolution can be insufficient for accurate calculations. In Fig. 11.5, we compare the direct simulation results for $\gamma(k)$ with the results of numerical solutions of the eigenvalue problem (11.6). As seen on inspection, the results of these two essentially independent methods of solution show satisfactory agreement. In particular, the results coincide for the values near to the cutoff wave numbers.

The direct simulation results have confirmed that all linearly unstable solitons, independent of the magnetic field strength and the perturbation wave number, in their nonlinear stage enter a collapse phase. This result is consistent with, e.g., work of Goldman et al. [242, 243] related to the collapse of Langmuir wave packets in a weak magnetic field with full ion dynamics. In order to illustrate this, we show typical time snapshots in Fig. 11.6, of the early collapse, which exhibit the basic collapse features: the initial localization, subsequent explosive amplitude growths connected with a rapid contraction of the spatial dimensions resulting in large wave

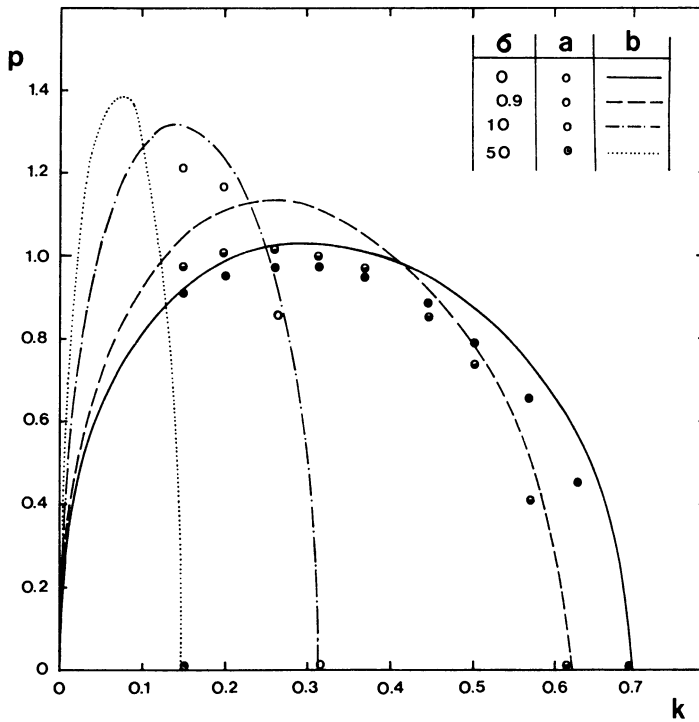


Fig. 11.5 Linear growth rate (p) versus transverse perturbation wavenumber (k) for different magnetic field strengths (σ): 0, 0.9, 10, and 50. (a) direct 2D simulation, (b) solution of the spectral problem

energy density. The earlier results on soliton break up and subsequent collapse in an unmagnetized plasma [248] were readily recovered in our simulations for $\sigma = 0$.

An important characteristic of the collapsing wave packet is its elongation (Fig. 11.7) i.e., the aspect ratio l_x/l_y . This quantity defines the energy content trapped inside the cavity, which is of considerable importance concerning the final collapse stage and the ultimate energy dissipation. Generally taken, there possibly exist solutions with various degrees of elongation [250,264], that are also depending on the way they were initially formed.

11.1.6 Self-Similarity and Collapse Regimes

Let us further discuss a highly nonlinear, developed stage of the collapse. A general scenario was proposed in the early work of Krasnosel'skikh and Sotnikov [271] which was based on an analytical study of a version of (11.4), which takes into account a full ion inertial response. The above is necessary in the so-called super-sonic regime ($E_0^2/8\pi nT > m/M$) of the magnetized Langmuir wave collapse.

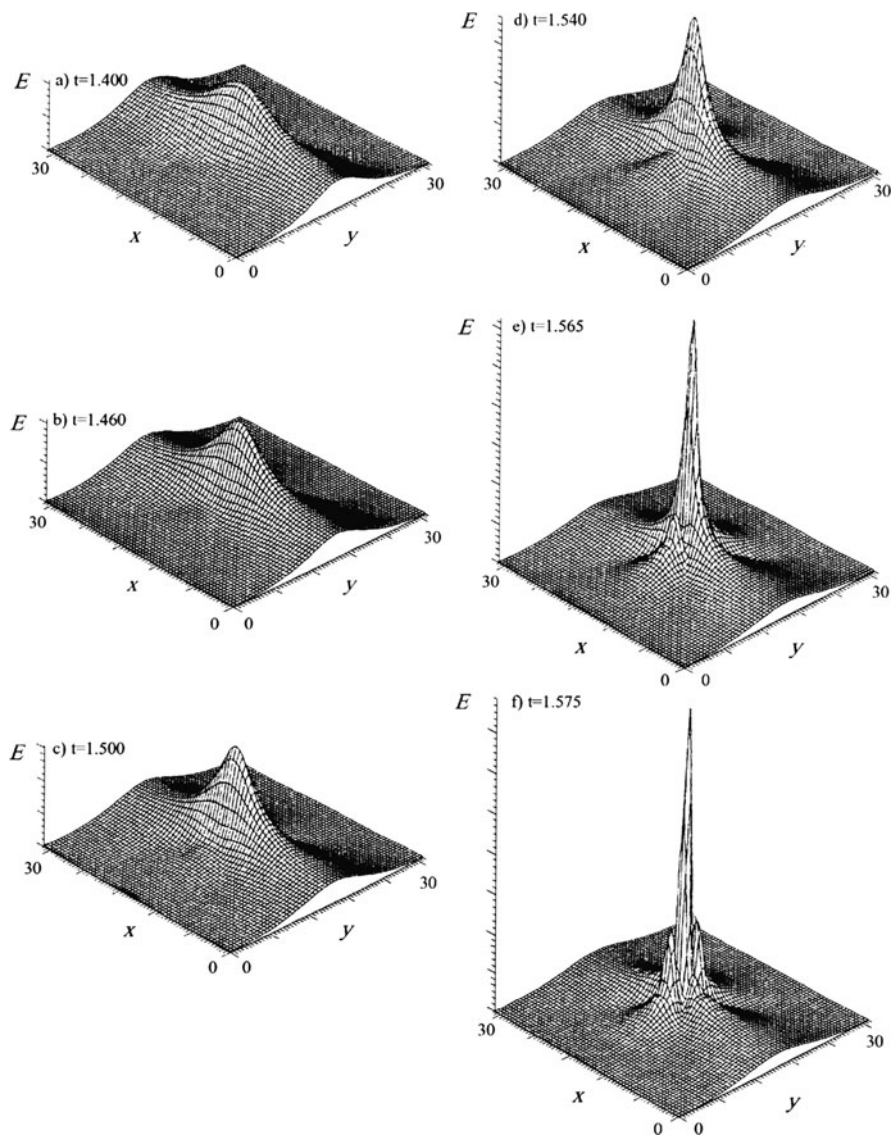


Fig. 11.6 Time snapshots of the 2D soliton electric field amplitude ($E(x, y)$) during Langmuir collapse. Initial soliton strength is $\lambda = 5$ and $\sigma = 10$

In the early stage of the soliton collapse, as long as $\omega_{ce}/\omega_{pe} \gg kr_{De}$, the transverse dimensions of the collapsing wave packet are substantially larger than the longitudinal ones, forming highly elongated, dipole field structures. During the collapse process, the transverse length scale of the cavity decreases more rapidly than the longitudinal one and therefore, when $kr_{De} \sim \omega_{ce}/\omega_{pe}$, two scales become of

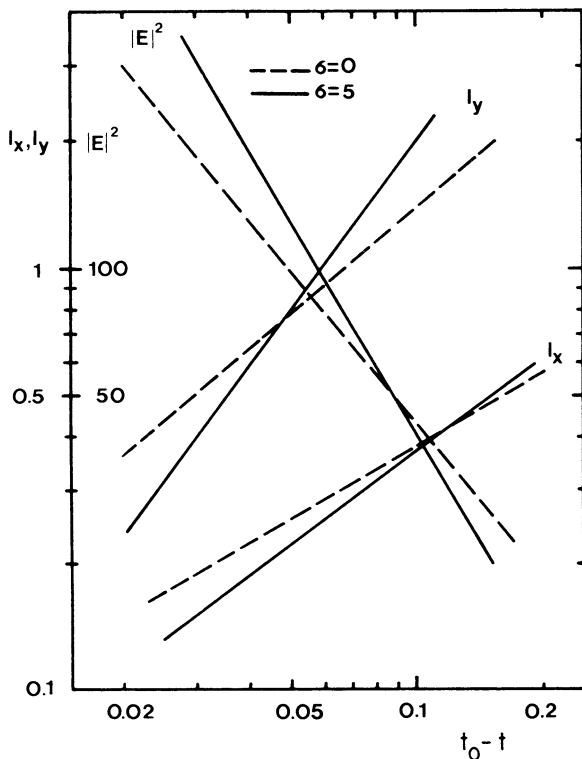


Fig. 11.7 Characteristic spatial scales (l_x, l_y) and the maximum electrostatic energy density ($|E|^2$) as a function of a time interval to the collapse time ($t - t_0$). Dashed lines denote $\sigma = 0$ and solid line denote $\sigma = 5$

the same order. Accordingly, in the later stage, the magnetic field was not expected to influence the collapse development [271]. However, it is still believed that in the final collapse stage, the magnetic field can possibly make an effect on the cavity structure and its energy content. Indeed, our simulations seem to point in the same direction (vide infra), i.e., that in the final stage, the cavity form and the trapped energy depend on the magnetic field strength.

Further, we investigate the possible self-similar time behavior of the wave collapse process in its developed stage, following Hadžievski et al., [264]. Studies of the wave collapse have shown that a collapse process, as a unique nonlinear phenomenon of the formation of a singularity in a finite time, can be developed through two different collapse regimes: *weak* and *strong*. Originally, Zakharov and Kuznetsov [267] for the NLS equation, followed by Kuznetsov and Škorić [266] for the nonlinear upper-hybrid and lower-hybrid waves have shown that during the strong collapse regime the trapped wave energy through the collapse stage remains finite and the wave radiation from the cavity is absent. On the other hand, in the weak regime, which formally preserves zero energy into the final collapse stage,

wave radiation is present. In the framework of (11.4), it has been shown that two regimes, weak and strong, exist. Both regimes near the singularity can be described by a general self-similar ansatz in the form [264]

$$\psi(\mathbf{r}, t) \rightarrow \frac{1}{(t_0 - t)^{a+ip}} f \left[\frac{z}{(t_0 - t)^b}, \frac{\mathbf{r}_\perp}{(t_0 - t)^c} \right], \quad (11.23)$$

where a, b, c , and p are the real parameters. This means that the cavity dimensions scale as

$$l_z \simeq (t_0 - t)^b, \quad l_\perp \simeq (t_0 - t)^c. \quad (11.24)$$

The number of waves localized in the cavity (N^{cav}) depends on time (in two dimensions) as

$$N^{\text{cav}} = \int \left| \frac{\partial \psi}{\partial x} \right|^2 d\mathbf{r} \sim (t_0 - t)^{2a-b+c}, \quad 2a - b + c \geq 0, \quad (11.25)$$

where for a strong collapse, N^{cav} should remain constant, while for a weak collapse, N^{cav} goes to zero as collapse reaches the singularity.

We shall return to the detailed classification of strong and weak collapse regimes in the next section, while here we just focus on the early collapse stage for $\omega_{ce}/\omega_{pe} \gg kr_{De}$. In this case, a simplified version of the nonlinear equations (11.4) is appropriate which accepts the following self-similar substitution [271]

$$\psi(\mathbf{r}, t) \rightarrow \frac{1}{(t_0 - t)^{ip}} f \left[\frac{z}{(t_0 - t)^{1/2}}, \frac{\mathbf{r}_\perp}{(t_0 - t)} \right], \quad (11.26)$$

where the corresponding cavity dimensions scale like

$$l_z \simeq (t_0 - t)^{1/2}, \quad l_\perp \simeq (t_0 - t).$$

However, the above self-similar ansatz describes the *weak* collapse process with a cavity plasmon number (N^{cav}) which decreases in time as

$$N^{\text{cav}} = \int \left| \frac{\partial \psi}{\partial x} \right|^2 d\mathbf{r} \sim (t_0 - t)^{1/2},$$

in the $2D$ case.

The numerical simulation of an axially symmetric version of the Krasnosel'skikh equation presented by Lipatov [250] supported the above general picture. However, conclusions concerning the late collapse stage, when the disappearing magnetic field effect supposingly switches the collapse to an isotropic type of evolution, are somewhat of a speculative nature.

In order to check the self-similar character of the collapse evolution consistent with (11.23) based on our 2D numerical simulation results, we vary t_0 to find the best fit for the time evolution of the maximum electric field amplitude, for different values of the magnetic field $\sigma = 0, 5, 10$, and 15. From (11.23), the maximum electric field amplitude squared, scales like

$$|E_{\max}|^2 \sim \frac{1}{(t_0 - t)^\alpha}, \quad \alpha = 2a + 2b. \quad (11.27)$$

Our results indicate that the self-similar evolution is exhibited also for smaller values of σ including $\sigma = 0$, with a slope $\alpha = 1.2$, which is in agreement with the results of Pereira et al. [248]. For $\sigma = 15$, they come close to the analytical predictions [266] for ($\alpha = 2$) based on (11.4). In Fig. 11.7, we plot in a double-logarithmic scale, the time variation of the maximum amplitude (11.27) and characteristic spatial dimensions of the collapsing cavity for $\sigma = 0$ and $\sigma = 5$. The self-similar behavior is evident, although with a changing slope:

$$\alpha_0 \simeq 1.20 \quad (\sigma = 0), \quad \alpha_5 \simeq 1.74 \quad (\sigma = 5).$$

Transverse length scales grow faster than longitudinal ones, resulting in that the initial dipole field structure tends to symmetrize. The above process gets more pronounced with the magnetic field increase. By calculating the parameters a , b , and c from Fig. 11.7, we readily find the time dependence of the plasma number (N^{cav}), which based on (11.25) turns out to decrease in time, as

$$N_0^{\text{cav}} \sim (t_0 - t)^{0.20}, \quad N_5^{\text{cav}} \sim (t_0 - t)^{0.36},$$

which defines a *weak* type of collapse.

However, the self-similar features of the described collapse processes, were studied in a time interval restricted to an increase of the energy density of just up to two orders of magnitude. Therefore, the later stages of the collapse, closer to the singularity, could exhibit somewhat a altered type of dynamics. As a general picture, our simulations indicate an early collapse development corresponding to a weak collapse regime. Possibly, this comes from the fact that in order to approach other, the so-called (ultra) strong collapse regime, much larger inertial interval seems necessary.

The energy content of the collapsing wave structure is of a considerable physical interest. Namely, the results indicate an altered effective absorption rate, i.e.,

$$[(k_0 r_{\text{De}}) / (\omega_{\text{ce}} / \omega_{\text{pe}})]^{1/2} \sim (E_0^2 / 8\pi nT)^{1/2} \omega_{\text{pe}} / \omega_{\text{ce}},$$

times the absorption rate for an isotropic plasma, corresponding to the increased level of the strong Langmuir turbulence. Still, a need for an experimental insight in such physical situation is of a great importance. Subsequently, it was understood that the ultimate “burn out” process of the collapsing wave packets is expected to

dissipate a finite amount of the wave energy. Namely, the structure of the weak collapse (conserving zero energy) seemed to indicate that the final stage should allow only an infinitely small value of the dissipated energy. However, in reality one expects a different situation. Further refinements in the wave-collapse theory [239] have indicated a possibility of a collapse process formed as a long-living spatially localized narrow core (hot spot) close to a singularity, which entrains the wave energy from the surrounding region (“black hole” effect). Such an effect was variably named as a “funnel effect” [239], “nucleation” or “distributed collapse” [252]. Accordingly, in our case, the solution (11.23) could supposedly be valid only in a thin region close to the singularity. At larger distances, different type of a solution could be formed providing a continuous energy influx into the singularity. More simply, in such a weak collapse dynamics a type of a “funnel” is formed entraining the energy from the outer regions. For the (11.6) this funnel looks as an anisotropic structure. Anisotropic bi-self-similar collapse solution [251] describing a finite energy capture into an anisotropic funnel has been also predicted.

11.2 Virial Theory of Wave Collapse

The virial theorem of collapse of the large amplitude Langmuir wavepacket is proved, in a model of Zakharov equations which in the static limit reduce to the nonlinear Schroedinger equation (see, an extensive review, [270]). The problem of plasma turbulence is of interest both from a theoretical point of view and from an experimental one for laboratory, fusion, and astrophysical plasmas (see, reviews [218, 254, 255], and references therein). The simplest case is that of Langmuir turbulence in an unmagnetized plasma which has been studied extensively, especially the transition from weak to strong turbulence in which the parametric modulational instability (MI), first suggested by Vedenov and Rudakov [75] and by Gailitis [257], plays a major role. In two- or three-dimensional plasmas, the MI ultimately leads to Langmuir collapse [215, 258], that is to the formation of cavitons – localized intense Langmuir wavepackets in regions of lowered plasma density – which appear unstable and can possibly collapse to the physical dimensions of a few Debye radii, when Landau damping becomes important. It should be borne in mind, however, that the Zakharov equations which describe the collapse, cease to be valid much before this final stage is reached. These equations are usually derived [215, 217, 226] on a two-fluid and two-timescale basis and they describe the evolution of nonlinear Langmuir waves coupled to ion sound by the ponderomotive force. It is the present aim to discuss the collapse dynamics of Langmuir waves and, by means of a general virial theorem, presented by Goldman and Nicholson [260], prove the condition for collapse existence. We only give an outline of our reasoning; with full details available elsewhere [188, 215, 217, 260, 261].

We consider a hot, collisionless, uniform, nonisothermal ($T_e \gg T_i$) plasma, for nonlinear Langmuir modes with frequencies $\sim \omega_0$ close to ω_{pe} . We follow the procedure by using the standard time-averaging over the fast Langmuir wave period

scale [188, 215, 217] to derive the following set of Zakharov equations for the Langmuir waves nonlinearly coupled to ion sound waves, which in the vector form, read [188, 215],

$$i \frac{\partial \mathbf{E}}{\partial t} + \frac{3}{2} \omega_0 r_{\text{De}}^2 \nabla (\nabla \cdot \mathbf{E}) - \left(\frac{c^2}{2\omega_0} \right) [\nabla \times [\nabla \times \mathbf{E}]] = \frac{1}{2} \omega_0 \left(\frac{\Delta n}{n_0} \right) \mathbf{E}, \quad (11.28)$$

$$\frac{\partial^2 \Delta n}{\partial t^2} - c_s^2 \nabla^2 \Delta n = \nabla^2 \frac{|\mathbf{E}|^2}{16\pi\pi M},$$

where $\mathbf{E}$ is the slowly-varying envelope of the fast-timescale electric field and Δn is the slow-timescale density variation.

Further, we restrict ourselves to a simplified version corresponding to weak amplitude case, which is valid under assumption of adiabatic ion response (neglecting the term $\partial^2 \Delta n / \partial t^2$ in the second equation) and potential (electrostatic) field $\mathbf{E}$, i.e., $(\nabla \times \mathbf{E} = 0)$, to obtain the following nonlinear Schroedinger type of equation

$$i \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{2} \nabla (\nabla \cdot \mathbf{E}) + |\mathbf{E}|^2 \mathbf{E} = 0, \quad (11.29)$$

where we have introduced, as usual, dimensionless units: $t \rightarrow t/\omega_0$, $r \rightarrow r r_{\text{De}} \sqrt{3}$, $\mathbf{E} \rightarrow (32\pi n_0 T_e)^{1/2} \mathbf{E}$.

From NLSE we can derive by the standard linear parametric instability analysis, the growth rate for the modulational instability (MI) [188, 215]. To discuss the fully nonlinear evolution, self-focusing and collapse of a Langmuir wave packet we now follow Goldman and Nicholson [260] (see also Kono et al. [259]). We first of all note that (11.29) can be derived from a Langrangian density $\mathcal{L}$, given by the formula (cf. Gibbons et al. [229])

$$\mathcal{L} = \frac{1}{2} i \left[\mathbf{E}^* \frac{\partial \mathbf{E}}{\partial t} - \mathbf{E} \frac{\partial \mathbf{E}^*}{\partial t} \right] - \frac{1}{2} |(\nabla \cdot \mathbf{E})|^2 + \frac{1}{2} |\mathbf{E}|^4. \quad (11.30)$$

From the Langrangian density, we readily get the energy and momentum conservation laws, in a form given in [259, 260].

$$\frac{\partial |\mathbf{E}|^2}{\partial t} + (\nabla \cdot \mathbf{s}) = 0, \quad \frac{\partial \mathbf{p}}{\partial t} + \left(\nabla \cdot \overset{\leftrightarrow}{\mathbf{T}} \right) = 0,$$

where now

$$\mathbf{p} = \frac{1}{2} i [(\mathbf{E} \cdot \nabla) \mathbf{E}^* - (\mathbf{E}^* \cdot \nabla) \mathbf{E}] = \mathbf{s}$$

$$T_{ij} = \frac{1}{2} [(\nabla \cdot \mathbf{E}) \nabla_i E_j^* + \text{c.c.}] - \frac{1}{2} \Delta_{ij} [\nabla \cdot \text{Re} \{ \mathbf{E} (\nabla \cdot \mathbf{E}) \}] + |\mathbf{E}|^4.$$

Proceeding as in [259, 260], we define the root mean square spatially averaged width $\langle \Delta r^2 \rangle^{1/2}$ ($\Delta r^2 \equiv |\mathbf{r} - \langle \mathbf{r} \rangle|^2$) of a localized wavepacket (i.e., Langmuir caviton)

(using as a probability weighting function $|\mathbf{E}|^2/\mathcal{N}$, where $\mathcal{N}$ is the plasmon number) to get the Ehrenfest theorem,

$$\frac{\partial}{\partial t} \langle \Delta \mathbf{r} \rangle \equiv \int \left(|\mathbf{E}|^2 / \mathcal{N} \right) \mathbf{r} d^3 \mathbf{r} = \frac{\mathbf{P}}{\mathcal{N}} = \text{const.} \quad (11.31)$$

Accordingly, it is straightforward to prove the general *virial theorem*

$$\frac{\partial^2}{\partial t^2} \langle \Delta r^2 \rangle = \left[\frac{2\mathcal{H}}{\mathcal{N}} - \frac{\mathcal{P}^2}{\mathcal{N}^2} \right] - \int |\mathbf{E}|^4 d^3 \mathbf{r}. \quad (11.32)$$

By integrating twice the above equation, we find

$$\langle \Delta r^2 \rangle = At^2 + Bt + C - (2 - D) \int_0^t dt_1 \int_0^{t_1} I(t_2) dt_2, \quad (11.33)$$

where, D is the number of spatial dimensions, A is a constant of motion and B and C are integration constants,

$$I(t) = \int |\mathbf{E}|^4 d^3 \mathbf{r}, \quad A = 2\mathcal{H}/\mathcal{N} - \mathcal{P}^2/\mathcal{N}^2, \quad (11.34)$$

$$\mathcal{H} = \frac{1}{2} \int \left[|(\nabla \cdot \mathbf{E})|^2 - |\mathbf{E}|^4 \right] d^3 \mathbf{r}, \quad \mathbf{P} = \int \mathbf{p} d^3 \mathbf{r}, \quad \mathcal{N} = \int |\mathbf{E}|^2 d^3 \mathbf{r}.$$

We note that if initially $A < 0$, for the dimensionality $D \geq 2$, then $\langle \Delta r^2 \rangle$ will collapse to zero in a finite time, or physical collapse will be stopped at few r_{De} , due to Landau damping. Of course, even before such a value is reached, above model equations cease to be valid [226].

11.3 Hierarchy of Collapse Regimes in a Magnetized Plasma

A collapse classification for upper-hybrid and lower-hybrid waves in a weakly magnetized plasma is presented. It is proved that in these nonlinear systems, three-dimensional soliton solutions do not exist. Further, it is demonstrated that the basic criterion for the existence of the wave collapse is the unboundness of the Hamiltonian from below due to nonlinear terms. Finally, we show that there exists a hierarchy of wave collapse regimes, starting from a *weak-collapse* case which formally preserves zero wave energy into the collapse stage, and concluding with the *strong-collapse*, when the trapped Langmuir wave energy remains finite.

11.3.1 Introduction

A theory of wave collapse reveals that the collapse [215, 218, 219, 267], as a known phenomenon of the formation of a singularity in a finite time, often appears in a multidimensional system, although a soliton solution is often found in an one-dimensional case. Such behavior is found to be due to the growing influence of nonlinear effects with an increase of spatial dimensionality. A similar situation takes place in the theory of phase transitions, where the phase transition is forbidden in a low-dimensional system although it is allowed for higher dimensional cases. In a seminal paper by Zakharov and Kuznetsov [267], followed by Kuznetsov and Škorić [266], it was shown that for the nonlinear Schroedinger equation (NLSE) [270] in three dimensions (3D) there exist a hierarchy of wave collapse regimes, starting from “weak” collapse, i.e., the most rapid regime described by the self-similar solution of the NLSE, which formally conserves zero energy into the final stage, and concluding with “strong” wave collapse in which the wave energy remains finite. Corresponding results were obtained in a work [266], which studied a different type of wave collapse in a weak dispersive medium, i.e., magnetosonic wave collapse. This situation differs from the Langmuir collapse because of a different nonlinearity of a hydrodynamic type.

In this section, we present the collapse classification for upper-hybrid (UH) and lower-hybrid (LH) waves in a weakly magnetized plasmas [266], i.e., when the electron plasma frequency ω_{pe} is larger than the electron cyclotron frequency ω_{ce} , namely, $\omega_{pe} \gg \omega_{ce}$. First, for upper-hybrid waves, we take into account only the influence of the magnetic field on the dispersion law and use the model equation of [271]. As the lower-hybrid collapse is concerned, we describe it with the help of a self-consistent system of equations, originally derived in [263]. Further, we shall show that in these nonlinear systems, three-dimensional soliton solutions do not exist. The reason, as will appear, is connected with a type of nonlinearity which is stronger than, for instance, that in the NLSE or in the Zakharov system [215], which describes the collapse of the nonlinear Langmuir waves in an unmagnetized plasma.

11.3.2 Model Equation

First, let us consider upper-hybrid waves in a weak magnetic field ($\omega_{pe} \gg \omega_{ce}$). The dispersion law is given in the form

$$\omega(k) = \omega_{pe} \left[1 + \frac{3}{2} k^2 r_{De}^2 + \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}^2} \frac{k_{\perp}^2}{k^2} \right],$$

where, r_{De} is the Debye radius, $k_{\perp}$ is the wavenumber component transverse to the magnetic field direction, and $\mathbf{B}_0$ is directed along the z -axis. In the dispersion relation [271], the first term describes the longitudinal electron plasma wave oscillations. Other terms are due to slower dispersive processes.

The nonlinear effect, in a weak amplitude region ($E^2/8\pi nT_e \equiv \overline{W} \ll 1$), appears to be slow with respect to fast oscillations at the plasma frequency. Therefore, we can obtain the envelope equation for the amplitude of *h.f.* oscillations, which in the dimensionless units has the form [271]

$$\Delta(i\psi_t + \Delta\psi) - \sigma\Delta_\perp\psi + \nabla \cdot (|\nabla\psi|^2 \nabla\psi) = 0, \quad (11.35)$$

where ψ is the slowly varying envelope of *h.f.* waves and $\sigma \equiv \frac{1}{2} \frac{\omega_{ce}^2}{\omega_{pe}^2}$. Here, we assume that $\overline{W} \ll m/M$, which means that the low-frequency density variations follow adiabatically the ponderomotive pressure induced by high-frequency waves,

$$n = -|\nabla\psi|^2. \quad (11.36)$$

Equation (11.35) can be further reduced under additional assumptions. Namely, it is known that due to weakly turbulent processes, such as induced scattering on ions or four-wave interaction, the energy transfers by cascades and condenses in the region $\omega_k \rightarrow \omega_{pe}$. In more details, it appears that these nonlinear interactions lead, in the first place, for waves with $k^2 r_{De}^2 \ll \sigma$, to a rapid decrease in $k_\perp$, and only subsequently, to a reduction of k_z down to a zero value. It means that the wave condensate must have characteristic longitudinal scales much smaller than the transverse ones, i.e., $k_\perp \ll k_z$. Under this assumption, (11.35) reduces to the following form (compare with [271])

$$\frac{\partial^2}{\partial z^2} \left[i\psi + \frac{\partial^2}{\partial z^2} \psi \right] - \Delta_\perp \psi + \frac{\partial}{\partial z} \left[\left| \frac{\partial \psi}{\partial z} \right|^2 \frac{\partial \psi}{\partial z} \right] = 0, \quad (11.37)$$

where without restriction we can put $\sigma = 1$ (related to a simple scaling transformation).

Analogous equations arise for *LH* waves near the LH frequency ω_{LH} . The dispersion relation for *LH* waves can be found, as

$$\omega(k) = \omega_{LH} \left[1 + k_\perp^2 R^2 + \frac{1}{2} \frac{m}{M} \frac{k_z^2}{k_\perp^2} \right],$$

where $R^2 \equiv \left[\frac{3}{4} + 3T_i/T_e \right] r_{ce}$, for $\omega_{ce} \ll \omega_{pe}$, while m and M are the electron and ion mass, respectively.

For weak intensity, as for the *UH* waves, the low-frequency density variation is related to the *h.f.* ponderomotive pressure [263], through

$$n = i |\nabla_\perp \psi \times \nabla_\perp \psi^*|_z. \quad (11.38)$$

Here, as in the *UH* case, we write (11.38) in dimensionless units; ψ stands for the slowly varying envelope of the *h.f.* electric potential, and $\times$ denotes the vector

product. The evolution equation for the envelope ψ can be found with the help of standard time-averaging over the high-frequency ω_{LH} , [215, 263].

$$\Delta_{\perp} (i\psi_t + \Delta_{\perp} \psi) - \nabla_{\perp} \cdot (|\nabla_{\perp} \psi \times \nabla_{\perp} \psi^*|_z |\hat{n} \times \nabla_{\perp} \psi|) - \phi \psi_{zz} = 0, \quad (11.39)$$

where the unit vector $\hat{n} \equiv \mathbf{B}_o/B_o$, and ϕ is a constant.

11.3.3 Nonexistence of Three-Dimensional Solitons

Now, let us study properties of (11.37) and (11.39) in more detail [219, 266]. As it is known, both equations are of the Hamiltonian type [262]. Accordingly, (11.37) can be represented in the form

$$i \frac{\partial^2}{\partial z^2} \psi_t = - \frac{\Delta H_1}{\Delta \psi^*}, \quad (11.40)$$

$$H_1 = \int \left(|\psi_{zz}|^2 + |\nabla_{\perp} \psi|^2 - \frac{1}{2} |\psi_z|^4 \right) d\mathbf{r},$$

while 11.39 is given by

$$i \Delta_{\perp} \psi_t = - \frac{\Delta H_2}{\Delta \psi^*}, \quad (11.41)$$

$$H_2 = \int \left(|\Delta_{\perp} \psi|^2 + |\psi_z|^2 + \frac{1}{2} |\nabla_{\perp} \psi \times \nabla_{\perp} \psi^*|^2 \right) d\mathbf{r}.$$

Besides H , the above nonlinear systems also conserve the plasmon number N : where for (11.40), $N_1 = \int |\nabla_z \psi|^2 d\mathbf{r}$, and for (11.41), $N_2 = \int |\nabla_{\perp} \psi|^2 d\mathbf{r}$. In both cases, N coincides up to a constant, with the wave energy. Equations (11.40) and (11.41) also possess conservation laws for the linear momentum and the longitudinal component of the angular momentum.

The possible stationary solutions of these equations should correspond to a soliton-like solution,

$$\psi = \psi_0(\mathbf{r}) \exp(i\lambda^2 t),$$

spatially localized, vanishing at infinity, with $\psi_0(\mathbf{r})$ being defined in the stationary limit of (11.40) and (11.41), as follows

$$\frac{\partial^2}{\partial z^2} \left[-\lambda^2 \psi_0 + \frac{\partial^2}{\partial z^2} \psi_0 \right] - \Delta_{\perp} \psi_0 + \frac{\partial}{\partial z} (|\psi_{0z}|^2 \psi_{0z}) = 0, \quad (11.42)$$

and

$$\Delta_{\perp} (-\lambda^2 \psi_t + \Delta_{\perp} \psi) - \nabla_{\perp} \cdot (|\nabla_{\perp} \psi \times \nabla_{\perp} \psi^*|_z |\hat{n} \times \nabla_{\perp} \psi|) - \phi \psi_{zz} = 0, \quad (11.43)$$

where, for convenience the subscript “0” has been dropped. On the other hand, it is easy to verify that the above equations can be represented in the following variational form [262]:

$$\Delta (H + \lambda^2 N) = 0, \quad (11.44)$$

that can be readily checked on inspection. Such a representation means that the soliton could be defined as the stationary point of H for fixed N , where λ^2 plays the role of the Lagrange multiplier.

For (11.42) and (11.43) and their variational representations, it is easy, for the soliton solution, to find relations between integrals I_{i1} , I_{i2} , and I_{i3} , which are constituent parts of the Hamiltonian H_i . For that purpose, we keep the following notation: $i = 1$, stands for UH , while $i = 2$, corresponds for LH . Therefore, for the UH system

$$\begin{aligned} I_{11} &= \int |\psi_{0z}|^2 d\mathbf{r}, \\ I_{12} &= \int |\nabla_{\perp} \psi_0|^2 d\mathbf{r}, \\ I_{13} &= \int |\psi_{0z}|^2 d\mathbf{r}, \end{aligned}$$

while for the second, LH system

$$\begin{aligned} I_{21} &= \int |\Delta_{\perp} \psi_0|^2 d\mathbf{r}, \\ I_{22} &= \int |\psi_{0z}|^2 d\mathbf{r}, \\ I_{23} &= - \int |\nabla_{\perp} \psi_0 \times \nabla_{\perp} \psi_0^*|^2 d\mathbf{r}. \end{aligned}$$

By definition, all above integrals are positive. Further, to check the existence of soliton solutions, we multiply (11.42) by ψ_0^* and integrate over the whole space. One gets

$$\lambda^2 N_1 + I_{11} + I_{12} - I_{13} = 0. \quad (11.45)$$

Equation (11.45) gives the first relation between I_{11} , I_{12} , and I_{13} . The other relations can be found with the help of (11.44). Let us further assume the trial function of a type $\psi = \psi_0(\alpha z, \beta \mathbf{r}_{\perp})$. It is evident that

$$\frac{\partial}{\partial \alpha} (H + \lambda^2 N) \big|_{\alpha=\beta=1} = \frac{\partial}{\partial \beta} (H + \lambda^2 N) \big|_{\alpha=\beta=1} = 0.$$

Indeed, e.g., for UH waves, we can readily prove the above formula, by

$$\begin{aligned}
& \frac{\partial}{\partial \alpha} (H + \lambda^2 N) \\
&= \int \left\{ \left(\frac{\partial \psi^*}{\partial \alpha} \psi_{zz} + \frac{\partial \nabla_{\perp} \psi^*}{\partial \alpha} \nabla_{\perp} \psi - \frac{\partial \psi_z^*}{\partial \alpha} \psi_z^2 \psi_z^* + \lambda^2 \frac{\partial \psi_z^*}{\partial \alpha} \psi_z + \text{c.c.} \right) \right\} \mathbf{dr} \\
&= \int \left\{ \frac{\partial \psi^*}{\partial \alpha} \left(\psi_{zzzz} - \nabla_{\perp}^2 \psi + \frac{\partial}{\partial z} (|\psi_z|^2 \psi_z) \right) + \text{c.c.} \right\} \mathbf{dr} = 0,
\end{aligned}$$

for $\alpha = 1$.

Applying the above procedure for (11.42), we get

$$\begin{aligned}
3I_{11} - I_{12} - \frac{3}{2}I_{13} + \lambda^2 N_1 &= 0, \\
-2I_{11} + I_{13} - 2\lambda^2 N_1 &= 0.
\end{aligned} \tag{11.46}$$

After some simple algebra, based on (11.45) and (11.46), we can show that

$$I_{12} = -2\lambda^2 N_1 < 0, \tag{11.47}$$

which contradicts the sign of I_{12} , which is positive definite. This contradiction basically implies that for the (11.42), stationary soliton solutions do not exist. Moreover, by the similar procedure, applied to (11.43), one can get corresponding relations between integrals I_{21} , I_{22} , and I_{23} for LH case. They are

$$I_{21} + I_{22} - I_{23} + \lambda^2 N_2 = 0 \tag{11.48}$$

$$-I_{21} + I_{22} + \frac{1}{2}I_{23} - \lambda^2 N_2 = 0, \tag{11.49}$$

$$2I_{21} - 2I_{22} - I_{23} = 0. \tag{11.50}$$

It is easy to check again that relations (11.48)–(11.50) appear to be contradictory.

11.3.4 Necessary Condition for Wave Collapse

What is the reason for the nonexistence of 3D solitons in our models? As will be shown below, this fact is basically connected with more pronounced nonlinear effects, as compared to the type of NLSE, i.e., Zakharov equations (static limit) which follows from (11.35), after relaxing the external magnetic field effect ($\sigma = 0$). In order to illustrate, let us consider the Hamiltonian under a scaling transformation, which conserves the number of waves N . For the system (11.40), they are

$$\psi(z, \mathbf{r}_{\perp}) \rightarrow \frac{a^{1/2}}{b} \psi \left[\frac{z}{a}, \frac{\mathbf{r}_{\perp}}{b} \right], \tag{11.51}$$

while for (11.41)

$$\psi(z, \mathbf{r}_\perp) \rightarrow \frac{1}{a^{1/2}} \psi\left[\frac{z}{a}, \frac{\mathbf{r}_\perp}{b}\right]. \quad (11.52)$$

Under such a transformation, the corresponding Hamiltonian becomes the function of scaling parameters a and b .

$$H_1(a, b) = \frac{I_{11}}{a^2} + \frac{I_{12}}{b^2} a^2 - \frac{1}{2} \frac{I_{13}}{ab^2}, \quad (11.53)$$

and

$$H_2(a, b) = \frac{I_{21}}{b^2} + \frac{I_{22}}{a^2} b^2 - \frac{1}{2} \frac{I_{23}}{ab^2}. \quad (11.54)$$

Further, it is easy to conclude that the function $H_1(a, b)$ is unbounded from below. For this purpose, let us consider the parabolic family $b = \gamma^2$, where γ is a constant. For this kind of dependence, the first two terms in $H_1(a, b)$ have the same (self-similar) dependence,

$$\frac{I_{11}}{a^2} + \frac{I_{12}}{b^2} a^2 = \frac{1}{a^2} \left[I_{11} + \frac{I_{12}}{\gamma^2} \right]. \quad (11.55)$$

As for the third nonlinear term, we get

$$\frac{1}{2} I_{13} \frac{1}{ab^2} = \frac{1}{a^5} \left[\frac{I_{13}}{2\gamma^2} \right]. \quad (11.56)$$

Under inspection, the above two terms, i.e., the dispersion and nonlinear term, show that $H_1(a, b)$ is unbounded from below. It should be stressed that the unboundedness of Hamiltonian $H_1(a, b)$ from below is due to the nonlinear term which grows with a scale decrease, more rapidly than, for instance, in the NLSE case. This is the main reason for the nonexistence of 3D solitons. Generally taken, it is the unboundedness of the Hamiltonian that is the main signature of collapse in the known collapsing systems (see [262]). The collapse, from this point of view, corresponds to a falling down of a particle in a self-consistent potential well, when the falling time to the center of the well is finite (compare with the 3D NLSE, [262]).

Similarly, it is clear that for (11.41), $H_2(a, b)$ is also unbounded from below. Here one needs to consider the curve $b = \gamma a^2$. It is evident that for this case, dispersive terms have the same self-similar behaviour and that unboundedness is due to the nonlinear term,

$$\frac{1}{2} I_{23} \frac{1}{ab^2} = \frac{1}{a^5} \left[\frac{I_{23}}{2\gamma^2} \right]. \quad (11.57)$$

Clearly, the nonlinearity is the main cause of the collapse. Moreover, it has been shown [262] that the role of nonlinear effects tend to grow with an increase of spatial dimensionality.

11.3.5 Classification of Wave Collapse Regimes

While the unboundedness of the Hamiltonian is the necessary condition for the existence of wave collapse, still, it is not the sufficient one. On the other hand, the analogy with the free falling particle dynamics while useful and illustrative is somewhat oversimplified. Firstly, during a wave collapse, it is possible that, e.g., the leaked radiation of the low-amplitude waves plays a role of effective dissipation for the cavity; i.e., spatially localized compressed density region with an intense growing wave structure. At the first sight, the radiation process seems to slow down the collapse. However, in reality we have an opposite situation. Instead of halting the collapse, the wave radiation, from the cavity with negative H , actually speeds up the collapse process.

Let us consider an isolated cavity with characteristic scales $l_z(t)$ and $l_\perp(t)$ with $H < 0$, emitting small amplitude waves. It is easy to verify that, for example, for system (11.40), the following estimates take place [266]:

$$\max_{\mathbf{r}} \left| \frac{\partial \psi}{\partial z} \right|^2 \geq \frac{1}{2} \frac{|H|}{N}, \quad (11.58)$$

which is a consequence of the mean-value theorem applied to I_{13} ,

$$I_{13} = \int |\psi_z|^4 d\mathbf{r} \leq \max_{\mathbf{r}} |\psi_z|^2 \int |\psi_z|^2 d\mathbf{r},$$

which is valid for an arbitrary region. Here, we may recall that for (11.40), $|\psi_z|^2$ represents the wave energy density. What happens with a cavity when we take into account the effect of wave radiation? Due to conservation of H inside the cavity, H^{cav} evidently has to decrease; this means that, due to radiation, $H^{\text{cav}} \rightarrow -\infty$. Similarly, N as a positive definite, will decrease until it vanishes to zero. Thus, the ratio H/N becomes infinite and so does the maximum of the wave amplitude, which according to (11.58), goes to infinity. The similar estimate takes place for the lower-hybrid waves with

$$\max_{\mathbf{r}} |\nabla_\perp \psi|^2 \geq \frac{1}{2} \frac{|H|}{N}. \quad (11.59)$$

From (11.59), it also follows that the small amplitude wave radiation (leakage) promotes the collapse, and that the collapse becomes more rapid.

Now we come to another interesting question about the classification of the wave collapse regimes. Clearly, the nonlinearity is the main cause of the collapse; moreover, it was shown [262, 266] that basically there exist two main types of wave collapse. The first one, the so-called *strong* collapse, is the case when the captured wave energy through the collapse stage remains finite and the wave radiation is absent. The second regime, *weak* collapse, is the one when the wave radiation from a cavity is present.

Let us now assume that for UH waves near the singularity, $\psi(\mathbf{r}, t)$ exhibits a more general self-similar behaviour,

$$\psi(\mathbf{r}, t) \rightarrow \frac{1}{(t_0 - t)^{a+ip}} f \left[\frac{z}{(t_0 - t)^b}, \frac{\mathbf{r}_\perp}{(t_0 - t)^c} \right],$$

where a, b, c , and p are real constants. This means that the cavity size scales are

$$l_z \simeq (t_0 - t)^b, \quad l_\perp \simeq (t_0 - t)^c.$$

Moreover, one should note that based on the above, corresponding electric field components scale like

$$E_z \equiv \nabla_z \psi \simeq \frac{1}{(t_0 - t)^b} \psi, \quad \text{while} \quad E_\perp \equiv \nabla_\perp \psi \simeq \frac{1}{(t_0 - t)^c} \psi. \quad (11.60)$$

Substituting (11.60) in N_1 shows that the number of waves in a cavity depends on time. For (11.40) we have

$$N_1^{\text{cav}} \sim (t_0 - t)^{-2\sigma+b+2c}, \quad \sigma = a + b.$$

In an isolated cavity, the plasmon number can only decrease in time, so

$$-2\sigma + b + 2c \geq 0. \quad (11.61)$$

The equality sign in this expression corresponds to a strong collapse, and inequality to a weak one. Then, we substitute (11.60) into Hamiltonian H_1 . This gives

$$\begin{aligned} I_{11} &\sim (t_0 - t)^{-2b} N_1^{\text{cav}}, \\ I_{12} &\sim (t_0 - t)^{-2c+2b} N_1^{\text{cav}}, \\ I_{13} &\sim (t_0 - t)^{-2\sigma} N_1^{\text{cav}}. \end{aligned}$$

The collapse condition (11.58) leads to a second restriction on indices a, b, c ,

$$2\sigma \geq \max(2b, 2c - 2b). \quad (11.62)$$

Therefore, with condition (11.61) for σ , we have

$$b + 2c \geq 2\sigma \geq \max(2b, 2c - 2b). \quad (11.63)$$

Here the first equality, $2\sigma = b + 2c$, corresponds to the strong collapse. For this regime, it is easy to see that for a fixed value of index b we get $\max \sigma = \frac{5}{2}b$ and $c = 2b$. It means that ψ obeys, in this case, the following asymptotics:

$$\psi \sim \frac{1}{(t_0 - t)^{3/2b+ip}} f \left[\frac{z}{(t_0 - t)^b}, \frac{\mathbf{r}_\perp}{(t_0 - t)^{1b}} \right]. \quad (11.64)$$

All other types of collapse belong to the weak one. The most rapid case is realized when radiation of small amplitude waves is at maximum. Such a regime corresponds to the case, for fixed index b , of the minimum of index a . Simple calculation gives $\min a = 0$ and $c = 2b$, or

$$\psi \sim \frac{1}{(t_0 - t)^{0+ip}} f \left[\frac{z}{(t_0 - t)^b}, \frac{\mathbf{r}_\perp}{(t_0 - t)^{2b}} \right], \quad (11.65)$$

It is important to note that the asymptotics (11.65) represents the self-similar solution of (11.40), with $b = \frac{1}{2}$, given in [271]. For lower-hybrid wave collapse, we have the following restriction on the indices:

$$b/2 \geq a \geq b - 2c. \quad (11.66)$$

From these inequalities, we can similarly obtain the relations between indices a , b and c for the strong collapse and the most rapid weak collapse. For the strong *LH* collapse, simple calculations give

$$\psi \sim \frac{1}{(t_0 - t)^{b/2+ip}} f \left[\frac{z}{(t_0 - t)^b}, \frac{\mathbf{r}_\perp}{(t_0 - t)^c} \right], \quad (11.67)$$

where for fixed b , $b/4 \leq c \leq b/2$, and $\max c = b/2$. As for the most rapid weak collapse, we have

$$\psi \sim \frac{1}{(t_0 - t)^{0+ip}} f \left[\frac{z}{(t_0 - t)^b}, \frac{\mathbf{r}_\perp}{(t_0 - t)^{b/2}} \right], \quad (11.68)$$

which appears to be a self-similar substitution in (11.41) for $b = 1$, (see [256, 262]).

In conclusion, we have to underline that analytical determination of the concrete value of parameter b for the strong collapse case, for both considered models, remains open. It is possible that the value for b can be possibly found in the semiclassical limit [262].

11.4 Weak and Strong Langmuir Collapse

The self-similar evolution of weak and strong Langmuir collapse is studied by two-dimensional simulation of a soliton instability. The simulation is based on Zakharov's model of magnetized strong turbulence (UH waves) including ion dynamics. For the parameters considered, consistency with self-similar weak collapse regimes is found with no evidence of a strong Langmuir collapse [272].

11.4.1 Preliminaries

Both for laboratory and space plasmas with a large energy content, it is typical that a state is reached where nonlinear wave effects compete with the dispersion [268, 269]. In such a physical environment, spatially localized, soliton-like waveforms can be formed. These structures evolve rapidly from an arbitrary initial plasma state to determine the basic features of an emerging strong plasma turbulence [225]. In one dimensional system, solitons are mostly stable, but in real plasmas, as a rule, solitary structures often appear to be unstable with respect to perturbations in a transverse direction [262, 264]. In its nonlinear stage, this instability often leads to a soliton collapse, a unique nonlinear wave phenomenon of the formation of a singularity in a finite time. In the physical sense, wave collapse corresponds to wave breaking and particle acceleration, thus playing the role of an effective dissipation-heating process in a strongly turbulent plasma. Above analytical analyses of the wave collapse, based on a self-similar analysis, have revealed the hierarchy of collapse regimes. The basic distinction is between the weak collapse which formally brings the zero wave energy to the final collapse stage, and strong collapse where the initially trapped energy remains finite during the collapse [266]. In the rest, we present a numerical study in two spatial dimensions in order to check the existence of weak and strong Langmuir wave collapse and validity of self-similar solutions [266]. Our simulations are based on the above Zakharov model of strong Langmuir (UH wave) turbulence for a plasma in a weak magnetic field with the full ion dynamics.

11.4.2 Nonlinear Model Equations

As said, the simplest example of strong plasma turbulence, thoroughly studied by theory, simulation, and experiments, is the phenomenon of strong Langmuir turbulence (SLT), where the interacting modes are of the high-frequency Langmuir (UH) and low-frequency ion sound wave. Zakharov's model of SLT in a weakly magnetized plasma is given by two time-averaged dynamical equations [vide supra], which describe a nonlinear coupling between the Langmuir wave potential amplitude (ψ) and the ion density variation (n). In convenient dimensionless units,

$$\begin{aligned} t &\rightarrow \frac{3}{2}\mu\omega_{pe}^{-1}t, & r &\rightarrow \mu^{1/2}r_{De}r, & \psi &\rightarrow \frac{T}{e}\mu\sqrt{12}\psi, \\ n &\rightarrow \frac{4}{3}\mu n_0n, & \sigma &\rightarrow \frac{3}{4}\mu\left(\frac{\omega_{ce}}{\omega_{pe}}\right)^2, \end{aligned} \quad (11.69)$$

the system reads [272]

$$\begin{aligned} \nabla^2 (\mathbf{i} \psi_t + \nabla^2 \psi) - \sigma \nabla_{\perp}^2 \psi - \nabla (n \nabla \psi) &= 0, \\ n_{tt} - \nabla^2 n &= \nabla^2 |\nabla \psi|^2 \end{aligned} \quad (11.70)$$

where ω_{ce} and ω_{pe} are the electron cyclotron and the electron plasma frequency, respectively; μ is the ion to electron mass ratio, r_{De} is the Debye radius and T is the electron temperature in energetic units. The system (11.70) is derived under an assumption that $\sigma \ll \mu$, corresponding to the physical condition of a weak magnetic field (in x -direction) $\omega_{pe} \gg \omega_{ce}$. We note that for $\sigma = 0$, the system reduces to the original set of the curl-free Zakharov equations [262]. The vectorial form of (11.70) readily simplifies to a scalar model by replacing $(-\text{grad } \psi)$ by a scalar electric field $E(r)$. In the small amplitude (static ions) limit, the set further reduces to a single equation of the nonlinear Schroedinger (NLSE) type (see above, 11.36). However, for large Langmuir fields, the inclusion of the full ion inertia is essential. A stationary, spatially localized solution of the system (11.70) in the form of a “standing” planar (1D) soliton, for the external magnetic field in the x -direction, is given by

$$\begin{aligned} \psi_s &= \sqrt{2} \arctan [\sinh (\lambda x)] \exp (\mathbf{i} \lambda^2 t), \\ n_e &= -|\psi_{sx}|^2. \end{aligned} \quad (11.71)$$

The problem of the stability, nonlinear dynamics, and collapse of Langmuir solitons was presented earlier in detail [262, 264]. In a linear regime, an agreement between direct simulation and eigenvalue problem results has been obtained [264]. In the nonlinear regime, linearly unstable solitons exhibit the wave collapse. In its developed stage, Langmuir collapse is expected to follow the self-similar evolution [262]. We note that much works on the Langmuir collapse scaling were restricted to a simpler, static limit of (11.70). In distinction, we treat the full set of (11.70) accounting for the ion dynamics, that is important for large amplitude Langmuir solitons. General self-similar solution for the Langmuir potential was proposed in a form [262]

$$\psi(\mathbf{r}, t) \rightarrow \frac{1}{(t_0 - t)^{a+ip}} f\left(\frac{x}{(t_0 - t)^b}, \frac{r_{\perp}}{(t_0 - t)^c}\right), \quad (11.72)$$

where t_0 is the collapse time and a, b, c , and p are real constants. Starting from (11.72), we find that the electric field components and maximum field energy density scale according to

$$\begin{aligned} E_x &= \nabla_x \psi \rightarrow \frac{1}{(t_0 - t)^b} \psi, \\ E_{\perp} &= \nabla_{\perp} \psi \rightarrow \frac{1}{(t_0 - t)^c} \psi, \\ |E_{\max}|^2 &\simeq \text{const.} \times \frac{1}{(t_0 - t)^{2a+2b}}, \end{aligned} \quad (11.73)$$

The characteristic dimensions of the collapsing soliton (caviton) contract like

$$l_x \rightarrow (t_0 - t)^b, \quad l_\perp \rightarrow (t_0 - t)^c,$$

while the caviton plasmon number (vide infra) scales as

$$N^{\text{cav}}(t) = \int |\nabla \psi|^2 d\mathbf{r} \simeq \int E_x^2 d\mathbf{r} \rightarrow (t_0 - t)^{-2a-b+2c}. \quad (11.74)$$

In an isolated cavity, the plasmon number is conserved or decreases in time, so the following inequality must hold,

$$-2a - b + 2c > 0,$$

which in the two-dimensional case becomes

$$-2a - b + c = p > 0.$$

Generally taken, the equality sign corresponds to a *strong*, while inequality stands for a *weak* collapse regime. For more details on the collapse hierarchy and the scaling analysis, we refer to earlier papers [262].

11.4.3 Simulation Results and Discussions

We have performed direct numerical simulations of nonlinear equations (11.70) based on the spectral Fourier method with respect to two spatial dimensions with an explicit time integration. The initial condition is chosen in a form of a standing planar soliton (11.71), perturbed in a transverse, y -direction [272], as given by

$$\psi_x(t=0) = \psi_{sx}(t=0)(1 + 2\epsilon \cos ky),$$

where the initial ion density is taken to satisfy the adiabatic matching, in order to shorten plasma transients. We have used periodic boundary conditions (L_x, L_y), a numerical grid 64×64 points (checked upon 128×128) and the perturbation level $\epsilon = 0.01$, performing a regular numerical check of conserved integrals of motion in ((11.70); the plasmon number N)

$$N = \int |\nabla \psi|^2 d\mathbf{r}, \quad (11.75)$$

and the Hamiltonian (H)

$$H = \int \left(|\nabla^2 \psi|^2 + \sigma |\nabla_\perp \psi|^2 + n |\nabla \psi|^2 + \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} n^2 \right) d\mathbf{r}, \quad n_t = \nabla^2 \phi. \quad (11.76)$$

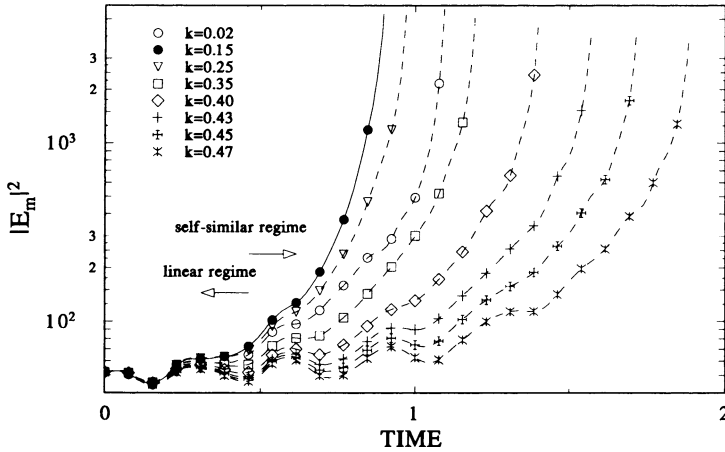


Fig. 11.8 Maximum soliton amplitude in time for different transverse perturbation wavenumbers. The soliton strength is $\lambda = 5$ and $\sigma = 3$

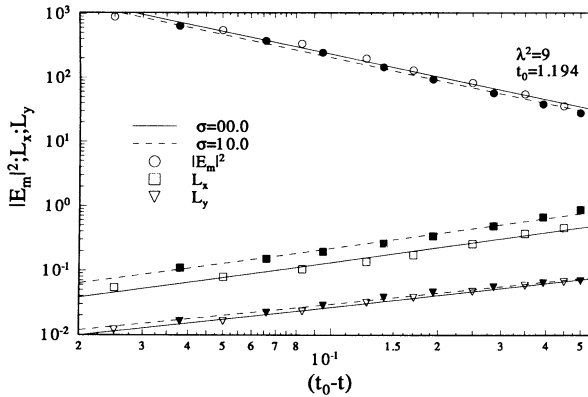


Fig. 11.9 Characteristic spatial scales (l_x, l_y) and the maximum soliton energy as a function of time interval $(t - t_0)$. Numerical fits (lines) of 2D simulation data (points)

To study the space–time dynamics of the soliton instability, we have performed runs with different values of k , λ , and σ . The simulations have confirmed that all linearly unstable solitons [264, 265] in the nonlinear stage enter the collapse phase. Typical space–time evolution of the soliton collapse was illustrated in Fig. 11.6. Further, we show temporal evolution of the soliton amplitude in Fig. 11.8. The initial, linearly unstable phase is followed by an explosive growth; entering a self-similar stage of a Langmuir collapse. The case, $k = 0.15$, corresponds to the most linearly unstable perturbation. Further, in order to check if the self-similar character of the collapse is consistent with (11.72), we vary t_0 to find the best fit with simulation data for the maximum electric field and corresponding soliton dimensions l_x and l_y (Fig. 11.9). We measure the values of the scaling parameters α ($\alpha = 2a + 2b$), b ,

and c to find β . In all considered cases, we have found good agreement with the self-similar solution (11.72), as indicated by the power law dependence in Fig. 11.9. In our simulations of the inertial phase of the Langmuir collapse, for various values of k , λ , and σ , weak collapse ($\beta > 0$) is regularly observed, with no evidence of the strong Langmuir collapse. Increase of the magnetic field speeds up the amplitude growth and the transverse contraction rate; however, these effects are suppressed for a larger soliton strength.

Apart from contributing to a general theory of strong Langmuir turbulence, the presented results can be readily applied to studies of SLT in ionospheric, auroral, and solar plasmas [268,269]. In these applications, varying signatures of anisotropic wave power spectra, heated particle tail, etc., are obtained by space observations and computer simulations [273–285]. In order to make estimates and certain predictions and to justify proposed application models, one can readily turn to the above results on self-similar properties of the magnetized SLT.

Chapter 12

Spatiotemporal Complexity in Plasmas

Nonlinear systems with an infinite number of degrees of freedom are readily described by partial differential equations (PDE). Behavior of such systems exhibits a rich variety of dynamical structures in space and time with coherent, as well as, chaotic and turbulent features [286]. Nearly conservative systems and dissipative systems represent two distinctive highly important categories. In dissipation dominated systems, often a limited number of equilibrium states is available resulting in formation of patterns due to rapid system relaxation under strong dissipative processes. However, for both, nearly conservative and dissipative systems, evolution in an infinitely dimensional phase space can approach attractors that are low-dimensional [287, 288]. An important family of PDE includes nonlinear evolution equations, such as KdV equation, NLS equation, Zakharov's equation, Ginzburg–Landau, Sine–Gordon, Kuramoto–Shivashinsky, as well as the three-wave interaction (3WI), which possess a remarkable nonlinear class of soliton solutions. These equations while including a lowest order of nonlinearity describe some of basic paradigmatic interactions in physics, and nature, in general [289, 290, 298].

12.1 Spatiotemporal Effects in Three-Wave Interaction

A nonlinear resonant interaction of three waves in space and time in weak-coupling (WKB) approximation is readily represented by a system of coupled equations

$$\begin{aligned}\frac{\partial a_0}{\partial t} + V_0 \frac{\partial a_0}{\partial x} &= -a_1 a_2 + i\delta_0 a_0, \\ \frac{\partial a_1}{\partial t} + V_1 \frac{\partial a_1}{\partial x} &= a_0 a_2^* + i\delta_1 a_1, \\ \frac{\partial a_2}{\partial t} + V_2 \frac{\partial a_2}{\partial x} &= a_0 a_1^* - \Gamma a_2 + i\delta_2 a_2 - i\sigma |a_2|^2 a_2.\end{aligned}\tag{12.1}$$

where V_i and δ_i , ($i = 0, 1, 2$) are group velocity and linear phase shift, respectively; while Γ and σ correspond to linear damping and nonlinear phase shift (detuning) of a_2 complex amplitude, respectively. It is a simplified version of the more general

system, assuming linear damping and nonlinear shift put to zero for first two waves, i.e., $\Gamma_1 = \Gamma_2 = 0$ and $\sigma_1 = \sigma_2 = 0$. A basic mathematical formulation and physical background of resonant wave interactions in plasmas have been explored before in details. Here, we further concentrate on space-time aspects of nonlinear 3WI and route to saturation and dynamical complexity and in a *finite* plasma system relevant to applications in laser plasmas.

12.1.1 Convective and Absolute Instability

Investigation of a system stability against perturbations in a finite region of space requires a solution of a boundary value problem [293, 319].

$$u(x, t) = \frac{1}{2\pi} \sum_{s=1}^{\infty} \int_{-\infty}^{\infty} u_s(k, 0) e^{[i\omega_s(k)t - ikx]} dk, \quad (12.2)$$

where $u_s(k, 0)$ is the wave number spectrum of an initial perturbation, with a summation done over all eigenmode wave numbers. Exponential growth in time of the particular k -components still do not guarantee that the perturbation grows at a specific point in space. Namely, perturbations can, while growing, propagate out of the unstable region. This is a *convective* instability. However, if, among exponentially growing perturbations, there are those that do not leave the finite region, i.e., which simultaneously grow at each point in space, this is a condition for an *absolute* instability. More formally, if

$$\lim u(x, t) \longrightarrow \infty, \quad x \in (x_1, x_2),$$

where $u(x, t)$ is a perturbation (x_1 and x_2 are boundaries of the unstable region), instability is absolute. However, if

$$\lim u(x, t) \longrightarrow 0, \quad x \in (x_1, x_2),$$

this instability is convective.

Obviously, the feature of an instability will depend on our choice of the frame of reference. If the observer travels together with a perturbation, in a new moving frame, such instability will appear as absolute. On the contrary, in a system with an absolute instability with a change to new variables, $t_n = t$, $x_n = x - v_0 t$, instability transits into a convective one. Determining the instability character by analyzing the dispersion relation is not an easy task. However, for a large class of systems described by the hyperbolic PDE, it will be sufficient to find in the (x, t) plane, the boundaries of the perturbation propagation which correspond to the characteristics of the *PDE* with a maximum and minimum slope. This is a basic of the method of characteristics [319]. The essence of the method of characteristics

is that characteristics are determined by the asymptotes of dispersion curves in a linear problem. Characteristics and asymptotes have the same slope in the phase space (x, t) and (ω, k) , respectively. Moreover, for hyperbolic systems, where the number of asymptotes with a finite slope is equal to the number of normal modes (eigen-modes), it is possible, based on the form of dispersion curves to determine the character of the instability. If the asymptotes point out in opposite directions, the instability is absolute and vice versa. For the basic system of equations, one gets

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^n a_{ik}(u) \frac{\partial u_k}{\partial x} + b_i(u) = 0, \quad i = 1, 2, \dots, n, \quad (12.3)$$

where u_i are system variables, $a_{ik}(u)$, $b_i(u)$ are nonlinear functions of u_i , equation of characteristics is therefore

$$\text{Det}(a_{ik} - V\delta_{ik}) = 0, \quad (12.4)$$

with V —*tangens* of the inclination angle on the t —axis.

Linearized system of (12.3) is described by the dispersion relation

$$\text{Det}\left(a_{ik} - \frac{\omega}{k}\delta_{ik} - \frac{1}{k}\delta_{ik}\right) = 0, \quad b_{ik} = \frac{\partial b_i}{\partial u_k} \Big|_{u_k=u_0}, \quad (12.5)$$

which coincides with (12.4) for $k \rightarrow \infty$, where the slope of the dispersion curves is identical to a slope of the characteristics. For a parametric instability in a plasma, important effect comes from the spatial localization of the pump in the unstable region controlled by the finite pump extent or phase mismatch due to a spatial non-uniformity. If an instability is of the convective type, locally growing instability can propagate out of the unstable region before the substantial amplitude growth is achieved. For the absolute instability, amplitude will grow exponentially until other nonlinear effects start to play their role.

Discussion on absolute versus convective instability for the decay instability in a uniform medium can be found in e.g., [318]. To determine the character of the instability, it is sufficient to observe the long-time behavior of the a_1 —scattered amplitude at $x = 0$, the point at which the instability was initiated. Furthermore, if $a_1(t \rightarrow \infty, 0) \rightarrow 0$, instability is convective; otherwise, for $a_1(t \rightarrow \infty, 0) \rightarrow \infty$, the instability is absolute. Long-time evolution is determined by the character of the poles in the dispersion relation $D(p, k) = 0$. Analysis has shown that $V_1 V_2 > 0$ gives the convective instability, while for $V_1 V_2 < 0$, absolute instability appears, if the additional condition is satisfied, namely

$$\frac{\gamma_0^2 l^2}{|V_1 V_2|} > \frac{\pi^2}{4}. \quad (12.6)$$

where γ_0 is the uniform linear parametric instability growth rate and l is the length of the system.

12.1.2 Space-Only Problem in Three Wave Interaction

In a stationary (steady state, $\frac{\partial}{\partial t} = 0$) case, 3WI system of (12.1), becomes

$$\begin{aligned} V_0 \frac{da_0}{dx} &= -a_1 a_2 + i\delta_0 a_0, \\ V_1 \frac{da_1}{dx} &= a_0 a_2^* + i\delta_1 a_1, \\ V_2 \frac{da_2}{dx} &= a_0 a_1^* + i\delta_2 a_2 - \Gamma a_2 - i\sigma |a_2|^2 a_2. \end{aligned} \quad (12.7)$$

This system of equations for space-only case (12.7) for $V_i > 0$ is analogous to the time-only case. However, e.g., a change of the sign with V_1 (propagation in the negative x-direction) transforms the initial value problem to a more demanding boundary value problem in a finite media. Introducing boundary conditions at two different boundaries results in a qualitative change in the eigen-value spectrum. More precisely, equations for amplitudes and phase become

$$\begin{aligned} \frac{du_0}{dx} &= -u_1 u_2 \cos \phi, \\ \frac{du_1}{dx} &= -u_0 u_2 \cos \phi, \\ \frac{du_2}{dx} &= u_0 u_1 \cos \phi - G u_2, \\ \frac{d\phi}{dx} &= \left(-\frac{u_1 u_2}{u_0} + \frac{u_0 u_2}{u_1} - \frac{u_0 u_1}{u_2} \right) \sin \phi + \delta + \Delta u_2^2, \end{aligned} \quad (12.8)$$

where above, V_1 is replaced by $-V_1$ and the following anzats is introduced

$$\begin{aligned} u_i(x) &= \frac{1}{\sqrt{V_j V_k}} a_i(x) \equiv u_i(x) e^{i\phi_i(x)}, \quad i, j, k = 0, 1, 2, \\ \phi(t) &= \phi_0(x) - \phi_1(x) - \phi_2(x), \\ \delta &= -\frac{\delta_0}{V_0} - \frac{\delta_1}{V_1} + \frac{\delta_2}{V_2}, \Delta = \frac{\sigma V_0 V_1}{V_2} \text{ and } G = \frac{\Gamma}{V_2}. \end{aligned} \quad (12.9)$$

Solution of the system for $n_i(t) = (u_i(t))^2$, with $G = \delta = \Delta = 0$, is given in terms of *Jacobian* elliptic functions

$$\begin{aligned}
n_0(x) &= dn^2(x, \sqrt{r}), \\
n_1(x) &= r cn^2(x, \sqrt{r}), \\
n_2(x) &= r sn^2(x, \sqrt{r}),
\end{aligned} \tag{12.10}$$

with a parameter r which corresponds to a reflectivity. Solutions appear to be oscillatory in space, with a period

$$\Lambda = 2K(\sqrt{r}), \tag{12.11}$$

where $K(r) \in (\frac{\pi}{2}, \infty)$ (complete elliptic integral of the first kind) for $r \in (0, 1)$, which for $r \rightarrow 1$, $\Lambda \rightarrow \infty$, that is the spatially aperiodic solutions, given by

$$\begin{aligned}
n_0(x) &= \sec h^2(x), \\
n_1(x) &= r \sec h^2(x) \\
n_2(x) &= r th^2(x).
\end{aligned} \tag{12.12}$$

If $G = \Delta = 0$, but $\delta \neq 0$, solutions are found as

$$\begin{aligned}
n_0(x) &= 1 - n_2(x), \\
n_1(x) &= r - n_2(x), \\
n_2(x) &= \frac{\alpha - \eta}{2} sn^2 \left(\sqrt{\frac{\alpha + \eta}{2}} x, \sqrt{\frac{\alpha - \eta}{\alpha + \eta}} \right)
\end{aligned} \tag{12.13}$$

where $\alpha = 1 + r + \delta^2/4$ and $\eta = \sqrt{\alpha^2 - 4r}$, with a period

$$\Lambda = \frac{2K\left(\sqrt{\frac{\alpha - \eta}{\alpha + \eta}}\right)}{\sqrt{\frac{\alpha + \eta}{2}}}, \tag{12.14}$$

which in the limit $\delta \rightarrow 0$, reduces to (12.11).

In a finite system, solutions have to satisfy the nonzero boundary conditions, given as

$$n_1(L) = \epsilon, \tag{12.15}$$

where L is the length of a system. In a general case, the spectrum of eigenvalues is countably finite. However, depending on whether $\epsilon = 0$ or $\epsilon \neq 0$, different implications follow.

In the case $\epsilon = 0$, it was shown that for $G = \delta = \Delta = 0$, of all possible solutions, only, the so-called fundamental mode exists and is asymptotically stable [300, 301] (it corresponds to 1/4 period of the *Jacobian* elliptic solution)

$$L = K(\sqrt{r}). \tag{12.16}$$

Based on L values, the reflectivity of the fundamental mode is calculated as

$$r = 4 \frac{L - L_0}{L_0}, \quad L \geq L_0 = \frac{\pi}{2} \frac{\sqrt{V_1 V_2}}{\gamma_0},$$

$$r = 1 - 16e^{-2L}, \quad L \geq 2L_0,$$

where $L \geq L_0$ is the condition for neglecting the convective amplification of the noise (ϵ) found in the linear parametric analysis [307]. Moreover, for δ and/or Δ values different from zero, it was found that solutions exist only for nonzero ϵ values. The system (12.8) with nonzero damping ($\delta = \Delta = 0, G \neq 0$) has stationary solution related to an aperiodic fundamental mode [313]. Condition for existence of nontrivial stationary state, which is a saturated decay instability, corresponds to instability threshold found from a linear parametric theory (constant pump). Marginal stability condition [296] is found as

$$L = \frac{\left[\frac{\pi}{2} + \sin^{-1} \left(\frac{\beta}{2} \right) \right]}{\left(1 - \frac{\beta^2}{4} \right)^{\frac{1}{2}}},$$

with $\beta = L_0/L_a$, where L_a is the longitudinal absorption rate. Instability is of the absolute or convective type, respectively, depending on whether $\beta < 2$ or $\beta > 2$.

12.1.3 Spatiotemporal Evolution in Three-Wave Interaction

The 3WI in space and time is written in a known form (12.1)

$$\begin{aligned} \frac{\partial a_0}{\partial t} + V_0 \frac{\partial a_0}{\partial x} &= -a_1 a_2 + i\delta_0 a_0, \\ \frac{\partial a_1}{\partial t} + V_1 \frac{\partial a_1}{\partial x} &= a_0 a_2^* + i\delta_1 a_1, \\ \frac{\partial a_2}{\partial t} + V_2 \frac{\partial a_2}{\partial x} &= a_0 a_1^* - \Gamma a_2 + i\delta_2 a_2 - i\sigma |a_2|^2 a_2. \end{aligned} \tag{12.17}$$

Assuming that $V_0, V_1, V_2 > 0$, the system (12.17) describes 3WI as an initial value problem at one point in space-time. However, the case of, e.g., $V_0, V_2 > 0$ and $V_1 < 0$ moves one condition to other boundary (finite or infinite), which leads to a two-point boundary problem.

In seminal papers, Fuchs [295, 296] has investigated stationary solution for (12.17) system ($\partial/\partial t = 0$) for a zero phase shift and zero damping. The solutions found are of the *Jacobian* elliptic function type, with the existence of particular solution defined by the condition (12.15). To exclude the convective noise growth (ϵ), i.e. to concentrate on the absolute instability regime, noise level is taken to zero,

so that condition (12.15) taken with boundary conditions,

$$n_0(0) = 1 \geq n_1(0) = \varepsilon \geq n_2(0) = 0, \quad (12.18)$$

becomes

$$cn(L, \sqrt{r}) = 0,$$

where L is the system length and r is the reflectivity.

From the characteristics of the *Jacobian* elliptic cosine function [320], it follows

$$L = (2n + 1)K(\sqrt{r}), \quad (12.19)$$

with $n = 0, 1, 2, \dots$ and $K(\sqrt{r})$ one quarter of the *Jacobian* cosine. Since $K(\gamma)$, $\gamma = \sqrt{r}$ is growing monotonically from $\pi/2$ to ∞ , for γ between 0 and 1, it is evident that if $L < \pi/2$ then $r = 0$. On the other hand, for $L > \pi/2$, unique solutions are found (12.17), such that

$$1 > r_0 > r_1 > r_2 > \dots > r_{N-1} > 0, \quad (12.20)$$

where $2N - 1$ is largest odd integer part of $2L/\pi$.

Fundamental solution is the only eigen-function which is definite continuous for all $L \in (0, \infty)$. For $L \in (0, \pi/2)$, one gets trivial solution ($r = 0$). For $L > \pi/2$, if $K(\sqrt{r}) \rightarrow \infty$ and $r \rightarrow \infty$, the fundamental mode continually evolves into

$$u_{0,1} \longrightarrow \sinh x, \quad u_2 \longrightarrow \tanh x,$$

that is a unique solution for the semi-infinite system case ($L \rightarrow \infty$). Quantities u_i ($i = 0, 1, 2$) are introduced through (12.20).

Stability analysis of the system (12.17) was given in [300] by using the perturbation treatment, for $\delta_{0,1,2} = \sigma = \Gamma = 0$. Transformation to a system of equations for amplitudes and phases of three waves is performed, which are real quantities, by substituting

$$a_i(x, t) = A_i(x, t) e^{i\phi_i(x, t)}$$

where A_i and ϕ_i are the amplitude and phase of the wave ($i = 0, 1, 2$), respectively.

Amplitude-phase equations are represented by

$$\begin{aligned} \frac{\partial A_0}{\partial t} + V_0 \frac{\partial A_0}{\partial x} &= -A_1 A_2 \cos \phi, \\ \frac{\partial A_1}{\partial t} + V_1 \frac{\partial A_1}{\partial x} &= A_0 A_2 \cos \phi, \\ \frac{\partial A_2}{\partial t} + V_2 \frac{\partial A_2}{\partial x} &= A_0 A_1 \cos \phi, \\ A_0 \left(\frac{\partial \phi_0}{\partial t} + V_0 \frac{\partial \phi_0}{\partial x} \right) &= A_1 A_2 \sin \phi, \end{aligned} \quad (12.21)$$

$$A_1 \left(\frac{\partial \phi_1}{\partial t} + V_1 \frac{\partial \phi_1}{\partial x} \right) = A_0 A_2 \sin \phi,$$

$$A_2 \left(\frac{\partial \phi_2}{\partial t} + V_2 \frac{\partial \phi_2}{\partial x} \right) = A_0 A_1 \sin \phi,$$

where the total phase shift is $\phi = \phi_0 - \phi_1 - \phi_2$.

The stability of (12.21) was analyzed by the Lyapunov stability theory. Linearized equations around the equilibrium (steady state), i.e., $\dot{A}_i = 0, \dot{\phi}_i = 0$ are analyzed. Stationary solutions are written as $Y_m^{(n)}, m = 1, \dots, 6$ (indices denote, wave amplitude (1, 2, 3) and phase (4, 5, 6), respectively, while $n = 0, 1, \dots, N-1$, for N given by (12.20). Actually, stationary solution (12.21) is defined as a vector

$$Y_m^{(n)} = (A_\alpha^{(n)}, \phi_\alpha^{(n)}), \alpha = 0, 1, 2,$$

where $A_\alpha^{(n)}$ are stationary amplitudes with the corresponding zero phase. Then, a general solution appears as

$$Y_m = (A_\alpha, \phi_\alpha), \text{ where } \alpha = 0, 1, 2, \quad m = 1, 2 \dots 6.$$

Departure from an equilibrium is given by

$$y_m = Y_m - Y_m^{(n)}, \quad m = 1, 2 \dots 6$$

which after substituting in (12.21) gives a set of equations for small perturbations of the wave amplitude and phase y_m , as

$$\frac{\partial y_m}{\partial t} + V_{m-1} \frac{\partial y_m}{\partial x} = f_m(Y^{(n)}, y), \quad (12.22)$$

where group velocities V_{m-1} are formally defined by $V_\alpha = V_{\alpha+3}$.

In [300], decoupling of amplitude and phase perturbations from equilibrium was revealed. Therefore, asymptotic stability of the fundamental mode is studied, by independently analyzing the stability of amplitude and phase to small perturbations in (12.22). Analysis of phase perturbations has shown that in a vicinity of the fundamental mode perturbation phases couple resulting in a stability, contrary to a case of unstable phase perturbations of the non-fundamental mode. Furthermore, stability of the fundamental mode to amplitude perturbations was performed by WKB (Wentzel–Kramers–Brillouin) procedure. Still, above perturbation analysis is not adequate for a system evolving far from the equilibrium state. Stability of the fundamental mode was confirmed in a classical paper by Harvey and Schmidt [301]. Time evolution is numerically calculated by perturbing the fundamental as well as non-fundamental mode solutions that satisfy the condition (12.15). In all considered cases, the system, in the time of γ_0^{-1} , saturates to the fundamental spatial mode.

12.2 Complexity in Laser Plasma Instabilities

The spatiotemporal evolution of stimulated Raman backscattering in a bounded, uniform, weakly dissipative plasma is analyzed. The nonlinear model of a 3WI involves a quadratic coupling of slowly varying complex amplitudes of the laser pump, the backscattered and the electron plasma wave (EPW). The corresponding set of coupled PDE with nonlinear phase detuning that is taken into account is solved numerically in space time with fixed nonzero source boundary conditions. The study of this open, convective, weakly confined system reveals a distinctive quasiperiodic transition to spatiotemporal chaos (STC) via spatiotemporal intermittency (STI). In the analysis of transitions and waves complexity, a dual scheme borrowed from fields of nonlinear dynamics and statistical physics is applied. An introduction of a nonlinear 3WI to a growing family of paradigmatic equations which exhibit a route to turbulence via STI is outlined [322].

12.2.1 Introduction to Stimulated Raman Scattering

The nonlinear 3WI as a physical concept, in its variety of appearances, has found its application in hydrodynamics, nonlinear optics, and plasma physics [291]. It occurs whenever waves encounter a resonance in a physical space, i.e., fulfill frequency and wave vector matching conditions. Stimulated Raman scattering (SRS) in plasma is a paradigm of a 3WI related to a nonlinear decay of an intense laser light (pump) to the EPW and the scattered light, shifted in wave number and frequency [291]. It has been studied to a great extent both experimentally and theoretically, largely because of its practical application in search of a future energy source based on inertial confinement thermonuclear fusion induced by intense laser beams [315]. That is, SRS belongs to a family of underdense plasma instabilities, which can have a detrimental effect on the efficiency of laser energy deposition into a fusion target. The well-developed parametric theory [314, 321] gives basic values of SRS instability threshold, growth rates in its initial stage, and some insight into long time spatiotemporal evolution in saturated regimes [302, 308, 321, 323–326]. However, in contemporary high-intensity laser plasma experiments, SRS has often displayed rich and exciting physics, not predicted by the parametric theory, such as a spiky-burstlike signal, anomalously low reflectivity, and spectral gaps and broadenings, as well as incoherent (irreproducible) dynamics [326]. With increasing evidence that SRS often transits from convective to absolute instability in typical laser fusion target-plasma experiments, the problem of nonlinear saturation has become a focus of many SRS studies. Various models, based on different physical mechanisms, have been recently attempted, pointing to a strongly nonlinear dynamics of the EPW as a key factor that determines the nature of saturated SRS states [327–330]. In particular, an inherent feature of strongly nonlinear SRS to transit from a coherent (regular) to chaotic (turbulent) dynamics has been anticipated recently [292, 302, 326]. In this Section, transition from a coherent to a strongly nonlinear incoherent regime, or

STC [331], is explored for saturated SRS in typical laser fusion conditions. This study follows the work by these authors [302, 309, 310, 313] on nonlinear saturation of backward SRS in a plasma layer as a paradigm of a ubiquitous 3WI in a dissipative, weakly confined spatially extended system, which can exhibit extensive chaos [292, 331]. The corresponding set of coupled PDEs, with nonlinear phase detuning of the EPW taken into account, is solved numerically in space and time with rigid nonzero source boundary conditions. Through the variation of physical (laser and plasma) parameters, in particular by increasing the pump strength, this open, convective system is driven toward a STC. A route via steady state and periodic regime with quasiperiodic transition to STI is observed.

The STI in which periodic or quasiperiodic, coherent (laminar) oscillations are interrupted by chaotic (turbulent) bursts is a widely observed phenomenon in spatially extended systems [331, 333, 334] with effectively many degrees of freedom, for example, in hydrodynamic systems (Rayleigh–Benard convection, surface waves, and open pipe flows [335]). This state should be contrasted with the weak or phase turbulence, where there is a competition between localized coherent structures in the sense that one mode dominates energetically, then another takes over, and so on. These localized structures occur at random within the physical domain, which retains a rather homogeneous structure as, for example, in Rayleigh–Benard convection, and such dynamics is usually low dimensional. As such, weak turbulence is closest to the so-called low dimensional chaos in which the system displays incoherence only in time while the spatial structure remains quasi frozen by the confined boundaries [336]. The STI, on the other hand, is further characterized by dominant macroscopic scales in space-time, which can, generally taken, be identified as the coherent (length, time) scales in distinction to fully developed turbulence, where there are no predominant macroscopic scales. In addition to experiments in hydrodynamics, the STI is frequently encountered in PDE simulations of routes to chaos generic to nonlinear dissipative extended systems [331, 333], modeled by, for example, the Kuramoto–Shivashinsky equation, the Swift–Hohenberg equation, and coupled map lattices [337], to name a few. The STI state displays the coexistence of patches of turbulence immersed in the rest of the structure still in the laminar state; the continuous transition amounts to a progressive increase of the turbulent fraction through the variation of control parameters [333]. In this case, turbulence is strong locally and affects only a part of a physical space, which can be very small, e.g., at the STI threshold [333]. For completeness, we are reminded that in nonlinear plasma physics an extensively developed theory of weak turbulence is available, starting from the early 1960s. This theory is built on a quasilinear concept of weakly interacting, weakly nonlinear plasma modes of random phases [13]. The rest of this Section is organized as follows. First, the one-dimensional model of the nonlinear SRS is presented. Next, classical diagnostics from dynamical systems theory to analyze the time-only aspect of the backscattered wave evolution is introduced. Further, the spatiotemporal aspect of the system and analysis of the corresponding patterns and the correlation functions are performed. Moreover, dimension and entropy that quantify the spatiotemporal behavior and identify the route to STI and chaos is introduced. Finally, coarse graining the degrees of freedom into binary

variables such that the local space-time regions are labeled as either chaotic or laminar, and the techniques from the theory of phase transitions and critical phenomena are used to identify the transition from STI to STC.

12.2.2 Nonlinear Saturation of SRS

Laser plasma instabilities are a useful test bed for exploring rich variety of strongly nonlinear plasma phenomena. As a rule, they include a number of important three-wave resonant instability models, involving the strong laser pump parametric coupling to plasma eigen-modes [314, 318, 357]. These instabilities are mostly of a decay type, where the energy transfer from the laser pump to lower frequency daughter waves typically involves the electrostatic plasma modes (i.e., EPW and ion sound wave). In laser fusion, great concern is related to stimulated Raman and Brillouin scattering on EPWs and ion sound waves, respectively, which can result in undesirable loss of laser energy at the target and production of energetic particles.

In propagation of an incident electromagnetic wave (laser) through an underdense plasma ($n \leq n_{cr}$, i.e., $\omega_0 \geq \omega_{pe}$), the pump can easily excite plasma eigen-modes which exist at the noise level. In an isotropic, unmagnetized warm plasma (vide infra), three basic linear eigen-modes are available:

- Electromagnetic (transverse) wave

$$\omega^2 = \omega_{pe}^2 + k^2 c^2,$$

- Electron plasma (longitudinal) wave, with

$$\omega^2 = \omega_{pe}^2 + 3k^2 v_{te}^2,$$

- Ion-acoustic (sound) wave

$$\omega^2 = c_s^2 k^2,$$

where, $\omega_{pe}^2 = e^2 n_0 / \varepsilon_0 m_e$, the electron plasma frequency, $v_{te}^2 = T_e / m_e$, electron thermal velocity, and $c_s^2 = z T_e / m_i$, ion sound velocity (z-ion charge).

Stimulated Raman backscattering is a 3WI resonant instability of the laser pump against excitation of an EPW and another electromagnetic (laser) wave which propagates in the backward direction. Therefore, being possible for laser frequency $\omega_0 \geq 2\omega_{pe}$, or in other words, at plasma density below the quarter critical, $n \leq n_{cr}/4$. SRS is of great relevance in contemporary laser plasma research and will be next studied in more detail.

The one-dimensional model of SRS, assumes a uniform plasma layer of thickness L , irradiated by a laser beam from $x < 0$, which enters the plasma at $x \geq 0$, boundary. The EPW and the scattered light are allowed to grow from their thermal noise levels (ε_1 and ε_2 , respectively). Moreover, the EPW is subjected to a weak dissipation characterized by the linear damping rate ν_e . The nonlinear 3WI

model derived here for the case of SRS describes the spatiotemporal evolution of complex amplitudes of the pump (a_0), scattered (a_1) and EPW (a_2) in a weakly coupling approximation. These equations are obtained from Maxwell's and fluid plasma equations in WKB approximation, assuming the resonant matching between frequencies and wave numbers of three waves ($\omega_0 = \omega_1 + \omega_2$, $\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2$) closely satisfying the corresponding linear dispersion relations

$$\omega_{0,1}^2 = \omega_{pe}^2 + k_{0,1}^2 c^2, \quad \omega_2^2 = \omega_{pe}^2 + 3k_2^2 v_{te}^2, \quad (12.23)$$

where indices 0, 1, and 2 stand for the pump, scattered, and EPW, respectively. For a case of *backscattering*, which is of most practical importance, the corresponding set of 3WI equations reads [309–312]

$$\begin{aligned} \frac{\partial a_0}{\partial \tau} + V_0 \frac{\partial a_0}{\partial \xi} &= -a_1 a_2, \\ \frac{\partial a_1}{\partial \tau} - V_1 \frac{\partial a_1}{\partial \xi} &= a_0 a_2^*, \\ \frac{\partial a_2}{\partial \tau} + V_2 \frac{\partial a_2}{\partial \xi} &= \beta_0^2 a_0 a_1^* - \Gamma a_2 + i\sigma |a_2|^2 a_2, \end{aligned} \quad (12.24)$$

with time and space variables $\tau = \omega_0 t$, $\xi = x/L$, where the dimensionless amplitudes of the coupled waves are related to the physical quantities, electric fields E_0 and E_1 of the two electromagnetic waves, and EPW-driven electron density fluctuation δn_e ,

$$\begin{aligned} a_0(\xi, \tau) &= \frac{ck_2}{4\omega_0} \left[\frac{\omega_{pe}}{\omega_1} \right]^{\frac{1}{2}} \frac{E_0(x, t)}{\mathcal{E}_0}, \\ a_1(\xi, \tau) &= \frac{ck_2}{4\omega_1} \left[\frac{\omega_{pe}}{\omega_0} \right]^{\frac{1}{2}} \frac{E_1(x, t)}{\mathcal{E}_0}, \\ a_2(\xi, \tau) &= \frac{\omega_{pe}^2}{4\omega_0 \sqrt{\omega_0 \omega_1}} \frac{\delta n_e(x, t)}{n_0}, \end{aligned} \quad (12.25)$$

$\mathcal{E}_0$ denotes the vacuum pump electric field amplitude, and n_0 the equilibrium plasma density. Normalized group velocities and the damping rate Γ are expressed by

$$V_0 = \frac{c^2 k_0}{\omega_0^2 L}, \quad V_1 = \frac{c^2 k_1}{\omega_0 \omega_1 L}, \quad V_2 = \frac{3k_2 v_{te}^2}{\omega_0 \omega_{pe} L}, \quad \Gamma = \frac{v_e}{2\omega_0}. \quad (12.26)$$

and the laser pump strength is given by the ratio of the electron quiver velocity in a laser pump field to the speed of light

$$\beta_0 \equiv \frac{v_{osc}}{c} = \frac{e\mathcal{E}_0}{m_e \omega_0 c}. \quad (12.27)$$

We note an additional self-modal nonlinearity in the equation for EPW, given in the form of a nonlinear phase detuning (shift) $\sim \sigma |a_2|^2$, which is due to large amplitude nonlinear EPWs excited through the SRS process [314, 327–330]. In present-day laser-plasma interaction experiments, high intensity lasers induce large amplitude EPWs. For short pulse, ultra high intensities, relativistic correction to the electron mass makes it necessary to include a nonlinear detuning term in (12.24) [297, 338, 339]. On the other hand, an analogous nonlinear term could be due to nonlinear density modulation involving pondermotive coupling of EPWs to ion sound in the saturation of long-pulse SRS via Langmuir decay instability. While relativistic correction to electron plasma frequency directly adiabatically induces nonlinear detuning of the wave resonance, density modulation has a more complex time-dependent effect due to ion dynamics [323–325]. For high-intensity relativistic laser plasma interaction, the corresponding model studied by some authors [297, 338, 339] evaluates a relativistic frequency shift σ , given as

$$\sigma = \frac{3\omega_0^2\omega_1}{c^2\omega_{pe}k_2^2}. \quad (12.28)$$

We also note the relevance of above model (12.24) to relativistic beat wave interactions [297, 338, 339].

To approach the absolute regime of the backward SRS instability, there is a minimum plasma length L_0 , i.e., basic amplification length or scattering length. In dimensionless units, the absolute instability condition for backward SRS can be written as

$$\frac{\omega_0 L}{c} \frac{\beta_0}{\beta_{te}} > \frac{2\pi}{\alpha} \sqrt{\frac{3\sqrt{11-2\alpha}}{\sqrt{11-\alpha^2} + \sqrt{11-2\alpha}}}, \quad (12.29)$$

where $\alpha = \omega_{pe}/\omega_0 = \sqrt{n_0/n_{cr}}$, and $\beta_{te} = \sqrt{T_e(\text{keV})/511}$.

As discussed, the standard conservative form ($\delta = 0$, $\Gamma = 0$) of 3WI in one dimension is integrable [291]. However, with an introduction of dissipation ($\Gamma > 0$), closed form analytical solutions are not available and a numerical solution is the only alternative. A spatially uniform (time-only) version of (12.24) has been studied in detail, and it was shown to exhibit a low-dimensional chaos under restricted conditions [67]. Naturally, as indicated above, the spatially extended model of 3WI is more difficult to investigate. Recently, the 3WI have been solved and studied in space time. These results have revealed rich physical behavior of saturated regimes corresponding to low-dimensional chaos as well as to STC [292, 302, 331].

The most useful information on the SRS is contained in the reflectivity R , which gives a fraction of incident laser intensity reflected backward [309],

$$R = \frac{V_0 |a_1(0)|^2}{V_1 |a_0(0)|^2} \quad (12.30)$$

with its maximum value normalized to unity in the stationary case. To solve (12.24), appropriate initial and boundary conditions are required. We choose physically

realistic boundary conditions, while the choice of the plasma slab length satisfies the criterion for the occurrence of the absolute instability. The wave amplitudes for backward SRS obey the initial and nonzero source fixed boundary conditions,

$$\begin{aligned} a_0(x, 0) &= 0 \quad (\text{for } x > 0), a_0(0, t) = A_0 \\ a_1(x, 0) &= a_1(L, t) = \varepsilon_1 A_0 \quad a_2(x, 0) = a_2(x, t) = 0 \end{aligned} \quad (12.31)$$

where A_0 follows from (12.25) for $E_0(0) = \mathcal{E}_0$.

12.2.3 Break-up of Manley–Rowe Invariants and Nonstationary SRS

Introducing the linear and nonlinear phase shift terms in the system (12.21) for the case of backscattering, in the steady state ($\partial/\partial t \rightarrow 0$), conserved quantities (Manley–Rowe relation) are readily calculated as

$$\begin{aligned} m_0 &= V_0 n_0(x) - V_1 n_1(x) = \text{const.}, \\ m_1 &= V_0 n_0(x) + V_2 n_2(x) = \text{const.}, \\ K(x) &= A_0 A_1 A_2 \sin \phi - \frac{\sigma}{4} A_2^4 - \frac{\delta}{2} A_2^2 = \text{const.} \end{aligned} \quad (12.32)$$

with $n_i(x) = A_i(x)^2$, $i = 0, 1, 2$. For boundary conditions

$$n_0(0) = 1, n_1(L) = 0, n_2(0) = 0 \quad (12.33)$$

the third invariant becomes $K(0) = 0$. However, at the rear boundary $x = L$, from (12.33), one calculates $K(L) \neq 0$, which breaks the invariance condition, i.e., $K(x) \neq \text{const.}$, hence, contradicts our basic assumption of the steady state. This simple argument, due to Škorić [302, 303, 306], based on a nonlinear phase mismatch, predicts an intrinsic onset of nonstationarity in 3WI backscattering, such as, e.g., nonlinear saturation of stimulated Raman (also Brillouin) instabilities in laser plasma interactions. At this point, we note relevance of above findings to numerous applications based on backward SRS (also SBS), such as, e.g., Raman amplification and compression of laser pulses [304, 305], where limitations in scaling to high intensities with coherence loss might be anticipated [303].

12.2.4 Bifurcations and Route to Low-Dimensional Chaos

A series of numerical simulations of model (12.24), by means of a central difference method, has been performed for different system (laser and plasma) parameters

within physically realistic values [294]. The accuracy of the centered-time, centered-space numerical scheme has been checked at each temporal step by using the modified Manley–Rowe conservation relations [291, 321], which follow from (12.24). The following parameters are chosen: $V_0 = 9.5 \times 10^{-3}$, $V_1 = 8.8 \times 10^{-3}$, $V_2 = 2.9 \times 10^{-4}$, $\Gamma = 1.6 \times 10^{-6}$, $\sigma = 3.5$ and $\varepsilon_1 = 10^{-2}$, corresponding to typical laser plasma conditions [309, 314, 324]: $n_0 = 0.1n_{\text{cr}}$, $T_e = 1 \text{ keV}$, $L = 100c/\omega_0$ and $\nu_e/\omega_{\text{pe}} = 10^{-5}$.

In the time-only aspect of the SRS, we follow the evolution of the backscattered wave. For the present exposition, we study the bifurcation sequence while the incident pump wave amplitude, as a control parameter, is varied. The justification for this choice of the control parameter lies in the fact that the variation of the plasma slab length, keeping the incident laser beam amplitude fixed, leads to a similar bifurcation sequence. The variation of the damping term, on the other hand, alters only the quasiperiodic regime as discussed below, and the same effect is observed when plasma density is changed. The robustness of the scenario was checked by varying the plasma slab length up to a factor of 5 by changing the damping term by two orders of magnitude [313], $(10^{-6} - 10^{-4}) \nu_e/\omega_{\text{pe}}$ and varying the plasma density within the interval $(0.001n_{\text{cr}} - 0.1n_{\text{cr}})$.

As the relative pump strength β_0 increases, starting from the value 0.01, the attractor is found to change according to the symbolic sequence

$$FP \rightarrow P \rightarrow QP \rightarrow I \rightarrow C$$

where FP stands for unimodal fixed point, P for periodic, QP for quasiperiodic, I for intermittent, and C for chaos. The quantitative boundaries in β_0 between successive attractors are depicted in Fig. 12.1.

The fixed point bifurcates to a stable limit cycle (Fig. 12.2c) through a supercritical Hopf bifurcation. As the relative pump strength is further increased, an additional spatial mode occurs, which is apparently associated with the second frequency in the quasiperiodic dynamics observed in the interval $(0.026 < \beta_0 < 0.029)$. The corresponding second frequency occurring for the EPW can be accounted for by the appearance of a traveling wave (Fig. 12.8b). Within this finite parameter range, a mode locking occurs in which two incommensurate frequencies become related by a ratio of integers and the winding number, defined as the ratio of the two frequencies, is equal to 5. The corresponding phase space representation, a 2-torus, is represented in Fig. 12.3c. Harmonics f_n , appearing in the power spectrum of the quasiperiodic evolution, are related to the two main frequencies by $f_n - f_0 = n(f_1 - f_0)$, where f_0 is the frequency of the peak to the left of the most energetic frequency

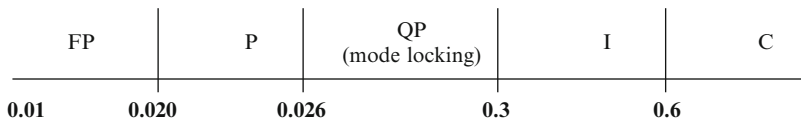


Fig. 12.1 The bifurcation sequence as a function of relative pump strength

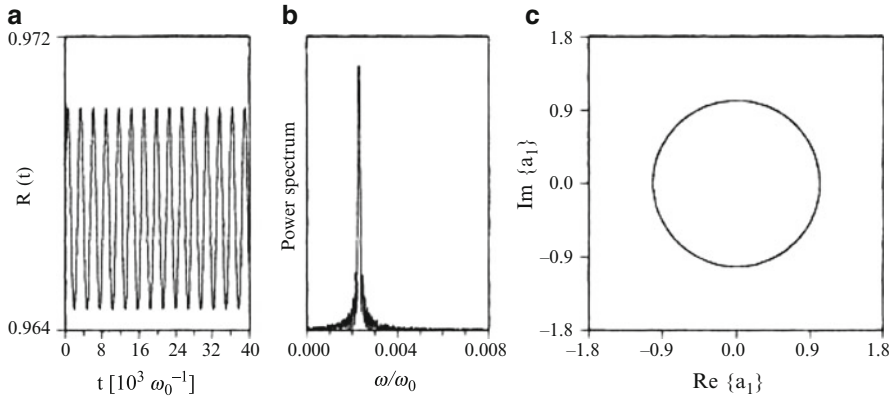


Fig. 12.2 (a) Stimulated Raman backscattering reflectivity, (b) power spectrum, and (c) phase diagram for $\beta_0 = 0.02534$

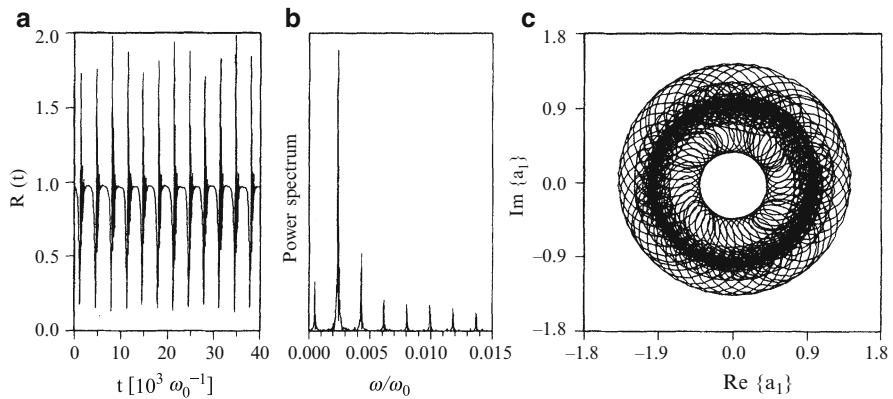


Fig. 12.3 (a) Reflectivity, (b) power spectrum, and (c) phase diagram for $\beta_0 = 0.027$

f_1 (Fig. 12.3b). The weak nonlinear effects stabilize the 2-torus trajectory in the vicinity of the former periodic trajectory, which although linearly unstable remains visible, disclosing a beautiful illustration of another supercritical Hopf bifurcation. At this point, it should be noted that the increase of the damping term leads to the bifurcation sequence in which the quasiperiodic regime occurs without frequency locking, which can be explained on the basis that increased damping suppresses the evolution of the traveling mode. With further increase in β_0 , the manifestation of an extra degree of freedom occurs through the destruction of torus without appearance of the third frequency (Figs. 12.4b and 12.4c). An interesting feature of this new regime is the existence of two metastable regions of the attractor, namely, “laminar” parts that retain almost unchanged a quasiperiodic nature and intermittent chaotic bursts. In the power spectra plots, the power spectrum of intermittent regime exhibits an increasing broadband noise level (Figs. 12.4b and 12.5b). As the

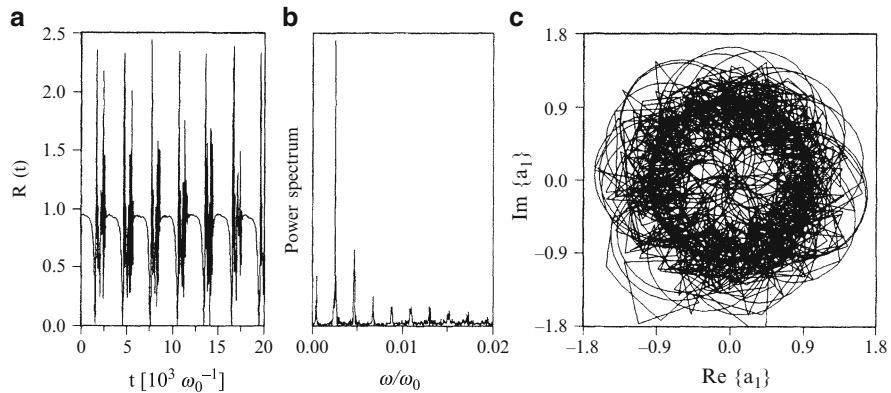


Fig. 12.4 (a) Reflectivity, (b) power spectrum, and (c) phase diagram for $\beta_0 = 0.03$

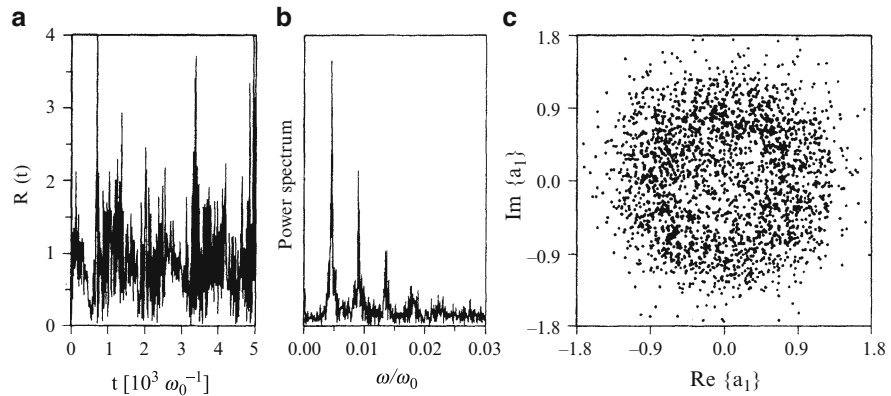


Fig. 12.5 (a) Reflectivity, (b) power spectrum, and (c) phase diagram for $\beta_0 = 0.05$

control parameter is further increased, the fraction of time spent in laminar regions decreases, and fully developed temporal chaos sets in, as reflected in the broadband power spectrum (Fig. 12.6b). Finally, the correlation dimension values obtained by the standard Grassberger–Procaccia algorithm [340] establish the low-dimensional nature of these attractors. The dimension increases from 1 (periodic) through 2.28 (quasiperiodic) to 3.7–4.7 (intermittent and chaotic).

All of the dynamical system diagnostics presented in this section analyzed the temporal aspect of the backscattered wave at the front boundary of the plasma slab. For a system of PDE (12.24), the temporal aspects must be correlated with spatial information, and more detailed insight is gained from the analysis of spatiotemporal patterns and the corresponding correlation functions. This is the subject of the next section.

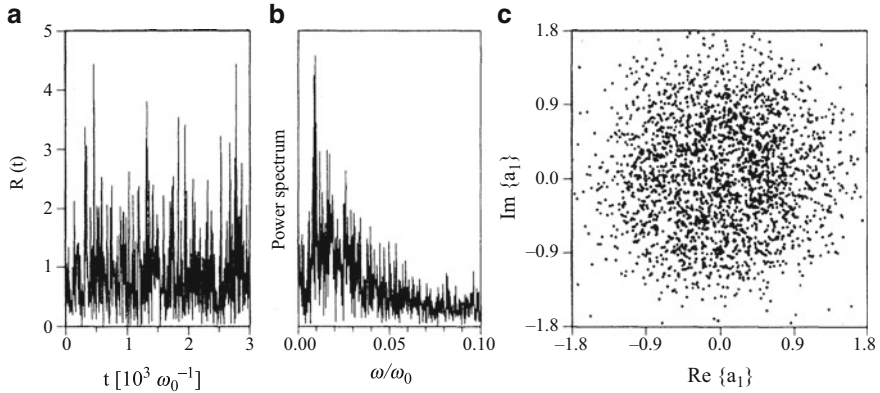


Fig. 12.6 (a) Reflectivity, (b) power spectrum, and (c) phase diagram for $\beta_0 = 0.1$

12.2.5 Complexity of Spatiotemporal Wave Patterns

The simulation results of the stimulated Raman backscattering dynamics exhibit STI and STC and are described in terms of correlation functions having spatial and temporal scales. The numerically observed spatial states involved in the bifurcation sequence include coherent states, traveling waves, and intermittent and chaotic states. In general, the qualitative space-time behavior of the pump wave is almost identical to the corresponding patterns of the scattered wave. This is due to the fact that the coherent structures of the backscattered wave are determined by the incident pump wave, while the features of the EPW represent the outcomes of non-linear interaction processes that take place between the pump and the backscattered wave. We give here a more detailed account of these spatiotemporal structures. For small values of the relative pump strength, the spatial structure resembles a semi-humplike structure that increases in size as the damping term (Γ) decreases. The mirror like symmetry of the scattered wave structure and the EPW structure with respect to each other is evident in Fig. 12.7. In next regime, a spatially periodic structure of the scattered wave superimposed on one-half of the hump occurs at regular time intervals, while no such structure can be noticed for the EPW. Further increase of the control parameter β_0 brings forth the appearance of the propagating EPW mode and flattening of the underlying spatial structure for both waves. The striking feature of the scattered wave pattern in this regime is the breatherlike coherent excitation oscillating in time (Fig. 12.8a). As β_0 increases and leaves the parameter range corresponding to the mode locking temporal behavior, the most interesting feature is the change in spatial symmetry, particularly for the EPW. As the relative pump strength increases, the temporal translational symmetry of the backscattered coherent structures breaks up more dramatically than the corresponding one of the EPW. A common feature of all four regimes is the same number of coherent structures in the time direction.

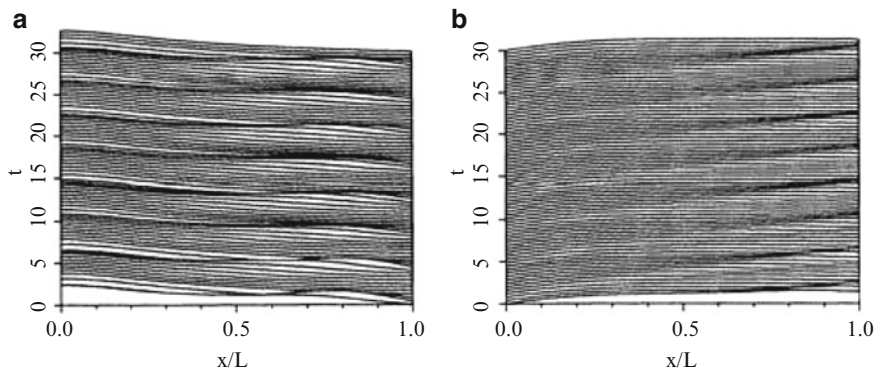


Fig. 12.7 Space-time patterns of (a) backscattered wave and (b) electron plasma wave for $\beta_0 = 0.02534$

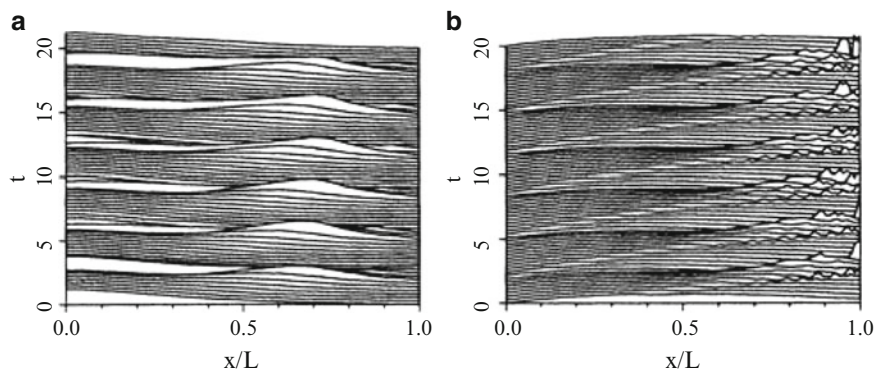


Fig. 12.8 Space-time patterns of (a) backscattered wave and (b) electron plasma wave for $\beta_0 = 0.027$

The transition from the STI to STC (Fig. 12.11) is continuous in the sense that, as the threshold is approached from below, the number of turbulent domains slowly increases, accompanied with the breakup of laminar domain fronts. The autocorrelation functions for the two regimes (Figs. 12.9, 12.10, and 12.12) clearly show that the correlations fall off gradually both in time and space. The well-defined correlation lengths (times) for these two regimes indicate that the dynamics is uncorrelated for lengths (times) greater than these characteristic values. The fact that the group velocity of the EPW is 30 times smaller than the backscattered wave velocity has important implications on the spatiotemporal characteristics of the corresponding wave patterns. The correlation length of the EPW is consequently approximately 30 times smaller than the correlation length of the backscattered wave (Fig. 12.9–12.12). Autocorrelation functions provide clear qualitative, although quantitatively not adequate, evidence of STC. Moreover, to determine the control parameter value at which the transition from intermittency to chaos occurs, methods from the theory

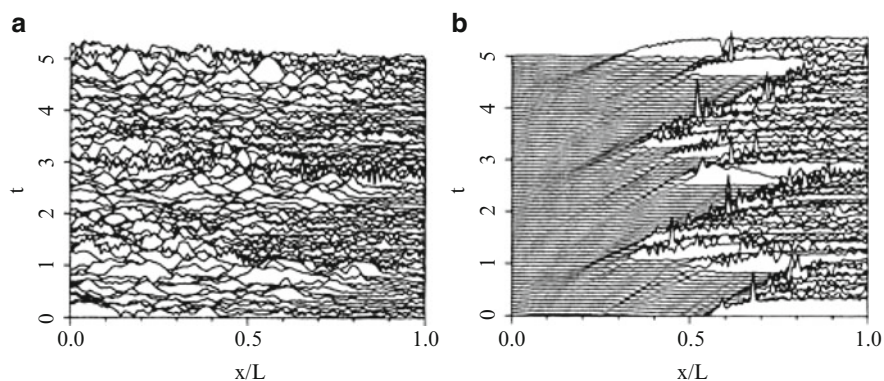


Fig. 12.9 Space-time patterns of (a) backscattered wave and (b) electron plasma wave for $\beta_0 = 0.06$

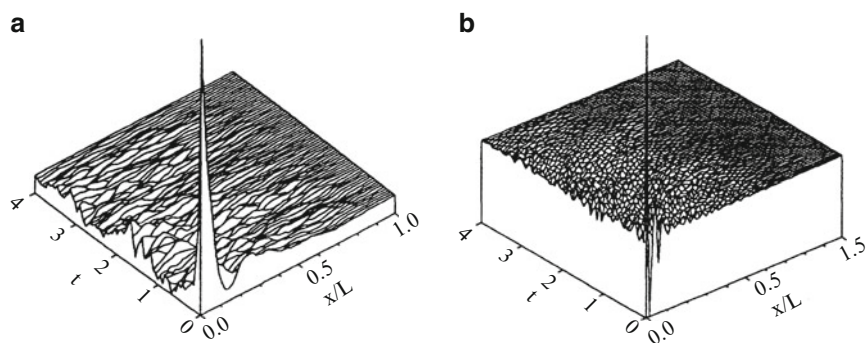


Fig. 12.10 Autocorrelation functions for (a) backscattered wave and (b) electron plasma wave for $\beta_0 = 0.06$

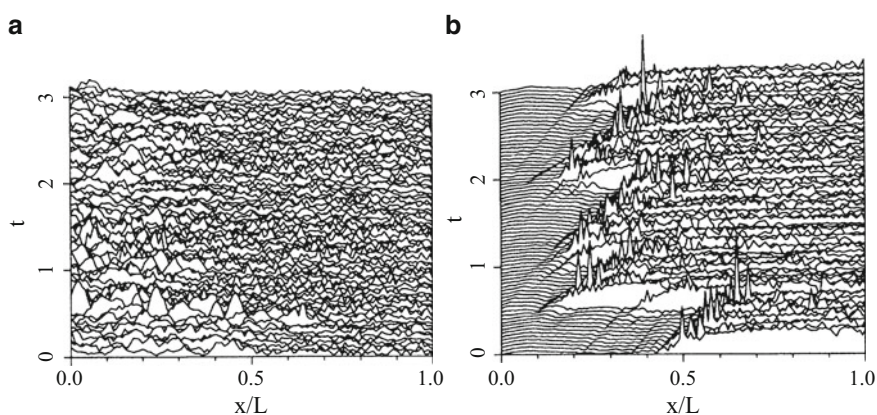


Fig. 12.11 Spatiotemporal patterns for (a) backscattered and (b) EPW for $\beta_0 = 0.1$

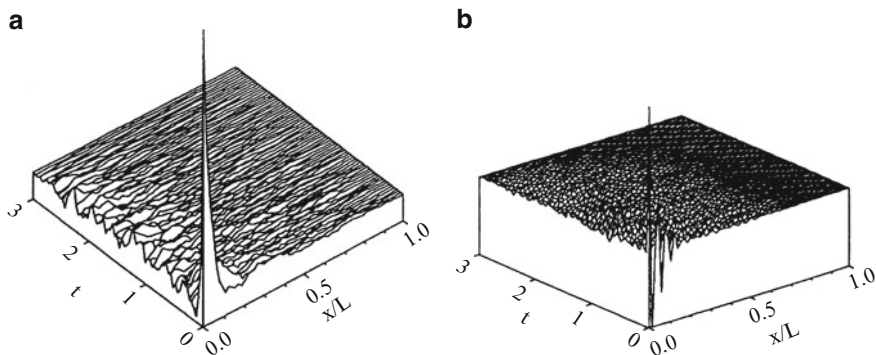


Fig. 12.12 Autocorrelation function for (a) backscattered and (b) EPW for $\beta_0 = 0.1$

of critical phenomena and phase transitions will be used. These aspects of the analysis are addressed in the next sections.

12.2.6 Quantitive Signatures of Spatiotemporal Regimes

To better understand the transition from regular to chaotic dynamics, the relationship between the spatial and temporal degrees of freedom in the system is essential. In this study, we choose the local approach on the attractor of the EPW in order to define the spatiotemporal quantities such as dimension and entropy. The approach is characterized by the determination of local orthogonal directions on the attractor (the local topological dimensionality) along which the local data points are distributed. The embedding space, and hence the attractor, is reconstructed with reference to spatial dependence. That is, embedding procedure consists in taking time series of each spatial location as one component of the embedding vector, and the number of spatial locations determines the embedding dimension. The centers defining local regions on the attractor are randomly selected in such a way that they nearly cover the entire surface of the attractor. The local dimension is determined by the rank of the local data matrix defined by the preselected number of nearest neighbors for each local center. The number of the nearest neighbors is determined by the condition that the local region they define is linear and their number usually varies between 30 and 50. The test for local linearity consists in successively decreasing the number of points in the local region until further decrease does not decrease the number of dominant orthogonal directions [341]. The rank of the local data matrix is determined by the singular value decomposition (SVD). If the local data matrix is labeled U , then the singular values λ_i of U are equal to the square roots of the eigenvalues of the data covariance matrix $R = U^T U$, and the number of nonzero singular values (eigenvalues) determines the local topological dimension of the attractor. Since it is almost impossible to obtain exactly zero eigenvalues

either in numerical simulations or from experimental data, the main challenge in this approach is to determine the threshold below which an eigenvalue should be considered to be zero. The topological dimension of the whole attractor is calculated as the weighted average of local dimensions, i.e.,

$$\langle D \rangle = n_1 d_1 + n_2 d_2 + \dots + n_s d_s,$$

where $n_i = m_i/N$, m_i is the number of local regions in which dimension d_i occurs, and N is the total number of local regions. Hence, as the local dimension is an integer value, the topological dimension of the whole attractor may be fractal in nature. Note that because of the nature of the embedding procedure, the dimension defined in such a way reflects the spatial and temporal dynamics simultaneously.

The separation of true signal from noise is accomplished using the method formulated on the basis of an information–theoretic criterion. Two such methods, the Akaike information criterion [342] and the minimum descriptive length of Risannen [343], have been extensively used in signal processing applications, particularly for the determination of the number of signals in high resolution arrays. However, since the Akaike information criterion tends to overestimate and the minimum descriptive length tends to underestimate the number of signal sources, active research is going on to overcome the shortcomings of these two criteria. In our approach, we have used the modified information-theoretic criterion, which does not show these deficiencies, and an interested reader can find a complete account of this method in Ref. [344]. The resulting topological dimensions for various spatiotemporal patterns of the SRS are presented in Fig. 12.13. As more and more active modes take part in the dynamics, the dimension correspondingly changes and various

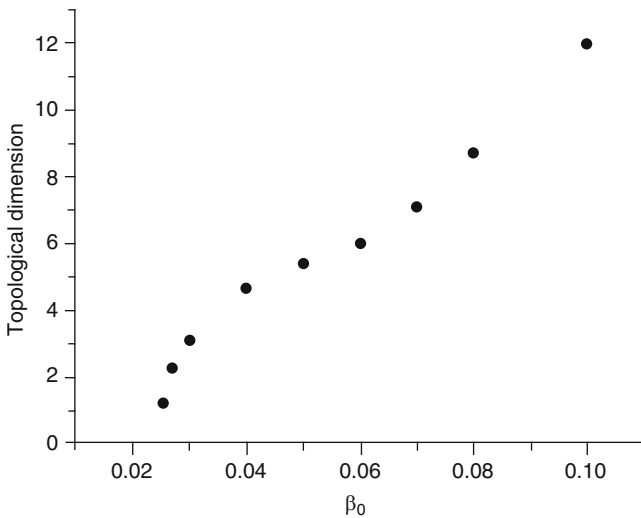


Fig. 12.13 Topological dimension of EPW versus β_0

spatiotemporal bifurcations may be identified through the dimension changes. Another important aspect of dimension calculations for extended systems, namely, density of dimension $\rho \equiv D(L)/L$, where L is the size of the system, has been verified as independent of L , and details of the growth rate of dimension as a function of the system size have been reported elsewhere [345].

Using the same embedding procedure, we have applied the Grassberger–Procaccia algorithm for calculating the widely used correlation dimension. The obtained results show very good agreement with the results for topological dimension for low-dimensional dynamics ($d < 8$). For higher dimensional dynamics, the correlation dimension is inaccurate because of the intrinsic limitations of the algorithm. That is, the upper bound permitted by the algorithm is $2 \log_{10} N$, where N is the time dimension of the embedding matrix (length of the time series) [346]. In general, however, the advantages of local analysis of topological properties of the attractor are particularly evident in identifying important topological features due to, for example, thin directions, which may indicate that the corresponding attractor cross section is a Cantor set, or the effects due to the limited amount of data [347]. Moreover, the criterion for separating signals from noise is the integral part of this approach, while the correlation dimension, as well as other metric and probabilistic dimensions, is quite sensitive to the presence of noise. Note that SVD is often used as a noise reduction procedure before applying the correlation dimension algorithm. That is, the few dominant eigenvalues in the singular value spectrum are retained based on certain arbitrary criterion, such as by counting those singular values (eigenvalues) that exceed a certain percentage of the largest singular value (eigenvalue), usually $\geq 95\%$ [348]. In either case, SVD is applied globally, while the information is lost on the local effects on noise, which can be of significant importance for thin directions on the attractor.

Important indicators of the sensitivity to initial conditions of the system are Lyapunov exponents, which represent the growth rates of edges of an infinitesimal tangent space to the trajectory of an attractor. In terms of singular values λ_i , the local Lyapunov exponents scale approximately as $\ln \lambda_i$. Hence, the number of local Lyapunov exponents is equal to the local topological dimension, and the number of positive local Lyapunov exponents averaged over the attractor increases with the increase of the control parameter, as well as with the system size.

The SVD of the spatiotemporal data matrix corresponds to the spectral decomposition of the signal into spatial and temporal orthogonal modes,

$$u(x, t) = \sum_{i=1}^N \lambda_i \phi_i(x) \psi_i(t), \quad (12.34)$$

with

$$\lambda_1 \geq \lambda_2 \geq \dots > 0,$$

and

$$(\phi_i, \phi_j) = (\psi_i, \psi_j) = \delta_{i,j},$$

which converges in norm. The pair (ψ_i, ψ_j) defines a spatiotemporal structure of energy λ_i^2 [349]. The relative energy of each structure is given as the ratio of one structure energy to the total energy of the signal [349, 350].

$$p_i = \lambda_i^2 / \sum_{i=1}^N \lambda_i^2 \quad (12.35)$$

Here, the characterization of the eigenvalue spectrum is based on the principle that the information contained in the eigenvalues can be interpreted as a probability distribution, so that each eigenvalue can be viewed as an indication of how likely it is to identify a spatiotemporal structure within the whole spectrum. On the basis of the information-theoretic definition of entropy, the normalized entropy can be defined as

$$H(u) = - \lim_{k \rightarrow \infty} \frac{1}{\ln k} \sum_{i=1}^k p_i \ln p_i. \quad (12.36)$$

Because of the normalizing factor $1/\log k$, the entropy is defined in the range between 0 and 1. If the dynamics of the system is such that only one eigenvalue is different from zero (i.e., the energy is concentrated in only one eigenvalue), the spatiotemporal entropy is equal to 0, indicating the lowest level of complexity. On the other hand, if the energy is equidistributed, i.e., all eigenvalues are the same, the global entropy is equal to 1, indicating the highest possible complexity. In analogy with the local topological dimension, we can calculate local entropy while the global entropy can be obtained by ensemble averaging over the attractor. The spatiotemporal entropy of the EPW as a function of relative pump strength is presented in Fig. 12.14. The entropy evolves from a value close to zero corresponding to the steady (laminar) state; increases as new coherent structure emerge and the energy spreads out on the eigenvalues; and approaches the value of 0.76, where many structures carry similar amounts of energy. Both the entropy and the topological dimension display a clear distinction between STI and STC, and additional work is necessary to relate the local aspects of these two quantities to the universal characteristics of STI and chaos in various physical systems.

12.2.7 *Transition from Spatiotemporal Intermittency to Spatiotemporal Chaos*

A method to locate a parameter threshold value is based on the coarse graining of the space-time data into binary values, according to which local spatiotemporal regions are labeled only as either laminar or chaotic [351, 352]. Since in the laminar (quasiperiodic) regions the local amplitude is lower than in turbulent regions, by setting an arbitrary cutoff, a binary representation may be obtained that easily distinguishes between chaotic and nonchaotic domains. The obtained representation

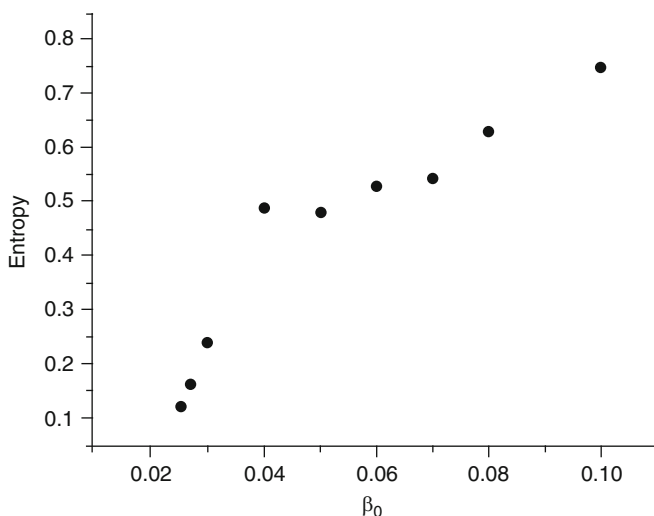


Fig. 12.14 Entropy of EPW for values of pump β_0

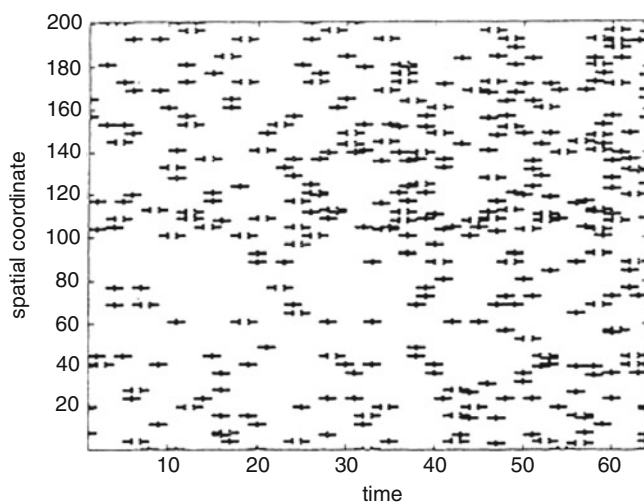


Fig. 12.15 Spatiotemporal pattern in binary reduction for $\beta_0 = 0.06$, near intermittency threshold

seems to be independent on the precise cutoff value within the accuracy of the calculation. The two-state representation for the regimes corresponding to $\beta_0 = 0.06$ and $\beta_0 = 0.1$ are presented in Figs. 12.14 and 12.15, respectively. Assuming an analogy to directed percolation [353], the laminar phase corresponds to a state where chaotic (turbulent) states percolate through the lattice until the size in the time direction, the time-correlation length, is reached. Directed percolation is known for exhibiting a continuous (second order) phase transition, usually characterized by

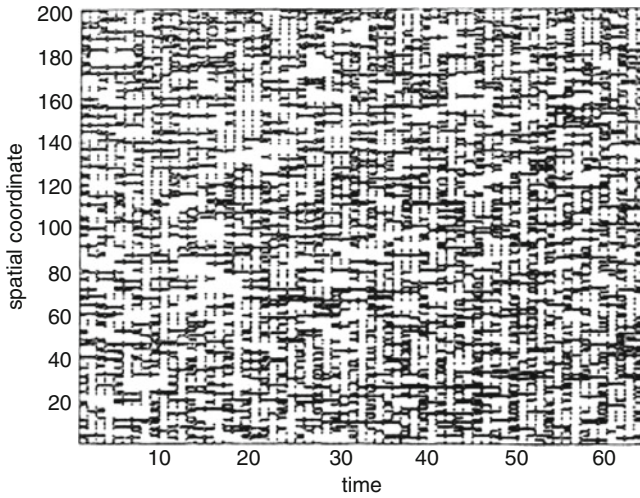


Fig. 12.16 Spatiotemporal pattern in binary reduction for $\beta_0 = 0.1$, well in the turbulent regime

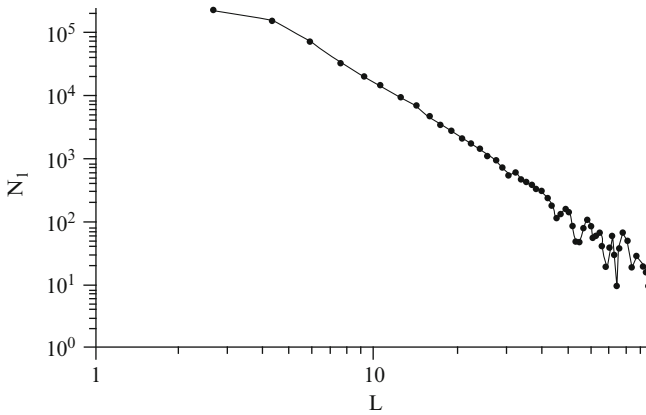


Fig. 12.17 Histogram of laminar domains sizes for $\beta_0 = 0.06$, near the threshold

a critical exponent that scales the variation of the order parameter near the transition point. The order parameter in this framework is defined as the mean number of laminar or turbulent domains. The distribution of sizes of laminar domains, or the corresponding distribution of sizes of clusters of laminar sites, defines a correlation length (or size) that characterizes the patterns; Figs. 12.16–12.18 clearly confirm the existence of a critical threshold value. Near the threshold ($\beta_0 = 0.06$), the distribution of sizes of laminar domains is characterized by the power law behavior (with the characteristic exponent of the order of 3.0), while deep in the chaotic region ($\beta_0 = 0.1$) the behavior follows the exponential behavior with the characteristic exponent 1.5. An analogous statistical analysis was performed for the time domain

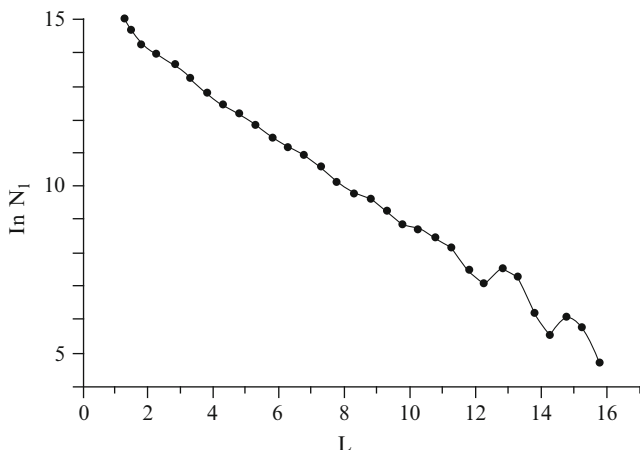


Fig. 12.18 Histogram of laminar domains sizes for $\beta_0 = 0.1$, well in the turbulent regime

distributions, and the temporal approach displays features similar to those obtained for the spatial distributions. The threshold in this case remains approximately the same ($\beta_0 = 0.06$) although the exponents are different. The distribution follows an algebraic decay, with a characteristic exponent 3.2, while above the threshold the decay is exponential with the characteristic exponent 1.2.

12.2.8 Summary

A study of the open, convective, weakly confined dissipative model of SRS as a paradigm of 3WI in an extended system was presented. The numerical simulation reveals a particularly beautiful example of a quasiperiodic route, accompanied by frequency locking, to STC via STI [309, 332]. The striking feature of this scenario is intermittency in both space and time scales, with laminar regions exhibiting the quasiperiodic nature of the preceding attracting state, as distinguished from the chaotic domains by the change in spatial symmetry. The occurrence of the second frequency in the power spectrum of the quasiperiodic regime is apparently due to the appearance of a new spatial mode in the case of the backscattered wave (traveling wave in the case of the EPW), suggesting a complex interplay between spatial and temporal degrees of freedom. Changes in the topological dimension of the chaotic attractor can be directly correlated with changes in the number of active modes, and a similar conclusion is valid for the spatiotemporal entropy. An important item of information provided by this analysis is that it supports the view that the route to low-dimensional chaos represents the main dynamical frame on which the route to STC is built. The coarse graining of the space-time data into binary variables enables the use of the methods from the theory of critical phenomena to draw qualitative

parallels between the transition from STI to STC and directed percolation. The final claim in this section establishes the place of 3WI in the growing family of physical phenomena that display the intermittent route to STC [331].

12.3 Self-Organization in a Dissipative 3WI-Saturated SRS Paradigm

Next, a nonlinear 3WI in an open dissipative plasma model of a stimulated Raman backscattering is studied. An anomalous kinetic dissipation due to electron trapping and plasma wave breaking is accounted for in a hybrid kinetic-fluid scheme. We simulate a finite plasma with open boundaries and vary a transport parameter to examine a route to spatio-temporal complexity. An interplay between self-organization (SO) at micro (kinetic) and macro (fluid) scales is found through quasi-periodic and intermittent evolution of dynamical variables, dissipative structures and related entropy rates. A consistency with a proposed scenario of SO is examined [357, 358].

12.3.1 Introduction

In recent papers on complexity and SO in plasmas, a profound underlying structure in strongly nonlinear and complex plasma phenomena was revealed [354, 355]. SO is a generic process of a creation of order in a nonlinear far-from-equilibrium system open to an environment [356]. Free energy supply, nonlinear instability, and structural bifurcation which result in dissipation, entropy production, and its subsequent removal from a system are key governing points [355, 356]. The above concept, as a working hypothesis, was successfully applied in studies of markedly diverse phenomena of the macro-scale MHD and micro-kinetic self-organization in plasmas. For a continual pumping of free energy and efficient excess entropy removal, generic SO to an intermittent state was found [355, 356].

In this section, we examine an open convective dissipative model of a stimulated Raman backscattering. In fluid simulations (*vide supra*), rich spatio-temporal complexity, which exhibits transition to intermittency and chaos following a quasi-periodic route was revealed [309]. Detailed analysis of spatiotemporal patterns, examining the partition of energy among coherent structures, has found a growing complexity and chaos as the pumping increases. However, based on general advancements in studies of plasma complexity [354, 356], it appears plausible that due to turbulence related anomalous dissipation, SO to a state of reduced complexity should be realized. To emulate the effect of entropy balance, a hybrid 3WI model that includes a phenomenological kinetic dissipation via particle trapping and wave breaking was originally proposed by [322, 357].

12.3.2 *Preliminaries on Nonlinear Kinetic SRS*

A resonant nonlinear 3WI, as a physical concept, is a paradigmatic phenomenon which has found applications in [18,291,331]. SRS involves parametric coupling of an electromagnetic pump to an EPW and a scattered electromagnetic wave [314]. Various applications in laboratory: laser and radio-frequency wave driven plasmas, as well as in space and astrophysical plasmas studied [359,363,364], find anomalous dissipation and heating related in nonlinear SRS. In advancing our previous model, here, a nonlinear evolution of a stimulated Raman backscattering in an open dissipative plasma system is examined. Generally taken, we have shown that invariants breaking points to an onset of nonstationarity for conditions of nonlinear phase detuning (12.32). Spatiotemporal complexity in a fluid model of stimulated Raman backscattering in a bounded weakly dissipative plasma was attempted in [309]. A continual increase in complexity with a control parameter (e.g., pump strength) was predicted by this model as shown above, thus establishing its place in a family of paradigmatic physical phenomena that display an intermittent route to spatiotemporal chaos [331]. However, the effects of anomalous Raman dissipation and plasma electron heating followed by entropy expulsion were omitted. It is a purpose of this section to introduce a plausible entropy inventory by a phenomenological modeling of anomalous kinetic dissipation related to Raman complexity, following [357]. In long saturated regimes, SO generic to an open dissipative system under a continuous free energy supply is expected [354].

Extensive studies of nonlinear stimulated Raman backscattering have been performed by analytics, fluid and kinetic-particle simulations [314,321,359], and references therein. In a strongly driven case, Raman instability exponentiates until arrested by nonlinear and dissipative effects. The saturation comes, basically, through pump depletion and/or higher-order nonlinearities as well as kinetic dissipation related to electron trapping and plasma wave breaking [314,321,359]. While pump depletion is readily included in fluid modeling, the latter effects are inherently kinetic. However, after more than three decades of intensive particle simulation studies, nonlinear Raman scattering is understood to possess relatively clear albeit anomalous overall features. As a result of electron trapping and breaking of large plasma waves, a hot tail-suprathermal electron population is generated. The corresponding velocity of hot (fast) electrons roughly equals the phase velocity of the EPW. As a general feature, two temperature (Maxwellian like) electron distribution is recorded, for the thermal -bulk and suprathermal -hot tail electron distributions. Energy exchange and hot component thermalization leads to an increase in the bulk temperature at the expense of plasma wave dissipation. However, actual details of this overall scenario are determined by complex wave turbulence and the electron transport, both influenced strongly by boundary and other plasma conditions. This qualitative understanding of anomalous Raman features has enabled useful scaling relations and semi-empirical formulas, typically extracted by averaging over time and shots of short-run particle simulation data. Generally taken, a realistic long time saturation (e.g., 10,000 plasma wave periods) does not appear to be assessable to even top performance particle simulations due to required computation time and

limitations of the numerical scheme involving large number of particles [311, 314]. It is this situation that has motivated us to address a problem of anomalous Raman in a long time evolution. A potential saturation to self-organizing plasma states using a general concept of complexity in plasmas in a system open to an environment is presented [357]. First, a phenomenological hybrid fluid model to try to emulate basic physics of anomalous Raman scattering, as a qualitative precursor of state-of-the-art future particle simulation with open boundaries, is introduced.

At this point, we may refer to an extensive analysis of nonlinear Raman saturation by Rose et al. [359–361]. In distinction to a simple adiabatic cubic nonlinearity (e.g., relativistic type) in our model equation for the EPW, these author(s) introduce nonlinearly coupled Langmuir-ion wave dynamics. Based on Zakharov's fluid model, processes such as Langmuir decay, wave collapse, and density modification are included in such description. One- and two-dimensional (1-2D) simulations allowed comparison with realistic laser-plasma experimental conditions. In comparison, a weakly coupled 3WI model which includes basic kinetic effects in 1D Raman model will be shown below. Although simple, kinetic dissipation gets self-consistently coupled with nonlinear Raman dynamics to study self organized saturated states. An important fact appeared, that both the mentioned approaches predict an intensity dependent spectral broadening and incoherence paradigm for Raman backscatter spectrum, as was observed in a number of experiments and simulations [316, 317, 359].

12.3.3 Dissipative SRS Saturation Model

Stimulated Raman backscattering in a plasma is a paradigm of a three-wave parametric decay interactions, and in the one-dimensional case, which is of main importance, the linear parametric Raman backscatter growth rate is given

$$\gamma_0 \approx 0.5\beta_0 \sqrt{\alpha/(1-\alpha)}\omega_0, \quad (12.37)$$

where α is the ratio between the electron plasma frequency and the frequency of the pump:

$$\alpha = \omega_{pe}/\omega_0 = \sqrt{n/n_{cr}} \quad (12.38)$$

and the quantity β_0 , as shown earlier, is a relative pump strength.

In a bounded, uniform, fully ionized plasma, the spatio-temporal evolution of the coupled waves' normalized, slowly-varying complex amplitudes $a_i(\xi, \tau)$ governed by the known set of PDE in dimensionless units, as presented in the previous section, namely

$$\frac{\partial a_0}{\partial \tau} + V_0 \frac{\partial a_0}{\partial \xi} = -a_1 a_2, \quad (12.39)$$

$$\frac{\partial a_1}{\partial \tau} - V_1 \frac{\partial a_1}{\partial \xi} = a_0 a_2^*, \quad (12.40)$$

$$\frac{\partial a_2}{\partial \tau} + V_2 \frac{\partial a_2}{\partial \xi} + \gamma a_2 + i\sigma |a_2|^2 a_2 = \beta_0^2 a_0 a_1^*, \quad (12.41)$$

where time and space coordinates are as usual, measured in units ω_0^{-1} and L^{-1} , respectively, with plasma wave damping rate given by γ . Ions are kept fixed to preserve the plasma quasineutrality. Damping of the light waves is neglected, while T designates the bulk electron temperature directly proportional to EPW group velocity, $V_2 \sim T_e$. An important feature of the system (12.39–12.41) is the self-modal cubic term in the plasma wave equation. It appears as a nonlinear phase shift due to a detuning of a large amplitude plasma wave. The spatially extended model has revealed rich complexity related to low-dimensional as well as spatio-temporal chaos [292, 309, 331].

The system (12.39–12.41) can be solved in space-time for standard initial and finite boundary conditions. In the loss-free steady state with zero phase shift, it predicts the well known elliptic function solutions, together with three (Manley–Rowe) conserved integrals [18]. However, it was shown above that for a nonzero phase shift ($\sigma > 0$) in finite boundaries, the third phase involving invariant is broken (12.32). Thus, a violation of the steady-state assumption points to a nonstationary Raman saturation. Indeed, subsequent evolution exhibits a quasiperiodic route to low dimensional intermittency to finally result in a fully developed STC in previous fluid model. However, we note that in a physical sense chaotic dynamics is related to plasma wave breaking followed at a kinetic level by a strong nonlinear electron acceleration and heating. In turn, hot electrons can Landau damp freshly SRS driven plasma waves to suppress and strongly alter Raman instability evolution and possibly limit the level of kinetic complexity. To perform studies in this direction, a phenomenological hybrid 3WI dissipative SRS model is introduced. The set of 3WI equations is solved simultaneously with simple model equations for hot and bulk plasma heating. In this way, effective damping $\gamma(t)$ and the electron temperature $T(t)$ in 3WI, appear as dynamical variables, in contrast with a standard model that assumes a constant plasma background. Therefore, dissipative effects on a long time Raman saturation and kinetic self-organization in an open system can be emulated [357].

12.3.4 Kinetic-Hybrid Scheme

It was chosen to simulate conditions relevant to anomalous Raman saturation in an open system, which means allowing an energy exchange between an interaction region and the plasma environment. To emulate basic kinetic effects missed by the earlier fluid 3WI model of (12.39–12.41), a phenomenological “hybrid” scheme which includes generation of hot electrons that are trapped and accelerated in large amplitude plasma wave was introduced [357]. Assuming that a part of plasma wave energy is transferred to electrons that are resonant with the forward propagating EPW, hot electron generation equation together with a 3WI set (12.39–12.41) can be numerically solved. As a consequence, the suppression of Raman instability by

hot electrons through a Landau damping is found. We further assume that the effective damping (γ term) of plasma waves is due to both linear Landau damping on hot electrons and nonlinear term, due to bulk electron acceleration via electron trapping in large EPW. Finally, a simple total energy balance equation to model bulk heating via the redistribution of the absorbed energy between bulk (thermal) and hot (suprathermal) electrons is added. It seems that, although straightforward, this idea appears to be a rare attempt to treat a rather complex, inherently kinetic regime of anomalous Raman by a simple fluid-based model. Simplifying the electron transport to spatially averaged dynamics a hybrid like coupled mode scheme that includes effects of both thermal and hot electrons on Raman instability is introduced. Open boundaries are carefully accounted for, thus enabling to model conditions of both current-free streaming and inhibited electron transport. A brief check of the model performance against particle simulation is done in order to try closer fits by adjusting a free transport parameter in the scheme. In that way, a longtime Raman saturation in an open system is addressed, an important question that remains difficult to answer even by highest performance particle simulations [314, 362].

We start by assuming the electron distribution function, including thermal (bulk) and hot component, to be approximated by bi-Maxwellian electron distribution [314, 357]:

$$F(x, t, v) = n_h(x, t) f_h(v) + n_b(x, t) f_b(v), \quad (12.42)$$

where n_h and n_b ($\gg n_h$) stand for slowly varying hot and bulk electron density, respectively, with each $f_i(v)$ normalized to unity. We assume that the total hot electron current includes a Source term due to trapped resonant electrons (in the thermal Maxwellian tail). Therefore, we write

$$\begin{aligned} j_h(x, t) = \int_{\text{hot}} v F(v) dv = n_h(x, t) \int_{-\infty}^{\infty} v f_h(v) dv \\ + n_b(x, t) \int_{v_{\text{ph}} - v_{\text{tr}}}^{v_{\text{ph}} + v_{\text{tr}}} v f_b(v) dv, \end{aligned} \quad (12.43)$$

where v_{ph} is the plasma wave phase velocity and v_{tr} is characteristic velocity of resonant electrons ($v \sim v_{\text{ph}}$) with orbits trapped in a trough of a large amplitude plasma wave (a_2) (in physical units, $v_{\text{tr}}(t) \sim (2eE_2/mk_2)^{1/2}$), where the maximum E_2 amplitude allowed is limited by the plasma wavebreaking condition [314, 327, 328]. We note that the above source term vanishes for small amplitude EPW ($v_{\text{tr}}(t) \sim 0$), since a contribution via inverse linear Landau damping on thermal electrons is negligible (since, $v_{\text{ph}} \gg v_{\text{te}}$).

Equation of continuity for hot electrons is written in a standard form

$$\frac{d}{dt} n_h(x, t) \equiv \frac{\partial}{\partial t} n_h(x, t) + \nabla \cdot j_h = 0, \quad (12.44)$$

or after performing the spatial average by integration, defined as

$$\langle \dots \rangle_L = \frac{1}{L} \int_0^L (\dots) dx, \quad (12.45)$$

where $n_h(t)$ is hot electron density averaged over the plasma length L , we obtain

$$\frac{d}{dt} n_h(x, t) + \frac{1}{L} j_h(x, t) \Big|_0^L = 0. \quad (12.46)$$

Using the electron current, one gets equation for the hot electron generation

$$\frac{d}{dt} n_h(t) = \frac{n_b(L, t)}{L} \int_{v_{ph}-v_{tr}(L, t)}^{v_{ph}+v_{tr}(L, t)} v f_b(v) dv - \kappa n_h(t). \quad (12.47)$$

We note that $v_{tr}(0, t) = 0$, due to zero boundary condition for a plasma wave. The Sink term is due to electrons lost through open plasma boundaries, with the transport coefficient $\kappa = k v_h / L \approx k v_{ph} / L$ and coefficient k , which is equal to 1 and 2 for a free streaming and a Maxwellian flow, respectively. We now proceed to self-consistently evaluate the effective damping rate in the plasma wave equation (12.41). We assume that a total damping is due to both linear Landau and nonlinear (trapping and wave breaking) effects, namely, the total damping rate: $\gamma(t) = \gamma_{\text{Landau}} + \gamma_{\text{nl}}$, where for the linear Landau term we shall use a standard formula appearing in the literature [314] Further, we introduce the spatially integrated plasma wave energy density, through

$$W(t) = \frac{1}{L} \int_0^L \frac{1}{8\pi} |E(x, t)|^2 dx. \quad (12.48)$$

The effective dissipation rate of the EPW energy through both linear and nonlinear processes is

$$2\gamma W(t) = 2\gamma_{\text{Landau}} W(t) + \frac{m n_b(L, t)}{2L} \int_{v_{ph}-v_{tr}(L, t)}^{v_{ph}+v_{tr}(L, t)} v^3 f_b(v) dv, \quad (12.49)$$

where an integral term gives a nonlinear contribution to a plasma wave damping, thus effectively determining the value of γ_{nl} .

Finally, equation for the total energy balance between the plasma wave, the thermal and hot electron component, by starting from a general conservation law (w_i denotes energy, ϕ_i denotes energy flux) is given

$$\frac{d}{dt} \sum_i w_i(x, t) = \frac{\partial}{\partial t} \sum_i w_i(x, t) + \text{div} \sum_i \phi_i(x, t) = 0. \quad (12.50)$$

For spatially averaged (integrated) quantities, one has to evaluate the energy flux at open boundaries

$$\frac{d}{dt} \sum_i W_i(x, t) + \frac{1}{L} \sum_i \Phi_i(x, t) \Big|_0^L = 0. \quad (12.51)$$

For our system, the plasma wave, thermal and hot electron component: $W_i(t) \Rightarrow W + E_{\text{th}} + E_{\text{h}}$, the electron bulk and hot energy densities are simply: $E_{\text{th}} = n_{\text{b}} T_{\text{b}}$ and $E_{\text{h}} = n_{\text{h}} T_{\text{h}}$; and the average energy flux is given by $\Phi_{\text{th}} = \pm v_{\text{th}} E_{\text{th}}$ and $\Phi_{\text{h}} = \pm v_{\text{h}} E_{\text{h}}$. For the Maxwellian distribution, the above energy flux has to be multiplied by a weighting factor $\simeq 0.65$. However, in real plasmas, an inhibited energy flux is modified by a parameter $k \sim (0 - 1)$ in a largely heuristic model of a highly complex electron energy transport [314]. The $\pm$ sign indicates the flux direction at the plasma boundary (at $x = 0, L$). By using the above expressions, the equation for the thermal energy variation is given as

$$\frac{d}{dt} [n_{\text{b}}(t) T_{\text{b}}(t)] = 2\gamma W(t) - \frac{d}{dt} [n_{\text{h}}(t) T_{\text{h}}(t)] - \frac{k}{L} [\Phi_{\text{th}} + \Phi_{\text{h}} - \Phi_{\text{q}}] \Big|_0^L, \quad (12.52)$$

where we have introduced Φ_{q} as the return flux of fresh (cold) ambient electrons through an open plasma boundary (*vide infra*). We note, in passing, a simplified form of (12.46) with respect to a lack of a finite relaxation time, that is normally required for a heat transfer via, e.g., hot-bulk electron collisions.

12.3.5 Open Boundary Model

A simulation model with boundaries open to electromagnetic waves and plasma electrons is introduced. Accordingly, a transport between an interaction region and a large surrounding plasma environment is allowed. For electrons which escape from an interaction layer (length L) fresh ambient electrons are re-injected in order to preserve the plasma quasineutrality. Accordingly, a long-time Raman saturation could be observed under conditions with physically realistic current free boundaries.

A straightforward procedure is briefly sketched. The total electron current at the boundary is written as an algebraic sum of the outgoing and the incoming components: $J_{\text{tot}} = J_{\text{out}} + J_{\text{in}}$, with $J_{\text{out}} = J_{\text{th}} + J_{\text{h}}$, for thermal and hot electron contributions. Further, $J_{\text{in}} = n_{\text{q}} v_0$, where $n_{\text{q}} v_0$ stands for a current of ambient electrons streaming into a plasma layer. By requiring the total current at the boundary to be zero, one readily evaluates J_{in} in terms of the thermal and hot components. The energy flux carried by ambient electrons (with temperature T_0) is simply $\Phi_{\text{q}} \simeq J_{\text{in}} T_0$, making the calculation of the loss term in the thermal balance equation an easy task.

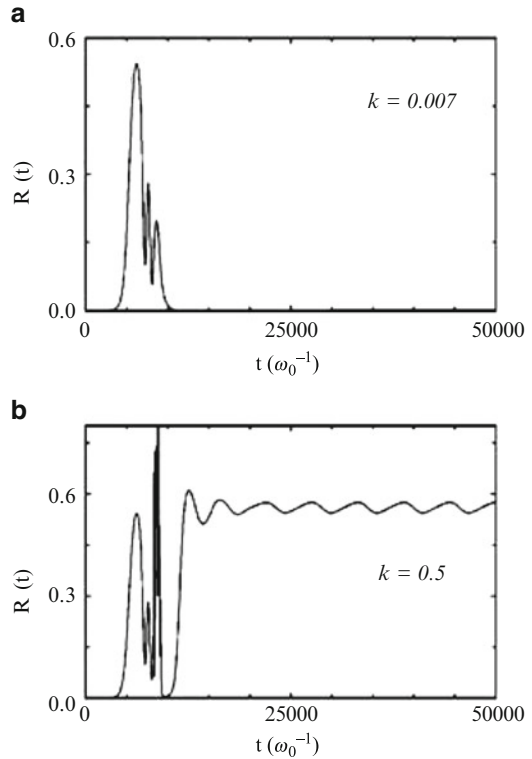
In further text, we refer to the above model as the “open” one, in contrast with the “closed” plasma model where an electron transport is inhibited by a build up of a space charge. The latter case corresponds to, e.g., plasma-vacuum boundary (e.g., exploding foil laser plasma), with the energy flux (Φ), coefficient restricted to low values (few percent). As for the Maxwellian with an open boundary, we get the factor, $k \sim 0.65$; one should expect a wide range of dynamical regimes between these two extreme transport cases [314].

12.3.6 Self-Organization at Micro- and Macro-Scales

The Raman complexity obtained in simulations is briefly analyzed. Parameters of the “standard case” studied in depth in a fluid 3WI model are chosen (vide supra) [309, 357]. Initial plasma parameters are as follows: the electron density is 0.1 of the critical density, the electron temperature of 0.5 keV and the plasma length $L = 100 c/\omega_0$. Further, the electron transport coefficient k is taken to be the basic *control* parameter. A continuous pumping is applied to observe different saturated states for an open and closed (isolated) system. Typically, the reflection and transmission coefficients in time, as well as the evolution of hot electron density and bulk temperature, are plotted. The pump strength is equal to 0.025. For two cases of an open ($k = 0.5$) and nearly closed ($k = 0.007$) plasma system, after transient pulsations, reflectivity saturates to a quasisteady state. As expected, the transmittivity follows the same scenario. However, while for the open system reflectivity saturates to a high-finite value, in a strongly confined-closed system, reflectivity quickly drops to zero due to a complete Raman suppression (see Figs. 12.19 and 12.20). More precise insight into a phase space dynamics finds out that, while the closed system saturates to an exact steady state (*fixed point*), in the open case, small periodic oscillations (*limit cycle*) are present. Moreover, in the latter case, the moderate hot electron population, which is locked to a finite-plasma wave as a source of hot electrons, saturates to a quasisteady state. In distinction, in the closed system, hot density is high and nonstationary, typically an order of magnitude higher than above rapidly generated during an abrupt dissipation of Raman driven large plasma waves (Fig. 12.21). It gradually relaxes in later times due to a convective cooling through the boundaries. Similar to hot population, important difference exists in the bulk temperature evolution. In the open system, temperature saturates to a steady-state, moderately above an initial-ambient temperature. This is due to a continual energy input via kinetic dissipation balanced by efficient convection losses. In the closed system, rapid and large temperature rise is observed, to reach its maximum by halting Raman instability, later to experience slow cooling through transport inhibited boundaries (Fig. 12.22).

Further, by carefully varying a control parameter k , generic structural bifurcations along the route to complexity are studied (see Figs. 12.23–12.26). For $k = 0.05$, in the system, a bifurcation to a new state of kinetic self-organization is revealed. Structural instability transits to a quasiperiodic dynamical state, observed

Fig. 12.19 Raman reflectivity in time versus transport parameter k , for (a) closed (inhibited transport) and (b) open system



readily as a train of temporal pulses in the reflectivity and transmittivity. Hot electron population follows with strong quasi-periodic pulsations peaked around 20% of the initial electron density. On the other hand, the bulk temperature, after its initial growth, exhibits strong sawtooth oscillations reminiscent of the MHD type observed in magnetically confined fusion plasmas. By further exploring a parameter space for the open system with $k = 0.9$, a transition to a quasiperiodic dynamics interrupted by chaotic bursts is found. Closer insight into the attractor space finds irregular portion of the dynamics, pointing to an intermittent nature of this regime. Indeed, hot electrons are intermittently ejected in a form of intense jet spikes ($T_h \sim 22$ keV) as a striking feature of this type of kinetic self-organization (Fig. 12.25). Bulk temperature follows an intermittent scenario, by exhibiting quasiperiodic (QP) fluctuations, somewhat above its initial value (Fig. 12.26). Finally, a temporal route to complexity by plotting a phase space attractor for the hot electrons is shown (Fig. 12.27). By varying a transport parameter k (0.7–0.9), a gradual onset of complexity and chaos is revealed, starting with typical “stretching and folding” features of the periodic trajectories.

Fig. 12.20 Raman reflectivity in time versus transport parameter k , for (a) closed (inhibited transport) and (b) open system

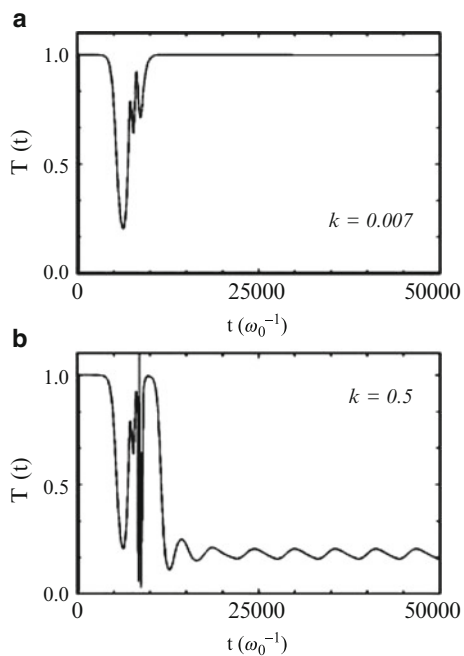


Fig. 12.21 Hot electron density variation in time for (a) closed and (b) open system

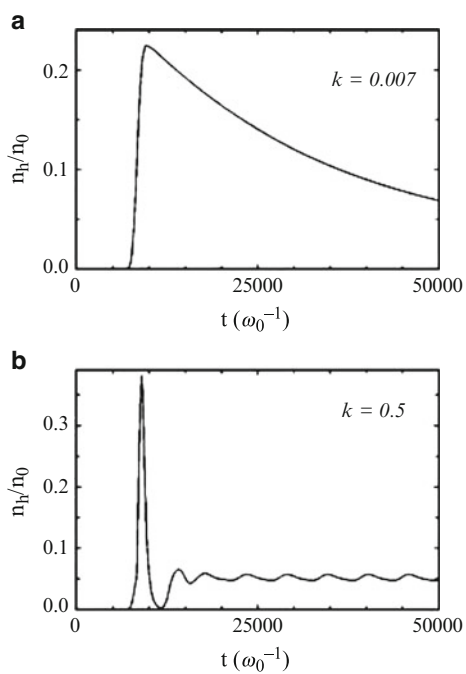


Fig. 12.22 Bulk electron temperature in time versus k for (a) closed and (b) open system

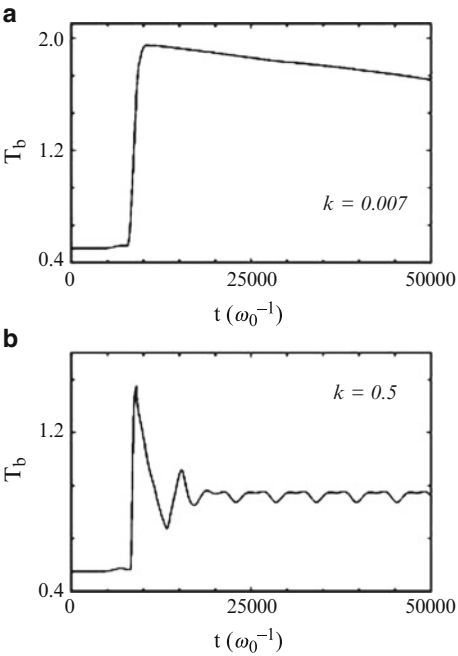


Fig. 12.23 Raman reflectivity in time shows complexity of (a) quasiperiodic and (b) intermittent type

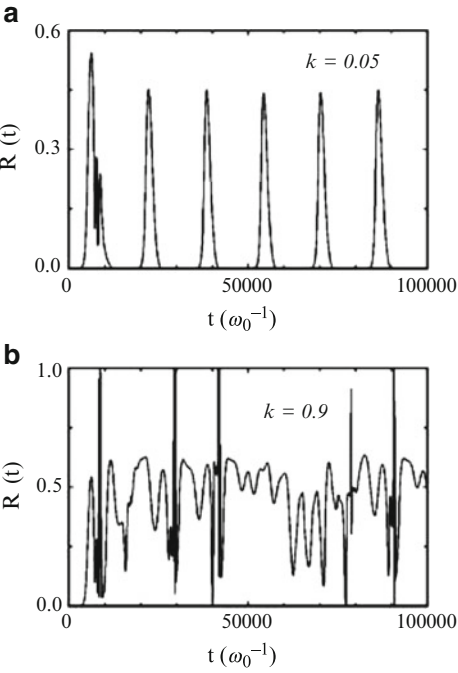


Fig. 12.24 Raman transmittivity in time for (a) quasiperiodic and (b) intermittent regime

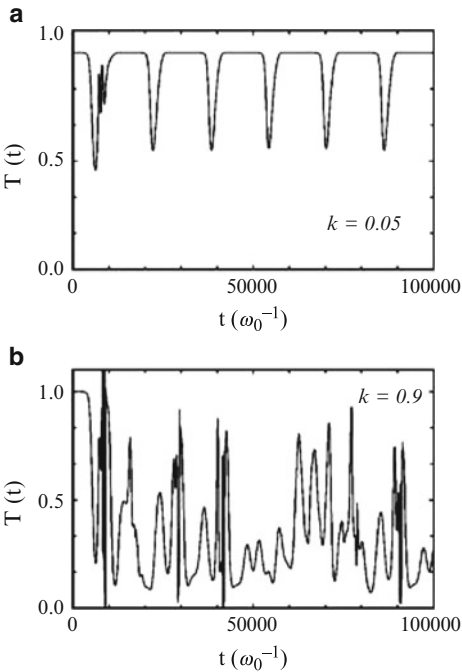


Fig. 12.25 Hot electron density in time shows (a) quasiperiodic and (b) intermittent pulsations

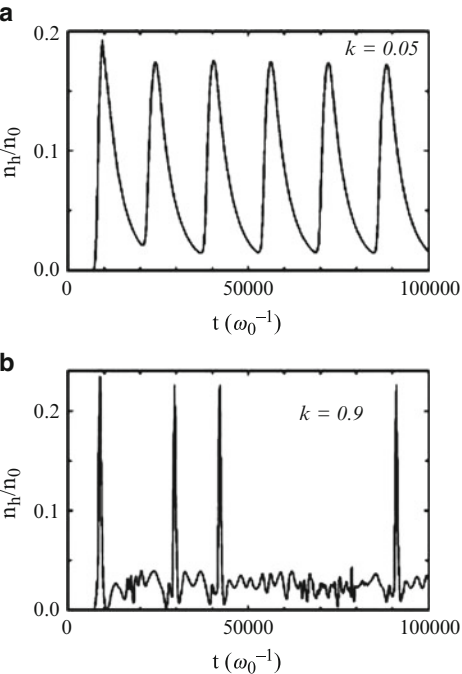
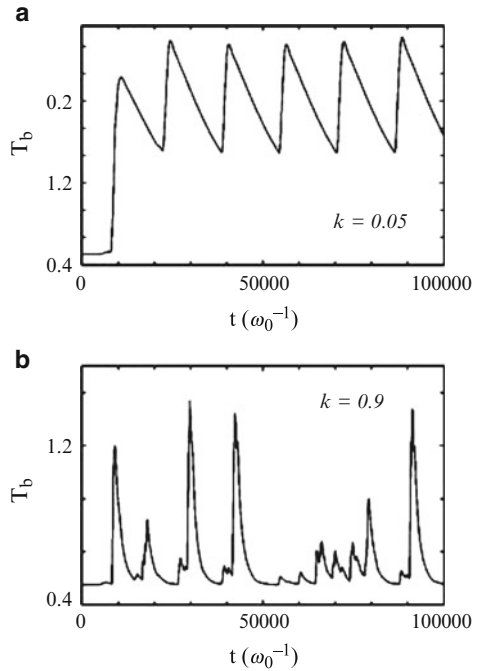


Fig. 12.26 Bulk electron temperature in time exhibits (a) sawtooth and (b) intermittent pulsations



12.3.7 Dissipative Structures and Entropy Rate

SO in strongly nonlinear far-from equilibrium systems can lead to a creation of ordered states that reflect an interaction of a given system with its environment. These dynamical structures or patterns, named dissipative structures to stress the crucial role of dissipation in their creation, have become a central theme of the science of complexity [354, 356]. On the other hand, there is a fundamental role of the entropy, in particular, the rate of entropy change in an open system. The rate of entropy production and its removal basically governs SO features of a system. A large amount of effort has been spent in attempts to relate the entropy rate extrema to structural bifurcations and transitions between different ordered states [354, 356].

First, focus is at self-organized dissipative structures developed at macro scales. Indeed, in above model, basic wave and fluid density variables were assumed to vary slowly in space-time. Therefore, we expect that original spatiotemporal profiles, found in simulations, should correspond to large dissipative structures, self-organized at macroscale levels. As an illustration, we plot the plasma wave profile (Fig. 12.28), in particular, to reveal a genuine spatiotemporal nature of an intermittent regime as compared to regular dynamical regimes of the steady-state and QP type. Spatiotemporal complexity of quasisteady and traveling wave patterns with regular and chaotic features is found in different states of SO.

However, a hybrid nature of this model will also allow to recover kinetic type properties of SO. By using an analytical dependance of the electron Maxwellian

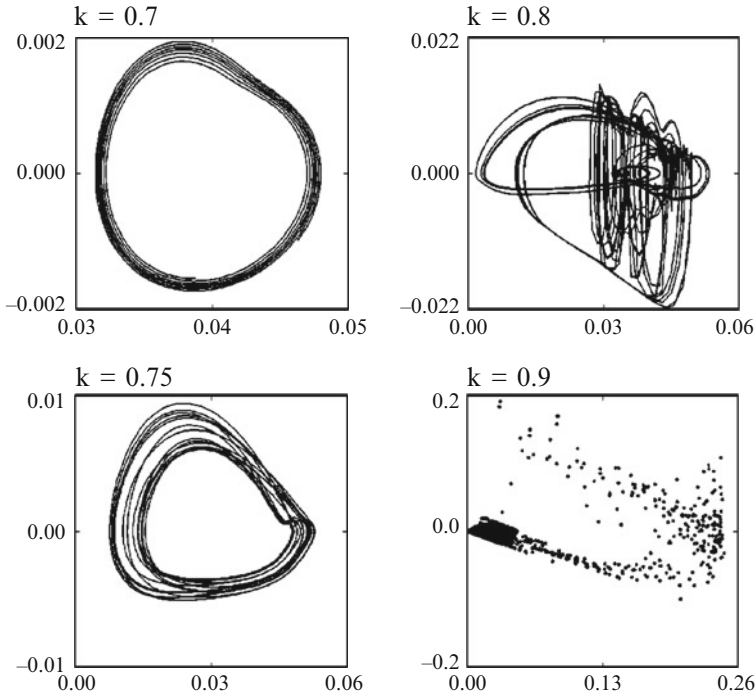


Fig. 12.27 Phase diagrams for hot electrons show stretching and folding of orbits reminiscent of chaos

distribution on time varying hot (bulk) temperature and density (12.42), a genuine picture of kinetic self-organization at plasma microscales is exposed. To show the SO featuring micro-levels, the electron velocity distribution function is plotted. In Fig. 12.29, a three-dimensional view of the electron velocity distribution in time for different saturated Raman regimes is shown, as indicated by values of parameter k . Kinetic self-organization of varying complexity is revealed in thermal and hot regions of the electron distribution. Furthermore, one can observe a complex connection and interplay between SO at *macro* and *micro* levels in a plasma.

Finally, in Fig. 12.30, the entropy rate $dS(t)/dt$ in time together with a spatio-temporal profile of the scattered wave energy are presented. To evaluate the entropy S related to electron distributions as: $S(t) = S_b(t) + S_h(t)$, one uses

$$S_i(t) = - \int_0^L dx \int_{-\infty}^{\infty} dv f_i(x, v, t) \ln f_i(x, v, t), (i = b, h). \quad (12.53)$$

For an intermittent regime, featuring an interchange between chaotic and laminar phases, clear evidence of structural transitions corresponding to the maximum (positive) and minimum (negative) entropy rate is found. As a striking example of

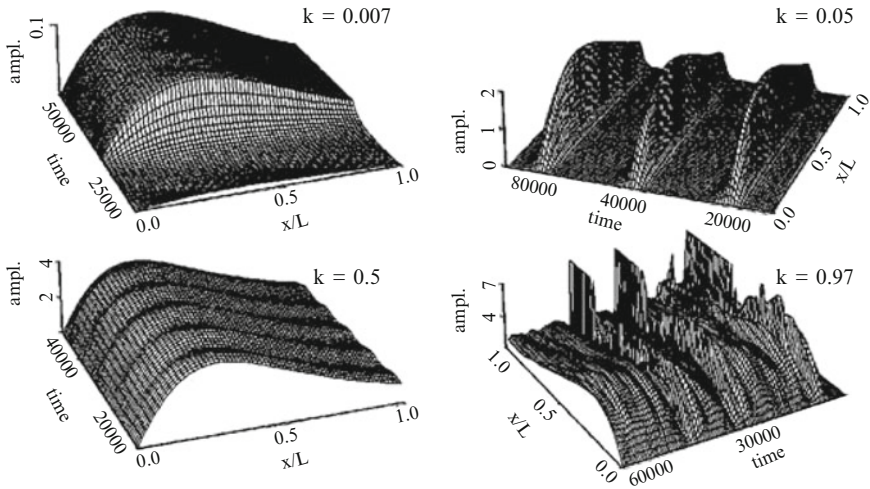


Fig. 12.28 Spatiotemporal dissipative structures of electron plasma wave for different k values exhibit a varying level of complexity; from the steady-state via quasi-periodic to intermittent regimes

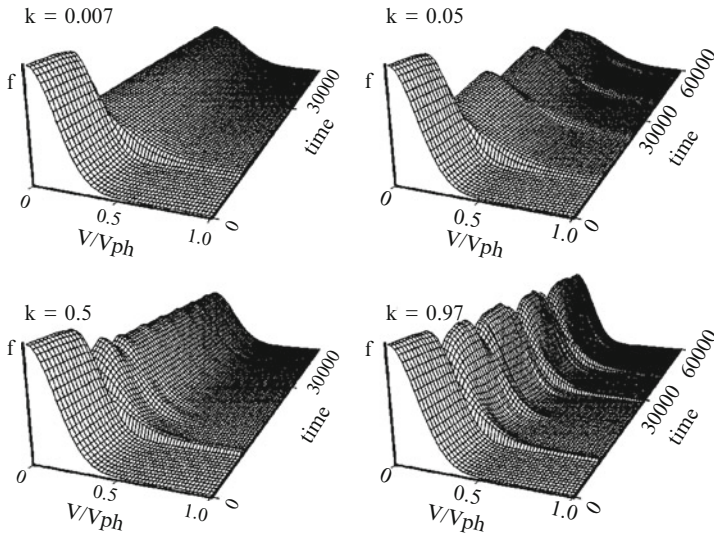


Fig. 12.29 3D view of $f_e(v)$ for different saturated regimes reveals micro-kinetic SO of both thermal and suprathermal electron components

SO in an open system, a rapid entropy increase which coincides with an onset of a chaotic phase is revealed. Subsequent anomalous dissipation and entropy growth are halted by a sudden entropy expulsion into the environment. Negative burst in entropy rate indicates a bifurcation from a chaotic, back to a laminar quasiperiodic

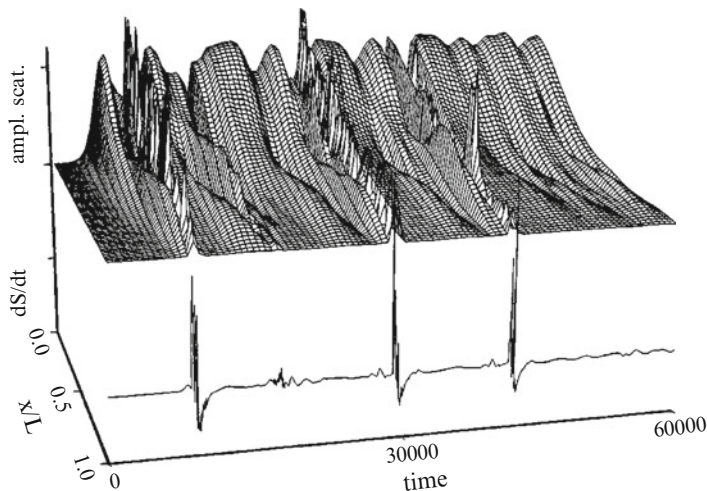


Fig. 12.30 Dissipative plasma wave structures vs. entropy rate in time. Positive entropy rate jumps coincide with onset of chaos, while negative bursts indicate transition to a laminar phase of SO at macro-scales

phase. An intermittent nature of this regime is shown through a repetitive pattern of behavior. We note that complex spatiotemporal dissipative wave structures are mapped onto a more simple entropy rate time series. Intervals of near zero entropy rate during a laminar phase mean a net balance between the entropy production and the expulsion. This appears to be an example of a stationary nonequilibrium state possibly realized in a strongly nonlinear open system [356] (Fig. 12.31).

12.3.8 Summary

In summary, above simple findings appear to be the early indication of a generic intermittent scenario in a kinetic self-organization of anomalous Raman instability. Although phenomenological rather than rigorous, above dissipative 3WI open model has self-consistently accounted for the entropy production and removal for both thermal and suprathermal electrons. In this way, rich transient Raman complexity gradually gets self-organized and attracted to definite saturated dynamical states, such as: steady-state, quasi-periodic, and intermittent ones. At this point, one is able to claim a type of consistency with the working hypothesis and general scenario of SO in plasmas [365]. As an illustration, we show in Fig. 12.32, some particle-in-cell simulation data for a model of an isolated plasma slab (thin foil plasma layer) in vacuum [311]. For the same parameters as above, particle simulations show an evident support for Raman reflectivity patterns (Fig. 12.19) obtained above for a nearly closed ($k = 0.007$) plasma system. Finally, above ideas may offer a qualitative

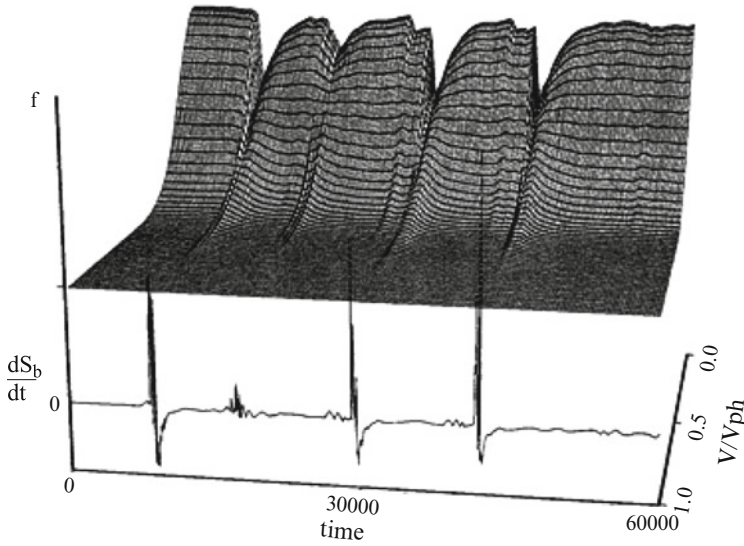


Fig. 12.31 Intermittent electron distribution function versus entropy rate in time. Similar to Fig. 12.30, structural bifurcations at micro-scales coincide with entropy rate extrema

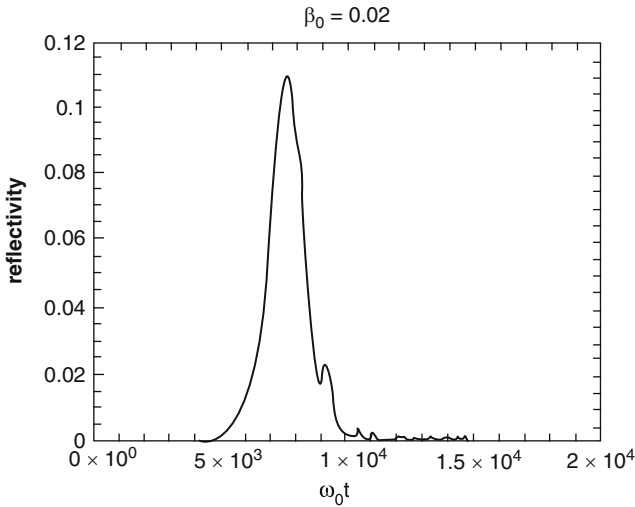


Fig. 12.32 Raman reflectivity in time from $1\frac{1}{2}D$ EM particle-in-cell simulation, for laser pump of 0.02 shows remarkable qualitative agreement with Fig. 12.19 (a) obtained from a simple 3WI hybrid model

insight into nonlinear saturation of laser plasma instabilities in physically realistic laser beam-target plasma configurations. Still, actively developed large comprehensive 3D fluid codes (e.g. pF3D) are mostly limited to linear parametric interaction regimes [366].

Chapter 13

Relativistic Laser Plasma Interactions

Ever since the much acclaimed paper of Akhiezer and Polovin [367], plasma theorists have been attempting to comprehend complex dynamics related to the propagation of high and ultra-high intensity electromagnetic (EM) radiation through a plasma. This topic was successfully revisited a number of years later by Kaw and Dawson [368] whose analysis threw more light on the propagation of coupled longitudinal-transverse waves of arbitrary intensity. The high phase velocity case was soon solved exactly by Max and Perkins [369]. Due to rapid progresses in ultrahigh intensity lasers over last two decades, based on the invention of the chirped pulse amplification (CPA) and advanced femtosecond techniques, it has become possible to drive electrons with relativistic energy opening up new avenues of relativistic nonlinear optics and plasma physics [371]. The problem of relativistic laser-plasma interactions is of particular interest concerning the fast ignition concept [370], relevant to contemporary laser inertial confinement fusion research. Moreover, the understanding of relativistic laser pulse evolution in a plasma is basic to many new applications, including optical-field-ionized x-ray lasers [372], plasma-based electron accelerator schemes [373, 378], and, the interpretation of some astrophysical phenomena (see [371], and references, therein). Currently, the rich field of relativistic laser matter interaction is diverging into two main broad directions: the first, related to laser fusion, high energy densities and laboratory astrophysics, and the second, related to ultra high field science, high energy particle and photon beam acceleration and ultra-fast attosecond phenomena [374–377]. Still, most of applications require stable beam guiding of intense laser pulses over longer distances, without significant energy losses. The pioneer workers in this field [367–369] did not consider the stability of the plane-wave solutions, related to nonlinear interaction between normal plasma modes. However, in a highly nonlinear system, we have to deal with nonlinearly interlinked wave modes and instabilities, which can evolve into nonlinear structures, chaos, and turbulence. [303, 309, 423].

13.1 Electronic Parametric Instabilities

In a plasma, there exists a number of instabilities that can be classified as parametric excitations of resonantly coupled waves [407]. This is a multi-wave process since in practice an externally driven pump wave can interact with a whole spectrum of waves in a plasma. However, in most cases the process can be well described as three wave interaction as long as certain resonant triplet of waves evolves (almost) independently of others. The three-wave interaction is the lowest-order nonlinear effect for a system that is approximately described by a linear superposition of discrete waves. In order for nonlinear 3WI to occur, the wave frequencies and wave vectors must satisfy matching conditions

$$\omega_- = \omega_0 - \omega_1, \quad \mathbf{k}_- = \mathbf{k}_0 - \mathbf{k}_1, \quad (13.1)$$

where modes “0” and “1” represent pump and plasma waves and mode “-” represent scattered wave (Stokes wave). When the pump amplitude exceeds a certain threshold, the remaining two waves which are initially at noise level start growing absorbing energy from the pump wave. In addition to the above wave triplet described by (13.1), another, usually weaker triplet having two waves in common with the first one can be observed in experiments and simulations. The resonant conditions for this wave triplet are

$$\omega_+ = \omega_0 + \omega_1, \quad \mathbf{k}_+ = \mathbf{k}_0 + \mathbf{k}_1, \quad (13.2)$$

where ω_+ denotes upshifted frequency by the frequency ω_1 . This wave is known as anti-Stokes wave. There is a rich experimental and theoretical support in laboratory and space plasmas for nonlinear wave–wave interactions that involve EM waves, Langmuir waves, and ion-acoustic waves. The decay schemes for these processes are

1. photon $\longrightarrow$ photon + Langmuir wave
2. photon $\longrightarrow$ photon + ion-acoustic wave

where the first type of the decay is known as stimulated Raman scattering (SRS) and the second type as stimulated Brillouin scattering (SBS) [314]. These scattering processes are of particular interest to inertial confinement fusion since the fusion targets are characterized by large regions of an underdense plasma. Much works have been devoted to stimulated Raman and SBS instabilities, concerning their ability to serve as a source of high energetic particles which may preheat the core of a fusion pellet. The stimulated scattering can be large enough to reflect a significant part of the laser light and thus to decrease the laser efficiency at the target. As has been shown by experiments and computer simulations, there can be a rich interplay between these two instabilities [424–426]. In present-day laser-fusion research, SRS instability is of a major concern [424, 427–429].

13.1.1 Stimulated Raman Scattering

As discussed, SRS is a parametric decay of an incident EM wave (ω_0, k_0) into a scattered light (ω_s, k_s) and an electron plasma (Langmuir) wave ($\omega_{\text{EPW}}, k_{\text{EPW}}$). The matching conditions [314, 430] for the frequencies and wave numbers are

$$\omega_0 = \omega_s + \omega_{\text{EPW}}, \quad k_0 = \pm k_s + k_{\text{EPW}}, \quad (13.3)$$

respectively, where

$$k_0 = \frac{\omega_0}{c} \left(1 - \frac{n}{n_{\text{cr}}} \right)^{1/2}, \quad (13.4)$$

$$k_s = \frac{\omega_s}{c} \left(1 - \frac{\omega_p^2}{\omega_s^2} \right)^{1/2}, \quad (13.5)$$

+ (−) denotes forward (backward) scattering, n is the density, $n_{\text{cr}} = n\omega_0^2/\omega_p^2$ is the critical density, and $\omega_p = (ne^2/(\epsilon_0 m))^{1/2}$ is the plasma frequency. In the low-temperature limit, the dispersion equation for Langmuir wave is $\omega_{\text{EPW}}^2 = \omega_p^2 + 3k_{\text{EPW}}^2 v_t^2$, where $v_t = (T/m)^{1/2}$ is the electron thermal velocity and T is the temperature. Since the minimum frequency of EM waves in a plasma is ω_p , the frequency matching condition (13.69) requires ($\omega_0 > 2\omega_p$), i.e., SRS instability can occur for electron densities $n < 0.25n_{\text{cr}}$. However, it turns out that above condition can be shifted to higher densities due to relativistic effects in a large amplitude electromagnetic waves (EMWs). The phase velocity of the plasma wave in the Raman parametric interaction can be slow enough to interact (Landau resonance) with the tail in the bulk electron distribution function. The electron tail can be further heated by becoming trapped in the large electron plasma wave (EPW). This dominantly occurs for stimulated Raman back-scattering (B-SRS) at low plasma densities $v_{\text{ph}} \approx \frac{\omega_p}{2k_0}$. For stimulated Raman forward scattering (F-SRS), the phase velocity is large, close to the velocity of light, and initially very small number of particles have initial energy to be trapped and accelerated. However, when the laser pump propagates in a plasma over a long distance, a large amplitude EPW can be driven which can generate a significant number of high energetic (hot) electrons.

13.1.2 Relativistic Dispersion Relation for Cold Plasma

In order to obtain insight into relativistic parametric instabilities, here we derive a hybrid dispersion equation for EM wave scattering in an unmagnetized homogeneous cold plasma [420, 431]. The ions are fixed as a cold plasma background and dynamics of electrons is relativistic. We assume that the radiation field is linearly polarized ($\mathbf{E} = E\mathbf{e}_y$) and consists of a pump wave

$$\mathbf{A}_0 = \frac{1}{2} A_0 \mathbf{e}_y e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + \text{c.c.} \quad (13.6)$$

and two daughter waves

$$\mathbf{A}_+ = \frac{1}{2} A_+ \mathbf{e}_y e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + \text{c.c.} \quad (13.7)$$

$$\mathbf{A}_- = \frac{1}{2} A_- \mathbf{e}_y e^{-i(\mathbf{k} \cdot \mathbf{x} - \omega^* t)} e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + \text{c.c.} \quad (13.8)$$

propagating in x direction. Here, $\mathbf{A}$ is the vector potential, index “0” denotes the laser wave, while “+” and “-” stand for daughter waves, Stokes ($\omega - \omega_0$, $\mathbf{k} - \mathbf{k}_0$) and anti-Stokes ($\omega + \omega_0$, $\mathbf{k} + \mathbf{k}_0$), respectively. The amplitudes of the scattered waves are assumed to be much smaller than the amplitude of the pump wave (A_+ , $A_- \ll A_0$). As one can see the frequency and wave vector matching conditions are included in expressions (13.7)–(13.8), ω and k represent, respectively, the frequency and wave vector of ES wave. The starting point for the description of the interaction of these waves in a cold plasma fluid is Maxwell equations

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad (13.9)$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (13.10)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (13.11)$$

the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0, \quad (13.12)$$

and the relativistic equation of motion

$$m \frac{d\gamma \mathbf{v}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (13.13)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, where $\mathbf{A}$ and ϕ are vector and scalar potential, ε_0 and μ_0 are permittivity and permeability of free space, $\mathbf{j}$ is the current density, n is the electron density, $\mathbf{v}$ is the velocity, and γ is the relativistic Lorentz factor.

Using vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ from Maxwell equations, one can obtain

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \nabla \left(\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} \right) = \mu_0 \mathbf{j}, \quad (13.14)$$

and, on the other hand, the equation of motion (13.13) can be rewritten as

$$\frac{\partial \gamma \mathbf{v}}{\partial t} = \frac{e}{m} \left(\nabla \phi + \frac{\partial \mathbf{A}}{\partial t} \right) - c^2 \nabla \gamma. \quad (13.15)$$

Here, (13.13), (13.14), and (13.15) will be examined order by order in the amplitudes of the incident waves¹, by writing,

$$\mathbf{j} = -en\mathbf{v} = \mathbf{j}^{[1]} + \mathbf{j}^{[2]} + \mathbf{j}^{[3]} + \dots \quad (13.16)$$

where

$$\mathbf{j}^{[1]} = -en_0\mathbf{v}^{[1]}, \quad (13.17)$$

$$\mathbf{j}^{[2]} = -en_0(\mathbf{v}^{[2]} + n^{[1]}\mathbf{v}^{[1]}), \quad (13.18)$$

$$\mathbf{j}^{[3]} = -en_0(\mathbf{v}^{[3]} + n^{[1]}\mathbf{v}^{[2]} + n^{[2]}\mathbf{v}^{[1]}), \quad (13.19)$$

and

$$\gamma \approx 1 + \frac{1}{2} \frac{v^2}{c^2} + \dots = 1 + \frac{1}{2} \frac{(\mathbf{v}_0 + \mathbf{v}_+ + \mathbf{v}_-)^2}{c^2} + \dots \quad (13.20)$$

Using renormalization $e\mathbf{A}/m \rightarrow \mathbf{A}$ and radiation gauge $\nabla \cdot \mathbf{A} = 0$ in first-order of perturbations, we have

$$\mathbf{v}_\perp^{[1]}(\omega_0, k_0) = \mathbf{A}_0, \quad (13.21)$$

$$\mathbf{v}_\perp^{[1]}(\omega + \omega_0, k + k_0) = \mathbf{A}_+, \quad (13.22)$$

$$\mathbf{v}_\perp^{[1]}(\omega - \omega_0, k - k_0) = \mathbf{A}_-^*, \quad (13.23)$$

where “*” denotes complex conjugate. No density fluctuations are produced in the first-order, $n^{[1]} = 0$.

In the second order from the continuity equation and equations (13.14) and (13.15), we write, respectively,

$$\frac{\partial n^{[2]}}{\partial t} + \nabla \cdot \mathbf{v}_\parallel^{[2]} = -\nabla(n^{[1]}v_\parallel^{[1]}) = 0, \quad (13.24)$$

$$\frac{1}{c^2} \nabla \frac{\phi^{[2]}}{\partial t} = -\frac{\mu_0 n_0 e^2}{m} \mathbf{v}_\parallel^{[2]}, \quad (13.25)$$

($e\phi/m \rightarrow \phi$), and

$$\frac{\partial \mathbf{v}_\parallel^{[2]}}{\partial t} = \nabla \phi^{[2]} - \frac{1}{2} \nabla_\parallel (\mathbf{v}_\perp^{[1]} \cdot \mathbf{v}_\perp^{[1]}), \quad (13.26)$$

so that we have

$$\mathbf{v}_\parallel^{[2]} = \frac{1}{2} \frac{k\omega}{\omega^2 - \omega_p^2} (\mathbf{v}_\perp^{[1]} \cdot \mathbf{v}_\perp^{[1]})_\omega, \quad (13.27)$$

$$\phi^{[2]} = -\frac{1}{2} \frac{\omega_p^2}{\omega^2 - \omega_p^2} (\mathbf{v}_\perp^{[1]} \cdot \mathbf{v}_\perp^{[1]})_\omega. \quad (13.28)$$

¹ This approach is in fact weakly-relativistic since it involves an expansion in the relativistic factor. A fully relativistic approach can be used for circularly polarized EM waves since the amplitude of the electric field and the electron quiver velocity remain constant.

$$n^{[2]} = \frac{1}{2} \frac{k^2}{\omega^2 - \omega_p^2} (\mathbf{v}_\perp^{[1]} \cdot \mathbf{v}_\perp^{[1]})_\omega. \quad (13.29)$$

The beating of the pump and scattered waves give perturbations of the plasma quantities at $(2\omega_0, 2k_0)$:²

$$\mathbf{v}_\parallel^{[2]}(2\omega_0, 2k_0) = \frac{2k_0\omega_0}{4\omega_0^2 - \omega_p^2} (\mathbf{A}_0 \cdot \mathbf{A}_0), \quad (13.30)$$

$$\phi^{[2]}(2\omega_0, 2k_0) = -\frac{1}{2} \frac{\omega_p^2}{4\omega_0^2 - \omega_p^2} (\mathbf{A}_0 \cdot \mathbf{A}_0), \quad (13.31)$$

$$n^{[2]}(2\omega_0, 2k_0) = \frac{2k_0^2}{4\omega_0^2 - \omega_p^2} (\mathbf{A}_0 \cdot \mathbf{A}_0), \quad (13.32)$$

at $(\omega + 2\omega_0, k + 2k_0)$:

$$\mathbf{v}_\parallel^{[2]}(\omega + 2\omega_0, k + 2k_0) = \frac{(k + 2k_0)(\omega + 2\omega_0)}{(\omega + 2\omega_0)^2 - \omega_p^2} (\mathbf{A}_0 \cdot \mathbf{A}_+), \quad (13.33)$$

$$\phi^{[2]}(\omega + 2\omega_0, k + 2k_0) = -\frac{\omega_p^2}{(\omega + 2\omega_0)^2 - \omega_p^2} (\mathbf{A}_0 \cdot \mathbf{A}_+), \quad (13.34)$$

$$n^{[2]}(\omega + 2\omega_0, k + 2k_0) = \frac{(k + 2k_0)^2}{(\omega + 2\omega_0)^2 - \omega_p^2} (\mathbf{A}_0 \cdot \mathbf{A}_+), \quad (13.35)$$

at $(\omega - 2\omega_0, k - 2k_0)$:

$$\mathbf{v}_\parallel^{[2]}(\omega - 2\omega_0, k - 2k_0) = \frac{(k - 2k_0)(\omega - 2\omega_0)}{(\omega - 2\omega_0)^2 - \omega_p^2} (\mathbf{A}_0^* \cdot \mathbf{A}_-^*), \quad (13.36)$$

$$\phi^{[2]}(\omega - 2\omega_0, k - 2k_0) = -\frac{\omega_p^2}{(\omega - 2\omega_0)^2 - \omega_p^2} (\mathbf{A}_0^* \cdot \mathbf{A}_-^*), \quad (13.37)$$

² From equations (13.30)–(13.32), one can obtain the following expressions for perturbations in the longitudinal direction at $(2\omega_0, 2k_0)$ [409, 410],

$$v_\parallel^{[2]}(2\omega_0, 2k_0) = -E_0 \frac{e^2}{m^2} \frac{k_0}{\omega_0} \frac{1}{4\omega_0^2 - \omega_p^2} \cos 2(k_0x - \omega_0t),$$

$$E^{[2]}(2\omega_0, 2k_0) = E_0^2 \frac{e}{m} \frac{\omega_p^2}{2\omega_0^2} \frac{k_0}{4\omega_0^2 - \omega_p^2} \sin 2(k_0x - \omega_0t),$$

$$n^{[2]}(2\omega_0, 2k_0) = -E_0^2 \frac{e^2}{m^2} \frac{k_0^2}{\omega_0^2} \frac{1}{4\omega_0^2 - \omega_p^2} \cos 2(k_0x - \omega_0t).$$

$$n^{[2]}(\omega - 2\omega_0, k - 2k_0) = \frac{(k - 2k_0)^2}{(\omega - 2\omega_0)^2 - \omega_p^2} (\mathbf{A}_0^* \cdot \mathbf{A}_-^*), \quad (13.38)$$

and at (ω, k) :

$$\mathbf{v}_{\parallel}^{[2]}(\omega, k) = \frac{k\omega}{\omega^2 - \omega_p^2} (\mathbf{A}_0^* \cdot \mathbf{A}_+ + \mathbf{A}_0 \cdot \mathbf{A}_-^*), \quad (13.39)$$

$$\phi^{[2]}(\omega, k) = -\frac{\omega_p^2}{\omega^2 - \omega_p^2} (\mathbf{A}_0^* \cdot \mathbf{A}_+ + \mathbf{A}_0 \cdot \mathbf{A}_-^*), \quad (13.40)$$

$$n^{[2]}(\omega, k) = \frac{k^2}{\omega^2 - \omega_p^2} (\mathbf{A}_0^* \cdot \mathbf{A}_+ + \mathbf{A}_0 \cdot \mathbf{A}_-^*). \quad (13.41)$$

Since $A_{\pm} \ll A_0$, the second (longitudinal) harmonics of the scattered waves $[2(\omega \pm \omega_0), 2(k \pm k_0)]$ can be neglected.

In third order, the total currents at the frequencies and wave vectors of the pump and scattered waves are derived. Since we have

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - c^2 \nabla^2 \mathbf{A} = -\omega_p^2 \mathbf{v}^{[1]} + \frac{e}{m\epsilon_0} \mathbf{j}^{[3]}, \quad (13.42)$$

one can obtain an amplitude dependent dispersion equation for the pump wave. At the peak of the wave amplitude, we have

$$\omega_0^2 = \omega_p^2 \left[1 - \frac{\beta^2}{4} \left(\frac{3}{2} - \frac{2c^2 k_0^2}{4\omega_0^2 - \omega_p^2} \right) \right] + c^2 k_0^2, \quad (13.43)$$

where $\beta = \frac{eE_0}{m\omega_0 c}$ is the laser strength. As we can conclude from equation (13.43), a high intensity wave introduces a relativistic shift in its natural frequency (or a shift in the wave number). For very low densities $\omega_0 \gg \omega_p$, the amplitude dependent part of equation (13.43) can be approximated as $\frac{\beta^2}{4}$, whereas near the critical density $\omega_0 \approx \omega_p$, this term is $\approx \frac{3\beta^2}{8}$.

For the scattered waves, on the other hand, we have

$$\mathcal{D}_+ A_+ = \Gamma \mathcal{D}_1^+ A_+ - \Gamma \mathcal{D}_2 A_-^*, \quad (13.44)$$

$$\mathcal{D}_- A_-^* = \Gamma \mathcal{D}_1^- A_-^* - \Gamma \mathcal{D}_2 A_+, \quad (13.45)$$

where $\Gamma = \frac{\omega_p^2 \beta^2}{4}$ and $\mathcal{D}_{\pm} = [(\omega \pm \omega_0)^2 - \omega_p^2 - c^2(k \pm k_0)^2]$,

$$\mathcal{D}_1^{\pm} = \left(\frac{c^2 k^2}{\omega^2 - \omega_p^2} + \frac{c^2 (k \pm 2k_0)^2}{(\omega \pm 2\omega_0)^2 - \omega_p^2} - 3 \right),$$

$$\mathcal{D}_2 = \left(\frac{c^2 k^2}{\omega^2 - \omega_p^2} + \frac{2c^2 k_0^2}{4\omega_0^2 - \omega_p^2} - \frac{3}{2} \right).$$

Finally, (13.44) and (13.45) can be combined to give the hybrid dispersion equation

$$\mathcal{D}_- \mathcal{D}_+ = \Gamma(\mathcal{D}_- \mathcal{D}_1^+ + \mathcal{D}_+ \mathcal{D}_1^-) + \Gamma^2(\mathcal{D}_2^2 - \mathcal{D}_1^+ \mathcal{D}_1^-). \quad (13.46)$$

Note that the term $\sim \Gamma^2$ can be neglected.

13.1.2.1 Solutions of Relativistic Dispersion Relation

The hybrid dispersion equation (13.46) is a sixth order equation in ω . It was solved numerically for a broad range of densities and laser strengths as a function of the real wave number k normalized to the wave number of the pump wave k_0 . The solutions of the dispersion equation [420], the real ω_r , and imaginary part ω_i (the temporal growth rate) of the ES wave frequency ω are normalized to the frequency of the pump wave ω_0 .

In Fig. 13.1, the solutions of the hybrid equation (13.46) are shown for plasma density $n = 0.1n_{\text{cr}}$ and laser strengths (a) $\beta = 0.3$ and (b) $\beta = 0.6$. The solution for $k/k_0 > 1$ is associated with the backward stimulated Raman scattering (B-SRS) instability, whereas the solution $k/k_0 < 1$ represents the forward stimulated Raman scattering (F-SRS) instability. The temporal growth rate for B-SRS and F-SRS instabilities are distinct for considered parameters. Moreover, instability branches are well separated in k space. For sufficiently high laser intensity, the relativistic correction to the mass of electrons oscillating in the incident electric field causes the relativistic modulational instability (RMI) [432]. The source of the RMI is the dependence of the wave dispersion on the amplitude of waves (see (13.43)). The consequent effect is that small perturbations of the original wave envelope become narrower and larger following an accumulation of the wave energy into a smaller space [432]. In Fig. 13.1b, the low-frequency, low-wave-number tail connected to F-SRS branch corresponds to this instability. The relativistic modulation propagates in the direction of the incident large-amplitude EM wave. In view of the Stokes and anti-Stokes sidebands, it should be pointed out that the contribution of the anti-Stokes wave (13.7) to B-SRS can be neglected, since this wave is non-resonant for backward scattering. For F-SRS and RMI, both waves, Stokes (13.8) and anti-Stokes (13.7), give the contribution. In particular for $k/k_0 \ll 1$, the amplitude of the anti-Stokes wave is of the order of the amplitude of the Stokes wave ($A_+ \approx A_-$) [450].

At higher electron densities, $n = 0.2n_{\text{cr}}$ (Fig. 13.2) and $n = 0.25n_{\text{cr}}$ (Fig. 13.3), the solutions are plotted for several laser strengths, (a) $\beta = 0.2$, (b) $\beta = 0.3$, (c) $\beta = 0.4$, and (d) $\beta = 0.6$. As we can see, for $n = 0.2n_{\text{cr}}$ (Fig. 13.2a), there is a gap between Raman branches that disappears with increasing the laser strength (Fig. 13.2b). The RMI branch appears for higher EM wave intensities with a gap that separates the RMI and SRS instability (Fig. 13.2c). Further increasing the incident wave intensity results in merging of all branches (Fig. 13.2d). The behavior of the

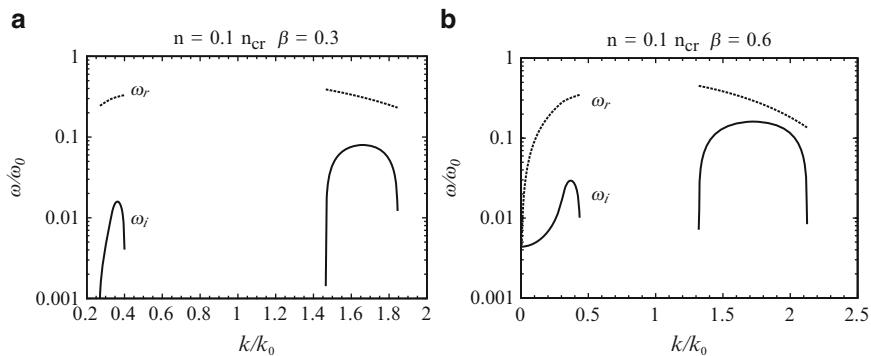


Fig. 13.1 Solutions of the hybrid dispersion equation (13.46) for $n = 0.1 n_{\text{cr}}$ and laser strengths (a) $\beta = 0.3$ and (b) $\beta = 0.6$. The *solid lines* are the growth rate (ω_i) and the *dashed lines* are the real part of the frequency (ω_r). The frequencies and wave numbers are normalized to ω_0 and k_0 , respectively

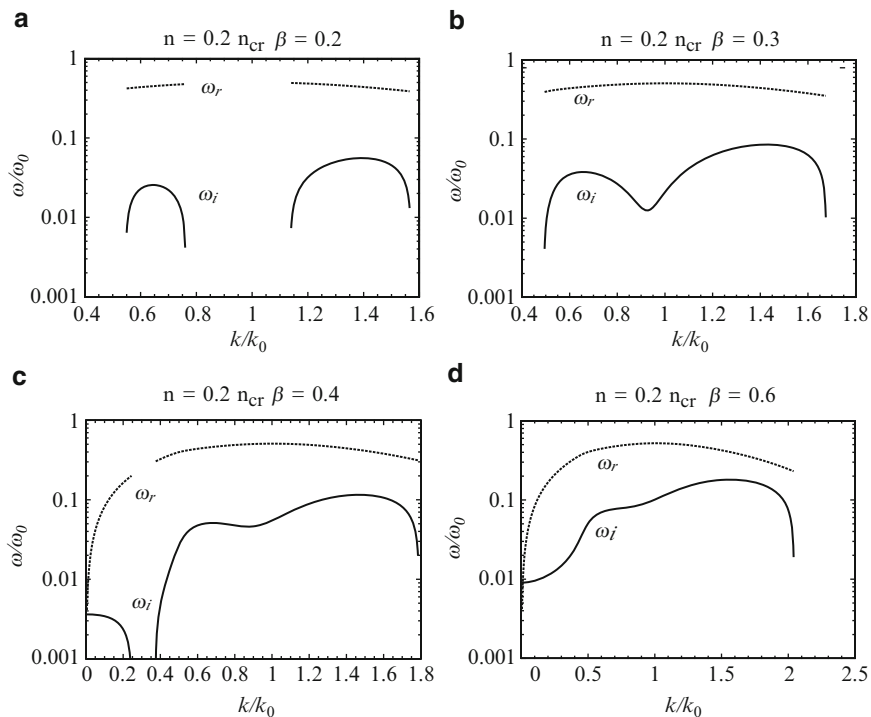


Fig. 13.2 Solutions of the hybrid dispersion equation (equation 13.46) for $n = 0.2 n_{\text{cr}}$ and laser strengths (a) $\beta = 0.2$, (b) $\beta = 0.3$, (c) $\beta = 0.4$, and (d) $\beta = 0.6$. The *solid lines* are the growth rate (ω_i) and the *dashed lines* are the real part of the frequency (ω_r). The frequencies and wave numbers are normalized to ω_0 and k_0 , respectively

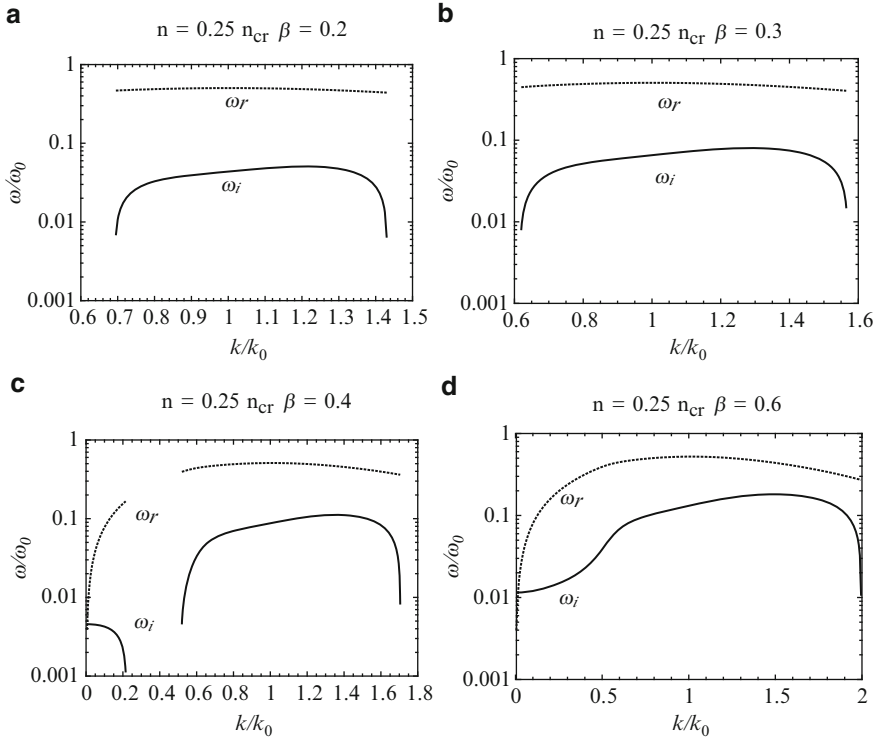


Fig. 13.3 Solutions of the hybrid dispersion equation (equation 13.46) for $n = 0.25n_{\text{cr}}$ and laser strengths (a) $\beta = 0.2$, (b) $\beta = 0.3$, (c) $\beta = 0.4$, and (d) $\beta = 0.6$. The solid lines are the growth rate (ω_i) and the dashed lines are the real part of the frequency (ω_r). The frequencies and wave numbers are normalized to ω_0 and k_0 , respectively

solutions shown in Fig. 13.3 is similar to the behavior described above, although it should be noted that the backward and forward Raman branches are already connected for $\beta = 0.2$.

In general, as one can conclude from Figs. 13.1–13.3, for $n \leq 0.25n_{\text{cr}}$, the RMI has a lower growth rate than the SRS instability. For low plasma densities, the B-SRS instability is the leading instability. With increasing density or/and laser strength the Raman branches get closer and merge. As is expected, an increase in the laser strength increases the growth rates of the instabilities within our analysis limited to the weakly-relativistic case. As an illustration of the solutions in the range of densities $n > 0.25n_{\text{cr}}$, in Fig. 13.4, we show unstable solutions of the hybrid dispersion equation for $n = 0.4n_{\text{cr}}$ and laser strengths $\beta = 0.4$ (Fig. 13.4a), $\beta = 0.422$ (Fig. 13.4b), $\beta = 0.45$ (Fig. 13.4c), and $\beta = 0.6$ (Fig. 13.4d). As we can see, in a plasma with density $> 0.25n_{\text{cr}}$, when SRS is halted, only the RMI is found (Fig. 13.4a). However, an increase in the laser strength results in the appearance of the B-SRS branch, relativistically shifted to high densities (Fig. 13.4b). For

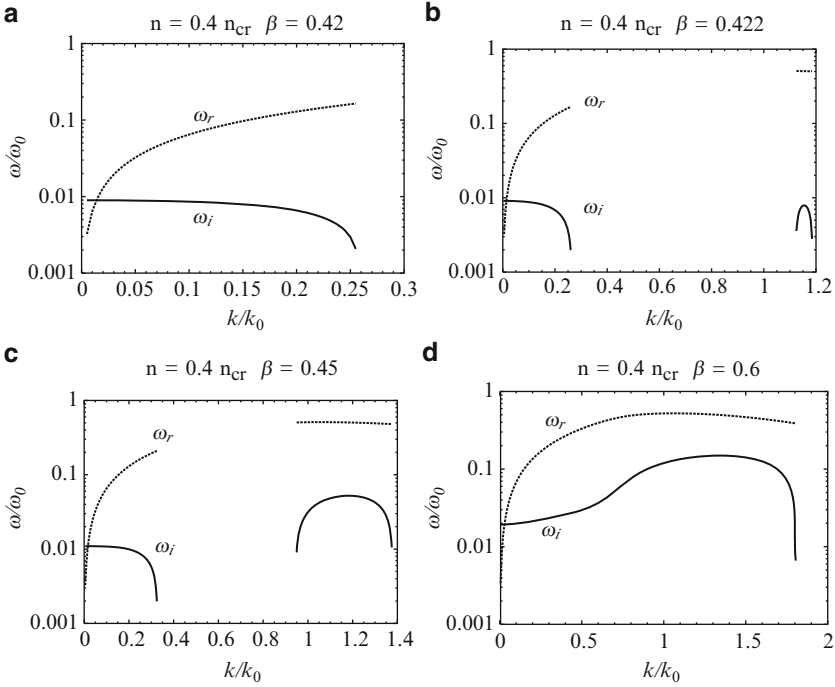


Fig. 13.4 Solutions of the hybrid dispersion equation (13.46) for $n = 0.4n_{\text{cr}}$ and laser strengths (a) $\beta = 0.42$, (b) $\beta = 0.422$, (c) $\beta = 0.45$, and (d) $\beta = 0.6$. The solid lines are the growth rate (ω_i) and the dashed lines are the real part of the frequency (ω_r). The frequencies and wave numbers are normalized to ω_0 and k_0 , respectively

higher laser strengths, the SRS instability again becomes the leading instability (Fig. 13.4c). Finally, the RMI and SRS instability branches merge (Fig. 13.4d).

13.1.3 Summary

As we conclude, a relativistic EM wave grows unstable in a plasma, with different regimes that depend on density (and other parameters) and EM wave intensity. In general, B-SRS is the leading instability. F-SRS competes with B-SRS at higher densities. When SRS is halted ($n > 0.25n_{\text{cr}} + \text{relativistic shift}$) only the RMI can grow. The relativistic description of parametric instabilities introduces a broadening of unstable regions in k space and merging of unstable branches of different instabilities. Although the laser intensity considered here are somewhat larger than allowed by expansion (13.20), obtained values show agreement with previously reported results. The majority of these results have been obtained under various approximations such as: a cold plasma, weak relativistic effects, circular polarization, very low

density plasma, the quasistatic approximation, one dimensional scattering geometry, etc. [411–417, 431, 433, 434, 450] three-dimensional approach to the problem of electron parametric instabilities of relativistically intense laser light in a cold plasma including harmonic generation is attempted in [435].

Finally, it should be pointed out that the hybrid dispersion equation (13.46) was solved for real values of the wave number (the initial value problem). This corresponds, strictly speaking, to the case when the propagating waves remain inside a plasma (an infinite or periodic plasma model) [321]. Complications arise, in the case of real (finite) plasmas the interacting waves escape from the interaction region. Therefore, we get the boundary value problem with additional complexity. In the following sections, mostly by using computer simulations, we shall study the propagation of intense relativistic EM waves in a finite homogeneous plasma system.

13.2 Computer Simulations of Relativistic Plasmas

Numerical modeling is crucial to our understanding of complex plasma phenomena. The basic idea of computer simulations is to numerically solve an evolution of a plasma system by using mathematical equations that basically describe the system on an accepted physical level. This approach to plasma complexity enables a numerical plasma experiment with detailed diagnostics. Such numerical experiments are of a great value for understanding results from the real ones. Simulations are also used for experiments that cannot be performed in the laboratory either due to their size or due to the characteristic time scale of certain physical processes. However, simulations are not a substitution for laboratory and space experiments and observations. They are restricted by computer performances and usually we can only sample limited set of initial conditions. Nevertheless, computer *simulation* fills a large gap between *theory* and *experiment*, and by combining these three approaches, it is possible to obtain detailed information on complex processes in plasmas. Today's supercomputers offer an excellent platform for various simulation methods, thus developing rapidly the third methodology of scientific research.

During the last decades, various simulation methods in plasmas have been introduced. Among these are a magneto-hydro-dynamics code, fluid code, Vlasov and particle-in-cell (PIC) codes, hybrid codes, etc. In general, these simulation codes are based on kinetic or/and fluid description of plasmas. Fluid simulations work by solving numerically the fluid (hydrodynamic) equations whereas kinetic simulations are based on the plasma kinetic equation (Vlasov equation) or on particle simulation. The particle simulation computes the motion of charged particles, interacting among each other and with externally applied EM fields. The hybrid code is a combination of these two approaches: e.g., particle model is used for ions and dynamics of electrons is treated by using the fluid model. In laser-fusion oriented research, two kinetic simulation approaches evolved and became widely used: Vlasov simulations and much prevailing particle-in-cell simulations (PIC). Vlasov

simulation method provides a much lower noise level than PIC simulations. However, Vlasov simulations are very expensive in CPU time for investigations of high intensity wave-plasma interactions in 2-3D, since the phase space required by this simulations becomes very large.

13.2.1 Particle-In-Cell Simulations

Electromagnetic relativistic PIC simulations [299,362] are a powerful tool for studying strong laser-plasma interaction. The simulation method for EM relativistic PIC is based on Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (13.47)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (13.48)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (13.49)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (13.50)$$

closed by the relativistic equation of particle motion

$$m \frac{d\gamma \mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (13.51)$$

where ρ is the charge density, $\mathbf{j}$ is the current density, ϵ_0 is the permittivity, and μ_0 is the magnetic permeability. In a PIC simulation, plasma is represented by a large number of quasi-particles each of which models the action of a large number of physical particles. While coordinate and particle velocities remain continuous quantities in this simulation scheme, the electric and magnetic fields and the charge and current densities are discretized on a spatial grid. The simulation cycle starts with some appropriate initial conditions for the particle positions and velocities. This initial choice of conditions is related to the “configuration” of the physical problem which is in a focus. In Fig. 13.5, a basic computational cycle is shown. From particle positions and velocities at each time step, we compute the charge and current densities (ρ , $\mathbf{j}$) on the spatial grid. These computations require some weighting methods at the grid points that are dependent on particle positions. Next, using Maxwell equations, we calculate electric and magnetic fields ($\mathbf{E}$, $\mathbf{B}$) and then use these fields, again performing a weighting, to find the Lorentz force $\mathbf{F}$ and advance in time the particle velocities $\mathbf{v}$ and positions $\mathbf{x}$.

The choice of code characteristics, i.e., time step and length of grids, should provide good accuracy and stability to make simulations correct over many computational cycles. The time step should be small enough to resolve the highest frequency in the problem and the choice of grid length should be fine enough to

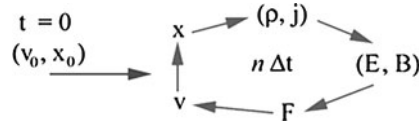


Fig. 13.5 The basic cycle of a particle simulation code

resolve the Debye length. Furthermore, to avoid nonphysical results due to the finite difference approximation used to solve equations (13.47)–(13.51), the time step Δt and the grid spacing Δx have to satisfy the following inequality (Courant condition) [299, 362, 376],

$$\Delta x > c \Delta t. \quad (13.52)$$

Thus, the time step and the grid spacing should be chosen in such a way that the particles during one time step cannot cross a distance larger than the size of the cell.

13.3 Relativistic Electromagnetic Solitons

Relativistic electromagnetic (EM) solitons in laser driven plasmas were analytically predicted and found by particle simulations [367, 382–387]. Relativistic solitons are EM coherent structures self-trapped by locally modified plasma refractive index via two effects: the relativistic electron mass increase and the electron density drop by the ponderomotive action of intense laser light [367, 383]. A large effort was put into studies of one-dimensional (1D) relativistic EM solitons in an ultraintense laser interaction with underdense and overdense plasmas [389–394]. For laser pulses longer than the EPW wavelength, spatially modulated depleted pulse, due to SRS, readily in nonlinear stage breaks-up into a slow train of ultra-short 1D relativistic EM solitons [385, 389, 390]. Moreover, soliton acceleration toward the plasma-vacuum interface can produce bursts of reflected low-frequency EM radiation [395]. Recently, circularly polarized subcycle relativistic EM solitons were found and studied in 2D and 3D by particle simulations [390, 396]; while the ion motion influence on dynamics and the lifetime of 1D and 2D relativistic solitons was investigated in [390, 391].

Evolution of a relativistic laser pulse in a long-scale moderately underdense plasma was studied analytically and by computer simulation [389–392]. While the circular polarization case was investigated in much detail [387, 388, 391–394], here, we treat a more complex linear polarization case. Linearly polarized laser light sets electrons into longitudinal motion by relativistic Lorentz force generating coupled longitudinal-transverse wave modes. In this situation, by relativistic fluid and particle simulations for a long laser pulse, we have recently analyzed nonlinear interplay between forward and backward stimulated Raman scattering and RMI [389]. Parametric down-cascade evolves into a weak turbulence, which saturates into a photon condensate at the bottom of the light spectrum. This phenomenon,

similar to Langmuir condensate, corresponds to strong energy depletion and a laser beam break-up, as observed in many simulations. In the final stage of saturation, behind the pulse front, the train of intense ultra-short standing relativistic solitons is formed. It was estimated, for ultra-short laser pulses [390], that 30–40% of the laser energy can be trapped inside these low-frequency electromagnetic solitons creating a significant channel for laser beam energy conversion. Below, the problem of existence, stability, and dynamics of linearly polarized electromagnetic solitons is studied by one-dimensional analytical model for a weak relativistic nonlinearity. A simple dynamical equation of the nonlinear Schrödinger (NLS) type, with two extra nonlocal (derivative) nonlinear terms, is derived. The conserved quantities, the photon number and the Hamiltonian, are calculated and a soliton solution in an implicit form is found analytically. Further, the soliton stability is examined and compared with the circular polarization case. Analytical results are confirmed by numerical simulations of the derived NLS model, as well as, of fully relativistic nonlinear cold fluid equations. Finally, the saturation amplitude and the frequency for large relativistic solitons observed in numerical simulations are analytically estimated.

13.3.1 Dynamical Equations

The relativistic wave equation and the cold electron momentum equation, in terms of vector and scalar potential ($\mathbf{A}$ and ϕ , respectively), in the *Coulomb* (radiation) gauge, where $\nabla \cdot \mathbf{A} = 0$, are written as [418]

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = -\frac{4\pi}{c} \mathbf{J}_\perp, \quad (13.53)$$

where $\mathbf{J}_\perp$ is the transverse component of the total electron current $\mathbf{J} = \mathbf{J}_\perp + \mathbf{J}_\parallel = e n \mathbf{v}$, where n is the electron density and ions assumed to be fixed, and

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{p} = e \nabla \phi + \frac{e}{c} \frac{\partial}{\partial t} \mathbf{A} - \frac{e}{c} \mathbf{v} \times (\nabla \times \mathbf{A}), \quad (13.54)$$

where $\mathbf{p} = \gamma m \mathbf{v}$ is the electron momentum, $\mathbf{v}$ is the electron velocity, and $\gamma = (1 + p^2/m^2 c^2)^{1/2}$ is the relativistic Lorentz factor. The relativistic electron momentum equation can be transformed by using the vector identity which after a straightforward algebra gives

$$\frac{\partial}{\partial t} \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right) = e \nabla \phi - m c^2 \nabla \gamma + \mathbf{v} \times \left[\nabla \times \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right) \right], \quad (13.55)$$

that is the equation for the generalized momentum $\left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right)$. By taking the *curl* of the above relation, one can readily conclude that if $\nabla \times \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right)$ is equal to zero initially, it will remain zero at later times. Therefore, an important relation,

$\mathbf{p}_\perp = \frac{e}{c} \mathbf{A}$, is found, which after a substitution in $\mathbf{J}_\perp$, simplifies the wave equation in terms of the vector potential $\mathbf{A}$.

Accordingly, in one dimensional case, relativistic wave equation, the continuity equation, and the electron momentum equation, read [414],

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) a = -\frac{\omega_p^2}{n_0} \frac{n}{\gamma} a, \quad (13.56)$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} \left(\frac{np}{m\gamma} \right) = 0, \quad (13.57)$$

$$\frac{\partial p}{\partial t} = -eE_\parallel - mc^2 \frac{\partial \gamma}{\partial x}, \quad (13.58)$$

where $a = eA/mc^2$ is the normalized vector potential in y direction, n is the electron density, p is the electron momentum in x direction, $\gamma = (1 + a^2 + p^2/m^2c^2)^{1/2}$, $E_\parallel$ is the longitudinal electric field, n_0 is the initial electron-ion density, and $\omega_p = (4\pi e^2 n_0/m)^{1/2}$ is the electron plasma frequency.

In a weakly relativistic limit for $|a| \ll 1$ and $|\delta n| \ll 1$, we expand the right hand side of the (13.56) and introduce the normalized perturbed electron density $\delta n = (n - n_0)/n_0$ and dimensionless variables $x \rightarrow (c\omega_p^{-1})x$ and $t \rightarrow (\omega_p^{-1})t$ to obtain [393],

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) a = - \left(1 + \delta n + \frac{a^2}{2} \right) a. \quad (13.59)$$

Further, by combining the linearized equations of continuity (13.57), the electron momentum (13.58), and the Poisson equation, we get for the perturbed electron density [393],

$$\frac{\partial^2 \delta n}{\partial t^2} + \delta n = \frac{1}{2} \frac{\partial^2}{\partial x^2} a^2. \quad (13.60)$$

In distinction to circular polarization [367, 393], linearly polarized waves have odd harmonics of the vector potential a and even harmonics of the electron density δn , [390, 393]. Further, we introduce the slow time varying complex envelopes in a form

$$a = \frac{1}{2} [Ae^{-it} + A^*e^{it}]; \quad \delta n = N_0 + \frac{1}{2} (N_2e^{-i2t} + N_2^*e^{i2t}). \quad (13.61)$$

and find the envelopes N_0 and N_2 by substituting (13.61) into (13.60) and collecting the zero and second harmonic terms (e^{-i2t})

$$N_0 = \frac{1}{4} (|A|^2)_{xx}; \quad N_2 = -\frac{1}{6} (A^2)_{xx}, \quad (13.62)$$

where subscripts x designates derivations with respect to x -coordinate.

By substituting (13.61) and (13.62) into the wave equation (13.59) and collecting first harmonic terms (e^{-it}), we obtain the wave equation for the vector potential envelope A ,

$$i \frac{\partial A}{\partial t} + \frac{1}{2} A_{xx} + \frac{3}{16} |A|^2 A - \frac{1}{8} (|A|^2)_{xx} A + \frac{1}{48} (A^2)_{xx} A^* = 0. \quad (13.63)$$

The (13.63) has a form of the generalized NLS equation [397] with two extra nonlocal (derivative) nonlinear terms. We can derive three conserved quantities [414, 440]: Photon number P

$$P = \int |A|^2 dx, \quad (13.64)$$

Hamiltonian H

$$H = \frac{1}{2} \int \left\{ |A_{xx}|^2 - \frac{3}{16} |A|^4 - \frac{1}{8} [(|A|^2)_x]^2 - \frac{1}{12} |A|^2 |A_x|^2 \right\} dx. \quad (13.65)$$

and linear Momentum M

$$M = i \int (A \cdot A_x^* - A^* \cdot A_x) dx, \quad (13.66)$$

We further look for a stationary and localized solution of (13.63) in a form

$$A = \alpha(x) e^{i\lambda^2 t}, \quad (13.67)$$

with the boundary conditions

$$\alpha(\pm\infty) = 0, \quad \alpha(x) < \infty. \quad (13.68)$$

Under the assumptions (13.67) and (13.68), the first integration of (13.63) gives

$$(\alpha_x)^2 = 2 \frac{\alpha^2 \lambda^2 \left(1 - \frac{3}{32} \frac{\alpha^2}{\lambda^2}\right)}{1 - \frac{\alpha^2}{3}}. \quad (13.69)$$

Additional integration of (13.69) gives a localized soliton solution in an implicit form

$$\pm \lambda x = \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{1 - \frac{3}{32} \frac{\alpha^2}{\lambda^2}} + \sqrt{1 - \frac{\alpha^2}{3}}}{\left| \sqrt{1 - \frac{3}{32} \frac{\alpha^2}{\lambda^2}} - \sqrt{1 - \frac{\alpha^2}{3}} \right|} - \frac{4}{3} \lambda \ln \frac{\frac{1}{3} \sqrt{32\lambda^2 - 3\alpha^2} + \sqrt{1 - \frac{\alpha^2}{3}}}{\sqrt{\left| 1 - \frac{32}{9} \lambda^2 \right|}}, \quad (13.70)$$

with a soliton amplitude $a_0 = \frac{4\sqrt{2}}{\sqrt{3}} \lambda$.

This is linearly polarized soliton with a vector potential a oscillating with the frequency $\omega = 1 - \lambda^2$, slightly below the plasma frequency. For the soliton strength λ above the critical value $\lambda \geq \lambda_c = \frac{3}{4\sqrt{2}}$ ($a_0 \geq \sqrt{3}$), the solution (13.67) has a form of a “cuspl” soliton [16]; the centrally highly pointed waveform. In the small

amplitude limit $\lambda \ll \lambda_c$, one neglects the non-local (ponderomotive) terms and the solution (13.67) becomes the well-known secant hyperbolic (NLS) soliton.

13.3.2 Relativistic Soliton Stability

To check the stability of the soliton (13.67), we use the Vakhitov–Kolokolov stability criterion [397, 398]

$$\frac{dP_0}{d\lambda^2} > 0, \quad (13.71)$$

where P_0 is the soliton photon number defined by (13.64). The function $P_0(\lambda)$ for the soliton solution (13.70) is calculated analytically as

$$P_0(\lambda) = \frac{16\sqrt{2}}{3}\lambda + 2\left(1 - \frac{32}{9}\lambda^2\right) \ln \frac{1 + \frac{4\sqrt{2}}{3}\lambda}{\left|1 - \frac{4\sqrt{2}}{3}\lambda\right|}. \quad (13.72)$$

The curve $P_0(\lambda)$, shown in Fig. 13.6, represents the stationary solutions of (13.63) which correspond to the minimum of the Hamiltonian H for the fixed photon number P . According to the condition (13.71), the soliton (13.70) turns out to be stable in the region $\lambda < \lambda_s \approx 0.44$ ($a_0 < a_s \approx 1.44$) indicating that cusp solitons are also unstable ($\lambda_s < \lambda_c$). More generally, we can now conclude that small amplitude linearly polarized solitons ($a_0 < 1$) within the weakly relativistic model (13.59)–(13.60) are stable.

On the other hand, for circularly polarized EM waves, fully nonlinear relativistic soliton solution of (13.56) and (13.57) has been analytically found [390, 394], as

$$a(x, t) = \frac{2\lambda \cosh(\lambda x)}{\cosh^2(\lambda x) - \lambda^2} \exp\left(i\sqrt{1 - \lambda^2}t\right), \quad (13.73)$$

with $a(x, t) = a_y + ia_z$, where indices denote y and z components of the vector potential. The soliton amplitude is $a_0 = \frac{2\lambda}{1 - \lambda^2}$ and frequency $\omega = \sqrt{1 - \lambda^2}$. However, in the circular polarization case, ω is the angular velocity of the rotation of the polarization plane (with constant amplitude) while for the linear polarization it stands for the frequency of the oscillating vector potential. A condition for the electron density greater than zero imposes a constraint on the maximum soliton amplitude, as $a_0 < \sqrt{3}$ and frequency $1 > \omega > \sqrt{\frac{2}{3}}$ ($0 < \lambda < \sqrt{3}/3$).

To further check the stability of the circularly polarized solitons (13.73), we can use the similar procedure as for the case of linearly polarized solitons. Therefore, the corresponding soliton photon number $P_0(\lambda)$ reads

$$P_0(\lambda) = \frac{4 \arctan\left(\frac{\lambda}{\sqrt{1 - \lambda^2}}\right)}{(1 - \lambda^2)^{3/2}} + \frac{4\lambda}{1 - \lambda^2}, \quad (13.74)$$

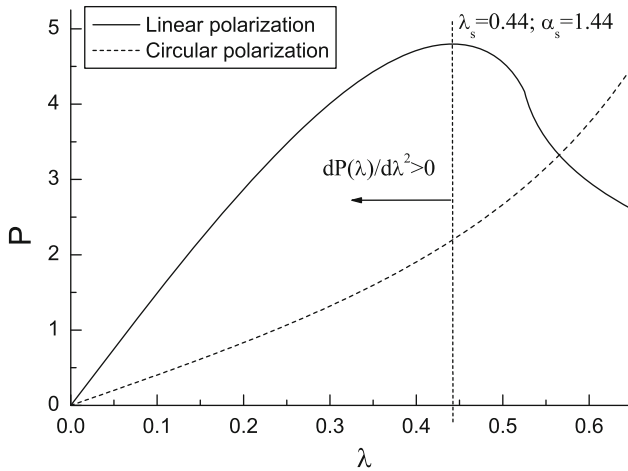


Fig. 13.6 Photon number $P_0(\lambda)$ variation for solitons with linear and circular polarization with an illustration of the Vakhitov–Kolokolov stability criteria

while according to the Vakhitov–Kolokolov stability criterion (13.72), we conclude that the circularly polarized solitons remain stable ($\frac{dP_0}{d\lambda^2} > 0$) inside the whole region $0 < a_0 < \sqrt{3}$ of existence (Fig. 13.6). However, this approach is valid only for the limited class of circularly polarized perturbations and does not give an answer for arbitrary perturbations. Our initial analytical investigation of the stability of the circularly polarized solitons under arbitrary (symmetry breaking) perturbations reveal that the circularly polarized perturbations appear as the special case given by a trivial solution of the corresponding eigen value problem.

The shape of the curve $P_0(\lambda)$ (Fig. 13.6) for the linearly polarized solitons predicts the existence of the soliton instability region $\lambda > \lambda_s \approx 0.44$. However, the Vakhitov–Kolokolov criterion just solves a linear stability problem for solitons indicating the presence of exponentially growing or decaying modes; therefore, giving no prediction about the subsequent nonlinear evolution of unstable solitons or about stability of the localized structures with arbitrary profiles. According to the nonlinear analysis [399], for the generalized NLS equation with a similar shape of the corresponding curve $P_0(\lambda)$, beside the stationary solution, there exist two other regimes of the soliton dynamics: (a) soliton collapse and (b) long-lived relaxation oscillations around the stable soliton amplitude. In our case, due to the local cubic nonlinear (NLS) term in (13.63), one can plausibly expect such dynamical regimes. However, the presence of two extra nonlocal nonlinear terms in (13.63) indicates the possible existence of some other dynamical states.

13.3.3 Soliton Dynamics

To further check analytical results and prediction of the regimes of the soliton dynamics, we perform direct numerical simulation of the model equation (13.63) using a numerical algorithm based on the split-step Fourier method, originally developed for the NLS equation [400, 414]. The main feature of the split-step Fourier method is numerical calculation of the spatial derivatives in the Fourier space in each integration time step. In our case, the spatial derivative of the additional nonlinear term (nonlocal term) is also calculated in the Fourier space.

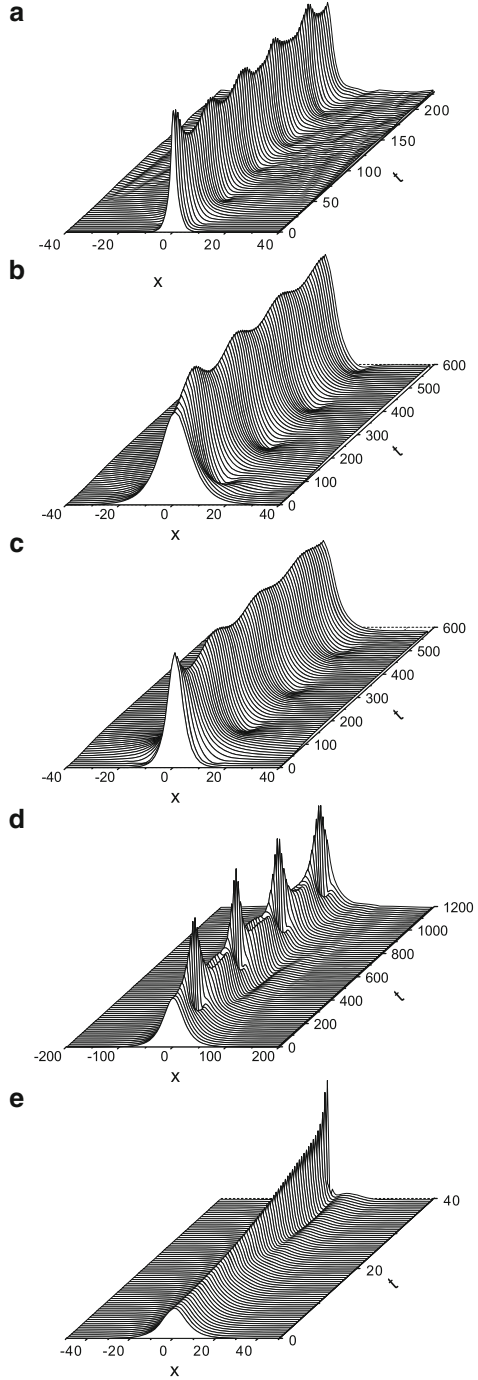
Numerical results prove that the initially launched solitons (13.70) with the soliton parameters inside the stability region $\lambda < \lambda_s \approx 0.44$ remain stable. Solitons with parameters outside the stability region evolve toward the corresponding stable soliton with long-lived relaxation oscillations. The evolution of the initially launched soliton with amplitude $a_0 \approx 1.6$ ($\lambda \approx 0.2$) outside the stability region is shown on Fig. 13.7a. A similar behavior exhibit initially perturbed solitons with photon number $P < P_{\max} = P_0(\lambda_s) \approx 4.79$ inside the stability region $\lambda < \lambda_s$ or different localized structures with small deviation from the stable equilibrium state. As an example of such dynamics, the time evolution of two Gaussian structures with different amplitudes but with the same photon number $P = 2.875$ are shown on Fig. 13.7b and Fig. 13.7c. The evolution in both cases is long-lived relaxation oscillations around the stable soliton amplitude (13.70) with $\lambda \approx 0.2$ ($a_0 \approx 0.653$) which corresponds to the exact value of the photon number $P = 2.875$. These dynamical regimes exist also in NLS equation and they are analytically predicted and numerically confirmed in [400]. However, when the initial perturbation increases, the period grows with oscillations becoming strongly nonlinear to exhibit new types of long-lived localized dynamical structures (Fig. 13.7d). Further deviation from the stable equilibrium state can lead to a rapid aperiodic growth of the amplitude and the soliton collapse (Fig. 13.7e).

The understanding of different dynamical regimes is important for an insight into the low-frequency process of formation of stable relativistic solitons behind the laser pulse front inside the photon condensate [389]. More detailed determination of the regions in the parameter plane (P, λ) or separatrix curves in the phase space for different dynamical regimes would require additional analytical and numerical study which is out of the scope of this paper.

13.3.4 Strongly-Relativistic Solitons

Above considerations of linearly polarized solitons were restricted to a weakly relativistic regime, therefore being not applicable for describing large amplitude relativistic solitons observed numerically [389]. For that purpose, we have to use a different approach. A direct numerical simulation of the fully nonlinear relativistic one-dimensional fluid-Maxwell system (13.56)–(13.58) was performed to study the propagation of a long, strong relativistic laser pulse in a uniform, underdense

Fig. 13.7 Spatiotemporal evolution of different initial structures **(a)** Soliton in the instability region with amplitude $A_0 = 1.6$; **(b)** Gaussian structure with amplitude $A_0 = 0.536$ and photon number $P = 2.875$; **(c)** Gaussian structure with amplitude $A_0 = 0.536$ and the same photon number as in **(b)**; **(d)** An example of oscillating dynamical structures, and **(e)** An example of collapse dynamics



plasma. Our algorithm, based on the second order accuracy Lax–Wendorff method and leap-frog central scheme [389], was exploited. Linearly polarized pico-second pulse was launched into a plasma, and evolution of the mean pulse energy and other plasma parameters was followed. Numerical results given in Fig. 13.9 show the spatiotemporal evolution of the pulse energy, revealing the generation of a train of intense ultra-short relativistic (almost) standing EM solitons inside the photon condensate ($\omega < \omega_p$). Large relativistic solitons at a late time, after propagation of the pulse, are shown in Fig. 13.8. In Fig. 13.10, two characteristic neighboring solitary structures $A_s(x)$ are presented, along with electron density $\delta n_s(x)$ and the associated longitudinal momentum $p_x(x)$. Simulations clearly reveal that apart from stable small amplitude solitons, the family of large relativistic solitons, saturating at $A_0 \approx 2.5$ exists as well. The saturated value of the soliton amplitude depends on the laser intensity a_0 as well as on other parameters, e.g., the laser pulse duration. However, our study has shown that the soliton amplitude typically saturates at $A_0 \approx 2.9$ for intensities $a_0 \approx 0.7 - 0.8$.

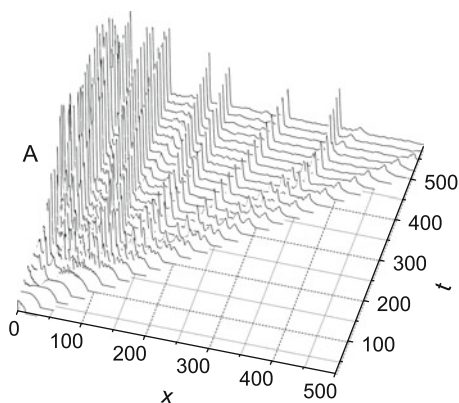


Fig. 13.8 Relativistic standing solitons formed behind the short laser pulse ($T = 1.5$ ps; $a_0 = 0.6$)

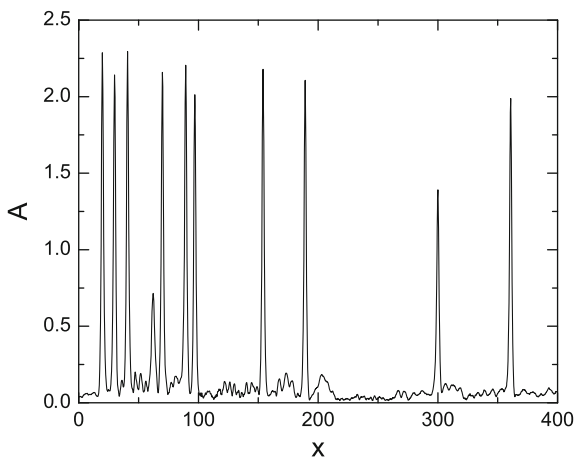
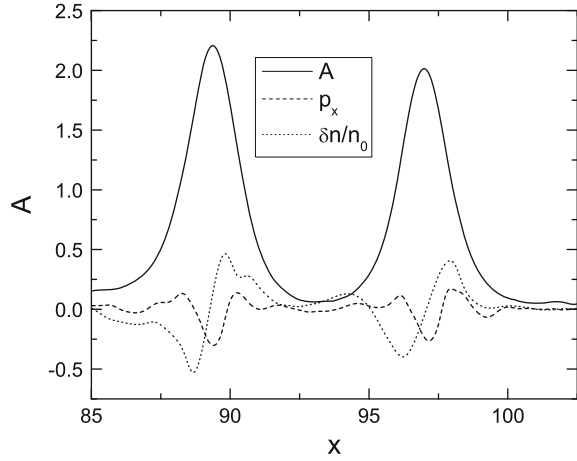


Fig. 13.9 Spatiotemporal EM energy density inside a photon condensate; from 1D fluid-Maxwell simulation of laser pulse ($a_0 = 0.5$) propagation (from *left to right*) in an underdense plasma ($n_0 = 0.1n_{cr}$) show slowing intense solitary structures

Fig. 13.10 Two neighboring solitons with the corresponding vector potential $A_s(x)$, density perturbation $\delta n_s(x)$, and linear momentum $P_x(x)$



As earlier analytical model fails for large amplitudes, we have used a different approach to estimate saturating values of the vector potential and its corresponding frequency. Namely, our numerical observations support the following substitution

$$\begin{aligned} a(x, t) &= A(x) \cos(\omega t); & p(x, t) &= P(x) \sin(2\omega t); \\ \delta n(x, t) &= N_0(x) + N_2(x) \cos(2\omega t) \end{aligned} \quad (13.75)$$

Neglecting p^2 in γ , but retaining p in the motion equation, after a simple trigonometric transformation, we get

$$\gamma = \sqrt{1 + a^2} = \sqrt{\left(1 + \frac{A^2}{2}\right) + \frac{A^2}{2} \cos(2\omega t)}. \quad (13.76)$$

The first term under the square root is larger than the absolute value of the second term, allowing the approximate expansion

$$\gamma \approx \gamma_A + \frac{A^2}{4\gamma_A} \cos(2\omega t), \quad (13.77)$$

where $\gamma_A = \sqrt{1 + A^2/2}$. Furthermore, under approximations (13.75–13.77), the following set of equations can be obtained

$$\begin{aligned} A_{xx} &= \left[\frac{5\gamma_A^2 - 1}{4\gamma_A} - \frac{\omega^2}{2} (3\gamma_A^2 - 1) + \frac{\gamma_A^2 + 2}{4\gamma_A^4} A_x^2 + \omega \frac{3\gamma_A^2 - 1}{2\gamma_A} P_x \right] A \\ N_0 &= \frac{A_{xx} A}{2\gamma_A} \left(1 - \frac{A^2}{8\gamma_A^2} \right) + \frac{A_x^2}{2\gamma_A^3} \left(1 - \frac{3A^2}{8\gamma_A^2} \right) \end{aligned}$$

$$\begin{aligned}
 N_1 &= \frac{A_{xx}A}{2\gamma_A} \left(1 - \frac{A^2}{4\gamma_A^2}\right) + \frac{A_x^2}{2\gamma_A^3} \left(1 - \frac{3A^2}{4\gamma_A^2}\right) + 2\omega P \\
 P &= \frac{\omega(1 + \gamma_A^2)AA_x}{2\gamma_A^2 \left(4\omega^2\gamma_A - N_0 + \frac{A^2}{16\gamma_A^2}N_1\right)}
 \end{aligned} \tag{13.78}$$

Solving the above system under assumption that maximum soliton amplitudes saturate at $\delta n = -1$ ($n = 0$) gives the result $A_0 = 2.67$ for $\omega_0 = 0.72$ close to the values obtained by direct numerical simulation of the fully relativistic system ($A_0 \approx 2.9$ and $\omega_0 \approx 0.73$). The difference in these results can come from, e.g., neglecting higher harmonics in A and consequently in γ . Nevertheless, we have demonstrated the existence of large linearly polarized electromagnetic solitons with an estimate for the soliton parameters A_0 and ω_0 .

In conclusion, above we have investigated existence and stability of 1D relativistic electromagnetic solitons in a cold underdense plasma, and have analytically found solutions and regions of stability of linearly polarized relativistic EM solitons in agreement with the relativistic fluid-Maxwell and particle simulations [389]. Analytical estimates gave the maximum amplitude and the frequency of large relativistic solitons close to simulation data. Difference in linear and circular polarization was singled out, e.g., in a role of the 2nd harmonic term in the relativistic γ -factor present for the linear polarization. The question of multi-dimensional effects, such as, e.g., transverse stability of 1D solitons for symmetry breaking perturbation [401] deserves further attention.

13.4 Stimulated Raman Cascade into Photon Condensation

The interaction of a strong relativistic laser pulse with an underdense plasma is investigated by relativistic fluid-Maxwell and PIC simulations. A nonlinear interplay between backward and forward stimulated Raman scattering instabilities produces a strong spatial modulation of the light pulse and the down cascade in its frequency spectrum. The Raman cascade saturates by a unique photon condensation at the bottom of the light spectrum near the electron plasma frequency, related to strong depletion and possible break-up of the laser beam. In the final stage of the cascade-into-condensate mechanism, the depleted downshifted laser pulse gets gradually transformed into a train of ultra-short relativistic light solitons [303, 389].

13.4.1 Introduction

Linearly polarized relativistic EM radiation sets plasma electrons into the longitudinal motion by the strong $(\mathbf{v} \times \mathbf{B})$ forces changing the nature of EMW to a coupled longitudinal-transverse mode. So are one dimensional (1D) electronic parametric

instabilities: stimulated Raman forward- and backward-scattering (F-SRS and B-SRS, respectively) and RMI [321, 430, 436, 437]. They do not appear isolated, but are often nonlinearly interlinked with other instabilities [435, 450]. Here, we shall mostly discuss F-SRS and B-SRS induced by picosecond, moderate to high intensity laser pulses in long length-scale plasmas. Particular attention will be paid to the effects of their mutual interplay and couplings to RMI. This results in rich 1D dynamics, reflecting in a spectral broadening and cascading process which transfers the incident laser energy to higher order scattering modes. Along the propagation beam, there is typically a Raman cascade in the light spectrum from fundamental (laser) frequency toward lower frequencies. The first Stokes line is significant; however, further with laser propagation it becomes a new pump, which decays via a secondary Raman scattering. This cascade process gets eventually halted around the plasma frequency, i.e., the cutoff frequency for light propagation in plasma. We find that continuing instability growth through stimulated Raman cascade downshifts a power maximum from the fundamental to the bottom of the light spectra. Thus, a unique stimulated photon condensation following a Raman cascade appears as a striking consequence of the relativistic laser-plasma interactions. This is similar to a situation with the Langmuir condensate in a weak turbulence theory [215, 218] where a condensate is due to a Langmuir decay spectral cascading down to small wave numbers (see above). As is well known, among these electronic instabilities, the SRS type is particularly a significant concern, because the excited EPW has a high phase velocity of the order of the velocity of light (F-SRS), it can produce very energetic electrons after the damping of EPW, such electrons can preheat the fuel in laser fusion applications [314].

The SRS instability can be most simply characterized as the resonant decay of an incident laser EM wave (ω_0, k_0) into a scattered *Stokes* EM wave (ω_s, k_s) plus an EPW ($\omega_{\text{EPW}}, k_{\text{EPW}}$) (Langmuir wave) with the following matching conditions for frequencies and wave numbers,

$$\omega_0 = \omega_s + \omega_{\text{EPW}}, \quad k_0 = \pm k_s + k_{\text{EPW}}$$

Here, $+k_s$ and $-k_s$ denote stimulated forward and backward Raman scattering (F-SRS/B-SRS), respectively. In addition to the above, wave triplet described by the above equation for SRS, usually, another weaker resonant three-wave coupling process with the first *anti-Stokes* scattered EM wave, can be observed in experiment and simulation.

$$\omega_s = \omega_0 + \omega_{\text{EPW}}, \quad \pm k_s = k_0 + k_{\text{EPW}}$$

By this three-wave coupling, one can obtain an upshifted frequency by the EPW frequency ω_{EPW} . An EM wave which propagates in a plasma has dispersion relation $\omega_0^2 = \omega_{\text{pe}}^2 + k_0^2 c^2$, EPW has dispersion relation $\omega_{\text{EPW}}^2 = \omega_{\text{pe}}^2 + 3k_{\text{EPW}}^2 v_{\text{the}}^2$, here, c , ω_{pe} , and v_{the} are speed of light, electron plasma frequency, and electron thermal velocity, respectively. The minimum frequency for EM wave propagating inside plasma is the local ω_{pe} , so it is clear that SRS instability requires that

$$\omega_0 \geq 2\omega_{\text{pe}} \implies n \leq \gamma n_{\text{cr}}/4 \quad (13.79)$$

where n and n_{cr} are the plasma density and the critical density for EM wave propagation in a plasma, respectively, and γ is relativistic Lorentz factor.

If intensity of the scattered (daughter) EM wave, including Stokes and anti-Stokes modes, which propagates backward or forward, exceeds the thresholds, it acts again as a new unstable pump that can drive fresh SRS instability. This mechanism can proceed in succession (cascade) as long as the corresponding threshold is exceeded. The so-called, stimulated Raman *cascade* process, with the following frequency and wave number matching conditions,

$$\omega_{s,j} = \omega_0 \pm j\omega_{\text{EPW}}; \quad \pm k_{s,j} = k_0 \pm jk_{\text{EPW}}$$

then takes place in relativistic laser-plasma interaction. The scattered EM waves include, not only the first-, second-, . . . , high-order Stokes modes ($j = -1, -2, \dots$), but also the first-, second-, . . . , high-order anti-Stokes modes ($j = 1, 2, \dots$), respectively. When a relativistic laser pulse propagates in underdense plasma, B-SRS and F-SRS can develop; still, they do not appear isolated but are often interconnected [389]. A nonlinear interplay between B-SRS and F-SRS produces a strong spatial modulation of the laser pulse and stimulated Raman cascade in its frequency spectra of scattered EM (light) waves. The continuing instability growth through stimulated Raman cascade downshifts the light energy maximum from the fundamental to the bottom of the EM wave spectra. It gets saturated by the *photon condensation* mechanism, related to strong depletion and possible break-up of the laser beam. In the final stage of the cascade-into-condensation mechanism, the depleted downshifted laser pulse gradually transforms into a train of ultra-short relativistic EM solitons [389, 438].

If making a comparison between homogeneous temporal instability growth rates of B-SRS and F-SRS in a weakly relativistic case, we find that their ratio is roughly $(n_e/n_c)^{3/4}$, where n_c is the critical electron density, which means that the Raman back-scatter growth time is 5–30 times faster than its forward-scatter counterpart, when the plasma density is varied from $0.1n_c$ to $0.01n_c$. However, it should be noted that B-SRS becomes saturated efficiently at a very early stage of evolution, while F-SRS and RMI can get close and even merge to a unique F-SRS/RMI instability, which competes with the B-SRS instability in the process of anomalous absorption of laser energy. B-SRS saturation can occur due to pump depletion or due to particle trapping, which could be dominant in a very low density plasmas. All of the above processes were intensively studied by theory and simulations [393, 412, 416, 434]. However, due to analytical difficulties related to nonlinear electron dynamics and anharmonicity in the plasma response for linearly polarized light as compared to circularly polarized laser light, the former has received less theoretical consideration. Therefore, below we solve a fully nonlinear system of cold electron-fluid and Maxwell equations for 1D propagation of relativistic, linearly polarized laser light in a uniform underdense plasma is presented. Further, detailed analysis is performed by 1D (2D) PIC simulations [389, 447]. The PIC method can follow the exact evolution of the laser light and plasma waves and particles on short time scales, much less than the laser period.

13.4.2 Relativistic Fluid-Maxwell Simulation

A long laser pulse enters the plasma and propagates in the x -direction, inducing both transverse and longitudinal electron motion. We consider only a purely one-dimensional case, in which the EM field is described by longitudinal (E_x) and transverse (E_y) electric field components and magnetic field B_z . For further treatment, it is convenient to normalize all the physical quantities to dimensionless values by introducing light-speed units. We express length x in $c\omega_0^{-1}$ and time t in ω_0^{-1} units, and normalize the electron fluid velocity v , momentum p , fields E and B , vector potential A , and density n_e to c , mc , $mc\omega_0 e^{-1}$, $m\omega_0 e^{-1}$, mce^{-1} , and n_0 , respectively. Here c , m , e , ω_0 , and n_0 designate, as usual, the vacuum speed of light, electron mass and (absolute) electron charge, laser light frequency, and initial uniform electron density, respectively. In this way, Maxwell equations in 1D reduce to [389],

$$\frac{\partial E_x}{\partial t} = \frac{n_e p_x}{\gamma}, \quad (13.80)$$

$$\frac{\partial E_y}{\partial t} + \frac{\partial B_z}{\partial x} = \frac{n_e A}{\gamma}, \quad (13.81)$$

$$\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} = 0, \quad (13.82)$$

$$\frac{\partial A}{\partial t} = -E_y, \quad (13.83)$$

in dimensionless units, while the cold electron fluid quantities are described by the continuity and electron motion equations

$$\frac{\partial n_e}{\partial t} = -\frac{\partial}{\partial x} \left(\frac{n_e p_x}{\gamma} \right), \quad (13.84)$$

$$\frac{\partial p_x}{\partial t} = -E_x - \frac{\partial \gamma}{\partial x}, \quad (13.85)$$

where $\gamma = (1 + p_x^2 + A^2)^{1/2}$, the transverse momentum p_y is replaced by the vector potential A , under Coulomb gauge; with immobile ions as a neutralizing background. Unlike the weakly relativistic case amplitude case (weak pump case and linear stability analysis) in the strong-pump limit considered here, the only exact analytical solution of the system of (13.80)–(13.85) can be found for idealized circularly polarized EM pump. Direct numerical solution of the above system has not been performed often in the past [379]. So, our effort has been invested into a construction of a stable numerical scheme for solving the 1D system (13.80)–(13.85) in a fully relativistic form, without any approximations. The fluid equations were treated by the second-order accuracy Lax–Wendroff method. As for the EM fields, a leap-frog-type central scheme was applied, with simultaneous calculation of all values, assisted by averaging exact electric and magnetic fields to approximate mesh node values at every space and time step.

In order to study the properties of backward and forward propagating EM waves, reflectivity R and transmittivity T are introduced at spatial point

$$R(x, t) = (F^-) / E_0^2, T(x, t) = (F^+) / E_0^2, \quad (13.86)$$

where F^+ and F^- are forward and backward propagating EM fields

$$F^\pm = E_y \pm B_z. \quad (13.87)$$

The aim of the simulation is to study penetration of a long ($> psec$) linearly polarized relativistic laser pulse into a long ($L > 1$ mm) moderately dense ($n_e < n_c/4$) plasma, as relevant for fast ignition in laser fusion. A relative laser intensity a_0 was varied in the interval $0.1 - 1$, where

$$a_0 = eE_0 / m_e \omega_0 c \simeq 8.85 \times 10^{-10} I^{1/2} \lambda_0, \quad (13.88)$$

laser intensity I is given in W/cm^2 , and λ_0 laser wavelength in microns (μm); therefore, roughly, the intensity range $10^{16} - 10^{18} W/cm^2$ was covered for laser light wavelength of $1 \mu m$. For lower pump intensity ($a_0 = 0.1$) and the plasma density $n_e < 0.1 n_c$, forward-propagating stimulated Raman scattering in a linear regime occurs, with Stokes component in the frequency spectrum far dominant over anti-Stokes one Fig. 13.11. Along the propagating beam, there is a cascade in the light spectrum from fundamental (laser) ω_0 — frequency toward lower frequencies. The first Stokes line is significant; however, along the propagation path, the Stokes mode further decays via a secondary Raman scattering. This cascade process is eventually halted near the electron plasma frequency — the cutoff for the light propagation.

Furthermore, the laser intensity a_0 was raised to higher values of 0.2 and 0.3. Figure 13.12 demonstrates the spatial profile of the laser beam energy (averaged over the laser period) at two moments of time. Although still a moderately relativistic case, typically strong spatial modulation, pulse depletion, and break up take

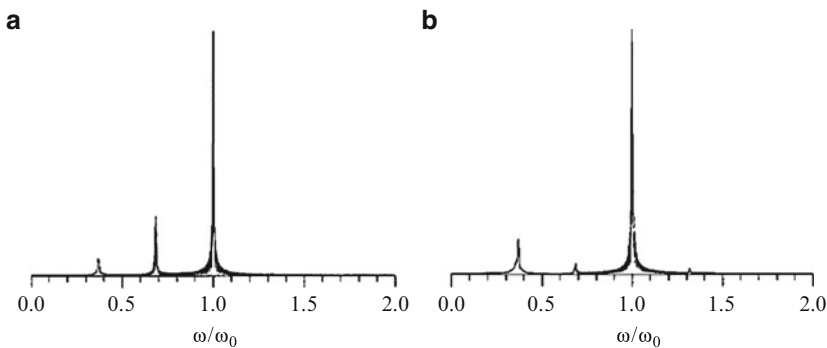


Fig. 13.11 Frequency spectra of forward propagating laser pulse at point in space, (a) $x = 60\lambda_0$ and (b) $x = 200\lambda_0$, respectively, inside plasma for $a_0 = 0.1$ and $n_e = 0.1n_c$

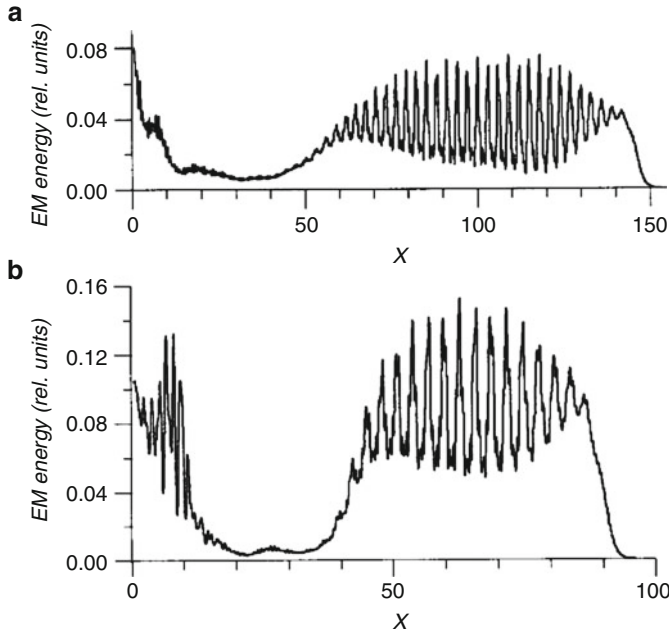


Fig. 13.12 Spatial distribution of laser EM energy density for (a) $a_0 = 0.2$, $n_e = 0.1n_c$ at $t = 160T_0$ and (b) $a_0 = 0.3$, $n_e = 0.1n_c$ at $t = 100T_0$, from fluid simulation

place. The characteristic nonlinear modulation spatial scale is $2\pi c/\omega_p$ or here, about ($\sim 3\lambda_0$) corresponding to F-SRS/RMI processes [389]. Similar relativistic modulation effects are regularly observed in particle simulations with ultra-short pulse high intensity laser pulses [376,380,381,393,412,416]. Having a significant growth rate, the predominantly convective B-SRS instability (in the pulse frame) develops at the pulse front and saturate over short distance in a back of the pulse. Strong backscattering depletes the body of a pulse by creating like a notch at some point in the body of the laser pulse [380].

In addition to 1D fluid-Maxwell results, fully relativistic 2D PIC simulations have been performed to search for an agreement with the above behavior [389]. The simulation box of $(200\lambda_0 \times 50\lambda_0)$ with uniform cold plasma and density $n_e = 0.1n_c$ is put in vacuum. In Fig. 13.13, the 2D laser pulse EM energy profile is plotted at three moments in time. Again, laser pulse break-up is observed at roughly the same spot as seen in earlier fluid simulations. Further, Fig. 13.14 gives a view of transmitted light spectrum, together with a spectral intensity at the central axis (top). Raman cascade containing first anti-Stokes and first and second Stokes sidebands is clearly revealed. Continuing the study by extensive 1D PIC simulations, remarkable spectral properties are observed, pointing to their general complex character (Fig. 13.15). Namely, B-SRS spectra reveal the SRS cascade toward lower frequency mode numbers, gradually down-shifting the maximum to the bottom of the spectrum. This unique stimulated photon condensation following the Raman cascade is a

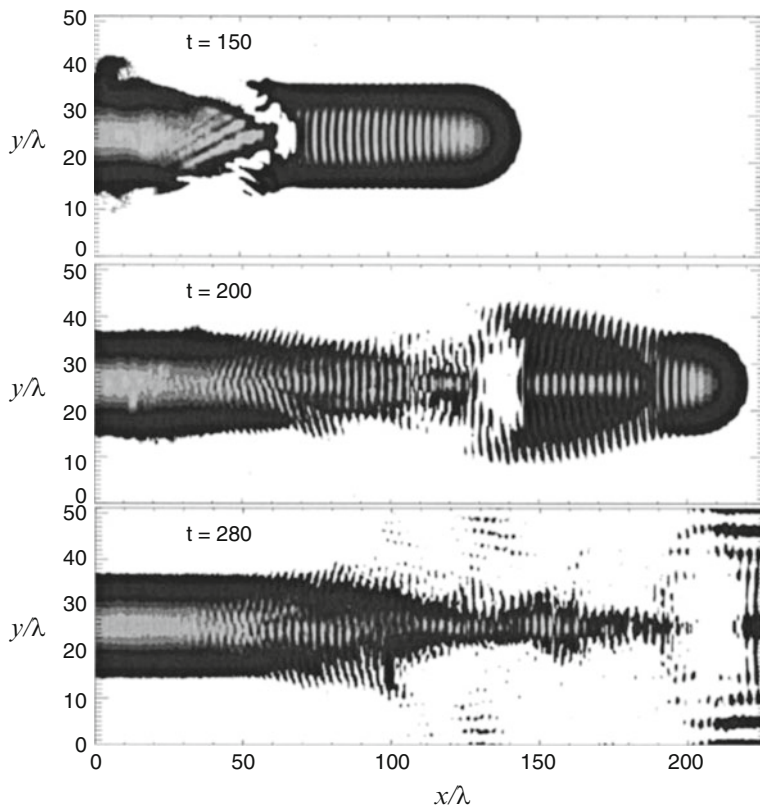


Fig. 13.13 2D distribution of EM energy density at three moments in time during pulse propagation ($a_0 = 0.2$, $n_e = 0.1n_c$) from 2D PIC simulation. Laser pulse break up is clearly observed

striking example of complexity in relativistic laser-plasma interactions, as was qualitatively predicted in the previous Chapter related to 3WI complexity [357]. Indeed, above scenario is well consistent with 3WI model of B-SRS with the relativistic frequency shift, which by increasing laser intensity can trigger nonlinear bifurcations on the route to spatio-temporal turbulence, as was first proposed by Škorić et al. [302, 309, 311, 357] (vide supra). In order to give a wider parameter survey, F-SRS and B-SRS spectra are shown in Fig. 13.16, revealing a unique paradigm of *Raman cascade into photon condensation* with a spectral energy accumulation at the cut-off frequency.

Next, fully relativistic 1D EM particle simulations of the SRS, including the Raman cascade, transition from Raman cascade into photon condensation and the generation of large amplitude relativistic EM soliton by a linearly-polarized intense laser interacting with an underdense uniform plasmas, are presented in more detail [447].

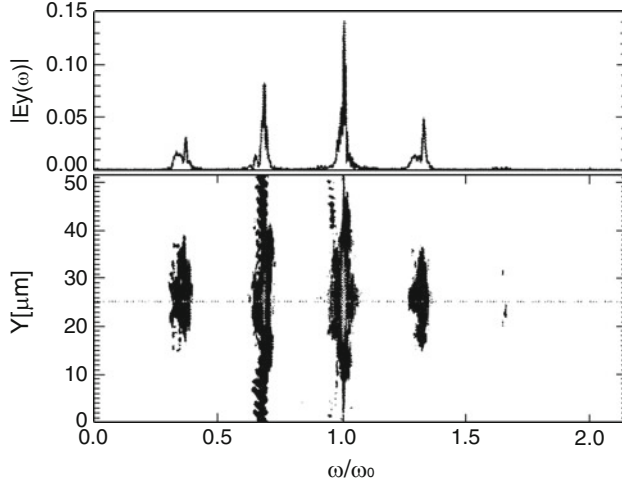


Fig. 13.14 Transmitted light spectrum variation in transverse direction (*bottom*) and spectral distribution on the axis (*top*) from 2D PIC simulation of Fig. 13.13

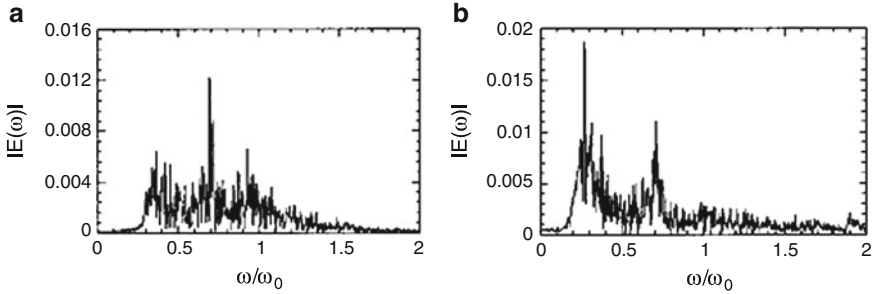


Fig. 13.15 B-SRS spectra from 1D PIC simulation for parameters of Fig. 13.14; (a) $a_0 = 0.2$ and (b) $a_0 = 0.3$ ($n_e = 0.1 n_c$)

13.4.3 Particle Simulations

One-dimension and three-velocity (all quantities depend on x -coordinate and the particle momenta have three components) fully relativistic EM particle-in-cell (1D3V-PIC) code is used. The total length of simulation system is $3L$ (c/ω_0), where c and ω_0 are the speed of light and the laser EM frequency in vacuum, respectively. The L long plasma begins at $x = 0$ and ends at L , in the front and rear side of the plasma layer, there are two L long vacuum regions. Ions are initially placed as a neutralizing background and are kept immobile. The number of cells is 10 per $1 c/\omega_0$ and 80 particles are put in each cell. The linearly-polarized laser with the electric field E_0 along the y -direction is launched at the position where $200c/\omega_0$ long distance before plasma. Its normalized amplitude is $a_0 = eE_0/m_e\omega_0c$, where e and m_e are the electron mass and charge, respectively. The electrons which reach

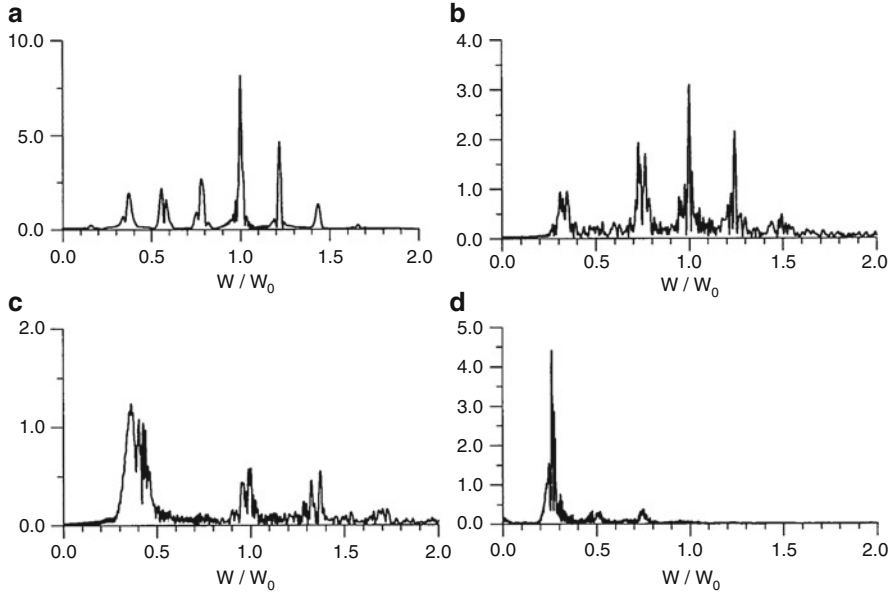


Fig. 13.16 Forward-SRS spectra from 1D PIC for (a) $a_0 = 0.2$, $n_e = 0.05n_c$, (b) $a_0 = 0.5$, $n_e = 0.07n_c$, (c) $a_0 = 0.3$, $n_e = 0.13n_c$ and backward-SRS spectrum, (d) $a_0 = 0.4$, $n_e = 0.07n_c$

vacuum region build a charge separation potential barrier that prevents more electrons from leaving the bulk plasma. For escaping electrons as well as for outgoing EM waves, two $100c/\omega_0$ long extra numerical damping regions are applied [447]. It should be noted that in the following text, by taking the EM fields at the front and rear sides at $100c/\omega_0$ away from the plasma, the frequency and wavenumber spectra of the reflectivity and transmissivity, and the temporal growth of all EM modes are calculated, respectively. The time, electric field, and magnetic field are normalized as usual, to the laser period $2\pi/\omega_0$, $m\omega_0 c/e$, and $m\omega_0/e$, respectively. The time is taken as zero, $\omega_0 t = 0$, when the laser arrives at the vacuum-plasma boundary.

13.4.3.1 Stimulated Raman Cascade into Photon Condensation

Our first simulations are performed by taking plasma density $n = 0.03n_{cr}$, plasma length $L = 1000c/\omega_0$, temperature $T_e = 1.0\text{keV}$, and the normalized laser amplitude $a = 0.3$, respectively.

The long plasma slab is initially uniform in density and the intense laser pulse is radiated continuously in our simulations. There is enough time for the growth of instabilities and rich interplay between relativistic electronic parametric instabilities, such as F-SRS, B-SRS, and RMI [389, 447]. In such a low plasma density, as shown in Fig. 13.17, in the early linear growth stage, B-SRS has lower growth rate and longer growth time than that of the F-SRS, it gets saturated efficiently at a very

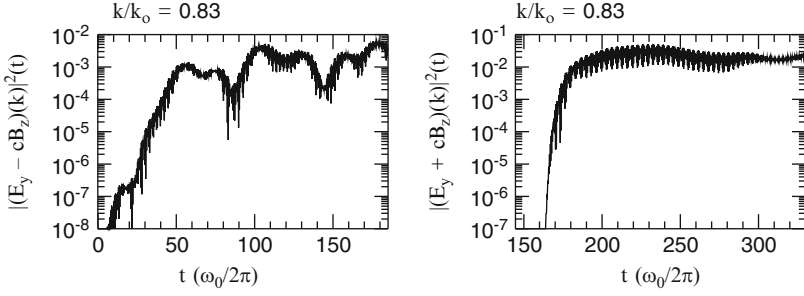


Fig. 13.17 The temporal growth for reflected EM wave (*left plot*) and transmitted EM wave (*right plot*) both measured in vacuum in the case of plasma $n = 0.03n_{cr}$, $L = 1000c/\omega_0$, $T_e = 1.0$ keV, and laser amplitude $a = 0.3$, respectively. From Fig. 13.18 to Fig. 13.22, the same simulation parameters are used

early stage of evolution due to plasma heating and trapping. As the time goes on, F-SRS and RMI can get close and merge to unique F-SRS/RMI instability, after which it can compete with the B-SRS instability.

The frequency spectra and wave numbers for reflected EM, transmitted EM, and ES waves are plotted in Fig. 13.18. One can see that, during the SRS process, the excited dominant EPW is with the frequency $\omega_{pe} \approx 0.17\omega_0$; the corresponding scattered Stokes EM wave has the frequency $\omega_s \approx 0.83\omega_0$. The wave numbers for both backward and forward scattered EM waves are $k_s \approx 0.83k_0$, the EPW develops two dominant peaks, the first peak with $k_{EPW} \approx 1.83k_0$ for B-SRS and the second peak with $k_{EPW} \approx 0.17k_0$ for F-SRS. Therefore, both B-SRS and F-SRS sidebands can be well-explained by the SRS matching conditions for the frequency $\omega_0 = \omega_s + \omega_{epw}$ and the wave number $k_0 = k_s + k_{EPW}$ are well-satisfied. It should be noted that, for very low plasma density, the wavenumbers measured in vacuum are nearly in equal values of those inside the plasma.

Figure 13.19 (top) shows the snapshots for the energy density of EM field, it demonstrates the spatial distribution of transverse EM energy density, in its early stage, B-SRS radiates its EM energy through soliton-like structures, as shown in Fig. 13.19, which corresponds to the peaks in the reflectivity plot. When the unique F-SRS/RMI instability dominates and can compete with the B-SRS instability, the strong spatial self-modulation of the order of $2\pi c/\omega_{pe} \approx 5.8\lambda_0$ which comes from F-SRS/RMI instability and the depletion of the laser pulse are taken place and observed from Fig. 13.19 (bottom). In a linear regime, B-SRS and F-SRS Stokes sidebands in the spectra are by far dominant over the anti-Stokes modes. Accordingly, there is a clear stimulated forward Raman cascade process, as nicely shown in Fig. 13.21. The scattered Stokes EM waves include the first- to fifth-Stokes harmonics. In addition, the first- to fourth- anti-Stokes harmonics can also be observed. As expected, at early times, laser driven cascade shows descending spectral intensity toward the higher harmonics. The stimulated forward Raman cascade is eventually halted at the cut-off, close to the background electron plasma frequency $0.15\omega_0$. The spectral gap between the laser fundamental and the lowest Stokes mode is apparent

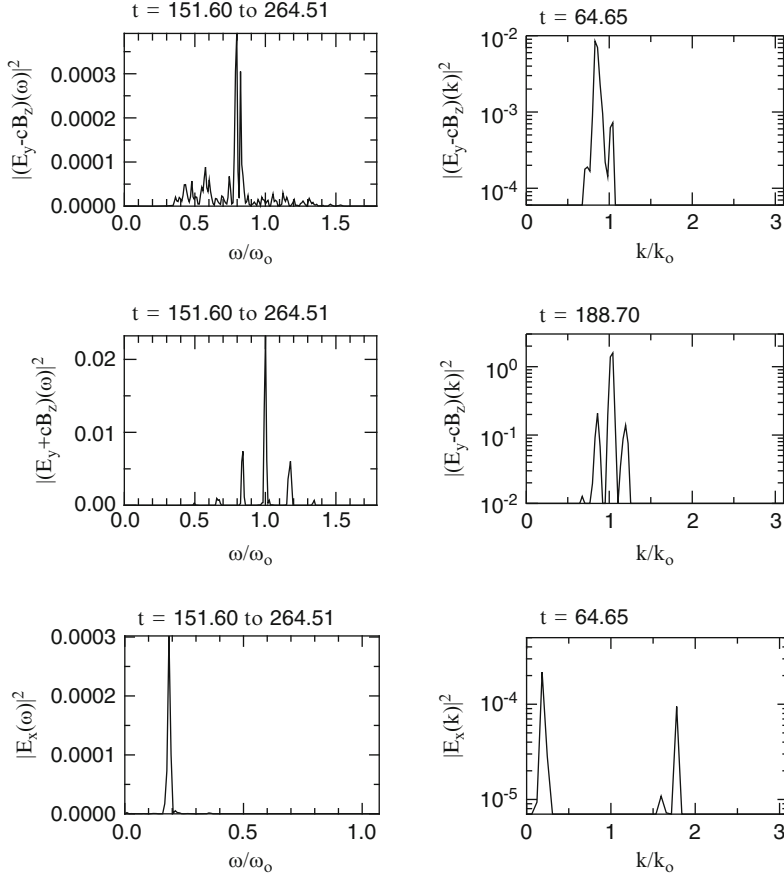


Fig. 13.18 The frequency spectra for the reflected and transmitted EM waves both measured in vacuum and for ES wave measured inside plasma. The wavenumbers for electric field E_y and for ES field E_x both measured inside plasma

(Fig. 13.21 and below). Continuous laser energy transfer by Raman cascade to low-frequency self-trapped EM mode further enhances the unique photon condensate. From the frequency and wavenumber spectra, stimulated backward Raman cascade can be recognized, which include both Stokes and anti-Stokes modes however, it does not reveal such picture, like shown for the Forward-SRS cascade due to already an onset of the intrinsic B-SRS complexity, as discussed above [309]. Consequently, an interesting feature is the broadening of the backscattered spectra, from the laser fundamental down to the relativistic electron plasma frequency. At the same time, the reflectivity exhibits an intense spiky behavior, as shown in Fig. 13.20 in the process of photon condensation and energy localization. [389].

To illustrate the onset and growth of the stimulate forward Raman cascade, the plots for the temporal evolution of dominant Stokes and anti-Stokes EM modes

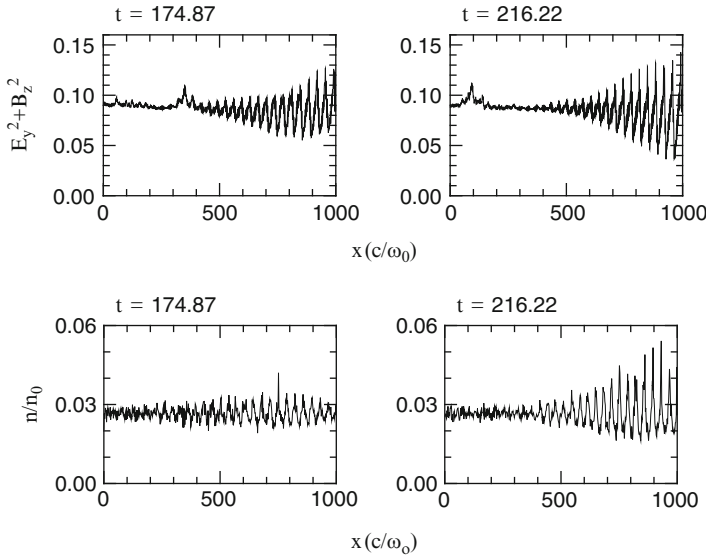


Fig. 13.19 The snapshots for energy density of EM field $E_y^2 + B_z^2$ (averaged over laser period) and plasma density n/n_0

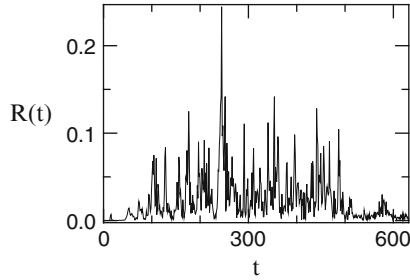


Fig. 13.20 The temporal reflectivity of EM wave measured in vacuum

measured in vacuum are given in Fig. 13.22. The fundamental laser mode shows a clear depletion of the EM energy which corresponds to the depression in the temporal evolution plot of the laser energy. Both, for Stokes and anti-Stokes modes, higher harmonics growth is slower and delayed in time in comparison to lower harmonics. In the later time, the continuing instability growth through Raman cascade downshifts a power maximum from the fundamental to the bottom of the EM spectra. They clearly reveal a tendency of a transition from the Raman cascade regime to an energy accumulation at the relativistically reduced electron plasma frequency, which is the cutoff frequency for EM wave propagating in a plasma, the so-called photon condensation.

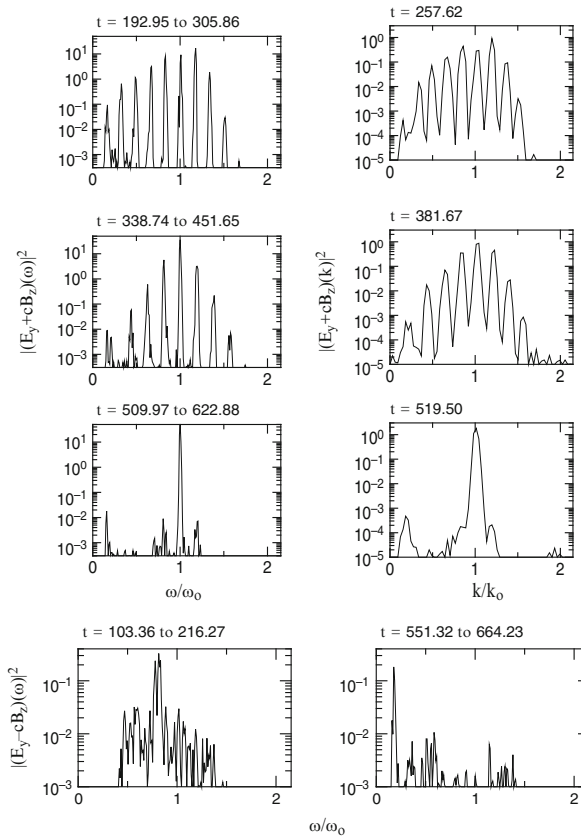


Fig. 13.21 The frequency and wavenumber spectra of transmitted EM wave measured in vacuum (*top*); the frequency spectra of reflected EM wave (*bottom*)

13.4.3.2 Effect of Laser Intensity on SRS Cascade into Photon Condensation

What role does the laser intensity play in SRS evolution and stimulated Raman cascade into photon condensation? To understand this question, several simulations have been performed only by varying the laser intensity. As shown in Fig. 13.23, the frequency spectra of transmitted EM wave for laser amplitude $a = 0.2$ and $a = 0.4$ are presented here.

It is clear that the cascade-to-condensation transition becomes more pronounced at higher laser intensity [389]. In the lower laser amplitude case, few Stokes and anti-Stokes EM modes are excited, further, with decreasing laser amplitude, the stimulated Raman cascade into photon condensation and even SRS do not take place. In the higher intensity case, more Stokes and anti-Stokes harmonics are excited with faster growth; moreover, the bandwidth of the modes becomes larger

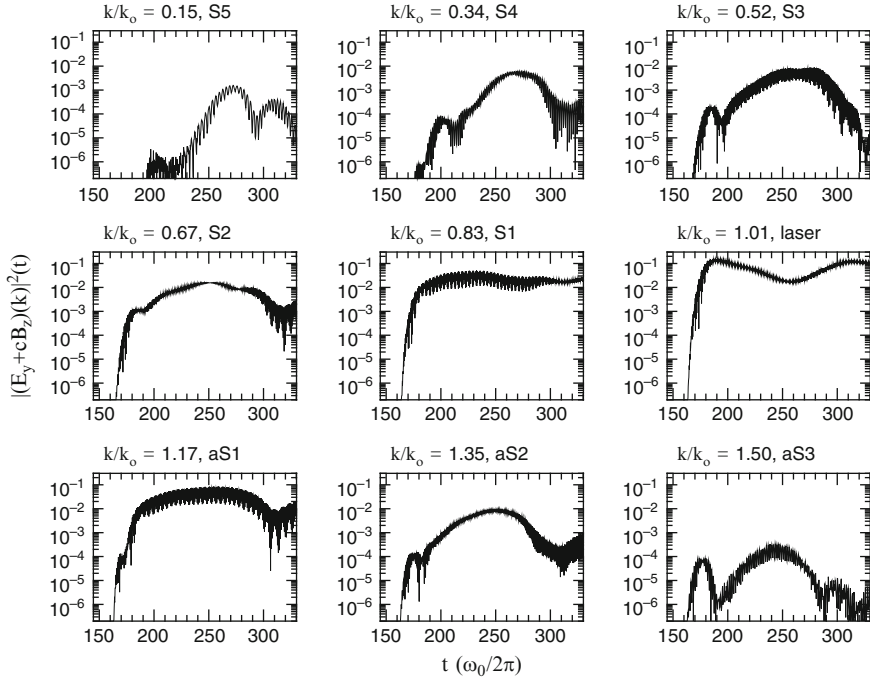


Fig. 13.22 The temporal growth of transmitted EM Stokes (S) and anti-Stokes (aS) modes

than that in the lower amplitude case. Further by increasing intensity, the modes can merge into incoherent continuum, which quickly suppresses the further growth of the instability. As for much lower laser intensity $a = 0.1$, as shown in Fig. 13.24, there are only few Stokes and anti-Stokes harmonics to be excited in comparison to higher intensities (Fig. 13.25).

In a low density plasma, in the regions of strong photon-condensation, first, large portion of electrons are pushed away due to the strong ponderomotive force of EM field, with a resulting electron density depletion due to nonlinearity; second, nonlinear dispersion plays an important role due to the relativistic electron mass increase in the intense EM field [392]. Therefore, in addition to nonlinearly interlinked electronic instabilities, above effects can result in a well-known nonlinear localization phenomenon, i.e., the generation of relativistic EM soliton [396,439,464]. As shown in Fig. 13.26, the soliton exhibits a regular EM and ES structure in space, i.e., its ES field E_x has one-cycle structure, the corresponding transverse electric field E_y is the half-cycle and the magnetic field B_z is the one-cycle periodic structure, respectively. Also the spatial EM field structure is oscillatory in time, but the ES field structure is not (zero harmonic). The explanation comes directly from Maxwell's equations. The Faraday law gives $B_z \sim \partial E_y / \partial x$; indeed, the x -derivative of the Gaussian-like soliton profile E_y gives B_z shown in Fig. 13.26. Similarly, from the Poisson equation, integration over x of the Gaussian density cavity leads to the ES field E_x in

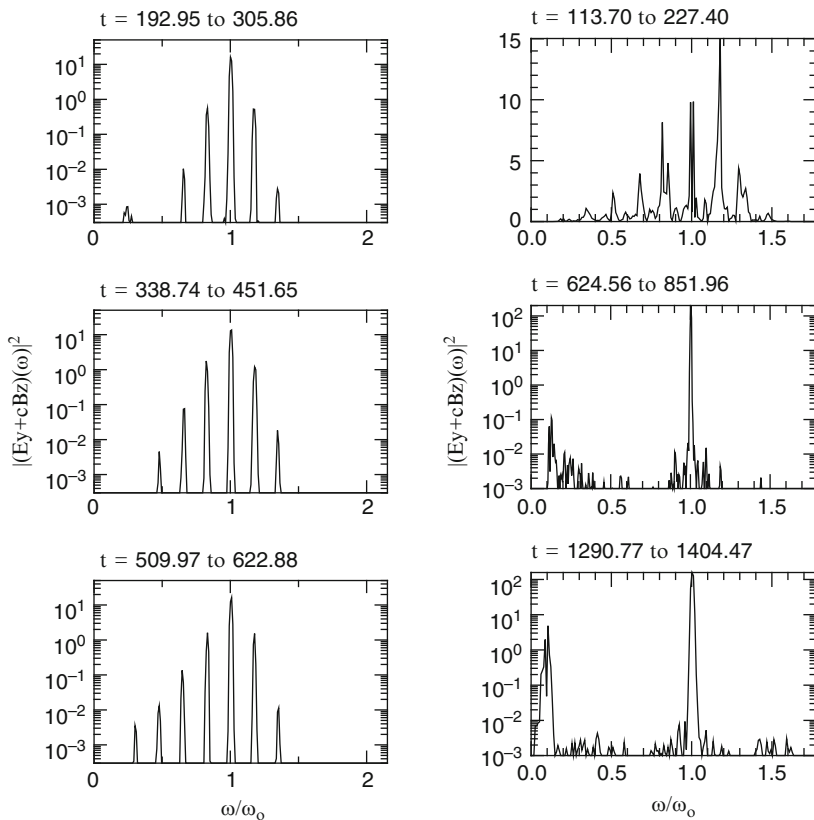


Fig. 13.23 The frequency spectra of transmitted EM wave, for $n = 0.03n_{cr}$, $L = 1000c/\omega_0$, $T_e = 1.0$ keV; for laser amplitude $a = 0.2$ (left) and $a = 0.4$ (right)

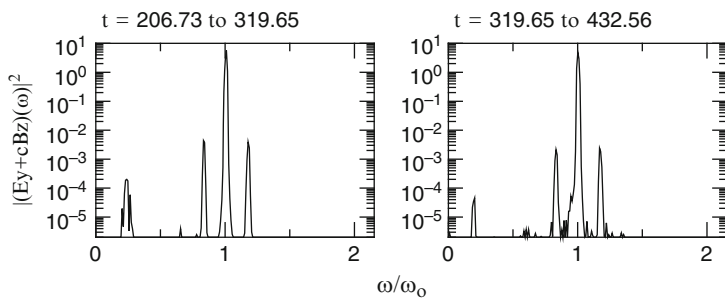


Fig. 13.24 The frequency spectra of transmitted EM wave, for $n = 0.03n_{cr}$, $L = 1000c/\omega_0$, $T_e = 1.0$ keV and $a = 0.1$

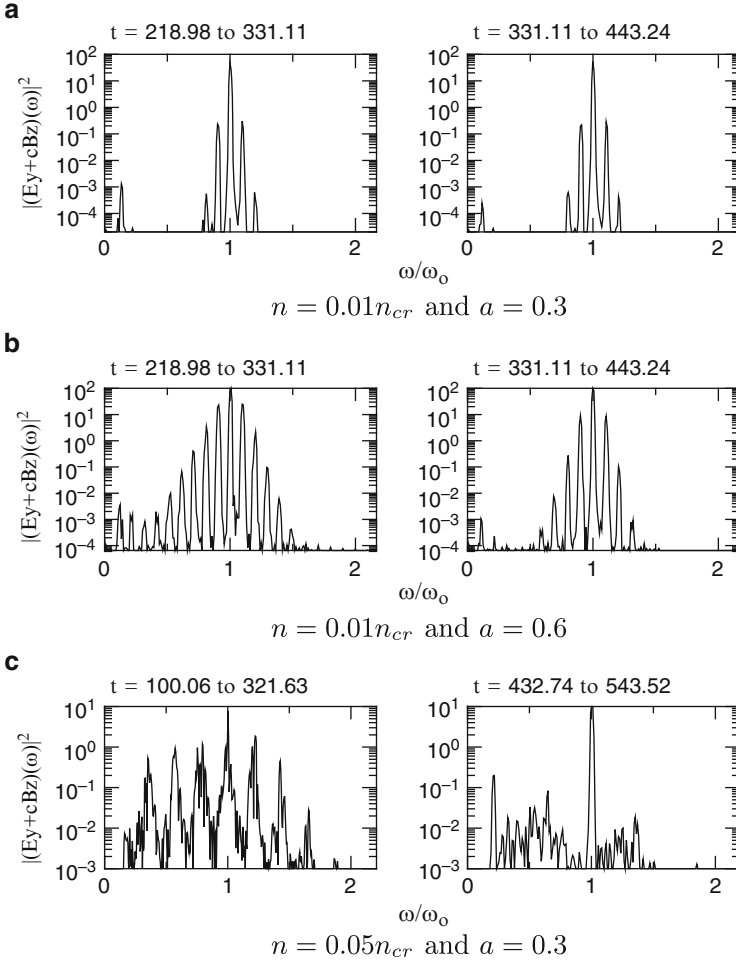


Fig. 13.25 The frequency spectra of transmitted EM wave for different plasma density and for fixed $L = 1000c/\omega_0$ and temperature $T_e = 1.0 \text{ keV}$. For (a) $n = 0.01n_{cr}$ and $a = 0.3$, (b) $n = 0.01n_{cr}$ and $a = 0.6$, (c) $n = 0.05n_{cr}$ and $a = 0.3$

Fig. 13.26. Moreover, PIC data and analytics, e.g., equations (13.62) of [414], show that zero-harmonic term dominates the electron density perturbation. Therefore, the Poisson equation gives the corresponding non-oscillatory ES field structure, such as E_x in Fig. 13.26; from which we find that, the size of EM soliton is about $5\lambda_0$, which is roughly equal to the electron plasma wavelength $\lambda_{pe} = c/\omega_{pe}$. More details on the ES and EM structures of the soliton, has been given in [441, 454]. Also, the soliton acceleration was discussed there, i.e., in addition to the density nonuniformity, the acceleration of relativistic EM solitons depends on the laser intensity and the plasma length.

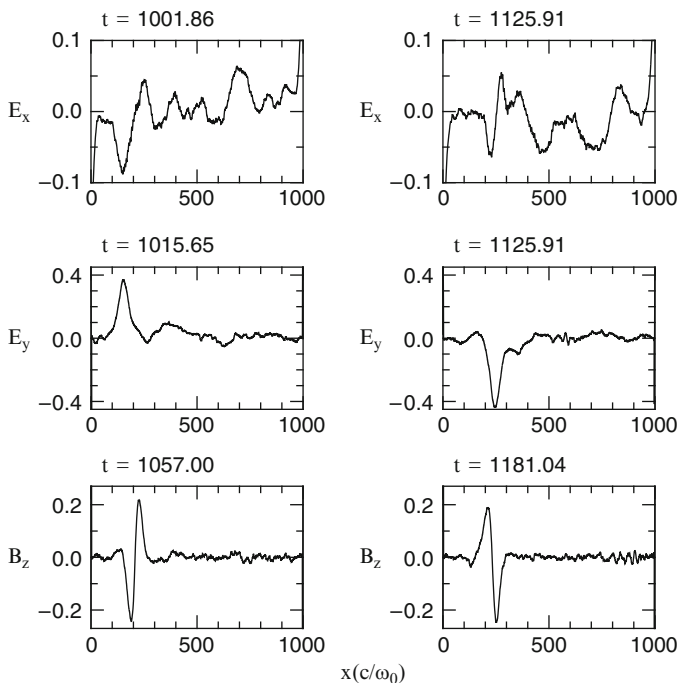


Fig. 13.26 The snapshots for ES field profile (averaged over EPW wavelength) and EM field structures (averaged over laser period) in the case of plasma $n = 0.03n_{cr}$, $L = 1000c/\omega_0$, $T_e = 1.0$ keV and laser amplitude $a = 0.5$, respectively

In conclusion, fully relativistic EM 1D-PIC simulation results on the SRS, stimulated Raman cascade, and the transition from Raman cascade into photon condensation induced by linearly-polarized intense laser in underdense homogeneous plasmas were presented. At appropriate laser intensity and plasma conditions, large relativistic EM solitons are readily formed inside the strong photon condensation.

13.5 Relativistic EM Solitons in a Low Density Plasma

Strongly relativistic electromagnetic solitons due to strong photon condensation, induced by a linearly polarized intense laser interacting with an underdense uniform collisionless plasma, are studied by particle simulations. In a homogeneous plasma, both standing and accelerated solitons are observed. It is found that the acceleration of the solitons depends upon not only the laser amplitude but also the plasma length. The electromagnetic frequency of the solitons is between half- and one-time of the unperturbed electron plasma frequency. The electrostatic field inside the soliton has a one-cycle structure in space, while the transverse electric and the magnetic fields

have half- and one-cycle structures, respectively. The acceleration of the solitons is briefly discussed.

13.5.1 Introduction

The mechanism of the relativistic EM soliton formation and its structure have been analytically investigated and observed by particle simulation of the interaction of intense laser radiation with underdense and overdense plasmas [387, 414, 452–465]. The EM solitons found in one-dimensional (1D) and two-dimensional (2D) particle simulations consist of slowly or non-propagating electron density cavities inside which an EM field is trapped and oscillates coherently with a frequency below the unperturbed electron plasma frequency, and with a spatial structure corresponding to half a cycle (subcycle soliton) [461]. In homogeneous plasmas, EM solitons have been found to exist for a long time, close to the regions where they were generated and eventually decay in the interaction with particles by converting EM energy into fast particles. In inhomogeneous plasmas, EM solitons are accelerated with the acceleration proportional to the gradient of the plasma density toward the low density side. When an EM soliton reaches some critical plasma region, for example, the plasma-vacuum interface, it radiates its energy away in the form of a short burst of low-frequency EM radiation [395]. The interaction of two solitons can lead to their merging with the resulting soliton retaining the total energy of the two. [390, 392, 462].

In a previous section, we mainly paid attention to SRS cascade and the transition from the Raman cascade into photon condensation [389, 447]. As a follow up, here, we concentrate on and present 1D-PIC simulations of a generation of the large, relativistic EM solitons inside a strong photon condensate due to the Raman cascade, induced by a linearly polarized intense laser pulse in long underdense plasma. The standing, backward-, and forward-accelerated EM solitons are observed. It is found that in a homogeneous plasma the acceleration of the EM solitons depends upon both the laser amplitude and the plasma length. The EM frequency of the solitons is between half- and one of the background electron plasma frequency. The ES field inside the soliton has a one-cycle structure in space, while the transverse electric and magnetic fields have half-cycle and one-cycle structures, respectively. The acceleration of the EM solitons is briefly addressed.

13.5.2 Relativistic EM Solitons

A one-dimensional and three-velocity fully relativistic EM particle-in-cell (1D3V-PIC) code is used in our simulations. The length of the plasma layer is L (c/ω_0), which begins at $x = 0$ and ends at $x = L$, where c and ω_0 are the speed of light and the laser frequency in vacuum, respectively; at the front and rear side of the plasma,

there are two L long vacuum regions. Ions are initially placed as a neutralizing background and are kept immobile. The number of cells is 10 per $1 c/\omega_0$ and 80 particles are put in each cell. The linearly polarized laser pulse, which with the electric field E_0 along the y direction and the normalized amplitude $a = eE_0/m_e\omega_0c$, is launched at the distance $200c/\omega_0$ from the plasma front interface, where e and m_e are the electron mass and electron charge, respectively. The plasma electrons which enter the vacuum region build a potential barrier that prevents more electrons leaving the plasma. For outgoing electrons as well as EM waves, two $100c/\omega_0$ long additional numerical damping regions are used. The time, electric field, and magnetic field are normalized to the laser period $2\pi/\omega_0$, $m_e\omega_0c/e$, and $m_e\omega_0/e$, respectively. The time is taken as zero, $\omega_0 t = 0$, when the unit-step laser pulse arrives at the front vacuum-plasma boundary.

All simulations are performed by introducing a homogeneous low density plasma, $n = 0.032n_{cr}$. The unit-step intense laser pulse is injected continuously; there is enough time for growth, development, and rich interplay between many relativistic electronic parametric instabilities, such as the stimulated backward Raman scattering (B-SRS), stimulated forward Raman scattering (F-SRS), and the RMI, etc. [389, 447]. In such laser plasma conditions, SRS can be fast excited by an intense laser EM wave coupling to a scattered EM wave plus an EPW. In an early, linear stage, B-SRS has a lower growth rate and longer growth time than that of the F-SRS. B-SRS irradiates its EM energy through spiky-like temporal structures and becomes saturated at an early stage due to electron heating by trapping and EPW breaking. At later times, F-SRS and RMI become closer and merge to a unique F-SRS/RMI instability, which can further dominate and compete with the B-SRS instability; as a result, a strong spatial self-modulation of the order of $2\pi c/\omega_{pe} \approx 5.6\lambda_0$ and depletion of the laser pulse will take place. Further in time, there is a clear SRS cascade with an excitation of higher-order Stokes and anti-Stokes harmonics[447]. At earlier times, the laser pump-driven Raman cascade develops a descending spectrum toward higher order harmonics. The Raman cascade is eventually halted at the lowest Stokes harmonic, close to the value of relativistic electron plasma frequency. In the later time, the continuing instability growth and energy transfer through Raman cascade downshifts EM energy from the fundamental to the bottom of the EM frequency spectrum, i.e., the low frequency trapped EM mode. It reveals a tendency of photon condensation [303, 389]. The cascade-to-condensation transition becomes more pronounced with increasing laser intensity and higher plasma density.

In the regions of strong photon-condensation, electron density decreases due to ponderomotive force and relativistically increased electron mass, both locally reduce the plasma refractive index which results in self-trapping of EM wave energy. Therefore, at appropriate conditions, the well-known physical phenomenon of non-linear EM energy localization in the form of the relativistic coherent EM soliton becomes possible. Since the photon number in the Raman cascade is conserved, the frequency downshift results in a corresponding large increase in the amplitude of the lowest harmonics [389, 396, 439, 447].

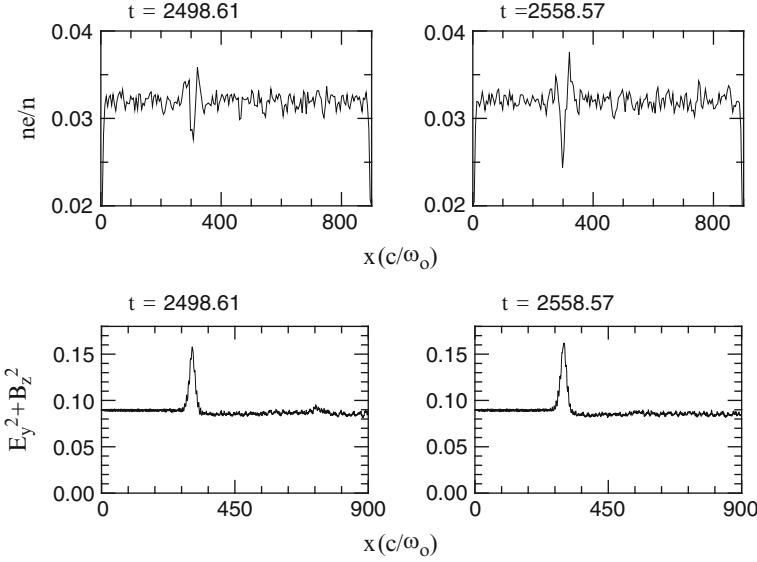


Fig. 13.27 The snapshots for electron density n/n_e and for EM energy density $E_y^2 + B_z^2$ averaged over laser wavelength λ_0 in the case of plasma $n = 0.032n_{cr}$, $L = 900c/\omega_0$, $T_e = 350$ eV and laser amplitude $a = 0.3$, respectively

13.5.2.1 Standing EM Solitons

In the case of plasma density $n = 0.032n_{cr}$, length $L = 900c/\omega_0$, temperature $T_e = 350$ eV, and laser amplitude $a = 0.3$, following SRS cascade and strong photon-condensation, the snapshots for electron plasma density n/n_e and EM energy density $E_y^2 + B_z^2$ are shown in Fig. 13.27. A spatially localized, non-propagating electron density cavity is created. Inside the density cavity, large EM field is trapped which oscillates coherently in time; i.e., a large amplitude relativistic EM soliton is formed. A standing EM soliton is observed within our simulation time, close to the region where it was born [395].

In Fig. 13.28, the frequency spectra for ES field E_x and EM field E_y , which are trapped locally inside the soliton, are plotted. In addition to the fundamental and the excited perturbed EPW, EM component with the frequency $\omega_{sol}^{EM} \approx 0.13\omega_0 \approx 0.72\omega_{pe}$ and the broadband ES component with the frequency $\omega_{sol}^{ES} \approx 0.87\omega_0$ are observed, respectively. If only from the spectral point of view, it appears that one can explain this phenomenon by a three-wave resonant wave coupling process during the existence of the EM soliton. The size of the EM soliton is about $\Delta x_{sol} \approx \lambda_{epw} \approx 5.6\lambda_0$, where λ_{epw} and λ_0 are the electron plasma wavelength and laser wavelength in vacuum, respectively. As shown in Fig. 13.29, the ES field E_x , inside the soliton, has a one-cycle structure in space, while the transverse electric field E_y has a half-cycle, and the magnetic field B_z a one-cycle structure, respectively. In our simulations, laser intensity was weakly to moderately relativistic ($a < 1$);

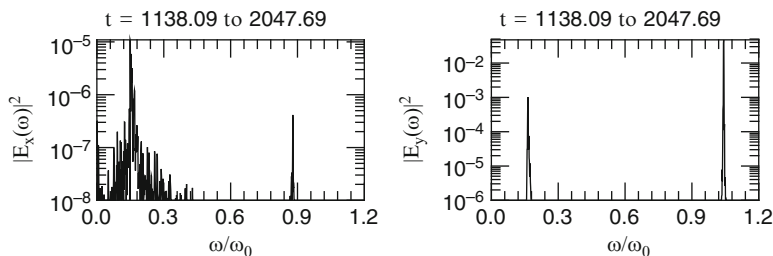


Fig. 13.28 The spectra of ES filed and EM field which trapped inside soliton region. The simulation parameters are the same as used in Fig. 13.1

however, the created low-frequency EM solitons are indeed strongly relativistic. The amplitude of the generated EM soliton scales with an increasing laser intensity; still, soliton saturates at a roughly equal maximum amplitude (strongly-relativistic) $a_{\text{sol}}^{\text{max}} \approx 5 - 6$ [447], clearly departing from a weakly relativistic soliton models, as discussed earlier.

It should be noted that the EM soliton frequency is $0.13\omega_0$, which is much smaller than the background electron plasma frequency $\omega_{\text{pe}} \sim 0.18\omega_0$. This comes from a strong relativistic effect, which by increasing the electron mass increase from m_e to γm_e , effectively reduces local electron plasma frequency from ω_{pe} to roughly $\omega_{\text{pe}}/\gamma^{1/2}$. Therefore, it became possible that nonlinear localized EM structure, with a frequency below the background ω_{pe} (undercritical), can exist and propagate inside a plasma.

However, keeping all plasma parameters fixed and increasing the laser intensity only, scenarios can change from the above standing EM soliton case. Both, the backward-accelerated and forward-accelerated EM solitons can be observed [454, 465].

13.5.2.2 Backward- and Forward Accelerated EM solitons

In two cases with the laser amplitude $a = 0.4$ and $a = 0.5$ (see Fig. 13.30 for laser amplitude $a = 0.4$), following Raman cascade and strong photon-condensation, a spatially localized large amplitude EM soliton due to strong photon-condensation begins to form. However, the soliton dynamics appears to be different in the standing soliton case; the observed EM soliton is now accelerated backward toward the plasma-vacuum interface. After arriving at the boundary, soliton radiates its energy away in the form of a short burst of low-frequency intense EM wave, due to a non-adiabatic interaction with the plasma-vacuum boundary. During the radiation of the EM soliton, as a result, one can observe a very high transient reflectivity, e.g., much larger than that of the B-SRS process (Fig. 13.30, top). For the higher amplitude $a = 0.5$, the behavior remains the same as for $a = 0.4$ case, i.e., a spatially localized backward accelerated large amplitude EM soliton can still be observed.

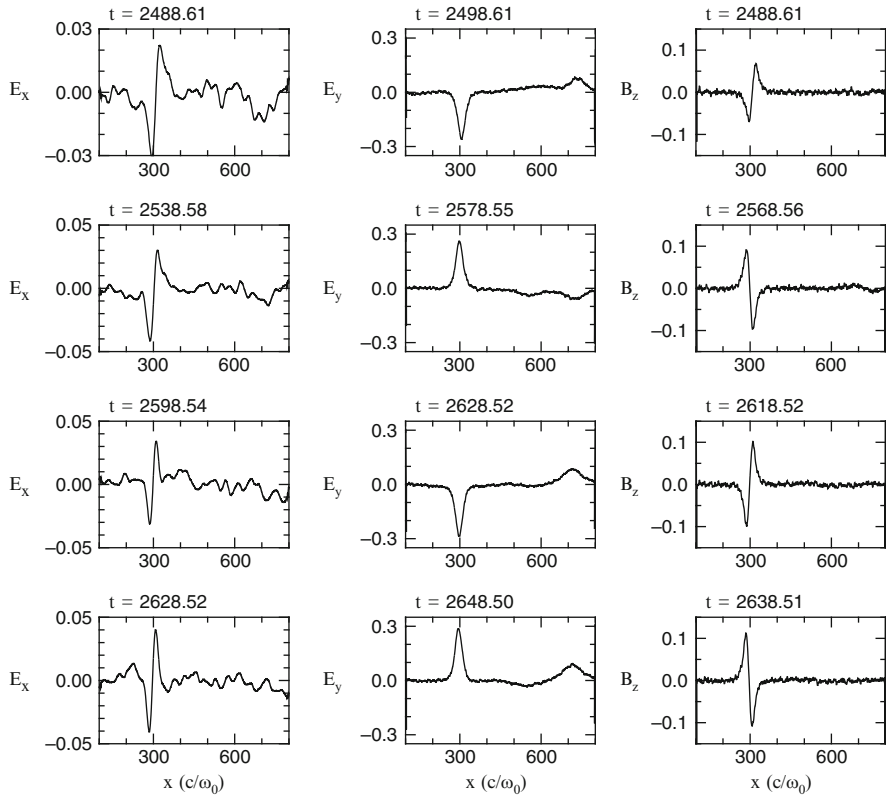


Fig. 13.29 The snapshots for ES field structure (averaged over the EPW wavelength λ_{epw}) and EM field structure (averaged over the laser wavelength λ_0). The simulation parameters are the same as used in Fig. 13.27

However, due to the higher acceleration, the “lifetime” of the EM soliton inside the plasma is shorter than that of the laser amplitude $a = 0.4$ case. Similarly, a very high transient reflectivity, can also be observed during the radiation of the EM soliton. By analyzing the frequency spectra in vacuum after the EM soliton irradiation, we find that frequencies are $\omega_{sol}^{EM} \approx 0.67\omega_{pe}$ for the laser amplitude $a = 0.4$ case, and $\omega_{sol}^{EM} \approx 0.61\omega_{pe}$ for the $a = 0.5$ case, respectively [454]. Further increase of the laser amplitude to $a = 0.6$ and $a = 0.7$, brings an interesting reversal, the large amplitude EM solitons is now accelerated forward. Again, after the EM soliton arrives at the plasma-vacuum interface, it irradiated energy away as a short intense burst of low-frequency EM waves. As expected, a very high transient transmissivity is detected as shown in Fig. 13.31 (top).

Similarly, by analyzing the frequency spectra in vacuum after the radiation of the EM soliton, we found that the EM frequencies of the soliton are $\omega_{sol}^{EM} \approx 0.56\omega_{pe}$ for the laser amplitude $a = 0.6$ case, and $\omega_{sol}^{EM} \approx 0.50\omega_{pe}$ for the laser amplitude $a = 0.7$ case, respectively [454].

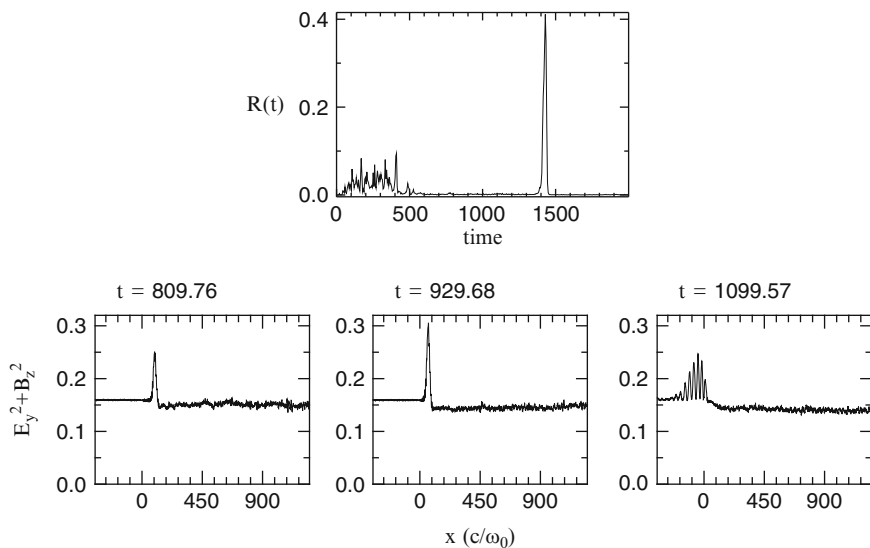


Fig. 13.30 The snapshots for reflectivity and EM energy density $E_y^2 + B_z^2$ (averaged over laser wavelength λ_0) in the case of plasma $n = 0.032n_{cr}$, $L = 900c/\omega_0$, $T_e = 350$ eV, and laser amplitude $a = 0.4$, respectively

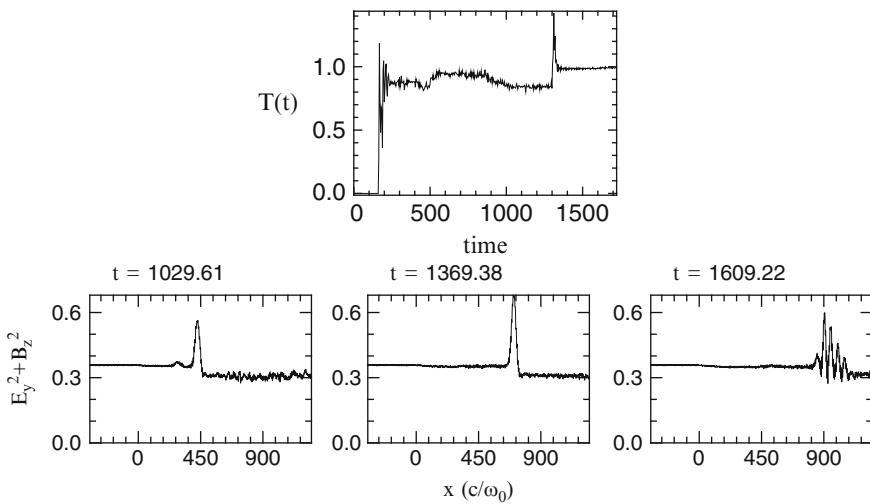


Fig. 13.31 The snapshots for transmissivity and EM energy density $E_y^2 + B_z^2$ (averaged over laser wavelength λ_0) in the case of plasma $n = 0.032n_{cr}$, $L = 900c/\omega_0$, $T_e = 350$ eV, and laser amplitude $a = 0.6$, respectively

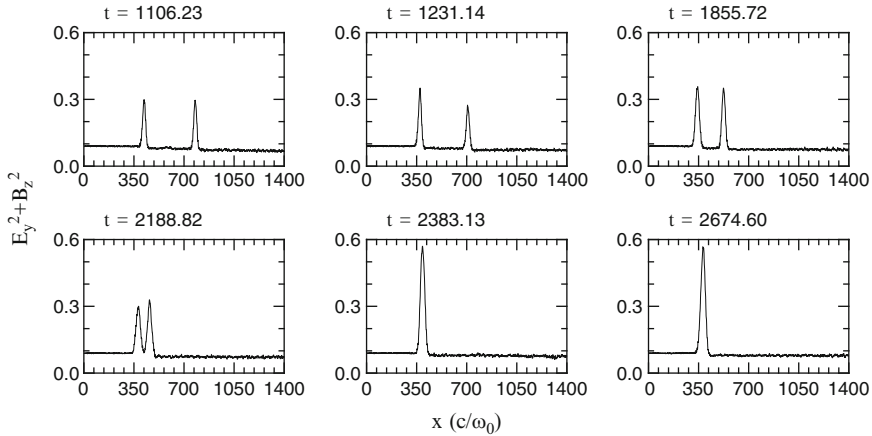


Fig. 13.32 The snapshots for EM energy density $E_y^2 + B_z^2$ averaged over laser wavelength λ_0 in the case of plasma density $n = 0.032n_{cr}$, length $L = 1400c/\omega_0$, temperature $T_e = 350eV$, and laser amplitude $a = 0.3$, respectively

13.5.2.3 Merging of Two Relativistic EM Solitons

By performing a number of simulation runs, we revealed that, for plasma density $n = 0.032n_{cr}$, length $L = 1400c/\omega_0$, temperature $T_e = 350eV$, and laser amplitude $a = 0.3$, as shown in the Fig. 13.32, two spatially localized large amplitude EM solitons can develop and coexist inside a plasma. After the formation, both solitons will be moving backward with different accelerations, in time they can overlap and merge, forming a new EM soliton, with an intensity roughly equal the sum of two solitons. This newly formed large amplitude EM soliton exists and does not separate again within a simulation time.

To conclude, the formation of both the standing and the accelerated large amplitude EM solitons due to strong photon condensation, induced by linearly polarized intense laser interacting with an underdense uniform collisionless plasma, is studied by 1D-PIC particle simulations. We found that, in addition to the inhomogeneity of the plasma density, the acceleration of the EM solitons depends upon both the laser amplitude and the plasma length. The EM frequency of the solitons is between half- and one of the unperturbed electron plasma frequency. The transverse electric field E_y , magnetic field B_z , and ES field E_x inside the soliton have half-, one-, and one-cycle structures in space, respectively.

13.6 Stimulated Electron Acoustic Scattering

The propagation of a laser light through an underdense plasma is an active topic. Much works have been devoted to stimulated Raman and SBS instabilities, concerning their ability to produce energetic particles which can preheat the core of

a fusion pellet. The stimulated scattering from an EPW (Raman scattering -SRS) or ion acoustic wave (IAW) (Brillouin scattering -SBS) can be large to reflect a significant part of the laser light and decrease the coupling efficiency at the target. As was shown by experiments and computer simulations, there can be a rich interplay between these two instabilities [424–426]. Although understanding of basic principles of laser parametric coupling to the EPW and IAW is quite satisfactory, the quantitative predictions are often in a disagreement with experimental observations. Moreover, there was an upsurge of interest on reportedly high SRS reflectivity related to experiments, emulating conditions of National Ignition Facility targets [402, 403].

13.6.1 *The Electron Acoustic Waves*

In 2001, D. S. Montgomery *et al.* reported observation of a novel stimulated electron-acoustic wave scattering (SEAS) to explain “single hot spot” experiments performed at the Trident laser facility [442, 443]. Namely, in the linear theory, the so-called electron-acoustic wave (EAW) exists, i.e., a Landau damped linear Vlasov–Maxwell (VM) branch with a phase velocity value between an EPW and an IAW [3]; often neglected in studies of wave-plasma instabilities. However, analytical studies of non-linear one-dimensional VM solutions have found that strong electron trapping can occur even for small amplitude electrostatic wave, resulting in undamped non-linear traveling waves (BGK-like) [405, 406] or, with an inclusion of small dissipation, in weakly damped traveling solutions [406]. The main difficulty in resolving SEAS from the standard SRS in laser-plasma experiments is that the backscattered light spectrum can cover the nearly continuous broad range of frequencies due to a simultaneous growth of instabilities at different spatial locations in a non-uniform plasma, complex wave-plasma dynamics due to the system length, etc. The first observation of backward SEAS and reinterpretation of earlier experimental results from low plasma densities [442, 443] have encouraged further investigation of domains and conditions for SEAS. However, under reported conditions, the intensity of the SEAS mode still remains well below (3,000 times) the observed backward SRS level.

13.6.2 *Stimulated Raman and Acoustic Wave Scattering*

Here, an excitation of SEAS and its interconnection with SRS instability is investigated by particle simulation of a linearly polarized laser beam interacting with a plasma layer placed in vacuum. An electromagnetic relativistic 1d3v PIC code was used [419]. The number of grids was 25 per $1c/\omega_0$ (ω_0 is the laser frequency), with minimum 50 particles/grid. The length of a simulation system was $220c/\omega_0$ and ions were kept immobile as a neutralizing background. The electrons which enter

vacuum build a potential barrier that prevents more plasma electrons of leaving the plasma. However, due to strong heating, some energetic electrons can even reach boundaries of the system. For these electrons, as well as for EM waves, additional damping regions were used. In a number of simulations, besides stimulated Raman backscattering, we have observed an intense reflection recognized as SEAS instability with a main contribution at regions with over-critical density for standard SRS. Strong SEAS reflection, which can several times exceed SRS reflectivity is followed by large heating of a plasma, which was the first report of such a plasma behavior given by [419–422].

In order to isolate possible SEAS instability from a conventional SRS, we choose a subcritical density ($n = 0.6n_{\text{cr}}$) much beyond a quarter critical n_{cr} . From our simulation data, SEAS is readily identified as a resonant three-wave parametric interaction [407] involving the laser pump (ω_0, k_0), the backscattered lightwave (ω_s, k_s), and the trapped electron-acoustic wave (EAW) (ω_a, k_a). In the linear instability stage, resonant conditions $\omega_0 = \omega_s + \omega_a$ and $k_0 = -k_s + k_a$ are well satisfied, while EMWs (pump and Stokes wave) satisfy standard dispersion equation $\omega_{0,s}^2 = \omega_p^2 + c^2 k_{0,s}^2$. The backscattered wave is always found to be driven near critical, i.e., $\omega_s \approx \omega_p$ which implies $k_s \approx 0$ and $V_s \approx 0$ ($\omega_p = (ne^2/(\epsilon_0 m \gamma))^{1/2}$ is the plasma frequency, γ is the relativistic factor, and $V_s = c^2 k_s / \omega_s$ is the light group velocity). Therefore, the Stokes sideband is a slowly propagating, almost standing EMW. The above decay scheme is observed for a wide range of laser intensities, plasma densities, and temperatures. It is known that high temperatures can significantly alter the growth rates and sometimes suppress parametric instabilities [408]. However, according to [406], efficient excitation of trapped EAW ($\omega_a < \omega_p$), is to be expected in the range $v_{\text{ph}}/v_t = 1 - 2$ (v_{ph} and $v_t = (T/m)^{1/2}$ are the phase and electron thermal velocities). Thus, for SEAS excitation at the threshold, high thermal velocity which closely match the EAW phase velocity is important. To illustrate an onset and growth of SEAS instability, spectra of electromagnetic-light (EM) waves and electrostatic (ES) waves are plotted in Figs. 13.33–13.34.

Figure 13.33 shows discrete spectra in an early phase of SEAS instability. The density and the plasma length are $n = 0.6n_{\text{cr}}$ ($n_{\text{cr}} = n(\omega_0/\omega_p)^{1/2}$) and $L = 40c/\omega_0$, respectively, the longitudinal thermal velocity is $v_t/c = 0.28$ and the laser strength is $\beta = (eE_0)/(mc\omega_0) = 0.3$ (E_0 is the amplitude of the electric field). The backscattered EM wave grows around the electron plasma frequency $\omega_p \approx 0.72\omega_0$ (laser fundamental at $\omega/\omega_0 = 1$ is not shown), while corresponding EAW is at $\omega_0 - \omega_p \approx 0.28\omega_0$. Note that apart from ES noise around a natural plasma mode ($\omega_p \approx 0.72\omega_0$), ponderomotively driven non-resonant modes are also present (not shown in Fig. 13.33) at 2-nd, $\omega = 2\omega_0$ and $k = 2k_0$ ($v_{\text{ph}}/c \approx 1.44$), as well as at zero-harmonic [409, 410]. From obtained data, it follows that the phase velocity of the EAW is $v_{\text{ph}}/c = (\omega_0 - \omega_p)/k_0 \approx 0.41$. In Fig. 13.34, nonlinearly broadened EM and ES spectra of fully developed SEAS are shown for a plasma with $n = 0.4n_{\text{cr}}$, $L = 40c/\omega_0$, $v_t/c = 0.20$, and laser strength $\beta = 0.3$. The instability growth results in the plasma frequency decrease and strong electron heating which tends to suppress the further growth. This is reconfirmed in the post -SEAS stage, after the instability was halted (vide infra). Once EAW has died off, dominant ES

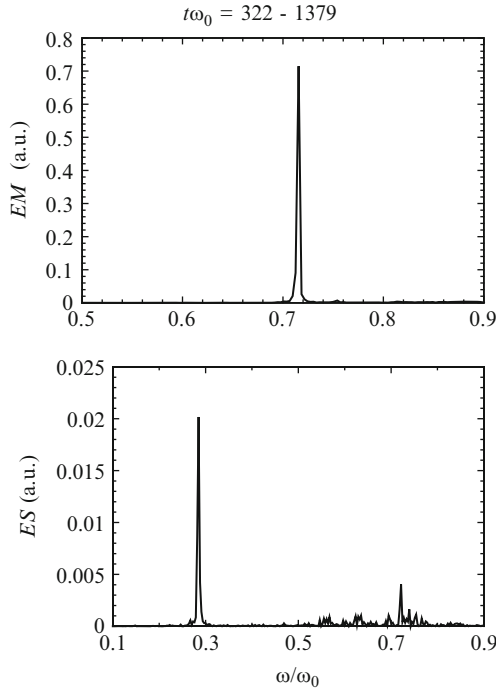


Fig. 13.33 Spectrum of electromagnetic (*top*) and electrostatic (*bottom*) waves in the plasma layer ($n = 0.6n_{cr}$, $L = 40c/\omega_0$) for time interval $t\omega_0 = 322 - 1379$. The initial electron thermal velocity is $v_i/c = 0.28$

response is weak EPW, which peaks at the perturbed $\omega_p \approx 0.54\omega_0$ (downshifted by 0.06 from the initial state), while the decreased Stokes sideband appears at $\omega_s \approx 0.58\omega_0$. Moreover, a blue-shifted, modulated and incoherently broadened EM spectrum (Fig. 13.33) seems consistent with the 3WI backscatter complexity induced by the nonlinear phase shift, predicted by these authors [309, 311, 403].

13.6.3 SEAS Model

We propose a SEAS model as a resonant parametric coupling of three waves $a_i(x, t) \exp[i(k_i x - \omega_i t)]$, in a weakly varying envelope approximation [309, 321],

$$\frac{\partial a_0}{\partial t} + V_0 \frac{\partial a_0}{\partial x} = -M_0 a_s a_a, \quad (13.89)$$

$$\frac{\partial a_s}{\partial t} - V_s \frac{\partial a_s}{\partial x} = M_s a_0^* a_a, \quad (13.90)$$

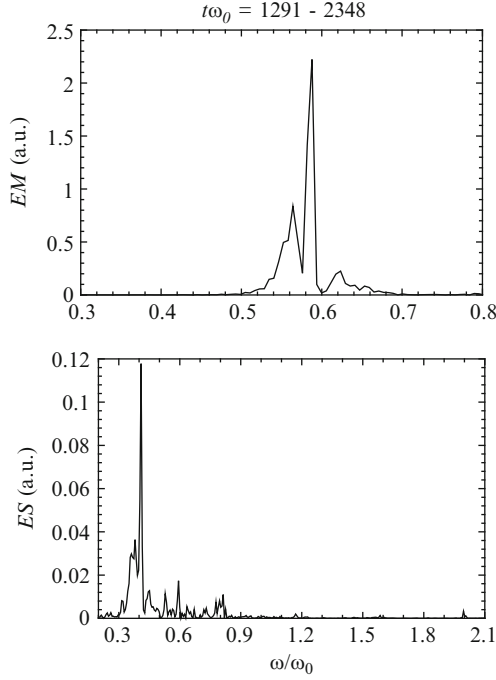


Fig. 13.34 Spectrum of electromagnetic (*top*) and electrostatic (*bottom*) waves in the plasma layer ($n = 0.4n_{cr}$, $L = 40c/\omega_0$) for time interval $t\omega_0 = 1291 - 2348$. The initial electron thermal velocity is $v_i/c = 0.2$

$$\frac{\partial a_a}{\partial t} + V_a \frac{\partial a_a}{\partial x} + \Gamma_a a_a = M_a a_0^* a_s, \quad (13.91)$$

where $V_i > 0$ are the group velocities, Γ_a is an effective damping rate for EAW ($\Gamma_0 = \Gamma_s = 0$ for light waves is used), $M_i > 0$ are the coupling coefficients and a_i are the wave amplitudes, where $i = 0, s, a$, stand for the pump, backscattered wave and EAW, respectively. Since considered model is a short plasma, in order to get high reflectivity, instability needs to be absolute. With standard boundary conditions $a_0(0, t) = E_0$, $a_s(L, t) = a_a(0, t) = 0$, the backscattering becomes an absolute instability if

$$L/L_0 > \pi/2, \quad (13.92)$$

[301, 321], where $L_0 = (V_s V_a)^{1/2}/\gamma_0$ is the interaction length and $\gamma_0 = E_0(M_s M_a)^{1/2}$ is the uniform growth rate. Since observed $V_s \approx 0$ for the backscatter, the condition (13.92) is readily satisfied ($L_0 \approx 0$). Explicit form of (13.89) and (13.90) is easy to get (light waves); however, for EAW (13.91), no linear (kinetic) dispersion relation in analytical form exists [404–406, 442, 446]. Since damping rate $\Gamma_a \neq 0$, the EAW is also characterized by the longitudinal absorption length $L_a = V_a/\Gamma_a$, SEAS-backscatter instability can become absolute under this extra condition [321],

$$L_0/L_a < 2. \quad (13.93)$$

In a linear theory, EAW is a highly kinetic damped mode, so the absorption length L_a is taking small values. However, as concluded earlier, the key factor for an onset and growth of SEAS appears a nearly critical “standing” backward Stokes wave ($L_0 \approx 0$), so that $V_s \approx 0$ satisfies (13.92) and as well as minimizes the threshold E_0 for SEAS instability [419], $\gamma_0 > 0.5\Gamma_a(V_s/V_a)^{1/2}$.

13.6.4 Simulations of SEAS

The temperature effect can be clearly seen near the threshold intensity for SEAS ($\beta \sim 0.3$). Since the longitudinal thermal velocity of electrons can readily increase due to, e.g., the Raman instability, the temperature in transverse direction was set to 500eV, with a longitudinal temperature taken as a control parameter. However, we note that the SEAS instability was readily observed for isotropic distribution as well. Just above the threshold, high electron temperature may be essential for an instability growth. This is illustrated by Fig. 13.35 in which reflectivity ($R = \langle S_r \rangle / \langle S_0 \rangle$, S_r , and S_i are Poynting vectors for reflected and incident wave, respectively, and $\langle \rangle$ denotes time averaged values) is shown for $\beta = 0.3$, $n = 0.4n_{cr}$, $L = 40c/\omega_0$ at several temperatures, $v_t/c = 0.19, 0.20, 0.28$, and 0.30 . There is an optimum temperature for perfect matching with an excited EAW, which results in a maximum SEAS reflectivity. For $v_t/c = 0.2$, observed reflectivity is very high – nearly 140% of the incident laser light. One calculates $v_{ph}/v_t \approx 2.64, 2.50, 1.84$ and 1.72 for $v_t/c = 0.19, 0.20, 0.28$, and 0.30 , respectively. For temperatures $\leq v_t/c = 0.18$ and $\beta = 0.3$, the instability was not observed during time period of $t\omega_0 = 5000$.

For laser intensity well above the threshold, there appears no need for high electron temperatures to excite SEAS. For example, already at $T = 500\text{eV}$, with a

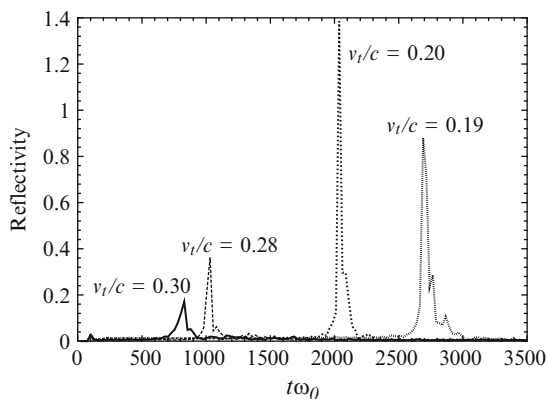


Fig. 13.35 Reflectivity in time from the plasma layer ($n = 0.4n_{cr}$, $L = 40c/\omega_0$) for different initial electron thermal velocities v_t/c and $\beta = 0.3$

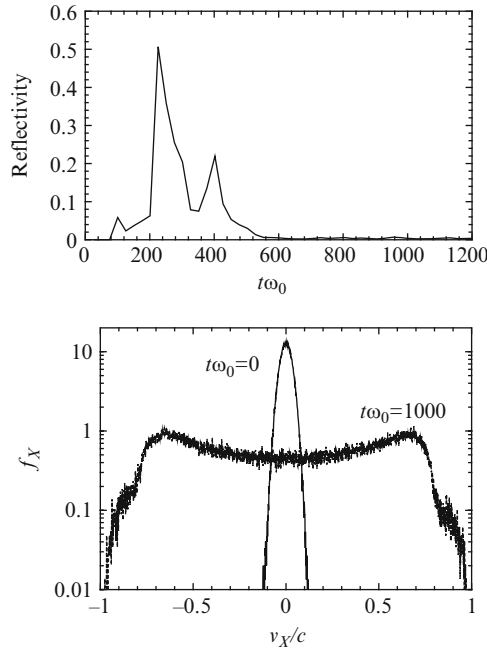


Fig. 13.36 Reflectivity in time (*top*) and electron velocity distribution (*bottom*) in the plasma layer ($n = 0.6n_{\text{cr}}$, $L = 40c/\omega_0$, $\beta = 0.6$) for $t\omega_0 = 0$ and $t\omega_0 = 1000$ (after the instability)

moderately relativistic pump $\beta = 0.6$, $n = 0.6n_{\text{cr}}$, and $L = 40c/\omega_0$ instability develops fast and quickly saturates within $t\omega_0 = 500$. In Fig. 13.36, time evolution of SEAS reflectivity and the electron distribution function $f(v_x/c)$ for the initial ($t\omega_0 = 0$) and the state after the instability ($t\omega_0 = 1000$) are plotted. First, estimates point out at the relativistic-nonlinear frequency shift (NLFS) of the electrostatic wave driven by a laser, as a possible cause [409, 410]. At relativistic intensity, large NLF generates broad ES harmonics which can cover resonant EAW frequency. SEAS resonance is broadened, while instability can grow rapidly, instead from a low background noise, directly from a finite ES harmonic, seeded by a laser. As seen in Fig. 13.36, SEAS produces large relativistic heating which deforms an initial Maxwellian into “water-bag” alike distribution and generates highly energetic electrons with main contribution near $v = \pm v_{\text{ph}}$ of the EAW.

Finally, we briefly address a question of coexistence and interrelation between SRS and SEAS. Now, the simulation system consists of two connected underdense plasma layers L_1 and L_2 of the length $L_1 = 20c/\omega_0$ and $L_2 = 80c/\omega_0$ with the density $n_1 = 0.2n_{\text{cr}}$ and $n_2 = 0.6n_{\text{cr}}$, respectively. Initial temperature is taken at 500eV. Our choice of densities makes L_1 strongly active for Raman instability, while L_2 (overdense for SRS) is practically in a role of a heat sink. Simulations show common picture, an excitation of strong SRS marked by intermittent reflectivity pulsations (see Fig. 13.37, $t\omega_0 < 1000$), [309, 311, 403]. The instability eventually

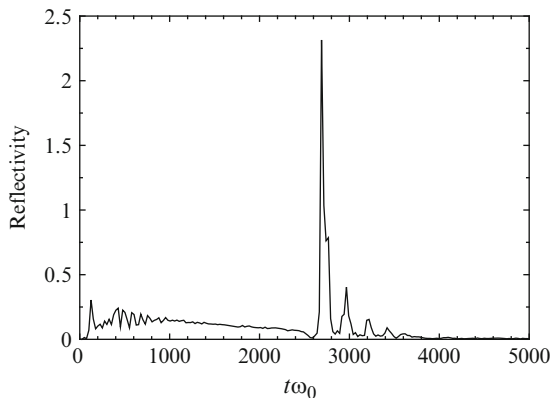


Fig. 13.37 Time history of the reflectivity from two connected plasma layers ($n_1 = 0.2n_{cr}$, $L_1 = 20c/\omega_0$, $n_2 = 0.6n_{cr}$, $L_2 = 80c/\omega_0$, $\beta = 0.3$). Initial reflectivity bursts from ordinary SRS in L_1 are followed at late times by a huge SEAS pulse generated in L_2 , which was heated by hot electrons from L_1

gets suppressed by strong heating of supra-thermal and bulk electrons. Since hot electrons quickly escape the Raman region (L_1), they enter and heat the sink (L_2). Moreover, a striking feature emerges at late times, with a generation of a second intense reflected pulse much larger than the original Raman signal (Fig. 13.33, $t\omega_0 \approx 2700$). This is readily identified as SEAS which originates from the large “sink”, once the temperature has grown to resonate with EAW and enable parametric excitation of SEAS. Therefore, SEAS mediated by SRS becomes a dominant process, as an example of a complex interplay possibly relevant to our understanding of future high intensity experiments.

In summary, in particle simulations in regions not accessible to SRS, strong transient SEAS reflection from trapped EAW was detected around the electron plasma frequency. A three-wave parametric model was exposed, in particular, a role of a standing Stokes sideband for excitation of an absolute SEAS instability. While in reported experiments [442] SEAS to SRS signal ratio was smaller than 10^{-3} , we find conditions in which SEAS dominates over standard SRS. Further study of SEAS role, e.g., in relativistic laser-plasmas deserves attention.

13.7 Trapped SEAS, EM Soliton and Ion-Vortices in Subcritical Plasmas

Stimulated trapped electron acoustic wave scattering by a linearly polarized intense laser in a subcritical plasma is studied by particle simulation. However, in distinction with a previous section limited to fixed ions model, here, we include full ion dynamics which enable rich relativistic complexity at plasma multi-scales. The scattering

process is a three-wave parametric decay of the laser pump into a critical Stokes electromagnetic sideband wave and the trapped electron acoustic wave. Now, as the ion acoustic wave also grows in time, it can break locally, followed by a generation of a large relativistic EM soliton. Further, a new phenomenon, MeV range vortex in the ion phase-space forms by local EM and ES fields inside the soliton.

13.7.1 Introduction

As indicated above, in addition to the standard SRS, a new type of stimulated back-scattering, involving the so-called electron acoustic waves (EAWs), was reported in single hot spot experiments [442, 443]. At relativistic intensities, it was first exposed by Nikolić and Škorić, et al. by PIC simulations in subcritical plasmas [419, 422]. The later PIC simulations by Valentini et al. have investigated the excitation of EAWs and the stability of the EAWs against decay, and found that the EAW is a non-linear wave with a carefully tailored trapped particle population, and the excitation process must create the trapped particle population [444]. As said, early calculated linear Vlasov dispersion relation, already noted two ES branches, corresponding to the well-known, high-frequency EPW plus a low-frequency Landau damped branch, termed electron acoustic waves (EAWs) [3, 445].

In this section, stimulated trapped electron acoustic wave scattering (T-SEAS) instability induced by a linearly polarized intense laser interacting with a plasma layer at a subcritical density range ($n_{cr}/4 < n/\gamma < n_{cr}$, which is overdense for standard SRS) is studied by 1D PIC simulations. We find that the instability takes place whether the ion dynamics is taken into account or not. Still, with ion dynamics, an excited EAW grows in time and breaks, accompanied by a large relativistic EM soliton. As a new phenomenon, a MeV energy ion-vortex structure in the momentum phase-space forms by localized EM and ES soliton fields. Ion-vortices both in homogeneous and in inhomogeneous plasmas are found. Earlier PIC simulations of subcritical plasmas by Adam et al. [449, 450] concentrated on electron effects in early stages of relativistic SRS parametric instability in 1D and 2D model, in particular. We found similar electron effects in 1D, still, that work lacks spectral data to test EPW/EAW nonlinear dispersion. We note, that, e.g., the electron hole detected in Fig. 3, [449, 450], could be related to a relativistic EM soliton structure as revealed in this work. Also, in the asymptotic stage, the reference to “. . . ES component being strongly Landau damped . . .” could be an actual evidence of EAWs (p. 4767, [449]). As will be discussed, our observed EAW nonlinear mode is similar to the kinetic electrostatic electron nonlinear waves (KEEN), which are stable, non-linear, multimode coherent structures in plasmas as was introduced by Afeyan et al. [451].

13.7.2 Stimulated Trapped Electron Acoustic Wave Scattering

In our simulations, fully relativistic EM 1D-PIC code is used. The $100 c/\omega_0$ long plasma begins at $x = 0$ and ends at $x = 100 c/\omega_0$; at the front and rear sides of the plasma layer, there are two $200c/\omega_0$ vacuum regions, where c and ω_0 is the speed and carrier frequency of laser pulse, respectively. The number of cells is 20 per $1 c/\omega_0$, 100 electrons and 100 ions are put in each cell. Plasma density is $n = 0.6n_{\text{cr}}$, where n_{cr} is the critical density. The electron and ion temperatures are $T_e = 5T_i = 1keV$, and the mass ratio is $m_i/m_e = 1836$. Linearly polarized laser plane wave, with electric field E_0 along the y direction and normalized laser amplitude $a = eE_0/m_e\omega_0c = 0.6$, is initialized at $x = -50 c/\omega_0$, where m_e and e are the electron mass and electron charge, respectively. The time, electric field, and magnetic field are normalized to $2\pi/\omega_0$, $m_e\omega_0c/e$, and $m_e\omega_0/e$, respectively; time is taken as zero, $t = 0$, when laser pulse arrives at the front vacuum-plasma boundary. In the plots, the ES field E_x is averaged over electron plasma wavelength $\lambda_{\text{pe}} = c/\omega_{\text{pe}}$, EM fields E_y , B_z , and EM energy density $E_y^2 + B_z^2$ are averaged over laser wavelength $\lambda_0 = c/\omega_0$, respectively.

Plasma density is taken beyond $n > 0.25n_{\text{cr}}$, so that the standard SRS is excluded. We found that when the laser amplitude $a > 0.4$, T-SEAS instability takes place. In a linear stage, as shown in Fig. 13.38, the spectrum is well explained by a resonant three-wave parametric decay of an intense laser pump into the slow backscattered Stokes EM sideband and the trapped EAW. The backscattered Stokes EM wave in vacuum region is found to be driven critical, i.e., near the relativistic electron plasma frequency $\omega_s = 0.62\omega_0 \approx \omega_{\text{pe}}/\gamma^{1/2}$, where $\gamma = \sqrt{1+a^2}$ is relativistic factor, while the corresponding EAW has the frequency $\omega_{\text{eaw}} \approx 0.40\omega_0 < \omega_{\text{pe}}$. The wavenumber for EM wave inside the plasma has two peaks; one with $k_s^P \approx 0.12k_0$, and another with $k_0^P \approx 0.80k_0$, which correspond to backscattered and laser EM waves, respectively; while the wavenumber for EAW is $k_{\text{eaw}} \approx 0.92k_0$, here, $k_0 = \omega_0/c$ is the wavenumber of laser light in vacuum. We further estimate the relativistic electron plasma frequency as $\omega_{\text{pe}}^* \approx 0.61\omega_0$, which is smaller than the initial laser pump induced $\omega_{\text{pe}}/\gamma^{1/2} \approx 0.69\omega_0$, since, in the region of intense EM and EAW waves, larger relativistic effect reduces the local electron plasma frequency. The EM waves for both pump and Stokes mode closely satisfy the standard dispersion relation $\omega_{0,s}^2 = \omega_{\text{pe}}^{*2} + c^2(k_{0,s}^P)^2$. Then, in the early T-SEAS instability, the matching conditions for both the frequency $\omega_0 = \omega_s + \omega_{\text{eaw}}$ and the wavenumber $k_0^P = -k_s^P + k_{\text{eaw}}$ appear to be satisfied. Note that the basic relativistic electron plasma eigen-mode $\omega_{\text{pe}}/\gamma^{1/2} \approx 0.69\omega_0$ is also weakly excited.

During the T-SEAS instability, as shown in Fig. 13.39 (top), large portion of electrons is trapped in the ES potential of the large EAW, this is why a term 'T-SEAS' is coined. In a nonlinear saturation, there is a rapid growth and strong localization of the Stokes EM wave by forming narrow intense spiky EM structures with frequency downshifted laser light trapped inside, as shown in Fig. 13.39 (bottom). The train of relativistic EM spikes gets eventually irradiated through the front vacuum-plasma boundary in a form of an intense reflection burst of the

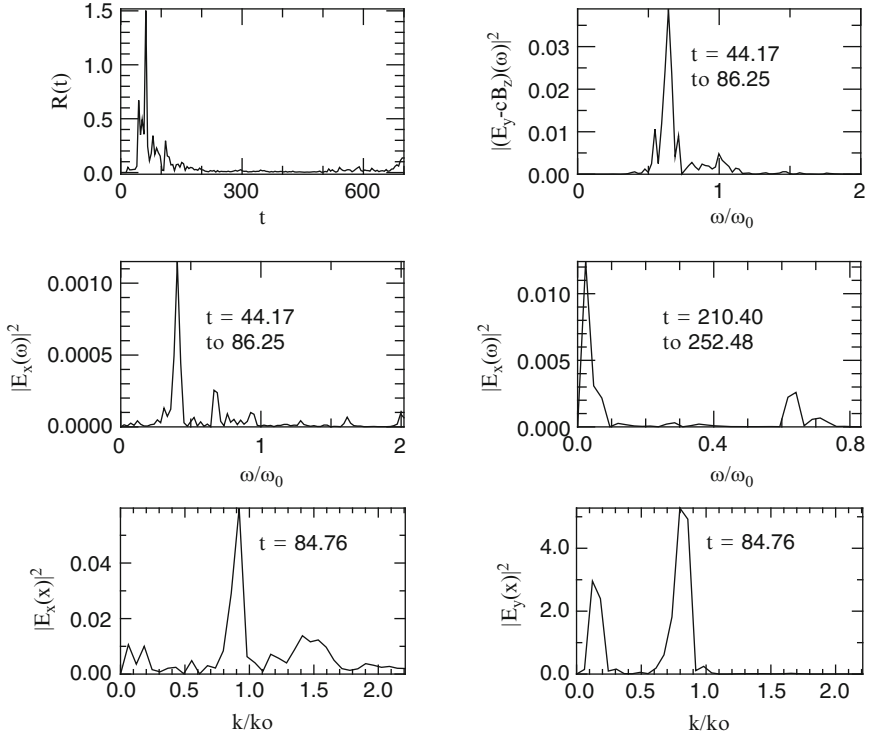


Fig. 13.38 Reflectivity and frequency spectrum for the reflected EM wave (*top*); frequency spectra of the ES wave inside plasma (*middle*); and wavenumber spectra for the electric fields E_x and E_y both measured inside plasma (*bottom*), respectively

downshifted laser light. The large trapped EAW excited inside a plasma with low phase velocity quickly heats up bulk electrons to relativistic energies, which will eventually suppress T-SEAS instability. To illustrate the onset and growth of the T-SEAS instability, in Fig. 13.40, the plots for the time evolution of the dominant backscattered EM mode $|E_y(k)|^2$ and EAW mode $|E_x(k)|^2$, both measured inside the plasma, are given. One can see that after an early linear phase a nonlinear process follows while further growth of EM and ES wave gets saturated. If we assume that both EM and ES mode harmonics roughly exhibit a linear parametric growth as $E_k(t) \sim E_k(0)e^{\gamma t}$, then by using our data, we can estimate the corresponding growth rate in the initial stage of T-SEAS instability. The growth rate for the backscattered EM mode is $\gamma_s \approx 0.176\omega_s = 0.113\omega_0$, the corresponding growth rate for EAW is $\gamma_{eaw} \approx 0.267\omega_{eaw} = 0.107\omega_0$, respectively. The difference can be attributed to an inherent damping which slows the growth of the EAW.

To further clarify the T-SEAS instability growth, we follow a proposed simple model [301, 309, 419, 422, 464] for a resonant parametric coupling between three waves $a_i(x, t)\exp[i(k_i x - \omega_i t)]$, that satisfy the frequency and wave number

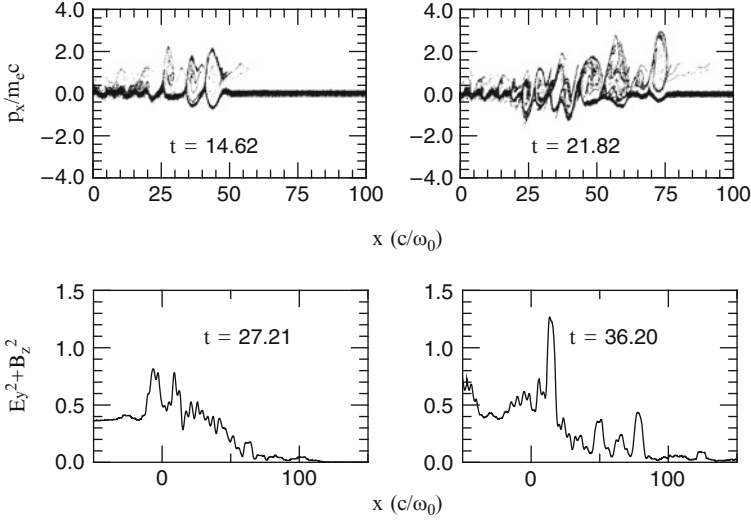


Fig. 13.39 Snapshots for electron phase space $x \sim p_x$ (top) and EM energy density $E_y^2 + B_z^2$ (bottom), respectively

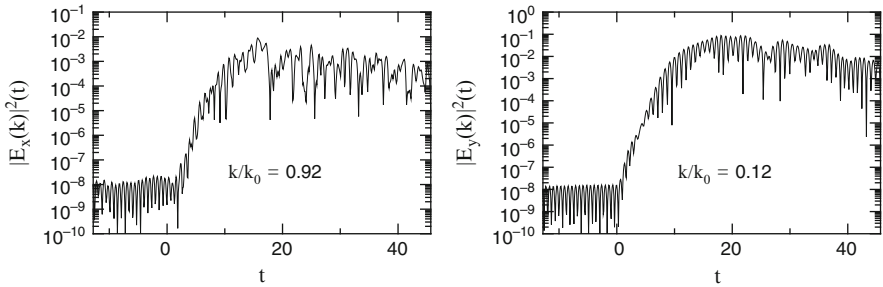


Fig. 13.40 Temporal growth for EAW mode E_x and backscattered EM mode E_y both measured inside plasma, respectively

matching conditions, which for weakly varying envelopes in dimensionless units reads:

$$\partial a_i / \partial t + V_i \partial a_i / \partial x + \Gamma_i a_i = M_i a_j^* a_k \quad (13.94)$$

where $V_i > 0$ are group velocity ($i, j, k = 0, s, eaw$, denotes pump EM, backscattered EM, and EAW modes, respectively), Γ_{eaw} is an effective damping rate for EAW; $\Gamma_0 = \Gamma_s = 0$ for EM waves is used. $M_i > 0$ are the coupling coefficients and a_i are the wave amplitudes. With standard open boundary conditions $a_0(0, t) = E_0$ and $a_s(L, t) = a_{eaw}(0, t) = 0$, the backscattering grows as an absolute instability, only if $L/L_0 > \pi/2$, where $L_0 = (V_s V_{eaw})^{1/2} / \gamma_0$ is the interaction length and $\gamma_0 = E_0 (M_s M_{eaw})^{1/2}$ is the uniform growth rate. The group velocity of the

backscattered EM wave is $-V_s = c^2 k_s / \omega_s \approx 0.19c$; even if we assume that the group velocity for EAW is so large that $V_{eaw} \approx c$ and a very small uniform growth rate $\gamma_0 \approx 0.1\omega_0$ (above growth rates for both backscattered EM wave and EAW are greater than $0.1\omega_0$), then the $L_0 \approx 4.3c/\omega_0$. So a posteriori, for plasma length $L = 100c/\omega_0$, the condition $L/L_0 > \pi/2$ for the absolute instability to take place, is readily satisfied. This means that intensive T-SEAS instability can indeed develop as observed for our simulation conditions.

The EAW mode found in our simulation is related to the kinetic electrostatic electron nonlinear (KEEN) waves. The KEEN waves, which were reported by Afeyan et al. [451], are stable, long-lived nonlinear, multimode coherent structures in plasmas, which can only be driven by sufficiently strong electric fields. The frequency of the excited KEEN waves is in the electron acoustic frequency range. In the case of KEEN waves, no flattened electron velocity distribution function need be invoked and no single mode behavior is observed [451]. However, in our case, we found that the observed EAW excited via SEAS is closely a single mode, as in Fig. 13.38 the peaked discrete frequency spectra of ES wave shows. Also, we find that kinetic effects play an important role in the stimulated EAW scattering, due to the large electron relativistic heating by T-SEAS process, the initial Maxwellian distribution gradually deforms into “water-bag” like distribution with highly energetic electrons with a main contribution near the EAW phase velocity $\pm v_{ph}$.

13.7.3 Electromagnetic Soliton and Ion-Vortices

We found that early T-SEAS instability behavior is similar, whether the ion dynamics is taken into account or not. However, with ion dynamics, for times longer than the ion period $2\pi/\omega_{pi}$, the IAW is excited and persists in time [448]. Already, before the IAW onset, electron trapped orbits (EAW) are well developed [449], (Fig. 13.40) to be rapidly destroyed by strong EM transverse and ES longitudinal fields and stochastic heating up to relativistic energies. The IAW peak in our early ES spectrum is hard to detect, as the strong trapped EAW is dominant. Once the T-SEAS instability is halted, as in Fig. 13.38 (middle) the ES spectrum shows, the IAW mode with frequency $\omega_{iaw} \approx 0.023\omega_0$, which is approximately equal to the ion plasma frequency $\omega_{pi} = (m_e/m_i)^{1/2}\omega_{pe} \approx 0.021\omega_0$, can be observed, clearly. The excited IAW propagates forward driven basically by two effects: direct laser ponderomotive force (PF) action on ions and by much stronger, indirect effect, due to relativistic electron dynamics, through fields (currents) created in the violation of charge quasi-neutrality. It is clear that a complete picture of such a complex paradigm of multi-scale nonlinear interactions can be revealed only by relativistic PIC simulations. Still, the direct PF effect is visualized in Fig. 13.41 (top, left), where early IAW is modulated by PF, i.e., wavenumber equal to the 2nd harmonic of laser light. Furthermore, a growing nonlinearity and slow finite velocity of driven IAWs quickly leads to a strong steeping of the wavefront and its eventual break-up. At later times, at $t > 21.82$, as observed in Fig. 13.41, ion orbits begin to overturn, close to

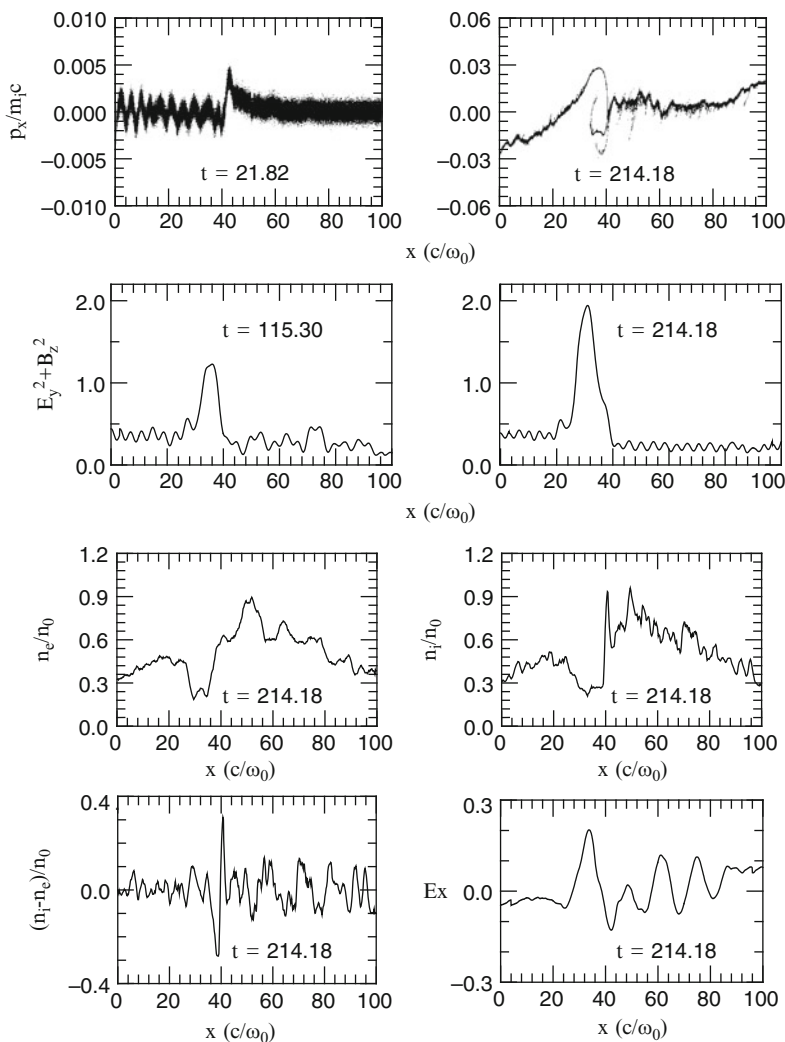


Fig. 13.41 Snapshots for ion phase-space, EM energy density $E_y^2 + B_z^2$, plasma density, and ES field E_x , respectively

$x = 40 c/\omega_0$ point, while the steep IAW density shock wave front breaks. After that, by EM energy localization in the density cavity, one EM soliton is nucleated at the same position and continues to grow in time to saturate at $t \approx 252.94$ with the maximum EM energy $E_y^2 + B_z^2 \approx 2.5$. During its growth, first, large number of electrons are expelled from the high EM field region by the relativistic ponderomotive force ($\sim \partial \gamma / \partial x$) [414]. At the front of the soliton ($x \approx 40 c/\omega_0$) at $t = 214.18$, because of large inertia, slow ions pile up and one sharp ion density peak is formed. Simultaneously, electrons do not accumulate at the front and rear side, because of their small

inertia. Then, at the narrow region around the soliton front edge $x \approx 40 c/\omega_0$, a net positive charge forms, as the $(n_i - n_e)/n_0$ plot in Fig. 13.41 shows. Behind the ion density peak, i.e., inside the soliton region, to preserve the charge neutrality, more electrons are pushed away to balance the ion density peak by the ES field of the charge separation. Thus, a net negative charge region forms. This charge redistribution results in a double layer structure, as in Fig. 13.41 the ES field E_x plot shows. Both the relativistic EM soliton field and the local large E_x sheath field via interplay can strongly accelerate or decelerate ions. Ions with initial negative velocity inside the soliton are reflected by sharp ion density peak, will first experience a deceleration process, and then acceleration again by the soliton field; while ions with positive velocity experience just the reverse processes. As a result, as the ion phase-space plot at $t = 214.18$ shows in Fig. 13.41, eventually, a remarkable large trapped ion-vortex (ion-hole) structure is formed. The size of the ion-vortex is close to the soliton width, i.e., $\Delta x_{vortex} \approx \Delta x_{soliton} \approx 1 \sim 2\lambda_{pe}$. The maximum ion-vortex energy is large in the range $1 \sim 2$ MeV. Qualitatively, an early excess charge in the electron density hole appears to trigger ion-vortex formation, analogous to a large water eddy caused by the hole at the river bottom. We can roughly estimate the maximum ion energy by simply assuming that ions are initially accelerated in the effective planar capacitor (capacity-C) with plates (area-A) at the soliton width (d) on the potential ($U = Q/C$) due to a locally expelled total electron (hole) charge (Q). This appears consistent with the ES profile (E_x in Fig. 13.41) in the soliton cavity region ($x \approx 25 - 40 c/\omega_0$). By using $C = \epsilon_0 A/d$, rough estimates for d and $Q = e\delta n_e A d$, readily predict the potential energy in a few MV range.

As an intense laser pulse propagates through the plasma layer, electrons spontaneously and via interactions with the laser spread out into vacuum by forming the charge separation ES fields at both boundaries; ions tend to follow electrons to keep the charge neutrality. As a result, the plasma density will decrease from the center to both boundaries gradually with time. As inhomogeneous plasma is formed, the EM soliton will be then accelerated down the density profile [395]. We found that similar large relativistic EM solitons often form in the front region of the plasma and are accelerated backward toward the front plasma-vacuum interface. As large amplitude EM soliton propagates into the boundary, as shown in Fig. 13.42 (top), by large EM field and ES field inside the soliton, the energy exchange between the soliton and ions continually takes place. Majority of ions are accelerated and trapped inside the soliton region. Along the soliton path, as shown in Fig. 13.42 (bottom), several trapped ion-vortices (ion-holes) in the phase-space form and persist with time. The EM soliton gradually loses energy and its amplitude decreases. After the EM soliton arrives at the plasma-vacuum boundary, related ion acceleration and trapping is completed, also the formation of new ion-vortices stops. At later times, created ion-vortices blur and become hard to discern and eventually disappear due to energy loss and thermalization inside the bulk plasma. We find again the EM soliton universal property, as shown in Fig. 13.43, the soliton transverse electric field E_y is a half-cycle structure in space, while the corresponding magnetic field B_z is a one-cycle structure. From Maxwell's equations, i.e., the Faraday law $B_z \sim \partial E_y / \partial x$, the x derivative of the Gaussian soliton profile E_y can lead to the B_z structure as

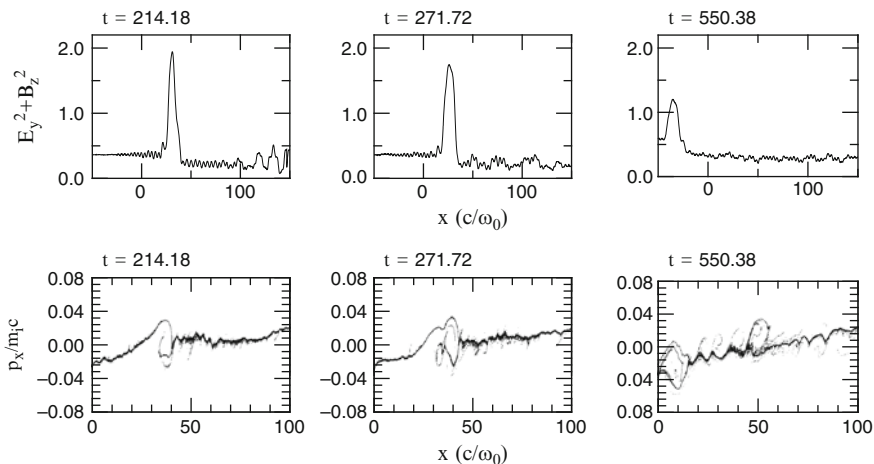


Fig. 13.42 Energy density of EM field (*top*) and ion phase-space snapshots (*bottom*) for laser amplitude $a = 0.6$

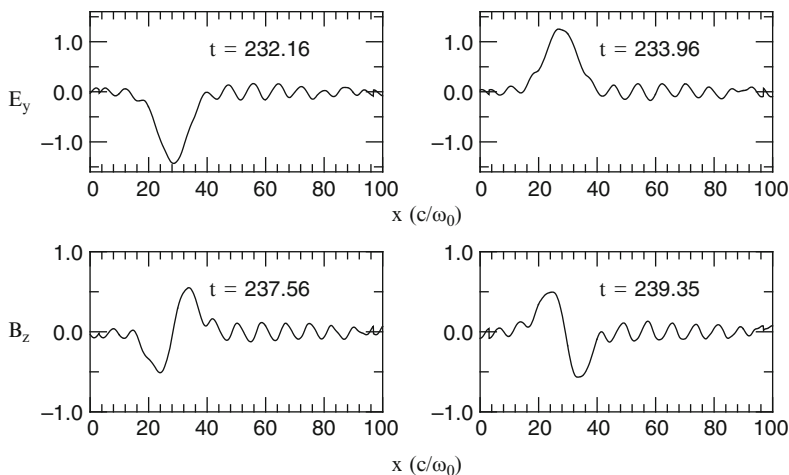


Fig. 13.43 The snapshots for electronic field E_y and magnetic field B_z in the case of laser amplitude $a = 0.6$

shown in Fig. 13.43. Moreover, the EM structure is periodic in time [454, 464]. We also performed a run for inhomogeneous plasma by taking a linear density profile, $n = 0.35 - 1.05n_{cr}$ and laser amplitude $a = 0.6$, other simulation parameters are kept fixed. The same scenario as before emerges, i.e., generation of large amplitude EM soliton which travels down the gradient of the plasma density, however, this time leaving behind in the wake the chain of ion-vortices.

In conclusion, the T-SEAS instability induced by a linearly polarized laser in a subcritical plasma which is overdense for standard SRS was studied by particle simulation. The spectrum in the early stage is well explained by a resonant three-wave parametric decay of the laser pump into the critical Stokes EM sideband and the trapped EAW, which takes place whether the ion dynamics is taken into account or not. However, when ion dynamics is considered, novel complex physical phenomena are observed: the excitation of large IAW, the generation of large EM soliton after the IAW front breaks up, and subsequent formation of the MeV range vortices in the ion phase-space. Ion-vortices are also found in simulations in nonuniform plasma. Also, it is found that the trapped EAW mode is similar to the KEEN waves.

Part IV

Multiscale Plasma Interactions

Chapter 14

Multifractal Characterization of Plasma Edge Turbulence

14.1 Introduction

Edge turbulence measurements represent an important object of current research efforts in understanding plasma confinement in magnetic fusion devices, and studies related to this issue focus, among other things, on quantification of intermittent aspects of the dynamics (e.g. [466]). Studies of MAST (Mega Ampere Spherical Tokamak at UKAEA Culham Laboratory) data are making important contributions to the study of plasma confinement in spherical as well as in toroidal devices. Recently, a study devoted to the analysis of self-similar aspects of the turbulence in the MAST device reported results based on the rescaled range analysis and its ability to distinguish between low and high operating regimes (L- and H-mode) with respect to the existence (or lack of) long-range dependence (LRD) [467]. In particular, this analysis concentrated on detecting long-range behavior in low and high confinement regimes. In recent years, several studies were reported addressing the same aspect of confined plasma turbulence in relation to the possible self-organized criticality (SOC) (see e.g. [468–470]). An important aspect of the LRD property is its relationship to the convective aspect of the dynamics since it is assumed that the avalanche-type transport induces LRD. Hence, detection of LRD property is important for understanding the intermittent convective transport, particularly in the scrape-off layer of magnetically confined plasmas. Most, if not all, of these research efforts assume that the processes representing turbulent plasma behavior are self-similar, in the sense that only one scaling parameter is sufficient to describe the self-similar properties of the dynamics. The purpose of this study¹ is to offer evidence that turbulent processes in an example based on low and high confinement regimes in the MAST device are multifractal, in the sense that many parameters are needed in order to adequately, and mathematically correctly, characterize the self-similar property. Moreover, we introduce several methods for quantifying multifractal behavior which offer new insights into the properties of turbulence of L and H modes. Several methods for the analysis of turbulent plasma regimes used

¹ M. Rajković and M.M. Škorić, Research Report NIFS-833 (2006), unpublished.

previously by the present authors may also yield some important information about the underlying dynamics; however, they are not well suited for the analysis of multifractal processes [471].

The chapter is organized in the following manner. Following a brief description of datasets, an overview of long-range dependent processes is given along with the results obtained for the MAST datasets. An important feature of this section is the statistical test of constancy in time of the scaling exponents. Next section gives a short overview of multifractal processes and its relationship with the wavelet transform. Results pertaining to this section contain discussion of multifractal properties of the L and H-mode turbulence in the MAST device. Finally, we discuss effects of coupling LRD with intermittency and the possibility of simultaneous determination of the strength of each of these processes. Concluding remarks are presented along with some suggestions for a possible future research.

14.2 Edge Turbulence Datasets

Three datasets are analyzed in this report, obtained by courtesy of R. O. Dandy (EURATOM/UKAEA Fusion Association, Culham Science Center, U.K.) and B.D. Dudson (University of Oxford, U. K.). These datasets consist of ion saturation current (I_{sat}) measurements obtained by a moveable Langmuir probe positioned at the outboard midplane on MAST device [467]. The advantage of studying ion saturation current lies in the fact that it carries information about bursts that carry large amounts of particles. Sampling frequency was 1 MHz. The datasets were taken during two confinement regimes: L-mode and a dithering H-mode. The L-mode is represented by two datasets labeled 6861 (high density L-mode) and 9035. Dataset 9031 is a dithering H-mode with heating power close to a threshold for L-H transition with intermittent high frequency edge localized modes. These signals are represented in Fig. 14.1. Further details pertaining to the datasets may be found in [467].

14.3 Quantification of Long-Range Dependence

Widely used methods for characterization of plasma turbulence include probability distribution function (PDF), autocorrelation function (ACF), and power spectrum, while recently several papers address the topic of possible LRD in the edge turbulence of toroidal magnetic confinement devices. Upon getting a Hurst exponent in the range $0.5 < H < 1$, the authors often make conclusions concerning the global self-similar properties of such signals, particularly in relationship with the SOC models (e.g. [466]). However, self-similarity is a strong statistical property which intuitively may be defined as a property of scale invariance. This property implies

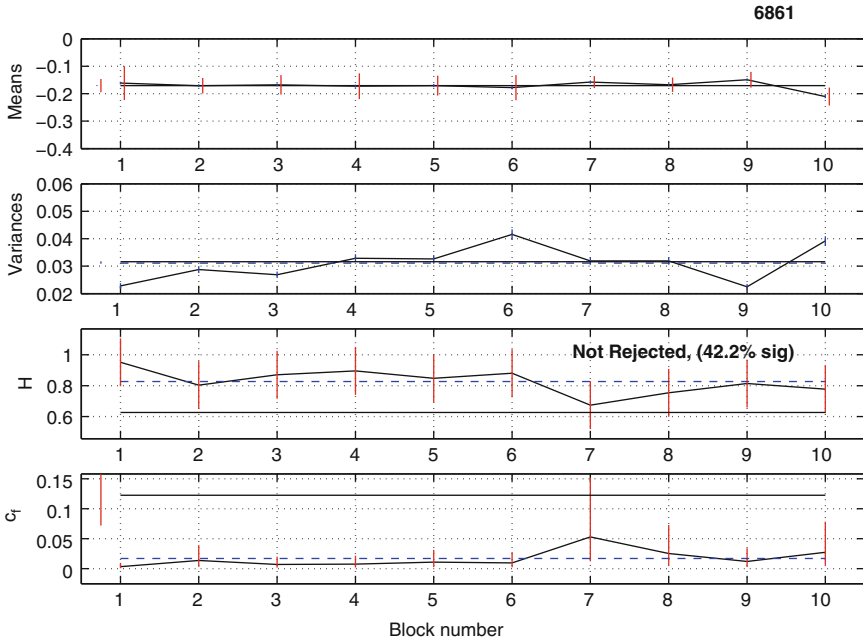


Fig. 14.1 Saturation current as a function of time for the low confinement regime (signals 6861 and 9035) and the high confinement (dithering H-mode) 9035 signal

that all scales have equal importance and hence, there is no characteristic scale controlling the dynamics. The notion of self-similar process is defined as follows.

Definition 1. A random process $X(t)$, $t > 0$ is called self-similar if for any $a > 0$, there exists $b > 0$, such that

$$X(at) \stackrel{d}{=} bX(t).$$

and the following property may be proved.

Theorem 1. (Lamperti). If random process $X(t)$, $t \geq 0$ is nontrivial, stochastically continuous at 0, and self-similar, then there exists unique $H \geq 0$ such that $b = a^H$. If $X(0) = 0$, then $H > 0$. A self-similar process with parameter H is usually denoted as H -ss.

Such a process, evidently, cannot be stationary. LRD, on the other hand, is associated with *stationary processes* and may be defined through spectral density. The stationarity issue may be avoided if, for example, linear filtering is used which produces stationary process. In such a case, one may define a quasi spectral density as a function of spectral density of filtered stationary process and transfer function of the filter. An example is the spectrum of the fractional Brownian motion.

A stationary process $X(t)$ is called long-range dependent (LRD) process if its ACF or spectral density behave as

$$r(k) \sim c_r k^{\alpha-1} \quad \text{as } k \rightarrow \infty, \alpha \in (0, 1) \quad (14.1)$$

or

$$\Gamma_X(v) \sim c_f |v|^{-\alpha} \quad \text{as } v \rightarrow 0, \alpha \in (0, 1). \quad (14.2)$$

Equations (14.1) and (14.2) imply that the covariances $r(k)$ decay so slowly, that

$$\sum_{k=-\infty}^{\infty} r(k) = \infty, \text{ or equivalently } \Gamma_X(0) = \infty.$$

There is a close relationship between LRD and self-similar process as increments of any finite variance H - *sssi* process (*sssi* stands for self-similar stationary increments) have LRD, as long as $1/2 < H < 1$, with H and α related through

$$\alpha = 2H - 1.$$

The self-similarity exponent H is usually called the Hurst exponent. However, a Hurst exponent $H > 1/2$ does not necessarily imply long time correlations such as those found in fractional Brownian motion. For example, Markov processes, which by construction have no memory, may also exhibit long-time correlations [472]. The statement that $H > 1/2$ (persistence), or $H < 1/2$ (antipersistence) imply that correlations may be deduced from a simple argument [473]. Calculating the autocorrelation as:

$$\begin{aligned} & 2 \langle \Delta x(t - \Delta t) \Delta x(t + \Delta t) \rangle \\ &= \langle (\Delta x(t - \Delta t) + \Delta x(t + \Delta t))^2 \rangle - \langle \Delta x^2(t - \Delta t) \rangle - \langle \Delta x^2(t + \Delta t) \rangle, \end{aligned}$$

where $\Delta x(t + \Delta t) = x(t + \Delta t) - x(t)$ and $\Delta x(t - \Delta t) = x(t) - x(t - \Delta t)$. If the process, which is assumed to be stochastic, has *stationary increments*, requiring that the mean square fluctuation from any $x(t)$ scales as [474]

$$\langle (\Delta x(t + \Delta t) - x(t))^2 \rangle = c \Delta t^{2H}.$$

Since the scaling relationship depends only on Δt and not on t , rescaling the ACF by the mean square fluctuation

$$C(-\Delta t, \Delta t) = \frac{\langle \Delta x(t - \Delta t) \Delta x(t + \Delta t) \rangle}{\langle \Delta x^2(\Delta t) \rangle},$$

the following relationship is obtained:

$$C(-\Delta t, \Delta t) = 2^\alpha - 1 = 2^{2H-1} - 1. \quad (14.3)$$

Hence, any $H \neq 1/2$ implies autocorrelations. The crucial part of the above derivation is that the autocorrelations may exist for $H \neq 1/2$ only if the increments are stationary. Hence, an empirical measurement (or theoretical prediction) of Hurst

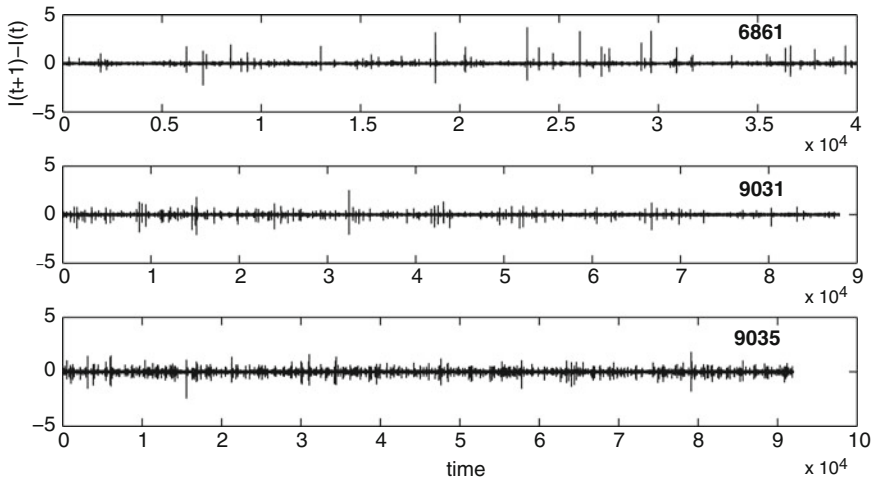


Fig. 14.2 Increment process of the L-mode 6861 and 9035 signals and H-mode signal 9035. Note high excursions in the 6861 signal. In spite of that, this signal also contains more stationary blocks than the other two signals

exponent, without evidence for stationarity of increments (or explicit evidence that the process possesses memory) cannot be accepted as evidence for autocorrelations in the data. Since the data for both L and H modes do not have stationary increments due to the extreme increment value excursions as evident in Fig. 14.2, the signals should be carefully inspected for stationarity, for example, by dividing the signal into blocks of equal length which are essentially stationary with respect to the first and second moments. It is interesting to notice that compared to the other two signals, signal 6861 has the largest number of segments with stationary increments in spite of having the largest bursts.

The LRD analysis presented here is based on the discrete wavelet technique of stochastic processes [477], while more details on the importance of the stationarity property may be found in [476]. Details of the wavelet techniques used in the determination of Hurst exponent may be found, for example, in [477] and [478], and we mention here only the most important aspects.

14.3.1 Wavelet Transform of Scaling Processes

Although the wavelet theory was originally developed for the analysis of deterministic finite energy processes, applications to stochastic processes, in particular to turbulence phenomena, have been very successful in recent years. Since the wavelet transform partitions the data into different frequency components and analyzes each component with a resolution matched to its scale, the coefficients may be used to

collect microscopic information about the scale-dependent properties of the data. It has been shown that for an H-ss process the wavelet coefficients $d_X(j, k)$ exactly reproduce the self-similarity property of the process. In particular, for sufficiently large scales j , the following relationship holds

$$\log_2 Ed_X^2(j, k) \sim j\alpha + C = j(2H - 1) + C, \quad (14.4)$$

where C is constant independent of location index k and E denotes the expectation value. The above property in the wavelet domain allows the analysis of stationary, short-range dependent (SRD) processes $d_X(j, \cdot)$ for each j . A quantity of central importance is the non-parametric, unbiased variance of the process $d_X(j, \cdot)$

$$\mu_j = \frac{1}{n_j} \sum_{k=1}^{n_j} |d_X(j, k)|^2, \quad (14.5)$$

where n_j is the number of coefficients at octave j available for the analysis. Based on the power-law expression (14.4), the scaling exponent α (and hence H) could be simply obtained by inspecting the slope of $\log_2 \mu_j$ vs. j . This scaling behavior is detected by means of the so called log-scale diagrams which display $\log \mu_j$ as a function of scale j . Confidence intervals about the $\log \mu_j$ increase monotonically with j as larger and larger scales are encountered and region of alignment in the log-scale diagram is detected where up to statistical variation, the $\log_2 \mu_j$ values fall on a straight line. Since possible LRD processes are analyzed, the alignment should be detected for large values of scales j (e.g. ≥ 6). Hence, a log-scale diagram may be considered a spectral estimator where large scales correspond to low frequencies.

14.3.2 Log-Scale Diagrams of Turbulent Datasets

The log-scale diagrams are presented in Figs. 14.3–14.5 corresponding to L-mode 6861, H-mode 9031, and L-mode 9035, respectively. According to these diagrams, L-mode turbulent signals 6861 and 9035 display LRD while H-mode is practically white noise with Hurst exponent almost equal to 1/2, hence not an LRD process. We also estimate coefficients c_f which take positive real values. Their importance lies in the property of LRD that the sum over all correlations is large (actually infinite), but individually their sizes (which can be arbitrarily small) at large lag is controlled by c_f . Moreover, confidence intervals around mean estimates of LRD are proportional to the square root of c_f . In Table 14.1, we present LRD parameter estimates for all three signals, along with the confidence intervals.

We have also performed the LRD parameter evaluation using the Allan variance and the obtained H values correspond well with the above values, being 0.620, 0.52, and 0.61 for signals 6861, 9031, and 9035, respectively. Characteristic feature of all log-scale diagrams obtained here is the large variability in top portion

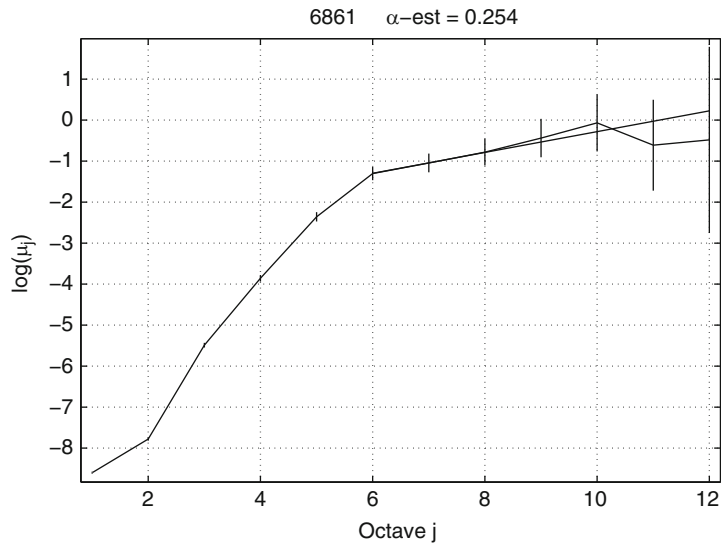


Fig. 14.3 Log-scale diagram displaying scaling of the variance of wavelet coefficients across scales for the L-mode 6861 signal

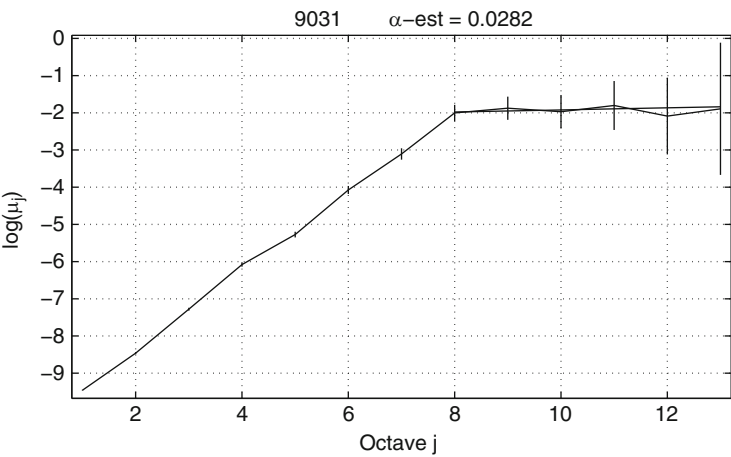


Fig. 14.4 Log-scale diagram for the H-mode 9031 signal. Note the zero slope for high scales indicating lack of long-range dependence

of the spectrum (large scales, usually greater or equal to 6), which suggests that particular care should be taken in interpreting the (possible) global Hurst exponent [478], requiring careful examination of the stationarity properties of each signal and possible evaluation of local Hurst exponents.

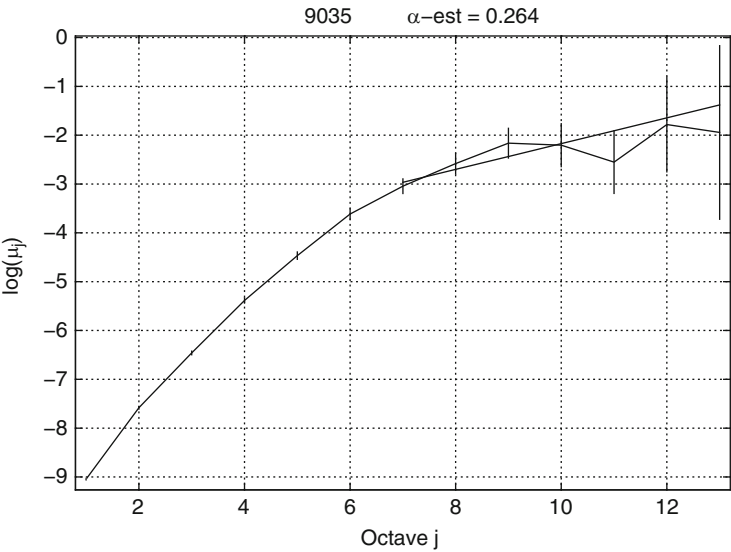


Fig. 14.5 Log-scale diagram displaying scaling of the variance of wavelet coefficients across scales for the L-mode 9035 signal

Table 14.1 Global indicators of self-similar character of the three signals. Confidence intervals are presented in square brackets

	L-mode 6861	H-mode 9031	L-mode 9035
α	0.254	0.03	0.264
H	0.627 [0.577 0.676]	0.515 [0.438 0.532]	0.632 [0.587 0.677]
c_f	0.1225 [0.0719 0.1956]	0.1775 [0.058 0.4187]	0.03 [0.023 0.120]

14.3.3 Testing Time Constancy of Scaling Exponents

Large variability in the scaling process of the log-scale diagram may easily yield erroneous detection of scaling regions when actually the data are not scaling but are non-stationary (in a non-scaling sense). Hence, any conclusion relating to the estimation of the global Hurst exponent (for the entire time series) calls for detecting constancy (or non constancy) of the scaling exponent. The test consists in dividing the dataset into non-overlapping blocks and estimation of scaling exponent for each of them [479]. The wavelet based estimates may be assumed as uncorrelated Gaussian variables with unknown means but known variances. The null hypothesis is that exponents for each block are equal, although unknown, and the test is so devised that if the null-hypothesis is rejected, one may conclude that the data are *both non-scaling and non-stationary*. The size of each block must be chosen in such a way so that the scaling region may be measurable over a sufficiently wide range of scales. Consequently, the number and size of the blocks should be large enough to see or follow precisely enough the variation in time of exponent H [479–481].

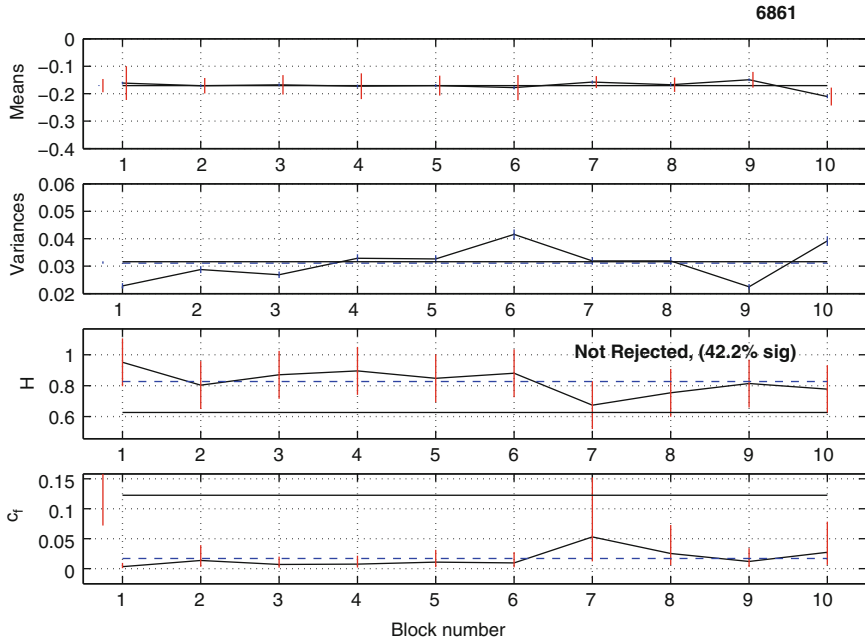


Fig. 14.6 From *top to bottom*: means, variances, Hurst exponents, and c_f 's evaluated over each block for signal 6861. On means and variances diagram, horizontal line gives the overall value for the entire series. On H and c_f diagram, solid horizontal line indicates the overall value and the dashed line gives the average value

For L-modes 6861 and 9035, we obtained very good constancy of exponent H for several sets of block sizes (between 5 and 30) and several scaling regions, as shown in Figs. 14.6 and 14.7. Hence, the null hypothesis was accepted in this case. In contrast to this, the null hypothesis was rejected in case of an H-mode dataset 9031 for some of the dataset partitioning as shown in Fig. 14.8, for the case of partitioning into 10 blocks. However, for the case of 5 blocks (small number of blocks), the null hypothesis is accepted due to the fact that the blocks are too wide to reveal the variation in H . The null-hypothesis is again rejected for small block sizes (≥ 20) because the statistical fluctuations of the estimates are large enough to mask the temporal variation of exponents H . Therefore, our final decision was to reject the hypothesis of constancy of exponent H over blocks. An important feature of the log-scale diagrams for each block (as well as for the entire signal) is that for large scales the $\log_2 \mu_j$ practically does not change as a function of scale j (the slope is practically equal to 0 within the confidence interval). But in order to deduce LRD, the scaling at large scales is absolutely necessary, so rejection of the null hypothesis is more a consequence of the non-scaling than the scaling variability. Based on these considerations, we conclude that the LRD is not present in this dataset. The non-stationarity (in the non-scaling sense) and inability to obtain the common scaling regions for the median number of blocks (10 in this case) for the

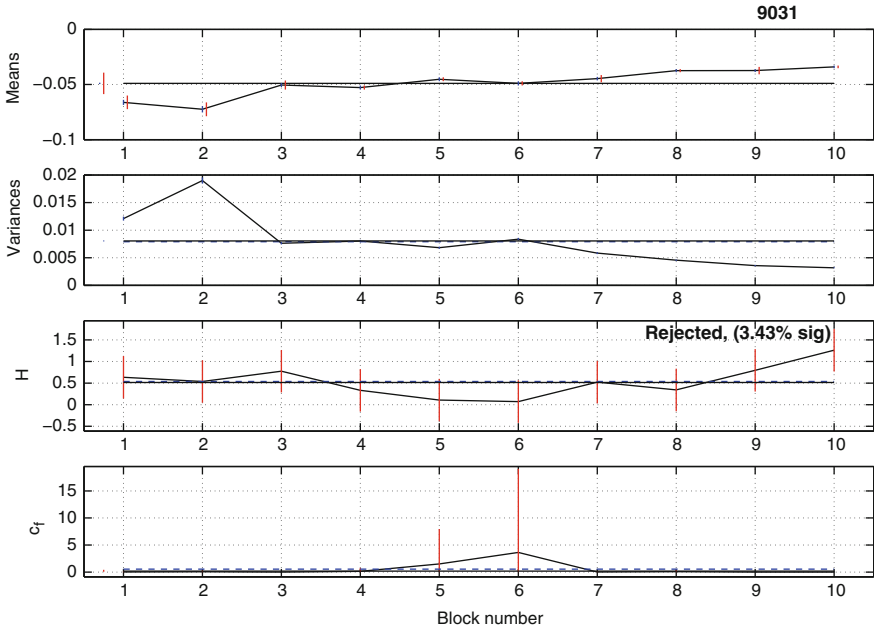


Fig. 14.7 From *top to bottom*: means, variances, Hurst exponents, and c_f 's evaluated over each block for signal 9031. On means and variances diagram, horizontal line gives the overall value for the entire series. On H and c_f diagram, solid horizontal line indicates the overall value and the dashed line gives the average value

H-mode dataset 9031 is probably due to the H-mode plasma being in the threshold region of the low to high confinement transition. This transition state and its influence on intermittency properties may be responsible for the high variability of the Hurst exponent.

14.3.4 Randomization Method for Long Range Correlations

The basic idea of a randomization method is to decouple the short-range from the long-range correlations in order to more clearly inspect the effects of the LRD. Following partitioning of the time series into a number of blocks of equal size, three types of randomization procedures are performed. The first one is the external randomization where the content of each block remains intact while the order of the block is randomly shuffled. If the series is sufficiently long, the autocorrelations should exhibit significant correlations beyond the block size. The next procedure is internal randomization, where the order of the blocks remains the same while the contents of each block are randomized. In this case, if the dataset has long memory, the ACF following such a procedure will still exhibit power-law behavior.

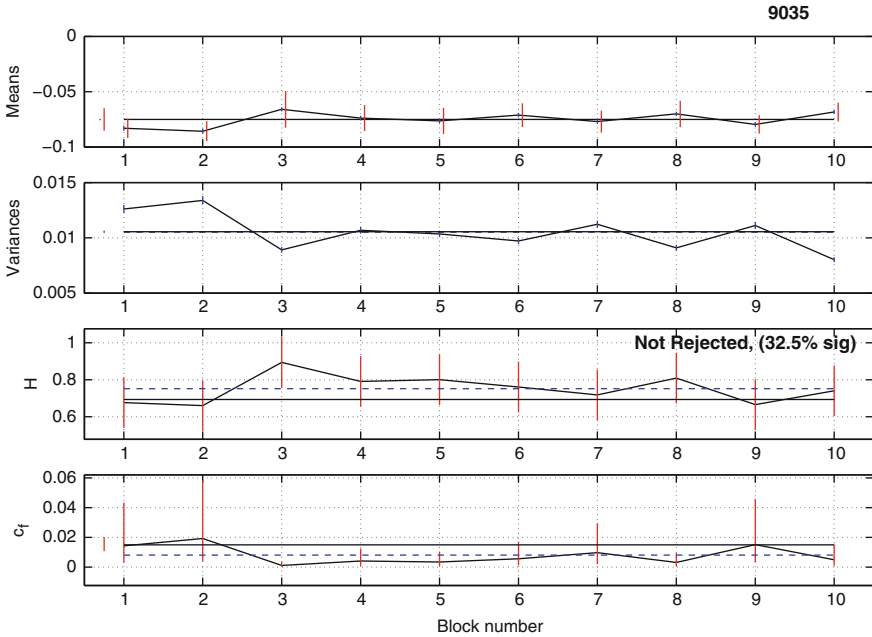


Fig. 14.8 From *top to bottom*: means, variances, Hurst exponents, and c_f 's evaluated over each block for signal 9035. On means and variances diagram, horizontal line gives the overall value for the entire series. On H and c_f diagram, solid horizontal line indicates the overall value and the dashed line gives the average value

Finally, there is a two level randomization where each block is further subdivided into smaller blocks and the randomization procedure is performed for the contents of each block as well as the order of the larger blocks is shuffled. As a result of this procedure, both short and long range correlations (across multiple blocks) are preserved, while medium range correlations (across multiple smaller blocks within the same block) are equalized. In our tests, the block size was set at 20 and the results are presented in Figs. 14.9–14.11. External randomizations for all signals causes elimination of all correlations beyond the block size. Internal randomization preserves power law behavior for the case of L-mode signals 6861 and 9035; however, this is not so clear for the case of H-mode 9035 which is almost exponential. Also two-level randomization distorts the medium-range correlations in the case of L-modes 6861 and 9035, while this is not the case for H-mode signal 9035.

Finally, we may conclude that the L-modes 6861 and 9035 definitely display long-range correlations, while the H-mode signal shows white noise like properties and the lack of LRD due to the non-stationarity and lack of constancy of the H exponent over time. In addition, we argue that the relationship of the power-law behavior at small frequencies with high variability of the block-data Hurst exponents at large scales (small frequencies) suggests that strong intermittent bursts at small scales (high frequencies) have a large impact on the dynamics at large scales.

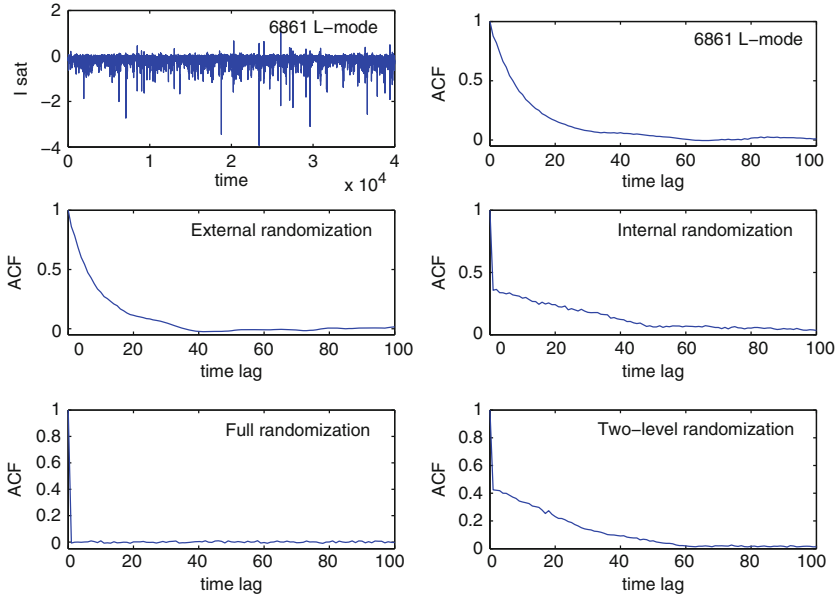


Fig. 14.9 *Top left:* signal 6861; *top right:* autocorrelation function (ACF). The other diagrams are obtained by performing internal, external, total, and two-level randomization as indicated

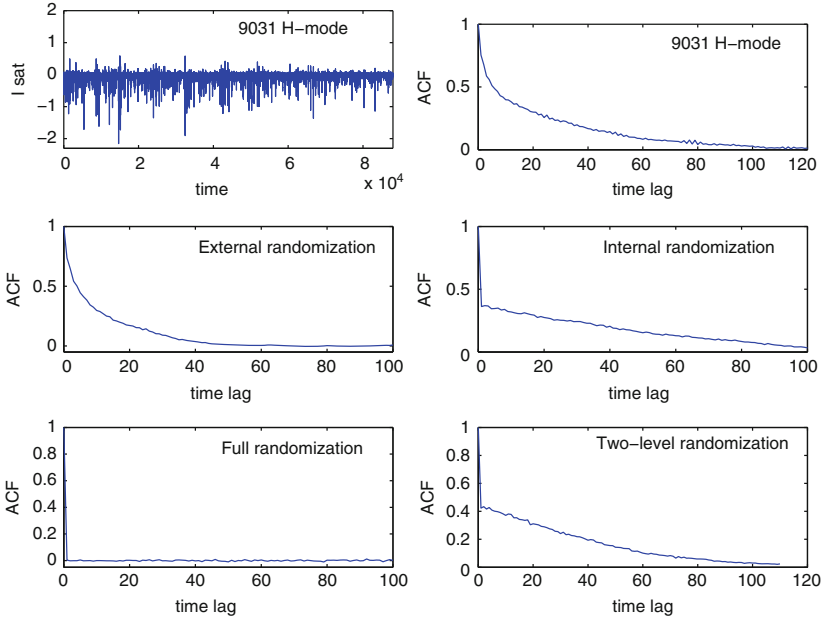


Fig. 14.10 *Top left:* signal 9031; *top right:* ACF. The other diagrams are obtained by performing internal, external, total, and two-level randomization as indicated

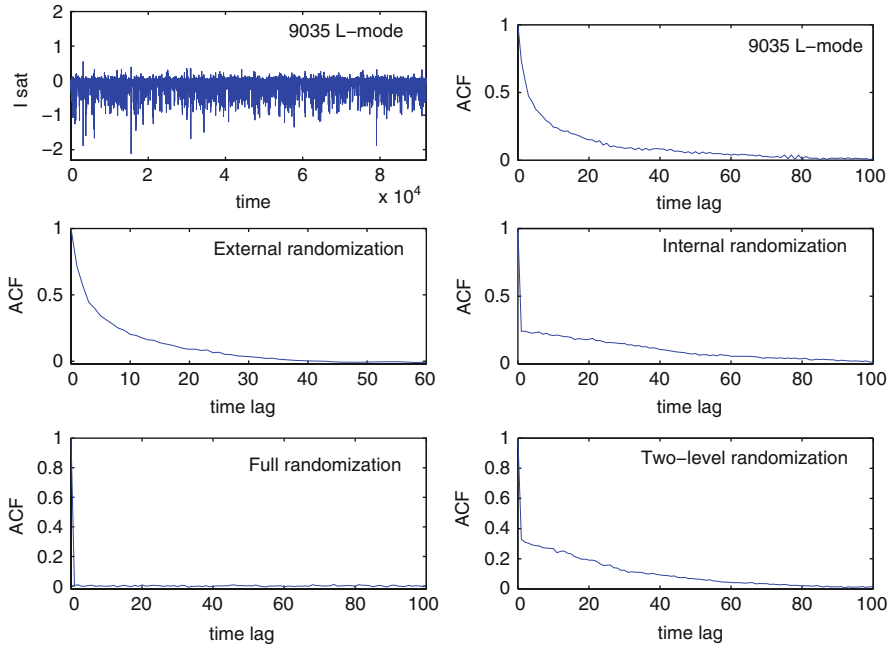


Fig. 14.11 Top left: signal 9035; top right: ACF. The other diagrams are obtained by performing internal, external, total, and two-level randomization as indicated

Hence, a simultaneous evaluation of the effects of long-range dependent effects and intermittency could give some more insight into this phenomenon.

14.4 Multifractal Properties of Datasets

14.4.1 Basic Properties of Multifractal Processes

Definition 2. A random process $X(t)$, $t > 0$ is called Multifractal process (mf), if for any $a > 0$, there exists a random function $M(a)$ such that

$$X(at) \stackrel{\circ}{=} M(a)X(t).$$

Here, the scaling (self-similar) factor $M(a)$ is a random variable, whose distribution does not depend on the particular time instant t . Exact self-similar process is a degenerate example of a multifractal, with $M(a) = ah$ and sometimes is referred to as a mono-fractal process. The generalized self-similarity index is defined as $h(a) = \log_a M(a)$. Therefore, the above relationship may be rewritten as

$$X(at) \stackrel{\circ}{=} a^{h(a)} X(t). \quad (14.6)$$

In contrast to self-similar processes, the index $h(a)$ is a random function of a . The exponent $h(a)$ is referred to as the Hölder exponent.

Definition 3. A random process $X(t)$, $t > 0$ is called Multifractal process if it has stationary increments and satisfies

$$E(|X(t)|^q) = C(q)^{\tau(q)+1}, \quad \text{for all } t \in T; q \in Q,$$

where T and Q are intervals on the real line, $\tau(q)$ and $C(q)$ are functions with domain Q . Moreover, we assume that T and Q have positive lengths, and that $0 \in t, [01] \subseteq q$.

Function $\tau(q)$ is the scaling function of a multifractal process. All $\tau(q)$ has the intercept -1 , which is implied by $E(|X(t)|^q) = 0$ at $q = 0$. As a special case, monofractal has the linear scaling function $\tau(q) = Hq - 1$. It is also shown that $\tau(q)$ is always a concave function for all multifractal functions.

Based on the expression for the self-similar process (strict sense),

$$X(at) \doteq a^H X(t) \quad (14.7)$$

one may question whether this relationship holds for the datasets under study and consequently question into the meaning of the determined Hurst exponent in the analysis of the LRD. As an initial step in this direction, our aim here is to determine whether the above expression is valid for the L and H mode datasets of the MAST confinement regimes.

Local exponents $h(a)$ are evaluated through the modulus of the maxima values of the wavelet transform at each point in the time series. Then, the scaling partition function $Z(q)$ is defined as the sum of the q -th powers of the local maxima of the modulus of the wavelet transform coefficients at scale a . For small scales, the following relationship is expected

$$Z(q) \sim a^{\tau(q)}. \quad (14.8)$$

For certain values of q , the exponents $\tau(q)$ have familiar meanings. In particular, $\tau(2)$ is related to the scaling exponent of the power spectra, $\Gamma(\nu) \sim 1/\nu^\beta$, as $\beta = 2 - \tau(2)$. For positive q , $Z(q)$ reflects the scaling of the large fluctuations and strong singularities, while for negative q , $Z(q)$ reflects the scaling of the small fluctuations and weak singularities [475]. Hence, the scaling exponent $\tau(q)$ may reveal much about the underlying dynamics. Monofractal signals display linear $\tau(q)$ spectrum,

$$\tau(q) = qH - 1, \quad (14.9)$$

where H is the global Hurst exponent. For multifractal signals, $\tau(q)$ is a nonlinear function $\tau(q) = qh(q) - D(h)$, where

$$h(q) \equiv d\tau(q)/dq \quad (14.10)$$

is non-constant Hölder exponent (local Hurst exponent) and $D(h)$ is the fractal dimension

$$D(h) = qh - \tau(q). \quad (14.11)$$

This function, also known as the Legendre multifractal spectrum since it is obtained by taking the Legendre transform of $\tau(q)$, is very useful to characterize multifractals. It is smooth and continuous and shows a single maximum. It is also universal in the sense that the same general type of function characterizes many different types of multifractal phenomena, or that an identical function characterizes a whole range of phenomena. The maximum value of $D(h)$ is the capacity dimension D_0 of the multifractal support, hence it may be an integer. The $D(h) = h$ line is tangent to the $D(h)$ vs. h plot, and the point of contact gives the information dimension D_1 . The other generalized dimensions are arranged around the $D(h)$ vs. h curve, positive values to the left of the maximum, and negative values to the right. The spread of the $D(h)$ vs. h curve is a measure of clustering of the data. Large positive $D(h)$ (low h values) correspond to points having small higher moments, so that large data values (or concentrations of data points) are clustered around these points. Large negative $D(h)$ (high h values) correspond to points with higher moments, so that low data values (or low concentrations of data points) are found close to these points. In a typical multifractal, there is a strong clustering of the data, whereas in a monofractal the $D(h)$ vs. h plot would be a single spike (all generalized dimensions are equal to D_0) indicating that clustering is no more than would be expected from the simple generating mechanism, or from a random process. The curve $D(h)$ vs. h is not necessarily symmetric. Most commonly the left side is steeper than the right one. This indicates that dense clusters, or clusters of exceptional large values, are rare relative to sparse concentrations, or low values.

14.4.2 Multiscale Diagrams and Wavelet Coefficients

In the wavelet transform formalism, the partition function is defined as

$$Z(q) = \lim_{j \rightarrow -\infty} \log E |d_X(j, k)|^q.$$

The log-scale diagrams used to inspect the scaling of the variance of the wavelet coefficients, (14.5), may be generalized to the study of higher order statistics so that the generalized (14.5) takes the form

$$\mu_j = \frac{1}{n_j} \sum_{k=1}^{n_j} |d_X(j, k)|^q.$$

This expression may be related to the Definition 4 of the multifractal process in the following manner. From the definition of self-similarity, the moments of the random process $X(t)$ satisfy

$$E(|X(t)|^q) = E(|X(1)|^q) \cdot |t|^{qH}, \quad \forall t.$$

The property of wavelet coefficients

$$E|d_X(j, k)|^q = E|d_X(0, k)|^q \cdot 2^{j(\tau(q)+q/2)},$$

implies that

$$E\mu_j^q = C(q)2^{j(\tau(q)+q/2)}, \quad \forall j,$$

with

$$\tau(q) = qH.$$

This relationship indicates that self-similarity (and multifractality) may be inferred by testing the linearity of $\zeta(q)$ with q . For multifractal processes,

$$\int |T_X(a, t)|^q dt \approx a^{(\tau(q)+q/2)} \quad (a \rightarrow 0),$$

where $T_X(a, t)$ are continuous wavelet coefficients

$$T_X(a, t) = \langle X | \psi_{a,t} \rangle, \quad a \in R^+, \quad t \in R,$$

and where $\psi_{a,t}$ are dilatations and translations of the mother wavelet ψ_0 . From these expressions, $\tau(q)$ may be measured and the Legendre multifractal spectrum may be obtained. Naturally, one is first interested whether $\tau(q)$ takes a simple form $\tau(q) = qH$. For example, self-similar processes, for which

$$\mu_j^q \approx 2^{j(\tau(q)+q/2)},$$

for all scales satisfy $\tau(q) = qH$, and are therefore fractal processes with $h = H$. The multiscale diagram is obtained by plotting $\tau(q) = hq - (1/2)(q - 1)$ against q , (together with confidence intervals about $\tau(q)$). If there is no alignment, i.e., the relationship $\tau(q) = qH$ does not hold, a multifractal scaling is deduced. Using the same wavelet formalism, we may obtain the $\tau(q)$ vs. q and the Legendre spectrum ($D(h)$ vs. h), and as discussed above, they may be used to qualitatively and quantitatively describe the multifractal properties of the signals.

14.4.3 Multiscale Diagrams and Multifractal Spectra for L and H Modes

Test for multifractal property of the signals are presented in Figs. 14.12–14.14.² Diagrams on the left side of each figure show scaling of $\tau(q)$ with q , while diagrams

² Matlab routines developed by P. Abry and D. Veitch were used for this purpose.

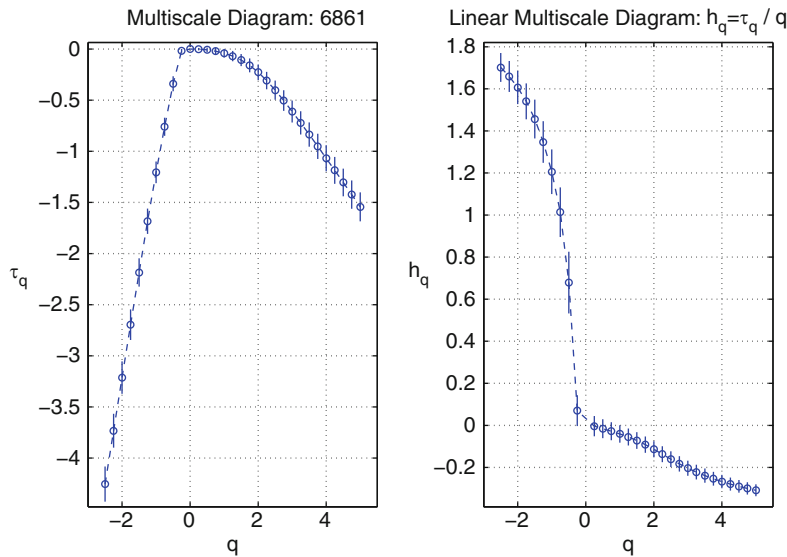


Fig. 14.12 Multiscale and Linear multiscale (LM) diagrams for the 6861 L-mode regime. Lack of the flat region in the LM diagram indicates that the process is not globally self-similar

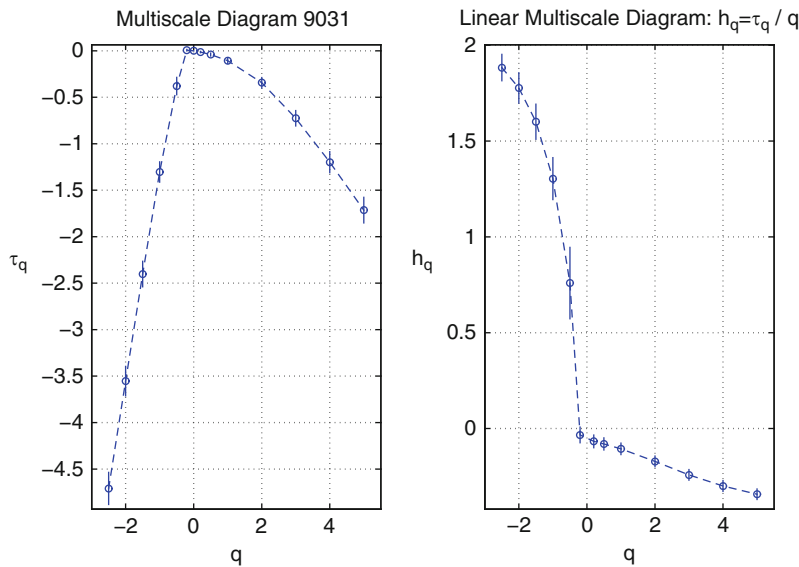


Fig. 14.13 Multiscale and Linear multiscale (LM) diagrams for the 9031 L-mode regime. Lack of the flat region in the LM diagram indicates that the process is not globally self-similar

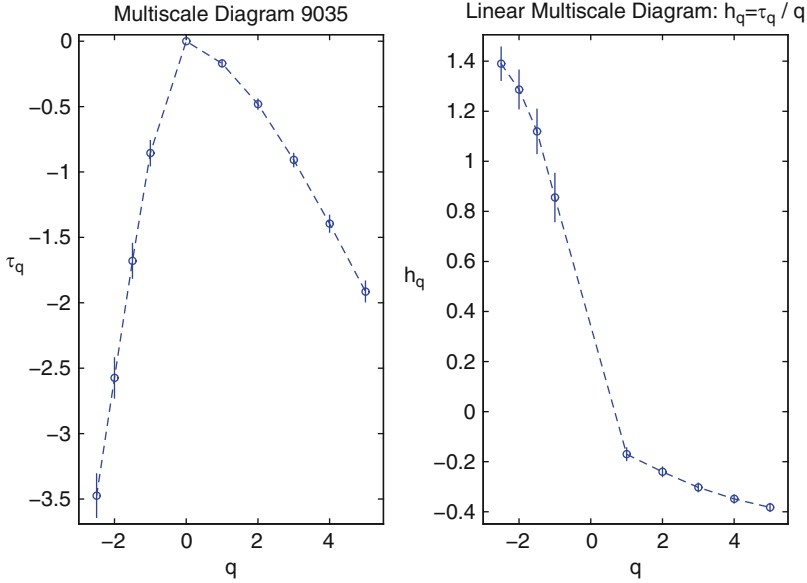


Fig. 14.14 Multiscale and Linear multiscale (LM) diagrams for the 9035 L-mode regime. Lack of the flat region in the LM diagram indicates that the process is not globally self-similar

on the right show q -dependence of $h_q (= \tau(q)/q)$. Diagrams clearly illustrate that all three confinement regimes (6861 L-mode, 9031 H-mode and 9035 L-mode) are multifractal processes, and hence cannot be characterized by a single Hurst exponent. Specifically, none of the diagrams on the right (Linear multiscale diagrams) have approximately constant $h(q)$ for positive q (a sign of global scaling). In order to compare this with the monofractal process, in Fig. 14.15, we show the same diagrams for the *fractal Gaussian noise* where a flat region for positive q is a clear indication of global self similarity. Hence, signals of both the low and high confinement regimes are *multifractal processes*.

In order to quantify the multifractal properties, we present in Figs. 14.16–14.18 the $\tau(q)$ spectra and the corresponding singularity spectra. The singularity spectra (or the Legendre spectra) are obtained from the scaling exponents $\tau(q)$ of the partition function by means of the Legendre transformation. Essentially, the singularity spectrum describes the statistical distribution of the singularity exponents by associating with any given exponent the Hausdorff dimension of the set of points which have the same singularity exponent [475]. It is simple to deduce three key attributes of the multifractal spectrum, namely the left slope, mode, and the width spread. By the arguments given earlier, one can deduce that the value corresponding to the most frequent singularity is the information dimension D_0 (mode). This quantity represents actually the most sensitive indicator of the mentioned three geometrical attributes of the multifractal spectrum. Information dimension for the L-mode signal 9035 and the H-mode signal 9031 are very close to 1; however, it is somewhat smaller for the L-mode 6861 signal ($D_0 \approx 0.95$). The left side in all three cases is

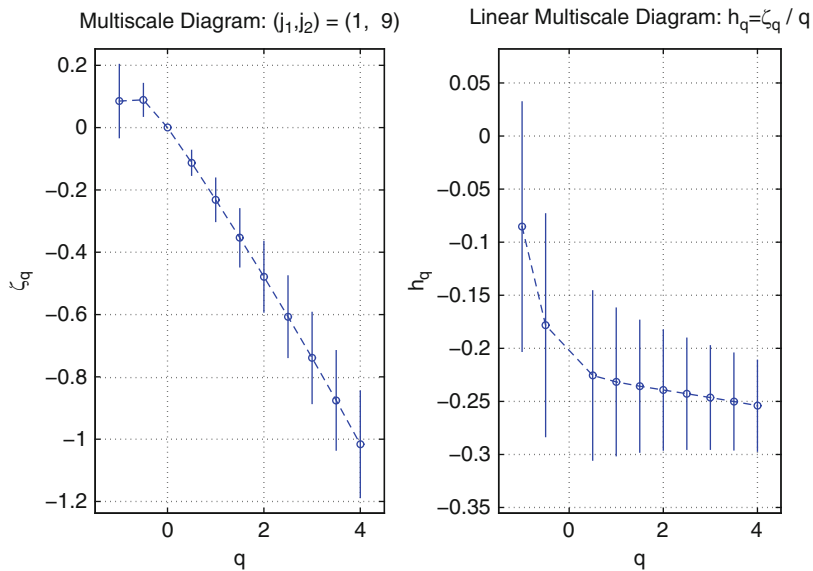


Fig. 14.15 Multiscale and Linear multiscale (LM) diagrams for the fractional Gaussian noise. An almost flat region in the LM diagram for positive q indicates that the process is self-similar (a mono-fractal process)

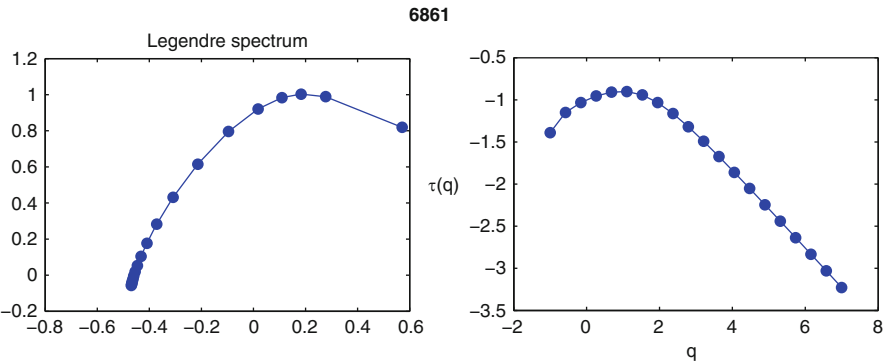


Fig. 14.16 Multifractal spectrum (*left*) and $\tau(q)$ vs. q diagram (*right*) for the 6861 signal, obtained using the wavelet transform method

steeper than the right one, indicating that dense clusters, or clusters of exceptional large values are rare relative to spares (low value) concentrations. The mode for the L-mode 6861 is the most positive indicating slightly weaker intermittency compared to other two datasets. Recalling that according to Kolmogorov K41 theory the mode is 1/3 and since all three signals have modes below this K41 value, the intermittency effects are strong in all three datasets. This suggest that it is important to study the coupling effects of LRD together with intermittency in order to more

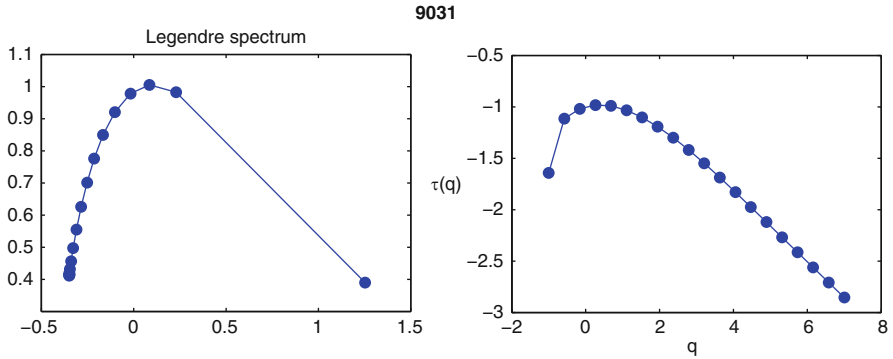


Fig. 14.17 Multifractal spectrum (*left*) and $\tau(q)$ vs. q diagram (*right*) for the 9031 signal, obtained using the wavelet transform method

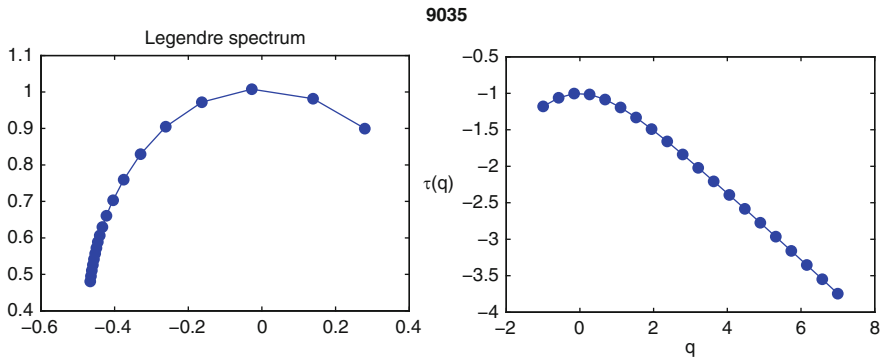


Fig. 14.18 Multifractal spectrum (*left*) and $\tau(q)$ vs. q diagram (*right*) for the 9035 signal, obtained using the wavelet transform method

effectively interpret the dynamics of the two regimes. This issue will be discussed in somewhat more detail later on.

In contrast to the power spectra, which describe the distribution of energy of the signal, the multifractal spectrum describes the distribution of local singularities expressed in terms of the so-called Hölder exponents. Formally, the function X is Hölder continuous with exponent α , $0 < \alpha < 1$, at t_0 if as $|t - t_0| = |\Delta t| \rightarrow 0$,

$$|X(t_0 + \Delta t) - X(t_0)| \leq C |\Delta t|^\alpha.$$

Geometrically, a local singularity at time t_0 can be visualized as a relation between neighborhood fluctuations of a function $X(t)$ and two bounding curves as shown in Fig. 14.19. To estimate such a singularity, two curves expressed through $X(t_0) \pm C |t - t_0|^\alpha$ may be constructed (where C is a constant). The maximum value of α that locally bounds $X(t)$ in the neighborhood of t_0 between these two curves is the local singularity (bottom-left diagram of Fig. 14.19). When α is large, the curvatures are narrow thus limiting local fluctuations (bottom-right diagram of

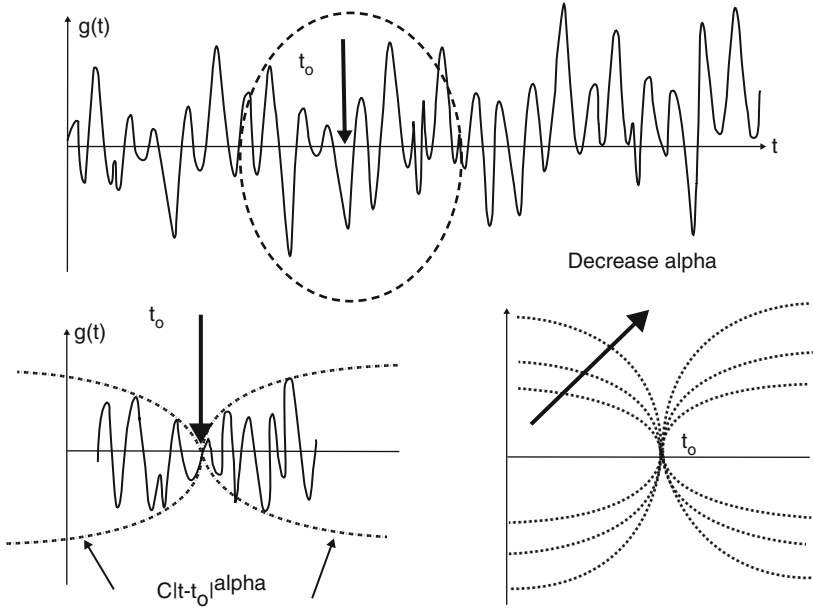


Fig. 14.19 Geometric interpretation of local scaling exponent. Distribution of scaling exponents evaluated for the complete time series represents the multifractal spectrum

Fig. 14.19). When α is small, two curves have small curvature, thus allowing $X(t)$ to take on large local irregular behavior and when t_0 slides across the time series, the distribution of the resulting α is described by the multifractal spectrum. It has been shown that the local singularity strength can be measured in terms of the wavelet coefficients as [482]

$$h(t) = \lim_{k2^j \rightarrow t} \frac{1}{j} \log_2 |d_X(j, k)|.$$

With the h determined, the multifractal spectrum, sometimes denoted by $f(\alpha)$ because α is used instead of h , measures the distribution of $h(t)$ within a time-series and can also be obtained using the standard box-counting technique. The Hölder exponent may be thus interpreted as a local Hurst exponent, and in the manner that global Hurst exponent carries information about self-similar functions, it characterizes the regularity of a function at a given point of the time-series. A Hölder exponent between 0 and 1 indicates that the signal is continuous but not differentiable at the considered point, and the closer this exponent is to zero, the less regular the function is. Pointwise Hölder exponents, measuring the scaling behavior at infinite resolution, for the two regimes are presented in diagrams of Figs. 14.20–14.22.³

³ Calculations were performed with the FracLab software available at INRIA website.

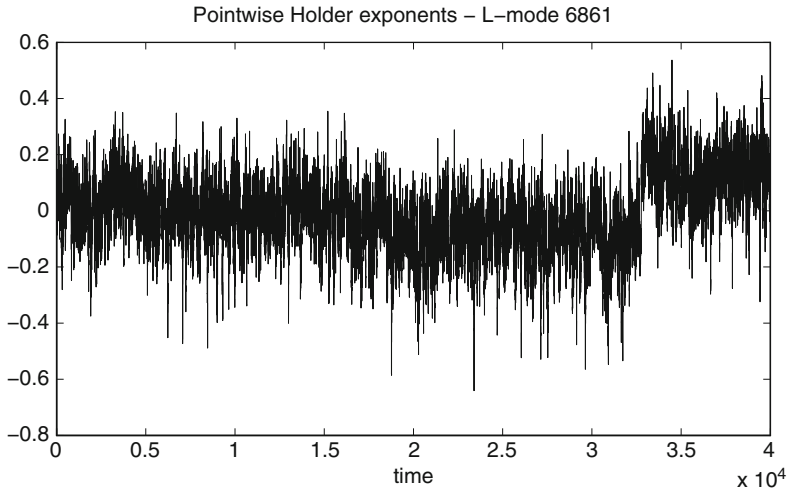


Fig. 14.20 Pointwise Hölder exponents for the 6861 L-mode

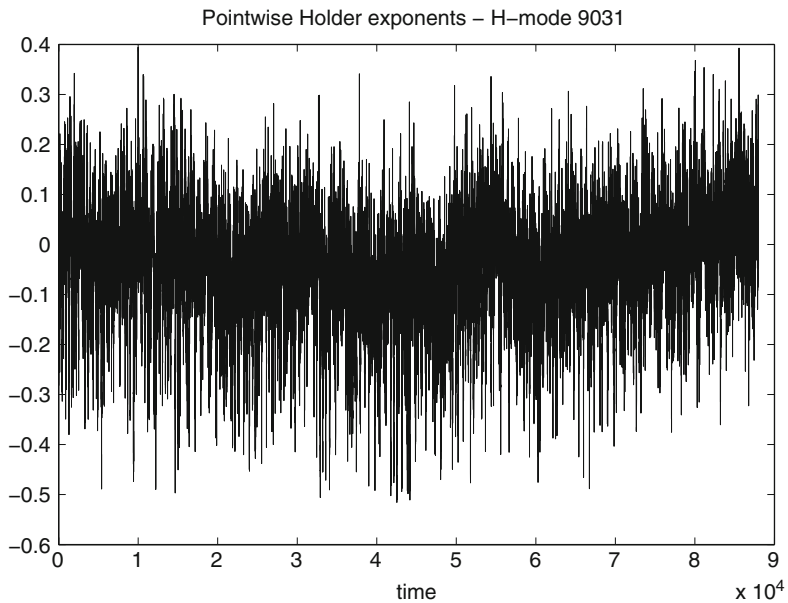


Fig. 14.21 Pointwise Hölder exponents for the 9031 H-mode

Hence, all modes are characterized by continuous but not differentiable signals. For comparison purposes, Hölder exponents for the three signals are presented using the large deviation spectra. A large deviation spectrum (LDS) represents coarse grained Hölder exponents, which measure scaling behavior at finite resolution. The large deviation spectra which point to discernible differences between the signals

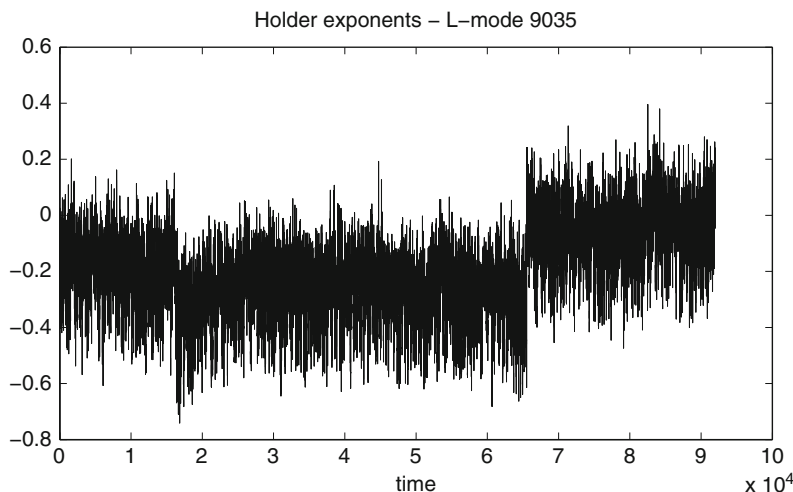


Fig. 14.22 Pointwise Hölder exponents for the 9035 L-mode

for the three confinement regimes are presented in Fig. 14.23. The x-axis in this diagram represents the Hölder exponents of signals and the y-axis reflects the number of points with the corresponding exponent, i.e., the probability of finding the particular Hölder exponent within the signal. The 6861 L-mode is considerably more regular than the other two, while the 9035 H-mode and 9035 L-mode exhibit surprisingly similar spectra with the L-mode 9035 being slightly more irregular. Recalling the almost stationary increments of the 6861 signal, it may be inferred that intermittent burst do not change much the regularity of this signal. Based on these results of the analysis alone, it is clear that all signals are the product of small-scale stochastic plasma turbulence, without large-scale events.

Few words should be devoted to the relationship between the singularity (Legendre) spectrum and the LDS. The singularity spectrum represents a concave approximation to the LDS. The LDS yields more robust estimates, however, at the expense of a loss of information. Hence, both spectra should be used in the analysis along with the Hausdorff spectrum, which is the most precise spectrum from a mathematical aspect, however, very demanding as far as computational time is concerned. The complete analysis of the three spectra will be given elsewhere [483,484].

14.5 Coupled Effects of Long-Range Dependence and Intermittency

The scaling of the energy spectrum in the high frequency range in all three processes is different from the scaling in the low frequency range. Since the scaling in the low frequency range determines long-range behavior and the scaling in the high

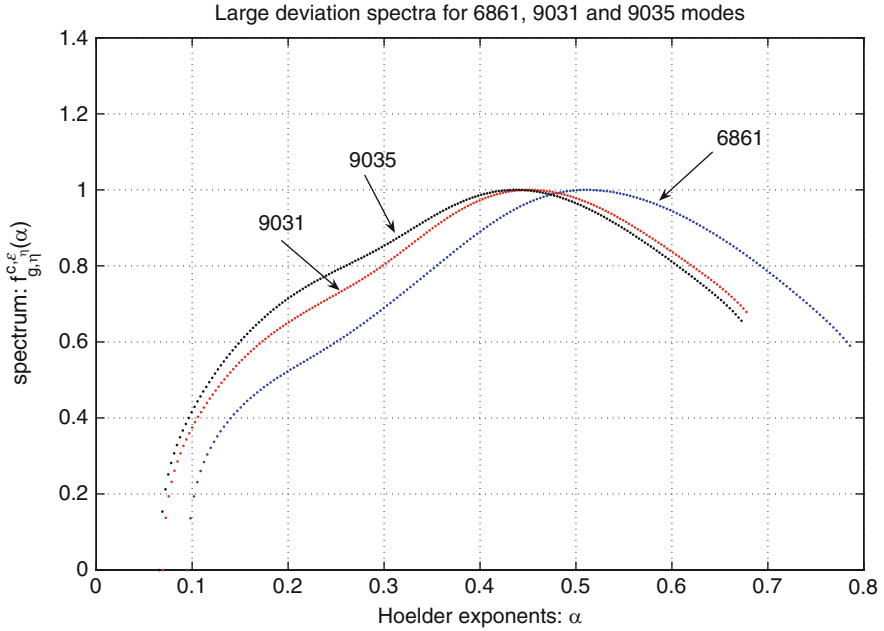


Fig. 14.23 Large deviation spectra for the L-modes 6861 and 9035 and an H-mode 9031

frequency range is determined by intermittency effects, it is of great importance to study both effects simultaneously on the basis of the model presented in [485] and [486]. Namely, it has been argued that the fractional Riesz–Bessel motion (fRB), which represents a Gaussian process which has stationary increments and spectral density of the form

$$\Gamma(v) = \frac{cf}{|v|^{2\beta}} \frac{1}{(1+v^2)^\gamma}, \quad (14.12)$$

with the two fractional parameters satisfying

$$\beta \in (1/2, 3/2) \text{ and } \gamma \geq 0,$$

or

$$\Gamma(v) = \frac{cf}{|v|^{2\beta}} \frac{1}{(1+v^2)^\gamma} \frac{v^2}{1+v^2}, \quad (14.13)$$

with the two fractional parameters being

$$\beta \in (1/2, 3/2) \text{ and } \beta + \gamma \geq 1/2,$$

may be appropriate to model combined effects of LRD and intermittency. Synergistic action of these two effects is reflected in the scaling of the energy spectrum. It may be easily noticed that the fractional Brownian Motion (fBM) is the limiting case of

the expression (14.12) as $\gamma \rightarrow 0$ and $H = (2\beta - 1)/2$. The importance of these two expressions (14.12) and (14.13) in the analysis of turbulent signals stems from the physical meaning of these expressions. The component $|v|^{-2\beta}$ signifies the LRD with fractional parameter $\beta = H + 1/2$, while the component $(1 + v^2)^{-\gamma}$ indicates second-order intermittency. Therefore, based on the model of the fractional Riesz–Bessel motion, it is possible to study both effects simultaneously by determining the corresponding exponents β and γ . To illustrate graphically these two effects, we show the periodograms for the three processes under study in Figs. 14.24–14.26. The periodogram I_N of the signal $X(t)$ represents the power spectrum of the *entire signal* according to the expression

$$I_N(v) = \frac{1}{2\pi N} \left| \int_0^N e^{-ivt} X(t) dt \right|^2,$$

where $N > 0$ is the upper bound of the interval $[0, N]$, on which each $X(t)$ is observed.

Naturally, periodograms of segments of the time series may be averaged together to form the power spectral density. The main advantage of the periodogram with respect to the power spectral density, in this particular case, is the ability to clearly disclose the scaling behavior both in the low and in the high frequency range.

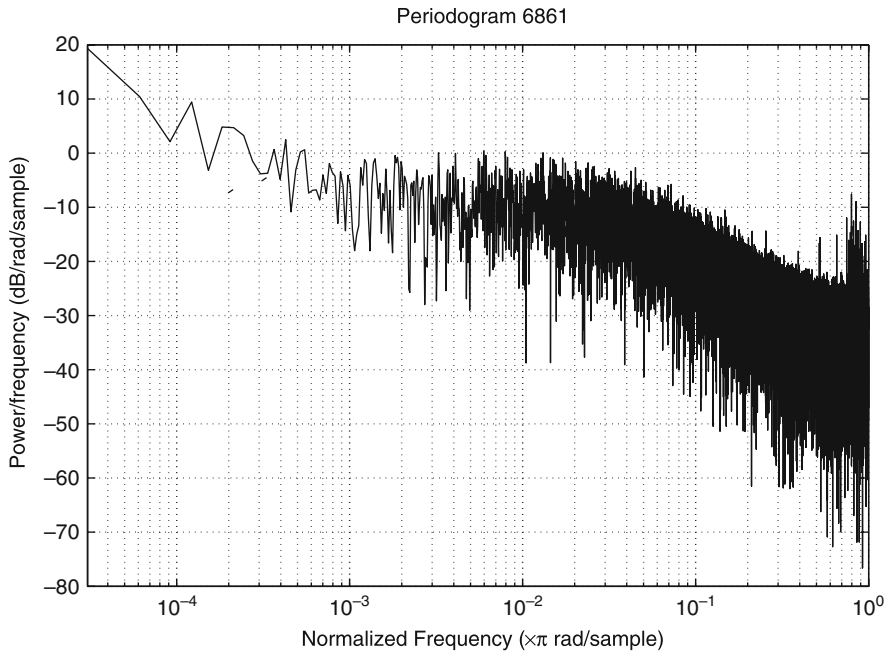


Fig. 14.24 Periodogram of the L-mode 6861 signal

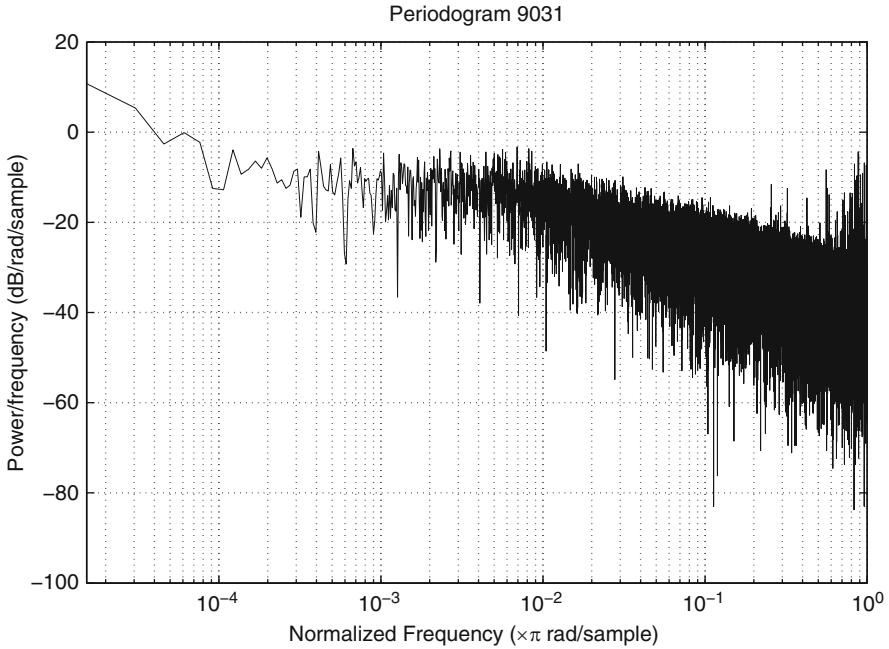


Fig. 14.25 Periodogram of the H-mode 9031 signal

The negative slope of power-law behavior in the low-frequency range determines the LRD while the steeper negative slope in the high frequency range determines the intermittency strength. Note that the flat region in the low frequency range for the H-mode signal 9035 suggest possible lack of LRD. Also it is evident that the slopes in the high frequency region are different, corresponding to the different intermittency effects in the low and high confinement regimes. An estimation procedure based on the wavelet transform, as suggested in [485] and [486], may be used for determining both fractional exponents simultaneously.

14.6 Conclusion

One of the main features of the H-mode signal 9035, pertaining to the lack of LRD, is that it shows large temporal variability in the Hurst exponent values evaluated over signal partitioned into equal segments. This variability cannot be inferred from the distribution of means and variances corresponding to the blocks, however, may be expected from the distribution of increments over time. The fact that this signal has been recorded close to the threshold of the low-to-high confinement regimes is of particular relevance with respect to the issue of exponents variability and suggests further inquiries into the mechanism of the loss of LRD. On the other hand,

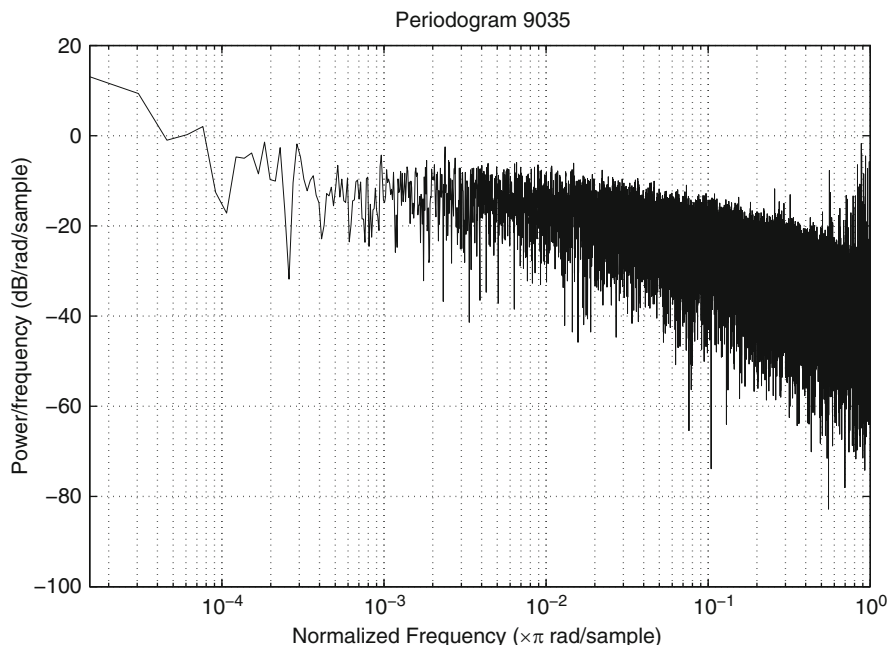


Fig. 14.26 Periodogram of the L-mode 9035 signal

almost stationary segments in the temporal dynamics of the 6861 increments suggest LRD process. The possible LRD in the signals may be anticipated from the slope of the periodogram at small frequencies (large scales). All turbulent signals are multifractal, which indicates that they are only locally self-similar and that many Hurst (Hölder) exponents are necessary in order to quantify the self-similar features. Singularity spectra, obtained with the wavelet transform method, yield information about the distribution of singularities and may be used to estimate several types of dimensions. Information dimension, pertaining to the maximum of the singularity spectrum, is slightly lower for the L-mode signal 6861 due to its higher regularity as compared to signals 9031 and 9035. Detailed analysis of the multifractal and intermittent properties in turbulent plasmas can benefit from advanced methods already developed in other fields of application such as the level-crossing analysis [487].

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Chapter 15

Multi-Scale Modelling of Nonlinear Plasmas

We discuss novel modeling frameworks for multi-scale plasma interactions, in particular, the so-called, Equation-Free Macro Projective Integration (EFMPI) which are attempting to perform macro-scale simulations while still taking into account the effects of micro-scale physics. We demonstrate a recently proposed primal-EFMPI scheme to simulate the paradigmatic nonlinear ion-sound wave evolution, which includes nonlinear wave steepening and kinetic effects of particle trapping.

15.1 Basics on Multi-Scale Modelling

Currently, there is an upsurge of activities in natural sciences and engineering in a quest to comprehend complex systems with nonlinear interactions involving multiple spatial and temporal scales, with researchers being increasingly capable to model, simulate, analyze, control, and predict behavior of systems of large complexity over the previously unprecedented range of scales. Generally taken, the macroscopic, e.g., coherent behavior in the complex systems emerges in the interactions of microscopic constituents – atoms, molecules, cells, individuals of a population – among themselves and with the environment. As a consequence, the macroscopic behavior should in some appropriate way be deduced from the microscopic one [488, 489]. For some problems such as Newtonian fluid mechanics, the Navier–Stokes equation predated its microscopic derivation from the kinetic theory. However, in many problems in physics, chemistry, biology, ecology, material science, engineering, etc., the closures required to transform from the microscopic (atomistic) level to a high-level macroscopic description are often unknown or incomplete ([488, 489]; Weinan et al. 2007, private communication). Severe limitations often arise in trying either to find closures or to solve these problems at the scale at which the questions of interest are asked, e.g., by using microscopic simulations only. Several numerical methods have been attempted with a varying success in many different applications. One of the supposedly first attempts aimed at the generalization of multi-grid methods was due to A. Brandt [490]. In its original form, a goal was to efficiently solve algebraic equations derived by discretizing partial differential equations (PDE) in order to find accurate PDE solutions. In an extended

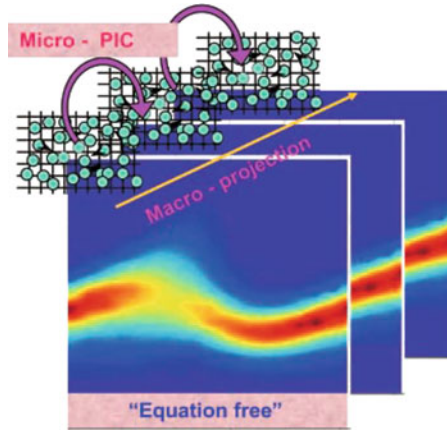


Fig. 15.1 Macro-scale projection EFREE method for multi-scale plasmas

version, one is basically focused at just capturing the macro-scale behavior. The so-called Equation-free (EFREE) method was recently proposed by Yanis Kevrekidis et al. [488], as the systematic framework for directly extracting from microscopic simulations the information which would be obtained from macroscopic models had these been available in closed form [488, 491–493]. This is a system based procedure processing the results of short bursts of appropriately initialized microscopic simulations. In distinction, in another proposed alternative, Heterogenous Multi-scale Methods (HMM), general strategy is to couple the macro (where available) and micro models, such that the macro-state provides the constraints for the micro model while the micro model supplies the necessary (missing) constitutive data for the macro model. Above methods all exploit scale separation and posses basic similarity with the multi-grid ideas of coupling at each macro-step, which requires that all algorithms explicitly move back and forth between the macro and micro states. Namely, micro-simulations have to be reloaded (initialized) at every macro step, which is not a simple task (see Fig. 15.1).

15.2 Multi-Scale Plasma Models

Multi-scale modeling of plasmas consisting of systems of charged particles are among the most challenging problems in the contemporary simulation science. Multi-scale problems such as, e.g., magnetic reconnection and plasma turbulence are rather difficult to simulate because of the strongly interconnected physics of micro- and macro-scales which basically defies direct numerical computations [495, 496]. For example, multi-scale modelings in magnetically confined fusion plasmas are among the most complex tasks, since the multi-physics processes governing fusion plasmas span over a huge range of temporal and spatial scales resulting in intractability of direct brute force simulations, in any foreseen future. In the extreme

cases, one finds the ratio of the transport time scale to the electron cyclotron time scale of $O(10^{14})$ and the device radius to the electron gyroradius of $O(10^4)$ [496]. At the same time, extreme anisotropy in the mean free path along and across the magnetic field can reach $O(10^{10})$, which together with high-dimensionality ($3 + 3 = 6D$) in real and velocity space, intrinsic nonlinearity and sensitivity to geometry, present fundamental challenges to fusion simulations [495,496]. Currently, scale separation is exploited to a large extent by solving the separate physical (hierarchy) models (see Fig. 15.2). Still, in reality, plasma phenomena are all coupled since sharing same particle distributions and electromagnetic fields. Accordingly, a large number of critical cases identified in fusion plasmas, where essentially coupling between multi-scale and multi-physics models must be performed [495,496].

Generally taken, with the construction of the International Thermonuclear Experimental Reactor (ITER) in Cadarache, France, an impressive multi-national, multi-billion dollar effort, as the first attempt in the burning fusion plasma experiments toward next generation Demo fusion reactors, a need for comprehensive plasma simulations with capability for integrated predictive whole machine modeling on all relevant scales, both space and time is becoming crucial. Still in future, toward developing sophisticated inter-linked simulation codes, advances in fusion physics modeling, applied mathematics, computer science, and high-performance software are envisaged [495,496].

To start, a plausible idea of interlinking the micro- and macro- scale plasma models has already meet some success [497–499]; while appealing, in reality meets obstacles, both at the basic conceptual level and at the more practical- numerical scheme realization. The former is related to a fundamental question of uniqueness and hence accuracy, as in communicating between microscopic and macroscopic models, different micro worlds can yield the same macro state, while embarking from a macro state requires some prior knowledge (assumption) about a micro state.

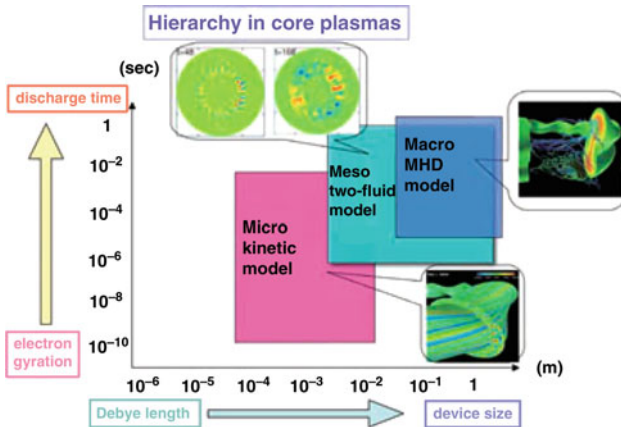


Fig. 15.2 Multi-scale physical hierarchies in magnetic fusion plasmas [495]

Next, basic ideas related to a novel simulation framework, the EFMPI methods for multi-scale plasmas will be illustrated in more detail [500, 502–504]. It is based on the “first principle”, micro-physics plasma simulator (kinetic, i.e., particle-in-cell code-PIC), while the macro-scale properties are determined by repeatedly extrapolating forward macro- variables calculated from short bursts of micro- simulations. Different schemes for reconstruction and mapping between high-dimensional (micro) and low-dimensional (macro) phase space, using moment representation [500], cumulative particle distributions [504, 505], wavelets, etc., can be attempted. Starting with proof-of-principle studies of simple one-dimensional nonlinear plasma paradigms, e.g., the kinetic ion sound waves, it was expected to be able to proceed toward more realistic multi-scale plasma models. A general potential, some basic limitations, and critical view of this novel approach to multi-scale plasma simulation will be briefly addressed.

15.3 Equation-Free Macro-Projective Integration Method

Over the past few years, a simulation framework, the so-called Equation-Free (EFREE) projective integration has been proposed and applied to a variety of multi-scale phenomena in engineering problems in which coarse-scale (macro) behavior can be obtained through short-time simulations within the fine-scale models (microscopic, stochastic, etc.) [488, 491–494]. In EFREE, the simulation acts on two scales. The macro scale dynamics is determined by repeatedly extrapolating forward coarse-scale estimates obtained from the short micro scale simulations. In the Equation-free framework, the main tool that allows the performance of numerical tasks at the macroscopic level using the microscopic (e.g., stochastic) simulation codes is the so-called coarse time-stepper. It consists of three parts: lifting (mapping from coarse-macroscopic to fine-microscopic level), short time micro calculations around which the macroscopic calculations are wrapped, and restriction (mapping from fine-micro scale to macroscopic level) [500]. The details about each part are presented in many papers ([488]; Weinan et al. 2007, private communication). The coarse time stepper is combined with time projection at macroscopic level, i.e., time projection of the coarse observables on the macroscopic scale. This structure is schematically presented in Fig. 15.1 (vide infra). The significant premise in the EFMPI framework, like in most current multi-scale methods, is a clear separation between micro and macro time scales.

The complexity of nonlinear multi-scale plasma phenomena has challenged researchers to try to implement the multi-scale approaches developed in other scientific fields. One of the first attempts is the implementation of the EFMPI procedure by Shay et al. [500], in the context of the nonlinear ion-acoustic wave, claimed as a preparatory step for the intriguing task to solve a problem of the magnetic reconnection. The ion-acoustic wave propagation and steepening are originally followed by the modified three-dimensional electromagnetic particle-in-cell (3D EM PIC) code [362]. In their EFMPI scheme, the electrons are assumed to be adiabatic, both

the electron and ion velocity distributions are assumed to be the shifted Maxwellian and quasi-neutrality is proposed. The results of the multi-scale EFMPI calculations are discussed with respect to the full micro-PIC simulations. At the first step, the coarse observables are determined. First three moments: ion density, ion velocity, and pressure are taken as the ‘active’ coarse observables, i.e., those macro variables which are directly computed forward in time. On the other hand, the electron density, electron velocity and electric field are taken as the ‘passive’ coarse variables, i.e., variables which are not calculated directly but from active macro observables. These observables are defined on the coarse mesh by the linear interpolation procedure. The micro quantities, the ion and electron positions, are obtained through the lifting from corresponding densities, and the ion and electron velocities are lifted from corresponding velocity distributions (approximated by the shifted Maxwellian). The PIC solver is then applied for the short time in order to ensure the system to stay near the so-called slow manifold. In other words, the implementation of the micro solver has to ensure the reconstruction of the values of the macro quantities which would be obtained under the same conditions but using only the micro solver. The coarse-macro observables are generated by the reverse operation- restriction. After the linear interpolation, they are projected in time. Approximately, time interval for the micro calculation is around 20 micro time steps and the macro (projection) time step is two order of magnitude larger than the micro time step [500]. However, above procedure basically neglects kinetic effects in a plasma. The problems which appeared in the reconstruction of the ion-acoustic wave when the wave steepening is noticed were related to the particle trapping, non-Maxwellian features, and violated quasi-neutrality. Instead of moments, the wavelet technique for reconstructing the particle probability distribution function (PDF) was indicated as a possible solution [501]. More recent schemes to test an applicability of the EFMPI framework, attempting to include the correlation effects in the nonlinear ion-sound wave paradigm was undertaken by one of these authors [504]. That approach uses the marginal (1-dimensional) and conditional (2-dimensional) cumulative particle distribution functions in the $(x - v_x)$ ion phase space as the macro (coarse) scale observables [504, 505], which results in a better agreement with fully kinetic electrostatic particle-in-cell (1D ES PIC) solver. The results of the EFMPI simulations are compared with the results obtained by the one-dimensional electrostatic particle-in-cell (1D ES PIC) solver [504].

In further text, we shall demonstrate a feasibility of the EFMPI ideas in nonlinear ion-sound paradigm by introducing an original scheme [502], which for its simplicity named the “primal EFMPI” (p-EFMPI). The basic platform for micro-simulation is the standard version of 1D ES PIC code [362]. However, our working hypothesis is simply that the ion motion could be assumed inherently coarse grained or “smoothed” as compared to electron-scale dynamics, to possibly describe the macro scale plasma evolution sufficiently well. Accordingly, we track individual ion orbits in time and directly extrapolate to project. In contrast to the initial EFMPI scheme [500, 504], here, ions are not to be restricted via PDF or the corresponding moments, rather, ions are kept “as they are,” i.e., in theory, preserving full ion kinetic effects.

15.4 The Nonlinear Ion-Sound Wave

The theoretical modeling of the ion-sound (acoustic) waves usually starts with the well known set of fluid equations for electrons and ions, in a form [500]

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{V}_i) = 0, \quad (15.1)$$

$$m_i \left(\frac{\partial}{\partial t} + \mathbf{V}_i \cdot \nabla \right) \mathbf{V}_i = -\frac{\nabla P_i}{n_i} + e \mathbf{E}, \quad (15.2)$$

$$0 = -\frac{\nabla P_e}{n_e} - e \mathbf{E}, \quad (15.3)$$

$$\frac{\partial P_i}{\partial t} + \mathbf{V}_i \cdot \nabla P_i + \gamma_i P_i \nabla \cdot \mathbf{V}_i = 0, \quad (15.4)$$

$$\nabla \cdot \mathbf{E} = 4\pi e(n_i - n_e), \quad (15.5)$$

where n is density, $\mathbf{V}$ is velocity, P is pressure, and the subscript “i” and “e” refer to positive ions and negative electrons, respectively. In above equations the electron inertia term (m_e) is ignored and because the electron thermal velocity is much larger than the ion-sound speed, the isothermal electrons ($\gamma_e = 1$) are used. Initially, the electron temperature is uniform in space, which allows to keep relation $\nabla P_e = T_e \nabla n_e$, for all time. The dispersion relation for ion-sound waves is readily obtained as

$$\omega^2 = \frac{k^2 C_{se}^2}{1 + k^2 \lambda_{de}^2} + k^2 C_{se}^2, \quad (15.6)$$

$$C_{se}^2 = \frac{T_e}{m_i}, \quad C_{si}^2 = \frac{\gamma_i T_i}{m_i}, \quad \lambda_{de} = \frac{C_{se}}{\omega_{pi}}, \quad (15.7)$$

by linearizing the above system of equation for small density perturbations and weak Landau damping in non-isothermal plasmas, $\frac{T_e}{T_i} \gg 1$.

15.5 Electrostatic Particle-In-Cell Code

We shall employ a one-dimensional electrostatic Particle-in-Cell (PIC) code based on well-known es1 code [362]. The ion and electron ensembles are initialized in the position – velocity space by the quiet start procedure assuming the Maxwellian velocity distribution for both species. The value of the particle charge densities are sampled by the first order weighting smoothness or the cloud-in-cell (CIC) procedure [362]. This procedure significantly reduces noise in the system. The new mesh is built of n_x grids (in x direction). The charge density is used for determination of the electrostatic field on the corresponding mesh through the Poisson equation

$$E = -\frac{\partial\phi}{\partial x}, \quad (15.8)$$

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{\rho}{\epsilon_0}, \quad (15.9)$$

where ρ is the charge density and E is the electric field. In numerical calculations the Poisson equation is solved by the fast Fourier transform. The field values at particle positions are determined by the interpolation procedure. The particle equations ($i = N$ equations) of motion

$$m \frac{dV_i}{dt} = F_i = q_i E, \quad (15.10)$$

$$\frac{dx_i}{dt} = v_i, \quad (15.11)$$

are replaced by the finite-difference equations and solved by the leap-frog method [12]. All quantities are normalized to the ion scale: the ion plasma frequency, the ion Debye length, and the ion-sound wave period.

Further one, the set of parameters (normalized) in this illustration is $L = 1.2$, $dt = 0.0001$, $N_e = N_i = N = 252144$, $\omega_{pe} = 5091$, $q_e/m_e = -1.0$, $v_{te} = 42.5$, $\omega_{pi} = 120$, $q_i/m_i = 0.00056$, $v_{ti} = 0.22$, $nx = 512$, the particle drift velocities are initially taken as zero. Note that above parameter set corresponds to the similar parameter set as in [500]. The macro (coarse) observables are the total energy and the particle densities. Their values obtained by the 1D ES PIC code and EFMPI procedure are compared and used to estimate the possible benefits of this novel approach.

15.6 Primal Macro-Projective Simulation Method

As noted, recently, Shay et al. made the first application of EFMPI to plasmas [500]. This application studies the propagation and steepening of 1D ion sound waves using a particle-in-cell (PIC) code as a microscopic simulator. First, to initialize the PIC, the macro (coarse) variables are lifted to a fine microscopic representation. The PIC code is stepped forward for a short time, and kinetic results are restricted and smoothed back to macro (coarse) space. The time derivatives are estimated by numerical extrapolation, and coarse variables are projected in large steps. The process is then repeated. Originally in [500], the macro-step forward in time was performed using only the first three moments of the ion velocity probability density function (PDF). It is claimed that EFMPI can reproduce PIC results, but large differences arise due to the physics assumptions made in the lifting algorithm (macro to micro). In particular, it was assumed that the ion PDF is Maxwellian and that electrons are adiabatic. A more recent suggestion has been to generalize Shay's projective integration scheme [500] to remove the restriction on the velocity PDF. It was

proposed to estimate the joint $(x - v_x)$ phase space PDF with a nonlinear wavelet approximation [501]. A limited number of wavelet coefficients, which represent the coarse grained structure of the joint PDF, are introduced as the macroscopic observables. More recently, some of these authors proposed a new EFMPI method, based on the marginal and conditional cumulative PDFs as macro-scale (coarse) observables, with the potential of representing nonlinear and kinetic plasmas, tested on the ion sound paradigm [504]. Here, as an illustration aimed at exploring the feasibility of the EFMPI method, we implement an original scheme, called, primal-EFMPI to account for nonlinear and kinetic plasma effects [502]. The basic platform for micro-simulation is the standard 1D Electrostatic PIC. Still, the working hypothesis is that the separation of scales allows us to assume that the ion dynamics is inherently coarse grained, or macroscopic, compared to electron micro-scale motion. Hence, individual ion orbits are tracked and extrapolated in time to make the projection. In contrast to the original EFMPI [500], here, ions are not restricted via three PDF moments. Rather, ions are kept as they are, i.e., theoretically preserving nonlinear kinetic effects fully. Indeed, by inspecting ion orbits for an ensemble of test particles, we found that most projected individual orbits agree well with the original PIC prediction. We also note that a typical coarse projection time step (Δt_p) of, e.g., 100 times the micro-step (Δt) is still close to the intrinsic ion time step. Further, we find a non-uniform ion density from the projected ion orbits, whereas to lift ions we actually just restart the ion motion. Next, we track the electric potential and coarse grain, i.e., the average over the electron plasma period ($2\pi/\omega_{pe}$) to smooth micro-scale fluctuations in order to extrapolate and project. However, a simplifying (adiabatic) approximation for electrons is not used. Instead, we self-consistently solve for nonuniform electron density from the Poisson equation and use corresponding projected values for potential and ion density. Strictly, above procedure in solving the coarse grained Poisson equation appears to depart from exact EFREE methodology by borrowing features based on the HMM procedure, where both micro and macro-scale models are utilized (vide supra). By sampling the electron phase space, we find the standard electron velocity PDF, which appears rather smooth and easy to interpolate and project. Finally, to lift electrons, we use two projected PDFs, called marginal DFs, representing the velocity and real space distributions (number density). We performed a number of simulation runs within the same plasma variable ranges as in [500], by varying the other p-EFMPI scheme parameters. Comparative snapshots at an earlier time $t = 0.46$ (in ion periods) are given in Fig. 15.3, to present the PIC data and the p-EFMPI code results (red curve). Snapshots of the electron (1) and ion (2) density and PDFs, electric potential, and electric field at $t = 0.46$ are provided along with ion phase space plots, PIC versus p-EFMPI (bottom right). As expected, a discrepancy appears in the potential and electric field (non-coarse, both macro- and micro-scale) as a phase mismatch, due to the interruptive nature of the p-EFMPI simulation cycle.

This stems from a difficulty in accurately reconstructing the phase relation in coherent particle dynamics, in particular, with intrinsically noisy PIC data. Our very recent optimization of the numerical scheme has improved the phase space matching. However, the smoothed variables, similar to particle density, PDFs, and even

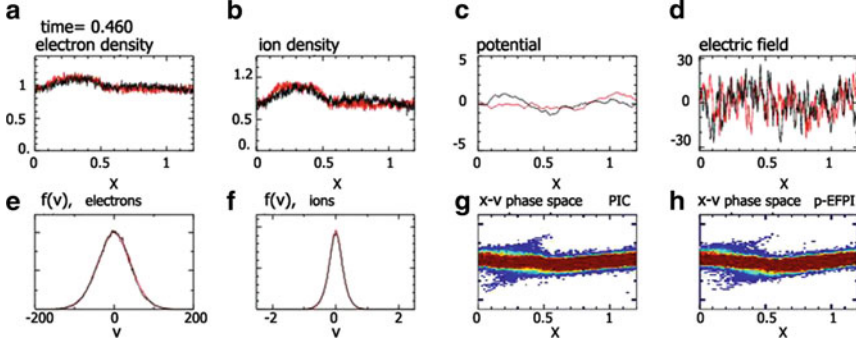


Fig. 15.3 Snapshots of (a) electron, (b) ion density, (c) electric potential, (d) electric field and (e) electron, (f) ion PDF, for PIC (red) and p-EFMPI (black) at $t = 0.46$; and ion phase space, PIC (g) versus p-EFMPI

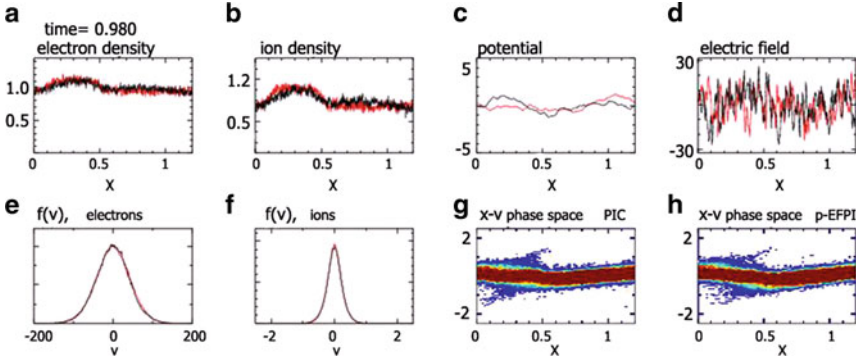


Fig. 15.4 Snapshots as above, however, at later time, $t = 0.98$

the ion phase space, compare well. Snapshots repeated at a later time, $t = 0.98$, shown in Fig. 15.4, show less agreement in the nonlinear kinetic regime. We also note that a difference in particle density defies simple electron adiabaticity. Finally, to check the important energy conservation of the scheme, Fig. 15.5 shows a plot of the comparative time evolution of the ES field energy, electron and ion kinetic and drift energies, and the total energy for PIC and p-EFMPI, in total energy units. A phase-mismatch time lead in the ion wave kinetics compared to PIC is typically observed. While the p-EFMPI projection step was modest ($20\text{--}30\Delta t$), the actual agreement with full PIC is reasonable, which gives a speed-up factor of 2 in the 1D case, or theoretically may scale as 2^3 in a 3D problem. For EFMPI feasibility as a multi-scale code, the speed-up depends on the smallest number of micro-steps (PIC) combined with the largest projection step possible. However, fundamental stability, as stated by the Courant condition, requires that $\Delta x / \Delta t > c_s$ for both micro- (PIC) and macro-projection grids, where c_s is the characteristic speed in a problem (here, the ion sound). Unexpectedly, it was found to appear as a physical effect, even if

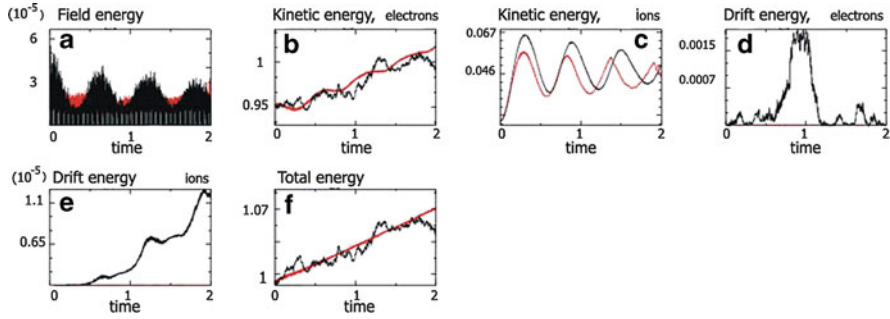


Fig. 15.5 Time evolution of electric field, particle, and total energies for PIC (red) and p-EFMPI (black)

no explicit macro-scale equation was solved [500, 502]. For a large projection time step, it is necessary to change the fine (micro) to a coarse (macro) spatial grid. However, with p-EFMPI, we maintained the original PIC fine resolution (512 points in space). Although we are in an early stage, the preliminary results seem promising. We point out that, as opposed to PIC with standard numerical heating proportional to a number of time steps, in p-EFMPI, the total energy fluctuated around the initial level. Some of the above ideas and methods concerning EFMPI, in particular, those relating to reconstructing and interlinking between macro- and micro-scale dynamic models, could be relevant to other attempts to make efficient multi-scale plasma simulations.

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