

Malte Henkel
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Theoretical and Mathematical Physics

Non-Equilibrium Phase Transitions

Volume 2

Ageing and Dynamical Scaling
Far from Equilibrium



Springer

Theoretical and Mathematical Physics

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Malte Henkel • Michel Pleimling

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Volume 2: Ageing and Dynamical Scaling
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Pour nos épouses, Maria do Rosário et Claudine

Preface

“The importance of knowledge consists not only in its direct practical utility but also in the fact the it promotes a widely contemplative habit of mind; on this ground, utility is to be found in much of the knowledge that is nowadays labelled ‘useless’.”

Bertrand Russel, *In Praise of Idleness*, London (1935)

“Why are scientists in so many cases so deeply interested in their work ? Is it merely because it is useful ? It is only necessary to talk to such scientists to discover that the utilitarian possibilities of their work are generally of secondary interest to them. Something else is primary.”

David Bohm, *On creativity*, Abingdon (1996)

In this volume, the dynamical critical behaviour of many-body systems far from equilibrium is discussed. Therefore, the intrinsic properties of the dynamics itself, rather than those of the stationary state, are in the focus of interest.¹ Characteristically, far-from-equilibrium systems often display dynamical scaling, even if the stationary state is very far from being critical. A

¹ As an example of a non-equilibrium phase transition, with striking practical consequences, consider the allotropic change of metallic β -tin to brittle α -tin. At equilibrium, the gray α -Sn becomes more stable than the silvery β -Sn at 13.2°C. Kinetically, the transition between these two solid forms of tin is rather slow at higher temperatures. It starts from small islands of α -Sn, the growth of which proceeds through an auto-catalytic reaction. The process is commonly referred to as **tin disease**. The maximal speed of the transformation is reached for temperatures lower than -30°C , with typical time-scales reaching down to hours.

It is often related that in order to save money, the buttons of the coats for the soldiers in Napoléon’s army were made of tin. When the army was *en route* to Moscow, the originally metallic β -Sn slowly transformed itself into powder in the cold Russian winter, and the buttons fell to pieces.

paradigmatic example for this is furnished by *physical ageing*, which has been recognised empirically since prehistoric times in the change of properties of materials over time, *even if ‘nothing special’ is done with the material*. Engineers then apply many tricks and twists to materials, in order to achieve certain desired properties. As theoretical physicists, we aim at a more formal characterisation and the *dynamical symmetry properties of physical ageing* will be one of the main topics of this volume. Although ageing phenomena occur predominantly in glassy systems, it has been recognised in recent years that non-disordered and non-frustrated systems can also show slow relaxations which share many properties with what is going on in glasses, albeit without being exactly identical. For this motive, we shall in this volume mainly concentrate on the ageing of such simpler systems.

The contents of this volume is presented in such a way that one may read it almost independently of Volume 1.

The material is organised into two parts, containing Chapters 1-3 and 4-6, respectively. In contrast to equilibrium and absorbing phase-transitions, it is possible to introduce physical ageing through an analysis of classic experimental results on glass-forming systems. Chapter 1 starts therefore with a phenomenological introduction, which tries to identify, from experiments on the mechanical deformation of plastics after a quench from the liquid phase to the glassy phase, the main properties of physical ageing, namely (i) slow relaxational dynamics, (ii) breaking of time-translation-invariance and (iii) dynamical scaling. We shall afterwards take these as the defining properties of our notion of *physical ageing*. One of the most simple examples for this has been found long ago in *phase-ordering* kinetics which already occurs in simple, i.e. non-frustrated and non disordered, magnets when ‘quenched’ rapidly from a very hot initial state to a low temperature, deep inside the coexistence region of the equilibrium phase diagram. We shall use this as a paradigm to discuss scaling and universal behaviour, as it is predicted by the celebrated *Allen-Cahn equation*. Afterwards, we continue with a more general discussion of the dynamical scaling, both at and below the equilibrium critical temperature. The intrinsic *non-Markovian* nature of the effective long-time behaviour, even if initially formulated, at short times, through explicitly Markovian Langevin and master equations, is made clear through an analysis of the scaling of the global persistence probability. Chapter 2 discusses a couple of exactly solvable systems. Exact solutions of non-trivial systems have throughout the entire history of equilibrium and non-equilibrium critical phenomena served as useful points of reference which allow for precise tests of more general

Some people also maintain that the provisions for the British South Pole expedition in 1910-1912 were transported in boxes soldered using pure β -tin. These became leaky, however, in the antarctic autumn temperatures of about -50°C , leading to the loss of essential fuel. This ultimately helped cost the lives of the expedition leader Sir Robert F. Scott and all the members of his crew, when on their way back from the South Pole.

frameworks. Chapter 3 gives a review on many aspects of ageing in a large variety of systems. This includes systems with a reversible microscopic dynamics and relaxing towards equilibrium steady states, for which an irreversible non-equilibrium dynamics results. For these, disordered as well as ordered initial states, the effects of a conserved order-parameter and/or a conserved energy density, frustrations and several kinds of disorder are explored. Ageing phenomena near surfaces are also described. On the other hand, we also consider the analogue of ageing phenomena in systems which are modelled by a fundamentally irreversible microscopic description, which might be a bridge towards studies of chemical ageing. The systems discussed in this book are in the steady-state universality classes as the absorbing phase-transitions studied in Volume 1, notably directed percolation. We then complete the circle by considering ageing in reversible reaction-diffusion processes and finally describe ageing effects in some simple models of surface growth. An effort has been made to collect the available values of universal exponents and amplitudes in the scaling description of ageing, widely scattered in the literature, for easy reference. On the other hand, a selection of the topics to be included had to be made, since it was not our intention, and beyond our capabilities, to achieve encyclopedic completeness.

The second part of this volume tries to introduce a broader perspective. In the next two chapters (4 and 5), we describe recent work which attempts to identify a larger dynamical structure in physical ageing than mere dynamical scaling. Initially, this undertaking was motivated by a formal analogy with the extension of scale-invariance to conformal invariance in equilibrium critical phenomena. For a large number of many-body systems at an equilibrium critical point, hence scale-invariant, it can be shown that for sufficiently short-ranged interactions (plus the rather obvious-looking requirements of translation- and rotation-invariance, at least in the continuum limit), conformal invariance follows. This suggests to inquire whether or not a similar result might be found in the context of dynamical scaling. Another motive is to see whether the classification of equilibrium universality classes might possess an analogue in dynamics. In other words, we seek an extension of dynamical scaling, for a given value of the dynamical exponent z , to a local, space-time-dependent form, which we shall call **local scale-invariance** (LSI).

A century ago, the impetus for such a programme might have been formulated as follows:

“La science est édifée sur des faits comme une maison est construite avec des pierres. Mais une collection de faits n’est pas plus de la science qu’un tas de pierres n’est une maison.”

Henri Poincaré, *La science et l’hypothèse*, Paris (1902)

Undertakings of this kind can be disappointingly difficult and tedious, as has been beautifully formulated as follows:

“O esforço é grande e o homem é pequeno. [...] A alma é divina e a obra é imperfeita.”

Fernando Pessoa, *Padrão*, Lisboa (1935)

One should not be dissuaded from an ambitious project by such difficulties, however:

“Man darf sich die Mühsal nicht ersparen, nicht die Vergeblichkeit scheuen, sinnvoll zu handeln.”

Benedikt Marnier, *Relationen und Perspektiven*, St. Ottilien (2007) [502]

Therefore, we describe in Chap. 4 the mathematically less difficult situation when the dynamical exponent $z = 2$. In this case, there is a well-known group of local scale-transformations, conventionally called the *Schrödinger group*. In many respects, it is quite analogous to the conformal group. However, since it is not a semi-simple group, its representations are *projective* and this is indeed an important ingredient for the physical applications, as we shall show. In particular, having projective representations will become an essential tool since the basic equations for the order parameter are *stochastic* Langevin equations, and not simply noiseless deterministic equations, as it was the case for equilibrium systems. Since the noise term in Langevin equations generically destroys any non-trivial dynamical symmetry, it is an important question how to solve this apparent paradox. The solution comes from a combination of the representation theory of the Schrödinger group and non-equilibrium field-theory and will be one of the central results in Chap. 4. However, since ageing is a non-equilibrium phenomenon, the Schrödinger group itself cannot be a dynamical symmetry group for ageing, and we shall rather have to restrict to the so-called **ageing group** by leaving out the time-translations. In consequence, this allows for a slightly more general form of co-variantly transforming scaling operator, called *quasi-primary* by analogy with conformal invariance. We shall see that the extra freedom gained will lead us to admit the possibility that the theoretically analysed quasi-primary scaling operators Φ and the associated physical observables ϕ are no longer identical (and we shall see that this ingredient is already needed in a model as simple as the one-dimensional Glauber-Ising model), in contrast to what one may have become used to from standard conformal invariance, valid at equilibrium phase-transitions. It might therefore come as a little surprise, that one might extend the symmetry group from the ageing group to a non-standard form of the conformal group in $d + 2$ dimensions. Specifically, we proceed in a way equivalent to what string theorists have recently re-discovered as a non-relativistic version of what they call the ‘holographic construction’ in the context of the AdS/CFT correspondence, by dualising the inverse diffusion constant (or ‘mass’ in our terminology), followed by the identification of the conformal symmetry. These mathematical constructions can be brought back

to physics by deriving explicit forms for the two-time response and correlation functions. Tests of these will be presented for the wide range of ageing physical systems with a dynamical exponent $z = 2$, analysed previously in the three chapters of part 1 in this volume.

Next, in Chap. 5, we describe an attempt to generalise the treatment to general values of z . Our starting point is a set of axioms, which we believe a physically sensible theory of local scale-invariance should satisfy. We arrived at these axioms by trying to isolate what we perceived as the key properties of the two cases of local scaling known around 1995, namely conformal invariance (with $z = 1$) and Schrödinger-invariance (with $z = 2$) [328].² The main new feature with respect to the cases $z = 1$ or $z = 2$ appears to be that the infinitesimal generators of local scale-transformation become generically non-local, which is technically expressed through fractional derivatives. Similarly, the scaling functions whose form is sought will now be found as solutions from certain fractional differential equations. Significantly, the non-semi-simple algebras of local scale-invariance lead to a generalisation of the Bargman superselection rule which takes the form of the conservation of an infinite set of powers of the momenta in the co-variant n -point functions. This looks to us reminiscent to what occurs in integrable relevant perturbations of $2D$ conformal field-theory (the most celebrated example of these is probably the $2D$ Ising model at the critical temperature $T = T_c$, but in an external magnetic field). Again, the theory can be tested by comparing the explicitly calculated two-time responses and correlators with explicit results from models. Regrettably, a precise comparison of experimental data with local scale-invariance has not yet been reported.

Finally, in Chap. 6, an analogous approach is presented for the description of the spatially strongly anisotropic scaling found in certain multicritical phenomena at equilibrium, referred to as *Lifshitz points*. We shall first give a modern presentation of the scaling near a Lifshitz point before describing the specificities of local scale-invariance in this context and how to test them in practise, mainly through the ANNNI model.

A book of this kind requires ingredients from several specialised fields, not necessarily closely related. To help the reader, we have provided numerous exercises meant to provide some simple worked-out examples or else to expand on more basic material, usually known to the experts. Occasionally, we give there further recent developments which will not be followed in the main text, e.g. on the conformal Galilean algebra. Their solutions or at least precise

² To this, we could now add *conformal Galilean invariance*, for which most space-time representations give $z = 1$. It was discovered slowly, and independently, during the last dozen of years or so as a physically interesting symmetry, by several research programmes in mathematical physics, including our symmetry analysis of ageing systems.

A further example of a closed Lie algebra of local scaling, with $z = \frac{3}{2}$, is briefly mentioned in Chap. 5.

hints are given in a special section at the end of this volume. Further background material is presented in several appendices – in particular we give in appendix A (expanded with respect to Volume 1) precise information on the equilibrium critical behaviour of the models whose ageing is studied in this volume. In order to read this book, some previous exposure to scaling and universality at phase-transitions and the basic notions of the renormalisation-group would be helpful; the necessary background can be found in standard textbooks such as [743] or else is presented compactly in Chap. 2 in Volume 1, including what we think will be sufficient here to know about conformal invariance. Knowledge of surface critical phenomena at equilibrium will be an useful analogy in the sections treating ageing or Lifshitz points near surfaces and can, again, be found in Chap. 2 in Volume 1. Background on absorbing phase-transitions will merely be required in those sections which discuss the ageing of directed percolation and the non-equilibrium kinetic Ising model. Certainly, many fascinating topics have not been included, which of course we regret. However, we shall not give here an apologetic list of topics which might have been included as well, at the expense of seeing this volume growing even more in size. By decision, this is a volume on the dynamics of classical particles and quantum effects are not considered here.

In writing this volume, we have enjoyed the support of many friends and colleagues, be it through scientific collaboration, direct advice or some other help. We gratefully thank all of them, whether or not their contribution becomes explicit here. It is a pleasure to express our gratitude to F. Baumann, B. Berche, Ch. Binek, A.J. Bray, C. Chatelain, R. Cherniha, L.F. Cugliandolo, X. Durang, S.B. Dutta, J.-Y. Fortin, M. Ebbinghaus, V. Elgart, T. Enss, A. Gambassi, C. Godrèche, H. Grandclaude, H. Hinrichsen, D. Karevski, R. Kenna, S. Lübeck, J.-M. Luck, M. Lücke, C. Maes, S.N. Majumdar, D. Minic, R. Monasson, D. Mukamel, J.D. Noh, H. Park, M. Passens, A. Picone, I.R. Pimentel, J. Ramasco, J. Richert, A. Röhlein, C. Roger, P. Ruelle, C.A. da Silva Santos, M.A.P. Santos, G.M. Schütz, B. Schmittmann, U. Schollwöck, R. Schott, S. Stoimenov, U.C. Täuber, L. Turban, E. Vincent, C. Wagner, J. Unterberger, R. K. P. Zia, J.-B. Zuber.

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for warm hospitality. Special thanks are due to H. Hinrichsen and S. Lübeck for their contributions in our joint effort in writing the first volume.

As seen in the first volume, we can but confess our inability to keep track of every single contribution to the exploding literature. The references included are those we needed in writing this volume and we sincerely apologise to any authors who might feel that we might have covered their important contribution inadequately.

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Nancy and Blacksburg,
February 2010

Malte Henkel
Michel Pleimling

Notations:

We use the same notations as in Volume 1 in order to characterise the asymptotic behaviour of physical quantities. Specifically:

$$f(x) \propto g(x) \iff \frac{f(x)}{g(x)} = c, \quad \forall x \quad (\text{proportional}),$$

$$f(x) = g(x) + O(x^n) \iff \left| \frac{f(x) - g(x)}{x^n} \right| < c, \forall x > x_0 \quad (\text{order of}),$$

$$f(x) \sim g(x) \iff \lim_{x \rightarrow x_c} \frac{f(x)}{g(x)} = c \quad (\text{asymptotically proportional}),$$

$$f(x) \simeq g(x) \iff \lim_{x \rightarrow x_c} \frac{f(x)}{g(x)} = 1 \quad (\text{asymptotically equal}).$$

Usually, the mathematical limit $x \rightarrow x_c$ corresponds to the physical situation that a phase transition is approached (usually $x \rightarrow 0$ or $x \rightarrow \infty$).

For the specification of numerical estimates and their error bars, we use the standard bracket notation $x(y)$, where y denotes the expected statistical error in the last digit. For example, the estimate $0.2765(3) = 0.2765 \pm 0.0003$ means that the true value is expected to be between 0.2762 and 0.2768.

Our notation distinguishes between d -dimensional space vectors $\mathbf{r} \in \mathbb{R}^d$ and n -component order parameters $\vec{\phi} \in \mathbb{R}^n$ (often subject to the constraint $|\vec{\phi}| = 1$).

We always use units such that the Boltzmann constant $k_B = 1$.

In mathematical statements, ‘iff’ abbreviates ‘if and only if’.

References to chapters, sections and equations are always understood to refer to this volume, unless the relationship with volume 1 is indicated explicitly.

For easy reference, frequently used symbols and abbreviations are listed on pages 487 and 488.

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Chapter 1

Ageing Phenomena

“Que serait un fait sans un souffle d’idée ?”

Marcel Otte, *Cro Magnon: aux origines de notre humanité*, Paris (2008)

The concept of ‘ageing phenomena’ colloquially refers to a change of properties of materials over time, even without any apparent forces acting on them. Ageing arises when the relaxation processes of a system, brought out of equilibrium by a sufficiently rapid change of its thermodynamic state variables, are governed by large fluctuation effects which prevent a rapid return to the stationary state. This may happen quite independently of whether the equilibrium state of the system is itself at an equilibrium critical point or not. Since glassy systems furnish many paradigmatic examples of ageing behaviour, we shall begin with a phenomenological description of mechanical ageing in polymer glasses, use this as a motivation to formulate a generic scaling description (essentially adapted to the description of non-glassy systems) and review results for non-equilibrium exponents and scaling functions. Although we shall initially use experimental information from glassy systems to enter the subject, the focus of this volume will nevertheless be on the ageing in non-glassy systems, which we hope to be more accessible to theoretical analysis, especially from the perspective of understanding their dynamical symmetries.

1.1 Introduction

For a long time, ageing effects had the reputation of depending strongly on the sample history and by implication being essentially irreproducible. In groundbreaking work, Struik’s classic experiments [684, 687] showed that the ageing in the mechanical properties can indeed be characterised in a fully reproducible way. Furthermore, he identified many universal aspects in ageing phenomena. In this section, devoted essentially to a qualitative discussion, we begin by recalling some of his findings. The scaling description motivated by

these experiments will then be cast in a form useful for the discussion of ageing in non-glassy systems. The intrinsic non-equilibrium nature of the ageing phenomenon will be discussed through the breaking of typical equilibrium properties, such as Kubo's fluctuation-dissipation theorem [454]. In Sect. 1.2, we shall look at the coarsening process as it occurs in the phase-ordering kinetics of a simple ferromagnet quenched into its phase-coexistence regime as an example where non-equilibrium dynamical scaling and universality naturally arise. We then take up the phenomenology of ageing in simple magnets, characterised by a simple time-dependent length-scale $L = L(t)$, which for sufficiently large times is assumed to scale as $L(t) \sim t^{1/z}$, from which one defines the **dynamical exponent** z . In Sect. 1.3, several rigorous results are derived, before analysing in Sect. 1.4 the scaling behaviour of several kinds of response functions as they may arise in simulations and/or experiments. In Sect. 1.5 we list results for the values of the non-equilibrium exponents found in ageing phenomena of simple magnets. Finally, in Sect. 1.6 we discuss large fluctuations of the global order-parameter via its persistence probability in order to demonstrate the presence of important memory effects in the ageing of simple, non-glassy magnets.

1.1.1 Ageing in Mechanically Deformed Polymers

It has been observed since prehistoric times that the properties (mechanical, electric, magnetic, ...) of many materials change over time even without any apparent forces acting on them. In many instances this comes about from slow reactions with the surroundings (thermal degradation, photo-oxidation, etc.) which are known as *chemical* ageing and are not studied in this volume. We shall rather consider a different type of behaviour which is called **physical ageing**. It is convenient to introduce the phenomenon through the classical experiments performed by Struik [687] in his study on the slow dynamics of certain glass-forming systems. He chose certain polymeric glasses, e.g. PVC. When cooling down a glass-forming liquid the system first remains in thermal equilibrium until the temperature T is reduced below some characteristic 'glass transition' temperature T_g below which the system's evolution slows down so much that it *falls out of equilibrium*. We shall give a precise explanation of the meaning of this later; for the moment, let it be sufficient to note that when approaching T_g from above, the viscosity of many glass-forming materials increases by over 10 orders of magnitude until the response of the material becomes so slow that it is experimentally no longer detectable [18]. Below T_g , there is a wide range of temperatures characterised by the slow evolution of the material towards some stationary state and where physical ageing occurs.¹ Many properties of glass-forming systems, e.g. their small-strain mechanical properties, change considerably as a function of the *ageing*

¹ For example, for PVC this is in the range from about -50°C to $+70^\circ\text{C}$.

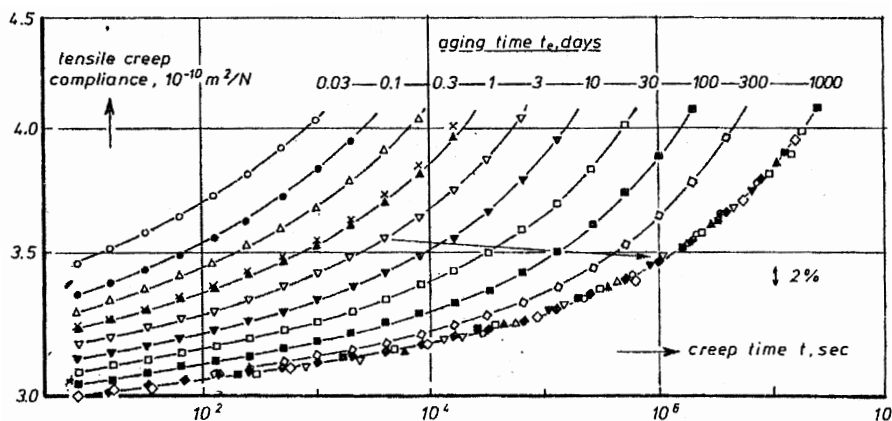


Fig. 1.1. Creep curves of rigid PVC quenched from 90°C to 20°C. After the ageing time t_e has passed (with values $t_e = [0.03, 0.1, 0.3, 1, 3, 10, 30, 100, 300, 1000]$ days from left to right), a small constant stress was applied and the compliance $J(t)$ was measured as a function of time t , in the range from 10^0 to 10^8 seconds. Reprinted from [687] with permission from Elsevier.

time. Being able to predict their long-time changes from short-time experiments would be of obvious practical value. Indeed, ageing is one of the basic features of the glassy state and has been observed in materials as distinct as polymers, colloidal gels, molecular glasses, spin glasses and glassy systems. We shall detail below that very similar behaviour is also found in non-disordered, non-frustrated simple systems such as simple ferromagnets. Furthermore, as we shall illustrate shortly, ageing proceeds in all glassy materials (and also non-glassy systems, see later) in an essentially similar way. This means that the ageing behaviour is not so much determined by the material's particular chemical composition but rather from the fact that the material was brought to its glassy state – which for the theorist kindles the hope to be able to explain ageing phenomena in a simple way.

The equilibrium response of a many-body system to an external field has been described in Sect. 2.4 in Volume 1. Here, we are interested in the response of non-equilibrium systems. The basic experimental set-up is as follows:

1. Prepare the system in a high-temperature state.
2. Lower the temperature ('**quench**') to a value low enough such that several equilibrium states exist and perform the quench rapidly enough such that the system is forced out of equilibrium.
3. Fix the temperature and observe the evolution of the system, for example its response in reaction to an external stimulation.

In Fig. 1.1 we show the mechanical response to an applied stress of a glass-forming PVC, rapidly cooled or quenched from the liquid phase into its glassy state [684, 687].² Because of the quench, the system is brought out of equilibrium and we are considering here a non-equilibrium relaxation phenomenon. The non-equilibrium nature of the observed behaviour is seen as follows. After having waited after the quench for a time t_e , a small, constant mechanical stress is applied and the linear response (‘compliance’) is measured through the creep curve as a function of the time t since the stress was applied. In Table 1.1, reproduced from Volume 1, the terminology used in these experiments and their analogies in magnetic and dielectric relaxations is listed. Clearly, the compliance changes slowly, provided that the experimentalist has the necessary patience to see it happen! This observation of a *slow* change of the material’s properties is the first characteristic of ageing. Furthermore, and in contrast with the linear response around an equilibrium system as discussed in Chap. 2 in Volume 1, for different values of t_e quite different t -dependent curves are found. Hence the compliance depends on *both* the ageing time t_e and the measuring time t . Together with a slow dynamics, such a *breaking of time-translation-invariance* is a second characteristic for ageing. Third, one observes that the individual curves for fixed values of t_e can be superimposed onto a single *master curve* by shifting them almost horizontally in Fig. 1.1, as indicated by the arrow. This is a clear indication of a **dynamical scaling** behaviour underlying the ageing process. We summarise this by the following **Definition:** *A physical many-body system is said to undergo ageing if the relaxation process towards its stationary state(s) obeys the properties:*

1. *slow dynamics (i.e. non-exponential relaxation)*
2. *breaking of time-translation-invariance*
3. *dynamical scaling*

This volume presents a detailed discussion of these properties, especially dynamical scaling and its possible generalisations. We add that it seems useful to insist on the simultaneous presence of all three characteristics for the definition of ageing. Indeed, the state of any system not in a stationary state will not be invariant under time-translations and dynamical scaling together with time-translation invariance occurs at *equilibrium* critical points. It is conceivable that different classes of ageing systems are distinguished by different types of dynamical scaling.

A further important aspect of **physical ageing** is its **thermoreversibility**. To test this, the above PVC sample [687], after having measured its ageing for over 1000 days, was reheated for 20 minutes to 90°C, well above the glass

² For the theorist, an infinitely rapid quench is trivially carried out by assigning a different value to the bath temperature T . Experimentally, temperature changes can only be performed with a *finite* rate. Also, it is not obvious that the entire sample should have exactly the same temperature as the external bath.

symbol	magnetic	dielectric	an-elastic
force	magnetic field $\mathbf{H}_z$	electric field $\mathbf{E}_z$	stress tensor $V\sigma_{zz}$
conjugate operator	magn. moment $\boldsymbol{\mu}_z$	el. dipole moment $\mathbf{d}_z$	strain ε_{zz}
interaction Hamiltonian	$-\mathbf{H}_z \cdot \boldsymbol{\mu}_z \cos \omega t$	$-\mathbf{E}_z \cdot \mathbf{d}_z \cos \omega t$	$-V\sigma_{zz}\varepsilon_{zz} \cos \omega t$
macroscopic observable	magnetisation M	polarisation P	an-elastic strain ε_{an}
response function	magnetic response	polarisation response	creep function
relaxation function	magnetic aftereffect		elastic aftereffect
alternating susceptibility	magn. susc. $\chi(\omega)$	permittivity $D(\omega)$	compliance $J(\omega)$
power absorbed	$\chi''(\omega)$	$D''(\omega)$	internal friction $F(\omega)$

Table 1.1. Glossary for magnetic, dielectric and an-elastic relaxation, listing quantities corresponding to the forces, their canonically conjugate operator and the associated macroscopic observables, along with response and relaxation functions and the alternating susceptibility.

transition point T_g , before quenching it again to 20°C and measuring its ageing afresh after a waiting time t_e of 1 day. The results are shown by crosses in Fig. 1.1. Clearly, the reheating to the liquid phase has erased all memory of the previous ageing procedure. This thermoreversibility, which has been found for all glassy systems undergoing physical ageing, is of course absent in processes of chemical/biological ageing and may serve as a distinguishing criterion between these two distinct types of ageing. In this way, physical ageing shows reproducible features which in view of the breaking of time-translation invariance might have been unexpected.

We briefly pause to illustrate a different aspect of the ageing in spin glasses which becomes apparent in **temperature-cycling experiments**. The classic example is obtained by performing an experiment using the **Kovacs protocol** [451]: quench the system from an initial high-temperature state to a low temperature $T_1 < T_c$ and let it evolve with T_1 fixed until a time t_1 , say, its time-dependent order-parameter $\langle \phi(t; T_1) \rangle = \langle \phi(T_2) \rangle_{\text{eq}}$ arrives at its *equilibrium* value at a different temperature T_2 , with $T_1 < T_2 < T_c$. Then heat the system instantly to the temperature T_2 and observe the temporal evolution of the time-dependent order-parameter $m(t) := \langle \phi(t; T_2) \rangle$ for times $t > t_1$, with the initial condition thus set at time $t = t_1$. The **Kovacs effect** [451] states that $m(t)$ will *not* remain constant at its equilibrium value, but rather will display a hump, starting at the equilibrium value $m_{\text{eq}}(T_2)$ and only returning to this value after some time. Therefore, systems which display a Kovacs effect, even if the value of one observable at the temperature T_1 coincides with its equilibrium value in another heat-bath with temperature $T_2 > T_1$, and even if they are suddenly brought into contact with that heat-bath, are *not* at equilibrium.³ An even more striking example of this kind is shown in Fig. 1.2 [462, 716, 714]. The dissipative part of the alternating susceptibility $\chi''(\omega, t)$

³ This comes about if the value of at least one distinct observable at time t_1 and temperature T_1 is unequal to its equilibrium value at temperature T_2 . See exercise 2.19 for an analogue using a magnetic field as control parameter.

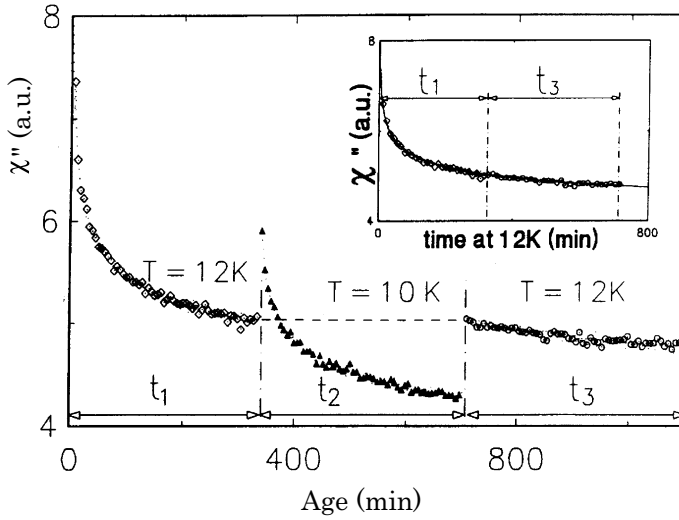


Fig. 1.2. Memory effect in the dissipative part $\chi''(\omega, t)$ of the alternating susceptibility ($\omega = 0.01[\text{Hz}]$) in the spin glass $\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$ through a temperature-cycling experiment. Times are in minutes since the quench. The inset shows the relaxation with the intermediate time-interval spent at $T = 10[\text{K}]$ eliminated. Reprinted from [714].

is observed as a function of the time t for the spin glass $\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$. In the first step, one quenches the system from the high-temperature phase to the temperature $T = 12[\text{K}] < T_c = 16.7[\text{K}]$. After having kept the system at this temperature for a time t_1 , the temperature is suddenly shifted to the lower value $T = 10[\text{K}]$ and again a relaxation process is seen. Remarkably, if after a time t_2 , the temperature is increased again to $T = 12[\text{K}]$, the relaxation process returns essentially to where it had been before the temperature-cycling, as is clearly demonstrated in the inset in Fig. 1.2. It appears that the spin glass is capable of keeping a memory of the state it had reached before the thermal cycling started. This phenomenon is known as the **memory effect**. For a more detailed review of the fascinating rejuvenation and memory effects one can observe, e.g. in ageing spin glasses, see [714] and references therein.

Returning to the ageing effects in the mechanical responses of polymers, we now discuss a key aspect. Namely, the physical ageing phenomenon as described so far apparently shows **universality** between the mechanical response of quite different materials. This is illustrated in Fig. 1.3, where the master creep curves, obtained for several materials with different thermal ranges of their ageing regime, are superposed [687]. The full curve gives a fit according to the relation $J(t) = J_0 \exp[(t/t_0)^{1/3}]$. It is not at all anticipated

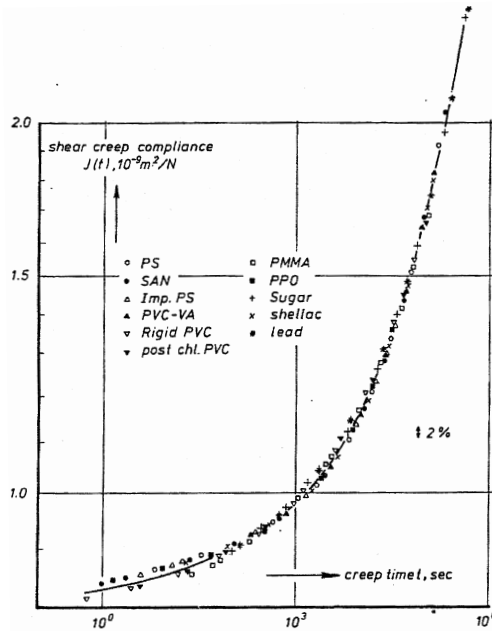


Fig. 1.3. Mechanical response of several materials: master creep curve as an example of universality. Reprinted from [687] with permission from Elsevier.

that the mechanical ageing of materials as distinct as lead and PVC should be describable in terms of a common scaling law. Observations of this kind are good news indeed for the theorist, who is relieved from the sort of nightmarish task to understand a peculiar, history-dependent phenomenon and can now rather afford to think in terms of reproducible, universal scaling functions. Clearly, the shape of these scaling functions can only depend on a few very global characteristics of ageing systems, in this respect quite analogous to the universal scaling functions at stationary-state (or equilibrium) phase transitions. In attempting to explain such a finding, the thought of some underlying dynamical symmetry is suggestive. We stress that in this discussion we focussed exclusively on the dynamical behaviour and disregarded completely whether the ‘equilibrium state’ of these materials were critical or not. In particular, dynamical scaling of the kind just described is found for a large range of temperatures (or other external control parameters) and there is no need to fine-tune certain parameters in order to observe dynamical scaling. In this respect, ageing phenomena are different from the scaling behaviour found at equilibrium or at the critical point of non-equilibrium steady states and studied in chapters 2–5, in the first volume.

In this second volume, we shall concentrate on the scaling properties of ageing systems but leave the complex, interesting and controversial physics of

the ‘glass transition’ aside. Detailed information on that can be found e.g. in [447, 166] and in a recent lecture notes volume [351].

Of course, engineers have worked on ageing phenomena essentially since prehistoric times. According to [103], one may distinguish matter from **materials** by stating that “... *matériaux: une matière qui doit remplir une fonction.*” (materials: matter which must have a function.) Among the mechanisms which may damage a material [103], one includes those which are reversible, at least in principle and on a microscopic scale: (i) *fluage*: deformation under stress, (ii) *fatigue*: alternating stress, leading to rupture. In this volume, we shall be essentially concerned with these and their macroscopically irreversible consequences, albeit in the language of magnetic systems, see the glossary in [Table 1.1](#). Furthermore, in materials science/engineering one considers irreversible modifications of materials, including the following: (iii) *usure*: consecutive to contacts in a system of given geometry, (iv) *corrosion*, (v) *irradiations*, (vi) *chemical or mechanical degradation*. In the terminology of this volume, this should probably be described by explicitly time-dependent interactions in the models and processes of such a kind will not be considered here. The scaling description of ageing, as they will be developed in this volume, emphasises the collective aspects of ageing, based on the properties of mesoscopic models, and allows in principle the extrapolation from relatively short-time data to much longer time-scales, and without having to destroy the sample.⁴

1.1.2 Correlations and Responses

It is accepted folklore that **spin glasses** may be characterised by the combined effects of **disorder** and **frustration** [518, 530], see appendix A. For the sake of notational simplicity, we shall largely consider ferromagnetic systems with neither disorder nor frustrations which should be more simple to understand than glass-forming systems. The differences in the ageing behaviour of glassy systems and simple magnets have recently been discussed in [140, 139] through the study of spatial-temporal fluctuations out of equilibrium. It has been argued that the observed difference in fluctuation pattern in ferromagnets and in glassy systems is due to more restricted dynamical symmetries in the former as compared to the latter. For a deeper discussion of dynamical scaling and dynamical symmetries, which are central questions studied in this volume, a more precise terminology is needed which we now introduce. We denote the value of the **order-parameter** (magnetisation) at time t and at the space point $\mathbf{r}$ by $\phi(t, \mathbf{r})$. The breaking of time-translation invariance as seen above suggests that the definition of two-time quantities might be a convenient tool for the study of ageing. Consider the two-time **correlation function**

⁴ A typical practical question with respect to ageing would be: what is the best moment to change a piece of equipment ? The ‘obvious’ answer: just before it breaks [103]!

$$C(t, s; \mathbf{r}) := \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{0}) \rangle - \langle \phi(t, \mathbf{r}) \rangle \langle \phi(s, \mathbf{0}) \rangle \quad (1.1)$$

of the order-parameter $\phi(t, \mathbf{r})$ at time t and position $\mathbf{r}$, where t, s with $t \geq s$ are two times, both measured since the quench. Since we shall rapidly restrict to simple ferromagnets without disorder or frustrations, we have already assumed here spatial translation-invariance and we shall do so throughout, unless explicitly stated otherwise. For a *fully disordered initial state*, one expects $\langle \phi(t, \mathbf{r}) \rangle = \langle \phi(0, \mathbf{r}) \rangle = 0$ and we shall implicitly admit this in this book, unless stated explicitly otherwise. We also define the (linear) two-time spatio-temporal **response function**

$$R(t, s; \mathbf{r}) := \left. \frac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{0})} \right|_{h=0}. \quad (1.2)$$

where h is the external (magnetic) field canonically conjugated to the order-parameter ϕ . **Causality** requires that $R(t, s; \mathbf{r}) = 0$ for $t < s$. We point out immediately that, since the system is far away from equilibrium, C and R are *not* related by a fluctuation-dissipation theorem, see eq. (2.125) in Volume 1, and must be studied independently from each other. We shall return to this below.

In ageing systems, two-time quantities such as $C(t, s; \mathbf{r})$ or $R(t, s; \mathbf{r})$ generically depend on *both* times t and s . Usually, s is called the **waiting time** while t is called the **observation time**.⁵ In many situations, it is convenient to work with the **autocorrelator** (or **autocorrelation function**) and the **autoresponse** defined through⁶

$$C(t, s) := C(t, s; \mathbf{0}) \quad , \quad R(t, s) := R(t, s; \mathbf{0}) \quad (1.3)$$

In spin glass experiments, (auto)correlation functions are difficult to measure because of a low signal/noise ratio and reliable data on $C(t, s)$ in spin glasses have only been obtained fairly recently [360, 361]. On the other hand, in numerical simulations data for $C(t, s)$ are much easier to obtain than for the (auto)response function, since the functional derivative is much affected by stochastic noise. In practice, rather than looking at the response to a short impulse, one preferentially studies a so-called **integrated response function** [40]. We make this explicit for the magnetic response. Two protocols are frequently used and differ in the time-history of the time-dependent magnetic field $h = h(t)$, see also Fig. 1.21 below.

1. In the **thermoremanent** protocol (TRM) a small field h is turned on right at the moment $t = 0$ of the quench and kept until the waiting time

⁵ In the literature, one often writes $t_w = s$ for the waiting time and we warn the reader that sometimes the time difference $\tau = t - s$ is denoted by t . If that is done, our time t would have to be re-expressed as $t_w + t$. Throughout this book, times will be measured starting at the quench.

⁶ Exercise 1.12 compares autocorrelators and -responses and their scaling forms to *global* correlations and responses.

s has elapsed. Then the field is turned off⁷ and at a later time t the so-called thermoremanent magnetisation $M_{\text{TRM}}(t, s)$ is measured – we have a relaxation experiment in the terminology of Sect. 2.4 in Volume 1.

2. In the **zero-field cooled** protocol (ZFC) the quench is performed without a magnetic field which is only turned on after the waiting time s has elapsed. The zero-field cooled magnetisation $M_{\text{ZFC}}(t, s)$ is then measured at a later time t – an example of a response experiment.

The measured time-dependent magnetisations are related to the autoresponse as follows

$$\begin{aligned} M_{\text{TRM}}(t, s) &= h\chi_{\text{TRM}}(t, s) = h \int_0^s du R(t, u) \\ M_{\text{ZFC}}(t, s) &= h\chi_{\text{ZFC}}(t, s) = h \int_s^t du R(t, u) \end{aligned} \quad (1.4)$$

where we also wrote the corresponding susceptibilities for the linear-response regime where h is small enough. The following sum rule is obvious

$$\chi_{\text{FC}}(t) = \chi_{\text{TRM}}(t, s) + \chi_{\text{ZFC}}(t, s) = \int_0^t du R(t, u) \quad (1.5)$$

where $\chi_{\text{FC}}(t) = \mathcal{R}_{\text{MM}}(t)$ (in the notation of Sect. 2.4 in Volume 1) is called the **field-cooled susceptibility** and M is the magnetisation.

1.1.3 Ageing in Spin Glasses

After these preparations, we can discuss in some more detail the ageing behaviour of a spin glass, taking as an example $\text{Ag}_{0.973}\text{Mn}_{0.027}$ [716, 212, 714]. In Fig. 1.4a, the thermoremanent magnetisation is shown. The plot of $M_{\text{TRM}}(t, s)$ over against the time difference $t - s$ and after a quench to $T = 0.87 T_c$, illustrates the dependence of $M_{\text{TRM}}(t, s)$ on *both* times and hence the breaking of time-translation-invariance. We draw attention to the fact that even for time scales up to 10^4 seconds, perceptible changes in $M_{\text{TRM}}(t, s)$ are seen. In addition, the data clearly depend on both the waiting time s and the observation time t . Clearly, the system becomes increasingly stiff with increasing waiting time s . In discussing the dynamical scaling behaviour, which in practise almost always refers to the case $T < T_c$, one often assumes [716, 166, 212] a decomposition $M_{\text{TRM}}(t, s) = M_{\text{stat}}(t - s) + M_{\text{age}}(t, s)$ into a so-called ‘stationary part’, usually taken to be a power-law $M_{\text{stat}}(\tau) = M_0(\tau_0/\tau)^\alpha$ and an ageing part $M_{\text{age}}(t, s)$ which satisfies dynamical scaling. Typical values of the parameters of the stationary part are $\tau_0 \approx 10^{-12}[\text{s}]$, the exponent $\alpha \approx 0.02$

⁷ While theorists can easily turn fields on or off (or perform quenches) instantaneously and homogeneously throughout the sample, this may be difficult to realise experimentally.

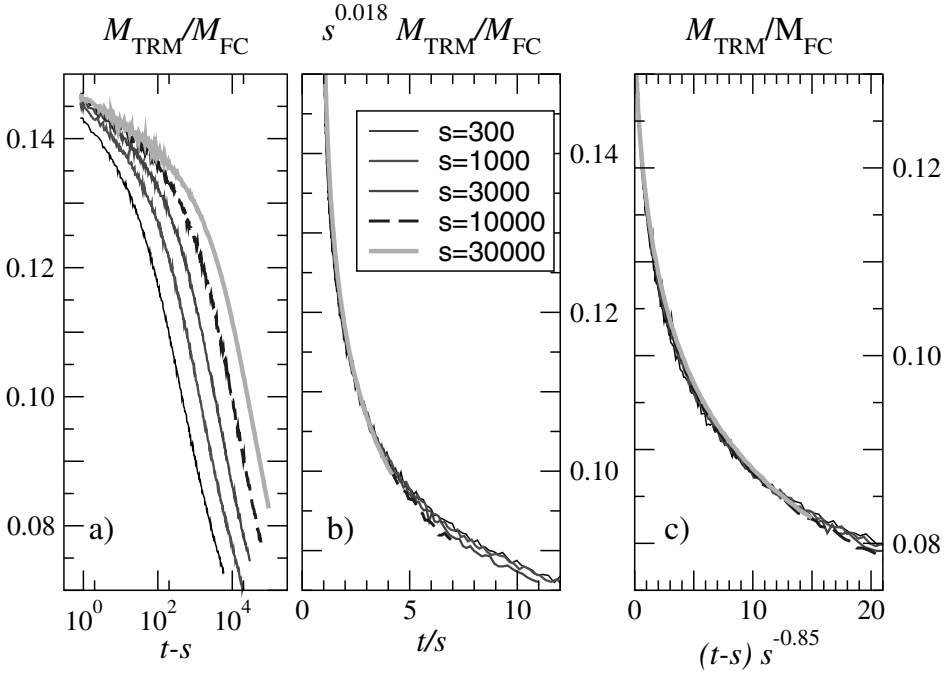


Fig. 1.4. Ageing in the spin glass $\text{Ag}_{0.973}\text{Mn}_{0.027}$, quenched to $T = 0.87 T_c$ from an initial high-temperature state. The times t and s are measured in seconds. Data courtesy M. Ocio, J. Hammann and E. Vincent.

is usually tiny and $A \approx 0.1$. In the case at hand, the stationary part only gives a minor contribution to the full thermoremanent magnetisation and we compare in Fig. 1.4b,c the data to two assumed scaling forms, namely

$$M_{\text{TRM}}(t, s) = \begin{cases} f((t-s)s^{-\mu}) & \text{subageing} \\ f(t/s) & \text{full ageing} \end{cases} \quad (1.6)$$

where $\mu < 1$ [457] is a free parameter [687, 551, 12] which may be called **subageing exponent**. In the limit $\mu \rightarrow 1$, one recovers the second scaling form (1.6) of simple or full ageing from the first and for $\mu \rightarrow 0$, there is no ageing at all. We shall come back later to the question how to motivate these scaling forms; for the moment we wish to discuss which of them, if any, is borne out by experimental data.

In the present example, the simple scaling behaviour is quite well respected by the data, see Fig. 1.4b, but the introduction of the subageing exponent as a further free fitting parameter may be used to improve the quality of the data collapse as shown in Fig. 1.4c. At present, there does not seem to exist a general consensus which of the above two scaling forms (1.6) should be preferred for spin glasses. If $\mu \approx 1$, one might believe that subageing

Material	μ	quantity	Ref.
cytoskeleton (human airway smooth muscle)	0.32	compliance	[701]
cytoskeleton (human muscle cell)	0.4	compliance	[110]
colloidal glass (PMMA)	0.48(1)	autocorrelator	[725]
	0.48(1)	ZFC-response	
polyelectrolyte microgel	~ 0.8	compliance	[150]
multilamellar vesicles	0.78(9)	compliance	[611]
	0.77(4)	intensity autocorrelation	
$\text{Fe}_{0.5}\text{Mn}_{0.5}\text{TiO}_3$	0.84	M_{TRM}	[212, 577]
	~ 1	frequency-dependent susceptibility $\chi(t, \omega)$	[214]
$\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$	0.87	M_{TRM}	[360, 361, 577]
	0.87	autocorrelator	
	~ 1	frequency-dependent susceptibility $\chi(t, \omega)$	see [714]
$\text{Au}_{0.92}\text{Fe}_{0.08}$	0.91	M_{TRM}	[577]
$\text{Ag}_{0.973}\text{Mn}_{0.027}$	0.97	M_{TRM}	see [716]
$\text{Cu}_{0.94}\text{Mn}_{0.06}$	0.999	M_{TRM}	[622]
$\text{SrCr}_{8.6}\text{Ga}_{3.4}\text{O}_{19}$	0.85	M_{TRM}	[459]

Table 1.2. Measured values of the subageing exponent μ in some materials, measured from several physical quantities. In some spin glasses, different values of μ were reported for different cooling protocols [622, 577]. We list here the largest value of μ found in any of the protocols considered.

behaviour could reflect an unrecognised correction to (simple) scaling, but for smaller values of μ it is more difficult to maintain such a point of view. Indeed, while in most spin glasses $\mu \approx 0.8 - 0.9$, considerably lower values have been reported in soft matter or from frustrated systems. For illustration, we collect in Table 1.2 some experimentally measured values of the subageing exponent. The materials range from living biological matter to soft matter, several spin glasses and finally to the frustrated, but non-disordered magnet $\text{SrCr}_{8.6}\text{Ga}_{3.4}\text{O}_{19}$. We also indicate which physical quantity was used to obtain an estimate of μ .

In several cases, the results for μ could be controlled by measuring the scaling behaviour of a further distinct quantity, besides the (integrated) linear response. While most of the results in Table 1.2 refer to the case of a constant external field, a few results coming from the response to an external oscillating field, of angular frequency ω , were also included. In almost all such cases, one finds quite a good ωt scaling behaviour, in contrast to the findings $\mu < 1$ in the same probe. Are these different results a reflection of the distinct regimes $\omega t \lesssim 1$ and $t/s > 1$ probed in these two kinds of experiment? In the case of soft/biological matter, the initially disordered state is not reached by raising

the temperature but rather through a strong shear. We also note that the observed values of μ are roughly the same for Ising and Heisenberg spin glasses, as well as for frustrated, but non-disordered systems.

While initial observations and qualitative arguments [687] suggested that μ should be close to unity, results such as collected in Table 1.2 raise the question of a possible explanation for $\mu < 1$. Indeed, in [212, 577], three main experimental limitations which may lead to $\mu < 1$ were discussed: (i) *Finite-size limitations*. In finite systems composed of many subsystems (grains), ageing will stop early in the smaller systems which would lead to an effective exponent μ decreasing with time. So far, there appears to exist no conclusive experimental proof for this. (ii) *Amplitude of the applied field*. Too strong amplitudes may cause the system to leave the regime of linear response. Indeed, it has been seen in several systems that for stronger external fields, μ decreases [611, 213, 360, 361, 714] with an increasing external field h . On the other hand, for very low fields $h \rightarrow 0$, $\mu(h)$ saturates, but usually at a value $\mu(0) < 1$ (for $\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$, fields down to 10^{-3} [Gauß] were probed [360, 361]). (iii) *Effects of slow cooling*. Since the sample cooling time is always large compared to the microscopic time scales, an instantaneous cooling is not possible experimentally. Hence the system spends some time (which is large compared to the microscopic times of order 10^{-12} [s]) at all temperatures above the final temperature T . Indeed, numerical simulations [72] in the Ising spin glass show that a slower cooling rate leads to reduced values of $\mu \approx 0.96$. Since even those slower cooling rates are still much faster than what is experimentally feasible, this result might provide an explanation for the observed values $\mu < 1$. However, Rodriguez, Kenning and Orbach [622] carefully studied the effects of different cooling protocols on the scaling behaviour and reported that in their $\text{Cu}_{0.94}\text{Mn}_{0.06}$ sample, a sufficiently rapid cooling to a temperature slightly below the desired final temperature, followed by a small reheating, raises indeed the subageing exponent to values arbitrarily close to unity. Even in this case, $\mu \approx 1$ could only be found for the temperature $T = 0.86 T_c$, while for different temperatures, fits to the subageing scheme produced values of μ slightly smaller than 1. However, their observation could not be reproduced for neither the $\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$, nor the $\text{Fe}_{0.5}\text{Mn}_{0.5}\text{TiO}_3$ nor the $\text{Au}_{0.92}\text{Fe}_{0.08}$ samples of the Saclay group, as discussed in [212, 577]. Although the Saclay group finds that a slight re-heating before turning off the field leads to a small increase in μ , they do not see any evidence that μ would go to 1 for very short cooling times.

For the time being, it remains open why and when subageing with $\mu < 1$ will hold in real materials and under what experimentally realisable conditions this might go over to simple ageing with $\mu \rightarrow 1$, if this happens at all.

Theoretically, subageing exponents $\mu < 1$ can be derived in certain trap models. We refer to the literature [87, 29, 524] for details, since this volume focuses on systems with simple ageing.

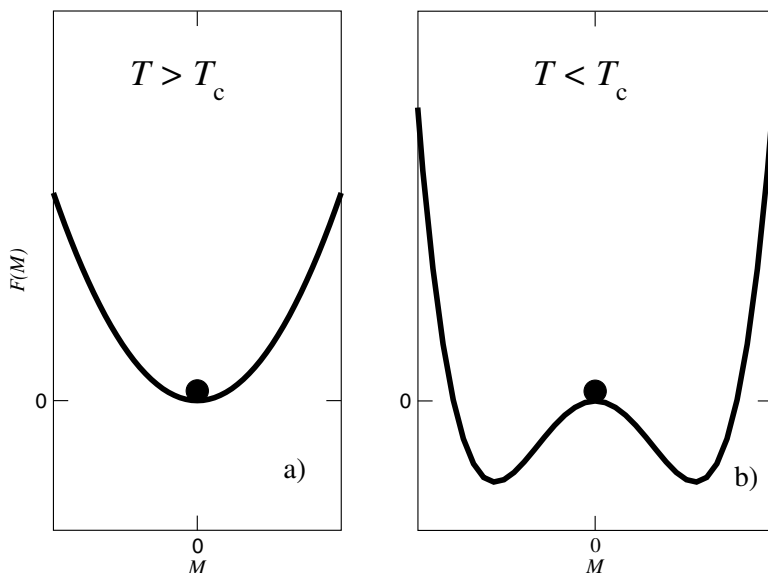


Fig. 1.5. Schematic equilibrium free energies $F(M)$ of an uniaxial magnet for (a) $T > T_c$ and (b) $T < T_c$. The ball indicates the state of the system, see text. Reprinted with permission from [594].

1.1.4 Ageing in Simple Magnets

The experimentally motivated discussion presented so far might have led the reader to presume that ageing would only occur for glassy systems. However, *simple*, non-disordered and non-frustrated ferromagnets show ageing as well. This is very fortunate, since presumably the dynamics of simple magnets should be much easier to understand than the one of complex glassy systems. This will especially become important in our quest for underlying dynamical symmetries where simple models with an exactly solved ageing behaviour will play an important rôle. As an example we consider the celebrated Ising model,⁸ in a setting which is paradigmatic for the kinetics of **phase-ordering**. Quite analogously to glassy systems, we initially prepare the system in a fully disordered state (infinite initial temperature) and then quench it to a final temperature T far below the critical temperature T_c . In Fig. 1.5 we illustrate the system's behaviour through the equilibrium free energy before and immediately after the quench. Initially, there is a single equilibrium state and the system has had enough time to relax to it, hence it is at the global minimum of the free energy as indicated by the ball in Fig. 1.5a. Directly after the quench, the state of the system has not yet evolved, but it is now at a local maximum of the free energy (see Fig. 1.5b) and should in principle

⁸ See appendix A for the definitions of the models used in this book and their equilibrium critical properties.

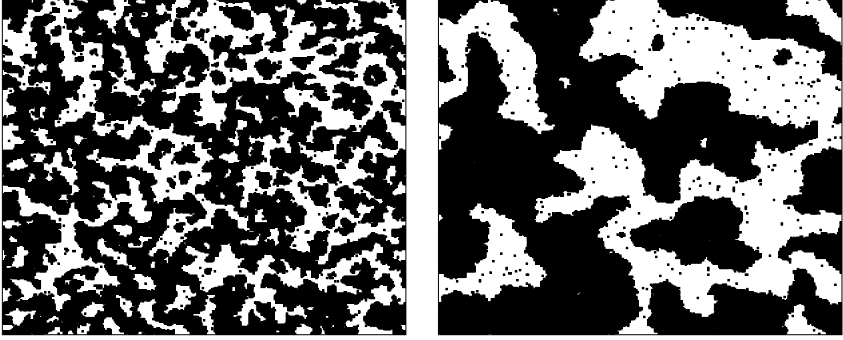


Fig. 1.6. Two snapshots of spin configurations (black and white) of the 2D Ising model at $T = 1.5 < T_c$, taken at times (a) $t = 25$ and (b) $t = 275$ Monte Carlo steps after the quench.

relax towards one of the several stable and equivalent equilibrium states. In the absence of a global magnetic field, the distinct equilibrium states are competing with each other. Locally, however, each spin variable (magnetic moment of a mesoscopic cell) is subject to the magnetic fields of its neighbours, which will induce a preferential relaxation towards one of the equilibrium states. Hence the spins will order, but only locally and the system should decompose into distinct ordered **domains**. In the interior of these domains, the state of the system does not change, but the domain walls move slowly in such a way that the typical linear size $L(t)$ increases as a function of time. This slow motion creates the slow evolution of the state of the system.

These qualitative ideas are illustrated for the 2D Ising model quenched to a temperature $T = 1.5 < T_c$ in the next two figures. In Fig. 1.6 we show two snapshots of the local state of the system, at two different times t after the quench. It can be seen that already shortly after the quench, ordered domains have formed, the size of which is slowly increasing with time. In Fig. 1.7 we show the magnetic autocorrelator $C(t, s) = \sum_{\mathbf{r} \in \mathbb{Z}^2} \langle M(t, \mathbf{r}) M(s, \mathbf{r}) \rangle$, where $M(t, \mathbf{r})$ is the time-dependent magnetisation of the cell at location $\mathbf{r}$. Clearly, $C(t, s)$ shows a slow evolution and does not relax within some finite time to a stationary value. Furthermore, it does not merely depend on the time difference $\tau = t - s$, see Fig. 1.7a. In addition, when the same data are re-plotted as a function of t/s , a clear collapse results, provided the waiting times are sufficiently large. This is shown in Fig. 1.7b. Hence the three defining conditions for ageing, see p. 4, are satisfied.

In summary, we have observed some aspects of ageing which will be very important for our attempts to describe and to understand this phenomenon.

1. Ageing may arise when a many-body system is rapidly brought out of its equilibrium state and into a coexistence region of its equilibrium phase diagram, such that there are at least two stable, equivalent and competing

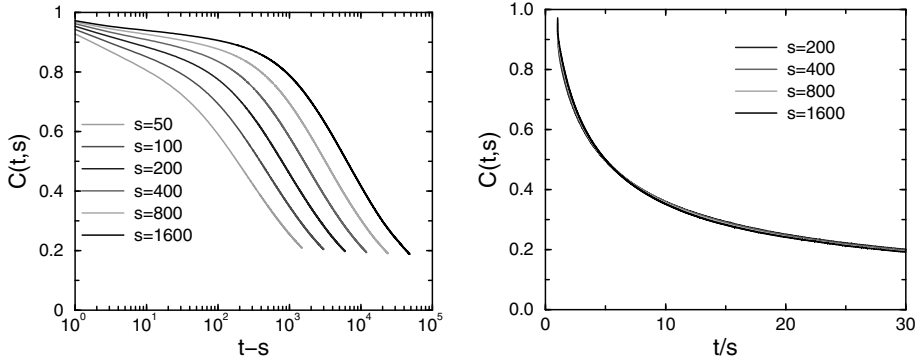


Fig. 1.7. Ageing in the 2D Ising model quenched to $T = 1.5 < T_c$. In (a) the magnetic autocorrelator $C(t, s)$ is shown as a function of $t - s$ for several waiting times s and in (b) the collapse of these same data is shown when re-plotted as a function of t/s .

equilibrium states where it can relax to. For simple magnets, the conceptually most simple way of doing this might be to quench the temperature from a very large initial value $T_{\text{ini}} \gg T_c > 0$ to a value $T < T_c$.

2. Ageing may also arise when a many-body system is quenched onto its critical point, i.e. by rapidly changing its temperature to $T = T_c > 0$. In contrast to the case $T < T_c$, here the slow dynamics of a many-body system *at* its critical point generates the ageing behaviour.
3. Time-translation-invariance is broken: at least one multi-time observable does not only depend on the time differences.
4. There is slow, non-exponential dynamics. Independently of the properties of the individual stationary states, one observes dynamical scaling.
5. Often the formation of ordered (or correlated) domains with a time-dependent length-scale $L(t)$ occurs.

In this book, we shall almost exclusively consider systems with a power-law growth of the size of the ordered domains, for sufficiently large times

$$L(t) \sim t^{1/z} \quad (1.7)$$

which defines the **dynamical exponent** z . Eq. (1.7) is satisfied for either **phase-ordering** or **phase-separation** (i.e. quenches to $T < T_c$ with a non-conserved or a conserved order-parameter, respectively) or else the **non-equilibrium critical dynamics** (i.e. quenches to $T = T_c$) of simple magnets, in simple models of surface growth or in reaction-diffusion systems, but it apparently does *not* hold in many glassy systems (for example spin glasses

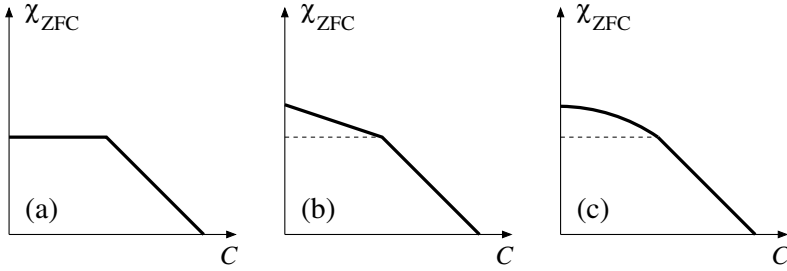


Fig. 1.8. Schematic relationship between the integrated response χ_{ZFC} and the autocorrelator, which illustrates the breaking of the fluctuation-dissipation theorem. In the quasi-stationary regime, represented by the straight line of slope $-1/T$ in each panel, the fluctuation-dissipation theorem is still valid. If the ageing regime is reached, the system departs from the fluctuation-dissipation theorem: (a) pure magnets quenched to $T < T_c$, (b) pure magnets quenched to $T = T_c$ or structural glasses at $T < T_c$, (c) spin glasses quenched to $T < T_c$.

quenched to $T < T_c$, where the relevant length scale grows more slowly than algebraically with time).⁹

The rapid quench onto a critical point or into a coexistence phase moves the system far away from its equilibrium state(s). As a consequence, correlations and responses are no longer related by a fluctuation-dissipation theorem, as it holds *at equilibrium* and discussed in Volume 1, Chap. 2. At equilibrium, both two-time responses and correlators only depend on the time difference $\tau = t - s$ and are related by Kubo's **fluctuation-dissipation theorem** eq. (2.125) in Volume 1 [454, 455] which we repeat explicitly

$$TR(\tau) = -\frac{\partial C(\tau)}{\partial \tau} . \quad (1.8)$$

Out of equilibrium, however, Cugliandolo and Kurchan [169, 171] proposed to characterise the extent of breaking of the fluctuation-dissipation theorem by the **fluctuation-dissipation ratio**

$$X(t, s) := TR(t, s) \left(\frac{\partial C(t, s)}{\partial s} \right)^{-1} . \quad (1.9)$$

For equilibrium systems, the fluctuation-dissipation theorem holds and hence $X_{\text{eq}}(t, s) = 1$. Consequently, the difference $X(t, s) - 1$ can be used to measure the distance from equilibrium.¹⁰ Experimentally, since it is difficult to

⁹ For glassy systems, one may define a time-scale t such that there exists a length scale $L(t)$ and two constants $C_{1,2}$ such that the bounds $C_1 L(t) \leq t \leq \exp(C_2 L(t)^d)$ hold true [524].

¹⁰ It is not uncommon to define an *effective temperature* $T_{\text{eff}}(t, s) := T/X(t, s)$ and to use the ratio $T_{\text{eff}}(t, s)/T$ as a measure for the distance from equilibrium. As in Volume 1, we shall use always units such that the Boltzmann constant $k_B = 1$.

measure the response function $R(t, s)$ directly, one rather considers the zero-field-cooled susceptibility $\chi_{\text{ZFC}}(t, s)$ which is plotted over against the auto-correlator $C(t, s)$, as it is illustrated schematically in Fig. 1.8. While in pure magnets the passage from the quasi-stationary state to the ageing regime is often quite rapid and the fluctuation-dissipation ratio is almost constant inside the ageing regime, this does not need to be the case in glassy systems where this passage is often considerably more gradual.¹¹ An excellent and detailed review on the fluctuation-dissipation theorem and its breaking out of equilibrium was provided by Crisanti and Ritort [163]; see also the recent survey by Leuzzi [465].

1.1.5 Mean-field Theory

For a first semi-quantitative overview, consider a mean-field-description of ageing as it may occur in a simple ferromagnet. We shall use here a phenomenological formulation, which centres on the coarse-grained order-parameter $m = m(t, \mathbf{r})$, which is conceived as the average magnetisation of aggregates consisting of many individual ‘spins’, which simultaneously should be large enough such that a continuum description becomes applicable but also small enough with respect to typical length scales that they may be considered point-like. Effectively, the thermodynamics of the order-parameter is assumed to be given in terms of the Ginzburg-Landau functional $\mathcal{H}[m]$, of the same qualitative form as the free energy in Fig. 1.5. In principle, a deterministic mechanical equation of motion could then be of the form $\partial_t^2 m = -\frac{\delta \mathcal{H}}{\delta m}$ where the right-hand terms describes the forces which are simply derived by assuming that the system should ‘slide down’ along the gradient of the effective potential $\mathcal{H}$, but such an ansatz would neglect the interactions of the aggregates with their environment. It seems more reasonable to admit that (i) these interactions should generate some dissipation, the qualitative effects of which might be included into the equation of motion by a formal ‘friction’ force $-\zeta \partial_t m$; and (ii) that one has furthermore on the aggregates the effects of many forces which come from the direct interaction with the smaller particles in the environment. It only seems feasible to attempt a statistical description which means that the net force should be the sum over many independent forces which are treated as random variables. It is then tempting to conclude that because of the central limit theorem, this random force should have a Gaussian distribution. The variance of this Gaussian distribution will be fixed by requiring the eventual relaxation towards equilibrium. This is treated in many textbooks, e.g. [710, 761, 512, 490], see also exercise 1.1. Finally, it is usually enough to consider the *over-damped* case when the ‘friction coefficient’ ζ is sufficiently large such that one may neglect the inertial term $\sim \partial_t^2 m$ on sufficiently long time-scales. This kind of (admittedly hand-waving) reasoning leads one to consider as the basis of the description a **Langevin equation**

¹¹ Finite-time effects may further affect these simplistic schemes.

$$\frac{\partial m(t, \mathbf{r})}{\partial t} = -D \frac{\delta \mathcal{H}[m]}{\delta m(t, \mathbf{r})} + \eta(t, \mathbf{r}) \quad (1.10)$$

where $D = 1/\zeta$ and the **white noise** $\eta(t, \mathbf{r})$ is assumed to be Gaussian, centred $\langle \eta(t, \mathbf{r}) \rangle = 0$ and of variance $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2TD\delta(t - t')\delta(\mathbf{r} - \mathbf{r}')$ such that there is relaxation towards the equilibrium state with a probability distribution $P_{\text{eq}}[m] \sim e^{-\mathcal{H}[m]/T}$.

For a simple magnet, the Ginzburg-Landau functional is conventionally assumed to have the form $\mathcal{H}[m] = \int d\mathbf{r} \left[\frac{1}{2} (\nabla m)^2 + \mathcal{V}[m] \right]$. Then the Langevin equation becomes (Δ_L being the spatial Laplacian)

$$\frac{\partial m(t, \mathbf{r})}{\partial t} = D\Delta_L m(t, \mathbf{r}) - D \frac{\delta \mathcal{V}[m]}{\delta m(t, \mathbf{r})} + \eta(t, \mathbf{r}) \quad (1.11)$$

from which one understands the interpretation of D as **diffusion constant** and the potential $\mathcal{V}[m]$ is identified with the free-energy as sketched in [Fig. 1.5](#).

In this book, we shall be essentially interested in the non-equilibrium relaxation processes of relatively simple, i.e. in general non-glassy, systems. In order to formulate for this kind of system a simple mean-field theory with an ageing behaviour, we shall for the time being ignore the spatial dependence in the order-parameter, hence $m = m(t)$. Rescaling the parameters, we consider the Langevin equation

$$\frac{dm(t)}{dt} = 3\lambda^2 m(t) - m(t)^3 + \eta(t) \quad (1.12)$$

where the centred Gaussian noise $\eta(t)$ has the variance $\langle \eta(t)\eta(t') \rangle = 2T\delta(t - t')$. Here, the control parameter $\lambda^2 \sim T_c - T$ distinguishes between ordered, critical and disordered equilibrium states.¹² We consider a totally disordered initial state such that $m_0 = m(0) = 0$ and quench at the initial time $t = 0$ the temperature to a value $T \leq T_c$. This means that for $T < T_c$ (or $\lambda^2 > 0$) one starts from the unstable fixed point of the eq. (1.12) after noise-averaging and asks for the long-time evolution of the system. Since the order-parameter stays fixed at its initial value $m(t) = m(0) = 0$ for all times $t \geq 0$, we consider the fluctuations around this value. These are measured by the variance $v(t) = \langle m(t)^2 \rangle$ or even better by the two-times autocorrelator $C(t, s)$ and autoresponse $R(t, s)$ which are defined by

$$\begin{aligned} C(t, s) &= \langle m(t)m(s) \rangle \\ R(t, s) &= \left. \frac{\delta \langle m(t) \rangle}{\delta h(s)} \right|_{h=0} = \frac{1}{2T} \langle m(t)\eta(s) \rangle \end{aligned} \quad (1.13)$$

The relationship (1.13) of $R(t, s)$ with a noise correlator is a generic property of this formalism. We leave the proof as exercise 1.3.

¹² Usually, one considers the approach of $m(t)$, starting from some non-vanishing initial value $m_0 \neq 0$, towards the stationary solutions $m_\infty = \pm\sqrt{3}\lambda, 0$, but this does not lead to ageing phenomena for $\lambda^2 > 0$. See exercise 1.2.

In order to derive equations of motion for the autoresponse and then the autocorrelator, we multiply eq. (1.12) with $\eta(s)$ and apply eq. (1.13). In order to carry out the average, recall the cumulant $\langle\langle m^3\eta \rangle\rangle = \langle m^3\eta \rangle - 3\langle m^2 \rangle \langle m\eta \rangle$. In mean-field theory, the cumulant $\langle\langle m^3\eta \rangle\rangle \stackrel{!}{=} 0$ is assumed to vanish. Using a similar argument for the autocorrelator, we arrive at the mean-field equations of motion (see exercise 1.4)

$$\begin{aligned}\partial_t R(t, s) &= 3(\lambda^2 - v(t)) R(t, s) + \delta(t - s) \\ \partial_s C(t, s) &= 3(\lambda^2 - v(s)) C(t, s) + 2TR(t, s).\end{aligned}\quad (1.14)$$

This depends only on the **variance** $v(t) = \langle m^2(t) \rangle$ which in turn satisfies the self-consistent equation of motion

$$\dot{v}(t) = 6(\lambda^2 v(t) - v^2(t)). \quad (1.15)$$

The solution of eqs. (1.14,1.15) has to be found with the initial conditions $m(0) = m_0 = 0$ and $v(0) = v_0 \neq 0$. At this point, an important difference between the disordered high-temperature phase ($T > T_c$ or $\lambda^2 < 0$) and the ordered or critical cases ($T \leq T_c$ or $\lambda^2 \geq 0$) appears. Namely, for $\lambda^2 < 0$, it follows from eq. (1.15) that the variance $v(t)$ vanishes exponentially fast, while on the contrary the variance $v(t) \rightarrow \lambda^2 > 0$ in the ordered phase and $v(t)$ merely vanishes according to an algebraic law at the critical point. Already at this stage, even without having performed any explicit calculations, we see that because of the non-vanishing variance, *fluctuations are likely to play an important rôle for $T \leq T_c$* , in contrast to the disordered phase. Explicitly, we find for sufficiently large times $t, s \gg \tau_{\text{micro}}$, where the Theta-function expresses the **causality** of the response function (the details are left as exercise 1.5)

$$R(t, s) \simeq \Theta(t - s) \begin{cases} 1 + O(e^{-6\lambda^2 s}) & ; \text{ if } T < T_c \\ \sqrt{s/t} & ; \text{ if } T = T_c \\ \exp(-3|\lambda^2|(t - s)) & ; \text{ if } T > T_c \end{cases} \quad (1.16)$$

$$C(t, s) \simeq T \begin{cases} 2 \min(t, s) & ; \text{ if } T < T_c \\ s \sqrt{s/t} & ; \text{ if } T = T_c \\ (3|\lambda^2|)^{-1} \exp(-3|\lambda^2|(t - s)) & ; \text{ if } T > T_c \end{cases} \quad (1.17)$$

For $T > T_c$, correlator and response relax within a finite time towards zero, and no ageing occurs. On the other hand, we observe for $T \leq T_c$:

- (i) a slow, non-exponential dynamics,
- (ii) a breaking of time-translation invariance and
- (iii) dynamical scaling according to simple ageing.

Hence our three defining criteria for physical ageing are satisfied. Furthermore, one may use the fluctuation-dissipation ratio (1.9) to characterise the distance with respect to equilibrium. The result is, both t and s being sufficiently large and also suppressing correction terms,

$$X(t, s) = X_\infty = \begin{cases} 1/2 & ; \text{ if } T < T_c \\ 2/3 & ; \text{ if } T = T_c \\ 1 & ; \text{ if } T > T_c \end{cases} . \quad (1.18)$$

Therefore, ageing behaviour for $T \leq T_c$ is correlated with the system *never* reaching an equilibrium state, while for $T > T_c$, the system relaxes exponentially fast, and within a finite time, towards equilibrium.¹³

The spatial dependence of the order-parameter $m = m(t, \mathbf{r})$ can be discussed similarly, see exercise 1.5 for the detailed computation.

1.1.6 Breaking of the Fluctuation-dissipation Theorem: Experiments

The treatment of the previous section has illustrated that ageing should generically occur far from equilibrium, where typical equilibrium results such as the (Kubo) fluctuation-dissipation theorem are no longer valid. We now quote several experimental examples for further illustration of this point.

1. As an illustration on the behaviour of responses and correlations from an ageing real material, in Fig. 1.9 the two-time correlation $C(t, s)$ (normalised by $C(t, t)$) and the zero-field cooled susceptibility $\chi(t, s) = \chi_{\text{ZFC}}(t, s)$ are shown for the insulating spin glass $\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$ obtained from SQUID measurements, performed by Hérisson and Ocio [360, 361].¹⁴ In the left panel, one clearly sees that (i) the time scales considered are large compared to microscopic time-scale, such that the relaxation process may be considered slow, (ii) time-translation-invariance is broken since the data depend on both the waiting time s and the time difference $t - s$ and (iii) dynamical scaling is observed, here according to subageing. Again, the three defining properties of physical ageing are met.

In the right panel, responses and correlations are compared (for technical reasons, it is common to subtract a stationary part, which is of no concern for us for the moment). For equilibrium systems, one could integrate the fluctuation-dissipation theorem and would find, because time-translation-invariance holds at equilibrium and all observables only depend on $\tau = t - s$, that $T\chi(\tau) = 1 - C(\tau)$, where the initial condition $C(0) = 1$ was used. In the other extreme case, one can assume that in eq. (1.9), the fluctuation-dissipation ratio $X(t, s) \rightarrow X_\infty$ has effectively become constant. If that is the case, integration gives, using now the initial condition $C(s, s) = 1$, the relation $(T/X_\infty)\chi(t, s) = 1 - C(t, s)$. Therefore, as it is done in the right panel of Fig. 1.9, when plotting $\chi(t, s)$ over against $C(t, s)$, one expects for small time differences $\tau = t - s$, when $C(s + \tau, s)$ is still relatively large, a linear relationship where the slope $-1/T$ is related to the temperature of the

¹³ This mean-field treatment over-estimates the values of $X(t, s)$ and hence under-estimates the distance from equilibrium when $T \leq T_c$.

¹⁴ This is one of the best known realisations of the Heisenberg spin glass.

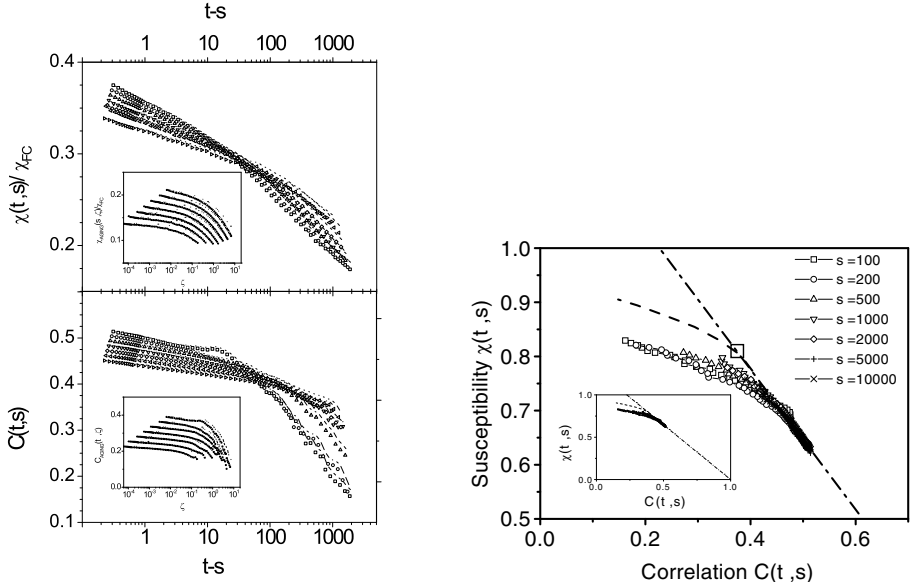


Fig. 1.9. Ageing, dynamical scaling and the breaking of the fluctuation-dissipation theorem in the spin glass $\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$ quenched to $T = 13.3 \text{ K} = 0.8 T_c$. In the left panel, the susceptibility (top) and the correlator (bottom) are shown for the waiting times $s = [100, 200, 500, 1000, 2000, 5000, 10000]$ (in seconds) from bottom to top. The insets show subageing scaling with a subageing exponent $\mu = 0.87$. In the right panel, the deviation of the fluctuation-dissipation theorem (straight dash-dotted line) is shown and the inset gives the data in the whole range. The dashed line is the expected extrapolation for $s \rightarrow \infty$. All times are measured in seconds. After [360, 361].

heat bath. Once the system starts to deviate from (quasi-)equilibrium, which essentially coincides with the autocorrelator $C(t, s)$ leaving its ‘plateau’ value (see left panel) for larger values of τ , one observes a deviation of the $\chi - C$ data in the fluctuation-dissipation plot from the above straight line. Deep inside the non-equilibrium regime, when $X(t, s) \rightarrow X_\infty$, one may again find a linear relationship, but now characterised by the quantity $T_{\text{eff}} := T/X_\infty$, as it is the case of the example at hand. In order to emphasise an analogy with equilibrium systems, the quantity T_{eff} is often called an **effective temperature** [171], which can be motivated in the context of mean-field theory of spin glasses.

Diagrams of this kind contain a lot of further information. Since a discussion of the finer details of spin glasses is beyond the scope of this book, we refer to [361] for a detailed comparison of these data with domain growth and spin glass models.

2. A different way of looking at the breaking of the fluctuation-dissipation ratio uses the response to a *periodic* external perturbation. As a thought ex-

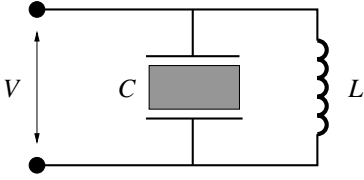


Fig. 1.10. Schematic LC circuit for the measurement of the power spectrum of a dielectric material.

periment, consider an electric RC circuit which contains a capacitor, whose dielectric will be the material under study, in parallel with a lossless inductor, see Fig. 1.10, where the fluctuations of the voltage are measured. At equilibrium, the integrated voltage noise of the oscillator $\langle V^2 \rangle = k_B T / C$. The power spectrum is given by a generalised **Nyquist formula**. We give explicitly the form in the case of an LC circuit [301]

$$\begin{aligned} S(\omega) &:= \frac{1}{2\pi} \int_{\mathbb{R}} dt \langle V(t)V(t+t') \rangle e^{-i\omega t} = \frac{2k_B T}{\pi} \operatorname{Re} Z(\omega) \\ &= -\frac{2k_B T}{\pi} \frac{\omega^3 L^2 C''}{(1 - \omega^2 L C')^2 + \omega^4 L^2 C''^2}, \end{aligned} \quad (1.19)$$

where $Z(\omega)$ is the circuit impedance, L the inductance and $C = C' + iC''$ the complex capacitance which all depend on the material's properties. If the system is out of equilibrium, the bath temperature T is replaced by an effective temperature, either $T_{\text{eff}}(t, s)$ or $T_{\text{eff}}(s; \omega)$ as in the case at hand, and a deviation of either of these from the bath temperature T is interpreted as a breaking of the fluctuation-dissipation theorem. The typical power spectrum

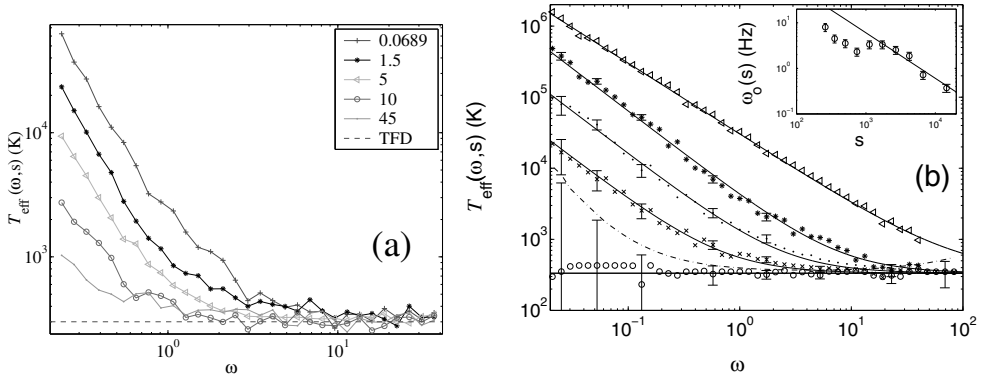


Fig. 1.11. (a) Effective temperature in laponite. The time differences $\tau = t - s$ are measured in hours. The line TFD is the prediction of the fluctuation-dissipation theorem. (b) Effective temperature in polycarbonate, for $s = [200, 260, 2580, 6542]$ seconds from top to bottom. The roughly horizontal data correspond to $s = 1$ day and the horizontal full line is the fluctuation-dissipation prediction. Reprinted from [107, Fig.4, Fig.10b]. Copyright (2003) Institute of Physics Publishing.

material	condition	X_∞	Ref.
$\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$	$T < T_c$	$0.2 - 0.4$	[360, 361]
PMMA	$\phi \simeq \phi_c$	$0.43(6)$	[725]
liquid crystal 5CB	$\epsilon \simeq 0$	$\simeq 0.31$	[412]

Table 1.3. Experimentally measured values of the limit fluctuation-dissipation ratio X_∞ in a spin glass, a colloidal glass and a liquid crystal.

contains a strong peak whose position, width and height fix the three parameters C and L , and a single measurement is sufficient. Indeed, in glycerol quenched to slightly below its glass-transition temperature, T_{eff}/T was seen to be larger than unity by a few percent, over time scales up to $\omega s \gtrsim 10^5$ [301]. More substantial deviations of T_{eff} from the bath temperature T as measured in a CR circuit, from the dielectric behaviour of the colloïd laponite and polycarbonate, are shown in Fig. 1.11. Clearly, for times very long with respect to microscopic relaxation times the system remains far from equilibrium but a relaxation towards equilibrium is seen within the experimentally accessed times. On the other hand, if the micromechanical properties of the same materials are studied, is it not yet clear whether or not an ageing behaviour can be found in laponite [107, 63, 1, 299, 394], while ageing has been found in polycarbonate [107, 63] and a PMMA colloïdal glass (where the effective temperature is found to be independent of the waiting time and about twice as large as the bath temperature T) [725].

Very similar results were recently also obtained by measuring in laponite for the orientational two-time correlation function $C(t, s)$ via depolarised dynamic light scattering and the corresponding response function via the electric field-induced birefringence [489]. For not too large waiting times ($s \lesssim 225[\text{min}]$), there is a passage from an early-time regime which is still in equilibrium to an ageing regime, where the fluctuation-dissipation relation is broken. The limit fluctuation-dissipation ratio X_∞ increases with s , however, from values around ≈ 0.2 for $s = 90[\text{min}]$ and the ageing regime is no longer observable for waiting times $s \gtrsim 225[\text{min}]$ [489]. This again points to a transient ageing regime, although in the relevant time window, this shares many aspects with ageing as defined in this volume.

Since a whole function such as $X(t, s)$ is sometimes a little difficult to handle, a simple quantitative measure of the distance to equilibrium can be helpful. Such a quantity is the **limit fluctuation-dissipation ratio**

$$X_\infty := \lim_{s \rightarrow \infty} \left(\lim_{t \rightarrow \infty} X(t, s) \right) \quad (1.20)$$

where the order of the limits is crucial, since in the case of the opposite order, of course such that $t > s$, one has trivially $\lim_{t \rightarrow \infty} (\lim_{s \rightarrow \infty} X(t, s)) = 1$. For phase-ordering systems, one expects from scaling theory that $X_\infty = 0$, represented by the horizontal segment in the left part of Fig. 1.8a. On the

other hand, for the non-equilibrium dynamics of systems with a simple ageing behaviour such that the physically relevant length scale $L(t) \mapsto \xi(t)$ is the time-dependent correlation length, Godrèche and Luck [286] have argued that since X_∞ is the ratio of two scaling amplitudes, it should be a *universal* quantity capable to characterise non-equilibrium universality classes in the same way as non-equilibrium critical exponents. In Fig. 1.8b we had shown a schematic fluctuation-dissipation plot for a simple magnet quenched to $T = T_c$. Therein, the slope of the straight line to the left of the break is $-X_\infty/T$. The expectation of universality of X_∞ has been confirmed in many studies as will be described later; for the moment we just mention that the universality of X_∞ has been confirmed in field-theoretic calculations at $T = T_c$ using the ε -expansion in the $O(n)$ model [121]. Theoretical values for X_∞ in non-equilibrium critical dynamics of simple magnets are given in Table 1.7 on p. 77 below and Table 1.3 lists known experimental results.

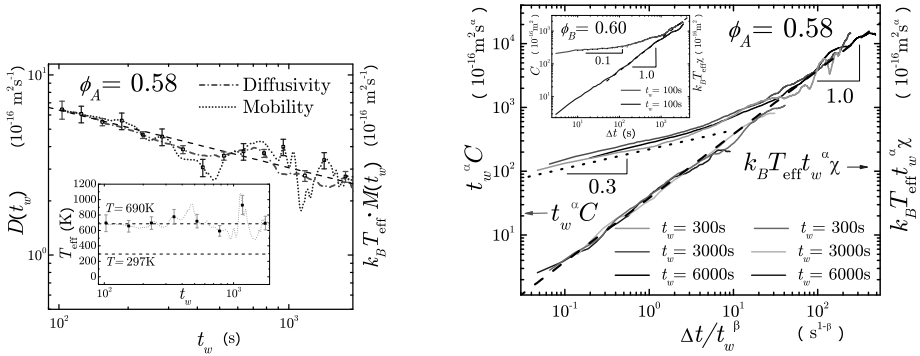


Fig. 1.12. Left panel: diffusivity (left axis) and mobility (right axis) in colloidal PMMA as a function of the waiting time s . The inset shows the effective temperature with average $T_{\text{eff}} = 690 \text{ K}$ and the bath temperature $T_{\text{bath}} = 297 \text{ K}$ for comparison. Right panel: subageing scaling of the correlation (upper curve, left axis) and the response (lower curve, right axis). The cross-over from $f_C(y) \sim y$ for $y \gg 30$ to $f_C(y) \sim y^{0.3}$ for $y \ll 30$ is indicated and the dotted line illustrates again the effective temperature $T_{\text{eff}} = 690 \text{ K}$. Times are measured in seconds. Reprinted by permission from Macmillan Publishers Ltd: Nature Physics [725], copyright (2006).

3. A more recent experiment carried out by Wang, Song and Makse [725, 726] considered a colloïd suspension of PMMA plus a small fraction of superparamagnetic beads as tracer particles. The motion of the particles was constrained to the plane and an external magnetic force F was applied perpendicular to it for measuring a ZFC-type integrated response. The sample

considered had a concentration $\phi = 0.58(1)$ slightly above the glass-transition concentration $\phi_g \sim 0.57 - 0.58$. The system was homogenised by stirring and the initial time is defined by the end of stirring. Following the positions of the tracers, the two-time quantities

$$\begin{aligned} C(t, s) &:= \frac{1}{2} \left\langle [x(t, s) - x(s, s)]^2 \right\rangle \sim 2D(s)(t - s) \\ \chi(t, s) &:= \frac{1}{F} \left\langle x(t, s) - x(s, s) \right\rangle \sim M(s)(t - s) \end{aligned} \quad (1.21)$$

were measured and the asymptotic forms were observed for $t - s > \tau_{\text{rel}}(s)$, where $\tau_{\text{rel}}(s)$ is the waiting-time-dependent relaxation time [725]. For large waiting times, the diffusivity $D(s)$ and the mobility $M(s)$ both scale as $D(s) \sim M(s) \sim s^{-0.32(8)}$, see Fig. 1.12. The system rapidly relaxes to a state where the relation between $D(s)$ and $M(s)$ is described in terms of a roughly constant effective temperature $T_{\text{eff}} = (690 \pm 100)$ K, see the inset.¹⁵ The right panel of Fig. 1.12 shows the subageing scaling of correlations and responses according to $C(t, s) = s^{-a} f_C((t - s)s^{-\mu})$ and $\chi(t, s) = s^{-a} f_\chi((t - s)s^{-\mu})$ and where $\mu = 0.48(1)$ and $a + \mu = 0.34(1)$ [725, 726], in agreement with the direct asymptotic measurement noticed above. The measured value of X_∞ is given in Table 1.3. Remarkably, the scaling functions $f_C(y) \sim f_\chi(y) \sim y$ increase with growing y . A similar scaling has been seen in models describing the roughness of elastic lines in a random environment [112, 114]. The simplest of these is the Edwards-Wilkinson growth model. Some aspects of the ageing of this model will be studied later.

4. The last example to be discussed here treats ageing close to the Fréedericksz transition [185] in the liquid crystal 5CB (Merck), carried out by Joubaud, Percier, Petrosyan and Ciliberto [412]. The material is placed between two plates, with distance $\ell = 9[\text{cm}]$. In this geometry, the molecules of the liquid crystal align in a unique direction parallel to the surface (chosen as the x -axis). The sample is subjected to a perpendicular electric field E (z -axis), with a frequency of $1[\text{kHz}]$ in order to avoid electrical polarisation. If the field exceeds a threshold value, the molecules tend to align with the field. This transition has the properties of a conventional second-order phase-transition. Near to criticality, the angle φ of the director of the liquid crystal molecules has the form $\varphi(t, x, y, z) = \varphi_0(t, x, y) \sin(\pi z/\ell)$. As long as the amplitude φ_0 remains small, and neglecting the spatial dependence, its dynamic is described by a Ginzburg-Landau equation

$$\tau_0 \frac{d\varphi_0}{dt} = \epsilon \varphi_0 - \frac{1}{2} (\kappa + \epsilon + 1) \varphi_0^3 + \eta \quad (1.22)$$

where $\epsilon = E/E_c - 1$ is the reduced control parameter, τ_0, κ are material-dependent constants and η is a thermal white noise. The spatially averaged

¹⁵ It can be checked that for densities much below ϕ_g , T_{eff} always remains close to T_{bath} , whereas for $\phi \gtrsim \phi_g$ the dynamics slows down so much that the scaling behaviour can no longer be analysed [725].

alignment of the molecules is measured by the variable ζ , which for φ_0 small enough becomes

$$\zeta \simeq \frac{2}{\ell} \frac{1}{\mathcal{A}} \int_{\mathcal{A}} dx dy \varphi_0^2 \quad (1.23)$$

where $\mathcal{A}$ is the area of the measuring region (a few $[\text{mm}]^2$), see [413, 411] for the experimental details. The average $\langle \zeta \rangle \sim \epsilon$, its relaxation time $\sim \epsilon^{-1}$ and its fluctuation spectrum has a Lorentzian form (see also Fig. 1.16). The experiment considers quenches from $\epsilon \simeq 0.3$ to close to criticality, with $\epsilon \simeq 0.01$, which can be realised within a [ms] [412] and the relaxation process is followed over 4 orders of time. The results for the two-time correlation and integrated ZFC-response are shown in Fig. 1.13. Clearly, the defining requirements of an ageing behaviour are met, in particular one observes a dynamical scaling behaviour compatible with simple ageing. Examining the fluctuation-dissipation plot, the experimental results nicely reproduce the theoretically expected scenario of Fig. 1.8b for a system quenched to its critical point. Table 1.3 gives the value of X_∞ extracted from these data [412].

1.1.7 Breaking of the Fluctuation-dissipation Theorem: Two Simple Solvable Models

Having described to some extent the phenomenology of the fluctuation-dissipation ratio $X(t, s)$ as defined in eq. (1.9) and its long-time limit X_∞ as defined in eq. (1.20), we consider some more theoretical aspects of these quantities. Two simple models, conceived for technical simplicity, in view of being able to analyse their ageing behaviour in detail, will be considered and will allow us to better appreciate the content of the following two questions:

1. in what sense do $X(t, s)$ or X_∞ serve as a measure of the distance to the equilibrium state ?
2. under what conditions can the ratio $T_{\text{eff}}(t, s) := T/X(t, s)$ or its limit value $T_{\text{eff}} = T/X_\infty$ be considered as a genuine, thermodynamic temperature ?

Although the simple models to be considered should not be considered as being realistic by themselves, their properties are such that many of their qualitative features should also arise in more realistic models which might in the end become comparable to experiments.

Question 1. In order to address the first question, following Villamina *et al.* [713], one considers a simple oscillator coupled to two heat reservoirs with temperatures $T_{r,\ell}$ such that the Langevin equations contains a memory term

$$m\ddot{x} = -kx - \int_{-\infty}^t dt' \Gamma(t-t')\dot{x}(t') + \eta(t). \quad (1.24)$$

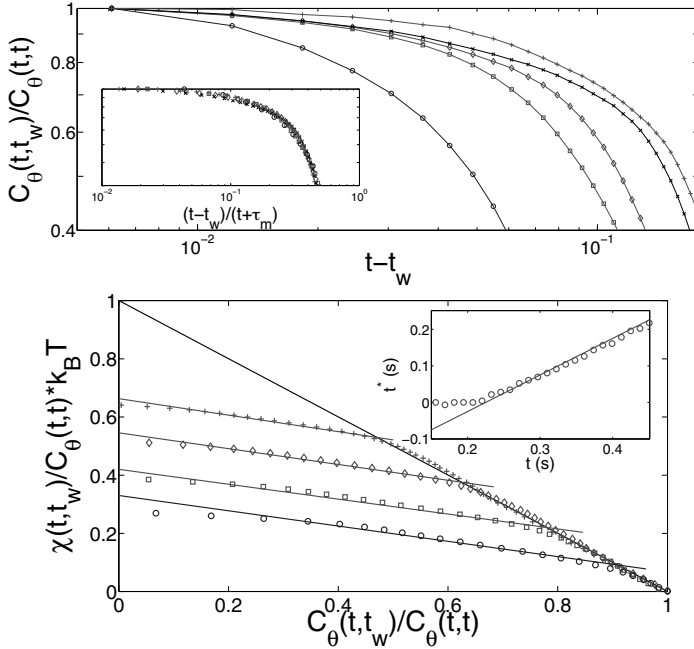


Fig. 1.13. Upper panel: measured normalised angular autocorrelation diffusivity in the liquid crystal 5CB as a function of the time difference $t - s$, for $t = [0.06, 0.2, 0.26, 0.28, 0.31]$ from left to right. The breaking of time-translation-invariance and dynamical scaling (see inset) is seen. τ_m is a microscopic reference time to improve the scaling. Lower panel: comparison of the ZFC integrated response with the correlation function, for $t = [0.4, 0.5, 0.6, 0.7]$ from bottom to top. The straight line indicates the prediction of the FDT. The inset shows the passage time towards the ageing regime. All times are measured in seconds. Reprinted with permission from [412]. Copyright (2009) by the American Physical Society.

Here, the **memory kernel** is assumed to read $\Gamma(t) = 2\gamma_r\delta(t) + \frac{\gamma_\ell}{\tau_\ell}e^{-t/\tau_\ell}$, where $\gamma_{r,\ell}$ are the couplings to the ‘rapid’ and ‘slow’ heat baths and τ_ℓ is a time-scale. The noise $\eta(t) = \eta_r(t) + \eta_\ell(t)$ is the sum of two centred Gaussian noises with the variances $\langle\eta_r(t)\eta_r(t')\rangle = 2\gamma_r T_r \delta(t - t')$ and $\langle\eta_\ell(t)\eta_\ell(t')\rangle = (\gamma_\ell/\tau_\ell) T_\ell e^{-|t-t'|/\tau_\ell}$. Although at first sight, this kind of model might appear to be quite artificial, the introduction of memory terms is potentially of considerable physical relevance. Indeed, as we shall see towards the end of this chapter from an analysis of the global persistence probability, quantities such as the global order-parameter in ageing systems should be generically described by a Gaussian, but non-Markovian process [493, 165].¹⁶

¹⁶ Equations such as (1.24) have been discussed in the context of driven glassy systems.

A standard trick [761] allows us to formally relate this kind of linear equation to a Markov process by adding further variables. Denoting the set of $\mathcal{N}$ variables by the vector $\mathbf{x}(t)$, one obtains an equation of the form $\dot{\mathbf{x}} = -A\mathbf{x} + \boldsymbol{\eta}$ where A is a real matrix whose eigenvalues have positive real parts. The noises $\boldsymbol{\eta}$ are centred Gaussian processes such that $\langle \eta_i(t)\eta_j(t') \rangle = D_{ij}\delta(t-t')$. The equilibrium probability distribution of the $\mathbf{x}$ is given by

$$\mathcal{P}_{\text{eq}}(\mathbf{x}) = (2\pi)^{-\mathcal{N}/2} (\det \sigma)^{-1/2} \exp \left[-\frac{1}{2} \sum_{i,j=1}^{\mathcal{N}} x_i \sigma_{ij}^{-1} x_j \right] \quad (1.25)$$

where the symmetric covariance matrix σ is obtained from $D = A\sigma + \sigma A^T$. What follows is restricted to the equilibrium situation, as defined by this probability distribution [713]. Next, define the matrix of correlators $C_{ij}(t) := \langle x_i(t)x_j(0) \rangle$. Defining the response function $R_{ij}(t)$ of the average of the variable $x_i(t)$ by taking the derivative with the respect to the conjugate field h_j of the variable $x_j(0)$, one readily finds for the matrix of responses $R(t) = C(t)\sigma^{-1}$, or written explicitly in components [713]

$$R_{ij}(t) = \sum_{n=1}^{\mathcal{N}} C_{in}(t) (\sigma^{-1})_{nj} . \quad (1.26)$$

This equation is referred to as a **generalised fluctuation-dissipation theorem**. Exercise 1.7 considers the relation of eq. (1.26) with the formulation *à la* Kubo of the fluctuation-dissipation theorem, as usually employed in this volume.

To make this more explicit, one returns to eq. (1.24), and furthermore focuses on the over-damped case when one may let $m = 0$. Introducing the auxiliary variable

$$u(t) := \frac{1}{\tau_\ell} \int_{-\infty}^t dt' e^{-(t-t')/\tau_\ell} \left[x(t') + \tau_\ell \sqrt{\frac{2T_\ell}{\gamma_\ell}} \eta_2(t') \right] \quad (1.27)$$

one finds the formally Markovian system

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = - \begin{pmatrix} k/\gamma_r + \gamma_\ell/\gamma_r\tau_\ell & -\gamma_\ell/\gamma_r\tau_\ell \\ -1/\tau_\ell & 1/\tau_\ell \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} \sqrt{2T_r/\gamma_r} \eta_1 \\ \sqrt{2T_\ell/\gamma_\ell} \eta_2 \end{pmatrix} \quad (1.28)$$

where $\eta_{1,2}(t)$ are independent white noises, i.e. $\langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij}\delta(t-t')$. The particular form of the auxiliary variable u implies that the generalised fluctuation-dissipation theorem (1.26) for the correlator $C_{xx}(t)$ at equilibrium takes the form $\partial_t C = -RA\sigma$. Explicitly, this leads to [713]

$$\left. \frac{\delta \langle x(t) \rangle}{\delta h(0)} \right|_{h=0} = \frac{1}{\gamma_r} R_{xx}(t) = -\frac{k\tau_r}{\gamma_r} \frac{1}{T_r} \frac{dC_{xx}(t)}{dt} - \frac{k\tau_\ell}{\gamma_\ell} \frac{\Gamma}{T_\ell - T_r} \frac{dC_{ux}(t)}{dt} \quad (1.29)$$

where $\Gamma = 1 + \gamma_r/\gamma_\ell + \tau_\ell/\tau_r$ and the natural ‘rapid’ time-scale $\tau_r := \gamma_r/k$ was introduced. Now, by analogy with experiments, one considers the (time-dependent) effective temperature $T_{\text{eff}}(t)$, defined as the inverse slope when plotting the susceptibility $\chi_{xx}(t) = \int_0^t dt' R_{xx}(t')$ against $C_{xx}(t)$. One finds for the extreme cases [713]

$$T_{\text{eff}}(t) := - \left(\frac{d\chi_{xx}(t)}{dC_{xx}(t)} \right)^{-1} \simeq \begin{cases} T_r & ; \text{ for } t \rightarrow 0 \\ T_\ell - \kappa(T_\ell - T_r) & ; \text{ for } t \rightarrow \infty \end{cases} \quad (1.30)$$

with $\kappa := (2 + \Gamma - \sqrt{\Gamma^2 - 4\tau_\ell/\tau_r})/2\Gamma$. At least in these limits, this reproduces the qualitative behaviour of the usual fluctuation-dissipation plots, see Fig. 1.8. This example illustrates that although the *phenomenological* fluctuation-dissipation relation between $\chi_{xx}(t)$ and $C_{xx}(t)$ may no longer be satisfied, the *generalised* fluctuation-dissipation relation eqs. (1.26, 1.29) which takes the contribution involving the ‘hidden’ variable u into account, does remain valid. If explicit information on how to construct these auxiliary variables is not available, one can still consider *phenomenological* fluctuation-dissipation plots in comparing $\chi_{xx}(t)$ with $C_{xx}(t)$, but a breaking of the expected Kubo-like fluctuation-dissipation relation $T_{\chi_{xx}}(t) = -\partial_t C_{xx}(t)$ might be better viewed as a measure of the relative importance of ‘hidden’ variables such as u in the above example. According to [713], the following typical situations can arise, which illustrate the complexity of the correct interpretation of fluctuation-dissipation plots and highlight the difficulties of giving a real physical meaning to the phenomenological quantity $T_{\text{eff}}(t)$:

1. $\tau_\ell \approx \tau_r$: Then the fluctuation-dissipation plot is non-linear and $T_{\text{eff}}(t)$ is a complicated function, not simply related to neither T_r nor T_ℓ .
2. $\tau_\ell \gg \tau_r$: In this ‘glassy limit’, with the constraint $(T_r/T_\ell)(\gamma_\ell/k\tau_\ell) \approx \frac{1}{2}$, the fluctuation-dissipation plot consists of two straight lines, as in Fig. 1.8b, and the slopes correspond to T_r and T_ℓ , respectively. The contribution of $\partial_t C_{ux}$ in eq. (1.29) is negligible in the first regime, but not in the second.
3. $\tau_\ell \ll \tau_r$ and $\gamma_\ell \approx \gamma_r$: The ratio of the contributions of $\partial_t C_{xu}$ and of $\partial_t C_{xx}$ in (1.29) becomes essentially constant and the fluctuation-dissipation plot consists of a *single* straight line, but $T_{\text{eff}} \neq T_r, T_\ell$.
4. $\tau_\ell \ll \tau_r$ and $\gamma_\ell \ll \gamma_r$: As in the case before, but now the contribution of $\partial_t C_{xu}$ is negligible and $T_{\text{eff}} = T_r$.

It is not yet understood how these considerations can be extended to genuinely non-equilibrium systems, but similar results are found for granular gases [713].

Question 2. Following Leuzzi and Nieuwenhuizen [466, 465], one accepts, for the time being, that the fluctuation-dissipation ratios $X(t, s)$ and X_∞ are measures of the freezing-in of some degrees of freedom and rather inquires about their thermodynamic interpretation. They formulate the following con-

ditions for a definition of a non-equilibrium **effective temperature** T_{eff} , to be defined in a suitable sub-class of non-equilibrium systems:¹⁷

1. all definitions of $T_{\text{eff}} \geq 0$ should coincide;
2. there exists some kind of effective ‘thermometer’ to measure T_{eff} ;
3. if T is the temperature of the heat-bath, there should be a heat flow from modes with $T_{\text{eff}} > T$ to modes thermalised at T ;
4. processes evolving on similar time-scales should have the same effective temperature.

In many spin-glasses quenched to $T < T_c$, based on exact solutions of their mean-field ageing behaviour, Cugliandolo, Kurchan, Parisi and Peliti [169, 171, 170, 172, 166, 458] argued that the long-time ageing behaviour was well captured by considering a single effective temperature T_{eff} , defined in (1.30) from the generalised fluctuation-dissipation theorem. Their proposal is indeed well-confirmed in many numerical simulations, see [166, 465] and refs. therein. In the previous section we listed the known experimental examples confirming this scenario. At first sight, the often observed values $T_{\text{eff}} > T$ (or $X_\infty < 1$) might appear surprising in view of the slow dynamics, but could be explained if the energy landscape is explored less frequently but where the amplitude of the jumps has increased [725]. Indeed, in a model of silica (a strong glass) values $X_\infty < 1$ were found [640]. Conceptually, in the framework of such a **two-temperature thermodynamics** [466, 465], the first and second laws become

$$dU = TdS + T_{\text{eff}}dS_c - pdV \quad , \quad \delta Q \leq Td(S + S_c) \quad (1.31)$$

where U is the internal energy, S the entropy, S_c the complexity, p the pressure and the heat exchange $\delta Q = TdS + T_{\text{eff}}dS_c$. Besides the heat-bath temperature T , here T_{eff} arises as a second, independent, temperature and state variable. The second law may also be written as $(T_{\text{eff}} - T)dS_c \leq 0$. The Gibbs functional reads $G = U - TS - T_{\text{eff}}S_c + pV$.

In order to assess the universality of the effective temperature T_{eff} so defined (possible alternatives include, but do not exhaust, the characteristic temperature from the detailed balance condition, an effective non-equilibrium canonical ensemble, or the Legendre-transform of S_c) one considers $\mathcal{N}$ harmonic oscillators x_i coupled to ‘spherical spins’ $s_i \in \mathbb{R}$, according to the Hamiltonian [465]

$$\mathcal{H} = \sum_{i=1}^{\mathcal{N}} \left[\frac{K}{2} x_i^2 - H x_i - J x_i s_i - B s_i \right] \quad , \quad \sum_{i=1}^{\mathcal{N}} s_i^2 \stackrel{!}{=} \mathcal{N} \quad (1.32)$$

¹⁷ A practical question: in cooling, silica vitrifies at $T_g \approx 1800 - 2100[\text{K}]$. The effective temperature is related to the falling out of equilibrium at those temperatures and should remain there when the glass is cooled to room temperature, such that there remain modes with $T_{\text{eff}} \lesssim T_g$. So why can one hold a glass in one’s hands ? [465]

and with the couplings K, J and the magnetic fields H, B . Integrating out the spins s_i and introducing a *parallel* Monte Carlo dynamics for the x_i , one obtains an explicit closed system of equations of motion for the first two moments $m_k = \mathcal{N}^{-1} \sum_i x_i^k$ with $k = 1, 2$ and also for the two-time correlators and responses. These are readily integrated numerically. Depending on the chosen variance in the Monte Carlo updates of the x_i , one can arrange for a glassy relaxation behaviour (in spite of the trivial static behaviour), with either an Arrhenius or a Vogel-Fulcher behaviour, which are often-used forms to fit experimental/simulational data. A Kovacs effect is also observed, which underlines the non-equilibrium nature of the relaxation.

Now, considering a long list of possible definitions of effective temperatures T_{eff} , it is shown [465] that these only agree for sufficiently long times if a certain characteristic function decays with time more slowly than $1/\ln t$. This illustrates that already in this simple model, constraints exist which are necessary conditions for an ageing behaviour compatible with a two-temperature thermodynamics. Furthermore, it was argued that the observed Kovacs effect is incompatible with a two-temperature thermodynamics [465]. Generic criteria which inform under which easily testable conditions a two-temperature thermodynamics is admissible, at least for certain glassy systems, remain to be found.

While the argument outlined so far applies to glassy systems, difficulties of a different nature have been raised against a thermodynamic interpretation of $T_{\text{eff}}(t, s) = T/X(t, s)$ for simple magnets without disorder. Here, in several models it was shown by explicit calculations that the value of T_{eff} either depends on the chosen observable [120, 507] or/and may become negative [73, 588, 506, 724]. This directly violates the first of the criteria for a thermodynamic interpretation of T_{eff} , stated on p. 31 above.

Summarising, the examples presented in this section are probably best viewed as reminders of the conceptual and practical difficulties of correctly appreciating the thermal behaviour of non-equilibrium systems.

1.1.8 Outline

One of the objectives of this book is to present a detailed analysis of the dynamical scaling which underlies ageing phenomena. In order to do so, we shall largely restrict ourselves to simple spin systems without disorder, leaving the more complicated topic of spin glasses to existing excellent reviews, e.g. [166, 163, 431, 140, 139]. The developing of a physical understanding might be helped by beginning with one of the simplest possible examples: domain growth in simple magnets quenched into their ordered phase with at least two coexisting macroscopic states. One of the central questions concerns the form of the growth law $L = L(t)$. We shall review the Bray-Rutenberg theory [101, 102] which, starting from the assumption (1.7) of dynamical scaling and the phenomenological Porod's law, predicts the value of the dynamical

exponent z . We shall then come back to general phenomenological questions, review several exact results for possible scaling forms before we treat the values of the various non-equilibrium exponents which must be defined in order to describe ageing. The next chapter will present some exactly solvable models with simple ageing behaviour and what kind of information can be obtained from them. Afterwards, we shall present an overview of various situations of simple ageing, which range from non-equilibrium critical dynamics, ageing effects near free surfaces, the rôle of a global conservation law of the order-parameter, ageing in frustrated systems, but without disorder and critical spin glasses. More general are systems without detailed balance which no longer relax to an equilibrium state. These chapters are mainly concerned with critical exponents and universal amplitude ratios such as X_∞ . We shall postpone until chapters 4-6 the general calculation of universal scaling functions which requires an extension of dynamical scaling to *local* scale-transformations.

1.2 Phase-ordering Kinetics

As an entry towards the theoretical description of ageing phenomena, we now describe some basic aspects of dynamical scaling. Dynamical scaling for late times is related to the existence of a single relevant length-scale $L = L(t)$ which grows with the time t . In this section, we shall be only concerned with the specific situation of phase-ordering kinetics of ferromagnets without disorder, but we mention that there is recent experimental evidence [71] for a growing dynamical length in spin glasses. This section is largely inspired by the reviews [93, 94].

Phase-ordering kinetics arises for instance in simple magnets which are prepared initially in a disordered state and are then quenched to a temperature T below the critical temperature T_c . We begin with an outline of how a formal description is set up and only at the end we shall add some more comments on the physical assumptions which go into this. The equilibrium free energy is sketched in Fig. 1.5. In principle, such a curve should be considered for each lattice site, where local magnetic fields created by the surrounding spins will lead to a small tilting of the free energy, such that *locally*, equilibrium will be rapidly reached. However, that is not the case globally, since both steady-states are equivalent. One expects that a *coarse-grained* description in terms of a continuum order-parameter $\phi(t, \mathbf{r})$ will be sufficient and the ordered phase of simple magnets may be described in terms of a Ginzburg-Landau functional

$$\mathcal{H}[\phi] = \int d\mathbf{r} \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi) \right] \quad (1.33)$$

where the essential feature of the ‘potential’ $V = V(\phi)$ is its double-well structure, see Fig. 1.5. Occasionally, when we shall need an explicit form, we

shall use $V(\phi) = (1 - \phi^2)^2$, but we shall show below that *the precise form of $V(\phi)$ is not essential*. In fact, this is one of the main results of this Section.

In what follows, we shall always assume that the variables are rescaled such that the minima of $V(\phi)$ occur at $\phi = \pm 1$ and that $V(\pm 1) = 0$. A suitable equation of motion may be obtained by admitting that the field ‘slides down’ the slope $\delta\mathcal{H}/\delta\phi$ and it is common to write a stochastic Langevin equation for the coarse-grained order-parameter

$$\frac{\partial\phi}{\partial t} = -D \frac{\delta\mathcal{H}}{\delta\phi} + \eta = D(\Delta\phi - V'(\phi)) + \eta \quad (1.34)$$

where D is a kinetic coefficient which for the purposes of this Section can be set to unity, $D = 1$, $V'(\phi) = dV/d\phi$ and η is a Gaussian white noise¹⁸ with zero mean and variance $\langle\eta(t)\eta(t')\rangle = 2T\delta(t-t')$. We restrict here attention to the case of a *non-conserved order-parameter*. A renormalisation-group analysis would show that there are stable fixed points at $T = 0, \infty$ and an unstable one at $T = T_c$. Hence, for $T < T_c$, temperature-dependent terms will merely contribute to metric factors and corrections to the leading scaling behaviour. The universal scaling properties are those found for $T = 0$.¹⁹ Similarly, short-ranged correlations in the initial state should be irrelevant and it is enough to consider white-noise initial conditions $\langle\phi(0, \mathbf{r})\rangle = 0$ and

$$\langle\phi(0, \mathbf{r})\phi(0, \mathbf{r}')\rangle = a_0\delta(\mathbf{r} - \mathbf{r}') \quad (1.35)$$

where a_0 is a constant.

The task is to solve the equations of motion (1.34) and then average over the initial conditions according to (1.35), in order to obtain the long-time behaviour of the solutions and also of correlations and responses. Following [93, 94], this analysis proceeds in several steps, see below.

Let us pause a moment in order to appreciate better the physical content of the equations written down in this Section. The form of the free energy for $T < T_c$ mimics the essential behaviour of an Ising model with its two distinct ordered states. From *equilibrium* statistical physics, recall the well-known fact that Ising models can be used to describe the equilibrium properties of uniaxial magnets or liquid-gas transitions or else the behaviour of binary liquids or alloys. When considering the *dynamics*, however, the main difference between

¹⁸ η describes the sum of the many random forces created by the microscopic particles interacting with the coarse-grained order-parameter ϕ . From the central limit theorem one expects that η should have a Gaussian distribution and for a non-conserved order-parameter the variance $2T$ is the celebrated Einstein coefficient such that the model relaxes towards equilibrium, see e.g. [710, 761]. See [183] for more details on the derivation of such coarse-grained equations of motion from various mean-field forms of the free energy.

¹⁹ In the classification of Hohenberg and Halperin of dynamical critical phenomena, this is referred to as **model A** [370, 253]. For $T = 0$, eq. (1.34) is known as time-dependent **Ginzburg-Landau equation** (TDGL).

magnets/liquid-gas transitions and binary liquids/alloys on the other hand is that for the first ones it is physically admissible to change one kind of local state at a given site to the other (flip an ‘up’ spin to a ‘down’ spin) while in the second kind of system only alchemists could do that and only exchanges between spins at different sites are admissible. Eq. (1.34) is valid for a non-conserved dynamics. For that reason, the question of whether the order-parameter is non-conserved (as it is in the first case) or conserved (as it is in the second) is central for the dynamical behaviour, even if the equilibrium states are the same. Although the simple scaling behaviour (1.7) should remain valid in both cases, the value of the dynamical exponent z does depend on whether the order-parameter is non-conserved, as it is for **phase-ordering kinetics** when $z = 2$ (as we shall see shortly), or conserved, as it is for **phase separation**, when one must replace eq. (1.34) by the Cahn-Hilliard equation, see later, and obtains $z = 3$ or $z = 4$ for quenches to $T < T_c$. Clearly, the dynamics with a conserved order-parameter are considerably slower than for the non-conserved case.

An important aspect completely skipped in this discussion is the possibility of hydrodynamic transport, required for the discussion of phase-separation in binary *fluids*. The effects of viscosity lead to new growth regimes with modified scaling properties, in particular with changed values of the dynamical exponent, but whose description is beyond the scope of this book. We refer to [93, 94, 183] for further information and add that in two dimensions, the competition between the diffusive and hydrodynamics growth modes may lead to a breakdown of dynamical scaling in symmetric binary fluids, but depending on the initial droplet morphology [720, 719].

In what follows, we shall almost exclusively discuss the simpler case of a non-conserved order-parameter but the corresponding results for a conserved order-parameter will be quoted at the end.

1.2.1 Linear Stability Analysis

One begins by inquiring about the stability of the spatially homogeneous solution $\phi = \text{constant}$. The linearised form of (1.34) is $\partial_t \phi \simeq \Delta \phi - V''(0)\phi + \dots$ where $V''(0) < 0$. Going over to momentum space, the solution becomes $\hat{\phi}_{\mathbf{k}}(t)$ and reads

$$\hat{\phi}_{\mathbf{k}}(t) = \hat{\phi}_{\mathbf{k}}(0) \exp(\omega_{\mathbf{k}} t) \quad , \quad \omega_{\mathbf{k}} = |V''(0)| - k^2. \quad (1.36)$$

Consequently, modes with $\omega_{\mathbf{k}} > 0$, or equivalently $k < k_c = |V''(0)|$, are unstable. In particular, the homogeneous mode $\mathbf{k} = \mathbf{0}$ is the first one to become unstable, which means that the system spontaneously loses its spatial homogeneity. This instability is related to the formations of ordered domains. Their formation is related to the breakdown of the linear approximation. For the most unstable mode $\mathbf{k} = \mathbf{0}$, the linear approximation is no longer applicable for times t such that $|V''(0)|t \gtrsim 1$.

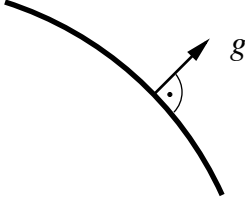


Fig. 1.14. Definition of the normal coordinate g of a domain wall.

1.2.2 Domain Walls

Having seen how the spatially homogeneous solution already becomes unstable for very short times, one now looks at the stationary solution of (1.34) in order to understand what happens in the $t \rightarrow \infty$ limit. One assumes that the walls separating the domains are sufficiently smooth. Then a normal coordinate g can be defined such that $\phi \simeq \phi(g)$ will effectively only depend on g and $\lim_{g \rightarrow \pm\infty} \phi(g) = \pm 1$, and the stationarity condition $\partial_t \phi = 0$ becomes $d^2\phi/dg^2 = V'(\phi)$. Multiplying this with $d\phi/dg$ and integrating, one has with $V(\pm 1) = 0$

$$\left(\frac{d\phi}{dg}\right)^2 = 2V(\phi) \quad (1.37)$$

and, given $V(\phi)$, one can find the shape of the domain wall explicitly.

Let us look at this for $|g| \gg 1$. One expects $\phi(g) \simeq \pm 1 + \delta\phi$, hence from the stationarity condition $\delta\phi'' \simeq V''(\pm 1)\delta\phi$ which leads to

$$\delta\phi \sim \exp\left(-\sqrt{V''(\pm 1)}|g|\right) \quad ; \quad |g| \rightarrow \infty. \quad (1.38)$$

We see that one can define a *finite saturation length* $\xi_{\text{sat}} := 1/\sqrt{V''(\pm 1)}$.

The domains are maintained since there is a **surface tension** σ between them. Using the spatial translation-invariance perpendicularly to the surface, this can be calculated as follows

$$\begin{aligned} \sigma &:= \int dg \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi) \right] = \int dg \left[\frac{1}{2} \left(\frac{d\phi}{dg}\right)^2 + V(\phi) \right] \\ &= \int dg \left(\frac{d\phi}{dg}\right)^2 = \int_{-1}^{+1} d\phi \sqrt{2V(\phi)}, \end{aligned} \quad (1.39)$$

where the Ginzburg-Landau functional was used and in the second line eq. (1.37) was applied twice, together with a change of variables. It is clear that the value of σ depends on the precise form of the potential $V(\phi)$.

1.2.3 The Allen-Cahn Equation

We now come to a very important insight which serves as the foundation for all attempts to formulate a universal theory for phase-ordering kinetics. This

is based on the simple physical observation that a surface tension is a force per surface area and should hence be proportional to the mean curvature of that surface [14].

As a simple example, consider a three-dimensional sphere of radius R . If F is the force per surface unit, the work done to reduce the radius R by dR is $W = 4\pi FR^2 dR$. Since this must be equal to the surface energy $W = 8\pi\sigma R dR$, one has $F = 2\sigma/R$. This force will cause the domain walls to move and in the over-damped limit, one expects an equation of motion of the form $\eta dR/dt = -2\sigma/R$, where η is a viscosity coefficient.

One can re-derive the same result directly from the equations of motion (1.34). Consider a spherical domain, where $\phi = -1$, in an infinite ocean where $\phi = +1$. Because of the spherical symmetry, one has in d -dimensional spherical coordinates

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial r^2} + \frac{d-1}{r} \frac{\partial\phi}{\partial r} - V'(\phi) . \quad (1.40)$$

Hence, for distances $r \gg \xi_{\text{sat}}$, large with respect to the domain wall thickness given by ξ_{sat} , one expects solutions of the form $\phi = \phi(t, r) = f(r - R(t))$, where $R = R(t)$ is the time-dependent radius of the domain. This leads to the equation

$$f'' + \left(\frac{d-1}{r} + \frac{dR}{dt} \right) f' - V'(f) = 0. \quad (1.41)$$

Since one knows that $\phi \simeq +1$ if $r > R(t) + \xi_{\text{sat}}$ and $\phi \simeq -1$ if $r < R(t) - \xi_{\text{sat}}$, $f = f(x)$ should change from -1 to $+1$ in a small region around $x = 0$ of width $\sim \xi_{\text{sat}}$, which means that $f'(x)$ should have a strong peak close to $x = 0$ (i.e. $r = R(t)$) and $f'(x) \approx 0$ otherwise. Multiplying (1.41) with f' and integrating across the interface, one has

$$\int dx \frac{d}{dx} \left(\frac{1}{2} f'(x)^2 - V(f(x)) \right) + \int dx \left(\frac{d-1}{r} + \frac{dR}{dt} \right) f'(x)^2 = 0 . \quad (1.42)$$

Because of the properties of $f'(x)$ and since $V(f) = V(-f)$, the first term in (1.42) vanishes. In the second term, a sizeable contribution only occurs for $x \approx 0$. Therefore (1.42) implies

$$\frac{d-1}{R} + \frac{dR}{dt} = 0 . \quad (1.43)$$

This is the **Allen-Cahn equation** [14] for spherical domains. It must be stressed that all explicit reference to any specific property of the system (but for the dimension d) has disappeared from the equation of motion of the linear domain size $R = R(t)$. It therefore nicely illustrates the **universality** of the growth law of domains and one might hope that similar universality might be seen in other quantities. In particular, we see that the surface tension σ , although essential in the *formation* of the domain walls, does not enter the equation of motion for $R = R(t)$.

The spherical Allen-Cahn equation (1.43) is readily solved and leads to $R(t)^2 = R(0)^2 - 2(d-1)t$ which means that a spherical domain of initial size $R(0)$ will collapse within the time $t_{\text{coll}} \sim R(0)^2$. These results have been directly confirmed experimentally in several distinct systems [576, 727].

For non-spherical domains, the above discussion can easily be generalised. We shall not repeat the calculation [14], see also [93, 94], and merely state the result

$$v = -(d-1)K \quad (1.44)$$

where v is the mean velocity of the domain walls and K is the mean curvature. Of course (1.43) is recovered for spherical domains. Again, we note the completely *universal* nature of the **Allen-Cahn equation** (1.44), which states that the motion of the domain walls depends merely on their geometry, as described through K . Again, the details of the form of the potential $V(\phi)$ do not enter, the only important feature is the existence of two equivalent minima.

Finally, the Allen-Cahn equations (1.43, 1.44) are perfectly consistent with the physics-motivated calculation at the beginning of this discussion. Consistency is achieved if $\sigma = \eta$. To see how this comes about, recall that the energy dissipation is the product of the frictional force $-\eta \mathbf{v}$ and the wall velocity $\mathbf{v}$, hence $dE/dt \simeq -\eta v^2$. On the other hand

$$\begin{aligned} \frac{dE}{dt} &= \int_{-\infty}^{\infty} dg \frac{\delta \mathcal{H}}{\delta \phi} \frac{\partial \phi}{\partial t} = - \int_{-\infty}^{\infty} dg \left(\frac{\partial \phi}{\partial t} \right)^2 \\ &= -v^2 \int_{-\infty}^{\infty} dg \left(\frac{\partial f}{\partial g} \right)^2 \simeq -v^2 \sigma \end{aligned} \quad (1.45)$$

up to higher orders in v and where the normal form $\phi(t, g) = f(g - vt)$ was used and the announced equality $\sigma = \eta$ follows. Therefore, even if the surface tension σ controls the domain-wall dynamics, its value cancels in the equation of motion of the linear domain size $R(t)$.

1.2.4 Topological Defects

Domain walls are merely the most simple example of topological structures which can arise in phase-ordering. In order to achieve a fuller understanding, it is useful to extend the consideration to $O(n)$ -symmetric models²⁰ where the order-parameter $\vec{\phi}$ has n components²¹ and the Ginzburg-Landau functional reads

²⁰ In appendix A some equilibrium properties of models with the orthogonal group $O(n)$ in n dimensions in spin space are collected.

²¹ We shall distinguish between the d -dimensional spatial vector $\mathbf{r}$ and the n -dimensional component vector $\vec{\phi} = \vec{\phi}(t, \mathbf{r})$.

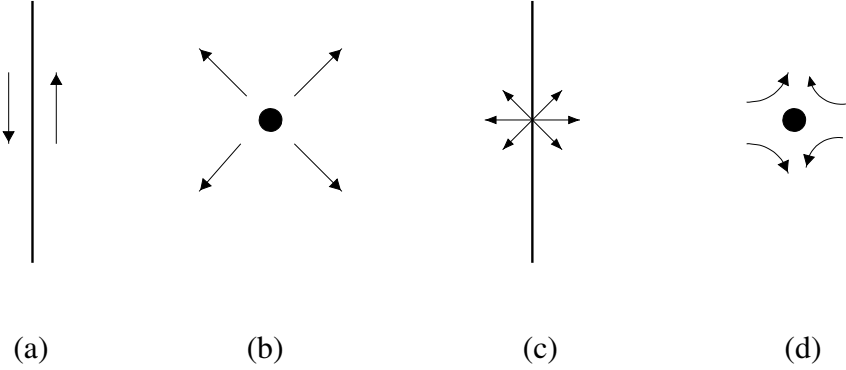


Fig. 1.15. Some simple examples of topological defects in the $O(n)$ model. Shown are (a) a domain wall ($d = 2$, $n = 1$), (b) a vortex ($d = 2$, $n = 2$), (c) a vortex line ($d = 3$, $n = 2$) and (d) an anti-vortex. The arrows indicate the local orientation of the order-parameter.

$$\mathcal{H}[\vec{\phi}] = \int d\mathbf{r} \left[\frac{1}{2} \left(\nabla \vec{\phi} \right)^2 + V(|\vec{\phi}|) \right] \quad (1.46)$$

such that the potential V only depends on the norm of the vector $\vec{\phi} \in \mathbb{R}^n$. The associated n Langevin equations read for a non-conserved order-parameter, using the same scalings and normalisations as before

$$\partial_t \phi_a = \Delta \phi_a - \frac{\partial V}{\partial \phi_a} + \eta_a, \quad a = 1, \dots, n. \quad (1.47)$$

Again, the thermal noise as described through $\vec{\eta}$ is assumed to be Gaussian and will be irrelevant for the leading scaling behaviour. We shall simply drop it from now on.

While the phase-ordering of the scalar case $n = 1$ was governed by the motion of the domain walls, which can be defined as solutions of the condition $\phi = 0$, **topological defects** are analogously defined by the n simultaneous conditions $\vec{\phi} = \vec{0}$. This set of n equations for n functions depending on d variables defines a surface of dimension $d - n$. One must require $n \leq d$, otherwise this set of equations would be over-determined. Topological defects have a stability property: local changes in the order-parameter can move, but not destroy them. In Fig. 1.15 some simple examples of topological defects are illustrated. Topological defects are created from the disordered initial state. Their evolution in time is described by eq. (1.47) as follows. If $n < d$, topological defects are spatially extended and the dynamics proceeds by smoothing them and effacing small closed loops of topological defects. On the other hand, if $n = d$, defects are point-like and the dynamics proceeds via the annihilation of defects and antidefects, which for the case $n = d = 2$ are also illustrated in Fig. 1.15bd.

One needs to characterise the ‘width’ of a defect. Since the examples of Fig. 1.15 are all radially symmetric, we concentrate on this case. As for $n < d$, the field $\vec{\phi}$ only varies in the n directions orthogonal to the defect ‘core’ and are uniform in the remaining $d - n$ ‘parallel’ directions, we may write $\vec{\phi}(\mathbf{r}) = \vec{e}_r f(r)$, where $\vec{e}_r$ is a radial unit vector. The stationary solution for radially symmetric defects then follows from the equation

$$f''(r) + \frac{n-1}{r} f'(r) - \frac{n-1}{r^2} f(r) - V'(f(r)) = 0 \quad (1.48)$$

together with the boundary conditions $f(0) = 0$ and $f(\infty) = 1$. The approach to saturation for r large can be found by writing $f(r) = 1 - \epsilon(r)$ and to leading order one has [93, 94]

$$\epsilon(r) \simeq \frac{n-1}{V''(1)} \frac{1}{r^2} ; \quad r \rightarrow \infty \quad (1.49)$$

and which is quite different from the exponential approach found above for the scalar case $n = 1$, see (1.38). For $n > 1$, a finite ‘core’ size ξ_{core} may be defined through $f(r) \simeq 1 - (\xi_{\text{core}}/r)^2$ for r large, hence $\xi_{\text{core}} = \sqrt{(n-1)/V''(1)}$.

In conclusion, topological defects have for all values $n \geq 1$ a finite width ξ_d , namely $\xi_d = \xi_{\text{sat}}$ for $n = 1$ and $\xi_d = \xi_{\text{core}}$ for $n > 1$.

1.2.5 Porod’s Law

We now discuss the scaling behaviour of the equal-time correlation function

$$C_t(\mathbf{r}) = \left\langle \vec{\phi}(t, \mathbf{r}_0 + \mathbf{r}) \cdot \vec{\phi}(t, \mathbf{r}_0) \right\rangle \quad (1.50)$$

and the average is over the initial conditions. Spatial translation-invariance will be assumed here and later. If $L = L(t)$ is the typical radius of curvature of the topological defects, we are interested in the region

$$\xi_d \ll |\mathbf{r}| \ll L . \quad (1.51)$$

The first condition assures that the order-parameter is saturated and the second guarantees that the defects are essentially unaffected by the other defects, at a typical distance L .

First, consider scalar fields. Then $\phi(t, \mathbf{r}_0 + \mathbf{r})\phi(t, \mathbf{r}_0)$ has the value -1 if a domain wall passes between the space points $\mathbf{r}_0$ and $\mathbf{r}_0 + \mathbf{r}$ and has the value $+1$ if there is none. The probability to find more than two walls is negligibly small, since $r \ll L$. The probability that a randomly-placed line of length r cuts a domain walls should be $\mathfrak{P}_1 \approx \mathfrak{p}r/L$ for $r \ll L$, with some constant $\mathfrak{p}$. Hence, for $r \ll L$

$$C_t(\mathbf{r}) \simeq (-1)\mathfrak{P}_1 + (+1)(1 - \mathfrak{P}_1) \approx 1 - 2\mathbf{p}\mathbf{r}/L . \quad (1.52)$$

Remarkably, this result is *non-analytic* in $\mathbf{r}$, or rather more precisely, as $\mathbf{r}/L \rightarrow 0$. It has been observed both in simulations and experiments many times, see Fig. 1.16 below. A tangible consequence is the following power-law tail in the **structure factor**

$$S_t(\mathbf{k}) := \left\langle \widehat{\phi}_{\mathbf{k}}(t) \widehat{\phi}_{-\mathbf{k}}(t) \right\rangle \sim \frac{1}{L k^{d+1}} \quad (1.53)$$

which is known as **Porod's law** and is expected to hold in the regime $L^{-1} \ll k \ll \xi_d^{-1}$. The factor $1/L$ in (1.53) is proportional to the total surface of domain walls per unit volume.²²

For the $O(n)$ model, the dimension of the defects is $d - n$, hence the total surface of defects per unit volume is L^{-n} . One then expects the following scaling behaviour

$$\begin{aligned} S_t(\mathbf{k}) &:= \left\langle \vec{\widehat{\phi}}_{\mathbf{k}}(t) \cdot \vec{\widehat{\phi}}_{-\mathbf{k}}(t) \right\rangle = L^d g(kL) \\ g(y) &\sim y^{-d-n} ; \quad y \rightarrow \infty \end{aligned} \quad (1.54)$$

usually referred to as **generalised Porod's law**. This result can also be derived for non-conserved fields from approximate treatments of the equations of motion, see [93, 94] and references therein, but the simple heuristic argument reproduced here suggests that eq. (1.54) may indeed be very general and its validity should extend considerably beyond the specific context treated here.

In practice, Porod's law is usually observed in momentum space. In Fig. 1.16 we show the change in the shape of the structure factor $S_t(\mathbf{k})$ measured from X-ray scattering along (010) direction in the binary alloy Cu_3Au , quenched to $T = 365.8^\circ\text{C}$ well below the critical temperature $T_c = 390^\circ\text{C}$ [662]. For relatively short times, a Gaussian shape is observed which changes its form once domain growth with its dynamical scaling sets in for sufficiently large times. In this later regime, where a Lorentzian-squared form

$$S_t(\mathbf{k}) = S_0 \left[1 + \left(\frac{\mathbf{k} - \mathbf{k}_0}{\Gamma(t)} \right)^2 \right]^{-2} \quad (1.55)$$

describes the data well and for $|\mathbf{k}|$ large, Porod's law (1.53) is recovered. In 3D direct space, a structure factor with the form of a Lorentzian squared leads to

$$C_t(\mathbf{r}) = \int_{\mathbb{R}^3} d\mathbf{r} e^{i\mathbf{k} \cdot \mathbf{r}} S_t(\mathbf{k}) = C_t(\mathbf{0}) e^{-r\Gamma(t)} \quad (1.56)$$

²² For $kL \gg 1$ a scattering experiment will probe structures much smaller than the radius of curvature of the domain-wall spacing and the structure factor, which is proportional to the scattered intensity, should then scale with the total surface of domain walls.

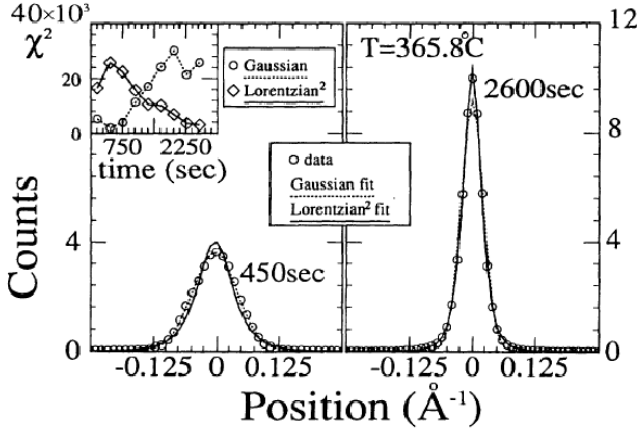


Fig. 1.16. Change in the shape of the structure factor measured along the (010) direction in Cu_3Au . The passage from a Gaussian shape at $t = 450$ s (left panel) to a square Lorentzian shape at late times ($t = 2600$ s, right panel) is clearly seen. The inset shows the χ^2 for the two fits. Reprinted with permission from [662]. Copyright (1992) by the American Physical Society.

from which one can identify $\Gamma(t) \sim L(t)^{-1}$ with the inverse linear domain size. This form shows the non-analytic peak at $|\mathbf{r}| = 0$, in agreement with (1.52).

Of course, the simple scaling functions just discussed should merely be regarded as parametrisations which can be applied to experimental or numerical data. In order to illustrate the analytic complexity of the correlation function, we now quote a scaling function due to Bray, Puri and Toyoki [100, 700] in the $O(n)$ model. Their result generalises an approximate non-linear transformation found by Ohta, Jasnow and Kawasaki [556] which reduces the phase-ordering problem for a scalar order-parameter to the diffusive motion of the interfaces. It reads

$$C_t(\mathbf{r}) = \frac{n}{2\pi} e^{-\mathbf{r}^2/L^2(t)} B\left(\frac{n+1}{2}, \frac{1}{2}\right)^2 {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{n+2}{2}; \exp\left(-\frac{2\mathbf{r}^2}{L^2(t)}\right)\right) \quad (1.57)$$

where $L(t)$ is the domain size, $B(x, y)$ is Euler's beta function and ${}_2F_1$ is a hypergeometric function [4]. In the scalar case $n = 1$, one has explicitly $L^2(t) = 8(d-1)/d \cdot t$ and $C_t(\mathbf{r}) = (2/\pi)^{-1} \arcsin(\exp[-\mathbf{r}^2/L^2(t)])$ which reproduces the result of [556]. The main assumption in the derivation of the formula (1.57) is that the defect core size vanishes. Although this assumption is never strictly satisfied in simulations, the above result gives good description of many numerical data, in particular if one merely takes $L(t) \simeq L_0 t^{1/2}$ for the domain size and fits L_0 to the data. The effects of non-zero core size (i.e. interface thickness for the scalar case $n = 1$) have been added by Oono

and Puri [565].²³ On the other hand, Blundell, Bray and Sattler [86] have pointed out that if one also considers the effects of phase-ordering on different observables (i.e. the energy density) a good agreement with the analogue of eq. (1.57) is only found when L_0 takes a *different* value for each observable. In addition, they consider parameter-free ways of testing the prediction eq. (1.57) and find systematic deviations with simulational data in soft spin models which are thought to be in the same universality class as the kinetic Ising model.

1.2.6 Bray-Rutenberg Theory for the Growth Law

It has been shown by Bray and Rutenberg [101, 102] that the growth law $L = L(t)$ can be found under the following conditions: (i) dynamical scaling prevails, (ii) the generalised Porod's law eq. (1.54) is valid, and (iii) phase-ordering is dominated by the motion of the typical topological defects structures and not the disappearance of small bubbles. We shall give a simplified sketch of their theory and refer to their original papers [101, 102] and [93, 94] for a fuller discussion.

The equation of motion for $T = 0$ of the Fourier components is

$$\partial_t \vec{\hat{\phi}}_{\mathbf{k}} = -k^\mu \frac{\delta \mathcal{H}}{\delta \vec{\hat{\phi}}_{-\mathbf{k}}} \quad (1.58)$$

where the non-conserved case treated so far is recovered for $\mu = 0$ while the case of a *conserved* order-parameter is described by $\mu = 2$ (see below). The total energy per volume V is $\mathfrak{h} = \mathcal{H}/V$ and the rate of energy dissipation is given by

$$\frac{d\mathfrak{h}}{dt} = \frac{1}{(2\pi)^d} \int d\mathbf{k} \left\langle \left(\frac{\delta \mathcal{H}}{\delta \vec{\hat{\phi}}_{\mathbf{k}}} \right) \cdot \partial_t \vec{\hat{\phi}}_{\mathbf{k}} \right\rangle = -\frac{1}{(2\pi)^d} \int d\mathbf{k} k^{-\mu} \left\langle \partial_t \vec{\hat{\phi}}_{\mathbf{k}} \cdot \partial_t \vec{\hat{\phi}}_{-\mathbf{k}} \right\rangle \quad (1.59)$$

where the equation of motion was used. This relates the energy dissipation to the dynamics of the correlation function and will yield the law for $L(t)$.

²³ For example, choosing a Gaussian form for the *slope* of the interface profile, they studied the scalar case $n = 1$ [565]

$$C_t(\mathbf{r}) = \arcsin \left(\alpha e^{-\mathbf{r}^2/L^2(t)} \right) / \arcsin \alpha, \quad \alpha := (1 + w_i^2 t^{-1})^{-1}$$

where w_i is a constant proportional to the interface width. This expression agrees with Porod's law and also with the Tomita sum rule [698, 699], from which the small-distance behaviour $C_t(\mathbf{r}) = 1 - \mathbf{c}_1(|\mathbf{r}|/L(t)) + \mathcal{O}((|\mathbf{r}|/L(t))^3)$ can be derived for a scalar order-parameter. Recall, however, that above the roughening temperature, the interface width actually grows in time according to $w_i \sim t^{1/4}$ in the 2D Ising model and $w_i \sim \ln^{1/2} t$ in the 3D Ising model, respectively [2, 3].

First, one considers the right-hand side of (1.59). One may extend the hypothesis of dynamical scaling to the two-time correlation function

$$\left\langle \vec{\widehat{\phi}}_{\mathbf{k}}(t) \cdot \vec{\widehat{\phi}}_{-\mathbf{k}}(t') \right\rangle = k^{-d} \mathbf{g}(kL(t), kL(t')) \quad (1.60)$$

and hence

$$\left\langle \partial_t \vec{\widehat{\phi}}_{\mathbf{k}}(t) \cdot \partial_t \vec{\widehat{\phi}}_{-\mathbf{k}}(t) \right\rangle = \frac{\partial^2}{\partial t \partial t'} \left\langle \vec{\widehat{\phi}}_{\mathbf{k}}(t) \cdot \vec{\widehat{\phi}}_{-\mathbf{k}}(t') \right\rangle \Big|_{t=t'} = \dot{L}^2 L^{d-2} g_1(kL)$$

with a scaling function g_1 . Next, one has to show that

$$\int d\mathbf{k} k^{-\mu} \left\langle \partial_t \vec{\widehat{\phi}}_{\mathbf{k}} \cdot \partial_t \vec{\widehat{\phi}}_{-\mathbf{k}} \right\rangle \sim \begin{cases} \dot{L}^2 L^{-n} \xi_d^{n+\mu-2} & ; \text{ if } n + \mu < 2 \\ \dot{L}^2 L^{-n} \ln(L/\xi_d) & ; \text{ if } n + \mu = 2 \\ \dot{L}^2 L^{\mu-2} & ; \text{ if } n + \mu > 2 \end{cases} \quad (1.61)$$

The behaviour of the above integral depends on whether the integral converges as $k \rightarrow \infty$. If the integral is divergent, one needs in principle the form of the scaling function g_1 for $kL \gg 1$, which is hard to come by.

A simple heuristic argument runs as follows. Consider a scalar field for simplicity. In a frame co-moving with the domain wall, one has $d\phi/dt = 0$, hence $\partial_t \phi = -\mathbf{v} \cdot \nabla \phi$, where $\mathbf{v}$ is the velocity of the interface and one sees that $\partial_t \phi$ is non-vanishing only near to the interface. Hence, in direct space

$$\begin{aligned} \langle \partial_t \phi(t, \mathbf{r}_1) \partial_t \phi(t, \mathbf{r}_2) \rangle &= \langle \mathbf{v}_1 \cdot \nabla \phi(t, \mathbf{r}_1) \mathbf{v}_2 \cdot \nabla \phi(t, \mathbf{r}_2) \rangle \\ &= \frac{\langle v^2 \rangle}{d} \langle \nabla \phi(t, \mathbf{r}_1) \cdot \nabla \phi(t, \mathbf{r}_2) \rangle \end{aligned}$$

since for $r \ll L$, the two points $\mathbf{r}_{1,2}$ must belong to the *same* interface and v is varying slowly along the essentially flat section of the interface considered. Then the averages over positions and velocities can be performed independently. In Fourier space this gives

$$\left\langle \partial_t \widehat{\phi}_{\mathbf{k}}(t) \partial_t \widehat{\phi}_{-\mathbf{k}}(t) \right\rangle = \frac{\langle v^2 \rangle}{d} k^2 S_t(\mathbf{k}) \sim \frac{\langle v^2 \rangle}{L k^{d-1}} \quad (1.62)$$

for $kL \gg 1$ because of Porod's law (1.53). Since the typical interface velocity is $\dot{L}$, assumption (iii) made above leads to $\langle v^2 \rangle \sim \dot{L}^2$. Finally, repeating the above heuristics which led to the generalised Porod's law (1.54), Bray and Rutenberg extended the above argument and arrive for the $O(n)$ model at

$$\left\langle \partial_t \vec{\widehat{\phi}}_{\mathbf{k}}(t) \cdot \partial_t \vec{\widehat{\phi}}_{-\mathbf{k}}(t) \right\rangle \sim \frac{\dot{L}^2}{L^n k^{d+n-2}} ; \quad kL \gg 1 . \quad (1.63)$$

According to this argument, the integral (1.61) diverges if $n + \mu \leq 2$. Subject to the assumptions made, notably assumption (iii), this is also the region

where the result (1.63) is expected to hold [101, 102]. The integral is then dominated by k near to the cutoff $1/\xi_d$ and the first two lines in (1.61) follow. On the other hand, if $n + \mu > 2$, the integral is convergent for $k \rightarrow \infty$ which leads immediately to the last line in (1.61).

Second, one must consider the left-hand side of (1.59). One calculates the scaling behaviour of the energy density

$$\mathfrak{h} \sim \left\langle (\nabla \vec{\phi})^2 \right\rangle = \frac{1}{(2\pi)^d} \int d\mathbf{k} k^2 L^d g(kL) . \quad (1.64)$$

Because of (1.54), the integral converges for $k \rightarrow \infty$ if $n > 2$ and $\mathfrak{h} \sim L^{-2}$. For $n \leq 2$, one imposes a cut-off at $k \sim 1/\xi_d$ and obtains

$$\mathfrak{h} \sim \begin{cases} L^{-n} \xi_d^{n-2} & ; \text{ if } n < 2 \\ L^{-2} \ln(L/\xi_d) & ; \text{ if } n = 2 \\ L^{-2} & ; \text{ if } n > 2 \end{cases} . \quad (1.65)$$

Third, one must compare the results eqs. (1.61,1.65) through the basic relation (1.59) and solve the resulting differential equations for $L = L(t)$. The Bray-Rutenberg result for the late-times behaviour of $L(t)$ is listed in Table 1.4, with $\mu = 0$ for a non-conserved order-parameter.

	$n < 2$	$n = 2$	$n > 2$
$n + \mu < 2$	$t^{1/2}$		
$n + \mu = 2$	$(t/\ln t)^{1/2}$	$t^{1/2}$	
$n + \mu > 2$	$t^{1/(n+\mu)}$	$(t \ln t)^{1/(2+\mu)}$	$t^{1/(2+\mu)}$

Table 1.4. Growth laws $L = L(t)$ according to Bray and Rutenberg [101, 102] for coarsening systems quenched to $T < T_c$, for both non-conserved ($\mu = 0$) and conserved ($\mu = 2$) order-parameter in the $O(n)$ model.

Some comments are in order. First, for non-conserved systems with $d > n$ or $n > 2$, one always has $L(t) \sim t^{1/2}$, hence the dynamical exponent $z = 2$. For $d > n = 2$ there are important logarithmic corrections of the form $L(t) \sim t^{1/2}(1 + O(1/\ln t))$. For non-conserved scalar fields with $d > 1$ both the energy density (1.65) and the dissipation (1.61) have the same dependence on the core size, which therefore disappears from $L(t) \sim t^{1/2}$. This simple result might be regarded as fortuitous. Bray and Rutenberg carefully point out the crucial rôle played by their assumption (iii) and clearly state that their results need no longer hold true if that assumption is not satisfied. Indeed, this situation occurs for $n = d \leq 2$. There are two physical models in this class: the 1D scalar theory at $T = 0$ where the methods of Bray and Rutenberg cannot be applied and results may even depend on the detailed form of the potential $V(\phi)$. While one usually expects that the scalar theory should correspond to

Material	T_c (K)	$1/z$	λ_C	Ref.
Cu ₃ Au	658.5	0.50(3)		[662] first order
Fe ₃ Al	825	0.5		[576]
ZLI-5014-100 (Merck)	343.6	0.5(1)		[200] liquid crystal
Merck (CCH-501)	310.2	0.515(26)	1.246(79)	[505] 2D liquid crystal
8OBE	372.4	0.44(7)		[189] chiral liquid crystal
Cu _{0.79} Pd _{0.21}	751	0.5		[727] long-period superlattice

model	d	$1/z$	Ref
TDGL	3	0.51(10)	[106]
TDGL	2	0.498(6)	[154]
Potts-8	2	0.48(4)	[626] first order
XY	2	0.51(1)	[487] ordered init. state
	2	0.5 (log)	[625, 95] disord. init. state

Table 1.5. Dynamical exponent $1/z$ in phase-ordering with non-conserved order-parameter, as directly estimated from $L(t)$ in several materials and a few models. If available, experimental values of the autocorrelation exponent λ_C are also given, see eq. (1.77) for the definition. Critical temperatures T_c are given in kelvin and TDGL is the time-dependent Ginzburg-Landau equation. The numbers in brackets give the estimated error in the last given digit(s).

the kinetic Ising model (with Glauber-like dynamics) an exact calculation has shown that at least in $1D$ this correspondence does no longer hold [96], see also Table 1.7 below. We shall see later from the exact solution of the $1D$ Glauber-Ising model that $z = 2$ there. In the $2D$ XY model a fuller study has shown that the linear domain size scales for large times as $L(t) \sim (t/\ln t)^{1/2}$; hence dynamical scaling is broken by an additional logarithmic factor [95]. We shall return to the XY model in Chap. 2.

In summary, the Bray-Rutenberg theory [101, 102] gives the dynamical exponent defined from $L(t) \sim t^{1/z}$ as $z = 2$ for a non-conserved order-parameter (where $\mu = 0$), up to possible logarithmic factors. For a conserved order-parameter, they predict $z = 3$ for a scalar order-parameter and $z = 4$ for vector order-parameters ($n > 1$).

Having seen the assumptions which went into the Bray-Rutenberg theory in order to calculate the dynamical exponent z , it is interesting to compare the predictions of Table 1.4 with simulational and experimental results. It is fair to say that indirect confirmations for the values of z abound. Direct tests, however, are relatively rare. In Table 1.5 we collect some experimental results for the growth law at the late stages of phase-ordering and some direct simulational confirmations for the form $L(t) \sim t^{1/z}$, for a non-conserved order-parameter. The agreement with the prediction $z = 2$ is impressive.

The growth law of the q -states Potts model is also described by the Bray-Rutenberg theory. Since there are q distinct ordered states, in principle either two or several of these can meet in a defect. But since the Porod tail and the

energy density are dominated by the domain walls, the Potts model behaves in this respect as a scalar system ($n = 1$). On the other hand, for bulk nematic liquid crystals the dominant defects are strings, hence $n = 2$ and for a dynamics without conservation $L(t) \sim t^{1/2}$.

1.2.7 Exact Result in Two Dimensions

In Bray-Rutenberg theory dynamical scaling equivalent to the form $L(t) \sim t^{1/z}$ had to be assumed. Recently, a geometric argument has been proposed by Arenzon, Bray, Cugliandolo and Sicilia, which permits this to be derived for a *non*-conserved scalar order-parameter, in *two* dimensions [27]. Consider the area A of the interior, the **hull**, of a domain boundary. According to the Allen-Cahn equation (1.44), the velocity v of the domain boundary is proportional to the local curvature $\kappa \sim L^{-1}$, viz. $v = -K$ in two dimensions and using the normalisations from above, with the kinetic coefficient arbitrarily set to unity. The change in the area is found by integrating the velocity around the hull

$$\frac{dA}{dt} = \oint v d\ell = - \oint K d\ell = -2\pi \quad (1.66)$$

where first the Allen-Cahn equation and then the Gauß-Bonnet theorem was used. Therefore, at time t hulls with an enclosed area less than $2\pi t$ will have disappeared and if $N_h(A, t)$ is the number of hulls per unit area of the system with enclosed area A at time t , one has

$$N_h(A, t) = N_h(A + 2\pi t, 0) \quad (1.67)$$

Shortly after the quench from the infinite-temperature initial state, the system must pass through the critical point of continuum percolation, where it was shown by Cardy and Ziff [128] that for $A \rightarrow \infty$ the number of percolation hulls per unit area with area greater than A is $N_p(A) = (8\pi\sqrt{3})^{-1}A^{-1}$. Hence [27]

$$N_h(A, t) = 2N_p(A + 2\pi t) = \frac{1}{4\pi\sqrt{3}} \frac{1}{A + 2\pi t} \quad (1.68)$$

where the factor 2 arises since in the domain growth problem there are two types of hull, while the percolation argument [128] only counts the clusters of occupied sites. This has the scaling form $N_h(A, t) = t^{-1}f_h(A/t)$, hence $L(t) \sim A(t)^{1/2} \sim t^{1/2}$ and in agreement with Bray and Rutenberg. In this case, the power-law form for the single relevant length scale $L(t)$ no longer needs to be given as an input.

On the other hand, if an initial state at criticality is used, it can be shown that the end result (1.68) must be multiplied by $\frac{1}{2}$ [128]. For both values of the initial temperature, $T_{\text{ini}} = \infty$ and $T_{\text{ini}} = T_c$, simulations in the 2D Ising model at $T = 0$ have confirmed the scaling function predicted in (1.68)

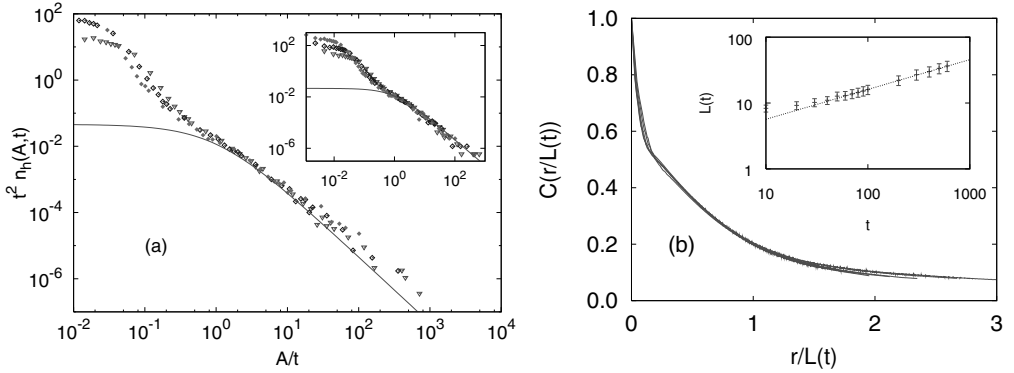


Fig. 1.17. (a) Scaling of number density $n_h(A, t)$ of hull-enclosed area in a liquid crystal, at three different times, where the lines are the prediction (1.69). In the inset domains which touch the border are excluded. (b) Scaling of the spatial correlation function for times $t = [100, 200, 300, 400, 500]$ seconds after the quench. The inset shows the time-dependent length-scale $L(t)$ as a function of time, where the straight line gives $1/z = 0.45(10)$. Reprinted with permission from [671]. Copyright (2008) by the American Physical Society.

[27]. Recent numerical evidence has suggested the validity of (1.68) even for *disordered* Ising models quenched to $T < T_c$ [670].

A recent experiment in the achiral liquid crystal JVC KY-F1030 [671] has measured the single-time correlator $C_t(\mathbf{r})$ and the hull-enclosed areas per unit area $n_h(A, t)dA$, with

$$n_h(A, t) = -\frac{\partial N_h(A, t)}{\partial A} = \frac{1}{4\pi\sqrt{3}} \left(\frac{1}{A + \lambda_h t} \right)^2 \quad (1.69)$$

where λ_h is a material-dependent constant which relates the local velocity v of an interface and the local curvature K via the Allen-Cahn equation $v = -(\lambda_h/2\pi)K$. When cooled from the isotropic liquid, liquid crystals such as the one considered in this experiment form an optically isotropic phase but upon the application of an external electric field, **chiral deracemisation** occurs where domains of opposite handedness grow in size as a function of time, as can be seen from snapshots of the growing structures [671]. Identifying these two states with the two states of a simple Ising model, the equal-time correlator $C_t(\mathbf{r}) = \mathcal{C}(r/L(t))$ can be measured. In Fig. 1.17b the observed data collapse with a time-dependent length-scale $L(t) \sim t^{1/z}$ and the estimated exponent $z \approx 2$ is as expected for domain coarsening without conservation laws, after some initial transient. The experimentally observed scaling of the number density $n_h(A, t) = (\lambda_h t)^{-2} f(A/\lambda_h t)$ of the hull-enclosed area²⁴ is shown

²⁴ Note that the initial ‘magnetisation’ is not strictly zero, but it remains roughly conserved during the experiment [671].

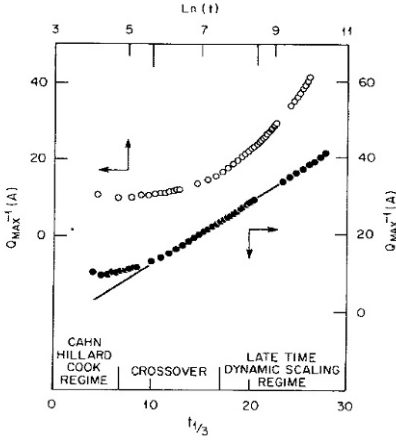


Fig. 1.18. Temporal evolution of the growth law in the phase-separating binary alloy $\text{Mn}_{0.67}\text{Cu}_{0.33}$. The open circles test a hypothetical relation $Q_{\max}^{-1}(t) \sim \ln t$. The full circles test, with the same data, the growth law $L(t) \sim Q_{\max}^{-1}(t) \sim t^{1/3}$. Reprinted with permission from [272]. Copyright (1987) by the American Physical Society.

in Fig. 1.17a, with the universal scaling function $f(x) = (4\pi\sqrt{3})^{-1}(x+1)^{-2}$. In this way, the relevant time-dependent length scale $L(t) = \sqrt{\lambda_h t}$ can be identified (the upward deviation of the data with respect to the asymptotic form $1/A^2$ should come from the finite image size). This gives a nice experimental confirmation of the scaling picture and the exact predictions for domain growth.

1.2.8 Conserved Order-parameter: Phase-separation

We now briefly discuss the dynamics with a conserved order-parameter. This describes **phase-separation** which occurs for example in the demixing of binary alloys. The equation of motion is here a continuity equation, $\partial_t \phi = -\nabla \cdot \mathbf{j}$ and with a current $\mathbf{j} = -\Gamma \nabla \delta \mathcal{H} / \delta \phi$. In momentum space, one recovers for $T = 0$ the equation of motion (1.58) with $\mu = 2$ (which is usually referred to as **Cahn-Hilliard equation**). We shall not describe here how the arguments of this section which have led us to the Bray-Rutenberg theory [101, 102] can be carried over to the conserved case and refer to the excellent reviews [93, 94]. The result of these calculations is also included in Table 1.4, with $\mu = 2$.²⁵

The Bray-Rutenberg prediction $z = 3$ for a scalar order-parameter with a dynamics described by the Cahn-Hilliard equation, is in agreement with a rigorous bound, due to Kohn and Otto [448]. Considering the interfacial area per unit volume $E(t) \sim 1/L(t)$, they establish that there is a positive constant C such that for sufficiently large times t

$$\frac{1}{t} \int_0^t dt' E^2(t') \geq \frac{C}{t} \int_0^t dt' t'^{-2/3} \quad (1.70)$$

²⁵ A recent study includes a finite-size scaling analysis and obtains $1/z = 0.334(4)$ in the 2D Ising model [497].

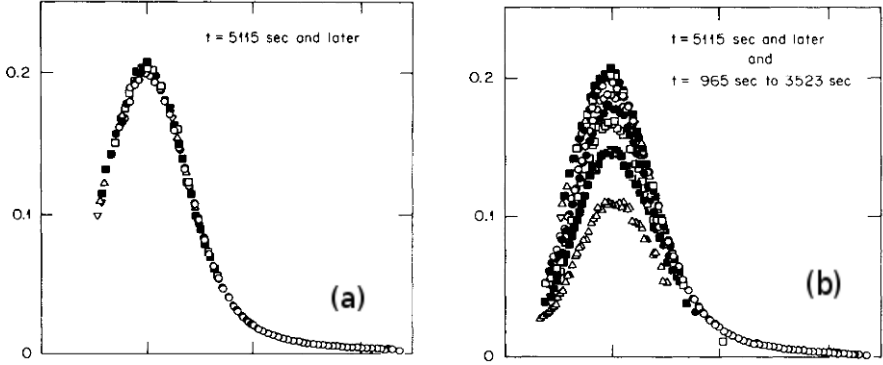


Fig. 1.19. Dynamical scaling of the rescaled structure factor $|Q_{\max}(t)|^3 S_t(Q)$ over against $|Q|/|Q_{\max}(t)|$ in the phase-separating binary alloy $\text{Mn}_{0.67}\text{Cu}_{0.33}$. (a) Dynamical scaling is realised for sufficiently long times $t \geq 5115$ seconds. (b) For times $t = [965, 1602, 2239, 2886, 3523]$ seconds from bottom to top, there is not yet a data collapse but a slow evolution towards the scaling regime. Reprinted with permission from [272]. Copyright (1987) by the American Physical Society.

which is a time-averaged version of the yet unproven bound $E(t)^{-1} \leq C' t^{1/3}$.

A nice experimental illustration of the growth law $L(t) \sim t^{1/3}$ for phase-separation with a scalar order-parameter (e.g. Ising model) comes from a neutron-scattering experiment on the 3D binary alloy $\text{Mn}_{0.67}\text{Cu}_{0.33}$ [272]. The sample was initially prepared at a temperature of 800°C , before it was quenched to the measurement temperature of 450°C , where there is a clear miscibility gap in Mn-rich alloys such that one is well in the coexistence phase. The scattering experiments give directly the Fourier transform $S_t(Q)$ of the single-time correlator $C_t(\mathbf{r}) = \langle \phi(t, \mathbf{r}) \phi(t, \mathbf{0}) \rangle$ and for each fixed time t , the data show a clear maximum at some momentum $Q_{\max}(t)$. In the context of Bray-Rutenberg theory, the domain size $L(t) \sim |Q_{\max}(t)|^{-1} \sim t^{1/3}$ for sufficiently long times. In Fig. 1.18 the several temporal regimes which are experimentally observed are indicated. The emergence of the late-time scaling regime is clearly seen. This may also be seen more directly by looking at the scaling form $S_t(Q) = |Q_{\max}(t)|^{-d} \mathcal{S}(|Q|/|Q_{\max}(t)|)$. This is illustrated in Fig. 1.19, which shows a clear data collapse for sufficiently large times t , but one also sees that dynamical scaling only occurs in the late-time regime. This is analogous to the change in the shape of the structure factor seen for the phase-ordering system Cu_3Au in Fig. 1.16.

Although the entire discussion was presented here in the language of magnetic systems, similar effects can also be observed in completely different non-magnetic systems. As an example, we consider phase-separation in a **diblock copolymer** [564]. A diblock copolymer is a long chain molecule $a - b$, which consists of two different molecular chains a and b attached by a covalent chemical bond to form a single chain. A melt of such block copolymers will mix

homogeneously above the critical temperature T_c , but if below T_c the two blocks tend to separate, a lamellar phase can form, with the a 's and b 's segregated. It can be shown that the lamellar thickness $D \sim N^{2/z}$, where N is the molecular weight [564]. The experimental finding $2/z \simeq 2/3$ [319] is in agreement with the prediction of the Bray-Rutenberg theory.

1.2.9 Critical Dynamics

In critical dynamics, the natural length scale is set by the correlation length ξ which describes the size of the correlated, but no longer ordered clusters. In particular, the assumption (iii) of the Bray-Rutenberg theory is no longer satisfied and although $L(t) \sim \xi(t) \sim t^{1/z}$ still holds, the values of z are different and can be determined from *equilibrium* critical dynamics. For example, in the $O(n)$ model, a field-theoretical renormalisation-group analysis gives in $4 - \varepsilon$ dimensions [759, 253]

$$z = z(T_c) = \begin{cases} 2 + (6 \ln \frac{4}{3} - 1)\eta + O(\varepsilon^3) & ; \text{ non-conserved (model A)} \\ 4 - \eta & ; \text{ conserved (model B)} \end{cases} \quad (1.71)$$

depending on whether the order-parameter is conserved or not and where η is a standard equilibrium critical exponent. For the $O(n)$ -model in $4 - \varepsilon$ dimensions, its value is $\eta = \frac{1}{2} \frac{n+2}{(n+8)^2} \varepsilon^2 + O(\varepsilon^3)$. Results for z obtained from direct numerical simulations at $T = T_c$ are listed in Table 1.7 below. Further results will be reviewed in Chap. 3.

1.3 Phenomenology of Ageing

We now consider general results for phenomenological scaling behaviour. Throughout, we consider a fully disordered initial state with vanishing initial magnetisation. In their classic review, Hohenberg and Halperin [370] gave a classification of the various kinds of *equilibrium* critical dynamics. Their terminology has become generally used. In this book, we shall merely consider three of their dynamic classes:

1. **Model A**, without any conservation law. A typical example is the Glauber-Ising model. This will be discussed in the rest of this chapter (unless explicitly stated otherwise) and in Sections 3.1.1 and 3.2.
2. **Model B**, where the order-parameter is conserved. A typical example is the Kawasaki-Ising model. See Sect. 3.3.
3. **Model C**, where the order-parameter is not conserved, but the energy density is conserved. A typical example is the *antiferromagnetic* kinetic Ising model with a conserved magnetisation. See Sect. 3.1.2.

An excellent recent review of the other classes, especially with numerous results for the dynamical exponent z , is given in [253]. Ageing phenomena arise when considering the relaxation from a non-equilibrium initial state. This is habitually referred to as follows.

1. For a quench to the temperature $T = T_c$, one speaks of **non-equilibrium critical dynamics**, further distinguished as model A, B or C.
2. For a quench to temperatures $T < T_c$, one speaks of **phase-ordering** for model A dynamics and of **phase-separation** for model B dynamics.

1.3.1 Scaling Forms

From now on, we shall consider merely the most simple case which arises in situations where there is a single time-dependent length scale of the form

$$L(t) \sim t^{1/z} \quad (1.72)$$

where z is the **dynamical exponent**. This situation occurs for instance in the kinetics of phase-ordering [93] or in the critical dynamics (in and out of equilibrium) of simple magnets [287] or *critical* spin glasses [333]. One of the main aspects of ageing, namely the breaking of time-translational invariance, is made explicit in two-time observables and we choose to focus on them here. Indeed, one finds that there is a **scaling regime** for large times t, s such that

$$t \gg \tau_{\text{micro}} \quad , \quad s \gg \tau_{\text{micro}} \quad \text{and} \quad t - s \gg \tau_{\text{micro}} \quad (1.73)$$

where τ_{micro} is some microscopic reference time-scale. In the scaling regime (1.73), the autocorrelation and autoresponse functions should satisfy²⁶

$$C(t, s) = \langle \phi(t) \phi(s) \rangle = s^{-b} f_C(t/s) \quad (1.74)$$

$$R(t, s) = \left. \frac{\delta \langle \phi(t) \rangle}{\delta h(s)} \right|_{h=0} = s^{-1-a} f_R(t/s) \quad (1.75)$$

where a and b are so-called **ageing exponents** and $f_{C,R}(y)$ are scaling functions. They can be precisely defined through the following **scaling limit**

$$\begin{aligned} f_C(y) &= \lim_{s \rightarrow \infty} s^b C(ys, s) \Big|_{y>1} \\ f_R(y) &= \lim_{s \rightarrow \infty} s^{1+a} R(ys, s) \Big|_{y>1} \end{aligned} \quad (1.76)$$

with both $t, s \rightarrow \infty$ but such that $y = t/s$ is kept fixed at some value $y > 1$.

Physically, the first two conditions (1.73) mean that the lengths $L(t)$ and $L(s)$ are both large with respect to the lattice constant, while the third one

²⁶ See exercise 1.12 for a comparison with the *global* correlation/response functions.

guarantees that also $L(t-s)$ is large with respect to microscopic length scales, which means that the local state must have had enough time to become statistical again, between the two measurements. We stress the importance of the third condition (1.73) for obtaining scaling and shall see below that not taking this into account can lead to serious misinterpretations of data.

Usually, one expects that for large arguments $y \rightarrow \infty$, the scaling functions should follow a simple power law

$$f_C(y) \sim y^{-\lambda_C/z} \quad , \quad f_R(y) \sim y^{-\lambda_R/z} \quad (1.77)$$

which defines the **autocorrelation exponent** λ_C [245] and the **autoresponse exponent** λ_R [588].²⁷ Alternatively, one often proposes to extract these exponents from the asymptotic behaviour $C(t, 0) \sim t^{-\lambda_{C'}/z}$ and $R(t, 0) \sim t^{-\lambda_{R'}/z}$ for $t \rightarrow \infty$. A formal definition of the autocorrelation and autoresponse exponents requires to consider the double limits

$$\frac{\lambda_C}{z} := - \lim_{y \rightarrow \infty} \left(\lim_{s \rightarrow \infty} \frac{\ln(s^b C(ys, s))}{\ln y} \right) \quad , \quad \frac{\lambda_{C'}}{z} := - \lim_{t \rightarrow \infty} \left(\lim_{s \rightarrow 0} \frac{\ln C(t, s)}{\ln t} \right) \quad (1.78)$$

$$\frac{\lambda_R}{z} := - \lim_{y \rightarrow \infty} \left(\lim_{s \rightarrow \infty} \frac{\ln(s^{1+a} R(ys, s))}{\ln y} \right) \quad , \quad \frac{\lambda_{R'}}{z} := - \lim_{t \rightarrow \infty} \left(\lim_{s \rightarrow 0} \frac{\ln R(t, s)}{\ln t} \right). \quad (1.79)$$

It is usually taken for granted that these limits should have the same value and this is indeed so for almost all known cases. This depends, however, on the *assumption* that the two limits (i) $t, s \rightarrow \infty$ with $y = t/s > 1$ fixed and then $y \rightarrow \infty$ and (ii) $s \rightarrow 0$ and then $t \rightarrow \infty$ in eqs. (1.78, 1.79) commute. If these limits do not commute, the exponents λ_C and $\lambda_{C'}$ (and λ_R and $\lambda_{R'}$) are different. This occurs for example for model B non-equilibrium critical dynamics [284, 674],²⁸ see Chap. 3. Of course, if these two different limits do not commute, it remains to be understood what physical reason lies behind such a mathematical statement.

In writing these scaling forms, we have made no particular distinction between systems with an ordered and those with a critical equilibrium state, although the physical mechanism responsible for ageing are very different in these cases. At the level of formal scaling description used here, this is admissible, provided one allows that the values of the exponents depend on whether the system is or is not at a critical point.²⁹

When considering *glasses* below their glass-transition temperature T_g , the situation is less clear. While theoretically, one usually admits the existence of growing clusters of linear size $L(t)$, experimental evidence in favour of this has

²⁷ In the literature, one often considers only a fully disordered initial state, when $\lambda = \lambda_C = \lambda_R$. In early work, the notation λ was sometimes used for $d - \lambda = \bar{\lambda}$.

²⁸ In these references, the notations $\lambda = \lambda_{C'}$ and $\lambda' = \lambda_C$ are used.

²⁹ Therefore, notations such as z_c , a_c , b_c , λ_c for the values of these exponents at criticality will not be used in this book.

only very recently become available [71]. For sufficiently large time, several distinct forms for the growth law have been proposed

$$L(t) \sim \begin{cases} t^{1/z} \\ (\ln t)^{1/\psi} \end{cases} \quad (1.80)$$

At present, there is apparently no clear consensus which of the two scaling forms, if any, is to be preferred for glassy systems and it has also been suggested that there might be a cross-over from the first form, valid at moderately large times, to the second one, supposedly valid at truly large times, see [747]. Practically, since the dynamical exponent $z \approx 6 - 12$ is huge in these systems, the available length scales are in any case only of the order of a few lattice constants at the end of the experiments or simulations and it may become very hard indeed to make an objective distinction between these forms.

Consequently, it should not come as too large a surprise that alternative scaling forms with respect to eqs. (1.74,1.75) have been considered for systems with a dynamic glassy behaviour. For example, if one considers the two-time autocorrelation function, the following ansatz is often used

$$C(t, s) = C_{\text{st}}(t - s) + C_{\text{age}}(t, s) \quad (1.81)$$

such that the ‘stationary part’ satisfies $\lim_{t \rightarrow \infty} C_{\text{st}}(t) = 0$ and furthermore $\lim_{t-s \rightarrow \infty} (\lim_{s \rightarrow \infty} C_{\text{st}}(t - s)) = q_{\text{EA}}$, where $q_{\text{EA}} = M_{\text{eq}}^2$ is the Edwards-Anderson order-parameter for glasses and M_{eq} is the equilibrium value of the magnetisation. On the other hand, the ageing part is assumed to read

$$C_{\text{age}}(t, s) = \mathcal{C} \left(\frac{h(t)}{h(s)} \right) \quad , \quad h(t) = h_0 \exp \left[\frac{1}{A} \frac{t^{1-\mu} - 1}{1 - \mu} \right] \quad (1.82)$$

where $\mathcal{C}$ is a scaling function, μ is a free parameter and h_0 and A are constants. We have seen above, see Fig. 1.4 and Table 1.2, that using this extra parameter allows one to achieve nice data collapses in scaling plots of many physically very different materials. If $\mu = 0$, one recovers the time-translation-invariant situation familiar from equilibrium, whereas for $\mu \rightarrow 1$ one is back to the scaling description (1.74,1.75) used for simple magnets.

With respect to the scaling behaviour (1.82), one distinguishes commonly three types of ageing behaviour:

1. if $0 < \mu < 1$, this is called **subageing**;
2. if $\mu = 1$, this is called **full ageing** or simple ageing;
3. if $\mu > 1$, this is called **superageing**.

1.3.2 Passage into the Ageing Regime

In order to understand the physical origin of these scaling forms, we must first consider the passage from the initial state, supposedly at equilibrium

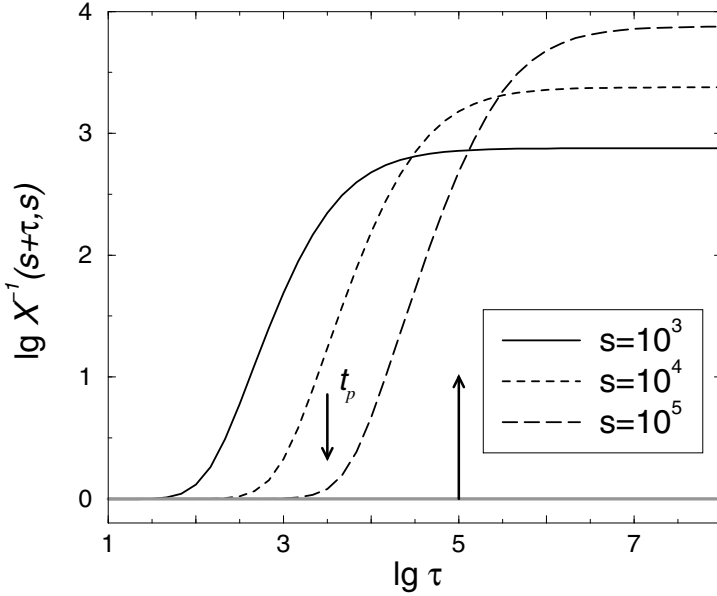


Fig. 1.20. Passage in the fluctuation-dissipation ratio $X(t, s)$, plotted as a function of $\tau = t - s$ for the 3D spherical model at $T = 0.5T_c$, for three different waiting times s . The two arrows indicate, for $s = 10^5$, the waiting time (right, up) and the cross-over time $t_p \sim s^\zeta$ (left, down). The thick gray line indicates the equilibrium value $X_{\text{eq}} = 1$.

at the initial temperature T_{ini} , towards the ageing regime, which is far from equilibrium. In consequence, the fluctuation-dissipation theorem eq. (1.8) is no longer valid. Cugliandolo, Kurchan and Parisi [171] proposed to measure the extent of its breaking from the fluctuation-dissipation ratio (FDR)

$$X(t, s) := TR(t, s) \left(\frac{\partial C(t, s)}{\partial s} \right)^{-1}. \quad (1.83)$$

Only at equilibrium, the FDT holds and one recovers $X(t, s) = 1$. Therefore, tracing $X(s + \tau, s)$ for s fixed and large as a function of τ provides a convenient tool to understand how a system quenched to $T < T_c$ departs from its quasi-equilibrium realised for τ small.³⁰ In Fig. 1.20, following the careful study of Zippold, Kühn and Horner [760], we illustrate the cross-over towards the ageing regime in the example of the 3D spherical model (see the following chapter and appendix A for the precise definition of the model) quenched to $T = 0.5T_c$. For several large values of the waiting time s , it can be seen that

³⁰ To this effect, one often concentrates on the universal [285, 286] **limit fluctuation-dissipation ratio** $X_\infty := \lim_{y \rightarrow \infty} (\lim_{s \rightarrow \infty} X(y s, s))$ [171], see exercises 1.16, 1.17.

model	condition	d	ζ	Ref.
XY	$T = 0$	1	2/3	[284]
spherical	$T < T_c$	> 2	$4/(d+2)$	[760]
p -spin spherical glass $p = 2$	$T < T_c$	–	4/5	[760]
$p = 3$	$T < T_c$	–	≈ 0.68	[438]
spherical	$T = T_c$	3	0.33(1)	[215]
		5	0.40(1)	
BCPD	$\lambda = \mu$	3	0.31(1)	[215]
		3.5	0.40(1)	
		4	0.47(1)	
		5	0.53(1)	
		6	0.58(1)	

Table 1.6. Values of the passage exponent ζ , for several models, either in the coexistence phase or at criticality. The XY and spherical models are defined in Chap. 2, the p -spin spherical glass in Chap. 5 and the bosonic contact process with diffusion (BCPD) is studied in Chap. 3.

$X(s + \tau, s) \approx 1$ for τ small and that its value decreases starting from time differences $\tau \sim t_p(s) \sim s^\zeta$ and finally crosses over to some other stationary value depending on s . Here the **passage exponent** ζ with $0 < \zeta < 1$ describes this change of behaviour [760], known values of which are listed in Table 1.6. Remarkably, the departure from the quasi-stationary state does *not* occur at time scales $\tau \sim s$, but rather at the *earlier* times $\tau \sim t_p(s) \ll s$ [760]. One can see in Fig. 1.20 that for increasing s (i) the passage time $t_p(s)$ also increases and (ii) that $t_p(s)/s$ decreases.

When comparing this passage to the behaviour of the two-time correlation $C(t, s)$, see Fig. 1.7a, the quasi-stationary regime corresponds to the plateau where $C(t, s) \approx q_{\text{EA}} = M_{\text{eq}}^2$ while the decrease of $X(t, s)$ is coupled to the onset of the ageing behaviour with its dynamical scaling, see Fig. 1.7b.³¹

These elements can be nicely combined [17] in the following argument which explains the origin of the form (1.81,1.82) and in particular allows us to derive the form of the function $h(t)$. One now considers how the passage between the plateau and the ageing regime takes place and following the above reasoning [760], one can assume that close to the end of the plateau one has

$$C(t, s) = q_{\text{EA}} + t^{-\alpha} g_1((t-s)t^{-\zeta}) + \dots \quad (1.84)$$

where α is some exponent and g_1 is a scaling function. In writing this, one uses the fact that the passage towards ageing starts at time scales $\tau = t - s \sim t_p(s) \sim s^\zeta$ as seen above in Fig. 1.20. Next, one assumes that there is some scaling according to the left equation (1.82). Taking $t - s = xt^\zeta$ and performing first the limit $t \rightarrow \infty$ and then $x = t/s \gg 1$ gives, to leading order

³¹ See Fig. 1.18 for the different temporal regimes in the case of phase-separation.

$$C(t, s) = \mathcal{C} \left(\frac{h(t)}{h(s)} \right) \simeq \mathcal{C} \left(1 + xt^\zeta \frac{d \ln h(t)}{dt} \right) \simeq q_{\text{EA}} + c_{\text{age}}^{(1)} \left(xt^\zeta \frac{d \ln h(t)}{dt} \right)^{\mathfrak{b}} \quad (1.85)$$

where $\mathfrak{b}$ is a further exponent. Its value has been worked out for certain model spin glasses, see [17], but we shall not require it here. Comparing eqs. (1.84,1.85), the dependences on t and on x can be separated which in particular leads to $d \ln h(t)/dt = A^{-1}t^{-\mu}$, where $\mu = \zeta + \alpha/\mathfrak{b}$ and A is a separation constant. This is readily integrated and the right part of eq. (1.82) follows directly [17].

The content of the above result [17] might be further appreciated by rewriting it slightly, as follows

$$\begin{aligned} C_{\text{age}}(t, s) &= \mathcal{C} \left(\frac{h(t)}{h(s)} \right) = \mathcal{C} \left(\exp \left[\frac{1}{A} \frac{t^{1-\mu} - s^{1-\mu}}{1 - \mu} \right] \right) \\ &= F_{\mathcal{C}} \left(\int_s^t d\tau \tau^{-\mu} \right) = F_{\mathcal{C}} \left(\frac{t - s}{\tau_*^\mu} \right) \end{aligned} \quad (1.86)$$

where the exponential and the constant A were absorbed in the definition of the scaling function $F_{\mathcal{C}}$ and in the last step the mean-value theorem of integral calculus was used. The intermediate point $\tau_* = \tau_*(t, s)$ has the dimension of a time. If the dependence of τ_* on t could be dropped, then from dimensional analysis one would have $\tau_*(s) \sim s$. If that would be true, and absorbing several constants into the scaling function, the ageing part of $C(t, s)$ could then be written as $C_{\text{age}}(t, s) = g((t-s)s^{-\mu})$. This kind of scaling form is indeed often used in analysing experimental data, but for $\mu \neq 1$ it is *not* identical to the theoretically justified form (1.84,1.85) [17].

We point out that for quenches to $T = T_c$, the same phenomenological description for the passage to the ageing regime seems to work as well [215], see exercise 1.8 and Table 1.6 for known values of ζ . Having understood how the usually considered scaling forms come about, we now turn to several general results which should be model-independent.

1.3.3 Kurchan's Lemma

Depending on the value of μ , we had defined above three types of ageing behaviour. A basic result of Kurchan [457] states roughly that *superageing should be impossible*. A precise formulation, which makes explicit reference to a certain type of scaling, is as follows.

Lemma. (Kurchan [457]) *Consider a two-time autocorrelation with the following assumed scaling behaviour*

$$C(t_1, t_2) = C_{\text{st}}(t_1 - t_2) + M_{\text{eq}}^2 C_{\text{age}}(t_1, t_2) \quad , \quad C_{\text{age}}(t_1, t_2) = g \left(\frac{t_2 - t_1}{t_1^\mu} \right) \quad (1.87)$$

where $t_2 > t_1$, μ is a free parameter and $g(u)$ is a scaling function with $g(0) = 1$ and which decreases strictly monotonously with u . Assume further that the ‘stationary’ part $C_{\text{st}}(t) \rightarrow 0$ if $|t| \rightarrow \infty$. Then $\mu > 1$ is impossible.

Proof: One proceeds in three steps. In the first step, one derives a basic autocorrelation inequality. Consider the normalised autocorrelators at times t_i and t_j

$$N_{i,j} = N(t_i, t_j) = \frac{\langle \phi(t_i) \phi(t_j) \rangle}{\sqrt{\langle \phi^2(t_i) \rangle \langle \phi^2(t_j) \rangle}}$$

such that

$$N(t_i, t_j) = \frac{C(t_i, t_j)}{\sqrt{C(t_i, t_i) C(t_j, t_j)}} .$$

Next, construct the non-negative quadratic form

$$0 \leq \left\langle \left(\sum_{i=1}^r \frac{\phi(t_i) v_i}{\sqrt{\langle \phi^2(t_i) \rangle}} \right)^2 \right\rangle = \sum_{i,j} N_{i,j} v_i v_j$$

Consequently, the $r \times r$ matrix $\hat{N}_r$ with elements $N_{i,j}$ $i, j = 1, \dots, r$ is positive semi-definite, hence $\det \hat{N}_r \geq 0$. For $r = 2$, a moment’s thought shows that this merely implies $|N_{1,2}| \leq 1$, as it should be. However, a slightly less trivial result is found for $r = 3$

$$\begin{aligned} \det \hat{N}_3 &= \begin{vmatrix} 1 & N_{1,2} & N_{1,3} \\ N_{1,2} & 1 & N_{2,3} \\ N_{1,3} & N_{2,3} & 1 \end{vmatrix} \\ &= 1 + 2N_{1,2}N_{2,3}N_{1,3} - N_{1,2}^2 - N_{1,3}^2 - N_{2,3}^2 \geq 0 \end{aligned}$$

Rearranging and completing the square with respect to $N_{1,3}$ leads to the inequality

$$|N_{1,3} - N_{1,2}N_{2,3}| \leq \sqrt{(1 - N_{1,2}^2)(1 - N_{2,3}^2)} \quad (1.88)$$

which is a quantitative formulation of the following intuitively obvious fact: if the state of the system between the times t_1 and t_2 are correlated and if there also is a correlation between the states at times t_2 and t_3 , then there must be a correlation between times t_1 and t_3 , viz. $N_{1,3} \neq 0$.

In the second step, one considers the scaling form (1.87) in the special case where $C_{\text{st}} = 0$. This means $C(t_1, t_2) = M_{\text{eq}}^2 g(u_{12})$, with $u_{12} = (t_2 - t_1)/t_1^\mu \geq 0$. One now chooses three times $t_1 < t_2 < t_3$ such that

$$0 < N_{1,2} = g(u_{12}) < 1 \quad , \quad 0 < N_{2,3} = g(u_{23}) < 1$$

which is possible because of the assumptions made. According to the assumed scaling form, this means that one is working in the limit $t_1 \rightarrow \infty$ such that

$$u_{12} = \frac{t_2 - t_1}{t_1^\mu} = \text{fixed} \quad , \quad u_{23} = \frac{t_3 - t_2}{t_2^\mu} = \text{fixed}$$

Now, consider the following transformation

$$\begin{aligned}
 \frac{t_3 - t_1}{t_1^\mu} &= \frac{(t_3 - t_2) + (t_2 - t_1)}{t_1^\mu} \\
 &= \frac{t_3 - t_2}{t_2^\mu} \cdot \left(\frac{t_2 - t_1 + t_1}{t_1^\mu} \right)^\mu \cdot \frac{t_1^{\mu^2}}{t_1^\mu} + \frac{t_2 - t_1}{t_1^\mu} \\
 &= \left(\frac{t_3 - t_2}{t_2^\mu} \right) \left(\frac{t_2 - t_1}{t_1^\mu} \right)^\mu \left(1 + \frac{t_1}{t_2 - t_1} \right)^\mu t_1^{\mu(\mu-1)} + \frac{t_2 - t_1}{t_1^\mu}
 \end{aligned}$$

One now assumes that $\mu > 1$. Then, in the $t_1 \rightarrow \infty$ limit

$$u_{13} \stackrel{t_1 \rightarrow \infty}{=} u_{23} u_{12}^\mu t_1^{\mu(\mu-1)} + u_{12} \rightarrow +\infty$$

and consequently, $N_{1,3} = g(u_{13}) \xrightarrow{!} \min$. It is always possible to arrange such that $t_2 - t_1$ and $t_3 - t_2$ are sufficiently close that $|N_{1,2} - 1|$ and $|N_{2,3} - 1|$ are as small as desired. From Kurchan's inequality (1.88) it then follows that $|N_{1,3} - 1|$ must be arbitrarily close to zero, which is in contradiction with the assumed strict monotonicity of $g(u)$. Hence the assumption $\mu > 1$ is inadmissible.

In the third step, one now considers the full scaling form (1.87). Because of the decrease of $C_{\text{st}}(t)$ with t , one has

$$M_{\text{eq}}^2 C_{\text{age}}(u_{12}) = \lim_{t_1, t_2 \rightarrow \infty; u_{12} \text{ fixed}} C(t_1, t_2)$$

and one is back to the case studied in the second step. q.e.d.

In consequence, evidence for superageing, either numerical or experimental, with an effective value $\mu > 1$, should be carefully checked before such a result could be accepted at face value. For example, effective superageing with $\mu > 1$ might have come from unrecognised corrections to the leading scaling behaviour, in particular if the estimates of μ are only slightly larger than 1.

The same argument should also work for other experimentally considered quantities, such as the thermoremanent magnetisation.

1.3.4 The Yeung-Rao-Desai Inequalities

We now derive another general result for ageing systems, which holds within the context of full ageing of the correlation function, as described by eq. (1.74). In a simple form, this can be stated as follows.

Lemma. (Yeung, Rao and Desai [744]) *Consider a ferromagnetic system with a non-conserved order-parameter, quenched to a temperature $T \leq T_c$. Assume that the correlation function satisfies the scaling form (1.74) with the auto-correlation exponent (1.78). Then: (i) for short-ranged initial correlations or an uncorrelated initial state and a quench to $T < T_c$ one has³²*

³² A heuristic argument for this inequality was given in [245].

$$\lambda_{C'} \geq \frac{d}{2} ; \quad (1.89)$$

(ii) for relevant long-ranged initial correlations of the form $C_{\text{init}}(\mathbf{r}) \sim |\mathbf{r}|^{-d-\alpha}$ and a quench to $T < T_c$ one has

$$\lambda_{C'} \geq \frac{d + \alpha}{2} ; \quad (1.90)$$

(iii) for a quench to T_c or more generally a long-range model with equilibrium correlations $C_{\text{eq}}(\mathbf{r}) \sim |\mathbf{r}|^{-(d-2+\eta)}$ one has

$$\lambda_C \geq d - 2 + \eta . \quad (1.91)$$

We point out that some of these bounds apply to the exponent $\lambda_{C'}$, rather than the autocorrelation exponent λ_C , see eq. (1.78). Although in most cases $\lambda_C = \lambda_{C'}$, these exponents can be distinct, see p. 155. The conserved case is discussed below.

Proof: Let $m_0 = V^{-1} \int_V d\mathbf{r} \phi(t, \mathbf{r})$ be the mean magnetisation and consider the fluctuation $\delta\phi(t, \mathbf{r}) = \phi(t, \mathbf{r}) - m_0$. Its Fourier components are

$$\widehat{\delta\phi}_{\mathbf{k}}(t) = \frac{1}{\sqrt{V}} \int_V d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \delta\phi(t, \mathbf{r}) .$$

Then one has the following estimate

$$\begin{aligned} C(t_1, t_2) &= \langle \delta\phi(t_1, \mathbf{r}) \delta\phi(t_2, \mathbf{r}) \rangle \\ &= \int d\mathbf{k} \langle \widehat{\delta\phi}_{\mathbf{k}}(t_1) \widehat{\delta\phi}_{-\mathbf{k}}(t_2) \rangle \\ &\leq \int d\mathbf{k} \sqrt{\left\| \widehat{\delta\phi}_{\mathbf{k}}(t_1) \right\| \left\| \widehat{\delta\phi}_{-\mathbf{k}}(t_2) \right\|} \\ &= \int d\mathbf{k} \sqrt{S_{t_1}(\mathbf{k}) S_{t_2}(\mathbf{k})} \end{aligned}$$

with the structure factor $S_t(\mathbf{k}) = S(t, t, \mathbf{k})$ and the two-time structure factor $S(t_1, t_2, \mathbf{k}) := \langle \widehat{\delta\phi}_{\mathbf{k}}(t_1) \widehat{\delta\phi}_{-\mathbf{k}}(t_2) \rangle$. Here, in the second line the average at fixed momentum $\mathbf{k}$ was considered as a scalar product which was then estimated via the Cauchy-Schwarz inequality, using the natural norm $\left\| \widehat{\delta\phi}_{\mathbf{k}}(t) \right\| := \langle \widehat{\delta\phi}_{\mathbf{k}}(t) \widehat{\delta\phi}_{-\mathbf{k}}(t) \rangle$. One now analyses this in the scaling regimes and also sets $t_2 > t_1$ for the sake of clarity.

First, one considers quenches to $T < T_c$. One expects that $\widehat{\delta\phi}_{\mathbf{k}}(t_1)$ and $\widehat{\delta\phi}_{\mathbf{k}}(t_2)$ should be uncorrelated if the interfaces between the ordered domains has moved beyond the length scale $\Delta L > (2\pi)/|\mathbf{k}|$ which means that one expects $S(t_1, t_2, \mathbf{k}) \rightarrow 0$ rapidly if $(L(t_2) - L(t_1))k \gg 1$. From the scaling

of the correlation function, one expects the following scaling form for the structure factor

$$S_t(\mathbf{k}) = L(t)^d f(kL(t))$$

and the following long-distance behaviour

$$\lim_{k \rightarrow 0} S_t(\mathbf{k}) \sim k^\beta .$$

Then, in the scaling limit where also $t_2 \gg t_1$

$$\begin{aligned} \lim_{t_2/t_1 \rightarrow \infty} C(t_1, t_2) &\sim L(t_2)^{-\lambda_{C'}} \\ &\leq L(t_2)^{d/2} \int_0^{2\pi a/L(t_2)} dk k^{d-1} k^{\beta/2} f(kL(t_2)) S_d \\ &= \text{constant} \cdot L(t_2)^{-d/2-\beta/2} \end{aligned}$$

from which the bound

$$\lambda_{C'} \geq \frac{\beta + d}{2} \quad (1.92)$$

follows. For a disordered state, one has initially $\lim_{k \rightarrow 0} S_0(\mathbf{k}) \sim O(1)$, hence $\beta = 0$ and the assertion (1.89) follows. On the other hand, for a long-range initial correlation, one rather has $\beta = \alpha$ which implies (1.90).

Next, one considers quenches to the critical point $T = T_c$. One must return to the basic estimate of $C(t_1, t_2)$ since the scaling of the structure factor now becomes $S_t(\mathbf{k}) = k^{-2+\eta} g(kL(t_2))$. This implies along the same lines as before $\lambda_{C'} \geq d - 1 + \frac{1}{2}(\eta + \beta)$. If in addition also t_1 is already in the scaling regime one has $\beta = -2 + \eta$ together with $\lambda_C = \lambda_{C'}$ and the last inequality (1.91) follows. q.e.d.

For a *conserved* order-parameter, the estimates depend on whether t_1 can be considered to be in the scaling regime. If this is not the case, one has e.g. (1.89) for a quench to $T < T_c$. But if t_1 is in the scaling regime, one has the sharper estimates [744]

$$\lambda_C \geq \begin{cases} \frac{1}{2}d + 2 & ; \text{ if } d \geq 2 \\ \frac{3}{2} & ; \text{ if } d = 1 \end{cases} \quad (1.93)$$

1.4 Scaling Behaviour of Integrated Responses

While the measurement of two-time correlation functions in a simulation does not present any particular difficulty and the scaling interpretation appears to be straightforward, obtaining reliable data for the response is more involved and requires a detailed discussion and their interpretation requires some care. In what follows, we shall limit ourselves to systems with a simple power-law

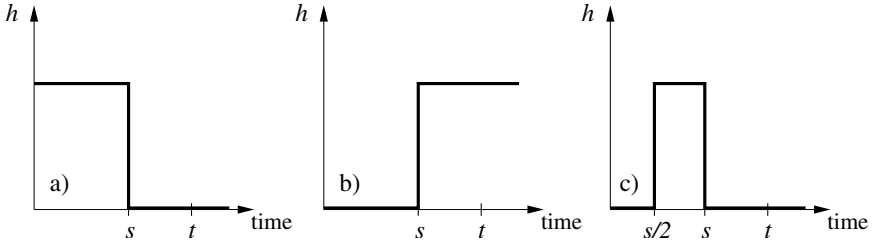


Fig. 1.21. Several protocols for measuring integrated susceptibilities. Shown is the time-dependence of the amplitude h_0 of the spatially random magnetic field h . (a) thermoremanent (TRM) (b) zero-field cooled (ZFC) and (c) intermediate (Int).

scaling $L(t) \sim t^{1/z}$ of the domain size, but it will not be necessary to specify the nature of these domains.

First, it is clear that a direct measurement of the response function $R(t, s; \mathbf{r})$, defined in (1.2) as a functional derivative, is not feasible since it would lead to a very small signal dominated by the noises (statistical, thermal, initial) of the simulation. To avoid these difficulties, Barrat [40] proposed to study certain *integrated* two-time responses instead, but applying a time-dependent magnetic field h , whose time-dependence is chosen according to certain protocols, some of which are indicated in Fig. 1.21. Appendix G in Volume 1 describes numerical methods for the calculation of linear responses, following [141, 616]. The integrated response will then be given in terms of an integral $\int du R(t, u)$, with the integration limits to be read off from Fig. 1.21. For spin-glasses, where the method was originally proposed, the intrinsic spatial disorder of the usually studied spin-glass models permits the use of a spatially constant magnetic field, but for simple magnets one should better use a *spatially random* field, in order not to prefer one of the equilibrium states over the other(s).

Given the scaling form eq. (1.75), one might expect that the integrated response should simply scale as $s^{-a} f(t/s)$, but the actual behaviour is considerably more complicated, as we now show.

1.4.1 Thermoremanent Susceptibility

It is important to realise that the scaling behaviour (1.75) of the two-time response function $R(t, s)$ is determined by the difference $\tau = t - s$. We have already discussed, see Fig. 1.20 [760], that there is an intermediate time-scale $t_p(s) \sim s^\zeta$ with $0 < \zeta < 1$ such that

$$R(t, s) = \begin{cases} R_{\text{gg}}(\tau) & ; \text{ if } \tau_{\text{micro}} \ll \tau \lesssim t_p \\ R_{\text{age}}(t, s) = s^{-1-a} f_R(t/s) & ; \text{ if } t_p \lesssim \tau \end{cases} \quad (1.94)$$

where $R_{\text{gg}}(\tau)$ is an equilibrium response. In the spherical model with $2 < d < 4$, one has $\zeta = 4/(d+2)$ for $T < T_c$ [760], and we refer to Table 1.6 for values

in other models. In the following discussion it is assumed that (i) this result can be taken over to any simple ageing magnet (allowing for a different value of $0 < \zeta < 1$) and (ii) the passage between the equilibrium regime and the ageing regime with a scaling behaviour is very rapid. In principle, this last point can be controlled through a fluctuation-dissipation plot (see Fig. 1.8), where one should find a sharp kink between the equilibrium regime and the ageing regime (for that reason, the following discussion cannot be taken over straightforwardly to glassy systems).

Furthermore, a scaling behaviour of the response function $R(t, s)$ will only set in for waiting times large with respect to some microscopic reference time (and, of course, only if $t - s \gg t_p(s) \sim s^\zeta$, as discussed above on p. 54). Rather, for very small s one expects the following behaviour $R(t, \tau_{\text{micro}}) \sim t^{-\lambda_{R'}/z}$. While it is usually *assumed* that $\lambda_{R'} = \lambda_R$ [383], and this indeed seems to be true in the large majority of cases, it is on the other hand not entirely obvious why the two distinct limits used in the definition of these two autoresponse exponents (see pp. 53, 155) should always commute. So one needs to introduce a further time-scale t_ϵ such that $s - t_\epsilon \simeq \tau_{\text{micro}}$ and which describes the cross-over between the full scaling regime and the regime of initial scaling, again assuming that the cross-over is rapid. After these preparations, we have in the ageing regime, slightly generalising [338]

$$\begin{aligned} \chi_{\text{TRM}}(t, s) &= \int_0^s du R(t, u) = \int_0^s d\tau R(t, s - \tau) \\ &\simeq \int_0^{t_p} d\tau R_{\text{age}}(t, s - \tau) + \int_{t_p}^{t_\epsilon} d\tau R_{\text{age}}(t, s - \tau) + \int_{t_\epsilon}^s d\tau R(t, \tau_{\text{micro}}) \\ &= s \int_0^{t_p/s} dv R_{\text{age}}(t, s(1 - v)) + s \int_{t_p/s}^{t_\epsilon/s} dv R_{\text{age}}(t, s(1 - v)) + t^{-\lambda_{R'}/z} c_\infty \\ &\stackrel{s \rightarrow \infty}{\simeq} s^{-a} \int_0^1 dv (1 - v)^{-1-a} f_R\left(\frac{t/s}{1 - v}\right) + c_\infty t^{-\lambda_{R'}/z} \end{aligned}$$

with $c_\infty = c_\infty(\tau_{\text{micro}})$ and where the term coming from initial scaling was estimated from the mean-value theorem. In the last line, the limit $s \rightarrow \infty$ was taken. Any terms which might have come from the transition regions between the three regimes mentioned above were neglected.³³

In summary, one expects for the limit of large waiting times $s \rightarrow \infty$ the following scaling behaviour of the thermoremanent magnetisation $M_{\text{TRM}}(t, s) = h\chi_{\text{TRM}}(t, s)$ and its susceptibility

$$\chi_{\text{TRM}}(t, s) = s^{-a} f_M(t/s) + c_\infty t^{-\lambda_{R'}/z} \quad (1.95)$$

³³ For $t - s \ll t_p(s)$ one rather has that $\chi_{\text{TRM}}(t, s) \simeq \int_0^{t_p} d\tau R_{\text{gg}}(t - s + \tau) + \int_{t_p}^s d\tau R(t, \tau_{\text{micro}}) \simeq T^{-1} (C_{\text{gg}}(0) - C_{\text{gg}}(t - s))$ reduces to the equilibrium correlator plus a time-dependent correction.

where c_∞ is a constant. The scaling function $f_M(y)$ can be calculated once the function $f_R(y)$ is known.³⁴

In practice, it has turned out that in the phase-ordering of simple magnets the first correction to the leading scaling behaviour of χ_{TRM} is quantitatively very important. We shall come back to this later several times.

1.4.2 Zero-field Cooled Susceptibility

For the case of the zero-field cooled susceptibility, it turns out that the integration may become singular close to one of the limits of integration (see Fig. 1.21b). In certain cases, this may even generate terms which dominate over the ageing behaviour but merely reflect the slow dynamics, since they depend algebraically on the observation time t only. For a quantitative analysis, terms of such a kind are much more significant and disturbing than those found for the case of the thermoremanent susceptibility.

To analyse this, it is best to use first the sum rule

$$\chi_{\text{FC}}(t) = \int_0^t du R(t, u) = \chi_{\text{TRM}}(t, s) + \chi_{\text{ZFC}}(t, s) \quad (1.96)$$

where $\chi_{\text{FC}}(t)$ corresponds to a steady magnetic field and is known as the **field-cooled susceptibility**. In what follows, we shall analyse $\chi_{\text{FC}}(t)$ and then deduce $\chi_{\text{ZFC}}(t, s)$ from the sum rule (1.96) and the result (1.95) for the thermoremanent susceptibility.

At this point, an important distinction should be made, according to the behaviour of the *equilibrium* spin-spin correlator $C_{\text{eq}}(\mathbf{r}) = \langle \sigma_{\mathbf{r}} \sigma_{\mathbf{0}} \rangle_{\text{eq}}$ [338, 339].

1. If $C_{\text{eq}}(\mathbf{r}) \sim \exp(-|\mathbf{r}|/\xi)$ is *short-ranged* with a finite correlation length ξ , we say the system is in **class S**. Typical members of this class are kinetic Glauber-Ising models in $d > 1$ dimensions quenched to $T < T_c$.
2. If $C_{\text{eq}}(\mathbf{r}) \sim |\mathbf{r}|^{-(d-2+\eta)}$ is *long-ranged* where η is some exponent, we say the system is in **class L**. All simple magnets quenched to $T = T_c > 0$ (then η is the standard equilibrium critical exponent, see Volume 1, Chap. 2) are in this class, but also the spherical model quenched to any temperature $T \leq T_c$ is in class L, with $\eta = 0$.³⁵

The attentive reader will have remarked that 1D systems, where $T_c = 0$, have been left out. It is of course conceivable that future work might show that the above simple scheme should be refined. We now return to the discussion of $\chi_{\text{FC}}(t)$ which must be made separately for these two classes [339].

³⁴ In the applications, we shall always assume $\lambda_R = \lambda_{R'}$ unless explicitly stated otherwise.

³⁵ All known exactly solved spin systems with ageing behaviour can be interpreted to fall into class L, see Chap. 2.

We begin with class S. Physically, it is well-accepted [94, 89, 166] that the evolution of systems in this class proceeds through the formation of ordered domains at an early stage and subsequent slow motion of the domain walls between these domains, see Fig. 1.6 for illustration in the Glauber-Ising model, and the linear size of the domains is $L(t) \sim t^{1/z}$. We now perturb with a random field of zero mean $\overline{h_i} = 0$ [40], for example a binary field $h_i = \pm h$. For clarity, consider first the case when $T = 0$. Then, because the spins deep inside the cluster are fully ordered, the only non-vanishing contribution to χ_{FC} comes from the spins from near the interfaces between the ordered clusters. Since the interfaces are $(d-1)$ -dimensional, the interface density is $\rho_I(t) \sim (L(t))^{d-1}/(L(t))^d = L(t)^{-1}$. From [40], one has on a lattice $\Lambda \in \mathbb{Z}^d$ with $\mathcal{N}$ sites where the brackets denote the thermal average and the over-bar the average over the random field

$$\chi_{\text{FC}}(t) = \frac{1}{\mathcal{N}h^2} \overline{\left\langle \sum_{i \in \Lambda} \sigma_i(t) h_i \right\rangle} = \frac{1}{\mathcal{N}h^2} \overline{\left\langle \sum_{i \in \text{interfaces}} \sigma_i(t) h_i \right\rangle} \sim L(t)^{-1} w(t) \quad (1.97)$$

with $w(t)$ denoting the width of the interface. For a finite temperature $T > 0$, the order deep inside the clusters is not perfect and there remains a residual contribution to the susceptibility. We then have from the static fluctuation-dissipation theorem, for large times

$$\chi_{\text{FC}}(t) \simeq \chi_0 + L(t)^{-1} w(t) \quad , \quad T\chi_0 = 1 - M_{\text{eq}}^2 \quad (1.98)$$

where M_{eq} is the equilibrium magnetisation (for the 2D Ising model, $M_{\text{eq}} = (1 - \sinh(2/T)^{-4})^{1/8}$, see [54]). The dynamics of a $(d-1)$ -dimensional interface in a d -dimensional system ($d \geq 2$) can be described by the dynamics of a height model [3] of continuous height variables $v_j \in \mathbb{R}$ with the with nearest-neighbour interaction equilibrium Hamiltonian $\mathcal{H}[v] = -\frac{\tau}{2} \sum_{\langle j, j' \rangle} (v_j - v_{j'})^2$ where τ is the effective interfacial tension [3]. In adopting this description, we tacitly assume that the system is above its ‘roughening temperature’ T_R [712], such that the fluctuations of the interface are unbounded (rough interface) for $T > T_R$ and with bounded fluctuations for $T < T_R$ (smooth interface). This condition is always satisfied for $d = 2$, since then $T_R = 0$ and indeed the description adopted here can be derived rigorously for the 2D Ising model [2]. On the other hand, for the 3D Ising model one has $T_R \approx 0.5 T_c$ and finally, $T_R = \infty$ for $d \geq 4$. If the (non-conserved) dynamics of the height model is described by a Langevin equation, the squared interface width was shown by Abraham and Upton to scale for large times as [3]

$$w(t)^2 = \langle v_0(t)^2 \rangle \sim \begin{cases} t^{1/2} & ; \text{ if } d = 2 \\ \ln t & ; \text{ if } d = 3 \end{cases} \quad (1.99)$$

Inserting this into (1.97), we finally have the following leading dynamical scaling for χ_{FC} for models in the class S (e.g. the Glauber-Ising model for $d > 1$)

$$\chi_{\text{FC}}(t) - \chi_0 \sim \begin{cases} t^{-1/4} & ; \text{ if } d = 2 \\ t^{-1/2}(\ln t)^{1/2} & ; \text{ if } d = 3 \end{cases} \quad (1.100)$$

for $t \rightarrow \infty$ and provided that $T > T_R$. For dimensions $d \geq 4$, and generically if $T < T_R$, one expects a flat interface with $w(t) \sim \text{constant}$.

Next, we consider class L by adapting the above heuristic argument. Indeed, the main physical difference with respect to systems of class S appears to be that although correlated (but not necessarily ordered) clusters of size $L(t) \sim t^{1/z}$ form, fluctuations persist in the interior of the clusters on all length scales up to $L(t)$. Hence it is natural to consider that the width $w(t) \sim L(t)$ which in turn leads to $\chi_{\text{FC}}(t) \sim \text{constant}$.

Summarising this discussion, we have seen that for large times [339]

$$\chi_{\text{FC}}(t) = \chi_0 + \chi_1 t^{-A} \quad , \quad A = \begin{cases} z^{-1} - \kappa & ; \text{ class S} \\ 0 & ; \text{ class L} \end{cases} \quad (1.101)$$

where $\kappa \geq 0$ describes the interface width $w(t) \sim t^\kappa$ for systems in class S, see (1.99). It has also become apparent that the behaviour of $\chi_{\text{FC}}(t)$ is independent of the waiting time s . We now insert this into the sum rule (1.96). The constant terms (χ_0 for class L, $\chi_0 + \chi_1$ for class S) can be found from the static fluctuation-dissipation theorem. The final result is [347]

$$\begin{aligned} \chi_{\text{ZFC}}(t, s) &= \chi_{\text{FC}}(t) - \chi_{\text{TRM}}(t, s) \\ &= \begin{cases} \frac{1}{T} (1 - M_{\text{eq}}^2) + \chi_1 t^{-A} - s^{-a} f_M(t/s) & ; \text{ for class S} \\ \frac{1}{T} (1 - M_{\text{eq}}^2) - s^{-a} f_M(t/s) & ; \text{ for class L} \end{cases} \end{aligned} \quad (1.102)$$

We emphasise that for systems in class S, the rather trivial non-ageing term $\sim t^{-A}$ can completely mask the ageing contributions one usually wishes to study. The mere observation of dynamical scaling of $\chi_{\text{ZFC}}(t, s)$ is not necessarily enough to be able to exclude this.³⁶

³⁶ In glassy systems, motivated from procedures applied in the analysis of experimental data, it is common to decompose $\chi_{\text{ZFC}}(t, s) = \chi_{\text{st}}(t - s) + \chi_{\text{age}}(t, s)$ into a ‘stationary’ and an ‘ageing’ term. For simple magnets of class L, this is compatible with (1.102), but into which one of those should one include the term $\sim t^{-A}$, which dominates over the ageing term for systems in class S, since there $A < a$? Furthermore, motivated by the fluctuation-dissipation theorem, often a decomposition of the autocorrelator $C(t, s) = C_{\text{st}}(t - s) + C_{\text{age}}(t, s)$ is tried, in such a way that $\chi_{\text{st}}(t)$ and $C_{\text{st}}(t)$ satisfy a FDT. This implies that $C_{\text{st}}(t - s)$ acts at most as a finite-time correction to the scaling term $C_{\text{age}}(t, s)$ – both for $T < T_c$ and $T = T_c$ – and it is not always obvious that other possible corrections terms are not larger. Finally, it is sometimes even attempted to push these decompositions to the limit by writing $R(t, s) = R_{\text{st}}(t - s) + R_{\text{age}}(t, s)$ instead of the more conservative eq. (1.94), but as we have shown, this decomposition of $R(t, s)$ is not needed to derive the scaling form (1.102) of the ZFC susceptibility, which was the main motivation for doing so. Hence this artificial separation of $R(t, s)$ should have little meaning at least for the ageing of simple magnets.

1.4.3 Intermediate Susceptibility

Having seen that both $\chi_{\text{TRM}}(t, s)$ and $\chi_{\text{ZFC}}(t, s)$ can present serious difficulties for their correct interpretation, a natural solution would be to avoid the tricky limits $s \approx 0$ and $s \approx t$ altogether. A possible way to do this is the intermediate protocol sketched in Fig. 1.21c. Indeed, in this case the integrations can be formally carried out, with the result [339, 347]

$$\chi_{\text{Int}}(t, s) = s^{-a} f_{\text{Int}}(t/s) + o\left(t^{-\lambda_{R'}/z}\right). \quad (1.103)$$

We remind the reader that in the above discussion, any corrections to scaling to the response function $R(t, s)$ itself have been neglected. Such scaling corrections, if noticeable, will likely first be observed in quantities such as χ_{Int} .

1.4.4 Alternating Susceptibility

A different way of studying ageing is through the long-term behaviour of the linear response with respect to a harmonic perturbation with angular frequency ω . Indeed, using analogous techniques as applied for the TRM susceptibility, it is easy to see that the dissipative part $\chi''(\omega, t)$ scales as

$$\chi''(\omega, t) = \int_0^t du R(t, u) \sin(\omega(t-u)) = \chi''_{\text{st}}(\omega) + t^{-a} \chi''_2(\omega t) + O\left(t^{-\lambda_{R'}/z}\right) \quad (1.104)$$

where χ''_{st} is a stationary part and the scaling function is given by [346]

$$\chi''_2(y) = \int_0^1 dv f_R\left(\frac{1}{1-v}\right) \frac{\sin yv}{(1-v)^{1+a}}. \quad (1.105)$$

It is an important open problem to what extent the scaling considerations of this section for the dynamic susceptibilities can be taken over to glassy systems.

Practical methods for the numerical calculation of integrated susceptibilities from Monte Carlo simulations are described in appendix G in Volume 1.

1.5 Values of Non-equilibrium Exponents

1.5.1 Values of the Ageing Exponents a and b

We now give the values of the ageing exponents a and b , for systems with a disordered initial state and with non-conserved (model A) dynamics. The cases of long-ranged initial conditions and/or of a non-vanishing initial order-parameter $\langle \phi(0, \mathbf{r}) \rangle \neq 0$ will be considered in Chap. 3. For the exponent b , the following is generally accepted [93, 287]

$$b = \begin{cases} 0 & ; \text{ for class S} \\ 0 & ; \text{ for class L and } T < T_c \\ a & ; \text{ for class L and } T = T_c \end{cases} . \quad (1.106)$$

Here the result $b = 0$ for all systems (S or L) quenched to $T < T_c$ expresses the empirical fact³⁷ that the autocorrelation function obeys the scaling form $C(t, s) = f_C(t/s)$, see also Fig. 1.7. On the other hand, at $T = T_c$ the result $a = b$ follows from the fact that at criticality there is a non-vanishing and finite limit fluctuation-dissipation ratio X_∞ .

On the other hand, an explicit result can be given for the exponent a [338]

$$a = \begin{cases} 1/z & ; \text{ for class S} \\ (d - 2 + \eta)/z & ; \text{ for class L} \end{cases} \quad (1.107)$$

for dimensions $d < d_c$, where d_c is the upper critical dimension.

Proof: For systems of class S, one requires merely a standard argument, e.g. [89], which asserts that for simple short-ranged ferromagnets, observables such as the alternating susceptibility should decompose into a stationary part and an ageing part, viz.

$$\chi''(\omega, t) = \chi''_{\text{eq}}(\omega) + L(t)^{-1} \chi''_{\text{age}}(\omega t) . \quad (1.108)$$

Here χ''_{age} is meant to represent the response of a single domain wall. Since the domains should have a linear size $L(t)$, they occupy a volume $\sim L(t)^d$ and a surface $L(t)^{d-1}$, hence their density should scale as $1/L(t)$. Comparison with (1.104) and $L(t) \sim t^{1/z}$ then gives the first part of the assertion (1.107).

For systems of class L, one can no longer argue around the presence of well-defined domain walls. Rather, as we shall now show, one should anticipate³⁸

$$\chi''(\omega, t) = \chi''_{\text{eq}}(\omega) + L(t)^{-(d-2+\eta)} \chi''_{\text{age}}(\omega t) \quad (1.109)$$

from which the second part of the assertion follows. We again start from the hypothesis that all length scales are measured through the domain size $L(t)$. It then appears natural to postulate that the spin-spin correlator should be expressible through the quasi-static correlator as $C(t, \mathbf{r}) \approx C_{\text{qs}}(L(t), \mathbf{r}) \sim r^{-(d-2+\eta)} c(r/L(t))$, where c is some scaling function. The time-dependent contribution to the susceptibility per volume V should be $\chi_{\text{qs}} \sim V^{-1} \int_V d\mathbf{r} C_{\text{qs}}(L(t), \mathbf{r}) \sim L(t)^{-(d-2+\eta)}$ which leads to the result stated above. q.e.d.

In these arguments, it is assumed throughout that the leading finite-time correction, of order $O(t^{-\lambda_R/z})$, does not become dominant. However, it is known that for the spherical model in $d > d^* = 4$ dimensions, the rôle of these terms is exchanged [339].

³⁷ We are not aware of any model-independent derivation of this statement.

³⁸ Possible logarithmic factors are suppressed.

The above argument for class L made quite a few plausible, but after all still heuristic, assumptions. We therefore recall all tests from specific models known to us (see chapters 2 and 3 for the definitions). First, for non-equilibrium critical dynamics, it is known [287, 121] that $a = b = 2\beta/\nu z = (d - 2 + \eta)/z$, in agreement with (1.107). Second, the exact solution of the spherical model quenched to $T < T_c$ [286] gives $a = d/2 - 1$, which together with the known exponents $z = 2$ and $\eta = 0$ reproduces (1.107). Third, one may also consider the spherical model with long-ranged interactions of the form $J(\mathbf{r}) \sim |\mathbf{r}|^{-d-\sigma}$ which leads to new non-trivial results if either $0 < \sigma < d$ and $d < 2$ or else $0 < \sigma < 2$ and $d > 2$. In particular, $\eta = 2 - \sigma$ is known [414]. For quenches to $T < T_c$, as well as to $T = T_c$, one finds $z = \sigma$ and $a = d/\sigma - 1$ [127, 45], in agreement with (1.107). Fifth, in the spherical model with a conserved order-parameter quenched to $T = T_c$ one has $z = 4$, $\eta = 0$ and $a = b = (d - 2)/4$, as it should be [47]. Finally, for the 2D XY model, a spin-wave approximation [73] gives $a = b = \eta/2$, again in agreement with (1.107) since $z = 2$. These results apparently remain valid even if the spin-wave approximation does not hold true any more, as has been checked through simulations [73, 5, 463].

It must be mentioned here that the above results, notably eq. (1.107) for the exponent a , are not generally accepted. Based on their analysis of the ZFC susceptibility, Corberi, Lippiello and Zannetti [156, 157, 158] have since some time advocated a different expression. They admit throughout the scaling form $\chi_{\text{ZFC}}(t, s) = \chi_0 + s^{-a_\chi} f(t/s)$ and in particular do not recognise the separation into classes S and L we have introduced above, see p. 64. From the results of their numerical studies and from several exactly solved models they proposed the form

$$a_\chi = \frac{n}{z} \frac{d - d_L}{d_U - d_L} \quad (1.110)$$

where $d_{U,L}$ are upper and lower critical dimensions and $n = 1$ resp. $n = 2$ for a scalar resp. a vector order-parameter. While a_χ does indeed describe phenomenologically the leading power-law behaviour $\sim t^{-A}$ of χ_{ZFC} in (1.102), they assert furthermore that $a_\chi \stackrel{!}{=} a$ [156, 157, 158] should also describe the scaling of the response function. If their proposal were indeed correct, it would imply that the simple and straightforward arguments used to derive (1.107) would be in error and hence the physical picture standing behind them would be incorrect; namely, that phase-ordering kinetics is driven by the slow motion of the domain walls. For that reason, the proposal of Corberi *et al.* [156, 157, 158] deserves a detailed quantitative discussion.

Clearly, since their assumed scaling of χ_{ZFC} agrees with the scaling (1.102) for systems in class L and their proposal for the value of a is abstracted from the result (1.107) for class L, one has agreement with all known systems in our class L. A discussion of their proposal is therefore only necessary for those systems which are in class S.

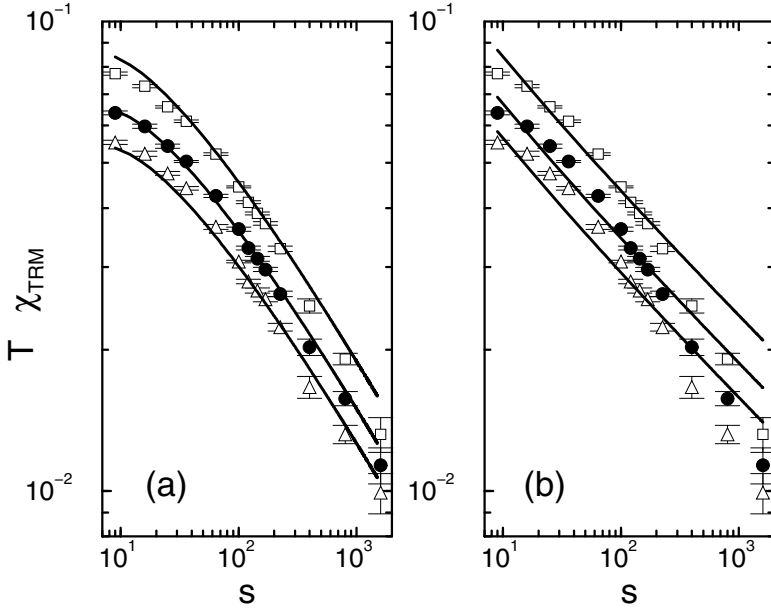


Fig. 1.22. Scaling of the thermoremanent susceptibility of the 2D Ising model quenched to $T = 1.5$ as a function of the waiting time s , for several values of $y = t/s$ (squares: $y = 5$, circles: $y = 7$ and triangles $y = 9$). The autoresponse exponent $\lambda_R = 1.26$ was used and we assumed $a = 1/2$ in (a) and $a = 1/4$ in (b). After [338].

Indeed, a closer inspection of systems in the class S reveals several difficulties with the identification $a \stackrel{?}{=} a_\chi$ proposed by Corberi *et al.*. First, consider systems of class S above their roughening temperature such that the interface width exponent κ , defined on p. 66, is positive, $\kappa > 0$, and hence $A < a = 1/z$, see (1.101). One might be tempted to interpret the first time-dependent term $\sim t^{-A}$ in the scaling form (1.102) as a scaling function $f(t/s)$ of a two-time response. However, since $t^{-A} = s^{-A}(t/s)^{-A}$, this interpretation would lead to the further conclusion $\lambda_R/z = A < 1/z$, in contradiction with the Yeung-Rao-Desai inequality (1.89) for all phase-ordering systems with $d \geq 2$, since $\lambda_C = \lambda_R$ for the disordered initial state used here. Therefore, there is a conceptual reason that the interpretation of Corberi *et al.* of χ_{ZFC} cannot be maintained for class S systems such as the 2D Ising or Potts model quenched to $T < T_c$.

Second, we now describe quantitative tests in the 2D Ising model, simulated through heat-bath dynamics, where $a = 1/2$ as opposed to $a_\chi = 1/4$ should allow for a clear decision. A first explicit test uses the thermoremanent magnetisation $M_{TRM}(t, s) = h\chi_{TRM}(t, s)$ and from (1.95) the effect of the leading finite-time correction should be taken into account. As we shall show in Chap. 4, the theory of local scale-invariance (LSI) fixes the form of the

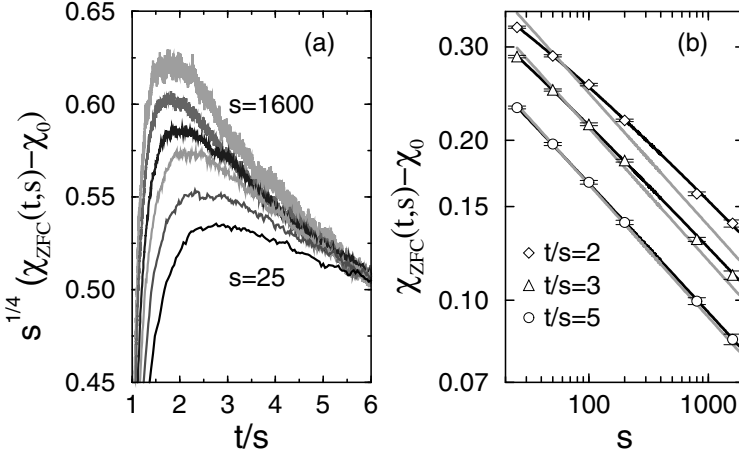


Fig. 1.23. (a) Scaling of $\chi_{ZFC}(t)$ over against t/s in the 2D Ising model, at $T = 1.5$, for waiting times $s = 25, 50, 100, 200, 800, 1600$ (bottom to top). (b) Comparison of the data with a fit $\chi_{ZFC}(t, s) - \chi_0 = a_0 s^{-1/4} + a_1 s^{-1/2}$ (black lines) and $\chi_{ZFC}(t, s) - \chi_0 = a_0 s^{-1/4}$ (grey lines). The data for $t/s = 2$ and $t/s = 3$ were shifted upwards by factors 1.4 and 1.2, respectively. Reprinted with permission from [347]. Copyright (2005) by the American Physical Society.

scaling function $f_M(y)$ [330, 338]

$$f_M(y) = r_0 y^{-\lambda_R/z} {}_2F_1 \left(1 + a, \frac{\lambda_R}{z} - a; \frac{\lambda_R}{z} - a + 1; y^{-1} \right) \quad (1.111)$$

where ${}_2F_1$ is a hypergeometric function and r_0 a non-universal normalisation constant and the leading correction term is written in the form $r_1 s^{-\lambda_R/z} (t/s)^{-\lambda_R/z}$, where r_1 is another non-universal constant. We now fix a value of $y = t/s$ and fit the constants $r_{0,1}$ to the data, with either $a = 1/2$ or else $a = 1/4$ assumed. Having done so, we have a *parameter-free prediction* for $\chi_{TRM}(t, s)$ for any other value of y . The result is shown in Fig. 1.22, where we used the $y = 7$ data for the initial fit ($r_0 \simeq 1.76, r_1 \simeq -1.84$ for $a = 1/2$ and $r_0 \simeq 0.22, r_1 \simeq 0.09$ for $a = 1/4$). The full lines give the LSI-prediction for the scaling function. We clearly see in Fig. 1.22a that for $a = 1/2$ the data are very well-described by the theory while if we take $a = 1/4$, not even the initial fit for $y = 7$ produces satisfactory results and Fig. 1.22b shows large deviations between the LSI prediction and the data. Similar tests were also done for the spherical model [338] and the 2D q -states Potts model [481].

Next, we turn to an analysis of $\chi_{ZFC}(t, s)$. Because of the sum rule (1.96), it is sufficient to analyse the behaviour of the field-cooled susceptibility $\chi_{FC}(t)$; see Fig. 1 in [339] for an explicit test of (1.96) and for the confirmation that $\chi_{FC}(t) - \chi_0 \sim t^{-1/4}$. The scaling behaviour of $\chi_{ZFC}(t)$ is displayed in Fig. 1.23a which clearly shows that besides the ‘field-cooled’ term $\sim t^{-1/4}$, a second term

is clearly recognisable, for values up to $y \approx 5$. The order of that correction term is studied in Fig. 1.23b, with clear evidence pointing towards this term being of order $s^{-1/2}$, just as expected from (1.95) with $a = 1/2$, whereas the simple scaling advocated by Corberi *et al.* [158] does not reproduce the data for $y = 2$ or $y = 3$.

Finally, we show in Fig. 1.24 the scaling of the intermediate response (1.103). While for $a = 1/2$ a good collapse is seen, the data fail to collapse for $a = 1/4$. The good scaling behaviour of $\chi_{\text{Int}}(t, s)$ was also tested for the spherical model [339].

Last but not least, the value of a can be tested by calculating the response function $R(t, s)$ directly, for which by now several good methods exist [141, 616, 477] and see appendix G in volume 1. In Fig. 1.25, we show data for the autoresponse function $R(t, s)$ in the 2D Ising model quenched to $T = 1.667 < T_c$. While for $a = 1/2$ even for the relatively small values of s used, there is clear data collapse. We also observe that finite-time corrections, especially for smaller values of t/s , are still notable. On the other hand, the same data do not scale if one assumes $a = 1/4$.

In conclusion, the numerical data in the 2D Ising model clearly show that the leading term in $\chi_{\text{ZFC}}(t, s)$ for large times, although superficially compatible with scaling, is not straightforwardly related to the scaling of the two-time response function. The exponent a_χ studied by Corberi *et al.* should rather be identified with the exponent $A \neq a$, as defined in eq. (1.101). This is an illustrative example and warning that even obtaining a scaling plot is not always sufficient for an unambiguous interpretation of data for integrated responses.

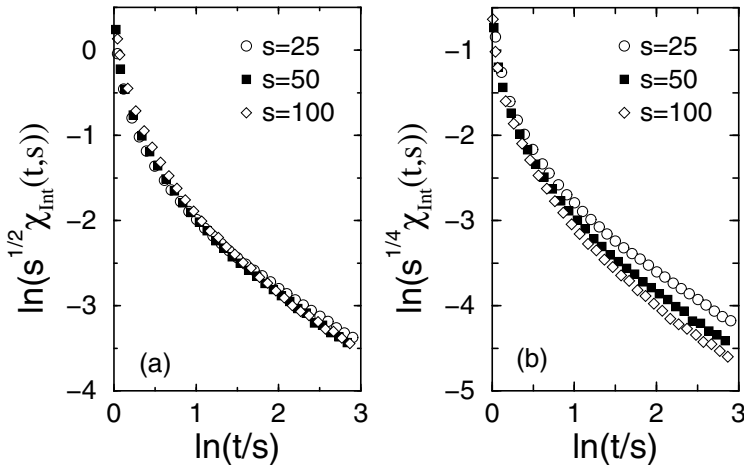


Fig. 1.24. Scaling of $\chi_{\text{Int}}(t)$ over against t/s in the 2D Ising model, at $T = 1.5$, for (a) $a = 1/2$ and (b) $a = 1/4$. Reprinted with permission from [347]. Copyright (2005) by the American Physical Society.

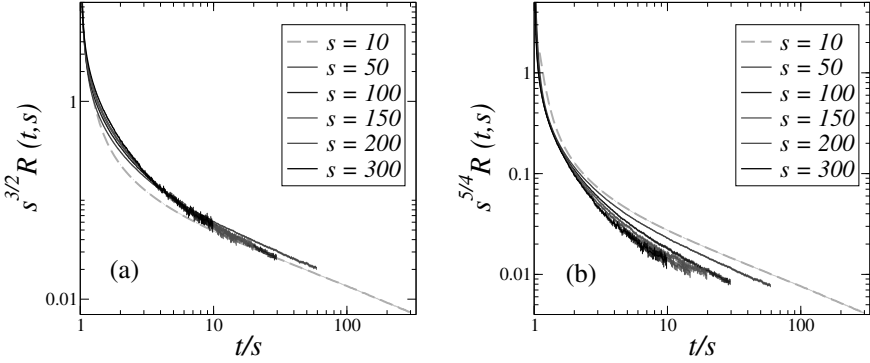


Fig. 1.25. Scaling of the response function for the 2D Ising model quenched to $T = 1.666$, with an assumed ageing exponent (a) $a = 1/2$ and (b) $a = 1/4$. The waiting times are $s = [10, 50, 100, 150, 200, 300]$. Data courtesy C. Chatelain.

1.5.2 Values of the Critical Autocorrelation Exponent

We now review existing knowledge of the values of the autocorrelation and autoresponse exponents, for model A dynamics. Before we look at model-specific results, we discuss a non-trivial scaling relation.

Theorem: (Janssen, Schaub, Schmittmann) [404, 402] *For a critical system which relaxes from a disordered initial state and obeys a dynamics without macroscopic conservation laws (model A), one has*

$$\lambda_C = d - \Theta z. \quad (1.112)$$

This relates the critical autocorrelation exponent $\lambda_C = \lambda_C(T_c)$ to the critical initial **slip exponent** Θ , originally identified [404] in the early stages of critical dynamics. Indeed, the existence of an universal early-time regime is by now well-established and is routinely used as an analytic tool (early results are reviewed in [755]).

The qualitative generic behaviour of a magnetic system which begins its evolution with an infinitesimal initial magnetisation is sketched in Fig. 1.26. Since small ordered domains tend to increase in size, the initially very small magnetisation first increases as a function of time according to $m(t) \sim m_0 t^\Theta$ (we implicitly assume here that $\Theta > 0$ which is the case for most systems studied so far). Only at later times, when clusters up to the size $L(t) \sim t^{1/z}$ have become correlated, does the magnetisation cross over to the late-time decrease $m(t) \sim t^{-\beta/\nu z}$ towards equilibrium. Despite its apparent simplicity, the derivation of eq. (1.112) requires a not-so-simple argument. We point out that (1.112) only holds for fully disordered initial states [123], *without* spatially long-range correlations [449]. We shall discuss various implicit conditions for the validity of (1.112) in Chap. 3.

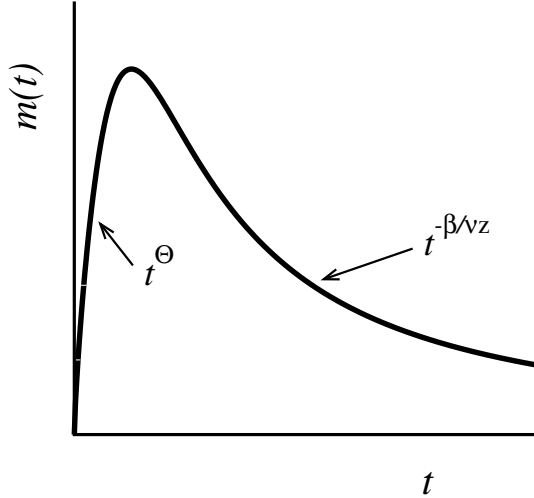


Fig. 1.26. Schematic early-time scaling and initial slip in critical magnets, starting from an infinitesimal initial magnetisation, and the cross-over to the relaxation behaviour at later times.

Proof: The outline of the derivation of (1.112) follows [402]. Consider the field-theoretical formulation for the dynamics of a simple ferromagnet, relaxing from some disordered initial state. Appendix D recalls the main steps for the derivation of the effective action, or Janssen-de Dominicis functional, eq. (D16). Denoting the order-parameter by ϕ and the associated response field by $\tilde{\phi}$, and their initial values at time $t = 0$ by ϕ_0 and $\tilde{\phi}_0$, respectively, one is interested in the combined correlation and response functions

$$G_{n,\tilde{n}}^m = \left\langle \phi^n \tilde{\phi}^{\tilde{n}} \tilde{\phi}_0^m \right\rangle. \quad (1.113)$$

For a simple relaxing ferromagnet, at a distance $\tau := (T - T_c)/T_c$ from criticality, the following scaling behaviour can be derived

$$\begin{aligned} G_{n,\tilde{n}}^m(t_i, \mathbf{r}_i; \tau, \tau_0^{-1}) &= \ell^{\delta(n,\tilde{n},m)} G_{n,\tilde{n}}^m(\ell^z t_i, \ell \mathbf{r}_i; \ell^{1/\nu} \tau, \ell^z \tau_0^{-1}) \\ 2\delta(n, \tilde{n}, m) &= n(d - 2 + \eta) + \tilde{n}(d + 2 + \tilde{\eta}) + m(d + 2 + \tilde{\eta} + \tilde{\eta}_0) \end{aligned} \quad (1.114)$$

where the exponents $\eta, \tilde{\eta}, \tilde{\eta}_0$ are found as usual from the fixed-point values of the renormalisation-group functions.³⁹ Define the slip exponent [402]

$$\begin{aligned} \Theta &:= -\frac{1}{2z} (\eta + \tilde{\eta} + \tilde{\eta}_0) \\ &= \frac{n+2}{n+8} \frac{\varepsilon}{4} \left(1 + \frac{6}{n+8} \left(\frac{n+3}{n+8} + \ln \frac{3}{2} \right) \varepsilon + O(\varepsilon^2) \right) \end{aligned} \quad (1.115)$$

³⁹ At equilibrium, the fluctuation-dissipation theorem would hold, from which the relation $\tilde{\eta} - \eta = 2z - 4$ can be derived [759], but we cannot use this for the situation of *non-equilibrium* critical dynamics at hand.

where the second line follows from an ε -expansion in the $O(n)$ model, with $\varepsilon = 4 - d$.

The next important ingredient involves a short-time expansion which in turn is motivated from what is known in equilibrium critical phenomena close to a free surface, see [197, 388, 593] for reviews. Indeed, from the Gaussian part of the action (D16) alone, one finds the response and correlation function in momentum space

$$\begin{aligned}\hat{R}_{\mathbf{q}}(t, s) &= \Theta(t - s) e^{-(\tau + q^2)(t - s)} \\ \hat{C}_{\mathbf{q}}(t, s) &= \frac{1}{\tau + q^2} \left(e^{-(\tau + q^2)|t - s|} - e^{-(\tau + q^2)(t + s)} \right) + \frac{1}{\tau_0} \hat{R}_{\mathbf{q}}(t, 0) \hat{R}_{\mathbf{q}}(s, 0)\end{aligned}$$

where the first term for $\hat{C}_{\mathbf{q}}$ is referred to as the **Dirichlet correlator**. The Θ -function is defined by $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ otherwise and expresses the causality in the response function. If one now considers correlations where the earlier time $s \rightarrow 0$, the Dirichlet correlator vanishes and recalling that both $\hat{C}_{\mathbf{q}}$ and $\hat{R}_{\mathbf{q}}$ may be rewritten as correlators, one observes the following suggestive relation between the initial order-parameter and its response function.

$$\phi_0(\mathbf{r}) = \frac{1}{\tau_0} \tilde{\phi}_0(\mathbf{r}) . \quad (1.116)$$

It can be shown [404, 402] that eq. (1.116) is invariant under renormalisation and should be interpreted as the leading-term in the short-time expansion starting from $t = 0$.

One can now derive the scaling of the correlation with the initial state. Indeed, because of (1.116) and (1.115) one has

$$C(t, 0) = \langle \phi(t, \mathbf{r}) \phi(0, \mathbf{r}) \rangle = \frac{1}{\tau_0} \left\langle \phi(t, \mathbf{r}) \tilde{\phi}_0(\mathbf{r}) \right\rangle \sim t^{\Theta - d/z} . \quad (1.117)$$

The exponent in this power law can be identified with the critical autocorrelation exponent $-\lambda_{C'}/z$, from which (1.112) follows if $\lambda_C = \lambda_{C'}$.⁴⁰ q.e.d.

It remains to explain the relation of Θ as defined in (1.115) with the initial slip. To do so, consider the average magnetisation with a homogeneous initial magnetisation m_0

$$\begin{aligned}m(t, \mathbf{r}) &= \langle \phi(t, \mathbf{r}) \rangle = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \phi e^{-\mathcal{J}_{\text{eff}}[\phi, \tilde{\phi}]} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \int \prod_{i=1}^k d\mathbf{r}_i G_{1,0}^k(t_i, \mathbf{r}_i; \tau) m_0^k \\ &= \ell^{\beta/\nu} \sum_{k=1}^{\infty} \frac{1}{k!} \int \prod_{i=1}^k d\mathbf{r}_i G_{1,0}^k(\ell^z t_i, \mathbf{r}_i; \ell^{1/z} \tau) \left(\ell^{-\beta/\nu - z\Theta} m_0 \right)^k \\ &= t^{\Theta} m_0 f \left(t^{\Theta + \beta/\nu z} m_0, \tau t^{-1/\nu z} \right)\end{aligned} \quad (1.118)$$

⁴⁰ Exercise 5.13 describes an alternative proof using local scale-invariance.

where in the third line, one has expanded in m_0 before applying the scaling (1.114) and lastly setting $\ell^z t = 1$ in order to obtain the desired scaling form of the magnetisation. Whenever one has dynamical scaling, the second argument can be dropped and one has the asymptotic limits, see Fig. 1.26,

$$m(t) \sim \begin{cases} t^\Theta & ; \text{ for } t \rightarrow 0 \\ t^{-\beta/\nu z} & ; \text{ for } t \rightarrow \infty \end{cases} . \quad (1.119)$$

In the generic case when $\Theta > 0$, one sees from (1.119) that the **slip exponent** Θ describes the initial *increase* of the magnetisation from a small initial value before at later times the relaxation towards the equilibrium state takes over. That increase comes entirely from the cooperative action between the individual spins and is absent in mean-field theories. The scaling form for t large is easily found by realising that the late stages should be independent of the initial magnetisation m_0 .⁴¹

While in this way the critical autocorrelation and slip exponents have become related, there is no further relation between them and other known static or dynamic exponents.

In Table 1.7, we collect values for the critical dynamical exponent $z(T_c)$, the critical initial slip exponent Θ , the critical autocorrelation exponent $\lambda_C(T_c)$ and the universal limit fluctuation-dissipation ratio X_∞ , defined in exercises 1.16 and 1.17.⁴² Appendix A recalls the precise definitions of these models and serves as a reminder of some of their equilibrium critical properties.⁴³ The data in Table 1.7 are meant to represent typical values as currently accepted but we did not attempt an exhaustive collection of the many individual results in this active field. Also, we have only collected here simulational (and a few exact) results. At the time of writing, the estimates eqs. (1.71, 1.115) coming from field-theory of the $O(n)$ model for the dynamical exponent z and for Θ tend to be slightly lower ($\lesssim 1 - 2\%$) than the simulational values, see [121] and references therein. We point out that in many cases extensive tests of the universality of these exponents with respect to the underlying lattice (see especially [142]) and/or the choice of dynamics (e.g. heat bath vs Metropolis [560, 755, 306]) were carried out. In general, the scaling relation (1.112) is confirmed to within a few percent.

⁴¹ Since it was realised that there exists an *early*-time dynamical scaling regime, its study has become an important tool. For example, early-time dynamical scaling was used in order to study the order of the helix-coil transition in polypeptides [25].

⁴² We quote the results and their error bars as given in the sources, but there still is considerable scatter in many of these results, often up to an order of magnitude larger than the estimated errors, see e.g. [174] for a collection of values of z in the Potts- q models.

⁴³ We only include here simple, non-glassy, spin systems. Data for the ageing of some critical glassy systems are collected in Table 3.5 on p. 178. Further data on fully frustrated systems without disorder are given in Table 3.2 on p. 162.

model	d	$z(T_c)$	Θ	$\lambda_C(T_c)$	X_∞	Refs.
TDGL	1	2	0.199969 ...	0.600616 ...		E [96]
Ising - KDH	1	4		1		E [219]
Ising - Glauber	1	2	0	1	1/2	E [285, 478]
	2	2.172(6)	0.191(3)	1.585 [#]		[294, 295]
	2	2.155(3)	0.191(1)	1.588(2)*	0.33(1)	[560, 639]
	2	2.1667(5)	0.20 [#]	1.60(4)*	0.328(1)	[548, 142]
	3	2.032(4)	0.104(3)	2.79 [#]		[294, 295]
	3	2.042(6)	0.108(2)	2.78(4)*		[405]
	3	2.055(10)			≈ 0.4	[391, 286]
majority voter	2	2.170(5)	0.191(2)	1.595(10)*		[515]
Potts-3	2	2.196(8)	0.075(3)	1.836(2)*		[560]
	2	2.197(3)	0.072(1)	1.85(4)*	0.406(1)	[174, 177, 142]
		2.2 [#]		1.76(4)*	0.403(8)	AT [142]
Potts-4	2	2.290(3)	-0.047(3)	2.27(5)*	0.459(8)	[174, 177, 142]
		2.296(5)	-0.046(9)	2.11 [#]		[26]
		2.294(3)	-0.046(9)	2.11 [#]		[239]
	2	2.383(4)	-0.03(1)	2.32(5)*	0.466(3)	[672, 142]
Turban-3		2.292(4)	-0.047(8)	2.11 [#]		[26]
	2	2.05(10)	-1.00(5)	—	—	1 st [600]
Baxter-Wu	2	2.294(6)	-0.186(2)	2.6(1)*	0.548(15)	[24, 142]
	2	1.994(24)	-0.185(2)	2.369(2)*		[306]
Blume-Capel	2	2.215(2)	-0.53(2)	3.17 [#]		[175]
Ising FF	2	1.999(8)*	0 [#]	2.006(10)*	0.33(1)	[722, 723, 424]
diluted Ising	3	2.62(7)	0.10(2)	2.75(7)*	$\frac{1}{2} - \sqrt{\frac{3\varepsilon}{424}}$	[644, 582]
		2.2(2)	0.10(3)	2.73(30)		[553]
clock-6	2	2.16(4)	0.254(5)	1.45 [#]		T_+
		2.24(2)	0.314(2)	1.29 [#]		T_- [173]
XY	2	2 (log)		$1 + \eta(T)$	$\approx \frac{1}{2} \frac{T}{T_{KT}}$	[5]
			0.250(1)	1.48(1)		[559, 463]
		2 (log)	0.245(2)	1.494(5)	0.215(15)	[756, 724]
	3	≈ 2	0.16 [#]	2.68(10)*	0.43(4)	[406, 6]
XY FF	2	2 (log)	0.13 [#]	1.74(6)*	0.385(15)	T_- [724]
		2 (log)	0.02 [#]	1.96(10)*	0.405(5)	T_+ [724]
Heisenberg	3	1.976(9)	0.482(3)	2.04(3)		[240]
double exchange	3	1.975(10)	0.480(7)	2.05 [#]		[242]
spherical	< 4	2	$1 - d/4$	$\frac{3}{2}d - 2$	$1 - 2/d$	E
	> 4	2	0	d	1/2	E [286]
long-range spherical, $\sigma < 2$	< 2 σ	σ	$1 - d/(2\sigma)$	$\frac{3}{2}d - \sigma$	$1 - \sigma/d$	E
	> 2 σ	σ	0	d	1/2	E [47]

Table 1.7. Non-equilibrium parameters for non-conserved critical spin models, with an uncorrelated initial state. A star * indicates that only λ_C/z is quoted in the source and λ_C was recalculated using the value of z given in the same line. A quantity calculated via a scaling relation from other information is indicated by [#]. Abbreviations: TDGL = time-dependent Ginzburg-Landau equation, AT = realised through the Ashkin-Teller model, FF = fully frustrated, $T_\pm$ = upper/lower transitions in the clock-6 and XYFF models, 1st = first-order transition, E = exact solution. The Blume-Capel model is understood to be at its tricritical point. In the diluted Ising model, X_∞ is given up to terms of first order in $\varepsilon = 4 - d$.

Some further tests of universality should be noted. It is generally accepted that the critical behaviour of the 2D Ising model and of the majority voter model are the same and comparison of the entries in the Table illustrates that this also extends to the dynamics. Similarly, the behaviour of the Potts-3 model and its variant realised as a special case of the Ashkin-Teller model seem to be the same, in and out of equilibrium, even if the available numerical evidence for that agreement looks a little marginal.

Next, we take up three different realisations of the Potts-4 equilibrium universality class, namely the Potts-4 model itself, a multispin Ising model (MSI) and the Baxter-Wu model. Remarkably, the Baxter-Wu model shows a distinct dynamical behaviour from the other two. This becomes particularly clear from the slip exponent Θ as well as the limit fluctuation-dissipation ratio X_∞ , and also from the autocorrelation exponent λ_C .⁴⁴ We see that calculating Θ , λ_C or X_∞ can be an important diagnostic tool in distinguishing dynamic universality classes. Such differences may even exist if the equilibrium behaviour (and possibly even the exponent z) is the same.

When discussing the two-dimensional clock and XY models, one must recall that the standard scaling $L(t) \sim t^{1/z}$ of the domain sizes does not hold in general. Rather, the characteristic length scale behaves as [625, 95]

$$\xi(t) \sim \begin{cases} (t/\ln t)^{1/2} & ; \text{ if } T_{\text{init}} > T_{\text{KT}} \\ t^{1/2} & ; \text{ if } T_{\text{init}} < T_{\text{KT}} \end{cases} \quad (1.120)$$

and this makes the interpretation of numerical data more difficult. Present results do not seem to agree with the scaling relation (1.112) and one may wonder whether this reflects that in some studies [173, 559] the extra logarithmic factors as suggested in (1.120) were not taken into account. However, there seems to be some consistency, as expected, between the six-states clock and the XY model.

Turning to the 1D Glauber-Ising model, we point out that its exactly known dynamical behaviour [285, 478, 356] (see Chap. 2 for the derivation) is quite distinct from the one derived exactly from the time-dependent Ginzburg-Landau equation by Bray and Derrida [96]. In the previous section, in discussing phase-ordering kinetics, we have followed the habitual folklore and took for granted that the long-time dynamics of spin systems described by a master equation can be modelled through a Ginzburg-Landau equation [93, 94]. It is not clear whether the non-equivalence of the 1D Ising model and the 1D TDGL equation at $T = 0$ is merely a kind of ‘exotic’ behaviour coming from the fact that the critical point of the 1D Ising model is at $T_c = 0$.

We finish the discussion of the critical autocorrelation exponent by mentioning three examples of *non-universal* behaviour.

⁴⁴ Any outcome of the present debate about the value of z in the Baxter-Wu model [24, 306] will not change this conclusion.

1. 2D Ising model with a defect line [673, 599].

It is well-known that the equilibrium behaviour of the magnetisation close to a line of ‘defects’ with a modified coupling constant J' depends continuously on the defect strength $\kappa = J'/J$ [38, 513, 388, 329]. Numerical studies have shown that also Θ becomes κ -dependent, but such that the initial scaling dimension of the magnetisation

$$x_0 = \frac{\eta(\kappa)}{2\nu} + z\Theta(\kappa) \simeq 0.53 \quad (1.121)$$

remains constant at the value of the pure system (where $J = J'$).

2. Ising model with competing interactions [418].

One considers an Ising model with competition between nearest neighbours and next-nearest neighbours, respectively, as described by the Hamiltonian

$$\mathcal{H} = -J_1 \sum_{(i,j)} \sigma_i \sigma_j - J_2 \sum_{((i,j))} \sigma_i \sigma_j \quad (1.122)$$

where the first sum runs over the nearest neighbours and the second over the next-nearest neighbours. For $J_2/J_1 < -1/2$, there is a superantiferromagnetic ground state where a row of up spins alternates with a row of down spins. The phase-transition towards the paramagnetic phase is non-universal and the exponents depend on J_2/J_1 . This non-universality was also observed for both z and Θ .

3. XY non-collinear magnets [61].

Consider an antiferromagnetic XY model on stacked three-dimensional triangular lattices with a Hamiltonian

$$\mathcal{H} = \sum_{(i,j)} J_{i,j} \vec{S}_i \cdot \vec{S}_j + r \sum_{\text{plaquettes}} \left(\vec{A} + \vec{B} + \vec{C} \right)^2 \quad (1.123)$$

where $J_{i,j}$ are antiferromagnetic within the triangular layers and ferromagnetic between the layers and $|J_{i,j}| = 1$. The indices $\vec{A}, \vec{B}, \vec{C}$ label the spin vectors on the three sublattices on the corners of each elementary triangle. Varying the parameter r controls the importance of the constraint on the spin orientation on the three sublattices. Again, numerical studies suggest that the effective static and dynamic exponents, notably Θ , depend continuously on r . Models of this kind have been used to describe a great number of stacked triangular materials such as CsXCl_3 , with $\text{X}=\text{Cu}, \text{Ni}, \text{Mn}$ or else XY helimagnets such as Ho , Dy or Tb .

1.5.3 Values of the Autocorrelation Exponent Below T_c

We now turn to systems with a stationary state in the coexistence phase, usually reached by temperature quenches to $T < T_c$. In Table 1.8 we list known values for the autocorrelation exponent for non-conserved systems quenched to *below* T_c , again for uncorrelated initial conditions. We remark that in two dimensions, the numerical values found are actually quite close to their rigorous lower bound as given by the Yeung-Rao-Desai inequality (1.89). This is further borne out by the following estimate in the $O(n)$ -model [99]

$$\lambda_C = \frac{d}{2} + \left(\frac{4}{3}\right)^d (d+2) \frac{2d}{9} B\left(1 + \frac{d}{2}, 1 + \frac{d}{2}\right) \frac{1}{n} + O(n^{-2}) \quad (1.124)$$

and where $B(x, y)$ is Euler's beta function [4].

There is an instructive heuristic argument which produces the upper bound $\lambda_{C'} \leq \frac{5}{4}$ in the 2D Glauber-Ising model, due to Fisher and Huse [245]. Consider the two-time autocorrelator $C(t, 0) \sim L(t)^{-\lambda_{C'}}$. One may arbitrarily fix the initial spin to be 'up', and let denote $\frac{1}{2} + f$ the fraction of 'up' spins at time t . They expect that the average fraction of 'up' spins in a region of linear size $L(t)$, denoted by $\mathcal{P}(L(t), f)$, should satisfy, for $L(t)$ large enough, a scaling form $\mathcal{P}(L(t), f) - \frac{1}{2} = g(fL(t)^a)$ with $g(x) \sim x$ for $x \rightarrow 0$ and where the exponent a remains to be determined. Furthermore, they assume that the number of 'up' spins at time t should be uncorrelated from the number of 'up'

model	d	λ_C	a	Refs.
Ising	2	1.25	–	[245]
		$\simeq 1.24$	–	[383]
		1.246(20)	–	[479]
		1.24(2)	1/2	[481]
	3	$\simeq 1.59$	–	[385]
		1.60(2)	0.5	[344]
Potts-3	2	1.19(3)	0.49	[481]
		1.22(2)	–	[159]
Potts-8	2	1.25(1)	0.51	[481]
XY	3	1.7(1)	1/2	[6]
spherical	> 2	$d/2$	$d/2 - 1$	[536, 286]
spherical, long-range	$> \sigma$	$d/2$	$d/\sigma - 1$	[127]

Table 1.8. Autocorrelation exponent λ_C and ageing exponent a for spin systems quenched to $T < T_c$ from an uncorrelated initial state and with a non-conserved order-parameter. The upper part contains systems in class S and the lower part systems in class L. The value of the exponent $a = 1/2$ in the 2D Ising model is discussed in detail in Sect. 1.5. With the only exception of the spherical model with long-range interactions, where $z = \sigma$, the dynamical exponent $z = 2$ in all models.

spins at time 0 and describe the initial random distribution of the spins by a normal distribution in f , with variance $L(t)^d$. In this way, they obtain the lower bound

$$C(t, 0) \geq \text{constant} \cdot \int_{\mathbb{R}} df e^{-f^2/(2L(t)^d)} L(t)^{d/2} f g(fL(t)^a) \sim L(t)^{a-d} \quad (1.125)$$

which leads to $\lambda_{C'} \leq d - a$. For the $2D$ Ising model on a square lattice, both the clusters of ‘up’ and ‘down’ spin are at their isotropic percolation threshold. Fisher and Huse propose to identify $f \sim p - p_c$, where $p_c = 0.5$ is the percolation threshold. Since for $2D$ isotropic percolation, the correlation length $\xi \sim (p - p_c)^{-4/3}$, they conclude that $a = \frac{3}{4}$, hence $\lambda_{C'} \leq \frac{5}{4}$. Although a couple of non-trivial assumptions have been made, the estimates listed in [Table 1.8](#) support this inequality, if $\lambda_C = \lambda_{C'}$.⁴⁵

1.5.4 Values of the Autoresponse Exponent

Having dealt so far exclusively with the autocorrelation exponent λ_C , it remains to discuss the values of the autoresponse exponent λ_R . It is convenient, following [99, 93], to generalise slightly and to consider initial conditions with a *Gaussian* distribution and with variance

$$\langle \hat{\phi}_{\mathbf{k}}(t) \hat{\phi}_{-\mathbf{k}}(0) \rangle = \Delta(\mathbf{k}) \quad , \quad \Delta(\mathbf{k}) = \Delta_0 + \Delta_1 k^\alpha \quad (1.126)$$

in momentum space and where $\hat{\phi}_{\mathbf{k}}(t)$ is the spatial Fourier transform of the order-parameter field $\phi(t, \mathbf{r})$. The second term in the initial correlator gives in coordinate space a decay $\sim r^{-d-\alpha}$. Now, because of the Gaussian property in the initial correlator it is easily seen that

$$\hat{R}_{\mathbf{k}}(t, 0) = \left\langle \frac{\delta \hat{\phi}_{\mathbf{k}}(t)}{\delta \hat{\phi}_{\mathbf{k}}(0)} \right\rangle = \Delta(\mathbf{k}) \langle \hat{\phi}_{\mathbf{k}}(t) \hat{\phi}_{-\mathbf{k}}(0) \rangle = \Delta(\mathbf{k}) \hat{C}_{\mathbf{k}}(t, 0) \quad . \quad (1.127)$$

Furthermore, the standard scaling forms of R and C become in Fourier space $\hat{R}_{\mathbf{k}}(t, 0) = t^{(d-\lambda_R)/z} g_R(\mathbf{k} t^{1/z})$ and similarly for $\hat{C}$ with λ_C replacing λ_R and a new scaling function g_C . When inserting into (1.127), the result depends on which of the two terms of $\Delta(\mathbf{k})$ in (1.126) is the more relevant in the renormalisation group sense. If the first term dominates, one remains with effectively *short-ranged* initial correlations while if the second one dominates, one passes over to the case of *long-ranged* initial correlations.⁴⁶

⁴⁵ On the other hand, the numerical precision achieved so far is not yet sufficient to test the stronger conjecture $\lambda_C = 5/4$ [245] in the $2D$ Glauber-Ising model.

⁴⁶ In the $O(n)$ model, the critical value for α is: $\alpha_c = -\left(\frac{4}{3}\right)^{d+1} \frac{d}{3} (d+2) B \left(1 + \frac{d}{2}, 1 + \frac{d}{2}\right) \frac{1}{n}$ to leading order in $1/n$, where B is Euler’s beta function [99].

Then comparison of the scaling forms of $\widehat{R}_{\mathbf{k}}(t, 0)$ and $\widehat{C}_{\mathbf{k}}(t, 0)$ gives [99]

$$\begin{aligned}\lambda_C &= \lambda_R & , \text{ short-ranged} \\ \lambda_C &= \lambda_R + \alpha & , \text{ long-ranged}\end{aligned}\tag{1.128}$$

We shall see later that this result is reproduced in several exactly solved models, see for instance Tables 2.4, 2.8, 2.10, as well as in numerical simulations [383] and field-theory [511].

In Chap. 4, we shall give another proof, based on local scale-invariance with $z = 2$, that $\lambda_C = \lambda_R$ for short-ranged initial conditions, in phase-ordering kinetics [589].

1.6 Global Persistence

In connection with ageing, it is of interest to consider, besides the correlators and response functions, a different kind of observables, namely the **persistence** probabilities [190]. Since a full discussion of the actively studied field of persistence is beyond the scope of this book, we concentrate on a single aspect and refer to [491] for a fuller discussion. Consider the global order-parameter $\phi_g(t) \sim \int d\mathbf{r} \phi(t, \mathbf{r})$, which for magnetic systems is the total magnetisation, and ask for the probability $P_g(t)$ that the global order parameter did not change its sign until the time t . When considering a system with its thermodynamic parameters chosen to be at the values of an equilibrium critical point, one expects for a large system a power-law behaviour [190]

$$P_g(t) \sim t^{-\theta_g} \quad ; \quad t \rightarrow \infty \tag{1.129}$$

where θ_g is called the **global persistence exponent**. One of the motives to study this quantity comes from the following beautiful result of Majumdar, Bray, Cornell and Sire.

Theorem: [493] *Consider a d -dimensional system undergoing non-equilibrium critical dynamics with a fully disordered initial state. If furthermore the underlying stochastic process is Markovian, one has*

$$\theta_g z = \lambda_C - \frac{1}{2}\eta - d + 1 \tag{1.130}$$

where η is a standard equilibrium critical exponent and $\lambda_C = \lambda_C(T_c)$ is the value of the autocorrelation exponent at criticality.

This result is remarkable since it relates a qualitative property of the physical system, here the Markov property of its underlying stochastic process, to a quantitative relationship. Later on, we shall turn this relation around and use simulational and/or exact data from specific models in order to test eq. (1.130) and *a fortiori* the Markov property in these models.

Proof: The proof of this result requires several distinct ingredients which are arranged in several steps [492].

Step 1. Physically, non-equilibrium critical dynamics starts from a non-critical initial state, so that one can assume that initially, the correlation length ξ_0 is very small, e.g. of the order of the lattice constant or less. The dynamics will lead to the formation of mutually independent correlated regions, of linear size $L(t) \sim t^{1/z}$. Dynamical scale-invariance asserts that $L(t)$ should be the only physically relevant length scale. This also requires that the volume of the system $V \gg L(t)^d$ is sufficiently large. Construct the global order-parameter

$$\hat{\phi}_0(t) = \frac{1}{\sqrt{V}} \int_V d\mathbf{r} \phi(t, \mathbf{r})$$

by summing the local order parameter $\phi(t, \mathbf{r})$ in many (almost) independent cells. Because of the disordered initial state, it has by definition a vanishing first moment $\langle \hat{\phi}_0(t) \rangle = 0$ and the second moment $\langle \hat{\phi}_0^2(t) \rangle \sim L(t)^{2-\eta}$ is finite for each finite time. Therefore, the conditions of the central limit theorem are met. Hence, $\hat{\phi}_0(t)$ has a Gaussian probability distribution, in the limit $V \rightarrow \infty$.

Step 2. One analyses the scaling of the autocorrelator. Since in Fourier space

$$\hat{\phi}_{\mathbf{k}}(t) = \frac{1}{\sqrt{V}} \int_V d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \phi(t, \mathbf{r}),$$

the two-time autocorrelator takes the form, with the convention that $t_1 > t_2$

$$\begin{aligned} \langle \hat{\phi}_0(t_1) \hat{\phi}_0(t_2) \rangle &= \lim_{\mathbf{k} \rightarrow 0} \langle \hat{\phi}_{\mathbf{k}}(t_1) \hat{\phi}_{-\mathbf{k}}(t_2) \rangle \\ &= \lim_{\mathbf{k} \rightarrow 0} \frac{1}{V} \int_{V^2} d\mathbf{r}_1 d\mathbf{r}_2 e^{i\mathbf{k} \cdot (\mathbf{r}_2 - \mathbf{r}_1)} \langle \phi(t_1, \mathbf{r}_1) \phi(t_2, \mathbf{r}_2) \rangle \\ &= \lim_{\mathbf{k} \rightarrow 0} \frac{1}{V} \int_{V^2} d\mathbf{r}_1 d\mathbf{r}_2 e^{-i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} t_2^{-(d-2+\eta)/z} F_C \left(\frac{t_1}{t_2}, \frac{\mathbf{r}_1 - \mathbf{r}_2}{L(t_1 - t_2)} \right) \\ &= \lim_{\mathbf{k} \rightarrow 0} \int_V d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} t_2^{-(d-2+\eta)/z} F_C \left(\frac{t_1}{t_2}, \frac{\mathbf{r}}{L(t_1 - t_2)} \right) \\ &= t_2^{(2-\eta)/z} \left(\frac{t_1}{t_2} - 1 \right)^{d/z} \int_V d\mathbf{r} F_C \left(\frac{t_1}{t_2}, \mathbf{r} \right) \\ &= t_2^{(2-\eta)/z} \hat{f} \left(\frac{t_1}{t_2} \right) ; \quad \text{where } \hat{f}(y) \stackrel{y \rightarrow \infty}{\sim} y^{(d-\lambda_C)/z}. \end{aligned}$$

Here the asymptotic formula follows from the usual scaling of the two-time correlator.

Step 3. To make use of this scaling form, define the normalised autocorrelator

$$\hat{N}(t_1, t_2) := \frac{\langle \hat{\phi}_0(t_1) \hat{\phi}_0(t_2) \rangle}{\sqrt{\langle \hat{\phi}_0^2(t_1) \rangle \langle \hat{\phi}_0^2(t_2) \rangle}} = \hat{f}_N \left(\frac{t_1}{t_2} \right)$$

with the asymptotic behaviour $\hat{f}_N(y) \sim y^{(d-\lambda_C-1+\eta/2)/z}$ for $y \rightarrow \infty$. By the change of variable $T = \ln t$, one has $\hat{N}(t_1, t_2) = \bar{N}(T_1, T_2) = n(T_1 - T_2)$. Therefore, *the Gaussian process describing $\hat{\phi}_0(T)$ is stationary*. Asymptotically for $T \rightarrow \infty$, one has $n(T) \sim e^{-\mu T}$, with $\mu = (\lambda_C - d + 1 - \eta/2)/z$.

Step 4. One can now use a classical result, due to Doob:

Lemma 1: [205] *Consider a stochastic process $X(t)$ with $\langle X(t) \rangle = 0$ and which is assumed to be Gaussian and stationary. This process is Markovian, if and only if the autocorrelator has exactly an exponential form*

$$\langle X(t_1)X(t_2) \rangle = X_0 e^{-\mu|t_1-t_2|}.$$

where μ is a constant and X_0 a normalisation. A well-known example of such a Markovian, stationary and Gaussian stochastic process is the Ornstein-Uhlenbeck process [710]. A proof is outlined in exercise 1.18.

The stochastic process of the global order-parameter satisfies the conditions of the lemma. If one furthermore assumes it to be Markovian, the known asymptotic form implies that for all $T \in \mathbb{R}_+$ one must have exactly $n(T) = e^{-\mu T}$, with $\mu = (\lambda_C - d + 1 - \eta/2)/z$.

Step 5. The relationship with the global persistence probability is obtained from the following result, due to Slepian.

Lemma 2: [675] *Consider a stochastic process which is stationary and Gaussian, and with an autocorrelator $\langle X(T)X(0) \rangle = e^{-\mu|T|}$. Then the global persistence probability for $X(t)$ is given by*

$$P_g(T) = \frac{2}{\pi} \arcsin(e^{-\mu T}). \quad (1.131)$$

Because of Doob's lemma 1, the conditions for the application of lemma 2 to the physical system undergoing non-equilibrium critical dynamics are met, provided the process is Markovian. Recall that $T = T_1 - T_2 = \ln(t_1/t_2)$, hence for T sufficiently large

$$P_g(t_1, t_2) = P_g(T) \sim e^{-\mu T} \sim \left(\frac{t_1}{t_2}\right)^{-(\lambda_C - d + 1 - \eta/2)/z}$$

so that one has rather found a two-time persistence probability. It is now sufficient to send the earlier time t_2 to some microscopically small 'initial time-scale' and then one finds for the global single-time persistence probability $P_g(t) \sim t^{-(\lambda_C - d + 1 - \eta/2)/z}$ for t sufficiently large. Comparison with (1.129) gives the assertion. q.e.d.

One can test the assumed Markov property by comparing eq. (1.130) with explicit data, which is done in Table 1.9. First, we consider several magnetic systems with an equilibrium stationary state, non-conserved dynamics (model A) and a fully disordered initial state. The exponents listed are averages we

model	d	z	Θ	λ_C	η	θ_g		Ref.
						Markovian	numeric	
Ising	1	2	0	1	1	1/4	1/4	[493]
	2	2.1667(5)	0.191(1)	1.588(2)	1/4	0.214(1)	0.237(3)	[647]
						0.214(1)	0.235(5)	[579]
	3	2.043(6)	0.106(2)	2.78(4)	0.0364(5)	0.374(1)	0.41(2)	[681]
Potts-3	2	2.197(3)	0.072(2)	1.836(4)	4/15	0.321(2)	0.350(2)	[647]
Potts-4	2	2.293(3)	-0.046(8)	2.15(7)	1/4	0.43(1)	0.474(7)	[239]
Blume -Capel	2	2.215(2)	-0.53(2)	3.17	3/80	0.97(2)	1.080(4)	[176]
diluted Ising	3	2.62(7)	0.10(2)	2.75(7)	0.037(4)	0.28(2)	0.35(1)	[582]
double exchange	3	1.975(10)	0.480(7)	2.05	0.0375(5)	0.017(10)	0.335(9)	[241]
spherical, long-range	$< 2\sigma$ σ		$1 - d/2\sigma$	$\frac{3}{2}d - \sigma$	$2 - \sigma$	$d/2\sigma - \frac{1}{2}$	$d/2\sigma - \frac{1}{2}$	
spherical	< 4	2	$1 - d/4$	$\frac{3}{2}d - 2$	0	$d/4 - \frac{1}{2}$	$d/4 - \frac{1}{2}$	[493]
mean-field	> 4	2	0	d	0	1/2	1/2	
NEKIM	1	1.75(1)		1.51(1)	1	0.58(1)	0.67(1)	[517]

Table 1.9. Test of the Markovian scaling relation (1.130) for the global persistence exponent θ_g in non-equilibrium critical dynamics and for the non-equilibrium kinetic Ising model (NEKIM) with a non-equilibrium stationary state. The Blume-Capel model is at its tricritical point. The long-range (LR) spherical model has a non-mean-field critical point for $\sigma < d < 2\sigma$ and $0 < \sigma < 2$. The quoted sources refer to the direct calculation of θ_g , labelled ‘numeric’.

took from Table 1.7 or come from appendix A. In those cases when no direct estimate of λ_C was available, the scaling relation eq. (1.112) was used. In this way the predicted value of θ_g , referred to as ‘Markovian’, was found. Unless the model is exactly solvable, as the 1D Glauber-Ising model or the spherical model, one clearly sees that the consequence eq. (1.130) of an assumed effective Markovian dynamics is *not* valid. Next, we consider the non-equilibrium kinetic Ising model NEKIM (see also Chap. 3), which has two simultaneous, competing dynamics, namely (i) Glauber dynamics at temperature $T = 0$ and (ii) Kawasaki dynamics at infinite temperature such that the stationary state is no longer an equilibrium state. The persistence measured is the one of the global magnetisation [517].

Summarising, the available evidence clearly suggests that *the effective stochastic process which describes the long-time behaviour of the global order-parameter at criticality, albeit being Gaussian and related to a stationary process, is in general not a Markov process.* The only known exceptions occur in exactly solvable models, where the equations of motions are manifestly Marko-

vian (see exercises 2.9 and 2.17). In general, one observes $z\theta_g \geq \lambda_C - \frac{1}{2}\eta - d + 1$. Therefore, the exponent θ_g should be considered as an independent exponent whose value can also be used as a diagnostic tool to relate a given, specific model to an universality class.⁴⁷

This conclusion has also been reached in several analytic studies, which compute the global persistence exponent θ_g from field-theoretic methods by expanding up to one or two loops. These calculations are based on a systematic expansion around a Markovian process. If the autocorrelator is $\langle X(T)X(0) \rangle = e^{-\mu|T|} + \epsilon f(T)$, then one can show that the global persistence exponent is to leading order in ϵ [496, 554]

$$\theta_g = \mu \left(1 - \epsilon \frac{2\mu}{\pi} \int_0^\infty dT f(T) [1 - e^{-2\mu T}]^{-3/2} + o(\epsilon) \right). \quad (1.132)$$

Applying this to $O(n)$ models, let $\Delta\theta_g := \theta_g - (\lambda_C - \frac{1}{2}\eta - d + 1)$ be the difference of the persistence exponents with respect to the Markovian prediction (1.130). Similarly, let $\Delta\theta_g := \theta_g - 1 - d/(2z)$ for directed percolation, see also Chap. 3. In $d = 4 - \epsilon$ dimensions, one finds

$$\Delta\theta_g = \begin{cases} \frac{3}{8} \frac{n+2}{(n+8)^2} \mathbf{a} \epsilon^2 + O(\epsilon^3) & ; \text{ } O(n) \text{ - model A} & [554] \\ \frac{7-4\sqrt{2}}{24} \epsilon + O(\epsilon^2) & ; \text{ } O(n) \text{ - model C} & [554] \\ \frac{\sqrt{12}-\sqrt{6}}{\sqrt{212}} \epsilon^{1/2} + O(\epsilon) & ; \text{ diluted Ising - model A} & [582] \\ \frac{1}{3\pi} \mathbf{b} \epsilon + O(\epsilon^2) & ; \text{ directed percolation} & [555] \end{cases} \quad (1.133)$$

with the numerical constants $\mathbf{a} = 0.271577604975\dots$ and $\mathbf{b} = 0.630237\dots$. Numerical estimates derived from these agree well with the results quoted in Table 1.9 for non-conserved (model A) dynamics.⁴⁸

⁴⁷ For example, consider the probabilistic three-states cellular automaton with a spin variable $\sigma_i = \pm 1, 0$ attached to each site and the local rates $\pm 1 \xrightarrow{1-r} \mp 1$, $\pm 1 \xrightarrow{r} 0$, $0 \xrightarrow{p} 0$ and finally $0 \xrightarrow{p} \pm 1$ if $s = \pm 1$ and $0 \xrightarrow{3p/2} \pm 1$ if $s = 0$, where $s = \text{sign}(\sum_i \sigma_i)$ is the sign of the local spin variables surrounding the spin to be flipped. This model has been proposed to describe the response of T-helper cells under parasitic infections (i.e. for HIV infections). Comparison of the values of $z = 2.097(8)$ and of $\theta_g = 0.247(4)$ [178] with those of Table 1.9 suggests that the model should be in the 2D Ising universality class. This is expected, because of the global symmetry of the rates under the map $\sigma_i \mapsto -\sigma_i$ [268].

Similarly, the value $\theta_g = 0.705(6)$ found in a model describing a helix-coil transition [25] might become of diagnostic value in the future.

⁴⁸ The global persistence exponent in a 2D antiferromagnetic Ising model with a conserved magnetisation, which should be a representative of model C dynamics, was found to depend on the value m_0 of the magnetisation [635].

model	d	condition	z	λ_C	θ_g		Ref.
					Markovian	numeric	
Ising	1	$T = 0$	2	1	1/4	1/4	[493]
Ising	2	$T = 0$	2	1.24(2)	0.12(1)	$\simeq 0.09$	[164]
		$T = 1.0$	2	1.24(2)	0.12(1)	0.062(2)	[350]
		$T = 1.5$	2	1.24(2)	0.12(1)	0.065(2)	[350]
TDGL	2	$T = 0$	2	1.24	0.12	$\simeq 0.06$	[165]
spherical	> 2	$T < T_c$	2	$d/2$	0	0	[350]
long-range spherical	$> \sigma$	$T < T_c$	σ	$d/2$	0	0	[350]

Table 1.10. Test of the Markovian scaling relation (1.134) for the global persistence exponent θ_g in phase-ordering kinetics for several models. The value of λ_C in the time-dependent Ginzburg-Landau equation (TDGL) is taken from the assumed universality with the Glauber-Ising model. The long-ranged spherical model is considered for $0 < \sigma < 2$. The quoted sources refer to the direct calculation of θ_g , labelled ‘numeric’.

The scaling behaviour of the global persistence $P_g(t)$ has also been studied for systems undergoing *phase-ordering kinetics* after a quench to $T < T_c$. Indeed, the reasoning described above for the critical case [493] can be taken over with minor modifications. In particular, for any finite time t , the global order-parameter $\hat{\phi}_0(t)$ is described by a Gaussian process. Eq. (1.130) is replaced by the following statement [165]: *if the stochastic process underlying the evolution of the global order-parameter in a d -dimensional system undergoing phase-ordering kinetics, with a disordered initial state, is Markovian, one has*

$$\theta_g = \frac{1}{2} \left(\lambda_C - \frac{d}{2} \right), \quad (1.134)$$

where λ_C is the autocorrelation exponent. In Table 1.10, we list some results testing this prediction. The values of θ_g are clearly consistent with $T < T_c$ being an irrelevant parameter. While the exactly solvable 1D Glauber-Ising and spherical models are explicitly Markovian and (1.134) holds true as expected, it clearly fails in the 2D Glauber-Ising model or the time-dependent Ginzburg-Landau equation, whose phase-ordering dynamics hence cannot be described in terms of a Markovian process. In Fig. 1.27 [350] the normalised global autocorrelator $\hat{N}(t, s)$ is shown. While the available data converge well towards the expected asymptotic limit for $t/s \rightarrow \infty$, the form of $\hat{f}_N(t/s)$ is *not* a simple power law, and one of the conditions for the validity of Doob’s lemma, namely the Markov property of the stochastic process which describes the global order-parameter, cannot be correct. We point out that this is much more clear evidence *against* the Markov property than the numerically rather small difference between the measured value of θ_g and the ‘Markovian’ prediction eq. (1.134).

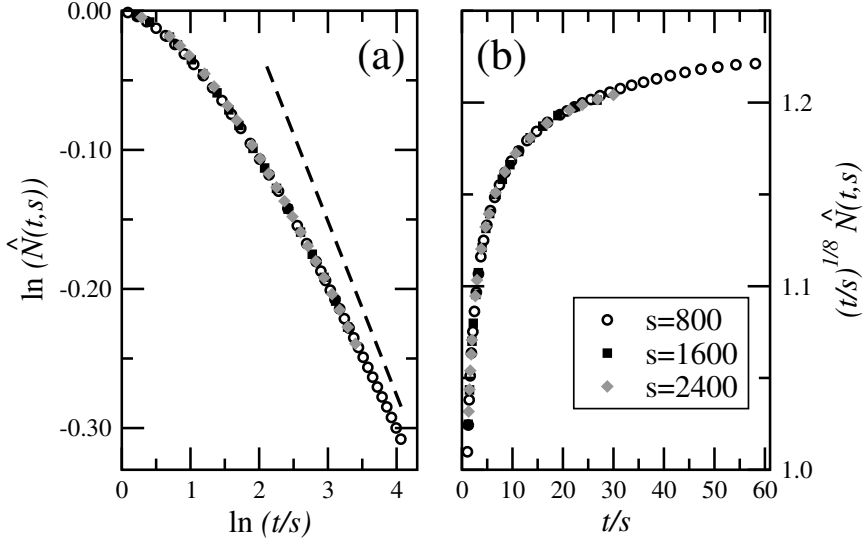


Fig. 1.27. Normalised correlator $\hat{N}(t,s)$ for the 2D Glauber-Ising model with $T_c \approx 2.27$, quenched to $T = 1.5$. (a) Data collapse for different waiting times. The effective exponent for t/s large is ≈ 0.115 , and the slope of the dashed line indicates the expected value 0.125. (b) Plotting $(t/s)^{0.125} \hat{N}(t,s)$ over against t/s shows the convergence towards the asymptotic power-law regime. Modified after [350].

The presently available results clearly indicate that the global order-parameter should be generically described by a *non-Markovian process*, for both non-equilibrium critical dynamics and phase-ordering kinetics. The examples already available clearly illustrate that the value of the dynamical exponent z is not related to the absence of the Markov property. Rather, the only known examples with a Markovian global order-parameter are systems which are exactly solvable and indeed whose solution can be expressed in terms of an underlying free field-theory (either fermionic as in the 1D Glauber-Ising model or bosonic as in the spherical models). Curiously, for quenches to $T = T_c$ all available data for the global persistence exponent θ_g are larger than their ‘Markovian’ estimate (1.130), while the opposite might be true for quenches to $0 < T < T_c$.⁴⁹

Computationally, it can be of interest to analyse the persistence behaviour of the order parameter integrated over blocks of linear size ℓ [164, 165]. This interpolates between the behaviour of the global persistence $P_g(t)$ if $\ell \gg L(t)$ and the local persistence $P_\ell(t) \sim t^{-\theta_\ell}$ in the opposite case $\ell \ll L(t)$, see exercise 1.19.

⁴⁹ In the 1D TDGL quenched to $T = 0$, the numerical estimate $\theta_g \simeq 0.165$ [165] is larger than the Markovian expectation, viz. $z\theta_g \simeq 0.10$, but one should recall the unusual scaling properties of this system, see [165] and references therein.

In this section, the discussion has been limited to systems with a fully disordered initial state such that the averaged global order-parameter $\langle \hat{\phi}_0(0) \rangle$ vanishes. In Chap. 3, we shall consider the case of a non-vanishing initial order-parameter. Again, we shall see that in general non-Markovian effects have to be taken into account for a correct description of the long-time behaviour, see Table 3.1.

Problems

1.1. Consider the Brownian motion of a particle of mass m , submitted to an external force $F(t)$, in one dimension. The velocity satisfies the Langevin equation of motion

$$\frac{dv(t)}{dt} = -\gamma v(t) + \eta(t) + \frac{1}{m}F(t) \quad (1.135)$$

where $\eta(t)$ is a centred Gaussian noise with variance $\langle \eta(t)\eta(t') \rangle = 2B\delta(t-t')$. For the case $F = 0$ without an external force, use the condition that at equilibrium $\langle v^2 \rangle_{\text{eq}} = T/m$ to derive the **Einstein relation** $B = \gamma T/m$. Calculate at equilibrium the two-time correlation function $C = C(t-t') = \langle v(t)v(t') \rangle$ and the linear response function $R = R(t-t') = \delta \langle v(t) \rangle / \delta F(t')|_{F=0}$. Show that they satisfy the fluctuation-dissipation theorem $\Im \hat{R}(\omega) = (\omega/(2\gamma T)) \hat{C}(\omega)$ and verify the equivalence with (1.8). Explain the relationship with the Einstein relation.

1.2. The relaxation of magnetic systems towards their steady-state is often described in terms of a stochastic Langevin equation for the coarse-grained time-dependent order-parameter (magnetisation) $m(t)$

$$\frac{dm(t)}{dt} = -\frac{\delta F[m]}{\delta m(t)} + \eta(t)$$

where $\eta(t)$ is a Gaussian centred noise with variance $\langle \eta(t)\eta(t') \rangle = 2T\delta(t-t')$ and $F[m]$ is the equilibrium free energy for which we take here the mean-field form $F[m] = -\frac{3}{2}\lambda^2 m^2 + \frac{1}{4}m^4$. Averaging over the noise, one has the mean-field equation $\dot{m} = 3\lambda^2 m - m^3$. Find the stationary solutions and discuss the relaxation from an initial magnetisation $m(0) = m_0 \neq 0$ towards them.

1.3. Consider a Langevin equation of the form

$$\dot{m}(t) = -F(m(t)) + h(t) + \eta(t)$$

where F is some functional, $\eta(t)$ is a centred Gaussian noise with variance $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$ and $h = h(t)$ is a time-dependent external field. Derive eq. (1.13), that is show that in the $h \rightarrow 0$ limit, one has

$$\langle m(t)\eta(s) \rangle = 2TR(t, s)$$

where $R(t, s) = \delta\langle m(t) \rangle / \delta h(s)|_{h=0}$ is the two-time response function.

1.4. Derive eq. (1.14).

1.5. Consider the Langevin equation

$$\partial_t m = \Delta_L m + 3\lambda^2 m - m^3 + \eta \quad (1.136)$$

of a space-time-dependent magnetisation $m = m(t, \mathbf{r})$ where the centred Gaussian noise η has the variance $\langle \eta(t, \mathbf{r})\eta(t', \mathbf{r}') \rangle = 2T\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$. Assuming spatial translation- and rotation-invariance, derive mean-field equations for the response function $R(t, s; \mathbf{r} - \mathbf{r}') = \langle m(t, \mathbf{r})\eta(s, \mathbf{r}') \rangle / (2T)$ and the correlator $C(t, s; \mathbf{r} - \mathbf{r}') = \langle m(t, \mathbf{r})m(s, \mathbf{r}') \rangle$ and analyse their long-time behaviour.

1.6. Use the potential $\mathcal{V}(\phi) = (1-\phi^2)^2$ to calculate the profile (1.37) explicitly and find the corresponding surface tension.

Compare with the results which follow from the following forms, which come from different levels of mean-field approximations, namely the Bragg-Williams (BW) and the Glauber (GL) forms [183]

$$\begin{aligned} \mathcal{V}_{\text{BW}}(\phi) &= \frac{T}{2} \left[-\frac{T_c}{T} \phi^2 + (1+\phi) \ln(1+\phi) + (1-\phi) \ln(1-\phi) \right] \\ \mathcal{V}_{\text{GL}}(\phi) &= \frac{T}{2} \left[\phi^2 - \frac{2T}{T_c} \ln \left(\cosh \frac{T_c}{T} \phi \right) \right] \end{aligned}$$

1.7. For an equilibrium system with dynamic variables $x_i(t)$, correlations and responses are defined as

$$C_{x_i x_j}(t) := \langle x_i(t)x_j(0) \rangle, \quad R_{x_i x_j}(t) := \frac{\delta \langle x_i(t) \rangle}{\delta x_j(0)}.$$

As reviewed in [499], in very general contexts the following generalised fluctuation-dissipation theorem (FDT) holds

$$R_{x_i x_j}(t) = C_{x_i s_j}(t) = - \left\langle x_i(t) \frac{\partial \rho(\mathbf{x})}{\partial x_j} \right\rangle, \quad s_j = - \frac{\partial \rho(\mathbf{x})}{\partial x_j} \quad (1.137)$$

where $\rho(\mathbf{x})$ is the invariant measure of phase-space.

Show that in the following two cases: (i) a Hamiltonian system in the canonical ensemble at temperature T and (ii) a Langevin equation with a conservative force and a small external perturbation h canonically conjugate to x , the generalised FDT reduces to the usual Kubo form $TR_{xh}(t) = -\partial_t C_{xx}(t)$ of the FDT, as used in this book [713].

1.8. Extend the argument of Andreanov and Lefèvre [17], outlined in the text, for the derivation of the admissible scaling form (1.82) of the ageing part of the two-time correlation function to critical quenches $T = T_c$.

1.9. Consider a quench from a critical system with initial correlator $C_{\text{init}}(\mathbf{r}) \sim |\mathbf{r}|^{-(d-2+\eta_1)}$ to another critical system with equilibrium correlator $C_{\text{eq}}(\mathbf{r}) \sim |\mathbf{r}|^{-(d-2+\eta_2)}$. If the order-parameter is non-conserved, show that $\lambda_C \geq d-2 + \frac{1}{2}(\eta_1 + \eta_2)$. Compare with the spin-wave approximation in the XY model.

1.10. Calculate the time-dependent correlation function in $d = 1, 2$ and 3 dimensions for a system where the structure factor has a Lorentzian-squared form

$$S_t(\mathbf{k}) = S_0 \left[1 + \left(\frac{\mathbf{k} - \mathbf{k}_0}{\Gamma(t)} \right)^2 \right]^{-2}$$

Why is a simple Lorentzian $S_t(\mathbf{k}) \sim [1 + (\mathbf{k} - \mathbf{k}_0)^2 / \Gamma(t)^2]^{-1}$ not acceptable ?

1.11. Given the scaling behaviour (1.75) of the two-time response function, derive the asymptotic behaviour of the scaling function $f_{\text{Int}}(y)$ of the intermediate susceptibility (1.103), as $y = t/s \rightarrow \infty$.

1.12. Consider a lattice system undergoing simple ageing, such that the two-time order-parameter correlator and the conjugate response, as defined on a lattice of $\mathcal{N}$ sites, satisfy the scaling forms

$$C(t, s; \mathbf{r}) := \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \langle \phi(t, \mathbf{r} + \mathbf{r}_i) \phi(s, \mathbf{r}_i) \rangle = L(s)^{-bz} F_C \left(\frac{L(t)}{L(s)}, \frac{\mathbf{r}}{L(t-s)} \right)$$

$$R(t, s; \mathbf{r}) := \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \left. \frac{\delta \langle \phi(t, \mathbf{r} + \mathbf{r}_i) \rangle}{\delta h(s, \mathbf{r}_i)} \right|_{h=0} = L(s)^{-(1+a)z} F_R \left(\frac{L(t)}{L(s)}, \frac{\mathbf{r}}{L(t-s)} \right)$$

where $\mathbf{r}_i$ runs over the sites of the lattice, spatial translation-invariance is assumed and $L(t) \sim t^{1/z}$ is the characteristic length scale. Compare the scaling of the *autocorrelator* $C(t, s; \mathbf{0})$ and the *autoresponse* $R(t, s; \mathbf{0})$ with the scaling form of the **global correlation function** $\hat{C}_0(t, s)$ and the **global response function** $\hat{R}_0(t, s)$, which are defined by

$$\hat{C}_0(t, s) := \frac{1}{\mathcal{N}^2} \sum_{i,j=1}^{\mathcal{N}} \langle \phi(t, \mathbf{r}_i) \phi(s, \mathbf{r}_j) \rangle = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \langle \phi(t, \mathbf{r}_i) \phi(s, \mathbf{0}) \rangle$$

$$\hat{R}_0(t, s) := \frac{1}{\mathcal{N}^2} \sum_{i,j=1}^{\mathcal{N}} \left. \frac{\delta \langle \phi(t, \mathbf{r}_i) \rangle}{\delta h(s, \mathbf{r}_j)} \right|_{h=0} = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \left. \frac{\delta \langle \phi(t, \mathbf{r}_i) \rangle}{\delta h(s, \mathbf{0})} \right|_{h=0} \quad (1.138)$$

where in the second relations spatial translation-invariance is assumed.

1.13. For a system in contact with a heat-bath at temperature T , prove the **Katzav-Schwartz inequality**

$$2T \left| \widehat{R}_{\mathbf{k}}(t, t) \right|^2 \leq \widehat{C}_{\mathbf{k}}(t, t). \quad (1.139)$$

1.14. Consider a master equation, written in the usual form $\partial_t P(\{\sigma\}; t) = \sum_{\sigma'} W(\sigma, \sigma') P(\{\sigma'\}; t)$, where $\{\sigma\}$ is a configuration of spin variables $\sigma_i = \pm 1$ such that the transition rates W have an up-down symmetry

$$W(\{-\sigma\}, \{-\sigma'\}) = W(\{\sigma\}, \{\sigma'\}) .$$

Define furthermore the average of the magnetisation by

$$M(t) = \frac{1}{\mathcal{N}} \left\langle \sum_{i \in A} \sigma_i(t) \right\rangle = \frac{1}{\mathcal{N}} \sum_{\sigma} \sum_{i \in A} \sigma_i P(\{\sigma\}; t)$$

where $\mathcal{N}$ is the total number of sites of the lattice A . Consider the following initial state with initial magnetisation m_0

$$P(\{\sigma\}; 0) = \prod_{i \in A} \frac{1}{2} (1 + m_0 \sigma_i) .$$

Then show that for early times [697]

$$\lim_{m_0 \rightarrow 0} \frac{1}{\mathcal{N}} \frac{M(t)}{m_0} = \frac{1}{\mathcal{N}^2} \left\langle \sum_{i, j \in A} \sigma_i(t) \sigma_j(0) \right\rangle \sim t^{\Theta} \quad (1.140)$$

where Θ is the critical initial slip exponent. For extensions to models without an up-down symmetry, such as antiferromagnetic system, the Baxter-Wu model or the contact process, see [696].

1.15. In an attempt to study ageing in an electric Anderson insulator (such as polycrystalline $\text{In}_2\text{O}_{3-x}$ films), the following experimental protocol is applied: after sample preparation, the sample is cooled to the measurement temperature T (typically a few K) with a fixed voltage V_{g0} held at the gate and is allowed to evolve for at least a day. For calibration purposes, one measures the conductance G_0 of the sample towards the end of this time. Then, with T being kept fixed, the voltage is switched to a value $V_{gn} > V_{g0}$ and is kept there for a ‘waiting time’ t_w . Finally, the voltage is switched back to V_{g0} and the conductance $G(t)$ is measured, where t denotes the time since the return of the voltage to its initial value V_{g0} . In particular, it is found that $\Delta G(t, t_w) = G(t) - G_0 = g(t/t_w)$ satisfies a dynamical scaling form.

Can one interpret this result as evidence of ageing, as defined in the text ? Assuming that the difference $V_{gn} - V_{g0}$ is small enough that the signal $\Delta G(t, t_w)$ is within the regime of linear response, and that simple power-law scaling can be used, try to write a phenomenological scheme to calculate the function $g(y)$.

1.16. Use the scaling forms of correlations and responses to derive the non-equilibrium scaling of the fluctuation-dissipation ratio (FDR) eq. (1.83). What conditions are needed that for large values of the scaling variable $y = t/s$ the FDR goes to a finite value X_∞ , usually called the **limit fluctuation-dissipation ratio**? Show that for non-equilibrium critical dynamics with $T_c > 0$, and if the autocorrelator and the autoresponse have the same sign, one must have $X_\infty > 0$. The limit fluctuation-dissipation ratio X_∞ is thought to be an universal quantity whose value can be used to characterise a non-equilibrium universality class [285, 286].

1.17. The limit fluctuation-dissipation ratio is defined as the double limit

$$X_\infty = \lim_{s \rightarrow \infty} \left(\lim_{t \rightarrow \infty} X(t, s) \right) = \lim_{y \rightarrow \infty} \left(\lim_{s \rightarrow \infty} X(y s, s) \right). \quad (1.141)$$

Intuitively, the value of X_∞ states if the response to an external perturbation of the order-parameter is equal to what one expects at equilibrium at temperature T , given that the order-parameter correlator is known. Why does one not consider the opposite order of these limits? Analyse the quantity

$$\overline{X}_\infty := \lim_{t \rightarrow \infty} \left(\lim_{s \rightarrow \infty} X(t, s) \right) = \lim_{\tau \rightarrow \infty} \left(\lim_{s \rightarrow \infty} X(\tau + s, s) \right). \quad (1.142)$$

1.18. A stochastic process $X(t)$ is characterised by the joint probabilities $P_n(t_1, \dots, t_n; x_1, \dots, x_n)$ that the random variable $X(t)$ takes the value x_1 at time t_1 , and the value x_2 at time t_2 and so on. For a **Gaussian process**, one has

$$\begin{aligned} P_n(t_1, \dots, t_n; x_1, \dots, x_n) &= \left(\frac{\det A}{(2\pi)^d} \right)^{1/2} \\ &\times \exp \left[-\frac{1}{2} \sum_{i,j=1}^n (x_i - \langle X(t_i) \rangle) A_{ij} (x_j - \langle X(t_j) \rangle) \right] \\ \text{where } (A^{-1})_{ij} &= \langle (x_i - \langle X(t_i) \rangle) (x_j - \langle X(t_j) \rangle) \rangle =: \langle\langle X(t_i) X(t_j) \rangle\rangle. \end{aligned} \quad (1.143)$$

For a **stationary process**, the joint probabilities are time-translation-invariant

$$P_n(t_1 + \tau, \dots, t_n + \tau; x_1, \dots, x_n) = P_n(t_1, \dots, t_n; x_1, \dots, x_n) \quad (1.144)$$

Finally, a **Markov process** is completely characterised by $P_1(t; x)$ and the **conditional probability** $P(t_1, x_1 | t_2, x_2) = P_2(t_1, t_2; x_1, x_2) / P_1(t_2, x_2)$ which must satisfy, for $t_3 > t_2 > t_1$

$$P(t_3, x_3 | t_1, x_1) = \int dx_2 P(t_3, x_3 | t_2, x_2) P(t_2, x_2 | t_1, x_1) \quad (1.145)$$

$$P_1(t_2; x_2) = \int dx_1 P(t_2, x_2 | t_1, x_1) P_1(t_1; x_1). \quad (1.146)$$

The first of those is the **Chapman-Kolmogorov equation**, see [710].

Prove Doob's lemma: *For any stationary, Gaussian and Markovian stochastic process, the autocorrelator is given by an exponential*

$$\langle\langle X(t_1)X(t_2) \rangle\rangle = X_0 e^{-\mu|t_1-t_2|}. \quad (1.147)$$

and conversely, a stationary and Gaussian process with the covariance (1.147) is Markovian.

1.19. Consider a hypercubic volume Ω of linear size ℓ such that $|\Omega| = \ell^d$. Derive via a scaling argument the behaviour of the **block persistence** $P_b(t, \ell)$, that is the probability that the block order parameter $|\Omega|^{-1} \int_{\Omega} d\mathbf{r} m(t, \mathbf{r})$ has not changed its sign up to time t [164, 165].

Chapter 2

Exactly Solvable Models

In this chapter, we look at the properties of several exactly solved models with simple ageing behaviour.

2.1 One-dimensional Glauber-Ising Model

The study of relaxations of the 1D Ising model has since Glauber's classical article [281] served as a paradigmatic example. The classical Ising Hamiltonian is $\mathcal{H} = -J \sum_n \sigma_n \sigma_{n+1}$, where $\sigma_n = \pm 1$ are the Ising spin variables and the sum runs over the $\mathcal{N}$ sites of a one-dimensional chain (since we shall take the limit $\mathcal{N} \rightarrow \infty$ at the end, it will not be necessary to specify boundary conditions in detail). Since classical spin variables do not have a natural dynamics of their own, the dynamics is defined by coupling the system to an external reservoir. Here, we shall concentrate on the celebrated **Glauber dynamics** [281], which we shall present in two slightly different variants.¹

A. Using the **heat-bath** formulation (see appendix G in volume 1), one selects, at each time step $\delta t = 1/\mathcal{N}$, randomly a spin σ_n and updates its value to $\sigma_n(t + \delta t) = \pm 1$ with the probabilities

$$P(\sigma_n(t + \delta t) = \pm 1) = \frac{1}{2} \left(1 \pm \tanh \left(\frac{1}{T} h_n(t) \right) \right) \quad (2.1)$$

$$h_n(t) := J(\sigma_{n-1}(t) + \sigma_{n+1}(t))$$

where $h_n(t)$ is the local field, generated by its nearest neighbours, on the site n at time t . The remarkable property of the heat-bath rule (2.1) is that *closed*

¹ At first sight, this might appear as a purely theoretical exercise. We point out, however, that the slow relaxation dynamics in real systems such as the single-chain magnet $\text{Mn}_2(\text{saltmen})_2\text{Ni}(\text{pao})_2(\text{py})_2(\text{ClO}_4)_2$ has been explicitly compared to Glauber dynamics in 1D [162] *en route* to meta-stable magnetism and these authors state that “*The design of new slow-relaxing magnetic nanosystems is a challenging goal for applications (as information storage).*” [162].

systems of equations of motion for the time-dependent magnetisation $M_n(t) = \langle \sigma_n(t) \rangle$ and other correlators can be derived, see exercise 2.1. In particular, the two-time correlation function $C_{n-m}(t, s) := \langle \sigma_n(t) \sigma_m(s) \rangle$ satisfies on an infinite, unbounded chain the following set of equations of motion [281, 285, 478]

$$\frac{\partial C_n(t, s)}{\partial t} = -C_n(t, s) + \frac{\gamma}{2} (C_{n-1}(t, s) + C_{n+1}(t, s)) \quad (2.2)$$

where $\gamma = \tanh(2J/T)$. Here, the initial value $C_n(t, t) = C_n(t) = \langle \sigma_n(t) \sigma_0(t) \rangle$ is specified by the equal-time correlation function. These also satisfy a closed system of equations of motion which is derived in exercise 2.1 as well.

B. More traditionally, one uses a **master equation**

$$\partial_t P(\{\sigma\}; t) = \sum_j [w_j(-\sigma_j) P(\{\sigma^{(j)}\}; t) - w_j(\sigma_j) P(\{\sigma\}; t)], \quad (2.3)$$

where $w_j(\sigma_j)$ is the rate of flipping the single spin $\sigma_j \mapsto -\sigma_j$ at site j such that $\{\sigma\} = (\sigma_1, \dots, \sigma_j, \dots, \sigma_N) \mapsto (\sigma_1, \dots, -\sigma_j, \dots, \sigma_N) =: \{\sigma^{(j)}\}$ and $P(\{\sigma\}; t)$ is the probability of the configuration $\{\sigma\}$ at time t . Glauber's choice was [281]

$$w_j(\sigma_j) = \frac{\alpha}{2} \left(1 - \frac{\gamma}{2} \sigma_j (\sigma_{j-1} + \sigma_{j+1}) \right), \quad (2.4)$$

where the normalisation constant α will be set to unity in what follows. The condition of *detailed balance* then implies that (see exercise 2.2)

$$\gamma = \tanh(2J/T). \quad (2.5)$$

Now, any N -point correlator $\langle \sigma_{n_1} \cdots \sigma_{n_N} \rangle(t) := \sum_{\{\sigma\}} \sigma_{n_1} \cdots \sigma_{n_N} P(\{\sigma\}; t)$, where the n_i are pairwise distinct ($n_i \neq n_j$ if $i \neq j$), may be found from

$$\partial_t \langle \sigma_{n_1} \cdots \sigma_{n_N} \rangle(t) = -2 \left\langle \sigma_{n_1} \cdots \sigma_{n_N} \sum_{j=1}^N w_{n_j}(\sigma_{n_j}) \right\rangle(t), \quad (2.6)$$

see exercise 2.3. Then the same equations of motion as found above are recovered, see exercise 2.4.

For the *response function*, recall that to leading order in an external site-dependent magnetic field $H_n(t)$ one has

$$M_n(t) = \langle \sigma_n(t) \rangle = \frac{1}{T} \int_0^t du \sum_m R_{n-m}(t, u) H_m(u) + \cdots \quad (2.7)$$

Generalising the derivation for the equation of motion (2.174) in exercise 2.1, one finds to linear order in the local field H_n

$$\begin{aligned} \frac{dM_n(t)}{dt} = & -M_n(t) + \frac{\gamma}{2} (M_{n-1}(t) + M_{n+1}(t)) \\ & + \frac{H_n(t)}{T} \left[1 - \frac{\gamma^2}{2} (1 - C_{n-1, n+1}(t)) \right]. \end{aligned} \quad (2.8)$$

Hence, the (linear) response function $R_{n-m}(t, s) = \delta M_n(t) / \delta H_m(s) |_{H=0}$ also satisfies the same equation of motion (2.2) as does $C_n(t, s)$, but with a different initial condition [285]

$$R_n(s, s) = \left(1 - \frac{\tanh^2(2J/T)}{2} [1 + C_2(s, s)] \right) \delta_{n,0}. \quad (2.9)$$

See exercise 2.8 for an alternative route towards equations of motion for the response function.

It is well-known that the critical point of the one-dimensional Ising model is at $T_c = 0$. Although the Glauber-Ising model can be solved for any temperature, in our analysis of the ageing behaviour, we shall restrict to $T = 0$ throughout. For any $T > 0$, there is only a single stable equilibrium state to which the model relaxes exponentially fast, as determined by the *finite* relaxation time $\tau_{\text{eq}} \approx O(e^{4J/T})$. Indeed, from the exact solution one may distinguish three regimes with different dynamical behaviour, see [285] for details

$$\begin{cases} \tau \approx \tau_{\text{eq}} \ll s & \text{equilibrium} \\ \tau \approx s \ll \tau_{\text{eq}} & \text{dynamic scaling and ageing} \\ \tau \ll \tau_{\text{eq}} \approx s & \text{early-time or temporal Porod regime} \end{cases}. \quad (2.10)$$

Common ways of solving the above equations of motion are either through generating functions as already used by Glauber [281] or via Fourier/Laplace transforms as carried out by Godrèche and Luck [285] and Lippello and Zannetti [478]. These methods work particularly well for uncorrelated initial states. Here we shall present a derivation which keeps the intuitively more appealing time variables and which has the additional advantage that initial long-range correlations of the form $C_{\text{init}}(r) \sim |r|^{-\mathbb{N}}$ can be treated [356].

2.1.1 Two-time Correlation Function

It is convenient to write the two-time correlation function in a Liouvillian (or quantum Hamiltonian) formalism already introduced in Chap. 3 in Volume 1, see also [648, 331] for recent reviews. One has, with $\tau = t - s$

$$C_n(s + \tau, s) = C_n(s; \tau) = \langle s | \sigma_n^z e^{-\mathcal{L}_0 \tau} \sigma_0^z e^{-\mathcal{L}_0 s} | P_0 \rangle \quad (2.11)$$

where $\sigma_n^{x,y,z}$ are the Pauli matrices acting on the n^{th} site of the chain and

$$\mathcal{L}_0 = \frac{1}{2} \sum_n (1 - \sigma_n^x) \left(1 - \frac{\gamma}{2} \sigma_n^z (\sigma_{n-1}^z + \sigma_{n+1}^z) \right) \quad (2.12)$$

is the Liouville operator (quantum Hamiltonian) introduced in Volume 1, Chap. 3 for Glauber dynamics [237]. Furthermore, $|P_0\rangle$ is the probability vector representing the initial distribution of spins and the constant summation

vector $\langle s|$ is the left steady state. The generator $\mathcal{L}_0$ is constructed such that it satisfies detailed balance with respect to the equilibrium distribution at temperature T of the 1D zero-field Ising model with interaction strength J . We introduce the shorthand

$$C_n := C_n(0, 0) ; C_n(t) := C_n(t; 0) = \langle \sigma_n(t) \sigma_0(t) \rangle \quad (2.13)$$

for the initial correlations and for the equal-time correlation function, respectively. For an uncorrelated initial state, one has $C_n = \delta_{n,0}$. Notice that for all $t \geq 0$

$$C_n(t) = C_{-n}(t). \quad (2.14)$$

For Glauber dynamics the time-evolution of the magnetisation at $T = 0$ (hence $\gamma = 1$) and without an external field satisfies a lattice diffusion equation for a 1D random walk with hopping rate $1/2$ [281]. The Green's function

$$G_n(y) = e^{-y} I_n(y) \quad (2.15)$$

of the lattice diffusion equation ($I_n(y)$ is a modified Bessel function [4], see exercise 2.1) describes the probability of moving a distance of n lattice units after time y . Hence

$$\langle s | \sigma_n^z e^{-\mathcal{L}_0 \tau} = \sum_{m=-\infty}^{\infty} e^{-\tau} I_{n-m}(\tau) \langle s | \sigma_m^z \quad (2.16)$$

which immediately yields the following reduction formula from the two-time to equal-time correlators

$$C_n(s; \tau) = \sum_{m=-\infty}^{\infty} e^{-\tau} I_{n-m}(\tau) C_m(s). \quad (2.17)$$

It remains to calculate the equal-time two-point correlation function $C_m(s)$, which for special initial conditions was first done by Glauber [281] (see exercise 2.6). In order to be able to do this for quite general initial conditions, we observe that the total correlation function may be split into a so-called *interaction part* $C_m^{\text{int}}(s)$ and a *correlation part* $C_m^{\text{corr}}(s)$. The latter one vanishes for uncorrelated (infinite-temperature) initial states. Then, using again (2.16)

$$\begin{aligned} C_m(s) &= e^{-2s} I_m(2s) + 2 \overbrace{\sum_{k=1}^{\infty} e^{-2s} I_{|m|+k}(2s)}^{=: C_m^{\text{int}}(s)} \\ &\quad + \underbrace{\sum_{k=1}^{\infty} e^{-2s} [I_{|m|-k}(2s) - I_{|m|+k}(2s)] C_k}_{=: C_m^{\text{corr}}(s)}. \end{aligned} \quad (2.18)$$

The two-time autocorrelation function then follows from (2.17) by setting $n = 0$. Combining (2.18) and (2.17), we can also define $C_n^{\text{int}}(s; \tau)$ and $C_n^{\text{corr}}(s; \tau)$ which will be calculated separately and $C_n(s; \tau) = C_n^{\text{int}}(s; \tau) + C_n^{\text{corr}}(s; \tau)$.

Using the completeness property $\sum_{n \in \mathbb{Z}} G_n = 1$ of the lattice Green's function, the interaction part can be split into a contribution which is large at early times and a second contribution which dominates the late-time behaviour. This gives

$$\begin{aligned} C_0^{\text{int}}(s; \tau) &= e^{-(\tau+2s)} I_0(\tau+2s) + e^{-\tau} I_0(\tau) [1 - e^{-2s} I_0(2s)] \\ &\quad + 4 \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} e^{-2s} I_{|m|+k}(2s) e^{-\tau} I_m(\tau). \end{aligned} \quad (2.19)$$

For $s, \tau \gg 1$ only the late-time part (containing the double sum) plays a role. Using the Greens's function asymptotic form

$$G_n(y) = \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{n^2}{2y}\right) (1 + O(y^{-1})) \quad (2.20)$$

the above sums can be converted into integrals. One has the scaling form

$$C_0^{\text{int}}(s; \tau) = \frac{4}{\pi} \int_0^\infty du \int_0^\infty dv e^{-u^2 + (u\alpha + v)^2} = \frac{2}{\pi} \arctan \sqrt{\frac{2s}{t-s}}. \quad (2.21)$$

This is the result derived in [285] and [478] for an uncorrelated initial state.

Here and in what follows we shall use the scaling variable

$$\alpha := \sqrt{\frac{\tau}{2s}}. \quad (2.22)$$

Any possible contribution from initial long-range correlation must be found from the correlation part. For its analysis, consider the integral representation

$$G_n(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \cos(qn) \exp(-\epsilon_q y) \quad (2.23)$$

of the Green's function with the **dispersion relation** $\epsilon_q = (1 - \cos q)$. This yields the exact expression

$$\begin{aligned} C_0^{\text{corr}}(s; \tau) &= \sum_{m=-\infty}^{\infty} e^{-\tau} I_m(\tau) C_m^{\text{corr}}(s) \\ &= \sum_{m=1}^{\infty} G_m(\tau) \sum_{n=1}^{\infty} C_n \frac{2}{\pi} \int_{-\pi}^{\pi} dq \sin(qm) \sin(qn) \exp(-\epsilon_q s) \\ &\simeq \frac{4\alpha}{\pi^{3/2} s^{1/2}} \sum_{n=1}^{\infty} C_n \int_0^\pi dq q \sin\left(\frac{qn}{\sqrt{s}}\right) {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \alpha^2 q^2\right) \exp(-q^2(1+\alpha)^2) \end{aligned} \quad (2.24)$$

where the last line gives the leading term in the scaling regime $s, \tau \gg 1$ where the sum over m may be replaced by an integral, (2.20) was used and ${}_1F_1(a; b; x)$ is a confluent hypergeometric series.

To analyse eqs. (2.21, 2.24) for waiting times $s \gg 1$ one must distinguish three cases, which depend on the form of the initial correlation. The initial correlator is assumed to be of the form

$$C_n(0, 0) = C_n \sim \frac{B}{n^{\aleph}} \quad \text{for } n \rightarrow \infty \quad (2.25)$$

where $\aleph \geq 0$ and B are parameters which in principle can be non-universal. We also define the unnormalised first moment

$$A := \sum_{n=1}^{\infty} n C_n \quad (2.26)$$

of the initial correlation function and then have the following three cases [356].

Case 1: $A < \infty$ converges

The series A converges to a *finite* value in situations such as antiferromagnetic alternating-sign correlations or for rapidly decaying ferromagnetic correlations with $\aleph > 2$. For large waiting times s the leading contribution to the integral in (2.24) comes from small values of the integration variable q (which describe the long wave-length fluctuations of the spin system). Expanding the sine and re-summing the resulting series of Gaussian integrals leads to the exact asymptotic expression, in leading order of s

$$C_0^{\text{corr}}(s; \tau) = \frac{A}{\pi} \frac{\alpha}{1 + \alpha^2} \cdot \frac{1}{s} \quad (2.27)$$

where α was defined in (2.22). Because of the extra factor $1/s$, this is asymptotically smaller than the interaction part (2.21). Hence the non-universal amplitude A does not contribute to the leading late-time asymptotics of the two-time autocorrelation function.

Case 2: Slowly decaying ferromagnetic correlations, $0 < \aleph < 2$

If $0 < \aleph < 2$ such that A diverges, one has

$$\begin{aligned} \frac{1}{\sqrt{s}} \sum_{n=1}^{\infty} C_n \sin\left(\frac{qn}{\sqrt{s}}\right) &\rightarrow \int_0^{\infty} dy C(y\sqrt{s}) \sin qy \\ &= B s^{-\aleph/2} \Gamma(1 - \aleph) \cos\left(\frac{\pi\aleph}{2}\right) |q|^{\aleph-1} \text{sign}(q). \end{aligned} \quad (2.28)$$

This yields $C_0^{\text{corr}}(s; \tau) \sim s^{-\aleph/2}$, with a known proportionality constant and shows that also in this case the correlation part of the two-time autocorrelation function is asymptotically small compared to the interaction part.

Case 3: Partial ferromagnetic long range order, $\aleph = 0$

The case $\aleph = 0$ corresponds to an asymptotically constant spin-spin correlation function in the initial state, mimicking (partial) ferromagnetic long range order. Such initial states may for example be obtained by quenching from a uniformly magnetised initial state to zero temperature and zero field. From the same steps as in case 2 one finds $C_0^{\text{corr}}(s; \tau) = (2B/\pi) \arctan \alpha$ and the total correlation function becomes [588]

$$C_0(s; \tau) = 1 - (1 - B) \frac{2}{\pi} \arctan \alpha. \quad (2.29)$$

For later examination of the fluctuation-dissipation ratio, we note that for t fixed one has

$$-s \frac{\partial}{\partial s} C_0(t, s) = \frac{1 - B}{\pi} \frac{y}{1 + y} \sqrt{\frac{2}{y - 1}} \quad (2.30)$$

which is of the same form as in the uncorrelated case, up to the non-universal amplitude $1 - B$.

2.1.2 Two-time Response Function

Now we consider the time evolution of the local magnetisation

$$M_n(t) = \langle \sigma_n(t) \rangle = \langle \sigma_n^z(t) \rangle = \langle s | \sigma_n^z e^{-\mathcal{L}t} | P_0 \rangle \quad (2.31)$$

for an initial distribution with initial magnetisation $m_0 \neq \pm 1$. In zero field, $M_n(t) = m_0$ for $T = 0$ Glauber dynamics. To study the linear response of the system to a small localised perturbation by a magnetic field, we let an external field act at site 0 of the lattice. In a Liouvillian formulation, this perturbation of the zero-field dynamics is represented by the perturbed Liouvillian

$$\mathcal{L} = \mathcal{L}_0 + V(h). \quad (2.32)$$

The perturbation $V(h)$ is determined by the requirement that the full generator $\mathcal{L}$ satisfies detailed balance with respect to the equilibrium distribution

$$P_{\text{eq}}[\{\sigma\}] \sim \exp \left[\frac{1}{T} \sum_n (J \sigma_n \sigma_{n+1} + h \sigma_0) \right] \quad (2.33)$$

of the ferromagnetic Ising system with interaction strength J and local magnetic field h at site 0. This requirement, on which the usual equilibrium fluctuation-dissipation theorem is based, does not uniquely fix V , as different dynamical rules may lead to the same equilibrium distribution (2.33). Here we do not follow the heat-bath method outlined above but rather define a minimally perturbed dynamics by [356]

$$V = \frac{1}{2} (1 - \sigma_0^x) \left[1 - \frac{\gamma}{2} \sigma_0^z (\sigma_{-1}^z + \sigma_1^z) \right] \left[e^{-(h/T)\sigma_0^z} - 1 \right] \quad (2.34)$$

which is close in spirit to Glauber's approach (see exercise 2.8) [281]. At zero temperature, one has $\gamma = 1$. A dimensionless field strength h/T will be used throughout.

Following the standard procedures of quantum theory, the Liouvillian $\mathcal{L}$ acts as time-evolution operator. For an external field h acting at time s we calculate the linear response function (in units of T)

$$R_n(t, s) = R_n(s; \tau) = \frac{\delta}{\delta h(s)} M_n(t) \Big|_{h=0} \quad (2.35)$$

at observation time t . As before $\tau = t - s \geq 0$ is the time elapsed after the perturbation. By expanding the full time evolution operator $\exp(-\mathcal{L}t)$ in powers of h one finds from (2.31, 2.35)

$$R_n(s; \tau) = -\langle s | \sigma_n^z e^{-\mathcal{L}\tau} V' e^{-\mathcal{L}s} | P_0 \rangle. \quad (2.36)$$

Here V' is the derivative of V with respect to h/T taken at $h = 0$. Using (2.16) it can be shown that the autoresponse function ($n = 0$) factorises

$$R_0(s; \tau) = e^{-\tau} I_0(\tau) [1 - C_1(s)] \quad (2.37)$$

in terms of the Green's function $G_0(\tau) = e^{-\tau} I_0(\tau)$ and a contribution involving the two-point correlation function at time s .²

Similarly, the *space-time* response can be expressed by the Green's function as $R_n(s; \tau) = w(s) G_n(\tau)$ where $w(s)$ is the same factor as found for the autoresponse.

To calculate the interaction part of the response function we deduce by analogy with (2.19)

$$1 - C_1^{\text{int}}(t) = e^{-2t} (I_0(2t) + I_1(2t)). \quad (2.38)$$

For large waiting times $s \gg 1$ and time differences $t - s \gg 1$ the interaction part of the space-time response becomes

$$R_n^{\text{int}}(s; \tau) = \frac{\sqrt{2}}{\pi} \sqrt{\frac{1}{s(t-s)}} \exp\left(-\frac{n^2}{2(t-s)}\right). \quad (2.39)$$

When expressed in terms of the scaling variable $y = t/s$, we recover the same asymptotic result as found for heat-bath dynamics [285], up to normalisation.

The calculation of the correlation part of the autoresponse function proceeds along the same lines as the calculation of the autocorrelation function.

² The heat-bath method mentioned at the beginning of this section leads to $R_0(s; \tau) = e^{-\tau} I_0(\tau) (1 - C_2(s))/2$ [285]. The scaling behaviour is the same as in the main text. Compare with exercise 2.8

The results are:

Case 1: $A < \infty$

$$C_1^{\text{corr}}(s) = \frac{A}{4\pi^{1/2}s^{3/2}} \quad (2.40)$$

for s large enough. Comparison with (2.38) shows that this term merely generates a finite-time correction.

Case 2: $0 < \aleph < 2$

$$C_1^{\text{corr}}(s) = \frac{B}{2^\aleph \sqrt{\pi}} \Gamma\left(1 - \frac{\aleph}{2}\right) s^{-(\aleph+1)/2}, \quad (2.41)$$

corresponds again to sub-leading contribution to the scaling behaviour. Hence initial correlations decaying to zero do not change the asymptotic behaviour of the autoresponse function.

Case 3: $\aleph = 0$

One has $C_1^{\text{corr}}(s) = B/\sqrt{\pi s}$, hence

$$R_0(s; \tau) = \frac{(1-B)}{\pi\sqrt{2\tau s}} = \frac{1-B}{2\pi s} \frac{1}{\alpha}. \quad (2.42)$$

We see from equations (2.30), (2.42) that the same non-universal amplitude enters the two-time correlation function and the response function respectively.

The results of this analysis of the scaling forms in the 1D Glauber-Ising model may be summarised as follows: *provided that $\aleph > 0$, one has in the ageing regime, where $s \gg 1$ and also $t - s \gg 1$*

$$C(t, s) = \frac{2}{\pi} \arctan \sqrt{\frac{2s}{t-s}}, \quad R(t, s) = \sqrt{\frac{2}{\pi^2} \frac{1}{s(t-s)}} \quad (2.43)$$

in agreement with simple ageing. In [Table 2.1](#) we collect some universal non-equilibrium quantities. In particular, we have seen that *for all values $\aleph \geq 0$ in the power-law initial correlations (2.25) the fluctuation-dissipation ratio $X(t, s)$ is in the scaling regime given by*

$$X(t, s) = R(t, s) \left(\frac{\partial C(t, s)}{\partial s} \right)^{-1} = \frac{y+1}{2y}. \quad (2.44)$$

The same expression for $X(t, s)$ was obtained in [285] for different microscopic dynamics and uncorrelated initial states. The limit fluctuation-dissipation ratio $X_\infty = 1/2$, as well as the exponents $a, b, \lambda_C, \lambda_R$ are independent of the

	a	a'	b	λ_C	λ_R	z	X_∞
power	0	$-\frac{1}{2}$	0	1	1	2	$\frac{1}{2}$
island	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	2	0

Table 2.1. Ageing exponents and limit fluctuation-dissipation ratio for the 1D Glauber-Ising model quenched to $T = 0$ for power-law initial correlations (2.25) and for the island initial state (2.45).

decay of the initial correlations as described by $\aleph \geq 0$. This constitutes a nice confirmation of the Godrèche-Luck conjecture of the universality of X_∞ .³

2.1.3 Low-temperature Initial States

In our discussion of the 1D Glauber-Ising model we have so far assumed a translation-invariant initial state with a *finite density* of domain walls. However, at very low temperatures it may become more relevant to study the time evolution of an almost ordered system with only *finitely many* domain walls at the initial time. For definiteness we consider two domain walls located at sites $-L$ and L respectively of the lattice. This corresponds to the initial configuration

$$P_0 = \dots \downarrow\downarrow\downarrow\uparrow\uparrow \dots \uparrow\uparrow\downarrow\downarrow \dots \quad (2.45)$$

where the inversions of the spins occur at the positions $-L$ and $+L$, respectively. Although these initial ('island') conditions break spatial translation-invariance, we have chosen the coordinate system such that reflection symmetry with respect to the origin is maintained. With these initial conditions, the autocorrelation and autoresponse functions can be shown to become in the scaling regime $s \gg L^2$ [356]

$$\begin{aligned} -\frac{\partial}{\partial s} C_0(t, s) &= \frac{2L+1}{\sqrt{2\pi^3 s^3}} \left(\frac{1}{\sqrt{2\alpha}} + \arctan(\sqrt{2\alpha}) \right), \\ R(t, s) &= \frac{2L+1}{\sqrt{2\pi^3}} \frac{1}{\sqrt{2\alpha}} \cdot s^{-3/2}. \end{aligned} \quad (2.46)$$

The corresponding exponents are listed in Table 2.1 (it is easy to check that the values of λ_C and λ_R are in agreement with the prediction (1.128)). They are different from those found for a power-law decay (2.25) of the initial correlations and, not surprisingly, the fluctuation-dissipation ratio

$$X(t, s) = \left(1 + \sqrt{y-1} \arctan(\sqrt{y-1}) \right)^{-1} \quad (2.47)$$

is a universal function of the scaling variable $y = t/s$ but takes on a form distinct from (2.44).

³ For complete initial antiferromagnetic order $C_n = (-1)^n$ the same form of the FDR is found.

process	rates		dual process
	Glauber	KDH	
$\uparrow\downarrow\uparrow \longrightarrow \uparrow\uparrow\uparrow$	$\frac{1}{2}(1 + \gamma)$	$\frac{1}{2}(1 + 3\delta)$	$AA \longrightarrow \emptyset\emptyset$
$\uparrow\uparrow\uparrow \longrightarrow \uparrow\downarrow\uparrow$	$\frac{1}{2}(1 - \gamma)$	$\frac{1}{2}(1 - \delta)$	$\emptyset\emptyset \longrightarrow AA$
$\uparrow\uparrow\downarrow \longrightarrow \uparrow\downarrow\downarrow$	$\frac{1}{2}$	$\frac{1}{2}(1 - \delta)$	$\emptyset A \longrightarrow A\emptyset$
$\uparrow\downarrow\downarrow \longrightarrow \uparrow\uparrow\downarrow$	$\frac{1}{2}$	$\frac{1}{2}(1 - \delta)$	$A\emptyset \longrightarrow \emptyset A$

Table 2.2. Comparison of flip rates in the 1D kinetic Ising model, for either Glauber dynamics with $\gamma = \tanh(2J/T)$, or else for the KDH dynamics with $\delta = \tanh(2J/T)/(2 - \tanh(2J/T))$. The dual reaction-diffusion processes are also shown.

2.1.4 Comparison With the 1D Ginzburg-Landau Equation

Earlier in Chap. 1, we had looked at the time-dependent Ginzburg-Landau equation (TDGL) at zero temperature

$$\frac{\partial \phi(t, \mathbf{r})}{\partial t} = D \Delta \phi(t, \mathbf{r}) - \frac{\delta \mathcal{V}[\phi]}{\delta \phi(t, \mathbf{r})} \quad (2.48)$$

with a Ginsburg-Landau potential scaled to $\mathcal{V}[\phi] = (1 - \phi^2)^2$ and with the dynamics driven by the fully disordered initial conditions.

Although it is usually admitted that a description through the TDGL should give the same results for the long-time scaling behaviour as the kinetic Ising model, this is not true in one spatial dimension, where $T_c = 0$. Indeed, the autocorrelation exponent, after a quench from a disordered initial state to $T = 0$, can be calculated exactly and the result $\lambda_C = 0.60616\dots$ [96] is distinct from either of the 1D Glauber-Ising results quoted in Table 2.1. Therefore, the 1D TDGL and the 1D Glauber-Ising model are in different non-equilibrium universality classes.

2.2 A Non-Glauberian Kinetic Ising Model

2.2.1 Definition

We now discuss a second case of a kinetic Ising model with spin-flip dynamics where exact results can be found. This case was identified by Kimball [441] and Deker and Haake [188]. We shall refer to it as **KDH dynamics**.

To define the model, consider the flip rates

$$w_n(\sigma_n) = \frac{1}{2} \left(1 - \frac{\gamma}{2} \sigma_n (\sigma_{n-1} + \sigma_{n+1}) + \delta \sigma_{n-1} \sigma_{n+1} \right). \quad (2.49)$$

These are the most general rates which are invariant under a general spin reversal $\sigma_n \mapsto -\sigma_n$, are left-right symmetric and only depend explicitly on the

spin σ_n to be flipped and its nearest left and right neighbours $\sigma_{n\pm 1}$. Detailed balance imposes the condition $\gamma = (1 + \delta) \tanh(2J/T)$ and KDH dynamics is defined by the additional condition [441, 188]

$$\delta = \frac{\gamma}{2} = \frac{\tanh 2J/T}{2 - \tanh 2J/T}. \quad (2.50)$$

If $J > 0$, then the low-temperature limit $T \rightarrow 0$ implies that $\delta \rightarrow 1$, and if $J < 0$, one has $\delta \rightarrow -\frac{1}{3}$ as $T \rightarrow 0$. The admissible reactions and their rates are shown for both Glauber and KDH dynamics in Table 2.2. It is sometimes more instructive to go over to a dual description [608] and to associate with a ‘kink’ ($\uparrow\downarrow$ or $\downarrow\uparrow$) on a pair of neighbouring sites a particle (A) on the corresponding site of the dual lattice and with a pair ($\uparrow\uparrow$ or $\downarrow\downarrow$) a vacancy ($\emptyset$) on the dual lattice. In this way, the correspondence of kinetic Ising models with reaction-diffusion processes is clearly seen. In particular, one sees that in the zero-temperature limit (with $J > 0$), the Glauber and KDH dynamics show a different pattern of stationary states. While there are two steady-states for Glauber dynamics, which correspond to the two equivalent thermodynamic states, one has for KDH dynamics of the order $O(g^L)$ stationary states for a lattice of L sites, where $g = (\sqrt{5} + 1)/2 \simeq 1.618 \dots$ is the golden number [133], see also Chap. 3 in Volume 1.

2.2.2 Calculation of the Dynamical Exponent

This difference leads to a different dynamical behaviour, as we now show. Consider the local spin variables σ_n and the three-spin $q_n := \sigma_{n-1}\sigma_n\sigma_{n+1}$. Their equations of motion are straightforwardly found, see exercise 2.10. Then define the *global* magnetisation and global three-spin

$$\widehat{M}(t) := \frac{1}{L} \sum_n \langle \sigma_n \rangle(t) \quad , \quad \widehat{T}(t) := \frac{1}{L} \sum_n \langle q_n \rangle(t), \quad (2.51)$$

which satisfy the closed system of equations of motion

$$\frac{d}{dt} \begin{pmatrix} \widehat{M}(t) \\ \widehat{T}(t) \end{pmatrix} = \begin{pmatrix} 2\delta - 1 & -\delta \\ 3\delta & -3 \end{pmatrix} \begin{pmatrix} \widehat{M}(t) \\ \widehat{T}(t) \end{pmatrix}. \quad (2.52)$$

The global magnetisation $\widehat{M}(t)$ and the global three-spin $\widehat{T}(t)$ become

$$\begin{aligned} \widehat{M}(t) &= \frac{1}{2\Delta} \left[\left(\alpha_+ \widehat{M}(0) - \delta \widehat{T}(0) \right) e^{-\lambda_- t} - \left(\alpha_- \widehat{M}(0) - \delta \widehat{T}(0) \right) e^{-\lambda_+ t} \right] \\ \widehat{T}(t) &= \frac{1}{2\Delta\delta} \left[\alpha_- \left(\alpha_+ \widehat{M}(0) - \delta \widehat{T}(0) \right) e^{-\lambda_- t} - \alpha_+ \left(\alpha_- \widehat{M}(0) - \delta \widehat{T}(0) \right) e^{-\lambda_+ t} \right] \end{aligned} \quad (2.53)$$

where

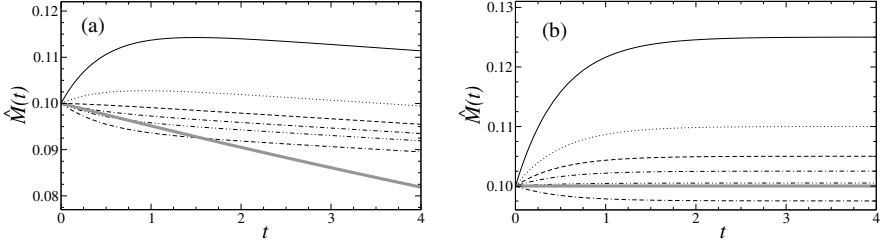


Fig. 2.1. Time-dependence of the global magnetisation $\widehat{M}(t)$ in the 1D Ising model with KDH dynamics for the initial values $\widehat{T}(0)/\widehat{M}(0) = [0.50, 0.80, 0.90, 0.95, 0.99, 1.05]$ from top to bottom and for (a) $\delta = 0.90476\dots$ and (b) $\delta = 1$. The thick grey lines give the time-dependent global magnetisation of the Glauber-Ising model with the same values of J/T . Reprinted from [219]. Copyright (2009) Institute of Physics Publishing.

$$\lambda_{\pm} = 2 - \delta \pm \Delta, \quad \alpha_{\pm} = 1 + \delta \pm \Delta, \quad \Delta = (1 + 2\delta - 2\delta^2)^{1/2}. \quad (2.54)$$

Since the equilibrium correlation-length ξ has at low temperatures the form

$$\xi^{-1} = -\ln \tanh(J/T) \approx 2e^{-2J/T} + \frac{2}{3}e^{-6J/T}, \quad (2.55)$$

the relation (2.50) implies for sufficiently large ξ

$$\delta \approx 1 - \xi^{-2} + \frac{11}{12}\xi^{-4}. \quad (2.56)$$

The dynamical exponent is found by comparing the largest relaxation time τ_- with the correlation length ξ , which gives $\tau_- = \lambda_-^{-1} \approx \frac{2}{3}\xi^4$ and one reads off [188]

$$z = 4, \quad (2.57)$$

which is different from the result $z = 2$ for Glauber dynamics. We note (see exercise 2.4) that the global magnetisation $\widehat{M}^{(\text{Glaue})}(t) = \widehat{M}(0)e^{-(1-\gamma)t}$ of Glauber dynamics always decays monotonously towards its stationary value $\widehat{M}(\infty) = 0$. For KDH dynamics, however, $\widehat{M}(t)$ only decays monotonously if the initial conditions are such that $\alpha_-\delta^{-1} \leq \widehat{T}(0)/\widehat{M}(0) \leq \alpha_+\delta^{-1}$. This is illustrated in Fig. 2.1.

2.2.3 Global Response Functions

In order to analyse the ageing behaviour of the model, we perturb the system with a small space-time-dependent magnetic field $h = h_n(s) = h(s)$ and define the global response functions

$$\widehat{R}(t, s) := \frac{T}{L} \sum_n \frac{\delta \langle \sigma_n \rangle(t)}{\delta h(s)} \Big|_{h=0}, \quad \widehat{Q}(t, s) := \frac{T}{L} \sum_n \frac{\delta \langle q_n \rangle(t)}{\delta h(s)} \Big|_{h=0}. \quad (2.58)$$

The fluctuations of the global spin and three-spin are described by

$$C_m^{gf}(t) := \frac{1}{L} \sum_n C_{n,n+m}^{gf}(t) \quad , \quad \widehat{C}^{gf}(t) := \frac{1}{L^2} \sum_{m,n} C_{m,n}^{gf}(t), \quad (2.59)$$

where g and f are either σ or q , and the equal-time correlation functions are

$$C_{m,n}^{\sigma\sigma}(t) := \langle \sigma_m \sigma_n \rangle(t) \quad , \quad C_{m,n}^{q\sigma}(t) := \langle q_m \sigma_n \rangle(t) \quad , \quad C_{m,n}^{qq}(t) := \langle q_m q_n \rangle(t). \quad (2.60)$$

The response functions may be found by adding a field-dependent term $\delta w_n(\sigma_n)$ to the rates (2.49), which reads to linear order in h

$$\delta w_n(\sigma_n) = \frac{1}{2} \tanh(h_n/T) \left(\sigma_n - \frac{\gamma}{2} (\sigma_{n-1} + \sigma_{n+1}) + \delta \sigma_{n-1} \sigma_n \sigma_{n+1} \right), \quad (2.61)$$

We then obtain the following system

$$\partial_t \widehat{R}(t, s) = (2\delta - 1) \widehat{R}(t, s) - \delta \widehat{Q}(t, s) \quad , \quad \partial_t \widehat{Q}(t, s) = 3\delta \widehat{R}(t, s) - 3\widehat{Q}(t, s), \quad (2.62)$$

together with the initial conditions

$$\widehat{R}(s, s) = 1 - 2\delta C_1^{\sigma\sigma}(s) + \delta C_2^{\sigma\sigma}(s) \quad (2.63)$$

$$\widehat{Q}(s, s) = \delta + 2(1 - \delta) C_1^{\sigma\sigma}(s) + (1 - 2\delta) C_2^{\sigma\sigma}(s) + 2\delta C_3^{\sigma\sigma}(s) - 2\delta C_2^{q\sigma}(s).$$

In general, this requires some non-global correlators, for which closed systems of equations of motion are unknown. However, one may analyse the response with respect to an explicitly given initial state. For example, consider an initial state with temperature $T_{\text{ini}} > 0$. The required correlators are promptly computed and we find, after a quench to $T = 0$ at the initial time $t = 0$ [219]

$$\widehat{R}(t, 0) = (1 - \tanh \eta)^3 + \tanh \eta (1 - \tanh \eta)^2 e^{-2t} \quad (2.64)$$

where we have set $\eta := J/T_{\text{ini}}$. The system never reaches equilibrium, since stationary value $\widehat{R}_{\text{stat}} = \lim_{t \rightarrow \infty} \widehat{R}(t, 0) = (1 - \tanh \eta)^3$ is distinct from the equilibrium value $\widehat{R}_{\text{eq}} = 0$, which is only reached in the limit $T_{\text{ini}} \rightarrow 0$.

For an interpretation in terms of dynamical scaling, recall that

$$\widehat{R}(t, 0) = \widehat{R}_0(t, 0) = \left. \int_{\mathbb{R}} dr e^{-iqr} R(t, 0; r, 0) \right|_{q=0} \sim t^{(1-\lambda_R)/z} \quad (2.65)$$

is the Fourier transform of the space-time response, at vanishing momentum $q = 0$. We compare this with the expected scaling behaviour, given by $R(t, 0; r) = t^{-\lambda_R/z} \Phi_R(r^z/t)$. Comparison with (2.64) for t sufficiently large then suggests that $\lambda_R = 1$. Provided that $\alpha_+ \widehat{R}(0, 0) - \delta \widehat{Q}(0, 0) \neq 0$, this result is generic. The independence of the initial state confirms the expected universality of the autoresponse exponent λ_R .

2.2.4 Global Correlation Functions

The global two-time correlation functions are defined by

$$\widehat{C}^{gf}(t, s) := \frac{1}{L^2} \sum_{m,n} C_{n,m}^{gf}(t, s) = \widehat{C}^{fg}(s, t), \quad (2.66)$$

where g, f stand for either σ or q . Their equations of motion for $t > s$ read

$$\frac{\partial}{\partial t} \begin{pmatrix} \widehat{C}^{\sigma f}(t, s) \\ \widehat{C}^{qf}(t, s) \end{pmatrix} = \begin{pmatrix} 2\delta - 1 & -\delta \\ 3\delta & -3 \end{pmatrix} \begin{pmatrix} \widehat{C}^{\sigma f}(t, s) \\ \widehat{C}^{qf}(t, s) \end{pmatrix} \quad (2.67)$$

such that for equal times $\lim_{t \rightarrow s+} \widehat{C}^{gf}(t, s) = \widehat{C}^{gf}(s)$ we return to the global single-time correlators. In the limit $L \rightarrow \infty$, their equations of motion close

$$\frac{d}{dt} \begin{pmatrix} \widehat{C}^{\sigma\sigma}(t) \\ \widehat{C}^{q\sigma}(t) \\ \widehat{C}^{qq}(t) \end{pmatrix} = \begin{pmatrix} 4\delta - 2 & -2\delta & 0 \\ 3\delta & 2\delta - 4 & -\delta \\ 0 & 6\delta & -6 \end{pmatrix} \begin{pmatrix} \widehat{C}^{\sigma\sigma}(t) \\ \widehat{C}^{q\sigma}(t) \\ \widehat{C}^{qq}(t) \end{pmatrix} \quad (2.68)$$

provided the initial conditions are chosen such that the initial values $\widehat{C}^{gf}(0)$ are all of order $O(1)$ [219]. We leave the proof of this statement as exercise 2.12.

The explicit expressions for the single-time correlators are

$$\begin{aligned} \widehat{C}^{\sigma\sigma}(t) &= B_- e^{-2\lambda_- t} + B_0 e^{-(\lambda_- + \lambda_+)t} + B_+ e^{-2\lambda_+ t}, \\ \widehat{C}^{q\sigma}(t) &= \frac{1}{\delta} \left(\alpha_- B_- e^{-2\lambda_- t} + (1 + \delta) B_0 e^{-(\lambda_- + \lambda_+)t} + \alpha_+ B_+ e^{-2\lambda_+ t} \right), \\ \widehat{C}^{qq}(t) &= 3 \left(\frac{\alpha_-}{\alpha_+} B_- e^{-2\lambda_- t} + B_0 e^{-(\lambda_- + \lambda_+)t} + \frac{\alpha_+}{\alpha_-} B_+ e^{-2\lambda_+ t} \right), \end{aligned} \quad (2.69)$$

where $\lambda_{\pm}$ and $\alpha_{\pm}$ are given in (2.54) and

$$B_{\mp} := \frac{1}{4\Delta^2} \left(\alpha_{\pm}^2 \widehat{C}^{\sigma\sigma}(0) - 2\delta\alpha_{\pm} \widehat{C}^{q\sigma}(0) + \delta^2 \widehat{C}^{qq}(0) \right), \quad (2.70)$$

$$B_0 := -\frac{1}{2\Delta^2} \left(\alpha_+ \alpha_- \widehat{C}^{\sigma\sigma}(0) - 2\delta(1 + \delta) \widehat{C}^{q\sigma}(0) - \delta^2 \widehat{C}^{qq}(0) \right). \quad (2.71)$$

The global correlations at large times in general relax as $\exp(-2t/\tau_-)$. Since the leading correction, coming from the non-global terms, to these correlations will be of order $O(t/L)$, the solution is valid for any time $t < \tau_-$ in the large- L limit, but *provided* that B_0 and $B_{\pm}$ are of order $O(1)$.

The general solution for the two-time correlation functions are for $t > s$

$$\begin{aligned} \widehat{C}^{\sigma f}(t, s) &= A^{\sigma f}(s) e^{-\lambda_- (t+s)} + E^{\sigma f}(s) e^{-(\lambda_- t + \lambda_+ s)} \\ &\quad + F^{\sigma f}(s) e^{-(\lambda_+ t + \lambda_- s)} + B^{\sigma f}(s) e^{-\lambda_+ (t+s)}, \end{aligned} \quad (2.72)$$

$$\begin{aligned} \widehat{C}^{qf}(t, s) &= \frac{1}{\delta} \alpha_- \left(A^{\sigma f}(s) e^{-\lambda_- (t+s)} + E^{\sigma f}(s) e^{-(\lambda_- t + \lambda_+ s)} \right) \\ &\quad + \frac{1}{\delta} \alpha_+ \left(F^{\sigma f}(s) e^{-(\lambda_+ t + \lambda_- s)} + B^{\sigma f}(s) e^{-\lambda_+ (t+s)} \right), \end{aligned} \quad (2.73)$$

where in principle $A^{\sigma f}(s)$, $B^{\sigma f}(s)$, $E^{\sigma f}(s)$, and $F^{\sigma f}(s)$ are arbitrary functions of s . Since we must require that for $t = s$ the two-time correlators reduce to the single-time correlators given above, we find

$$\begin{aligned} A^{\sigma\sigma} &= B_- , \quad B^{\sigma\sigma} = B_+ , \quad A^{\sigma q} = \frac{1}{\delta} \alpha_- B_- , \quad B^{\sigma q} = \frac{1}{\delta} \alpha_+ B_+ , \\ E^{\sigma\sigma} &= F^{\sigma\sigma} = \frac{1}{2} B_0 , \quad E^{\sigma q} = \frac{1}{2\delta} \alpha_+ B_0 , \quad F^{\sigma q} = \frac{1}{2\delta} \alpha_- B_0 . \end{aligned} \quad (2.74)$$

Then the solution (2.72,2.73) can be extended to $t < s$ as well.

The equations of motion (2.67) are exact for periodic boundary conditions, but the system (2.68) for the equal-time correlators only holds true in the infinite-size limit $L \rightarrow \infty$ and when at least one of the global correlators is much larger than of order $O(1/L)$. In particular, the global correlations with the initial state read

$$\hat{C}^{\sigma f}(t, 0) = \frac{1}{2\Delta} \left(A_-^{\sigma f} e^{-\lambda_- t} - A_+^{\sigma f} e^{-\lambda_+ t} \right) \quad (2.75)$$

$$\hat{C}^{qf}(t, 0) = \frac{1}{2\Delta\delta} \left(\alpha_- A_-^{\sigma f} e^{-\lambda_- t} - \alpha_+ A_+^{\sigma f} e^{-\lambda_+ t} \right) \quad (2.76)$$

where $A_{\mp}^{\sigma f} = \alpha_{\pm} \hat{C}^{\sigma f}(0) - \delta \hat{C}^{qf}(0)$. We discuss two examples for different initial states.

1. For a fully disordered state the equations of motion do not close and (2.68) are not valid. However, the global correlator with the initial state may be read off from (2.75) and we find $C^{\sigma\sigma}(t, 0) = 1/L$. Similar results may be found for any initial temperature $T_{\text{ini}} > 0$.
2. For the partially ordered initial states $\cdots \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \cdots$ and $\cdots \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \cdots$ we find at the critical point $\delta = 1$ that $B_- = 1$, $B_0 = -1/3$ and $B_+ = 4/9$. Hence

$$\hat{C}^{\sigma\sigma}(t, s) = 1 - \frac{1}{6} (e^{-2s} + e^{-2t}) + \frac{4}{9} e^{-2(t+s)} \quad (2.77)$$

which relaxes exponentially towards equilibrium. For the global spin-spin correlator with this initial state we find from (2.75)

$$\hat{C}^{\sigma\sigma}(t, 0) = \frac{1}{3} - \frac{2}{9} e^{-2t} \quad (2.78)$$

which is distinct from the formal $s \rightarrow 0$ limit of the two-time result (2.77).

A correlation with respect to the initial state is interpreted as

$$\hat{C}^{\sigma\sigma}(t, 0) = \int_{\mathbb{R}} dr e^{-iqr} C(t, 0; r, 0) \Big|_{q=0} = t^{(1-\lambda_C)/z} \int_{\mathbb{R}} du \Phi_C(u) \quad (2.79)$$

Comparison with the explicit result (2.78) for large times gives $\lambda_C = 1$.

Exercise 2.13 illustrates the calculation of z in a 1D kinetic Ising model with Kawasaki dynamics [762].

2.3 The Free Random Walk

The following examples of solvable models undergoing ageing will be formulated in terms of a Langevin equation, which was introduced in Chap. 1 and combines a ‘deterministic’ self-interaction described via a Ginzburg-Landau functional and a ‘stochastic’ part. The essence is already captured by the free random walk [171]. Here the order-parameter is assumed to satisfy a Langevin equation without any ‘deterministic’ part

$$\partial_t \phi(t) = \eta(t) \quad (2.80)$$

and η is a Gaussian noise with the first two moments

$$\langle \eta(t) \rangle = 0 \quad , \quad \langle \eta(t) \eta(t') \rangle = 2T \delta(t - t') \quad . \quad (2.81)$$

Equations of motion for the autocorrelator $C(t, t') = \langle \phi(t) \phi(t') \rangle$ and the autoresponse $R(t, t') = \delta \langle \phi(t) \rangle / \delta h(t') = (2T)^{-1} \langle \phi(t) \eta(t') \rangle$ (see exercise 1.3) are easily found by multiplying eq. (2.80) first by $\eta(t')$ before averaging and also by setting $t \mapsto t'$ and multiplying with $\phi(t)$ before averaging. The solutions were already given in Chap. 1 and read

$$C(t, t') = 2T \min(t, t') \quad , \quad R(t, t') = \Theta(t - t') \quad (2.82)$$

where $\Theta(x)$ stands for the Heaviside function and expresses the causality of the response.

In consequence, the fluctuation-dissipation ratio is $X(t, t') = 1/2$ and clearly $X_\infty = 1/2$. Hence the system never reaches equilibrium. The usual scaling forms for the autocorrelation and autoresponse functions are satisfied and we read off the exponents $a = b = -1$, $\lambda_C = \lambda_R = 0$. This simple model is in the same non-equilibrium universality class as the mean-field model with $T < T_c$ discussed in Chap. 1, see also exercise 1.5.

2.4 The Spherical Model

2.4.1 Definition and Formalism

The **spherical model** is one of the most-studied systems, both in and out of equilibrium because it is one of the very few models which may be solved in any dimensions and in a large variety of physical settings. It may be either introduced as the $n \rightarrow \infty$ limit of the $O(n)$ model, which involves a n -component order-parameter field $\vec{\phi}(t, \mathbf{r})$ satisfying an equation of motion of the form

$$\partial_t \vec{\phi} = D \nabla^2 \vec{\phi} + \vec{\phi} - \frac{1}{n} \left(\vec{\phi}^2 \right) \vec{\phi} \quad (2.83)$$

at zero temperature, see [99, 404, 536, 443, 152, 117]. Now, for $n \rightarrow \infty$, $\vec{\phi}^2/n$ may be replaced by its average and the equation of motion become, for any component of $\vec{\phi}$

$$\partial_t \phi_i = D \nabla^2 \phi_i + \mathbf{a}(t) \phi_i, \quad \mathbf{a}(t) = 1 - \langle \phi_i^2 \rangle \quad (2.84)$$

which can be solved exactly.⁴

Here we rather prefer to define the spherical model in terms of real spin variables $S_{\mathbf{r}}$ attached to the sites of a d -dimensional hypercubic lattice and subject to the **spherical constraint**

$$\sum_{\mathbf{r}} S_{\mathbf{r}}^2 = \mathcal{N} \quad (2.85)$$

where $\mathcal{N}$ is the total number of sites, and the usual nearest-neighbour Hamiltonian $\mathcal{H} = - \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} S_{\mathbf{r}} S_{\mathbf{r}'}$ [167, 760, 286, 127, 156, 588, 265]. The (non-conserved) dynamics is given by the stochastic Langevin equation

$$\frac{dS_{\mathbf{r}}}{dt} = \sum_{\mathbf{s}(\mathbf{r})} S_{\mathbf{s}} - (2d + \mathfrak{z}(t)) S_{\mathbf{r}} + \eta_{\mathbf{r}}(t) \quad (2.86)$$

where $\mathbf{s}(\mathbf{r})$ are the nearest neighbour sites of the site $\mathbf{r}$, the Gaussian white noise $\eta_{\mathbf{r}}(t)$ has the correlation $\langle \eta_{\mathbf{r}}(t) \eta_{\mathbf{r}'}(t') \rangle = 2T \delta_{\mathbf{r}, \mathbf{r}'} \delta(t - t')$ and $\mathfrak{z}(t)$ is determined by satisfying the spherical constraint (2.85) in the mean.⁵ The solution is straightforward. By a Fourier transformation

$$\widehat{f}(\mathbf{q}) = \sum_{\mathbf{r}} f_{\mathbf{r}} e^{-i\mathbf{q} \cdot \mathbf{r}}, \quad f_{\mathbf{r}} = (2\pi)^{-d} \int_{\mathcal{B}} d\mathbf{q} \widehat{f}(\mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}} \quad (2.87)$$

where the integral is over the first Brillouin zone $\mathcal{B}$, eq. (2.86) is transformed into

$$\frac{\partial \widehat{S}(\mathbf{q}, t)}{\partial t} = - [\omega(\mathbf{q}) + \mathfrak{z}(t)] \widehat{S}(\mathbf{q}, t) + \widehat{\eta}(\mathbf{q}, t) \quad (2.88)$$

where in addition, together with the $|\mathbf{q}| \rightarrow 0$ limit

⁴ This choice of dynamics corresponds of course to a non-conserved order-parameter (model A). The *conserved* case (model B) is also solvable, but leads for $T < T_c$ to a multi-scaling behaviour of the single-time correlation function [153], which falls outside the kind of dynamical scaling considered in this book. This multi-scaling arises as a peculiarity of the $n \rightarrow \infty$ limit of the conserved $O(n)$ model [98]. This solution has been extended to the two-time correlator and response and re-interpreted in terms of a subaging behaviour [69]. The case $T = T_c$ is reviewed in Chap. 3.

⁵ Taking either the strict or the mean spherical constraint leads to the same results for the long-time dynamics [265].

$$\omega(\mathbf{q}) = 2 \sum_{i=1}^d (1 - \cos q_i) \simeq q^2, \quad (2.89)$$

$$\langle \hat{\eta}(\mathbf{q}, t) \hat{\eta}(\mathbf{q}', t') \rangle = 2T(2\pi)^d \delta^d(\mathbf{q} + \mathbf{q}') \delta(t - t').$$

The equation of motion is readily solved and gives

$$\begin{aligned} \hat{S}(\mathbf{q}, t) &= \frac{\exp(-\omega(\mathbf{q})t)}{\sqrt{g(t)}} \left[\hat{S}(\mathbf{q}, 0) + \int_0^t dt' e^{\omega(\mathbf{q})t'} \sqrt{g(T, t')} \hat{\eta}(\mathbf{q}, t') \right], \\ g(t) &:= \exp \left(2 \int_0^t dt' \mathfrak{z}(t') \right) \end{aligned} \quad (2.90)$$

which forms the basis for all subsequent calculations.

The Lagrange multiplier $g(t)$ is determined from the spherical constraint which implies

$$C_0(t) = \langle S_{\mathbf{r}}(t) S_{\mathbf{r}}(t) \rangle = 1 \quad (2.91)$$

In Fourier space, the equal-time correlator $\tilde{C}(\mathbf{q}, t)$ can now be obtained from $\langle \hat{S}(\mathbf{q}, t) \hat{S}(\mathbf{q}', t) \rangle = (2\pi)^d \delta^d(\mathbf{q} + \mathbf{q}') \hat{C}(\mathbf{q}, t)$ and is given by

$$\hat{C}(\mathbf{q}, t) = \frac{\exp(-2\omega(\mathbf{q})t)}{g(t)} \left[\hat{C}(\mathbf{q}, 0) + 2T \int_0^t dt' e^{2\omega(\mathbf{q})t'} g(t') \right], \quad (2.92)$$

hence (2.91) means in Fourier space that $\int d\mathbf{q} (2\pi)^{-d} \hat{C}(\mathbf{q}, t) = 1$ and finally one arrives at a **Volterra integral equation** for $g(t)$ [167, 286, 588]

$$g(t) = A(t) + 2T \int_0^t dt' f(t - t') g(t'). \quad (2.93)$$

Hence, the solution of the kinetic spherical model is exactly reduced to solving a single equation for the function $g(t)$. In (2.93), the auxiliary functions $f(t)$ and $A(t)$ are

$$\begin{aligned} f(t) &= \frac{1}{(2\pi)^d} \int_{\mathcal{B}} d\mathbf{q} e^{-2\omega(\mathbf{q})t} = (e^{-4t} I_0(4t))^d, \\ A(t) &= \frac{1}{(2\pi)^d} \int_{\mathcal{B}} d\mathbf{q} e^{-2\omega(\mathbf{q})t} \hat{C}(\mathbf{q}, 0), \end{aligned} \quad (2.94)$$

and I_0 is a modified Bessel function. Equation (2.93) is promptly solved via Laplace transformations $\bar{f}(p) = \int_0^\infty dt f(t) e^{-pt}$ and reads

$$\bar{g}(p) = \frac{\bar{A}(p)}{1 - 2T \bar{f}(p)}. \quad (2.95)$$

The so far unspecified initial condition enters through $\bar{A}(p)$.

Once we know $g(t)$ explicitly, we can specify the two-time quantities of interest. The two-time correlation function is $C_{\mathbf{r}-\mathbf{r}'}(t, s) = \langle S_{\mathbf{r}}(t) S_{\mathbf{r}'}(s) \rangle$ and reads in Fourier space

$$\widehat{C}(\mathbf{q}, t, s) = \widehat{C}(\mathbf{q}, s) e^{-\omega(\mathbf{q})(t-s)} \sqrt{\frac{g(s)}{g(t)}} \quad (2.96)$$

and where the expression (2.92) for the single-time correlator has been used. The two-time autocorrelation function then becomes

$$\begin{aligned} C(t, s) = C_0(t, s) &= (2\pi)^{-d} \int_{\mathcal{B}} d\mathbf{q} \widehat{C}(\mathbf{q}, t, s) \\ &= \frac{1}{\sqrt{g(t)g(s)}} \left[A\left(\frac{t+s}{2}\right) + 2T \int_0^s ds' f\left(\frac{t+s}{2} - s'\right) g(s') \right]. \end{aligned} \quad (2.97)$$

The response function is obtained by adding a small magnetic field term $\delta\mathcal{H} = -\sum_{\mathbf{r}} h_{\mathbf{r}}(t) S_{\mathbf{r}}(t)$ to the Hamiltonian. This leads to an extra term $h_{\mathbf{r}}(t)$ on the right-hand side of the Langevin equation (2.86). Carrying out the functional derivative with respect to h leads to (with $t > s$ assumed)

$$\widehat{R}(\mathbf{q}, t, s) = \left. \frac{\delta \langle \widehat{S}(\mathbf{q}, t) \rangle}{\delta \widehat{h}(\mathbf{q}, s)} \right|_{h_{\mathbf{r}}=0} = e^{-\omega(\mathbf{q})(t-s)} \sqrt{\frac{g(s)}{g(t)}}, \quad (2.98)$$

and the autoresponse function finally is

$$R(t, s) = R_0(t, s) = (2\pi)^{-d} \int_{\mathcal{B}} d\mathbf{q} \widehat{R}(\mathbf{q}, t, s) = f\left(\frac{t-s}{2}\right) \sqrt{\frac{g(s)}{g(t)}}. \quad (2.99)$$

The ageing behaviour of the spherical model can now be read from eqs. (2.97, 2.99). The initial correlator is taken to be [588]⁶

$$\widehat{C}(\mathbf{q}; 0, 0) = c_0 + c_1 |\mathbf{q}|^\alpha \quad (2.100)$$

with normalisation constants $c_{0,1}$. The first term alone would give an uncorrelated initial state $C_{\mathbf{r}}(0) = c_0 \delta_{\mathbf{r}, \mathbf{0}}$ [286] while the second term describes a long-range initial correlator (for $|\mathbf{r}|$ large enough)

$$C_{\mathbf{r}}(0) \sim |\mathbf{r}|^{-d-\alpha} = |\mathbf{r}|^{-(D-2)}, \quad (2.101)$$

and we also define an equivalent initial dimension $D = d + \alpha + 2$ of the initial state [588]. Then for $t \rightarrow \infty$ one has for the auxiliary function $A(t) \sim t^{-(d+\alpha)/2}$ and the uncorrelated case is recovered by formally setting $\alpha = 0$.

⁶ The initial magnetisation is assumed to vanish. Otherwise, a more elaborate formalism is needed [19], which takes the fluctuations in the Lagrange multiplier into account.

2.4.2 Solution of the Volterra Integral Equation

The long-time behaviour of the solution is contained in the properties of the function $\bar{g}(p)$ for small p . In turn, this depends on the well-known small- p behaviour of $\bar{f}(p)$ which reads (see exercise 2.14)

$$\bar{f}(p) \simeq \begin{cases} Bp^{-(1-d/2)} & ; \text{ if } d < 2 \\ A_1 - Bp^{d/2-1} & ; \text{ if } 2 < d < 4 \\ A_1 - (8\pi)^{-2}(C_E p + p \ln p) & ; \text{ if } d = 4 \\ A_1 - A_2 p - Bp^{d/2-1} & ; \text{ if } d > 4 \text{ (and } d < 6) \end{cases} \quad (2.102)$$

where $B = |\Gamma(1 - d/2)|(8\pi)^{-d/2}$, $C_E = 0.5772 \dots$ is Euler's constant and the d -dependent constants $A_{1,2}$ are given by

$$\begin{aligned} A_1 &= \frac{1}{(2\pi)^d} \int_B \frac{d\mathbf{q}}{2\omega(\mathbf{q})} = \int_0^\infty du (e^{-4u} I_0(4u))^d = \frac{1}{2T_c} \\ A_2 &= \frac{1}{(2\pi)^d} \int_B \frac{d\mathbf{q}}{4\omega(\mathbf{q})^2} = \int_0^\infty du u (e^{-4u} I_0(4u))^d \end{aligned} \quad (2.103)$$

from which they may be calculated numerically. For $d > 2$, A_1 is finite and a phase transition exists with a positive critical temperature $T_c = (2A_1)^{-1} > 0$.

For illustration, we briefly recall how to find the solution explicitly in the special case of a fully disordered initial state [286, 314, 224]. Then $\hat{C}(\mathbf{q}, 0) = 1$, hence $A(t) = f(t)$. We first consider quenches to $T < T_c$. Inserting eq. (2.103) into (2.95), one finds the leading behaviour for $p \rightarrow 0$

$$\bar{g}(p) \simeq \begin{cases} \frac{A_1}{M_{\text{eq}}^2} - \frac{B}{M_{\text{eq}}^4} p^{d/2-1} & ; \text{ if } 2 < d < 4 \\ \frac{A_1}{M_{\text{eq}}^2} + \frac{C_E}{64\pi^2 M_{\text{eq}}^4} p + \frac{p \ln p}{64\pi^2 M_{\text{eq}}^4} & ; \text{ if } d = 4 \\ \frac{A_1}{M_{\text{eq}}^2} - \frac{A_2}{M_{\text{eq}}^4} p + \frac{B}{M_{\text{eq}}^4} p^{d/2-1} & ; \text{ if } d > 4 \text{ (and } d < 6) \end{cases} \quad (2.104)$$

where the exact equilibrium result $M_{\text{eq}}^2 = T_c - T$ [67] was used. This in turn leads to the following form for $g(t)$, which combines the asymptotic form for $t \rightarrow \infty$ with important singular contributions at $t = 0$

$$g(t) = \begin{cases} \frac{A_1}{M_{\text{eq}}^2} \delta(t) + \frac{f(t)}{M_{\text{eq}}^4} & ; \text{ if } 2 < d < 4 \\ \frac{A_1}{M_{\text{eq}}^2} \delta(t) + \frac{C_E}{64\pi^2 M_{\text{eq}}^4} \delta'(t) + \frac{f(t)}{M_{\text{eq}}^4} & ; \text{ if } d = 4 \\ \frac{A_1}{M_{\text{eq}}^2} \delta(t) - \frac{A_2}{M_{\text{eq}}^4} \delta'(t) + \frac{f(t)}{M_{\text{eq}}^4} & ; \text{ if } d > 4 \text{ (and } d < 6) \end{cases} \quad (2.105)$$

where $\delta(t)$ is the Dirac delta function. Similarly, for quenches to $T = T_c$ one calculates first $\bar{g}(p)$ for $p \rightarrow 0$ and then inverts the Laplace transformation. For $t \rightarrow \infty$, the asymptotic behaviour is [286, 314, 224]

$$g(t) \sim \begin{cases} t^{-(2-d/2)} & ; \text{ if } 2 < d < 4 \\ 1/\ln t & ; \text{ if } d = 4 \\ A_1^2/A_2 & ; \text{ if } d > 4 \end{cases} \quad (2.106)$$

where singular contributions at $t = 0$ as well as the known prefactors (which do not enter anyway into the physical observables) are suppressed.

2.4.3 Dynamical Scaling Behaviour

The two-point functions can now be read off from eqs. (2.97, 2.99). For the calculation of the scaling of the response function, the singular contributions to $g(t)$ can be discarded, but they need to be taken into account for the correlator, see also exercise 2.15. For the autoresponse function, one recovers for all $d > 2$ the expected scaling behaviour $R(t, s) = s^{-1-a} f_R(t/s)$ where $a = d/2 - 1$ and the scaling function reads for $T < T_c$

$$f_R(y) = (4\pi)^{-d/2} (y-1)^{-d/2} y^{-d/4}, \quad T < T_c$$

whereas at the critical point

$$f_R(y) = \begin{cases} (4\pi)^{-d/2} (y-1)^{-d/2} y^{1-d/4} & ; \text{ if } 2 < d < 4 \\ (4\pi)^{-d/2} (y-1)^{-d/2} \left[1 + \frac{\ln y}{\ln s}\right]^{1/2} & ; \text{ if } d = 4 \\ (4\pi)^{-d/2} (y-1)^{-d/2} & ; \text{ if } d > 4 \end{cases}, \quad T = T_c.$$

At the upper critical dimension $d = 4$, and in contrast to equilibrium critical behaviour, dynamical scaling is not broken by logarithmic factors and the scaling functions depend continuously on d , although strong additive logarithmic corrections to scaling are present. The same kind of behaviour is found for the autocorrelator at $d = 4$ [314, 224]. In what follows, the scaling behaviour at those special dimensions where $\bar{f}(p)$ has logarithmic contributions can therefore be obtained from a continuity argument in the dimension d and needs not be treated explicitly.

This is also reflected in the behaviour of the time-dependent length scale $L(t) \sim t^{1/2}$, *without* any logarithmic corrections for all dimensions $d > 2$ and all temperatures $T \leq T_c$ [224], see exercise 2.16.

We now return to the discussion of the ageing behaviour for long-ranged initial conditions (2.100). Following the same steps as before, the following types of behaviour can be found [588].

1. If $T > T_c$, the system relaxes rapidly to equilibrium, since $\bar{g}(p)$ has a pole at a finite positive value $p = 1/\tau_{\text{eq}}$, hence $g(t) \sim e^{t/\tau_{\text{eq}}}$ diverges exponentially for large times. Near to T_c , the relaxation time τ_{eq} diverges

$$\tau_{\text{eq}} \sim \begin{cases} (T - T_c)^{-2/(d-2)} & ; \text{ if } 2 < d < 4 \\ (T - T_c)^{-1} \ln |T - T_c| & ; \text{ if } d = 4 \\ (T - T_c)^{-1} & ; \text{ if } d > 4 \end{cases}. \quad (2.107)$$

Regime	conditions	F
I	$2 < d < 4$ $2 < D < 4$	$-1 - \alpha/2$
II	$4 < d$ $2 < D < 4$	$1 - (d + \alpha)/2$
III	$2 < d < 4$ $4 < D$	$d/2 - 2$
IV	$4 < d$ $4 < D$ $\alpha > -2$	0
V	$4 < d$ $4 < D$ $\alpha < -2$	0
$T < T_c$	$2 < d$	$-(d + \alpha)/2$

Table 2.3. Values of the exponent $F(\alpha, d)$ which describes the long-time behaviour of $g(t)$ in five regimes at $T = T_c$ where $D = d + \alpha + 2$, and in the low-temperature regime $T < T_c$.

2. For $T < T_c$, one has the long-time algebraic behaviour $g(t) \sim t^{-(d+\alpha)/2}$ [536].
3. At $T = T_c$, one has again a power-law behaviour $g(t) \sim t^F$. The values of the exponent F are listed in Table 2.3 and five distinct regimes of ageing behaviour of non-equilibrium critical dynamics can be recognised.

From these results, we have the following scaling forms

$$\begin{aligned}
 C(t, s) &= (4\pi)^{-d/2} s^{-d/2+1} \left[s^{-1-F-\alpha/2} M_0(t/s) + K_0(t/s) \right] \\
 R(t, s) &= (4\pi)^{-d/2} s^{-d/2} (t/s - 1)^{-d/2} (t/s)^{-F/2}
 \end{aligned} \tag{2.108}$$

with the auxiliary functions

$$\begin{aligned}
 M_0(y) &= m_0 y^{-F/2} \left(\frac{y+1}{2} \right)^{-(d+\alpha)/2} \\
 K_0(y) &= 2T_c y^{-F/2} \int_0^1 dw w^F (y+1-2w)^{-d/2}
 \end{aligned} \tag{2.109}$$

where m_0 is a constant. In the low-temperature regime the scaling function is obtained by letting $T \rightarrow 0$, quite in agreement with what is expected in phase-ordering kinetics where the temperature is an irrelevant variable for $T < T_c$ [93]. The values for the ageing exponents are listed in Table 2.4.

We see that while for quenches to $T < T_c$, the initial correlations are felt for all times, for critical quenches the behaviour is more subtle and is conveniently characterised by the spatial dimension d and the effective dimension $D = d + \alpha + 2$ in the initial correlations. It is not yet known whether long-range initial correlations will lead to such a rich variety of ageing behaviours in other models. We also see that λ_C and λ_R need no longer to be equal to each other, in agreement with the prediction (1.128), valid for $T < T_c$ [99], but we find here the same kind of relation for critical quenches as well.

We have only considered here the correlations and responses of the order-parameter. The analogous results for several other observables (i.e. the energy-

Regime	a	b	λ_C	λ_R	X_∞
I	$d/2 - 1$	$d/2 - 1$	$d + \alpha/2 - 1$	$d - \alpha/2 - 1$	0 (if $\alpha \neq -2$)
II	$d/2 - 1$	1	$(d + \alpha)/2 + 1$	$(d - \alpha)/2 + 1$	0
III	$d/2 - 1$	$d/2 - 1$	$3d/2 - 2$	$3d/2 - 2$	$1 - 2/d$
IV	$d/2 - 1$	$d/2 - 1$	d	d	$1/2$
V	$d/2 - 1$	$d/2 + \alpha/2$	$d + \alpha$	d	0
$T < T_c$	$d/2 - 1$	0	$(d + \alpha)/2$	$(d - \alpha)/2$	0

Table 2.4. Values of the critical autocorrelation and autoresponse exponents a , b , λ_C and λ_R of the spherical model, in the five critical ageing regimes and in the low-temperature regime $T < T_c$. The limit fluctuation-dissipation ratio X_∞ is also included. In regime I, one has $X_\infty = 1$ for $\alpha = -2$.

density may be recovered as special cases from the long-range spherical model to which we now turn.

2.5 The Long-range Spherical Model

2.5.1 Definition and Composite Observables

We now consider an extension of the spherical model where the dynamical exponent no longer is $z = 2$. We consider the Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} J(\mathbf{r} - \mathbf{r}') S_{\mathbf{r}} (S_{\mathbf{r}'} - S_{\mathbf{r}}) \quad (2.110)$$

where the sum extends over all pairs $(\mathbf{r}, \mathbf{r}')$ of a hypercubic lattice such that $\mathbf{r} - \mathbf{r}' \neq \mathbf{0}$ and the exchange coupling reads

$$J(\mathbf{r}) = \left(\sum_{\mathbf{s}}' |\mathbf{s}|^{-(d+\sigma)} \right)^{-1} |\mathbf{r}|^{-d-\sigma} \quad (2.111)$$

when $\mathbf{r} \neq \mathbf{0}$ and $J(\mathbf{0}) = 0$. Again, the sum is over all lattice vectors $\mathbf{s} \neq \mathbf{0}$. The equilibrium behaviour of the model depends in an essential way on the control parameter σ , see [414] for a classic review. When $\sigma > 2$, the long-range behaviour is identical to the spherical model with nearest-neighbour interactions discussed in the previous Section. On the other hand, for $0 < \sigma < 2$, the model has a second-order phase-transition in dimensions $d > \sigma$ with critical exponents which are in general different from those found for $\sigma > 2$. In this section, we restrict ourselves exclusively to this latter case.

Here we are interested in the non-equilibrium dynamics after a quench to $T \leq T_c$ from a fully disordered initial state [127, 47]. As for the short-ranged model, the dynamics is given by the Langevin equation (2.86) with the same

Gaussian noise as before. The solution proceeds again via a Fourier transform, but the dispersion relation now becomes

$$\omega(\mathbf{q}) = \widehat{J}(\mathbf{0}) - \widehat{J}(\mathbf{q}) \simeq J_0 |\mathbf{q}|^\sigma \quad (2.112)$$

in the $|\mathbf{q}| \rightarrow 0$ limit, for $0 < \sigma < 2$ and where J_0 is a calculable constant. From this dispersion relation it is clear that the physically relevant length scale will for sufficiently large times scale as $L(t) \sim t^{1/z}$, where the dynamical exponent is given by

$$z = \sigma . \quad (2.113)$$

The formal solution is obtained exactly as before, in particular, the autocorrelation function is given by (2.97), the autoresponse function by (2.99) and the full response function by (2.98) in momentum space where the function $g(t)$ is again the solution of the Volterra integral equation (2.93)

$$g(t) = f(t) + \int_0^t dt' f(t-t')g(t')$$

(recall that $\widehat{C}(\mathbf{q}, 0) = 1$ because of the chosen initial condition) but where the function $f(t)$ now reads $f(t) := f(t, \mathbf{0})$ and

$$f(t, \mathbf{r}) = (2\pi)^{-d} \int_{\mathcal{B}} d\mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{r} - 2\omega(\mathbf{k})t) . \quad (2.114)$$

The critical temperature is given by $T_c = \frac{1}{2} (\int_0^\infty dt f(t))^{-1}$ [414, 127].

It is of interest to consider not only observables built directly from the order-parameter $S(t, \mathbf{r})$,

$$\mathcal{O}_1(t, \mathbf{r}) := S(t, \mathbf{r}) , \quad (2.115)$$

but also some composite fields, specifically the spin-squared (spin²) operator

$$\mathcal{O}_2(t, \mathbf{r}) := S^2(t, \mathbf{r}) - \langle S^2(t, \mathbf{r}) \rangle \quad (2.116)$$

and the energy-density operator

$$\begin{aligned} \mathcal{O}_\epsilon(t, \mathbf{r}) &:= \mathcal{E}(t, \mathbf{r}) - \langle \mathcal{E}(t, \mathbf{r}) \rangle , \\ \mathcal{E}(t, \mathbf{r}) &:= \sum_{\mathbf{r}'}' J(\mathbf{r} - \mathbf{r}') S(t, \mathbf{r}) (S(t, \mathbf{r}') - S(t, \mathbf{r})) . \end{aligned} \quad (2.117)$$

These composite operators are defined in such a way that their average value is zero, hence their correlation functions are *connected* correlation functions. Since the energy is defined only up to a constant there is no unique definition of the energy-density operator.

The distinction between $\mathcal{O}_2$ and $\mathcal{O}_\epsilon$ might be better understood as follows, by reconsidering the continuum limit of the energy-density for short-range model [47]. Then, the expression for the energy is usually taken as

$$\mathcal{H}_{\text{sr}} = -J \sum_{\mathbf{r}, \boldsymbol{\mu}(\mathbf{r})} S_{\mathbf{r}} S_{\mathbf{r}+\boldsymbol{\mu}(\mathbf{r})}, \quad (2.118)$$

where $\mathbf{r} + \boldsymbol{\mu}(\mathbf{r})$ runs over the neighbouring sites of $\mathbf{r}$. In such a case, the energy density could be defined as $\tilde{\epsilon}(\mathbf{r}) = -J \sum_{\boldsymbol{\mu}} S_{\mathbf{r}} S_{\mathbf{r}+\boldsymbol{\mu}}$, which in the continuum limit would reduce to $\tilde{\epsilon}(\mathbf{r}) = -J (2S_{\mathbf{r}}^2 + \mathbf{a}^2 S_{\mathbf{r}} \nabla^2 S_{\mathbf{r}})$, where $\mathbf{a}$ is the lattice constant. But if we had added an overall constant $E_0 = \mathcal{N} = \sum_{\mathbf{r}} S_{\mathbf{r}}^2$ to the Hamiltonian, then the energy density could be defined as

$$\epsilon(\mathbf{r}) = -J \sum_{\boldsymbol{\mu}} S_{\mathbf{r}} (S_{\mathbf{r}+\boldsymbol{\mu}} - S_{\mathbf{r}}) \rightarrow -J \mathbf{a}^2 S_{\mathbf{r}} \nabla^2 S_{\mathbf{r}} (1 + \text{O}(\mathbf{a})). \quad (2.119)$$

Hence $\mathcal{H}_{\text{sr}} = \sum_{\mathbf{r}} \epsilon(\mathbf{r}) = J \mathbf{a}^2 \sum_{\mathbf{r}} (\nabla S_{\mathbf{r}})^2$, up to boundary terms. In conclusion, one must distinguish between $\mathcal{O}_2(t, \mathbf{r})$ and $\mathcal{O}_{\epsilon}(t, \mathbf{r})$.

We define the connected two-point correlation functions of the composite operators [47]

$$\mathcal{C}_{ab}(t, s; \mathbf{r} - \mathbf{r}') := \langle \mathcal{O}_a(t; \mathbf{r}) \mathcal{O}_b(s; \mathbf{r}') \rangle \quad (2.120)$$

where $\mathcal{C}_{11}$ is the usual spin-spin correlator and $\mathcal{C}_{22}$ is the energy-energy correlator. The response functions of the fields $\{\mathcal{O}_a(t, \mathbf{r})\}$ with respect to their conjugate fields $\{h_a(t, \mathbf{r})\}$ are defined by

$$\mathcal{R}_{ab}(t, s, \mathbf{r} - \mathbf{r}') := \left. \frac{\delta \langle \mathcal{O}_a(t, \mathbf{r}) \rangle_{\{h\}}}{\delta h_b(s, \mathbf{r}')} \right|_{\{h\}=\{0\}}, \quad (2.121)$$

and are obtained by linearly perturbing the Hamiltonian as follows: $\delta \mathcal{H} = \sum_{a,t,\mathbf{r}} h_a(t, \mathbf{r}) \mathcal{O}_a(t, \mathbf{r})$. Here $\mathcal{R}_{11}$ is the response of the order-parameter with respect to its conjugate magnetic field. We also define responses of the fields $\{\mathcal{O}_a(t, \mathbf{r})\}$ to local temperature fluctuations by perturbing the noise strength $T \rightarrow T + \delta T(t, \mathbf{r})$ and have

$$\mathcal{R}_a^{(T)}(t, s, \mathbf{r} - \mathbf{r}') := \left. \frac{\delta \langle \mathcal{O}_a(t, \mathbf{r}) \rangle_{\delta T}}{\delta T(s, \mathbf{r}')} \right|_{\delta T=0}. \quad (2.122)$$

The scaling behaviour for the autocorrelation function $\mathcal{C}_{ab}(t, s) := \mathcal{C}_{ab}(t, s; \mathbf{0})$ and the autoresponse functions $\mathcal{R}_{ab}(t, s) := \mathcal{R}_{ab}(t, s; \mathbf{0})$ and $\mathcal{R}_a^{(T)}(t, s) := \mathcal{R}_a^{(T)}(t, s; \mathbf{0})$ is expected to be

$$\mathcal{C}_{ij}(t, s) = s^{-b_{ij}} f_C^{ij}(t/s), \quad f_C^{ij}(y) \stackrel{y \rightarrow \infty}{\sim} y^{-\lambda_C^{ij}/z}, \quad (2.123)$$

$$\mathcal{R}_{ij}(t, s) = s^{-a_{ij}-1} f_R^{ij}(t/s), \quad f_R^{ij}(y) \stackrel{y \rightarrow \infty}{\sim} y^{-\lambda_R^{ij}/z}, \quad (2.124)$$

$$\mathcal{R}_i^{(T)}(t, s) = s^{-a_i^{(T)}-1} f_R^{(T)i}(t/s), \quad f_R^{(T)i}(y) \stackrel{y \rightarrow \infty}{\sim} y^{-\lambda_R^{(T)i}/z}, \quad (2.125)$$

in the scaling regime where t, s and $t-s$ are simultaneously large. This defines the non-equilibrium critical exponents $a_{ij}, b_{ij}, a_i^T, \lambda_R^{ij}, \lambda_C^{ij}, \lambda_R^{(T)i}$.

Function	b		λ_C		
	$T < T_c$	$T = T_c$	$T < T_c$	$T = T_c$	
				$\sigma < d < 2\sigma$	$d > 2\sigma$
$\mathcal{C}_{11}$	0	$d/\sigma - 1$	$d/2$	$3d/2 - \sigma$	d
$\mathcal{C}_{22}$	0	$2d/\sigma - 2$	d	$3d - 2\sigma$	$2d$
$\mathcal{C}_{2\epsilon}$	1	$2d/\sigma - 1$	$d + \sigma$	$3d - \sigma$	$2d + \sigma$
$\mathcal{C}_{\epsilon\epsilon}$	2	$2d/\sigma$	$d + 2\sigma$	$3d$	$2d + 2\sigma$

Table 2.5. Non-equilibrium exponents b , λ_C , for several non-equilibrium autocorrelation functions in the long-range spherical model. The exponents for the short-range model can be recovered by taking the limit $\sigma \rightarrow 2$.

Function	a		λ_R		
	$T < T_c$	$T = T_c$	$T < T_c$	$T = T_c$	
				$\sigma < d < 2\sigma$	$d > 2\sigma$
$\mathcal{R}_{11}$	$d/\sigma - 1$	$d/\sigma - 1$	$d/2$	$3d/2 - \sigma$	d
$\mathcal{R}_{22}$	$d/\sigma - 1$	$2d/\sigma - 2$	d	$3d - 2\sigma$	$2d$
$\mathcal{R}_{\epsilon 2}$	d/σ	$2d/\sigma - 1$	$d + \sigma$	$3d - \sigma$	$2d + \sigma$
$\mathcal{R}_{2\epsilon}$	d/σ	$2d/\sigma - 1$	$d + \sigma$	$3d - \sigma$	$2d + \sigma$
$\mathcal{R}_{\epsilon\epsilon}$	$d/\sigma + 1$	$2d/\sigma$	$d + 2\sigma$	$3d$	$2d + 2\sigma$
$\mathcal{R}_2^T$	$2d/\sigma - 1$	$2d/\sigma - 1$	d	$3d - 2\sigma$	$2d$
$\mathcal{R}_\epsilon^T$	$2d/\sigma$	$2d/\sigma$	$d + \sigma$	$3d - \sigma$	$2d + \sigma$

Table 2.6. Non-equilibrium exponents $a = a'$ and λ_R , for several scaling operators in the long-range spherical model. The exponents for the short-range model can be obtained by taking the limit $\sigma \rightarrow 2$.

The calculation of these scaling forms follows the same lines as in the short-ranged spherical model [127, 47]. Solving the Volterra integral equation leads to the asymptotic behaviour of $g(t)$, when $t \rightarrow \infty$

$$g(t) \sim t^F, \quad F = \begin{cases} -d/\sigma & ; \text{ if } T < T_c \text{ and } d > \sigma \\ -2 + d/\sigma & ; \text{ if } T = T_c \text{ and } \sigma < d < 2\sigma \\ 0 & ; \text{ if } T = T_c \text{ and } 2\sigma < d \end{cases} \quad (2.126)$$

We list the exponents for the correlation functions in Table 2.5 and those for the response functions in Table 2.6. Explicit expressions for all scaling functions can be found in [47]. For later use in Chap. 4, we list the spin-spin autocorrelator, in the scaling limit and in Fourier space for the sake of brevity

Regime	conditions	F
I	$2 < d < 4$ $\sigma < D < 2\sigma$	$-1 - \alpha/\sigma$
II	$4 < d$ $\sigma < D < 2\sigma$	$1 - (d + \alpha)/\sigma$
III	$2 < d < 4$ $2\sigma < D$	$d/\sigma - 2$
IV	$4 < d$ $2\sigma < D$ $\alpha > -\sigma$	0
V	$4 < d$ $2\sigma < D$ $\alpha < -\sigma$	0
$T < T_c$	$2 < d$	$-(d + \alpha)/\sigma$

Table 2.7. Values of the exponent $F(\alpha, \sigma, d)$ describing the long-time behaviour of $g(t)$ in five regimes at $T = T_c$ where $D = d + \alpha + \sigma$, and in the low-temperature regime $T < T_c$, for the long-range spherical model.

$$\widehat{C}(ys, s; \mathbf{q}) \sim \begin{cases} s^{d/\sigma} y^{d/2\sigma} \exp[-J_0 |\mathbf{q}|^\sigma s(y+1)] & ; T < T_c \\ T_c s y^{1-d/2\sigma} \int_0^1 dz z^{-2+d/\sigma} \\ \quad \times \exp[-J_0 |\mathbf{q}|^\sigma s(y+1-2z)] & ; T = T_c \text{ and } \sigma < d < 2\sigma \\ T_c/J_0 \cdot |\mathbf{q}|^{-\sigma} (\exp[-J_0 |\mathbf{q}|^\sigma s(y-1)] \\ \quad - \exp[-J_0 |\mathbf{q}|^\sigma s(y+1)]) & ; T = T_c \text{ and } 2\sigma < d \end{cases} \quad (2.127)$$

and up to normalisation. On the other hand, the response of the order-parameter with respect to an external magnetic field is

$$R_{11}(t, s; \mathbf{r}) = R_{11}(t, s) G(|\mathbf{r}|[(t-s)/2]^{-1/\sigma}) \quad (2.128)$$

where the spatial part is given by $G(|\mathbf{v}|) = G(0)f(1, \mathbf{v})$ where $f(t, \mathbf{r})$ was given in eq. (2.114) and the autoresponse is, up to normalisation,

$$s^{d/\sigma} R_{11}(ys, s) \sim \begin{cases} (y-1)^{-d/\sigma} y^{d/2\sigma} & ; \text{ if } T < T_c \\ (y-1)^{-d/\sigma} y^{1-d/2\sigma} & ; \text{ if } T = T_c \text{ and } \sigma < d < 2\sigma \\ (y-1)^{-d/\sigma} & ; \text{ if } T = T_c \text{ and } 2\sigma < d \end{cases} \quad (2.129)$$

We shall discuss the form of these scaling functions from the point of view of local scale-invariance in chapters 4 and 5.

2.5.2 Long-range Initial Correlations

Similarly to the case of short-range interactions discussed in the previous section, one may include [217] the case of *initial long-range correlations*, of the form

$$\widehat{C}(\mathbf{q}; 0, 0) = c_0 + c_1 |\mathbf{q}|^\alpha. \quad (2.130)$$

As before, the second term describes in real space correlations decaying according to eq. (2.101), and one merely has to adapt the initial dimension, which now becomes $D = d + \alpha + \sigma$. Provided one restricts attention to local observables, the fluctuations in the Lagrange multiplier can be neglected in the

Regime	a	b	λ_C	λ_R	X_∞
I	$d/\sigma - 1$	$d/\sigma - 1$	$d + \alpha/\sigma - 1$	$d - \alpha/\sigma - 1$	0
II	$d/\sigma - 1$	1	$(d + \alpha)/\sigma + 1$	$(d - \alpha)/\sigma + 1$	0
III	$d/\sigma - 1$	$d/\sigma - 1$	$3d/2 - 2\sigma$	$3d/2 - \sigma$	$1 - \sigma/d$
IV	$d/\sigma - 1$	$d/\sigma - 1$	d	d	$1/2$
V	$d/\sigma - 1$	$d/\sigma + \alpha/\sigma$	$d + \alpha$	d	0
$T < T_c$	$d/2 - 1$	0	$(d + \alpha)/2$	$(d - \alpha)/2$	0

Table 2.8. Values of the critical autocorrelation and autoresponse exponents a , b , λ_C and λ_R of the long-range spherical model, in the five critical ageing regimes defined in Table 2.7 and in the low-temperature regime $T < T_c$. The limit fluctuation-dissipation ratio X_∞ is also included. In regime I, one has $X_\infty = 1$ for $\alpha = -\sigma$.

thermodynamic limit and the discussion of the previous section is readily generalised [217]. The solution $g(t)$ of the Volterra integral equation satisfies the long-time behaviour $g(t) \sim t^F$, which can be used to distinguish five different scaling regimes at criticality and a further one for quenches to $T = T_c$, with the definitions and the values of F listed in Table 2.7. The various exponents describing the ageing behaviour of the magnetic correlators and responses are listed in Table 2.8 [217].

The form of the scaling functions is obtained from the case of short-ranged interactions, upon substitution of the exponents.

2.5.3 Magnetised Initial State

In the present formalism, one may also discuss the effects of a non-vanishing initial magnetisation. This is described by the initial conditions

$$\frac{1}{\mathcal{N}} \sum_{\mathbf{r}} \langle S_{\mathbf{r}}(0) \rangle = m_0 \quad , \quad \langle S_{\mathbf{r}}(0) S_{\mathbf{r}'}(0) \rangle = (1 - m_0^2) \delta_{\mathbf{r}, \mathbf{r}'} + m_0^2. \quad (2.131)$$

A non-trivial behaviour is expected only for quenches onto the critical point $T = T_c$ on which we focus from now on. Since we are merely interested in local observables, of which the autoresponse and autocorrelation functions are simple examples, any fluctuations in the Lagrange multiplier can still be neglected [19, 20]. Hence the formalism developed so far can be applied, with the main change that the initial autocorrelator now takes the form [20, 217]

$$A(t) = (1 - m_0^2) f(t) + m_0^2. \quad (2.132)$$

Solving the Volterra integral equation (2.93) leads to

$$g(t) = g_M t^F \left(1 + \frac{t}{t_M} \right) \quad , \quad F = \begin{cases} 0 & ; \text{ if } d > 2\sigma \\ d/\sigma - 2 & ; \text{ if } d < 2\sigma \end{cases} \quad (2.133)$$

where g_M is a known non-universal constant and the new time-scale $t_M := (1+F)(1-m_0^2)/(2T_c m_0^2)$. In order to find the time-dependent magnetisation, one may rewrite eq. (2.90) in the form

$$\hat{S}(\mathbf{q}, t) = \hat{R}(\mathbf{q}, t, 0) \hat{S}(\mathbf{q}, 0) + \int_0^t dt' \hat{R}(\mathbf{q}, t, t') \hat{\eta}(\mathbf{q}, t') \quad (2.134)$$

where the propagator $\hat{R}(\mathbf{q}, t, s)$ is given by eq. (2.98). Hence the average magnetisation is found by letting $\mathbf{q} \rightarrow \mathbf{0}$ and reads

$$m(t) = \frac{m_0}{\sqrt{g(t)}} \simeq \frac{m_0}{\sqrt{g_M}} t^{-F/2} \left(1 + \frac{t}{t_M}\right)^{-1/2}. \quad (2.135)$$

This illustrates explicitly that in the two regimes $t \ll t_M$ and $t \gg t_M$ the time-dependent behaviour is different, as qualitatively sketched in Fig. 1.26. One reads off the slip exponent $\Theta = -F/2 = \frac{1}{2} - \beta/(\nu z)$.

Similarly, the (connected) two-time autocorrelator is now given by

$$\begin{aligned} C(t, s) &= \frac{1}{\sqrt{g(t)g(s)}} \left[(1 - m_0^2) f\left(\frac{t+s}{2}\right) + 2T_c \int_0^s ds' f\left(\frac{t+s}{2} - s'\right) g(s') \right] \\ &= s^{1-d/\sigma} f_C\left(\frac{t}{s}, \frac{s}{t_M}\right) \end{aligned} \quad (2.136)$$

which replaces eq. (2.97). The autoresponse function is

$$R(t, s) = s^{-d/\sigma} f_R\left(\frac{t}{s}, \frac{s}{t_M}\right) \quad (2.137)$$

and the corresponding scaling functions read explicitly

$$\begin{aligned} f_C(y, u) &= f_{C,0} \frac{y^{-F/2}}{\sqrt{(1+uy)(1+u)}} \int_0^1 dw (y+1-2w)^{-d/\sigma} w^F (1+uw) \\ f_R(y, u) &= f_{R,0} \sqrt{\frac{1+u}{1+uy}} (y-1)^{-d/\sigma} y^{-F/2}. \end{aligned} \quad (2.138)$$

The case of short-ranged interactions can be recovered from these by taking the limit $\sigma \rightarrow 2$.

While in the limit $u \rightarrow 0$ one recovers the same expressions as found above for a fully disordered initial state, for finite values of u one has a cross-over to a distinct scaling regime. The ageing behaviour for an ordered initial state is reached in the limit $u \gg 1$. In Table 2.9 the exponents and the limit fluctuation-dissipation ratio X_∞ describing the ageing in the two extreme regimes are listed [20, 217]. In Chap. 3 we shall quote some results for global observables, the calculation of which does require to include the non-Gaussian fluctuations in the Lagrange multiplier [19, 20] and which is beyond the scope of this book.

conditions		$a = b$	$\lambda_C = \lambda_R$	X_∞
$s \ll t_M$	$d < 2\sigma$	$d/\sigma - 1$	$3d/2 - \sigma$	$1 - d/\sigma$
	$d > 2\sigma$	$d/\sigma - 1$	d	$1/2$
$s \gg t_M$	$d < 2\sigma$	$d/\sigma - 1$	$(3d - \sigma)/2$	$1 - d/\sigma$
	$d > 2\sigma$	$d/\sigma - 1$	$d + \sigma/2$	$2/3$

Table 2.9. Values of the critical ageing, autocorrelation and autoresponse exponents, as well as the limit fluctuation-dissipation ratio, in the spherical model with initial magnetisation m_0 .

2.6 XY Model in Spin-wave Approximation

2.6.1 Outline of the Method and Applicability

Another system which can be exactly analysed is the kinetic XY model in spin-wave approximation [171, 633, 73, 589]. The variables of the XY model are planar spins

$$\vec{S}(\mathbf{r}) = \cos(\phi(\mathbf{r})) \vec{e}_1 + \sin(\phi(\mathbf{r})) \vec{e}_2 = \begin{pmatrix} \cos \phi(\mathbf{r}) \\ \sin \phi(\mathbf{r}) \end{pmatrix} \quad (2.139)$$

which are attached to the sites $\mathbf{r}$ of a d -dimensional hypercubic lattice and $\phi(\mathbf{r})$ is the phase. The XY Hamiltonian is

$$\mathcal{H}[\phi] = - \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \vec{S}(\mathbf{r}) \cdot \vec{S}(\mathbf{r}') = - \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \cos(\phi(\mathbf{r}) - \phi(\mathbf{r}')) \simeq \frac{1}{2} \int d\mathbf{r} (\nabla \phi(\mathbf{r}))^2 \quad (2.140)$$

where the coupling constant J has been set to unity and the sum runs over nearest neighbours. In the last line we have approximated $\mathcal{H}$ by its spin-wave form (up to an unimportant additive constant). The relaxational dynamics is described by a Langevin equation

$$\frac{\partial \phi(t, \mathbf{r})}{\partial t} = - \frac{\delta \mathcal{H}[\phi]}{\delta \phi(t, \mathbf{r})} + \eta(t, \mathbf{r}). \quad (2.141)$$

where η represents an uncorrelated Gaussian noise with zero mean and variance

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2T_f \delta(t - t') \delta(\mathbf{r} - \mathbf{r}') \quad (2.142)$$

and T_f is the final temperature to which the system is quenched at time $t = 0$, starting from an initial temperature T_i . The spin-wave approximation works in the low-temperature regime where (see exercise 2.20)

$$T_i, T_f \ll T_c(d), \quad (2.143)$$

$T_c(d)$ being the critical temperature of the XY model in d dimensions (if $d = 2$, $T_c(2) = T_{KT}$ is the Kosterlitz-Thouless temperature of the transition).

In writing this, we have implicitly absorbed the spin-wave stiffness [450, 633] into a redefinition of the temperatures.⁷

We are interested in the properties of the two-point functions, either for the magnetic variables

$$\begin{aligned}\Gamma(t, s; \mathbf{r}, \mathbf{r}') &:= \langle \vec{S}(t, \mathbf{r}) \cdot \vec{S}(s, \mathbf{r}') \rangle = \langle \cos(\phi(t, \mathbf{r}) - \phi(s, \mathbf{r}')) \rangle, \\ \rho(t, s; \mathbf{r}, \mathbf{r}') &:= \lim_{\vec{h} \rightarrow 0} \frac{\partial \langle \vec{S}(t, \mathbf{r}) \rangle}{\partial \vec{h}(s, \mathbf{r}')},\end{aligned}\quad (2.144)$$

where the response is found by adding a term $\delta\mathcal{H}_{\text{mag}} = \sum_{\mathbf{r}} \vec{h} \cdot \vec{S}$ to $\mathcal{H}$, or else for the angular variables

$$\begin{aligned}C(t, s; \mathbf{r}, \mathbf{r}') &:= \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \rangle, \\ R(t, s; \mathbf{r}, \mathbf{r}') &:= \lim_{h \rightarrow 0} \frac{\partial \langle \phi(t, \mathbf{r}) \rangle}{\partial h(s, \mathbf{r}')}\end{aligned}\quad (2.145)$$

(where a perturbation $\delta\mathcal{H}_{\text{ang}} = \sum_{\mathbf{r}} h\phi$ should have been added).

In order that the spin-wave approximation be applicable, we must start from an (almost) ordered initial state of the system. In Fourier space, the initial correlator reads

$$\hat{a}(\mathbf{q}) = \hat{C}(\mathbf{q}, 0, 0) = \frac{2\pi\eta(T_i)}{q^2} = \frac{T_i}{q^2} \quad (2.146)$$

where $\eta(T_i)$ is the standard equilibrium critical exponent and the relation $2\pi\eta(T_i) = T_i$ valid in the spin-wave approximation was used, see [450, 171, 633, 73, 65, 589] (the rather unusual scaling properties of the 1D case [537] are discussed in exercise 2.22).

2.6.2 Two-time Correlations

Practically, the calculations are made easier by going over to a field-theoretic formulation given by the Janssen-de Dominicis functional

$$\begin{aligned}\mathcal{J}[\phi, \tilde{\phi}; \vec{h}] &= \int dt d\mathbf{r} \tilde{\phi} \left[\frac{\partial \phi}{\partial t} - \Delta \phi + \sin(\phi)h_1 - \cos(\phi)h_2 \right] \\ &\quad - T \int dt d\mathbf{r} \tilde{\phi}(t, \mathbf{r}) \tilde{\phi}(t, \mathbf{r}) - \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \tilde{\phi}(0, \mathbf{r}) a(\mathbf{r} - \mathbf{r}') \tilde{\phi}(0, \mathbf{r}')\end{aligned}\quad (2.147)$$

⁷ In 2D, it is known that for initial states with a temperature T_{KT} , any vortices present will be tightly bound and for distances larger than the characteristic pair size, the XY model renormalises to the spin-wave Hamiltonian [633]. For quenches from *above* T_{KT} in 2D, however, vortex configurations also become important and this leads to logarithmic scaling [625, 95].

where the function $a(\mathbf{r})$ describes the initial conditions according to eq. (2.146) and $\vec{h} = h_1 \vec{e}_1 + h_2 \vec{e}_2$ is the magnetic field.

While the angular correlation function is readily written down in momentum space

$$\begin{aligned}\widehat{C}(\mathbf{q}, t, s) &= \langle \widehat{\phi}_{\mathbf{q}}(t) \widehat{\phi}_{-\mathbf{q}}(s) \rangle \\ &= \frac{T_i - T_f}{q^2} \exp(-q^2(t+s)) + \frac{T_f}{q^2} \exp(-q^2(t-s))\end{aligned}\quad (2.148)$$

the magnetic correlator Γ should first be further simplified since for a vanishing magnetic field the functional $\mathcal{J}[\phi, \tilde{\phi}; \vec{0}]$ is a quadratic form in the fields $\phi, \tilde{\phi}$. These are therefore Gaussian fields and well-known methods of Gaussian integration give [589]

$$\begin{aligned}\Gamma(t, s; \mathbf{r}, \mathbf{r}') &= \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \cos(\phi(t, \mathbf{r}) - \phi(s, \mathbf{r}')) \exp(-\mathcal{J}[\phi, \tilde{\phi}]) \\ &= \frac{1}{2} \left[\langle \exp[i(\phi(t, \mathbf{r}) - \phi(s, \mathbf{r}'))] \rangle + \langle \exp[-i(\phi(t, \mathbf{r}) - \phi(s, \mathbf{r}'))] \rangle \right] \\ &= \frac{1}{2} \left\langle \exp \left[i \int du d\mathbf{R} j(u, \mathbf{R}) \phi(u, \mathbf{R}) \right] \right\rangle \\ &\quad + \frac{1}{2} \left\langle \exp \left[-i \int du d\mathbf{R} j(u, \mathbf{R}) \phi(u, \mathbf{R}) \right] \right\rangle \\ &= \exp \left[-\frac{1}{2} \langle (\phi(t, \mathbf{r}) - \phi(s, \mathbf{r}'))^2 \rangle \right] \\ &= \exp \left[C(t, t; \mathbf{r}, \mathbf{r}') - \frac{C(t, t; \mathbf{r}, \mathbf{r}) + C(s, s; \mathbf{r}', \mathbf{r}')}{2} \right] \\ &\simeq 1 + C(t, s; \mathbf{r}, \mathbf{r}') - \frac{1}{2} [C(t, t; \mathbf{r}, \mathbf{r}) + C(s, s; \mathbf{r}', \mathbf{r}')] \end{aligned}\quad (2.149)$$

where in the third line the current $j(u, \mathbf{R}) := \delta(u-t)\delta(\mathbf{R}-\mathbf{r}) - \delta(u-s)\delta(\mathbf{R}-\mathbf{r}')$ was introduced, in the forth line the standard free-field identity

$$\begin{aligned}&\left\langle \exp \left[i \int du d\mathbf{R} j(u, \mathbf{R}) \phi(u, \mathbf{R}) \right] \right\rangle \\ &= \exp \left[-\frac{1}{2} \int du du' d\mathbf{R} d\mathbf{R}' j(u, \mathbf{R}) j(u', \mathbf{R}') \langle \phi(u, \mathbf{R}) \phi(u', \mathbf{R}') \rangle \right]\end{aligned}$$

was used, in the fifth line the argument of the exponential was expanded and in the sixth line an expansion to lowest order in T_i, T_f was made. This expresses Γ in terms of angular correlators C . A more suggestive form of this is found as follows. We have

$$\begin{aligned}\langle \vec{S}(t, \mathbf{r}) \rangle \cdot \langle \vec{S}(s, \mathbf{r}') \rangle &= \langle \cos \phi(t, \mathbf{r}) \rangle \langle \cos \phi(s, \mathbf{r}') \rangle + \langle \sin \phi(t, \mathbf{r}) \rangle \langle \sin \phi(s, \mathbf{r}') \rangle \\ &\simeq 1 - \frac{1}{2} \langle \phi(t, \mathbf{r})^2 \rangle - \frac{1}{2} \langle \phi(s, \mathbf{r}')^2 \rangle + \langle \phi(t, \mathbf{r}) \rangle \langle \phi(s, \mathbf{r}') \rangle + \dots\end{aligned}$$

where in the second line we performed a low-temperature expansion which must be kept to second order in ϕ in order to be of first order in the temperature, since $C = \langle \phi \phi \rangle = O(T)$. Furthermore, because of the $\phi \mapsto -\phi$ inversion symmetry, $\langle \phi \rangle = 0$. Inserting this into (2.149) we find within the spin-wave approximation

$$\langle \vec{S}(t, \mathbf{r}) \cdot \vec{S}(s, \mathbf{r}') \rangle - \langle \vec{S}(t, \mathbf{r}) \rangle \cdot \langle \vec{S}(s, \mathbf{r}') \rangle = C(t, s; \mathbf{r}, \mathbf{r}'). \quad (2.150)$$

Therefore, we have seen that the *angular* two-point correlator C is the same as the *connected magnetic* spin-spin correlator Γ_{conn} . An analogous result is well-known at equilibrium [392].

2.6.3 Two-time Responses

Next, we consider the response functions. For the angular response one has the standard form

$$R(t, s; \mathbf{r}, \mathbf{r}') = \langle \phi(t, \mathbf{r}) \tilde{\phi}(s, \mathbf{r}') \rangle. \quad (2.151)$$

It remains to consider the response of the spin vector $\mathbf{S}$ at time t and position $\mathbf{r}$ to some magnetic field $\mathbf{h}(s, \mathbf{r}')$ at time s and position $\mathbf{r}'$. From the definition (2.144) we have

$$\rho(t, s; \mathbf{r}, \mathbf{r}') = \lim_{\mathbf{h} \rightarrow \mathbf{0}} \left[\frac{\langle \langle \cos \phi(t, \mathbf{r}) \rangle \rangle - \langle \cos \phi(t, \mathbf{r}) \rangle}{h_1(s, \mathbf{r}')} + \frac{\langle \langle \sin \phi(t, \mathbf{r}) \rangle \rangle - \langle \sin \phi(t, \mathbf{r}) \rangle}{h_2(s, \mathbf{r}')} \right] \quad (2.152)$$

where the average $\langle \langle \cdot \rangle \rangle$ is to be taken with a magnetic field. Expanding the functional (2.147) to first order in both components h_1, h_2 of the magnetic field, one has

$$\begin{aligned} \langle \langle \cos \phi(t, \mathbf{r}) \rangle \rangle &= \langle \cos \phi(t, \mathbf{r}) \rangle + h_1(s, \mathbf{r}') \langle \cos \phi(t, \mathbf{r}) \sin \phi(s, \mathbf{r}') \tilde{\phi}(s, \mathbf{r}') \rangle, \\ \langle \langle \sin \phi(t, \mathbf{r}) \rangle \rangle &= \langle \sin \phi(t, \mathbf{r}) \rangle - h_2(s, \mathbf{r}') \langle \sin \phi(t, \mathbf{r}) \cos \phi(s, \mathbf{r}') \tilde{\phi}(s, \mathbf{r}') \rangle. \end{aligned}$$

By techniques similar to those used above, the response function can be expressed as [589]

$$\rho(t, s; \mathbf{r}, \mathbf{r}') = \left\langle \tilde{\phi}(s, \mathbf{r}') \sin(\phi(t, \mathbf{r}) - \phi(s, \mathbf{r}')) \right\rangle = R(t, s; \mathbf{r}, \mathbf{r}') \Gamma(t, s; \mathbf{r}, \mathbf{r}') \quad (2.153)$$

and we see explicitly that the relationship between R and ρ is non-trivial. That no higher correlators than the magnetic two-point correlation function Γ enter is a consequence of the Gaussian nature of the theory at hand.

For notational simplicity, we now list only the autocorrelator and autoreponse functions explicitly. First, the angular correlation function is

angular correlation and response				
	a	b	λ_C	λ_R
$d = 2$	0	0	0	2
$d > 2$	$d/2 - 1$	$d/2 - 1$	d	d
magnetic correlation and response				
	$a^{(\prime)}$	$b^{(\prime)}$	$\lambda_C^{(\prime)}$	$\lambda_R^{(\prime)}$
$d = 2$	$\eta_f/2$	$\eta_f/2$	$(\eta_i + \eta_f)/2$	$2 + (\eta_i + \eta_f)/2$
$d > 2$	$d/2 - 1$	$d/2 - 1$	d	d

Table 2.10. Ageing exponents of the XY model in $d \geq 2$ dimensions in the spin-wave approximation. Here $\eta_{i,f} = \eta(T_{i,f}) = T_{i,f}/(2\pi)$ describe the initial and final correlation exponents.

$$C(t, s) = \begin{cases} c_+ [(T_i - T_f)(t + s + \Lambda^2)^{1-d/2} + T_f(t - s + \Lambda^2)^{1-d/2}] & ; \text{ if } d > 2 \\ (4\pi)^{-1} [(T_f - T_i) \ln(t + s + \Lambda^2) - T_f \ln(t - s + \Lambda^2)] & ; \text{ if } d = 2 \end{cases} \quad (2.154)$$

with $c_+ := \frac{2(4\pi)^{-d/2}}{d-2}$ and from which the magnetic correlations are readily reconstructed (see exercises). Second, the autoresponse functions are given by

$$\begin{aligned} \rho(t, s) &= R(t, s) \left(1 + C(t, s) - \frac{1}{2} [C(t, t) + C(s, s)] \right), \\ R(t, s) &= [4\pi(t - s + \Lambda^2)]^{-d/2}. \end{aligned} \quad (2.155)$$

Ageing occurs when $T_i \neq T_f$ and when time-translation invariance is broken. In order to describe the scaling behaviour of angular and magnetic variables, we shall use the following non-standard notation. The exponents $a, b, \lambda_C, \lambda_R$ refer to the angular two-point functions C and R and the exponents and $a^{(\prime)}, b^{(\prime)}, \lambda_C^{(\prime)}, \lambda_R^{(\prime)}$ refer to the magnetic two-point functions Γ and ρ . From the above results, these exponents are straightforwardly read off in the ageing regime where $t, s, t - s \gg \Lambda$ and we collect the results in [Table 2.10](#).

We see that the exponents satisfy the equalities $a = b$ and $a^{(\prime)} = b^{(\prime)}$ expected for non-equilibrium critical dynamics and point out that in $2D$, the autocorrelation and autoresponse exponents are different: $\lambda_R - \lambda_C = \lambda_R^{(\prime)} - \lambda_C^{(\prime)} = 2$. Because of the non-disordered initial conditions (2.146) this is consistent with the relation eq. (1.128) between the autocorrelation and autoresponse exponents. In $d > 2$, however, these long-range correlations are no longer strong enough to modify the autocorrelation/autoresponse exponents.

2.6.4 Numerical Tests and Extensions

For an initially ordered state ($T_i = 0$) the results of this section, although only derived for small temperatures T_f , apparently describe well the entire

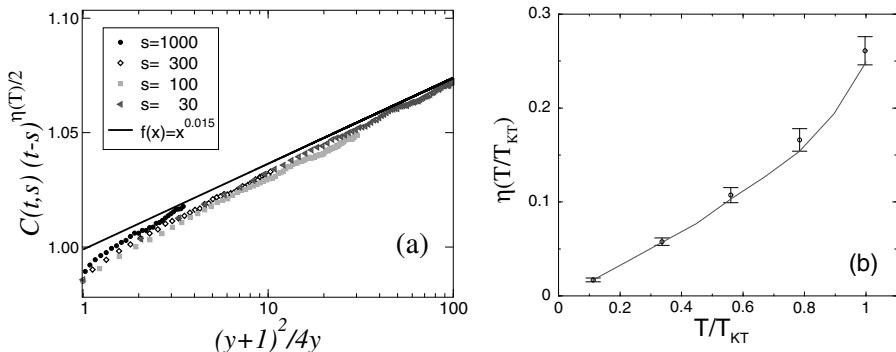


Fig. 2.2. (a) Test of the prediction (2.156), given by the straight black line, of spin-wave theory in the $2D$ XY-model for $T = 0.3$ and $\eta(0.3)/4 = 0.015$. (b) Comparison of the values of $\eta(T)$ determined from the scaling of $C(t, s)$ with the high-precision determination of [65] (full curve). Reprinted from [5]. With kind permission of the European Physical Journal (EPJ)

low-temperature phase of the $2D$ XY-model. In Fig. 2.2, we illustrate a test [5] of the magnetic autocorrelator [73]

$$C(t, s) = (t - s)^{-\eta(T)/2} \left[\frac{(t/s + 1)^2}{4t/s} \right]^{\eta(T)/4} \quad (2.156)$$

where $\eta(T)$ is the temperature-dependent equilibrium correlation exponent [450]. The form of (2.156) can be checked by plotting $C(t, s)(t - s)^{\eta(T)/2}$ over against $(y + 1)^2/(4y)$ where $y = t/s$ is the scaling variable. It is apparent from Fig. 2.2a that the data collapse nicely and are furthermore quite well described by eq. (2.156), although $T/T_{KT} \approx 0.3$ is not extremely small. The slope in this plot can be used to fix $\eta(T)$. Fig. 2.2b shows that the $\eta(T)$ so computed agree very well with the results of another high-precision method (at equilibrium, see appendix A) [65], for all $T \leq T_{KT}$.

These results have been recently re-confirmed [463] for $T \gtrsim 0.8T_c$. One generalises the spin-wave result (2.156) to

$$C(t, s) = L(s)^{-\eta} F_C(L(t)/L(s)) , \quad (2.157)$$

but because of the closeness to T_c , the leading corrections to scaling are taken into account by extending the growth law to include a correction term. One sets $L(t) \sim t^{1/2}(1 + c/t)^{-2/\eta}$ and $F_C(y) = y^{-\eta/2}(1 + c'y^{-2})$. From the spin-wave approximation one expects $c' = 4$, which is in good agreement with the numerical data. In Table 2.11, we list further numerical estimates of non-stationary exponents in the $2D$ XY model, obtained for both a fully disordered as well as a fully ordered state, as a function of the bath-temperature T [756].

If one adds to the Hamiltonian random frustrations, the model can be used to describe a vortex-glass phase transition which is of interest for the

	ordered				random			
T	0.9	0.89	0.8	0.7	0.9	0.89	0.8	0.7
z	2.00(2)	2.01(1)	2.00(1)	2.01(1)	2.00(2)	2.01(1)	2.00(1)	2.01(1)
η	0.246(3)	0.234(2)	0.176(2)	0.144(1)	0.28(2)	0.25(2)	0.21(2)	0.16(2)
λ_C/z					0.756(5)	0.738(4)	0.711(5)	0.695(6)
Θ					0.241(2)	0.249(2)	0.263(4)	0.280(4)

Table 2.11. Estimates of some equilibrium and non-equilibrium exponents in the 2D XY model, for both ordered and random initial spin configurations, as a function of the temperature T . The data are from [756].

T/J	Θ	λ_C/z	z	η	Ref.
0.60	0.355(5)	0.517(6)	2.29(3)	0.08(5)	[173]
0.68	0.314(2)	0.577(6)	2.24(2)	0.08(4)	[173]
		0.57(1)			[159]
0.76	0.297(2)	0.622(11)	2.18(3)	0.17(5)	[173]
0.84	0.273(6)	0.644(19)	2.18(6)	0.19(7)	[173]
0.92	0.254(5)	0.670(10)	2.16(4)	0.23(6)	[173]

Table 2.12. Estimates of some non-equilibrium and equilibrium critical exponents as a function of temperature in the 2D six-states clock model.

description of high- T_c superconductors [627]. We shall describe this briefly in Chap. 3.

Finally, if one quenches the 2D XY model from a fully *disordered* initial state to a temperature $T \leq T_{KT}$, one finds $L(t) \sim (t/\ln t)^{1/2}$ [95]. The ageing is described by the scaling forms $C(t, s) = (t - s)^{-\eta(T)/2} f_C(t \ln s / s \ln t)$ and $R(t, s) = (t - s)^{-1-\eta(T)/2} f_R(t \ln s / s \ln t)$ [5].

2.6.5 Comparison With the Clock Model

The two-dimensional p -state clock model undergoes for $p > 4$ two phase-transitions, at the critical temperatures T_1 and $T_2 > T_1$, but without a local order-parameter. Rather, both phases with $T \leq T_2$ show a behaviour quite similar to the 2D XY model. In Table 2.12, we illustrate this for the example $p = 6$, which shows the T -dependence of the four exponents Θ , λ_C , z and η across the low-temperature and the intermediate phases (recall the estimates $T_1 = 0.68(2)$ and $T_2 = 0.92(1)$ [137]).

2.6.6 Fluctuation-dissipation Relations in the XY Model

We now discuss the distance from equilibrium through the fluctuation-dissipation ratios [171]. Since we have seen that in the XY model angular and

magnetic observables behave quite differently, it is convenient to define two distinct fluctuation-dissipation ratios, namely

$$\begin{aligned}\Xi(t, s) &:= T_f \rho(t, s) \left(\frac{\partial \Gamma(t, s)}{\partial s} \right)^{-1} \\ X(t, s) &:= T_f R(t, s) \left(\frac{\partial C(t, s)}{\partial s} \right)^{-1}\end{aligned}\tag{2.158}$$

For magnetic variables, one has [589]

$$\frac{1}{\Xi(y)} = 1 + \left(1 - \frac{T_i}{T_f} \right) \left[\left(\frac{y-1}{y+1} \right)^{d/2} - \left(\frac{y-1}{2} \right)^{d/2} \right] \simeq \frac{T_i - T_f}{T_f} \left(\frac{y-1}{2} \right)^{d/2}\tag{2.159}$$

and therefore $\Xi_\infty = 0$, indeed a universal constant. It is not immediately obvious why the asymptotic value of $\Xi(y)$ should be independent of d and why it should agree with the value $\Xi_\infty = 0$ of phase-ordering kinetics. This kind of result should be more typical of a ferromagnet relaxing towards an ordered equilibrium state as it occurs for $d > 2$ but is not really expected for $d = 2$ since the equilibrium $2D$ XY model is critical even below T_{KT} .

On the other hand, for angular variables the fluctuation-dissipation ratio reads [589]

$$\frac{1}{X(y)} = 1 + \left(1 - \frac{T_i}{T_f} \right) \left(\frac{y+1}{y-1} \right)^{-d/2}\tag{2.160}$$

but, remarkably, its $y \rightarrow \infty$ limit

$$X_\infty = \left(2 - \frac{T_i}{T_f} \right)^{-1}\tag{2.161}$$

depends *continuously* on T_i/T_f . We recall from [Table 2.10](#) that the non-equilibrium exponents of the *angular* variables are all independent of both T_i and T_f and although the exponents do depend on d , we see from (2.161) that X_∞ does not. Taken literally, this would be an example of a non-universal value of the limit fluctuation-dissipation ratio.

2.7 OJK Approximation

We briefly consider a simple approximation scheme which is close in spirit to the classical Ohta-Jasnow-Kawasaki (OJK) approximation, since it readily gives another explicit result for the response function and furthermore allows us to illustrate some of the subtleties which one may encounter when analysing integrated responses, notably using the zero-field cooled protocol.

The OJK approximation [556] starts from the physical idea that for quenches to $T < T_c$ the domain walls should be rather sharp and therefore

it should be admissible to replace the order-parameter field $\phi(t, \mathbf{r})$ (which on a lattice takes the values ± 1) by a smooth auxiliary field $m(t, \mathbf{r})$. This can be achieved by using a non-linear function $\phi(m)$ with a ‘sigmoid’ shape, such as $\tanh m$. Following [93], consider the dynamics of the domain walls, as described by the Allen-Cahn equation $v = -(d-1)K = -\nabla \cdot \mathbf{n}$, where $\mathbf{n} = \nabla m / |\nabla m|$ is a unit vector normal to the wall and K is the curvature. Hence, one has for the normal velocity of the domain wall

$$v = \frac{-\nabla^2 m + n_a n_b \nabla_a \nabla_b m}{|\nabla m|}. \quad (2.162)$$

On the other hand, in a frame of reference co-moving with the interface, one has $0 = \frac{dm}{dt} = \partial_t m + \mathbf{v} \cdot \nabla m$ and since $\mathbf{v}$ and ∇m are parallel, $\mathbf{v} \cdot \nabla m = v |\nabla m|$, which gives for the normal velocity

$$v = -\frac{1}{|\nabla m|} \frac{\partial m}{\partial t}. \quad (2.163)$$

Eliminating the normal velocity, and also adding an external field, one arrives at an equation of motion for the auxiliary field at temperature $T = 0$ [70]:

$$\frac{\partial m}{\partial t} = \nabla^2 m - n_a n_b \nabla_a \nabla_b m + h |\nabla m|. \quad (2.164)$$

Concentrate now on the ZFC integrated response by turning on a spatially random field h , of magnitude h_0 , at time s . Then the zero-field-cooled (ZFC) susceptibility reads [70]

$$\chi_{\text{ZFC}}(t, s) = \frac{\langle h(\mathbf{r}) \phi(t, s; \mathbf{r}) \rangle}{h_0^2} \simeq \sqrt{\frac{2}{\pi}} \frac{\langle h(\mathbf{r}) m(t, s; \mathbf{r}) \rangle}{h_0^2 \sqrt{\langle m^2 \rangle}} \quad (2.165)$$

where ϕ is the order-parameter. Here, it is assumed that for late times, one can approximate $\phi \sim \text{sign}(m)$. The OJK-approximation consists of making two further simplifications. Namely, one performs (i) a circular average, which amounts to the replacement $n_a n_b \mapsto \delta_{ab}/d$ and (ii) one replaces $|\nabla m| \mapsto \langle (\nabla m)^2 \rangle^{1/2}$. Then one arrives at the OJK equation of motion, which constitutes the model and in our context becomes [556, 93, 70]

$$\partial_t m = D \nabla^2 m + h \langle (\nabla m)^2 \rangle^{1/2} \quad (2.166)$$

with $D = (d-1)/d$. With the abbreviation $\mathbf{a}(t) := \langle (\nabla m)^2 \rangle^{1/2}$, the solution of this equation reads in Fourier space

$$\widehat{m}(t, s; \mathbf{k}) = \widehat{m}(0, \mathbf{k}) + \widehat{h}(\mathbf{k}) \int_s^t dt' \mathbf{a}(t') e^{-D \mathbf{k}^2 (t-t')} \quad (2.167)$$

with random initial conditions. Self-consistency may be used to show that for large times $\mathbf{a}(t) \sim t^{-(d+2)/4}$. From all this, the ZFC susceptibility is readily found [70]

$$\begin{aligned}
\chi_{\text{ZFC}}(t, s) &= \int_s^t du \frac{(Dt)^{d/4}}{(Du)^{(d+2)/4}} \int_{\mathbb{R}^d} d\mathbf{k} e^{-k^2 D(t-u)} \\
&= \text{constant} \cdot \int_s^t du u^{-(d+1)/2} \left(\frac{t}{u}\right)^{d/4} \left(\frac{t}{u} - 1\right)^{-d/2} \\
&\stackrel{!}{=} \int_s^t du R(t, u).
\end{aligned} \tag{2.168}$$

This integral becomes singular near the upper limit $u \approx t$ and we shall analyse this below. Before we shall do this, we read off the autoresponse function

$$R(t, u) = u^{-(d+1)/2} f_{\text{OJK}}(t/u) \quad , \quad f_{\text{OJK}}(y) = f_0 y^{d/4} (y-1)^{-d/2} \tag{2.169}$$

and identify the ageing exponents of the OJK approximation, along with $z = 2$,

$$a_{\text{OJK}} = \frac{d-1}{2} \quad , \quad a'_{\text{OJK}} = \frac{d-2}{2} \quad , \quad \lambda_{R,\text{OJK}} = \frac{d}{2}. \tag{2.170}$$

From an analysis of the two-time correlations, it follows further that $\lambda_{C,\text{OJK}} = d/2$ and $b_{\text{OJK}} = 0$ [93]. The ageing exponent $a' \neq a$ will be defined in Chap. 4, where it will become important for the interpretation of these results in terms of local scale-invariance. Eq. (2.169) is also obtained in the Gaussian theory of phase-ordering [511].

However, this was not the conclusion reached in [70], which rather quote $\chi_{\text{ZFC}}(t, s) \sim s^{-1/2} F(t/s)$ for $d > 2$ which would mean $a = 1/2$, provided of course that the scaling law $\chi_{\text{ZFC}}(t, s) = s^{-a} f_\chi(t/s)$ could be used in a straightforward way. In order to understand the true scaling of χ_{ZFC} , we reconsider the singularity. Following [70], one introduces a cut-off parameter Λ^2 (which should be sent to zero at the end) and writes instead of (2.168)

$$\begin{aligned}
\chi_{\text{ZFC}}(t, s) &= \int_s^t du \frac{(Dt)^{d/4}}{(Du)^{(d+2)/4}} \int_{\mathbb{R}^d} d\mathbf{k} e^{-k^2 D(t-u+\Lambda^2)} \\
&\sim t^{(1-d)/2} \int_{s/t}^1 dv v^{-(d+2)/4} (1-v+\Lambda^2/t)^{-d/2}
\end{aligned} \tag{2.171}$$

and one must analyse the contribution of the integrand near $v \approx 1$. We decompose the domain of integration $\int_{s/t}^1 = \int_{1-\epsilon}^1 + \int_{s/t}^{1-\epsilon}$. In the first factor of the first term, we may set $v \simeq 1$ and in the second term, we let $\Lambda^2 \rightarrow 0$. Then, for $d > 2$ it follows⁸ that [346]

$$\chi_{\text{ZFC}}(t, s) \simeq s^{-1/2} \cdot \chi_\infty(t/s)^{-1/2} + s^{-a} f_\chi(t/s) \tag{2.172}$$

where the value $a = (d-1)/2$ as required from (2.170) was used, χ_∞ is a constant and f_χ is a scaling function. Rather than a simple conventional scaling behaviour with $a = 1/2$, we see that we have indeed *two* scaling terms. By

⁸ In $d = 2$ a similar argument produces logarithmic corrections.

analogy with our previous discussion of the scaling of the ZFC susceptibility, see eq. (1.102), the first term, of order $\sim s^{-1/2}$ and with an associated trivial scaling function, dominates for large times but should be considered as coming from the singular integration while it is rather the second term, of order $\sim s^{-(d-1)/2}$, which reflects the non-trivial ageing scaling behaviour. Consequently, ageing exponents should be read from this second, non-dominant term.

This example again illustrates the potentially difficult interpretation of $\chi_{\text{ZFC}}(t, s)$ and underlines that the mere observation of a dynamical scaling behaviour in this quantity is not always sufficient to meet this end.

2.8 Further Solvable Models

Another solvable system is the **SOS model** (solid on solid) which can be defined in terms of a discrete height variable $n_i \in \mathbb{Z}$ by the Hamiltonian

$$\mathcal{H} = \sum_{(i,j)} (n_i - n_j)^2 . \quad (2.173)$$

At equilibrium, the model has a roughening transition at a temperature T_r , separating a float phase at low T from a (logarithmically) rough phase above T_r . In the continuum limit, this reduces to a free field-theory and the resulting correlation and response functions can be read off from the previous Sections [171]. We leave this as an exercise.

Other soluble models with simple ageing behaviour include models of ballistic aggregation [261, 257, 259], or the non-equilibrium critical dynamics of the spherical model with a conserved order-parameter or the bosonic contact and pair-contact processes, see Chap. 3.

A much-studied solvable non-equilibrium model is the **zero-range process**, with a spatial structure quite distinct from the models considered in this Chapter. Depending on the models' parameters, particles may condensate in momentum space. See e.g. [16, 232, 288, 283, 312, 649, 302, 369] for an entry into the very rich literature.

Problems

2.1. Consider the 1D Glauber-Ising model, as defined by the heat-bath rule (2.1). Show that the time-dependent magnetisation $M_n(t) := \langle \sigma_n(t) \rangle$ satisfies the equation of motion

$$\frac{dM_n(t)}{dt} = -M_n(t) + \langle \tanh h_n(t)/T \rangle = -M_n(t) + \frac{\gamma}{2} (M_{n-1}(t) + M_{n+1}(t)) \quad (2.174)$$

where $\gamma = \tanh(2J/T)$. In order to solve this equation, consider first the initial condition $M_n(0) = \delta_{n,0}$. This solution is called the **Green's function** and will be denoted by $G_n(t)$ throughout these exercises. Use $G_n(t)$ to solve (2.174) for an arbitrary initial distribution $M_n(0)$.

Derive similarly the equations of motion for the single-time correlator $C_{n-m}(t) := \langle \sigma_n(t) \sigma_m(t) \rangle$ and for the two-time correlator $C_{n-m}(t, s) := \langle \sigma_n(t) \sigma_m(s) \rangle$ (in both cases, spatial translation-invariance is assumed).

2.2. Use detailed balance in order to determine the constant γ in the Glauber transition rates (2.4).

2.3. Consider a kinetic Ising model with single-spin-flip dynamics described by rates $w_j(\sigma_j)$. Prove (2.6).

2.4. Use Glauber's transition rates (2.4) and the relation (2.6) to re-derive the equation of motion (2.174) for the time-dependent magnetisation $M_n(t)$.

2.5. Find the time-dependent magnetisation $M_n(t)$ in the 1D Glauber-Ising model on an infinite chain where one spin is fixed, e.g. $M_0(t) = 1$ [281].

2.6. Use the result of the previous exercise 2.5 to find the two-point correlation function $C_n(t)$ for an arbitrary spatially translation-invariant initial distribution $C_n(0)$ [281].

Discuss the relaxation towards equilibrium. Can one define a dynamical exponent z ?

2.7. Consider the two-time correlation function $C_{n-m}(t, s) = \langle \sigma_n(t) \sigma_m(s) \rangle$ in the 1D Glauber-Ising model. If the system is initially at thermal equilibrium with some finite temperature T (what does this mean for the correlation functions ?), check that $C_{n-m}(t, s) = C_{n-m}(t - s)$ is time-translationally invariant [281].

2.8. Include a local and time-dependent magnetic field $H_n(t)$ into the Ising model Hamiltonian. Derive Glauber transition rates which are compatible with detailed balance. Find the equations of motion, together with any required boundary conditions, for the two-time response function $R_{n,m}(t, s)$.

2.9. Consider the 1D Glauber-Ising model quenched to temperature $T = 0$ from a fully disordered initial state. Derive the value of the global persistence exponent θ_g , by using Slepian's formula eq. (1.131)[493, 165].

2.10. Consider the kinetic Ising model with the rates (2.49) and the KDH condition $\gamma = 2\delta$. Derive the coupled set of equations of motion for the averages of the spin variable σ and the three-spin $q_n = \sigma_{n-1} \sigma_n \sigma_{n+1}$.

2.11. Generalise the result of the previous exercise 2.10 and find the equations of motion for the averages $\langle \sigma_n \sigma_m \rangle$, $\langle \sigma_n q_m \rangle$ with $|n - m| > 1$ and $\langle q_n q_m \rangle$ with $|n - m| > 2$ for the 1D Ising model with KDH dynamics [219]. Do you obtain a closed system of equations?

2.12. Derive the equations of motion (2.68) for the global two-point correlators $\hat{C}^{gf}(t)$ for KDH dynamics [219].

2.13. Consider a 1D kinetic Ising model with the transition rates

$$w_n(\sigma_n, \sigma_{n+1}) = \alpha(1 - \sigma_n \sigma_{n+1}) \left(1 - \frac{\gamma}{2}(\sigma_{n-1} \sigma_n + \sigma_{n+1} \sigma_{n+2})\right). \quad (2.175)$$

Because of the first factor, these rates describe a **spin-exchange dynamics** rather than a spin-flip dynamics as in the Glauber-Ising model. Detailed balance implies that $\gamma = \tanh 2J/T$. Analyse the linear response to a small external field and derive a continuity equation for the space-time-dependent magnetisation. Deduce the value of the dynamical exponent z [762].

2.14. Consider the kinetic spherical model. As a preparation to the derivation of the long-time behaviour of the solutions of the Volterra integral equation, prove eq. (2.102).

2.15. Consider the spherical model quenched to $T < T_c$ from a fully disordered initial state. Show explicitly how the singular terms in eq. (2.105) contribute to the scaling of the autocorrelator $C(t, s)$.

2.16. An estimate of the time-dependent domain size $L(t)$ in the kinetic spherical model can be obtained from the second moment of the single-time correlation function

$$L^2(t) := \lim_{|A| \rightarrow \infty} \frac{\sum_{\mathbf{r} \in A} |\mathbf{r}|^2 C(t, \mathbf{r})}{\sum_{\mathbf{r} \in A} C(t, \mathbf{r})} = - \left. \frac{\partial^2 \hat{C}(\mathbf{q}, t) / \partial q^2}{\hat{C}(\mathbf{q}, t)} \right|_{\mathbf{q}=0}. \quad (2.176)$$

Derive the long-time behaviour of $L(t)$ and show that for all temperatures $T \leq T_c$ and for all dimensions $d > 2$, one has $L(t) \sim t^{1/z}$ without additional logarithmic correction factors and $z = 2$ [224].

2.17. Calculate the global persistence exponent θ_g , by using Slepian's formula eq. (1.131), in the spherical model quenched to either $T < T_c$ or $T = T_c$, from a totally disordered initial state (neglect fluctuations in the Lagrange multiplier $\mathfrak{z}(t)$).

2.18. Consider the Langevin equation of the spherical model and add an additional spatially random magnetic field $h_{\mathbf{x}}$, with zero mean and the variance $\overline{h_{\mathbf{x}} h_{\mathbf{y}}} = 2\Gamma \delta_{\mathbf{x}, \mathbf{y}}$. Show directly that on a lattice A with N sites the field-cooled susceptibility can be written as [339]

$$\chi_{\text{FC}}(t) = \frac{1}{\mathcal{N}} \frac{1}{2\Gamma} \overline{\left\langle \sum_{\mathbf{x} \in A} S_{\mathbf{x}}(t) h_{\mathbf{x}} \right\rangle}.$$

2.19. Consider the mean spherical model in $2 < d < 4$ dimensions and in an external magnetic field H . Show that the model can be used to describe a magnetic analogue of the Kovacs effect. Prepare the system, at a fixed temperature $T < T_c$ and with a magnetic field H such that (i) the mean magnetisation $\langle m \rangle = m_{\text{eq}}(T, H)$ is equal to its equilibrium thermodynamic average but (ii) spins on different sites are uncorrelated. Calculate the time-dependent magnetisation $m(t)$ and give a heuristic explanation [570].

2.20. Consider the XY model in the spin-wave approximation. First, using eq. (2.148) with the initial conditions (2.146), show that a correlation function $\langle \phi \phi \rangle$ is of order $O(T_i, T_f)$ whereas a response function $\langle \phi \tilde{\phi} \rangle$ is of order $O(1)$ in the initial and final temperatures. Next, consider the first term beyond the spin-wave approximation in the Hamiltonian

$$\mathcal{H}[\phi] = \mathcal{H}_0 + \frac{1}{2} \int d\mathbf{r} \left[(\nabla \phi(\mathbf{r}))^2 + g_4 (\nabla \phi(\mathbf{r}))^4 \right]$$

where g_4 is a constant. Estimate that to first order in g_4 , the corrections to the spin-wave approximations are of the order

$$\delta C(t, s; \mathbf{r}, \mathbf{r}') \sim O \left(\left(\frac{T_i}{T_c} \right)^2, \left(\frac{T_f}{T_c} \right)^2 \right)$$

where T_c is the critical temperature. Consequently, a spin-wave approximation is exact to first order in the initial and final temperatures T_i, T_f .

2.21. Consider the 2D XY-model in the spin-wave approximation, with $T_i = 0$ and $T_f = T$. Prove the following expressions [73] for the magnetic autocorrelation function

$$\Gamma(t, s) = \langle \vec{S}(t, \mathbf{r}) \cdot \vec{S}(s, \mathbf{r}) \rangle = \left(\frac{\Lambda^4 (t + s + \Lambda^2)^2}{(2t + \Lambda^2)(2s + \Lambda^2)(t - s + \Lambda^2)^2} \right)^{\eta(T)/4}$$

and the magnetic autoresponse

$$\begin{aligned} \rho(t, s) &= \frac{1}{4\pi(t - s + \Lambda^2)} \left(\frac{\Lambda^4 (t + s + \Lambda^2)^2}{(2t + \Lambda^2)(2s + \Lambda^2)(t - s + \Lambda^2)^2} \right)^{\eta(T)/4} \\ &= \frac{\Gamma(t, s)}{4\pi(t - s + \Lambda^2)}. \end{aligned}$$

Numerical studies of the 2D XY-model with an ordered initial state ($T_i = 0$) and $T_f < T_c$ indicate that these expressions remain very accurate almost up to $T = T_{\text{KT}}$ [5]. Generalise for arbitrary T_i [633].

2.22. Consider the 1D XY model quenched to zero temperature and with a non-conserved order-parameter. Use the Hamiltonian eq. (2.140) to analyse the dynamical scaling behaviour of the two-time angular correlation function $C(t, s; r)$ of eq. (2.145), where spatial translation-invariance is assumed. Does one always recover the conventional dynamical scaling of simple ageing ? Discuss this for different choices of initial conditions [537, 93].

2.23. In d spatial dimensions, the **voter model** is defined in terms of configurations $\{\sigma\}$ of Ising spins $\sigma_{\mathbf{r}} = \pm 1$, and a master equation with the transition rates

$$w_{\mathbf{r}}(\sigma_{\mathbf{r}}) = \frac{\alpha}{2} \left[1 - \frac{\gamma}{2d} \sigma_{\mathbf{r}} \sum_{j=1}^d (\sigma_{\mathbf{r}+\mathbf{e}_j} + \sigma_{\mathbf{r}-\mathbf{e}_j}) \right], \quad (2.177)$$

where $\mathbf{e}_j$ is the unit vector in direction j and γ, α are constants. Does this model satisfy detailed balance ? Derive the equation of motions and analyse their solutions, see e.g. [260, 206, 207, 126, 315, 316].

Chapter 3

Simple Ageing: an Overview

We now review several situations where simple ageing occurs. We shall emphasise the scaling description and the extraction of universal exponents, amplitudes and scaling functions.

3.1 Non-equilibrium Critical Dynamics

3.1.1 Purely Relaxational Dynamics (Model A)

When considering the non-equilibrium behaviour of a system quenched *onto* its critical point, one of the main differences with phase-ordering, which we mainly considered up to now, is that there is no surface tension. Although the initial disorder will no longer spawn ordered clusters, it still creates correlations whose typical size grows with time. In the case of a non-conserved order-parameter (**model A** dynamics in the terminology of [370]), one may write down the Langevin equation

$$\partial_t \phi(t, \mathbf{r}) = -D \frac{\partial \mathcal{H}[\phi]}{\delta \phi(t, \mathbf{r})} + \eta(t, \mathbf{r}) \quad (3.1)$$

where $\mathcal{H}$ is the equilibrium Ginzburg-Landau functional and η is a centred Gaussian noise with variance $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2DT_c \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$.

In Fig. 3.1 we illustrate the growth of correlated (but not ordered) clusters in the critical 2D Ising model, where the dynamics was created by a heat-bath algorithm. In contrast with phase-ordering, for which typical configurations have been shown in Fig. 1.6, we see that fluctuations occur inside the growing clusters. Therefore, at least for conventional phase transitions, the single relevant length scale should be the time-dependent correlation length

$$\xi(t) \sim t^{1/z} . \quad (3.2)$$

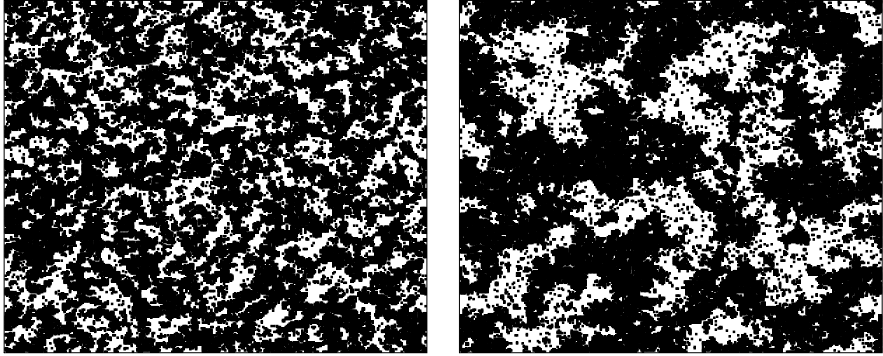


Fig. 3.1. Two snapshots of spin configurations (black and white) of the 2D Ising model at $T = T_c$, taken at times (a) $t = 25$ and (b) $t = 275$ Monte Carlo steps after the quench.

It follows that the dynamical exponent z is already determined by the equilibrium critical properties of the system. For *non-conserved* dynamics (model A), the value of the dynamical exponent $z = z(T_c)$ is non-trivial. Precise determinations have been obtained in exactly solved models, from field-theoretical renormalisation group studies or from numerical simulations, and numerical values are listed in Table 1.7. In $4 - \varepsilon$ dimensions, the compact formula $z = 2 + (6 \ln \frac{4}{3} - 1)\eta + O(\varepsilon^3)$ holds for the $O(n)$ -model [759, 253], where η is an equilibrium critical exponent (see Chap. 2 in Volume 1 and Chap. 1 in this Volume).

If $\xi = \xi(t)$ is indeed the only physically relevant length-scale, one expects that the dynamical scaling behaviour eqs. (1.74, 1.75) should still be valid. Since now $a = b = (d - 2 + \eta)/z$, the breaking of the fluctuation-dissipation theorem, see (1.8), is described by the fluctuation-dissipation ratio [171]

$$X(t, s) := TR(t, s) \left(\frac{\partial C(t, s)}{\partial s} \right)^{-1} = \mathfrak{X}(t/s) \quad (3.3)$$

where the last relation is valid only in the scaling regime and provided that $a = b$. In particular, the limit fluctuation-dissipation ratio

$$X_\infty := \lim_{s \rightarrow \infty} \left(\lim_{t \rightarrow \infty} X(t, s) \right) = \lim_{y \rightarrow \infty} \mathfrak{X}(y) \quad (3.4)$$

is expected to be universal [286]. This expectation has been generally confirmed. Values of X_∞ and more can be found in Table 1.7.

At the critical point, field-theoretical methods can be used in order to calculate the ageing exponents, response and correlation functions and the fluctuation-dissipation ratio, including two-loop effects [118, 121]. In particular, one considers the following space-dependent fluctuation-dissipation ratio $X(t, s; \mathbf{r}) = TR(t, s; \mathbf{r}) / \partial_s C(t, s; \mathbf{r})$ with $X_\infty = \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} X(t, s; \mathbf{0})$.

Since field-theoretical calculations are best carried out in momentum space, one introduces the analogous definitions in momentum space

$$\mathcal{X}_{\mathbf{q}}(t, s) = \frac{T \widehat{R}_{\mathbf{q}}(t, s)}{\widehat{C}_{\mathbf{q}}(t, s)} \quad , \quad \mathcal{X}_{\infty} := \lim_{s \rightarrow \infty} \left(\lim_{t \rightarrow \infty} \mathcal{X}_{\mathbf{0}}(t, s) \right). \quad (3.5)$$

Obviously, $\mathcal{X}_{\mathbf{q}}(t, s)$ is not the Fourier transform of $X(t, s; \mathbf{r})$, but the limit ratios in direct and in momentum space can be related as follows [117]

$$\begin{aligned} \frac{1}{X(t, s; \mathbf{0})} &= \frac{\int d\mathbf{q} \partial_s \widehat{C}_{\mathbf{q}}(t, s)}{T \int d\mathbf{q} \widehat{R}_{\mathbf{q}}(t, s)} \\ &= \frac{\int d\mathbf{q} \widehat{R}_{\mathbf{q}}(t, s) (\partial_s \widehat{C}_{\mathbf{q}}(t, s) / \widehat{R}_{\mathbf{q}}(t, s))}{T \int d\mathbf{q} \widehat{R}_{\mathbf{q}}(t, s)} = \left\langle \frac{1}{\mathcal{X}_{\mathbf{q}}(t, s)} \right\rangle_{\widehat{R}_{\mathbf{q}}} \end{aligned} \quad (3.6)$$

and $1/X(t, s; \mathbf{0})$ can be seen as the average of $1/\mathcal{X}_{\mathbf{q}}(t, s)$ with weight $\widehat{R}_{\mathbf{q}}(t, s)$. If $\widehat{R}_{\mathbf{q}}(t, s)$ has a strong peak around $\mathbf{q} = \mathbf{0}$ with a variance which vanishes for $t \gg s \rightarrow \infty$ (this is often the case for critical systems) the main contribution to the integral comes from the region where $|\mathbf{q}|$ is small, such that the universal limit fluctuation-dissipation ratio

$$X_{\infty} = \mathcal{X}_{\infty} \quad (3.7)$$

is indeed the same in direct and momentum space [117]. Indeed, explicit calculations in the Cardy-Ostlund model [642, 643] (see p. 167) have confirmed eq. (3.7), even if the argument leading to (3.7) is no longer applicable in these models. We shall see later when considering fully magnetised initial states that counter-examples to (3.7) may be found [20, 217]. In what follows, we shall admit that (3.7) is valid, unless explicitly stated otherwise, but the reader should be aware that field-theoretical calculations usually find $\mathcal{X}_{\infty}$ and that X_{∞} must afterwards be obtained from eq. (3.7).

In $4 - \varepsilon$ dimensions, the two-loop estimate of the limit fluctuation-dissipation ratio is for the $O(n)$ -model [118]

$$\frac{1}{X_{\infty}} = 2 + \frac{n+2}{2(n+8)}\varepsilon + \frac{n+2}{(n+8)^2} \left[\frac{n+2}{4} + \frac{3}{2} \frac{3n+14}{n+8} + \mathbf{c}_2 \right] \varepsilon^2 + O(\varepsilon^3) \quad (3.8)$$

with a numerical constant $\mathbf{c}_2 = -0.0415\dots$. The numerical predictions extracted from this are in very good agreement with simulational studies or exact solutions in many specific models, see Table 1.7 and [121, 142]. Explicit expressions for the scaling functions of order-parameter responses and correlations have also been derived [118, 121].¹

¹ We limit ourselves here to the discussion of the order-parameter and refer to [121] for the many results for other physical observables (energy density, ...) for model A non-equilibrium critical dynamics. We remark that the notation used in many field-theoretical studies is different from ours. For example, one has $a_{\text{CG}} = (2 - \eta - z)/z$ and $\theta'_{\text{CG}} = a_{\text{CG}} + \theta_{\text{CG}} = \Theta = (d - \lambda_C)/z$, where the index CG refers to the exponents as used in [121].

3.1.2 Conserved Energy-density (Model C)

The non-equilibrium critical dynamics for a conserved order-parameter (model B) will be treated in a separate Section. We now briefly consider the case of a non-conserved order-parameter coupled to a conserved energy-density (**model C** in the terminology of [370]), quenched to $T = T_c$. The dynamics is now given by a pair of coupled Langevin equations

$$\partial_t \phi(t, \mathbf{r}) = -D \frac{\partial \mathcal{H}[\phi, \varepsilon]}{\delta \phi(t, \mathbf{r})} + \eta_\phi(t, \mathbf{r}) \quad , \quad \partial_t \varepsilon(t, \mathbf{r}) = D' \nabla_{\mathbf{r}}^2 \frac{\partial \mathcal{H}[\phi, \varepsilon]}{\delta \varepsilon(t, \mathbf{r})} + \eta_\varepsilon(t, \mathbf{r}) \quad (3.9)$$

with kinetic coefficients D, D' . $\mathcal{H}[\phi, \varepsilon] = \mathcal{H}[\phi] + \frac{1}{2} \int d\mathbf{r} (\varepsilon^2 + g_0 \varepsilon \phi^2)$ is the Ginzburg-Landau functional which extends the usual functional $\mathcal{H}[\phi]$ for short-ranged interactions in the $O(n)$ model and with a coupling constant g_0 . Finally, $\eta_{\phi, \varepsilon}$ are centred Gaussian noises with second moments

$$\begin{aligned} \langle \eta_\phi(t, \mathbf{r}) \eta_\phi(t', \mathbf{r}') \rangle &= 2DT_c \delta(t - t') \delta(\mathbf{r} - \mathbf{r}') \\ \langle \eta_\varepsilon(t, \mathbf{r}) \eta_\varepsilon(t', \mathbf{r}') \rangle &= -2D'T_c \nabla_{\mathbf{r}}^2 \delta(t - t') \delta(\mathbf{r} - \mathbf{r}') \end{aligned} \quad (3.10)$$

such that the equilibrium distribution is $\exp(-\mathcal{H}[\phi, \varepsilon])$, up to normalisation. Addressing the question whether the long-time dynamics described by the system (3.9, 3.10) is really different from the case of a purely dissipative dynamics (model A), it turns out [759, 252] that the corresponding renormalisation group fixed point is only stable if the specific heat exponent at equilibrium $\alpha > 0$. Then the dynamical exponent is given exactly in terms of standard equilibrium exponents by $z = 2 + \alpha/\nu$. In $4 - \varepsilon$ dimensions, one has to leading order for the $O(n)$ model $\alpha = \frac{\varepsilon}{2}(4 - n)/(n + 8)$ which would imply model C behaviour for $n < 4$ but more precise studies showed that in fact in $d = 3$ dimensions, $\alpha < 0$ for $n \geq 2$. Hence only the scalar $n = 1$ (Ising) case will be described by the model C fixed point for $d = 3$. In $2D$, the q -state Potts models have $\alpha > 0$ for $q > 2$ and are good candidates for studying non-equilibrium critical dynamics of model C type.

We list some results for the $O(n)$ -model in $4 - \varepsilon$ dimensions [116]. The slip exponent is, to one-loop order,

$$\Theta = \frac{n^2 - 7n + 9}{(n - 2)(n + 8)} \varepsilon + O(\varepsilon^2) \quad (3.11)$$

from which $\lambda_C = \lambda_R = d - \Theta z$ can be found. The limit fluctuation-dissipation ratio is

$$X_\infty = \frac{1}{2} \left[1 + \frac{4 - n}{n + 8} \left[\frac{n - 1}{(4 - n)(2 - n)} + \frac{n(2 - n)}{4(n - 1)^2} \ln(n(2 - n)) \right] \varepsilon \right] + O(\varepsilon^2) \quad (3.12)$$

which for $n = 1$ reduces to $X_\infty = \frac{1}{2}(1 - \varepsilon/12) + O(\varepsilon^2)$. Explicit scaling functions have been obtained as well.

3.1.3 Effects of Initial Long-range Correlations

The effects of initial long-range correlations on the ageing behaviour with a non-conserved order-parameter have been studied recently [449]. In this study, the ageing behaviour of the 2D Ising model is considered, evolving through a standard heat-bath algorithm, but starting from initial states with long-range correlations. As initial states, they use the critical state of either the Baxter-Wu model or else the critical three-states or four-states Turban model, see appendix A. Since these models are all self-dual, they have the *same* critical point. The well-known equilibrium behaviour of both the Baxter-Wu model and the three-states Turban model are in the same universality class as the four-states Potts model. However, their non-equilibrium properties are different, see Table 1.7. The four-states Turban model has a first-order transition at equilibrium and its relaxation is best described in terms of stretched exponentials [600]. Analysing the scaling behaviour of the autocorrelation function with initial states from these distinct models at their equilibrium critical point therefore allows us to test the role of global symmetries on the ageing behaviour. Simulations of the autocorrelator in the non-conserved 2D critical Ising model give a nice power-law scaling behaviour and lead to the numerical estimates, according to the different initial conditions [449],

$$\Theta = \begin{cases} 0.17(1) & ; \text{Baxter-Wu} \\ 0.165(10) & ; \text{Turban-3} \\ 0.475(10) & ; \text{Turban-4} \end{cases}, \quad \frac{\lambda_C}{z} = \begin{cases} 0.18(1) & ; \text{Baxter-Wu} \\ 0.18(1) & ; \text{Turban-3} \end{cases}. \quad (3.13)$$

From these results, one might conjecture the scaling relation $\Theta = \lambda_C/z$ for non-equilibrium critical dynamics with initial long-ranged correlations [449] but further confirmations would of course be needed. In any case, these results already show that the scaling relation (1.112) [404, 402] *cannot* be extended beyond the case of short-ranged initial conditions for which it was derived.

3.2 Ordered Initial States

In most discussions on ageing it is assumed that the initial state is fully disordered, and only the case of an initially vanishing magnetisation is usually considered (sometimes referred to as **down-quench** in the literature). In many experiments, however, the given initial state can be the ordered one before the temperature is suddenly raised (**up-quench**). Alternatively, one may place the system into an external magnetic field before the quench. In the following we shall look into this situation where the initial magnetisation is not vanishing.

3.2.1 Scaling Theory

To be specific, we shall limit ourselves to the $O(n)$ -model with a non-conserved order-parameter (model A). Theoretical studies have so far been limited to field-theoretical studies at $T = T_c$, sometimes controlled by numerical simulations [271, 123, 122, 235, 579], or to the exact solutions of the spherical model [19] or the XY model in spin-wave approximation [633, 73, 589].² The usual field-theoretical setup (see appendix D) for model A dynamics can be reused and the effect of a macroscopic initial condition is taken into account by adding an extra term

$$\mathcal{H}_0[\phi] = \int d\mathbf{r} \frac{\tau_0}{2} [\phi_0(\mathbf{r}) - m_0]^2 \quad (3.14)$$

to the Ginzburg-Landau functional $\mathcal{H}$, where $\phi_0(\mathbf{r}) := \phi(0, \mathbf{r})$ is the initial coarse-grained order-parameter and m_0 stands for the macroscopic initial magnetisation. By assumption, the correlations of this initial state are Gaussian short-ranged with a width proportional to $1/\tau_0$ and the average $M_0 = \langle \phi(0, \mathbf{r}) \rangle$, which for simplicity is assumed to be spatially constant. In what follows, we shall restrict ourselves to the scalar case $n = 1$ unless stated otherwise.

The starting point for a field-theoretical analysis is the Gaussian approximation. In this approximation, the two-time response and (connected !) correlation functions in momentum space read [123]

$$\begin{aligned} \hat{R}_{\mathbf{q}}(t, s) &= \exp \left[-D \left(\mathbf{q}^2(t-s) + \int_s^t dt' m^2(t') \right) \right] \\ \hat{C}_{\mathbf{q}}(t, s) &= 2D \int_0^\infty dt' \hat{R}_{\mathbf{q}}(t, t') \hat{R}_{\mathbf{q}}(s, t') \end{aligned} \quad (3.15)$$

where D is the diffusion constant and $m(t)$ is the time-dependent magnetisation the scaling of which was already given in (1.118, 1.119). The equation of motion of the magnetisation, including the one-loop terms, is for a ϕ^4 theory, at the critical point [123]

$$\frac{dm(t)}{dt} + \frac{D}{3} m^3(t) + \frac{Dg}{2(2\pi)^d} \int d\mathbf{q} \hat{C}_{\mathbf{q}}(t, t) + O(g^2) = 0, \quad (3.16)$$

where g is the coupling constant of the interaction term. These mean-field results are equivalent to the solution of the Glauber-Ising model for either (i)

² For quenches to $T < T_c$, ageing behaviour is only expected for systems with a higher symmetry such that several equivalent competing ground states can be reached even if the starting point is close to one of the minima of the equilibrium free energy. That is not the case for models with a discrete global symmetry such as Ising/Potts models.

a fully connected lattice with an effective infinite range of the interactions or (ii) the limit of large dimension $d \rightarrow \infty$ [271].

Turning to a scaling analysis, one considers the generalised correlator $G_{n,\tilde{n}}^m := \langle \phi^n \tilde{\phi}^{\tilde{n}} \tilde{\phi}_0^m \rangle$ and by extending the scaling analysis already carried out for the time-dependent magnetisation and using eqs. (1.114, 1.115), one has at the critical point [123]

$$G_{n,\tilde{n}}^m(\{t, \mathbf{r}\}; m_0) = \ell^{\delta(n,\tilde{n},m)} G_{n,\tilde{n}}^m(\{\ell^z t, \ell \mathbf{r}\}; \ell^{-\beta/\nu - \Theta z} m_0) \quad (3.17)$$

where $\{t, \mathbf{r}\}$ is meant as a short-hand for all time and space coordinates of the fields averaged over and m_0 denotes the initial value of the magnetisation. From this, the asymptotic limits of the cross-over between the small-time regime, where the scaling functions must depend linearly on m_0 , and the long-time regime which should be independent of m_0 , is readily obtained. The non-trivial new element is that the (connected) autocorrelation and autoresponse exponents $\lambda_C = \lambda_C(T_c)$ and $\lambda_R = \lambda_R(T_c)$ are no longer independent, but rather satisfy the remarkable scaling relation found by Calabrese, Gambassi and Krzakala [123] (and conjectured independently by Fedorenko and Trimper [235])

$$\lambda_C = \lambda_R = d + z + \frac{\beta}{\nu} . \quad (3.18)$$

Proof: We concentrate here on the autoresponse exponent λ_R . One needs to analyse carefully the short-time limit $s \rightarrow 0$, which is done by a short-time expansion [123]. For small times, one expects for the response field

$$\tilde{\phi}(s, \mathbf{r}) \simeq Q(s, m_0) \tilde{\phi}(0, \mathbf{r}) = Q(s, m_0) \tilde{\phi}_0(\mathbf{r}) \quad , \quad \text{if } s \rightarrow 0 .$$

Inserting this into a generalised correlator $G_{n,\tilde{n}}^m$, one finds, using again (1.114), the scaling form $Q(s, m_0) = s^{\tilde{\eta}_0/(2z)} \mathcal{Q}(s m_0^{-1/(\Theta + \beta/\nu z)})$. While for small arguments $\mathcal{Q}(u) \simeq \text{constant}$ reproduces the well-known scaling behaviour for vanishing initial magnetisation, the case of interest here is described by the opposite limit $u \rightarrow \infty$ of large arguments. This limit behaviour of $\mathcal{Q}(u)$ is obtained by considering the response function (in Fourier space, where the volume factor $\mathcal{V}$ regularises the delta function $(2\pi)^d \delta(\mathbf{0})$ arising from spatial translational invariance)

$$\begin{aligned} \mathcal{V} \hat{R}_{\mathbf{q}}(t, s) &= \left\langle \hat{\phi}_{\mathbf{q}}(t) \hat{\tilde{\phi}}_{-\mathbf{q}}(s) \right\rangle \stackrel{s \rightarrow 0}{\simeq} s^{\tilde{\eta}_0/2z} \mathcal{Q}(s m_0^{1/\kappa}) \left\langle \hat{\phi}_{\mathbf{q}}(t) \hat{\tilde{\phi}}_{0,-\mathbf{q}} \right\rangle \\ &\sim s^{\tilde{\eta}_0/2z} \mathcal{Q}(s m_0^{1/\kappa}) \mathcal{V} \frac{\delta m(t)}{\delta m_0} \end{aligned}$$

with the short-hand $\kappa := \Theta + \beta/(\nu z)$ and one also recalls the definition of the response function $\left\langle \hat{\phi}_{\mathbf{q}}(t) \hat{\tilde{\phi}}_{0,-\mathbf{q}} \right\rangle$. Now, the time-dependent magnetisation can be written in the scaling form $m(t) = t^{-\beta/\nu z} \mathbf{m}(t m_0^{1/\kappa})$, see (1.119), where

one *assumes* that, for large arguments, the scaling function behaves as $\mathbf{m}(u) = \mathbf{m}_0 + \mathbf{m}_1 u^{-1} + \mathcal{O}(u^{-2})$ and where $\mathbf{m}_{0,1}$ are constants. Calculating the derivative, one has

$$\widehat{R}_{\mathbf{q}}(t, s) \sim s^{\bar{\eta}_0/(2z)} \mathcal{Q}(s m_0^{1/\kappa}) t^{-1-\beta/\nu z} m_0^{-1-1/\kappa}.$$

Since one expects on physical grounds that in the $s \rightarrow 0$ limit the leading scaling behaviour should be independent of m_0 ,³ this fixes $\mathcal{Q}(u) \sim u^{\kappa+1}$ for u large. Hence, $\widehat{R}_{\mathbf{q}}(t, s) \sim t^{-1-\beta/\nu z}$, up to a s -dependent amplitude. Comparing with the expected scaling behaviour and recalling (1.112), one finally reads off $\Theta = -1 - \beta/\nu z = (d - \lambda_R)/z$ from which the assertion follows.

In order to treat the *connected* correlation function, the short-time expansion of $\phi(s, \mathbf{r})$ is required which finally leads to $\lambda_C = \lambda_R$ [123].⁴ q.e.d.

3.2.2 Application to the Ising Model

Explicit expressions for the scaling functions of both responses and *connected* correlators⁵ have been obtained to one-loop order [123] but will not be reproduced here. We merely quote the limit fluctuation-dissipation ratio in $4 - \varepsilon$ dimensions [123] (for the Ising case $n = 1$)

$$X_\infty = \frac{4}{5} - \left(\frac{73}{600} - \frac{\pi^2}{100} \right) \varepsilon + \mathcal{O}(\varepsilon^2). \quad (3.19)$$

In a study of the non-linear σ -model with $\mathcal{O}(n)$ -symmetry, one finds from an expansion in $2 + \bar{\varepsilon}$ dimensions about the lower critical dimension that $X_\infty = 1/2 + \mathcal{O}(\bar{\varepsilon}^2)$ [235].

Since the above argument depends on the assumed properties of the scaling functions of the time-dependent magnetisation, it is important to check directly the scaling relation (3.18). Indeed, direct simulations in the 2D Glauber-Ising model quenched to $T = T_c$ from a fully *ordered* initial state are for both the autocorrelator and the autoresponse completely consistent with the expectation $\lambda_C/z = \lambda_R/z \simeq 1.98$ [123].⁶ Eq. (3.18) is also confirmed, to one-loop order, in the field-theoretical renormalisation group [123]. Finally, the existing numerical estimate $X_\infty = 0.73(1)$ [123] is in good agreement with the one-loop prediction (3.19).

³ One can show by dimensional analysis that m_0 should be irrelevant.

⁴ Heuristically, this should have been expected because the limit fluctuation-dissipation ratio is finite.

⁵ Only for the *connected* autocorrelator $C(t, s) = \mathcal{N}^{-1} \langle \sum_{\mathbf{i} \in \Lambda} S_{\mathbf{i}}(t) S_{\mathbf{i}}(s) \rangle - \mathcal{N}^{-2} \langle \sum_{\mathbf{i} \in \Lambda} S_{\mathbf{i}}(t) \rangle \langle \sum_{\mathbf{i} \in \Lambda} S_{\mathbf{i}}(s) \rangle$ one has the asymptotic scaling function $f_C(y) \sim y^{-\lambda_C/z}$ with λ_C given by (3.18). The *uncorrelated* autocorrelator trivially scales as $\mathcal{N}^{-1} \langle \sum_{\mathbf{i} \in \Lambda} S_{\mathbf{i}}(t) S_{\mathbf{i}}(s) \rangle \sim s^{-2\beta/\nu z} (t/s)^{-\beta/\nu z}$, see [464] for a recent check in the 2D Ising model.

⁶ Eq. (3.18) has been generalised to stochastic systems not relaxing to an equilibrium state [46]. Tests of this generalised relation (3.99) are listed in Table 3.10.

3.2.3 Vector Order-parameter With $n \geq 2$ Components

When considering systems with a continuous symmetry such as the $O(n)$ -model with $n \geq 2$ [122], in order to account for a non-vanishing mean order-parameter⁷ $\vec{M}(t) = \langle \vec{\phi}(t, \mathbf{r}) \rangle$ during the time evolution, one should consider a longitudinal component $\sigma(t, \mathbf{r})$ and $n - 1$ transverse components $\vec{\pi}(t, \mathbf{r})$ as follows

$$\vec{\phi}(t, \mathbf{r}) = \begin{pmatrix} \sigma(t, \mathbf{r}) + M(t) \\ \vec{\pi}(t, \mathbf{r}) \end{pmatrix}, \quad \vec{\tilde{\phi}}(t, \mathbf{r}) = \begin{pmatrix} \tilde{\sigma}(t, \mathbf{r}) \\ \vec{\tilde{\pi}}(t, \mathbf{r}) \end{pmatrix} \quad (3.20)$$

and similarly for the response field. In consequence, responses and correlations must be defined for longitudinal and transverse modes separately. The autoresponse and autocorrelation exponents of the longitudinal and transverse modes are now different [235, 122]

$$\lambda_{C,\text{long}} = \lambda_{R,\text{long}} = d + z + \frac{\beta}{\nu}, \quad \lambda_{C,\text{tran}} = \lambda_{R,\text{tran}} = z + \frac{3\beta}{\nu}. \quad (3.21)$$

Both longitudinal and transverse response and correlation functions were computed to one-loop order. For the sake of brevity, we merely quote here the limit fluctuation-dissipation ratio in $4 - \varepsilon$ dimensions [122] and $2 + \bar{\varepsilon}$ dimensions [235], respectively

$$\begin{aligned} X_{\infty,\text{long}} &= \frac{4}{5} - \frac{1895 - 162\pi^2 + 76n}{1800(8+n)}\varepsilon + O(\varepsilon^2) \\ X_{\infty,\text{tran}} &= \frac{2}{3} - \frac{49 + 6n}{108(8+n)}\varepsilon + O(\varepsilon^2) \\ X_{\infty,\text{tran}} &= \frac{1}{2} + \frac{1}{8} \frac{n-1}{n-2} \bar{\varepsilon} + O(\bar{\varepsilon}^2) \end{aligned} \quad (3.22)$$

No tests of these results through simulations are known to us. In contrast to what usually happens for an initially disordered state, the limit values X_{∞} can exceed unity, when the initial state is (partially) ordered, as we shall explain further below.

Recall, however, that the spin-wave analysis of the XY model [633, 73, 589] is an expansion around the totally ordered state which is valid to leading order in the temperature T . For example, the connected spin-spin correlator, which is related to the angular correlation function, see (2.150), may be decomposed via (3.20) into $C(t, s; \mathbf{r}, \mathbf{r}') = C_{\text{long}} + C_{\text{tran}}$ and reproduces the Gaussian behaviour. Fig. 2.2 illustrates that the spin-wave approximation remains largely valid through the entire low-temperature phase in $2D$ [5].

⁷ As before, the n -dimensional vector $\vec{\phi}$ represents the components of the order-parameter, while $\mathbf{r}$ is a d -dimensional space vector.

The spherical model with an initially fully ordered state was already discussed in Chap. 2. Here we extend the presentation given there to *global* observables, like the autoresponses and autocorrelators in momentum space, following the detailed studies by Annibale and Sollich [19, 20]. From the usual spherical Hamiltonian $\mathcal{H} = \frac{1}{2} \sum_{(\mathbf{r}, \mathbf{r}')} (S(\mathbf{r}) - S(\mathbf{r}'))^2$, with $S(\mathbf{r}) \in \mathbb{R}$ and where the sum runs over all nearest-neighbour pairs of the hypercubic lattice $\Lambda \subset \mathbb{Z}^d$ with $\mathcal{N}$ sites, they consider the extended Langevin equation

$$\begin{aligned} \partial_t S(t, \mathbf{r}) = & -\frac{\delta \mathcal{H}}{\delta S(t, \mathbf{r})} + \eta(t, \mathbf{r}) \\ & - \frac{1}{\mathcal{N}} \sum_{\mathbf{b} \in \Lambda} \left[S(t, \mathbf{b}) \left(-\frac{\delta \mathcal{H}}{\delta S(t, \mathbf{b})} + \eta(t, \mathbf{b}) \right) S(t, \mathbf{r}) \right] \end{aligned} \quad (3.23)$$

where η is the usual centred Gaussian noise with variance $2T$. With respect to the treatment given earlier, the extra term in the Langevin equation allows for fluctuations in the Lagrange multiplier $\mathfrak{z}(t)$ and guarantees the validity of the spherical constraint $\sum_{\mathbf{r} \in \Lambda} S(t, \mathbf{r})^2 = \mathcal{N}$. Indeed, it is easy to see that, labelling the spherical spins at all sites as $S_1, \dots, S_{\mathcal{N}}$, it follows from (3.23) that the two vectors $(\partial_t S_1, \dots, \partial_t S_{\mathcal{N}})$ and $(S_1, \dots, S_{\mathcal{N}})$ are orthogonal, hence the spherical constraint holds for all times if imposed as an initial condition.⁸ Of course, since now the solution of the model can no longer be made in terms of Gaussian variables with the effects of the interactions summed up by a single equation coming from the spherical constraint, the analysis of the model is a formidable task. The scaling behaviour of the global observables can be given in terms of the two scaling variables $y = t/s$ and $u = s/t_M$, where t_M is the finite reference time-scale introduced by the non-vanishing initial magnetisation m_0 , see Chap. 2. A relatively simple form can be given for the scaling limit of the global autoresponse $\hat{R}_0(t, s) = \hat{f}_R(y, u)$ with [20]

$$\hat{f}_R(y, u) = \begin{cases} y^{1-d/4} \sqrt{\frac{1+u}{1+uy}} \left[1 - \frac{uy}{1+uy} \left(\frac{y-1}{y} \right)^{(d-2)/2} \right] & ; \text{ if } 2 < d < 4 \\ \left(\frac{1+u}{1+uy} \right)^{-3/2} & ; \text{ if } d > 4 \end{cases} \quad (3.24)$$

up to normalisation. This interpolates between an unmagnetised initial state ($u \ll 1$) and a fully magnetised initial state ($u \gg 1$). For brevity, we shall not list the correlation functions explicitly, but shall illustrate the complexity of the answer by giving the limit fluctuation-dissipation ratio for a fully magnetised initial state [20]

$$\mathcal{X}_\infty = \begin{cases} 4/5 & ; \text{ if } d > 4 \\ 4/5 - \frac{19}{450}\varepsilon + \mathcal{O}(\varepsilon^2) & ; \text{ if } d = 4 - \varepsilon \\ 1/2 + \mathcal{O}(\bar{\varepsilon}) & ; \text{ if } d = 2 + \bar{\varepsilon} \end{cases} \quad (3.25)$$

and remark that an explicit, albeit tedious, expression for $\mathcal{X}_\infty$ for all values of d is known which smoothly interpolates between the cases $d \searrow 2$ and

⁸ In [19, 20], the Stratanovich convention is used throughout.

model	d	θ_g		Ref.
		Markovian	direct	
Ising	2	1.462(1)	1.7(1)	[579]
	$4 - \varepsilon$	$2 - \varepsilon/4$	$2 - \varepsilon \cdot 0.195$	[579]
$O(n)$	$4 - \varepsilon$	$2 - \varepsilon/4$	$2 - \varepsilon \frac{0.02+n \cdot 1.74}{8+n}$	[579]
$O(\infty)$	$2 < d < 4$	$\frac{d}{4} + 1$	$\frac{d}{4} + 1$	[579]
spherical	$2 < d < 4$	$\frac{d}{4} + 1$	$\frac{d}{4} + 1$	[579]
mean-field	> 4	2	2	[579]
DP	1	1.3163	$\geq 1.50(4)$	[367]
	2	1.566(1)	2.5(5)	[13]
	$4 - \varepsilon$	$2 - \varepsilon/4$	$2 - \varepsilon \cdot 0.183$	[555]
CLG	2	1.327(3)	1.58(5)	[482]
	3	1.549(6)	1.54(5)	[482]

Table 3.1. Global persistence exponents for two classes of systems: (a) in the upper part, systems with a non-conserved order-parameter and for which the global order-parameter is initially non-vanishing and (b) in the lower part, systems with an absorbing phase transition, namely directed percolation (DP) and the conserved lattice gas (CLG). The Markovian predictions are from eq. (3.26) and the results of the direct calculation of θ_g are from the indicated sources. Analytical results are given up to terms of order $O(\varepsilon^2)$.

$d \nearrow 4$.⁹ On the other hand, while the interpolation between the regimes $u \ll 1$ and $u \gg 1$ is smooth for $d \approx 4$, this is no longer so when d is getting nearer to the lower critical dimension when $\mathcal{X}_\infty(u)$ may develop poles in u and can even become negative [20]. Comparison of the results (3.25) for $\mathcal{X}_\infty$ in momentum space with the corresponding ones for X_∞ in direct space, as listed in Table 2.9, shows that at least in this case $X_\infty \neq \mathcal{X}_\infty$ and eq. (3.7) does not hold.

3.2.4 Global Persistence

The global persistence for systems with an ordered initial state can be studied along the lines discussed in Sect. 1.4, see exercise 3.1. The argument [493] which was used to show that the global order-parameter should be described by a Gaussian process which is based on the central limit theorem remains valid. Because of the scaling relation (3.18), the prediction (1.130) of the global persistence exponent θ_g in the case of a Markov process becomes $\theta_g = 1 + d/(2z)$ for systems with a scalar order-parameter. For vector order-parameters, one must distinguish between longitudinal and transverse

⁹ After the correction of some errors [19] in a subsequent article [20], these results are fully consistent with the $n \rightarrow \infty$ limit of the results quoted above for the $O(n)$ -model, when considering the longitudinal mode [122, 235].

fluctuations and hence one must define two distinct global persistence probabilities, $P_{g,\text{long}}(t) \sim t^{-\theta_{g,\text{long}}}$ and $P_{g,\text{tran}}(t) \sim t^{-\theta_{g,\text{tran}}}$ and using the scaling relations (3.21), the predictions for these global persistence exponents for Markov processes become

$$\theta_g = \theta_{g,\text{long}} = 1 + \frac{d}{2z} \quad , \quad \theta_{g,\text{tran}} = 1 - \frac{d}{2z} + \frac{2\beta}{\nu z} \quad . \quad (3.26)$$

In Table 3.1 results of the global persistence exponent θ_g (or $\theta_{g,\text{long}}$ in the case of the $O(n)$ -model) are collected. As it was already seen for the case of a vanishing initial order-parameter in Sect. 1.4, the prediction eq. (3.26) coming from *the assumption of an underlying effective Markovian process for the long-time behaviour of the global order-parameter is in general not satisfied*. For the $O(n)$ -model one may also calculate the transverse global persistence, with the result [579]

$$\theta_{g,\text{tran}} = 1 - \varepsilon \frac{1.945 + n 0.25}{8 + n} + O(\varepsilon^2) \quad (3.27)$$

which is distinct from the Markovian prediction $\theta_{g,\text{trans}}^{(\text{mark})} = 1 - \varepsilon/4$ which follows from (3.26). In the $n \rightarrow \infty$ limit, one has $\theta_{g,\text{tran}} = d/4$.

We have also included in Table 3.1 results for systems with an absorbing phase transition, whose properties have been discussed in much detail in Volume 1. **Directed percolation** is the most simple of these universality classes and the **conserved lattice gas** is for $d \geq 2$ a member of the so-called **Manna universality class** which is the paradigmatic case for systems with self-organised criticality. The scaling relations used above for the $O(n)$ model relaxing towards equilibrium may be readily generalised by letting $\nu \mapsto \nu_\perp$, see eq. (3.99) [46], and $2\beta/\nu_\perp = (d - 2 + \eta_\perp)$, such that again $\theta_g = 1 + d/(2z)$ [555]. The data needed for the calculation of the ‘Markovian’ values of θ_g come from Tables 4.3 and 5.5 of Volume 1, respectively. The ageing properties of systems at an absorbing phase-transition, including directed percolation, will be taken up in Sect. 3.8.

Comparison with Table 1.9 shows that the value of θ_g depends on the initial magnetisation m_0 , viz. $\theta_g = \theta_g(m_0)$. It is useful to distinguish the two extreme cases and to write $\theta_{g,0} := \theta_g(0)$ for the case of a disordered initial state, see table 1.9 and $\theta_{g,\infty} := \theta_g(1)$ for the case of a fully ordered initial state, see table 3.1. The cross-over between these two regimes may be described by the scaling form $P_g(t; m_0) = t^{-\theta_{g,0}} \mathcal{P}(t/\tau_0)$, where the initial magnetisation m_0 enters via $\tau_0 \sim m_0^{-1/(\Theta + \beta/(\nu z))}$. The scaling function must obey $\mathcal{P}(t) = O(1)$ for $t \ll 1$ and $\mathcal{P}(t) \sim t^{\theta_{g,0} - \theta_{g,\infty}}$ for $t \gg 1$. This has been checked in the $2D$ Ising model as shown in figure 3.2 [579], which clearly illustrates how one goes over from the regime with $m_0 \ll 1$ and an effective global persistence exponent $\theta_{g,0} \simeq 0.235$ towards the regime with $m_0 \sim O(1)$ and an effective global persistence exponent $\theta_{g,\infty} \simeq 1.7$. As far as we know, an analytic calculation for the scaling function $\mathcal{P}(t)$ has not yet been carried out.

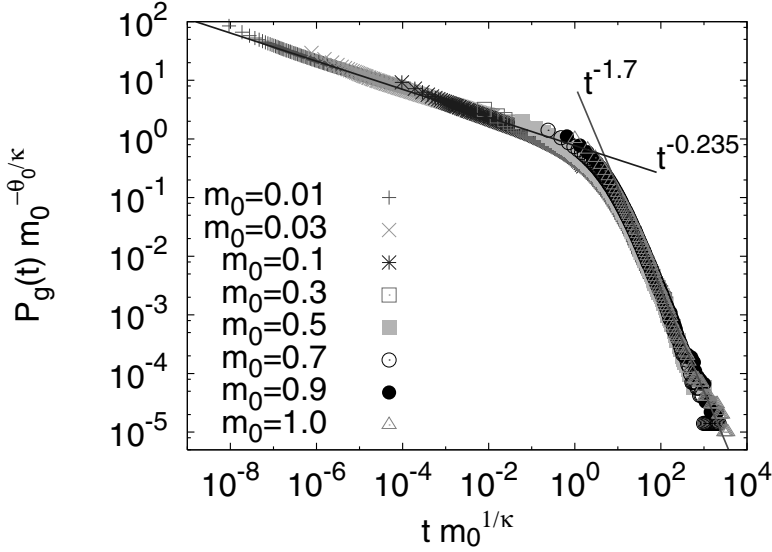


Fig. 3.2. Scaling of the global persistence $P_g(t)$ in the 2D critical Ising model with several values of the initial average magnetisation m_0 and the cross-over scaling function. Here $\kappa = \Theta + \beta/(\nu z)$ and $\theta_0 = \theta_{g,0}$. The straight lines indicate the asymptotic limits. After [579].

3.2.5 Semi-ordered Initial States

Semi-ordered initial states were recently studied in the 2D Ising model quenched to $T = T_c$ [757]. They considered a rectangular lattice with $2L \times L$ sites and chose the spin in the left half of the lattice to be fully ordered $\sigma = +1$ while the spins in the right half are fully ordered in the opposite sense $\sigma = -1$. After the quench, the two interfaces, initially at $r_x = 0$ and $r_x = L$ (periodic boundary conditions in both directions were used), will broaden and their width should increase as $w \sim t^{1/z}$. For quantitative statements, consider the position-dependent magnetisation and the unconnected autocorrelator

$$m(t; r_x) = \frac{1}{L} \left\langle \sum_{r_y=1}^L \sigma_{r_x, r_y}(t) \right\rangle = t^{-\beta/\nu z} \mathcal{M} \left(t^{1/z} r_x^{-1} \right) \quad (3.28)$$

$$A(t, s; r_x) = \frac{1}{L} \left\langle \sum_{r_y=1}^L \sigma_{r_x, r_y}(t) \sigma_{r_x, r_y}(s) \right\rangle = s^{-\beta/\nu z} \mathcal{A} \left(t/s, t^{1/z} r_x^{-1} \right)$$

Extensive simulations confirm this scaling description and suggest the asymptotic limits

$$\mathcal{M}(u) \sim \begin{cases} \text{constant} & ; \text{ if } u \rightarrow 0 \\ u^{-\beta_0/\nu} & ; \text{ if } u \rightarrow \infty \end{cases} \quad (3.29)$$

with the estimate $\beta_0 = 0.998(5)$. From this and unpublished data from the 3D Ising model, the conjecture $\beta_0/\nu = 1$ is drawn [757]. Similarly, the scaling ansatz for the autocorrelator A is confirmed, with the estimate $\beta_0 = 1.00(8)$. It is pointed out that the evolution inside the domain interface is described by a surface magnetisation exponent $\beta_1 = \beta + \beta_0$, distinct from the bulk exponent β . In this respect the behaviour appears to be analogous to the relaxation at a surface, but starting from a fully ordered initial state.

3.3 Conserved Order-parameter (Model B)

Throughout this chapter, we have studied the purely relaxational behaviour of many-body systems where the microscopic dynamics was such that a direct transition from one of the local equilibrium states to another one (usually represented by the different orientations of Ising spins) was possible. However, there are physical situations (e.g. relaxations in binary alloys or liquids) where the change of the local order-parameter rather is realised through *transport*. In other words, one has a continuity equation

$$\partial_t \phi(t, \mathbf{r}) + \nabla \cdot \mathbf{J}(t, \mathbf{r}) = 0 \quad (3.30)$$

where the ‘current’ $\mathbf{J}$ is a driving force. This means that the time-dependent ‘charge’ $\bar{\phi}(t) = \int_V d\mathbf{r} \phi(t, \mathbf{r})$ in a volume V is conserved if there is no transport across the boundary ∂V of V : $\mathbf{J}|_{\partial V} = \mathbf{0}$. A common phenomenological ansatz is the assumption of a ‘linear law’ $\mathbf{J} = -\nabla \bar{\mu}$ where $\bar{\mu} = \delta \mathcal{H}[\phi]/\delta \phi$ is a chemical potential. Finally, the effect of the ‘fast’ degrees of freedom on the ‘slow’ one $\phi(t, \mathbf{r})$ is included through a random force η and one writes down the Langevin equation for **model B** dynamics [370]

$$\partial_t \phi(t, \mathbf{r}) = -\Delta_L^2 \phi(t, \mathbf{r}) + \nabla^2 \frac{\delta \mathcal{V}[\phi]}{\delta \phi(t, \mathbf{r})} + \eta(t, \mathbf{r}) \quad (3.31)$$

where $\mathcal{V}[\phi]$ is a Ginzburg-Landau potential and the noise is assumed to be Gaussian, centred and of variance

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = -2T \nabla_{\mathbf{r}}^2 \delta(t - t') \delta(\mathbf{r} - \mathbf{r}'). \quad (3.32)$$

This choice of the noise is compatible with the conservation law for ϕ and with the expected equilibrium state as stationary solution. In lattice simulations based on the master equation, this corresponds to Kawasaki dynamics.

Consider the *non-equilibrium critical dynamics* of the Ising model with Kawasaki dynamics, after a quench from an initially completely disordered state. It is well-known that the dynamical exponent $z = 4 - \eta$ [759], where η is an equilibrium critical exponent. It turns out that for an analysis of this system one must recall the distinction between the exponents λ_C and $\lambda_{C'}$

$$\frac{\lambda_C}{z} := - \lim_{y \rightarrow \infty} \left(\lim_{s \rightarrow \infty} \frac{\ln(s^b C(y s, s))}{\ln y} \right), \quad \frac{\lambda_{C'}}{z} := - \lim_{t \rightarrow \infty} \left(\lim_{s \rightarrow 0} \frac{\ln C(t, s)}{\ln t} \right) \quad (3.33)$$

which we introduced in Sect. 1.3. Using the by now habitual scaling forms, the values

$$z = 3.75(20) \quad , \quad a = b = 0.107(1) \quad , \quad \lambda_{C'} = 1.95(11)$$

have been reported in the 2D Ising model, see [553]. A careful numerical study [284] of the Ising model obtained the values $\lambda_C \simeq 2.5$ and $\lambda_{C'} \simeq 1$ in 1D and $\lambda_C \simeq 3.5$ and $\lambda_{C'} \simeq 2$ in 2D from Monte Carlo simulations with an improved efficiency in order to speed up the calculation. These results are in agreement with the earlier prediction $\lambda_{C'} = d$ [494]. The fact that $\lambda_C \neq \lambda_{C'}$ means that the two limits in (3.33) do not commute, in contrast to what is seen for ageing with a non-conserved order-parameter. The same conclusion could already have been obtained by comparing the prediction $\lambda_{C'} = d$ with the Yeung-Rao-Desai inequalities (1.93). In order to account for these findings, Godrèche, Krzakala and Ricci-Tersenghi propose the following scaling form for the autocorrelation [284]

$$C(t, s) = s^{\lambda_{C'}/z} \mathfrak{C} \left(\frac{t}{s^\phi} \right), \quad \mathfrak{C}(u) \sim \begin{cases} \mathfrak{C}_\infty & ; \text{ if } u \rightarrow \infty \\ u^{(\lambda_{C'} - \lambda_C)/z} & ; \text{ if } u \rightarrow 0 \end{cases},$$

$$\phi = 1 + \frac{2 - \eta}{\lambda_C - \lambda_{C'}} \quad (3.34)$$

and the expected behaviour of the scaling function $\mathfrak{C}(u)$ is checked against their numerical data.¹⁰ However, the value of the exponent λ_C was later revised by Sire [674] who considered even longer time scales and found a cross-over from the values reported in [284] to $\lambda_C = d + 2$ for both the 1D and 2D Ising model with Kawasaki dynamics.

A further clarification of the situation comes from the exact solution of the non-equilibrium critical dynamics of the spherical model with a conserved order-parameter.¹¹ For nearest-neighbour interactions, the Langevin equation reads

$$\partial_t S(t, \mathbf{r}) = -\nabla_{\mathbf{r}}^2 \left[\nabla_{\mathbf{r}}^2 S(t, \mathbf{r}) + \mathfrak{z}(t) S(t, \mathbf{r}) + h(t, \mathbf{r}) \right] + \eta(t, \mathbf{r}) \quad (3.35)$$

and where units are chosen such that kinetic coefficients are set to unity. In eq. (3.35) the Lagrange multiplier $\mathfrak{z}(t)$ is introduced to enforce the spherical constraint and $h(t, \mathbf{r})$ is an external field. Some remarks are in order:

¹⁰ For the 1D case, the scaling regime sets in for time differences $\tau = t - s \sim s^{2/3}$, consistent with the general expectation [760].

¹¹ The spherical model with conserved order-parameter quenched into the coexistence phase $T < T_c$ shows a multiscaling [153] which is not described by simple ageing as considered in this book. It has been shown that this is a peculiarity of the $n \rightarrow \infty$ limit of the conserved $O(n)$ vector-model [509, 98].

- The Lagrange multiplier is included in a slightly different way than in the non-conserved case, namely with an additional $-\nabla_{\mathbf{r}}^2$ in front of it.
- The second moment of the centred Gaussian noise is now

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = -2T_c \nabla_{\mathbf{r}}^2 \delta(t - t') \delta(\mathbf{r} - \mathbf{r}') \quad (3.36)$$

such that the noise does not destroy the conservation law.

- The perturbation by an external field is included as $-\nabla_{\mathbf{r}}^2 h(t, \mathbf{r})$ into the Langevin equation, which means that the perturbation also respects the conservation law.

Essentially the same equation for the coarse-grained order-parameter is found for the $n \rightarrow \infty$ limit of the $O(n)$ model with model B dynamics [443, 493, 674]. Since $\eta = 0$ in this model, one expects $z = 4$.

It is well-known that this model shows scaling behaviour for quenches onto the critical point [443, 493, 674, 47]. The solution of the Langevin equation follows essentially the same lines as for the non-conserved case. The solution of the Langevin equation (including the external field) is found by a Fourier transform

$$\begin{aligned} \widehat{S}(t, \mathbf{k}) &= \left[\widehat{S}(0, \mathbf{k}) + \int_0^t d\tau \exp(\omega(\tau, \mathbf{k})) \right. \\ &\quad \left. \times \left(\widehat{\eta}(\tau, \mathbf{k}) + k^2 \widehat{h}(\tau, \mathbf{k}) \right) \right] \exp(-\omega(t, \mathbf{k})) \end{aligned} \quad (3.37)$$

where $\widehat{S}(t, \mathbf{k})$ denotes the spatial Fourier transformation of $S(t, \mathbf{r})$ and similarly for η and h . The dispersion relation is for $h \rightarrow 0$ [443, 493, 674, 47]

$$\omega(t, \mathbf{k}) = k^4 t - k^2 \int_0^t d\tau \mathfrak{z}(\tau) \simeq k^4 t - \mathfrak{g}_d k^2 \quad (3.38)$$

where the last relation follows in the large-time limit from the analysis of the spherical constraint which fixes $\mathfrak{z}(t)$. Here the constant $\mathfrak{g}_d$ is found from the condition [443]

$$\int_0^\infty dx x^{d-1} \left\{ 2x^2 \int_0^1 dy \exp[-2x^4(1-y) + 2\mathfrak{g}_d x^4(1-\sqrt{y})] - \frac{1}{x^2} \right\} = 0. \quad (3.39)$$

In particular, $\mathfrak{g}_d = 0$ for $d > 4$ and asymptotic forms for $d \rightarrow 2$ and $d \rightarrow 4$ are as follows [443]

$$\mathfrak{g}_d = \begin{cases} \sqrt{-\frac{1}{8} \ln(\frac{d}{2} - 1)} & ; \text{ for } d \rightarrow 2 \\ \sqrt{\pi/8} (1 - \frac{d}{4}) & ; \text{ for } d \rightarrow 4 \end{cases} \quad (3.40)$$

and $\mathfrak{g}_3 \approx 0.254 \dots$. From this the Fourier transform of the response function is computed as $\widehat{R}(t, s, \mathbf{k}) = \delta \langle \widehat{S}(t, \mathbf{k}) \rangle / \delta h(s, -\mathbf{k})$ and in direct space, we have for $t > s$ in the scaling limit [47]

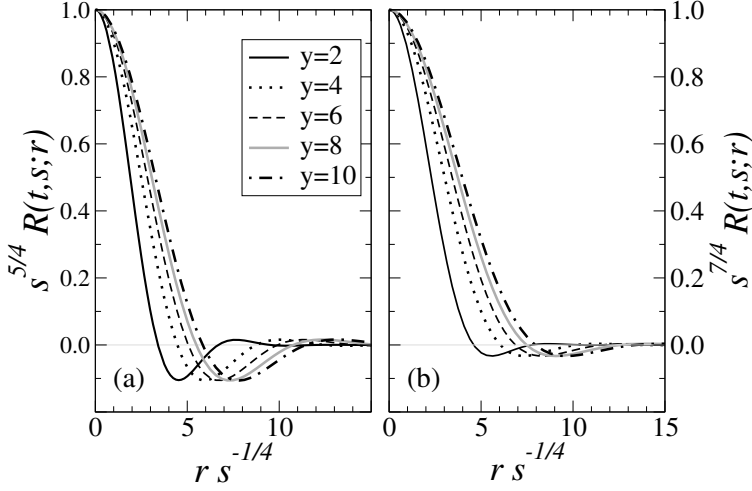


Fig. 3.3. Space-time response function in the spherical model with a conserved order-parameter quenched to $T = T_c$ in (a) 3D and (b) 5D, for several values of $y = t/s$. [47]

$$R(ys, s; \mathbf{r}) = (ys - s)^{-(d+2)/4} \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} k^2 \exp\left(-\frac{i\mathbf{k} \cdot \mathbf{r}}{(t-s)^{1/4}}\right) \times \exp\left(-k^4 + \mathfrak{g}_d k^2 \frac{y^{1/2} - 1}{(y-1)^{1/2}}\right). \quad (3.41)$$

For $d > 4$, one may rewrite this more explicitly in terms of hypergeometric functions

$$R(t, s; \mathbf{r}) = \frac{\sqrt{\pi}}{2^{3d/2} \pi^{d/2} \Gamma(d/4)} (t-s)^{-(d+2)/4} \left[{}_0F_2\left(\frac{1}{2}, \frac{d}{4}; \frac{r^4}{256(t-s)}\right) - \frac{8}{d} \frac{\Gamma(\frac{d}{4} + 1)}{\Gamma(\frac{d}{4} + \frac{1}{2})} \left(\frac{r^2}{16\sqrt{t-s}}\right) {}_0F_2\left(\frac{3}{2}, \frac{d}{4} + \frac{1}{2}; \frac{r^4}{256(t-s)}\right) \right]. \quad (3.42)$$

The scaling behaviour of the (normalised) space-time response is illustrated in Fig. 3.3. In contrast to the non-conserved case, they are decaying in an oscillatory way as a function of $|\mathbf{r}|/s^{1/4}$ for each fixed value of y . For certain values of the scaling variable the response function is even *negative*, such that the system responds against the direction of the external stimulation. Similar decaying oscillations are a well-known characteristic of the correlation function in systems with a conserved order-parameter, see e.g. [47] for examples in the critical conserved spherical model.

In the same way, the Fourier-transformed two-time correlator $\hat{C}(t, s, \mathbf{k}) := \langle \hat{S}(t, -\mathbf{k}) \hat{S}(s, \mathbf{k}) \rangle$ is readily found

$$\begin{aligned}\widehat{C}(t, s, \mathbf{k}) &= \left\langle \widehat{S}(0, -\mathbf{k}) \widehat{S}(0, \mathbf{k}) \right\rangle \exp \left(-\omega(t, \mathbf{k}) - \omega(s, \mathbf{k}) \right) \\ &+ 2T_c \int_0^s du k^2 \exp \left(-\omega(t, \mathbf{k}) - \omega(s, \mathbf{k}) + 2\omega(u, \mathbf{k}) \right).\end{aligned}\quad (3.43)$$

In contrast to the non-conserved case where the first term coming from the noise of the initial configuration is not only irrelevant but also numerically unimportant even for relatively small times, this has been shown to be different in the conserved critical spherical model [674]. Indeed, if one takes, instead of the usual scaling limit, s to be fixed and sends $t \rightarrow \infty$, then the first term in (3.43) dominates and one finds $C(t, 0) \sim t^{-d/z}$, hence $\lambda_{C'} = d$, in agreement with the expectation of [494].

On the other hand, in the scaling limit and at criticality ($T = T_c$), the scaling form of the two-time correlation function is given by [674, 47]

$$\begin{aligned}C(y, s, s; \mathbf{r}) &= 2T_c s^{-(d-2)/4} \int_0^1 d\theta (y + 1 - 2\theta)^{-(d+2)/4} \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} k^2 \\ &\times \exp \left(-\frac{i\mathbf{k} \cdot \mathbf{r}}{(s(y + 1 - 2\theta))^{1/4}} \right) \exp \left(-k^4 + \mathfrak{g}_d k^2 \left(\frac{y^{1/2} + 1 - 2\theta^{1/2}}{(y + 1 - 2\theta)^{1/2}} \right) \right)\end{aligned}\quad (3.44)$$

and this term dominates in the time regime $L(s) \ll L(t) \ll L(s)^\phi$ where $L(t)$ is the linear size of the correlated domain and ϕ is given in (3.34) [674]. In general, the full long-time behaviour of the autocorrelator is described as follows [674]

$$C(t, s) = C(t, 0) + s^{-(d-2+\eta)/z} f_C(t/s) \quad (3.45)$$

instead of (3.34) and such that $f_C(y) \sim y^{-\lambda_C/z}$ for $y \rightarrow \infty$ and $C(t, 0) \sim t^{-\lambda_{C'}/z}$ for $t \rightarrow \infty$.

Comparison with the scaling forms (1.74, 1.75) of simple ageing and recalling the definition (3.33) yields the *exact exponents of model B non-equilibrium critical dynamics*

$$z = 4 - \eta, \quad a = b = \frac{d - 2 + \eta}{4 - \eta}, \quad \lambda_R = \lambda_C = d + 2, \quad \lambda_{C'} = d \quad (3.46)$$

(recall that $\eta = 0$ in the spherical model). A heuristic argument due to Sire [674] shows that *the values (3.46) of the exponents λ_C and $\lambda_{C'}$ should hold true for the $O(n)$ model with conserved order-parameter in any dimension d* .¹²

We also note that for $d > 4$, the scaling function of the autocorrelator has the following simple form, up to normalisation,

$$f_C(y) \sim \frac{8T_c}{2-d} \left[(y-1)^{(2-d)/4} - (y+1)^{(2-d)/4} \right]. \quad (3.47)$$

¹² This implies $\phi = z/2$ in (3.34).

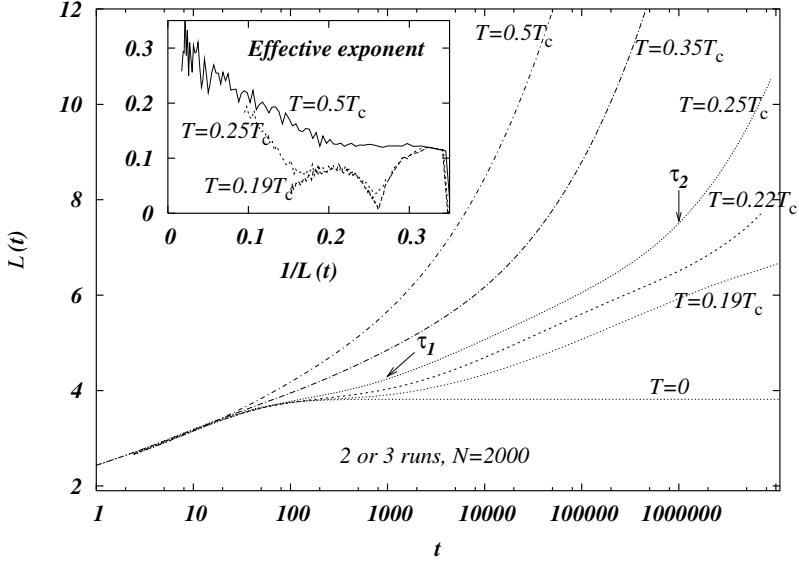


Fig. 3.4. Time-dependent domain size $L(t)$ of the 2D Kawasaki-Ising model quenched to several $T < T_c$ on a 2000^2 lattice. The two activation times $\tau_{1,2}$ are indicated for $T = 0.25T_c$. The inset shows the effective growth exponent $1/z_{\text{eff}}$ over against $1/L(t)$ and illustrates the convergence towards $1/3$ as $L(t) \rightarrow \infty$. Reprinted with permission from [453]. Copyright (2005) by the American Physical Society.

Since the usual requirements $\lambda_C = \lambda_R$ and $a = b$ for the existence of a non-trivial fluctuation-dissipation ratio are satisfied, this quantity and in particular its limit for $y = t/s \rightarrow \infty$ can be calculated. For the limit fluctuation-dissipation ratio, one finds in the spherical model $X_\infty = 1/2$ [47]. This is in agreement with the one-loop result $X_\infty = 1/2 + O(\varepsilon^2)$ from field-theory [121]. Numerical simulations in the 1D and 2D Ising model with Kawasaki dynamics [284] are also consistent with X_∞ being different from the one found for Glauber dynamics.

The form of the two-time scaling functions (3.41,3.44), which do not contain an explicit dependence on the initial correlations (provided these are sufficiently short-ranged), will be explained in the Chap. 5 from local scale-invariance.

We finish this section with a remark for quenches to $T < T_c$, that is *phase-separation*. It can be shown that $\lambda_{C'} = d$ in general when the continuum description (3.31,3.32) of model B is applicable [494]. The comparison of this kind of prediction with simulational data is far from straightforward, however. For example, simulations in the 2D Ising model with Kawasaki dynamics and quenched to a temperature $T \ll T_c$ show evidence for three distinct temporal regimes [453]: (i) a zero-temperature regime, where after finite time,

the domain size $L(t)$ reaches a plateau value, (ii) the so-called glassy regime, which is entered after an activation time $\tau_1 \sim e^{4J/T}$ after the quench and with an extremely slow (sub-logarithmic) domain growth and (iii) the asymptotic ageing regime, reached after an activation time $\tau_2 \sim e^{8J/T}$ and where $L(t) \simeq A + Bt^{1/3}$. The behaviour of $L(t)$ is illustrated in Fig. 3.4. Analysis of the fluctuation-dissipation ratio¹³ shows that in the intermediate ‘glassy’ regime the system becomes very sensitive to the existence of meta-stable states, quite analogous to what is found in many glassy systems. It is conceivable that the behaviour exemplified here is quite generic.

Finally, in [105] the autocorrelation exponent has also been determined for the case of conserved dynamics and $T < T_c$. Simulating the two-dimensional Cahn-Hilliard-Cook model, Brown *et al.* obtained the value $\lambda_C \approx 4.47(3)$ which is consistent with the Yeung-Rao-Desai lower bound $\lambda_C \geq (d/2) + 2$ for conserved dynamics [744], but violates the upper bound $\lambda_C \leq d$ conjectured by Fisher and Huse [245].

3.4 Fully Frustrated Systems

Since it is accepted folklore knowledge [518, 530] that the main ingredients of a glassy material are disorder and frustration, it is of interest to investigate the ageing behaviour of fully frustrated models without disorder.¹⁴ Such models are defined in such a way that on each plaquette of the defining lattice one spin cannot align with the others in the ground state. This leads to a rich, but yet only partially understood behaviour; see [474, 742] and appendix A for a short survey on the equilibrium properties of these systems.

An analysis of the dynamics of the fully frustrated q -states Potts model (here throughout on the square lattice) must begin with a short look at its many ground states [722, 723]. In Fig. 3.5a the possible distinct spin configurations of a single plaquette are indicated. In total, there are $q + 3q(q - 1)$ possible configurations from which the ground states are built. Even if three of the spins of a plaquette are fixed, there may exist several possibilities to choose the last spin, leading to a highly degenerate ground state with a *finite* entropy even at $T = 0$. Spin configuration can be described in terms of domain walls, but such that at most one domain wall crosses any given plaquette. Hence domain walls cannot intersect, and in each plaquette they must cross the antiferromagnetic bond and one of the ferromagnetic ones. Examples for

¹³ A considerable increase in simulational efficiency can be achieved by using the techniques to calculate $R(t, s)$ directly without explicitly using an external field [141, 616, 284], also described in appendix G of Volume 1, and furthermore by using a continuous-time method [501, 535] which keeps track of all broken bonds and updates the time needed for the system to find it.

¹⁴ Partially frustrated models with either weakened or strengthened antiferromagnetic bonds have the same asymptotic behaviour as pure, unfrustrated models.

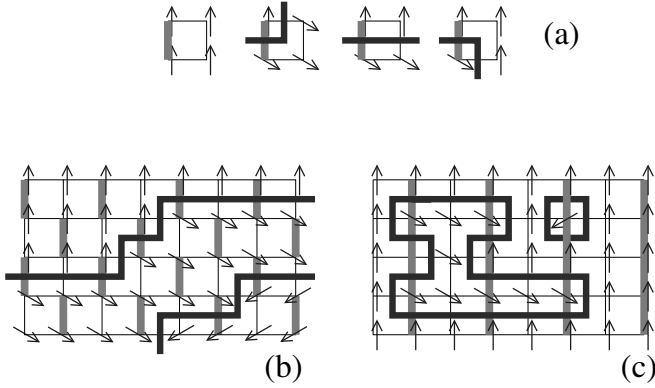


Fig. 3.5. (a) Possible spin configurations on a plaquette in the ground state of a fully frustrated Potts model on the 2D square lattice. The thick grey lines indicate the antiferromagnetic bonds and the dark lines indicate the domain walls between the different ferromagnetic states. Examples of ground-state spin configurations and their domain walls are shown in (b) for the checkerboard and in (c) for the domino-tile modulation.

spin configurations for both checkerboard and domino-tile arrangement of the antiferromagnetic bonds are shown in Fig. 3.5bc. For checkerboard modulation, domain walls can form neither overhangs nor loops and therefore cross the entire lattice from one boundary to the other. Single spin flips do not cost energy and in general the system is not frozen at $T = 0$. For domino-tile modulation, loops are possible which may enclose a single spin. Isolated spins may be flipped without a cost in energy and are referred to as **loose spins**.

First, we describe results for the fully frustrated Ising model [722, 723]. Here, the checkerboard and the domino-tile modulation are equivalent. An ageing behaviour is only expected at the critical point $T = 0$, after a quench from an initially fully disordered lattice. From the equilibrium scaling, one has $a = b = (d - 2 + \eta)/z = 0.2495(5)$ which together with the known result $\eta = 1/2$ [683, 254] in 2D gives $z = 2.004(4)$. However, there is evidence that the time-dependent length scale behaves as $L(t) \sim (t/\ln t)^{1/2}$, either from numerical data on the scaling behaviour of the single-time correlation function $C_t(\mathbf{r})$ [527] or else from the numerical data for both the spin-spin autocorrelation as well as the autoresponse function [722, 723, 721] which show the best scaling behaviour for the assumed forms

$$C(t, s) = s^{-a} f_C \left(\frac{t}{\ln t} \frac{\ln s}{s} \right) , \quad R(t, s) = s^{-1-a} f_R \left(\frac{t}{\ln t} \frac{\ln s}{s} \right) . \quad (3.48)$$

This is analogous to the ageing behaviour of the 2D XY model after a quench from a disordered initial state, see Chap. 2. The asymptotic behaviour

model	T	H	a	λ_C/z	X_∞	Ref.
Ising FF	0	0	0.2495(5)	1.02(2)	0.33(1)	[723, 721]
Ising TAF	0	0	0.2501(10)	1.003(10)		[424]
Ising TAF	0	H_c	0.229(5)	0.93(3)		[424]
Potts-3 FFCB	0	0	0.165(2)	$\simeq 1$	$\gtrsim 0.35(1)$	[722, 721]
Potts-3 FFDT	$0.35T_c$	0	1/2	$\simeq 1$	0.150(12)	[722, 721]
XY FFDT	T_{KT}	0	0.13	0.87(3)	0.385(15)	[724]
	T_χ	0	0.10(2)	0.98(5)	0.405(5)	[724]

Table 3.2. Exponents and limit fluctuation-dissipation ratio for the ageing in several fully frustrated systems in $2D$. Here FF refers to fully frustrated, FFCB and FFDT indicate checkerboard and domino-tile frustrations, TAF refers to the triangular antiferromagnet. The bath temperature T after the quench and the external field H are indicated. In all systems $z = 2$. The Potts-3 model with domino-tile frustrations is the only one quenched into the low-temperature phase. In the XY model, one has two distinct transitions: a Kosterlitz-Thouless-like transition at T_{KT} and a chiral transition at T_χ .

$f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$ for $y \rightarrow \infty$ gives $\lambda_C/z = \lambda_R/z = 1.02(2)$. The limit fluctuation-dissipation ratio is estimated as $X_\infty = 0.33(1)$, surprisingly close to the value of the non-frustrated $2D$ Ising model [722].

Recently, these results have been confirmed in a study of the ageing of the $2D$ antiferromagnetic Ising model on a triangular lattice [424]. In order to have a direct access to the growth law $L = L(t)$ of the relevant time-dependent length scale, they measured, for finite systems of linear size N , the sample-dependent relaxation time t_r for the return of the system to the ground state. Their result $\langle t_r \rangle \sim N^2 \ln N$ points to the importance of logarithmic corrections to scaling. Although the effective dynamical exponents extracted from a short section of the data might fall somewhat above 2, it is carefully argued in [424] that in the $N \rightarrow \infty$ limit one should have $z = 2$. Analysis of the two-time autocorrelation function gives $a = (d - 2 + \eta)/z = 0.2501(10)$ and $\lambda_C/z = 1.003(10)$ [424]. The logarithmic scaling form (3.48) is fully confirmed. All this is in full agreement with the expectation that the fully frustrated Ising model and the antiferromagnetic triangular Ising model should be in the same universality class, see Table 3.2 for a summary.

Eq. (3.48), as found by [722, 723] and confirmed by [424], is in disagreement with earlier data from the triangular Ising antiferromagnet quenched to $T = 0$. There it was rather suggested that $L(t) \sim t^{1/z}$, without any logarithmic corrections, and $z = 2.33(5)$ and a sub-ageing scaling $C(t, s) = s^{-\eta/z} f((t-s)s^{-\mu})$ with $\mu \simeq 0.89$ was reported [439, 440]. However, in [424] it was pointed out through a careful comparison of these scaling forms that these exponent values arise as effective exponents, from not having taken into account the logarithmic corrections to simple scaling in the argument of the scaling function.

Next, we turn to the fully frustrated Potts-3 model. In the case of checkerboard modulation, where the critical point is still at $T = 0$, one finds again the

scaling behaviour (3.48) but now with the exponent $a = 0.165(2)$ [722]. The autocorrelation exponent is estimated as $\lambda_C/z \simeq 1$ but the limit fluctuation-dissipation ratio $X_\infty \gtrsim 0.35(1)$ appears to be slightly larger than in the Ising case.

In the case of domino-tile modulation of the Potts-3 model, a quench to $T/J = 0.13 \ll 0.37 = T_c/J$ again reproduces the scaling (3.48), with $a = 1/2$. Data are consistent with $\lambda_C/z \simeq 1$ and $X_\infty = 0.150(12)$ is non-vanishing in the low-temperature phase [722]. This last feature is shared with spin glasses or the 2D XY model, where a simple power-law scaling of the time-dependent length scale $L(t)$ also receives a logarithmic correction.

From the available numerical data, the conjectures $\lambda_C = d$ and $z = 2$, for both the Ising and the Potts model, and for $T < T_c$ and $T = T_c$, are formulated [722, 723].

When one considers the antiferromagnetic triangular Ising model in an external field, one expects that at $T = 0$ there is an endpoint H_c which should be in the Kosterlitz-Thouless universality class. Numerical data are consistent with $z = 2$ and give $a(H_c) = 0.229(5)$ and $\lambda_C(H_c)/z = 0.93(2)$ [424]. The value of $a(H_c)$ nicely agrees with the value $\eta(H_c)/z = 2/9$, derived from conformal-invariance methods [83].

Finally, we consider the fully frustrated XY model [724]. The two distinct phase transitions at equilibrium occur at $T_{KT} = 0.4461(2)$ and $T_\chi = 0.4545(2)$. The properties of the topological Kosterlitz-Thouless-like transition may be analysed in terms of the spin correlators and responses. For a disordered initial state, the available numerical evidence is consistent with the scaling form (3.48) and the exponent estimates are listed in Table 3.2. If on the other hand, the initial state is one of the ground states of the model, the simulational data agree well with the following form suggested from spin-wave theory [724]

$$C(t, s) = \left\langle \sum_i \vec{\sigma}_i(t) \cdot \vec{\sigma}_j(s) \right\rangle \sim (t-s)^{-\eta/2} \left(\frac{t+s}{\sqrt{ts}} \right)^{\eta/2} \quad (3.49)$$

$$R(t, s) = \frac{\delta \langle \vec{\sigma}_i(t) \cdot \vec{e}_1 \rangle}{\delta \vec{h}_i(s) \cdot \vec{e}_1} \bigg|_{h=0} + \frac{\delta \langle \vec{\sigma}_i(t) \cdot \vec{e}_2 \rangle}{\delta \vec{h}_i(s) \cdot \vec{e}_2} \bigg|_{h=0} \sim (t-s)^{-1-\eta/2} \left(\frac{t+s}{\sqrt{ts}} \right)^{\eta/2}$$

where $\vec{e}_{1,2}$ denote the unit basis vectors in spin space and for $\eta = \eta(T)$ the temperature-dependent equilibrium estimates are used (see appendix A). This leads to a singularity in the fluctuation-dissipation ratio, quite analogously to what is also observed in the XY model or the spherical model [73, 588]. The properties of the chiral transition may be analysed by using a chirality correlator $C_\chi(t, s) = \langle \chi(t) \chi(s) \rangle$, where the local chirality is defined by

$$\chi(t) := \text{sign} \left[\sum_{i,j \in \mathbb{P}} J_{i,j} \sin(\theta_i - \theta_j) \right] \quad (3.50)$$

where the sum extends over the four bond of the square plaquette $\mathbb{P}$. For a disordered initial state, the scaling is again best described by (3.48) and this leads to the estimates included in Table 3.2.

Summarising, the ageing in the $2D$ fully frustrated XY model is well-described in terms of simple ageing, with a logarithmic correction in the time-dependent length scale $L(t) \sim (t/\ln t)^{1/2}$ for a disordered initial state. The computed values of the non-equilibrium exponents strongly suggest that both transitions are distinct from the unfrustrated Ising, Potts and XY universality classes [724].

3.5 Disordered Systems I: Ferromagnets

In this and the next section, we return to the origin of systematic studies on physical ageing and consider the effects of disorder. The paradigmatic example will again be the Ising model, which we now take as a **random-bond Ising model** defined by the classical Hamiltonian

$$\mathcal{H}_{\text{dis}} = - \sum_{(i,j)} J_{i,j} \sigma_i \sigma_j \quad (3.51)$$

where the $J_{i,j}$ are random variables. We shall consider two examples:

1. a disordered, but unfrustrated system where the $J_{i,j} \geq 0$ are uniformly distributed over the interval $[1 - \varepsilon/2, 1 + \varepsilon/2]$ with $0 < \varepsilon \leq 2$;
2. an Ising spin glass with both disorder and frustration where the random couplings $J_{i,j}$ can take both positive and negative values with equal probabilities. Several probability distributions will be considered.

In Fig. 3.6, the evolution of configurations of kinetic Ising models with different kinds of disorder and frustration are compared. Clearly, there are visible similarities between the evolution in simple, non-disordered Ising models and disordered, but unfrustrated systems, although the evolution is slower in the second case. This might suggest that similar scaling descriptions might be used and we shall present evidence that this should be the case. On the other hand, there is no obvious similarity with the evolution in spin glasses.

3.5.1 Phenomenological Description

The effects of disorder may depend sensibly on how the disorder is introduced. For example, the **random-site Ising model**, described by the classical Hamiltonian $\mathcal{H} = -J \sum_{(i,j)} \varepsilon_i \varepsilon_j \sigma_i \sigma_j$ where $\varepsilon_i \in \{0, 1\}$ are random variables selected according to a control parameter p , may behave quite differently from the random-bond model (3.51). Provided that on average $[J_{i,j}]_{\text{av}} = 1$,

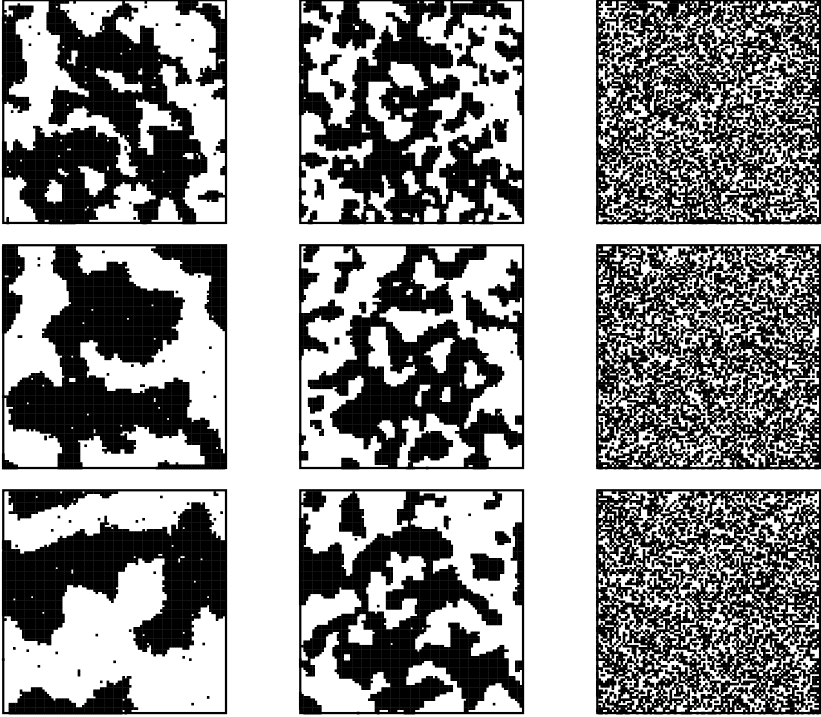


Fig. 3.6. Snapshots illustrating the growth of clusters in three variants of the kinetic Ising model, at times $t = 25, 100$ and 225 after the quench, from top to bottom. The left column shows the two-dimensional Glauber-Ising model without disorder quenched to $T = 1.5 < T_c$ and the middle column shows the random bond Ising model in two dimensions with $\varepsilon = 2$ and $T = 0.7 < T_c$. The right column shows a section of the three-dimensional Ising spin glass with binary disorder $J_{i,j} = \pm 1$ at $T = 0.8 < T_c$. After [348].

the critical temperature in the presence of bond disorder should not change much. For a site-diluted model, however, the phase-transition should disappear once the non-magnetic sites (where $\varepsilon_i = 0$) will begin to percolate. Throughout this section, we shall restrict ourselves to quenched (immobile) random-bond disorder.

We first have to understand dynamical scaling, as expressed through the time-dependence $L = L(t)$ of the linear size of the clusters. Comparing the left and the middle column of Fig. 3.6, one arrives at the following physical picture, due to Henley and Huse [386]. Initially, small clusters will form and start to grow. These early stages are still unaffected by the presence of the disorder which will only begin to be felt once the average cluster size has become of the same order as the mean distance between two impurities. Then the domain walls will become trapped close to the disorder-induced defects and

material	model	$L(t)$	ψ	$T_c[K]$	Ref.
$\text{Rb}_2\text{Co}_{0.60}\text{Mg}_{0.40}\text{F}_4$	diluted Ising antiferromagnet	$A + B(\ln t)^{1/\psi}$	$0.28(15[K]/T)$ $\text{if } T \lesssim 15[K]$	20	[390]
$\text{Rb}_2\text{Cu}_{0.89}\text{Co}_{0.11}\text{F}_4$	random-bond Ising	$(\ln t)^{1/\psi}$	$0.20(5)$ $T = 3.9[K]$	4.93(5)	[646]
ZLI 4792 (Merck) (liquid crystal)	random-bond Ising	$(\ln t)^{1/\psi}$	$1/4$		[666]
tryglycine sulfate (TGS)	ferroelectric <i>conserved</i>	$\sim (t - t_0)^{1/z}$ $\sim (\ln t)^{1/\psi}$	$z \approx 3$ $1/4$	322	[475] [476]

Table 3.3. Experimental results on the growth of the domain size $L(t)$ in some disordered, but non-glassy systems in two dimensions. For $\text{Rb}_2\text{Co}_{0.60}\text{Mg}_{0.40}\text{F}_4$ a different temperature-dependent exponent ψ is found close to T_c . For TGS the order-parameter is conserved.

their motion will slow down correspondingly. The impurities are thought to act as energy barriers to the domain growth. Hence the pinning centres are localised in energetically favourable positions, which explains the slowing-down of the kinetics. Phenomenologically, for a non-conserved order-parameter one expects

$$\frac{dL(t)}{dt} = \frac{D(L, T)}{L(t)} \quad (3.52)$$

which is a minimal extension of the Allen-Cahn equation where the diffusion constant $D = D(L, T)$ now depends on the domain scale $L = L(t)$ and the temperature T . For a constant D , one is back to the non-disordered case, with $L(t) \sim t^{1/2}$. For thermally activated motion, one should have

$$D(L, T) \simeq D_0 \exp(-E_B/T) \quad (3.53)$$

where $E_B = E_B(L)$ is the barrier energy. Huse and Henley [386] have argued that the barrier energy should depend on the domain size algebraically $E_B(L) \simeq E_0 L^\psi$ with an exponent $\psi = (2\zeta + d - 3)/(2 - \zeta)$ where ζ is the domain-wall roughness exponent. For example, in two dimensions it is known that $\zeta = \frac{2}{3}$, hence $\psi = \frac{1}{4}$. Inserting this into eq. (3.52) leads to [386]

$$L(t) = \left(\frac{T}{E_0}\right)^{1/\psi} \mathcal{L}\left(\frac{t}{t_0}\right), \quad \mathcal{L}(\tau) = \begin{cases} \frac{2}{\psi}\tau & ; \tau \ll 1 \\ (\ln \tau)^{1/\psi} & ; \tau \gg 1 \end{cases} \quad (3.54)$$

which describes a qualitative change of behaviour between the initial regime and a second regime of slower growth for larger times.

This result agrees qualitatively with the available experimental evidence, see Table 3.3, in the sense that one does indeed observe a change from a

relatively fast kinetics seen at not too late times [390, 475, 476] to a slowing-down of the kinetics at later times. On the other hand, the results of some experiments, if they can be taken at face value,¹⁵ are different [390] and give an effective exponent ψ which depends on temperature, in strong contrast to the constant $\psi = 1/4$ predicted theoretically in two dimensions [386]. Further, in most published studies the data were usually only compared to a logarithmic law $L(t) \sim (\ln t)^{1/\psi}$ and alternatives were not explored. Indeed, it has been argued that in *site-diluted* Ising models, because of the fractal nature of the domain boundaries the energy barriers should rather depend logarithmically on $L(t)$, viz. $E_B(L) = \epsilon \ln(1 + L)$ [359, 610, 580]. Inserting this into (3.52) leads to an algebraic law $L(t) \sim t^{1/z}$ where the effective z crosses over from $z = 2$ for short times to

$$z = z(T, \epsilon) = 2 + \epsilon/T \quad (3.55)$$

for late times (see exercise 3.2). In at least one experiment, data were explicitly seen to be also compatible with an algebraic growth law [475, 476].

For a conserved order-parameter (model B), a similar discussion can be carried out. It can be shown that for algebraic energy barriers the result (3.54) remains unchanged [386], whereas for logarithmic barriers one finds again algebraic growth $L(t) \sim t^{1/z}$ with $z = z(T, \epsilon) = 3 + \epsilon/T$ [580, 581].

3.5.2 Exact Results

The temperature-dependence of the dynamical exponent z can be checked in the 2D solid-on-solid (SOS) model on a disordered substrate. The Hamiltonian is [645]

$$\mathcal{H} = \sum_{(i,j)} (h_i - h_j)^2, \quad h_i = n_i + d_i \quad (3.56)$$

where (i, j) runs over the pairs of nearest neighbours, $n_i \in \mathbb{Z}$ and $0 \leq d_i < 1$ are uniformly distributed random variables and uncorrelated between different sites. In the presence of disorder, the roughening transition of the pure model becomes a so-called **super-roughening transition** at a temperature $T_g = 2/\pi$. For $T > T_g$ the surface is logarithmically rough and the disorder is irrelevant, while for $T < T_g$ the pinning disorder induces a stronger roughness. In the continuum limit, the behaviour near T_g is in the same universality class as the **Cardy-Ostlund model** [132] (or the sine-Gordon model with random phase shifts) with Hamiltonian

$$\mathcal{H} = \int d\mathbf{r} \left[(\nabla \phi(\mathbf{r}))^2 - g \cos(2\pi[\phi(\mathbf{r}) - \xi(\mathbf{r})]) \right] \quad (3.57)$$

with a continuous field $\phi(\mathbf{r})$ and a random-phase variable $\xi(\mathbf{r})$, uniformly distributed in $[0, 1)$.

¹⁵ Indeed, the fit used in [390] does not take the Porod's law in 2D into account.

Studying the connected two-time height-height correlation functions and the linear response of the heights in the disordered SOS model (3.56) with a non-conserved order-parameter, Schehr *et al.* [642, 645, 643] have shown analytically by one-loop renormalisation-group calculations, both for $T \ll T_g$ and for $T \lesssim T_g$, and confirmed numerically that to leading order ($C_E = 0.5772\dots$ is Euler's constant)

$$z \simeq \begin{cases} 2 + \frac{2\pi^2}{9} \frac{T_g}{T} & ; \text{ if } T \ll T_g \\ 2 + 2e^{C_E} \frac{T_g - T}{T_g} & ; \text{ if } T \approx T_g \end{cases}, \quad (3.58)$$

whereas $\lambda_C = \lambda_R \approx 2$ throughout the low-temperature phase. For the time being, the disordered SOS model is the only one where a consequence of the assumed existence of logarithmic barrier heights can be verified analytically, through the characteristic dependence $z(T) \sim T^{-1}$ for $T \ll T_g$.

3.5.3 Simulation Studies

For random-bond Ising models (see the definition (3.51) above), where a non-conserved dynamics was created via a Metropolis or heat-bath algorithm, a precise numerical study suggests a more subtle picture of the phase-ordering kinetics in disordered ferromagnets [349]. Before discussing these results, however, we first recall that for disordered systems prepared in an initial disordered state and then quenched below the critical point one needs to average the data not only over the realisations of the noise but also over the realisations of the disorder. Thus, the two-time spin-spin autocorrelator is obtained as

$$C(t, s) = \left[\left\langle \frac{1}{N} \sum_{i \in \Lambda} S_i(t) S_i(s) \right\rangle \right]_{\text{av}} \quad (3.59)$$

where $\langle \dots \rangle$ and $[\dots]_{\text{av}}$ indicate the average over the noise (and the initial disordered state) and over the disorder, respectively. As a consequence, numerical studies of disordered systems require considerably more computer time than investigations of perfect systems.

The first question we address here concerns the scaling form to be used. In Fig. 3.7 we show data for the autocorrelation $C(t, s)$ over against t/s , for several typical values of ε and $T < T_c$. A nice data collapse is seen, which is fully consistent with simple ageing. These scaling plots imply that the exponent $b = 0$, analogously to what is found in the phase-ordering of pure systems. In [583], a ‘super-ageing’ scaling form (1.82) $C(t, s) = \mathcal{C}(\exp[\frac{(t-s)^{1-\mu} - s^{1-\mu}}{1-\mu}])$ was considered for the random-site Ising model where the exponent $\mu > 1$ is fitted to the data. Simple ageing is recovered in the $\mu \rightarrow 1$ limit. However, the values of $\mu \approx 1.03$ reported in [583] are so close to unity that a careful study of possible finite-time corrections to scaling appears to be required

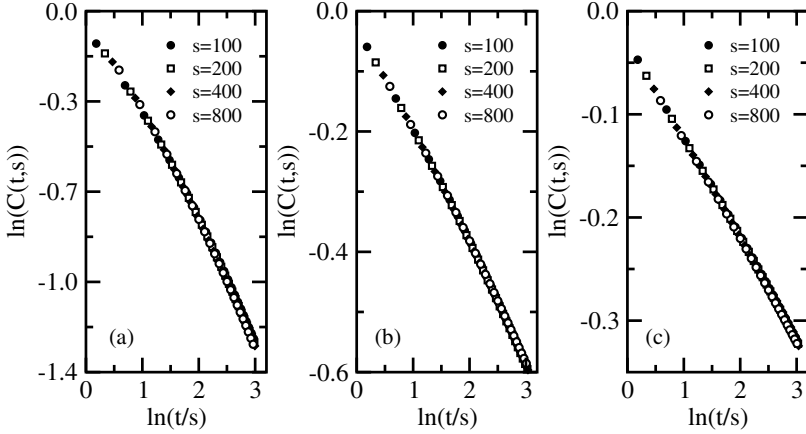


Fig. 3.7. Scaling of the autocorrelation $C(t, s)$ in the 2D random-bond Ising model eq. (3.51) for (a) $\varepsilon = 0.5$, $T = 1$, (b) $\varepsilon = 1$, $T = 0.4$ and (c) $\varepsilon = 2$, $T = 0.4$, for several values of s . Here and in the following error bars are smaller than the sizes of the symbols, unless explicitly stated otherwise. Reprinted with permission from [349]. Copyright (2008) by the American Physical Society.

before such a conclusion could be accepted. For all $\varepsilon < 2$, the data for the random-bond model show no hint for a ‘super-ageing’ behaviour, in contrast to the findings in [583]. Note that the observed scaling form of simple ageing would be incompatible with a non-power-law form of $L(t)$ in the range of times considered.

Having in this way checked that the relevant length scale $L = L(t)$ should indeed scale algebraically with time, one can then determine the dynamical exponent $z = z(T, \varepsilon)$ from the criterion [670] involving the single-time correlator

$$C_t(L(t)) \stackrel{!}{=} \frac{1}{2}, \quad L(t) \sim t^{1/z(T, \varepsilon)}. \quad (3.60)$$

The results for z are shown in Fig. 3.8a and are listed in Table 3.4. One observes that the values of z obtained in fact only depend on the dimensionless ratio ε/T , to within the numerical accuracy.¹⁶ The function $z = z(\varepsilon/T)$, however, is *non*-linear and only becomes an approximately linear function in a relatively small region of values of z .

Fig. 3.9 shows data for the scaling of the thermoremanent magnetisation, expected to be of the form

$$M_{\text{TRM}}(t, s) = s^{-a} f_M(t/s). \quad (3.61)$$

¹⁶ This might suggest that one could identify the phenomenological parameter ϵ from p. 167 with the model parameter ε as defined in eq. (3.51) as was suggested in the past when early data appeared even to indicate the possibility of an empirical relationship $\epsilon \approx \varepsilon$.

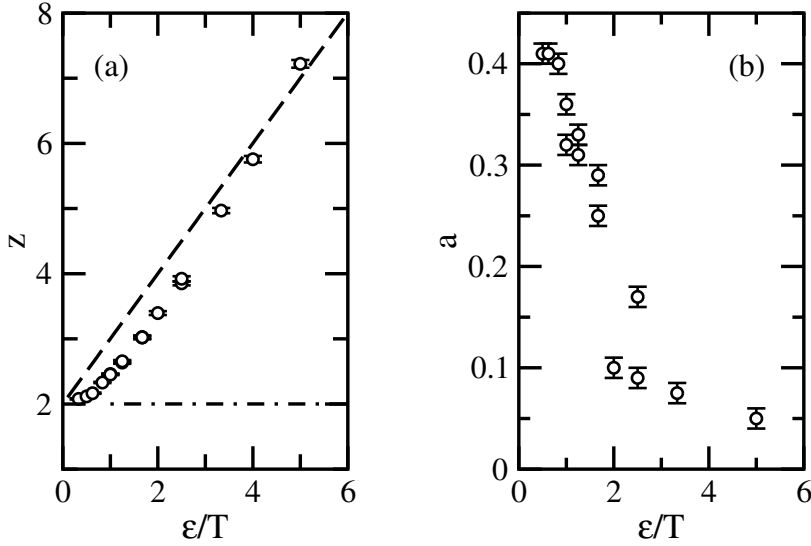


Fig. 3.8. Exponents for the two-dimensional random-bond Ising model eq. (3.51). Left panel: Dynamical exponent z , determined from the condition $C_t(L(t)) \stackrel{!}{=} \frac{1}{2}$, as a function of ϵ/T . The dashed line corresponds to the equation (3.55), with the identification $\epsilon = \epsilon$. Right panel: ageing exponent a , as determined from the scaling of $M_{\text{TRM}}(t, s)$. In some cases more than one data point is shown for a given value of ϵ/T . These data points correspond to different values of ϵ and T with ϵ/T constant, see Table 3.4. Reprinted with permission from [349]. Copyright (2008) by the American Physical Society.

The exponent a is obtained by plotting the thermoremanent magnetisation as a function of the waiting time s for fixed values of the ratio t/s [345]. The resulting power-law decay, see eq. (3.61), then yields the value of the exponent a . The numerical values are given in Table 3.4 and shown in Fig. 3.8b. One observes that the estimates $a = a(T, \epsilon)$ scatter considerably more than those for z which suggests that a cannot be reduced to a function of the single variable ϵ/T . Furthermore, considering in detail the numerical values from Table 3.4, we see that the relation $a = 1/z$, eq. (1.107), known from the phase-ordering of pure ferromagnets [166, 89, 338], no longer seems to be valid.

In order to try to understand this finding, recall from p. 68 how the relation $a = 1/z$ was derived for pure ferromagnets. Consider a pure ferromagnet in an external oscillating magnetic field of angular frequency ω . The dissipative part of the linear response is given as the imaginary part of the dynamic susceptibility and reads

$$\chi''(\omega, t) = \int_0^t du R(t, u) \sin(\omega(t - u)) = \chi_1''(\omega) + t^{-a} \chi_2''(\omega t) + \dots \quad (3.62)$$

ε	T	z	a	λ_C/z	λ_R/z
0.5	1.5	2.08(1)	—	0.59(1)	—
0.5	1.0	2.11(1)	0.41(1)	0.565(10)	0.61(2)
0.5	0.8	2.16(1)	0.41(1)	0.56(1)	0.61(3)
0.5	0.6	2.33(1)	0.40(1)	0.54(1)	0.60(2)
0.5	0.5	2.46(1)	0.36(1)	0.48(1)	0.55(2)
0.5	0.4	2.64(2)	0.33(1)	0.46(1)	0.50(2)
0.5	0.3	3.02(2)	0.29(1)	0.385(10)	0.46(2)
1.0	1.0	2.45(1)	0.32(1)	0.49(1)	0.51(2)
1.0	0.8	2.65(2)	0.31(1)	0.445(10)	0.50(2)
1.0	0.6	3.02(2)	0.25(1)	0.38(1)	0.43(2)
1.0	0.4	3.85(3)	0.17(1)	0.29(1)	0.34(1)
1.5	0.9	3.02(3)	—	0.375(10)	—
2.0	1.0	3.39(3)	0.10(1)	0.315(10)	0.35(2)
2.0	0.8	3.92(4)	0.09(1)	0.270(5)	0.32(2)
2.0	0.6	4.97(4)	0.075(10)	0.217(3)	0.28(1)
2.0	0.5	5.76(5)	—	0.189(3)	—
2.0	0.4	7.22(6)	0.05(1)	0.155(3)	0.21(1)

Table 3.4. Dynamical exponent z , non-equilibrium response exponent a , autocorrelation exponent λ_C/z and autoresponse exponent λ_R/z in the 2D random-bond Ising model eq. (3.51) for different values of ε and T . After [349].

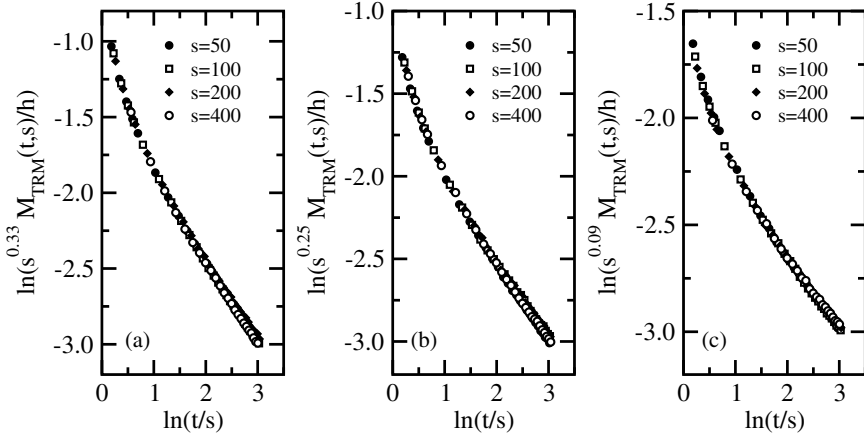


Fig. 3.9. Scaling of the thermoremanent magnetisation $M_{\text{TRM}}(t, s)$ in the 2D random-bond Ising model eq. (3.51) for several values of s , with (a) $\varepsilon = 0.5$, $T = 0.4$, (b) $\varepsilon = 1$, $T = 0.6$ and (c) $\varepsilon = 2$, $T = 0.8$.

where the last relation follows from the usually assumed scaling (1.75) of the autoresponse function $R(t, s)$. On the other hand, motivated from the physical picture that the dynamics in phase-ordering should only come from the motion of the domain walls between the ordered domains, one would expect to find

[89]

$$\chi''(\omega, t) = \chi''_{\text{st}}(\omega) + L(t)^{-1} \chi''_{\text{age}}(\omega t) + \dots \quad (3.63)$$

from which one may identify the stationary and the ageing part with the terms in eq. (3.62) coming from the scaling analysis. Since only the domain boundaries contribute to the dynamics, the leading time-dependent part should be proportional to the surface area of the domain divided by the total volume, hence to $L(t)^{d-1}/L(t)^d = 1/L(t)$ which accounts for the factor $1/L(t)$ in (3.63). Comparison of eqs. (3.62, 3.63), together with $L(t) \sim t^{1/z}$, then gives $a = 1/z$.

The empirical observation that $az < 1$ suggests that the above argument should no longer apply to *random* ferromagnets. Since (3.62) only depends on the dynamical scaling assumption (1.75), and given that the numerical results appear to be compatible with it, one expects that (3.62) should remain valid for disordered ferromagnets. Since also for disordered ferromagnets, the contribution to the ageing behaviour should come from the boundary region between ordered domains (this is also suggested by looking at the microscopic spin configurations, see Fig. 3.6), one expects it to be proportional to $N_d(L)/N_b(L)$ where $N_{b,d}(L)$ denote the number of mobile spins in the bulk and at the domain boundaries, respectively. While one should still have $N_b(L) \sim L^d$, disorder may cause the domain boundary to become fractal and, hence, $N_d(L) \sim L^{d_f}$ with d_f the fractal dimension (and $d_f = d - 1$ for the pure case). Then eq. (3.63) would be replaced by

$$\chi''(\omega, t) = \chi''_{\text{st}}(\omega) + L(t)^{-(d-d_f)} \chi''_{\text{age}}(\omega t) + \dots \quad (3.64)$$

and comparison with (3.62) would now imply

$$a = \frac{d - d_f}{z}. \quad (3.65)$$

The empirical results (Table 3.4) imply that $d_f > d - 1$, that is, the disorder should modify the domain boundaries into fractal curves. From eq. (3.65), since a depends on the dynamical exponent $z = z(\varepsilon/T)$ as well as the fractal dimension d_f , it may appear more natural that a *cannot* be written as a function of the single variable ε/T .

Using the scaling forms (1.74, 1.75) in the limit of large $y = t/s$ for the autocorrelation and autoresponse functions, one can also extract the exponents λ_C/z and λ_R/z , see Table 3.4. In contrast to the pure case, where for fully disordered initial conditions one may show that $\lambda_C = \lambda_R$, see pp. 82 and 260 [93, 589], the values of the autocorrelation exponent λ_C are different from those of the autoresponse exponent λ_R . In particular, one finds that within the numerical accuracy, λ_C/z is a function of the single variable ε/T , at least for $\varepsilon < 2$, while λ_R/z cannot be expressed in this way. These data suggest that $\lambda_R \geq \lambda_C$ and they are consistent with the rigorous Yeung-Rao-Desai inequality $\lambda_C \geq d/2$ [744]. Furthermore, one observes that $\lambda_R/z - a$ should be practically constant (again for $\varepsilon < 2$ and with a value in the range $\approx 0.17 - 0.20$).

3.5.4 Superuniversality

We have seen in Chap. 1 that dynamical scaling was developed from an analysis of the long-time behaviour of the single-time correlator of the order-parameter $\phi(t, \mathbf{r})$. Dynamical scaling arises if there exists a unique reference length scale $L = L(t)$ such that the single-time correlator

$$C_t(\mathbf{r}) = \langle \phi(t, \mathbf{r}) \phi(t, \mathbf{0}) \rangle = t^{-b} f\left(\frac{|\mathbf{r}|}{L(t)}\right) \quad (3.66)$$

can cast into a scaling form when the time t is large enough. Furthermore, it has been suggested since a long time that for disordered ferromagnets without frustration, the resulting scaling function f should become **superuniversal** in the sense that its form should be *independent* of the disorder [245]. This observation of superuniversality has indeed been confirmed for the single-time spin-spin correlator, see [97, 606, 80, 28]. We illustrate it here in Fig. 3.10a for the 2D bond-disordered Ising model eq. (3.51) where indeed numerical data obtained for several temperatures T and several values of the parameter ε which describes the disorder, fall on top of each other. That the domain-size $L(t)$ is indeed the only physically relevant length scale has been shown in phase-ordering by considering the hull-enclosed area and domain perimeter length [670], as already described in Chap. 1. It should be noted that superuniversality is in qualitative agreement with the experimental observation in several distinct polymers and metals that the linear response to a small mechanical stress can be described in terms of an universal master curve which is independent of the material studied [687], see Fig. 1.3.

In Fig. 3.10bc this test of superuniversality is extended to the case of *space-time*-dependent two-time correlators and responses. The collapse of data for different values of T and of ε suggests that one take qualitatively the same conclusion as for the single-time correlator. The data appear to confirm the superuniversality hypothesis, that is when all length scales are expressed in terms of $L(t)$, the form of the scaling function is independent of both the disorder ε and the temperature T .

On the other hand, superuniversality cannot hold for all spatial distances down to $|\mathbf{r}| = 0$, since one then goes over to the well-known scaling behaviour of the autocorrelation and autoresponse functions, the form of which is described by the autocorrelation exponent λ_C and the autoresponse exponent λ_R . The data in Table 3.4 indicate that the values of these exponents should depend on both T and ε . Returning to the scaling functions themselves, some small but systematic deviations from superuniversality are observed for small spatial distances, the largest deviations being seen for the autocorrelation and autoresponse functions with $|\mathbf{r}| = 0$. The value of $|\mathbf{r}|/L(s)$ where the deviations set in seems to depend slightly on the value of s , but a larger range of s values than accessed in [349] is needed for a more quantitative discussion of this point. That means that although dynamical scaling does hold true

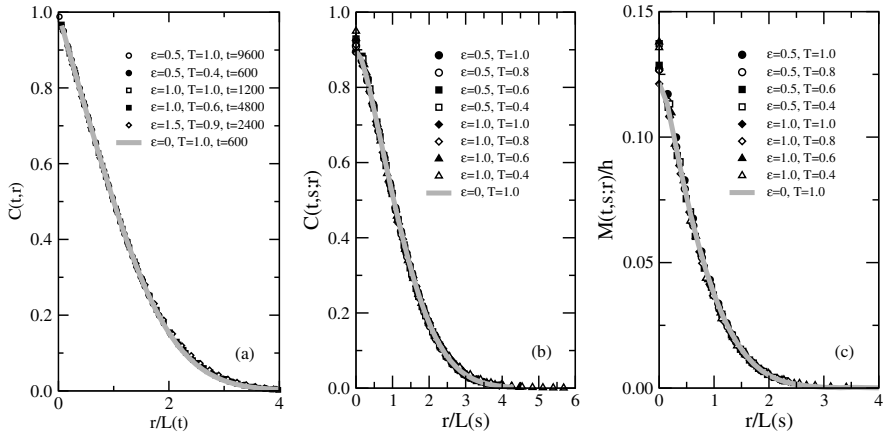


Fig. 3.10. Test of the superuniversality in the 2D random-bond Ising model of (a) the single-time correlator, (b) the two-time correlation function, and (c) the two-time thermoremanent magnetisation. Data for $\varepsilon = 0, 0.5, 1, 1.5$ are shown to fall onto the same rescaled curve. For (b), (c) $t/s = 4$ and $s = 100$. The grey curves show the data of the pure system. Reprinted with permission from [349]. Copyright (2008) by the American Physical Society.

even down to the autocorrelators and autoresponses, correlators and responses taken over a spatial distance of at least a typical cluster size $L(t)$ show yet a larger degree of universality.

3.6 Disordered Systems II: Critical Glassy Systems

Spin glasses form a very remarkable class of systems that have been intensively studied during the last three decades, see [78, 431] for some reviews. Due to the nature of the couplings between magnetic moments, which are both random and frustrating, a complex behaviour emerges which still is not fully understood. Many studies address the controversial low-temperature behaviour of spin glasses or the intriguing critical behaviour at the spin glass transition. In the context of this book it is important to note that spin glasses are considered to be prototypical systems displaying ageing. We already have discussed some of the experiments that so beautifully illustrate the concepts and consequences of dynamical scaling and ageing.

The disordered ferromagnets studied in the previous section can be viewed as intermediary between the simple, non-disordered magnets and the disordered and highly frustrated spin glasses. We now ask whether the results obtained for the disordered ferromagnets can be extended at least to certain glassy systems. We shall discuss two physically distinct kinds of systems, yet whose ageing behaviour share many common qualitative features: first we con-

sider Ising spin glasses quenched to their critical point. Second, the ageing of gauge glasses is analysed which is considered as a paradigmatic model for a vortex-glass phase transition. In relationship with the physics of superconductors, the comparison with results of a model of interacting flux lines will give interesting insight.

3.6.1 Critical Ising Spin Glasses

In the Ising spin glass (3.51), also known as (**Edwards-Anderson model**), the nearest-neighbour couplings $J_{i,j}$ are random variables. We shall consider three different distributions of the couplings: (i) the **bimodal distribution** with

$$P_B(J_{i,j}) = [\delta(J_{i,j} - J) + \delta(J_{i,j} + J)]/2, \quad (3.67)$$

(ii) the **Gaussian distribution** with

$$P_G(J_{i,j}) = \exp(-J_{i,j}^2/2J^2)/(J\sqrt{2\pi}), \quad (3.68)$$

and (iii) the **Laplacian distribution** with

$$P_L(J_{i,j}) = \exp(-\sqrt{2} |J_{i,j}/J|)/(J\sqrt{2}). \quad (3.69)$$

All distributions are symmetric with zero mean and variance $[J_{i,j}^2]_{\text{av}}/J^2 = 1$. The dynamics of the model is given by a master equation where the rates are chosen according to heat-bath dynamics. It is by now established, see e.g. [431] for a review, that for $d > 2$ dimensions this model undergoes an equilibrium phase-transition between a paramagnetic and a frustrated spin-glass phase. There has been considerable debate on the precise relationship between the relevant time and length scales for quenches below the spin-glass critical temperature T_c . It has been attempted to summarise the present state of knowledge into the form [88]

$$t(L) \sim L^z \exp \left(\frac{\Delta_0}{T} \left(\frac{L}{\xi_{\text{eq}}(T)} \right)^\psi \right) \quad (3.70)$$

where Δ_0 is an energy scale of order T_c , ψ is the barrier exponent and $\xi_{\text{eq}}(T)$ is the equilibrium correlation length at temperature T . This form has been used to fit successfully simulational data in the three-dimensional and in the four-dimensional Edwards-Anderson model [88, 72]. Although the typical length scales are merely of the order of a few lattice sizes, see e.g. [747, 714], the relaxation times are sufficiently large for a dynamical scaling to set in. While the expression (3.70), if correct, points towards a cross-over behaviour between a simple power-law scaling and an exponential scaling for $T < T_c$ in spin glasses as would follow from the droplet model, it also suggests that at criticality simple power-law scaling should prevail.

Analogously to simple, non-glassy, ferromagnets, the magnetic autocorrelation function is measured via

$$C(t, s) = \mathcal{N}^{-1} \sum_i [\langle \sigma_i(t) \sigma_j(s) \rangle]_{\text{av}} = s^{-b} f_C(t/s) \quad (3.71)$$

where we also indicate the expected scaling form at the critical point, valid in the double limit $t, s \rightarrow \infty$ such that $y = t/s > 1$ is kept fixed. Since the response function is too noisy to be measured directly, we find it by quenching the system in a small magnetic field h which is kept until the waiting time s has elapsed, when it is turned off, see also appendix G in volume 1. The thermoremanent magnetisation $M_{\text{TRM}}(t, s) = h \chi_{\text{TRM}}(t, s)$ measured at time t is an integrate autoreponse

$$\chi_{\text{TRM}}(t, s) = \int_0^s du R(t, u) = s^{-a} f_M(t/s). \quad (3.72)$$

For critical spin glasses, it is expected that the ageing exponents a and b are related to the equilibrium exponents β_{EA} and η_{EA} of the Edwards-Anderson order-parameter by

$$a = b = \frac{\beta_{\text{EA}}}{\nu z} = \frac{d - 2 + \eta_{\text{EA}}}{2z}. \quad (3.73)$$

On the other hand, the autocorrelation and autoreponse exponents $\lambda_{C,R}$, defined from the limit behaviour $f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}$ as $y \rightarrow \infty$ (and similarly, $f_M(y) \sim y^{-\lambda_R/z}$) are independent of the equilibrium critical exponents.

From now on, we consider the three- and the four-dimensional Ising spin glass quenched to $T = T_c$ from a fully disordered initial state and discuss the dynamical scaling behaviour of critical two-time quantities.

Let us first verify that the standard scaling forms (1.74, 1.75) for the autocorrelation and the autoreponse are indeed satisfied in *critical* spin glasses. From the data shown in Fig. 3.11 for the case of a bimodal distribution of the couplings in three dimensions we conclude that both the correlation (Fig. 3.11a) and integrated response (Fig. 3.11b) are consistent with a power-law scaling. Similar results are obtained [345, 595] for the two other distributions, both in three and four space dimensions.

From the slopes of Fig. 3.11 we obtain the estimates for the exponents a and b gathered in Table 3.5. Remarkably, one sees that $a = b$ within our numerical errors for a fixed choice of the distribution of the couplings, but that the results for different distributions are different. Even more intriguingly, other critical non-equilibrium dynamical quantities, as for example the exponent λ_C/z describing the decay of the correlations in the long-time limit or the limit value of the fluctuation-dissipation ratio (3.4), also vary strongly and systematically with the form of the interaction distribution, see Table 3.5 and Fig. 3.11c [595]. Here we emphasise the diagnostic usefulness of the *universal* limit fluctuation-dissipation ratio X_∞ [286] whose numerical estimates vary up to a factor of 2 between the various distributions.

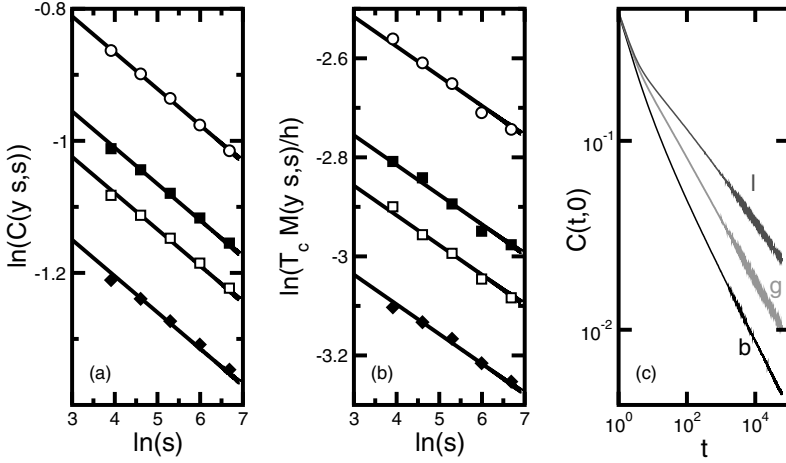


Fig. 3.11. Scaling of (a) the autocorrelation function $C(y, s)$ and (b) the thermoremanent magnetisation $M(y, s)$ of the critical three-dimensional Edwards-Anderson spin glass with a bimodal distribution of the couplings for (from top to bottom) $y = 5, 8, 10, 15$. The full curves are $C(y, s) = c_0 s^{-b}$ and $M(y, s) = m_0 s^{-a}$, where c_0, m_0 were fitted to the numerical data. (c) Temporal evolution of the autocorrelation $C(t, 0)$ in the three-dimensional cases (b: bimodal, g: Gaussian, l: Laplacian). Reprinted from [348].

We recall that there exists a standing controversy whether exponents in spin glasses should be independent of the distribution of the coupling constants, as one would expect from the renormalisation group [607], or whether the values of the critical exponents should depend on the distribution of the coupling constants [68]. Recent high temperatures series expansions [179] and extensive Monte Carlo simulations seem to indicate that the static critical exponent of Ising spin glasses are independent of the specific form of the distributions of the couplings. For example for the exponent η_{EA} recent studies [428, 409, 318] yielded consistent values $\eta_{\text{EA}} = -0.375(10)$ and $\eta_{\text{EA}} = -0.275(25)$ in three and four space dimensions, respectively. The situation is far less clear when it comes to dynamic exponents and amplitudes [408, 430, 345, 595]. We summarise in Table 3.6 the best estimates for the dynamical critical exponent z for the three- and four-dimensional Ising spin glass with various coupling distributions [595]. Taken at face value, these estimates, which are still not very precise, indicate that the critical dynamics is strongly affected by the nature of the disorder. This is also consistent with the distribution-dependent values of the non-equilibrium quantities listed in Table 3.5. Thus, insertion of the values for η_{EA} and z into equation (3.73)

Ising glass, bimodal					
d	a	b	λ_C/z	λ_R/z	X_∞
3	0.060(4)	0.056(3)	0.362(5)	0.38(2)	0.12(1)
4	0.18(1)	0.180(5)	0.615(10)	—	0.20(1)
Ising glass, Gaussian					
d	a	b	λ_C/z	λ_R/z	X_∞
3	0.044(1)	0.043(1)	0.320(5)	0.33(2)	0.09(1)
4	0.169(4)	0.171(2)	0.58(1)	—	0.175(10)
Ising glass, Laplacian					
d	a	b	λ_C/z	λ_R/z	X_∞
3	0.033(3)	0.032(2)	0.259(2)	—	0.055(2)
4	0.143(5)	0.140(3)	0.54(1)	—	0.13(1)
gauge glass, uniform					
d	a	b	λ_C/z	λ_R/z	X_∞
3	0.06(1)	0.06(1)	0.49(2)	0.52(2)	0.12(1)

Table 3.5. Non-equilibrium quantities of critical glassy systems. The critical Ising spin glass is considered for bimodal, Gaussian, and Laplacian distributions of the nearest-neighbour couplings in $d = 3$ and $d = 4$ dimensions [345, 595]. In the critical 3D gauge glass the random vector potential is uniformly distributed [627].

d	bimodal	Gaussian	Laplacian
3	5.7(2)	6.2(1)	8.6(2)
4	4.45(10)	5.1(1)	6.05(10)

Table 3.6. Values for the dynamical critical exponent z for three- and four-dimensional critical Ising spin glasses. Bimodal, Gaussian, and Laplacian distributions of the nearest-neighbour couplings are considered [595].

yield values for $a = b$ which roughly agree with the values that are directly obtained from the scaling of the correlation and response functions.¹⁷

Let us now turn to the scaling functions themselves. Plotting the rescaled autocorrelator $s^b C(t, s)$ versus the scaling variable t/s yields a nice data collapse compatible with a simple power-law scaling $L(t) \sim t^{1/z}$ [345, 595]. In Fig. 3.12 we show the two-time scaling of the integrated response where again a collapse of the data in terms of a simple power-law scaling is observed. The fact that both the autocorrelator and the thermoremanent magnetisation can be described in terms of such a power-law scaling is evidence in favour of the time-dependent length-scale (3.70). In principle, one would like to extract an exponent λ'_R/z from the slopes of the thermoremanent magnetisation. It turns out that the values of λ'_R/z thus obtained are significantly different

¹⁷ While systematic errors may easily affect estimates of critical exponents, universal amplitude ratios such as X_∞ should be much less sensitive to this problem and the dependence of X_∞ on the coupling constants might have diagnostic value.

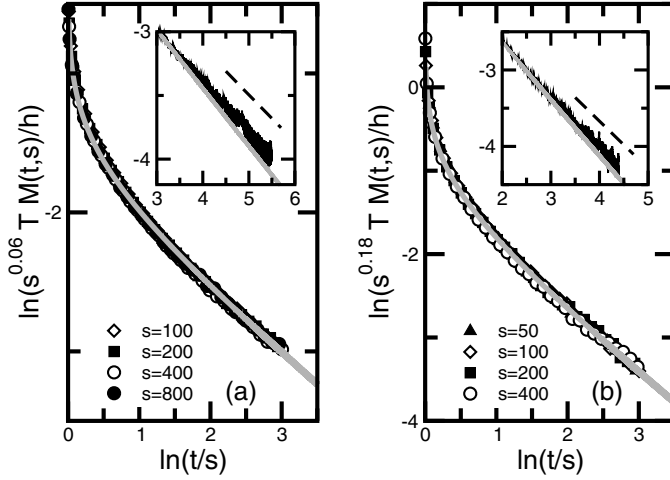


Fig. 3.12. Scaling of the thermoremanent magnetisation in the Edwards-Anderson spin glass at criticality with a bimodal distribution of the couplings in (a) three and (b) four dimensions. The full curve is the prediction of local scale-invariance, see Chap. 5. The results of very long runs for a single waiting time are shown in the insets where (a) $s = 100$ in three dimensions and (b) $s = 25$ in four dimensions. After [345]

from the ones found for λ_C/z . For example, for the Ising spin glass with a bimodal distribution of the couplings one finds $\lambda'_R/z = 0.45$ (resp. 0.72) in three resp. four dimensions. On the other hand, since X_∞ is finite, one must have $\lambda_R = \lambda_C$, but it may be necessary to go to very large values of $y = t/s$ in order to see this. Indeed, if one considers larger values of $y = t/s$ as is shown in the insets of Fig. 3.12ab, one observes a passage from an effective exponent λ'_R/z at intermediate values of the scaling variable y to the truly asymptotic value λ_R/z at larger values of y .

3.6.2 Gauge Glass

The **gauge glass** model (also called **randomly frustrated XY model**) is defined by the Hamiltonian [627]

$$\mathcal{H} = -J \sum_{(i,j)} \cos(\theta_i - \theta_j - A_{i,j}) \quad (3.74)$$

with the usual nearest-neighbour interactions on a simple hyper-cubic lattice. The vector potentials $A_{i,j} = -A_{j,i}$ are quenched random variables which are uniformly distributed over the interval $[0, 2\pi)$ and $J > 0$ is the exchange coupling. In the context of applications of this model to superconductivity, the variable θ_i represents the superconducting phase at site i and can be arranged as the angle of a classical two-dimensional vector $\vec{S}_i = (\cos \theta_i, \sin \theta_i)$.

It is thought that in $d > 2$ dimensions, the model undergoes a continuous phase transition. Values of $T_c(d)$ and some equilibrium critical exponents are listed in Table A9 in appendix A. The non-equilibrium properties of the model (3.74) may be analysed through the two-time autocorrelation

$$C(t, s) = \frac{1}{\mathcal{N}} \left[\left\langle \sum_{j=1}^{\mathcal{N}} \cos(\theta_j(t) - \theta_j(s)) \right\rangle \right]_{\text{av}} = s^{-b} f_C(t/s) \quad (3.75)$$

where the two averages to be performed are made explicit. The response is studied through the thermoremanent susceptibility, realised by perturbing the system by a small magnetic field $\vec{h}$ until the waiting time s has passed. Then, measuring at time t

$$\chi_{\text{TRM}}(t, s) = \frac{1}{h^2 \mathcal{N}} \left[\left\langle \sum_{j=1}^{\mathcal{N}} \vec{h}_j \cdot \vec{S}_j \right\rangle \right]_{\text{av}} = s^{-1-a} f_M(t/s) . \quad (3.76)$$

Quenching the 3D system from a fully disordered initial state to its critical temperature $T = T_c$, the values of the non-equilibrium exponents $a, b, \lambda_C/z, \lambda_R/z$ can be obtained [627] and are listed in Table 3.5, along with an estimate of the limit fluctuation-dissipation ratio X_∞ . It can be seen that the numerical values found are not far from those found in the critical Ising spin glass. On the other hand, although the same qualitative features apply to the ageing of the gauge glass model as for the non-glassy simple 3D critical XY model [6] (see Chap. 1), the values of the exponents and of X_∞ are quite distinct, see Table 1.7. Therefore, the 3D gauge glass and the 3D XY model are in different non-equilibrium universality classes [627].

3.6.3 Interacting Flux Lines

The physics of interacting vortex lines in disordered type-II superconductors is remarkably complex and has been a major research focus in condensed matter physics in the past two decades. A variety of distinct phases have been shown to exist in the temperature vs. magnetic field phase diagram [81]. By now the existence of glassy phases in vortex matter has been well-established. The Abrikosov lattice of the pure system is destroyed already by weak point-like disorder such as oxygen vacancies. The first-order vortex lattice melting transition of the pure system is then replaced by a continuous transition into a disorder-dominated vortex glass phase [249, 244, 236, 531]. Here, the vortices are collectively pinned, displaying neither translational nor orientational long-range order. In addition, there is now mounting evidence for a topologically ordered dislocation-free Bragg glass phase at low magnetic fields or for weak disorder [531, 276, 277, 444, 437, 243] .

Signatures of ageing in disordered vortex matter have been identified experimentally. For example, Du *et al.* recently demonstrated that the voltage response of a 2H-NbSe₂ sample to a current pulse depended on the pulse duration [210] (see also [325]). Out-of-equilibrium features of vortex glass systems relaxing towards their equilibrium state have been studied some time ago by Nicodemi and Jensen through Monte Carlo simulations of a two-dimensional coarse-grained model system [539, 540, 541, 538, 542]; however, this model applies to very thin films only since it disregards the prominent three-dimensional flux-line fluctuations. More recently, three-dimensional Langevin dynamics simulations of vortex matter have been employed by Olson *et al.* [561] and by Bustingorry, Cugliandolo, and Domínguez [112, 113] in order to investigate non-equilibrium relaxation kinetics, with quite intriguing results and unambiguous indications of ageing behaviour in quantities such as the two-time density-density correlation function, the linear susceptibility, and the mean-square displacements. Romá and Domínguez extended these studies to Monte Carlo simulations of the three-dimensional gauge glass model at the critical temperature [627].

In the Langevin dynamics simulations discussed in [112, 113] three-dimensional elastic flux lines are considered. The effective model Hamiltonian $\mathcal{H}$ is given in terms of the flux line trajectories $\mathbf{r}_i(z)$, where i labels the flux lines, and contains three contributions: the elastic line energy, the repulsive vortex-vortex interaction, and the disorder-induced pinning potential. The dynamics is given by the Langevin equation (with $\mathbf{r}_{iz} = \mathbf{r}_i(z)$)

$$D^{-1} \frac{\partial \mathbf{r}_{iz}}{\partial t} = \frac{\delta \mathcal{H} [\{\mathbf{r}_{iz}(t)\}]}{\delta \mathbf{r}_{iz}} + \boldsymbol{\eta}_{iz}(t) \quad (3.77)$$

where D^{-1} is a friction coefficient (called the Bardeen-Stephen friction coefficient) and $\boldsymbol{\eta}_{iz}(t)$ is a δ -correlated thermal force with zero average. In absence of interaction and defect pinning this equation decouples into independent Edwards-Wilkinson equations (3.156) which are used also in the context of growth processes [227]. We shall come back to the Edwards-Wilkinson equation in more detail in Sect. 3.11.

Using equation (3.77), Bustingorry, Cugliandolo, and Domínguez [112, 113] studied two-times quantities and found ageing in the various quantities as for example the mean-squared-displacement correlation or the density-density correlation, see Fig. 3.13. Interestingly, this study revealed temperature- and disorder-intensity-dependent ageing exponents, similar to what is observed in the disordered ferromagnets undergoing phase-ordering. In a recent paper [389], these temperature-dependent exponents are interpreted as being *effective exponents that arise from crossover-induced finite-size and finite-time effects*, the crossover connecting an initial power-law regime with a slow logarithm growth regime at later times.

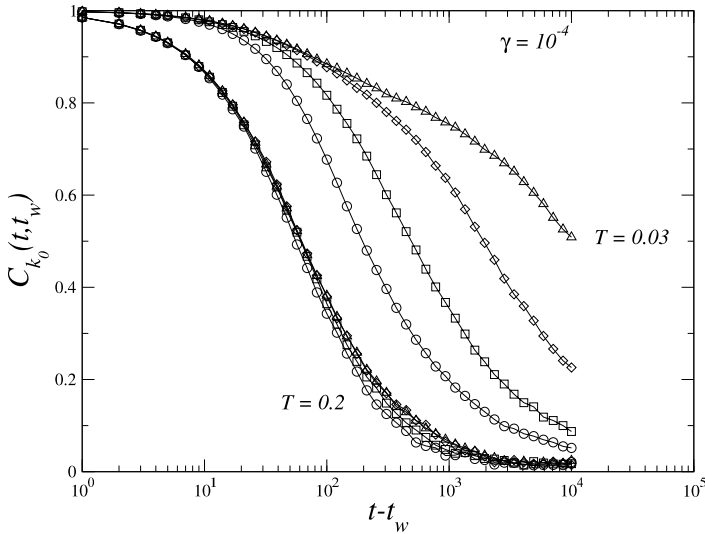


Fig. 3.13. Ageing of the density-density correlation in Langevin simulations using the dynamics (3.77). Shown is the Fourier component of the correlation with $k_0 = 0.27$. Two different temperatures are considered, and ageing is observed for the lower temperature. The waiting times are $s = t_w = 10$ (circles), 10^2 (squares), 10^3 (diamonds) and 10^4 (triangles). γ is a measure for the defect density. Reprinted with permission from [113]. Copyright (2007) by the American Physical Society.

Summarising, it seems fair to conclude that the current understanding of the dynamical scaling and ageing in critical spin glasses continues to evolve. People still appear to be far from a definite and final picture.

3.7 Surface Effects

Having studied ageing in spatially infinite systems, we now briefly review the consequences of introducing surfaces in the system. Surfaces are present in any real system and often play an essential role in the dynamics. In equilibrium critical phenomena and more generally in the critical behaviour of non-equilibrium steady-states it is well-known that the behaviour in the immediate vicinity of the boundary can be markedly different from the one in the bulk and furthermore is influenced by the boundary conditions used, see [129, 197, 388, 593, 269] and chapters 2 and 4 in Volume 1.

We consider the idealised situation of a d -dimensional semi-infinite system where position vectors are written in the form $\mathbf{r} = (\mathbf{r}_{\parallel}, r_{\perp})$ where $\mathbf{r}_{\parallel}$ is a

$(d - 1)$ -dimensional vector parallel to the surface and we restrict to $r_\perp \geq 1$, where $r_\perp = 1$ represents the surface layer. The *equilibrium* critical dynamics of such semi-infinite systems was studied long ago [201, 619, 493] and it has been established that *the dynamical exponent z , even for observables studied near the surface, retains the same value as in the bulk.*

Consider the time-dependent order-parameter and the following two-time correlation and response functions

$$\begin{aligned} M(t, r_\perp; m_0) &= \langle \phi(t; r_\perp, \mathbf{r}_\parallel) \rangle \\ C(t, s; r_\perp, r'_\perp, \mathbf{r}_\parallel - \mathbf{r}'_\parallel) &= \langle \phi(t; r_\perp, \mathbf{r}_\parallel) \phi(s; r'_\perp, \mathbf{r}'_\parallel) \rangle \\ R(t, s; r_\perp, r'_\perp, \mathbf{r}_\parallel - \mathbf{r}'_\parallel) &= \left. \frac{\langle \delta \phi(t; r_\perp, \mathbf{r}_\parallel) \rangle}{\delta h(s; r'_\perp, \mathbf{r}'_\parallel)} \right|_{h=0} \end{aligned} \quad (3.78)$$

and where translation-invariance in the direction parallel to the wall was already assumed. We explicitly added a possible dependence on an initial value m_0 of the order-parameter. Later, we shall focus on the local autocorrelation and autoresponse functions and their anticipated scaling forms

$$\begin{aligned} C_1(t, s) &:= C(t, s; 1, 1, \mathbf{0}) = s^{-b_1} f_{C_1}(t/s) \quad , \quad f_{C_1}(y) \sim y^{-\lambda_{C_1}/z} \\ R_1(t, s) &:= R(t, s; 1, 1, \mathbf{0}) = s^{-1-a_1} f_{R_1}(t/s) \quad , \quad f_{R_1}(y) \sim y^{-\lambda_{R_1}/z} \end{aligned} \quad (3.79)$$

which should hold true as usual when t, s and $t - s$ become much larger than any microscopic reference scale. In this way, surface analogues of the ageing, autocorrelation and autoresponse exponents are defined.

It turns out, however, that the short-time scaling regime has properties not yet seen deep in the bulk [598, 599]. In order to understand these, consider a quench to the critical point $T = T_c$, where one has the scaling form $M(t, r_\perp; m_0) = t^{-x/z} \tilde{M}(r_\perp t^{-1/z}, m_0 t^{x_0/z})$ of the order-parameter, where $x = \beta/\nu$ is the (bulk) scaling dimension of the order-parameter and x_0 is its initial scaling dimension, see exercise 3.3. Deep in the bulk ($r_\perp \rightarrow \infty$), one has, for a sufficiently small initial magnetisation (see exercise 3.4)

$$M(t, \infty; m_0) = t^{-x/z} \tilde{M}(m_0 t^{x_0/z}) \sim \begin{cases} m_0 t^{(x_0-x)/z} & ; \text{ if } m_0 t^{x_0/z} \ll 1 \\ t^{-x/z} & ; \text{ if } m_0 t^{x_0/z} \gg 1 \end{cases} \quad (3.80)$$

and by comparison with the discussion of short-time critical dynamics from Chap. 1, one identifies the slip exponent $\Theta = (x_0 - x)/z$. Conversely, since Θ is known in many models, see Table 1.7, values for the exponent x_0 may be found. Next, consider for a moment the *equilibrium* scaling behaviour near to a surface. Then one has (see exercise 3.3) the scaling form $M_{\text{eq}}(t, r_\perp) = t^{-x/z} \tilde{M}(r_\perp t^{-1/z}) \sim t^{-x/z} (r t^{-1/z})^{x_1-x}$ where $x_1 = \beta_1/\nu$ is the surface scaling dimension of the order-parameter. This result has been confirmed from a short-distance expansion in field-theory [197, 618, 620, 495].

In extending these considerations to the non-equilibrium case, notice that a cluster, starting from an initial seed, is growing in time with a factor $\sim t^{x_0/z}$ while the relaxation process will reduce its size by a factor $\sim t^{x_l/z}$, where the local exponent $x_l = x$ if one is deep in the bulk while $x_l = x_1$ close to the surface. Now an important distinction has to be made.

1. **Domain growth** regime, $x_0 > x_1$. In this case the non-equilibrium growth dominates over the relaxation near to the surface and the non-equilibrium ordering penetrates into the surface region. Then the short-distance behaviour of the equilibrium case remains valid, such that for short times one finds

$$M(t, r_\perp; m_0) \sim m_0 r^{x_1 - x} t^{(x_0 - x_1)/z} \quad (3.81)$$

from which one may identify a **surface slip exponent** [618, 495]

$$\Theta_1 = \frac{x_0 - x_1}{z} = \Theta - \frac{\beta_1 - \beta}{\nu z} = \Theta - \frac{\eta_\parallel - \eta}{2z} \quad (3.82)$$

2. **Cluster dilution** regime, $x_0 < x_1$. Now, at least for short times, relaxation dominates such that non-equilibrium order cannot penetrate to the surface. In consequence, no new clusters form and the existing ones slowly dissolve. Then the dynamics is governed by the large clusters, of weight $\sim \ell^d$, which are present in the initial configuration with the very small probability $P(\ell) = m_0^{\ell^d}$. After a time t , the initial weight of a cluster will be proportional to $t^{(x_0 - x_1)/z}$. A cluster dissolves if a domain wall, of weight $\sim \ell^{d-1}$, is created. Domain walls of an initial linear size ℓ can be created at essentially no cost after a time $t(\ell)$, where $t(\ell)^{(x_0 - x_1)/z} \ell^{d-1} = O(1)$. At times $t > t(\ell)$, only clusters exist which were initially larger than ℓ . Hence, the average magnetisation at time t is $M(t) \sim \sum_{\ell \geq \ell(t)} P(\ell) \ell^d$. For large enough times, the main contribution to the sum comes from the smallest possible clusters and this leads to the short-time behaviour [598, 599]

$$M(t, r_\perp; m_0) \sim t^{-x/z} \exp\left(-\mathfrak{c} (r_\perp t^{-1/z})^\kappa\right) \quad , \quad \kappa = \min\left(\frac{(x_1 - x_0)d}{d-1}, z\right) \quad (3.83)$$

and $\mathfrak{c}$ is some constant. This non-algebraic decay is characteristic for the short-time dynamics of cluster dilution.¹⁸

For larger times, both cases go over to the usual decay $M(t, 0; m_0) \sim t^{-\beta_1/\nu z}$ near to a free surface.

Analogously, the behaviour of the two-time correlator may be studied. First, in the case with *domain growth*, $x_0 > x_1$, one finds near to the surface and for $t \gg s$ limit $C(t, s) \sim s^{-\lambda_{C_1}/z}$ where [618, 495]

¹⁸ The condition $x_0 < x_1$ is for example satisfied at the ordinary transition of the 3D Ising model. Cluster dilution should therefore be observable experimentally.

$$\lambda_{C_1} = d - x_0 - x + 2x_1 = \lambda_C + \frac{2(\beta_1 - \beta)}{\nu} = \lambda_C + \eta_{\parallel} - \eta \quad (3.84)$$

which relates the surface autocorrelation exponent to known exponents. In the case of *cluster dilution*, $x_0 < x_1$, one has in the short-time regime $t - s \ll s$, and near to the surface [598, 599]

$$C(t, s) \sim \exp \left(-\mathfrak{c}' (t - s)^{\kappa/z} \right) \quad (3.85)$$

and where κ was given in (3.83) and $\mathfrak{c}'$ is again some constant.

The scaling relations (3.82, 3.84) and the cluster-dilution predictions (3.83, 3.85) have been confirmed [598, 599] in great detail for several distinct systems, namely

1. the semi-infinite Ising model with Hamiltonian $\mathcal{H} = -J_s \sum_{\text{surface}} \sigma_i \sigma_j - J_b \sum_{\text{bulk}} \sigma_i \sigma_j$ where the first sum is restricted to nearest-neighbour spins lying both in the surface layer whereas the second sum runs over all other nearest-neighbour pairs. The equilibrium critical behaviour close to a surface has been outlined in Volume 1, Chap. 2, and appendix A lists values of the independent surface critical exponents for several universality classes. For J_s small enough, the surface does not order at $T = T_c$ and has an ordinary transition. In $3D$, for couplings $J_s \gtrsim 1.50J_b$, the surface can order independently of the bulk and the meeting point with the ordinary transition is the so-called special transition.
2. the Hilhorst-van Leeuwen model [362] in the marginal case $\omega = 1$ where x_1 depends continuously on a control parameter.
3. a variant of the Ising model with an infinitely long defect line [38, 513, 340] such that the local exponent x_l depends continuously on the defect strength.

As an illustration, we show in Fig. 3.14 [599] the time-dependent magnetisation $m_1(t)$ near the surface for the $2D/3D$ semi-infinite Ising model (where $x_0 \simeq 0.53$ in $2D$ and $\simeq 0.74$ in $3D$) and for several values of the non-universal amplitude in the Hilhorst-van Leeuwen model, where again $x_0 \simeq 0.53$. The change of behaviour when going from the domain-growth regime with $x_0 < x_1$ to the cluster-dilution regime with $x_0 > x_1$ is clearly seen.

Having understood the scaling relations (3.82, 3.84), we now return to the scaling forms (3.78). For quenches to $T = T_c$, one further expects

$$a_1 = b_1 = (d - 2 + \eta_{\parallel})/z. \quad (3.86)$$

Since the surface fluctuation-dissipation ratio and its limit

$$X_1(t, s) := \frac{TR_1(t, s)}{\partial_s C_1(t, s)} \quad , \quad X_{1,\infty} := \lim_{s \rightarrow \infty} \left(\lim_{t \rightarrow \infty} X_1(t, s) \right) \quad (3.87)$$

are finite at $T = T_c$, one must also have $\lambda_{R_1} = \lambda_{C_1}$. This may be confirmed in several systems, both at the ordinary and at the special transition, as is

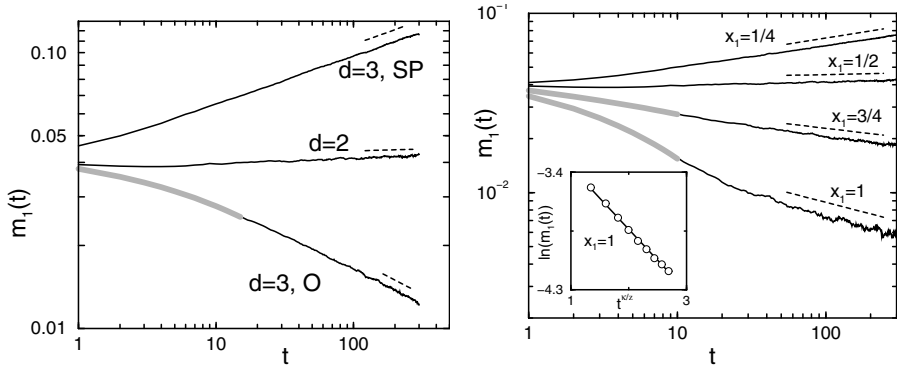


Fig. 3.14. Relaxation of the local magnetisation $m_1(t)$. Left panel: 2D/3D semi-infinite Ising model, at the ordinary (O) or special transition (SP). Right panel: Hilhorst-van Leeuwen model for the surface scaling dimension $x_1 = [1/4, 1/2, 3/4, 1]$ from top to bottom. The inset displays $\ln m_1(t)$ over against $t^{\kappa/z}$ for the case $x_1 = 1$ as a test of (3.83). The grey lines are fits to the stretched exponential (3.83) for early times in the cluster-dilution regime and the dashed lines indicate the power-law behaviour expected from (3.82) for late times. Reprinted with permission from [599]. Copyright (2005) by the American Physical Society.

model	d	$a_1 = b_1$	λ_{C_1}/z	λ_{R_1}/z	$X_{1,\infty}$	Ref.
Gaussian	d	$d/2$	$d/2 + 1$	$d/2 + 1$	$1/2$	[121]
spherical	< 4	$d/2$	$3d/4$	$3d/4$	$1 - 2/d$	[50]
	> 4	$d/2$	$d/2 + 1$	$d/2 + 1$	$1/2$	[50]
Ising	> 4	$d/2$	$d/2 + 1$	$d/2 + 1$	$1/2$	[51]
	3	1.24	2.10(1)	2.18(3)	0.59(2)	[592]
	2	0.46	1.09(1)	1.12(2)	0.37(1)	[592]
Hilhorst-van	2	0.85	0.84(1)	0.84(1)	0.31(1)	[592]
		0.96	0.96(1)	0.95(1)	0.33(1)	[592]
Leeuwen		1.19	1.22(2)	1.21(2)	0.40(2)	[592]
		1.31	1.30(2)	1.35(3)	0.43(2)	[592]

Table 3.7. Exponents and limit fluctuation-dissipation ratio for critical ageing systems ($T = T_c$), near to a plane surface, for the ordinary transition, which corresponds to Dirichlet boundary conditions. In the Hilhorst-van Leeuwen model, the parameter $A = [0.50, 0.25, -0.25, -0.50]$ from top to bottom.

shown in Tables 3.7 and 3.8 for $T = T_c$. In the analytically solved models, one has always $a_1 = b_1$ and $\lambda_{R_1} = \lambda_{C_1}$ and we also observe that in general $X_{1,\infty} \neq X_\infty$. A few results are also available for quenches to $T < T_c$, see Table 3.9.

model	d	$a_1 = b_1$	λ_{C_1}/z	λ_{R_1}/z	$X_{1,\infty}$	Ref.
spherical	< 4	$d/2 - 1$	$3d/4 - 1$	$3d/4 - 1$	$1 - 2/d$	[50]
	> 4	$d/2 - 1$	$d/2$	$d/2$	$1/2$	[50]
Ising	3	0.37	1.16(2)	1.24(2)	0.44(2)	[592]

Table 3.8. Exponents and limit fluctuation-dissipation ratio for critical ageing systems ($T = T_c$), near to a plane surface, for the special transition (Neumann boundary conditions).

Exact results are known in the spherical model [50].¹⁹ In the case of **Dirichlet boundary conditions**, the order-parameter $\phi|_{\partial\Lambda} = 0$ vanishes at the surface (the boundary of the lattice Λ), while for **Neumann boundary conditions**, its gradient $\nabla\phi|_{\partial\Lambda} = 0$ vanishes at the boundary. At criticality, Dirichlet boundary conditions correspond to the ordinary transition and Neumann boundary conditions correspond to the special transition. Let us decompose the position vector $\mathbf{r} = (\mathbf{r}_{\parallel}, r_{\perp})$ into parallel and perpendicular components with respect to the free surface. Then the response function is in the scaling limit given by

$$R^{(d,n)}(t, s; \mathbf{r}, \mathbf{r}') = R(t, s) \exp\left(-\frac{(\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel})^2}{4(t-s)}\right) \times \left[\exp\left(-\frac{(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp})^2}{4(t-s)}\right) \mp \exp\left(-\frac{(\mathbf{r}_{\perp} + \mathbf{r}'_{\perp})^2}{4(t-s)}\right) \right] \quad (3.88)$$

where $R(t, s)$ is the bulk autoresponse function derived in Chap. 2 and (d) respectively (n) stands for **Dirichlet** or **Neumann** boundary conditions. Near to the surface, one can admit that $|\mathbf{r}_{\perp}| = O(a_{\text{mic}})$ where a_{mic} is some microscopic length-scale such as a lattice constant. Expanding the second line in (3.88) and setting $\mathbf{r}_{\parallel} = \mathbf{r}'_{\parallel}$, one finds the **surface autoresponses**

$$R_1^{(d)}(t, s) \sim R(t, s)(t-s)^{-1}, \quad R_1^{(n)}(t, s) \sim 2R(t, s) \quad (3.89)$$

from which the scaling forms and the response exponents of the spherical model, listed in Tables 3.7 and 3.9, can be recovered from the bulk val-

model	d	Dirichlet			Neumann			Ref.
		b_1	a_1	$\lambda_{C_1} = \lambda_{R_1}$	b_1	a_1	$\lambda_{C_1} = \lambda_{R_1}$	
spherical	> 2	1	$d/2$	$d/2 + 2$	0	$d/2 - 1$	$d/2$	[50]
Ising	2	0	$1/2$	1.90(6)				[51]

Table 3.9. Exponents for ageing systems with $T < T_c$, near to a plane surface, for Dirichlet and Neumann boundary conditions.

¹⁹ The *surface* critical behaviour of the spherical model is *not* necessarily the same as the one of the $n \rightarrow \infty$ limit of the $O(n)$ -model near to a surface, see [75, 197].

ues, see Chap. 2. The surface autocorrelator has the scaling form $C_1(t, s) = s^{-b_1} f_{C_1}(t/s)$ where the scaling function reads for Dirichlet boundary conditions, up to normalisation

$$f_{C_1}^{(d)}(y) = \begin{cases} y^{d/4}(y+1)^{-1-d/2} & ; \text{ if } T < T_c \\ y^{1-d/4}(y+1)^{-1}(y-1)^{-d/2} & ; \text{ if } T = T_c, 2 < d < 4 \\ \times (1 - (4/d)(y+1)^{-1}) & \\ (y-1)^{-d/2} - (y+1)^{-d/2} & ; \text{ if } T = T_c, d > 4 \end{cases} \quad (3.90)$$

and for Neumann boundary conditions

$$f_{C_1}^{(n)}(y) = \begin{cases} y^{d/4}(y+1)^{-d/2} & ; \text{ if } T < T_c \\ y^{1-d/4}(y+1)^{-1}(y-1)^{1-d/2} & ; \text{ if } T = T_c, 2 < d < 4 \\ (y-1)^{1-d/2} - (y+1)^{1-d/2} & ; \text{ if } T = T_c, d > 4 \end{cases} \quad (3.91)$$

At $T = T_c$ and $d > 4$, these scaling forms are identical to the ones found in the exactly solved $d \rightarrow \infty$ limit of the semi-infinite Ising model [51] and the semi-infinite Gaussian model [121].

3.8 Ageing with Absorbing Steady-states I

So far, we have discussed the ageing behaviour of simple magnets. At least locally, these always relax towards an equilibrium state, although a global equilibrium state is never reached. When formulating these systems in terms of a master equation, relaxing towards equilibrium states was guaranteed because the dynamics was chosen to satisfy detailed balance. In this and the next section, we shall look at some examples of what may happen when detailed balance is no longer valid.

In Chap. 3 of Volume 1, we have used the **contact process** [313] as paradigmatic example of a system with an absorbing phase transition in the steady-state. This model may be viewed as a simple description of the spreading of an epidemic. Reformulated in terms of an equivalent reaction-diffusion process, one considers a system of particles of a single species A which may move diffusively by nearest-neighbour hopping on a lattice and undergo the reactions $A \rightarrow \emptyset$ and $A \rightarrow 2A$. On the level of mean-field theory, one considers the order-parameter $\phi(t, \mathbf{r})$, interpreted as the spatio-temporal distribution of infected individuals. The competition between extinction of isolated individuals, reproduction and diffusive motion leads to a mean-field equation

$$\partial_t \phi(t, \mathbf{r}) = \Delta_L \phi(t, \mathbf{r}) + \tau \phi(t, \mathbf{r}) - \phi(t, \mathbf{r})^2 \quad (3.92)$$

where for simplicity several rescalings have been made. The long-time behaviour depends on the value of the control parameter τ . If $\tau < 0$, the system is in the **absorbing phase** and evolves towards the **absorbing state**, characterised by $\phi(t, \mathbf{r}) = 0$, into which the system can enter but cannot leave. On

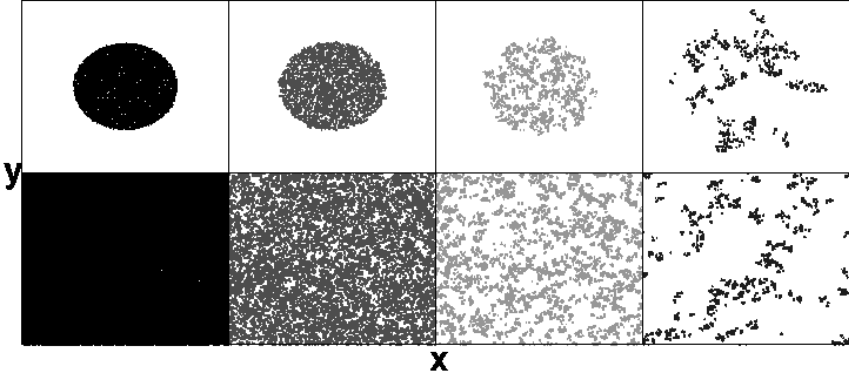


Fig. 3.15. Microscopic evolution of clusters in the critical 2D contact process, on a lattice of size 1000×1000 . The initial condition of the upper series is a full circle with radius 100 placed in the centre of the lattice, while in the lower series it is a full lattice. The times are $t = [2, 20, 200, 2000]$ for the upper series and $t = [20, 200, 2000, 20000]$ below. Reprinted from [609]. Copyright (2004) Institute of Physics Publishing.

the other hand, for $\tau > 0$, the stationary state is in the **active phase**, with an average non-vanishing particle density. At the critical point $\tau = 0$, the system undergoes an **absorbing phase transition**. As for equilibrium systems, these may be characterised by the values of the associated critical exponents. For example, the stationary order parameter behaves as $\phi_s \sim \tau^\beta$, with the value $\beta_{\text{MF}} = 1$ in mean-field theory. We refer to Table 4.1 in Volume 1 for the definition of the other exponents of absorbing phase transitions. For the contact process, the transition is in the universality class of **directed percolation** and the values of the exponents are listed in Table 4.3 in Volume 1.

Following [609, 230] it is therefore natural to ask if there occurs some analogue of the ageing phenomena we have studied in simple magnets, with the Ising model as paradigmatic example. We have seen before that ageing will only occur if there are at least two equivalent competing steady-states (or a critical steady-state). Since in the contact process both the active and the absorbing phase have only a single stable steady-state, one would expect that ageing only occurs *at* the critical point. This is indeed the case, as we shall see. We also remark that the kind of models studied here represents only one of several possible ways to suppress the relaxation towards an equilibrium state. For example, detailed balance in an Ising model with a conserved order-parameter may be removed by introducing an additional external driving (‘electrical’) field. Ageing phenomena in such systems, if they exist at all, have apparently not yet been studied.

3.8.1 Contact Process

A first qualitative idea of the dynamics of the critical contact process can be seen by looking at the temporal evolution of certain initial configurations, see Fig. 3.15. Recall that in phase-ordering, see Fig. 1.6, a non-vanishing surface tension creates and maintains ordered domains, whose linear size grows according to the law $L(t) \sim t^{1/z}$. Similarly, in non-equilibrium critical dynamics, there is no surface tension but the dynamics is still driven by growing correlated clusters, see Fig. 3.1. From Fig. 3.15 we see that in the critical contact process there is no apparent growing length-scale at all. Rather, the evolution proceeds via a different mechanism, namely **cluster dilution**. Cluster dilution has first been demonstrated to occur in several variants of the two-dimensional **voter model** [207, 182], but its occurrence is neither specific to two spatial dimensions nor to the voter-model universality class. Remarkably, cluster dilution also occurs in the early stages of surface ageing in simple magnets, as discussed above in Sect. 3.7.

In studying the dynamics, we first consider the average particle-density $N(t) := \langle n_i(t) \rangle$ which because of spatial translation-invariance is independent of the site i . Ageing is conveniently studied through the two-time (connected) autocorrelator²⁰ and autoresponse functions

$$\begin{aligned} C(t, s) &:= \langle n_i(t)n_i(s) \rangle - \langle n_i(t) \rangle \langle n_i(s) \rangle \\ R(t, s) &:= \left. \frac{\delta \langle n_i(t) \rangle}{\delta h_i(s)} \right|_{h=0} \end{aligned} \quad (3.93)$$

where $h_i(s)$ is the rate of the spontaneous creation process $\emptyset \rightarrow A$ in the site i at time s . While the calculation of the time-dependent averages $\langle n_i(t) \rangle$ and $\langle n_i(t)n_i(s) \rangle$ is straightforward, the calculation of the autoresponse is more involved, because of the implied functional derivative. However, for the contact process, its calculation is still not too complicated. One may work either in the setting of the LCTMRG and use the explicit expression (4.232) for $R(t, s)$ derived in Volume 1, Chap. 4. Since no random numbers are involved, noisiness of the data is not an issue. Alternatively, one may also measure $R(t, s)$ through a simulation in the contact process, since the spatio-temporal cluster that would have been generated without perturbation is always a subset of the actual cluster. This special structure of DP allows one to tag all inactive sites at time s by mutually exclusive labels and to monitor during the subsequent temporal evolution how they would have changed the pattern of active sites.²¹

The time-dependent average density converges exponentially fast towards its steady-state value when $p \neq p_c$ and $N(t) \sim t^{-\delta}$ for $p = p_c$. More interesting is the behaviour of the two-time quantities.

²⁰ We warn the reader that in [609, 230, 49, 552] the *unconnected* correlation function is denoted by C . Similar notations are also used for the respective exponents.

²¹ Integrated responses for the contact process are discussed in [609].

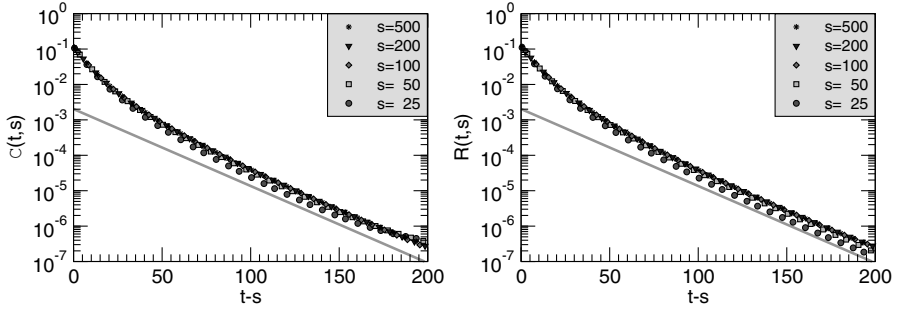


Fig. 3.16. Connected autocorrelator $C(t, s)$ and autoresponse $R(t, s)$ of the 1D contact process in the active phase ($p = 0.1$). The straight lines are proportional to $\exp(-0.05(t - s))$. Reprinted from [230]. Copyright (2004) Institute of Physics Publishing.

Active phase

In contrast with simple magnets, where there are two distinct stable ground states in the low-temperature phase, in the active phase of the contact process there is only a *single* stable steady state. Consequently, there is here *no* breaking of time-translation invariance and we illustrate this in 1D in Fig. 3.16. After a short transient, the data for both $C(t, s)$ and $R(t, s)$ collapse when plotted over against $t - s$ and their decay can be described in terms of a single characteristic time. This means that the contact process shows no ageing in its active phase.

Absorbing phase

From the comparison with the high-temperature phase of simple magnets, one would also expect to find time-translation invariance in the absorbing phase of the contact process. However, the correlation function shows a subtlety the origin of which is best understood by considering the case $p = 1$ first. If $p = 1$, particles on different sites are uncorrelated and simply decay with a fixed rate. For any fixed site i and with two times $t > s$, it is clear that $n_i(t)n_i(s) = n_i(t)$, since $n_i \in \{0, 1\}$. Hence $\langle n_i(t)n_i(s) \rangle = \langle n_i(t) \rangle$ and $C(t, s) = N(t)(1 - N(s))$. For sufficiently long times, $C(t, s)$ will then only depend on t . Indeed, this behaviour survives in the entire absorbing phase [230]. On the other hand, the expected time-translation invariance for the autoresponse function is readily checked.

Critical point

For the critical contact process we show in Fig. 3.17a that ageing does occur, that is, the autocorrelator (and the autoresponse as well) depend on *both* the

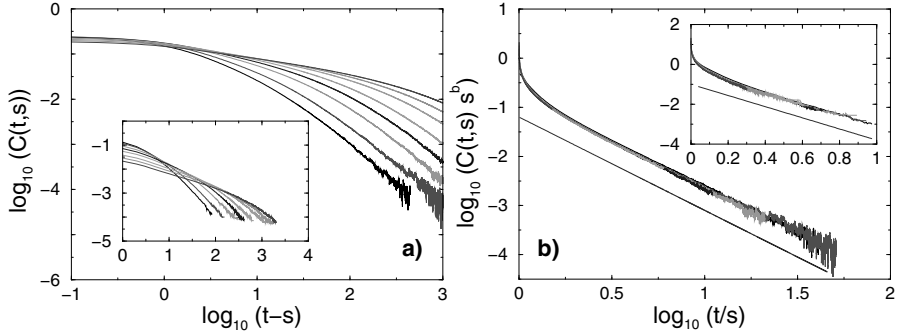


Fig. 3.17. Connected autocorrelator of the critical contact process in 1D (main plots) and 2D (insets). Panel (a) shows the ageing of the autocorrelator and panel (b) illustrates the scaling behaviour. The straight lines correspond to the exponents $\lambda_C/z = 1.9$ in 1D and 2.8 in 2D. Reprinted from [230]. Copyright (2004) Institute of Physics Publishing.

observation time t and the waiting time s . Furthermore, when the same data are re-plotted over against t/s , a data collapse after rescaling can be achieved, see Fig. 3.17b. As a starting configuration the LCTMRG used a completely filled lattice [230] whereas the Monte Carlo study started from configurations with initial density $n_0 = 0.8$ [609]. By analogy with simple magnets, one defines the ageing exponents a, b and the autocorrelation and autoresponse exponents $\lambda_{C,R}$ from

$$\begin{aligned} C(t, s) &= s^{-b} f_C(t/s) \quad , \quad f_C(y) \sim y^{-\lambda_C/z} \\ R(t, s) &= s^{-1-a} f_R(t/s) \quad , \quad f_R(y) \sim y^{-\lambda_R/z} \end{aligned} \quad (3.94)$$

where the asymptotic forms should hold for $y \rightarrow \infty$. Similarly, scaling can be observed for the autoresponse function as shown in Fig. 3.18. On the other hand, the unconnected correlator behaves for large times simply as $\langle n_i(t) n_i(s) \rangle \sim (ts)^{-b/2}$.

The results for the ageing exponents $a, b, \lambda_C, \lambda_R$ are collected in Table 3.10. The agreement between the results of the LCTMRG and Monte Carlo simulations serves as a useful control on the reliability of the results. Available experimental results [693] will be discussed in the next sub-Section. While the equality $\lambda_C = \lambda_R$ is fully analogous to what was seen in non-equilibrium critical dynamics, the exponents a and b are no longer equal but satisfy

$$1 + a = b = 2\delta \quad . \quad (3.95)$$

Some comments are in order.

1. The relation $1 + a = b$ can be understood [46] to be a consequence of the rapidity-reversal symmetry of Reggeon field-theory. The field-theoretical

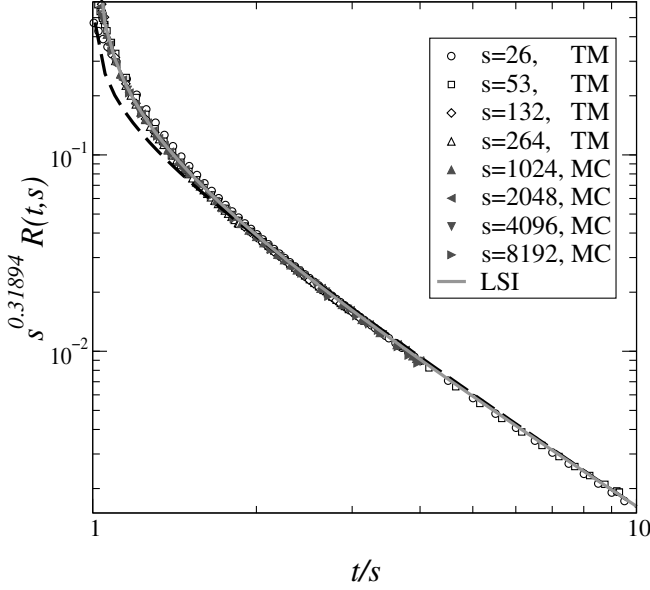


Fig. 3.18. Autoresponse function for the critical 1D contact process for several waiting times s . The data labelled TM come from the transfer matrix renormalisation group [230] and MC denotes Monte Carlo data [364]. The full line gives the LSI prediction (5.41) with $a - a'$ adjusted for an optimal fit and the dashed line corresponds to the assumption $a = a'$. Reprinted from [336]. Copyright (2006) Institute of Physics Publishing.

action in the Janssen-de Dominicis formulation reads *at* the critical point

$$\mathcal{J}[\phi, \tilde{\phi}] = \int dt d\mathbf{r} \left[\tilde{\phi} (\partial_t - D\Delta) \phi - u (\tilde{\phi} - \phi) \tilde{\phi} \phi - h \tilde{\phi} \right] \quad (3.96)$$

where ϕ and h are the coarse-grained particle-densities and creation rates for particles. For $h = 0$ and if the time $t \in \mathbb{R}$ is unbounded, the action is invariant under the **rapidity-reversal** transformation

$$\tilde{\phi}(t, \mathbf{r}) \mapsto -\phi(-t, \mathbf{r}) \quad , \quad \phi(t, \mathbf{r}) \mapsto -\tilde{\phi}(-t, \mathbf{r}) \quad . \quad (3.97)$$

In particular, it follows that the scaling dimensions $x_\phi = x_{\tilde{\phi}} = \beta/\nu$ of the order-parameter and the response field must be equal. This remains true even if rapidity-reversal is broken by initial conditions at time $t = 0$. For a rapidity-reversal-invariant action $\mathcal{J}$, the connected correlator is [46]

$$\begin{aligned} C(t, s; \mathbf{r} - \mathbf{r}') &= \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \rangle - \langle \phi(t, \mathbf{r}) \rangle \langle \phi(s, \mathbf{r}') \rangle \\ &= \langle \tilde{\phi}(-t, \mathbf{r}) \tilde{\phi}(-s, \mathbf{r}') \rangle - \langle \tilde{\phi}(-t, \mathbf{r}) \rangle \langle \tilde{\phi}(-s, \mathbf{r}') \rangle \\ &= 0 \end{aligned} \quad (3.98)$$

model	d	a	b	λ_C/z	λ_R/z	method	Ref.
CP	1	-0.68(5)	0.32(5)	1.85(10)	1.85(10)	LCTMRG	[230]
		-0.57(10)	0.3189	1.9(1)	1.9(1)	Monte Carlo	[609]
		-0.6810			1.76(5)	Monte Carlo	[364]
		-0.6810	0.3189	1.7921	1.7921	scaling	[46]
	2	0.3(1)	0.901(2)	2.8(3)	2.75(10)	Monte Carlo	[609]
		-0.198(2)	0.901(2)	2.58(2)	2.58(2)	scaling	[46]
			0.9(1)	2.5(1)		experiment	[693]
	> 4	$d/2 - 1$	$d/2$		$d/2 + 2$	mean-field	[609]
NEKIM	1	-0.430(4)	0.570(4)	1.9(1)	1.9(2)	Monte Carlo	[552]
		-0.430(4)	0.570(4)	1.86(1)	1.86(1)	scaling	
BCPD	≥ 1	$d/2 - 1$	$d/2 - 1$	$d/2$	$d/2$	E	[49]
BCPL	$\geq \eta$	$d/\eta - 1$	$d/\eta - 1$	d/η	d/η	E	[215]
BPCPD	> 2	$d/2 - 1$	$d/2 - 1$	$d/2$	$d/2$	E, $\alpha < \alpha_C$	[49]
	< 4	$d/2 - 1$	0	$d/2$	$d/2$	E, $\alpha = \alpha_C$	
	> 4	$d/2 - 1$	$d/2 - 2$	$d/2$	$d/2$	E, $\alpha = \alpha_C$	
BPCPL	$> \eta$	$d/\eta - 1$	$d/\eta - 1$	d/η	d/η	E, $\alpha < \alpha_C$	[215]
	$< 2\eta$	$d/\eta - 1$	0	d/η	d/η	E, $\alpha = \alpha_C$	
	$> 2\eta$	$d/\eta - 1$	$d/\eta - 2$	d/η	d/η	E, $\alpha = \alpha_C$	
EW2	≥ 1	$d/2 - 1$	$d/2 - 1 - \rho$	$d/2 - \rho$	$d/2$	E	[629]
MH1	≥ 1	$d/4 - 1$	$d/4 - 1$	$d/4$	$d/4$	E	[629]
MH2	≥ 1	$d/4 - 1$	$d/4 - 1 - \rho/2$	$d/4 - \rho/2$	$d/4$	E	[629]
MHC	≥ 2	$(d-2)/4$	$(d-2)/4$	$(d+2)/4$	$(d+2)/4$	E	[47]

Table 3.10. Non-equilibrium exponents for the following critical models: the contact process (CP), the non-equilibrium kinetic Ising model (NEKIM), the diffusive bosonic contact process (BCP) and the bosonic pair-contact process (BPCPD) and their analogues with Lévy flights (BCPL and BPCPL with the Lévy parameter $0 < \eta < 2$), where the results depend on the location on the critical line. Methods used are Monte Carlo simulation [609], LCTMRG [230], exact solution [49, 215] and experiment [693]. Several kinetic growth models based on the Edwards-Wilkinson (EW) and Mullins-Herring (MH) equations are also listed. The model EW1 is the same universality class as the BCPD. If possible, the exponents are compared with the scaling relations (3.95) and (3.99). An exact result is labelled by E.

where the second line comes from rapidity-reversal symmetry and the last line follows from causality. Hence $C(t, s; \mathbf{r}) = 0$ in the steady-state but for relaxations from an initial state $C(t, s; \mathbf{0}) = \langle \phi(t)\phi(s) \rangle_c$ and $R(t, s; \mathbf{0}) = \langle \phi(t)\tilde{\phi}(s) \rangle$ are *non*-vanishing and have the same scaling dimensions, which implies $b = a + 1$ and also $\lambda_C = \lambda_R$. The relationship $b = 2\delta$ follows from comparison with the scaling of the autocorrelator in the steady-state, see Chap. 3 in Volume 1.

2. Ageing in the contact process differs from the one found in magnets in that the order-parameter is initially non-vanishing. Then one has the scaling

relation [46]

$$\frac{\lambda_C}{z} = \frac{\lambda_R}{z} = 1 + \frac{d}{z} + \frac{\beta}{\nu_\perp z} \quad (3.99)$$

which relates $\lambda_{C,R}$ to known exponents of the absorbing phase transition. Equation (3.99) generalises the scaling relation (3.18) which holds for systems satisfying detailed balance and hence with equilibrium stationary states. The predictions following from (3.99) are listed in Table 3.10 with the method indicated as ‘scaling’.

3. As we have already discussed in Sections 1.4 and 3.2, the assumption that the stochastic process which describes the global order-parameter is a Markov process, relates the autocorrelation exponent to the global persistence exponent θ_g through the scaling relation [493]

$$\frac{\lambda_C}{z} = \theta_g - \frac{2(1-d) - \eta}{2z} . \quad (3.100)$$

Using the available estimates [368, 13] for θ_g , see also Table 3.1, from this relation one would find $\lambda_C/z = 1.98(2)$ in $1D$ and $\lambda_C/z = 3.5(5)$ in $2D$, in disagreement with the exact scaling relation (3.99). This indicates the existence of an effective non-Markovian dynamics and long-range temporal correlations in the critical contact process. Sometimes, the presence of such long-range interactions can be made explicit. Indeed, recent simulations [180, 181] yielded evidence that the non-equilibrium exponents of certain stochastic differential equations with delay may be the same as for directed percolation.

The importance of effective long-range interactions for the description of either the long-time and/or the long-range collective behaviour has already been commented on in Volume 1, when discussing directed percolation with spatio-temporal Lévy flights. We wish to draw to the attention of the reader the fact that the phase-diagram Fig. 5.6 in Volume 1 [7, 605] shows that even if the effective dynamical exponent $z = 2$, the critical behaviour need not be given by a simple, Markovian mean-field theory.

4. The non-equality of the exponents a and b means that no analogue of a fluctuation-dissipation ratio can be defined. To see this, consider first an equilibrium system. Using time-translation invariance in combination with the dynamical scaling forms would give

$$C(t, s) \sim (t - s)^{-b} , \quad R(t, s) \sim (t - s)^{-1-a} .$$

The fluctuation-dissipation theorem would then give $a = b$. Since equilibrium states may be found from master equations where the transition rates satisfies the detailed balance condition, one may view the equality $a = b$ as a necessary condition for detailed balance. More generally, the existence of a finite limit fluctuation-dissipation ratio $X_\infty \neq 0, \infty$ also requires that $a = b$, see exercise 1.16.

5. Is it possible to define a non-equilibrium temperature from the steady-states of systems without detailed balance? A recent attempt by Sastre, Dornic and Chat   [639] started from the observation that in simple magnets, the fluctuation-dissipation ratio $X(t, s) \rightarrow 1$ as $t \rightarrow \infty$ and $t - s \rightarrow 0$. From this observation, they define a *dynamical temperature* by

$$\frac{1}{T_{\text{dyn}}} := \lim_{t \rightarrow \infty} \left(\lim_{t-s \rightarrow 0} \frac{R(t, s)}{\partial C(t, s) / \partial s} \right). \quad (3.101)$$

By explicit calculation, they confirm that in the $2D$ critical voter model this limit exists, has a non-trivial value and is universal [639]. Still, their appealing idea has met with several criticisms. First, Mayer and Sollich [507] construct in the coarsening $1D$ Glauber-Ising model a defect-pair observable such that the fluctuation-dissipation ratio $X(t, s) \neq 1$ in the short time-regime (in particular they show $\lim_{s \rightarrow \infty} X(s, s) = 3/4$). Second, the example of the contact process already shows that the fluctuation-dissipation ratio itself is no longer defined, since $a \neq b$ [230, 609]. It appears that the proposal (3.101) relies on specific properties of the voter model.

6. Rather than the fluctuation-dissipation ratio $X(t, s)$, defined for systems with detailed balance, one may instead consider the ratio $\Xi(t, s)$, and its limit Ξ_∞ , which are defined by [230]

$$\Xi(t, s) := \frac{R(t, s; \mathbf{0})}{C(t, s; \mathbf{0})} = \frac{f_R(t/s)}{f_C(t/s)}; \quad \Xi_\infty := \lim_{s \rightarrow \infty} \left(\lim_{t \rightarrow \infty} \Xi(t, s) \right) \quad (3.102)$$

which are well-defined because of (3.95). The limit Ξ_∞ , being the ratio of two quantities with the same classical and scaling dimensions, is expected to be universal [46]. Baumann and Gambassi [46] argue that a value $\Xi(t, s)^{-1} \neq 0$ is a measure for the breaking of the rapidity-reversal symmetry (in the same way as $X \neq 1$ measures the distance from an equilibrium state) and show explicitly that only the zero-momentum modes contribute to a non-vanishing value of $\Xi(t, s)^{-1}$. In this respect, for systems with rapidity-reversal symmetry, Ξ appears to be the analogue of the fluctuation-dissipation ratio X . Of course, the applicability of their argument depends on the validity of the scaling relation $1 + a = b$. For the value of Ξ_∞ , they find in $4 - \varepsilon$ dimensions

$$\Xi_\infty = 2 \left[1 - \varepsilon \left(\frac{119}{480} - \frac{\pi^2}{120} \right) \right] + O(\varepsilon^2) \quad (3.103)$$

which in $1D$ is in semiquantitative agreement with the estimate $\Xi_\infty = 1.15(5)$ [230].

3.8.2 Experimental Results for Directed Percolation

As we have discussed in detail in Volume 1, obtaining reliable experimental results on systems with an absorbing phase-transition was for a long time a

observable	exponent	experiment		directed percolation
density order-parameter	β	0.59(4)		0.583(3)
density decay	α	0.48(5)		0.4505(10)
survival probability decay	δ	0.46(5)		0.4505(10)
spatial correlation length	$\nu_{\perp}$	0.75(6) ^x	0.78(9) ^y	0.733(7)
temporal correlation length	$\nu_{\parallel}$	1.29(11)		1.295(6)
spatial empty interval	$\epsilon_{\perp}/\nu_{\perp}$	1.08(18) ^x	1.19(12) ^y	1.204(2)
temporal empty interval	$\epsilon_{\parallel}/\nu_{\parallel}$	1.60(5)		1.5495(10)
number of active sites	Θ	0.22(5)		0.2295(10)
mean square spreading	$2/z$	1.15(9)		1.1325(10)
ageing exponent	b	0.9(1)		0.901(2)
autocorrelation exponent	λ_C/z	2.5(3)		2.58(2)
local persistence	θ_{ℓ}	1.55(7)		1.611(7)

Table 3.11. Measured critical exponents in thin films of the transition DSM1-DSM2 in the turbulent liquid crystal MBBA [693] and comparison with the values of $(2+1)D$ directed percolation, taken from Table 4.3 in volume 1. For the effective spatial exponents $\nu_{\perp}, \epsilon_{\perp}$ different results were found, depending on the spatial direction x and y , as indicated. In the lower part, results for some exponents describing critical ageing along with the estimates taken from Table 3.10 are given, as well as for exponent of the local persistence probability together with a recent theoretical estimate [297].

notoriously difficult problem and has been rightfully considered as being one of the most important open challenges in the field [296]. The ground-breaking work of Takeuchi, Kuroda, Chaté and Sano [692] on the continuous phase-transition between two turbulent states of electroconvection in thin films of the nematic liquid crystal MBBA offered for the first time a consistent set of critical exponents which agree within a few percent with the predictions of the directed percolation universality class in two spatial dimensions, see Tables 3.3 and 4.3 in Volume 1. Very recently, these authors have extended their studies to an even more comprehensive test of both static as well as dynamic exponents [693]. In Table 3.11, we list their results for the measured critical exponents and compare with the theoretical prediction of the directed percolation universality class, taken from Table 4.3 (vol. 1), Table 3.10 and [297]. For the purposes of this section only, we use herein the usual definitions of the exponents of absorbing phase transitions as summarised in Table 4.1.

Clearly, in this experiment one is able, for the first time, to reproduce reliably the whole set of steady-state as well as spreading exponents of two-dimensional directed percolation, to within a few percent.²² The large set of independently measured exponents also allows us to test several of the scaling relations between these exponents, as derived in Volume 1. Again, a good agreement, up to the experimental uncertainty of a few percent is observed

²² Intermittency effects, which may affect the values of $\epsilon_{\parallel, \perp}$ [323, 341], are still small with respect to the experimental error estimates.

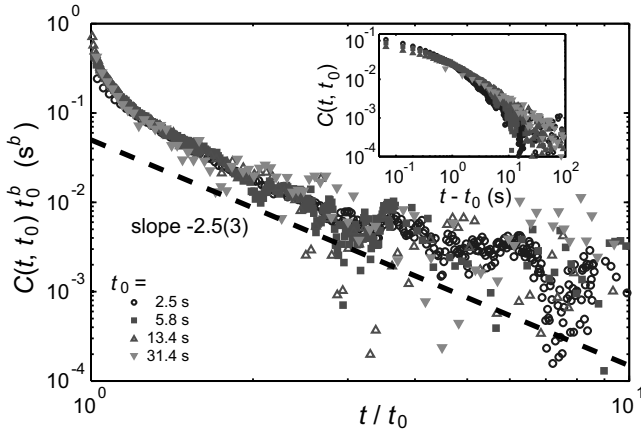


Fig. 3.19. Autocorrelator for the thin-film liquid crystal MBBA, quenched to the critical point and starting from a fully active initial state. The slope of the dashed line indicates the asymptotic decay of the scaling function $f_C(y) \sim y^{-\lambda_C/z}$. The inset shows the unscaled data. Reprinted with permission from [693]. Copyright (2009) by the American Physical Society.

[693]. In particular, the rapidity-reversal symmetry of the directed percolation universality class is tested through several scaling relations, especially $\alpha = \delta$. Furthermore, several non-trivial scaling functions have been measured and also agree with those found in numerical simulations of the contact process, with a similar accuracy. The success of this experiment, besides working with macroscopic units which considerably reduces the sensitivity to quenched disorder, appears to depend on the following three ingredients [693]: (i) the number of degrees of freedom $\sim 2.7 \cdot 10^6$ is several orders of magnitude larger than in all earlier experiments and occurs in combination with very short microscopic timescale ($\sim 10^{-2}$ s); (ii) absorbing state is almost perfect because of an unobservably small spontaneous nucleation rate, and (iii) a fluctuating absorbing state (DSM1) is used which largely eliminates long-range interactions.

It is therefore of interest to investigate the ageing properties of such a system. After preparing the system in the active (DSM2) state, the external control parameter was suddenly changed to its critical value and the evolution of the system was observed. While the mean density $N(t)$ scales as expected, in Fig. 3.19 the result of [693] for the connected two-time autocorrelator $C(t, s)$ is shown. The data can be collapsed using the usual scaling form (3.94) of simple ageing and the exponents b and λ_C/z have been extracted. Their values agree with the expectation from DP, within the experimental accuracy.

3.8.3 Non-equilibrium Kinetic Ising Model

Quite similar results can be found when studying the ageing in a non-equilibrium kinetic Ising model (NEKIM) where the parity of the total particle-number is conserved [552]. In this model [516, 553], one combines spin-flips as in zero-temperature Glauber dynamics with spin-exchanges as in Kawasaki dynamics. The model can be formulated either in terms of Ising spins ($\uparrow$, $\downarrow$) or else in terms of a particle-reaction model of the kinks with occupied or empty sites ($\bullet$, $\circ$). First, the Glauber-like part of the dynamics contains a diffusive motion

$$\uparrow\downarrow\downarrow\rightleftharpoons\uparrow\uparrow\downarrow \quad \text{or equivalently} \quad \bullet\circ\rightleftharpoons\circ\bullet ; \quad \text{with rate } D$$

and the pair-annihilation of nearest neighbours

$$\uparrow\downarrow\uparrow\rightarrow\uparrow\uparrow\uparrow \quad \text{or equivalently} \quad \bullet\bullet\rightarrow\circ\circ ; \quad \text{with rate } 2\alpha .$$

The Kawasaki-like part of the dynamics is described by

$$\uparrow\uparrow\downarrow\downarrow\rightleftharpoons\uparrow\downarrow\uparrow\downarrow \quad \text{or equivalently} \quad \circ\bullet\circ\rightleftharpoons\bullet\bullet\bullet ; \quad \text{with rate } k .$$

In full, this is a model describing branching and annihilating random walks with an even number of offspring. By increasing k , one finds a second-order phase-transition [516] where the kinks go from an absorbing to an active state. This phase-transition is in the so-called parity-conserving (PC) universality class [298, 131, 126] which is different from the one of the contact process.

Using the parametrisation $k = 1 - 2\Gamma$, $D = \Gamma(1 - \bar{\delta})/2$ and $2\alpha = \Gamma(1 + \bar{\delta})$, the critical point is located at $\Gamma = 0.35$, $k = 0.3$ and $\bar{\delta} = -0.3928$ [552]. Measuring the kink-density through an efficient cluster algorithm, and starting from a fully ordered kink state with spins being alternatingly $\uparrow$ and $\downarrow$, Odor finds a power-law scaling $\langle n_i(t) \rangle \sim t^{-0.285(2)}$.

Data for the unconnected kink-kink two-time correlation function are fully compatible with the scaling behaviour $\langle n_i(t)n_i(t') \rangle \sim t'^{-0.57}(t/t')^{-0.285}$ for $t/t' \rightarrow \infty$ and allows us to determine the ageing exponent b , see Table 3.10. This result is also consistent with earlier results on the spin-spin autocorrelator in the same model, see [552] for details. The connected autocorrelator was also calculated, with the result $\lambda_C/z = \lambda_R/z = 1.9(1)$. This is in agreement with the scaling relation (3.99). In Fig. 3.20 we show the scaling of the spin-autoresponse $R(t, s)$ with respect to a magnetic field. A clear data collapse with $a = b - 1$ is observed, quite analogous to the contact process, although here no analogue of the rapidity-reversal symmetry is known.

3.9 Ageing with Absorbing Steady-states II

3.9.1 Bosonic Contact and Pair-contact Processes

We now turn towards exactly solvable models which we shall use to further illustrate the ageing in systems without detailed balance. The exact results

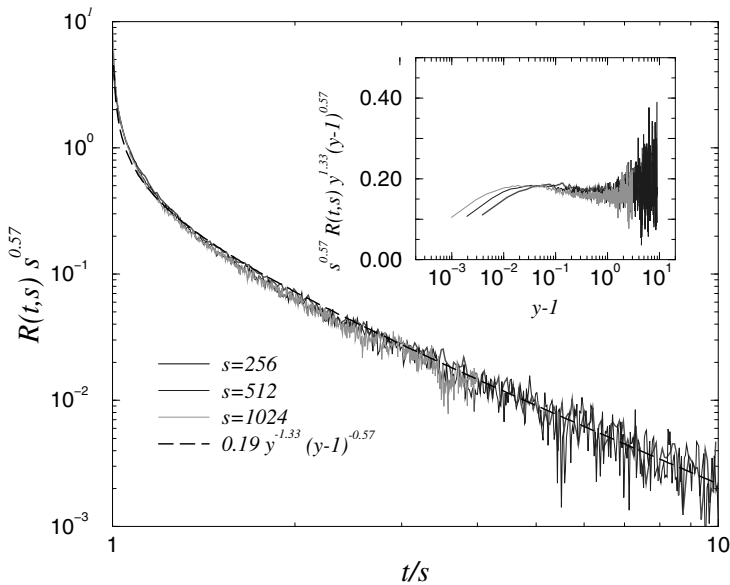


Fig. 3.20. Scaling of the autoresponse function for the critical 1D NEKIM, after [552], for several waiting times s . The dashed line is the LSI-prediction (5.41). The inset shows a rescaled response function, with $s = [256, 512, 1024]$ from right to left.

for the scaling functions which we shall obtain will be shown in chapters 4 and 5 to be examples of local scale-invariance.

The **bosonic contact process** was introduced in order to describe **clustering phenomena** in biological systems [376] whereas the **bosonic pair-contact process** was originally conceived [571] as a solvable variant of the usual (fermionic) pair-contact process, see Chap. 5 in Volume 1. These ‘bosonic’ models are defined as follows – we shall explain the terminology shortly. Consider a set of particles of a single species A which move on the sites of a d -dimensional hypercubic lattice. On any site one may have an *arbitrary* (non-negative) number of particles. This is the main distinguishing property with respect to the usual contact and pair-contact processes which obey a ‘fermionic’ constraint in that in their usual lattice-model definitions, they admit at most a single particle on any lattice site. For the ‘bosonic’ models to be considered now, it is on the contrary admissible that an *arbitrary* number of particles may be present at any site. In what follows, we use the labels ‘fermionic’ and ‘bosonic’ as a short-hand to distinguish them – of course both kinds of models are classical particle-reaction models in which quantum effects are completely discarded. Continuing with the definition, single particles may hop to a nearest-neighbour site with a rate $\sim D$ and in addition, the following single-site creation and annihilation processes are admitted

$$mA \xrightarrow{\mu} (m+1)A, \quad pA \xrightarrow{\lambda} (p-\ell)A; \quad \text{with rates } \mu \text{ and } \lambda \quad (3.104)$$

where ℓ is a positive integer such that $|\ell| \leq p$. We are interested in the following special cases:

1. *Critical bosonic contact process:* $p = m = 1$. Here only $\ell = 1$ is possible. Furthermore the creation and annihilation rates are equal, $\mu = \lambda$.
2. *Critical bosonic pair-contact process:* $p = m = 2$. We fix $\ell = 2$, set $2\lambda = \mu$ and define the control parameter²³

$$\alpha := \frac{3\mu}{2D}. \quad (3.105)$$

The master equation is written in a **Liouvillian** (or quantum Hamiltonian) formulation as $\partial_t |P(t)\rangle = -\mathcal{L} |P(t)\rangle$ [202, 203, 648] where $|P(t)\rangle$ is the time-dependent state vector and the Liouvillian $\mathcal{L}$ can be expressed in terms of annihilation and creation operators $a(\mathbf{r})$ and $a^\dagger(\mathbf{r})$. We define also the particle number operator as $n(\mathbf{r}) = a^\dagger(\mathbf{r})a(\mathbf{r})$. Then the Liouvillian of the model (3.104) reads [571]

$$\begin{aligned} \mathcal{L} = & -D \sum_{j=1}^d \sum_{\mathbf{r}} [a(\mathbf{r})a^\dagger(\mathbf{r} + \mathbf{e}_j) + a^\dagger(\mathbf{r})a(\mathbf{r} + \mathbf{e}_j) - 2n(\mathbf{r})] \\ & -\lambda \sum_{\mathbf{r}} \left[(a^\dagger(\mathbf{r}))^{p-\ell} (a(\mathbf{r}))^p - \prod_{i=1}^p (n(\mathbf{r}) - i + 1) \right] \\ & -\mu \sum_{\mathbf{r}} \left[(a^\dagger(\mathbf{r}))^{m+1} (a(\mathbf{r}))^m - \prod_{i=1}^m (n(\mathbf{r}) - i + 1) \right] - 2D \sum_{\mathbf{r}} h(t, \mathbf{r}) a^\dagger(\mathbf{r}) \end{aligned} \quad (3.106)$$

where $\mathbf{e}_j$ is the j^{th} unit vector. For later use in the calculation of response functions, we have also added an external field h , which describes the spontaneous creation of a single particle $\emptyset \rightarrow A$ with a site-dependent rate $h = h(t, \mathbf{r})$ on the site $\mathbf{r}$.

Single-time observables $g(t, \mathbf{r})$ can be obtained from the time-independent quantities $g(\mathbf{r})$ by switching to the Heisenberg picture. They satisfy the Heisenberg equation of motion $\partial_t b(t, \mathbf{r}) = [\mathcal{L}, b(t, \mathbf{r})]$,²⁴ from which the differential equations for the desired quantities are obtained.²⁵ For brevity, we shall in what follows always rescale times according to $t \mapsto t/(2D)$. The space-time-dependent particle-density $\rho(t, \mathbf{r}) := \langle a^\dagger(t, \mathbf{r})a(t, \mathbf{r}) \rangle = \langle a(t, \mathbf{r}) \rangle$ satisfies

²³ If instead we would treat a coagulation process $2A \rightarrow A$, where $\ell = 1$, the results presented in the text are recovered by setting $\lambda = \mu$ and $\alpha = \mu/D$.

²⁴ This analogy is one of the reason why the name ‘quantum Hamiltonian’ is also used for $\mathcal{L}$.

²⁵ Performing these calculations, recall that averages are written $\langle A(t) \rangle = \langle s | A | P(t) \rangle$ where the left steady-state $\langle s |$ has the properties $\langle s | \mathcal{L} = 0$, $\langle s | a^\dagger(\mathbf{r}) = \langle s |$.

$$\frac{\partial}{\partial t} \langle a(t, \mathbf{r}) \rangle = \frac{1}{2} \Delta_{\mathbf{r}} \langle a(t, \mathbf{r}) \rangle - \frac{\lambda \ell}{2D} \langle a(t, \mathbf{r})^p \rangle + \frac{\mu}{2D} \langle a(t, \mathbf{r})^m \rangle + h(t, \mathbf{r}) \quad (3.107)$$

where we have used the short-hand

$$\Delta_{\mathbf{r}} f(t, \mathbf{r}) := \sum_{j=1}^d \left(f(t, \mathbf{r} - \mathbf{e}_j) + f(t, \mathbf{r} + \mathbf{e}_j) - 2f(t, \mathbf{r}) \right) \quad (3.108)$$

which in the continuum limit goes over to the usual Laplacian. Similarly, the equal-time correlation functions satisfy the equations of motion, for a vanishing external field

$$\begin{aligned} \frac{\partial}{\partial t} \langle a(\mathbf{r}) a(\mathbf{r}') \rangle &= \frac{1}{2} \sum_{j=1}^d \left[\langle a(\mathbf{r}) a(\mathbf{r}' - \mathbf{e}_j) \rangle + \langle a(\mathbf{r}) a(\mathbf{r}' + \mathbf{e}_j) \rangle \right. \\ &\quad \left. + \langle a(\mathbf{r} - \mathbf{e}_j) a(\mathbf{r}') \rangle + \langle a(\mathbf{r} + \mathbf{e}_j) a(\mathbf{r}') \rangle - 4 \langle a(\mathbf{r}) a(\mathbf{r}') \rangle \right] \\ &\quad - \frac{\lambda \ell}{2D} \left(\langle a(\mathbf{r}) a(\mathbf{r}')^p \rangle + \langle a(\mathbf{r})^p a(\mathbf{r}') \rangle \right) + \frac{\mu}{2D} \left(\langle a(\mathbf{r}) a(\mathbf{r}')^m \rangle + \langle a(\mathbf{r})^m a(\mathbf{r}') \rangle \right) \end{aligned} \quad (3.109)$$

$$\begin{aligned} \frac{\partial}{\partial t} \langle (a(\mathbf{r}))^2 \rangle &= \sum_{j=1}^d \left[\langle a(\mathbf{r}) a(\mathbf{r} - \mathbf{e}_j) \rangle + \langle a(\mathbf{r}) a(\mathbf{r} + \mathbf{e}_j) \rangle - 2 \langle a(\mathbf{r})^2 \rangle \right] \\ &\quad + \frac{\lambda \ell}{2D} \left((1 + \ell - 2p) \langle a(\mathbf{r})^p \rangle - 2 \langle a(\mathbf{r})^{p+1} \rangle \right) + \frac{\mu}{2D} \left(2m \langle a(\mathbf{r})^m \rangle + 2 \langle a(\mathbf{r})^{m+1} \rangle \right) \end{aligned} \quad (3.110)$$

where in (3.109) $\mathbf{r} \neq \mathbf{r}'$ is understood. Since $\langle n(\mathbf{r})^2 \rangle = \langle a(\mathbf{r})^2 \rangle + \langle a(\mathbf{r}) \rangle$, the main equal-time quantity of interest, namely the variance $\sigma^2 := \langle n(\mathbf{r})^2 \rangle - \langle n(\mathbf{r}) \rangle^2$ can be found.

It turns out that for the bosonic contact process $p = m = 1$ the system (3.109, 3.110) of equations closes for arbitrary values of the rates. On the other hand, for the bosonic pair-contact process where $p = m = 2$ a closed system of equations is only found along the critical line given by [571]

$$\ell \lambda = \mu. \quad (3.111)$$

In both models, the spatial average of the local particle-density $\rho(t, \mathbf{r}) := \langle a(t, \mathbf{r}) \rangle$ remains constant in time,

$$\int d\mathbf{r} \rho(t, \mathbf{r}) = \int d\mathbf{r} \langle a(t, \mathbf{r}) \rangle = \rho_0, \quad (3.112)$$

where ρ_0 is the initial mean particle-density. Furthermore, the critical line (3.111) separates an active phase with a formally infinite particle-density in the steady-state from an absorbing phase where the steady-state particle-density vanishes. The phase diagrams are sketched in [Fig. 3.21](#).

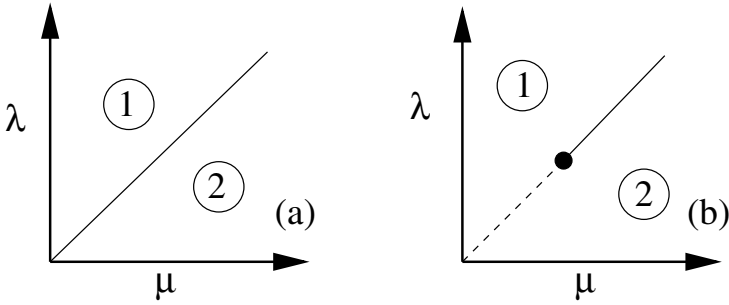


Fig. 3.21. Schematic phase-diagrams for $D \neq 0$ of (a) the bosonic contact process and the bosonic pair-contact process in $d \leq 2$ dimensions and (b) the bosonic pair-contact process in $d > 2$ dimensions. The absorbing region 1, where $\lim_{t \rightarrow \infty} \rho(t, \mathbf{r}) = 0$, is separated by the critical line eq. (3.111) from the active region 2, where $\rho(t, \mathbf{r}) \rightarrow \infty$ as $t \rightarrow \infty$. Clustering along the critical line is indicated in (a) and (b) by full lines, but in the bosonic pair-contact process with $d > 2$ the steady-state may also be homogeneous (broken line in (b)). These two regimes are separated by a multicritical point. Reprinted from [49]. Copyright (2005) Institute of Physics Publishing.

The physical nature of this transition becomes apparent when equal-time correlations are studied [376, 571]. For example, for the bosonic contact process at criticality one finds [376]

$$\langle a(t, \mathbf{r})^2 \rangle = \begin{cases} c_1 t^{-d/2+1} & ; \text{ if } d < 2 \\ c_2 \ln t & ; \text{ if } d = 2 \\ c_3 + c_4 t^{-d/2+1} & ; \text{ if } d > 2 \end{cases} \quad (3.113)$$

where $t \gg 1$ and $c_1, \dots, c_4$ are known positive constants. For $d \leq 2$, the *fluctuations* in the mean particle-density increase with time, although the mean particle-density itself remains constant. Physically, this means that the particle-number on relatively few sites will increase while many other sites will become empty. Only for $d > 2$ fluctuations will eventually die out. For the bosonic contact process, this critical behaviour is the same along the entire critical line.

For the bosonic pair-contact process, that is different. Rather, there exists a critical value α_C of the control parameter, given by

$$\alpha_C = \alpha_C(d) = \frac{1}{2A_1} \quad , \quad A_1 := \int_0^\infty du \left(e^{-4u} I_0(4u) \right)^d \quad (3.114)$$

and where $I_0(u)$ is a modified Bessel function. Specific values are $\alpha_C(3) \approx 3.99$ and $\alpha_C(4) \approx 6.45$ and $\lim_{d \searrow 2} \alpha_C(d) = 0$. Then the variance behaves as [571]

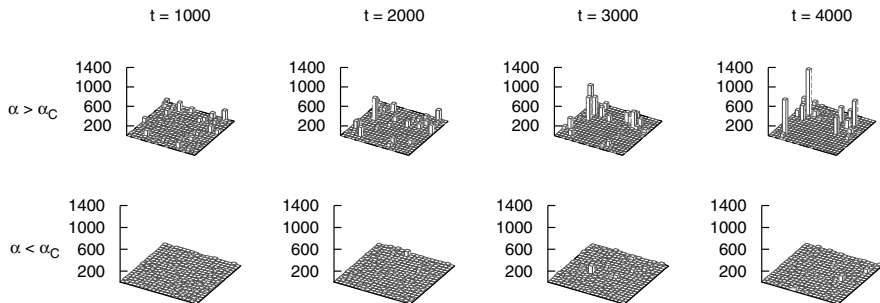


Fig. 3.22. Snapshots of a 2D section of configurations in the 3D critical bosonic pair-contact process, on a 20^3 lattice with periodic boundary conditions and for several times t . Initially, each site was occupied by ten particles. The upper row corresponds to $\alpha > \alpha_C$ and the lower row to $\alpha < \alpha_C$ [44].

$$\langle a(t, \mathbf{r})^2 \rangle = \begin{cases} f_0 & ; \text{ if } \alpha < \alpha_C \\ f_1 t^{d/2-1} & ; \text{ if } \alpha = \alpha_C \text{ and } 2 < d < 4 \\ f_2 t & ; \text{ if } \alpha = \alpha_C \text{ and } d > 4 \\ f_3 e^{t/\tau} & ; \text{ if } \alpha > \alpha_C \text{ (or } d < 2) \end{cases} \quad (3.115)$$

where the $f_0, \dots, f_3$ and τ are known positive constants. This means that at the multicritical point at $\alpha = \alpha_C$ there occurs a **clustering transition** such that for $\alpha < \alpha_C$ the systems evolves towards a more or less homogeneous state while for $\alpha \geq \alpha_C$ particles accumulate on very few lattice sites while the other ones remain empty. In contrast with the bosonic contact process, clustering occurs on some part of the critical line for all values of d . We illustrate this in Fig. 3.22 where for the 3D BPCPD we show a series of snapshots of the occupation of a two-dimensional section of the 3D lattice.

We are interested in studying the impact of this clustering transition on the two-time correlations and linear responses. In order to obtain the equations of motion of the two-time correlator, the time-ordering of the operators $a(\mathbf{x}, t)$ must be taken into account. This leads to the following equations of motion for the two-time correlator, after rescaling the times $t \mapsto t/(2D)$, $s \mapsto s/(2D)$, and for $t > s$, [281, 49]

$$\begin{aligned} & \frac{\partial}{\partial t} \langle a(t, \mathbf{r}) a(s, \mathbf{r}') \rangle \\ &= \frac{1}{2} \Delta_{\mathbf{r}} \langle a(t, \mathbf{r}) a(s, \mathbf{r}') \rangle - \frac{\lambda \ell}{2D} \langle a(t, \mathbf{r})^p a(s, \mathbf{r}') \rangle + \frac{\mu}{2D} \langle a(t, \mathbf{r})^m a(s, \mathbf{r}') \rangle . \end{aligned} \quad (3.116)$$

Consider the two-time connected correlation function

$$C(t, s; \mathbf{r}) := \langle a(t, \mathbf{x}) a(s, \mathbf{x} + \mathbf{r}) \rangle - \rho_0^2 \quad (3.117)$$

and take an uncorrelated initial state. The linear two-time response function is found by adding a particle-creation term $\sum_{\mathbf{r}} h(t, \mathbf{r}) (a^\dagger(t, \mathbf{r}) - 1)$ to the quantum Hamiltonian H and taking the functional derivative

$$R(t, s; \mathbf{r}) := \left. \frac{\delta \langle a(t, \mathbf{r} + \mathbf{x}) \rangle}{\delta h(s, \mathbf{x})} \right|_{h=0} \quad (3.118)$$

and for which the usual scaling behaviour (3.94) is anticipated. On the critical line (3.111), one may easily see [44], for both the BCPD and the BPCPD, that the connected density-density correlator

$$\langle n(t, \mathbf{r}_0) n(s, \mathbf{r} + \mathbf{r}_0) \rangle - \rho_0^2 = C(t, s; \mathbf{r}) + R(t - s, \mathbf{r}) \rho_0 \quad (3.119)$$

is related to the connected two-point function. In particular, whenever ageing with breaking of time-translation-invariance occurs, the second term will be negligible in the scaling limit, hence $C(t, s; \mathbf{r})$ describes the scaling behaviour of the density-density correlation and one has *Baumann's inequality* [44]

$$b \leq a + 1. \quad (3.120)$$

The proof of (3.119) is left as exercise 3.8.

The equations of motion (3.116) for the two-time quantities form a set of linear equations whose solution is straightforward, if just a little tedious. As in the spherical model, the essential step is the solution of a Volterra integral equation [571]. It can be shown [49] that the anticipated scaling behaviour exists along the critical line, *provided* $\alpha \leq \alpha_C$. For these cases, the exponents are listed in Table 3.10. In particular, we see that at the multicritical point $\alpha = \alpha_C$, the exponents a and b are different and, furthermore, do not satisfy the relation $1 + a = b$ found for the critical contact process. In addition, the explicit forms of the scaling functions can also be found and are listed in Table 4.2 [49]. While the form of the autoresponse scaling function $f_R(y) \sim (y - 1)^{-d/2}$ is remarkably simple, the results for the autocorrelator can be rendered as an integral

$$f_C(y) = C_0 \int_0^1 d\theta \theta^{a-b} (y + 1 - 2\theta)^{-d/2} \quad (3.121)$$

where the exponents a, b are taken from Table 3.10. In Chap. 4, we shall show how these results can be explained from the local scale-invariance of the bosonic contact and pair-contact processes [52].

3.9.2 Bosonic Particle-reaction Models with Lévy Flights

An interesting variant of the bosonic contact and pair-contact process is obtained if one replaces the nearest-neighbour hopping with a fixed rate D by long-range hoppings with rates distributed according to a Lévy distribution

$$D(\mathbf{r}) = \frac{D}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{q} \exp(i\mathbf{q} \cdot \mathbf{r} - c|\mathbf{q}|^\eta) \quad (3.122)$$

where $0 < \eta < 2$ is the Lévy control parameter and $c > 0$ is a normalisation constant. The reaction rates are still given by eq. (3.104) as before. In this way, one obtains the bosonic contact and pair-contact processes with Lévy flights, abbreviated as BCPL and BPCPL, respectively [215]. The equations of motions for particle-densities, correlators and responses can be written down by slightly updating those found before for the BCPD and BPCPD in that the nearest-neighbour hopping term are replaced by long-range jumps according to (3.122). The main difference is the modified form of the dispersion relation

$$\omega(\mathbf{q}) = 1 - \exp(-c|\mathbf{q}|^\eta) \stackrel{|\mathbf{q}| \rightarrow 0}{\simeq} c|\mathbf{q}|^\eta \quad (3.123)$$

such that in the limit $\eta \rightarrow 2$ one recovers the results of the BCPD and BPCPD. Therefore, the general structure of the phase diagrams and of the critical line is unchanged if one takes into account that the lower critical dimension and the upper critical dimension are now shifted to η and 2η , respectively (only the location α_C of the multicritical point is modified).

The non-equilibrium exponents and the corresponding scaling functions are listed in Tables 3.10 and 4.2, respectively. We shall analyse them from the point of view of local scale-invariance in Chap. 5.

The available exact results for the BCPD and BCPL allow to consider in detail the passage from the initial quasi-stationary regime to the ageing regime, which for quenches into the coexistence phase $T < T_c$ we have already considered in Chap. 1, following there [760]. A convenient quantity is the relative change in the autocorrelator [215]

$$\delta C(t, s) := \left| \frac{C(t, s) - C_{\text{stat}}(t - s)}{C_{\text{ageing}}(t, s) - C_{\text{stat}}(t - s)} \right| \quad (3.124)$$

where

$$C_{\text{stat}}(t) = C_0 \int_{t/2}^{\infty} d\tau \int_0^{\infty} dq q^{d-1} e^{\omega(q)\tau} \quad , \quad C_{\text{ageing}} = s^{-b} f_C(t/s) \quad (3.125)$$

are, respectively, the quasi-stationary and the ageing limits of the two-time autocorrelator. Clearly, when going from the quasi-stationary to the ageing regime, δC changes from zero to unity. Passage times between the regimes may be defined by the conditions

$$\delta C(\tau_{\text{stat}}(s) + s, s) = 0.1 \quad , \quad \delta C(\tau_{\text{age}}(s) + s, s) = 0.9 \quad (3.126)$$

such that one leaves the quasi-stationary regime at time-scale $\tau_{\text{stat}}(s)$ and enters into the ageing regime at the time-scale $\tau_{\text{age}}(s)$. It turns out that at least for the BCPD and BCPL, for sufficiently large waiting times s one finds a power-law behaviour,

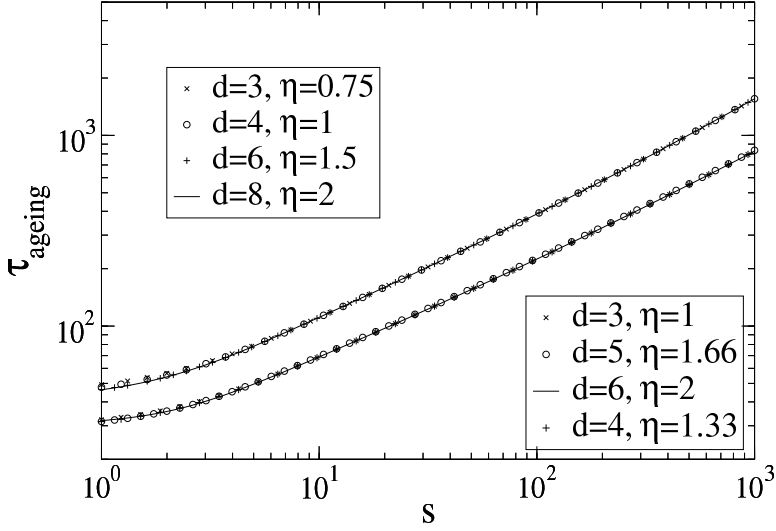


Fig. 3.23. Passage time $\tau_{\text{age}}(s)$ in the critical bosonic contact process, at dimension d and with Lévy parameter η . Upper curves: $d/\eta = 4$. Lower curves: $d/\eta = 3$. Reprinted from [215]. Copyright (2009) Institute of Physics Publishing.

$$\tau_{\text{age}}(s) \sim s^\zeta \quad (3.127)$$

where ζ is the **passage exponent**. This is illustrated in Fig. 3.23. Numerical estimates for ζ have already been given in Table 1.6. We observe that if one fixes d/η in the BCPD/BCPL, the scaling behaviour is the same such that the estimates given in Table 1.6 for the BCPD at dimension d also hold for the BCPL at dimension $\eta d/2$.

A similar analysis can be carried out for the non-equilibrium critical dynamics of the spherical model, with analogous conclusions [215], see Table 1.6 for numerical values of ζ .

These results can be understood physically as follows. At equilibrium, the fluctuation-dissipation theorem describes the expected size of the fluctuations, given the response to an external perturbation. If the FDT is broken in ageing phenomena, the inverse fluctuation-dissipation ratio (FDR) describes by how much this expectation is larger than the actually found result, for a given value of the temperature T [456]. Since for the BCPL, the ageing exponents $a = b$ are equal, define an analogue of the fluctuation-dissipation ratio [215]

$$X(t, s) := \frac{\frac{\partial C}{\partial s}(s, s)}{R(s, s)} \frac{R(t, s)}{\frac{\partial C}{\partial s}(t, s)} \quad (3.128)$$

but where now the initial FDR $X_{\text{ini}}(s)^{-1} := R(s, s)/(\partial_s C(t, s))|_{t \rightarrow s}$ takes over the role which the temperature plays in systems with an equilibrium stationary state. The exact solution of the BCPL gives, in the ageing regime

$$X_{\text{age}}(t, s) = \left(1 + \left(\frac{t-s}{t+s} \right)^{d/\eta} \right)^{-1} \stackrel{t/s \rightarrow \infty}{=} \frac{1}{2}. \quad (3.129)$$

Using this definition of $X(t, s)$, one can distinguish three different regimes, namely (i) a quasi-stationary regime with microscopic relaxation for $t - s \ll \tau_{\text{stat}}(s)$, (ii) a non-analytic transition regime for $\tau_{\text{stat}}(s) \ll t - s \ll \tau_{\text{age}}(s)$ and (iii) the ageing regime when $t - s \gg \tau_{\text{age}}(s)$.

3.10 Reversible Reaction-diffusion Systems

Reversible reaction-diffusion systems form another class of exactly solvable systems that provides interesting insights into the relaxation mechanism of diffusion-limited reactions out of equilibrium. Whereas in systems with irreversible reactions, slow dynamics have so far only been observed at the absorbing phase transition, *in reversible reaction-diffusion systems non-exponential relaxation is a generic property* that is observed without any fine-tuning of the system parameters. A power-law behaviour in the long-time limit was first predicted in [751] for the bi-molecular reversible reaction $A + B \rightleftharpoons C$, based on physical ideas involving spatial concentration fluctuations. Naïvely, from the chemical rate equations one would expect an exponential relaxation, but a slower, power-law relaxation arises due to fluctuations. This power-law behaviour was later verified through more elaborate analytical approaches [752, 422, 567], through numerical simulations [8, 9], and through some exactly solved models [615, 290, 291]. It has also been observed experimentally in excited-state proton transfer reactions [384, 677, 590], see Fig. 3.24. These studies show that two of the defining characteristics of physical ageing, namely (i) slow relaxational dynamics and (ii) breaking of time-translation-invariance, are typically encountered in reversible reaction-diffusion systems. We shall investigate here to what extent the third defining characteristic of ageing, namely dynamical scaling, can be realised.

Reaction-diffusion systems with irreversible reactions are characterised by the fact that some quantities are generically conserved. This raises the interesting question to what extent the presence of these conserved quantities might change the ageing properties of the system. As we shall see in this section, *the scaling forms for response functions crucially depend on whether a perturbation conserves or breaks these conservation laws*. In order to give an explicit example, we study here the ageing properties of the reversible reaction-diffusion system given by the reaction $A + A \rightleftharpoons C$ [228] which might become paradigmatic for a novel aspect of ageing systems.

In the **RAC-model** (reversible A-C model) which we are now going to define, we consider two species of particles (called A and C in what follows) which are placed on the sites of a d -dimensional hypercubic lattice. One allows for multiple occupancy of a lattice site and considers as an initial state a

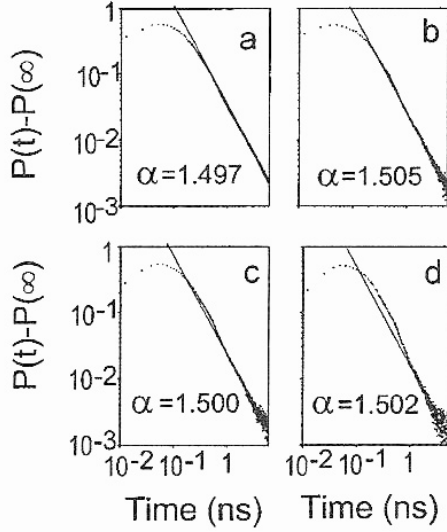


Fig. 3.24. Fluorescence decay of dye molecules HPTS (8-hydroxypyrene 1,3,6-trisulfonate) which dissociates in the excited state to produce two proton-excited anion pairs. After subtracting off the steady-state values, a power-law decay $P(t) - P(\infty) \sim t^{-\alpha}$ of the normalised fluorescence decay $P(t)$ is observed. The different panels correspond to different pH values of the solutions. Reprinted with permission from [590]. Copyright (2001) American Institute of Physics.

configuration of uncorrelated A - and C -particles with given densities, formally described by spatially uncorrelated Poisson distributions on each site and for each particle species. Single particles may diffuse on the lattice. They undergo, with other particles at the same site, the following reactions:

1. the reaction $A + A \longrightarrow C$ with rate λ_0 describes the fusion of two A -particles into a C -particle
2. the reverse reaction $C \longrightarrow A + A$ describes the dissociation of a C -particle, with rate μ .

As shown by Rey and Cardy [615], the dynamics of this model allows an exact description in terms of a set of coupled stochastic Langevin equations. Introducing $\lambda = \lambda_0 \ell^d$, where ℓ is the lattice constant, one obtains in the continuum limit a pair of Langevin equations

$$(\partial_t - D_a \Delta_L) a(t, \mathbf{r}) = -2\lambda a^2(t, \mathbf{r}) + 2\mu c(t, \mathbf{r}) + \eta(t, \mathbf{r}) \quad (3.130)$$

$$(\partial_t - D_c \Delta_L) c(t, \mathbf{r}) = \lambda a^2(t, \mathbf{r}) - \mu c(t, \mathbf{r}), \quad (3.131)$$

where η denotes a *complex* Gaussian noise with zero mean value, whose variance is given by

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2 \langle \mu c(t, \mathbf{r}) - \lambda a^2(t, \mathbf{r}) \rangle \delta(t - t') \delta(\mathbf{r} - \mathbf{r}'). \quad (3.132)$$

Inserting equation (3.131) into this expression yields

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = -2 \partial_t \langle c(t) \rangle \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (3.133)$$

Spatially translation-invariant initial conditions have been assumed here.

Note that the variables $a(t, \mathbf{r})$ and $c(t, \mathbf{r})$ do not represent the particle densities, as they are *complex* [615]. Of course, the mean densities $n_a(t) = \langle a(t, \mathbf{r}) \rangle$ and $n_c(t) = \langle c(t, \mathbf{r}) \rangle$ of the particles of types *A* and *C* are real-valued.

In the long-time limit $t \rightarrow \infty$, the density $n_c(t) \rightarrow c_\infty$ tends towards a stationary value. Hence the noise variance will vanish for large enough time, according to eq. (3.133). Therefore the *exact* values of the stationary densities $a_\infty = \lim_{t \rightarrow \infty} n_a(t)$ and $c_\infty = \lim_{t \rightarrow \infty} n_c(t)$ will be given by their mean-field values and satisfy

$$\lambda a_\infty^2 = \mu c_\infty, \quad (3.134)$$

as follows directly from equation (3.132).

Furthermore, the conservation of the total mass means that the quantity $n_a(t) + 2n_c(t)$ is conserved. From now on, we restrict to the case of equal diffusion constants $D_a = D_c =: D$. Then $\chi := a + 2c$ obeys the noisy diffusion equation

$$(\partial_t - D \Delta_L) \chi(t, \mathbf{r}) = \eta(t, \mathbf{r}), \quad (3.135)$$

as follows directly from equations (3.130) and (3.131). Since $\langle \eta \rangle = 0$ and the initial conditions are homogeneous, the average value of the field $\langle \chi(t, \mathbf{r}) \rangle$ is conserved. In particular we have $\chi_0 := \langle \chi(0, \mathbf{r}) \rangle = \langle \chi(\infty, \mathbf{r}) \rangle =: \chi_\infty$. From equation (3.134) the equilibrium densities read as a function of χ_0 [615]:

$$a_\infty = \frac{\mu}{4\lambda} \left(\sqrt{1 + \frac{8\lambda}{\mu} \chi_0} - 1 \right), \quad c_\infty = \frac{1}{2} (\chi_0 - a_\infty). \quad (3.136)$$

Starting from this Langevin description and exploiting the existence of the conserved quantity, Rey and Cardy [615] developed a systematic approximation scheme that enabled them not only to derive the power-law relaxation toward equilibrium but also to compute the corresponding amplitude exactly. In [228] this approach was extended to two-time quantities and exact expressions for correlation and response functions in the ageing regime were derived.

One of the quantities of interest is the connected correlation function

$$C(t, s; \mathbf{r}, \mathbf{r}') = \langle \delta c(t, \mathbf{r}) \delta c(s, \mathbf{r}') \rangle - \langle \delta c(t, \mathbf{r}) \rangle \langle \delta c(s, \mathbf{r}') \rangle \quad (3.137)$$

with $\delta c(t, \mathbf{r}) := c(t, \mathbf{r}) - c_\infty$. The autocorrelator is obtained as $C(t, s) = C(t, s; \mathbf{r}, \mathbf{r}) = C(t, s; \mathbf{0}, \mathbf{0})$, where spatial translation-invariance was also used.

In principle, we should define a similar quantity for the variable $a(t, \mathbf{r})$. However, as $\delta a(t, \mathbf{r}) = a(t, \mathbf{r}) - a_\infty = \delta \chi(t, \mathbf{r}) - 2\delta c(t, \mathbf{r})$ with $\delta \chi(t, \mathbf{r}) = \chi(t, \mathbf{r}) - \chi_\infty$, we can immediately derive the correlator for a once we know the correlators for the conserved quantity χ and for the variable c .

A straightforward but rather tedious calculation yields in the ageing regime, where not only t and s but also $|t - s|$ are large, the leading behaviour for the space-time correlation function

$$C(t, s; \mathbf{r}, \mathbf{r}') = \mathcal{C}_0 (4\pi D(t + s))^{-d/2} \exp \left[-\frac{1}{4D} \frac{(\mathbf{r} - \mathbf{r}')^2}{|t - s|} \right] \quad (3.138)$$

with $\mathcal{C}_0 = \frac{1}{2} \left(\frac{\sigma - \mu}{\sigma} \right)^2 (c_0 - c_\infty)$ and the short-hand $\sigma := 4\lambda a_\infty + \mu$. The derived expression can be cast in the usual scaling form and yields for the non-equilibrium exponents the values $b = d/2$ and $\lambda_C/z = d/2$.

In order to probe the response of the system to some perturbation one can proceed in different ways. One of the possibilities is to inject new particles (which can be of type A or C) at time s . The ‘injection’ process is assumed to be spatially random with the same small occurrence probability at each lattice site. The response of the system to that perturbation is then monitored at a later time t by measuring the densities of particles of type A or C . In this way one obtains different responses that one notes as $R_i^f(t, s)$, where i stands for the type of particles that are created whereas f indicates the type of particles whose density is measured. For example, the response function

$$R_C^A(t, s) := \left. \frac{\delta \langle a \rangle(t)}{\delta h_C(s)} \right|_{h_C=0} \quad (3.139)$$

measures the linear response of the density of A -particles to the additional creation of C -particles. We stress that the addition of C -particles does modify the value of the conserved quantity $\langle \chi \rangle$. In order to assess the impact of this modification on the response of the system, a different process that conserves the total mass of the particles should also be considered. As we shall see shortly, the response of a reversible diffusion-reaction system strongly depends on the chosen perturbation.

Let us start by injecting particles of type C into the system and by monitoring the subsequent change in the particle density of the same particle type. We might want to inject Ω_C additional C particles at time s , so that the particle injection probability $h_C(t)$ is given by

$$h_C(t) = \Omega_C \delta(t - s) . \quad (3.140)$$

However, as the final state of the evolution will again be a homogeneous state, the precise form of h_C is unimportant, provided only that the particle injection has ended before the measurement of the response at time t . The total number of injected particles is given by

$$\Omega_C = \int_0^t d\tau h_C(\tau) . \quad (3.141)$$

With this particle injection process, the equation of motion for the C -particles is now given by

$$(\partial_t - D\Delta_L)c^h(t, \mathbf{r}) = \lambda a^h(t, \mathbf{r})^2 - \mu c^h(t, \mathbf{r}) + h(t) , \quad (3.142)$$

where in the continuum limit we set

$$h(t) = \frac{h_C(t)}{\ell^d} , \quad \Omega = \frac{\Omega_C}{\ell^d} . \quad (3.143)$$

From now on, we suppress the index C for the quantities divided by the volume ℓ^d . We use in the following the index h in order to emphasise the presence of the additional creation process and to distinguish the corresponding quantities from those obtained in the absence of this process. The Langevin equation for the A -particles is unchanged by the creation of C -particles, but the noise-noise correlator (3.133) becomes

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2 [h(t) - \partial_t \langle c^h(t) \rangle] \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') , \quad (3.144)$$

for spatially translation-invariant initial conditions, due to the equations (3.132) and (3.142).

Then one readily derives the following conditions for the stationary state:

$$a_\infty^h + 2c_\infty^h = a_0 + 2c_0 + 2\Omega = \chi_0 + 2\Omega , \quad (3.145)$$

$$\lambda (a_\infty^h)^2 = \mu c_\infty^h . \quad (3.146)$$

Note that as Ω has the meaning of the average total change in the number of C -particles due to the creation process, equation (3.145) reflects the modification of the total mass due to this process. Equations (3.145) and (3.146) immediately yield the following expressions for the mean values a_∞^h and c_∞^h in the new stationary state:

$$a_\infty^h = \frac{\mu}{4\lambda} \left(\sqrt{1 + \frac{8\lambda}{\mu} (\chi_0 + 2\Omega)} - 1 \right) , \quad c_\infty^h = \frac{1}{2} (\chi_0 + 2\Omega - a_\infty^h) \quad (3.147)$$

which differ from the expressions (3.136) without the field h through the replacement of χ_0 by $\chi_0 + 2\Omega$.

At last, we find the two-time linear response function

$$\begin{aligned} R_C^C(t, s) &= \left. \frac{\delta \langle c^h(t) \rangle}{\delta h(s)} \right|_{h=0} \\ &= 2 \frac{\lambda \mu^2}{\sigma^3} \left[\frac{\mu}{\sigma} - 1 - \frac{24\lambda\mu}{\sigma^2} (c_0 - c_\infty) \right] (8\pi D t)^{-d/2} \Theta(t - s) \\ &\quad + 2 \frac{\lambda \mu^2}{\sigma^3} [8\pi D (t - s)]^{-d/2} \Theta(t - s) , \end{aligned} \quad (3.148)$$

where we used the fact that $\frac{\delta\Omega}{\delta h(s)} = \Theta(t-s)$. This expression can also be cast in the standard scaling form $R_C^C(t, s) = s^{-1-a} f_R(t/s) \Theta(t-s)$, with $a = d/2 - 1$ and $\lambda_R/z = d/2$. Interestingly, comparison with eq. (3.138) shows that although $\lambda_C = \lambda_R$ still holds, we have $a \neq b$.

The change in the density of particles A due to the injection of C particles, R_C^A , is related to the response R_C^C by

$$R_C^A(t, s) = 2\Theta(t-s) - 2R_C^C(t, s) , \quad (3.149)$$

which follows directly from $\langle a^h \rangle + 2\langle c^h \rangle = \langle \chi^h \rangle = \chi_0 + 2\Omega$. Thus the expression (3.149) is the sum of two terms with different scaling behaviours where the constant term is the leading one.

Next, we discuss the response of the system to an injection of A -particles. One possible way of doing this consists in injecting pairs of A -particles into the system with a rate h_{2A} : $\emptyset \longrightarrow 2A$. As these pairs can immediately react to form a C -particle, it is expected that this leads to the same behaviour as observed when injecting C -particles into the system. Indeed, the creation of pairs of A -particles on the one hand changes the noise-noise correlator which now reads (with $h = h_{2A}/l^d$)

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = [h(t) - 2\partial_t \langle c^h(t) \rangle] \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') , \quad (3.150)$$

and on the other hand modifies the Langevin equation for the A -particles:

$$(\partial_t - D_a \Delta_L) a(t, \mathbf{r}) = -2\lambda a^2(t, \mathbf{r}) + 2\mu c(t, \mathbf{r}) + \eta(t, \mathbf{r}) + 2h(t) . \quad (3.151)$$

Since the Langevin equation for the C -particles remains unchanged, it is then straightforward to show that we have indeed $R_{2A}^C = R_C^C$ and $R_{2A}^A = R_C^A$.

The situation changes if instead of injecting pairs of A -particles only *single* A -particles are created with rate h_A . In that case the additional A -particles do not automatically lead to the formation of additional C particles, but instead a newly created A -particle must first diffuse through the system in order to encounter another additional A -particle. Formally, the creation of single A -particles again leads in the Langevin equation (3.130) for the A -particles to an additional field term $h(t)$, but the noise-noise correlator is the same as for the $h = 0$ case:

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = -2\partial_t \langle c^h(t) \rangle \delta(t - t') \delta(\mathbf{r} - \mathbf{r}') . \quad (3.152)$$

One then readily obtains the expression (σ was defined on p. 211)

$$R_A^C(t, s) = \frac{\lambda\mu^2}{\sigma^3} \left[\frac{\mu}{\sigma} - 1 - \frac{24\lambda\mu}{\sigma^2} (c_0 - c_\infty) \right] (8\pi Dt)^{-d/2} \Theta(t-s) , \quad (3.153)$$

for the response of the C -particles to the creation of single A -particles. It is important to note that the expression (3.153) does not depend (at least in

leading order) on the excitation time s . This is a direct consequence of the fact that the rate of encounter of two newly created A -particles does not depend on the time s at which the particles have been created.

All the perturbations considered so far have in common that they modify the value of the quantity $\langle \chi \rangle = n_a + 2n_c$, that would be constant without the perturbation. In order to assess the importance of this conservation law, we also look at the response to a perturbation that keeps $\langle \chi \rangle$ unchanged. This can be achieved by the simultaneous creation of particles of one type and destruction of particles of the other type. For example, we can create pairs of A -particles with rate h_χ and at the same time remove C particles with rate h_χ . In principle, the removal procedure is an ill-defined process, since one may end up with negative number of particles on a given site. However, for large equilibrium concentrations a_∞ and c_∞ , we expect the number of particles on each site of the lattice to remain larger than zero following an infinitesimal excitation of the type just described.

It is important to note that the Langevin equation (3.135) for χ is unaffected by this perturbation, even so additional field terms are entering into the Langevin equations for a and c . Doing the calculations along the lines just sketched for the other perturbations, one remarks that, due to the fact that the asymptotic values c_∞^h and a_∞^h are independent of the field $h = h_\chi/l^d$, the response functions become time translational invariant. Thus for the response of the C -particles to this perturbation we get

$$R_\chi^C(t, s) = 2 \frac{\lambda \mu^2}{\sigma^3} [8\pi D(t-s)]^{-d/2} \Theta(t-s), \quad (3.154)$$

whereas for the response of the A particles we obtain, since $\Omega = 0$

$$R_\chi^A(t, s) = -2R_\chi^C(t, s) = -4 \frac{\lambda \mu^2}{\sigma^3} [8\pi D(t-s)]^{-d/2} \Theta(t-s). \quad (3.155)$$

In Table 3.12, we summarise the exponents found for the ageing of the autoresponse functions $R_i^f(t, s)$ in the RAC-model. For the comparison with local scale-invariance in Chap. 4, we have included, besides the standard exponents a and λ_R/z , also the further exponent a' , which will be defined for systems which satisfy not only dynamical scaling, but furthermore obey some form of local scale-invariance (LSI) of a single-component Langevin equation.²⁶ For the time being, it is enough to observe that for the perturbations which conserve χ , one always has $a = a'$, whereas a and a' can be different if the value of χ is changed through the perturbation. Again, we point out that *no fine-tuning of the reaction rates was required to obtain dynamical scaling*.

While we focused here on the simple reversible reaction $A + A \rightleftharpoons C$, this study can be straightforwardly extended to other systems like $A + B \rightleftharpoons C$ or

²⁶ The response function $R_C^C(t, s)$ is a linear combination of terms, each of which agrees with the LSI-prediction eq. (4.132) but with different values of a' . Hence no unique value of a' can be obtained in this case.

i	f	a	$a' - a$	λ_R/z
C	C	$d/2 - 1$		$d/2$
C	A	-1	0	0
A	C	$d/2 - 1$	$-d/2$	$d/2$
A	A	-1	0	0
χ	C	$d/2 - 1$	0	$d/2$
χ	A	$d/2 - 1$	0	$d/2$

Table 3.12. Values of the ageing exponents a and a' and of the autoresponse exponent λ_R/z for the autoresponse functions $R_i^f(t, s)$ in the reversible AC-model, as defined in the text.

$A + B \rightleftharpoons C + D$ [228]. It turns out that the scaling forms of correlators and responses are exactly the same as for the RAC-model considered in this section, provided the value of the constant σ is adapted accordingly. Therefore, all these systems show the importance of conserved quantities in the ageing regime: the observed dynamical scaling depends on whether conserved quantities are unchanged by the perturbation or whether the perturbation breaks some of these quantities. In the latter case, the steady state reached in the long-time limit will always differ from the steady state of the unperturbed system. Furthermore, it is found that the approach to stationarity strongly depends on the conservation or non-conservation of these quantities, yielding different scaling functions for the different cases.

3.11 Growth Processes

A different class of non-equilibrium models considers the ballistic deposition of particles on a surface, as occurs for example in thin film growth due to vapour deposition, see [514, 308] for reviews. The state of this surface may be described in terms of a height variable $h(t, \mathbf{r})$ which is usually assumed to be determined from a rather simple-looking Langevin equation. In many instances important aspects of the problem are already captured by *linear* Langevin equations [452] and we shall restrict here to this situation. For irreversible deposition, the system clearly never arrives at an equilibrium state. Working in the frame co-moving with the mean surface height, the simplest kinetic equation one may write in the case without mass conservation is the well-known **Edwards-Wilkinson (EW) model** [227]

$$\partial_t h(t, \mathbf{r}) = \nu_2 \nabla_{\mathbf{r}}^2 h(t, \mathbf{r}) + \eta(t, \mathbf{r}) \quad (3.156)$$

which may be used to describe film growth with a normal incidence of the incoming particles. However, if mass conservation must be taken into account, one might rather consider the **Mullins-Herring (MH) model**, see [529, 736]

$$\partial_t h(t, \mathbf{r}) = -\nu_4 (\nabla_{\mathbf{r}}^2)^2 h(t, \mathbf{r}) + \eta(t, \mathbf{r}) \quad (3.157)$$

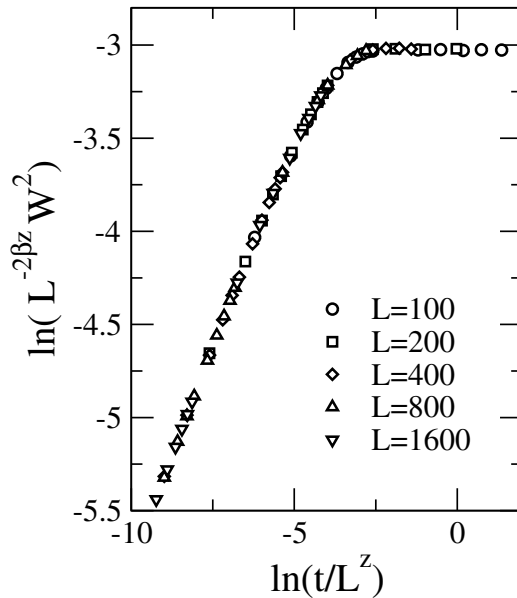


Fig. 3.25. Scaling of the width squared for the Family deposition model which belongs to the same universality class as the Edwards-Wilkinson model. The values of the exponents are $z = 2$ and $\beta = 1/4$.

which has been used to describe film growth via molecular beam epitaxy or tumor growth. These systems show naturally dynamical scaling as is for example illustrated through the time-dependent width distribution [21] or the mean-square width itself which for a substrate of linear size L scales as $W^2(t, L) = L^{2\beta z} F(tL^{-z})$, where the exponent β describes the roughness of the surface, through the increase of the width according to $W(t) \sim t^\beta$. This scaling of the width squared is illustrated in Fig. 3.25 where we show numerical data obtained for the Family deposition model [234], which will be defined below on p. 218. As we shall see shortly, the Family model belongs to the same universality class as the Edwards-Wilkinson model [452].

The linear stochastic equations (3.156) and (3.157) are the simplest representatives of a vast range of non-equilibrium growth models. Other, more realistic models, are given by the Kardar-Parisi-Zhang (KPZ) [423] and the molecular beam epitaxy (or conserved Kardar-Parisi-Zhang) [736, 638] equations which differ from the EW and MH equations by the presence of a non-linear term. These non-linear stochastic equations are thought to be relevant for most of the non-equilibrium growth processes encountered in nature.

Depending on the physical system, different types of Gaussian noise with vanishing first moment $\langle \eta(t, \mathbf{r}) \rangle = 0$ can be considered:

- (a) non-conserved, short-ranged $\langle \eta(t, \mathbf{r}) \eta(s, \mathbf{r}') \rangle = 2D \delta(\mathbf{r} - \mathbf{r}') \delta(t - s)$.

- (b) non-conserved, long-ranged $\langle \eta(t, \mathbf{r}) \eta(s, \mathbf{r}') \rangle = 2D |\mathbf{r} - \mathbf{r}'|^{2\rho-d} \delta(t-s)$ and $0 < \rho < d/2$.
- (c) conserved, short-ranged $\langle \eta(t, \mathbf{r}) \eta(s, \mathbf{r}') \rangle = -2D \nabla_{\mathbf{r}}^2 \delta(\mathbf{r} - \mathbf{r}') \delta(t-s)$.

In [629, 630] the following models were studied:

1. EW1: eq. (3.156) with the non-conserved noise (a).
2. EW2: eq. (3.156) with the non-conserved, long-ranged noise (b).
3. MH1: eq. (3.157) with the non-conserved noise (a).
4. MH2: eq. (3.157) with the non-conserved, long-ranged noise (b).
5. MHC: eq. (3.157) with the conserved noise (c) [47].

In the models EW1 and MHC the noise is in agreement with a fluctuation-dissipation theorem if the ratio ν_2/D (or ν_4/D) are chosen appropriately, and in this case relaxation towards an equilibrium state follows. For the other models, this is not the case.

Solving the linear equations (3.156) and (3.157) is straightforward and leads to the following results [629]. For the EW1 and EW2 model, the space-time response function is

$$R(t, s; \mathbf{r}) = \left. \frac{\delta \langle h(t, \mathbf{r}) \rangle}{\delta j(s, \mathbf{0})} \right|_{j=0} = r_0 (t-s)^{-d/2} \exp \left[-\frac{1}{4\nu_2} \frac{\mathbf{r}^2}{t-s} \right] \quad (3.158)$$

where $j(s, \mathbf{0})$ is a small perturbation, conjugate to the height h , and spatial translation-invariance was assumed from the outset. One reads off the exponents $a = d/2 - 1$, $\lambda_R = d$, $z = 2$ and the scaling function $f_R(y) = (y-1)^{-d/2}$. The space-time correlator is for a white noise (EW1 model)

$$\begin{aligned} C(t, s; \mathbf{r}) &= \langle h(t, \mathbf{r}) h(s, \mathbf{0}) \rangle \\ &= c_0 |\mathbf{r}|^{2-d} \left[\Gamma \left(\frac{d}{2} - 1, \frac{1}{4\nu_2} \frac{\mathbf{r}^2}{t+s} \right) - \Gamma \left(\frac{d}{2} - 1, \frac{1}{4\nu_2} \frac{\mathbf{r}^2}{t-s} \right) \right] \end{aligned} \quad (3.159)$$

where the normalisation constant c_0 is known in terms of the model's parameters and $\Gamma(d, x)$ is an incomplete Gamma function [4]. For spatially correlated noise, a series representation of $C(t, s; \mathbf{r})$ is found [629]. One has $b = d/2 - 1 + \rho$ and $\lambda_C = d - 2\rho$. For later comparison with local scale-invariance the autocorrelation scaling functions $C(t, s; \mathbf{0}) = s^{-b} f_C(t/s)$ are listed in Table 4.2. Whenever d and ρ are such that $b \rightarrow 0$, the autocorrelation scaling function reduces to $f_C(y) = \ln[(y+1)/(y-1)]$, up to normalisation.

Similarly, for the conserved case one finds in $2D$, for white noise (MH1 model) as well as correlated noise (MH2 model) [629]

$$\begin{aligned} R(t, s; \mathbf{r}) &= \frac{1}{8\pi\sqrt{\nu_4}} \left[\sqrt{\pi} {}_0F_2 \left(\frac{1}{2}, 1; \frac{1}{256\nu_4} \frac{|\mathbf{r}|^4}{t-s} \right) \right. \\ &\quad \left. - \frac{1}{4\sqrt{\nu_4}} \frac{|\mathbf{r}|^2}{(t-s)^{1/2}} {}_0F_2 \left(1, \frac{3}{2}; \frac{1}{256\nu_4} \frac{|\mathbf{r}|^4}{t-s} \right) \right] \end{aligned} \quad (3.160)$$

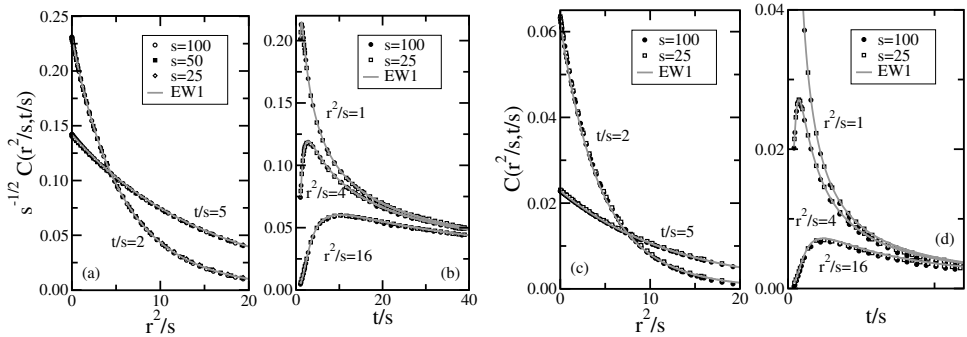


Fig. 3.26. Dynamical scaling of the space-time correlation function $C(\mathbf{r}^2/s, t/s)$ of the Family model in (a,b) $(1+1)D$ and (c,d) $(2+1)D$. In panels (a,c) the spatial behaviour for $y = t/s$ fixed to 2 or 5 is shown while in (b,d) the time-dependent scaling for the fixed values $\mathbf{r}^2/s = 1, 4, 16$ is shown. The full lines labelled EW1 give the prediction eq. (3.159) from the EW1 model. Reprinted with permission from [630]. Copyright (2007) by the American Physical Society.

where ${}_0F_2$ is a hypergeometric function. A similar explicit expression is obtained in $1D$ [629]. We point out that the contributions exponentially growing with $|\mathbf{r}|$ from each term cancel such that the total response function decreases with $|\mathbf{r}|$, as it should. The scaling exponents are $a = d/4 - 1$, $\lambda_R = d$ and $z = 4$ and the scaling function of the autoresponse function reads $f_R(y) = (y - 1)^{-d/4}$, up to normalisation. The space-time correlations are also explicitly known, for brevity we merely list the scaling function of the autoresponse functions in Table 4.2.

Finally, the model MHC is simply obtained from the critical conserved spherical model where the Lagrange multiplier $\mathfrak{z}(t) = 0$, or equivalently $\mathfrak{g}_d = 0$ so that the results for R and C in the conserved spherical model in the conserved spherical model (see p. 155) can be taken over and naïvely continued in d in order to obtain the corresponding results for the MHC model [47], see again Table 4.2.

Lattice formulations of growth processes in the EW1 universality class are provided by the Family model [234] and a variant of it. The **Family model** is a ballistic deposition model, combined with surface diffusion. Its microscopic rules are as follows. Consider a plane surface (e.g. a lattice $\Lambda \subset \mathbb{Z}^2$) onto which particles of a single species are deposited. Select a site $\mathbf{r} \in \Lambda$ for the deposition of the particle. Next, consider the nearest neighbours of the site $\mathbf{r}$ and place the incoming particles at the site where before the adsorption the local height variable $h(\mathbf{r})$ was minimal. If several neighbours have the same value of h , choose one of them randomly.

There is a variant, the *restricted* Family model [572], which differs from the Family model in that the last step of the redeposition is only carried out if the local minimum of $h(\mathbf{r})$ is unique. In $(1+1)D$, it has been shown that the EW1 model can be derived from both the Family and restricted Family model by a coarse-graining procedure [717]. This already suggests that these models should be in the EW1 universality class and is indeed borne out in numerical simulations. In Fig. 3.26 results of a Monte Carlo simulation of the Family model in both $(1+1)D$ and $(2+1)D$ are shown [629, 630]. The agreement with the exact solution is perfect. The same conclusion is reached for the variant of the $(2+1)D$ model, although the values of the non-universal constants D and ν_2 are different [629, 630]. Therefore, the study of the ageing properties of growth processes such as the one discussed here can have diagnostic value for identifying the universality class. In the case at hand, this is of immediate relevance, since previous numerical studies tried to extract the dynamical exponent z and concluded [572, 573] that the variant Family model were in a different universality class than the EW1 model.

Unfortunately, it appears to be unknown how to implement the variable j , conjugate to the height variable h , in the Family models and its variants, so that responses have not yet been calculated.

Whereas some work has been done on ageing phenomena described by linear growth models, only few results are available for non-linear models. In [111] Bustingorry studied the out-of-equilibrium dynamics of the KPZ equation for weak non-linearities and found that also in this case two-time quantities as for example the two-time roughness display ageing behaviour similar to what is observed for the EW equation. A fuller analysis of non-linear growth models is beyond the scope of this book.

Problems

3.1. Consider the non-equilibrium critical dynamics of a system with a scalar and non-conserved order-parameter, where the global initial magnetisation $m_0 \neq 0$ is assumed to be non-zero. Use a mean-field approximation to calculate the global persistence probability $P_g(t; m_0)$ and in particular the exponent θ_g [579]. Discuss the influence of the value of m_0 .

3.2. Derive the asymptotic growth law (3.54) in the limit $t \rightarrow \infty$ of large times. What growth laws do you find for a barrier height of the form $E_B(L) \sim (\ln L)^\omega$?

3.3. Consider a semi-infinite equilibrium system at a distance $\tau = T - T_c$ from its critical point and derive the scaling of its position-dependent order-parameter [598, 599].

3.4. Use the result of the previous exercise to prove the scaling limits (3.80).

3.5. Derive the mean-field response function of Reggeon field-theory. The effective action reads

$$\mathcal{J} = \int dt d\mathbf{r} \left[\tilde{\phi} (\partial_t \phi - m\phi - D \nabla_{\mathbf{r}}^2 \phi) + g\phi^2 \tilde{\phi} - g\phi \tilde{\phi}^2 - n_0 \delta(t) \tilde{\phi} \right],$$

where $\tilde{\phi}$ is the response field, $m \sim p - p_c$ measures the distance from the critical point, D is the diffusion constant, g a coupling constant and n_0 the initial density. The mean-field approximation amounts to treating $\mathcal{J}$ as a classical action. First show by solving the classical equation of motion $\delta\mathcal{J}/\delta\tilde{\phi} = 0$ that for $m = 0$ the spatially homogeneous solution is $\phi(t) = n_0/(1 + gn_0 t)$. Next, the Fourier-transformed second equation of motion $\delta\mathcal{J}/\delta\phi = 0$ becomes

$$-\partial_t \hat{\tilde{\phi}}_{\mathbf{k}}(t) = m \hat{\tilde{\phi}}_{\mathbf{k}}(t) - D k^2 \hat{\tilde{\phi}}_{\mathbf{k}}(t) - 2g\phi(t) \hat{\tilde{\phi}}_{\mathbf{k}}(t).$$

Show that in momentum space the response function becomes

$$\left\langle \hat{\phi}_{-\mathbf{k}}(t) \hat{\tilde{\phi}}_{\mathbf{k}}(s) \right\rangle = \hat{R}_{\mathbf{k}}(t, s) = e^{-\mathbf{k}^2(t-s)} \left(\frac{1 + gn_0 s}{1 + gn_0 t} \right)^2$$

and rewrite this for real space. Find the mean-field values of the exponents a and λ_R [609].

3.6. Can one generalise the scaling relation (1.112) to systems undergoing non-equilibrium critical dynamics at an absorbing phase transition ?

3.7. Consider the bosonic contact and pair-contact processes *without* diffusion, i.e. $D = 0$. Then each site can be treated independently. Establish the equations of motion, at the critical point

$$\frac{\partial}{\partial t} \langle a(t) \rangle = 0 \quad , \quad \frac{\partial}{\partial t} \langle (a(t))^2 \rangle = \mu(1 + \ell) \langle a(t)^m \rangle$$

and find the particle-density and the variance [376].

3.8. Consider the bosonic contact or pair-contact process with diffusion on their critical line (3.111). Derive eq. (3.119).

Generalise your argument to Liouvillians (or quantum Hamiltonians) of the form

$$\mathcal{L} = \mathcal{L}_{\text{diff}}[a, a^\dagger] + \mathcal{L}_{\text{int}}[a, a^\dagger]$$

where $\mathcal{L}_{\text{diff}}$ describes diffusion of single particles and the interaction part $\mathcal{L}_{\text{int}}$ is such that $\langle s | [a, \mathcal{L}_{\text{int}}] = 0$ holds true. Therefore Baumann's inequality $b \leq a + 1$ will remain valid [44].

Chapter 4

Local Scale-invariance I: $z = 2$

We have seen in the first part of this volume that dynamical scaling plays a central role in the description of ageing phenomena. In this second part, we shall be concerned in particular with the *form* of the universal scaling functions which may arise in several physical observables. Are there general approaches, independent of the explicit treatment of a specific model, which would allow scaling functions of ageing systems to be predicted ?

4.1 Introduction

One of the central aspects of ageing systems is the dynamical self-similarity of averages. We illustrate this once more in [Fig. 4.1](#) for a two-dimensional Glauber-Ising model, quenched to its critical point from a fully disordered initial state. Although the structure of the typical configurations clearly depends on the time t elapsed since the quench, a rescaling of all length scales with the linear size $L(t) \sim t^{1/z}$ of correlated domains results in statistically similar configurations. This analogy with similar rescaling transformations from equilibrium critical phenomena will serve as a guide in what follows.

Moreover, the universality of the non-equilibrium scaling functions, as discussed in the previous chapters, suggests that a dynamical symmetry might be a fruitful way to proceed towards an answer. The example of conformal invariance in equilibrium critical phenomena will serve as a useful analogue. Indeed, as discussed in Chap. 2 in volume 1, conformal transformations can be defined as a particular kind of *local* spatial scale-transformations, where the dilation factor may be position-dependent, but in such a way that angles are kept unchanged. The enormous utility of conformal invariance for the analysis of two-dimensional equilibrium critical phenomena [62] was one of our main original motivations in attempting to explore possible generalisations of (time-dependent) dynamical scaling to some **local scale-invariance** (LSI) [326, 327, 328, 330, 48]. In contrast to equilibrium critical phenomena,

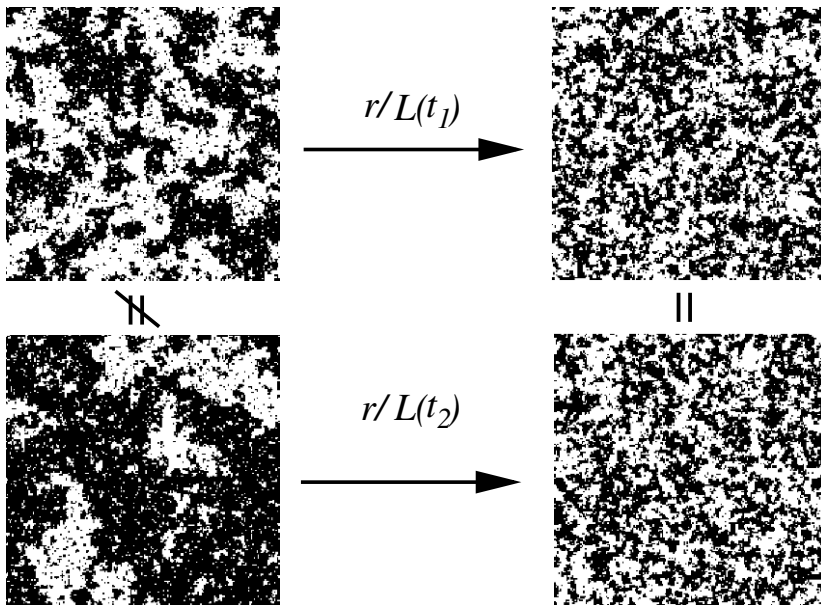


Fig. 4.1. Illustration of the dynamical self-similarity in the non-equilibrium critical dynamics of the 2D Glauber-Ising model, quenched to $T = T_c$ from a totally disordered initial state. The two Figures on the left show typical configurations at times t_1 and $t_2 \gg t_1$ after the quench. The two figures on the right are obtained by rescaling the distances r with the time-dependent correlation length $L(t) \sim t^{1/z}$ and are statistically similar. Courtesy J.-C. Walter [721].

time will play a special role. Since the dynamical exponent $z \neq 1$, of course one cannot relabel time as a further spatial direction and then simply and naïvely use ‘space-time’ conformal transformations. Rather, we shall have to construct Lie algebras of infinitesimal local space-time transformations which take the specific given value of z into account.

A definite and fully formalised theory of LSI does not yet exist. However, sufficient progress in this direction has been achieved that a comprehensive presentation of the present state of LSI seems justified. Such a full presentation has not yet been available, although partial expositions of LSI, usually focussed on a specific kind of application, have been given in a couple of reviews [331, 332, 333, 334, 335, 348]. One objective of such a presentation should be to familiarise the readers with the various techniques required as to be able to contribute themselves to the development of the subject.

In the next three chapters, an introduction to local scale-invariance will be given. In Chapters 4 and 5, we shall study extensions of dynamical scaling for non-equilibrium systems, especially in the context of ageing. In Chap. 6, a related exposition will be presented for a certain kind of equilibrium critical phenomena which contain a preferred direction (||) and are spatially so

strongly anisotropic that the **anisotropy exponent** $\theta = \nu_{\parallel}/\nu_{\perp} \neq 1$ can take over the role the dynamic exponent plays for non-equilibrium systems.

For technical reasons, we shall study, in this chapter, LSI for non-equilibrium systems with a dynamical exponent $z = 2$. In this case, a well-known Lie group, called the Schrödinger group and discovered by Lie in 1881 as a dynamical symmetry group of the free diffusion equation,¹ is a natural candidate for a group of local scale-transformations. We shall therefore begin to recall some properties of this group, of its Lie algebra and especially of its subgroups/subalgebras which appear to be the most relevant for applications to ageing.² The requirement of Galilei-invariance will lead to the conceptually important Bargman superselection rules. These will in turn become an essential tool for the discussion of the dynamic symmetries of *stochastic* Langevin equations, where at first sight the noise terms seem to exclude any interesting symmetries. A non-trivial reduction formula, based on the Bargman superselection rules, allows us to solve this problem and hence to calculate explicitly the scaling functions of two-time response and correlation functions, ready to be tested against concrete models. In Chap. 5, we describe an extension of this programme to the case of a generic dynamical exponent $z \neq 2$. Since no Lie algebras of local scale-transformations with $z \neq 2$ are available, a necessary first step is the construction of infinitesimal generators and of invariant equations. A closure procedure must then be introduced in a suitable way which leads to a discussion of Galilei-invariance extended to $z \neq 2$ and the generalised Bargman rules which follow. Remarkably, the structure obtained shows some similarity with the one found for conformally invariant systems perturbed by certain relevant scaling operators (of which a well-known example is the 2D critical Ising model in an external magnetic field) [748]. From this, explicit results for the two-time responses and correlations can be computed. These allow for explicit tests of these still hypothetical symmetries in concrete models, as we shall discuss.

In Chap. 6, an analogous programme is carried out for Lifshitz points, which provide a well-studied example of strongly anisotropic equilibrium critical points, characterised by an anisotropy exponent $\theta \approx \frac{1}{2}$.

¹ Jacobi noted *en passant* in his Königsberg *Vorlesungen über Dynamik* in 1842/43 that the Newtonian n -body equations of motion (where interactions with an inverse-square potential can be included) are invariant under what we consider today as the elements of the Schrödinger group [398, 322].

² See appendix L for a reminder of some basic definitions and facts from the theory of Lie groups and Lie algebras.

4.2 The Schrödinger Group

4.2.1 Dynamical Conformal Invariance

As defined in Volume 1, Chap. 2, conformal transformations may be viewed as angle-preserving scale-transformations with a spatially varying dilatation factor $b(\mathbf{r})$. Cardy [130] first discussed local dynamical scaling for *equilibrium* critical dynamics in the form of the conformal transformations $\mathbf{r} \mapsto b(\mathbf{r})\mathbf{r}$ and $t \mapsto b(\mathbf{r})^z t$ in two space and one time dimensions. The value of the dynamical exponent z is one of the data to put into the theory and cannot be predicted. Assuming that response functions transform co-variantly under these, Cardy considered the conformal map from $2D$ infinite space onto an infinitely long strip of finite width and argued that it was legitimate to use dynamic mean-field (van Hove) theory to find the response function. For a non-conserved order-parameter, this led to $R(t, \mathbf{r}) \sim t^{-1-(d-2+\eta)/2z} \exp(-|\mathbf{r}|^z/t)$, up to normalisation-constants [130]. However, this result apparently has never been reproduced in any non-trivial model calculation and different forms were found in slightly more general situations [64, 327]. In any case, Cardy's argument would be inapplicable for non-stationary systems, for which a different starting point should be sought.

4.2.2 Definition of the Schrödinger Group

In Volume 1, Chap. 2, we saw that conformal transformations act as dynamical symmetry on Laplace's equation $\Delta_L \phi = \nabla^2 \phi = 0$ since they map any solution of that equation to another solution of the same equation (exercise 4.2). As the most elementary example of a dynamic equation, we begin with the simple diffusion equation $2\mathcal{M}\partial_t \phi = \Delta_L \phi$ where $\mathcal{M} > 0$ is a constant kinetic coefficient.³ While the symmetries of free motion were already mentioned by Jacobi in 1842/43 [398, 322] and the symmetry Lie group of the free diffusion equation was found by Lie in 1881 [472, 473], it seems that in particular Lie's result may have been overshadowed by his other achievements. It was re-derived in mathematics at least twice, by Appell [23] and by Goff [289]. Even later, the same symmetries were, apparently independently, rediscovered in physical contexts by Kastrup [426], Niederer [544], Hagen [307], Jackiw [395] and Burdet and Perrin [108]. Indeed, it has become common to formulate results as symmetries of the free Schrödinger equation [544] and the symmetry group of that equation is now usually called the **Schrödinger group** $Sch(d)$.⁴

³ If $\mathcal{M} = im$ is imaginary, this becomes the free Schrödinger equation and for $\mathcal{M} = -m < 0$ one has a Hamilton-Jacobi equation.

⁴ A more appropriate name for $Sch(d)$ would be 'Lie group' but evidently this is no longer available. Rather, the Schrödinger-group was one of the first examples of the far more general mathematical structure which bears Lie's name today.

The Schrödinger-group in d spatial dimensions contains the following set of space-time transformations (Fig. 5.1 on p.293 gives a geometric illustration)

$$t \rightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta} ; \quad \mathbf{r} \rightarrow \mathbf{r}' = \frac{\mathfrak{D}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta} , \quad \alpha\delta - \beta\gamma = 1 \quad (4.1)$$

where $\alpha, \beta, \gamma, \delta, \mathbf{v}, \mathbf{a}$ are real (vector) parameters and $\mathfrak{D}$ is a rotation matrix in d spatial dimensions. The Schrödinger group contains the following transformations: (i) temporal and spatial translations which are parametrised by β and $\mathbf{a}$, respectively, (ii) spatial rotations $\mathfrak{D}$, (iii) **Galilei-transformations** parametrised by $\mathbf{v}$, (iv) **dilatations** (scale-transformations) with a dynamical exponent $z = 2$ and parametrised by δ and (v) **special Schrödinger-transformations** parametrised by γ . These transformations form a group and can be obtained as a semi-direct product of the Galilei-group with the group $Sl(2, \mathbb{R})$ of the real projective transformations in time. A faithful $d + 2$ -dimensional matrix representation is

$$\mathcal{L}_g = \begin{pmatrix} \mathfrak{D} & \mathbf{v} & \mathbf{a} \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix} , \quad \mathcal{L}_g \mathcal{L}_{g'} = \mathcal{L}_{gg'} . \quad (4.2)$$

A function-space representation of the Schrödinger-transformations (4.1), mapping any solution of the free Schrödinger equation

$$\left(i \frac{\partial}{\partial t} + \frac{1}{2m} \frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} \right) \phi = 0 \quad (4.3)$$

onto another solution of (4.3) through $(t, \mathbf{r}) \mapsto g(t, \mathbf{r})$, is given by $\phi \mapsto T_g \phi$

$$(T_g \phi)(t, \mathbf{r}) = f_g(g^{-1}(t, \mathbf{r})) \phi(g^{-1}(t, \mathbf{r})) \quad (4.4)$$

with $g \in Sch(d)$ and where the **companion function** reads [544, 587]

$$f_g(t, \mathbf{r}) = \frac{1}{(\gamma t + \delta)^{\frac{d}{2}}} \exp \left[-\frac{im}{2} \frac{\gamma \mathbf{r}^2 + 2\mathfrak{D}\mathbf{r} \cdot (\gamma \mathbf{a} - \delta \mathbf{v}) + \gamma \mathbf{a}^2 - t\delta \mathbf{v}^2 + 2\gamma \mathbf{a} \cdot \mathbf{v}}{\gamma t + \delta} \right] \quad (4.5)$$

It is then natural to include also arbitrary phase-shifts of the wave function ψ within the **Schrödinger group** $Sch(d)$. The Schrödinger group so defined is the largest group which maps *any* solution of the free Schrödinger equation onto another solution.⁵

Furthermore, non-relativistic free field-theory is Schrödinger-invariant and this statement can even be extended to higher-spin fields [307, 358]. Hagen's argument uses the explicit construction of conserved currents from certain symmetry properties of the energy-momentum tensor and we shall describe

⁵ Niederer generalised this statement to Schrödinger equations with quite general potentials [545, 546], see exercise 4.5.

this later. It is often stated, following an argument by Barut [42], that the Schrödinger-group could be obtained as the non-relativistic limit of a conformal group in $d + 1$ space dimensions. This was believed sufficient motivation to refer to the Schrödinger-group as a ‘non-relativistic conformal group’. However, the limit procedure invoked in [42] only applies to the Galilei subgroup, and an extension to the non-Galilean transformations gives as the non-relativistic limit a different Lie group, which in this book will be called the **conformal Galilean group** $CGG(d)$ [322], and which was also re-discovered several times under different names such as the **non-relativistic conformal group** [533, 532] or **altern group** $Alt(d)$ [328, 357]. Formally, one may go over to the diffusion equation and thereby make contact with Lie’s results by letting the non-relativistic mass become $m = (2iD)^{-1}$, where D is the diffusion constant.

In contrast with conformal transformations, both the Galilei- and the special Schrödinger-transformations act on the wave function ϕ only *projectively*, that is ϕ as well as the scaling operators ϕ_i of Galilei- or Schrödinger-invariant theories pick up a complex phase as described by $f_g \neq 0$ and characterised by the **mass** m [37, 468, 469]. It is useful to restrict ourselves to the infinitesimal transformations which make up the **Schrödinger algebra**

$$\mathfrak{sch}(d) := \text{Lie}(\text{Sch}(d)). \quad (4.6)$$

In $d = 1$ spatial dimensions, the Schrödinger algebra may be taken as $\mathfrak{sch}(1) = \langle X_{-1,0,1}, Y_{-\frac{1}{2},\frac{1}{2}}, M_0 \rangle$ and is spanned by the infinitesimal generators with the explicit form [327]

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \frac{n+1}{2}t^n r \partial_r - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}r^2 - \frac{x}{2}(n+1)t^n \\ Y_m &= -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right)\mathcal{M}t^{m-1/2}r \\ M_n &= -\mathcal{M}t^n. \end{aligned} \quad (4.7)$$

Here x is the scaling dimension and $\mathcal{M} = im$ the **mass** of the scaling operator ϕ on which these generators act.⁶ For $x = \mathcal{M} = 0$, we recover the infinitesimal transformations of the Schrödinger group (4.1). The non-vanishing commutation relations are

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{2} - m\right)Y_{n+m} \\ [X_n, M_{n'}] &= -n'M_{n+n'} \quad , \quad [Y_m, Y_{m'}] = (m - m')M_{m+m'}. \end{aligned} \quad (4.8)$$

Since M_0 commutes with the algebra $\mathfrak{sch}(1)$, its eigenvalue $M_0 = -\mathcal{M}$ can be used along with the eigenvalue $\mathcal{Q}$ of the quadratic Casimir operator [587]

⁶ For readers with a background in high-energy physics, we emphasise that $\mathcal{M}$ does not measure the distance to a critical point but rather plays the role of a charge.

$$Q := \left(4M_0X_0 - 2\{Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}\}\right)^2 - 2\left\{2M_0X_{-1} - Y_{-\frac{1}{2}}^2, 2M_0X_1 - Y_{\frac{1}{2}}^2\right\} \quad (4.9)$$

(with the anticommutator $\{A, B\} := AB + BA$) to characterise the unitary irreducible (projective) representations of the Lie algebra (4.8) of the Schrödinger group [587]. One can show that the representations with $Q = 0$ realised on scalar functions reproduce the transformation (4.4,4.5) [587].⁷

The invariance of the free 1D Schrödinger equation under $\mathfrak{sch}(1)$ can be seen in a compact way by introducing the **Schrödinger operator**

$$\mathcal{S} := 2M_0Y_{-1} - Y_{-1/2}^2 \quad (4.10)$$

which takes under the function-space representation (4.7) the familiar form $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2$. It satisfies the following commutators with the generators of the Schrödinger-algebra $\mathfrak{sch}(1)$:

$$\begin{aligned} [\mathcal{S}, X_{-1}] &= [\mathcal{S}, Y_{\pm 1/2}] = [\mathcal{S}, M_0] = 0 \\ [\mathcal{S}, X_0] &= -\mathcal{S} \\ [\mathcal{S}, X_1] &= -2t\mathcal{S} - (2x - 1)M_0 \end{aligned} \quad (4.11)$$

Therefore, for any solution of the 1D Schrödinger equation $\mathcal{S}\phi = 0$ with scaling dimension $x = 1/2$, the infinitesimally transformed solution $\mathcal{X}\phi$ with $\mathcal{X} \in \mathfrak{sch}(1)$ also satisfies the Schrödinger equation $\mathcal{S}(\mathcal{X}\phi) = 0$.

We have been making two important physical assumptions here.

1. Borrowing terminology from conformal invariance [62] (see Chap. 2 in Volume 1), only so-called **quasi-primary** scaling operators ϕ will transform under the action of the Schrödinger group such that the change under an infinitesimal transformation, generated by $\mathcal{X} \in \mathfrak{sch}(1)$, is simply given by $\mathcal{X}\phi$ [327, 330]. It is non-trivial that a given physical observable should be represented by a quasi-primary scaling operator, although this is usually admitted for the order-parameter,⁸ which at the level of classical field-theory will satisfy an equation of motion $\mathcal{S}\phi = 0$.
2. The free diffusion/Schrödinger equation is of first order in the time-derivative. We shall see later that the equation of motion for the order-parameter has to be completed by a suitable noise term. One then has to analyse a stochastic Langevin equation of first order in ∂_t , which corresponds to a *Markov process*.

The Schrödinger-algebra $\mathfrak{sch}(1)$ can be extended [327] to an infinite-dimensional Lie algebra $\mathfrak{sv}(1) := \langle X_n, Y_m, M_n \rangle$ with $n \in \mathbb{Z}$ and $m \in \mathbb{Z} + \frac{1}{2}$. The

⁷ In the mathematical literature (see appendix L), $\mathfrak{sch}(d)$ with $M_0 \neq 0$ is referred to as a centrally extended algebra. Since in most physical applications one needs $\mathcal{M} \neq 0$ anyway, we simplify the terminology and refer to (4.8) as the Schrödinger Lie algebra *tout court* and avoid talking of any ‘central extensions’ in this context.

⁸ If $\phi = \phi(t, \mathbf{r})$ is quasi-primary, neither of the derivatives $\partial_t\phi$ nor $\partial_r\phi$ is.

commutation relations (4.8) extend to this infinite range of indices, hence $\mathfrak{sv}(1)$ contains a Virasoro algebra (in the chosen representation with vanishing central charge) as a subalgebra. Only the finite-dimensional subalgebra $\mathfrak{sch}(1) \subset \mathfrak{sv}(1)$ is a dynamical symmetry of the Schrödinger equation $\mathcal{S}\phi = 0$. Exercise 4.3 looks at the group transformations, generalising (4.1).

Because of the importance of the notions of quasi-primary and primary scaling operators, we define them formally, generalising from the conformal case.

Definition: Let $\mathcal{L}(V)$ be the set of linear operators on a Hilbert space V and let $\mathfrak{R}^{(n)} : \mathfrak{g} \rightarrow \mathcal{L}(V)^{\otimes n}$ be the tensor representation of a finite-dimensional Lie algebra $\mathfrak{g}$. If ϕ_i are scaling operators on V and if $\langle \cdots \rangle : V^{\otimes n} \rightarrow \mathbb{C}$ denotes the average, then an n -point function $F_n := \langle \phi_1 \otimes \cdots \otimes \phi_n \rangle$ is said to be **$\mathfrak{g}$ -covariant** under the representation $\mathfrak{R}$, if for all generators $\mathcal{X} \in \mathfrak{g}$

$$\begin{aligned} & \langle \mathfrak{R}^{(n)}(\mathcal{X})(\phi_1 \otimes \cdots \otimes \phi_n) \rangle \\ &= \langle (\mathfrak{R}(\mathcal{X})\phi_1) \otimes \phi_2 \otimes \cdots \otimes \phi_n \rangle + \cdots + \langle \phi_1 \otimes \cdots \otimes \phi_{n-1} \otimes (\mathfrak{R}(\mathcal{X})\phi_n) \rangle = 0 \end{aligned} \quad (4.12)$$

Then the ϕ_i are **$\mathfrak{g}$ -quasi-primary** with respect to $\mathfrak{R}$. If $\mathfrak{g}$ can be extended to an infinite-dimensional algebra which includes a Virasoro algebra such that the Ward identity (4.12) still holds, the ϕ_i are **$\mathfrak{g}$ -primary** under the representation $\mathfrak{R}$. For brevity, we shall often use simply ‘quasi-primary’ and ‘primary’.

Example: we illustrate the transformation of a Schrödinger-primary scaling operator ϕ with scaling dimension x . Suppressing the spatial coordinates for simplicity, the transformation law under $t \mapsto t'$ reads, see exercise 4.3,

$$t = \beta(t') \quad , \quad \phi(t) = \left(\frac{d\beta(t')}{dt'} \right)^{-x/2} \phi'(t') \quad (4.13)$$

where $\dot{\beta}(t') \geq 0$. When $x \mapsto 2x$ and if t may stand for one of the $2D$ complex coordinates $z, \bar{z}$, this becomes the transformation (2.169) in Volume 1 of a scalar conformal primary operator in $2D$. For a Schrödinger-*quasi*-primary operator, (4.13) needs only to hold if $\beta(t)$ is a projective transformation (4.1) in time.

The extension of these results to $d > 1$ dimensions presents no difficulty. We give the result below, but in most calculations in this Chap., we shall restrict ourselves to the $1D$ case, which already contains the essential information. The generators of the Schrödinger algebra $\mathfrak{sch}(d) = \langle X_{0,\pm 1}, Y_{\pm 1/2}^{(j)}, M_0, R_0^{(jk)} \rangle$ in d spatial dimensions read [330, 623]

$$\begin{aligned} X_n &= -t^{n+1} \partial_t - \frac{n+1}{2} t^n \mathbf{r} \cdot \boldsymbol{\partial} - \frac{n(n+1)}{4} \mathcal{M} t^{n-1} \mathbf{r}^2 - \frac{x}{2} (n+1) t^n \\ Y_m^{(j)} &= -t^{m+1/2} \partial_j - \left(m + \frac{1}{2} \right) \mathcal{M} t^{m-1/2} r_j \\ M_n &= -t^n \mathcal{M} \\ R_n^{(jk)} &= -t^n (r_j \partial_k - r_k \partial_j) \end{aligned} \quad (4.14)$$

where $\partial_j = \partial/\partial r_j$ and $j, k = 1, \dots, d$, $\mathcal{M}$ is a constant, $n \in \mathbb{Z}$ and $m \in \mathbb{Z} + \frac{1}{2}$. In comparison with the $1D$ case, there are now d sets of (time-dependent) Galilei-transformations $Y_m^{(j)}$ and a new family of generators $R_n^{(jk)}$ of spatial rotations with time-dependent angles. The non-vanishing commutators are

$$\begin{aligned}
[X_n, X_{n'}] &= (n - m)X_{n+n'} \\
[X_n, Y_m^{(j)}] &= \left(\frac{n}{2} - m\right) Y_{n+m}^{(j)} \\
[X_n, M_{n'}] &= -n' M_{n+n'} \\
[X_n, R_{n'}^{(jk)}] &= -n' R_{n+n'}^{(jk)} \\
[Y_m^{(i)}, Y_{m'}^{(j)}] &= \delta_{i,j} (m - m') M_{m+m'} \\
[R_n^{(ij)}, R_{n'}^{(kl)}] &= \delta_{i,k} R_{n+n'}^{(jl)} + \delta_{j,l} R_{n+n'}^{(ik)} - \delta_{i,l} R_{n+n'}^{(jk)} - \delta_{j,k} R_{n+n'}^{(il)} \\
[R_n^{(ij)}, Y_m^{(k)}] &= \delta_{i,k} Y_{n+m}^{(j)} - \delta_{j,k} Y_{n+m}^{(i)}
\end{aligned} \tag{4.15}$$

The infinite-dimensional Lie algebra $\mathfrak{sv}(d) := \langle X_n, Y_m^{(j)}, M_n, R_n^{(jk)} \rangle$ with $j, k \in \{1, \dots, d\}$, $n \in \mathbb{Z}$ and $m \in \mathbb{Z} + \frac{1}{2}$ is called the **Schrödinger-Virasoro algebra** in d spatial dimensions, with the non-vanishing commutators (4.15). The Schrödinger operator now reads, in an obvious vector notation,

$$\mathcal{S} = 2M_0 X_{-1} - \mathbf{Y}_{-1/2} \cdot \mathbf{Y}_{-1/2} = 2\mathcal{M}\partial_t - \nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}} \tag{4.16}$$

and has the finite-dimensional subalgebra $\mathfrak{sch}(d) \subset \mathfrak{sv}(d)$ as a dynamical symmetry. Invariant solutions of the free diffusion equation must have the scaling dimension $x = d/2$ (see exercise 4.4).

The mathematical theory of $\mathfrak{sv}(d)$, its central extensions, deformations and representations, is developed in [623, 706, 705, 707, 624]. We shall return in Chap. 5 to the question of a geometric interpretation of Schrödinger-transformations [221].

4.2.3 Physical Examples of Schrödinger-invariance

Schrödinger-invariance (meaning that either the finite-dimensional algebra $\mathfrak{sch}(d)$ or even the infinite-dimensional algebra $\mathfrak{sv}(d)$ is a dynamical symmetry) has been found in quite distinct physical contexts, some of which will now be listed.

1. The Schrödinger group in one spatial dimension, as dynamical symmetry of the $1D$ diffusion equation, is one of the earliest examples of physically relevant Lie groups [472, 473], dating to the last quarter of the 19th century. The free diffusion/Schrödinger equation may be seen as the Euler-Lagrange equation of motion of free non-relativistic free-field theory, the Schrödinger-invariance of which led physicists to rediscover the

Schrödinger group about a century later [426, 544, 307, 108]. We have already derived above the Schrödinger-invariance of the free diffusion equation and shall discuss below the corresponding properties of the energy-momentum tensor, which can be done by following closely the lines of conformal invariance, as shown in Volume 1, Chap. 2 [307, 357].

This may be generalised naturally, from the scalar case considered so far, to fields with spin $\frac{1}{2}$ when the diffusion equation is replaced by the **Lévy-Leblond equations** [469, 307, 358] and further to fields of higher spin.

At first sight, there seems to arise an important difference between the Schrödinger and diffusion equations in that the solutions of the first are complex-valued (necessary for consistency with Galilei-invariance) while those of the second are real-valued. However, in the context of the physical response-field formalism outlined in appendix D (both at and far from equilibrium), one may identify the so-called ‘response field’ $\tilde{\phi} = \tilde{\phi}(t, \mathbf{r})$ as playing the role of the complex conjugate of the real-valued solution $\phi = \phi(t, \mathbf{r})$ such that Galilei- and Schrödinger-invariance are indeed satisfied for the diffusion equation as well.

2. The dynamical symmetries of the equation $(i\partial_t - \Delta_L - V(t, \mathbf{r}))\phi = 0$ were analysed in detail by Niederer [545, 546]. For example, for a freely falling particle or a harmonic oscillator, the dynamical symmetry group of the associated Schrödinger equation is isomorphic to $Sch(d)$, but the representations to be used are different from that of the free Schrödinger equation. Indeed, the potentials $V = V(t, \mathbf{r})$ admitting a non-trivial dynamical symmetry, which can be as large as a group isomorphic to $Sch(d)$, are characterised as solutions of a certain differential equation [546], see exercise 4.5.

Many Schrödinger equations (or systems) with a potential admit non-trivial dynamical symmetries which are subgroups of $Sch(d)$, see e.g. [90, 267, 617, 147, 148, 149, 549] for an entry into this very rich literature.

3. Schrödinger-invariance also holds in many non-linear equations. We recall Niederer’s classical result on the non-linear 1D diffusion equation [547]

$$\partial_t u - D\partial_r^2 u + F(u, \partial_r u) = 0 \quad (4.17)$$

where he considered the following affine form of the function-space representations of the Schrödinger group

$$(T_g u)(t, r) = f_g[g^{-1}(t, r)]u[g^{-1}(t, r)] + h_g[g^{-1}(t, r)] \quad (4.18)$$

with *two* companion functions f_g and h_g . In Table 4.1 we list the four classes of differential equations, characterised by the function $F(u, \partial_r u)$, which are Schrödinger-invariant with a group action of the form (4.18). These equations can all be reduced to a simple diffusion equation $\partial_t \psi = D\partial_r^2 \psi$ by a generalised **Cole-Hopf transformation**, these are also listed.

	$F(u, v)$	transformation $u = u[\psi]$	
1	$k + cv$	$\exp\left[-\frac{c^2}{4D}t + \frac{c}{2D}r\right]\psi - kt$	
2	$k + bv + cv^2$	$-\frac{D}{c} \ln \psi - \left[k - \frac{b^2}{4c}\right]t - \frac{b}{2c}r$	$c \neq 0$
3	$b(u + q) + cv + D(1 - \lambda)\frac{v^2}{(u + q)}$	$\exp\left[-\left[b + \frac{c^2}{4D\lambda}\right]t + \frac{c}{2D\lambda}r\right]\psi^{1/\lambda} - q$	$\lambda \neq 0$
4	$bv + cuv$	$-\frac{2D}{c} \frac{\partial_r \psi}{\psi} - \frac{b}{c}$	$c \neq 0$

Table 4.1. The four classes of 1D Schrödinger-invariant non-linear diffusion equations (4.17), characterised by the function $F(u, v)$ with $v = \partial_r u$ and transforming according to eq. (4.18), where b, c, k, q, λ are constants. The transformation laws $\psi \mapsto u = u[\psi]$ map a solution $\psi = \psi(t, r)$ of the 1D diffusion equation $\partial_t \psi = D \partial_r^2 \psi$ to a solution of each class [547].

Explicit expressions for the companion functions and also the non-trivial co-cycles can be found in Niederer's article [547], but may of course be computed from the transformation laws $\psi \mapsto u[\psi]$ in Table 4.1 and the action of $Sch(1)$ on ψ as given in (4.4) and $D^{-1} = 2mi$. It has also been shown that even more general forms of the function-space representations do not lead to new types of invariant equations [547]. Equations of this kind might be used as mean-field equations of motion of potentially Schrödinger-invariant theories.⁹

4. An example of a non-linear equation where the Schrödinger symmetry can be extended to an infinite-dimensional Schrödinger-Virasoro symmetry is given by **Burgers' equation**, see class 4 in Table 4.1 [547, 393]. In one spatial dimension, it reads $\partial_t u + \frac{1}{2} \partial_r u^2 - \nu \partial_r^2 u = f$ and is used for simplified descriptions of turbulence or shocks of a fluid flow $u = u(t, r)$ in dependence of an external force $f = f(t, r)$. If one has $f = 0$, the unforced Burgers equation is an example of a non-linear Schrödinger-invariant equation. However, the solutions $u = u(t, r)$ do *not* transform according to eqs. (4.4, 4.5) with $\mathcal{M} = (2\nu)^{-1}$, as one might have conjectured by comparing with the linear Schrödinger equation. Rather, one needs different representations of $\mathfrak{sch}(1)$, see eq. (4.18), with $\mathcal{M} = 0$, and which also include additive terms (and not merely multiplicative ones as in (4.4)) in the transformation of u [547]. Furthermore, these representations may be extended to representations of $\mathfrak{sv}(1)$ which relate Burgers' equations with different external forces [393]. Explicitly

⁹ Class 2 describes the burning of a gas in rocket. The 1D Navier-Stokes equation with homogeneous pressure is either in class 4 with u the velocity, or else in class 2 with u the velocity potential.

$$\begin{aligned}
X_n &= -t^{n+1}\partial_t - \frac{n+1}{2}t^n r \partial_r - \frac{n+1}{2}(nr - u)\partial_u \\
&\quad - \frac{(n+1)n}{2}((n-1)r - 3f)\partial_f \\
Y_m &= -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right)t^{m-1/2}\partial_u - \left(m^2 - \frac{1}{4}\right)\partial_f
\end{aligned} \tag{4.19}$$

and where, as usual, $n \in \mathbb{Z}$ and $m \in \mathbb{Z} + \frac{1}{2}$.

On the other hand, Schrödinger-invariance is broken down to Galilei-invariance, with a Lie algebra spanned by $\langle X_{-1}, Y_{\pm 1/2} \rangle$, if f is taken to be a *random* force with a Gaussian distribution [393]. Indeed, the breaking of extended scale-symmetry by noise is a generic phenomenon. We shall discuss in Sect. 4.5 how to identify *hidden* non-trivial symmetries in stochastic systems.

5. **Euler's hydrodynamical equations** for a perfect, viscous fluid are given by

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\
&\rho(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = \nu \Delta_L \mathbf{v} + \left(\zeta + \frac{\nu}{3}\right) \nabla(\nabla \cdot \mathbf{v}),
\end{aligned} \tag{4.20}$$

where $\rho = \rho(t, \mathbf{r})$ is the fluid density whose flow is described by the velocity field $\mathbf{v} = \mathbf{v}(t, \mathbf{r})$, $p = p(t, \mathbf{r})$ is the pressure and ζ and ν denote the bulk and shear viscosities. The adiabatic motion of the fluid is expressed by the conserved transport of the entropy $S = S(p, \rho)$, described by $(\partial_t + \mathbf{v} \cdot \nabla)S(p, \rho) = 0$. The functional form of $S = S(p, \rho)$ gives the equation of state. In case of a so-called *polytropic equation of state* $p \sim \rho^{\gamma_p}$ with a **polytropic exponent** $\gamma_p = 1 + 2/d$, Euler's equations are Schrödinger-invariant in the inviscid case $\zeta = \nu = 0$, with a 'massless' representation $\mathcal{M} = 0$. This fact was known to mathematicians already in the 1960s [568] and was later rediscovered by physicists [320, 321, 566]. Here, the so-called 'special' transformation (parametrised in eq. (4.1) by γ) relates collapsing and exploding solutions.¹⁰ This has been applied to relate supernova explosions to laboratory experiments on plasma implosions [209].

6. An equilibrium field-theoretical description of a system at a multi-critical point of order p may be given in terms of a Ginzburg-Landau functional $\mathcal{H}[\phi] = \int d\mathbf{r} \left(\frac{1}{2}(\nabla \phi)^2 + g' \phi^{2+2p} \right)$, where the field ϕ describes the space-time-dependent order parameter. Here $p = 1$ corresponds to

¹⁰ For non-vanishing viscosities, this 'special' transformation is no longer a symmetry. In the incompressible case $\nabla \cdot \mathbf{v} = 0$, the dynamical symmetry of the Euler equations consists of the (massless) Galilei group, combined with dilatations with an arbitrary dynamical exponent z for $\nu = 0$ or with $z = 2$ for $\nu > 0$ [754].

a usual critical point, $p = 2$ to a tri-critical point and so on. The equilibrium critical dynamics of such a system is often captured by a **time-dependent Ginzburg-Landau equation** of motion of the form $\partial_t \phi = -\frac{\delta \mathcal{H}}{\delta \phi} = \Delta_L \phi + g \phi^{1+2p}$ and $g = 2(p+1)g'$ is proportional to g' .

By definition, the system is at its **characteristic dimension** $d = d^{(c)}$, when the coupling constant g is dimensionless. A dimensional analysis of the Janssen-de Dominicis functional $\mathcal{J}[\tilde{\phi}, \phi] := \int dtd\mathbf{r} [\tilde{\phi} \partial_t \phi + \tilde{\phi} \frac{\delta \mathcal{H}}{\delta \phi}]$ shows that this is achieved here for $p = 2/d$. It is mathematical textbook knowledge that the time-dependent Ginzburg-Landau equation is Schrödinger-invariant if and only if $d = d^{(c)}$, see [90, 267].

Generalising this argument to the **Gross-Pitaevskii equation** (with a complex-valued order-parameter ϕ), applications of Schrödinger-invariance to the Bose-Einstein condensation have been discussed [275].

From a physical point of view, it might appear unnatural to have Schrödinger-invariance only for $d = d^{(c)}$. Indeed, this question can be addressed by considering generalised representations of $\mathfrak{sch}(d)$ which also include the transformation of g [686], as will be described in more detail below, see p. 256.¹¹

7. As an example of a Schrödinger-invariant *gauge theory*, consider the gauged non-linear 2D Schrödinger equation

$$i\partial_t \psi = \left(-\frac{1}{2m} (\nabla - ie\mathbf{A})^2 + eA^0 - \Lambda \psi^* \psi \right) \psi, \quad (4.21)$$

where $(A^0, \mathbf{A})$ is an electromagnetic vector potential in (1+2) dimensions and Λ is a constant. The electromagnetic density $\rho = \psi^* \psi$ and current $\mathbf{J} = \frac{1}{2im} [\psi^* (\nabla - ie\mathbf{A})\psi - \psi ((\nabla - ie\mathbf{A})\psi)^*]$ are defined as usual. However, the electric and magnetic fields do not satisfy Maxwell's equations but rather obey the field-current relations

$$B = \nabla \wedge \mathbf{A} = -\frac{e}{\kappa} \rho, \quad \mathbf{E} = -\nabla A^0 - \partial_t \mathbf{A} = \frac{e}{\kappa} {}^* \mathbf{J} \quad (4.22)$$

with the abbreviation ${}^* J^i = \epsilon^{ij} J_j$ where ϵ^{ij} is the totally antisymmetric tensor and $i, j = 1, 2$. For a given value of Λ , the static solutions can be interpreted as Chern-Simons vortices [397, 222, 321].

8. Many differential equations acquire dynamical symmetries if one rather looks for so-called *non-classical* or **conditional symmetries** which are valid *modulo* an auxiliary condition [84, 85, 266, 267, 467]. For example [145], one may show that the Schrödinger-Virasoro algebra $\mathfrak{sv}(2)$ with $\mathcal{M} = 0$ is a conditional dynamical symmetry of the non-linear equation

¹¹ Appendix K describes the construction of an improved energy-momentum tensor (equivalent to the Belinfante tensor [191]) for *conformally* invariant field-theories from similar symmetry arguments.

$$\det \begin{pmatrix} u_t & u_{x_1} & u_{x_2} \\ u_{tx_1} & u_{x_1x_1} & u_{x_1x_2} \\ u_{tx_2} & u_{x_2x_1} & u_{x_2x_2} \end{pmatrix} = \frac{\partial}{\partial x_1} \left(\frac{u_{x_1}}{u^2} \right) + \frac{\partial}{\partial x_2} \left(\frac{u_{x_2}}{u^2} \right), \quad (4.23)$$

where the abbreviations $u_t = \partial_t u$, $u_{x_j} = \partial u / \partial x_j$ etc. are used. The auxiliary condition is in this case the celebrated **Monge-Ampère equation** $u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2 = 0$ which arises in many applications in differential geometry.¹²

9. Recently, non-relativistic analogues of the AdS/CFT correspondence, which relates field-theories on a curved background metric to conformal field-theories, have been intensively discussed. We shall return to them below, in Sect. 4.8, after having introduced the mathematical tools of Schrödinger-invariance.
10. The relevance of the Schrödinger algebra (or rather some of its subalgebras) to *ageing* phenomena with a dynamical exponent $z = 2$ will be the main theme of this chapter.

4.2.4 Simple Consequences of Schrödinger-invariance

We now show how Schrödinger-covariant two- and three-point functions can be found [327]. The method is close in spirit to the one used for conformal invariance in Chap. 2 in Volume 1. From (4.7) we see that a quasi-primary scaling operator ϕ is characterised by the pair $(x, \mathcal{M})$. Consider a **two-point function** $F = F(t_1, t_2; \mathbf{r}_1, \mathbf{r}_2) = \langle \phi_1(t_1, \mathbf{r}_1) \phi_2(t_2, \mathbf{r}_2) \rangle$ of two Schrödinger-quasi-primary scaling operators $\phi_{1,2}$. If these are scalars under rotations, it is enough to consider the $1D$ case, since any two spatial points $\mathbf{r}_1, \mathbf{r}_2$ can be brought to lie on a fixed line. The covariance conditions (4.12) for the generators $\mathcal{X} \in \mathfrak{sch}(1)$ will give a set of linear differential equations for F , to be written down shortly.

Clearly, space- and time-translation-invariance imply that $F = F(t, \mathbf{r})$ where $t = t_1 - t_2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. In order to analyse the other conditions, we use the short-hand $\partial_i := \partial / \partial t_i$ and $D_i := \partial / \partial r_i$. Then the requirement of Galilei-invariance leads to

$$\begin{aligned} Y_{1/2} F &= [-t_1 D_1 - \mathcal{M}_1 r_1 - t_2 D_2 - \mathcal{M}_2 r_2] F \\ &= [(-t \partial_r - \mathcal{M}_1 r) - r_2 (\mathcal{M}_1 + \mathcal{M}_2)] F \stackrel{!}{=} 0 \end{aligned} \quad (4.24)$$

This is only consistent with spatial translation-invariance if all reference to the individual space coordinates disappears. In particular, any explicit reference to r_2 must disappear. We therefore have the two separate conditions

$$(-t \partial_r - \mathcal{M}_1 r) F(t, \mathbf{r}) = 0 \quad (4.25)$$

$$\mathcal{M}_1 + \mathcal{M}_2 = 0 \quad (4.26)$$

¹² Generalisations of this result to any spatial dimension $d \geq 2$ and to arbitrary dynamical exponents z exist [145].

where the first one fixes the scaling function and the second one relates the two ‘masses’. This is a first example of the well-known **Bargman superselection rules** which quite generally follow from the combination of Galilei- and spatial translation-invariance [37]. Similarly, dilatation-invariance leads to

$$\begin{aligned} X_0 F &= \left[-t_1 \partial_1 - \frac{1}{2} r_1 D_1 - t_2 \partial_2 - \frac{1}{2} r_2 D_2 - \frac{1}{2} (x_1 + x_2) \right] F \\ &= \left[-t \partial_t - \frac{1}{2} r \partial_r - \frac{1}{2} (x_1 + x_2) \right] F \stackrel{!}{=} 0 \end{aligned} \quad (4.27)$$

and finally co-variance under the special transformation gives

$$\begin{aligned} X_1 F &= \left[-t_1^2 \partial_1 - t_2^2 \partial_2 - t_1 r_1 D_1 - t_2 r_2 D_2 - \frac{\mathcal{M}_1}{2} r_1^2 - \frac{\mathcal{M}_2}{2} r_2^2 - x_1 t_1 - x_2 t_2 \right] F \\ &= \left[\left(-t^2 \partial_t - t r \partial_r - \frac{\mathcal{M}_1}{2} r^2 - x_1 t \right) - \frac{1}{2} r^2 (\mathcal{M}_1 + \mathcal{M}_2) \right. \\ &\quad \left. + 2t_2 \left(-t \partial_t - \frac{1}{2} r \partial_r - \frac{1}{2} (x_1 + x_2) \right) + r_2 (-t \partial_r - \mathcal{M}_1 r) \right] F \\ &= \left[-t^2 \partial_t - t r \partial_r - \frac{\mathcal{M}_1}{2} r^2 - x_1 t \right] F(t, r) \stackrel{!}{=} 0 \end{aligned} \quad (4.28)$$

where we used the decompositions

$$\begin{aligned} t_1^2 - t_2^2 &= (t_1 - t_2)^2 + 2t_2(t_1 - t_2) \\ t_1 r_1 - t_2 r_2 &= (t_1 - t_2)(r_1 - r_2) + t_2(r_1 - r_2) + r_2(t_1 - t_2) \end{aligned} \quad (4.29)$$

and the last line in (4.28) follows from the three conditions (4.25, 4.26, 4.27), derived from $Y_{1/2} F = X_0 F = 0$. Now, the function F of two variables is determined by the three linear equations (4.25, 4.27, 4.28). To simplify this further, multiply eq. (4.27) by $-t$ and add to eq. (4.28) and then multiply eq. (4.25) with $-r/2$ and also add. The result is the condition

$$tr(x_1 - x_2)F(t, r) = 0 \quad (4.30)$$

which implies that $x_1 = x_2$. Using this condition, the solution of the remaining system (4.25, 4.27) is elementary and gives [327], where f_0 is a normalisation constant,

$$\langle \phi_1(t_1, \mathbf{r}_1) \phi_2(t_2, \mathbf{r}_2) \rangle = \delta_{x_1, x_2} \delta_{\mathcal{M}_1 + \mathcal{M}_2, 0} f_0 (t_1 - t_2)^{-x_1} \exp \left[-\frac{\mathcal{M}_1}{2} \frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{t_1 - t_2} \right] \quad (4.31)$$

which is essentially the heat-kernel solution (Green’s function) of the diffusion equation. Compared with the analogous result (2.159) in Volume 1 of conformal invariance, we have the same condition $x_1 = x_2$ and a similar time-dependence, taking into account that $z = 2$ in the case at hand. Specific to Schrödinger-invariance is the spatio-temporal scaling function and the Bargman rule $\mathcal{M}_1 + \mathcal{M}_2 = 0$. There are two standard physical interpretations:

1. For a Schrödinger equation (with real time), $\mathcal{M} = im$ is imaginary and one rather considers two-point functions $\langle \psi \psi^* \rangle$ built from the wave function ψ and its complex conjugate ψ^* . Then naturally $\mathcal{M}^* = -\mathcal{M}$.
2. For a diffusion equation with $\mathcal{M}$ real, one must also consider the response field $\tilde{\phi}$ associated with the order-parameter field ϕ . Then the Bargman rule means that $\tilde{\mathcal{M}} = -\mathcal{M} \leq 0$ which we shall recover from the Langevin equation.

In a similar way, the Schrödinger-covariant **three-point function** can be found [327]

$$\begin{aligned} \langle \phi_1(t_1, \mathbf{r}_1) \phi_2(t_2, \mathbf{r}_2) \phi_3(t_3, \mathbf{r}_3) \rangle &= \delta_{\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3, 0} \exp \left[-\frac{\mathcal{M}_1}{2} \frac{\mathbf{r}_{13}^2}{t_{13}} - \frac{\mathcal{M}_2}{2} \frac{\mathbf{r}_{23}^2}{t_{23}} \right] \\ &\times t_{13}^{-x_{13,2}/2} t_{23}^{-x_{23,1}/2} t_{12}^{-x_{12,1}/2} \Psi_{12,3} \left(\frac{(\mathbf{r}_{13} t_{23} - \mathbf{r}_{23} t_{13})^2}{t_{12} t_{13} t_{23}} \right) \end{aligned} \quad (4.32)$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, $t_{ij} = t_i - t_j$, $x_{ij,k} := x_i + x_j - x_k$ and $\Psi_{12,3}$ is an arbitrary differentiable function. Again, the time-dependence of this is quite similar to the conformal three-point function (2.161) in Volume 1. We also find the Bargman superselection rule $\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 = 0$. In contrast to conformal invariance, already the three-point function is not completely determined.

It is also possible to study the consequences of co-variance under Schrödinger subalgebras, for example the subalgebra $\langle X_{-1}, X_0, X_1, Y_{\pm 1/2}^{(j)}, M_0, R_0^{(jk)} \rangle \subset \mathfrak{sch}(d)$ with $j, k = 1, \dots, d-1$. This algebra applies to the study of systems *without* spatial translation-invariance in one spatial direction, as may occur for example in studies of dynamical scaling with $z = 2$ close to a plane surface. Then the position vector $\mathbf{r} = (\mathbf{r}_{\parallel}, r_{\perp})$ is decomposed into the components parallel and perpendicular to the surface. The co-variant two-point function is [327]

$$\begin{aligned} \langle \phi_1(t_1, \mathbf{r}_1) \phi_2^*(t_2, \mathbf{r}_2) \rangle &= \delta_{x_1, x_2} \delta_{\mathcal{M}_1, \mathcal{M}_2^*} (t_1 - t_2)^{-x_1} \chi_{12} \left(\frac{r_{\perp,1} r_{\perp,2}}{t_1 - t_2} \right) \\ &\times \exp \left[-\frac{\mathcal{M}_1}{2} \frac{r_{\perp,1}^2}{t_1 - t_2} - \frac{\mathcal{M}_2^*}{2} \frac{r_{\perp,2}^2}{t_1 - t_2} \right] \exp \left[-\frac{\mathcal{M}_1}{2} \frac{(\mathbf{r}_{\parallel,1} - \mathbf{r}_{\parallel,2})^2}{t_1 - t_2} \right] \end{aligned} \quad (4.33)$$

where the scaling function $\chi_{12}(u)$ remains undetermined and we also assumed that the masses $\mathcal{M}_{1,2}$ are not direction-dependent. In $1D$, if invariance under the generator M_0 is not required, the constraint $\mathcal{M}_1 = \mathcal{M}_2^*$ can be dropped.

If one wishes to describe the cross-over from a dynamical scaling close to a surface to bulk dynamical scaling with $z = 2$, one expects that $\chi_{12}(u) \sim \exp(\mathcal{M}_1 u)$ for $u \rightarrow \infty$. Furthermore, if the order-parameter vanishes at the surface (as occurs for example for the ordinary transition), one expects $\chi_{12}(0) = 0$. These two conditions can be recovered exactly from the method of images, see exercise 4.8.

4.3 From Schrödinger-invariance to Ageing

As it stands, the Schrödinger algebra cannot be applied to ageing phenomena, since it contains the time-translations generated by X_{-1} . We now show how to adapt the ideas contained in Schrödinger-invariance to this case and shall find in the next 3 Sections three new ingredients which structurally go beyond what is needed in conformal invariance.

4.3.1 Ageing-invariance

For applications to ageing, we must consider the **ageing algebra**, defined as

$$\mathbf{age}(d) := \left\langle X_{0,1}, Y_{-\frac{1}{2}, \frac{1}{2}}^{(j)}, M_0, R_0^{(jk)} \right\rangle \subset \mathbf{sch}(d) \quad ; \quad j, k = 1, \dots, d \quad (4.34)$$

which is the subalgebra of $\mathbf{sch}(d)$ without time-translations. The generators $Y_m^{(j)}, M_n, R_n^{(jk)}$ retain their forms (4.14), known from $\mathbf{sch}(d)$, but the generators X_n now only exist for $n \geq 0$. They now read in $d = 1$ spatial dimensions [589, 336]

$$\begin{aligned} X_n = & -t^{n+1} \partial_t - \frac{n+1}{2} t^n r \partial_r - \frac{(n+1)n}{4} \mathcal{M} t^{n-1} r^2 \\ & - \frac{x}{2} (n+1) t^n - \xi n t^n \quad ; \quad n \geq 0 \end{aligned} \quad (4.35)$$

where ξ is a new scaling dimension, peculiar to the ageing algebra, and associated with the field ϕ on which the generators X_n act. When $\xi \neq 0$, the generator X_1 only belongs to a function-space representation of $\mathbf{age}(1)$, but not of $\mathbf{sch}(1)$. This is only possible for systems out of a stationary state. Otherwise, the requirement of time-translation-invariance and $[X_1, X_{-1}] = 2X_0$ would lead back to $\xi = 0$. It is readily checked that the commutators (4.8) remain unchanged. However, in the commutators (4.11) which express the dynamical symmetries of the free Schrödinger operator $\mathcal{S}$, the last relation is the only one to be modified and now reads

$$[\mathcal{S}, X_1] = -2t\mathcal{S} - (2x + 2\xi - 1)M_0. \quad (4.36)$$

This leads to the weaker condition $x + \xi = 1/2$ in order that the representation of $\mathbf{age}(1)$ can act as a dynamical symmetry of $\mathcal{S}\phi = 0$. Generalisations to $d \geq 1$ are obvious and will not be spelt out.

The meaning of this new scaling dimension ξ of an **age**-quasi-primary scaling operator is more easy to understand when we consider the extension of $\mathbf{age}(1)$ to the infinite-dimensional algebra $\mathbf{agev}(1) \subset \mathbf{sv}(1)$ spanned by $(X_n)_{n \in \mathbb{N}_0}$, $(Y_m)_{m \in \mathbb{Z}+1/2}$ and $(M_n)_{n \in \mathbb{Z}}$, such that eq. (4.8) remains valid [327, 336]. We call this algebra **ageing-Virasoro algebra**. It is well-known that Y_m and M_n generate time-dependent translations (with an additional

phase) and phase shifts, respectively. On the other hand, the generators $(X_n)_{n \geq 0}$ eq. (4.35) are the infinitesimal generators of the transformation $(t, \mathbf{r}) \mapsto (t', \mathbf{r}')$ where

$$t = \beta(t') \quad , \quad \mathbf{r} = \mathbf{r}' \sqrt{\frac{d\beta(t')}{dt'}} \quad (4.37)$$

such that $\beta(0) = 0$ (and of course $\dot{\beta}(t') \geq 0$) and the scaling operator ϕ transforms as [336]

$$\phi(t, \mathbf{r}) = \left(\frac{d\beta(t')}{dt'} \right)^{-x/2} \left(\frac{d \ln \beta(t')}{d \ln t'} \right)^{-\xi} \exp \left[-\frac{\mathcal{M} \mathbf{r}'^2}{4} \frac{d}{dt'} \ln \frac{d\beta(t')}{dt'} \right] \phi'(t', \mathbf{r}') \quad (4.38)$$

For $\xi \neq 0$, this transformation is different from that of a Schrödinger quasi-primary operator, derived in exercise 4.3. However, the scaling operator defined as

$$\Phi(t, \mathbf{r}) := t^{-\xi} \phi(t, \mathbf{r}) \quad (4.39)$$

is indeed primary, not only under the ageing-Virasoro algebra $\mathbf{agev}(d)$, but also under the Schrödinger-Virasoro algebra $\mathbf{sv}(d)$, but with a modified scaling dimension $x_\Phi = x_\phi + 2\xi_\phi$ [336].

Therefore, in order to describe the local scale-invariance of ageing systems with $z = 2$, it should be enough to *study first Schrödinger-quasi-primary operators Φ and only at the end, one moves over to ageing scaling operators (order-parameter) ϕ* using eq. (4.39). In this way, our previous results for the two- and three-point function derived from Schrödinger-invariance can be readily extended to ageing-invariance. The absence of time-translation-invariance has led us to identify a novel feature, namely the characterisation of a quasi-primary scaling operator by the two *independent* scaling dimensions x, ξ . This feature does not occur in conformal/Schrödinger-invariance.

4.3.2 Example: Application to Mean-field Theory

For further illustration of this aspect, which will become important later when comparing the predictions of ageing-invariance with exactly solved models or simulational data, we reconsider the mean-field approach described in Chap. 1. As we shall now see, already in mean-field theory there is evidence that the Schrödinger-quasi-primary scaling operator Φ cannot always be identical with the physical order-parameter ϕ . Recall the mean-field equation

$$\partial_t \phi(t, \mathbf{r}) = \Delta_L \phi(t, \mathbf{r}) + 3(\lambda^2 - v(t)) \phi(t, \mathbf{r}) \quad (4.40)$$

where $v(t) = \langle \phi(t, \mathbf{r})^2 \rangle$ is the variance, which was calculated in exercise 1.5 in a self-consistent way. The control parameter $\lambda^2 \sim T_c - T$ distinguishes between the ordered, critical and disordered equilibrium states. Letting $\phi(t, \mathbf{r}) =$

$\exp[-3 \int_0^t d\tau (\lambda^2 - v(\tau))]\Phi(t, \mathbf{r})$, it follows that $\partial_t \Phi = \Delta_L \Phi$. Therefore, the Schrödinger group should act co-variantly on the solution $\Phi(t, \mathbf{r})$, or in other words, $\Phi(t, \mathbf{r})$ should be a $\mathfrak{sch}$ -quasi-primary scaling operator. This already proves the ageing-invariance of our mean-field theory. We use the explicitly known solution for $v(t)$ to show that for a sufficiently large time t , the order-parameter $\phi(t, \mathbf{r})$ and the quasi-primary scaling operator $\Phi(t, \mathbf{r})$ are related as follows

$$\phi(t, \mathbf{r}) \sim \Phi(t, \mathbf{r}) \cdot \begin{cases} 1 & ; \text{ if } T < T_c \\ t^{-1/2} & ; \text{ if } T = T_c \end{cases} \quad (4.41)$$

At least in the context of our mean-field equation, one may indeed identify the order-parameter ϕ with a quasi-primary scaling operator Φ in a straightforward way in the ordered phase $T < T_c$. On the other hand, at criticality $T = T_c$, a power-law prefactor, exactly as required in (4.39), arises already in mean-field theory in the relationship between the order-parameter ϕ and the $\mathfrak{sch}$ -quasi-primary scaling operator Φ . Comparing with (4.39), we read off $\xi = 0$ for $T < T_c$ and $\xi = -\frac{1}{2}$ for $T = T_c$.¹³

4.4 Conformal Invariance and Ageing

Although it might appear surprising at first sight, there is a physically interesting relationship between the Schrödinger and ageing algebras and a non-standard representation of the conformal algebra in $d+2$ dimensions [109]. We shall construct this by *considering the ‘mass’ $\mathcal{M}$ not as some constant, but rather as an additional variable* [280]. We shall then recall the classification of the parabolic subalgebras of the conformal algebra before turning to physical applications, which in particular will allow us to relate the n -point functions already calculated to non-equilibrium *response* functions [357].

4.4.1 Conformal Invariance of the Free Diffusion Equation

We consider the Fourier transform with respect to $\mathcal{M}$ of the order-parameter and define a new field ψ as follows

$$\phi(t, \mathbf{r}) = \phi_{\mathcal{M}}(t, \mathbf{r}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\zeta e^{-i\mathcal{M}\zeta} \psi(\zeta, t, \mathbf{r}) \quad (4.42)$$

where ζ is the new ‘dual’ coordinate. Provided $\lim_{\zeta \rightarrow \pm\infty} \psi(\zeta, t, \mathbf{r}) = 0$, the diffusion equation becomes

$$2i \frac{\partial^2 \psi}{\partial \zeta \partial t} + \frac{\partial^2 \psi}{\partial \mathbf{r}^2} = 0 \quad (4.43)$$

¹³ This feature is not captured in the habitual field-theoretical approach to simple ageing of ferromagnets at $T = T_c$, which expands around the Gaussian theory [121] and hence implicitly assumes $\xi = 0$ from the beginning.

which for brevity we shall also call a diffusion/Schrödinger equation. The transformed generators of $\mathfrak{sv}(1)$ read (generalisations to $d \geq 1$ and/or to $\mathfrak{agev}(d)$ are trivial)

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \frac{n+1}{2} t^n r \partial_r - (n+1) \frac{x}{2} t^n + i \frac{n(n+1)}{4} t^{n-1} r^2 \partial_\zeta \\ Y_m &= -t^{m+1/2} \partial_r + i \left(m + \frac{1}{2} \right) t^{m-1/2} r \partial_\zeta \\ M_n &= i t^n \partial_\zeta \end{aligned} \quad (4.44)$$

This change of variables trades the complicated phases acquired by the field ϕ under a Schrödinger/ageing transformation for a time-dependent translation of the new internal coordinate ζ , as first observed for the Galilei sub-algebra [280]. The resulting global group transformations are computed in exercise 4.10.

The passage to the new variable ζ implies an extension of the dynamical symmetry algebra of the diffusion equation (4.43) [109, 357] which we illustrate for $d = 1$. Rewrite the physical coordinates ζ, t, r as the components of a three-dimensional vector $\boldsymbol{\xi} = (\xi_{-1}, \xi_0, \xi_1)$ where

$$t = \frac{1}{2}(-\xi_0 + i\xi_{-1}) \quad , \quad \zeta = \frac{1}{2}(\xi_0 + i\xi_{-1}) \quad , \quad r = \sqrt{\frac{i}{2}} \xi_1 \quad (4.45)$$

and $\psi(\zeta, t, r) = \Psi(\boldsymbol{\xi})$. Then the diffusion equation (4.43) becomes a three-dimensional massless Klein-Gordon/Laplace equation

$$\partial_\mu \partial^\mu \Psi(\boldsymbol{\xi}) = 0 \quad , \quad \mu = -1, 0, 1. \quad (4.46)$$

The Lie algebra of the maximal dynamical symmetry group of this equation is the conformal algebra $\mathfrak{conf}(3)$, with generators¹⁴

$$\begin{aligned} P_\mu &= \partial_\mu \\ M_{\mu\nu} &= \xi_\mu \partial_\nu - \xi_\nu \partial_\mu \\ K_\mu &= 2\xi_\mu \xi^\nu \partial_\nu - \xi_\nu \xi^\nu \partial_\mu + 2x \xi_\mu \\ D &= \xi^\nu \partial_\nu + x \end{aligned} \quad (4.47)$$

($\mu, \nu \in \{-1, 0, 1\}$) which represent, respectively, translations, rotations, special transformations and the dilatation. Hence the generators of $\mathfrak{sch}(1)$ are linear combinations (with complex coefficients) of the above $\mathfrak{conf}(3)$ generators. Generalising to any $d \geq 1$, *one has an inclusion of the complexified Lie algebra $(\mathfrak{sch}(d))_{\mathbb{C}}$ into $(\mathfrak{conf}(d+2))_{\mathbb{C}}$* [109], and called the *holographic principle* in string theory. Explicitly, for $d = 1$ the correspondence is

$$\begin{aligned} X_{-1} &= i(P_{-1} - iP_0) \quad , \quad X_0 = -\frac{1}{2}D + \frac{i}{2}M_{-10} \quad , \quad X_1 = -\frac{i}{4}(K_{-1} + iK_0) \\ Y_{-1/2} &= -\sqrt{\frac{2}{i}} P_1 \quad , \quad Y_{\frac{1}{2}} = -\sqrt{\frac{i}{2}} (M_{-11} + iM_{01}) \quad , \quad M_0 = P_{-1} + iP_0. \end{aligned} \quad (4.48)$$

¹⁴ See appendix L for non-standard notations, especially Table L1.

The four remaining generators needed to get the full conformal Lie algebra $(\mathfrak{conf}(3))_{\mathbb{C}}$ can be taken in the form

$$\begin{aligned} N &= -t\partial_t + \zeta\partial_\zeta && \text{time-phase symmetry} \\ V_- &= -\zeta\partial_r + ir\partial_t && \text{'dual' Galilei transformation} \\ W &= -\zeta^2\partial_\zeta - \zeta r\partial_r + \frac{i}{2}r^2\partial_t - x\zeta && \text{'dual' special transformation} \\ V_+ &= -2tr\partial_t - 2\zeta r\partial_\zeta - (r^2 + 2i\zeta t)\partial_r - 2xr && \text{transversal inversion.} \end{aligned} \quad (4.49)$$

The generators V_- and W are, up to constant coefficients, the complex conjugates of $Y_{1/2}$ and X_1 , respectively, in the coordinates ξ^μ , hence their names. The complex conjugation becomes the exchange $t \leftrightarrow \zeta$ in the physical coordinates (ζ, t, r) . Transforming the generators (4.49) back to the function space with fixed $\mathcal{M}$, we see that this representation of the conformal algebra is unusual in the sense that space-time rotations with $\mathcal{M}$ fixed or not occur and that the generators are not always of first order in $\partial_{\mathcal{M}}$.

The new generators (4.49) are indeed dynamical symmetries of the free diffusion/Schrödinger equation. To see this, consider the 1D Schrödinger operator

$$\mathcal{S} = 2M_0X_{-1} - Y_{-1/2}^2 = 2i\partial_\zeta\partial_t + \partial_r^2. \quad (4.50)$$

It is straightforward to check that

$$\begin{aligned} [\mathcal{S}, V_-] &= 0 = [\mathcal{S}, N] \\ [\mathcal{S}, V_+] &= 2(1 - 2x)\partial_t - 4r\mathcal{S} \\ [\mathcal{S}, W] &= i(1 - 2x)\partial_r - 2\zeta\mathcal{S} \end{aligned} \quad (4.51)$$

which complement eq. (4.11). We summarise our result.

Proposition: [357] *Any solution ψ of the 1D diffusion/Schrödinger equation $\mathcal{S}\psi = 0$ with the Schrödinger operator (4.50) with a scaling dimension $x = 1/2$ is mapped onto another solution of the same equation under the action of the conformal algebra $(\mathfrak{conf}(3))_{\mathbb{C}}$ represented by eqs. (4.44, 4.49).*

The generalisation to dimensions $d \geq 1$ is left to the reader.

4.4.2 Parabolic Subalgebras

The algebraic results of the previous subsection can be better understood by restating them in Fig. 4.2 in terms of a root diagram. As explained in appendix L, to each generator $\mathcal{X} \in \mathfrak{conf}(3)$ one may associate a two-dimensional vector $\mathfrak{x} \in \mathbb{Z}^2$ which links the origin to one of the points in the root diagram such that forming the commutator $[\mathcal{X}_1, \mathcal{X}_2] = c_{12}^3 \mathcal{X}_3$ corresponds to vector addition $\mathfrak{x}_1 + \mathfrak{x}_2 = \mathfrak{x}_3$ inside the diagram and $c_{12}^3 = 0$ if the result of that addition falls outside the root diagram. The correspondence between the generators of $\mathfrak{conf}(3) \supset \mathfrak{sch}(1)$ and the roots is indicated in Fig. 4.2. It is easily

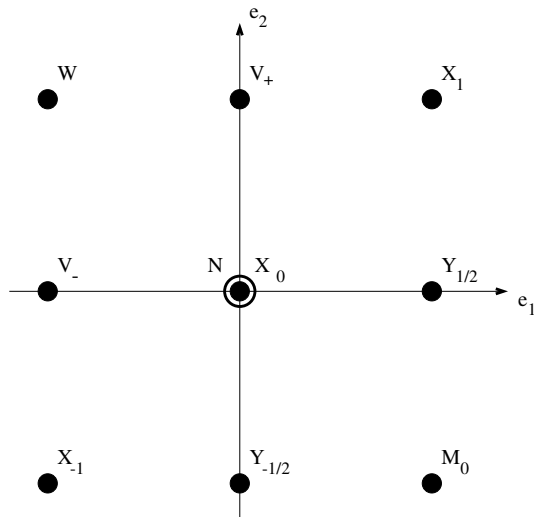


Fig. 4.2. Roots of the complex simple Lie algebra B_2 and their relation with the generators of the Schrödinger algebra $\mathfrak{sch}(1)$ and the conformal algebra $\mathfrak{conf}(3)$. The double circle in the centre denotes the Cartan subalgebra $\mathfrak{h}$. Reprinted from [357] with permission from Elsevier.

checked that this correspondence reproduces correctly the conformal commutation relations. Since the origin in the root diagram is doubly occupied, the Cartan subalgebra $\mathfrak{h}$ is two-dimensional, hence $\mathfrak{conf}(3)$ is of rank 2. The root diagrams of the simple Lie algebras of rank 2 are shown in Fig. L2. Comparison with the root diagram Fig. 4.2 establishes the isomorphism with the roots of the complex Lie algebra $B_2 \cong (\mathfrak{conf}(3))_{\mathbb{C}}$, in the terminology of the Cartan classification.

An advantage of using root diagrams is that one can quickly obtain an overview of the algebraic structure of a Lie algebra [446]. For example, any convex subset in a root diagram indicates a sub-algebra. Possible isomorphisms between subalgebras are described by the **Weyl group** which is defined as the finite group of symmetry transformations of the root diagram onto itself. For the Lie algebra B_2 , the Weyl group is generated by two simple point symmetries in Fig. 4.2, namely

$$w_1 : (e_1, e_2) \mapsto (-e_2, -e_1) \quad , \quad w_2 : (e_1, e_2) \mapsto (-e_1, e_2). \quad (4.52)$$

It is then not difficult to write down the list of non-isomorphic maximal subalgebras of B_2 . They are related to parabolic subalgebras of B_2 . By definition, a **parabolic subalgebra** is the sum of the Cartan subalgebra $\mathfrak{h}$ and of the positive root spaces [446]. To illustrate this definition graphically, consider placing a straight line on the root diagram which passes through the origin. Those roots which are on the line or on one side of it are said to be **posi-**

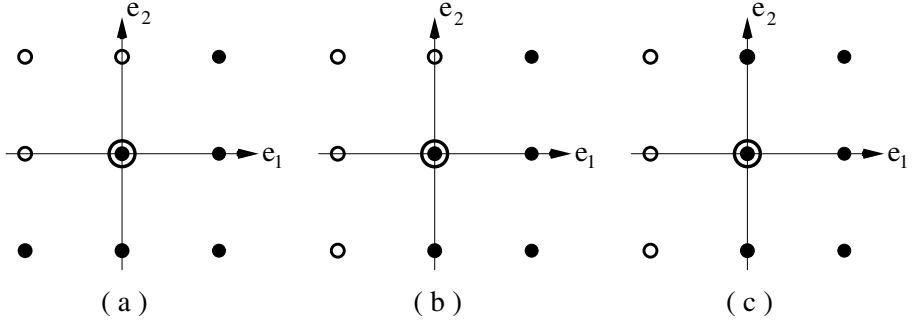


Fig. 4.3. Root diagrams of the three non-isomorphic parabolic subalgebras of the conformal algebra $\text{conf}(3)$ which are indicated by the full points, namely (a) $\widetilde{\text{sch}}(1)$, (b) $\widetilde{\text{age}}(1)$ and (c) $\widetilde{\text{alt}}(1)$. The double circle denotes the Cartan subalgebra.

five. The non-isomorphic maximal subalgebras of B_2 are the following three distinct parabolic subalgebras [357]

1. The extended ageing algebra $\widetilde{\text{age}}(1) := \text{age}(1) \oplus \mathbb{C}N$, obtained from $\text{age}(1)$ by adding the generator N . This algebra is the *minimal standard parabolic subalgebra*.
2. The extension $\widetilde{\text{sch}}(1) := \text{sch}(1) \oplus \mathbb{C}N$ of the Schrödinger algebra.
3. The extended algebra $\widetilde{\text{alt}}(1) := \text{alt}(1) \oplus \mathbb{C}N$ where $\text{alt}(1) := \langle D, Y_{-1/2}, M_0, Y_{1/2}, V_+, X_1 \rangle = \text{Lie}(\text{Alt}(d))$ is either called the **altern algebra**¹⁵ [328, 357] or better the **conformal Galilean algebra** $\text{CGA}(d) = \text{Lie}(\text{CGG}(d))$ [322, 533, 532]. More details on $\text{alt}(d)$, its extension to the infinite-dimensional altern-Virasoro algebra $\text{altv}(d)$, possible central extensions and several representations will be given in Chap. 5 and in [569, 483, 484, 354, 355, 705, 707, 503, 32, 754, 146].

In Fig. 4.3, we illustrate these subalgebras through their root diagrams. Acting with the elements of the Weyl group, it is intuitively clear that these three algebras are the only non-isomorphic parabolic subalgebras, see also exercise 4.9.

In many applications, only one of the generators of the Cartan subalgebra is retained so that one has the algebras $\text{age}(1)$, $\text{sch}(1)$ and $\text{alt}(1)$. These might be called *almost* parabolic subalgebras [686].

4.4.3 Non-relativistic Limits

Since the Schrödinger equation may be obtained by taking a non-relativistic limit of the Klein-Gordon equation, one may wonder whether a similar re-

¹⁵ The name ‘altern’ (= german for ‘ageing’) was chosen in the context of possible applications to ageing.

lationship can exist between the Schrödinger and conformal groups. Indeed, Barut [42] attempted long ago to obtain the Schrödinger group as a non-relativistic limit of the conformal group. He started from the massive Klein-Gordon equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} - \mathcal{M}^2 c^2 \right) \varphi_{\mathcal{M}}(t, \mathbf{r}) = 0 \quad (4.53)$$

where c is the speed of light. This equation is invariant under the conformal group in $(d+1)$ dimensions, *provided* the mass $\mathcal{M}$ is transformed as well [42]. After the substitution

$$\partial_t \mapsto \mathcal{M}c + \frac{1}{c} \partial_t \quad (4.54)$$

the non-relativistic limit $c \rightarrow \infty$ reduces (4.53) to the Schrödinger equation. However, the $c \rightarrow \infty$ limit of the conformal generators (4.47) does not yield dynamical symmetries of the Schrödinger equation which motivated Barut to argue that one might “... *fix $\mathcal{M}$, but change the transformation properties of t and $\mathbf{r}$ in such a way that we obtain symmetry operations for the Schrödinger operator ...*” [42]. Barut claimed that a group contraction $\mathfrak{conf}(d+1) \rightarrow \mathfrak{sch}(d)$ could be achieved in this way and this statement has been uncritically reproduced many times in the literature.¹⁶

We find this argument rather unconvincing, as it involves such ill-defined transfers which are never spelt out explicitly. It might be more reliable to try to follow Barut’s ideas as outlined above as closely as possible, but to keep strict control over each step. Starting again from (4.53), we define a new function $\chi(u, t, \mathbf{r})$ through [357]

$$\varphi_{\mathcal{M}}(t, \mathbf{r}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} du e^{-i\mathcal{M}u} \chi(u, t, \mathbf{r}) \quad (4.55)$$

which satisfies the equation of motion (if $\lim_{u \rightarrow \pm\infty} \chi(u, t, \mathbf{r}) = 0$)

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}} + c^2 \frac{\partial^2}{\partial u^2} \right) \chi(u, t, \mathbf{r}) = 0 \quad (4.56)$$

which is related to the $(d+2)$ -dimensional massless Klein-Gordon equation and the associated conformal generators (4.47) by defining $\Psi(\xi) := \chi(u, t, \mathbf{r})$ where $\xi_{-1} = u/c$, $\xi_0 = ct$ and $\xi_a = r_a$ with $a = 1, \dots, d$. Next, we define the wave function

$$\psi(\zeta, t, \mathbf{r}) := \chi(u, t, \mathbf{r}) \quad , \quad \zeta := u + ic^2 t \quad (4.57)$$

which is the exact analogue of Barut’s substitution (4.54). It follows from (4.56) that

¹⁶ The standard contraction of the Poincaré algebra in $d+1$ space-time dimensions to the Galilei-algebra in d space dimension is completely well-defined. We look here at the effect of the $c \rightarrow \infty$ limit on the special conformal transformations.

$$\left(2i\frac{\partial^2}{\partial\zeta\partial t} + \frac{\partial}{\partial\mathbf{r}} \cdot \frac{\partial}{\partial\mathbf{r}}\right)\psi(\zeta, t, \mathbf{r}) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\zeta, t, \mathbf{r}) = \mathcal{O}(c^{-2}) \quad (4.58)$$

which reduces to the Schrödinger equation (4.43) in the $c \rightarrow \infty$ limit. Next, we rewrite the generators (4.47) of $\mathfrak{conf}(d+2)$ and for brevity, we specialise to $d=1$. For the translations, we find

$$P_{-1}\Psi = -icM_0\psi, \quad P_0\Psi = c[M_0\psi + \mathcal{O}(c^{-2})], \quad P_1\Psi = -Y_{-1/2}\psi \quad (4.59)$$

For the rotations, we obtain

$$\begin{aligned} M_{01}\Psi &= -c[Y_{1/2}\psi + \mathcal{O}(c^{-2})], \quad M_{-11}\Psi = ic[Y_{1/2}\psi + \mathcal{O}(c^{-2})], \\ M_{-10}\Psi &= iN\psi + \mathcal{O}(c^{-2}). \end{aligned} \quad (4.60)$$

The dilatation becomes $D\Psi = (-2X_0 + N)\psi$ and for the special conformal transformations we find

$$\begin{aligned} K_{-1}\Psi &= 2ic[X_1\psi + \mathcal{O}(c^{-2})], \quad K_0\Psi = -2c[X_1\psi + \mathcal{O}(c^{-2})], \\ K_1\Psi &= -[V_+\psi + \mathcal{O}(c^{-2})] \end{aligned} \quad (4.61)$$

Therefore, *when taking the non-relativistic limit $c \rightarrow \infty$, we do not obtain a group contraction* (which would have kept the number of independent generators unchanged), *but rather we have a projection of the complexified algebras $\mathfrak{conf}(3) \rightarrow \widetilde{\mathfrak{alt}}(1) \not\cong \mathfrak{sch}(1)$ [357].*¹⁷ It follows that Schrödinger-invariance cannot be simply obtained as a non-relativistic limit of conformal invariance and it is rather the ‘other’ parabolic subalgebra $\mathfrak{alt}(1) \equiv \widetilde{\text{CGA}}(1)$ which is found.

These projections and the group contraction from the conformal algebra to the Schrödinger and conformal Galilean algebras have recently been rediscovered in the context of the study of non-relativistic versions of the AdS/CFT correspondence [678, 33, 521, 503, 30, 264], see Sect. 4.8.

4.4.4 Causality

We now use the concept of variable masses to derive a causality condition for the conformally covariant n -point functions. Physically, this implies that *Schrödinger- or conformal covariance determines response functions, rather than correlation functions.* This observation will be borne out fully when the noise terms are included in Sect. 4.5.

Causality is readily derived for the two-point function [357]. From Chap. 2 in Volume 1, we recall the conformally covariant two-point function [603]

¹⁷ On the other hand, $\mathfrak{alt}(1) \equiv \text{CGA}(1)$ and its infinite-dimensional extension $\mathfrak{alt}\mathfrak{v}(1)$ can be obtained from a group contraction, realised as a non-relativistic limit, of a pair of commuting Virasoro algebras [355], see exercise 5.5.

$$\begin{aligned}
\langle \psi_1(\zeta_1, t_1, \mathbf{r}_1) \psi_2(\zeta_2, t_2, \mathbf{r}_2) \rangle &= \langle \Psi_1(\boldsymbol{\xi}_1) \Psi_2(\boldsymbol{\xi}_2) \rangle = \Psi_0 \delta_{x_1, x_2} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^{-2x_1} \\
&= \psi_0 \delta_{x_1, x_2} (t_1 - t_2)^{-x_1} \left(\zeta_1 - \zeta_2 + \frac{i}{2} \frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{t_1 - t_2} \right)^{-x_1} \quad (4.62)
\end{aligned}$$

where $\psi_0 = 4^{-x_1} \Psi_0$ and Ψ_0 is a normalisation constant. In the second line, we reverted to the ‘physical’ coordinates $(\zeta, t, \mathbf{r}) = (\boldsymbol{\xi})$. The meaning of this result for the dynamical applications we have in mind becomes clear when rewriting it in terms of scaling operators $\phi_a(t, r)$ with fixed, non-negative masses $\mathcal{M}_a \geq 0$, $a = 1, 2$, using eq. (4.42). As shown in exercise 4.11, provided only that $x_1 > 0$, we have

$$\begin{aligned}
\langle \phi_1(t_1, \mathbf{r}_1) \phi_2^*(t_2, \mathbf{r}_2) \rangle &= \int_{\mathbb{R}^2} d\zeta_1 d\zeta_2 e^{-i\mathcal{M}_1 \zeta_1 + i\mathcal{M}_2 \zeta_2} \langle \psi_1(\zeta_1, t_1, \mathbf{r}_1) \psi_2(\zeta_2, t_2, \mathbf{r}_2) \rangle \\
&= \phi_0 \delta_{x_1, x_2} \delta_{\mathcal{M}_1, \mathcal{M}_2} \mathcal{M}_1^{1-x_1} \Theta(t_1 - t_2) (t_1 - t_2)^{-x_1} \exp \left(-\frac{\mathcal{M}_1}{2} \frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{t_1 - t_2} \right) \quad (4.63)
\end{aligned}$$

where ϕ_0 is again a normalisation constant (proportional to ψ_0) and Θ is the Heaviside function. Similarly, for the three-point function we have, provided $x_1 > 0$ and $x_2 > 0$ [357]

$$\begin{aligned}
\langle \phi_1(t_1, \mathbf{r}_1) \phi_2(t_2, \mathbf{r}_2) \phi_3^*(t_3, \mathbf{r}_3) \rangle &= \mathcal{C}_{12,3} \delta(\mathcal{M}_1 + \mathcal{M}_2 - \mathcal{M}_3) \\
&\times \Theta(t_1 - t_3) \Theta(t_2 - t_3) (t_1 - t_2)^{-x_{12,3}/2} (t_1 - t_3)^{-x_{13,2}/2} (t_2 - t_3)^{-x_{23,1}/2} \\
&\times \exp \left[-\frac{\mathcal{M}_1}{2} \frac{(\mathbf{r}_1 - \mathbf{r}_3)^2}{t_1 - t_3} - \frac{\mathcal{M}_2}{2} \frac{(\mathbf{r}_2 - \mathbf{r}_3)^2}{t_2 - t_3} \right] \quad (4.64) \\
&\times \Phi_{12,3} \left(\frac{1}{2} \frac{[(\mathbf{r}_1 - \mathbf{r}_3)(t_2 - t_3) - (\mathbf{r}_2 - \mathbf{r}_3)(t_1 - t_3)]^2}{(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)} \right)
\end{aligned}$$

where $\mathcal{C}_{12,3}$ is a constant related to the conformal OPE coefficient C_{123} and $\Phi_{12,3}(v)$ is a scaling function for which integral representations, and sometimes explicit forms, are known. Below, we shall use conformal invariance to restrict its form.

The causal prefactors $\Theta(t_1 - t_2)$ in (4.63) and $\Theta(t_1 - t_3)\Theta(t_2 - t_3)$ in (4.64) would have to be put in by hand if only $\mathfrak{sch}(d)$ -covariance had been required. They are consistent (i) with the requirement of a decay to zero for large spatial distances and (ii) with the *identification of the complex conjugate scaling operator ϕ^* with the response operator $\tilde{\phi}$* , conjugate to the order parameter scaling operator ϕ , in the context of non-equilibrium field-theory (see appendix D). We shall use this identification throughout.

4.4.5 Spinors and Supersymmetric Generalisations

Similarly to the scalar case, extended local scaling symmetries can be found for higher spin fields. In the same way as, in a relativistic context, the Dirac

equation is the spin- $\frac{1}{2}$ analogue of the Klein-Gordon equation for scalar fields, one can construct a non-relativistic spin- $\frac{1}{2}$ analogue of the Schrödinger equation for scalar fields. These are the **Dirac-Lévy-Leblond equations** [468]. The relationship between the scalar representations of the Schrödinger algebra $\mathfrak{sch}(d)$ and the conformal algebra $\mathfrak{conf}(d+2) \supset \mathfrak{sch}(d)$ generalises in a natural way to the spin- $\frac{1}{2}$ case and is again described by the root diagram Fig. 4.2 [358]. This is treated in the exercises 4.15 to 4.18, which lead up to explicit expressions for two-point (response) functions. Remarkably, if ϕ, ψ are the two components of a spin-1/2 doublet, the $\mathfrak{sch}(d)$ -covariant two-point functions $\langle \phi \psi \rangle$ do not vanish even if the scaling dimensions satisfy $x_\phi = x_\psi \pm 1$.

Furthermore, Schrödinger-covariant scalar and spinorial fields can be combined to a supermultiplet of *non-relativistic supersymmetry*.¹⁸ For example, the solutions of the free Schrödinger equation, together with the two components of the solutions of the Dirac-Lévy-Leblond equations, belong to the same multiplet of a larger algebraic structure, which upon taking scale- and special Schrödinger and conformal symmetries into account, can be extended to the so-called ortho-symplectic Lie superalgebra $\mathfrak{osp}(2|4)$ [358], such that the influence of a gradual breaking of the supersymmetries on the form of the co-variant two-point functions can be studied. Non-relativistic supersymmetries, with N supercharges have been constructed systematically as extensions of the Schrödinger algebra $\mathfrak{sch}(d)$ by Duval and Horváthy [220], whereas the application of the supersymmetries mentioned above has been explored in the context of supersymmetric quantum mechanics [60, 58, 59]. In principle, supersymmetries arise naturally in either disordered or dynamical field-theories such that a more systematic exploration of supersymmetric extensions of Schrödinger or related symmetries might lead to new insight, notably in contexts where non-relativistic strings are important.¹⁹

Analogously to the Virasoro algebra, the Schrödinger-Virasoro algebra may have central extensions which have been classified in [623], together with a study of possible deformations. Vertex operator representations of these algebras were constructed in [704, 706].

Physical applications of these new infinite-dimensional mathematical structures remain to be explored. A promising arena for this might be non-relativistic strings, but unfortunately this fascinating topic is essentially orthogonal to the content of this book.

¹⁸ Lie superalgebras contain so-called ‘even’ generators, which satisfy commutator relations, and ‘odd’ generators, whose anticommutators give even generators.

¹⁹ Supersymmetric extensions of the Schrödinger-Virasoro algebra with N supercharges, analogous to the superconformal Neveu-Schwarz symmetries, have been constructed and classified [358].

4.5 Galilei-invariance

Besides dynamical scaling, **Galilei-invariance** is the other main ingredient for local scaling with $z = 2$. In the first subsection, we shall use free field-theory as an example to illustrate some important properties of Galilei-invariant deterministic systems [357]. This prepares the treatment of space-time symmetries in the stochastic Langevin equation [589], to be taken up in the following subsection. In particular, how can we compute *correlations*, rather than responses, in ageing systems ?

4.5.1 Galilei-invariance in Deterministic Systems

For a scalar non-relativistic free field with a fixed mass $\mathcal{M} \geq 0$, the action is $\mathcal{J}_{(a)} = \int dt d\mathbf{r} \mathcal{L}_a$ with the Lagrangian density

$$\mathcal{L}_a = \mathcal{M} \left(\phi^\dagger \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^\dagger}{\partial t} \right) + \frac{\partial \phi^\dagger}{\partial \mathbf{r}} \cdot \frac{\partial \phi}{\partial \mathbf{r}} \quad (4.65)$$

The equation of motion for $\mathcal{J}_{(a)}$ gives back the free diffusion equation, whereas the conjugate field $\phi^\dagger$ satisfies the same equation with $\mathcal{M}$ replaced by $-\mathcal{M}$. The ‘mass’ $\mathcal{M}$ plays therefore the role of a quantum number and we associate with ϕ a ‘mass’ $\mathcal{M}$ and with $\phi^\dagger$ a mass $\mathcal{M}^\dagger := -\mathcal{M}$.

First, we generically consider the Schrödinger/ageing **Ward identities**. Denote by $\boldsymbol{\rho} = (t, \mathbf{r})$ the vector of space-time coordinates. Consider an arbitrary infinitesimal coordinate transformation $\boldsymbol{\rho} \mapsto \boldsymbol{\rho}' = \boldsymbol{\rho} + \boldsymbol{\epsilon}(\boldsymbol{\rho})$ such that a field operator $\phi = \phi(t, \mathbf{r})$ may also pick up a phase $\eta = \eta(t, \mathbf{r})$. Then the simplest possible way a local translation-invariant action $\mathcal{J}_{(a)}$ may transform to leading order in ϵ , according to $\mathcal{J}_{(a)} \mapsto \mathcal{J}'_{(a)} = \mathcal{J}_{(a)} + \delta \mathcal{J}_{(a)}$, is given by

$$\delta \mathcal{J}_{(a)} = \int dt d\mathbf{r} (T_{\mu\nu} \partial_\mu \epsilon_\nu + J_\mu \partial_\mu \eta) + \int_{(t=0)} d\mathbf{r} (U_\nu \epsilon_\nu + V \eta) \quad (4.66)$$

which defines the **energy-momentum tensor** $T_{\mu\nu}$ and the **current** vector J_μ , with $\mu, \nu = 0, 1, \dots, d$.²⁰ In addition, we allow for the breaking of time-translation invariance and then have to include the terms U_ν, V concentrated on the initial line $t = 0$ into the transformation law.

From spatial translation-invariance and phase-shift invariance, one has $U_a = 0$ for $a = 1, \dots, d$ and $V = 0$ since the bulk term is translation-invariant by construction. In the remaining transformations of **age**(d), U_0 will

²⁰ In conformally invariant systems, any phases can be absorbed into changes of the generators and hence no current J_μ arises in $\delta \mathcal{J}$. But for non semi-simple algebras such as the Galilei- or Schrödinger algebras, this is no longer possible and J_μ must be included. The present discussion neglects the possibility of *spatial* boundary terms. Appendix K discusses how to include them for conformal field-theories.

never occur. Furthermore, scale invariance implies the modified ‘trace’ identity [307, 396, 327]

$$2T_{00} + T_{11} + \dots + T_{dd} = 0 \quad (4.67)$$

For Galilei transformations, the phase $\eta = -\mathcal{M}\mathbf{r} \cdot \boldsymbol{\epsilon}$ and invariance of the action S implies

$$T_{0a} + \mathcal{M}J_a = 0 \quad ; \quad a = 1, \dots, d \quad (4.68)$$

Furthermore, using rotation invariance, one gets $T_{ab} = T_{ba}$ with $a, b = 1, \dots, d$. Now, quite analogously to conformal invariance, see Chap. 2 in Volume 1, the Ward identities (4.67, 4.68) imply full ageing-invariance of the action. Take $d = 1$ for simplicity, then $\epsilon_0 = \epsilon t^2$, $\epsilon_1 = \epsilon t r$ and $\eta = \frac{1}{2}\mathcal{M}r^2\epsilon$. Thus for a special Schrödinger transformation

$$\delta_{X_1}\mathcal{J}_{(a)} = \epsilon \int dt dr [(2T_{00} + T_{11})t + (T_{01} + \mathcal{M}J_1)r] = 0 \quad (4.69)$$

Schematically, we can summarise this as follows, in close analogy to the conformal-invariance result (2.166) in Volume 1 [124]

$$\left. \begin{array}{l} \text{spatial translation-invariance} \\ \text{phase-shift invariance} \\ \text{Galilei-invariance} \\ \text{scale-invariance with } z = 2 \\ \text{locality} \end{array} \right\} \implies \text{special Schrödinger-invariance} \quad (4.70)$$

where locality is understood in the sense of (4.66).²¹ We see that (i) time-translation invariance is not needed for special Schrödinger-invariance and (ii) *Galilei-invariance appears as an independent requirement*, unrelated to dynamical scaling with $z = 2$.²²

It is possible that a system shows dynamical scaling with $z = 2$, yet is not Galilei-invariant. If that occurs, local scale-invariance as developed here is not applicable.

Next, we list how the action $\mathcal{J}_{(a)}$ should transform under the generators X_n, Y_m of the Schrödinger-Virasoro algebra. Straightforward calculations lead to the transformation $\mathcal{J}_{(a)} \mapsto \mathcal{J}'_{(a)} = \mathcal{J}_{(a)} + \delta\mathcal{J}_{(a)}$, where

$$\begin{aligned} \delta_X \mathcal{J}_{(a)} &= \int dt' dr' \frac{1}{2} \mathcal{M}^2 r'^2 \{\beta(t'), t'\} \phi'^\dagger \phi' \\ \delta_Y \mathcal{J}_{(a)} &= \int dt' dr' \mathcal{M}^2 (\alpha(t') - 2r') \ddot{\alpha}(t') \phi'^\dagger \phi' \end{aligned} \quad (4.71)$$

respectively. Here,

²¹ This conclusion is only valid when boundary terms can be neglected, as discussed in [602, 621] and appendix K for conformal fields.

²² For theories with $\mathcal{M} > 0$, dynamical scaling is only compatible with Galilei-invariance if $z = 2$.

$$\{\beta(t), t\} = \frac{\ddot{\beta}(t)}{\dot{\beta}(t)} - \frac{3}{2} \left(\frac{\ddot{\beta}(t)}{\dot{\beta}(t)} \right)^2 \quad (4.72)$$

is the **Schwarzian derivative**. Consequently, $\delta\mathcal{J}_{(a)} = 0$ if $\alpha(t)$ is at most linear in t and $\beta(t)$ is a Möbius transformation. These transformations make up exactly the 1D Schrödinger group, as expected [544, 307]. On the other hand, by using directly (4.66), one expects

$$\delta_{X(\epsilon)}\mathcal{J}_{(a)} = \int dt dr \frac{1}{4} \mathcal{M} r^2 J_0 \ddot{\epsilon} \quad , \quad \delta_{Y(\epsilon)}\mathcal{J}_{(a)} = - \int dt dr \mathcal{M} r J_0 \ddot{\epsilon} \quad (4.73)$$

where

$$X(\epsilon) = -\epsilon(t)\partial_t - \frac{1}{2}\dot{\epsilon}(t)r\partial_r - \frac{\mathcal{M}}{4}\ddot{\epsilon}(t)r^2 - \frac{x}{2}\dot{\epsilon}(t) \quad , \quad Y(\epsilon) = -\epsilon(t)\partial_r - \mathcal{M}\dot{\epsilon}(t)r \quad (4.74)$$

and by comparison with the infinitesimal transformations of S_a which can be read from eq. (4.71) we identify

$$J_0 = 2\mathcal{M}\phi^\dagger\phi \quad . \quad (4.75)$$

For free fields, the canonical energy-momentum tensor $T_{\mu\nu}$ and the current J_μ can be explicitly found. Standard textbook recipes from classical field-theory, e.g. [759, 191], would give us

$$\tilde{T}_{\mu\nu} = -\delta_{\mu\nu}\mathcal{L}_a + \frac{\partial\mathcal{L}_a}{\partial(\partial^\mu\phi)}\partial_\nu\phi + \frac{\partial\mathcal{L}_a}{\partial(\partial^\mu\phi^\dagger)}\partial_\nu\phi^\dagger \quad , \quad J_\mu = \frac{\partial\mathcal{L}_a}{\partial(\partial^\mu\phi)}\phi - \frac{\partial\mathcal{L}_a}{\partial(\partial^\mu\phi^\dagger)}\phi^\dagger \quad (4.76)$$

Using the equations of motion, these may be shown to satisfy the conservation laws $\partial^\mu\tilde{T}_{\mu\nu} = \partial^\nu J_\nu = 0$ and all Schrödinger Ward identities with the only exception of the ‘trace’ condition (4.67). This can be remedied [124, 357] by constructing the improved tensor

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + \partial^\lambda B_{\lambda\mu\nu} \quad (4.77)$$

where B is antisymmetric in the two first variables, $B_{\lambda\mu\nu} = -B_{\mu\lambda\nu}$. If we take $B_{a00} = \frac{d}{4}(\phi^\dagger\partial_a\phi + \phi\partial_a\phi^\dagger)$ with $a = 1, \dots, d$ and $B_{\lambda\mu\nu} = 0$ unless $(\lambda\mu\nu) = (a00)$ (up to symmetries), then we get a classically conserved energy-momentum tensor, satisfying all required Ward identities, which reads [357]

$$\begin{aligned} T_{00} &= \frac{d\mathcal{M}}{2} (\phi^\dagger\partial_t\phi - \phi\partial_t\phi^\dagger) + \left(\frac{d}{2} - 1\right) \partial_{\mathbf{r}}\phi \cdot \partial_{\mathbf{r}}\phi^\dagger \\ T_{a0} &= \left(1 - \frac{d}{4}\right) (\partial_a\phi^\dagger\partial_t\phi + \partial_a\phi\partial_t\phi^\dagger) - \frac{d}{4} (\phi^\dagger\partial_{a,t}\phi + \phi\partial_{a,t}\phi^\dagger) \\ T_{0a} &= \mathcal{M} (\phi^\dagger\partial_a\phi - \phi\partial_a\phi^\dagger) \\ T_{aa} &= 2\partial_a\phi\partial_a\phi^\dagger - \partial_{\mathbf{r}}\phi \cdot \partial_{\mathbf{r}}\phi^\dagger - \mathcal{M} (\phi^\dagger\partial_t\phi - \phi\partial_t\phi^\dagger) \\ T_{ab} &= \partial_a\phi\partial_b\phi^\dagger + \partial_b\phi\partial_a\phi^\dagger. \end{aligned} \quad (4.78)$$

The current J_μ needs not be improved and we have

$$J_0 = 2\mathcal{M}\phi^\dagger\phi \ , \ J_a = \phi\partial_a\phi^\dagger - \phi^\dagger\partial_a\phi. \quad (4.79)$$

In particular, we recover (4.75). For a physical interpretation, it is better to divide $\mathcal{L}_a$ by $2\mathcal{M}$. We then recover the usual interpretation of J_0 as a probability density, J_a as a probability current, T_{00} as an energy density, T_{0a} as a momentum density and T_{a0} and T_{ab} as energy and momentum currents, respectively.

An analogous discussion can be carried out for variable masses which allows us also to check the full conformal invariance of the action. For free fields $\mathcal{J}_{(b)} = \int d\zeta dt d\mathbf{r} \mathcal{L}_b$ with the Lagrangian density

$$\mathcal{L}_b = 2i \frac{\partial\psi}{\partial\zeta} \frac{\partial\psi}{\partial t} + \left(\frac{\partial\psi}{\partial\mathbf{r}} \right)^2. \quad (4.80)$$

It is useful to construct a vector $\boldsymbol{\xi}$ with components

$$\xi_{-1} = \zeta \ , \ \xi_0 = t \ , \ \xi_1 = r_1 \ , \ \dots \ , \ \xi_d = r_d \quad (4.81)$$

and write the derivatives $\partial^\mu\psi = \partial\psi/\partial\xi_\mu$. Under an infinitesimal transformation $\boldsymbol{\xi} \mapsto \boldsymbol{\xi}' = \boldsymbol{\xi} + \boldsymbol{\epsilon}(\boldsymbol{\xi})$, the action is assumed to transform to leading order as (we leave it to the reader to discuss the case of the maximal subalgebra $\widetilde{\mathfrak{alt}}_1$ without time-translations)

$$\delta\mathcal{J}_{(b)} = \int d\zeta dt d\mathbf{r} T_\mu^\nu \partial^\mu \epsilon_\nu. \quad (4.82)$$

Again, $\mathcal{J}_{(b)}$ is translation-invariant by construction and dilatation-invariance implies $2T_0^0 + T_1^1 = 0$. Invariance of $\mathcal{J}_{(b)}$ under the three generators N , $Y_{1/2}$ and V_- coming from the 3D conformal rotations, see Fig. 4.2, leads to the following Ward identities, respectively

$$T_{-1}^{-1} - T_0^0 = 0 \ , \ T_0^1 - iT_1^{-1} = 0 \ , \ T_{-1}^1 - iT_1^0 = 0 \quad (4.83)$$

and it follows that T has 5 independent components. It is left as exercise 4.12 to check that indeed

$$\delta_{X_1}\mathcal{J}_{(b)} = \delta_W\mathcal{J}_{(b)} = \delta_{V_+}\mathcal{J}_{(b)} = 0. \quad (4.84)$$

This merely translates the well-known result (2.166) in Volume 1 of conformal invariance to the formulation at hand. The construction of the (improved) energy-momentum tensor is left as exercise 4.13, see also [357] where the transformations of the action $\mathcal{J}_{(b)}$ is also derived.

From the discussion in this section, it has become clear that for sufficiently local theories, dynamical scaling together with Galilei-invariance should be sufficient to imply the complete structure of ageing- or Schrödinger-invariance, under rather weak conditions.

4.5.2 Galilei-invariance in Langevin Equations

So far, we have considered the dynamical symmetries of simple diffusion equations but we should really have been interested in the possible symmetries of stochastic Langevin equations. For a non-conserved order-parameter (model A) one usually writes [370]

$$2\mathcal{M}\frac{\partial\phi}{\partial t} = \Delta_L\phi - \frac{\delta\mathcal{V}[\phi]}{\delta\phi} + \eta \quad (4.85)$$

where $\mathcal{V}$ is the Ginzburg-Landau potential and η is a centred Gaussian noise which describes the coupling to an external heat-bath and the initial distribution of ϕ . At first sight, there do not appear to be *any* non-trivial symmetries (besides the obvious translation and rotation invariance and possibly dynamical scaling), since the noise η does break Galilei-invariance. This may be seen from an easy calculation, but in order to understand this physically, consider a magnet which is at rest with respect to a homogeneous heat-bath at temperature T . If the magnet is moved with a constant velocity with respect to the heat-bath, the effective temperature will now appear to be direction-dependent, and the heat-bath is no longer homogeneous.²³

Still, there are hidden non-trivial symmetries in eq. (4.85) which come about as follows [589]: *split the Langevin equation into a ‘deterministic’ part with non-trivial dynamic symmetries and a ‘noise’ part and then show using these symmetries that all averages can be reduced exactly to averages within the deterministic, noiseless theory.* Technically, one first rewrites the equation (4.85) as the equation of motion of a field-theory, following the standard Janssen-de Dominicis procedure, recalled in appendix D. The resulting **action** or **Janssen-de Dominicis functional** $\mathcal{J}[\phi, \tilde{\phi}]$ where $\tilde{\phi}$ is the response field associated to the order-parameter ϕ , is decomposed into two parts

$$\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_0[\phi, \tilde{\phi}] + \mathcal{J}_b[\tilde{\phi}]. \quad (4.86)$$

Here

$$\mathcal{J}_0[\phi, \tilde{\phi}] = \int_{\mathbb{R}_+ \times \mathbb{R}^d} dt d\mathbf{r} \tilde{\phi} \left(2\mathcal{M}\partial_t\phi - \Delta_L\phi + \frac{\delta\mathcal{V}}{\delta\phi} \right) \quad (4.87)$$

contains the terms coming from the ‘deterministic’ part of the Langevin equation ($\mathcal{V}$ is the self-interacting ‘potential’) whereas

$$\mathcal{J}_b[\tilde{\phi}] = -T \int_{\mathbb{R}_+ \times \mathbb{R}^d} dt d\mathbf{r} \tilde{\phi}(t, \mathbf{r})^2 - \frac{1}{2} \int_{\mathbb{R}^{2d}} d\mathbf{r} d\mathbf{r}' \tilde{\phi}(0, \mathbf{r}) a(\mathbf{r} - \mathbf{r}') \tilde{\phi}(0, \mathbf{r}') \quad (4.88)$$

²³ Consider a well-known example from cosmology: the cosmic 3K microwave background, which should be totally homogeneous and uniform. Since the Earth is actually moving with respect to the frame defined by the cosmic background, the apparent temperature of the latter is observed to be direction-dependent. This ‘dipole-term’ must be subtracted before any more profound interpretation of the background signal might be possible.

contains the ‘noise’ (**bruit**) terms coming from (4.85). It was assumed here that $\langle \phi(0, \mathbf{r}) \rangle = 0$ and $a(\mathbf{r})$ denotes the initial two-point correlator

$$a(\mathbf{r}) := C(0, 0; \mathbf{r} + \mathbf{r}', \mathbf{r}') = \langle \phi(0, \mathbf{r} + \mathbf{r}') \phi(0, \mathbf{r}') \rangle = a(-\mathbf{r}) \quad (4.89)$$

while the last relation follows from spatial translation-invariance which we shall admit throughout.

It is instructive to consider briefly the case of a free field, where $\mathcal{V} = 0$. Variation of (4.86) with respect to $\tilde{\phi}$ and ϕ , respectively, then leads to the equations of motion

$$2\mathcal{M}\partial_t\phi = \Delta_L\phi + T\tilde{\phi} \quad , \quad -2\mathcal{M}\partial_t\tilde{\phi} = \Delta_L\tilde{\phi} \quad . \quad (4.90)$$

The first one of those might be viewed as a Langevin equation if $\tilde{\phi}$ is interpreted as a noise (and which is manifestly not Galilei-invariant if $T \neq 0$). Comparison of the two equations of motion (4.90) shows that if the order-parameter ϕ is characterised by the ‘mass’ $\mathcal{M}$ (which by physical convention is positive), then the associated response field $\tilde{\phi}$ is characterised by the *negative* mass $-\mathcal{M}$. This characterisation remains valid beyond free fields.

We now concentrate on actions $\mathcal{J}_0[\phi, \tilde{\phi}]$ which are Galilei-invariant. The Galilei algebra for $d = 1$ is $\mathfrak{gal}(1) = \langle X_{-1}, Y_{-1/2}, Y_{1/2}, M_0 \rangle$. Galilei-invariance of the action $\mathcal{J}_0$ implies that any average $\langle \mathcal{A}[\phi, \tilde{\phi}] \rangle_0$ taken with $\mathcal{J}_0$ alone, viz.

$$\langle \mathcal{A}[\phi, \tilde{\phi}] \rangle_0 := \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \mathcal{A}[\phi, \tilde{\phi}] e^{-\mathcal{J}_0[\phi, \tilde{\phi}]} \quad (4.91)$$

should also be Galilei-invariant, or simply $\mathcal{X} \langle \mathcal{A}[\phi, \tilde{\phi}] \rangle_0 \stackrel{!}{=} 0$ for any $\mathcal{X} \in \mathfrak{gal}(1)$. Consider the n -point function $F_n := \langle \Phi_1(t_1, r_1) \cdots \Phi_n(t_n, r_n) \rangle_0$, where the Φ_i have a mass $\mathcal{M}_i$ and can be built from order-parameter fields ϕ_i or response fields $\tilde{\phi}_i$. Because of time- and space-translation invariance, generated by X_{-1} and $Y_{-1/2}$, one has

$$F_n = F_n(t_1 - t_n, t_2 - t_n, \dots, t_{n-1} - t_n; r_1 - r_n, r_2 - r_n, \dots, r_{n-1} - r_n).$$

For brevity, we write $u_i := r_i - r_n$ and $D_i := \partial/\partial u_i$. Then Galilei-invariance $Y_{1/2}F = 0$ implies

$$\begin{aligned} & [(t_1 - t_n)D_1 + \dots (t_{n-1} - t_n)D_n + \mathcal{M}_1 r_1 + \dots + \mathcal{M}_n r_n] F_n = \\ & [(t_1 - t_n)D_1 + \dots (t_{n-1} - t_n)D_n + \mathcal{M}_1 u_1 + \dots + \mathcal{M}_{n-1} u_{n-1} \\ & \quad + (\mathcal{M}_1 + \dots + \mathcal{M}_{n-1} + \mathcal{M}_n) r_n] F_n \stackrel{!}{=} 0 \end{aligned}$$

The last term in the above equation is incompatible with spatial translation-invariance. Hence it must vanish, which implies

$$\mathcal{M}_1 + \mathcal{M}_2 + \dots + \mathcal{M}_{n-1} + \mathcal{M}_n = 0. \quad (4.92)$$

This mass conservation is a consequence of the fact that the Galilei algebra acts projectively and is a special case of the celebrated **Bargman superselection rules** [37, 468]. In the context of the field-theoretic action $\mathcal{J}_0$, see eq. (4.87), non-vanishing averages must contain both ϕ and $\tilde{\phi}$. Using the masses of ϕ and $\tilde{\phi}$ derived above, the result (4.92) may be restated as

$$\left\langle \underbrace{\phi \cdots \phi}_n \underbrace{\tilde{\phi} \cdots \tilde{\phi}}_m \right\rangle_0 = \delta_{n,m} \mathcal{F}_n[\phi] \quad (4.93)$$

and the functional $\mathcal{F}_n[\phi]$ remains to be calculated. The extension to any $d > 1$ is immediate. Of course, any responses or correlations of interest in ageing studies are exactly of the form considered here, but see exercise 4.14.

Theorem. [589] *If a physical system coupled to a heat bath and a distribution of initial conditions and its action permits a decomposition $\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_0[\phi, \tilde{\phi}] + \mathcal{J}_b[\tilde{\phi}]$ such that $\mathcal{J}_0$ is Galilei-invariant and $\mathcal{J}_b$ does not contain the order-parameter ϕ , then all response and correlation functions can be exactly reduced to correlation functions which are calculated by an average which only involves the deterministic part of the action.*

This result answers the question raised at the beginning of this Section. Rather than looking for symmetries of the whole Langevin equation, it is enough to identify a ‘deterministic’ part $\mathcal{J}_0$ of the action which satisfies Galilei-invariance. We shall then proceed to study the additional symmetries of the ‘deterministic’ part of the system. The proof of the Galilei-invariance of $\mathcal{J}_0$ is considerably more difficult and we shall return to this below.

Proof. We illustrate the proof of the above statement through two typical examples. First, we consider the autoresponse function (we suppress for notational simplicity the spatial coordinates)

$$\begin{aligned} R(t, s) &= \left. \frac{\delta \langle \phi(t) \rangle}{\delta h(s)} \right|_{h=0} = \left\langle \phi(t) \tilde{\phi}(s) \right\rangle \\ &= \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \phi(t) \tilde{\phi}(s) \exp \left(-\mathcal{J}_0[\phi, \tilde{\phi}] - \mathcal{J}_b[\tilde{\phi}] \right) \\ &= \left\langle \phi(t) \tilde{\phi}(s) e^{-\mathcal{J}_b[\tilde{\phi}]} \right\rangle_0 \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left\langle \phi(t) \tilde{\phi}(s) \left(\mathcal{J}_b[\tilde{\phi}] \right)^k \right\rangle_0 \\ &= \left\langle \phi(t) \tilde{\phi}(s) \right\rangle_0 = R_0(t, s) \end{aligned} \quad (4.94)$$

where in the third line the ‘noise’ part of the action was included in the observable to be averaged while in the fifth line the Bargman superselection

rule (4.93) was used. In other words, the two-time response function does not depend explicitly on the ‘noise’ at all.²⁴

As a second example, we consider the two-time correlation function

$$\begin{aligned}
 C(t, s; \mathbf{r}) &= \langle \phi(t, \mathbf{r} + \mathbf{y}) \phi(s, \mathbf{y}) \rangle \\
 &= \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \phi(t) \tilde{\phi}(s) \exp \left(-\mathcal{J}_0[\phi, \tilde{\phi}] - \mathcal{J}_b[\tilde{\phi}] \right) \\
 &= T \int_{\mathbb{R}_+ \times \mathbb{R}^d} du d\mathbf{R} \left\langle \phi(t, \mathbf{r} + \mathbf{y}) \phi(s, \mathbf{y}) \tilde{\phi}(u, \mathbf{R})^2 \right\rangle_0 \\
 &\quad + \frac{1}{2} \int_{\mathbb{R}^{2d}} d\mathbf{R} d\mathbf{R}' a(\mathbf{R} - \mathbf{R}') \left\langle \phi(t, \mathbf{r} + \mathbf{y}) \phi(s, \mathbf{y}) \tilde{\phi}(0, \mathbf{R}) \tilde{\phi}(0, \mathbf{R}') \right\rangle_0
 \end{aligned} \tag{4.95}$$

where again the Bargman superselection rule (4.93) was applied to the second line and we used the specific form (4.88). The two-time correlation function is hence reduced to ‘noiseless’ three- and four-point response functions. Remarkably, only terms which depend explicitly on the ‘noise’ remain, as the ‘noiseless’ two-point function $\langle \phi(t) \phi(s) \rangle_0 = 0$ because of the Bargman superselection rule.

It is now clear how the reduction proceeds for any n -point correlation or response function. q.e.d.

The explicit formulæ (4.94, 4.95) will become very important below, in practical tests of Schrödinger- or ageing-invariance.

4.5.3 Extensions

Finally, we discuss two extensions of the present formalism which will be needed for the planned applications to phase-ordering.

1. Consider the following extension of (4.85)

$$2\mathcal{M} \frac{\partial \phi}{\partial t} = \Delta_L \phi - \frac{\delta \mathcal{V}[\phi]}{\delta \phi} - v(t) \phi + \eta. \tag{4.96}$$

For example, the time-dependent potential $v(t)$ arises naturally in spherical models and we have also met it in mean-field theory in Chap. 1. In more general models, it might be viewed as a Lagrange multiplier to enforce that $C(t, t)$ is finite. If $\mathcal{V} = 0$, one may through the gauge transformation

$$\phi(t, \mathbf{r}) \mapsto \phi(t, \mathbf{r}) k(t) \quad ; \quad k(t) := \exp \left(-\frac{1}{2\mathcal{M}} \int_0^t dt' v(t') \right) \tag{4.97}$$

²⁴ This does not imply that $R(t, s)$ should be temperature-independent, since T may enter into $\mathcal{J}$, besides the noise correlator, also through control parameters in the deterministic part $\mathcal{J}_0$.

reduce eq. (4.96) to the more simple eq. (4.85) [589]. If for sufficiently long times $k(t)$ has an asymptotic algebraic behaviour, we are back to the relationship between the order-parameter and the Schrödinger-quasi-primary scaling operator discussed in Sect. 4.3 for the ageing algebra $\mathfrak{age}(d)$.

2. Almost all deterministic equations of the form (4.85) one encounters are semi-linear, with a potential $\mathcal{V} = \mathcal{V}[\phi, \tilde{\phi}]$. There is a well-established procedure to find the symmetries of such equations, see e.g. [90, 267, 563] which is recalled in exercises 4.19-4.21. However, the results are too restrictive to be useful in the context we have in mind.

The last ingredient we need comes from recognising that the coupling constant g of a non-linear term will in general have some scaling dimension y_g . This implies in turn that *under local scale-transformations g must be transformed* as well and consequently, the infinitesimal generators of local scale-invariance have to be generalised accordingly.²⁵ This is in practise a tedious construction [52, 686]. Then the standard methods quoted above [90, 267, 563] can be used to compute the Schrödinger-invariant semi-linear equations, in the generalised sense we have explained. In view of the length of the calculations, we merely quote the results:

1. For fixed masses, the invariant semi-linear Schrödinger equations are of the form

$$\mathcal{S}\phi = \phi \left(\phi \tilde{\phi} \right)^{1/x} f \left(g^x \left(\phi \tilde{\phi} \right)^{y_g} \right) \quad (4.98)$$

where x is the scaling dimension of ϕ and f is an arbitrary differentiable function [52].

2. If one works with the dual variables ζ for variable masses, the $1D$ invariant semi-linear Schrödinger equations with real-valued solutions are of the form

$$\mathcal{S}\psi = \psi^5 \bar{f} \left(g \psi^{4y_g} \right) \quad (4.99)$$

where $\bar{f}$ is an arbitrary differentiable function [686, 685]. We point out that in this way one can obtain Schrödinger-invariant diffusion equations with *real-valued* solutions, at the expense that after inverse Laplace transformation with respect to ζ products $\psi \cdot \psi \dots$ are replaced by convolutions $\phi * \phi * \dots$ with respect to $\mathcal{M}$.

Slightly more general forms are possible if only ageing-invariance is required and can be classified [52, 686]. We shall appeal to this construction to justify the Schrödinger-invariance in the deterministic part of the action $\mathcal{J}$ in the practical calculations of responses and correlators which now follow.

²⁵ In appendix K, we show that for conformally invariant theories this procedure gives back the Belinfante tensor as an improvement over the canonical energy-momentum tensor, which has the required symmetry properties [219].

4.6 Calculation of Two-time Response and Correlation Functions

We now turn to the explicit LSI-calculation of the two-time responses and autocorrelators in ageing systems with $z = 2$, using all of the ingredients prepared in the last three Sections. The starting points are the exact reduction formulæ eqs. (4.94,4.95). As the initial condition, we use a fully disordered initial state, with $a(\mathbf{r}) = a_0\delta(\mathbf{r})$. Then the calculation of $R(t, s)$ will be immediate, while that of $C(t, s)$ reduces to find a three-point response function $\langle \phi(t)\phi(s)\tilde{\phi}^2(u) \rangle_0$ within the deterministic part of the theory, and local scale-invariance will, again, become useful. The results will be summarised at the beginning of Sect. 4.7.

4.6.1 Ageing-invariant Response

At first sight, since there is no time-translation-invariance, it might appear that the only dynamical symmetry available would be $\mathbf{age}(d)$, with rather limited predictive capacities. Because of the relation between $\mathbf{age}(d)$ and $\mathbf{sch}(d)$ described in Sect. 4.3, it is better to proceed as follows, which defines what we understand by **local scale-invariance** for $z = 2$ [342, 336, 335]:

1. We break time-translation-invariance *explicitly* by considering a Langevin equation of the form (4.96) with a time-dependent potential $v(t)$. We can reduce this to the standard form (4.85) via the gauge transformation (4.97)

$$k(t) \sim t^F \quad \text{or equivalently} \quad v(t) = -\frac{2\mathcal{M}F}{t}. \quad (4.100)$$

2. Recall that for ageing-invariance the scaling operators ϕ (with scaling dimension x) do not transform as conventional quasi-primary scaling operators, but are related via (4.39) to *bona fide* quasi-primary scaling operators $\tilde{\Phi}$, with scaling dimension $x + 2\xi$.
3. We then use full Schrödinger-invariance for the calculation of the required three-point function $\langle \Phi(t)\Phi(s)\tilde{\Phi}^2(u) \rangle_0$.
4. Since this procedure still leaves a scaling function of a single variable undetermined, we use the further extension from Schrödinger to $(d + 2)$ -dimensional conformal invariance. This requires us to consider the masses as further variables, however.

We first consider the autoresponse function $R(t, s) = \langle \phi(t)\tilde{\phi}(s) \rangle_0$. According to the procedure outlined above, ϕ and $\tilde{\phi}$ are characterised by the two exponents x, ξ and $\tilde{x}, \tilde{\xi}$, respectively, in addition to the parameter F coming from the gauge transformation (4.97,4.100). This leads to [336]

$$\begin{aligned}
R(t, s) &= \left\langle t^\xi \tilde{\Phi}(t, \mathbf{r}) s^{\tilde{\xi}} \tilde{\Phi}(s, \mathbf{r}) \right\rangle_0 \\
&= t^\xi s^{\tilde{\xi}} \frac{k(t)}{k(s)} (t-s)^{-(x+2\xi)} \Theta(t-s) \delta_{x+2\xi, \tilde{x}+2\tilde{\xi}} \\
&= s^{-(x+\tilde{x})/2} \left(\frac{t}{s}\right)^{\xi+F} \left(\frac{t}{s} - 1\right)^{-x-2\xi} \Theta(t-s) \delta_{x+2\xi, \tilde{x}+2\tilde{\xi}} \quad (4.101)
\end{aligned}$$

which reproduces the expected scaling form $R(t, s) = s^{-1-a} f_R(t/s)$ (1.75), including its causality. We identify the ageing exponents as follows (we use $z = 2$ here)

$$\lambda_R = 2(x + \xi - F) \quad , \quad a' - a = \xi + \tilde{\xi} \quad , \quad 1 + a = \frac{1}{2}(x + \tilde{x}) \quad (4.102)$$

Here, we introduce a further **ageing exponent** a' which arises through the form of the scaling function $f_R(y) = y^{-\lambda_R/z+1+a'}(y-1)^{-1-a'}$. If one has $\xi + \tilde{\xi} = 0$, then $a = a'$, which is a frequent hidden assumption in the literature. Alternatively, the gauge exponent F may be expressed as $F = \frac{1}{2}(\tilde{x} - x) + \tilde{\xi} - \xi$.

Next, we derive the spatio-temporal response function

$$\begin{aligned}
R(t, s; \mathbf{r}) &= \left\langle t^\xi \tilde{\Phi}(t, \mathbf{r}) s^{\tilde{\xi}} \tilde{\Phi}(s, \mathbf{0}) \right\rangle_0 \\
&= R(t, s) \exp \left[-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s} \right] \quad (4.103)
\end{aligned}$$

which indeed follows from our previous discussion of Schrödinger-invariance, see (4.31, 4.63).

4.6.2 Ageing-invariant Autocorrelators

Similarly, the autocorrelation function can be written as

$$\begin{aligned}
C(t, s) &= \frac{a_0}{2} \int_{\mathbb{R}^d} d\mathbf{R} R_0^{(3)}(t, s, \tau_{\text{ini}}; \mathbf{R}) \\
&\quad + \frac{T}{2\mathcal{M}} \int_0^\infty du \int_{\mathbb{R}^d} d\mathbf{R} R_0^{(3)}(t, s, u; \mathbf{R}) \quad (4.104)
\end{aligned}$$

$$\begin{aligned}
R_0^{(3)}(t, s, u; \mathbf{r}) &:= \left\langle \phi(t; \mathbf{y}) \phi(s; \mathbf{y}) \tilde{\phi}(u; \mathbf{r} + \mathbf{y})^2 \right\rangle_0 \\
&= (ts)^\xi u^{2\tilde{\xi}_2} \left\langle \tilde{\Phi}(t; \mathbf{y}) \tilde{\Phi}(s; \mathbf{y}) \tilde{\Phi}(u; \mathbf{r} + \mathbf{y})^2 \right\rangle_0 \\
&= (ts)^\xi u^{2\tilde{\xi}_2} \frac{k(t)k(s)}{k^2(u)} \mathcal{R}_0^{(3)}(t, s, u; \mathbf{r}) \quad (4.105)
\end{aligned}$$

where the composite field $\tilde{\phi}^2$ is characterised by $2\tilde{x}_2$ and $2\tilde{\xi}_2$. Here τ_{ini} is some small ‘initial’ time which sets the beginning of the scaling regime. At a later stage, the limit $\tau_{\text{ini}} \rightarrow 0$ should be taken.

Throughout, we consider $\tilde{\Phi}^2$ as a composite field with scaling dimension $2\tilde{x}_2 + 4\tilde{\xi}_2$ (only for free fields, one would have $\tilde{x} = \tilde{x}_2$ and $\tilde{\xi} = \tilde{\xi}_2$). Recall from the previous Sections that Schrödinger-invariance gives the three-point response function $\mathcal{R}_0^{(3)}$ for $v(t) = 0$ [327]

$$\begin{aligned} \mathcal{R}_0^{(3)}(t, s, u; \mathbf{r}) &= \mathcal{R}_0^{(3)}(t, s, u) \exp \left[\frac{\mathcal{M}(2u - t - s)}{2(s - u)(t - u)} \mathbf{r}^2 \right] \Psi \left(\frac{t - s}{2(t - u)(s - u)} \mathbf{r}^2 \right) \\ \mathcal{R}_0^{(3)}(t, s, u) &= \Theta(t - u) \Theta(s - u) \\ &\quad \times (t - u)^{-\tilde{x}_2 - 2\tilde{\xi}_2} (s - u)^{-\tilde{x}_2 - 2\tilde{\xi}_2} (t - s)^{-x - 2\xi + \tilde{x}_2 + 2\tilde{\xi}_2} \quad (4.106) \end{aligned}$$

where $\Psi = \Psi(\rho)$ is an arbitrary scaling function, to be determined. Of course, and we shall come back to this below, this is only applicable within the ageing regime t, s and $t - s \gg \tau_{\text{micro}}$. It is useful to write down the autocorrelation function in the form $C(t, s) = C_{\text{th}}(t, s) + C_{\text{init}}(t, s)$. The ‘thermal’ part C_{th} and the ‘initial’ part C_{init} (where the limit $\tau_{\text{ini}} \rightarrow 0$ was taken) are given by

$$\begin{aligned} C_{\text{th}}(t, s) &= C_{0,\text{th}} s^{d/2+1-x-\tilde{x}_2} \left(\frac{t}{s} \right)^F \left(\frac{t}{s} - 1 \right)^{\tilde{x}_2 + 2\tilde{\xi}_2 - x - 2\xi - d/2} \\ &\quad \times \int_0^1 dv v^{2\tilde{\xi}_2 - 2F} \left[\left(\frac{t}{s} - v \right) (1 - v) \right]^{d/2 - \tilde{x}_2 - 2\tilde{\xi}_2} \Psi \left(\frac{t/s + 1 - 2v}{t/s - 1} \right) \\ C_{\text{init}}(t, s) &= C_{0,\text{init}} s^{d/2+2F-\tilde{x}_2-2\tilde{\xi}_2-x} \\ &\quad \times \left(\frac{t}{s} \right)^{d/2+\xi+F-\tilde{x}_2-2\tilde{\xi}_2} \left(\frac{t}{s} - 1 \right)^{\tilde{x}_2-x+2\tilde{\xi}_2-2\xi-d/2} \Psi \left(\frac{t/s + 1}{t/s - 1} \right) \\ \Psi(w) &:= \int_{\mathbb{R}^d} d\mathbf{R} \exp \left[-\frac{\mathcal{M}w}{2} \mathbf{R}^2 \right] \Psi(\mathbf{R}^2) \quad (4.107) \end{aligned}$$

and we have explicitly used $s < t$. Here $C_{0,\text{th}}$ is a normalisation constant proportional to the temperature T and $C_{0,\text{init}}$ is a rescaled normalisation depending on the initial time-scale τ_{ini} .

For the interpretation of these forms, we concentrate on simple magnets relaxing towards equilibrium stationary states. From renormalisation arguments [93, 121] one expects that the thermal part C_{th} will be the leading one for quenches *onto* the critical temperature $T = T_c$ while the preparation part C_{init} should be dominant for quenches *into* the ordered phase ($T < T_c$). Comparison with the expected scaling forms (1.74, 1.75) then gives

1. $T = T_c$: for relaxation to equilibrium, one has²⁶ $a = b$ and from (4.107) we read off

$$a = b = x + \tilde{x}_2 - 1 - \frac{d}{2}, \quad \lambda_C = 2(x + \xi - F). \quad (4.108)$$

²⁶ Applying this kind of argument to systems relaxing to critical *non-equilibrium* steady states, one must take into account that $a \neq b$ in general, see Chap. 3.

The autocorrelation function scaling function becomes, up to normalisation

$$f_C^{(\text{th})}(y) = y^{1+a'-\lambda_C/2} (y-1)^{b-2a'+2\mu-1} \quad (4.109)$$

$$\times \int_0^1 dv v^{\lambda_C+2\mu-2a'-2} [(y-v)(1-v)]^{a'-b-2\mu} \Psi\left(\frac{y+1-2v}{y-1}\right)$$

where $\mu := \xi + \tilde{\xi}_2$ is a further parameter whose value is not predicted by the theory.

2. $T < T_c$: one has $b = 0$ and from the asymptotic scaling function we find

$$\lambda_C = 2(x + \xi - F) . \quad (4.110)$$

The scaling function reads simply, up to normalisation,

$$f_C^{(\text{init})}(y) = y^{\lambda_C/2} (y-1)^{-\lambda_C} \Psi\left(\frac{y+1}{y-1}\right) . \quad (4.111)$$

Since $\tilde{x}_2 - x + 2\tilde{\xi}_2 - 2\xi = \frac{1}{2}d - \lambda_C \leq 0$ because of the Yeung-Rao-Desai inequality [744], this exponent combination will only vanish for free fields. This illustrates again that $\tilde{x}_2$ and ξ_2 must be treated as independent quantities and cannot *a priori* be related to other exponents.

Comparing the above results for λ_C with eq. (4.102) giving λ_R , we see that *for short-ranged initial conditions and a deterministic part $\mathcal{J}_0$ of the action which is Schrödinger-invariant, the autocorrelation and autoresponse exponents are equal, $\lambda_C = \lambda_R$* [589]. Having discussed this well-known result [93] on p. 82 in Chap. 1, we now see that it can also be understood as a consequence of a dynamical symmetry.

Schrödinger-invariance by itself does not determine the form of $\Psi(\rho)$. However, the related three-point response function should be non-singular as $t - s \rightarrow 0$. This requirement fixes the asymptotic behaviour for $w \rightarrow \infty$ [589, 336]

$$\Psi(w) \sim w^{-\varphi} , \quad \varphi = \begin{cases} \lambda_C & ; \text{ if } T < T_c \text{ and } b = 0 \\ 1 + 2a' - b - 2\mu & ; \text{ if } T = T_c \text{ and } a = b \end{cases} . \quad (4.112)$$

This suggests the following approximate forms of the scaling function, with $y = t/s$,

$$f_C(y) \approx \begin{cases} [(y+1)^2/(4y)]^{-\lambda_C/2} & ; \text{ if } T < T_c \\ y^{1+a'-\lambda_C/2} \int_0^1 dv v^{\lambda_C-2-2a'-2\mu} [(y-v)(1-v)]^{a'-b-2\mu} & ; \text{ if } T = T_c \\ \quad \times (y+1-2v)^{b-2a'-1+2\mu} & \end{cases} \quad (4.113)$$

At least, these forms are consistent with the required asymptotic behaviour of $f_C(y)$ as $y \rightarrow \infty$. For free field-theories, (4.112) holds for all values of w and then (4.113) becomes exact.

4.6.3 Conformal Invariance

In order to find the last remaining scaling function Ψ , we now propose [342, 335] to extend Schrödinger-invariance further towards conformal invariance in $d + 2$ dimensions as described in Sect. 4.4. This implies that we must consider the ‘mass’ $\mathcal{M}$ of the scaling operator Φ as a further variable. Consider three Schrödinger-quasiprimary scaling operators $\Phi_\alpha = \Phi_\alpha(t_\alpha, \mathbf{r}_\alpha, \mathcal{M}_\alpha)$ where $\alpha = a, b, c$. Using the variables

$$\tau = t_a - t_c, \quad \sigma = t_b - t_c, \quad \mathbf{r} = \mathbf{r}_a - \mathbf{r}_c, \quad \mathbf{s} = \mathbf{r}_b - \mathbf{r}_c \quad (4.114)$$

the three-point function is

$$\langle \Phi_a \Phi_b \Phi_c \rangle_0 = \delta \left(\sum_{\alpha=a,b,c} \mathcal{M}_\alpha \right) F(\tau, \sigma, \mathbf{r}, \mathbf{s}; \mathcal{M}_a, \mathcal{M}_b, \mathcal{M}_c) \quad (4.115)$$

where the delta function expresses the Bargman superselection rule, with $\tau > 0$ and $\sigma > 0$ because of causality. Schrödinger-invariance almost fixes the form of F :

$$F = \tau^{-\gamma_1} \sigma^{-\gamma_2} (\tau - \sigma)^{-\gamma_3} \exp \left[-\frac{\mathcal{M}_a}{2} \frac{\mathbf{r}^2}{\tau} - \frac{\mathcal{M}_b}{2} \frac{\mathbf{s}^2}{\sigma} \right] \Psi(\rho, \mathcal{M}_a, \mathcal{M}_b)$$

$$\rho = \frac{(\mathbf{r}\sigma - \mathbf{s}\tau)^2}{2\tau\sigma(\tau - \sigma)} \quad (4.116)$$

and

$$\gamma_1 = \frac{1}{2}x_{ac,b} + \xi_{ac,b}, \quad \gamma_2 = \frac{1}{2}x_{bc,a} + \xi_{bc,a}, \quad \gamma_3 = \frac{1}{2}x_{ab,c} + \xi_{ab,c} \quad (4.117)$$

with $x_{ab,c} := x_a + x_b - x_c$ and $\xi_{ab,c} := \xi_a + \xi_b - \xi_c$.

The last function $\Psi = \Psi_{abc}$ will now be found by extending invariance under $\mathfrak{sch}(d)$ to invariance under the conformal algebra $\mathfrak{conf}(d+2)$. For the explicit calculation, it is enough to consider the case $d = 1$. From the root diagram Fig. 4.2 it is clear that if one has invariance under the generators N and V_- , then invariance under the other generators follows. The explicit single-particle form of the generators N and V_- is

$$N = -t\partial_t - 1 - \mathcal{M}\partial_{\mathcal{M}}$$

$$V_- = -i\partial_{\mathcal{M}}\partial_r + ir\partial_t \quad (4.118)$$

or $d = 1$. Then the covariance of the three-point function under N and V_- leads to

$$(\tau\partial_\tau + \sigma\partial_\sigma + 2 + \mathcal{M}_a\partial_{\mathcal{M}_a} + \mathcal{M}_b\partial_{\mathcal{M}_b})F = 0$$

$$(r\partial_r + s\partial_s - \partial_{\mathcal{M}_a}\partial_r - \partial_{\mathcal{M}_a}\partial_s)F = 0 \quad (4.119)$$

In turn, these imply the following conditions for the function Ψ

$$[(2 - \gamma_1 - \gamma_2 - \gamma_3) - \rho\partial_\rho + \mathcal{M}_a\partial_{\mathcal{M}_a} + \mathcal{M}_b\partial_{\mathcal{M}_b}]\Psi = 0 \quad (4.120)$$

$$\left[\frac{r}{\tau} [1 - \gamma_1 + \mathcal{M}_a\partial_{\mathcal{M}_a} + \partial_{\mathcal{M}_a}\partial_\rho] + \frac{s}{\sigma} [1 - \gamma_2 + \mathcal{M}_b\partial_{\mathcal{M}_b} + \partial_{\mathcal{M}_b}\partial_\rho] \right. \\ \left. + \frac{r-s}{\tau-\sigma} [-\gamma_3 - \rho\partial_\rho - \partial_{\mathcal{M}_a}\partial_\rho - \partial_{\mathcal{M}_b}\partial_\rho] \right] \Psi = 0 \quad (4.121)$$

In eq. (4.121), it can be seen that if any two of the inner square brackets vanish when applied to Ψ , the vanishing of the third one follows. Consequently, we have the following linear system of three equations for the function Ψ which depends on three variables

$$\begin{aligned} [(2 - \gamma_1 - \gamma_2 - \gamma_3) - \rho\partial_\rho + \mathcal{M}_a\partial_{\mathcal{M}_a} + \mathcal{M}_b\partial_{\mathcal{M}_b}]\Psi &= 0 \\ [1 - \gamma_1 + \mathcal{M}_a\partial_{\mathcal{M}_a} + \partial_{\mathcal{M}_a}\partial_\rho]\Psi &= 0 \\ [1 - \gamma_2 + \mathcal{M}_b\partial_{\mathcal{M}_b} + \partial_{\mathcal{M}_b}\partial_\rho]\Psi &= 0 \end{aligned} \quad (4.122)$$

The first of these leads to the scaling form

$$\Psi(\rho, \mathcal{M}_a, \mathcal{M}_b) = \rho^{2-\gamma_1-\gamma_2-\gamma_3} K\left(\frac{\mathcal{M}_a + \mathcal{M}_b}{2}\rho, \frac{\mathcal{M}_a - \mathcal{M}_b}{2}\rho\right) \quad (4.123)$$

and the other two give the following system of equations for $K = K(\eta, \zeta)$

$$\begin{aligned} [2 - \gamma_1 - \gamma_2 + (3 - \gamma_1 - \gamma_2 - \gamma_3)\partial_\eta + \eta\partial_\eta + \zeta\partial_\zeta + \eta\partial_\eta^2 + \zeta\partial_\eta\partial_\zeta] K(\eta, \zeta) &= 0 \\ [\gamma_2 - \gamma_1 + (3 - \gamma_1 - \gamma_2 - \gamma_3)\partial_\zeta + \eta\partial_\zeta + \zeta\partial_\eta + \zeta\partial_\zeta^2 + \eta\partial_\eta\partial_\zeta] K(\eta, \zeta) &= 0 \end{aligned}$$

which in principle can be solved through a double series in η and ζ .

For our purpose, namely to find the two-time autocorrelation function, we merely need the special three-point function $\langle \Phi\Phi\tilde{\Phi}^2 \rangle_0$. In this case, we have

$$\gamma_1 = \gamma_2 = \tilde{x}_2 + 2\tilde{\xi}_2, \quad \gamma_3 = x - \tilde{x}_2 + 2\xi - 2\tilde{\xi}_2, \quad (4.124)$$

$\mathcal{M}_a = \mathcal{M}_b = \mathcal{M}$ and $\mathcal{M}_c = -2\mathcal{M}$. Therefore, the requested solution becomes $\Psi = \Psi_{\Phi\Phi\tilde{\Phi}^2} = \rho^{2-x-\tilde{x}_2-2\xi-2\tilde{\xi}_2} K(\mathcal{M}\rho, 0)$ where, up to normalisation, $K(\eta, 0)$ is the solution of

$$[2 - 2\gamma_1 + (3 - 2\gamma_1 - \gamma_3)\partial_\eta + \eta\partial_\eta + \eta\partial_\eta^2] K(\eta, 0) = 0. \quad (4.125)$$

This is readily solved in terms of confluent hypergeometric functions and the final result for the scaling function Ψ reads, where $\psi_{0,1}$ are arbitrary constants

$$\begin{aligned} \Psi(\rho, \mathcal{M}, \mathcal{M}) &= \\ \psi_0 \rho^{2-x-\tilde{x}_2-2\xi-2\tilde{\xi}_2} {}_1F_1\left(2 - 2\tilde{x}_2 - 4\tilde{\xi}_2, 3 - x - \tilde{x}_2 - 2\xi - 2\tilde{\xi}_2; -\mathcal{M}\rho\right) \\ + \psi_1 \mathcal{M}^{x+\tilde{x}_2+2\xi+2\tilde{\xi}_2-2} {}_1F_1\left(x - \tilde{x}_2 + 2\xi - 2\tilde{\xi}_2, x + \tilde{x}_2 + 2\xi + 2\tilde{\xi}_2 - 1; -\mathcal{M}\rho\right) \end{aligned} \quad (4.126)$$

The main applications of eq. (4.126) will be to phase-ordering kinetics ($T < T_c$), where $z = 2$ for simple magnets without disorder, see Chap. 1, and furthermore $b = 0$. Then

$$\begin{aligned} \Psi(\rho, \mathcal{M}, \mathcal{M}) &= \psi_0 \rho^{\lambda_C - 2a' - d/2} {}_1F_1(2\lambda_C - d - 2a', \lambda_C + 1 - 2a' - d/2; -\mathcal{M}\rho) \\ &+ \psi_1 \mathcal{M}^{2a' + d/2 - \lambda_C} {}_1F_1(\lambda_C - d/2, 1 + 2a' + d/2 - \lambda_C; -\mathcal{M}\rho) . \end{aligned} \quad (4.127)$$

On the other hand, for non-equilibrium critical dynamics (quenches to $T = T_c$) with dynamical exponent $z = 2$ and also $a = b$, we have

$$\begin{aligned} \Psi(\rho, \mathcal{M}, \mathcal{M}) &= \\ &\psi_0 \rho^{1-b-2\mu-d/2} {}_1F_1(2-d+2a'-2b-4\mu, 2-b-2\mu-d/2; -\mathcal{M}\rho) \\ &+ \psi_1 \mathcal{M}^{d/2+b+2\mu-1} {}_1F_1(1-d/2+2a'-b-2\mu, d/2+b-2\mu; -\mathcal{M}\rho) . \end{aligned} \quad (4.128)$$

We point out that the result eq. (4.101) derived above for the response functions does remain valid for this type of conformal invariance. This follows since conformal invariance merely imposes causality for the two-point response functions and does not alter its functional form, as we have seen in the previous Section. Furthermore, the exponent a' of the autoresponse function plays a rôle in fixing the form of $C(t, s)$.

Before we insert these explicit results into the scaling forms eqs. (4.111) and (4.109), we should consider under which physical conditions their derivation is valid. We used from the outset dynamical scaling and therefore restricted ourselves to the ageing regime. In particular the condition $t - s \gg t_{\text{micro}}$ must be satisfied, see [760] for a careful discussion of this point. In this context, this means that for small arguments $\rho \rightarrow 0$, the form of the function $\Psi(\rho)$ is *not* given by local scale-invariance. Rather, for $\rho \ll 1$ one should expect that the two-time autocorrelation function $C(t, s) = C(s, t)$ should be symmetric in t and s and especially $C(t, s)$ should be non-singular in the limit $t - s \rightarrow 0$. The requirement of the absence of a singularity in $C(t, s)$ as $t - s \rightarrow 0$ is equivalent to an analogous requirement for the three-point function $\langle \Phi(t)\Phi(s)\tilde{\Phi}^2(u) \rangle_0$ and leads to

$$\Psi(\rho) \sim \rho^{x-\tilde{x}_2+2\xi-2\tilde{\xi}_2} ; \text{ as } \rho \rightarrow 0 . \quad (4.129)$$

Specifically, for the two physical applications under study, one recovers eq. (4.112). Since this condition is not satisfied by the solution given in eq. (4.126), one might consider the possibility that local scale-invariance only holds for sufficiently large arguments, say $\rho \geq \epsilon$, and write

$$\Psi(\rho) \simeq \begin{cases} \Psi_0 \rho^{x-\tilde{x}_2+2\xi-2\tilde{\xi}_2} & ; \text{ if } \rho \leq \epsilon \\ \Psi_{\text{LSI}}(\rho) & ; \text{ if } \rho \geq \epsilon \end{cases} \quad (4.130)$$

where $\Psi_{\text{LSI}}(\rho)$ is the expression given by eq. (4.126) and the constant ϵ sets the scale which separates the two regimes. Finally, the constant Ψ_0 is found from the condition that $\Psi(\rho)$ is continuous at $\rho = \epsilon$.

In this way, one may obtain an explicit scaling function which at least satisfies the basic boundary conditions required for a physically sensible scaling function. For models with an underlying free-field theory, the approximation eq. (4.112) becomes exact and leads to eq. (4.113), in agreement with the available exact solutions, as we shall review in the next Section. For the case of phase-ordering, it can be shown that [335]

$$\begin{aligned}
\Psi(w) = & A\Gamma(\lambda_C - 2a') w^{2a' - \lambda_C} \\
& \times {}_2F_1(2\lambda_C - d - 2a', \lambda_C - 2a'; \lambda_C + 1 - 2a' - d/2; -1/w) w^{2a' - \lambda_C} \\
& + B\Gamma(d/2) {}_2F_1(\lambda_C - d/2, d/2; 1 + 2a' + d/2 - \lambda_C; -1/w) w^{-d/2} \\
& + AE^{1-2a'} \frac{2\lambda_C - d - 2a'}{\lambda_C + 1 - 2a' - d/2} w^{-\lambda_C} \\
& \times \left[(Ew)^{2a'-1} \gamma(\lambda_C + 1 - 2a', Ew) - \gamma(\lambda_C, Ew) \right] \\
& + AE^{-2a'} w^{-\lambda_C} \gamma(\lambda_C, Ew) - A\gamma(\lambda_C - 2a', Ew) w^{2a' - \lambda_C} \\
& + BE^{d/2 - \lambda_C} w^{-\lambda_C} \gamma(\lambda_C, Ew) - B\gamma(d/2, Ew) w^{-d/2}
\end{aligned} \tag{4.131}$$

where ${}_2F_1$ is a hypergeometric function and $\gamma(a, z)$ is an incomplete gamma-function [4], whereas $E = M\epsilon$ and A, B are constants related to $\psi_{0,1}$. The first two lines would give the scaling function Ψ if the above-described procedure were unnecessary. For free fields, one has $\lambda_C = d/2$. Setting $A = 0$, we then have $\Psi(w) = B\Gamma(d/2) w^{-d/2}$ and recover the case $T < T_c$ of (4.112).

We have the final result [351, 335]: *eqs. (4.111, 4.131) together give our prediction for the autocorrelation function $C(t, s)$ of phase-ordering kinetics.*

We shall not give here the analogous expression for quenches to $T = T_c$, since the known exactly solvable cases with $z = 2$ (such as the spherical model or the XY model in spin-wave approximation) are described by a free-field theory and in more general cases the dynamical exponent $z \neq 2$ [121]. This will be considered in Chap. 5.

4.7 Tests of Ageing- and Conformal-invariance for $z = 2$

Having presented the ingredients and some of the predictions of local scale-invariance (LSI) with $z = 2$, in the form of ageing-invariance and its extension to conformal invariance, we now must see to what extent the predictions obtained are compatible with explicit model results. In carrying out such studies, three different tests are performed:

1. Tests of the prediction (4.101) of the *autoresponse* function $R(t, s) = R(t, s; \mathbf{0})$ (r_0 is a normalisation constant)

$$R(t, s) = r_0 s^{-1-a} \left(\frac{t}{s} \right)^{1+a' - \lambda_R/2} \left(\frac{t}{s} - 1 \right)^{-1-a'} \Theta(t - s) \tag{4.132}$$

which depends on the three exponents λ_R , a and a' , see exercise 4.7. Such tests may be carried out directly on R itself or on one of the various forms of integrated responses. For these, the scaling forms were derived in Chap. 1. Tests on these quantities look for the presence of the special Schrödinger transformation generated by X_1 and verify to what extent the Möbius-transformation $t \mapsto t/(1 + \gamma t)$ is a dynamical symmetry.

2. Tests of the *spatio*-temporal response $R(t, s; \mathbf{r})$ as given in the factorised form of eq. (4.103)

$$R(t, s; \mathbf{r}) = R(t, s) \Phi \left(\frac{\mathbf{r}^2}{t - s} \right) \quad , \quad \Phi(u) = \exp \left(-\frac{\mathcal{M}}{2} u \right) \quad (4.133)$$

and $R(t, s)$ is the autoresponse function given above. While this factorisation is a general property of LSI, the special function form of $\Phi(u)$ is peculiar to the case $z = 2$. This prediction allows us in particular to check explicitly for Galilei-invariance.

3. Tests of the autocorrelation function $C(t, s)$, for which we predict either eq. (4.113) for models with an underlying free-field theory or else eqs. (4.111, 4.131) or (4.109, 4.131) if that is not the case. Besides the three-point functions of Schrödinger-invariance, this also tests the stronger hypothesis of conformal invariance.

Of course, local scale-invariance as formulated here is based on the assumptions of an algebraic growth law $L(t) \sim t^{1/2}$ and the possibility that the quasi-primary scaling operators $\Phi(t, \mathbf{r})$ are *not* identical to the physical observables $\phi(t, \mathbf{r})$, but are related through $\Phi(t, \mathbf{r}) \sim t^{-\xi} \phi(t, \mathbf{r})$, and similarly for the associated response operators. The remaining essential hypotheses of ageing- and conformal invariance with $z = 2$ are again summarised as follows.

hypothesis	tested through
Möbius transformation-invariance	autoresponse $R(t, s)$
Galilei-invariance	space-time response $R(t, s; \mathbf{r})$
conformal invariance	two-time correlation function

While the functional form of the response function is remarkably simple, this is not so for the correlation function. In [Table 4.2](#), we collect results for $f_C(y)$ of some models, solved analytically in Chapters 2 and 3, which we now proceed to analyse. Afterwards, we shall examine simulational results.

4.7.1 One-dimensional Glauber-Ising Model

Our first example is the 1D Ising model with Glauber dynamics [281], at zero temperature and with nearest-neighbour Hamiltonian $\mathcal{H} = -J \sum_i \sigma_i \sigma_{i+1}$, $\sigma_i = \pm 1$. For a totally disordered initial state, recall that the exact two-time autocorrelation and space-time response functions are [285, 478]

			z	$f_C(y)$
1D Glauber-Ising, $T = 0$			2	$(2/\pi) \arctan(\sqrt{2/(y-1)})$
spherical model, $T < T_c$			2	$\left((y+1)^2/4y\right)^{-d/4}$
spherical, $T = T_c$	$2 < d < 4$	2	$(y-1)^{-\frac{d}{2}+1}y^{1-d/4}(y+1)^{-1}$	
spherical, $T = T_c$	$d > 4$	2	$(y-1)^{-\frac{d}{2}+1} - (y+1)^{-\frac{d}{2}+1}$	
contact process BCP			η	$(y-1)^{-\frac{d}{\eta}+1} - (y+1)^{-\frac{d}{\eta}+1}$
pair	$\alpha < \alpha_C$	$d > \eta$	η	$(y-1)^{-\frac{d}{\eta}+1} - (y+1)^{-\frac{d}{\eta}+1}$
contact process	$\alpha = \alpha_C$	$\eta < d < 2\eta$	η	$(y+1)^{-\frac{d}{\eta}}{}_2F_1\left(\frac{d}{\eta}, \frac{d}{\eta}; \frac{d}{\eta} + 1; \frac{2}{y+1}\right)$
		$d > 2\eta$	η	$\frac{\eta}{2}\left[(y+1)^{2-\frac{d}{\eta}} - (y+1)(y-1)^{1-\frac{d}{\eta}}\right] + (d-\eta)(y-1)^{1-\frac{d}{\eta}}$
Edwards-Wilkinson EW1			2	$(y-1)^{-\frac{d}{2}+1} - (y+1)^{-\frac{d}{2}+1}$
Edwards-Wilkinson EW2			2	$(y-1)^{-\frac{d}{2}+1+\rho} - (y+1)^{-\frac{d}{2}+1+\rho}$
Mullins-Herring MH1			4	$(y-1)^{-\frac{d}{4}+1} - (y+1)^{-\frac{d}{4}+1}$
Mullins-Herring MH2			4	$(y-1)^{-\frac{d}{4}+1+\rho/2} - (y+1)^{-\frac{d}{4}+1+\rho/2}$
Mullins-Herring MHC			4	$(y-1)^{-(d-2)/4} - (y+1)^{-(d-2)/4}$

Table 4.2. Dynamical exponents and scaling functions (up to normalisation) of the autocorrelator of some analytically solved models with simple ageing behaviour, see text for the definitions. For the bosonic contact and pair-contact processes, the cases $0 < \eta < 2$ correspond to Lévy-flight transport and the limit $\eta \rightarrow 2$ corresponds to diffusive transport.

$$C(t, s) = \frac{2}{\pi} \arctan \sqrt{\frac{2}{t/s - 1}} \quad , \quad R(t, s; r) = \sqrt{\frac{1}{2\pi^2} \frac{1}{s(t-s)}} \exp \left[-\frac{1}{2} \frac{r^2}{t-s} \right] . \quad (4.134)$$

As shown in Chap. 2, these results remain valid for long-ranged initial conditions $\langle \sigma_r \sigma_0 \rangle \sim r^{-\aleph}$ with $\aleph > 0$ [356].

Comparing the autoresponse $R(t, s; 0)$ with the LSI-prediction eq. (4.132) we read off $a = 0, a' = -1/2$ and $\lambda_R/z = 1/2$ [589], see also Table 2.1. The full space-time response $R(t, s; r)$ is also completely consistent with the LSI-expectation (4.133) and one identifies $\mathcal{M} = 1$.

The autocorrelation function does not depend on the range of the initial conditions. This independence of the scaling behaviour of the initial conditions is a typical property of critical systems. Therefore, we believe it should be reasonable to consider $T = 0$ as the critical point of the 1D Ising model. Since $b = a = 0$ and $\lambda_C = \lambda_R = 1$ and the 1D Glauber model is described in terms of free fermions, we may use (4.112) at $T = T_c = 0$ and then have from

eq. (4.113)

$$C(t, s) = C_0 \int_0^1 dv v^{2\mu} \left[\left(\frac{t}{s} - 1 \right) (1 - v) \right]^{-2\mu-1/2} \left(\frac{t}{s} + 1 - 2v \right)^{2\mu} \quad (4.135)$$

Choosing $\mu = -1/4$ and $C_0 = \sqrt{2}/\pi$, the exact result quoted above is reproduced [336].

We remark that also in the case of an ‘island’ initial state (2.45), the response function, with the exponents as listed in Table 2.1, completely agrees with LSI.

4.7.2 XY Model in Spin-wave Approximation

The XY model describes the nearest-neighbour interactions of planar spins $\vec{S} \in \mathbb{R}^2$ and with $|\vec{S}| = 1$. In the spin-wave approximation, the usual nearest-neighbour Hamiltonian is approximated in terms of the phases $\phi(\mathbf{r})$ of each spin $\vec{S}$ and reads

$$\mathcal{H} = - \sum_{(\mathbf{r}, \mathbf{r}')} \vec{S}(\mathbf{r}) \cdot \vec{S}(\mathbf{r}') \simeq \frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{r} (\nabla \phi(\mathbf{r}))^2. \quad (4.136)$$

In Chap. 2, we have seen that angular responses and correlations are different from the magnetic ones. The exponents describing the non-equilibrium scaling behaviour are listed in Table 2.10. The expression (2.155) for the angular response fully agrees with LSI. For an initial temperature $T_{\text{ini}} = 0$ and final temperature $T \ll T_c$, the result (2.154) for the scaling function of the autocorrelation function can be summarised for $d \geq 2$ as [589]

$$f_C(y) = f_0 \int_0^1 d\theta (y + 1 - 2\theta)^{-d/2} \quad (4.137)$$

where f_0 is a known normalisation constant. This agrees with the free-field LSI-prediction (4.113) if one identifies $\mu = 0$. If $T_{\text{ini}} > 0$, the thermal noise gives rise to an extra term of the same order, such that in this case both sources of noise contribute, as long as the spin-wave approximation is valid.

While we can conclude that angular responses and correlations are fully described by LSI, this is different for the spin responses and correlations [73]. This is not surprising since the spin-wave approximation considers the small deviations from a fully ordered state, for which LSI as formulated in this book is not applicable. The extension of LSI to systems with a non-vanishing initial value of the order-parameter $m(0) \neq 0$, and specifically an explanation of the time-dependent magnetisation $m(t)$ as sketched in Fig. 1.26, remains an open problem.

4.7.3 Mean-field Theory and the Free Random Walk

We have already seen in Sect. 4.3 that the simple mean-field theory introduced in Chap. 1, as specified by eqs. (1.14) for responses and correlators, along with the self-consistency condition (1.15) for the time-dependent potential $v(t)$ and finally eq. (4.40) for the order-parameter, is completely consistent with LSI, for both critical quenches to $T = T_c$ and for phase-ordering with $T < T_c$.

In particular, at criticality the order-parameter ϕ and the quasi-primary scaling operator Φ are distinct from each other, see eq. (4.41).

The free random walk, which may be viewed as a degenerate case of an ageing system, agrees with mean-field theory for $T < T_c$, up to exponentially small corrections. Logically, its two-time functions (2.82) are in full agreement with the predictions of LSI.

4.7.4 Spherical Model

The spherical model (here with short-ranged interactions in order to have $z = 2$) is described by a set of continuous spin variables $\phi \in \mathbb{R}$ subject to the spherical constraint $\sum_{\mathbf{r}} \langle \phi(t, \mathbf{r})^2 \rangle = \mathcal{N}$ where $\mathcal{N}$ is the number of sites of the lattice. In the continuum limit, the equation of motion is the following stochastic Langevin equation [286]

$$\partial_t \phi(t, \mathbf{r}) = \Delta_L \phi(t, \mathbf{r}) - v(t) \phi(t, \mathbf{r}) + \eta(t, \mathbf{r}), \quad (4.138)$$

where η is a standard centred Gaussian noise with variance $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2T \delta(t-t') \delta(\mathbf{r}-\mathbf{r}')$. The time-dependent potential $v(t)$ arises from the spherical constraint. Since we are interested in *local* observables here, there is no need to study the effects of fluctuations in $v(t)$ [19, 20]. As discussed in Chap. 2, $v(t)$ can be found from the function $g(t) = \exp(2 \int_0^t d\tau v(\tau))$ where $g(t)$ solves the Volterra integral equation, yielding for short-ranged initial correlations

$$g(t) = f(t) + 2T \int_0^t dt' f(t-t') g(t') \quad , \quad f(t) = \Theta(t) (e^{-4t} I_0(4t))^d \quad (4.139)$$

and I_0 is a modified Bessel function [4]. This is exactly the kind of equation of motion of the type (4.96) considered above and it only remains to check that indeed $v(t) \sim t^{-1}$, or equivalently $g(t) \sim t^F$ for large times. The detailed analysis of various kinds of the spherical model in Chap. 2 has shown this to be the case, both for quenches to $T = T_c$ and to $T < T_c$, for short-ranged interactions,²⁷ initial short-ranged or long-ranged states and even for initial magnetised states. Values for F in these situations have been collected in Tables 2.3 and 2.7 and eqs. (2.126, 2.133). It is therefore clear that the

²⁷ This statement also applies to long-ranged interactions, but since then $z = \sigma < 2$, the analysis from the point of view of LSI must be postponed to Chap. 5.

predictions (4.101, 4.103) and (4.113) of local scale-invariance will be satisfied *both* below and at criticality (since $z = 2$ throughout) as indeed has been checked explicitly [589], see also Table 4.2.

The attentive reader might wonder why one always finds $a = a'$ in the spherical model, although the presence of a time-dependent potential $v(t)$ in the Langevin equation (4.138) implies that the second scaling dimension $\xi \neq 0$. However, when considering the corresponding action $\mathcal{J}$, it is easy to see that the corresponding quantity in the equation of motion of the response field $\tilde{v}(t) = -v(t)$ always cancels that effect, so that one has $\xi + \tilde{\xi} = a' - a = 0$ (which also occurs in mean-field theory). That the order-parameter ϕ and the quasi-primary scaling operator Φ differ in the spherical model should not surprise, in view of its long-range correlations (class L) for all temperatures $T \leq T_c$.

Similarly, we can discuss the LSI prediction for the *surface* responses and correlations in the spherical model. Because the free surface breaks translation-invariance in the spatial direction perpendicular to it, one has to consider the corresponding sub-algebra of $\mathfrak{age}(d)$. While this leaves one scaling function undetermined, see (4.33), one may try to fix this function by invoking the method of images, see exercise 4.8. For example, for Dirichlet boundary conditions the exact spatio-temporal response (3.88) completely agrees with the prediction eq. (S.46) obtained as just indicated. The analogous statement for Neumann boundary conditions is readily checked, too. Similarly, the consistency of the surface autocorrelators C_1 in the spherical model with LSI adapted to free surfaces may be verified.

The discussion has so far been limited to the order-parameter ϕ , its correlation and its response to the conjugate magnetic field h . Similarly, one may identify [45] an energy-like quasi-primary scaling operator ε , which in the spherical model corresponds to the composite field $\varepsilon = \phi^2$, which is conjugate to the temperature T , as defined through the noise correlator. If ϕ is characterised by the three constants $(x, \xi, \mathcal{M})$, then $\varepsilon = \phi^2$ is characterised by $(2x, 2\xi, 2\mathcal{M})$. The resulting exponents for the correlator $C_{22} = 2C_{11}^2 = 2C(t, s; \mathbf{r})^2$ and the thermal responses $\mathcal{R}_2^T = 2\mathcal{R}_{11}^2 = 2R(t, s; \mathbf{r})^2$ are listed in Tables 2.5 and 2.6. On the other hand, as discussed on p. 120, the natural-looking definition of an energy-density ϵ from the spin Hamiltonian (2.118) reduces in the continuum limit to $\epsilon \sim -\nabla_{\mathbf{r}}^2 \phi$. Being a derivative of a quasi-primary scaling operator, ϵ cannot be quasi-primary itself.²⁸

After all, in view of the linearity of the Langevin equation of the order-parameter, it was to be expected that the spherical model is a model case for the validity of local scale-invariance, in a variety of physical situations.

²⁸ For comparison, in the Ginzburg-Landau classification of primary operators in the ‘minimal models’ (Ising, Potts, ...) of 2D conformal field-theory, along with the order-parameter ϕ only the first few powers : ϕ^ℓ : are primary as well.

Indeed, it did serve more than once as a source of inspiration in our trying to identify the necessary ingredients of LSI [589, 52, 45].

4.7.5 Ising Model in Two and Three Dimensions

We now turn to systems where no exact solution is known and which are not described by a Langevin equation, but by a master equation. In this subsection we present a test of LSI through the space-time response functions of the 2D and 3D Ising models undergoing phase-ordering kinetics after a quench to $T < T_c$, from a fully disordered initial state [344]. Correlation functions in the 2D Ising model will be studied in the next subsection. Simulation of these models is through the heat-bath algorithm, outlined in appendix G in volume 1.

We first consider the thermoremanent susceptibility $\chi_{\text{TRM}}(t, s)$. As we have seen in Chap. 1, the scaling analysis of this quantity requires the careful subtraction of the leading finite-time correction which for simple ferromagnets quenched to $T < T_c$ turns out to be particularly large. Having done this, we obtain the scaling function of the thermoremanent susceptibility $\chi_{\text{TRM}}(t, s) = s^{-a} f_M(t/s)$. From the explicit form (4.132) of the two-time autoresponse function, we have the LSI-prediction, derived in exercise 4.23

$$f_M(y) = f_0 y^{-\lambda_R/z} {}_2F_1 \left(1 + a', \frac{\lambda_R}{2} - a; 1 + \frac{\lambda_R}{2} - a; \frac{1}{y} \right) \quad (4.140)$$

and where ${}_2F_1$ is a hypergeometric function and f_0 is a normalisation constant. Any tests of the LSI-prediction of the linear response, based on numerical simulations, which we shall describe now use this prediction.

While in principle a and a' are independent exponents, *all presently available evidence in systems undergoing phase-ordering kinetics is compatible with $a = a'$* , which we shall admit from now on. However, we shall see in Chap. 5 that for non-equilibrium critical dynamics, consistency with the available numerical data can in certain systems only be achieved if a and a' are different.

The practical calculation of thermoremanent susceptibilities followed the method proposed by Barrat [40], using a spatial random magnetic field $h_i = \pm \mathfrak{h}$ with amplitude $\mathfrak{h} = 0.05$, on hypercubic lattices with $\mathcal{N} = 300^2$ sites in 2D and 60^3 in 3D. This turned out to be sufficient in order to reach the dynamical scaling regime, after averaging over the disordered initial states [344].

In Fig. 4.4, estimates are shown for the scaling function $f_M(t/s)$ which describe the phase-ordering in the Glauber-Ising model in both two and three dimensions, for several waiting times s . Clearly, after having subtracted off the leading correction to scaling, as discussed in Chap. 1, a nice scaling behaviour is found. Using the known values $a = 1/z = 1/2$ and of λ_R , and making the additional assumption $a = a'$, the form of the scaling function is completely fixed and only the normalisation remains to be determined. The result is

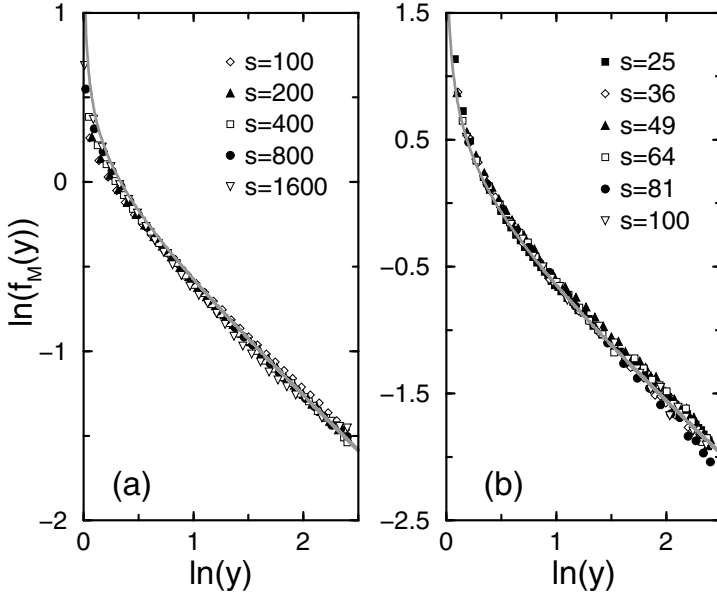


Fig. 4.4. Scaling function $f_M(y)$ of the thermoremanent susceptibility of the (a) 2D Glauber-Ising model quenched to $T = 1.5 < T_c$ and (b) 3D Glauber-Ising model quenched to $T = 3 < T_c$. The full curves give the prediction (4.140) of LSI. Reprinted with permission from [344]. Copyright (2003) by the American Physical Society.

presented as the full curves and an excellent agreement between the LSI-prediction eq. (4.140) and the data is seen. This example, together with the 1D case discussed above, strongly suggests that *the validity of LSI should not depend on the space dimension d* .

For reference, we collect in Table 4.3 the values of the non-universal parameters r_0, r_1 and $\mathcal{M}$ extracted from the data. It can be seen, see also Chap. 1, that for ferromagnetic phase-ordering systems in 2D the amplitude of the leading finite-time correction is of the order and of the opposite sign of the leading scaling term. Since furthermore $\lambda_R = \lambda_C \gtrsim d/2$ is not far from the lower YRD bound (1.89) and hence $a = 1/z \lesssim \lambda_R/z$ are quite close in 2D, the careful subtraction of this correction term is necessary in order to obtain meaningful results. From the scaling behaviour shown in Fig. 4.4, it becomes also clear that no further corrections to scaling need to be considered, at the present level of numerical precision.

Next, in order to check directly for Galilei-invariance, we consider the space-time response. Following the same lines as used in Chap. 1 for the autoresponse function, we derive the appropriate scaling form, where $\mathbf{r}$ varies along a fixed direction on the lattice [344]

model	d	T	r_0	r_1	$\mathcal{M}_{(10)}$	$\mathcal{M}_{(11)}$	
Potts-2	2	0.75	1.24(2)	-1.18(2)	1.94(8)	1.95(5)	[481]
Ising	2	1.5	2.65(5)	-2.76(5)	4.08(4)	4.08(4)	[344]
Potts-3	2	0.4975	1.01(1)	-0.91(1)	2.19(6)	2.23(4)	[481]
Potts-8	2	0.3725	0.55(1)	-0.50(2)	3.30(8)	3.41(5)	[481]
Ising	3	3.0	0.31(2)	0.61(2)	4.22(5)	4.22(5)	[344]

Table 4.3. Parameters for the linear response functions in the Ising model (random update) and q -states Potts models (checkerboard update) as well as values for the masses. In the 2D models, these are $\mathcal{M}_{(10)}$ and $\mathcal{M}_{(11)}$, according to the direction in which the spatial integration is performed. For the 3D Ising model, the entries for the masses correspond to $\mathcal{M}_{(10)} \mapsto \mathcal{M}_{(100)}$ and $\mathcal{M}_{(11)} \mapsto \mathcal{M}_{(110)}$, respectively.

$$\rho_0(t, s; \mathbf{r}) = r_0 s^{-a} F_0\left(\frac{t}{s}, \mathcal{M} \frac{r^2}{s}\right) + r_1 s^{-\lambda_R/z} G_0\left(\frac{t}{s}, \mathcal{M} \frac{r^2}{s}\right) \quad (4.141)$$

where the universal scaling functions read

$$\begin{aligned} F_0(y, u) &= \int_0^1 dv \exp[-u/2(y-1+v)] (1-v)^{-1-a} f_R\left(\frac{y}{1-v}\right) \\ G_0(y, u) &\simeq y^{-\lambda_R/z} e^{-u/2y} \end{aligned} \quad (4.142)$$

and where the mass $\mathcal{M}$, when used as a phenomenological parameter, is in principle direction-dependent. Such anisotropies may arise, for example, from the fact that the model is defined on a discrete lattice and this will become important for low temperatures [632, 634]. For tests of LSI it is best to choose T such as to keep these anisotropy effects as small as possible but also to avoid the cross-over to critical dynamics at $T = T_c$. The values of T finally selected are believed to be a good compromise between these opposing considerations. As before, the expression for G_0 mainly serves as an approximate parametrisation of the leading corrections to scaling which allows us to remove these corrections efficiently from the data. The scaling behaviour is illustrated in Fig. 4.5 for the 2D Ising model at $T = 1.5$ where the scaling function F_0 is plotted over against r^2/s . For the displayed data points r varies between 1 and $\sqrt{s}$ when going along the (10) direction and between $\sqrt{2}$ and $\sqrt{2s}$ when going along the (11) direction. At the considered temperature any anisotropies are very small and the values of $\mathcal{M}$, determined separately in the two directions, coincide within the numerical errors, $\mathcal{M}_{(10)} = \mathcal{M}_{(11)} = \mathcal{M}$. Similar results also hold in the 3D Ising model at $T = 3$, see Table 4.3. One may determine $\mathcal{M}$ by fixing $y = 7$ and then use (4.141), as is illustrated in the inset of Fig. 4.5. One then obtains a parameter-free prediction for any other value of y . Indeed, the inset of Fig. 4.5 compares the LSI-prediction to numerical data for two additional values of y and two different waiting times. Since by now all non-universal parameters are fixed, this constitutes by itself a quantitative test of local scale-invariance. In particular, this is a clear direct

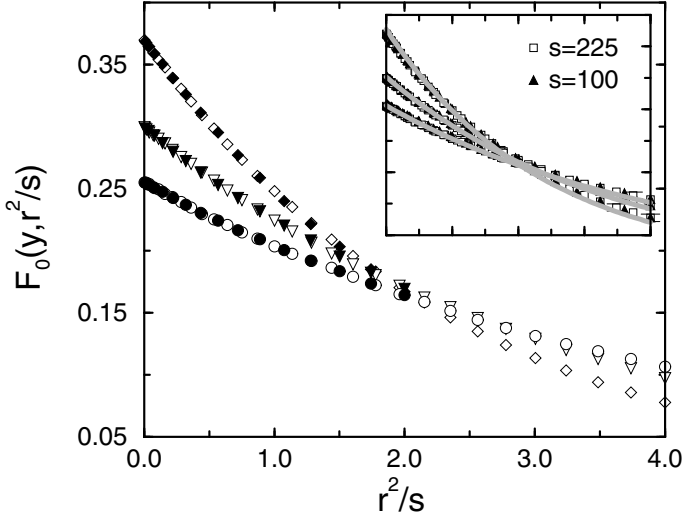


Fig. 4.5. Temporally integrated space-time response of the 2D Glauber-Ising model, at $T = 1.5$ and for $s = 225$. Data obtained by integrating in the (10) direction are indicated by open symbols and those for the (11) direction are given by closed symbols. The values of y correspond to: diamonds $y = 5$, triangles $y = 7$, circles $y = 9$. In the inset, the comparison with the scaling function F_0 from eq. (4.141) determines the value of $\mathcal{M}$. Reprinted with permission from [344]. Copyright (2003) by the American Physical Society.

evidence for the validity of Galilei-invariance in the phase-ordering kinetics of the Glauber-Ising model.

A fuller, but computationally expensive, test of the spatio-temporal response considers the spatially and temporally integrated response function

$$\frac{d\rho(t, s; \mu)}{d\Omega} = T \int_0^s du \int_0^{\sqrt{\mu s}} dr r^{d-1} R(t, u; \mathbf{r}) \quad (4.143)$$

where the space integral is along a straight line of length $\Lambda = \sqrt{\mu s}$. In principle, this test still allows for an anisotropy as a function of the solid angle Ω , although in the case at hand the directional independence of $\mathcal{M}$ checked above makes it clear that the effects of such an anisotropy, if they occur at all, are smaller than the numerical precision of this study. Again using the methods of Chap. 1, we find the scaling form [344], which depends on the control parameter μ

$$\frac{d\rho(t, s; \mu)}{d\Omega} = r_0 s^{d/2-a} \rho^{(2)}(t/s, \mu) + r_1 s^{d/2-\lambda_R/z} \rho^{(3)}(t/s, \mu) \quad (4.144)$$

with the explicit scaling functions

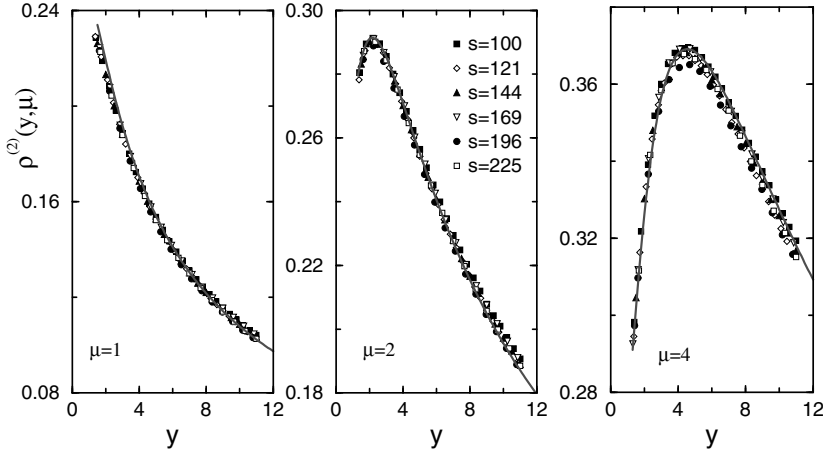


Fig. 4.6. Scaling function $\rho^{(2)}$ of the spatio-temporally integrated space-time response of the 2D Glauber-Ising model at $T = 1.5$ and comparison with LSI for three values of the control parameter μ and several values of s . Reprinted with permission from [344]. Copyright (2003) by the American Physical Society.

$$\begin{aligned}
 \rho^{(2)}(y, \mu) &= \frac{\mu^{d/2}}{d} \int_0^1 dv (1-v)^{-1-a} f_R \left(\frac{y}{1-v} \right) \mathcal{F}_d \left(\frac{\mathcal{M}\mu}{y-1+v} \right) \\
 \rho^{(3)}(y, \mu) &\simeq \frac{\mu^{d/2}}{d} y^{-\lambda_R/z} \mathcal{F}_d \left(\frac{\mathcal{M}\mu}{y} \right) \\
 \mathcal{F}_d(y) &= e^{-y/2} {}_1F_1 \left(1, 1 + \frac{d}{2}; \frac{y}{2} \right)
 \end{aligned} \tag{4.145}$$

This is the general expression for the scaling of the spatio-temporally integrated response function. If we fix μ and let $y = t/s$ vary, the form of the scaling function of $s^{a-d/2} d\rho/d\Omega$ merely depends on μ . We stress that the exponent λ_R and the free parameters $r_0, r_1, \mathcal{M}$ are now all fixed such that there remains no free fitting parameter at all when comparing (4.144) with our numerical data.

Before making this comparison, let us compare the maximal distance $r_{\max}$ accessed by the simulation [344] with the physical length scale of the problem. We use here the domain size $L(s)$ at the instant when the magnetic field is turned off. From the correlation function, one estimates in 2D $L(s) \approx 3, 6, 9.5$ for $s = 25, 100, 225$, respectively, with $h_{(0)} = 0.05$. Similarly, in 3D, one finds $L(s) \approx 2.1, 3.3, 4$ for $s = 25, 64, 100$. As expected $L(s) \sim s^{1/z}$, with $z \approx 2$. The value $\mu = 4$ ($\mu = 2$) corresponds to $r_{\max}/L(s) \approx 3.2$ (3.4) in 2D (3D), respectively and we see that the data probe the large-distance region, well beyond the domain size $L(s)$.

The scaling function $\rho^{(2)}$ of the integrated spatio-temporal response in 2D at $T = 1.5$ is shown in Fig. 4.6 for three values of μ . One has a nice

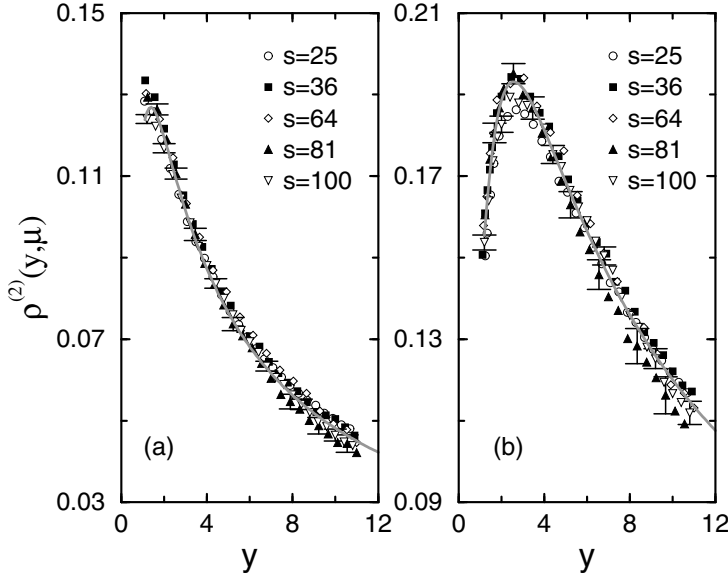


Fig. 4.7. Scaling function $\rho^{(2)}$ of the spatio-temporally integrated space-time response of the 3D Glauber-Ising model at $T = 3$. The control parameter is (a) $\mu = 1$ and (b) $\mu = 2$. The full curves give the LSI-prediction. The error bars give the size of the statistical noise in the data. Reprinted with permission from [344]. Copyright (2003) by the American Physical Society.

scaling behaviour over the whole range of waiting times available. We stress that the agreement between local scale-invariance and our data for several values of μ is a real test, since no free parameter remains. In particular, the height, the position of the maximum as well as the width around the maximum of the scaling function for all three values of μ are completely fixed. This constitutes a non-trivial confirmation of LSI and in particular yields a very clean confirmation of Galilei-invariance in a system which would correspond to a non-linear Langevin equation.

Similarly, the integrated spatio-temporal response in 3D at $T = 3$ is shown in Fig. 4.7. Again, a nice scaling behaviour in perfect agreement with LSI is found, even if the calculation is harder to perform in 3D than in 2D and consequently the stochastic error in the data is larger.

These tests of LSI, and in particular for the explicit Galilei-invariance, has been repeated for the 2D q -states Potts model with $q = 2, 3, 8$ [480, 481]. As in the Ising model, the thermoremanent susceptibility is calculated by using a small spatially random magnetic field with amplitude $h = 0.05$ in order to remain within the linear response regime, on square lattices with $N = 800^2$ sites [480, 481]. Since autocorrelations are considerably more easy to simulate than response functions, the autoresponse exponent was fixed

by using the expected relation $\lambda_R = \lambda_C$ and the known values of λ_C , see Table 4.4 below. From a consideration of the scaling function (4.141) the isotropy of the model was checked and a nice agreement between the LSI-prediction and the numerical data was found. The parameters extracted are collected in Table 4.3 where the difference of the values of the non-universal parameters $r_{0,1}$ and $\mathcal{M}$ come from the differences of the lattice realisations between the Ising and the two-states Potts model and the different update schemes (random vs checkerboard). Since with respect to the Ising model simulations, the Potts model data with $q = 3, 8$ come from slightly lower values of T/T_c , it is not surprising that small anisotropies in the mass parameter begin to be visible [632, 634] which provides practical information about the range of temperatures where LSI can be tested in simple spin models without having to correct explicitly for lattice anisotropies.

Besides the exactly solved models considered above, these are the only cases of non-linear models with $z = 2$ where Galilei-invariance has been checked explicitly so far.

4.7.6 XY Model in Two and Three Dimensions

Simulational data for the autoresponse function $R(t, s)$ from the 3D XY model quenched to several values of $T < T_c$ from a fully disordered initial state are in good agreement with the LSI-prediction (4.132), with $a = a' = 1/2$ and $\lambda_R = 1.7(1)$ [6].

In the 2D XY model with $T < T_{KT}$, simple scaling is found for a completely ordered initial state when the two-time responses and correlators are very well described by the spin-wave approximation [5, 463, 589]. On the other hand, for a totally disordered initial state, the domain size $L(t) \sim (t/\ln t)^{1/2}$ acquires a logarithmic correction. The resulting scaling forms have not yet been explained from the point of view of LSI.

4.7.7 Two-dimensional Ising and Potts Models

We now turn to a conceptually even more demanding test of LSI, which not only requires the validity of local scale-invariance as considered so far (that is, ageing-invariance) but furthermore its extension to the non-conventional conformal invariance, described in Sections 4.4 and 4.6, in models with non-linear Langevin equations. On the other hand, the numerical calculation of two-time correlation functions in these systems quenched to $T < T_c$ is considerably simpler than the determination of response functions considered so far.

To this end, the two-time autocorrelation function was calculated in the 2D kinetic Ising model with heat-bath dynamics [342] as well as for the q -states Potts model, with $q = 2, 3, 8$ [481]. The two-time autocorrelations were

model	T/T_c	λ_C	A	B	E	update	Ref.
Ising	0.0	1.25(1)	-0.601	3.94	0.517	R	[342]
Ising	0.6610	1.25(1)	-5.41	18.4	1.24	R	[342]
Potts-2	0.6610	1.24(2)	-5.41	18.4	1.24	C	[481]
Potts-3	0.5000	1.19(3)	-0.05	2.15	0.6	C	[481]
Potts-8	0.5001	1.25(1)	-0.07	1.98	0.4	C	[481]

Table 4.4. Parameters describing the autocorrelation function of 2D Ising and q -states Potts models, according to LSI enhanced by conformal invariance, eq. (4.131). The letters R and C refer to random and checkerboard update, respectively.

measured as

$$\begin{aligned}
 C(t, s)_{\text{Ising}} &= \frac{1}{\mathcal{N}} \sum_i \langle \sigma_i(t) \sigma_i(s) \rangle \\
 C(t, s)_{\text{Potts}} &= \frac{1}{q-1} \left(\frac{q}{\mathcal{N}} \sum_i \langle \delta_{\sigma_i(t), \sigma_i(s)} \rangle - 1 \right)
 \end{aligned} \tag{4.146}$$

where the sums run over the entire lattice with $\mathcal{N}$ sites. In the first case, the $\sigma_i = \pm 1$ are usual Ising spins, whereas in the second case $\sigma_i = 1, 2, \dots, q$ are Potts spins. Another difference between the two simulations is that either random or checkerboard update was used. The results for the parameters describing $C(t, s)$ from the two simulations are shown in Table 4.4 and allow some explicit tests of universality. Throughout, the relation $a = a'$ was assumed, which seems natural for non-critical systems. First, the estimates for the exponent λ_C in the $q = 2$ universality class, in agreement with earlier results [245, 104], appear to be independent of the lattice realisation and of the temperature, as long as $T < T_c$, which is of course expected from renormalisation arguments [93]. It is also satisfying to see that the same values of the parameters A, B, E of the LSI formula (4.111, 4.131) can be used for two different update schemes. One observes that the term parametrised by B is much more important than the other one, but the ratio A/B apparently depends on temperature.

In Figures 4.8 and 4.9, the degree of agreement between the numerical data and the LSI-prediction (4.111, 4.131) is shown. Clearly, the system ages, as can be seen in Fig. 4.8c,d, and there is excellent dynamical scaling for all models considered. For the Ising model, we show in Fig. 4.8ab the free-field approximation (4.113), labelled ‘app’, which only describes the data for rather large values of $y = t/s$. This is hardly surprising, since the model is known not to be described by a free field. On the other hand, we see that in all models the LSI predictions eqs. (4.111, 4.131) give a good overall description over the whole range of the scaling variable, with the only exception of the relatively small region $t/s \lesssim 2-3$. It is conceivable that the patching procedure outlined in Sect. 4.6 is not precise enough and we shall return to this in Chap. 5. The

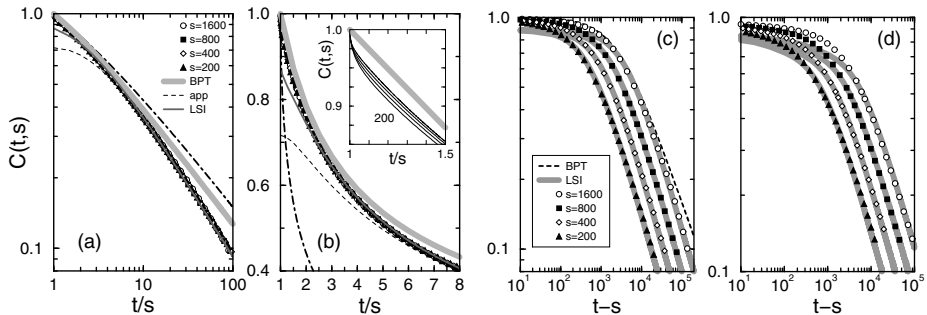


Fig. 4.8. Two-time autocorrelation function $C(t, s)$ for the 2D Ising model and comparison with the LSI prediction (4.111, 4.131). In (a) the scaling behaviour for $T = 0$ is shown for a large range of values of $y = t/s$ and (b) redisplayes the same data for smaller values of y . The inset compares the data for $y \approx 1$ and $s = [200, 400, 800, 1600]$ with the closed approximation of BPT [100, 700]. The dash-dotted line in (a) is the second-order calculation from [510] and the dash-dotted line in (b) gives the result from [479], see text. The autocorrelation is re-plotted as a function of $t - s$ and compared to LSI for $T = 0$ in (c) and for $T = 1.5$ in (d) [342].

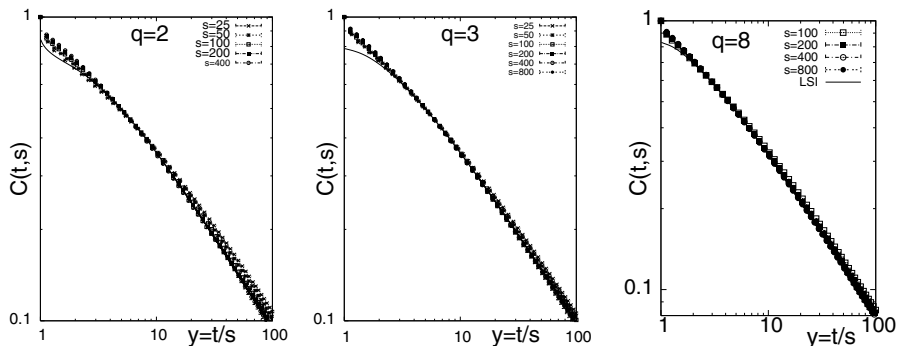


Fig. 4.9. Two-time autocorrelation function for the 2D q -states Potts model with $q = 2$ (left), $q = 3$ (middle) and $q = 8$ (right) [481, 400]. The different data symbols correspond to different waiting times. The solid lines show the fits to the scaling prediction (4.131) based on local scale-invariance enhanced by conformal invariance.

problem of how to derive a quantitatively precise formula which reproduces the numerical data over the entire range $1 < y < \infty$ is still not fully solved. In Chap. 5, we shall describe recent progress in this direction.

On the other hand, the difficulty of achieving any quantitative agreement of an analytical approach and the numerical data is illustrated by comparing with the results of earlier attempts to find $C(t, s)$ in phase-ordering kinetics. Using Gaussian closure procedures, Bray, Puri and Toyoki (BPT) find for the

$O(n)$ -model the scaling function [100, 700, 625]

$$f_{C,\text{BPT}}(y) = \frac{n}{2\pi} \left[B\left(\frac{1}{2}, \frac{n+1}{2}\right) \right]^2 \times \left(\frac{4y}{(y+1)^2} \right)^{d/4} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{n+2}{2}; \left(\frac{4y}{(y+1)^2} \right)^{d/2}\right) \quad (4.147)$$

where B is Euler's beta function [4]. While this form does follow the general trend of the data, especially if y is not too large, but without reproducing them entirely, it mainly suffers from the disadvantage that it predicts the autocorrelation exponent $\lambda_C = d/2$ in disagreement with the data, see also Fig. 4.8c. Liu and Mazenko [479] and Mazenko [510] tried to remedy this by constructing more elaborate analytical schemes of which the BPT formula represents the lowest order. In particular, this leads for the 2D Ising model to fairly accurate values $\lambda_C = 1.246(20)$ [479] and $\lambda_C = 1.105\dots$ [510] and it can further be shown that within this scheme, one has $\lambda_R = \lambda_C$, as it should be [511]. However, the detailed behaviour for $f_C(y)$ deviates significantly from the data, as can be seen from the dash-dotted lines in Fig. 4.8b for the prediction of [479] – in agreement with the earlier tests performed by Brown *et al.* [104] – and in Fig. 4.8a for the prediction of [510]. These examples illustrate the extreme difficulty of describing $C(t, s)$ theoretically. We consider it remarkable that local scale-invariance achieves, for the first time, a precise representation of the numerical data over almost the entire range of the scaling variable y . Still, for relatively small values of y systematic discrepancies remain.

A similar behaviour to that of the Ising model is found for the Potts models with $q = 3$ as well as $q = 8$ [481]. This is the first time that the ageing behaviour and LSI of the autocorrelator was tested for a system which undergoes a first-order transition.

The examples presented here are the only ones which presently allow us to test the extension $\mathbf{age}(d) \rightarrow \mathbf{sch}(d) \rightarrow \mathbf{conf}(d+2)$ of the local dynamical symmetry.

4.7.8 Bosonic Contact Processes

We now turn to two models without detailed balance whose stationary state therefore is not an equilibrium state.

As discussed in Chap. 3, solving the equations of motion with uncorrelated initial conditions for the BCPD and the BPCPD yields the exact two-time autocorrelator²⁹ (see Table 4.2 and note that in the BPCPD there is no scaling for $\alpha > \alpha_C$) which may also be written in an integral form

²⁹ In the critical linear voter model [186, 639], the exact two-time autocorrelator [206] agrees in 1D with that of the 1D Glauber-Ising model and for $d > 2$ with that of the BCP.

field	scaling dimension	mass
ϕ	x	$\mathcal{M}$
$\tilde{\phi}$	$\tilde{x}$	$-\mathcal{M}$
$\tilde{\phi}^2$	$\tilde{x}_2$	$-2\mathcal{M}$
$\Upsilon := \tilde{\phi}^2 \phi$	x_Υ	$-\mathcal{M}$
$\Sigma := \tilde{\phi}^3 \phi$	x_Σ	$-2\mathcal{M}$
$\Gamma := \tilde{\phi}^3 \phi^2$	x_Γ	$-\mathcal{M}$

Table 4.5. Scaling dimensions and masses of some composite fields in the bosonic contact and pair-contact processes.

$$f_C(y) = C_0 \int_0^1 d\theta \theta^{a-b} (y+1-2\theta)^{-d/2} \quad (4.148)$$

quite similar to (4.113). While the relation $a = b$ still holds true for the BCPD and the BPCPD with $\alpha < \alpha_C$, this is no longer true for the BPCPD at its multicritical point $\alpha = \alpha_C$ [49].

We now review, following [52], how the formalism of LSI can be adapted to these models. We begin with the BCPD in this subsection. The analysis of the BPCPD requires a further effort which we shall describe in the next subsection.

From a field-theoretic point of view, the BCPD can be described in terms of an order-parameter field ϕ and a response field $\tilde{\phi}$, defined such that $\langle \phi(t, \mathbf{r}) \rangle = \langle \tilde{\phi}(t, \mathbf{r}) \rangle = 0$. The action [379, 695, 694] is again decomposed $\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_0[\phi, \tilde{\phi}] + \mathcal{J}_b[\phi, \tilde{\phi}]$ into a ‘deterministic’ part

$$\mathcal{J}_0[\phi, \tilde{\phi}] := \int_{\mathbb{R}^d} d\mathbf{R} \int_0^\infty du \left[\tilde{\phi} (2\mathcal{M} \partial_u - \nabla^2) \phi \right] \quad (4.149)$$

which is manifestly Galilei-invariant and a ‘noise’ part that now contains two contributions and is described by

$$\mathcal{J}_b[\phi, \tilde{\phi}] := -\mu \int_{\mathbb{R}^d} d\mathbf{R} \int_0^\infty du \left[\tilde{\phi}^2 (\phi + \rho_0) \right]. \quad (4.150)$$

where uncorrelated initial conditions are implied. We notice that the structure of the noise term is different from the one used for magnetic systems which reflects the fact that the stationary state of the bosonic contact and pair-contact processes is absorbing.

In what follows, some composite fields will be needed. Together with their scaling dimensions and their masses, they are listed in Table 4.5. For free fields one has $\tilde{x}_2 = 2\tilde{x}$, $x_\Upsilon = 2\tilde{x} + x$, $x_\Sigma = 3\tilde{x} + x$ and $x_\Gamma = 3\tilde{x} + 2x$, but not necessarily so for interacting fields. On the other hand, from the Bargman superselection rules we expect that the masses of the composite fields as given in Table 4.5 should remain valid for interacting fields as well.

As for magnets, it is an easy exercise to reduce the response function to a form which does not contain the noise explicitly and which reproduces the

exact solution derived in Chap. 3 [52]. Here, we concentrate on the correlator which becomes

$$\begin{aligned}
 C(t, s; \mathbf{r} - \mathbf{r}') &= \left\langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \exp \left(-\mu \int d\mathbf{R} \int du \tilde{\phi}^2(u, \mathbf{R}) \phi(u, \mathbf{R}) \right) \right. \\
 &\quad \left. \times \exp \left(-\mu \rho_0 \int d\mathbf{R} \int du \tilde{\phi}^2(\mathbf{R}, u) \right) \right\rangle_0 \\
 &= -\mu \rho_0 \int d\mathbf{R} \int du \left\langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \tilde{\phi}^2(u, \mathbf{R}) \right\rangle_0 \\
 &\quad + \frac{\mu^2}{2} \int d\mathbf{R} d\mathbf{R}' \int du du' \left\langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \Upsilon(u, \mathbf{R}) \Upsilon(u', \mathbf{R}') \right\rangle_0
 \end{aligned} \tag{4.151}$$

using the field Υ , see Table 4.5. Hence the connected correlator is determined by three- and four-point functions of the noiseless theory.

The noiseless three-point response can be found from its covariance under the ageing algebra $\mathfrak{age}(d)$ [52]

$$\begin{aligned}
 \left\langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \tilde{\phi}^2(u, \mathbf{R}) \right\rangle_0 &= (t-s)^{x-\frac{1}{2}\tilde{x}_2} (t-u)^{-\frac{1}{2}\tilde{x}_2} (s-u)^{-\frac{1}{2}\tilde{x}_2} \\
 \times \exp \left(-\frac{\mathcal{M}(\mathbf{r}-\mathbf{R})^2}{2} \frac{1}{t-u} - \frac{\mathcal{M}(\mathbf{r}'-\mathbf{R})^2}{2} \frac{1}{s-u} \right) &\Psi_3(u_1, v_1) \Theta(t-u) \Theta(s-u)
 \end{aligned} \tag{4.152}$$

with

$$\begin{aligned}
 u_1 &= \frac{u}{t} \frac{[(s-u)(\mathbf{r}-\mathbf{R}) - (t-u)(\mathbf{r}'-\mathbf{R})]^2}{(t-u)(s-u)^2} \\
 v_1 &= \frac{u}{s} \frac{[(s-u)(\mathbf{r}-\mathbf{R}) - (t-u)(\mathbf{r}'-\mathbf{R})]^2}{(t-u)^2(s-u)}
 \end{aligned} \tag{4.153}$$

and an undetermined scaling function Ψ_3 . As the critical BCPD is described by a free field-theory, one can expect that $x = \tilde{x} = d/2$, hence $\tilde{x}_2 = d$, $x_\Upsilon = \frac{3}{2}d$ for the composite fields. Therefore the autocorrelator takes the general form $C(t, s) = s^{1-d/2} g_1(t/s) + s^{2-d} g_2(t/s)$ and for $d > 2$ the second term, which comes from the four-point function, merely furnishes a finite-time correction to scaling. Now, if one chooses in eq. (4.152) [589]

$$\Psi_3(u_1, v_1) = \Xi \left(\frac{1}{u_1} - \frac{1}{v_1} \right) \tag{4.154}$$

where Ξ remains an arbitrary function, then

$$\begin{aligned}
 C(t, s) &= -\mu \rho_0 s^{\frac{d}{2}+1-x-\frac{1}{2}\tilde{x}_2} (t/s-1)^{\frac{1}{2}\tilde{x}_2-x-\frac{d}{2}} \\
 &\quad \times \int_0^1 d\theta [(t/s-\theta)(1-\theta)]^{\frac{d}{2}-\frac{1}{2}\tilde{x}_2} \Psi \left(\frac{t/s+1-2\theta}{t/s-1} \right)
 \end{aligned} \tag{4.155}$$

where the function Ψ is defined by

$$\Psi(w) = \int_{\mathbb{R}^d} d\mathbf{R} \exp\left(-\frac{\mathcal{M}w}{2}\mathbf{R}^2\right) \Xi(\mathbf{R}^2) \quad (4.156)$$

and the LSI-prediction for the critical point (4.113) is indeed recovered. The specific result for the BCP as listed in Table 4.2 follows for $\Psi(w) = \Psi_0 w^{-1-a}$.

4.7.9 Bosonic Pair-contact Process

In this case, a new ingredient will be needed. The action now reads [379] $\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_0[\phi, \tilde{\phi}] + \mathcal{J}_b[\phi, \tilde{\phi}]$ with the ‘deterministic’ part

$$\mathcal{J}_0[\phi, \tilde{\phi}] := \int_{\mathbb{R}^d} d\mathbf{R} \int_0^\infty dt \left[\tilde{\phi}(2\mathcal{M}\partial_t - \nabla^2)\phi - \alpha\tilde{\phi}^2\phi^2 \right]. \quad (4.157)$$

and the noise part

$$\mathcal{J}_b[\phi, \tilde{\phi}] = \int_{\mathbb{R}^d} d\mathbf{R} \int_0^\infty du \left[-\alpha\rho_0^2\tilde{\phi}^2 - 2\alpha\rho_0\tilde{\phi}^2\phi - \mu\tilde{\phi}^3\phi^2 - 2\mu\rho_0\tilde{\phi}^3\phi - \rho_0^2\tilde{\phi}^3 \right]. \quad (4.158)$$

The standard representations of $\mathbf{age}(d)$ or $\mathbf{sch}(d)$ considered for the BCPD cannot be used, since the equation of motion associated to $\mathcal{J}_0$ is non-linear, viz.

$$2\mathcal{M}\partial_t\phi(t, \mathbf{r}) = \nabla^2\phi(t, \mathbf{r}) - g\phi^2(t, \mathbf{r})\tilde{\phi}(t, \mathbf{r}). \quad (4.159)$$

Rather, as mentioned in Sect. 4.5, new representations of $\mathbf{age}(d)$ and $\mathbf{sch}(d)$ must be constructed which take into account that g is a *dimensionful* quantity which transforms under local scale-transformations [686, 52], which is dealt with in more detail in exercise 4.22. It can then be shown that: (i) *eq. (4.159) is Schrödinger-invariant for any value³⁰ of d* and (ii) *the Bargman superselection rules (4.93) still apply*.

Using this new representation of $\mathbf{age}(d)$, it is easy to check that the two-time response function is indeed consistent with the exact solution [52].

Turning to the calculation of the autocorrelation function, one must consider five possible contributions. It turns out [52] that the results of Table 4.2 for $f_C(y)$ in the BPCPD can be reproduced from the single term

$$C(t, s) = \alpha\rho_0^2 \int_{\mathbb{R}^d} d\mathbf{R} \int_0^\infty du \left\langle \phi(t, \mathbf{r})\phi(s, \mathbf{r})\tilde{\phi}^2(u, \mathbf{R}) \right\rangle_0. \quad (4.160)$$

The required $\mathbf{age}(d)$ -invariant three-point function now reads, where $x = a + 1 = d/2$ is read off from the response function [52],

³⁰ If g is taken to be a dimensionless constant, it is a well-known mathematical fact that Schrödinger-invariance of (4.159) only holds true in $2D$ [267, 563], see exercise 4.21.

$$\begin{aligned} \left\langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \tilde{\phi}^2(u, \mathbf{R}) \right\rangle_0 &= (t-s)^{x-\frac{1}{2}\tilde{x}_2} (t-u)^{-\frac{1}{2}\tilde{x}_2} (s-u)^{-\frac{1}{2}\tilde{x}_2} \\ &\times \exp \left(-\frac{\mathcal{M}(\mathbf{r}-\mathbf{R})^2}{2} \frac{1}{t-u} - \frac{\mathcal{M}(\mathbf{r}'-\mathbf{R})^2}{2} \frac{1}{s-u} \right) \tilde{\Psi}_3(u_1, v_1, \beta_1, \beta_2, \beta_3) \end{aligned} \quad (4.161)$$

with

$$\begin{aligned} u_1 &= \frac{u}{t} \frac{[(s-u)(\mathbf{r}-\mathbf{R}) - (t-u)(\mathbf{r}'-\mathbf{R})]^2}{(t-u)(s-u)^2} \\ v_1 &= \frac{u}{s} \frac{[(s-u)(\mathbf{r}-\mathbf{R}) - (t-u)(\mathbf{r}'-\mathbf{R})]^2}{(t-u)^2(s-u)} \\ \beta_1 &= \frac{u}{t} \frac{\alpha^{1/y}}{(t-u)}, \quad \beta_2 = \frac{u}{t} \frac{(t-u)\alpha^{1/y}}{(s-u)^2}, \quad \beta_3 = \frac{t}{u} \frac{\alpha^{1/y}}{t-u} \end{aligned}$$

where y is the scaling dimension of g . Consider the following choice for $\tilde{\Psi}_3$ [52]

$$\tilde{\Psi}_3(u_1, v_1, \beta_1, \beta_2, \beta_3) = \Xi \left(\frac{1}{u_1} - \frac{1}{v_1} \right) \left[-\frac{(\sqrt{\beta_1} - \sqrt{\beta_2})\sqrt{\beta_3}}{\beta_3 - \sqrt{\beta_2}\beta_3} \right]^{(a-b)} \quad (4.162)$$

where the scaling function Ξ was already given in eq. (4.154), as for the bosonic contact process. We now have to distinguish the two different cases: first, if $\alpha < \alpha_C$ then $a = b$ [49] and we are back to the expressions found for the BCP. Second, if $\alpha = \alpha_C$ then $a - b = d/2 - 1$ for $2 < d < 4$ and $a - b = 1$ for $d > 4$, respectively [49]. Identifying $\tilde{x}_2 = 2(b-a) + d$, we finally obtain

$$\begin{aligned} C(t, s) &= s^{-b}(y-1)^{(b-a)-a-1} \int_0^1 d\theta [(t/s - \theta)(1-\theta)]^{a-b} \\ &\times \Psi \left(\frac{t/s + 1 - 2\theta}{t/s - 1} \right) \left[\frac{\theta(t/s - 1)}{(t/s - \theta)(1-\theta)} \right]^{a-b} \end{aligned} \quad (4.163)$$

The specific results of [Table 4.2](#) for the BPCP are recovered if one uses the free-field form $\Psi(w) = \Psi_0 w^{-1-a}$.

4.7.10 Reversible Reaction-diffusion Systems

For the RAC-model, one has a *pair* of coupled Langevin equations. We have not tried to present a generalised formulation of LSI which could treat *pairs* of equations of motion. Still, it is encouraging to see that the form of several of the autoresponses listed in [Table 3.12](#) is consistent with our LSI-prediction eq. (4.132). In particular, we recognise further examples of models with distinct ageing exponents $a \neq a'$.

4.7.11 Surface Growth: Edwards-Wilkinson Model

The deterministic part of the Langevin equation (3.156) of the Edwards-Wilkinson model is a simple diffusion equation. It is therefore only to be

expected that both responses and correlation functions agree with the predictions of LSI, where the consideration of short-ranged and long-ranged noise allows us to test the respective forms of the two-time correlator.

These expectations are completely borne out by the exact solution of the Edwards-Wilkinson equation. Furthermore, the same scaling forms for the two-time space-time correlators were also found in the Family model [629, 630]. We conclude that the Edwards-Wilkinson universality class is described by local scale-invariance.

This example might become useful since it suggests how one might identify quasi-primary scaling operators of LSI in more general models of surface growth. To what extent this can be practically done is still open.

4.8 Nonrelativistic AdS/CFT Correspondence

In Sect. 4.4, we discussed the embedding of the Schrödinger algebra $\mathfrak{sch}(d) \subset \mathfrak{conf}(d+2)$ into a conformal algebra. Besides the direct applications to aspects of ageing we have seen, it has been realised recently that this construction also serves as the main ingredient for a geometric formulation of Schrödinger-invariance, which is close in spirit to the celebrated **AdS/CFT correspondence**, much studied in string-theory contexts, which attempts to relate the behaviour of a conformal field-theory (CFT) to a ‘dual’ description of gravity in an Anti-de-Sitter (AdS) background [498, 11]. Here, we shall give a schematic overview of how a non-relativistic version of this correspondence might look [678, 33, 521], before briefly describing a possible application to the physics of cold atoms [264].

4.8.1 Holographic Construction

From our earlier discussion in Sect. 4.4, we recall that the extension of the Schrödinger algebra was carried out in practise through the formal Laplace transformation (4.42) by which we introduced an extra new coordinate ζ instead of the ‘mass’ $\mathcal{M}$. Together with the time t , ζ is viewed as one of the so-called ‘light-cone coordinates’ $(t, \zeta) = (\xi_-, \xi_+)$ with $\xi_{\pm} = (\xi_0 \pm \xi_{-1})/\sqrt{2}$, which belong to a $(d+2)$ -dimensional coordinate vector $\boldsymbol{\xi} = (\xi_{-1}, \xi_0, \xi_1, \dots, \xi_d)$. As we have seen, in these coordinates the Schrödinger/diffusion equation is recast as a Klein-Gordon/Laplace equation (4.46) in $d+2$ dimensions. The dynamical symmetry algebra of this Klein-Gordon equation is the conformal algebra $\mathfrak{conf}(d+2)$ and the Schrödinger algebra may be obtained as the sub-algebra of $\mathfrak{conf}(d+2)$ whose generators commute with the light-cone momentum operator $P_+ := (P_0 + P_{-1})/\sqrt{2}$ and where P_i stands for the momentum operator in the direction i . Of course $P_+ = i\partial_{\xi_+} \sim \partial_{\zeta}$ is nothing but the mass operator M_0 .

In $d = 1$ dimensions, we visualise this by reconsidering the root diagram Fig. 4.2. For the four generators $W, V_{\pm}, N$ which do not belong to $\mathfrak{sch}(1)$, $\text{ad } M_0$ applied to any of these will lead to another point in the root diagram, whereas the action of $\text{ad } M_0$ on the generators of $\mathfrak{sch}(1)$ would lead to a root vector outside of the diagram and the corresponding commutator vanishes.

Based on this algebraic embedding, one tries to make contact with the well-known geometric realisation of the conformal group in $d + 1$ dimensions in the **anti-de-Sitter space** AdS_{d+2} . This space is defined by the metric

$$ds^2 = \frac{1}{u^2} (du^2 + g_{IJ} d\xi^I d\xi^J), \quad (4.164)$$

where g_{IJ} is the metric tensor, the indices $I, J = -1, 0, 1, \dots, d$ and u is a further auxiliary coordinate. In order to obtain the Schrödinger algebra, one now deforms this metric in such a way that only the Schrödinger algebra with $z = 2$ survives as an algebra of isometries. This leads to the metric [678, 33]

$$ds^2 = \frac{1}{u^2} (du^2 + dr^j dr^j - 2d\zeta dt) - \beta^2 \frac{dt^2}{u^4} \quad (4.165)$$

with $j = 1, \dots, d$. The parameter β measures the deformation from the pure AdS-space. The isometries of this metric are given by (written in an Euclideanised form, directly comparable to the other explicit function-space representation in this volume)

$$\begin{aligned} M_0 &= -\partial_{\zeta} \quad , \quad X_{-1} = -\partial_t \quad , \quad R^{(jk)} = -(r_j \partial_k - r_k \partial_j) \\ Y_{-1/2}^{(j)} &= -\partial_j \quad , \quad Y_{1/2}^{(j)} = -t \partial_j - r_j \partial_{\zeta} \\ X_0 &= -t \partial_t - \frac{1}{2} r_j \partial_j - \frac{1}{2} u \partial_u \\ X_1 &= -t^2 \partial_t - t r_j \partial_j - t u \partial_u - \frac{1}{2} (r_j r_j + u^2) \partial_{\zeta} \end{aligned} \quad (4.166)$$

with the short-hand $\partial_j = \frac{\partial}{\partial r_j}$. These generators form indeed a representation of $\mathfrak{sch}(d)$. Representations of this kind can also be found when discussing an extension of Schrödinger-invariance to semi-linear differential equations $(2\mathcal{M}\partial_t - \Delta_L)\phi = g\phi^n$ by realising that g is a dimensionful constant and should be transformed as well. Representations of the three parabolic and almost parabolic sub-algebras of $\mathfrak{conf}(3)$ and their invariant equations have been systematically constructed (where the coupling g takes the place of the auxiliary coordinate u) [686], see also p. 256. Further extensions to several couplings and applications to Bose-Einstein condensation are outlined in [685].

Furthermore, from these correspondences one may try to establish a dictionary between the Schrödinger-invariant theory and the ‘dual’ gravity version on AdS-space. For the correlators, this correspondence is thought to be of the form [264]

$$\begin{aligned} & \langle \phi_1(t_1, \mathbf{r}_1) \dots \phi_n(t_n, \mathbf{r}_n) \rangle_{\text{Sch}} = \\ & \int \prod_{j=1}^n d\zeta_j e^{-i(\mathcal{M}_1 \zeta_1 + \dots \mathcal{M}_n \zeta_n)} \langle \phi_1(\zeta_1, t_1, \mathbf{r}_1) \dots \phi_n(\zeta_n, t_n, \mathbf{r}_n) \rangle_{\text{AdS}} \end{aligned} \quad (4.167)$$

This relationship is indeed completely natural from the point of view of the Fourier/Laplace transform (4.42) which we used before to relate the Schrödinger and conformal algebras. It has been confirmed at the *tree level* for an interacting complex scalar field, but cannot be extended to the one-loop level [264].

As long as (4.167) holds true, one may for example use the conformal invariance of the relativistic AdS theory to find the two- or three-point function at the tree level and then transform via (4.167) to the non-relativistic theory. We have already used this procedure in order to derive the *causal* forms (4.63, 4.64) of the Schrödinger-covariant two- and three-point functions [357], see exercise 4.11. As we have seen, the predictions of conformal invariance in $d + 2$ dimensions and the explicit two- and three-point functions found from Schrödinger-invariance in d dimensions [327] are perfectly consistent.

4.8.2 Relationship with Cold Atoms

As a specific illustration, we now consider a non-relativistic field-theory of *cold atoms* with contact interactions [264, 550]. The model is described in terms of a two-component fermion with the action (here written in a Euclidean time)

$$S[\psi] = \int dt d\mathbf{r} \left[\sum_{i=1}^2 \psi_i^* \left(\partial_t - \frac{\Delta_L}{2m} - \mu \right) \psi_i - c_0 \psi_1^* \psi_2^* \psi_2 \psi_1 \right] \quad (4.168)$$

where m is the fermion mass, μ denotes the chemical potential and c_0 is the interaction coupling. In order to carry out an explicit test of the holographic construction outlined above, one first performs a Hubbard-Stratonovich transformation, which leads to the action

$$S[\psi, \phi] = \int dt d\mathbf{r} \left[\sum_{i=1}^2 \psi_i^* \left(\partial_t - \frac{\Delta_L}{2m} - \mu \right) \psi_i + c_0^{-1} \phi^* \phi - (\phi^* \psi_1 \psi_2 + \phi \psi_1^* \psi_2^*) \right] \quad (4.169)$$

where ϕ is a composite boson field of mass $2m$. From this, through the Lippmann-Schwinger equations one may find the *exact* Euclidean propagators, with the result [550]

$$G_\psi(\omega, \mathbf{q}) = \frac{1}{i\omega + \varepsilon(\mathbf{q})} \quad , \quad G_\phi(\omega, \mathbf{q}) = \frac{g_0}{(i\omega + \varepsilon(\mathbf{q})/2)^{d/2-1}} \quad (4.170)$$

with the dispersion relation $\varepsilon(\mathbf{q}) = q^2/2m$ and g_0 is a known constant. In particular, one may read off the scaling dimensions $x_\psi = d/2$ and $x_\phi = 2$.³¹

³¹ For a free field, one would have expected $x_\phi = d$.

From these expressions, the two-point and especially the three-point functions can be found. In particular, the explicit calculation of the three-point function $\langle \psi \psi \phi^* \rangle$, which describes the scattering of two atoms ψ into a di-atom ϕ , can be fully carried out.³² Not only is the result in complete agreement with the prediction (4.64) of Schrödinger-invariance, but also the usually undetermined scaling function $\Psi_{\psi\psi,\phi}$ has been explicitly obtained [264]. The result confirms the holographic correspondence (4.167).

Problems

4.1. Show that the projective conformal transformations discussed in Volume 1, Chap. 2, when specialised to $2D$, are exactly the **Möbius transformations**

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta} \quad ; \quad \alpha\delta - \beta\gamma = 1$$

and similarly for $\bar{z}$.

4.2. Show that the $2D$ Laplace equation $\Delta_L \phi = 0$ is conformally invariant for scalar fields with scaling dimension $x_\phi = 0$.

4.3. Find the group transformations generated by the infinitesimal generators of the Schrödinger-Virasoro algebra $\mathfrak{sv}(d)$ [357].

4.4. Consider in d spatial dimension the Schrödinger operator

$$\mathcal{S} = 2M_0 X_{-1} - \mathbf{Y}_{-1/2} \cdot \mathbf{Y}_{-1/2} \cdot$$

Check that $\mathcal{S}$ is a Casimir operator of the Galilei subalgebra $\mathfrak{gal}(d) := \langle X_{-1}, \mathbf{Y}_{\pm 1/2}, R_0^{(jk)}, M_0 \rangle$. Then show that the Schrödinger/diffusion equation $\mathcal{S}\phi = 0$ has the algebra $\mathfrak{sch}(d)$ as a dynamical symmetry and in particular find the scaling dimension x of the solution ϕ .

4.5. Consider the dynamical symmetries of a d -dimensional Schrödinger equation with an external potential $V = V(t, \mathbf{r})$, of the form

$$\left(i\partial_t + \frac{1}{2m} \Delta_L - V(t, \mathbf{r}) \right) \phi(t, \mathbf{r}) = 0 \quad . \quad (4.171)$$

Show that the dynamical group with elements $g : (t, \mathbf{r}) \mapsto g(t, \mathbf{r})$ and the projective representation $\phi \mapsto T_g \phi$ with $(T_g \phi)(t, \mathbf{r}) = f_g(g^{-1}(t, \mathbf{r})) \phi(g^{-1}(t, \mathbf{r}))$ is given by [546]

$$g(t, \mathbf{r}) = \left(\int \frac{dt}{e(t)^2}, \frac{\mathfrak{D}\mathbf{r} + \mathbf{s}(t)}{e(t)} \right) \quad (4.172)$$

³² A similar test can be carried out for $\langle \phi \psi^* \psi^* \rangle$, see exercise 4.6.

where $\mathfrak{D}$ is a rotation matrix and $e(t)$ and $\mathbf{s}(t)$ are time-dependent scalars and vectors. Find the explicit expression for the companion function and show that the potential must satisfy

$$\begin{aligned} V(g(t, \mathbf{r})) - e^2 V(t, \mathbf{r}) &= \frac{m}{2} e \ddot{\mathbf{r}}^2 + m(\mathfrak{D} \mathbf{r}) \cdot (e \dot{\mathbf{s}} - e^2 \ddot{\mathbf{s}}) \\ &\quad + \frac{m}{2} (e \ddot{\mathbf{s}}^2 - e^2 \ddot{\mathbf{s}} \cdot \mathbf{s} + e^2 \dot{\ell}) \end{aligned} \quad (4.173)$$

which also determines the admissible form of $e = e(t)$. What is the maximal dimension of the dynamic group ?

For $V = 0$, $V = -m\mathbf{g} \cdot \mathbf{r}$ and $V = \frac{1}{2}\omega \mathbf{r}^2$, show that the resulting dynamical group is isomorphic to the Schrödinger group as defined in the text [546].

4.6. Check the explicit expression (4.32) of the three-point function of a Schrödinger-invariant system.

4.7. For **age**-quasi-primary scaling operators, derive the autoresponse (4.101) directly, without invoking a gauge transformation.

4.8. Calculate the two-time function for a Schrödinger-invariant system defined on a half-space $\mathbb{H}^{d-1,1} \subset \mathbb{R}^d$ with coordinates $\mathbf{r} = (\mathbf{r}_{\parallel}, r_{\perp})$ such that $r_{\perp} \geq 0$ and where furthermore the order-parameter vanishes at the surface, viz. $\phi(\mathbf{r}_{\parallel}, 0) = 0$. Use the bulk two-point function and the method of images [327]. Try to generalise to ageing-invariant systems.

4.9. Use the geometric definition of the parabolic subalgebra via the root diagram to give a rapid heuristic version of the classification of the parabolic subalgebras of $B_2 \cong (\mathfrak{conf}(3))_{\mathbb{C}}$, listed on p. 243.

4.10. After having introduced the ‘dual’ coordinate ζ through the Fourier transformation (4.43), derive the group transformations from the infinitesimal generators (4.44) of the Schrödinger-Virasoro algebra $\mathfrak{sv}(1)$ [357].

4.11. Derive the causal form (4.63) for the two-time function from its covariance under the conformal algebra $\mathfrak{conf}(d+2)$. Generalise to the three-point function (4.64).

4.12. Check (4.84).

4.13. Construct the improved energy-momentum tensor for a free-field theory with Lagrangian (4.80) which is Schrödinger- and conformally invariant.

4.14. Consider **age**-quasi-primary scaling operators ϕ_i with positive masses $\mathcal{M}_i > 0$. Show that any co-variant *correlation* function $\langle \phi_1 \cdots \phi_n \rangle = 0$.

4.15. In relativistic quantum mechanics, one way of introducing the **Dirac equation** considers a relativistic wave equation $\mathcal{K}\psi := (c^{-2}\partial_t^2 - \Delta_L)\psi = 0$. The Dirac equation $\mathcal{D}\psi = 0$ is then obtained by formally attempting to take the ‘square root’ $\mathcal{K} = \mathcal{D}^2$ of the Klein-Gordon operator which can be done only when $\mathcal{D}$ is interpreted as a matrix differential operator.

Apply an analogous idea to the free Schrödinger equation $\mathcal{S}\psi = 0$, with the Schrödinger operator $\mathcal{S} = 2\mathcal{M}\partial_t - \nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}}$ and construct the **Dirac-Lévy-Leblond equations** $\mathcal{R}\Psi = 0$ from the condition $\mathcal{R}^2 = \mathcal{S}$ [469]. Write explicitly the case $d = 1$. Is the wave function Ψ still a scalar ?

4.16. Show that the dynamical symmetry algebra of the Dirac-Lévy-Leblond equations derived in the previous exercise 4.15 is isomorphic to $\mathfrak{conf}(d+2)$. Discuss the case $d = 1$ explicitly [358].

4.17. Let $\Psi_i = \begin{pmatrix} \psi_i \\ \phi_i \end{pmatrix}$, $i = 1, 2$, be two quasi-primary spinors under the spinorial representation of $\mathfrak{conf}(3)$ constructed in the previous exercise 4.16, with scaling dimensions $\begin{pmatrix} x_i \\ x_i + 1 \end{pmatrix}$ of the component fields. Find the two-point functions of these which are co-variant under the extended Schrödinger algebra $\widetilde{\mathfrak{sch}}(1)$ or the conformal algebra $\mathfrak{conf}(3)$ [358].

From translation-invariance it is clear that these will only depend on $\zeta = \zeta_1 - \zeta_2$, $t = t_1 - t_2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

4.18. Convert the results for the Schrödinger-covariant spinorial two-point functions, derived in the previous exercise 4.17, to scaling operators with fixed masses $\mathcal{M} > 0$ and derive the causality conditions [358].

4.19. Consider the non-linear Schrödinger equation $\mathcal{S}\phi(t, \mathbf{r}) = F(t, \mathbf{r}, \phi, \phi^*)$, where furthermore the Schrödinger operator satisfies a dynamical symmetry, with the infinitesimal generator $\mathcal{X}\phi(t, \mathbf{r}) = (a(t, \mathbf{r})\partial_t + \mathbf{b}(t, \mathbf{r}) \cdot \partial_{\mathbf{r}} + c(t, \mathbf{r}))\phi(t, \mathbf{r})$, such that $[\mathcal{S}, \mathcal{X}] = \lambda(t, \mathbf{r})\mathcal{S}$. Show that the above non-linear Schrödinger equation has $\mathcal{X}$ as a dynamical symmetry if [90]

$$\begin{aligned} & [a(t, \mathbf{r})\partial_t + \mathbf{b}(t, \mathbf{r}) \cdot \partial_{\mathbf{r}} - c(t, \mathbf{r})\phi\partial_{\phi} - c^*(t, \mathbf{r})\phi^*\partial_{\phi^*} \\ & + (c(t, \mathbf{r}) + \lambda(t, \mathbf{r}))]F(t, \mathbf{r}, \phi, \phi^*) = 0. \end{aligned} \quad (4.174)$$

4.20. What is the most general form of a Galilei-invariant non-linear Schrödinger equation $\mathcal{S}\phi(t, \mathbf{r}) = F(t, \mathbf{r}, \phi, \phi^*)$, where one assumes that F does not contain any derivatives of ϕ, ϕ^* ? Discuss the case of fixed masses $\mathcal{M}$ as well as the dual formulation $\mathcal{S}\psi(\zeta, t, \mathbf{r}) = F(\zeta, t, \mathbf{r}, \psi, \psi^*)$ [90, 267, 563, 686].

4.21. How are the results of the previous exercise 4.20 modified for Schrödinger-invariant non-linear Schrödinger/diffusion equations ? [90, 267, 563] When working with the dual variable ζ , find the invariant forms also for the ageing and altern algebras $\mathfrak{age}$ and $\mathfrak{alt}$ [686].

4.22. Construct representations of the ageing algebra $\mathfrak{age}(1)$, where the dilatation generator contains an additional dimensionful coupling g and reads

$$X_0 = -t\partial_t - \frac{1}{2}r\partial_r - yg\partial_g - \frac{x}{2} \quad (4.175)$$

Can one extend this construction to the Schrödinger algebra $\mathfrak{sch}(d)$ and the conformal Galilean/altern algebra $\mathfrak{alt}(1) \equiv \mathfrak{cga}(1)$ [686, 52] ?

4.23. Prove eq. (4.140). For $z \neq 2$, this generalises to eq. (5.64).

4.24. Consider a field-theory given by the action

$$\mathcal{J}[\phi, \tilde{\phi}] = \int dt d\mathbf{r} \left[\tilde{\phi}(2\mathcal{M}\partial_t - \Delta_L)\phi + v(t)\tilde{\phi}\phi \right]$$

which contains a time-dependent potential $v(t)$. Which form of $v(t)$ is consistent with $\mathfrak{age}(d)$ -invariance ?

Chapter 5

Local Scale-invariance II: $z \neq 2$

“We are not satisfied with the hollow victory of falsification. Having a beautiful idea that nearly succeeds, we look to improve it, by finding a still more beautiful version that works in detail. We seek truthification.”

Frank Wilczek, *Anticipating a new golden age* (2008) [734]

Having considered an extension of dynamical scaling for the special case $z = 2$, we now inquire whether a similar extension might be possible, and of physical relevance, for an arbitrary value of z .

5.1 Axioms of Local Scale-invariance

Given the practical success of both conformal ($z = 1$) and Schrödinger/ageing ($z = 2$) invariance, we shall try to remain as close as possible to these. Specifically, our attempted construction is based on the following requirements which are the defining axioms of our notion of **local scale-invariance** (LSI) [330]. For simplicity of notation, we shall formulate these in $d = 1$ space dimensions, extensions to $d > 1$ being obvious.

1. For both conformal and Schrödinger-invariance, Möbius-transformations play a prominent role. We shall thus seek space-time transformations such that the time coordinate undergoes a Möbius-transformation

$$t \rightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta} \quad ; \quad \alpha\delta - \beta\gamma = 1 . \quad (5.1)$$

If we call the infinitesimal generators of these transformations X_n , ($n = -1, 0, 1$), we require that even after the transformations on the spatial coordinates $\mathbf{r}$ are included, the conformal commutation relations

$$[X_n, X_m] = (n - m)X_{n+m} \quad (5.2)$$

remain valid. Scaling operators which transform covariantly under (5.1) are called **quasi-primary**.

2. The generator X_0 of scale transformations is

$$X_0 = -t\partial_t - \frac{1}{z}r\partial_r - \frac{x}{z} \quad (5.3)$$

where x is the scaling dimension of the quasi-primary operator on which X_0 is supposed to act.

3. Spatial translation invariance is required.
4. If time-translations are to be included, their infinitesimal generator is $X_{-1} = -\partial_t$.
5. When acting on a quasi-primary operator ϕ , extra terms coming from the scaling dimension of ϕ must be present in the generators and be compatible with (5.3).
6. By analogy with the ‘mass’ terms contained in the generators (4.7) for $z = 2$, mass terms constructed such as to be compatible with $z \neq 1, 2$ should be expected to be present.
7. We shall test the notion of local scale invariance by calculating two-point functions of quasi-primary operators and comparing them with explicit model results. We require that the generators when applied to a quasi-primary two-point function will yield a *finite* number of independent conditions.

The simplest way to satisfy the last condition is the requirement that the generators applied to a two-point function provide a realisation of a finite-dimensional Lie algebra. However, more general ways of finding non-trivial two-point functions are possible.

By the second axiom, two important constraints are imposed: (i) typical time scales τ and typical length scales ξ must satisfy an algebraic relation $\tau \sim \xi^z$ which in particular means that time-dependent length scales grow algebraically $L(t) \sim t^{1/z}$; (ii) cross-over effects are not included and we always assume the most simple scaling behaviour. The geometric effects of local scale-transformations are illustrated in Fig. 5.1 for the case $z = 2$, but remains qualitatively similar for any value of $z \neq 1$. In contrast to conformal transformations, local scale-transformations generically produce a shear in space-time $(t, \mathbf{r})$.

5.2 Construction of the Infinitesimal Generators

5.2.1 Generators Without Mass Terms

The general ansatz [330] for the construction of the generators X_n is

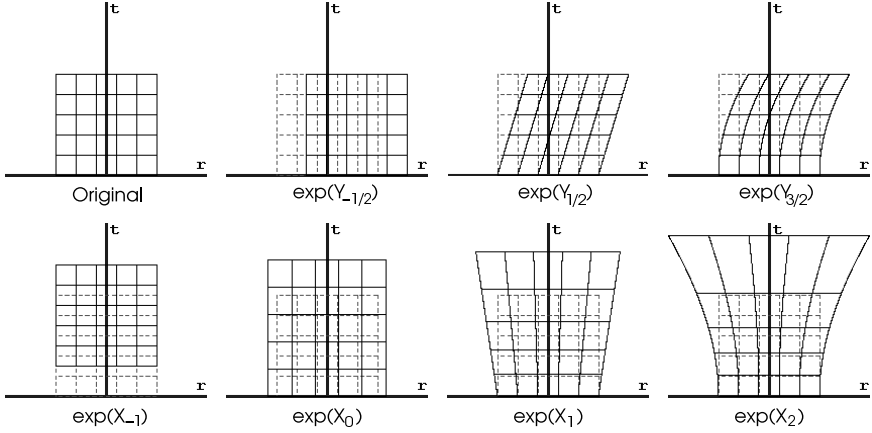


Fig. 5.1. Geometry of several Schrödinger-transformations in $d = 1$ space dimensions, as obtained in the integrated form. Reprinted from [366]. Copyright (2008) Institute of Physics Publishing.

$$X_n = X_n^{(I)} + X_n^{(II)} + X_n^{(III)} \quad (5.4)$$

where $X_n^{(I)} = -t^{n+1}\partial_t$ is the infinitesimal form of (5.2), $X_n^{(II)}$ will contain the action on $\mathbf{r}$ and the scaling dimension x and $X_n^{(III)}$ will contain the ‘mass’ terms. Likewise, spatial translations will generate a family of generators Y_m , for which we assume the form

$$Y_m = Y_m^{(II)} + Y_m^{(III)} \quad (5.5)$$

where $Y_m^{(II)}$ contains the action on $\mathbf{r}$ and $Y_m^{(III)}$ contains the ‘mass’ terms.

The parts $X_n^{(II)}$ and $Y_m^{(II)}$ with $m = k - 1/z$ are constructed from the above axioms, using the ansatz, again for $d = 1$ for simplicity,

$$X_n^{(II)} = a_n(t, r)\partial_r + b_n(t, r), \quad Y_{k-1/z}^{(II)} = -\frac{z}{k+1} \left(\frac{\partial a_k(t, r)}{\partial r} \partial_r + \frac{\partial b_k(t, r)}{\partial r} \right) \quad (5.6)$$

where $a_n(t, r)$ and $b_n(t, r)$ have to be determined and $n, k \in \mathbb{N}$. The following classification is then found.

Theorem. [330] *For a given dynamical exponent z , the commutators*

$$[X_n, X_{n'}] = (n - n')X_{n+n'}, \quad [X_n, Y_m] = \left(\frac{n}{z} - m \right) Y_{n+m} \quad (5.7)$$

hold true for $m = k - 1/z$ and $n, n', k \in \mathbb{Z}$. These commutators are indeed necessary for the axioms 1) – 4) of local scale-invariance to be valid. Then the only sets of generators which satisfy these four axioms are given in Table 5.1.

This is proven through a fairly direct, if lengthy, analysis of the commutators generated by the ansatz (5.6). A few comments are in order.

(i)	$X_n^{(I)} + X_n^{(II)} = -t^{n+1}\partial_t - \frac{n+1}{z}t^n r \partial_r - \frac{(n+1)x}{z}t^n - \frac{n(n+1)}{2}B_{10}t^{n-1}r^z$ $Y_{-1/z+k}^{(II)} = -t^k \partial_r - \frac{z^2}{2}k B_{10}t^{k-1}r^{-1+z}$	z
(ii)	$X_n^{(I)} + X_n^{(II)} = -t^{n+1}\partial_t - \frac{1}{2}(n+1)t^n r \partial_r - \frac{1}{2}(n+1)xt^n$ $- \frac{n(n+1)}{2}B_{10}t^{n-1}r^2 - \frac{(n^2-1)n}{6}B_{20}t^{n-2}r^4$ $Y_{k-1/2}^{(II)} = -t^k \partial_r - 2kB_{10}t^{k-1}r - \frac{4}{3}k(k-1)B_{20}t^{k-2}r^3$	2
(iii)	$X_n^{(I)} + X_n^{(II)} = -t^{n+1}\partial_t - A_{10}^{-1}[(t + A_{10}r)^{n+1} - t^{n+1}]\partial_r$ $- (n+1)xt^n - \frac{n+1}{2}\frac{B_{10}}{A_{10}}[(t + A_{10}r)^n - t^n]$ $Y_{k-1}^{(II)} = -(t + A_{10}r)^k \partial_r - \frac{k}{2}B_{10}(t + A_{10}r)^{k-1}$	1

Table 5.1. Generators X_n and $Y_{-1/z+k}$ of local scale-transformations with a given dynamical exponent z without mass terms, where $n, k \in \mathbb{Z}$ (here for $d = 1$) and B_{10}, B_{20} and A_{10} are constants. The last column gives the dynamical exponent.

1. In case (i), both the dynamical exponent z and the constant B_{10} can be freely chosen. The case $B_{10} = 0$ was already found in [328] and in this case the algebra closes, since then we have $[Y_m, Y_{m'}] = 0$.
2. In case (ii) we recover for $B_{20} = 0$ the Schrödinger-Virasoro algebra $\mathfrak{sv}(1)$ with $B_{10} = \mathcal{M}/2$. For $B_{20} \neq 0$, additional generators must be written down to close the Lie algebra, see exercise 5.1 [330].
3. Case (iii) gives an example of local scaling with $z = 1$ which is inequivalent to standard conformal invariance. We find $[Y_n, Y_m] = A_{10}(n - m)Y_{n+m}$ and the Lie algebra is isomorphic to the Lie algebra $\langle \ell_n, \bar{\ell}_n \rangle_{n \in \mathbb{Z}}$ of $2D$ infinitesimal conformal transformations, see exercise 5.2. For $A_{10} \neq 0$, exercise 5.3 gives an invariant differential equation, distinct from the conformally invariant Laplace equation, and the co-variant two- and three-point functions are derived in exercise 5.4.

The limit $A_{10} \rightarrow 0$ corresponds to a non-relativistic limit. For the finite-dimensional Lie subalgebra, a Lie algebra contraction leads to the algebra $\text{CGA}(1) = \mathfrak{alt}(1)$, which is either called the **conformal Galilean algebra** [322, 533, 532] or the **altern algebra** [328] (as we also did in Chap. 4). In this book, both names will be used synonymously. The infinite-dimensional extension of $\mathfrak{alt}(1)$ is called the **altern-Virasoro algebra** $\mathfrak{altv}(1)$, see exercise 5.5, with an obvious generalisations to $d \geq 1$. The analogue of conformal field-theory, but with $\mathfrak{altv}(1)$ in place of $\mathfrak{vir} \oplus \mathfrak{vir}$, is studied in [503, 31].

Exercises 5.5 to 5.7 treat the derivation of co-variant two- and three-point functions in several inequivalent representations of $\mathfrak{alt}(1) = \text{CGA}(1)$ and their extensions to $d \geq 1$. In two space dimensions, there exists a so-called ‘exotic’ extension of $\text{CGA}(2)$, which is called the **exotic conformal Galilean algebra** which we shall denote by **ECGA** [483, 484]. The precise definition is given in exercises 5.8 and 5.9 which also treat simple invariant linear equations and the co-variant two-point functions under this algebra. While the algebra **ECGA** certainly has an interesting mathematical structure which might become useful in string theory, it turned out that the standard equations of hydrodynamics do not accept **ECGA** as a dynamical symmetry [754, 146].¹

We point out that the ansatz (5.6) excludes the standard conformal transformations, but these can be recovered by adding suitable ‘mass’ terms to the generators [330], as we shall describe in Chap. 6.

5.2.2 On Geometrical Interpretations of Local Scaling

We now sketch the recent classification of symmetry algebras of non-relativistic space-time by Duval and Horváthy [221]. This provides a geometric setting for the local scale-transformations discussed above and also illustrates in what sense these transformations may be considered ‘conformal’. We shall restrict ourselves here to the flat case only and refer to [221] for the general case.

Recall that a **Newton-Cartan spacetime** is made from a smooth manifold $M = \mathbb{R} \times \mathbb{R}^d$, a symmetric tensor field $\gamma = \gamma^{jk} \partial_j \otimes \partial_k$, whose kernel is spanned by the one-form $\theta_j \text{dr}^j$, and a connection Γ . In the flat case, one has

$$\gamma^{jk} = \delta_j^k \delta_K^{JK} \quad , \quad \theta_j = \delta_j^0 \quad , \quad \Gamma_{jk}^\ell = 0 \quad (5.8)$$

with $j, k, \ell = 0, 1, \dots, d$ and $J, K = 1, \dots, d$. The tensor γ^{jk} is the metric tensor and θ_j acts as a Galilean clock. Starting from the conformal invariance of a Lorentzian manifold and then going over to the non-relativistic case, the ‘conformal invariance condition’ is taken in the form $L_X(\gamma^{\otimes m} \otimes \theta^{\otimes n}) = 0$, which can be rewritten as [221]

$$L_X \gamma = f \gamma \quad , \quad L_X \theta = g \theta \quad , \quad f + qg = 0, \quad (5.9)$$

where $q := n/m$. Here L_X is the **Lie derivative** along the **vector field** $X = X^j \partial_j$. Indeed, the condition $L_X \gamma = f \gamma$ is the familiar geometric formulation of conformal invariance in relativistic space-time, see e.g. [231].² The **dynamical**

¹ Considering **ECGA** as a *conditional* symmetry, several systems of invariant non-linear equations can be found, but whose physical interpretation remains to be discovered. For example, the system $\nabla \cdot \mathbf{v} = 0$, $(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} - q \nabla \wedge \boldsymbol{\omega} = \mathbf{0}$ with vectors $\mathbf{v} = (v_x, v_y, 0)$ and $\boldsymbol{\omega} = (0, 0, \omega)$ is, in two space dimensions, conditionally invariant under a certain representation of **ECGA** [146].

² Explicitly, $L_X \gamma^{jk} = X^\ell \partial_\ell \gamma^{jk} - 2 \partial_\ell X^j \gamma^{k\ell} - 2 \partial_\ell X^k \gamma^{j\ell}$ and $L_X \theta_j = \partial_j (\theta_k X^k)$.

exponent is defined by $z := 2/q = 2m/n$. The corresponding vector fields form the **conformal Galilei Lie algebra with dynamical exponent z** , denoted $\mathbf{cgal}_z(d)$, and read (with $\partial_J := \partial/\partial r^J$)

$$X = \beta(t)\partial_t + [\omega_K^J(t)r^K + z^{-1}\dot{\beta}(t)r^J + \alpha^J(t)]\partial_J. \quad (5.10)$$

This is the extension of case (i) in Table 5.1 with $B_{10} = 0$ to dimensions $d \geq 1$. The limit $z \rightarrow \infty$ can be formally taken in (5.10) which leads to the generators of the Lie algebra $\mathbf{cgal}_\infty(d)$.

Next, one considers the co-variance of the geodesics equation

$$\frac{d^2 r^\ell}{d\tau^2} + \Gamma_{jk}^\ell \frac{dr^j}{d\tau} \frac{dr^k}{d\tau} = \mu \frac{dr^\ell}{d\tau} \quad (5.11)$$

where μ is a smooth function and τ is a curve parameter (i.e. arc length). Geodesics are called **time-like** when $\frac{dt}{d\tau} = \theta_j \frac{dr^j}{d\tau} \neq 0$ and are called **light-like** when $\frac{dt}{d\tau} = \theta_j \frac{dr^j}{d\tau} = 0$.

A detailed analysis now shows that in the time-like case, one recovers the generators of the Schrödinger-Virasoro algebra $\mathfrak{sv}(d)$. On the other hand, in the light-like case one obtains the generators of the **conformal Newton-Cartan algebra**, denoted as $\mathbf{cnc}(d)$, which read

$$X = \beta(t)\partial_t + [\omega_K^J(t)r^K + \varepsilon(t)r^J + \alpha^J(t)]\partial_J. \quad (5.12)$$

The maximal finite-dimensional subalgebra is called the **conformal Milne algebra** and is denoted by $\mathbf{cmil}(d)$. Therein, time and space can still be transformed separately. If one furthermore requires a fixed dynamical exponent z , the corresponding Lie subalgebras are the Schrödinger algebra $\mathbf{sch}(d)$ for $z = 2$ and the conformal Galilean algebra $\mathbf{cga}(d) = \mathbf{alt}(d)$ for $z = 1$. The classification is completed by adding the algebra $\mathbf{cnc}_\infty(d)$ to the list [221].

Comparing this geometry-based classification with the results listed in Table 5.1, we note that while a fixed dynamical exponent z was an essential ingredient in the derivation of Table 5.1 [330], this was not a necessary element in the classification of [221], which in this sense is more general and also considers $d \geq 1$ space dimensions. On the other hand, the analysis in [221] is restricted to vector fields *only* and does not even allow terms which describe a scaling dimension or a phase (as needed for example in the Schrödinger algebra). Hence in case (iii), only the limit $A_{10} \rightarrow 0$ is included. It has also become clear that the above list of space-time algebras is in a certain sense already exhaustive, at least if one insists on using function-space representations in terms of vector fields. In exercise 5.10, we explicitly construct a further Lie algebra of space-time transformations which realises $z = \frac{3}{2}$, but the generators are no longer vector fields.

Indeed, it will be one of the conceptual results of this and the following chapter that for dynamical exponents $z \neq 1, 2$ one must go beyond the too restrictive setting of vector fields in order to be able to construct non-trivial Lie algebras of local scale-transformations.

5.2.3 Generators With Generalised Mass Terms

We now list the complete generators, which include generalised ‘mass terms’ for the ‘Galilei-’ and ‘special’ generators $Y_{1-1/z}$ and X_1 [47, 44, 48]

$$\begin{aligned}
 X_{-1} &:= -\partial_t && \text{time translation} \\
 X_0 &:= -t\partial_t - \frac{1}{z}(\mathbf{r} \cdot \partial_{\mathbf{r}}) - \frac{x}{z} && \text{scale dilatation} \\
 X_1 &:= -t^2\partial_t - \frac{2(x+\xi)}{z}t - \mu r^2 \nabla_{\mathbf{r}}^{2-z} - \frac{2}{z}t(\mathbf{r} \cdot \partial_{\mathbf{r}}) && \text{special LSI-} \\
 &\quad -2\gamma(2-z)(\mathbf{r} \cdot \partial_{\mathbf{r}})\nabla_{\mathbf{r}}^{-z} - \gamma(2-z)(d-z)\nabla_{\mathbf{r}}^{-z} && \text{transformation} \\
 Y_{-1/z}^{(i)} &:= -\partial_{r_i} && \text{space translation} \\
 Y_{-1/z+1}^{(i)} &:= -t\partial_{r_i} - \mu z r_i \nabla_{\mathbf{r}}^{2-z} - \gamma z(2-z)\partial_{r_i} \nabla_{\mathbf{r}}^{-z} && \text{generalised Galilei-} \\
 &&& \text{transformation} \\
 R^{(i,j)} &:= r_i \partial_{r_j} - r_j \partial_{r_i} && \text{rotation} \quad (5.13)
 \end{aligned}$$

Here and in what follows, we use the fractional derivative in terms of the pseudodifferential operator given in appendix J. Again, the infinitesimal change of a **quasi-primary** scaling operator ϕ is given by one of these generators, or its (iterated) commutators. Hence, *a quasi-primary operator is characterised by the quartet (x, ξ, μ, γ)* . In order to understand the parameter ξ , some remarks are in order. The parameters x, ξ are the *two* scaling dimensions which were already introduced in Sect. 4.3 for ageing invariance (where $z = 2$). For a system in a stationary state, where time-translation-invariance holds, the commutator $[X_1, X_{-1}] = 2X_0$ implies $\xi = 0$. On the other hand, the single ‘mass’ $\mathcal{M}$ of Schrödinger/ageing-invariance is replaced by two dimensionful ‘mass’ parameters μ, γ .

The physical meaning of ξ can be understood as follows [355], by generalising slightly our previous discussion for the case $z = 2$, in Sect. 4.3. Suppressing the spatial dependence of the observables for a moment, the exponentiation of the generators $\langle X_n \rangle_{n \geq 0}$ leads to the following transformation law of a primary scaling operator $\phi(t)$ with scaling dimensions $x = x_\phi$ and $\xi = \xi_\phi$

$$\phi(t) = \dot{\beta}(t')^{-x/z} \left(\frac{t' \dot{\beta}(t')}{\beta(t')} \right)^{-2\xi/z} \phi'(t') \quad (5.14)$$

where we have written $t = \beta(t')$ and also require that $\beta(0) = 0$. The dot denotes the derivative with respect to time. For quasi-primary scaling operators, we restrict to $\beta(t) = t\delta^{-1}/(\gamma t + \delta)$. A primary operator in the usual sense (i.e. $\xi = 0$) may be defined as

$$\Phi(t, \mathbf{r}) := t^{-2\xi/z} \phi(t, \mathbf{r}). \quad (5.15)$$

which has effective scaling dimensions $x_\Phi = x + 2\xi$ and $\xi_\Phi = 0$. As we already saw in the case $z = 2$, the order-parameter ϕ and the primary scaling operator Φ are not necessarily identical. Since for critical systems, one generically expects that $z \neq 2$, this distinction will be even more important than for Schrödinger- or ageing-invariance.

5.2.4 Some Basic Facts

Following essentially the same steps as in Chap. 4, we begin with an analysis of the ‘deterministic’ part for which we now list some basic properties. These can be proven by straightforward calculations.³ In the next section, we inquire how to generalise the Bargman superselection rules, needed for the treatment of stochastic systems.

Fact 1: [44, 48] *The generators (5.13) satisfy the following commutation relations, with $n \in \{-1, 0, 1\}$, $m \in \{-1/z, -1/z + 1\}$, and $i, j = 1, \dots, d$:*

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'} \\ [X_n, Y_m^{(i)}] &= \left(\frac{n}{z} - m\right)Y_{n+m}^{(i)} \\ [Y_m^{(i)}, R^{(i,j)}] &= -[Y_m^{(i)}, R^{(j,i)}] = Y_m^{(j)} \\ [R^{(i,j)}, X_n] &= 0, \quad [R^{(i,j)}, Y_m^{(k)}] = 0, \quad \text{if } k \neq i, j \end{aligned} \quad (5.16)$$

Fact 2: [44, 48] *Define the generalised Schrödinger operator*

$$\mathcal{S} := -\mu\partial_t + \frac{1}{z^2}\nabla_{\mathbf{r}}^z. \quad (5.17)$$

Then the commutator of $\mathcal{S}$ with any one of the generators (5.13) vanishes except for the following two cases:

$$\begin{aligned} [\mathcal{S}, X_0] &= -\mathcal{S} \\ [\mathcal{S}, X_1] &= -2t\mathcal{S} + \frac{1}{z}\left(2\mu(x + \xi) - \mu(z - 2 + d) - 2\gamma(2 - z)\right) \end{aligned} \quad (5.18)$$

This means that the generators (5.13) act as dynamical symmetry operators for the equation $\mathcal{S}\phi = 0$ provided that the relation

$$x + \xi = \frac{z - 2 + d}{2} + \frac{\gamma}{\mu}(2 - z) \quad (5.19)$$

for the scaling dimensions $x = x_\phi$ and $\xi = \xi_\phi$ of the solution ϕ holds true.

The invariant Schrödinger equation $\mathcal{S}\phi = 0$ is linear and of first order in the time-derivative. It is non-trivial and unknown at present if equations

³ An earlier formulation of LSI [330], albeit with a different fractional derivative, already allowed us to derive the first two facts but did not contain the generalised Bargman rules discussed below.

of the form $(\partial_t^{2/z} + \mu \partial_r^2)\phi = 0$ may or may not have a local scale-symmetry. Since systems undergoing non-equilibrium critical dynamics or phase-ordering kinetics are in general not Markovian, as illustrated in Chap. 1 through the analysis of the global persistence probability, this poses an important open problem. See [330] and Chap. 6 for some preliminary studies in this direction.

5.3 Generalised Bargman Superselection Rule

An important insight, which goes beyond our earlier results obtained for $z = 2$, comes from the consideration of the iterated commutators of the Y_m . For $d \geq 1$ dimensions we have

$$\begin{aligned} M_0 &:= [Y_{-1/z}^{(i)}, Y_{-1/z+1}^{(i)}] = \mu z \nabla_{\mathbf{r}}^{2-z} \\ N_0^{(i)} &:= [M_0, Y_{-1/z+1}^{(i)}] = -z^2(2-z)\mu^2 \partial_{r_i} \nabla_{\mathbf{r}}^{2-2z} \end{aligned} \quad (5.20)$$

and then recursively for $\ell \geq 1$

$$M_\ell := \sum_{i=1}^d [N_{\ell-1}^{(i)}, Y_{-1/z+1}^{(i)}] \quad , \quad N_\ell^{(i)} := [M_\ell, Y_{-1/z+1}^{(i)}] \quad (5.21)$$

Solving this operator recurrence [44, 48], we find, with $a_0 = z$,

$$\begin{aligned} M_\ell &= a_\ell \mu^{2\ell+1} \nabla_{\mathbf{r}}^{2\ell+2-(2\ell+1)z} \\ a_{\ell+1} &= -a_\ell z^2[(2\ell+1)z - (2\ell+2)][d + (2\ell+2)(1-z)] \end{aligned} \quad (5.22)$$

Fact 2 implies that $[\mathcal{S}, M_\ell] = 0$, hence the M_ℓ generate further dynamical symmetries of the equation $\mathcal{S}\phi = 0$.⁴ Applied to a n -point function $F^{(n)} = \langle \phi_1 \dots \phi_n \rangle$ of quasi-primary scaling operators ϕ_i , the co-variance conditions $M_\ell F^{(n)} = 0$, together with spatial translation-invariance $Y_{-1/z} F^{(n)} = 0$ lead to the following expression written for simplicity in Fourier space for all $\ell \in \mathbb{N}$:

$$\left(\sum_{i=1}^n \mu_i^{2\ell-1} |\mathbf{k}_i|^{2\ell-(2\ell-1)z} \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0 \quad (5.23)$$

$$\left(\sum_{i=1}^n \mathbf{k}_i \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0 \quad (5.24)$$

This kind of condition is quite similar to constraints found in factorisable scattering of relativistic particles [749, 748]. If one interprets an n -point function to describe a scattering process of m incoming and $n - m$ outgoing particles, the constraints eqs. (5.23, 5.24) mean that the total momentum and the sums

⁴ This set of dynamical symmetries is finite iff either (a) $z = (2N+2)/(2N+1)$ (with $N \geq 0$) or (b) $z = 1 + d/(2N+2)$ (for $N > 0$). Then $a_\ell = 0$ for $\ell > N$.

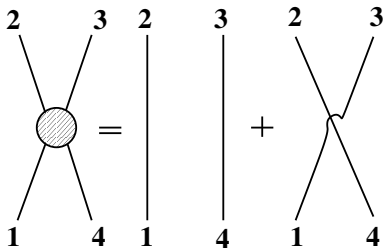


Fig. 5.2. Illustration of the generalised Bargman superselection for the four-point function $\langle \phi_1(1)\phi_2(2)\phi_3(3)\phi_4(4) \rangle$.

of certain powers of the momenta must be conserved. Furthermore, a classical result in relativistic factorisable scattering states that with a single conserved quantity of the above type, the scattering matrix factorises into the product of two-particle S -matrices [749, 748]. We use this result to arrange the mass factors $\mu_i^{2\ell+1}$ such that the conditions (5.23) and (5.24) are satisfied. A generalisation of the Bargman superselection rules follows:

Fact 3: [44, 48] *Let a system be given with dynamical exponent $z \neq 2$ and $\{\phi_i\}$ be a set of quasi-primary scaling operators transforming covariantly under the action of the generators $Y_{-1/z}^{(i)}$ and $Y_{-1/z+1}^{(i)}$ (for all $i = 1, \dots, d$). Let furthermore each scaling operator be characterised by the set $(x_i, \xi_i, \mu_i, \gamma_i)$. Then the $(2n)$ -point function*

$$F^{(2n)}(\{t_i\}, \{\mathbf{r}_i\}) := \langle \phi_1(t_1, \mathbf{r}_1) \dots \phi_{2n}(t_{2n}, \mathbf{r}_{2n}) \rangle \quad (5.25)$$

vanishes unless the μ_i form n distinct pairs $(\mu_i, \mu_{\tau(i)})$ ($i = 1, \dots, n$, $\tau(i) = n + 1, \dots, 2n$), such that for each pair

$$\mu_i = -\mu_{\tau(i)} . \quad (5.26)$$

This result is considerably more restrictive than the Bargman superselection rule for Galilei-invariant systems (where $z = 2$), which merely requires $\sum_{i=1}^{2n} \mu_i = 0$. As before, we expect that $\mu_i > 0$ if ϕ_i is an order-parameter field but $\mu_i < 0$ for response fields, and this can be proven as for the case $z = 2$ by considering the kinetic part of the equations of motion for the order-parameter ϕ and the conjugate response field $\tilde{\phi}$. Our result (5.26) was obtained here as a consequence of the *postulated* generalised Galilei-invariance. A formal proof of this invariance has still to be found. In what follows we shall derive some consequences of (5.26) in order to be able to carry out quantitative tests, in the hope of providing a physical motivation for such a deeper symmetry analysis.

The superselection rule (5.26) is illustrated in Fig. 5.2 for the four-point response function $\langle \phi_1(1)\tilde{\phi}_2(2)\tilde{\phi}_3(3)\phi_4(4) \rangle$ as it arises naturally in the Janssen-Dominicis formalism. It is then natural to consider the fields 1 and 4 as incoming and the other two as outgoing. Because of the (assumed) factorisation of the S -matrix, represented as the ball in the left part of the Fig., the total number of fields is conserved and there are only two ways to connect the

incoming and outgoing fields, respecting momentum-conservation because of the condition (5.24). We can then copy the treatment of stochastic systems from Chap. 4, by decomposing the action $\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_0[\phi, \tilde{\phi}] + \mathcal{J}_b[\tilde{\phi}]$ into a ‘deterministic’ part $\mathcal{J}_0[\phi, \tilde{\phi}]$ and a ‘noise’ part $\mathcal{J}_b[\tilde{\phi}]$ and defining again a deterministic average $\langle \cdot \rangle_0$. Then the Bargman superselection rule eq. (4.93) is expected to hold true for $z \neq 2$ as well.

A non-trivial consequence of this is a decomposition theorem analogous to the one derived in the previous chapter: *the reduction formulæ eqs. (4.94, 4.95), which relate the full response and correlation functions to two- and four-point functions of the deterministic part only, do remain valid for all values of z .* We shall show in the next two sections how these identities may be used to predict the two-time response and correlation functions.

5.4 Calculation of Two-time Responses

Starting from (4.94), for the calculation of the two-time response function we must find the two-point function of the quasi-primary scaling operators ϕ_i , $i = 1, 2$, and characterised by the four parameters $(x_i, \xi_i, \mu_i, \gamma_i)$. Because of the Bargman condition (5.24) we must have $\mu_1 = -\mu_2$. Requiring spatial translation-invariance from the outset, we set $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and consider

$$F^{(2)}(t_1, t_2, \mathbf{r}) := \langle \phi_1(t_1, \mathbf{r}_1) \phi_2(t_2, \mathbf{r}_2) \rangle = (t_1 - t_2)^{\hat{b}} \left(\frac{t_1}{t_2} \right)^{\hat{a}} t_2^{\hat{c}} G(t_1, t_2, \mathbf{r}) . \quad (5.27)$$

If the ϕ_i are scalars under rotations, it is enough to restrict ourselves to $d = 1$ dimensions, the generalisation to $d > 1$ being obvious. Setting

$$\hat{a} = \frac{2\xi_2 + x_2 - x_1}{z} , \quad \hat{b} = -\frac{(2\xi_1 + x_1) + (2\xi_2 + x_2)}{z} , \quad \hat{c} = \frac{2(\xi_1 + \xi_2)}{z} \quad (5.28)$$

covariance under $X_0, Y_{-1/z+1}$ and X_1 leads to the conditions

$$\begin{aligned} & \left(t_1 \partial_{t_1} + t_2 \partial_{t_2} + \frac{1}{z} \mathbf{r} \cdot \partial_{\mathbf{r}} \right) G = 0 \\ & \left((t_1 - t_2) \partial_{\mathbf{r}} + z \mu_1 \mathbf{r} \cdot \nabla_{\mathbf{r}}^{2-z} + (\gamma_1^\dagger - \gamma_2^\dagger) \partial_{\mathbf{r}} \nabla_{\mathbf{r}}^{-z} \right) G = 0 \\ & \left((t_1 - t_2)^2 \partial_{t_1} - t_2^2 \partial_{t_2} + \mu_1 \mathbf{r}^2 \nabla_{\mathbf{r}}^{2-z} + \frac{2}{z} (t_1 - t_2) \partial_{\mathbf{r}} \cdot \nabla_{\mathbf{r}}^{-z} \right. \\ & \quad \left. + (\gamma_1^* + \gamma_2^*) \nabla_{\mathbf{r}}^{-z} \right) G = 0 \end{aligned} \quad (5.29)$$

where we have defined the shorthands

$$\begin{aligned} \gamma^\dagger &= z(2-z)\gamma , \quad \hat{\gamma} = 2(2-z)\gamma , \quad \gamma^* = (2-z)(d-z)\gamma \\ \beta &= -(2-z) \left(1 - 2 \frac{\gamma}{\mu_1} \right) , \quad \alpha = \frac{1}{z^2 \mu_1 i^{2-z}} \end{aligned} \quad (5.30)$$

Solving the first equation in (5.29) leads to

$$G = H \left(\frac{\mathbf{r}}{(t_1 - t_2)^{1/z}}, \frac{t_1 - t_2}{t_2} \right), \quad (5.31)$$

while from the other two equations in (5.29) we find the following system of equations for the function $H = H(\mathbf{u}, v)$

$$\begin{aligned} & \left(\partial_{\mathbf{u}} + z\mu_1 \mathbf{u} \cdot \nabla_{\mathbf{u}}^{2-z} + (\gamma_1^\dagger - \gamma_2^\dagger) \partial_{\mathbf{u}} \nabla_{\mathbf{u}}^{-z} \right) H = 0 \\ & \left(z(v-1) \partial_v + (\gamma_2^\dagger + \gamma_1^\dagger) \partial_{\mathbf{u}} \nabla_{\mathbf{u}}^{-z} + z(\gamma_1^* + \gamma_2^*) \nabla_{\mathbf{u}}^{-z} \right) H = 0 \end{aligned} \quad (5.32)$$

The second of these was obtained by multiplying the second equation in (5.29) by $-\mathbf{u}$ and adding to the third. Transforming into Fourier space with respect to $\mathbf{u}$, we have

$$\begin{aligned} & \left(z\mu_1 i^{2-z} |\mathbf{k}|^{2-z} \partial_{\mathbf{k}} + \mathbf{k} + (z\mu_1 i(2-z) + (\gamma_1^\dagger - \gamma_2^\dagger)) \mathbf{k} |\mathbf{k}|^{-z} \right) \hat{H}_{\mathbf{k}}(v) = 0 \\ & \left(z(v-1) \partial_v - (\gamma_1^\dagger + \gamma_2^\dagger) \mathbf{k} |\mathbf{k}|^{-z} \partial_{\mathbf{k}} \right) \hat{H}_{\mathbf{k}}(v) = 0 \end{aligned} \quad (5.33)$$

The solution is

$$\hat{H}_{\mathbf{k}}(v) = f_0 \delta_{\gamma_1 + \gamma_2, 0} |\mathbf{k}|^{-(z\mu_1 i^{2-z}(2-z) + (\gamma_1^\dagger - \gamma_2^\dagger) i^{-z}) \alpha_1} \exp(-\alpha_1 |\mathbf{k}|^z) \quad (5.34)$$

where we note explicitly the constraint $\gamma_1 = -\gamma_2$. Here f_0 is an undetermined normalisation constant. Transforming back to real space and substituting all shorthands, we finally obtain

$$\begin{aligned} F^{(2)}(t_1, t_2; \mathbf{r}) &= f_0 \delta_{\alpha_1, -\alpha_2} \delta_{\beta_1, \beta_2} (t_1 - t_2)^{-((2\xi_1 + x_1) + (2\xi_2 + x_2))/z} \\ &\times \left(\frac{t_1}{t_2} \right)^{(2\xi_2 + x_2 - x_1)/z} t_2^{2(\xi_1 + \xi_2)/z} \mathcal{F}^{(\alpha_1, \beta_1)} \left(\frac{\mathbf{r}}{(t_1 - t_2)^{1/z}} \right). \end{aligned} \quad (5.35)$$

Hence, as found previously from ageing-invariance with $z = 2$, the two-point function $F^2(t_1, t_2; \mathbf{r}) = F(t, s) \mathcal{F}^{(\alpha_1, \beta_1)}(\mathbf{r}(t_1 - t_2)^{-1/z})$ factorises into a space-independent term $F(t, s)$ and a space-time part which is described by the function

$$\mathcal{F}^{(\alpha, \beta)}(\mathbf{u}) = \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{u} \cdot \mathbf{k}} |\mathbf{k}|^\beta \exp(-\alpha |\mathbf{k}|^z). \quad (5.36)$$

Since this integral is the Fourier transform of a function depending only on $|\mathbf{k}|$, one has $\mathcal{F}^{(\alpha, \beta)} = \mathcal{F}^{(\alpha, \beta)}(|\mathbf{r}|)$ and spatial rotation-invariance is explicit. Fig. 5.3 illustrates the behaviour of the space-dependent part.

In view of this result, it is useful to characterise a quasi-primary scaling operator by (x, ξ, α, β) , where α and β have been defined in (5.30). The Bargman superselection rule becomes $\delta_{\alpha_1, -\alpha_2} = \delta_{\mu_1, -\mu_2}$. The case $z = 2$ treated before is included as a special case where $\beta = 0$ and $\mu = 2\gamma$.

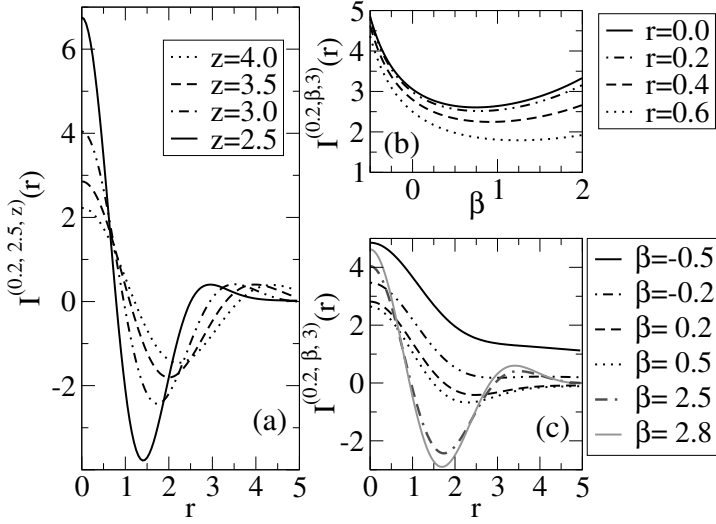


Fig. 5.3. Behaviour of the space-dependent part $\mathcal{F}^{(\alpha, \beta)}(|\mathbf{r}|) = I^{(\alpha, \beta, z)}(r)$, up to normalisation. (a) Dependence on r for $\alpha = 0.2$, $\beta = 2.5$ for several values of z . (b) Dependence on β for several values of r and $\alpha = 0.2$ and $z = 3$. (c) Dependence on r for several values of β and $\alpha = 0.2$ and $z = 3$.

For the calculation of physical response functions, consider the following dynamical equation which extends (4.96) to the case $z \neq 2$

$$\mu \partial_t \phi(t, \mathbf{r}) = \frac{1}{z^2} \nabla_{\mathbf{r}}^z \phi(t, \mathbf{r}) - v(t) \phi(t, \mathbf{r}) + \eta(t, \mathbf{r}) \quad (5.37)$$

which also contains a time-dependent potential $v(t)$ that naturally arises in many models, see chapters 2 and 3. The noise $\eta(t, \mathbf{r})$ is a centred Gaussian random variable, with the second moment

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2T \delta(t - t') b(\mathbf{R} - \mathbf{R}'). \quad (5.38)$$

The usual white noise is recovered if $b(\mathbf{r}) = \delta(\mathbf{r})$. The deterministic part of eq. (5.37) can be reduced to the invariant equation $\mathcal{S} \bar{\phi}(t, \mathbf{r}) = 0$ with $\mathcal{S}$ given by eq. (5.17),⁵ through the gauge transformation eq. (4.97)[589]

$$\phi(t, \mathbf{r}) = \bar{\phi}(t, \mathbf{r}) \exp \left(-\frac{1}{\mu} \int_0^t du v(u) \right) = g_\mu(t) \bar{\phi}(t, \mathbf{r}) \quad (5.39)$$

such that $g_\mu(t) \sim t^F$ for large times or, equivalently, $v(t) = -\mu F/t$. Breaking time-translation-invariance explicitly this way, we have $\langle \phi(t, \mathbf{r}) \bar{\phi}(s, \mathbf{r}') \rangle = F^{(2)}(t, s; \mathbf{r} - \mathbf{r}') g_\mu(t)/g_\mu(s)$ such that we can now use the result (5.35), valid

⁵ Physically, this approach involves the important hidden assumption that the equation of motion remains of first order in time, even after renormalisation.

for $v(t) = 0$. Expressing the various parameters in terms of the usual ageing exponent, the final LSI-prediction of the two-time response function is [48]

$$\begin{aligned} R(t, s; \mathbf{r}) &= R(t, s) \mathcal{F}^{(\alpha, \beta)} \left(\frac{\mathbf{r}}{(t-s)^{1/z}} \right) \\ &= r_0 s^{-a-1} \left(\frac{t}{s} \right)^{1+a'-\lambda_R/z} \left(\frac{t}{s} - 1 \right)^{-1-a'} \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^\beta \exp \left[\frac{i\mathbf{r} \cdot \mathbf{k}}{(t-s)^{1/z}} - \alpha |\mathbf{k}|^z \right] \end{aligned} \quad (5.40)$$

In particular the autoresponse reads

$$R(t, s) = r_0 s^{-a-1} \left(\frac{t}{s} \right)^{1+a'-\lambda_R/z} \left(\frac{t}{s} - 1 \right)^{-1-a'} \quad (5.41)$$

where r_0 is a normalisation constant, α is a non-universal dimensionful parameter, β is a universal exponent and the relationship between the parameters of the quasi-primary operators and the ageing exponents is given by

$$a = \frac{1}{z}(x+\tilde{x})-1, \quad a' = \frac{1}{z}\left((2\xi+x)+(2\tilde{\xi}+\tilde{x})\right)-1, \quad \frac{\lambda_R}{z} = -F + \frac{2}{z}(\xi+x) \quad (5.42)$$

5.5 Calculation of Two-time Correlators

We finally outline the calculation of the two-time autocorrelator [44, 48]. In view of the available data for testing the resulting prediction, we shall concentrate here on cases where the initial noise dominates over the thermal noise. This happens in systems quenched to $T < T_c$. Critical quenches will be treated in [48].

Because of the reduction formula (4.95), we require that the four-point function is given by

$$\begin{aligned} F^{(4)}(\{t_i\}, \{\mathbf{r}_i\}) &:= \left\langle \phi_1(t_1, \mathbf{r}_1) \dots \phi_4(t_4, \mathbf{r}_4) \right\rangle \\ &= \delta_{\mu_1, -\mu_2} \delta_{\mu_3, -\mu_4} G_1(\{t_i\}; \mathbf{r}_{12}, \mathbf{r}_{34}) + \delta_{\mu_1, -\mu_2} \delta_{\mu_3, -\mu_4} G_3(\{t_i\}; \mathbf{r}_{13}, \mathbf{r}_{24}) \\ &\quad + \delta_{\mu_1, -\mu_3} \delta_{\mu_2, -\mu_4} G_2(\{t_i\}; \mathbf{r}_{13}, \mathbf{r}_{24}) + \delta_{\mu_1, -\mu_3} \delta_{\mu_2, -\mu_4} G_4(\{t_i\}; \mathbf{r}_{12}, \mathbf{r}_{34}) \end{aligned}$$

where $\mathbf{r}_{ij} := \mathbf{r}_i - \mathbf{r}_j$. The factorisation written down follows from the generalised Bargman rules (5.26), see also Fig. 5.2. Requiring the covariance under local scale-transformations generated by X_1 , it can be shown [48] that it is enough to consider

$$F^{(4)}(\{t_i\}, \{\mathbf{r}_i\}) = \delta_{\mu_1, -\mu_2} \delta_{\mu_3, -\mu_4} G(\{t_i\}, \mathbf{r}, \tilde{\mathbf{r}}) \quad (5.43)$$

with $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$ and $\tilde{\mathbf{r}} := \mathbf{r}_3 - \mathbf{r}_4$. The other terms can then simply be obtained by permutation of the space-time points. Since spatial translation-invariance is already implemented, G is found by requiring the covariance

under the remaining local scale-transformation, generated by $X_0, X_1, Y_{1-1/z}$. Invariance under $Y_{-1/z}^{(i)}$ is already implemented. This calculation is quite tedious and we shall only quote here the special case relevant to the calculation of two-time correlators, namely four-point functions of the form $\langle \phi(t_1, \mathbf{r}_1) \phi(t_2, \mathbf{r}_2) \tilde{\phi}(t_3, \mathbf{r}_3) \tilde{\phi}(t_4, \mathbf{r}_4) \rangle$ such that the Bargman rules $\mu := \mu_1 = \mu_2 = -\mu_3 = -\mu_4$ and $\gamma := \gamma_1 = \gamma_2 = -\gamma_3 = -\gamma_4$ hold true. Then [48]

$$\begin{aligned} F^{(4)}(\{t_i\}, \{\mathbf{r}_i\}) &= (t_1 \cdot t_2)^{2\xi/z} (t_3 \cdot t_4)^{2\tilde{x}/z} (t_1 - t_2)^{-2(2\xi+x)/z+2(2\tilde{x}+\tilde{x})/z} \\ &\times (t_1 - t_3)^{-(2\tilde{x}+\tilde{x})/z} (t_1 - t_4)^{-(2\tilde{x}+\tilde{x})/z} (t_2 - t_3)^{-(2\tilde{x}+\tilde{x})/z} (t_2 - t_4)^{-(2\tilde{x}+\tilde{x})/z} \\ &\times \left[f_2 \mathcal{F}^{(\mu, \gamma)} \left(\frac{\mathbf{r}_1 - \mathbf{r}_3}{(t_1 - t_3)^{1/z}} \right) \mathcal{F}^{(\mu, \gamma)} \left(\frac{\mathbf{r}_2 - \mathbf{r}_4}{(t_2 - t_4)^{1/z}} \right) \right. \\ &\quad \left. + f_3 \mathcal{F}^{(\mu, \gamma)} \left(\frac{\mathbf{r}_1 - \mathbf{r}_4}{(t_1 - t_4)^{1/z}} \right) \mathcal{F}^{(\mu, \gamma)} \left(\frac{\mathbf{r}_3 - \mathbf{r}_2}{(t_3 - t_2)^{1/z}} \right) \right] \end{aligned} \quad (5.44)$$

with two undetermined functions $f_2 = f_2(u_2, v_2)$ and $f_3 = f_3(u_3, v_3)$, where

$$\begin{aligned} u_2 &= \frac{(t_3 - t_4)(t_1 - t_2)}{(t_1 - t_3)(t_2 - t_4)}, \quad v_2 = \frac{t_4(t_3 - t_1)}{t_1(t_3 - t_4)} \\ u_3 &= \frac{(t_4 - t_2)(t_1 - t_3)}{(t_1 - t_4)(t_3 - t_2)}, \quad v_3 = \frac{t_2(t_4 - t_1)}{t_1(t_4 - t_2)} \end{aligned} \quad (5.45)$$

Again, we include the possibility that time-translation-invariance is explicitly broken by a time-dependent potential $v(t)$ in the Langevin equation (5.37). By the same gauge transformation (4.97) as before we have

$$\langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \tilde{\phi}(u, \mathbf{R}) \tilde{\phi}(u', \mathbf{R}') \rangle = \frac{g_\mu(t) g_\mu(s)}{g_\mu(u) g_\mu(u')} F^{(4)}(t, s, u, u'; \mathbf{r}, \mathbf{r}', \mathbf{R}, \mathbf{R}') \quad (5.46)$$

In view of the specific applications we are going to consider, we limit ourselves here to quenches into the ordered phases $T \leq T_c$. Then, for the two-time autocorrelation function, we merely need $\langle \phi \phi \tilde{\phi} \tilde{\phi} \rangle|_{t_3=t_4}$ such that in (5.44) the functions f_2 and f_3 need not be distinguished and furthermore reduce to a constant. Then, using the identity [44]

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} d\mathbf{u} d\mathbf{v} \mathcal{F}^{(\alpha_1, \beta_1)}(c\mathbf{u} + \mathbf{a}) \mathcal{F}^{(\alpha_3, \beta_3)}(d\mathbf{v} + \mathbf{b}) g(\mathbf{u}, \mathbf{v}) \\ &= \int_{\mathbb{R}^{2d}} \frac{d\mathbf{k} d\mathbf{q}}{(2\pi)^{2d}} e^{i\mathbf{a} \cdot \mathbf{k} + i\mathbf{b} \cdot \mathbf{q}} \hat{g}(c\mathbf{k}, d\mathbf{q}) |\mathbf{k}|^{\beta_1} |\mathbf{q}|^{\beta_3} \exp(-\alpha_1 |\mathbf{k}|^z - \alpha_3 |\mathbf{q}|^z) \end{aligned} \quad (5.47)$$

where $\mathbf{a}, \mathbf{b}, c, d$ are arbitrary constants and $\hat{g}(\mathbf{k}, \mathbf{q})$ is the double Fourier transform of $g(\mathbf{u}, \mathbf{v})$, we have

$$\begin{aligned} C_{\text{init}}(t, s; \mathbf{r}) &= c_0 s^{2F-2(2\tilde{x}+x+\tilde{x})/z+2\beta/z+2d/z} y^{F-2(\tilde{x}+2\tilde{x}-\xi)/z+\beta/z+d/z} \\ &\times (y-1)^{-2(x-\tilde{x})/z-4(\xi-\tilde{\xi})/z} \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{2\beta} \exp(-\alpha |\mathbf{k}|^z (t+s)) e^{i\mathbf{r} \cdot \mathbf{k}} \hat{a}(\mathbf{k}) \end{aligned} \quad (5.48)$$

Here $\hat{a}(\mathbf{k})$ is the Fourier transform of the initial correlator $a(\mathbf{R} - \mathbf{R}') = \langle \phi(0, \mathbf{R}) \phi(0, \mathbf{R}') \rangle$. Since in lattice simulations one is often starting from a fully disordered lattice, a natural choice for the initial correlator is often believed to be $a(\mathbf{R}) = a_0 \delta(\mathbf{R})$, and if this assumption were correct, explicit scaling forms could be derived from (5.48), see exercise 5.14. Indeed, that would be the direct extension of the approach used in Chap. 4 to general dynamic exponents $z \neq 2$.

It is important to realise, however, that such a simple choice might be too simplistic. Recall from Chap. 1 (see page 54) that ageing only sets in on rather late time-scales $t_p(s) \sim s^\zeta$, where ζ was the *passage exponent*. It therefore can be more appropriate to generalise the structure of the theory and rather prescribe an ‘initial’ correlator at some time $u \leq s$. In this case, we have the spatio-temporal correlator

$$C_{\text{init}}(t, s; \mathbf{r}, \mathbf{r}') = \int_{\mathbb{R}^{2d}} d\mathbf{R} d\mathbf{R}' F^{(4)}(t, s, u, u; \mathbf{r}, \mathbf{r}', \mathbf{R}, \mathbf{R}') a(u, \mathbf{R}, \mathbf{R}') \quad (5.49)$$

where $a(u, \mathbf{R}, \mathbf{R}')$ is the equal-time correlator at time u and the four-point function is again given by (5.44) with $t_1 = t$, $t_2 = s$ and $t_3 = t_4 = u$. When we apply eq. (5.47), we find the general form

$$\begin{aligned} C_{\text{init}}(t, s; \mathbf{r}) &= (t \cdot s)^{2\xi/z+F} u^{4\tilde{x}/z-2F} (t-s)^{-2(2\xi+x)/z+2(2\tilde{\xi}+\tilde{x})/z} \\ &\times (t-u)^{-2(2\tilde{\xi}+\tilde{x})/z} (s-u)^{-2(2\tilde{\xi}+\tilde{x})/z} \tilde{f}\left(\frac{s}{t} \frac{u-t}{u-s}\right) \\ &\times \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{2\beta} \exp\left(i\mathbf{r} \cdot \mathbf{k} - \alpha |\mathbf{k}|^z (t+s-2u)\right) \hat{a}(u, \mathbf{k}) \end{aligned} \quad (5.50)$$

where $\tilde{f}$ is an undetermined function.

Now, one may consider two extreme cases. First, if we send $u \rightarrow 0$, we recover eq. (5.48), as expected. On the other had, if we let $u \rightarrow s$, we have

$$\begin{aligned} C_{\text{init}}(t, s; \mathbf{r}) &= C_0 (t \cdot s)^{2\xi/z+F} s^{4\tilde{x}/z-2F} (t-s)^{-2(2\xi+x)/z} \\ &\times \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{2\beta} \exp\left(i\mathbf{r} \cdot \mathbf{k} - \alpha |\mathbf{k}|^z (t-s)\right) \hat{a}(s, \mathbf{k}), \end{aligned} \quad (5.51)$$

where C_0 is a normalisation constant. In this way, we see how the equal-time correlator $C_t(\mathbf{r}) = a(t, \mathbf{r})$ will affect the scaling of the two-time correlator.

Below, we shall apply this to the phase-ordering kinetics of the 2D Glauber-Ising model.

5.6 Tests of Local Scale-invariance With $z \neq 2$

We now describe the conclusions of tests of the LSI-predictions for two-time responses and correlators in several spin systems which show simple ageing with

a dynamical exponent $z \neq 2$. For the sake of simplicity of the presentation, we shall in the exactly solved models limit ourselves to a detailed discussion of the response function only and refer to the quoted literature for the details of the tests of the autocorrelator. Unless explicitly stated otherwise, we shall concentrate on the response of the order-parameter with respect to its conjugate magnetic field and its autocorrelator. Conceptually, the consideration of different two-time quantities allows for a gradual test of the specific hypothesis which goes into local scale-invariance as formulated here:

1. the *autoresponse* $R(t, s)$ can be derived from the assumption of co-variance under the special Möbius-transformations $t \mapsto t/(1 + \gamma t)$, with the infinitesimal generator X_1 .
2. the *space-time response* $R(t, s; \mathbf{r})$ follows from the co-variance under the generalised Galilei-transformations, with the infinitesimal generator $Y_{1-1/z}$.
3. the form of the *autocorrelator* $C(t, s)$ is a consequence of the factorisation of the four-point response function and therefore may serve as a first check of the generalised Bargman rules and a possible much deeper structure underlying LSI.

5.6.1 Surface Growth: Mullins-Herring Model

In Chap. 3, we had introduced the Mullins-Herring equation (3.157) with either non-conserved or conserved noises, leading to the three models MH1, MH2 and MHC. In all three cases, the two-time response function is given by [629, 630]

$$R(t, s; \mathbf{r}) = \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{r}} e^{\nu_4 |\mathbf{k}|^4 (t-s)}, \quad (5.52)$$

see eq. (3.160) for a more explicit expression in terms of hypergeometric function. Clearly, this agrees with (5.40) if the choice

$$x = \tilde{x} = \tilde{x}_2 = \frac{d}{2}, \quad \xi = \tilde{\xi} = \tilde{\xi}_2 = 0, \quad \mu = -\frac{1}{16\nu_4}, \quad \frac{\gamma}{\mu} = \frac{1}{2} \quad (5.53)$$

is made. Similarly, the agreement of the exact autocorrelator, see Table 4.2, with LSI is readily established [629, 630].

The deterministic part of the Mullins-Herring equation (3.157) is identical to the LSI-invariant linear equation $\mathcal{S}\phi = 0$ with $z = 4$ and $\mathcal{S}$ given by eq. (5.17). Therefore, the agreement of the exactly solved Mullins-Herring equation with local scale-invariance is to be expected and simply reconfirms the reduction formulæ eqs. (4.94, 4.95) derived from the generalised Bargman rules (5.26). Similar remarks shall apply to the next two models as well, where the order-parameter is also described by a linear Langevin equation.

5.6.2 Spherical Model With Long-range Interactions

This model was defined as a spin model in eqs. (2.110, 2.111). In the continuum limit, eq. (4.138) for the order-parameter ϕ now becomes

$$\partial_t \phi(t, \mathbf{r}) = \nabla_{\mathbf{r}}^\sigma \phi(t, \mathbf{r}) - v(t) \phi(t, \mathbf{r}) + \eta(t, \mathbf{r}) \quad (5.54)$$

where the long-range interactions are described by the fractional derivative $\nabla_{\mathbf{r}}^\sigma$, characterised by $0 < \sigma < 2$ (see appendix J.2). Separating off the noise η leaves the deterministic part where then the potential term $v(t)$ is absorbed into the already familiar gauge transformation. We then arrive exactly at the kind of invariant Schrödinger equations $\mathcal{S}\phi = 0$, with $\mathcal{S}$ given by (5.17), analysed earlier and we can therefore expect LSI to be valid.

To see explicitly how this comes about, recall from the exact calculations described in Chap. 2 the two-time response function $R = \mathcal{R}_{11}$ of the order-parameter

$$R(t, s; \mathbf{r}) = s^{-d/\sigma} \left(\frac{t}{s} \right)^{-F/2} \left(\frac{t}{s} - 1 \right)^{-d/\sigma} \int_{\mathbb{R}^d} d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}(t-s)^{-1/\sigma}} e^{-Bk^\sigma} \quad (5.55)$$

where B is a known constant, whereas for short-ranged initial correlations the exponent F is given by [127, 45]

$$F = \begin{cases} -d/\sigma & ; \text{ if } T < T_c \\ -2 + d/\sigma & ; \text{ if } T = T_c \text{ and } \sigma < d < 2\sigma \\ 0 & ; \text{ if } T = T_c \text{ and } d > 2\sigma \end{cases} \quad (5.56)$$

Comparison with the LSI-prediction (5.40) shows full agreement with (5.55) for the following choice of parameters, besides the exponents of $\mathcal{R}_{11}$, already listed in Table 2.6, for all temperatures $T \leq T_c$

$$\begin{aligned} z = \sigma, \quad a = a' = \frac{d}{\sigma} - 1, \quad \lambda_R = d + \frac{F\sigma}{2} \\ \xi = -\frac{\sigma F}{4}, \quad \tilde{\xi} = \tilde{\xi}_2 = \frac{\sigma F}{4}, \quad \mu = (z^2 B i^{2-z})^{-1}, \quad \gamma/\mu = \frac{1}{2} \end{aligned} \quad (5.57)$$

In a similar way, the explicitly known results for the two-time correlator can be checked for complete agreement with LSI, for $T \leq T_c$ [45]. From this we conclude that the order-parameter does indeed correspond to a quasi-primary scaling operator of LSI.

For long-ranged initial correlations, the values of F , further dynamical universality classes, with modified values of F , as listed in Table 2.7, have to be distinguished [217]. Local scale-invariance continues to be valid, with the order-parameter ϕ being quasi-primary, as expected.

On the other hand, for $\sigma < 2$ the form of the response functions of the further observables, whose exponents are listed in Table 2.6, do not agree with the above prediction of LSI. Hence neither the squared order-parameter ε nor the energy density ϵ are any more quasi-primary [45].

5.6.3 Critical Conserved Spherical Model

The spherical model with a conserved order-parameter, quenched to $T = T_c$, may be described in terms of the Langevin equation

$$\partial_t S(t, \mathbf{r}) = -\nabla_{\mathbf{r}}^2 \left[\nabla_{\mathbf{r}}^2 S(t, \mathbf{r}) + \mathfrak{z}(t) S(t, \mathbf{r}) + h(t, \mathbf{r}) \right] + \eta(t, \mathbf{r}) \quad (5.58)$$

where the centred Gaussian noise η has the variance $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = -2T_c \nabla_{\mathbf{r}}^2 \delta(t-t') \delta(\mathbf{r}-\mathbf{r}')$. In order to compare the exact results eqs. (3.42, 3.44) with the predictions of local scale-invariance, we now have to use the following gauge transformation [47, 48]

$$\phi(t, \mathbf{r}) = \exp \left(-\frac{1}{z^2 \mu} \int_0^t d\tau v(\tau) \nabla_{\mathbf{r}}^2 \right) \bar{\phi}(t, \mathbf{r}) . \quad (5.59)$$

For the sake of brevity, we illustrate here the application to the two-time response function. Since the use of (5.59) eliminates the explicitly time-dependent potential in the equation of motion for $\bar{\phi}$, the two-time response function is given by [47]

$$R(t, s; \mathbf{r}) = \exp \left(-\frac{1}{16\mu} \int_s^t d\tau v(\tau) \nabla_{\mathbf{r}}^2 \right) \nabla_{\mathbf{r}}^2 F^{(2)}(t, s; \mathbf{r}) \quad (5.60)$$

where we assumed that $\int_0^t d\tau v(\tau) \stackrel{t \rightarrow \infty}{\sim} \kappa_0 t^{F_2}$, and $F^{(2)}$ is the LSI-covariant two-point function eq. (5.35). Then it is easy to see that the two factors $\nabla_{\mathbf{r}}^2$ in (5.60) lead to extra factors $\mathbf{k}^2$ such that the response function becomes

$$\begin{aligned} R(t, s; \mathbf{r}) &= s^{-a-1} \left(\frac{t}{s} \right)^{1+a'-\lambda_R/z} \left(\frac{t}{s} - 1 \right)^{-1-a'} \\ &\times \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{2+2\beta} \exp \left(-\frac{i\mathbf{k} \cdot \mathbf{r}}{(t-s)^{1/4}} \right) \exp \left(-\frac{1}{z^2 \mu_1 i^{2-z}} |\mathbf{k}|^z \right) \end{aligned} \quad (5.61)$$

where the ageing exponents are connected to the scaling dimensions via

$$\begin{aligned} a+1 &= \frac{2}{z}(x+\tilde{x}) + \frac{2}{z} , \quad a'+1 = \frac{1}{z}((2\xi+x) + (2\tilde{\xi}+\tilde{x})) + \frac{2}{z} \\ \frac{\lambda_R}{z} &= -F + \frac{2}{z}(\xi+x) + \frac{2}{z} \end{aligned} \quad (5.62)$$

This reproduces the exact solution eq. (3.41) if one identifies, in addition to the exponents (3.46), $\xi + \tilde{\xi} = 0$ and

$$z = 4 , \quad F_2 = 1/2 , \quad \gamma/\mu = \frac{1}{2} , \quad \kappa_0 = -\mathfrak{g}_d , \quad \mu = -\frac{1}{16} . \quad (5.63)$$

Similarly, the exactly known two-time correlator (3.44) can be shown to be fully consistent with the LSI-prediction at $T = T_c$ [47, 48].

5.6.4 Critical Ising Model

As a first example of an ageing system whose Langevin equation for the order-parameter is *non*-linear and which cannot be solved exactly, we consider the $2D/3D$ Ising model with Glauber dynamics quenched to $T = T_c$. The dynamical exponent z and the autoresponse exponent λ_R are listed in Table 1.7. Indeed, this system was one of the very first models where local scale-invariance could be confirmed [352] for the linear autoresponse. The thermoremanent susceptibility is expected to scale as $\chi_{\text{TRM}}(t, s) = \int_0^s du R(t, u) = s^{-a} f_M(t/s)$ where from the explicit form (5.41) we obtain, see exercise 4.23

$$f_M(y) = f_0 y^{-\lambda_R/z} {}_2F_1\left(1 + a', \frac{\lambda_R}{z} - a; \frac{\lambda_R}{z} - a + 1; \frac{1}{y}\right). \quad (5.64)$$

Comparison with early simulational data gave good agreement, where the assumption $a = a'$ was also made [352]. Somewhat later, Pleimling and Gambassi observed that the form of integrated responses in direct space only depends weakly on the difference $a - a'$ [596]. They decided therefore to measure the integrated response in *momentum* space. In distinction to the methods explained in appendix G, χ may be obtained from $M_{\text{Int}}(t, s) = \hbar \chi_{\text{Int}}(t, s) = (\hbar \mathcal{N})^{-1} \langle \sum_{i \in \Lambda} \sigma_i(t) \rangle$ where $\mathcal{N}$ is the number of sites of the lattice Λ . Specifically, they considered an intermediate integrated response, see Fig. 1.21, since a spatially constant magnetic field $\hbar/J = 0.05$ was switched on at time $s/2$ and turned off at time s , with measurements taken at times $t > s$.

Field-theoretical methods may be used to calculate the form of the expected scaling function. A two-loop calculation leads to the following second-order ε -expansion for the zero-momentum autoresponse, in the $O(n)$ -model and with $\varepsilon = 4 - d$ [118]

$$\begin{aligned} \hat{R}_0(t, s) &= A_R(t-s)^{-1-a+d/z} \left(\frac{t}{s}\right)^{1+a-\lambda_R/z} F_R\left(\frac{s}{t}\right) \\ A_R &= 1 + \frac{3(n+2)}{8(n+8)^2} (f(0) - 4C_E \ln \frac{4}{3}) \varepsilon^2 + O(\varepsilon^3) \\ F_R(v) &= 1 + \frac{3(n+2)}{8(n+8)^2} (f(v) - f(0)) \varepsilon^2 + O(\varepsilon^3) \end{aligned} \quad (5.65)$$

where the values of the exponents were given in Chap. 1, and explicit, if rather lengthy, expressions for $f(v)$ are quoted in [118].

Because of the product form (5.40) of the response function, the LSI prediction for the autoresponse $\hat{R}_{\mathbf{q}}(t, s) = \int_{\mathbb{R}^d} d\mathbf{r} R(t, s; \mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}}$ in momentum space becomes, up to a normalisation factor r_0 ,

$$\hat{R}_0(t, s) = r_0 s^{-1-a+d/z} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s} - 1\right)^{-1-a'+d/z} \quad (5.66)$$

such that the intermediate susceptibility becomes, in the scaling limit

$$\begin{aligned} \chi_{\text{Int}}(t, s) &:= \int_{s/2}^s du \hat{R}_0(t, u) = \chi_0 s^{-a+d/z} f_\chi(t/s) \\ f_\chi(y) &= y^{(d-\lambda_R)/z} \left[{}_2F_1 \left(1 + a' - \frac{d}{z}, \frac{\lambda_R}{z} - a; \frac{\lambda_R}{z} - a + 1; \frac{1}{y} \right) \right. \\ &\quad \left. - 2^{a-\lambda_R/z} {}_2F_1 \left(1 + a' - \frac{d}{z}, \frac{\lambda_R}{z} - a; \frac{\lambda_R}{z} - a + 1; \frac{1}{2y} \right) \right]. \end{aligned} \quad (5.67)$$

and where χ_0 is another normalisation constant. If one assumes in addition that $a = a'$, this prediction of LSI agrees with the result of the field-theoretic ε -expansion quoted above in (5.65), since to lowest order in ε one has $F_R(v) = 1 + O(\varepsilon^2)$ [117, 118]. On the other hand, deviations appear in the higher-order terms. From the smallness of the associated coefficients, one might expect that quantitatively, these corrections are not very easy to detect numerically, at least when ε is small enough.

In Fig. 5.4 we compare the non-perturbative numerical data for the intermediate integrated response in momentum space of the 2D/3D Glauber-Ising model quenched to $T = T_c$ with these theoretical predictions. This also allows us to assess *a posteriori* the applicability of non-equilibrium ε -expansions as a tool for numerical predictions. Clearly, for the values of s used the data fall into the scaling regime such that a comparison with the theoretical predictions is meaningful. Remarkably, the ε -expansion series (5.65), when truncated at second order [117], agrees quite well with the numerical data in 3D but there are strong deviations for $s/t \gtrsim 0.3$ in the two-dimensional case. Although field-theoretical ε -expansion methods can probably be relied on when the behaviour in the limit $t/s \rightarrow \infty$ of large separation between the two times is studied, this is no longer the case when one is interested in the precise form of the scaling function when t and s are comparable to each other. Hence reliable predictions for the exponents a and λ_R/z and the FDR limit X_∞ cannot be used to argue that the presently known perturbative field-theoretical methods capture the main features of the scaling functions. In order to achieve this, one may have to resum ε -expansion series for $F_R(v)$.⁶ However, the two non-vanishing terms presently known are not enough to be able to do this in a reliable way.

Similarly, if one uses the representation of LSI valid for stationary systems and hence with the assumption $a = a'$, the resulting prediction is in clear disagreement with the numerical data. Indeed, since then local scaling leads to the same form as a one-loop field-theoretical calculation, one has a prediction which is slightly farther away from the data than the result of the two-loop calculation. On the other hand, if one takes into account from the outset

⁶ Recall from Chap. 2 in Volume 1 that also for equilibrium systems the ε -expansion has to be resummed before its results can be used as numerical tools and a comparison with simulations or experiments becomes feasible.

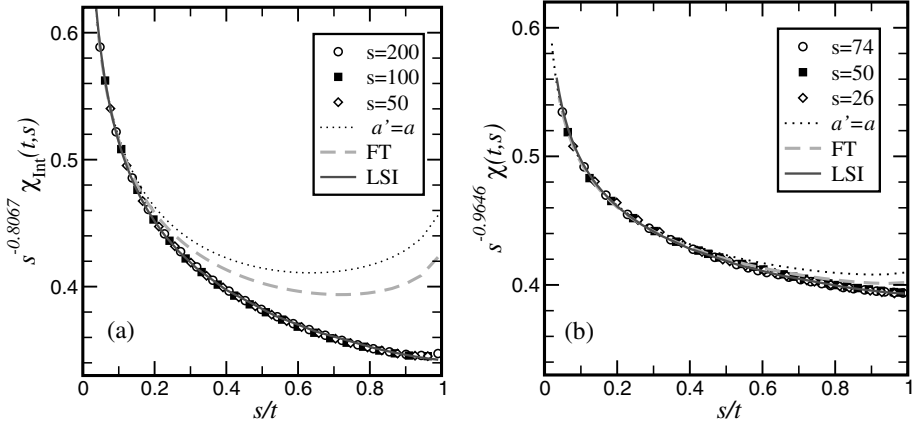


Fig. 5.4. Intermediate susceptibility $\chi_{\text{Int}}(t, s)$ in momentum space in the (a) 2D and (b) 3D critical Ising model, for several values of the waiting time s . The data points were calculated from the heat-bath algorithm [596], the dashed line labelled FT is obtained from the second-order ε -expansion [118, 596], the dotted line labelled $a' = a$ is obtained from (5.67) with the extra assumption $a = a'$ and the line labelled LSI is the prediction of local scale-invariance with $a' - a = -0.17$ in 2D and $a' - a = -0.022$ in 3D, respectively. Modified after [596, 336].

that time-translation-invariance is broken in ageing systems and therefore admits the possibility that $a \neq a'$, as it is completely natural to do in local scale-invariance, the new parameter $a' - a$ may be chosen such that an almost perfect agreement with the numerical data can be achieved.⁷ Especially in 2D, the improvement over the two-loop calculation is impressive, see Fig. 5.4a.

In Chap. 4, we have seen that $a - a' \neq 0$ becomes possible if the physical order-parameter ϕ is related to the corresponding quasi-primary scaling operator Φ via the time-dependent relation $\phi(t) \sim t^{2\xi/z} \Phi(t)$ [336]. Such a kind of relationship is already found in simple mean-field schemes for quenches to $T = T_c$ (see pp. 238, 257, 297), but does not seem to play a role for quenches to $T < T_c$ where one therefore expects that $a = a'$ should generically hold. The more indirect relationship between the physical observables and the corresponding quasi-primary scaling operators will only become manifest in the form of the response function if $\xi + \tilde{\xi} \neq 0$, however.

Having seen that the presently available results from the field-theoretical renormalisation group [118, 121] are not yet capable of reproducing the available numerical data, it seems premature to use results of these calculations in order to decide on the validity of general symmetry arguments such as LSI. On the other hand, LSI reproduces the available numerical data very well.

For reference, we list in Table 5.2 models where the possible difference between the exponents a and a' has been explicitly investigated. In numeri-

⁷ In [336], we extracted $a' - a = -0.187(20)$ in 2D from a slightly different plot.

model	d	a	$a' - a$	λ_R/z	Ref.
Ising	1	0	$-1/2$	$1/2$	[285, 589]
	2	0.115	$-0.17(2)$	$0.732(5)$	[596, 336]
	3	0.506	$-0.022(5)$	$1.36(2)$	[596, 336]
Ising spin glass	3	0.060(4)	$-0.76(3)$	$0.38(2)$	[346]
FA	1	1	$-3/2$	2	[506]
	> 2	$1 + d/2$	-2	$2 + d/2$	[506]
contact process	1	-0.681	$0.270(10)$	$1.76(5)$	[230, 336]
NEKIM	1	$-0.430(2)$	$0.00(1)$	$1.9(2)$	[552]
OJK model	≥ 2	$(d-1)/2$	$-1/2$	$d/4$	[511, 346]

Table 5.2. Known values of the exponents $a = (d - 2 + \eta)/z$, a' and λ_R/z which describe the autoresponse $R(t, s)$ for non-equilibrium critical dynamics; and also for the OJK-model at $T = 0$, see eq. (2.170). FA denotes the Fredrickson-Andersen model and NEKIM the non-equilibrium kinetic Ising model with conserved parity.

cal studies, the integrated autoresponse in direct space does apparently not depend very sensitively on the difference $a' - a$ and it is better to work in momentum space [596], although alternating time-dependent fields may be a useful alternative, see p. 317. For this reason, in older studies $a' = a$ was often simply assumed. In the several variants of the spherical model and in mean-field theory, although the order-parameter ϕ is no longer identical to the most simple scaling operator Φ , an analogous effect on the response operator compensates this, such that $a' = a$ in all models of this kind.

Presently, at least for model A dynamics, it seems that at $T = T_c$ the assumption of an independent exponent $a' \neq a$ sometimes gives a much better agreement with simulational data; for phase-ordering ($T < T_c$) the available data are consistent with $a = a'$, see Chap. 4. In several cases of non-equilibrium critical dynamics, LSI with the extra assumption $a = a'$ is ruled out by simulational data or an exact solution. This is consistent with what our earlier mean-field treatment, see eq. (4.41), had suggested.

5.6.5 Critical XY Model

The scaling function of the autoresponse $R(t, s)$ of the 3D XY model, quenched to $T = T_c$ from a fully disordered initial state and calculated in direct space, is shown in Fig. 5.5. Its form is seen to be precisely reproduced by local scale-invariance [6]. In this comparison, $a' = a$ was assumed.

5.6.6 Phase-ordering Kinetics in the 2D Ising Model

The general theory explained in the first section of this chapter can also be specialised to the case $z = 2$. This is fortunate, because it gives us the oppor-

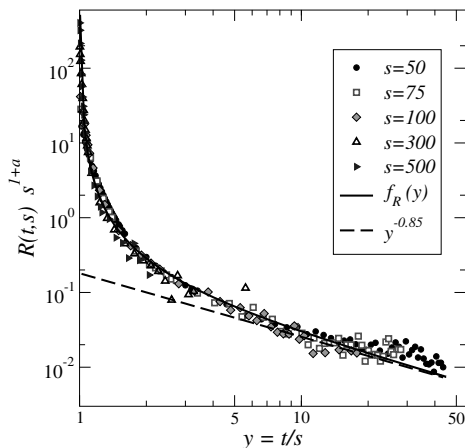


Fig. 5.5. Response function of the critical 3D XY model, for several values of s . The full line is the LSI prediction (5.41). Reprinted from [6]. With kind permission of the European Physical Journal (EPJ).

tunity to re-consider the scaling form of the two-time autocorrelation function in phase-ordering kinetics. In Chap. 4, we had used conformal invariance to determine the last remaining scaling function, but this was only possible because we uncritically had accepted that the correct initial single-time correlator was $C_0(\mathbf{r}) = a(0, \mathbf{r}) = a_0 \delta(\mathbf{r})$. Indeed, we had seen from Fig. 4.8 that although the predicted curve broadly agreed for all values of the scaling variable $y = t/s$ with the simulational data, there were systematic differences for smaller values $y \lesssim 2$.

We start from the consideration that since the passage towards the ageing regime occurs at rather large waiting times s , the initially uncorrelated system has had the time to evolve some short-ranged correlations, which must be taken into account. We propose to use the form (5.51), which in turn depends on the chosen single-time correlator $C_s(\mathbf{r})$. As we had seen in Chap. 1, the rigidity of the interfaces in the phase-ordering $O(n)$ -model was well taken into account by the BPT-form [100, 700] quoted in eq. (1.57). For the Ising model, this reduces to

$$C_s(\mathbf{r}) = \frac{2}{\pi} \arcsin \left[\exp \left(-\frac{\mathbf{r}^2}{L^2(s)} \right) \right], \quad (5.68)$$

where $L^2(s) \simeq s/\nu$ for large times. Inserting this into (5.51), it can be shown that in two spatial dimensions and for phase-ordering kinetics with $z = 2$, the autocorrelator takes the scaling form

$$C(y, s, s) = C_0 y^{(d-\lambda_C)/2} (y-1)^{-d/2-\lambda_C} \int_0^\infty dx e^{-x} f_\nu \left(\sqrt{\frac{x}{y-1}} \right) \\ f_\nu(\sqrt{u}) = \int_0^\infty dv \arcsin(e^{-\nu v}) J_0(\sqrt{uv}), \quad (5.69)$$

see exercise 5.15. Herein, J_0 is a Bessel function, C_0 is some undetermined normalisation constant and the parameter ν must be fitted to the data.

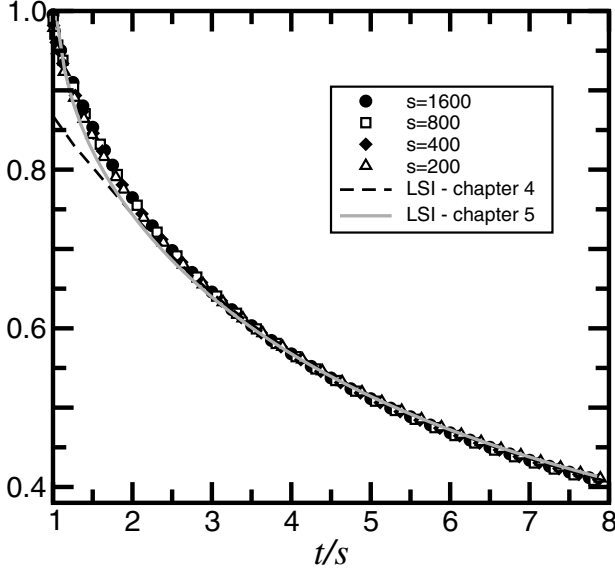


Fig. 5.6. Comparison of the predictions of local scale-invariance for the autocorrelator $C(t, s)$ in the $2D$ Ising model quenched to $T = 0$ from a totally uncorrelated initial state, for several values of the waiting time s . The data are compared to two predictions of LSI: the one labelled ‘chapter 4’ is given by eqs. (4.111,4.131) and assumes a fully uncorrelated state at the onset of ageing and the one labelled ‘chapter 5’ uses eq. (5.69), assuming the short-ranged correlator (5.68).

In Fig. 5.6 we compare the simulational data for the $2D$ Glauber-Ising model [342] for $C(t, s)$ with the two predictions of local scale-invariance, obtained in Chapters 4 and 5. The dashed curve is identical to what we had called ‘LSI-prediction’ in Fig. 4.8a,b and falls systematically below the simulational data for sufficiently small values of $y = t/s$. On the other hand, a very good overall agreement between the prediction (5.69) as given by the full line is seen. This indicates that the assumed delta-correlations in eqs. (4.111,4.131) do not reproduce faithfully enough the actual behaviour of the Ising model, while the assumed initial correlator (5.68) allows for a much better fit.⁸ Still, the agreement is not yet perfect. This may not come as a surprise, however, since it is known that the form used here does not take into account the increase of the interface width as a function of time [565], see p. 43 for a more precise form. We expect that using forms of this kind for $C_s(\mathbf{r})$, the agreement of LSI with the simulations can be even further improved.⁹

⁸ We point out that a simple exponential form $C_s(\mathbf{r}) \sim \exp(-(r/L(s))^2)$ is not sufficient in the $2D$ Ising model.

⁹ It would be interesting to re-analyse $C(t, s)$ in the phase-ordering of the Potts- q model in an analogous way, see Fig. 4.9 [481].

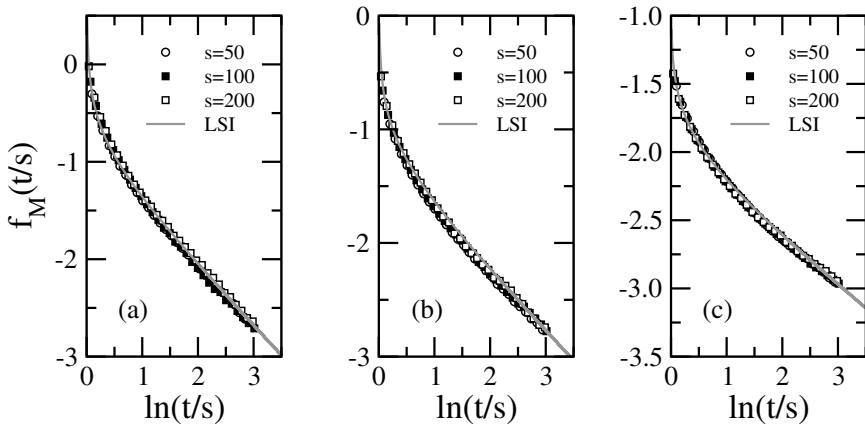


Fig. 5.7. Scaling function $f_M(t/s)$ of the thermoremanent susceptibility in the 2D disordered Ising model, for several values of s and with the parameters (a) $\varepsilon = 0.5$, $T = 1.0$, (b) $\varepsilon = 1.0$, $T = 0.8$ and (c) $\varepsilon = 2.0$, $T = 0.8$. The full line labelled LSI is the prediction (5.64), where $a = a'$ and λ_R/z are given in Table 3.4.

This is the first time that a quantitatively precise theory for the scaling of the two-time autocorrelator for the complete range of the scaling variable $y = t/s$ has been formulated.

5.6.7 Phase-ordering in the 2D Disordered Ising Model

For the phase-ordering in the disordered, but unfrustrated two-dimensional Ising model, we have seen in Sect. 3.5 that the dynamical exponent z depends continuously on the temperature T and the disorder. At the same time, the available evidence for *superuniversality*, which also applies to the spatio-temporal two-time correlators and responses, with the exception of their $\mathbf{r} \rightarrow \mathbf{0}$ limits, states that the form of their scaling functions should be independent of the disorder.

It remains to consider the form of the scaling function for $\mathbf{r} = \mathbf{0}$. In Fig. 5.7, we show data for the thermoremanent susceptibility $\chi_{\text{TRM}}(t, s) = s^{-a} f_M(t/s)$. For the available time-scales, an excellent collapse is seen and when using the data from Table 3.4 for the exponents $a = a'$ and λ_R/z , a very good agreement with the LSI-prediction eq. (5.64) is found.

It is not understood how to formulate in the context of LSI the passage between the autoresponses, whose scaling function depends on the disorder, and the super-universal spatio-temporal scaling, where the disorder merely enters into the time-dependent domain size $L = L(t)$.

5.6.8 Critical Ising Spin Glass I: Thermoremanent Susceptibilities

In Chap. 3, we have described the dynamical scaling behaviour of the thermoremanent susceptibility, for several choices (bimodal, Gaussian, Laplacian) of the distribution of the coupling constants, both for $d = 3$ and $d = 4$. In Fig. 3.12, we illustrate the comparison of the form of the scaling function $f_M(t/s)$ as obtained from our simulations with the prediction of local scale-invariance, for the case of a binary distribution of the couplings [346]. For the range of ratios $y = t/s$ available, we saw that the data are described in terms of an effective autoresponse exponent $\lambda'_R/z \neq \lambda_C/z$, in contrast to our observation of a finite limit fluctuation-dissipation ratio X_∞ . The LSI-predictions indicated by the full curves in Fig. 3.12 were obtained by using the values of λ'_R/z and do give a nice agreement with the numerical data. Similar results were obtained for the other distributions [346].

The meaning of this agreement with the numerical data and LSI in this intermediate regime remains to be understood. The fully asymptotic regime in the scaling function $f_M(y)$ is apparently not yet accessible to simulations.

5.6.9 Critical Ising Spin Glass II: Alternating Susceptibilities

The behaviour of critical spin glasses may also be analysed by using a time-dependent (oscillating) magnetic field, thereby studying simultaneously the dependence on time and on the imposed oscillation angular frequency ω . In experiments, one usually averages over at least one period of the oscillating field. In a great variety of glassy materials quenched to below or near to their glass-transition point one observes good but not always perfect evidence for an ωt -scaling behaviour, here for the period-averaged dissipative (imaginary) part [407, 688]

$$\overline{\chi''}(\omega, t) = \chi''_{\text{st}}(\omega) + \chi''_{\text{age}}(\omega, t) \quad , \quad \chi''_{\text{age}}(\omega, t) \simeq A''_{\text{age}} (\omega t)^{-b''} \quad (5.70)$$

where the amplitude A''_{age} and the exponent b'' are fitted to the experimental data. One expects a similar scaling for the dispersive part $\overline{\chi'}$ and it is usually thought that $b' = b''$.

While the exponents $b' = b''$ are usually treated as being independent of the other ageing exponents, local scale-invariance predicts the scaling relation

$$b' = b'' = a - a' \quad (5.71)$$

which allows us to measure the exponent difference $a - a'$ directly.

Proof: The starting point is the scaling relation (1.104). Inserting the prediction of LSI for the scaling function $f_R(y)$, we find, together with their leading behaviour as $y \rightarrow \infty$,

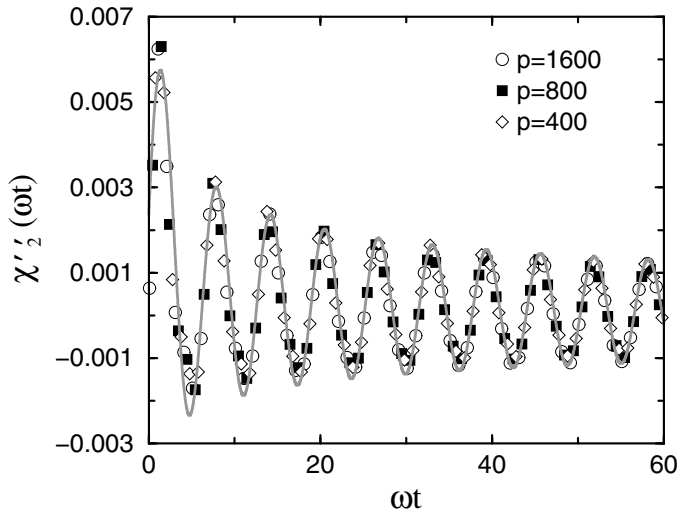


Fig. 5.8. Scaling of the dissipative part $\chi''_2(\omega t)$ of the alternating susceptibility as a function of the scaling variable ωt for different angular frequencies $\omega = 2\pi/p$ with $p = 1600, 800$, and 400 . The full curve is the theoretical prediction (5.72) with $f_0 = 0.002$ and $a' = -0.70$ but which has also been shifted horizontally by $y \rightarrow y + \Delta y$, with $\Delta y = -0.45$, see text. Reprinted from [346]. Copyright (2005) Institute of Physics Publishing.

$$\begin{aligned} \chi''_2(y) &= f_0 B \left(1 - a', \frac{\lambda_R}{z} - a \right) y^{1-a} \\ &\times {}_2F_3 \left(\frac{1-a'}{2}, \frac{2-a'}{2}; \frac{3}{2}, \frac{1-a-a'}{2} + \frac{\lambda_R}{2z}, \frac{2-a-a'}{2} + \frac{\lambda_R}{2z}; -\frac{y^2}{4} \right) \\ &\simeq f''_1 y^{a'-a} + f''_2 y^{-\lambda_R/z} \sin \left(y + \frac{\pi}{2} [a - \lambda_R/z] \right) \end{aligned} \quad (5.72)$$

where ${}_2F_3$ is a hypergeometric function and $f''_{1,2}$ are known constants. It follows that $\chi''_2(y)$ is the sum of a monotonously decreasing term and of an oscillating term. For y sufficiently large, the contribution of the oscillating term should vanish after averaging over at least one period. From the remaining algebraic term the exponent $a - a'$ can be extracted since $\chi''_2(y) \sim y^{a'-a}$ for $y \gg 1$ and the assertion follows. A similar argument works for b' . q.e.d.

This and the scaling function (5.72) have been checked in the 3D Ising spin glass with bimodal disorder, quenched to its critical point $T_c \approx 1.19$ from an uncorrelated initial state, and in an oscillating magnetic field $h(t) = h_0 \cos \omega t$. Then $\chi''(\omega, t) = \sum_i \sigma_i(t) \sin \omega t$, but the equilibrium parts of χ'' and χ' , see (1.104), must be subtracted off. The result is shown in Fig. 5.8.

For the larger values of p , corresponding to the smaller values of ω , one observes a very good data collapse, in agreement with the expected simple

Material	T_g [K]	T [K]	b''	b'	Ref.
Cu _{0.5} Co _{0.5} Cl ₂ -FeCl ₃ – GBIC	3.92(11)	3.25	0.01(4)	0.08(3)	[688] Ising spin glass
		3.5	0.017(32)	0.05(2)	
		3.75	0.16(3)	0.20(2)	
		3.85	0.15(3)		
		3.95	0.16(4)	0.20(2)	
Fe _{0.5} Mn _{0.5} TiO ₃	20.7	15	0.14(3)		[214, 211] Ising spin glass
		19	0.14(3)		
CdCr _{1.7} In _{0.3} S ₄	16.7	12	0.18(3)		[214, 211] Heisenberg spin glass
		14	0.18(3)		
CdCr _{2x} In _{2-2x} S ₄ $x = 0.95$	70	8	0.2		[213, 211] disordered ferro- magnet
		67	0.2		
		50	0.20		
$x = 0.90$	50	42	0.20		[715, 211]
Pb(Mg _{1/3} Nb _{2/3})O ₃	~ 220	$\lesssim 220$	0.17		[151] relaxor ferroelectric
3D Ising spin glass ($T = T_c$)			0.76(3)	0.76(3)	[346]

Table 5.3. Measured values of the exponents b'' and b' in several glassy materials, using the scaling form (5.70). Here T_g stands for the glass transition temperature and T is the temperature where the data were taken. For Fe_{0.5}Mn_{0.5}TiO₃ and CdCr_{1.7}In_{0.3}S₄ the relation $b' = b''$ was assumed. The simulational results in the critical Ising spin glass are also included.

ageing at $T = T_c$. For smaller values of p , the collapse is less good. Comparison with (5.72) shows a good agreement with the data, provided however that the scaling variable is shifted by $\Delta y \approx -0.45$ as compared to (5.72). The origin of this shift is not yet understood, but conceivably it might be related to the assumption of the rapid cross-over between the quasi-stationary and ageing regimes which was assumed in the derivation of (1.104). A similar description, with the *same* shift, also works for the dispersive part χ'_2 [346]. It appears that LSI already describes many features of this critical spin glass.

For comparison, we list in Table 5.3 some experimentally measured values of b' and b'' . There is quite good evidence confirming the relation $b' = b''$, as discussed in detail in [688]. Furthermore, the good experimental evidence for a pure ωt -scaling suggests that the exponent a should be quite small. On the other hand, errors on the exponent estimates are still too large to allow for a distinction between, say, Ising and Heisenberg systems. Surprisingly, the available experimental estimates $b' = b'' \approx 0.1 - 0.2$ are very far from the value $b'' \simeq 0.76$ found in the critical 3D Ising spin glass with binary disorder. Could this be seen as an indication that the spin glass models considered by theorists only capture imperfectly what is going on in real materials ?

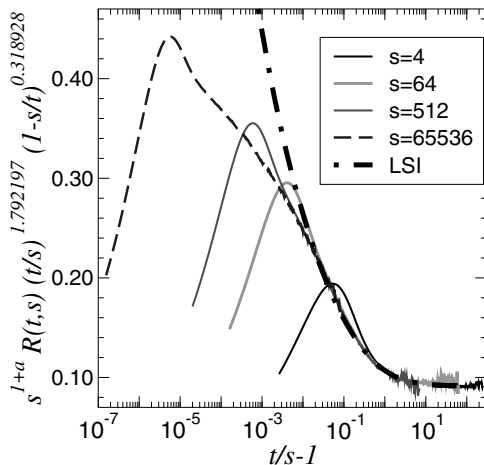


Fig. 5.9. Comparison of data of the autoresponse $R(t, s)$, obtained from the transfer-matrix renormalisation group, for the 1D critical contact process with the local scale-invariance prediction (5.41). Data courtesy T. Enns.

5.6.10 Critical Particle-reaction Models

The scaled autoresponse $s^{1+a}R(y, s)$ of the 1D critical *contact process* was shown in Fig. 3.18. We point out that these data were obtained [230, 364] starting from a non-vanishing initial value of the order-parameter, which is here the mean particle-density. Therefore, the LSI-prediction (5.41), derived for an initially vanishing order-parameter, is not directly applicable. Still, one may make a numerical experiment. In this spirit, comparing the data with the LSI-prediction, we see that if the additional hypothesis $a = a'$ is made (dashed line in Fig. 3.18), systematic deviations occur if $t/s \lesssim 2$. On the other hand, one may choose a' so that the data may be very well fitted, down to $y - 1 \approx 10^{-3}$ when systematic deviations are seen for s large enough that one is still in the scaling regime. It is remarkable that the simple LSI prediction approximates the data so well, but it is not understood why this is so. This point is further illustrated in Fig. 5.9, where a re-scaled autoresponse is plotted in order to investigate more closely the limit $t/s \rightarrow 1$. For very long waiting times, one observes that although dynamical scaling still appears to be valid, the data fall systematically below the prediction (dash-dotted line), which was derived for an initially vanishing order-parameter.

The field-theoretical study [46], to one-loop order, of the directed percolation universality class produces a systematic correction to the LSI prediction, if the initial order-parameter is non-vanishing. It is not clear, however, to what extent a truncated first-order ε -expansion series can be used as a tool for quantitative analysis. On the other hand, compatibility with LSI is found for an infinitesimally small initial particle-density.

The scaled spin autoresponse $s^{1+a}R(y, s)$ of the critical 1D *non-equilibrium kinetic Ising model* (NEKIM) [552] is shown in Fig. 3.20 as a function of $y = t/s$, for several values of the waiting time. Clearly, the data are fully consistent

with the LSI-prediction over the entire range of values of y considered, with the ageing exponents listed in Table 5.2. In the inset of Fig. 3.20, a considerably more demanding test of LSI is performed by plotting $R(t, s)/R^{\text{LSI}}(t, s)$ over against t/s , where any deviation from a constant would signal a departure from LSI. Indeed, within the numerical accuracy (and in contrast to what is found for the contact process), the data are consistent with being constant and departures only occur (around $y - 1 \lesssim 0.1$) when finite-time corrections to the scaling behaviour become sizeable.

5.6.11 Bosonic Particle-reaction Models

In Chap. 3, we outlined the exact solution of the critical bosonic contact process with Lévy-flight transport of single particles (BCPL) [215]. The creation and annihilation operators used to set up the Liouvillian become related in the continuum limit to the order-parameter field and a conjugate response field $\phi(t, \mathbf{r}) := a(t, \mathbf{r}) - \rho_0$ and $\tilde{\phi}(t, \mathbf{r}) := a^\dagger(t, \mathbf{r}) - 1$ with vanishing averages. Then the Janssen-de Dominicis functional $\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_0[\phi, \tilde{\phi}] + \mathcal{J}_b[\phi, \tilde{\phi}]$ is decomposed into a ‘deterministic’ term and a ‘noise’ term, where

$$\begin{aligned}\mathcal{J}_0[\phi, \tilde{\phi}] &= \int d\mathbf{R} \int du \left[\tilde{\phi}(2\mathcal{M}\partial_u - \Delta_L^{\eta/2})\phi \right], \\ \mathcal{J}_b[\phi, \tilde{\phi}] &= - \int d\mathbf{R} \int du \left[\mu \tilde{\phi}^2(\phi + \rho_0) \right],\end{aligned}\quad (5.73)$$

whereas ρ_0 is the average mean particle density and $\mathcal{M}$ is related to the Lévy-diffusion constant. The non-integral power of the Laplacian has to be interpreted as a fractional derivative, see appendix J.2.

The proof of local scale-invariance in this model [215] proceeds by combining the techniques applied for the long-range spherical model in subsection 5.6.2 [45, 217] with the method used in Chap. 4 for the BCPD where $z = 2$ [52]. For the space-time response function we readily recover the LSI result (5.40). For the two-time correlator, we have $C(t, s; \mathbf{r}, \mathbf{r}') = \langle \phi(t, \mathbf{r})\phi(s, \mathbf{r}')e^{-\mathcal{J}_b[\phi, \tilde{\phi}]} \rangle_0 =: C_1 + C_2$, where

$$\begin{aligned}C_1(t, s; \mathbf{r}, \mathbf{r}') &= -2\mu\rho_0 \int d\mathbf{R} \int du \left\langle \phi(t, \mathbf{r})\phi(s, \mathbf{r}')\widetilde{\phi^2}(u, \mathbf{R}) \right\rangle_0 \\ C_2(t, s; \mathbf{r}, \mathbf{r}') &= \mu^2 \int d\mathbf{R} d\mathbf{R}' \int du du' \left\langle \phi(t, \mathbf{r})\phi(s, \mathbf{r}')\mathcal{Y}(u, \mathbf{R})\mathcal{Y}(u', \mathbf{R}') \right\rangle_0\end{aligned}\quad (5.74)$$

where the Bargman superselection rules were used and $\mathcal{Y} := \widetilde{\phi^2}\phi$ is a composite field with a scaling dimension $x_{\mathcal{Y}} = 2\tilde{x} + x$. Applying the factorisation of the four-point functions, a dimensional analysis shows that $C_n(t, s; \mathbf{r}, \mathbf{r}') = s^{1-2nd/\eta} F_n(t/s, \mathbf{r}s^{-1/\eta}, \mathbf{r}'s^{-1/\eta})$. Hence in the $s \rightarrow \infty$ limit, only C_1 contributes. Explicit calculation of this integral, along the lines described in previous Sections, reproduces the exact correlation function [215].

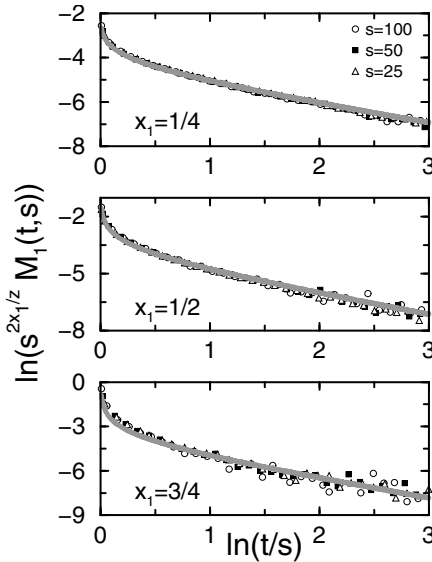


Fig. 5.10. Integrated surface thermoremanent magnetisation $M_{\text{TRM},1}(t, s)$ of the critical Hilhorst-van Leeuwen model with the surface scaling dimensions $x_1 = [1/4, 1/2, 3/4]$ from top to bottom. The full line gives the prediction (5.75). Reprinted with permission from [592]. Copyright (2004) by the American Physical Society.

From the point of view of LSI, this model provides a test with a different kind of noise term than those encountered in systems with detailed balance. In contrast to the case of the critical BCPD with $z = 2$, the factorisation of $2n$ -point responses functions is needed.

5.6.12 Surface Effects

Close to a free surface, the local thermoremanent magnetisation $M_{\text{TRM},1} = h\chi_{\text{TRM},1}$, obtained from the surface *autoresponse* functions, can be predicted from LSI by simply replacing the bulk ageing exponents by their surface counterparts, see Chap. 3. In this way, one may generalise eq. (5.64) as follows (in addition, $a_1 = a'_1$ was assumed)

$$\chi_{\text{TRM},1}(t, s) = \int_0^s du R_1(t, u) = s^{-2x_1/z} f_{M_1}(t/s) \quad (5.75)$$

$$f_{M_1}(y) = r_0 y^{-\lambda_{R_1}/z} {}_2F_1 \left(1 + \frac{2x_1}{z}, \frac{\lambda_{R_1}}{z} - \frac{2x_1}{z}; \frac{\lambda_{R_1}}{z} - \frac{2x_1}{z} + 1; \frac{1}{y} \right)$$

with the surface scaling dimension $x_1 = \beta_1/\nu$. $\lambda_{R_1} = \lambda_{C_1}$ was already given in (3.84) and r_0 is a normalisation constant.

The prediction (5.75) has been confirmed [592] (i) in the critical semi-infinite Ising model, both at the ordinary and at the special transition and (ii) in the critical Hilhorst-van Leeuwen model, as illustrated in Fig. 5.10.¹⁰

¹⁰ In view of the results in [596, 336], a reconsideration of these models in momentum space and allowing for the possibility $a_1 \neq a'_1$ would be of interest.

5.7 Global Time-reparametrisation-invariance

In order to obtain a somewhat broader view of possible extensions of dynamical scaling, we now outline a different and very interesting approach, called **global time-reparametrisation-invariance** (GTRI) which has been developed by Chamon, Charbonneau, Castillo, Cugliandolo, Iguain, Jaubert, Kennett, Picco, Reichman, Sellitto and Yoshino. We limit ourselves here to a mere outline of their main ideas and follow the review by Chamon and Cugliandolo [139]. The approach of GTRI is meant to explain the origin of dynamic fluctuations in glassy systems. This is a huge class of quite different looking systems such as polymer glasses, spin glasses, soft glassy materials or orientational glasses, see [139] and references therein. We add that it has become common to consider even totally unfrustrated, but kinetically constrained systems as useful illustrations of glassy behaviour. An example of this is the Fredrickson-Andersen model which is equivalent to a coagulation process $2A \rightarrow A$ of a single species of particles with single-particle diffusion, see [508] and references therein. Whether, or to what extent, all of these systems really could or should be considered under a single unifying perspective, is an open and exciting problem.

A starting point for introducing GTRI may be the study of spin glasses through the *p*-**spin models** [167] defined by the classical Hamiltonian

$$\mathcal{H} = - \sum_{j_1, j_2, \dots, j_p} J_{j_1, j_2, \dots, j_p} S_{j_1} S_{j_2} \cdots S_{j_p} \quad (5.76)$$

where the spin variables may be Ising spins $S_j = \pm 1$ or spherical spins $S_j \in \mathbb{R}$ satisfying the spherical constraint $\sum_{j \in \Lambda} S_j^2 = \mathcal{N}$, where $\mathcal{N}$ is the number of sites of the lattice Λ . The couplings $J_{j_1, j_2, \dots, j_p}$ are quenched random variables distributed according to the Gaussian law $P(J_{j_1, j_2, \dots, j_p}) \sim \exp(-p! J_{j_1, j_2, \dots, j_p}^2 / (2\mathcal{N}^{p-1}))$. Finally, the sum runs over all *p*-plets of sites, which gives the model a mean-field character since the partition function in the infinite-volume limit $\mathcal{N} \rightarrow \infty$ can be found exactly by saddle-point methods. The two-time disorder-averaged autocorrelation and autoresponse functions are defined as usual and satisfy in the limit $\mathcal{N} \rightarrow \infty$ the causal Schwinger-Dyson equations [139]

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathfrak{z}(t) \right) C(t, s) &= \int_0^t dt' \Sigma(t, t') C(t', s) + \int_0^s dt' D(t, t') R(s, t') \\ \left(\frac{\partial}{\partial t} - \mathfrak{z}(t) \right) R(t, s) &= \int_s^t dt' \Sigma(t, t') R(t', s) + \delta(t - s) \end{aligned} \quad (5.77)$$

where the vertex D and the self-energy Σ are given by

$$D(t, t') = \frac{p}{2} C(t, t')^{p-1}(t, t') \quad , \quad \Sigma(t, t') = \frac{p(p-1)}{2} C(t, t')^{p-2}(t, t') R(t, t') \quad (5.78)$$

and the Lagrange multiplier $\mathfrak{z}(t)$ is fixed by the condition $C(t, t) = 1$. For low temperatures, one admits a decomposition of the form

$$C(t, s) =: C_{\text{st}}(t - s) + C_{\text{ag}}(t, s) \quad , \quad \chi(t, s) =: \chi_{\text{st}}(t - s) + \chi_{\text{ag}}(t, s) \quad (5.79)$$

where $\chi(t, s) := \int_s^t du R(t, u)$ is the ZFC susceptibility. In the limit of long waiting times, the Schwinger-Dyson equations simplify since the variation of the correlation and the susceptibility become small with respect to the other terms and $\mathfrak{z}(t) \xrightarrow{t \rightarrow \infty} \mathfrak{z}_\infty$. Then the equation for R becomes [139]

$$\mathfrak{z}_\infty R_{\text{ag}}(t, s) \simeq \frac{p(p-1)}{2} \int_s^t dt' C_{\text{ag}}^{p-2}(t, t') R_{\text{ag}}(t, t') R_{\text{ag}}(t', s) \quad (5.80)$$

together with a similar equation for C_{ag} . Remarkably, and quite unexpectedly, these approximate equations are invariant under the global time-reparametrisations $t \mapsto h(t)$ where [139]

$$C_{\text{ag}}(t, s) \mapsto C_{\text{ag}}(h(t), h(s)) \quad , \quad R_{\text{ag}}(t, s) \mapsto \dot{h}(s) R_{\text{ag}}(h(t), h(s)) \quad (5.81)$$

where the function $h(t)$ as well as its derivative $\dot{h}(t)$ are positive. Indeed, this looks like an infinite-dimensional extension of LSI in the limit where the dynamical exponent $z \rightarrow \infty$. If the usual representations $C_{\text{ag}}(t, s) = \langle \phi(t) \phi(s) \rangle$ and $R_{\text{ag}}(t, s) = \langle \phi(t) \tilde{\phi}(s) \rangle$ are still applicable, and ϕ and $\tilde{\phi}$ could still be interpreted as quasi-primary scaling operators, one would infer the scaling dimensions $x_\phi = 0$ and $x_{\tilde{\phi}} = 1$. The consideration of $z = \infty$ in glassy systems may be a very sensible one, since it is widely, although not universally, believed that the linear size of correlated domains of such systems does not grow algebraically, but only logarithmically, see [747, 166, 431, 524].

As explained in [139], one may use the Janssen-de Dominicis action of the 3D Edwards-Anderson model of spin glasses where the average over the disorder can be carried out explicitly. It can then be shown that GTRI holds also for disorder-averaged action in the long-time limit and, indeed, this invariance of the action is considered to be the fundamental property of the theory. In principle, such a large symmetry should lead to far-reaching consequences but since it is only a symmetry of the large-time limit, the short-time parts of the action will select a single one of the functions $h(t)$ (the analogy with a small magnetic field breaking the global symmetry of a spin system is made here) so that one essentially reduces to dynamical scaling. So far, GTRI only involves changes in time but leaves the spatial coordinates unchanged. It is suggested that one might consider space-dependent reparametrisations $t \mapsto h(t, \mathbf{r})$ by perturbing around GTRI, in analogy with the spin-wave fluctuations in magnetic systems [139]. A formal proof of GTRI for glassy systems with two-body interactions at the renormalisation-group fixed point is given in [134].

While the approach of GTRI is meant to describe glassy systems (including structural glasses), apparently it does not extend to phase-ordering, see

[140, 155] and refs. therein. This has been checked in the $p = 2$ spherical Sherrington-Kirkpatrick model (which is in the same dynamical universality class as the 3D spherical model [167, 760]). An explicit calculation shows that the simplifications which led to (5.80) no longer apply in this case.¹¹ Hence GTRI does *not* hold for the $p = 2$ case [139] which therefore should not be considered as a ‘glassy’ system in the sense used above. The differences between glassy systems and phase-ordering have been carefully analysed [140].¹²

Among the predictions of GTRI, Chamon and Cugliandolo [139] list the qualitative properties of (i) a growing dynamical correlation length and (ii) dynamical scaling of the probability distribution function (PDF) of local two-time functions. These are general properties any scaling theory will have and are not specific to GTRI. Quantitatively, and specific to GTRI, is (iii) the functional form of the just mentioned PDF whose non-Gaussian and Gumbel-like form was successfully tested in the 3D Edwards-Anderson model, in certain non-disordered, but kinetically constrained systems and Lennard-Jones fluids (iv) certain triangular relations between two-time functions (which one might also recover from dynamical scaling) or (v) local fluctuation-dissipation relations. Certainly, GTRI is capable of producing falsifiable predictions which allow us to test it. The theory did pass some non-trivial tests, as described in detail in [135, 136, 138, 139, 140, 578].

It seems to be too early to formulate a definite appreciation of GTRI, let alone make a detailed comparison with LSI. Perhaps we might understand in the future that both GTRI and LSI captured some distinct aspects of a larger structure of extended dynamical scaling which at present is still unknown to us? In the absence of any deeper insight, probably the best that one might do presently is to list schematically some of the main properties of both, in order to make the similarities and the differences more clear. This is done in Table 5.4.

¹¹ This result is unsurprising, since for a simple magnet, the noise-average would bring us from the Janssen-de Dominicis functional $\mathcal{J}$ to the more complicated Onsager-Machlup functional, without any obvious non-trivial symmetries. As we have shown in Chap. 4 for phase-ordering kinetics (where $z = 2$), a non-trivial symmetry can *only* be found if one can split $\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_0[\phi, \tilde{\phi}] + \mathcal{J}_b[\tilde{\phi}]$ such that the properly identified ‘deterministic’ part $\mathcal{J}_0$ may possess non-trivial symmetry and the structure of the ‘noise’ part $\mathcal{J}_b$ is such that averages reduce to ‘deterministic’ averages in a simple way. This is perhaps the most essential conceptual difference between GTRI and LSI.

¹² Therein, properties of the non-disordered spherical model were compared with those of the p -spin spherical spin glass ($p \geq 3$) and other glassy systems. Some differences are: (i) for $t \rightarrow \infty$, the Lagrange multiplier $\mathfrak{z}(t) \rightarrow 0$ for the spherical model but $\mathfrak{z}(t) \rightarrow \mathfrak{z}_\infty \neq 0$ in the p -spherical glass; (ii) for quenches to $T < T_c$, the FDR is $X_\infty = 0$ in the spherical model but $X_\infty > 0$ in glasses; (iii) the probability distribution functions for the two-time correlator have a different form, compatible with a Gumbel-like form for glasses but not in the spherical model.

	LSI	GTRI
starting point	free diffusion equation ($z = 2$)	mean-field p -spin models ($z = \infty$)
	generalises Schrödinger-invariance	infinite-dimensional symmetry reduced to dilatations by initial conditions
symmetry	deterministic part of Janssen-de Dominicis action $\mathcal{J}$	disorder-averaged Janssen-de Dominicis action $\mathcal{J}$
invariant equation	deterministic part of Langevin equation	Schwinger-Dyson equation, $t \rightarrow \infty$
treatment of noise	exact reduction formulæ to deterministic part	noise integrated out
covariance	noiseless response functions	correlators and responses
spatial transformations	Galilei-invariance & Bargman superselection rule	none
predicts	two-time functions R & C	PDF for C local fluctuation-dissipation relations
tests	phase-ordering kinetics, noneq. critical dynamics, reaction-diffusion systems, growth processes, disordered models	3D Edwards-Anderson spin glass kinetically constrained systems Lennard-Jones glass disordered elastic lines

Table 5.4. Schematic comparison of some of the main features of local scale-invariance (LSI) and general time-reparametrisation invariance (GTRI).

5.8 Concluding Remarks

In the last two chapters, we have presented the current state of investigations on possible extensions of dynamical scaling to larger symmetries. Ageing phenomena were seen to provide a useful test case in order to check several of the structural properties of local scale-invariance. It should also have become clear that at present the theory of local scale-invariance is far from complete. Rather, the construction of such a theory remains a remote goal, still far away. From the very beginning, our formulation of *local scale-invariance* assumes *a power-law scaling of the single physically relevant length scale $L(t) \sim t^{1/z}$ with a finite value of z* . It therefore does not apply to systems with a different law for the time-dependence of the domain size $L(t)$.

At the time of writing, the following properties and ingredients of LSI appear to be the most important:

1. Co-variant transformation of response functions under the special conformal (or Möbius) transformation $t \mapsto t/(1 + \gamma t)$ in the time variable is assumed.
2. Galilei-invariance (for $z = 2$) and its generalisation for $z \neq 2$ are assumed.

3. The most important consequences of LSI, which arise from the algebras being non-semi-simple, appear to be the *Bargman superselection rules*. For $z \neq 2$, their generalisation further leads to the factorisation of the $2n$ -point response functions. We have interpreted this as a possible hint of some integrable structure underlying LSI. We stress that at the present level of understanding, this property follows from the *assumption* of a generalised Galilei-invariance for $z \neq 2$.
4. LSI is not a symmetry of the whole Langevin equation, but rather of the ‘deterministic part’ of that equation. This seems to depend crucially on the validity of Bargman superselection rules or a suitable generalisation thereof. LSI is therefore a *hidden* symmetry. The precise predictions of LSI strongly depend on the structure of the ‘noise terms’ in the Langevin equation.
5. Ageing phenomena arise far from equilibrium, where time-translation-invariance does not hold. Hence, the essential identity between the quasi-primary scaling operator $\tilde{\Phi}$ and the associated physical observable ϕ (and here response operators $\tilde{\phi}$ are to be included) should be replaced by a relation of the type $\tilde{\Phi}(t, \mathbf{r}) \sim t^{-2\xi/z} \phi(t, \mathbf{r})$.¹³

Time will tell if, and to what extent, these appreciations will be confirmed and if the programme of LSI can be realised in a physically relevant way.

A first confirmation of LSI consists in alternative proofs of some well-established results, namely

1. The relationship $\lambda_C = \lambda_R$ in phase-ordering kinetics and for short-ranged initial conditions (see p. 260) [589].
2. The scaling relation $\lambda_C(T_c) = d - \Theta z$ [404] in non-equilibrium critical dynamics, see eq. (1.112). Its derivation from LSI is treated in exercise 5.13.
3. To this, we add the new scaling relation eq. (5.71) for the exponents of the alternating susceptibility.

In any case, the first three assumptions can be tested systematically in models and/or experiments as follows, and in the same order as above:

1. The scaling form of the two-time *auto*response function of quasi-primary scaling operators of LSI reads

$$R(t, s) = r_0 s^{-a} \left(\frac{t}{s} \right)^{1+a'-\lambda_R/z} \left(\frac{t}{s} - 1 \right)^{-1-a'} \Theta(t-s) \quad (5.82)$$

(in agreement with the required causality) and tests for the assumed special conformal transformations in time.

¹³ Besides the non-semi-simple algebra and the Bargman superselection rules, this is the other presently recognised essential difference to conformal invariance or equilibrium critical dynamics (which include time-translation-invariance).

hypothesis	tested through
Möbius transformation-invariance	autoresponse $R(t, s)$
generalised Galilei-invariance $z \neq 2$	space-time response $R(t, s; \mathbf{r})$
factorisation	two-time correlation function

Table 5.5. Tests of assumptions underlying local scale-invariance.

- 2. The *space*-time response has the form $R(t, s; \mathbf{r}) = R(t, s)\Phi(|\mathbf{r}|/(t-s)^{1/z})$, where the function $\Phi(u)$ can be explicitly found. The form of $\Phi(u)$ can be used for testing generalised Galilei-invariance.
- 3. Through the exact reduction formulæ which follow from the (generalised) Bargman superselection rules, the two-time correlation function can be expressed in terms of four-time response functions which in turn factorise into products of two-time responses. Calculations of correlators can in principle be used to test this part of the theory.

Therefore, the analysis of the form of the scaling functions of $R(t, s)$, $R(t, s; \mathbf{r})$ and $C(t, s)$ allows us to test the main properties of LSI, one after another. This is summarised schematically again in [Table 5.5](#).

In [Table 5.6](#), we give a final overview to what extent, and in which systems, local scale-invariance has been tested, as far as we know. In giving the source (or a page reference within this volume), for brevity we usually only quote the most recent article which presents the current understanding of the specific results in terms of LSI and do not repeat the complete bibliography where the results for the scaling functions were obtained. The tests performed are either through simulational studies or via analytical calculations. The existing exact solutions (if available) and the numerical data were generally seen to be in agreement with LSI, where the numerical accuracy for the scaling functions is typically of the order of a few percent. *We consider the tests passed in the phase-ordering kinetics of the Ising and Potts models*, both for the parameter-free comparison of the space-time integrated response $R(t, s; \mathbf{r})$ and especially for the correlator $C(t, s)$ (where all previous attempts had failed to produce a theoretical expression valid for all values of the scaling variable $y = t/s$) *as a clear signal that at least for $z = 2$ the basic features of a local dynamical scaling should have been correctly identified*. When $z \neq 2$, tests have been mainly restricted to the autoresponse $R(t, s)$ and all models where tests have been carried further can be essentially reduced to a free bosonic field. The construction of a non-trivial exactly solvable model with $z \neq 2$, not simply related to free fields, and the analysis of its ageing behaviour would be very desirable. As described in Sect. 5.6, open questions remain about the interpretation of simulational data for the autoresponses of disordered systems. Furthermore, there exist also several field-theoretical renormalisation-group studies, mainly of simple magnets, and carried to one- or two-loop order, whose results in general are distinct from the LSI-predictions. Since the results of these latter

model	d	condition	$R(t, s)$	$R(t, s; \mathbf{r})$	$C(t, s)$
Ising	1	$T = 0$	[589]	[285]	[336]
	2	$T < T_c$	[344]	[344]	[337]
					p. 315
	2	$T = T_c$	[336]		
	3	$T < T_c$	[344]	[344]	
	3	$T = T_c$	[336]		
	3, s	$T = T_c$	[592]		
	$4 - \varepsilon$	$T = T_c$	[118]*		
Potts-2	2	$T < T_c$	[481]	[481]	[481]
Potts-3	2	$T < T_c$	[481]	[481]	[481]
Potts-8	2	$T < T_c$	[481]	[481]	[481]
XY	2	$T < T_c$	[589]		[5, 463]
	3	$T < T_c$	[6]		
	3	$T = T_c$	[6]		
spherical	> 2	$T < T_c$	[536, 588]	[286, 588]	[286, 589]
	> 2	$T = T_c$	[404, 588]	[286, 588]	[286, 589]
	$> 2, s$	$T \leq T_c$	[50]	[50]	
long-ranged	$> \sigma$	$T < T_c$	[127]	[127]	[45]
spherical	$> \sigma$	$T = T_c$	[45]	[45]	[45]
conserved spherical	> 2	$T = T_c$	[47]	[47]	[674, 47]
Hilhorst-van Leeuwen model	2	$T = T_c$	[592]		
disordered Ising	2	$T < T_c$	p. 316	[365]*	
	$4 - \varepsilon$	$T = T_c$	[116, 644]		
EA Ising spin glass	3	$T = T_c$	[345, 346]		
free random walk	≥ 1		[589]		[589]
contact process	1	$p = p_c$	[230, 336]		
			[364]*		
	$4 - \varepsilon$	$p = p_c$	[46]*		
	> 4	$p = p_c$	[609]		
non-equilibrium kinetic Ising	1	$p = p_c$	[552]		
voter	≥ 1	$\gamma = \gamma_c$			[589]
BCPD	≥ 1	$p = p_c$	[52]	[52]	[52]
BPCPD	≥ 2	$p = p_c$	[52]	[52]	[52]
		$\alpha \leq \alpha_C$			
BCPL	$\geq \eta$	$p = p_c$	[215]	[215]	[215]
Family	1				[629, 630]
	2				[629, 630]
Edwards-Wilkinson	≥ 1		[629, 630]	[629, 630]	[629, 630]
Mullins-Herring	≥ 1		[629, 630]	[629, 630]	[629, 630]

Table 5.6. Tests of local scale-invariance for the autoresponse $R(t, s)$, the space-time response $R(t, s; \mathbf{r})$ and the autocorrelator $C(t, s)$ were performed in the quoted reference(s). The order-parameter is non-conserved, unless explicitly stated otherwise. For empty entries, no tests of LSI are known to us. The conditions $T = T_c$, $p = p_c$ or $\gamma = \gamma_c$ indicate a model at a critical point. Surface scaling is indicated by s. A star * indicates disagreement with LSI.

studies also disagree with the available non-perturbative numerical data, we believe that these ε -expansion series ought to be resummed before they can become a tool for quantitative analysis. We emphasise the physically very different nature of the many systems already studied from the point of view of LSI.

Certainly, many important questions about LSI remain open, of which we list a few, in a more or less random order.

1. Simple ageing and instantaneous quenches are built into the present formulation of LSI from the very beginning. Does a similar extension of dynamical scaling exist for systems undergoing subageing, assuming of course that subageing is not an artefact of unrecognised corrections to scaling ?

How could one formulate an extension of dynamical scaling along the lines of LSI for systems with a non-algebraic growth law of the correlated clusters, as may be necessary in many glassy systems ?

2. Galilei-invariance and its generalisations to $z \neq 2$ are an essential ingredient of LSI. This is not a property of the full Langevin equation, but rather of a suitably identified ‘deterministic part’, and, provided that the remaining ‘noise part’ has a sufficiently simple structure, exact reduction formulæ follow which relate any average to an average calculated in the deterministic part only. For linear Langevin equations, Galilei-invariance of the deterministic part is formally proven but at best partial results exist for non-linear Langevin equations. Should one try to prove the Galilei-covariance of certain averages only, rather than of the deterministic part of the Langevin equation ? Is Galilei-covariance too strong a requirement and should better be replaced by some kind of asymptotic Galilei-invariance only valid in the scaling regime ?

On the other hand, several non-trivial consequences of (generalised) Galilei-invariance have been confirmed in various spin systems whose dynamics is defined in terms of a master equation, either from exact solution or numerically. The Langevin equation of the order-parameter in these models would be non-linear. It is as yet unknown how to prove Galilei-invariance of a general spin system with a dynamics given by a master equation, although the examples available strongly suggest that this might be possible.

3. For ferromagnetic spin systems quenched onto their critical point as well as in the critical contact process, the results for the two-time correlations and responses found from a field-theoretical ε -expansion in general do *not* agree with the predictions of LSI. These results obtained from the ε -expansion (see especially [118, 116, 119, 121, 46]) also do not agree with those obtained from non-perturbative simulational methods, even in those cases where LSI can be fitted to the numerical data. We recall that in several cases the agreement between LSI and the numerical simulations

is only obtained when a second ageing exponent $a' \neq a$ is introduced. Based on a novel mean-field scheme, we have argued in this book that one should indeed expect that for non-equilibrium critical dynamics, one should generically expect $a' \neq a$, while $a' = a$ should hold for phase-ordering kinetics and this seems to be supported by the presently available numerical data and several exactly solved models.¹⁴ Does this mean that one should first resum the ε -expansion series (as one does in equilibrium critical phenomena, see Chap. 2 in Volume 1) before attempting to compare with simulational data ?

In this context, recall that the unrenormalised (deterministic) action used as the starting point of the ε -expansion at $T = T_c$ is manifestly not Galilei-invariant. Does this mean that the local Langevin equation everyone is used to writing down and whose form is determined by physical ‘good sense’ is only good enough to give precise values of the critical exponents and related universal amplitudes, but subtly lacks the essential ingredients necessary for the calculation of scaling functions ? Or does this mean on the contrary that local scale-invariance is at best a good approximation at a critical point ?

4. For $z \neq 2$, the generalised Bargman rules imply the existence of several quantities related to the powers of the ‘momenta’ of the various scaling operators entering into the $2n$ -point response functions. Does this mean that there is an integrable structure underlying LSI ? While first tests seem to be compatible with at least some consequences of this, how could one test this more systematically ? Should one interpret this as a first hint of an underlying infinite-dimensional symmetry ?
5. Can we understand the limits of validity of LSI ? At present, it appears that whenever the system under study has a ‘reasonable’ spatial structure, (generalised) Galilei-invariance is found to hold true and hence co-variance under the full set of transformations of LSI should follow, as is proven so far for $z = 2$ and $z = 1$. Are there other physical criteria which have to be satisfied ?
6. Explicit LSI-calculations have been restricted to systems described by a single Langevin equation (or coupled equations related by a global symmetry, such as the $O(n)$ model). New representations of local scaling algebras, applicable to several coupled non-symmetric equations, must be constructed if one aims to test LSI for coupled systems, such as model C dynamics or reversible reaction-diffusion processes.

¹⁴ While one might think that the proposal $a' \neq a$ could be tested by looking at the cross-over towards the quasi-stationary regime $t \gtrsim s$, such a comparison may well be non-trivial since one also changes from a non-equilibrium regime with a FDR $X(t, s) \neq 1$ to a quasi-equilibrium critical regime with $X(t, s) \simeq 1$. Even in 2D conformal field-theory, the calculation of cross-over functions is still a non-trivial problem.

7. Is there an extension of LSI similar to *logarithmic* conformal field-theories and could this be used to treat the cases when $L(t) \sim (t/\ln t)^{1/z}$?
8. The present formulation of LSI lacks a feature which would allow it to distinguish between non-conserved and conserved dynamics, which at the current state of knowledge merely enters through the value of the dynamical exponent z . Furthermore, the *spatial* behaviour of the response function $R(t, s; \mathbf{r})$ is generically oscillating, in contrast to what is seen in the disordered kinetics Ising model [365], but in agreement with the exact solutions of the conserved or the long-ranged spherical models or the bosonic contact process with Lévy flights, described in this volume. Integrated responses average over these oscillations.
9. What is the algebraic structure underlying LSI for $z \neq 2, 1$? In particular, what is the interpretation of the generators of LSI, which certainly lead to phenomenologically reasonable expressions, but are not the simple differential operators of first order one is used to from the infinitesimal transformation of a Lie group?

For $z \neq 1, 2$, one generically must include some kind of fractional derivatives into the infinitesimal generators of local scale-transformations. Can one find physical criteria which should be the desirable properties of such fractional derivatives,¹⁵ since one cannot naively take over all of the ‘nice’ properties of ordinary derivatives ? What would be the meaning of the integrated form of these generators ?

10. Can one extend LSI from the Möbius transformations $t \mapsto \alpha t/(\gamma t + \delta)$ with $\alpha\delta = 1$ to the infinite-dimensional group $t \mapsto h(t)$ with $h(t) > 0$ and $\dot{h}(t) > 0$? If we restrict ourselves to the transformation of the coordinates $t, \mathbf{r}$ *only*, such an extension has been formulated in Table 5.1. It is not fully understood how to describe the transformation of the quasi-primary operators.¹⁶ If so, what would be physically observable consequences which one might use as a test ?
11. *A posteriori*, we have found through an analysis of the causality conditions that response functions transform co-variantly under local scale-transformations, while the co-variant part of correlation functions simply vanishes (correlators are found from a decomposition theorem, which relates them to response functions). Should one recast the theory by requiring that causality and co-variance in the response functions ought to be related ? Could this be a route towards a physically motivated extension

¹⁵ A very general attempt to construct such derivatives has been presented in [366]. It is correctly concluded that dynamical scaling and mathematical consistency *alone* are not sufficient to constrain the LSI-generators to an extent that definite predictions can be made. Rather, some further physical information, for example on invariant equations of motion, must be provided as well.

¹⁶ For $z = 2$, the mathematical theory of the Schrödinger-Virasoro algebra and of the associated Lie group will be presented in detail in a forthcoming book [624].

from the finite-dimensional algebras of local scale-invariance, which we have considered in this volume, towards an infinite-dimensional symmetry ?

12. Last, but not least, all present attempts of introducing local dynamical scaling make at some point the hidden assumption that the underlying process should be *Markovian*. Most importantly, the study of the global persistence has shown that neither at criticality, nor in the coexistence phase, the Markov property can be taken for granted and is in general *not* satisfied. If the difference of the value of the global persistence exponent θ_g from its predicted value for Markovian processes can be used as a quantitative measure of the non-Markovian nature of non-equilibrium ageing, these non-Markovian effects should be rather small.¹⁷ Unfortunately, all exactly solved models analysed until now are explicitly Markovian and by necessity only shed partial light on the long-time properties of ageing systems.

Developing physical criteria for appropriate non-Markovian effective descriptions and the analysis of their dynamical symmetries are important open problems.

On balance, local scale-invariance permits us to look at ageing systems in a way complementary to, but not fully independent of, traditional approaches using simple dynamical scaling. One of its strengths is that it relates the properties of different observables in a quantitative, testable way. The large number of successful tests already performed suggests that the LSI theory in its present state should already capture some relevant aspects in the ageing of non-glassy systems. We hope that a reader coming back to these pages in ten years or so will be able to appreciate this much better and more profoundly than we do now. *L'aventure continue !*

In the final chapter of this book, we consider a related equilibrium multicritical point, the so-called *Lifshitz point*, where ideas closely related to LSI can be applied and which were the first non-trivial examples of an extended strongly anisotropic local scaling behaviour [328, 597].

Problems

5.1. How do you close the generators as listed in [Table 5.1](#), case (ii), into a Lie algebra when $B_{20} \neq 0$?

¹⁷ The notable difference of θ_g from its ‘Markovian value’ in the 3D Heisenberg universality class might indicate that non-Markovian effects should be especially visible here.

5.2. The Lie algebra with generators as listed in Table 5.1, case (iii), is given by the commutators

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'} \quad , \quad [X_n, Y_{n'}] = (n - n')Y_{n+n'} \quad , \\ [Y_n, Y_{n'}] &= A_{10}(n - n')X_{n+n'} \end{aligned} \quad (5.83)$$

with $n, n' \in \mathbb{Z}$. If $A_{10} \neq 0$, show that it is isomorphic to the Lie algebra of $2D$ conformal transformations, without central charge. When extending this to a pair of commuting Virasoro algebras, what central extensions are implied in eq. (5.83) ?

5.3. Show that the Lie algebra $\langle X_n, Y_n \rangle_{n \in \mathbb{Z}}$ considered in the previous exercise 5.2 acts as a dynamical symmetry on the equation

$$\mathcal{S}\phi := (-A_{10}\partial_t + \partial_r)\phi = 0 \quad (5.84)$$

when the scaling dimension $x = x_\phi$ is correctly chosen.

5.4. Consider the finite-dimensional subalgebra $\langle X_{\pm 1,0}, Y_{\pm 1,0} \rangle$ of the algebra (5.83). From Table 5.1, case (iii), one has for $d = 1$ space dimensions the explicit representation $X_{-1} = -\partial_t$, $Y_{-1} = -\partial_r$ and

$$\begin{aligned} X_0 &= -t\partial_t - r\partial_r - x \\ X_1 &= -t^2\partial_t - 2tr\partial_r - \mu r^2\partial_r - 2xt - 2\gamma t \\ Y_0 &= -t\partial_r - \mu r\partial_r - \gamma \\ Y_1 &= -t^2\partial_r - 2\mu tr\partial_r - \mu^2 r^2\partial_r - 2\gamma t - 2\gamma\mu r \end{aligned} \quad (5.85)$$

with the notation of Table 5.1 being modified into $A_{10} =: \beta + \gamma =: \mu$ and $B_{10} =: 2\gamma$. The triplet of constants (x, μ, γ) characterises the quasi-primary scaling operators on which these generators act on.

Find the co-variant two- and three-point functions.

5.5. Show that if one performs in the algebra (5.83) a contraction by letting $A_{10} \rightarrow 0$, one obtains the **altern-Virasoro algebra** $\mathfrak{altv}(1)$ (without central charges), of which the altern/conformal Galilean algebra $\mathfrak{alt}(1) \equiv \text{CGA}(1)$ is the maximal finite-dimensional subalgebra [354, 355]. Try to generalise to $d \geq 1$.

What is the form of the central extensions and how many independent central extensions exist [569] ?

5.6. Find the two- and three-point functions built from $\mathfrak{alt}(1) \equiv \text{CGA}(1)$ -quasi-primary scaling operators with the representation (S.61) derived in the previous exercise 5.5.

5.7. Function-space representations of $\mathfrak{alt}(1)$, inequivalent to (S.61), can either be obtained by (i) recalling from Chap. 4 the definition $\mathfrak{alt}(1) := \langle D, Y_{\pm 1/2}, M_0, V_+, X_1 \rangle$ where the generators are given in eqs. (4.44, 4.49) and we suppress the second scaling dimension ξ for simplicity or else (ii) are given by [355]

$$\begin{aligned} X_n &= -t^{n+1} \partial_t - \frac{1}{2}(n+1)t^n r \partial_r + \frac{i}{4}(n+1)nt^{n-1}r^2 \partial_\zeta - \frac{x}{2}(n+1)t^n, \\ Y_n &= -t^{n+1} \frac{1}{r} \partial_r + \frac{i}{2}(n+1)t^n \partial_\zeta. \end{aligned} \quad (5.86)$$

Derive the two-point functions of quasi-primary scaling operators under each of these representations and compare with the result of the previous exercise 5.6 [355].

5.8. In two space dimensions, the conformal Galilean algebra admits a so-called *exotic* central extension, called the **exotic conformal Galilean algebra** ECGA $:= \langle X_n, Y_n^{(j)}, R_0^{(12)} = -R_0^{(21)}, \Theta \rangle$ with $n \in \{\pm 1, 0\}$ and $j \in \{1, 2\}$ [483, 484]. We write the non-vanishing commutation relations as follows

$$\begin{aligned} [X_n, X_{n'}] &= (n - n')X_{n+n'}, \quad [X_n, Y_{n'}^{(j)}] = (n - n')Y_{n+n'}^{(j)}, \\ [R_0^{(jk)}, Y_n^{(\ell)}] &= \delta^{j,\ell} Y_n^{(k)} - \delta^{k,\ell} Y_n^{(j)} \\ [Y_n^{(1)}, Y_{n'}^{(2)}] &= \delta_{n+n',0} (3\delta_{n,0} - 2) \Theta, \end{aligned} \quad (5.87)$$

where the new central generator Θ parametrises this central extension. Because of Schur's lemma, Θ may be replaced by its eigenvalue $\theta \neq 0$. An explicit function-space representation (along the lines of exercise 5.5) is given by

$$\begin{aligned} X_n &= -t^{n+1} \partial_t - (n+1)t^n \mathbf{r} \cdot \nabla_{\mathbf{r}} - (n+1)t^n x - (n+1)nt^{n-1} \boldsymbol{\gamma} \cdot \mathbf{r} \\ &\quad - (n+1)n \mathbf{h} \cdot \mathbf{r} \\ Y_n^{(j)} &= -t^{n+1} \partial_j - (n+1)t^n \gamma_j - (n+1)t^n h_j - (n+1)n \epsilon_{jk} r_k \theta \\ R_0^{(12)} &= -\epsilon_{jk} r_j \partial_k - \epsilon_{jk} \gamma_j \frac{\partial}{\partial \gamma_k} - \frac{1}{2\theta} \mathbf{h} \cdot \mathbf{h} = -R_0^{(21)}. \end{aligned} \quad (5.88)$$

Here, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ is a constant vector and the components of $\mathbf{h} = (h_1, h_2)$ satisfy $[h_1, h_2] = \Theta$. Finally, ϵ is the totally antisymmetric tensor.

Show that the ECGA acts as a dynamical symmetry on the equation

$$\mathcal{S}\phi = 0, \quad \mathcal{S} := -\Theta X_{-1} + \epsilon_{jk} Y_0^{(j)} Y_{-1}^{(k)} = \theta \partial_t + \epsilon_{jk} \gamma_j \partial_k + \epsilon_{jk} h_j \partial_k \quad (5.89)$$

and identify the required scaling dimension $x = x_\phi$.

5.9. An explicit representation of the Heisenberg algebra $\langle h_{1,2}, \Theta \rangle$ of exercise 5.8 is [503]

$$h_i = \frac{\partial}{\partial v_i} - \frac{1}{2} \epsilon_{ij} v_j \partial_\zeta, \quad \Theta = \partial_\zeta \quad (5.90)$$

where $v_{1,2}$ are auxiliary coordinates.

Use this in the explicit representation (5.88) of the ECGA to derive the co-variant two-point function.

5.10. Construct the infinitesimal generators of a Lie algebra $\mathfrak{a}$ of local scale-transformations, starting from

$$X_{-1} = -\partial_t, \quad X_0 = -t\partial_t - \frac{1}{z}r\partial_r - \frac{x}{z}, \quad Y_{-1/2} = -\partial_r$$

and try to find a generalised Galilei transformation $Y_{1/2}$. Construct $\mathfrak{a}$ such that X_0 acts as a counting operator, that is $[X_0, A] = aA$ for all elements $A \in \mathfrak{a}$. [330]. Use the ansatz $Y_{1/2} = -t\partial_r - \mathcal{M}(\partial_r)r$ and the singular fractional derivative (J7) from appendix J.

5.11. Use the generators constructed in the previous exercise 5.10 for $z = \frac{3}{2}$ to find the co-variant two-point function [330, app. C].

5.12. Gelfand and Shilov [274, p. 115] define a fractional derivative as follows. On the real line, they consider the generalised function

$$r_+^\alpha := \begin{cases} r^\alpha & ; \quad r > 0 \\ 0 & ; \quad r \leq 0 \end{cases}$$

and for generalised functions concentrated on the half-line $r > 0$, they define

$$\partial^a f(r) := f(r) * \frac{r_+^{-a-1}}{\Gamma(-a)} = \frac{1}{\Gamma(-a)} \int_0^r d\rho f(\rho) (r - \rho)^{-a-1}.$$

It is understood that the integral must be regularised [274]. Check that the conditions (J7) are satisfied, using $\int_0^1 dt t^{a-1} (1-t)^{b-1} = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ and analytic continuation.

5.13. Consider a magnetic system quenched to its critical point $T_c > 0$. For a small initial magnetisation $\langle \phi(0, \mathbf{r}) \rangle = m_0$, derive the time-dependent magnetisation $m(t)$ from local scale-invariance for times $t \ll t_M$, where $t_M \sim m_0^{-1/(\Theta+\beta/\nu z)}$ is the time-scale for the cross-over between the early-time regime of the initial critical slip and the late-time relaxation regime. Try to re-derive the scaling relation (1.112).

5.14. Starting from the general form (5.48) of the two-time correlation function in a system where the long-time behaviour is determined by the initial noise (as for quenches to $T < T_c$) and assuming that the ‘initial’ state at time $t = 0$ can be taken as fully uncorrelated, drive the scaling form $C(t, s) = s^{-b} f_C(t/s)$. Compute the ageing and autocorrelation exponent and the scaling function.

5.15. Derive the scaling form (5.69) for the autocorrelator in the phase-ordering of the 2D Ising model, assuming the equal-time correlator (5.68).

Chapter 6

Lifshitz Points: Strongly Anisotropic Equilibrium Critical Points

6.1 Phenomenology

In the previous chapters, we discussed ageing phenomena in non-equilibrium systems far from stationarity where obviously time plays a special role. Consequently, one way of looking at these ageing systems is to consider them as strongly anisotropic systems where the time-direction is fundamentally different to the spatial directions. Indeed, the presence of a single time-dependent length-scale $L(t) \sim t^{1/z}$ implies an anisotropic scaling behaviour where time has to be rescaled by a factor $t \mapsto b^z t$ when distances are rescaled by a factor $r \mapsto br$.

Interestingly, **strongly anisotropic scaling behaviour** is also observed in other physical situations, ranging from the critical behaviour in stationary states of non-equilibrium systems (for example encountered in driven diffusive systems) to anisotropic equilibrium critical points. Prominent examples of the latter class of systems, which of course can be fully studied in the well established framework of equilibrium statistical physics, are given by a particular class of anisotropic multicritical points called **Lifshitz points** (LP) [373, 374]. These special critical points, which have been studied intensively since the mid-1970s [655, 198], are characterised by the fact that one or more space directions display a fundamentally different scaling behaviour than the remaining space directions. This means that the anisotropy becomes so strong that the equilibrium critical behaviour of several observables has to be described by direction-dependent universal critical exponents.¹ Interestingly, due to the strongly anisotropic scaling behaviour, these static equilibrium critical points can be treated along similar lines as ageing systems far from equilibrium, as discussed in the following.

¹ On the contrary, one might call an anisotropy ‘*weak*’ if its effect merely leads to a direction-dependent redefinition of the non-universal metric factors.

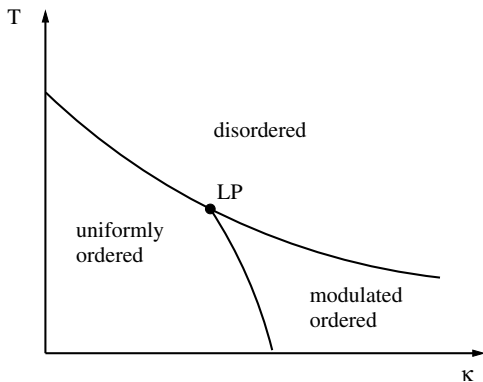


Fig. 6.1. Schematic phase diagram of a system possessing a multicritical Lifshitz point.

Fig. 6.1 schematically shows a typical phase diagram of a system with an LP. Here T is the temperature, whereas κ is a generalised non-ordering field. The LP is a multicritical point where three different phases meet: (i) a disordered phase, (ii) an uniformly ordered phase, and (iii) a modulated phase characterised by a modulation wave vector $\mathbf{q} = \mathbf{q}(\kappa)$ such that the corresponding modulation wave length is incommensurate with the lattice constant of the underlying lattice. The non-ordering field may for example be realised by additional interaction terms beyond the usually considered nearest-neighbour interactions. The most frequent case arises when these extra modulations only occur in a single space direction. Then the Lifshitz point, if it exists, is called **uniaxial**. Of course, there is *a priori* no restriction on the number m of non-vanishing components of $\mathbf{q}$ of this wave vector instability, i.e. $1 \leq m \leq d$ where d is the dimensionality of the system. The corresponding Lifshitz point is said to be **m -axial** and the uniaxial case corresponds to $m = 1$.² Along the critical line separating the disordered phase from the ordered phases, the approach to the LP, located at $T = T_L$ and $\kappa = \kappa_L$, is governed by a **crossover exponent** ϕ , and a **wave vector exponent** $\beta_{\mathbf{q}}$ such that

$$T_c(\kappa) - T_L \sim |\kappa - \kappa_L|^{1/\phi}, \quad |\mathbf{q} - \mathbf{q}_L| \sim |\kappa - \kappa_L|^{\beta_{\mathbf{q}}}, \quad (6.1)$$

where $\mathbf{q}_L$ is the modulation wave vector at the LP. Note that $\mathbf{q}_L$ does not necessarily vanish.

Over the years, the possibility of an LP has been discussed in a wide range of materials. In some instances the existence of an LP was convincingly demonstrated by a change of the values of the critical indices, see Table 6.1. In other systems it was solely inferred from the observation of a disordered phase, a uniformly ordered phase and a modulated phase. The diversity of systems for which an LP has been proposed is quite remarkable and includes *inter alia* magnetic systems like MnP and related compounds [57, 526, 663, 664, 79, 758,

² We shall be mainly interested in the case $m < d$, which displays an anisotropic scaling behaviour. If $m = d$, one is dealing with an **isotropic Lifshitz point**.

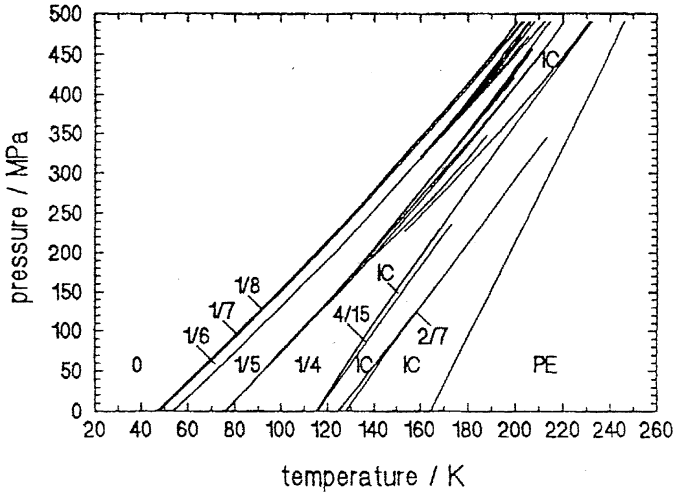


Fig. 6.2. Temperature-pressure phase diagram of BCCD (betaine calcium chloride dihydrate) which possesses a multitude of commensurately and incommensurately modulated phases. Only the most important phases, characterised by the wave number of the modulation, are shown. IC indicates an incommensurately modulated region, PE stands for the paraelectric phase and 0 for the unmodulated ferroelectric phase. Based on this phase diagram, a Lifshitz point was postulated to exist at $T_L = 346$ K and $p_L = 1.16$ GPa. Reprinted from [641] by permission of the publisher (Taylor & Francis Group).

56, 55] or CoNb_2O_6 [731], uniaxial ferroelectrics like the $\text{Sn}_2\text{P}_2\text{S}_6$ compounds [676, 718, 745], betaine calcium chloride dihydrate (BCCD) [641] or the A_2BX_4 [485, 486] family (as for example Rb_2WO_4 , K_2MoO_4 or K_2WO_4), ferroelectric liquid crystals [708] as well as mixtures of a homopolymer blend and diblock copolymer [43, 650]. As an example of the intriguing phase diagrams that can be observed, we show in Fig. 6.2 the phase diagram of BCCD which possesses a multitude of commensurately and incommensurately modulated phases [22].

In this chapter, we shall be interested in those properties of Lifshitz points which can be used to illustrate strongly anisotropic scaling behaviour. For simplicity, we shall from now on formulate everything in terms of a magnetic phase transition. In describing the Lifshitz critical behaviour, one must distinguish between spatially averaged thermodynamic quantities, like the specific heat C , the magnetisation m or the magnetic susceptibility χ , and on the other hand direction-dependent observables such as correlators. The first class obeys close to a Lifshitz point the conventional scaling behaviour (with $\kappa = \kappa_L$)

$$C \sim |T - T_L|^{-\alpha} \quad , \quad m \sim (T_L - T)^\beta \quad , \quad \chi \sim |T - T_L|^{-\gamma} \quad (6.2)$$

and the standard scaling relations such as $\alpha + 2\beta + \gamma = 2$ remain valid. The strongly anisotropic nature of the Lifshitz point becomes apparent in the spatial correlation functions. We decompose the d -dimensional space vector $\mathbf{r} = (\mathbf{r}_{\parallel}, \mathbf{r}_{\perp})$, where the m components of the ‘parallel’ vector $\mathbf{r}_{\parallel}$ point in the directions with spatial modulations, whereas the $d - m$ ‘perpendicular’ directions form the vector $\mathbf{r}_{\perp}$. With this, one has for the spin-spin correlation function $C(\mathbf{r}_{\parallel}, \mathbf{r}_{\perp}) = \langle \phi(\mathbf{r}_{\parallel}, \mathbf{r}_{\perp}) \phi(\mathbf{0}, \mathbf{0}) \rangle$ exactly at the LP the following behaviour:

$$C(\mathbf{0}, \mathbf{r}_{\perp}) \sim r_{\perp}^{-[d-m+\theta-2+\eta_{\perp}]} \quad \text{and} \quad C(\mathbf{r}_{\parallel}, \mathbf{0}) \sim r_{\parallel}^{-[(d-m+\theta-2)/\theta+\eta_{\parallel}]} , \quad (6.3)$$

with the **anisotropy exponent** being defined as

$$\theta := \eta_{\parallel} / \eta_{\perp} . \quad (6.4)$$

Equation (6.3) defines the direction-dependent critical exponents $\eta_{\parallel}$ and $\eta_{\perp}$. This strongly anisotropic critical behaviour is also visible through the existence of direction-dependent correlation lengths $\xi_{\perp}$ and $\xi_{\parallel}$ which on approaching the LP diverge with different critical exponents, with³ $\nu_{\parallel} = \theta \nu_{\perp}$:

$$\xi_{\perp} \sim |T - T_L|^{-\nu_{\perp}} \quad \text{and} \quad \xi_{\parallel} \sim |T - T_L|^{-\nu_{\parallel}} \quad (6.5)$$

This direction-dependence of the correlation function and of the correlation lengths has important consequences. For the correlation function one has, precisely at the Lifshitz point $T = T_L$ and $\kappa = \kappa_L$, the following anisotropic scaling form:

$$C(\mathbf{r}_{\parallel}, \mathbf{r}_{\perp}) = b^{-2x} C(b^{-\theta} \mathbf{r}_{\parallel}, b^{-1} \mathbf{r}_{\perp}) = r_{\perp}^{-2x} \Omega(r_{\parallel} r_{\perp}^{-\theta}) . \quad (6.6)$$

Here b is a scaling factor, whereas the value of the scaling dimension x can be read off from (6.3). $\Omega(v)$ is an universal scaling function that is not fixed by scale-invariance alone. It is obvious from the scaling form (6.6) that the anisotropy exponent θ plays for uniaxial systems the same role as the dynamical exponent z in the ageing systems discussed in the previous chapters. As another consequence of the observed direction-dependence some of the usual scaling relations have to be modified at an LP. With α and γ being the specific heat and susceptibility critical exponents, one has

$$2 - \alpha = m\nu_{\parallel} + (d - m)\nu_{\perp} , \quad \gamma = (2 - \eta_{\perp})\nu_{\perp} = (2/\theta - \eta_{\parallel})\nu_{\parallel} . \quad (6.7)$$

The first of these is a hyperscaling relation and will be valid for dimensions $d < d_c(m)$. The proof is left as exercise 6.1.⁴ As we shall discuss in more detail

³ An often-used alternative notation works with exponents [373, 195] $\nu_{\ell 2} = \nu_{\perp}$, $\nu_{\ell 4} = \nu_{\parallel}$, $\eta_{\ell 2} = \eta_{\perp}$ and $\eta_{\ell 4} = \eta_{\parallel} + 4 - 2/\theta$.

⁴ In the alternative notation of the previous footnote, the second relation (6.7) and the anisotropy exponent read

$$\gamma = (2 - \eta_{\ell 2})\nu_{\ell 2} = (4 - \eta_{\ell 4})\nu_{\ell 4} , \quad \theta = \frac{\nu_{\ell 4}}{\nu_{\ell 2}} = \frac{2 - \eta_{\ell 2}}{4 - \eta_{\ell 4}} .$$

in the next section, the values of critical exponents like α , β or γ differ at an LP from the values obtained for systems with an isotropic scaling behaviour. We also note the scaling relation $\beta_q \phi = \nu_{\parallel}$.

Many important insights into the intriguing behaviour at LPs have been gained through the theoretical study of classical spin models with competing interactions. The best studied model is the **ANNNI model** (axial next nearest neighbour Ising) [654]. On a three-dimensional cubic lattice, the Hamiltonian of the ANNNI model reads [229]

$$\begin{aligned} \mathcal{H} = & -J_0 \sum_{xyz} S_{xyz} (S_{(x+1)yz} + S_{x(y+1)z}) - J_1 \sum_{xyz} S_{xyz} S_{xy(z+1)} \\ & - J_2 \sum_{xyz} S_{xyz} S_{xy(z+2)} \end{aligned} \quad (6.8)$$

with Ising spins $S_{xyz} = \pm 1$. In the xy planes nearest-neighbour spins are coupled ferromagnetically with the coupling constant $J_0 > 0$, whereas in the single axial or z -direction competition takes place between ferromagnetic nearest neighbour couplings with strength $J_1 > 0$ and *antiferromagnetic* next-nearest neighbour couplings with strength $J_2 = -\kappa J_1 < 0$, leading to the appearance of spatially modulated phases in that direction. Fig. 6.3 displays the phase diagram obtained for this model in mean-field approximation as a function of temperature and competition parameter $\kappa = -J_2/J_1$ [657].⁵ A qualitatively similar phase diagram is expected beyond this approximation. This has partially been verified through low temperature series expansion [246, 247, 690, 248] and numerical simulations [658, 659, 612, 631]. It is obvious from Fig. 6.3 that this deceptively simple-looking model has a very complicated phase diagram with a multitude of modulated phases. Similar phases are also encountered for $J_1 < 0$ and $\kappa < 0$, i.e. for antiferromagnetic axial nearest- and next-nearest-neighbour couplings. The reader interested in the details of this phase diagram is referred to the reviews [654, 655].⁶ For our purposes it is important to note that the phase diagram of the 3D ANNNI model displays an LP whose location $T_L = 3.7475(5)$ and $\kappa_L = 0.270(4)$ has been determined with high accuracy through high-temperature series expansions [557] and Monte Carlo simulations [597]. As in this case spatial modulations only appear in one space direction, this LP is uniaxial.

Many generalisations of the ANNNI model exist and have been studied in the past. One of the most common generalisations consists of replacing the Ising spins with continuous spins with n components and global $O(n)$ symmetry. The resulting models are sometimes called **ANNNO(n) models**, the

⁵ In the following we always restrict ourselves to the case $J_0 = J_1$.

⁶ The disorder-modulated line beyond the LP is thought to be in the XY universality class [270, 753].

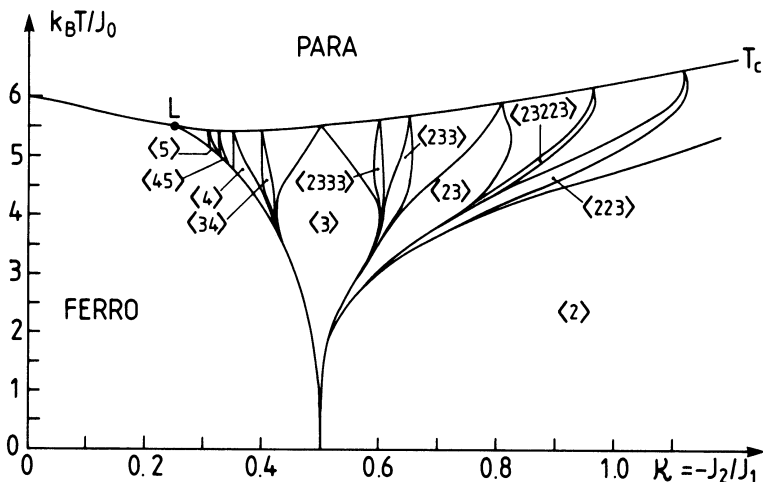


Fig. 6.3. Mean-field phase diagram of the ANNNI model. The symbols label the modulated phases and indicate the layer sequence that is repeated periodically. Thus $\langle 2 \rangle$ means that two layers with positive magnetisation are followed by two layers with negative magnetisation, a sequence which is then repeated periodically. The letter L indicates the location of the Lifshitz point. Reprinted from [657].

name reflecting the fact that these models are generalisations of the usual $O(n)$ models where axial next-nearest-neighbour interactions have been included.⁷

An exactly solvable variant is obtained by replacing the Ising spins at site i by a real variable S_i such that the spins verify the spherical constraint $\sum_i S_i^2 = \mathcal{N}$ where $\mathcal{N}$ is the total number of sites in the lattice. This model is called the **ANNNS model** (axial next-nearest neighbour spherical) and corresponds to the spherical limit ($n \rightarrow \infty$) of the ANNNO(n) model [375, 651, 614, 258]. An exact expression for the transition temperature $T_c(\kappa)$ to the paramagnetic phase in the ANNNS model reads as follows. In $d = 3$ space dimensions and expressed in units of J_1/k_B , $T_c(\kappa)$ is for a cubic lattice given by the integral [375]

$$\frac{1}{T_c(\kappa)} = \frac{2}{(2\pi)^3} \int_{\mathcal{B}} d^3\mathbf{k} [\lambda_m(\kappa) - \lambda(\mathbf{k}, \kappa)]^{-1}, \quad (6.9)$$

where $\mathcal{B}$ denotes the Brillouin zone, with

$$\lambda(\mathbf{k}, \kappa) = \cos k_x + \cos k_y + \cos k_z - \kappa \cos 2k_z \quad (6.10)$$

⁷ The cases $n = 2$ and $n = 3$ are also known as the ANNNXY (axial next-nearest-neighbour XY) and the ANNNH (axial next-nearest-neighbour Heisenberg) models, respectively [613, 614]. Still, we believe that ‘ANNNO model’ should be easier to pronounce.

and

$$\lambda_m(\kappa) = \begin{cases} 3 - \kappa & ; \text{ for } \kappa \leq 1/4 \\ (1 + 16\kappa + 8\kappa^2)/8\kappa & ; \text{ for } \kappa \geq 1/4 \end{cases} . \quad (6.11)$$

In the ANNNS model, however, the structure of the phase diagram is much simpler than in the ANNFI model and the rich set of phases with repeating structures seen in Fig. 6.3 is replaced by a single incommensurately modulated phase. At the (uniaxial) Lifshitz point, one has for the critical exponents

$$\gamma = \frac{4}{2d-5} , \quad \alpha = \frac{2d-9}{2d-5} , \quad \nu_{\perp} = \frac{1}{2}\gamma , \quad \theta = \frac{1}{2} , \quad \eta_{\parallel} = \eta_{\perp} = 0. \quad (6.12)$$

The ANNNS model remains soluble if one adds couplings to more distant neighbours along the axis of competing interactions [543, 651, 258]. For example, some studies investigated the extended ANNNS model where in addition to nearest ferromagnetic and next-nearest antiferromagnetic couplings also interactions between third-nearest neighbours with strength J_3 are considered. Introducing the competition parameters $\kappa_1 = -J_1/J_2$ and $\kappa_2 = -J_3/J_1$, a line of Lifshitz points is observed that ends in a **Lifshitz point of second order**.

Further variants that have been studied include the ANNFI model with interactions between third axial neighbours [656, 41] or the **BNNFI model** (biaxial next-nearest neighbour Ising) with competing interactions along two axes [558]. In addition to these spin models, regions with modulated phases and LPs have also been shown to exist in many other types of models. See [655, 534] for a discussion of these systems.

6.2 Critical Exponents at Lifshitz Points

As explained in the previous section, Lifshitz points are multicritical points belonging to universality classes that differ from the customary $O(n)$ -symmetric universality classes. In order to determine the universality class of a given system, one usually tries to measure a set of critical indices. Alternatively, one can also study scaling functions as these are also characteristic for a given universality class. We start in this section by considering critical exponents. Universal scaling functions will be discussed in the following Sections.

Let us choose as a starting point for our discussion the following continuum representation of the ANNNO(n) model, which generalises the ϕ^4 field-theory used for the description of the equilibrium phase transition in the $O(n)$ model [198]:

$$\mathcal{H} = \int d^d \mathbf{r} \left[\frac{1}{2} (\nabla_{\perp} \phi)^2 + \frac{\rho_0}{2} (\nabla_{\parallel} \phi)^2 + \frac{\sigma_0}{2} (\Delta_{\parallel} \phi)^2 + \frac{\tau_0}{2} \phi^2 + \frac{u_0}{4!} |\phi|^4 \right] \quad (6.13)$$

where ϕ is an n -component order-parameter field. As before, we split the d -dimensional vector $\mathbf{r} = (\mathbf{r}_{\parallel}, \mathbf{r}_{\perp})$ into an m -dimensional vector $\mathbf{r}_{\parallel}$ whose

components point into the spatially modulated directions and a $(d - m)$ -dimensional vector $\mathbf{r}_\perp$ pointing into the remaining directions. $\nabla_\perp$ and $\nabla_\parallel$ are the gradient operators in the 'perpendicular' and 'parallel' directions, respectively. In this representation (which may be derived from a Hubbard-Stratonovich transformation, see Chap. 2 in Volume 1), the LP is located at $\tau_0 = 0$ and $\rho_0 = 0$, such that the Laplacian in the 'parallel' directions, which results from the competition taking place in these directions, becomes relevant.

Equation (6.13) shows that an LP is characterised by the number of space dimensions, d , the number of order parameter components, n , and the dimensionality m of the subspace where the wave vector instability occurs. It is therefore customary to label an LP by the triple (m, d, n) . The LP encountered in the three-dimensional ANNO(1) = ANNI model is then characterised by the set $(m, d, n) = (1, 3, 1)$.

A dimensional analysis of (6.13) gives the upper critical dimension

$$d_c(m) = 4 + \frac{m}{2}, \quad m \leq 8 \quad (6.14)$$

above which the LP behaviour is described by the Gaussian theory with $u_0 = 0$. Note that the upper critical dimension depends on the number m of 'parallel' directions. For $m = 0$ we recover the upper critical dimension $d_c = 4$ of the usual ϕ^4 model, whereas the case $m = 8$ yields $d_c = m = 8$ which is the upper critical dimension of an isotropic LP.

Diehl and Shpot [195, 667, 196, 669] recently presented a systematic field-theoretical study of m -axial Lifshitz points, see [198] for a short review of these results. Using both ε -expansions as well as large- n expansions, they computed estimates for the different critical exponents. The consistency of these two distinct expansions has also been confirmed [668]. The result of most interest for the following discussion concerns the anisotropy exponent θ . In the Gaussian theory the value of this exponent is found to be $1/2$, and the same result is found in an expansion around the upper critical dimension, with $\varepsilon = d_c(m) - d$, to first order in ε . However, as shown by Diehl and Shpot, correction terms are present in second order of the ε -expansion, yielding $\theta = \frac{1}{2} - \mathbf{a}\varepsilon^2 + O(\varepsilon^3)$, where $\mathbf{a} \simeq 0.0054$ in the 3D ANNI model [667]. The same conclusion is reached from an $1/n$ expansion for the uniaxial LP in the 3D ANNO(n) model, where $\theta = \frac{1}{2} - 0.0487n^{-1} + O(n^{-2})$ is reported [669].

Driven by the need to provide reliable estimates for a comparison with experimental and/or simulational results, much effort has been devoted over the years to the computation of critical exponents. The methods used are the usual methods for the determination of critical exponents and encompass field-theoretical techniques, series expansions and numerical simulations. We collect in Table 6.1 exponent estimates for the two best studied cases, namely the ANNO(1) or ANNI model with $(m, d, n) = (1, 3, 1)$ and the ANNO(2) or ANNXY model with $(m, d, n) = (1, 3, 2)$ for which the most reliable esti-

n	$\nu_{\perp}$	$\nu_{\parallel}$	γ	β	α	ϕ	β_q	Method	Ref.
1	0.746	0.325	1.399	0.220	0.160	0.677	0.514	FT	[195, 667]
	—	—	1.6(1)	—	0.20(15)	—	—	SE	[522]
	—	—	1.36(3)	0.238(5)	0.18(2)	—	—	CM	[597]
	—	—	—	—	0.46(3)	0.63(4)	0.480(13)	EX	[57, 663]
	—	—	—	0.19(3)	—	—	—	EX	[526, 79]
2	0.757	0.372	1.495	0.276	-0.047	0.725	0.521	FT	[195, 667]
	0.805(15)	0.40(3)	1.535(25)	—	—	1.00(4)	0.40(2)	SE	[115]
	—	—	1.5(1)	0.20(2)	0.10(14)	—	—	MC	[653]
∞	2	1	4	1/2	-3	2	1/2		[651, 614]
MF	1/2	1/4	1	1/2	0	1/2	1/2		

Table 6.1. Critical exponents for the uniaxial Lifshitz point in the 3D ANNNO(n) model, according to renormalised field-theory (FT), high-temperature series expansions (SE), Monte Carlo simulations (MC) and cluster Monte Carlo simulations (CM). The mean-field values (MF) are also included. EX designates the available experimental data, obtained either for MnP [57, 663, 526, 79] or for $\text{Sn}_2\text{P}_2(\text{Se}_x\text{S}_{1-x})_6$ [676].

mates were obtained. For comparison the exact results of the 3d ANNNO(∞) or ANNNS model and the mean-field results are also included. In addition, we also include for the case $(m, d, n) = (1, 3, 1)$ some experimental values obtained in studies of MnP [57, 663, 526, 79] and of $\text{Sn}_2\text{P}_2(\text{Se}_x\text{S}_{1-x})_6$ [676]. For the following discussion of the LP critical exponents we avoid technicalities and instead focus on the results.

The calculation of the critical exponents at an m -axial LP using renormalisation-group techniques and ε -expansions was overshadowed for many years by a long-standing controversy. In the first papers on the subject, Hornreich, Luban, and Shtrikman [373, 374] computed the exponents $\nu_{\ell 2}$ and $\nu_{\ell 4}$ to order ε for all m and to order ε^2 for $m = 8$. The first attempts to obtain the exponents $\nu_{\ell 2}$, $\nu_{\ell 4}$ and β_q to order ε^2 for $m < 8$ were done by Mukamel [528] and Hornreich and Bruce [371]. The latter focused on the uniaxial LP and their results agreed with the results Mukamel obtained for all m . An independent calculation done by Sak and Grest for $m = 2$ and $m = 6$ [636], however, yielded results that differed from Mukamel's results. This problem was later revisited by Mergulhão and Carneiro [416, 417] who used renormalised field-theory instead of the Wilson-Fisher momentum-space technique of the earlier papers. Restricting themselves for technical reasons to the cases $m = 2$ and $m = 6$, they found agreement with the results obtained by Sak und Grest. In a series of papers Diehl and Shpot [195, 667, 196, 668] presented a complete two-loop renormalisation-group analysis of LP with arbitrary values of m , utilising dimensional regularisation and minimal subtraction of poles in position space. We include in Table 6.1 the values obtained by Diehl and Shpot in their field-theoretical approach. It is important to note that these calculations showed

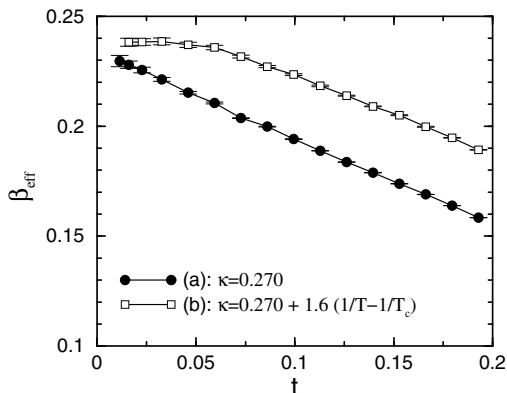


Fig. 6.4. Effective exponent β_{eff} versus τ for two different trajectories in the (T, κ) space, see text. Error bars include the uncertainty in $T_c(\kappa)$: $T_c(0.270) = 3.7475(5)$. Reprinted with permission from [597]. Copyright (2001) by the American Physical Society.

that the anisotropy exponent θ is not exactly $\frac{1}{2}$ but that correction terms appear at the order ε^2 .

The critical behaviour of LPs has also been studied through series expansions [613, 614, 557, 522, 115] and Monte Carlo simulations [652, 372, 653, 425, 597]. Since some of these studies were performed long ago, the numerical precision of the values of the critical exponents is not very high according to present standards. The only exceptions are a recent numerical investigation of the ANNNI model [597] and a recent series expansion study of the LP in the ANNNO(2) model [115]. We gather in Table 6.1 the best available estimates of critical exponents obtained with these two numerical methods. Obviously, there is a general agreement between the different methods, even though not all exponents have been measured through Monte Carlo simulations or series expansions.

We now illustrate briefly how to determine equilibrium critical exponents at an LP through Monte Carlo simulations in the 3D ANNNI model (a cluster algorithm which generalises the Wolff algorithms and which can be used for critical systems with competing interactions is described in appendix I).⁸ We thereby focus on the exponent β which describes the vanishing of the order-parameter (in our case this is the magnetisation) when the critical point is approached from lower temperatures: $m(t) \sim (T_L - T)^\beta$. There are two different ways for determining critical exponents through Monte Carlo simulations. In one often used approach systems of different sizes are simulated at the critical point and the values of critical exponents are extracted through finite-size scaling. Of course at an anisotropic critical point, the shape of the simulated

⁸ In the 2D ANNNI model, there is no LP with $T_L > 0$. Indeed, for the (m, d, n) -model, the lower critical dimension (where $T_L = 0$) is $d_- = 2 + m/2$ [195].

n	κ_L	T_L	reference
1	0.270(4)	3.7475(5)	[597]
2	0.2733(6)	1.778(2)	[115]
3	0.259(2)	—	[614]
ANNNS	1/4	eq. (6.9)	[375]

Table 6.2. Location of the Lifshitz point on a cubic lattice in 3D ANNNO(n) and ANNNS models. The critical temperature is given in units of J_1/k_B

systems has to take into account the anisotropic scaling behaviour [77, 691]. Fig. 6.4 shows the result of a different approach [597] where systems of different sizes are simulated at different temperatures in order to circumvent finite-size effects. The starting point thereby is the fact that the correlation length ξ is finite for temperatures $T \neq T_c$ such that the properties of systems of linear extent $L \gg \xi$ are indistinguishable from the properties of the infinite system. In order to avoid finite-size effects when approaching a critical point, systems of different sizes are simulated. At low temperatures rather small systems already yield data free of finite-size effects. At higher temperatures, however, data obtained for these systems may be affected by finite-size effects. One then switches to larger systems in order to approach the critical temperature more closely. At some temperature, finite-size effects again show up and even larger systems have to be simulated. This procedure is a little bit cumbersome but ensures that the properties studied are representative for the thermodynamic limit. Once data that are not affected by finite-size effects are available, one can determine critical exponents through the study of effective exponents. For example, in order to estimate the critical exponents of the magnetisation, an effective temperature dependent exponent $\beta_{\text{eff}}(\tau)$ can be defined by

$$\beta_{\text{eff}}(\tau) = d \ln(m(\tau)) / d \ln(\tau) \quad (6.15)$$

where the reduced temperature $\tau = (T_c - T)/T_c$ measures the distance to the critical point. In the limit $\tau \rightarrow 0$ the effective exponent yields the critical exponent β , provided finite-size effects can be neglected. The two sets of data in Fig. 6.4 correspond to two different paths in the temperature-interaction space of the ANNNO model, both ending at the point ($\kappa = 0.270$, $T_c = 3.7475$), setting $J/k_B = 1$. For set (a) κ was fixed at 0.270, whereas for set (b) $\kappa = 0.270 + 1.6(1/T - 1/T_c)$. The corrections to scaling for set (b) are small compared to set (a), resulting in a plateau for $\tau \leq 0.06$, thus making a very precise estimation of β possible, see Table 6.1. Based on the data from this numerical study, the Lifshitz point for the ANNNO model is located at $\kappa_L = 0.270 \pm 0.004$, $T_L = 3.7475 \pm 0.0005$, thus confirming a high temperature series estimate [557]. In Table 6.2 we give the best available estimates of the location of the Lifshitz point in the ANNNO(n) model.

Let us finish this short discussion of the values of critical exponents at Lifshitz points by mentioning that also *surface critical phenomena at a bulk*

Lifshitz point have been studied in semi-infinite systems [76, 263, 399, 591, 194, 192, 193], and especially in the semi-infinite ANNNI model. This generalises the discussion of surface critical phenomena and an isotropic equilibrium critical point in that one also expects the same kind of interplay between the effects of bulk and surface ordering found for isotropic critical behaviour (see Chap. 2 in Volume 1) which has led to the distinction of the ordinary, extraordinary, special and surface transitions. However, in the case at hand and due to the strong anisotropy, surfaces with different orientations are *not* equivalent, yielding two different sets of critical indices for different orientations. In the literature the following two surface orientations have been considered (see Fig. 6.5):

1. surfaces *perpendicular* to the axis of competing interactions (here referred to as **case A**)
2. surfaces *parallel* to this axis (here referred to as **case B**).

We shall discuss these cases in the following for the semi-infinite three-dimensional ANNNI model, using the indices b and s in order to distinguish between bulk and surface quantities. Thus the competing parameter in the bulk is denoted as κ_b and its value at the Lifshitz point as κ_b^L .

In principle, three different scenarios have to be distinguished, depending on the value of the bulk competing parameter κ_b . We restrict ourselves to cases where the bulk is at its Lifshitz point, with $\kappa_b = \kappa_b^L$, and refer the reader to the review [593] for an exhaustive discussion of all possible scenarios. Using a mean-field approach, Binder, Frisch, and Kimball [76, 263] studied for both

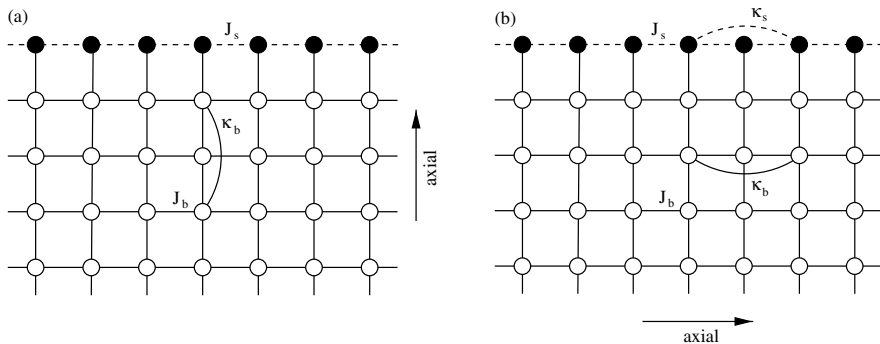


Fig. 6.5. Cross sections of semi-infinite three-dimensional ANNNI models showing two different types of surface orientations: (a) surfaces perpendicular to the axis of competing interactions, (b) surfaces parallel to this axis. J_b and J_s denote the nearest neighbour bulk and surface couplings, respectively, whereas the axial next-nearest neighbour interactions are labelled by the bulk, κ_b , and surface, κ_s , competing parameters. Surface lattice sites are represented by filled points. Reprinted from [593]. Copyright (2004) Institute of Physics Publishing.

	nature	method	β_1^L	γ_1^L	γ_{11}^L	Ref.
case A	OT	MF	1	1/2	-1/4	[76]
	OT	MC	0.62(1)	0.84(5)	-0.06(2)	[591]
	SP	MC	0.22(2)	1.28(8)	0.76(5)	[591]
case B	OT	MF	1	1/2	-1/2	[263]
	OT	MC	0.687(5)	0.82(4)	-0.29(6)	[591]
	OT	RG	0.697	0.947	-0.212	[192]
	SP	MC	0.23(1)	1.30(6)	0.72(4)	[591]
	SP	RG	0.23			[193]

Table 6.3. Surface critical exponents at a uniaxial bulk Lifshitz point, as described by the 3D ANNNI model. The surface layer is either oriented perpendicular (case A) or parallel (case B) to the axial direction. The ordinary transition (OT) or the special transition (SP) are considered. The methods used are Monte Carlo simulations (MC), mean-field theory (MF) or renormalisation-group methods (RG).

The surface exponents are defined in (6.16) in analogy with an isotropic critical point, see Chap. 2 in Volume 1.

cases the surface phase diagrams and determined the LP mean-field surface critical exponents at the ordinary transition, see Table 6.3. Interestingly, they obtained two different sets of critical exponents, depending on the orientation of the surface with respect to the direction of competition between interactions. This dependence of the ordinary transition critical exponents on the surface orientation is a direct consequence of the strongly anisotropic nature of the LP.

Formally, the surface critical exponents are defined analogously with respect to an isotropic critical point, see Chap. 2 in Volume 1. If m_1 denotes the magnetisation in the surface layer, h is the bulk magnetic field and h_1 is the surface magnetic field which acts only on the surface layer, one expects near to a Lifshitz point

$$m_1 \sim (T_L - T)^{\beta_1^L}, \quad \text{for } T \leq T_L \quad (6.16)$$

$$\chi_1 = - \left. \frac{\partial m_1}{\partial h} \right|_{h=h_1=0} \sim |T - T_L|^{-\gamma_1^L}, \quad \chi_{11} = - \left. \frac{\partial m_1}{\partial h_1} \right|_{h=h_1=0} \sim |T - T_L|^{-\gamma_{11}^L}$$

which defines the exponents β_1^L , γ_1^L and γ_{11}^L .

Extensive Monte Carlo simulations [591] showed that the predictions of mean-field theory [76, 263] are qualitatively correct, although the precise values of the critical exponents are in general different from those found within mean-field theory. As shown in Table 6.3, the values of the surface critical exponents at the ordinary transition in the vicinity of the Lifshitz point indeed depend on the surface orientation. It is worth noting that for both surface orientations the value β_1^L for the surface order parameter is clearly smaller than the corresponding value obtained in the semi-infinite Ising model. Mean-field theory yields for all these cases the same value $\beta_{1,\text{MF}}^L = 1$.

Estimates for the Lifshitz point surface critical behaviour at the special transition point have also been included in Table 6.3. Interestingly, the values of the critical exponents are very similar for the two different surface orientations. This is a strong indication that *at the bulk Lifshitz point, the surface critical behaviour at the special transition point may not depend on the orientation of the surface with respect to the competing axis*. One may relate this observation to the fact that both the bulk (diverging bulk correlation length) and the surface (diverging correlation length for correlations along the surface layer) are critical at the special transition point. It may then be argued that the surface critical behaviour is governed to a large extent by the critical fluctuations along the surface, so that the surface orientation with respect to the direction of competing interactions is only of minor importance. At the ordinary transition, however, the surface is not critical and the surface critical behaviour is governed exclusively by the critical bulk fluctuations, leading to orientation dependent critical exponents because of the anisotropic scaling at the bulk Lifshitz point.

Very recently Diehl and coworkers [194, 192, 193] analysed the surface critical behaviour at bulk Lifshitz points using renormalisation group methods. They thereby considered general m -axial Lifshitz points where the wave vector instability takes place in an m -dimensional subspace of the d -dimensional space. Thus for $m = 1$ one recovers the situation encountered in the ANNNI model. Restricting themselves to surfaces parallel to the modulation axes (i.e. to case B), they constructed the appropriate continuum $|\phi|^2$ models and computed the critical exponents at the ordinary transition to order ε^2 . Their results for $m = 1$ are included in Table 6.3. A very good agreement with the Monte Carlo results obtained in [591] can be seen.

6.3 A Different Type of Local Scale-transformation

In Chap. 5, we discussed the application of local scale-invariance to ageing systems with a dynamical exponent $z \neq 2$. In fact, a second type of local scale-transformation can be constructed along similar lines [330] and was shown to be applicable to the description of strongly anisotropic scaling of static systems, especially uniaxial Lifshitz points [328, 597]. In this section, we review this type of local scale-invariance and exploit it in order to compute scaling functions of two-point correlators in strongly anisotropic equilibrium critical systems.

6.3.1 Infinitesimal Generators

Writing the anisotropy exponent $\theta = 2/N$, we start by listing the infinitesimal generators for this alternative realisation [330]:

$$\begin{aligned}
 X_{-1} &:= -\partial_t && \text{time translation} \\
 X_0 &:= -t\partial_t - \frac{N}{2}(\mathbf{r} \cdot \partial_{\mathbf{r}}) - \frac{N}{2}x && \text{scale dilatation} \\
 X_1 &:= -t^2\partial_t - Nt\mathbf{r} \cdot \partial_{\mathbf{r}} - Nxt - \alpha\mathbf{r}^2\partial_t^{N-1} && \text{special LSI-} \\
 &&& \text{transformation} \\
 Y_{-1/z}^{(i)} &:= -\partial_{r_i} && \text{space translation} \\
 Y_{-N/2+1}^{(i)} &:= -t\partial_{r_i} - \frac{2\alpha}{N}r_i\partial_t^{N-1} && \text{generalised Galilei-} \\
 &&& \text{transformation} \\
 R^{(i,j)} &:= r_i\partial_{r_j} - r_j\partial_{r_i} && \text{rotation} \quad (6.17)
 \end{aligned}$$

with $i = 1, \dots, d$. For N non-integer, ∂_t^{N-1} has to be interpreted as a fractional derivative, for which we use the definition given in appendix J.1. If

$$\mathcal{S} = -\alpha\partial_t^N + \left(\frac{N}{2}\right)^2 \partial_{\mathbf{r}} \cdot \partial_{\mathbf{r}} \quad (6.18)$$

is the generalised **Schrödinger operator**, we have

$$[\mathcal{S}, X_{-1}] = [\mathcal{S}, Y_{-N/2}^{(i)}] = [\mathcal{S}, Y_{-N/2+1}^{(i)}] = [\mathcal{S}, R^{(i,j)}] = 0 \quad (6.19)$$

which shows that $\mathcal{S}$ is a Casimir operator of the ‘Galilei’-type sub-algebra generated from the minimal set $\langle X_{-1}, Y_{-N/2}^{(i)}, Y_{-N/2+1}^{(i)}, R^{(i,j)} \rangle$ by repeated application of the commutator. Furthermore,

$$[\mathcal{S}, X_0] = -N\mathcal{S}, \quad [\mathcal{S}, X_1] = -2Nt\mathcal{S} + \alpha N^2 \left(x - \frac{d}{2} + \frac{N-1}{N} \right). \quad (6.20)$$

In addition, since $[X_1, Y_{-N/2+k}^{(i)}] = (N-k)Y_{-N/2+k+1}^{(i)}$, it follows immediately that $[\mathcal{S}, Y_{-N/2+k}^{(i)}]\psi = 0$ for all $k \neq N$ if ψ solves $\mathcal{S}\psi = 0$, see exercise 6.3. Additional generators created from the commutators $[Y_m, Y_{m'}]$ are treated similarly. Therefore, we have the

Lemma: [330] *The realisation (6.17) generated from $\langle X_{-1}, X_1, Y_{-N/2}^{(i)} \rangle$, $i = 1, \dots, d$, sends any solution $\psi(t, \mathbf{r})$ with scaling dimension*

$$x = \frac{d}{2} - \frac{N-1}{N} \quad (6.21)$$

of the differential equation

$$\mathcal{S}\phi(t, \mathbf{r}) = \left(-\alpha\partial_t^N + \left(\frac{N}{2}\right)^2 \partial_{\mathbf{r}} \cdot \partial_{\mathbf{r}} \right) \phi(t, \mathbf{r}) = 0 \quad (6.22)$$

into another solution of the same equation. If we construct a free-field theory such that (6.22) is the equation of motion, then x as given in (6.21) is the scaling dimension of that free field ϕ .

From the different forms of the wave equations we see that the realisations of local scale-transformations in Chap. 5 and in this chapter describe physically distinct systems. The propagator resulting from (6.17), which in energy-momentum space is of the form $(\omega^N + p^2)^{-1}$, is typical for equilibrium systems with such strongly anisotropic interactions that the quadratic term $\sim \omega^2$ which is usually present is cancelled and the next-to-leading term becomes important. This occurs in fact at the Lifshitz point of spin systems with competing interactions. On the other hand, the propagator from the realisation discussed in Chap. 5 of the form $(\omega + p^z)^{-1}$ in energy-momentum space is that of a Langevin equation describing the time evolution of a non-equilibrium system. Indeed, as we saw in the previous chapter, aspects of ageing phenomena in different systems can be understood in this way.

6.3.2 Covariant Two-point Function

The main objective in the following is to calculate for strongly anisotropic critical systems two-point functions

$$C = C(t_1, t_2; r_1, r_2) = \langle \phi_1(t_1, r_1) \phi_2(t_2, r_2) \rangle \quad (6.23)$$

of scaling operators ϕ_i from their covariance properties under local scale transformations. We shall assume that spatio-temporal translation invariance holds and therefore

$$C = C(t, r) \quad , \quad t = t_1 - t_2 \quad , \quad r = r_1 - r_2 \quad . \quad (6.24)$$

Since for scaling operators ϕ invariant under spatial rotations, the two points can always be brought to lie on a given line, the case $d = 1$ is enough to find the functional form of the scaling function present in C . To do so, we have to express the action of the generators $X_{0,1}$ and Y_m on C . Each scaling operator ϕ_i is characterised by the pair (x_i, α_i) . By definition, two-point functions formed from *quasi-primary* scaling operators satisfy the covariance conditions

$$X_n C = Y_m C = 0 \quad . \quad (6.25)$$

Since all generators can be obtained from commutators of the three generators $X_{\pm 1}, Y_{-N/2}$, explicit consideration of a subset is sufficient.

For the realisation discussed in this chapter, the single condition

$$\alpha_2 = (-1)^{-N} \alpha_1 \quad (6.26)$$

is sufficient to guarantee that (6.25) is satisfied, together with the covariance under all commutators which can be constructed from the X_n, Y_m . We merely need to satisfy explicitly the following conditions

$$\begin{aligned}
X_0 C(t, r) &= \left(-t \partial_t - \frac{N}{2} r \partial_r - Nx \right) C(t, r) = 0 \\
X_1 C(t, r) &= \left(-t^2 \partial_t - Ntr \partial_r - 2Nx_1 t - \alpha_1 r^2 \partial_t^{N-1} \right) C(t, r) \\
&\quad + 2t_2 X_0 C(t, r) + Nr_2 Y_{-N/2+1} C(t, r) = 0 \\
Y_{-N/2+1} C(t, r) &= \left(-t \partial_r - \frac{2\alpha_1}{N} r \partial_t^{N-1} \right) C(t, r) = 0
\end{aligned} \tag{6.27}$$

where $2x = x_1 + x_2$. This makes it clear that time and space translation invariance are implemented. If we multiply the first equation in (6.27) by $-t$ and add it to the second one and then multiply the third equation in (6.27) by $-2/Nr$ and also add it, the condition $X_1 C(t, r) = 0$ simplifies to

$$\frac{1}{2} N t (x_1 - x_2) C(t, r) = 0 \tag{6.28}$$

which implies the constraint

$$x_1 = x_2 . \tag{6.29}$$

The two remaining equations in (6.27) may be solved by the ansatz

$$C(t, r) = \delta_{x_1, x_2} r^{-2x_1} \Omega \left(tr^{-2/N} \right) \tag{6.30}$$

which leads to the following equation for the scaling function $\Omega(v)$,

$$(\alpha_1 \partial_v^{N-1} - v^2 \partial_v - Nx_1) \Omega(v) = 0 , \tag{6.31}$$

with $v = tr^{-2/N}$. In addition, we need the physical boundary conditions

$$\Omega(0) = \Omega_0 , \quad \Omega(v) \simeq \Omega_\infty v^{-Nx_1} \quad \text{for } v \rightarrow \infty \tag{6.32}$$

where $\Omega_{0,\infty}$ are constants. Equations (6.30, 6.31, 6.32) together with the constraints (6.26, 6.29) determine the two-point function and its scaling function $\Omega(v)$. They contain what can be learned from LSI about this kind of scaling function.

6.3.3 Solution in the Case $N = 4$

We write eq. (6.31) in the form

$$(\alpha_1 \partial_v^{N-1} - v^2 \partial_v - v\zeta) \Omega(v) = 0 \quad , \quad \zeta = Nx_1 = \frac{2x_1}{\theta} . \tag{6.33}$$

It is useful to begin with the special case when N is an integer [328]. Then the anisotropy exponent $\theta = 2/N = 2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \dots$ and one merely has a finite number of generators Y_m , $m = -N/2, \dots, N/2$. For $N = 1$ and $N = 2$ we recover the scaling functions found from Schrödinger and conformal invariance. We concentrate on the new situations $N \geq 3$.

ζ	$\Omega(v)$	asymptotics (for $\beta' = 0$)
1	$\left[\frac{\Gamma(3/4)^2}{\pi^2} \sqrt{y} K_{1/4}^2\left(\frac{y}{2}\right) + \beta v I_{1/4}\left(\frac{y}{2}\right) K_{1/4}\left(\frac{y}{2}\right) \right] + \beta' v I_{1/4}^2\left(\frac{y}{2}\right)$	$4\sqrt{\alpha_1} \beta v^{-1}$
2	$\left[-\beta \sqrt{\pi} \Gamma\left(\frac{1}{4}\right) \left(\frac{y}{2}\right)^{1/4} [\mathbf{L}_{-1/4}(y) - \mathbf{I}_{1/4}(y)] \right. \\ \left. + \Gamma\left(\frac{3}{4}\right) \left(\frac{y}{2}\right)^{1/4} [\mathbf{I}_{-1/4}(y) - \mathbf{I}_{1/4}(y)] \right] + \beta' v^{1/2} \mathbf{I}_{1/4}(y)$	$4\sqrt{\alpha_1} \beta v^{-2}$
3	$\left[e^{-y} + \beta \left(\frac{\pi^2 \alpha_1}{64}\right)^{1/4} [e^y \operatorname{erf}(\sqrt{y}) - i e^{-y} \operatorname{erf}(i\sqrt{y}) - 2 \sinh y] \right] + \beta' \sinh y$	$\alpha_1 \beta v^{-3}$
4	$\left[1 + \sqrt{\pi} \Gamma\left(\frac{3}{4}\right) \left(\frac{y}{2}\right)^{3/4} [\mathbf{L}_{1/4}(y) - \mathbf{I}_{-1/4}(y)] \right. \\ \left. + \beta \left(\frac{y}{2}\right)^{3/4} [\mathbf{I}_{-1/4}(y) - \mathbf{I}_{1/4}(y)] \right] + \beta' v^{3/2} \mathbf{I}_{1/4}(y)$	$-2\alpha_1 v^{-4}$

Table 6.4. Some solutions $\Omega(v)$ of eq. (6.33) for $N = 4$ with $\Omega(0) = 1$ and their leading asymptotics for $\beta' = 0$ as $v \rightarrow \infty$. Here $\mathbf{I}_\nu, \mathbf{K}_\nu$ are modified Bessel functions, $\mathbf{L}_\nu$ is a modified Struve function, erf is the error function [4] and β, β' are constants. The abbreviation $y := v^2/(2\sqrt{\alpha_1})$ is used throughout.

For $N = 4$, some explicit solutions for a few integer values of ζ are given in Table 6.4. Given the boundary condition $\Omega(0) = 1$, these still depend on two free parameters β, β' . If $\beta' \neq 0$, these solutions diverge exponentially fast as $v \rightarrow \infty$ but if we take $\beta' = 0$, we find $\Omega(v) \sim v^{-\zeta}$ in agreement with the required boundary condition.

Using these examples as a guide, we now study the more general case with integer N and ζ arbitrary. The general solution of eq. (6.31) for integer $N \geq 2$ is readily found

$$\Omega(v) = \sum_{p=0}^{N-2} b_p v^p \mathcal{F}_p \quad (6.34)$$

with

$$\mathcal{F}_p = {}_2F_{N-1} \left(\frac{\zeta+p}{N}, 1; 1 + \frac{p}{N}, 1 + \frac{p-1}{N}, \dots, \frac{p+2}{N}; \frac{v^N}{N^{N-2}\alpha_1} \right) \quad (6.35)$$

where ${}_2F_{N-1}$ is a generalised hypergeometric function and the b_p are free parameters. To be physically acceptable, the boundary condition (6.32) must be satisfied. The leading asymptotic behaviour of the $\mathcal{F}_p$ for $v \rightarrow \infty$ can be found in the mathematical literature [738, 739]. The leading asymptotics of $\Omega(v)$ are given by

$$\Omega(v) \simeq \sqrt{\frac{4\pi^2 N}{N-2}} \left(\frac{v^{1/(N-2)}}{(\alpha_1 N)^{1/N}} \right)^{\zeta+1-N} \exp \left(\frac{N-2}{N\alpha_1^{1/(N-2)}} v^{N/(N-2)} \right) \times$$

$$\times \sum_{p=0}^{N-2} b_p \frac{\Gamma(p+1)}{\Gamma((p+1)/N)\Gamma((p+\zeta)/N)} \left(\frac{\alpha_1}{N^2}\right)^{p/N} \quad (6.36)$$

which grows exponentially as $v \rightarrow \infty$ if $N > 2$. Clearly, this leading term must vanish, which imposes the following condition on the b_p

$$\sum_{p=0}^{N-2} b_p \frac{\Gamma(p+1)}{\Gamma((p+1)/N)\Gamma((p+\zeta)/N)} \left(\frac{\alpha_1}{N^2}\right)^{p/N} = 0. \quad (6.37)$$

Remarkably, this condition is already sufficient to cancel not only the leading exponential term but in fact the *entire* series of exponentially growing terms [330]. Eliminating b_{N-2} , the final solution for N integer becomes

$$\begin{aligned} \Omega(v) &= \sum_{p=0}^{N-3} b_p \Omega_p(v) \\ \Omega_p(v) &= v^p \mathcal{F}_p - \frac{\Gamma(p+1)}{\Gamma(\frac{p+1}{N})\Gamma(\frac{p+\zeta}{N})} \frac{\Gamma(\frac{N-1}{N})\Gamma(1+\frac{\zeta-2}{N})}{\Gamma(N-1)} \times \\ &\quad \times \left(\frac{\alpha_1}{N^2}\right)^{(p+2-N)/N} v^{N-2} \mathcal{F}_{N-2}. \end{aligned} \quad (6.38)$$

Up to normalisation, the form of $\Omega(v)$ depends on ζ and on $N-3$ free parameters b_p , while α_1 merely sets a scale. The independent solutions Ω_p ($p = 0, 1, \dots, N-3$) satisfy the boundary conditions

$$\Omega_p(v) \simeq \begin{cases} v^p & ; v \rightarrow 0 \\ \Omega_{p,\infty} v^{-\zeta} & ; v \rightarrow \infty \end{cases} \quad (6.39)$$

where explicitly

$$\begin{aligned} \Omega_{p,\infty} &= - \left(\frac{\alpha_1}{N^2}\right)^{(\zeta+p)/N} \frac{\Gamma(\frac{1-\zeta}{N})}{\Gamma(1-\zeta)} \frac{\Gamma(p+1)}{\Gamma(\frac{p+1}{N})} \\ &\quad \times \frac{\pi \sin\left(\frac{\pi}{N}(p+2)\right)}{\Gamma(\frac{p+\zeta}{N}) \sin\left(\frac{\pi}{N}(p+\zeta)\right) \sin\left(\frac{\pi}{N}(\zeta-2)\right)}. \end{aligned} \quad (6.40)$$

Therefore, we have not only eliminated the entire exponentially growing series, but furthermore, the Ω_p satisfy exactly the physically required boundary condition for $v \rightarrow \infty$ [328]. In [Table 6.4](#) we listed a few closed solutions for $N = 4$ which satisfy the boundary condition $\Omega(0) = 1$. By varying the parameters β and β' one obtains the three independent solutions of the third-order differential equation (6.33). Furthermore, from the explicit form of the solutions we see that only the contribution parametrised by β' diverges exponentially as $v \rightarrow \infty$. There is a second solution which decays as $v^{-\zeta}$ for $v \rightarrow \infty$ and the third solution vanishes exponentially fast in the $v \rightarrow \infty$ limit. From the last

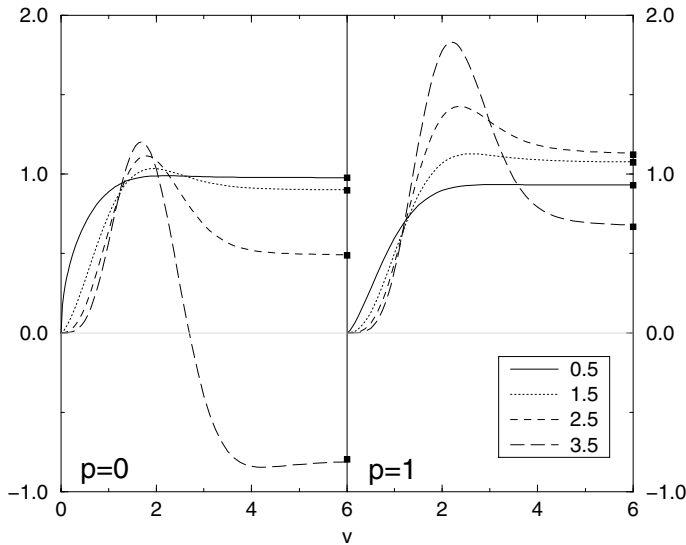


Fig. 6.6. Scaling functions $v^\zeta \Omega_p(v)$ for $N = 4$, $\alpha_1 = 1$ and $p = 0, 1$. The different curves belong to values of $\zeta = Nx_1$ as indicated and the squares are the values of $\Omega_{p,\infty}$. Reprinted from [330] with permission from Elsevier.

two solutions we can therefore construct scaling functions with the physically expected asymptotic behaviour.

We can illustrate the convergence of $v^\zeta \Omega_p(v)$ towards $\Omega_{p,\infty}$, as given in (6.40), by plotting $v^\zeta \Omega_p(v)$ as a function of v for several values of ζ . This is done for $N = 4$ in Fig. 6.6. Besides confirming the correctness of the asymptotic expressions (6.39,6.40) for $v \rightarrow \infty$, we also see that for a large range of values of ζ , the asymptotic regime is reached quite rapidly.

Below, we shall need the explicit expressions for $\Omega_{0,1}(v)$ for $N = 4$

$$\begin{aligned} \Omega_0(v) &= \frac{\Gamma(3/4)}{\Gamma(\zeta/4)} \sum_{n=0}^{\infty} \frac{\Gamma(n/2 + \zeta/4)}{n! \Gamma(n/2 + 3/4)} \left(-\frac{v^2}{2\sqrt{\alpha_1}} \right)^n \\ \Omega_1(v) &= \sqrt{\frac{\pi}{2}} \frac{v}{\Gamma((\zeta+1)/4)} \sum_{n=0}^{\infty} \frac{\Gamma((n+1+\zeta)/4) s(n)}{\Gamma(n/4+1) \Gamma((n+3)/2)} \left(-\frac{v}{\sqrt[4]{4\alpha_1}} \right)^n \end{aligned} \quad (6.41)$$

where $s(n) := \frac{1}{\sqrt{2}} (\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}) \cos \frac{n\pi}{4}$. These expressions will be encountered again in the next section for the correlators of the ANNNI and ANNS models at their Lifshitz points.

6.3.4 Solution in the Case $N \simeq 4$

We have seen above that the work of Diehl and Shpot [195, 667, 196, 669, 668] implies that the anisotropy exponent $\theta \approx \frac{1}{2}$, but not exactly. In particular, they established that the two-loop terms in the renormalised field-theory constructed for the ANNO(n) model yielded small corrections for the anisotropy exponent in second order of an ε -expansion, i.e. $\theta = \frac{1}{2} + O(\varepsilon^2)$, and similarly $\theta = \frac{1}{2} + O(n^{-1})$ for an $1/n$ expansion. Therefore we must study what happens when N is no longer exactly an integer [330]. It is useful to write the anisotropy exponent as

$$\frac{2}{\theta} = N = N_0 + \epsilon \quad , \quad \epsilon = \frac{p}{q} \quad (6.42)$$

where $N_0 \in \mathbb{N}$ and p, q are positive coprime integers.

For N integer, we have seen that there is a unique solution which decays as $\Omega(v) \sim v^{-\zeta}$ as $v \rightarrow \infty$. The presence of such a solution for arbitrary N may be checked by seeking solutions of the form

$$\Omega(v) = \sum_{n=0}^{\infty} a_n v^{-n/q+s} \quad , \quad a_0 \neq 0 \quad (6.43)$$

In making this ansatz, we concentrate on those solutions of eq. (6.33) which do *not* grow or vanish exponentially for v large. We find upon substitution

$$\begin{aligned} \sum_{n=p+qN_0}^{\infty} \frac{\alpha_1 \Gamma\left(\frac{1}{q}(-n+p+qN_0)+s+1\right)}{\Gamma\left(-\frac{n}{q}+2+s\right)} a_{n-p-qN_0} v^{-n/q} \\ + \sum_{n=0}^{\infty} \left(\frac{1}{q}(n-px_1-qN_0x_1)-s\right) a_n v^{-n/q} = 0 \end{aligned} \quad (6.44)$$

which must be valid for all positive values of v . Comparing the coefficients of v , we obtain

$$\begin{aligned} s &= -\frac{p}{q}x_1 - N_0x_1 = -(N_0 + \epsilon)x_1 = -\zeta \\ a_n &= 0 \quad ; \quad \text{for } n = 1, 2, \dots, p + qN_0 - 1 \\ \frac{n}{q}a_n &= -\frac{\alpha_1 \Gamma\left(-\frac{1}{q}(n-p(1+x_1)-qN_0(1+x_1))+1\right)}{\Gamma\left(-\frac{n}{q}-\frac{p}{q}x_1-N_0x_1+2\right)} a_{n-p-qN_0} \end{aligned} \quad (6.45)$$

In principle, there might be additional δ -function terms which come from the definition of the fractional derivative, see Appendix J. However, if we either restrict ourselves to $v > 0$ or else if ζ is distinct from the discrete set of values $\zeta_c = 2 + m - n/q$, where $n, m \in \mathbb{N}$, these terms do not occur.

We can now let $n = (p + qN_0)\ell$ and $a_n = a_{(p+qN_0)\ell} =: b_\ell$ and find the simpler recurrence

$$b_\ell = -\frac{\alpha_1}{N\ell} \frac{\Gamma(1+N-(\ell+x_1)N)}{\Gamma(2-(\ell+x_1)N)} b_{\ell-1} \quad , \quad \ell = 1, 2, \dots \quad (6.46)$$

The solution is [330]

$$b_\ell = b_0 \left(\frac{\alpha_1}{N^2} \right)^\ell \frac{\Gamma(1+(\zeta-1)/N)}{\ell! \Gamma(\ell+1+(\zeta-1)/N)} \frac{\Gamma(1-\zeta)}{\Gamma(1-\zeta-\ell N)} \quad . \quad (6.47)$$

Exercise 6.4 illustrates the evaluation of this for $N = 1$.

If we use the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ and define the function

$$\mathfrak{G}_{a,b}^{(N)}(z) := \Gamma(b)\Gamma(1-a) \sum_{\ell}^{\infty} \frac{\Gamma(\ell N+a)}{\ell! \Gamma(\ell+b)} z^\ell \quad (6.48)$$

the series solution with the requested behaviour $\Omega(v) \simeq b_0 v^{-\zeta}$ at $v \rightarrow \infty$ may be written as follows

$$\begin{aligned} \Omega(v) &= v^{-\zeta} \sum_{\ell=0}^{\infty} b_\ell v^{-N\ell} \\ &= \frac{b_0}{2\pi i} v^{-\zeta} \left[e^{\pi i \zeta} \mathfrak{G}_{\zeta, 1+\frac{\zeta-1}{N}}^{(N)} \left(\frac{\alpha_1}{N^2} e^{\pi i N} v^{-N} \right) - e^{-\pi i \zeta} \mathfrak{G}_{\zeta, 1+\frac{\zeta-1}{N}}^{(N)} \left(\frac{\alpha_1}{N^2} e^{-\pi i N} v^{-N} \right) \right] \end{aligned} \quad (6.49)$$

From these expressions, it follows that the radius of convergence of these series as a function of the variable $1/v$ is infinite for $N < 2$ and zero for $N > 2$. In the first case, we therefore have a convergent series for the scaling function, while in the second case we have obtained an asymptotic expansion.

In several applications, notably the 3D ANNNI model, the anisotropy exponent $\theta \simeq 1/2$ to a very good approximation. Therefore, consider fractional derivatives of order $a = N_0 + \epsilon$, where N_0 is an integer and ϵ is small. To study perturbatively the solutions for $\epsilon \ll 1$, we use the identity (J16), set $a = N = N_0 + \epsilon$ and expand to first order in ϵ . The result is [330]

$$\begin{aligned} \partial_r^N f(r) \Big|_{\text{reg}} &= \partial_r^\epsilon \partial_r^{N_0} f(r) \\ &\simeq f^{(N_0)}(r) + \epsilon \mathcal{L}_0 f^{(N_0)}(r) + O(\epsilon^2) \\ \mathcal{L}_0 g(r) &= \left[- (C_E + \ln r) + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell! \ell} r^\ell \frac{d^\ell}{dr^\ell} \right] g(r) \end{aligned} \quad (6.50)$$

where $C_E = 0.5772 \dots$ is Euler's constant. To find the first correction in ϵ with respect to the solution (6.38) when N is an integer, we set again $N = N_0 + \epsilon$ with $N_0 \in \mathbb{N}$ and $\epsilon > 0$ and consider

$$\Omega(v) = \Omega^{(0)}(v) + \epsilon \Omega^{(1)}(v) + O(\epsilon^2) \quad . \quad (6.51)$$

Then $\Omega^{(0)}(v)$ solves eq. (6.33) with $N = N_0$ and is consequently given by eq. (6.38), whereas $\Omega^{(1)}(v)$ satisfies the equation

$$(\alpha_1 \partial_v^{N_0-1} - v^2 \partial_v - \zeta v) \Omega^{(1)}(v) = \omega(v) := -\mathcal{L}_0 (v^2 \partial_v + \zeta v) \Omega^{(0)}(v) \quad (6.52)$$

which we now study.

First, we consider the limiting behaviour of $\Omega^{(1)}(v)$ for v either very large or very small. If $v \gg 1$, we see from eq. (6.49) that $\Omega^{(0)}(v) \sim v^{-\zeta}(1+O(v^{-N_0}))$. This implies in turn that $\omega(v) \simeq (A_\infty + B_\infty \ln v)v^{-\zeta-(N_0-1)}$, where A_∞, B_∞ are some constants. Therefore one must have $\Omega^{(1)}(v) \sim v^{-\zeta}(1+O(v^{-N_0}))$ in order to reproduce this result for $\omega(v)$. On the other hand, if $v \ll 1$, we have $\Omega^{(0)}(v) \simeq \text{constant}$ which leads to $\omega(v) \simeq (A_0 + B_0 \ln v)$ with some constants A_0, B_0 . This can be reproduced from the limiting behaviour $\Omega^{(1)}(v) \sim O(v, v^{N_0} \ln v)$. In conclusion, the first-order perturbation is compatible with the boundary condition (6.32) for the full scaling function $\Omega(v)$.

We now work out the first correction $\Omega^{(1)}(v)$ explicitly for $N_0 = 4$ [330]. We shall need this case later when discussing the scaling function at the Lifshitz point of the three-dimensional ANNNI model. There are two physically acceptable solutions $\Omega_0^{(0)}(v)$ and $\Omega_1^{(0)}(v)$ of zeroth order in ϵ which are given in (6.41). The general zeroth-order solution is then

$$\Omega^{(0)}(v) = \Omega_0^{(0)}(v) + \frac{\mathbf{p}}{\alpha_1^{1/4}} \Omega_1^{(0)}(v) \quad (6.53)$$

where $\mathbf{p} = b_1 \alpha_1^{1/4} / b_0$ is a universal constant. Then all metric factors in the scaling function are absorbed into the argument $v/\alpha_1^{1/4}$ and the form of $\Omega(v)$ is given by the two universal parameters ζ and $\mathbf{p}$.

Consider the first-order correction to $\Omega^{(0)}(v)$. From the explicit form of $\mathcal{L}_0$ and (6.41), a longish calculation shows [330]

$$\omega(v) = \sum_{n=0}^{\infty} [A_n v^{n+1} + B_n v^{n+1} \ln v] \quad (6.54)$$

where $A_n = -\psi(n+1)B_n$ and

$$B_n = \begin{cases} \frac{\Gamma(n+\zeta/4) \Gamma(3/4)}{\Gamma(n+3/4) \Gamma(\zeta/4)} \frac{4n+\zeta}{(2n)!} \left(\frac{1}{4\alpha_1}\right)^n & ; \quad n \equiv 0 \pmod{4} \\ \frac{1}{\alpha_1^{1/4}} \frac{\sqrt{\pi} \Gamma(n+(\zeta+1)/4)}{\Gamma((\zeta+1)/4) \Gamma(2n+3/2)} \frac{4n+1+\zeta}{n! 2} \left(\frac{1}{4\alpha_1}\right)^n \mathbf{p} & ; \quad n \equiv 1 \pmod{4} \\ -\frac{1}{\sqrt{\alpha_1}} \frac{\Gamma(n+(\zeta+2)/4)}{2 \Gamma(n+5/4)} \frac{4n+2+\zeta}{(2n+1)!} \left(\frac{1}{4\alpha_1}\right)^n \times \\ \quad \times \left[\frac{\Gamma(3/4)}{\Gamma(\zeta/4)} + \mathbf{p} \frac{1}{\Gamma((\zeta+1)/4)} \sqrt{\frac{\pi}{2}} \right] & ; \quad n \equiv 2 \pmod{4} \\ 0 & ; \quad n \equiv 3 \pmod{4} \end{cases} \quad (6.55)$$

Here $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function [4]. The solution of the third-order differential equation (6.52) is of the form

$$\Omega^{(1)}(v) = \sum_{n=0}^{\infty} [a_n v^n + b_n v^n \ln v] \quad (6.56)$$

where in addition

$$a_0 = a_3 = 0 \quad , \quad b_0 = b_1 = b_2 = b_3 = 0 \quad (6.57)$$

and, for all $n \geq 0$,

$$\begin{aligned} b_{n+4} &= \frac{1}{\alpha_1 (n+4)(n+3)(n+2)} [B_n + b_n (n + \zeta)] \quad , \\ a_{n+4} &= \frac{1}{\alpha_1 (n+4)(n+3)(n+2)} [A_n + a_n (n + \zeta) + b_n \\ &\quad - b_{n+4} \alpha_1 (3n^2 + 18n + 26)] \quad . \end{aligned} \quad (6.58)$$

The value of the constant a_0 is fixed because of the boundary condition $\Omega(0) = 1$. The first perturbative correction $\Omega^{(1)}(v)$ still depends on the free parameters $a_1 =: \mathbf{p}$ and $a_2 =: \mathbf{b}$. We also observe that because of (6.55), we have $a_{4n+3} = b_{4n+3} = 0$ for all $n \in \mathbb{N}$ and that the metric factor α_1 merely sets the scale in the variable $v/\alpha_1^{1/4}$ but does not otherwise affect the functional form of $\Omega^{(1)}(v)$.

We have seen above that for $v \gg 1$, we must recover $\Omega(v) \sim v^{-\zeta}$. We can therefore fix $\mathbf{a}$ and $\mathbf{b}$ such that the correction term $\Omega^{(1)}(v)$ goes to zero for v large. Furthermore, we see from Fig. 6.6 that the asymptotic regime is already reached for quite small values of v for a large range of values of ζ . To a good approximation, we can therefore determine $\mathbf{a}$ and $\mathbf{b}$ from the requirement that $\Omega^{(1)}(v_0) = 0$ and $d\Omega^{(1)}(v_0)/dv = 0$ if v_0 is finite, but chosen to be sufficiently large. The recursion (6.58) then gives a system of two linear equations for $\mathbf{a}$ and $\mathbf{b}$.

As an example, we illustrate this in Fig. 6.7 for $\zeta = 1.3$ [330]. From Fig. 6.6, we observe that $v_0 = 6$ is already far into the asymptotic regime. For the values $\mathbf{p} = \pm 0.11$, the scaling function (6.51), with the first-order correction included, is shown for several values of $\epsilon = N - 4$. The first-order perturbative corrections with respect to the solution found for $N = 4$ are quite substantial, even for small values of ϵ . This suggests that a non-integer value of N in the differential equation (6.33) might be detectable in numerical simulations. Of course, this is merely the first term of a new kind of perturbative expansion and it is unknown at present if a naive numerical application as performed here can be justified.

6.4 Application to Lifshitz Points

After these preparations, we can finally compare the predictions of local scale-invariance for the spin-spin correlator C_σ and the energy-energy correlator C_ϵ

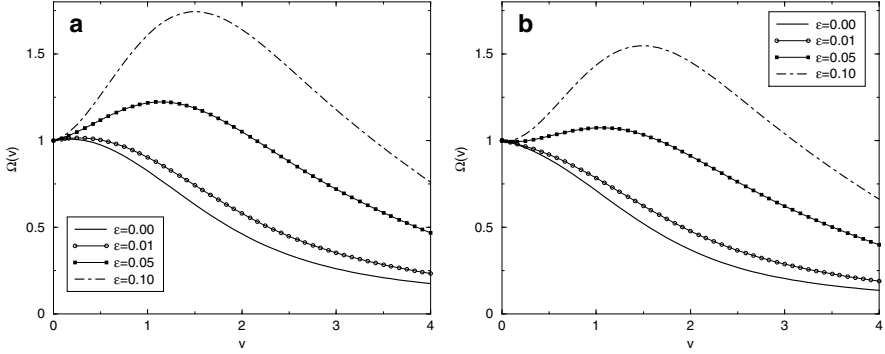


Fig. 6.7. Perturbative scaling functions $\Omega(v)$ eq. (6.51) for an uniaxial LP around $N_0 = 4$, with $\zeta = 1.3$, $\alpha_1 = 1$ and (a) $\mathbf{p} = +0.11$ and (b) $\mathbf{p} = -0.11$, for several values of ϵ . Reprinted from [330] with permission from Elsevier.

with the scaling forms obtained in specific models at a uniaxial Lifshitz point. From renormalisation-group arguments one expects the following anisotropic scaling of the correlation functions

$$C_{\sigma,\epsilon}(r_{\parallel}, \mathbf{r}_{\perp}) = b^{-2x_{\sigma,\epsilon}} C_{\sigma,\epsilon}(r_{\parallel} b^{-\theta}, \mathbf{r}_{\perp} b^{-1}) = r_{\perp}^{-\zeta_{\sigma,\epsilon}\theta} \Omega_{\sigma,\epsilon}(r_{\parallel}/r_{\perp}^{\theta}) \quad (6.59)$$

for both the spin-spin and the energy-energy correlators, respectively. For a Lifshitz point in $(d_{\perp} + 1)$ dimensions, we have

$$\zeta_{\sigma} = \frac{2(\theta + d_{\perp})}{\theta(2 + \gamma/\beta)} \quad , \quad \zeta_{\epsilon} = \frac{2(\theta + d_{\perp})(1 - \alpha)}{\theta(2 - \alpha)} \quad . \quad (6.60)$$

As already discussed one has at a Lifshitz point $\theta \simeq \frac{1}{2}$ at least to a good approximation. In terms of the notation of the previous section, this corresponds to $N = 2/\theta = 4$. For $N = 4$, we recall the two-point function

$$C(r_{\parallel}, \mathbf{r}_{\perp}) = r_{\perp}^{-\zeta/2} b_0 \left(\Omega_0(v) + \frac{\mathbf{p}}{\alpha_1^{1/4}} \Omega_1(v) \right) \quad , \quad v = tr^{-1/2} \quad (6.61)$$

where $\Omega_{0,1}(v)$ are explicitly given in eq. (6.41). The functional form of $\Omega(v)$ only depends on the universal parameters ζ and $\mathbf{p}$. The metric factor α_1 only arises as a scale factor through the argument $v \alpha_1^{-1/4}$.

6.4.1 ANNNS Model

We consider the ANNNS' model, obtained from the usual ANNNS model by adding a further third-neighbour coupling along the special axis, described by

a parameter κ_2 . The phase diagram of the ANNNS' model is well-known and uniaxial Lifshitz points occur along the line [543, 651, 258]

$$\kappa_2 = \frac{1}{9}(1 - 4\kappa_1) \quad , \quad \kappa_1 < \frac{2}{5} \quad (6.62)$$

with a known $T_L = T_L(\kappa_1, \kappa_2)$. The lower critical dimension is $d_- = \frac{5}{2}$. We need the following exactly known critical exponents in d dimensions [543, 651]

$$\beta = \frac{1}{2} \quad , \quad \gamma = \frac{4}{2d - 5} \quad , \quad \theta = \frac{1}{2} \quad , \quad \zeta_\sigma = 2 \left(d - \frac{5}{2} \right) \quad (6.63)$$

which means $N = 4$ in our notation. The exact spin-spin correlator along the line (6.62) of Lifshitz points is [258], see exercise 6.5

$$C_\sigma(r_\parallel, \mathbf{r}_\perp) = C_0 r_\perp^{-(d-d_*)} \Psi \left(\frac{d-d_*}{2}, \frac{1}{4} \sqrt{\frac{3}{2-5\kappa_1}} \frac{r_\parallel^2}{r_\perp} \right) \quad (6.64)$$

$$\Psi(a, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(k/2 + a)}{\Gamma(k/2 + 3/4)} x^k \quad (6.65)$$

where C_0 is a (known) normalisation constant. This reproduces the exponent ζ_σ from eq. (6.63). Comparing with the expected form (6.61) and the specific functions (6.41), we see that

$$\Omega_0(v) = \frac{\Gamma(3/4)}{\Gamma(\zeta_\sigma/4)} \Psi \left(\frac{\zeta_\sigma}{4}, \frac{v^2}{2\sqrt{\alpha_1}} \right) . \quad (6.66)$$

With the correspondence $t \leftrightarrow r_\parallel$, $r \leftrightarrow r_\perp$ and the non-universal metric factor $\alpha_1 = \frac{4}{3}(2 - 5\kappa_1)$, we therefore observe complete agreement. In particular, we identify the universal parameter $\mathbf{p} = 0$ [328].

We mention that the form of the spin-spin correlator in the case $N = 6$ has been considered at a Lifshitz point of second order in the context of the exactly solved extension of the ANNNS model [258, 330].

6.4.2 ANNNI Model

In the 3D ANNNI model, information on the scaling functions⁹ is available from simulations based on cluster algorithms for critical systems such as the Wolff algorithm [737] which allows us to study much larger lattices than possible with more traditional methods. In appendix I, we outline how to generalise the Wolff algorithm to systems with competing interactions beyond

⁹ Renormalised field-theory has not yet been pushed sufficiently far to be able to extract scaling functions.

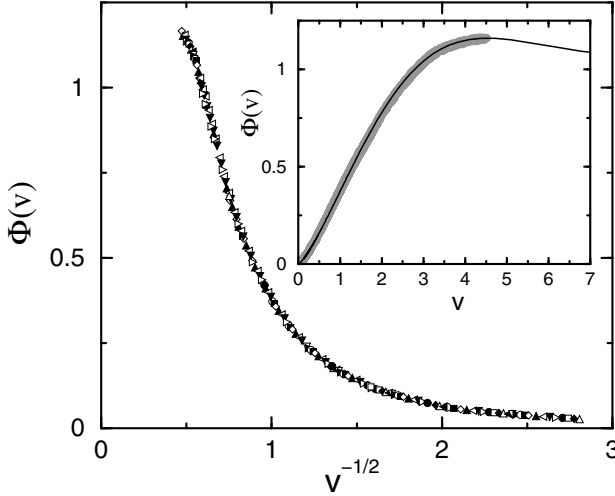


Fig. 6.8. Scaling function $\Phi(u)$ for the spin-spin correlator as defined in (6.68) for the 3D ANNNI model, for $\kappa_1 = 0.270$ and $T = 3.7475$. Selected data on a $200 \times 200 \times 100$ lattice are shown. The symbols correspond to several values of $r_\perp$. The inset shows the full data set (gray points) and the prediction (6.61) of local scale invariance with $N = 4$ and $\mathbf{p} = -0.11$, $\alpha_1 = 33.2$. The data are from [597].

nearest neighbours and, furthermore, how to include the Evertz-Linden algorithm [233] which permits the direct computation of correlation functions on an effectively infinite lattice [597, 343].

First, the anisotropy exponent θ and the scaling dimensions $\zeta_{\sigma,\epsilon}$ must be found. As discussed above, renormalised field-theory has shown that the anisotropy exponent θ is not exactly equal to $\frac{1}{2}$ [667, 669, 668]. However, the difference $|\theta - \frac{1}{2}|$ is in the 3D ANNNI model so small that as yet no simulational results with a precision high enough to identify this correction are available.

If we take the exponent estimates of [597] and in addition set $\theta = \frac{1}{2}$, we find from (6.60) for the 3D ANNNI model ($d_\perp = 2$)

$$\zeta_\sigma = 1.30 \pm 0.05 \quad , \quad \zeta_\epsilon = 4.5 \pm 0.2 \quad (6.67)$$

where the error follows from the quoted uncertainties in the determination of the exponents α, β, γ . If we now take $\theta = 0.48$ as suggested by two-loop results of [667], the resulting variation of both ζ_σ and ζ_ϵ stays within the error bars quoted in eq. (6.67). In conclusion, given the precision of the available exponent estimates, any effects of a possible deviation of θ from $\frac{1}{2}$ are not yet notable. We shall therefore undertake the subsequent analysis of the correlator by making the working hypothesis $\theta = \frac{1}{2}$ [597].

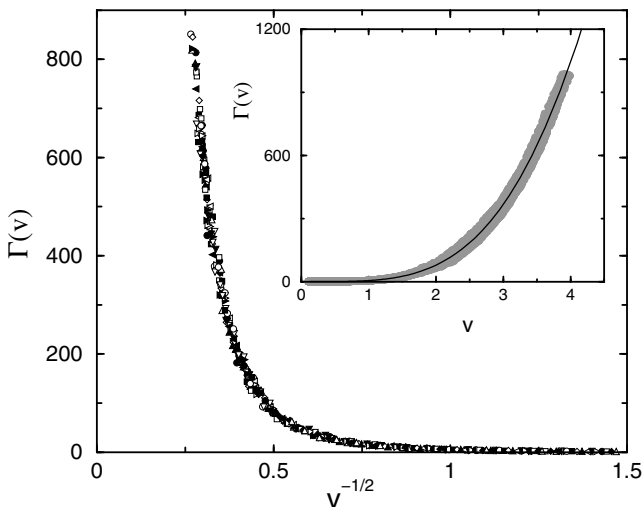


Fig. 6.9. Scaling function of the energy-energy correlator (6.69) in the 3D ANNNI model and with the same parameter values as in Fig. 6.8. The inset shows the full data set (gray points) and the prediction (6.61) of local scale invariance with $N = 4$ and $p = 0.24$, $\alpha_1 = 1400$

In this case, we can compare with the scaling prediction (6.61) obtained for $N = 2/\theta = 4$. In Figures 6.8 and 6.9 we show data [597] for the modified scaling function of the spin-spin correlator

$$\Phi(v) = v^{\zeta_\sigma} \Omega_\sigma(v) \quad (6.68)$$

and the energy-energy correlator

$$\Gamma(v) = v^{\zeta_\epsilon} \Omega_\epsilon(v) . \quad (6.69)$$

The clear data collapses establish scaling. It can be checked that there is no perceptible change in the scaling plots for values of θ slightly less than $\frac{1}{2}$ [597].

For a quantitative comparison with (6.61), one may consider for the scaling function of the spin-spin correlator the moments

$$M(n) = \int_0^\infty dv v^n \Phi(v) . \quad (6.70)$$

Then it is easy to show [64] that the moment ratios

$$J_k(\{m_i\}; \{n_j\}) = \prod_{i=1}^k M(m_i) / \prod_{j=1}^k M(n_j) , \quad \text{with} \quad \sum_{i=1}^k m_i = \sum_{j=1}^k n_j \quad (6.71)$$

and $k \geq 2$ are independent of b_0 and α_1 . They only depend on the functional form of $\Phi(u)$. This in turn is determined by ζ_σ and $\mathbf{p}$. Therefore, the determination of a certain moment ratio allows us, with ζ_σ given by (6.67), to find a value for $\mathbf{p}$. The Monte Carlo data will be consistent with (6.61) if the values of $\mathbf{p}$ found from several distinct ratios J_k coincide. In practise, these integrals cannot be calculated up to $v = \infty$ but only to some finite value v_0 and the moments retain a dependence on α_1 through the upper limit of integration. Then an iteration procedure must be used to find $\mathbf{p}$ and α_1 simultaneously [597]. The results are collected in Table 6.5.

Clearly, the two parameters can be consistently determined from different moment ratios. The final estimate is [597]

$$\mathbf{p} = -0.11 \pm 0.01 \quad , \quad \alpha_1 = 33.2 \pm 0.8 \quad . \quad (6.72)$$

Above, we have seen that $\mathbf{p} = 0$ for the spin-spin correlator of the ANNNI model. Hence the value of $\mathbf{p}$ is characteristic for the universality class at hand.

A similar analysis can be made for the scaling function of the energy-energy correlator. Resulting values of the parameters the $\mathbf{p}$ and α_1 are collected in Table 6.6. Our final estimate for these quantities is

$$\mathbf{p} = 0.24 \pm 0.03 \quad , \quad \alpha_1 = 1400 \pm 100 \quad . \quad (6.73)$$

These estimates are distinct from the ones found for the order-parameter, which illustrates that $\mathbf{p}$ and α_1 will in general depend on the chosen observable.

In the insets in Figures 6.8 and 6.9 the Monte Carlo data are compared with the resulting scaling function. The agreement between the data and the prediction (6.61) of local scale-invariance is very good.

k	$\{m_i\}$	$\{n_j\}$	$\mathbf{p}$	α_1
2	$\{0, -0.5\}$	$\{-0.25, -0.25\}$	-0.102	32.7
2	$\{-0.25, -0.75\}$	$\{-0.5, -0.5\}$	-0.125	34.0
2	$\{0.2, -0.9\}$	$\{0, -0.7\}$	-0.100	32.8
3	$\{0.2, -0.6, -0.8\}$	$\{-0.3, -0.4, -0.5\}$	-0.102	32.8
3	$\{-0.1, -0.6, -0.7\}$	$\{-0.4, -0.5, -0.5\}$	-0.117	33.5

Table 6.5. Values of the parameters $\mathbf{p}$ and α_1 , as determined from different moment ratios $J_k(\{m_i\}; \{n_j\})$, for the scaling function of the spin-spin correlator at the LP of the 3D ANNNI model.

k	$\{m_i\}$	$\{n_j\}$	$\mathbf{p}$	α_1
2	$\{0, -0.5\}$	$\{-0.25, -0.25\}$	0.22	1496.8
2	$\{1, -0.5\}$	$\{0, 0.5\}$	0.27	1296

Table 6.6. Values of the parameters $\mathbf{p}$ and α_1 , as determined from different moment ratios $J_k(\{m_i\}; \{n_j\})$, for the scaling function of the energy-energy correlator at the LP of the 3D ANNNI model.

To finish, we reconsider our working hypothesis $\theta = \frac{1}{2}$, or equivalently $N = 4$. In Fig. 6.7 we have shown how the form of the scaling function $\Omega(v)$ changes when $\epsilon = N - 4$ is increased, to first order in ϵ . In particular, rather pronounced non-monotonic behaviour is seen for values of $\epsilon \sim 0.1$ which is the order of magnitude suggested from the results of renormalised field theory [195, 667, 198]. Nothing of this is visible in the Monte Carlo data of Fig. 6.8. Given the different estimates for the exponents coming from renormalised field theory [667] and cluster Monte Carlo [597], a possible difference of θ from $\frac{1}{2}$ cannot yet be unambiguously detected. Direct precise estimates of θ from non-perturbative studies would be needed, but are difficult to obtain.

6.5 Conclusions

In this chapter, we have presented an attempt to derive the scaling functions of certain equilibrium systems with strongly anisotropic scaling as described by an anisotropy exponent $\theta \neq 1$, in terms of a postulated extension of scale-invariance. We have shown that an invariant linear fractional differential equation can be found and have used it to compute an explicit expression for the two-point covariant correlation function. Remarkably, these expressions can be matched to precise simulational data in systems as complex as the 3D ANNNI model and they also reproduce the exact results in the exactly solvable ANNS model, which can be described in terms of Gaussian fields. At the very least, the form of LSI discussed in this chapter leads to a very good quantitative description of some aspects of these models and is capable of taking into account that the anisotropy exponent does not need to take a simple numerical value. However, the two-loop contributions to the scaling functions computed within renormalised field-theory remain to be studied and it remains conceivable that LSI-predictions should not be compared term-by-term to the ϵ -expansion but rather to fully re-summed ϵ -expansion series. Also, at present the 3D ANNNI model is the only non-trivial system where the predictions of this form of LSI have been tested and the as yet available field-theoretical studies only treat the values of the exponents.

Looking now back at the programme realised so far, in our attempts to understand to what extent a generalisation of dynamical and/or strongly anisotropic scaling can be found, in physically interesting systems, it might be helpful, or simply amuse the reader, to recall some recent remarks by Dyson on string theory [223]:

“First, [it] is first-rate mathematics. [...] Second, the string theorists think of themselves as physicists rather than mathematicians. They believe that their theory describes something real in the physical world. And third, there is not yet any proof that the theory is

relevant to physics. [...] I consider it unlikely that string theory will turn out to be either totally successful or totally useless. By totally successful I mean that it is a complete theory of physics. By totally useless I mean that it remains a beautiful piece of pure mathematics. [...] I guess that it will be like the theory of Lie groups. [...] When the quantum revolution transformed physics, Lie algebras [...] became the key to understanding the central role of symmetries in the quantum world.”

Freeman Dyson, *Birds and frogs*, (2008)

Problems

6.1. Prove the scaling relations (6.7) at an m -axial Lifshitz point.

6.2. When analysing the influence of finite-size effects on conventional critical behaviour in Volume 1, Chap. 2, renormalisation-group arguments were given to derive the Privman-Fisher form of universal scaling for the singular part of the density of the Gibbs functional

$$g^{(sing)}(\tau, h) = L^{-d} Y(C_1 \tau L^{1/\nu}, C_2 h L^{\Delta/\nu})$$

where $\tau = (T_c - T)/T_c$ measures the distance from the critical point, h is a reduced magnetic field, $C_{1,2}$ are non-universal metric factors and Y is an universal scaling function.

Generalise that derivation to strongly anisotropic critical points, such as (uniaxial) Lifshitz points [353].

6.3. The mean-field free energy $F(T, \mathcal{N})$ for a system of $\mathcal{N}$ spins with the Hamiltonian $\mathcal{H}$ can be derived from the inequality

$$F \leq \Phi = F_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_0, \quad (6.74)$$

$$F_0 = -T \ln \left[\sum_{\{s\}} \exp(-\mathcal{H}_0[s]/T) \right].$$

Here, the sum is over all spin configurations, $\mathcal{H}_0$ is a trial Hamiltonian, and the average is taken with respect to $\mathcal{H}_0$. Choose the free trial Hamiltonian

$$\mathcal{H}_0 = - \sum_{x,y,z} \eta_z S_{xyz}, \quad (6.75)$$

where η_z is a variational parameter associated with the layer z , and derive the mean-field free energy for the ANNNI model (6.8) by minimising the right-hand side of eq. (6.74) with respect to η_z . Discuss the transitions lines between the disordered high-temperature phase and the ordered low-temperature phases and derive the location of the Lifshitz point.

6.4. Consider the generalised Schrödinger operator $\mathcal{S} = -\alpha\partial_t^N + \partial_{\mathbf{r}} \cdot \partial_{\mathbf{r}}$. If $\phi = \phi(t, \mathbf{r})$ solves $\mathcal{S}\phi = 0$, use the commutation properties of $\mathcal{S}$ with the LSI generators X_n and $Y_m^{(i)}$ derived in the text to show that $[\mathcal{S}, Y_m^{(i)}]\phi = 0$.

6.5. For an anisotropy exponent $\theta = 2/N = 2$, one has $N = 1$. Re-sum the series for the LSI scaling function (6.43) with the explicit coefficients (6.47) and verify that one recovers the prediction of Schrödinger-invariance.

6.6. Knowing that the spin-spin correlation function in the (mean) spherical model with uniaxial competing interactions (ANNNS model) is given at the critical point by the expression

$$C(\mathbf{r}) = \frac{T_c}{(2\pi)^d} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} d\phi_1 \dots d\phi_d \frac{\cos(\mathbf{r} \cdot \phi)}{\hat{J}(0) - \hat{J}(\phi)},$$

where $\hat{J}(\phi)$ is the Fourier transform of the exchange integral

$$\hat{J}(\phi) = \sum_{\mathbf{r}} J(\mathbf{r}) \exp(i\mathbf{r} \cdot \phi),$$

derive the exact expressions (6.64) and (6.65) for the correlator at the Lifshitz point of the d -dimensional ANNNS model, where $\kappa_1 = 1/4$. In particular, deduce the scaling form $C(\mathbf{r}) = C_0 |\mathbf{r}_{\perp}|^{-(d-d_-)} \Psi(\frac{d-d_-}{2}, C_2 r_{\parallel}^2 / |\mathbf{r}_{\perp}|)$, where $C_{0,2}$ are constants and d_- is the lower critical dimension. The form of the scaling function $\Psi(a, x)$ is listed for some values of a in the following Table, together with their leading asymptotics for $x \rightarrow \infty$ [258].

a	$\Psi(a, x)$	$x \rightarrow \infty$
1/4	$x^{1/2} [I_{-1/4}(x/2) + I_{1/4}(x/2)] K_{1/4}(x/2)$	$2 \cdot x^{-1/2}$
1/2	$\sqrt{\pi} (x/2)^{1/4} [I_{-1/4}(x) - L_{-1/4}(x)]$	$(2/\Gamma(1/4)) \cdot x^{-1}$
3/4	e^{-x}	
1	$1/\Gamma(3/4) + \sqrt{\pi} (x/2)^{3/4} [L_{1/4}(x) - I_{1/4}(x)]$	$-1/(2\Gamma(3/4)) \cdot x^{-2}$

Here, I, K are modified Bessel functions and L is a modified Struve function [4]. Compare with Table 6.4. One may generalise this to m -axial extensions of the spherical model.

Appendices

A. Equilibrium Models

We recall briefly the definition and some basic properties of the models of equilibrium critical behaviour discussed in this volume. For the convenience of the reader, this appendix repeats and extends the contents of Appendix A in Volume 1. The definitions of the equilibrium critical exponents can be found in Volume 1 or any textbook on equilibrium critical phenomena, e.g. [743]. In two dimensions, a lot of information about these models has been found either from exact solutions [54] or from conformal invariance methods [329]. We refer to the sources quoted for more detailed information.

The labelling of the sections of this appendix continues from Volume 1.

A.1 Potts Model

The q -state **Potts model** is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} \delta(\sigma_i, \sigma_j), \quad \delta(a, b) = \begin{cases} 1 & ; \text{ if } a = b \\ 0 & ; \text{ if } a \neq b \end{cases} \quad (\text{A1})$$

where the local variables σ_i can take the values $\sigma_i = 0, 1, \dots, q-1$, J is the coupling constant, and the sum is taken over the nearest neighbours of a hypercubic (or any other) lattice.¹ The global symmetry of the Potts model is the permutation group $\mathbb{S}_q$ of q elements. The **Ising model** is usually formulated in terms of the spin variables $\sigma_i = \pm 1$ and is given by the Hamiltonian $\mathcal{H} = -J \sum_{(i,j)} \sigma_i \sigma_j$. For $q = 2$, one may recover from the Potts model (A1) the Ising model, by first changing the range of values of σ_i to $\{\pm 1\}$ and then

¹ For brevity, we shall sometimes refer to the q -states Potts model as the *Potts- q model*. Similar conventions will apply to other spin models with a discrete global symmetry.

q	α	ν	β	γ	δ	η	$\eta_{\parallel}$	β_1
1	$-2/3$	$4/3$	$5/36$	$43/18$	$91/5$	$5/24$	$2/3$	$8/9$
2	$0(\log)$	1	$1/8$	$7/4$	15	$1/4$	1	$1/2$
3	$1/3$	$5/6$	$1/9$	$13/9$	14	$4/15$	$4/3$	$5/9$
4	$2/3$	$2/3$	$1/12$	$7/6$	15	$1/4$	2	$2/3$

Table A1. Some equilibrium bulk and surface critical exponents (ordinary transition) of the q -state Potts model in two dimensions.

using the identity $\delta(\sigma, \sigma') = (1 + \sigma\sigma')/2$. The case $q = 1$ corresponds to the universality class of isotropic **percolation**.

In two dimensions, the Potts- q model is self-dual and its critical point is given by $J/T_c = \ln(1 + \sqrt{q})$. The phase-transition is of second order if $q \leq 4$ and of first order if $q > 4$. The values of the critical exponents are listed in Table A1. In three dimensions, the transition remains of second order if $q = 2$ but is of first order already for $q = 3$. A very detailed review can be found in [740, 741].

We also include, besides the usual bulk critical exponents, *surface exponents* such as $\eta_{\parallel}$ which describes the decay of the two-point correlation function close to a plane free surface, and the local magnetisation exponent β_1 . Here and throughout we assume that the surface does *not* order before the bulk (i.e. we have the case without an external field located at the surface) and are therefore dealing with the **ordinary transition**.

Although there is no obvious up-down symmetry, an order-parameter is readily defined in numerical simulations by

$$\langle M(t) \rangle = \frac{1}{N_s} \frac{1}{(q-1)L^d} \sum_{n=1}^{N_s} \sum_{i=1}^{L^d} [q\delta(\sigma_{i,[n]}(t), 1) - 1] , \quad (\text{A2})$$

where $\sigma_{i,[n]}(t)$ denotes the i^{th} spin of sample number n at the Monte Carlo sweep t ; N_s is the number of Monte Carlo samples and $\mathcal{N} = L^d$ is the total number of lattice sites. See appendix G in volume 1 for simulational methods.

A.2 Clock Model

The **clock model** (sometimes abbreviated as the clock- p model) is a variant of the Potts model and is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} \cos(\vartheta_i - \vartheta_j) , \quad \vartheta_i = \frac{2\pi}{p} n_i \quad (\text{A3})$$

where $n_i = 0, 1, \dots, p-1$. Here the global symmetry is the cyclic group $\mathbb{Z}_p$ of p elements, which is a true subgroup of $\mathbb{S}_p$ if $p \geq 3$. For $p = 2$ and $p = 3$ one

recovers the Ising and Potts-3 models, respectively, whereas in the limit $p \rightarrow \infty$ one retrieves the XY model. In two dimensions, for $p \leq 4$, there is a single equilibrium phase-transition with conventional power-law behaviour. On the other hand, for $p > 4$, there exist two distinct transitions at temperatures T_1 and $T_2 > T_1$. In two dimensions, the system is paramagnetic for $T > T_2$ and ferromagnetic for $T < T_1$ whereas one has a Kosterlitz-Thouless phase with a non-local order parameter in between. Both phase transitions at T_1 and at T_2 are of Kosterlitz-Thouless type with exponentially diverging correlation lengths, susceptibilities etc.

An approximate real-space renormalisation-group calculation gives the $2D$ critical temperatures as $T_1/J \approx 23p^{-2}$ and $T_2/J \approx 0.95$. For $T \leq T_2$, the exponent $\eta = \eta(T)$ is temperature-dependent, with $\eta(T_1) \approx 4p^{-2}$ and $\eta(T_2) = \frac{1}{4}$ [410].

In three dimensions, the partition function for $p = 4$ factors into two Ising model partition functions with the replacement $T \mapsto 2T$. For $p > 4$, the model is in the XY universality class [378].

A.3 Turban Model

In this model, also referred to in the literature as the **multispin Ising model**, one studies the combined effect of the standard nearest-neighbour interactions and a higher-order term in one direction [703, 702]. The Hamiltonian is

$$\mathcal{H}^{(m)} = - \sum_{(i,j)} \left(J_y s_{i,j} s_{i+1,j} + J_x \prod_{\ell=0}^{m-1} s_{i,j+\ell} \right), \quad (\text{A4})$$

where $s_{i,j} = \pm 1$ is an Ising spin on the site (i, j) of a square lattice. The model is self-dual with a critical line given by $\sinh(2J_x/T) \sinh(2J_y/T) = 1$ [703, 702, 187]. A symmetry analysis suggests that the model has the same equilibrium properties as the q -state Potts model, where $q = 2^{m-1}$. In particular, for $m = 3$ one recovers the four-state Potts model [187, 586]. On the other hand, for $m \geq 4$, one expects a first-order transition. This conjecture has been thoroughly confirmed numerically, see [672, 600, 449] and references therein.

A.4 Baxter-Wu Model

This model is named after its exact solution and is defined in terms of Ising spins $s_i = \pm 1$ on a *triangular* lattice with the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j,k)} s_i s_j s_k, \quad (\text{A5})$$

where the sum involves the three spins on each triangle of the lattice. Since the triangular lattice can be decomposed into three sublattices, $\mathcal{H}$ is invariant

under reversal of all spins belonging to two of the sublattices and the ground state is hence four-fold degenerate. The model is self-dual and undergoes a second-order phase-transition at $J/T_c = \frac{1}{2} \ln(1+\sqrt{2})$. The exact solution gives the same equilibrium critical exponents (e.g. $\alpha = \nu = 2/3$ and $\eta = 1/4$) as for the Potts-4 model, see [54].

A.5 Blume-Capel Model

This model is defined in terms of spin variables $S_i = -1, 0, 1$ and the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} S_i S_j + D \sum_i S_i^2, \quad (\text{A6})$$

where in addition to the exchange coupling J a crystal field D is introduced. The model shows a line of phase transitions which are of second order if D/J is small (and which is in the Ising equilibrium universality class) but which become of first order for D sufficiently large. The meeting point of first- and second-order transitions is a **tricritical point** with a completely different critical behaviour [461] which for a square lattice occurs at $D/J \simeq 1.9655$ and $T_c/J = 0.610$ [175].

In two dimensions, the tricritical exponents can be found from conformal-invariance methods [329]. For example, one has in the two-dimensional tricritical Ising model the bulk and surface (ordinary transition) exponents

$$\frac{\beta_t}{\nu_t} = \frac{3}{40}, \quad \frac{1 - \alpha_t}{\nu_t} = \frac{1}{5}, \quad \frac{\beta_{t,1}}{\nu_t} = \frac{3}{2} \quad (\text{A7})$$

The upper critical dimension for tricritical points is $d_c = 3$.

A.6 XY Model

The XY model (or *planar rotator* model) is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} \cos(\vartheta_i - \vartheta_j) = -J \sum_{(i,j)} \vec{S}_i \cdot \vec{S}_j, \quad (\text{A8})$$

where $\vartheta_i \in [0, 2\pi]$ or, equivalently, $\vec{S}_i \in \mathbb{R}^2$ is a two-dimensional unit vector. In more than two dimensions, the model undergoes a conventional second-order phase-transition between a paramagnetic and a ferromagnetic phase but in two dimensions, the **Mermin-Wagner theorem** asserts that a spontaneous magnetisation is impossible. Rather, the whole low-temperature phase in two dimensions remains critical in that the magnetic correlation functions decay algebraically as $C(\mathbf{r}) \sim |\mathbf{r}|^{-\eta(T)}$ with a temperature-dependent exponent $\eta(T)$. Using conformal-invariance techniques, precise numerical estimates

T/J	η	η_{sw}	$\eta_{\parallel}$	$\eta_{\parallel, \text{sw}}$
0.1	0.017(1)	0.016	0.032(1)	0.032
0.2	0.036(3)	0.032	0.069(1)	0.064
0.3	0.052(5)	0.048	0.103(2)	0.095
0.4	0.074(6)	0.064	0.143(4)	0.127
0.5	0.100(8)		0.186(5)	
0.6	0.122(13)		0.231(8)	
0.7	0.154(14)		0.298(12)	
0.8	0.188(22)		0.374(17)	
0.893	0.250(28)		0.542(22)	

Table A2. Equilibrium bulk exponent η and surface exponent $\eta_{\parallel}$ in the low-temperature phase of the 2D XY model as a function of temperature T . The corresponding results from the spin-wave approximation (sw) are also listed. After [65].

for the bulk exponent $\eta(T)$ and also the surface exponent $\eta_{\parallel}(T)$ were found [65], see Table A2. The transition occurs at $T_{\text{KT}}/J \simeq 0.893$ [432] and is of **Kosterlitz-Thouless** type, meaning that the correlation length, susceptibility and other quantities diverge according to a stretched exponential, viz. $\xi(T) \sim \exp(b/(T - T_{\text{KT}})^{1/2})$. At $T = T_{\text{KT}}$, the spin-spin correlator decays as $C(\mathbf{r}) = \langle \vec{S}(\mathbf{r}) \cdot \vec{S}(\mathbf{0}) \rangle \sim |\mathbf{r}|^{-1/4} \ln(|\mathbf{r}|)^{1/8}$ [523], which implies $\eta(T_{\text{KT}}) = \frac{1}{4}$.

A.7 $O(n)$ Model

These systems are defined in terms of unit spin vectors $\vec{S}_i \in \mathbb{R}^n$ and $|\vec{S}_i| = 1$ and the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} \vec{S}_i \cdot \vec{S}_j . \quad (\text{A9})$$

For $n = 1$ and $n = 2$ one recovers the Ising and the XY model, respectively, while the case $n = 3$ defines the Heisenberg universality class. In $d > 2$ dimensions, $O(n)$ models have a conventional second-order ferromagnetic phase-transition while for $d = 2$ dimensions, the Mermin-Wagner theorem excludes a phase-transition if $n > 2$. Equilibrium critical exponents have been determined by several different methods, see [759, 585]. In Table A3 conservative estimates for values of critical exponents, as well as for $T_c(n)$, are listed for some values of n , but the reader should be aware that a lot of effort continues to be invested in further improving the exponent values. We refer to the very extensive reviews of Pelissetto and Vicari [585] and of Barmartz *et al.* [39] for details, in particular for universal amplitude combinations and for the comparison with experimental results. Re-summed variational perturbation theory with lists of exponents for $0 \leq n \leq 28$ is presented in [445]. A recent review for surface exponents is [199].

n	T_c/J	α	β	γ	ν	η	$\eta_{\parallel}$	β_1
1	4.5115279(6)	0.110(1)	0.3265(3)	1.2375(5)	0.6301(4)	0.0364(5)	1.528	0.796
2	2.20183(1)	-0.0146(8)	0.3485(3)	1.3177(3)	0.6715(3)	0.0380(4)	1.422	0.810
3	1.44299(1)	-0.134(2)	0.3689(3)	1.3960(9)	0.7112(5)	0.0375(5)	1.338	0.824
4	1.068535(9)	-0.247(6)	0.388(3)	1.471(4)	0.749(2)	0.0365(10)		
5	0.8559(3)	-0.298	0.396	1.506	0.766	0.034		
10	0.41187(1)	-0.61(2)	0.44(1)	1.721(14)	0.871(7)	0.025(20)		
∞		-1	$\frac{1}{2}$	2	1	0	1	1
SM	3.956776...	-1	$\frac{1}{2}$	2	1	0	2	$\frac{3}{2}$

Table A3. Critical temperatures and equilibrium critical exponents of the 3D $O(n)$ model. The bulk exponents for $1 \leq n \leq 5$ are from [585], the estimates of $T_c(n)$ for $n = 1, 2, 3, 10$ and the bulk exponents for $n = 10$ are from [730], $T_c(4)$ is from [34] and $T_c(5)$ is from [381]. The estimates of surface exponents $\eta_{\parallel}$, β_1 at the ordinary transition are from [75, 199]. SM denotes the 3D spherical model.

In the limit $n \rightarrow \infty$, the $O(n)$ -model reduces to the exactly solvable **spherical model**. Rather than through infinite-dimensional unit spin vectors, one may also define the spherical model in terms of real spin variables $S_i \in \mathbb{R}$ attached to each site i of a d -dimensional lattice Λ with $\mathcal{N}$ sites and subject to the **spherical constraint** $\sum_{i \in \Lambda} S_i^2 = \mathcal{N}$. The Hamiltonian is usually chosen to describe the habitual nearest-neighbour interactions $\mathcal{H} = -J \sum_{(i,j)} S_i S_j$ [67] but it is one of the attractive features of the model that it may be solved for considerably more general interactions and that one may even include external fields. A classical review of many of the equilibrium properties is [414], for more recent discussions see [54, 92].² Calculations are considerably simplified if the spherical constraint is only imposed on average [470, 471]. The equilibrium *bulk* critical behaviour of the spherical model is the same as in the $n \rightarrow \infty$ limit of the $O(n)$ model [680] but we warn the reader that this no longer holds true for the *surface* critical behaviour [75, 197].

A.8 Double Exchange Model

This model plays a role in the study of perovskite manganites and is defined in terms of classical spin vectors $\vec{S}_i \in \mathbb{R}^3$ by the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} \sqrt{1 + \vec{S}_i \cdot \vec{S}_j} \quad (\text{A10})$$

² While the spherical model is usually considered as being exclusively theoretically motivated, we mention that critical exponents close to those of the 3D spherical model have been found in diluted magnets such as the $\text{Eu}_x\text{Sr}_{1-x}\text{S}_{0.50}\text{Se}_{0.50}$ family [732] or for $\text{Eu}_{0.65}\text{La}_{0.35}\text{S}$, where $\alpha \approx -1$, $\beta \approx 0.5$, $\gamma \approx 2.1$ and $\delta \approx 4.7$ were reported [733].

with nearest-neighbour interactions on a simple hypercubic lattice. It undergoes in $3D$ a second-order phase transition at $T_c/J = 0.74515$. Some $3D$ equilibrium exponents are $\nu = 0.68(2)$ and $\beta = 0.356(6)$, see [242] and references therein. It is thought that the equilibrium phase transition is in the universality class of the $3D$ Heisenberg model.

A.9 Hilhorst-van Leeuwen Model

This model is a variant of the $2D$ Ising model on a semi-infinite square lattice. The Hamiltonian is [362]

$$\mathcal{H} = - \sum_{r_{\parallel} \in \mathbb{Z}} \sum_{r_{\perp} \in \mathbb{N}} [J_1 \sigma_{r_{\parallel}, r_{\perp}} \sigma_{r_{\parallel}+1, r_{\perp}} + J_2(r_{\perp}) \sigma_{r_{\parallel}, r_{\perp}} \sigma_{r_{\parallel}, r_{\perp}+1}] \quad (\text{A11})$$

and the coupling perpendicular to the surface varies as follows [82]

$$J_2(r_{\perp}) = J_2(\infty) + \frac{T_c \sinh(2J_2(\infty)/T_c)}{4} \frac{A}{r_{\perp}^{\omega}} \quad (\text{A12})$$

While the bulk critical behaviour is the same as in the usual $2D$ Ising model, the surface critical behaviour depends on the form of $J_2(r_{\perp})$, see [388] for a review. For $\omega > 1$ the modified couplings are irrelevant, while for $\omega < 1$ they are relevant. The case $\omega = 1$ is marginal. Then the surface exponent of the order-parameter depends continuously on the amplitude A . If one takes $J_1 = J_2(\infty)$ and $A < 1$, this reads simply

$$x_1 = \frac{\beta_1}{\nu} = \frac{1}{2}(1 - A) \quad (\text{A13})$$

and one recovers the usual semi-infinite $2D$ Ising model for $A = 0$.

A.10 Frustrated Spin Models

A different kind of very rich behaviour is found in spin systems with a non-random competition such that there is a macroscopic number of equilibrium states [474, 742]. The best-known example is probably the **triangular Ising antiferromagnet**. On the square lattice, one may consider the **fully frustrated Ising/Potts model** which is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{i,j} [\delta(\sigma_{i,j}, \sigma_{i+1,j}) + (-1)^{f(i,j)} \delta(\sigma_{i,j}, \sigma_{i,j+1})] \quad (\text{A14})$$

where the frustration variable $f(i, j)$ is chosen such that on each plaquette of the square lattice one bond is antiferromagnetic while the other three are ferromagnetic. The choices $f(i, j) = i + j$ and $f(i, j) = i$, respectively, led to the checkerboard and domino-tile pattern of frustration, see Fig. A1.

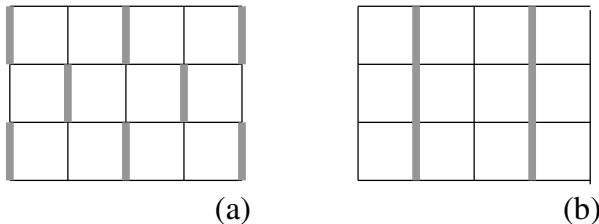


Fig. A1. Patterns of coupling constants for (a) checkerboard and (b) domino-tiled modulations of the square lattice. The antiferromagnetic bonds are indicated by the thick grey lines.

T/T_{KT}	Monte Carlo	spin wave
0.2	0.0212(12)	0.0200
0.4	0.045(2)	0.0402
0.6	0.072(2)	0.0602
0.8	0.107(4)	0.0804
1.0	0.27(2)	—

Table A4. Values of the equilibrium critical exponent η in the 2D fully frustrated XY model, as obtained [724] by Monte Carlo simulation and from spin-wave theory.

The triangular Ising antiferromagnet [377, 728, 729] and the fully frustrated Ising model (both with checkerboard and domino-tile modulation) [711] are disordered for any $T > 0$ but the zero-temperature state, which has a finite entropy, is critical. From an exact mapping onto the Baxter model, the equilibrium exponent $\eta = 1/2$ [683, 254] and the surface exponent $\eta_{\parallel} = 1$ if $J_s/J = \frac{1}{2}$ and $\eta_{\parallel} = 2$ if $J_s/J > \frac{1}{2}$ [304] can be found, where J_s is the coupling constant at the surface. The finite-size scaling of the correlation length and of the free energy are in agreement with conformal invariance, and a central charge $c = 1$ is found [304].

The fully frustrated three-states Potts model with checkerboard modulation is disordered for any $T > 0$ but critical at $T = 0$. The exponent $\nu = 1.139$ was estimated from transfer matrix calculations [256]. On the other hand, for a domino-tile modulation, one has a second-order phase-transition between a paramagnetic and a ferromagnetic state at $T_c/J = 0.365(1)$, while the exponent ν may be compatible with the pure Potts-3 result $\nu = \frac{5}{6}$ [255].

The **fully frustrated XY model** is defined by the Hamiltonian in terms of the angular variables $\theta_{\mathbf{r}_i} = \theta_{(x_i, y_i)} \in [0, 2\pi)$

$$\mathcal{H} = -J \sum_{(i,j)} (-1)^{x_i x_j} \cos(\theta_{(x_i, y_i)} - \theta_{(x_j, y_j)}) \quad (\text{A15})$$

with nearest-neighbour interactions between the sites $\mathbf{r}_i = (x_i, y_i)$ of a two-dimensional square lattice and the modulation is domino-tiled, see Fig. A1.

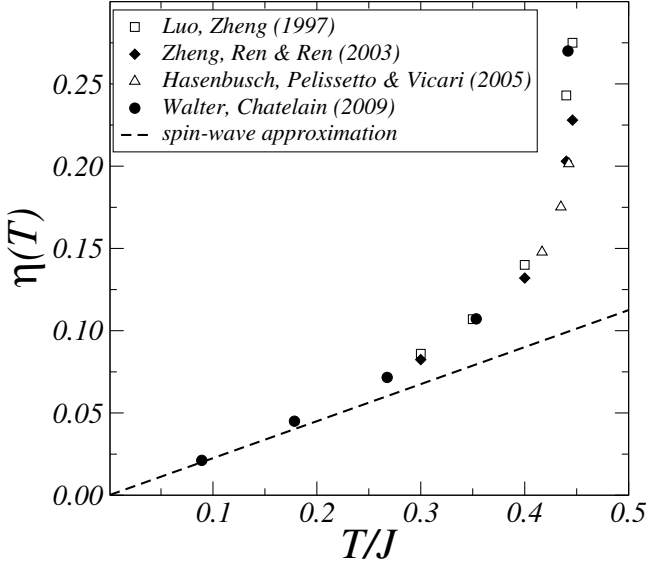


Fig. A2. Comparison of estimates for the equilibrium exponent η of the 2D fully frustrated XY model in the low-temperature phase $T \leq T_{KT}$, according to Luo and Zheng [488], Zheng, Ren and Ren [756], Hasenbusch, Pelissetto and Vicari [317] and Walter and Chatelain [724]. The dashed line indicates the result of a spin-wave approximation. Modified after [724].

The equilibrium critical properties of this model have been debated for a long time, see [317] for a detailed recent review. It is now thought that the model undergoes *two* distinct phase transitions: (i) a Kosterlitz-Thouless-like topological transition at $T_{KT}/J = 0.4461(2)$ where the global rotation-invariance of the spins is broken and (ii) and a conventional second-order transition (estimates for the critical point are $T_c/J = 0.4545(2)$ [487] or $T_c/J = 0.453243(2)$ [317]) where the phase chiralities of the spins around each plaquette order antiferromagnetically. In Table A4 some estimates [724] for the static exponent η in the critical low-temperature phase are listed. These estimates agree well with other available estimates, as illustrated in Fig. A2.

A.11 Weakly Random Spin Systems

Frequently, there are good motives to consider spin systems where the exchange interactions are frozen random variables. A typical example might be formulated as follows: consider a q -states Potts model on a hypercubic lattice, with a Hamiltonian

$$\mathcal{H} = - \sum_{(i,j)} J_{i,j} \delta(\sigma_i, \sigma_j) \quad (\text{A16})$$

where $\sigma = 1, 2, \dots, q$ and the couplings $J_{i,j} \geq 0$ are chosen from the random distribution

$$\mathcal{P}(J_{i,j}) = p\delta(J_{i,j} - J^{(1)}) + (1-p)\delta(J_{i,j} - J^{(2)}) \quad (\text{A17})$$

and where p is a control parameter. If both $J^{(1)}$ and $J^{(2)}$ are non-vanishing, one has a **binary disorder**. In this case, the critical line in $2D$ is known, for $p = \frac{1}{2}$, from a self-duality argument [442]

$$\left(e^{J_c^{(1)}/T} - 1\right) \left(e^{J_c^{(2)}/T} - 1\right) = q, \quad \text{if } p = \frac{1}{2}. \quad (\text{A18})$$

On the other hand, if $J^{(2)} = 0$, one has a **diluted Potts model** which is not self-dual. Here, a phase transition only exists if $p > p_c$, where p_c is the percolation threshold of the underlying lattice. Ising models with binary disorder or the diluted Ising model are recovered for $q = 2$. The celebrated **Harris criterion** [310] states that *a weak random perturbation around a critical point is irrelevant in the renormalisation group sense, if the specific heat exponent of the non-random (pure) system satisfies $\alpha < 0$. On the other hand, for $\alpha > 0$ one expects that the effects of disorder drive the system to a new, random fixed point.* Disordered systems have been studied enormously in the literature. Excellent recent reviews include [585, 251, 66] and cover simulational, field-theoretical, conformal invariance and experimental studies; and treat in detail the many subtleties of disordered systems whose exposition would require a book in itself.

We limit ourselves to listing in Table A5 some typical results for equilibrium critical exponents in disordered Potts models. It is seen that the disorder softens the first-order transition of the non-random Potts model with $d = 2$ and $q > 4$ or $d = 3$ and $q \geq 3$ into a second-order transition. Note that $\alpha < 0$ and $\nu > 2/d$, as is required for the stability of the random fixed point. Since corrections to scaling are sizeable in disordered systems, their careful inclusion is essential in order to obtain reliable estimates which no longer depend on the ratio $J^{(1)}/J^{(2)}$ [35].

In systems such as the $2D$ Ising model, the Harris criterion does not give an unambiguous answer, since there $\alpha = 0$. It is thought, however, that the disorder in the $2D$ Ising model should be marginally relevant such that sufficiently close to the random critical point, the power-law scaling forms described in Chap. 2 in Volume 1 are for the random $2D$ Ising model replaced by

$$\begin{aligned} C(\tau) &\sim \ln|\ln|\tau||, \quad m(\tau) \sim \tau^{1/8} [\ln|\tau|]^{-1/16}, \\ \chi(\tau) &\sim |\tau|^{-7/4} [\ln|\tau|]^{7/8}, \quad \xi(\tau) \sim |\tau|^{-1} [\ln|\tau|]^{1/2}. \end{aligned} \quad (\text{A19})$$

A.12 Logarithmic Sub-scaling Exponents

A systematic theory of the logarithmic correction factors to scaling, which may arise for marginal perturbations to scaling, and for which an example was given

q	d	α	β	γ	ν	η	$\eta_{\parallel}$	β_1	Ref.
2	3	-0.03(3)	0.349(5)	1.33(2)	0.678(10)	0.030(3)			[585]
	3	-0.05(1)*	0.354(3)	1.342(10)	0.684(5)	0.04(3)*			[35]
3	2	-0.06(6)*	0.136(1)	1.79(6)*	1.03(3)*	0.26(11)*	1.05(1)	0.54(1)	[574]
4	2	-0.04(8)*	0.142(2)	1.76(8)*	1.02(4)*	0.27(15)*	1.10(1)	0.56(2)	[574]
	3	-0.24(4)*	0.55(3)	1.15(4)	0.75(1)	0.47(9)*			[143]
5	2	-0.02(2)*	0.147(3)*	1.74(3)*	1.01(1)	0.27(5)*			[575]
6	2	-0.04(8)*	0.149(3)	1.74(9)	1.02(4)*	0.29(15)*	1.15(1)	0.58(3)	[574]
	8	-0.02(6)*	0.151(1)	1.72(6)*	1.01(3)*	0.30(11)*	1.17(1)	0.60(2)	[574]

Table A5. Equilibrium bulk and surface exponents (ordinary transition) of random q -state Potts models. An entry marked by * is calculated via a scaling relation from other information in the quoted source. For the 3D Ising model ($q = 2$), we quote both the results of field-theory [585] and of simulation [35], for comparison.

in eq. (A19), was developed by Kenna, Johnston and Janke [434, 435] through an analysis of the complex-temperature zeroes of the partition function. In Table A6 their definitions of the **logarithmic sub-scaling exponents** are listed where an exponent with a ‘hat’ refers to a sub-logarithmic exponent. They satisfy the following scaling relations

$$\hat{\beta}(\delta - 1) = \delta\hat{\delta} - \hat{\gamma} \quad , \quad 2\hat{\beta} - \hat{\gamma} = d(\hat{q} - \hat{\nu}) \quad , \quad \hat{\eta} = \hat{\gamma} - \hat{\nu}(2 - \eta) \quad (\text{A20})$$

$$\hat{\alpha} = d(\hat{q} - \hat{\nu}) + \begin{cases} 0 & ; \text{ if } \alpha \neq 0 \\ 0 & ; \text{ if } \alpha = 0 \text{ and } \varphi = \pi/4 \\ 1 & ; \text{ if } \alpha = 0 \text{ and } \varphi \neq \pi/4 \end{cases} \quad (\text{A21})$$

Here φ denotes the impact angle of the line of complex-temperature zeroes on the real axis. In Table A7, known values of logarithmic sub-scaling exponents are collected for several models. Beyond those already defined,

quantity	logar. exponent	relation	conditions
specific heat	$\hat{\alpha}$	$C \sim \tau ^{-\alpha} \ln \tau ^{\hat{\alpha}}$	$ \tau \neq 0 \quad h = 0 \quad L \rightarrow \infty$
magnetisation	$\hat{\beta}$	$m \sim \tau^{\beta} \ln \tau ^{\hat{\beta}}$	$\tau > 0 \quad h = 0 \quad L \rightarrow \infty$
	$\hat{\delta}$	$m \sim h^{1/\delta} \ln h ^{\hat{\delta}}$	$ \tau = 0 \quad h \neq 0 \quad L \rightarrow \infty$
susceptibility	$\hat{\gamma}$	$\chi \sim \tau ^{-\gamma} \ln \tau ^{\hat{\gamma}}$	$ \tau \neq 0 \quad h = 0 \quad L \rightarrow \infty$
correlation length	$\hat{\nu}$	$\xi \sim \tau ^{-\nu} \ln \tau ^{\hat{\nu}}$	$ \tau \neq 0 \quad h = 0 \quad L \rightarrow \infty$
	$\hat{q}$	$\xi \sim L \ln L ^{\hat{q}}$	$ \tau = 0 \quad h = 0 \quad L \text{ finite}$
correlation function	$\hat{\eta}$	$G \sim \mathbf{r} ^{-(d-2+\eta)} \ln \mathbf{r} ^{\hat{\eta}}$	$ \tau = 0 \quad h = 0 \quad L, \mathbf{r} \rightarrow \infty$

Table A6. Definition of several of the critical logarithmic sub-scaling exponents at equilibrium, according to [434, 435] and with $\tau = (T_c - T)/T_c$.

model	d	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\nu}$	$\hat{q}$	$\hat{\eta}$
Ising	2	1	0	0	0	0	0	0
random Ising	2	0	$-\frac{1}{16}$	$\frac{7}{8}$	0	$\frac{1}{2}$	0	0
	4	0.5	0.25	0	0.167	0	0.125	0
$O(n)$	4	$\frac{4-n}{n+8}$	$\frac{3}{n+8}$	$\frac{n+2}{n+8}$	$\frac{1}{3}$	$\frac{n+2}{2(n+8)}$	$\frac{1}{4}$	0
Potts-4	2	-1	$-\frac{1}{8}$	$\frac{3}{4}$	$-\frac{1}{15}$	$\frac{1}{2}$	0	$-\frac{1}{8}$
$O(n)$ -LR	$\frac{2}{\sigma}$	$\frac{4-n}{n+8}$	$\frac{3}{n+8}$	$\frac{n+2}{n+8}$	$\frac{1}{3}$	$\frac{n+2}{\sigma(n+8)}$	$\frac{1}{2\sigma}$	0
$O(n)$ -SG	6	$-\frac{3n+1}{2n-1}$	$-\frac{n+1}{2(2n-1)}$	$\frac{2n}{2n-1}$	$\frac{3n-1}{4(2n-1)}$	$\frac{n/2}{2n-1}$	$\frac{1}{6}$	$\frac{5n/6}{2n-1}$
percolation	6	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{5}{42}$	$\frac{1}{6}$	$-\frac{1}{21}$
Yang-Lee	6	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{5}{18}$	$\frac{1}{6}$	$\frac{1}{9}$
n -AT	2	$-\frac{n}{n-2}$	$-\frac{n-1}{8(n-2)}$	$\frac{7}{4}\frac{n-1}{n-2}$	0	$\frac{n-1}{n-2}$	0	0

Table A7. Values of logarithmic sub-scaling exponents defined in Table A6 in several models [10, 434, 435, 436, 433, 292], see text for the definitions.

the long-range (LR) $O(n)$ -model has a distance-dependent exchange coupling $J(\mathbf{r}) \sim |\mathbf{r}|^{-d-\sigma}$, in the $O(n)$ spin glass (SG) [226, 311] the nearest-neighbour couplings $J_{i,j}$ are glassy random variables with both positive and negative values with Hamiltonian $\mathcal{H} = -\sum_{(i,j)} J_{i,j} \vec{S}_i \cdot \vec{S}_j$, where $\vec{S}_i$ are n -component classical spin vectors of unit length. Percolation is here ordinary, undirected percolation [682], Yang-Lee refers to the Yang-Lee singularities found in complex external fields [433] and the n -colour Ashkin-Teller model (n -AT) with $n > 2$ has the nearest-neighbour Hamiltonian $\mathcal{H} = -J_2 \sum_{(i,j)} \sum_{a=1}^n \sigma_i^{(a)} \sigma_j^{(a)} - J_4 \sum_{(i,j)} \sum_{a \neq b}^n \sigma_i^{(a)} \sigma_i^{(b)} \sigma_j^{(a)} \sigma_j^{(b)}$ that couples n sets of Ising spin variables $\sigma_i^{(a)} = \pm 1$ [300, 420].³

The controversial case of the random four-dimensional Ising model requires special attention. Since the model is at the upper critical dimension, one has $\alpha = 0$ and there is no prediction from the Harris criterion. Indeed, existing results are all different from each other but can be cast into two groups, namely [10, 419, 36, 250, 292] and [661, 273], which are mutually incompatible. The scaling is not always of the form listed in Table A6 but the consensus seems to be that one rather should have [324, 292]

$$\begin{aligned}
 C(\tau) &\simeq A - B|\tau|^{-\alpha} \epsilon(\tau)^{-2} |\ln |\tau||^{\hat{\alpha}} \quad , \quad m(\tau) \simeq \tau^{\beta} \epsilon(\tau)^{-1/2} |\ln |\tau||^{\hat{\beta}} \\
 \chi(\tau) &\simeq |\tau|^{-\gamma} \epsilon(\tau) |\ln |\tau||^{\hat{\gamma}} \quad , \quad \xi(\tau) \simeq |\tau|^{-\nu} \epsilon(\tau)^{1/2} |\ln |\tau||^{\hat{\nu}} \quad (\text{A22})
 \end{aligned}$$

³ For $n = 2$, it reduces to the usual Ashkin-Teller model. See [329] and refs. therein for the conformal invariance results on the very rich critical behaviour of these models in $2D$.

with $\mathfrak{e}(\tau) := \exp[\sqrt{6|\ln|\tau||/53}]$. The values included in Table A7 for the $4D$ random Ising model are those from [10], which are numerically very close to the results found in [419, 36, 292].

A.13 Ising Spin Glasses

Since their first introduction in 1975 by Edwards and Anderson [226] in order to describe dilute magnetic alloys, **Ising spin glasses** have been the subject of numerous studies. The interest into these deceptively simple looking models comes from the fact that they display a very rich and complicated behaviour [78, 431], due to the combination of disorder- and frustration-effects.

The Hamiltonian of an Ising spin glass is given by

$$\mathcal{H} = - \sum_{(i,j)} J_{i,j} s_i s_j , \quad (\text{A23})$$

with $s_i = \pm 1$. This looks similar to the Hamiltonian of weakly random spin systems, see Appendix A.11, but the random couplings are now either ferromagnetic or antiferromagnetic. Commonly used distributions for the random couplings include (i) the **bimodal distribution** with

$$P_B(J_{i,j}) = [\delta(J_{i,j} - J) + \delta(J_{i,j} + J)]/2, \quad (\text{A24})$$

(ii) the **Gaussian distribution** with

$$P_G(J_{i,j}) = \exp(-J_{i,j}^2/2J^2)/(J\sqrt{2\pi}) \quad (\text{A25})$$

and (iii) the **Laplacian distribution** with

$$P_L(J_{i,j}) = \exp(-\sqrt{2} |J_{i,j}/J|)/(J\sqrt{2}). \quad (\text{A26})$$

These distributions are all symmetric and all have zero mean and variance $[J_{i,j}^2]_{\text{av}}/J^2 = 1$. Sometimes, distributions with a mean value different from zero are also considered.

As in spin glasses the magnetisation is not an order parameter, a different quantity has to be defined in order to distinguish between the disordered high temperature and the ordered low temperature phases. A commonly used choice for an order parameter is the so-called **Edwards-Anderson order parameter** defined by

$$q_{\text{EA}} := \lim_{\mathcal{N} \rightarrow \infty} [\langle q \rangle]_{\text{av}} = \lim_{\mathcal{N} \rightarrow \infty} \left[\left\langle \frac{1}{\mathcal{N}} \sum_i s_i^\alpha s_i^\beta \right\rangle \right]_{\text{av}} , \quad (\text{A27})$$

where α and β label two copies of the system with the same disorder. Hereby the brackets $\langle \cdot \rangle$ indicate the equilibrium expectation value for a fixed distribution of the couplings, whereas $[\cdot]_{\text{av}}$ denotes an average over the different bond

distributions. Another often-used observable is the **spin-glass susceptibility** $\chi_{\text{SG}} := \lim_{N \rightarrow \infty} \mathcal{N} [\langle q^2 \rangle]_{\text{av}}$.

In spin glasses, the Edwards-Anderson order-parameter takes over the role played by the magnetisation in ferromagnets, as it is non-zero below the critical point but vanishes for temperatures larger than the critical temperature. One can now define spin-glass exponents for $T < T_c$ as follows:

$$q_{\text{EA}} \sim (T_c - T)^{\beta_{\text{EA}}} \quad , \quad \chi_{\text{SG}} \sim (T_c - T)^{-\gamma_{\text{EA}}} \quad . \quad (\text{A28})$$

In addition, one can introduce also for spin glasses the analogues of the other usual static exponents. For example, if one defines a spin glass correlator $C_{\text{SG}}(\mathbf{r}) = [\langle q(\mathbf{r})q(\mathbf{0}) \rangle]_{\text{av}}$ from an order-parameter $q(\mathbf{r})$ localised at position $\mathbf{r}$, the decay of spatial correlations at criticality is described by the exponent η_{EA} , according to $C_{\text{SG}}(\mathbf{r}) \sim |\mathbf{r}|^{-(d-2+\eta_{\text{EA}})}$. For $T < T_c$, this decay is exponential, with a reference scale $\xi_{\text{SG}} \sim (T_c - T)^{-\nu}$.

Even after more than thirty years of research, the physics of Ising spin glasses remains controversial. Thus, the low temperature properties are not yet established and competing theories have been proposed, the best known being a mean-field like explanation, based on replica symmetry breaking, and the droplet picture [78, 431]. These theories yield very different predictions that in principle should be verifiable through numerical simulations. However, the complexity of the spin glasses, which is best illustrated through a very complex free energy landscape, has until now precluded clear-cut answers when using numerical simulations. Indeed, due to the complex free energy landscape, the dynamics is extremely slow in these random frustrated systems, making it very hard to equilibrate even samples of rather modest size. This slow dynamics is illustrated by the following estimates for the temperature-dependent dynamical exponent $z = z(T)$, obtained for a Gaussian distribution of the couplings with zero mean and unit variance [430]

$$z(T) = \frac{1}{T} \cdot \begin{cases} 3.9(1) & ; \text{ if } d = 2 \\ 6.40(15) & ; \text{ if } d = 3 \\ 9.15(20) & ; \text{ if } d = 4 \end{cases} \quad (\text{A29})$$

where, of course, T is measured in units of J .

Another subject of controversy concerns the critical properties at the spin glass transition. Due to the restriction to small system sizes, reliable estimates of critical temperatures and, concomitantly, of critical exponents and critical amplitudes are very difficult to obtain. In Table A8 we gather some of the published values for the critical temperatures and the critical exponents of various three-dimensional Ising spin glasses. Whereas for many years the published values of these quantities differed substantially, which led to speculations that equilibrium critical properties in spin glasses might depend on the distribution of the random bonds, a more universal picture has emerged in recent large-scale numerical studies [428, 318]. The discussion, however, is

distribution	T_c/J	η_{EA}	ν	Ref.
binomial	1.175			[382]
	1.109(10)	-0.375(10)	2.45(15)	[318]
	1.120(4)	-0.395(17)	2.39(5)	[428]
		-0.40(4)	2.72(8)	[125]
Gaussian	0.951(9)	-0.37(5)	2.44(9)	[428]
	0.92(1)	-0.42(2)		[595]
	0.95(4)	-0.36(6)	2.00(15)	[500]
Laplacian	0.72(2)	-0.55(2)		[595]

Table A8. Critical temperatures and equilibrium critical exponents of three-dimensional Ising spin glasses with various distributions of the couplings.

not yet completely settled, as shown by the recent discussion of the intriguing and prominent corrections to scaling observed in these systems [382].

A.14 Gauge Glass

This model is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{(i,j)} \cos(\theta_i - \theta_j - A_{i,j}) \quad (\text{A30})$$

with the nearest-neighbour exchange interactions ($J > 0$) on a simple hyper-cubic lattice. The ‘vector potentials’ $A_{i,j} = -A_{j,i}$ are quenched random variables which are uniformly distributed over the interval $[0, 2\pi)$. The phase θ_i can be viewed as the angle of a classical two-dimensional vector $\vec{S}_i = (\cos \theta_i, \sin \theta_i)$.

The Edwards-Anderson order parameter for this model is defined as

$$q_{\text{EA}} = \frac{1}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \exp \left(i(\theta_j^\alpha - \theta_j^\beta) \right) \quad (\text{A31})$$

where α, β indicate two copies of the system with the same disorder and $\mathcal{N}$ is the number of sites of the lattice. Observables, such as the spin-glass susceptibility, are then defined as $\chi_{\text{SG}} = \mathcal{N} [\langle |q^2| \rangle]_{\text{av}}$, where $\langle . \rangle$ denotes the thermal average and $[.]_{\text{av}}$ is the average over the disorder (or different samples).

It is thought that the model has a continuous phase transition at some $T_c > 0$, whereas $T_c = 0$ in two dimensions. Estimates for $T_c(d)$ and for some equilibrium critical exponents are listed in Table A9.

The temperature-dependent dynamical exponent $z = z(T)$ is estimated as follows [430]

$$z(T) = \begin{cases} 2.18(9) + T^{-1} \cdot 0.41(2) & ; \text{ if } d = 2 \\ 1.95(8) + T^{-1} \cdot 1.17(4) & ; \text{ if } d = 3 \\ 1.75(13) + T^{-1} \cdot 2.4(1) & ; \text{ if } d = 4 \end{cases} \quad (\text{A32})$$

d	T_c/J	η_{EA}	β_{EA}	ν	$z(T_c)$	Ref.
3	0.47(3)	-0.47(7)	0.37(5)	1.39(20)	4.7(1)	[562]
	0.460(1)	-0.47(2)	0.37(1)	1.39(5)	4.5(1)	[429, 430]
4	0.89(1)	-0.74(3)	0.44(2)	0.70(1)	4.50(5)	[429]

Table A9. Critical temperatures and equilibrium critical exponents of the gauge glass model in 3D and 4D on hyper-cubic lattices, with uniform disorder.

where T is measured in units of J . For the given estimates of $T_c(d)$, this is consistent with the values of $z(T_c)$ quoted in Table A9.

D. Langevin Equations and Path Integrals

Although stochastic processes are often defined in terms of a Langevin equation, it is often convenient to use a path integral representation instead. This allows for simple expressions for both correlation and response functions as averages in terms of path integrals, see eqs. (D12,D13) below [401, 53, 204]. Here, we sketch the equivalence of both approaches, considering stochastic processes as described by the **Langevin equation**

$$\lambda^{-1} \partial_t \phi(t, \mathbf{r}) = F[\phi(t, \mathbf{r})] + \eta(t, \mathbf{r}), \quad (\text{D1})$$

with the noise correlator

$$\langle \eta(t, \mathbf{r}) \eta(\mathbf{r}', t') \rangle = \lambda^{-1} \kappa N[\phi(t, \mathbf{r})] \delta^d(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (\text{D2})$$

Here, $F[\phi(t, \mathbf{r})]$ denotes the deterministic part of the Langevin equation. For example, the Langevin equation of DP discussed in detail in volume 1 is obtained for

$$F[\varrho(t, \mathbf{r})] = \left(\tau - g \varrho(t, \mathbf{r}) + \nabla^2 \right) \varrho(t, \mathbf{r}) + h, \quad (\text{D3})$$

$$N[\varrho(t, \mathbf{r})] = \varrho(t, \mathbf{r}). \quad (\text{D4})$$

Similarly, for a relaxing ferromagnet without any conservation laws but coupled to a bath of temperature T one has

$$F[\phi(t, \mathbf{r})] = -\frac{\delta \mathcal{H}[\phi]}{\delta \phi} + h, \quad \kappa N[\phi(t, \mathbf{r})] = 2T \quad (\text{D5})$$

where $\mathcal{H}$ is the equilibrium Ginzburg-Landau functional. Note that both $F[\phi(t, \mathbf{r})]$ and $N[\phi(t, \mathbf{r})]$ may include in general differential operators. For example, $N[\phi(t, \mathbf{r})]$ involves a Laplacian operator in the case of conserved fields.

The generating functional of the stochastic process becomes a functional integral over all realisations of the field $\phi(t, \mathbf{r})$ and the noise $\eta(t, \mathbf{r})$ which satisfy the above Langevin equation

$$Z = \int \mathcal{D}\eta P(\eta) \int \mathcal{D}\phi \delta\left(\lambda^{-1} \partial_t \phi(t, \mathbf{r}) - F[\phi(t, \mathbf{r})] - \eta(t, \mathbf{r})\right), \quad (\text{D6})$$

where the noise field is assumed to have a Gaussian distribution, that is

$$P(\eta) \sim \exp\left(-\int dt d\mathbf{r} \frac{\eta^2}{2\kappa N[\phi]}\right). \quad (\text{D7})$$

In order to carry out the integration over the noise η , auxiliary **response fields** $\tilde{\phi}(t, \mathbf{r})$ are introduced [504, 401], making use of the identity

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} d\tilde{\phi} \exp\left(i\tilde{\phi}x\right). \quad (\text{D8})$$

Applying this to the functional δ -function in (D6) one finds after an appropriate Wick rotation within the complex plane

$$Z \sim \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} e^{-\int dt d\mathbf{r} \tilde{\phi} (\lambda^{-1} \partial_t \phi - F[\phi])} \int \mathcal{D}\eta P(\eta) e^{-\int dt d\mathbf{r} \tilde{\phi} \eta}. \quad (\text{D9})$$

Performing the integration over η by completing the square yields

$$Z \sim \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} e^{-\mathcal{J}[\phi, \tilde{\phi}]} \quad (\text{D10})$$

with the so-called **Janssen-de Dominicis functional**

$$\mathcal{J}[\phi, \tilde{\phi}] = \int dt d\mathbf{r} \tilde{\phi} \left(\lambda^{-1} \partial_t \phi - F[\phi] - \frac{\kappa}{2} N[\phi] \tilde{\phi}\right). \quad (\text{D11})$$

This allows one to derive correlation functions from the path integral as follows

$$\langle \phi(t, \mathbf{r}) \phi(t', \mathbf{r}') \rangle = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \phi(t, \mathbf{r}) \phi(t' \mathbf{r}') e^{-\mathcal{J}[\phi, \tilde{\phi}]}. \quad (\text{D12})$$

The normalisation of the path integral is here implicitly assumed to be chosen such that $\langle 1 \rangle = 1$. The auxiliary field $\tilde{\phi}$ has the following physical meaning. Applying the external conjugated field h the linear response of the order-parameter is given by

$$\left. \frac{\delta \langle \phi \rangle}{\delta h} \right|_{h=0} = \langle \phi \tilde{\phi} \rangle. \quad (\text{D13})$$

For this reason, the auxiliary field is often referred to as the **response field**. Clearly, the two-point correlation function (D12) and the response function $\langle \phi \tilde{\phi} \rangle$ are different. For non-equilibrium systems this is reflected through the breaking of the fluctuation-dissipation theorem.

Non-stationary initial conditions can be included in the present formalism as follows [404, 403]. The paradigmatic example is a distribution of initial states with mean order-parameter $\langle \phi(0, \mathbf{r}) \rangle = m_0(\mathbf{r})$ and short-ranged Gaussian fluctuations

$$\left\langle [\phi(0, \mathbf{r}) - m_0(\mathbf{r})][\phi(0, \mathbf{r}') - m_0(\mathbf{r}')] \right\rangle = \tau_0^{-1} \delta^d(\mathbf{r} - \mathbf{r}'). \quad (\text{D14})$$

The corresponding contribution to the action will be a Gaussian of the form

$$\mathcal{J}_{\text{ini}}[\varrho] = \int d\mathbf{r} \frac{\tau_0}{2} [\phi(0, \mathbf{r}) - m_0(\mathbf{r})]^2. \quad (\text{D15})$$

Since the canonical dimension of the parameter τ_0 is 2, the only fixed-point value which leads to a normalisable distribution is $\tau_0 = \infty$. Since corrections coming from a finite value of τ_0 will be irrelevant, one can set from the outset $\tau = \infty$, unless in situations where terms vanish in this limit. Using the short-time relation $\phi(0, \mathbf{r}) = \tau_0^{-1} \tilde{\phi}(0, \mathbf{r})$ [403], the Janssen-de Dominicis functional now becomes

$$\mathcal{J}_{\text{eff}}[\phi, \tilde{\phi}] = \mathcal{J}[\phi, \tilde{\phi}] + \int d\mathbf{r} \left(\frac{1}{2\tau_0} \tilde{\phi}^2(0, \mathbf{r}) - m_0 \tilde{\phi}(0, \mathbf{r}) \right). \quad (\text{D16})$$

I. Cluster Algorithms: Competing Interactions

Close to a continuous phase transition, local update algorithms, involving for example single-spin flips in the case of classical spin models, greatly suffer from critical slowing down as temporal correlations become very large. Thus, close to the critical point the autocorrelation $\tau \sim L^z$ for a system of linear extent L and dynamical exponent z . Noting that $z \approx 2$ in many cases, it is obvious that local update algorithms become very ineffective when approaching the phase transition point.

In order to alleviate this problem, Swendsen and Wang [689] considered non-local updates in which clusters of spins are flipped. A simpler variant was proposed shortly thereafter by Wolff [737] where a single cluster is flipped at a time. Both algorithms have the important property that the dynamical exponent is much smaller than for the algorithms using local updates, with the typical values of z lying between 0.15 and 0.5. It follows that critical slowing down is strongly reduced when using cluster algorithms, making them the method of choice when studying the properties of static quantities at a critical point (see [460] for an in-depth discussion of these algorithms).

Cluster algorithms are routinely used nowadays for ferromagnetic spin models, but applications to systems with competing interactions are scarce. In this appendix we discuss the variants of the Wolff cluster algorithm that have been used in studies of the ANNNI model with competition between ferromagnetic and antiferromagnetic interactions in one spatial direction [597, 343, 660].

Consider first the Ising model with only ferromagnetic nearest-neighbour couplings of strength $J > 0$ at some temperature T . For this case the creation of a cluster with the **Wolff algorithm** [737] consists of the following steps:

- 1) Choose randomly a lattice site, the seed, and put the coordinates of that site on a stack.
- 2) For the first site on the stack, check for all its nearest neighbours whether they are already part of the cluster. If they are not part of the cluster, add them with probability

$$p = \frac{1}{2}(1 + \text{sign}(s_i s_j))(1 - \exp[-2J/T]) \quad (\text{I1})$$

to the cluster and put their coordinates on the stack. If all of its nearest neighbours have been checked, remove the original site from the stack.

- 3) Repeat 2) until the stack is empty.

In this way, one ends up with a cluster of spins all having the same sign, see Fig. I1a, and which is flipped as a whole. This kind of same-sign cluster is obviously not adapted to models with competing interactions.

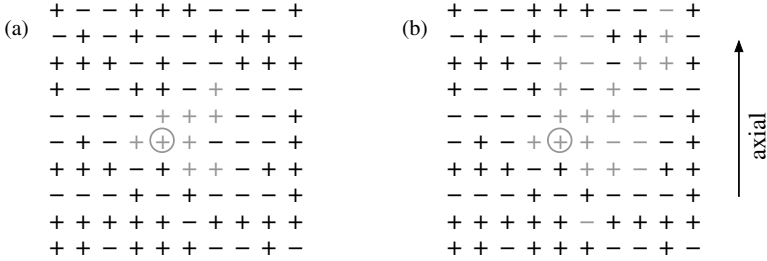


Fig. I1. Typical clusters (gray spins) obtained (a) by the Wolff algorithm and (b) by the modified algorithm for systems with competing interactions, here shown in two dimensions for simplicity. The competition takes place in the axial direction. The spin enclosed by a circle is the seed. [343]

A modified cluster algorithm has been proposed for spin models with competition between ferromagnetic nearest-neighbour and antiferromagnetic next-nearest-neighbour couplings in the axial direction, the paradigmatic example being the three-dimensional ANNNI model with the Hamiltonian

$$\mathcal{H} = -J \sum_{xyz} s_{xyz} (s_{(x+1)yz} + s_{x(y+1)z} + s_{xy(z+1)}) + \kappa J \sum_{xyz} s_{xyz} s_{xy(z+2)} \quad (\text{I2})$$

where $s_{xyz} = \pm 1$, whereas $J > 0$ and $\kappa > 0$ are coupling constants. This algorithm also starts with a randomly chosen seed but builds clusters that contain spins of both signs. The second step in the algorithm is then restated in the following way:

- 2') For the first site on the stack, check for all its nearest neighbours and axial next-nearest neighbours whether they are already part of the cluster. If a nearest neighbour is not part of the cluster, add it with probability

$$p = \frac{1}{2} (1 + \text{sign}(s_i s_j)) (1 - \exp[-2J/T]) \quad (\text{I3})$$

to the cluster and put its coordinates on the stack. If an axial next-nearest neighbour is not part of the cluster, add it with probability

$$p_a = \frac{1}{2} (1 - \text{sign}(s_i s_k)) (1 - \exp[-2J_k/T]) \quad (\text{I4})$$

to the cluster and put its coordinates on the stack. If all of its nearest neighbours and axial next-nearest neighbours have been checked, remove the original site from the stack.

Thus the final cluster, which will be flipped as a whole, contains spins of both signs, as shown in Fig. I1b. At variance with the traditional Wolff method, the flipped spins are not necessarily connected by nearest-neighbour couplings. This algorithm works very well in the paramagnetic phase, in the ferromagnetic phase and in the vicinity of the Lifshitz point, and has also been used with success for the study of modulated structures in thin films.

Evertz and Linden [233] proposed a variant for the Wolff cluster algorithm that can be used for the direct computation of two-point functions of an infinite lattice. In their approach, they modify step 1) and no longer select the seed at random, but instead they start each cluster at the *same* lattice site x_0 . It follows from this simple modification that after N cluster flips all spins within a radius $r(N)$ around the seed will be equilibrated. Increasing the number of iterations, this equilibrated region will increase accordingly. The advantage of this approach is that only the spin configuration within the area reached by a cluster has to be stored. In addition, once the region of radius $r(N)$ has been equilibrated, one can immediately measure the space-dependent correlation function $C(\mathbf{r}) = \langle s_0 s_{\mathbf{r}} \rangle$, where s_0 is the spin at site x_0 and the brackets indicate the usual Monte Carlo average.

A generalisation of this method to systems with competing interactions is done in a straightforward way by replacing the same-sign clusters with the modified clusters described above. This approach, which enables one to compute with high precision two-point correlators in spin systems with competing interactions, has been used for the computation of the correlation functions in the ANNNI model that are discussed in Chap. 6.

J. Fractional Derivatives

In several situations encountered in this book it is necessary to consider derivatives of a non-integer order. Derivatives of *integer* order are linear operators

which have several useful properties such as

$$\begin{aligned}
 &\text{composition: } \partial_r^n \partial_r^m = \partial_r^{n+m} \\
 &\text{commutation: } [\partial_r^n, r] = n \partial_r^{n-1} \\
 &\text{exponential: } \partial_r^n e^{ikr} = (ik)^n e^{ikr} \\
 &\text{Cauchy integral: } f^{(n)}(z) := \frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_C dw \frac{f(w)}{(w-z)^{n+1}}. \quad (\text{J1})
 \end{aligned}$$

In all these cases, $n, m \in \mathbb{N}$. However, when trying to define a ‘generalised derivative’ ∂_r^α of real order $\alpha \in \mathbb{R}$, it turns out that at least one of the above properties cannot be extended to the case $\alpha \notin \mathbb{N}$. This has led to a large variety of inequivalent notions of fractional derivatives. For reviews, see [274, 520, 637, 750, 601, 225].

One of the most frequently used definitions is the **Riemann-Liouville operator**. A convenient way to introduce it was given by Hadamard, which we follow here. Consider functions of a single variable r , which may be expressed as formal series $f(r) = \sum_e f_e r^e$ and the sum is assumed to be such that the set of indices e does not contain negative integers. The Riemann-Liouville derivative of order $a \in \mathbb{R}$ of r^e is ($\Gamma(x)$ denotes Euler’s Γ -function [4])

$$D^a r^e := \frac{\Gamma(e+1)}{\Gamma(e-a+1)} r^{e-a} \quad (\text{J2})$$

and furthermore, the Riemann-Liouville operator D^a is assumed to be linear. While it does generalise the second of the properties (J1), the third one does not extend, since with $a = k + \alpha$ and $k \in \mathbb{N}$ and $0 \leq \alpha < 1$

$$D^{k+\alpha} r^e = \sum_{n=-k}^{\infty} \frac{r^{n-\alpha}}{\Gamma(n+1-\alpha)} = r^{-\alpha} E_{1,1-\alpha}(r) \quad (\text{J3})$$

where $E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} z^k / \Gamma(\alpha k + \beta)$ is a Mittag-Leffler type function. The first property of (J1) does not generalise either. This is illustrated by the following classical example [520]. Take $N \in \mathbb{N}$ and let

$$\mathfrak{e}(r) := \sum_{k=0}^{N-1} \alpha^{N-1-k} r^{-k/N} E_{1,1-k/N}(\alpha r). \quad (\text{J4})$$

It is easily verified that $D^{1/N} \mathfrak{e}(t) = \alpha \mathfrak{e}(t)$ but

$$D^{2/N} \mathfrak{e}(r) = \alpha^2 \mathfrak{e}(r) + \frac{r^{-1-1/N}}{\Gamma(-1/N)} \neq \alpha^2 \mathfrak{e}(r) = D^{1/N} \left(D^{1/N} \mathfrak{e}(r) \right). \quad (\text{J5})$$

In the mathematical literature this fact is taken into account by introducing the rather awkward *sequential* fractional derivatives [520]. These examples show that neither composition nor Fourier/Laplace transforms of Riemann-Liouville operators will be easy to handle. Another difficulty arises since

$D^a 1 = (\Gamma(1-a)r^a)^{-1} \neq 0$ which makes the definition of an initial-value problem difficult.⁴

J.1 Singular Fractional Derivatives

One way around these difficulties is to include singular (i.e. distributional) terms into the definition of a fractional derivative [330]. This definition is used in Chap. 6 (and occasionally in Chap. 5).

Consider a set E of numbers e such that any $e \in E$ is not a negative integer, $e \neq -(n+1)$ with $n \in \mathbb{N}$. We call such a set an **E-set**. Let I be some (possibly infinite) positive real interval. We define the **$\mathcal{M}$ -space** of generalised functions associated with the E-set E as

$$\begin{aligned} \mathcal{M} &:= \mathcal{M}_E(I, \mathbb{R}) \\ &= \left\{ f : I \subset \mathbb{R} \rightarrow \mathbb{R} \left| f(r) = \sum_{e \in E} f_e r^e + \sum_{n=0}^{\infty} F_n \delta^{(n)}(r) ; f_e \in \mathbb{R}, F_n \in \mathbb{R} \right. \right\} \end{aligned} \quad (\text{J6})$$

where $\delta^{(n)}(r)$ is the n^{th} derivative of the Dirac delta function. The part of $f \in \mathcal{M}$ parametrised by the constants f_e is called the **regular part** of f and the part parametrised by the F_n is call the **singular part** of f .

Theorem: [274, p. 81] *Any generalised function $f(r)$ concentrated at $r = 0$ is given by a finite sum $\sum_n F_n \delta^{(n)}(r)$, where $\delta(r)$ is the Dirac distribution.*

Hence we can assume that only finitely many of the coefficients F_n in (J6) are non-vanishing. Below, we shall state sufficient conditions for the associated series to converge. For example, one may take $E = \mathbb{N}$ and let $\mathcal{M}$ be the space of analytic functions on I .

Definition: Let $a \in \mathbb{R}$, E be an E-set and let $E' := \{e' | e' = e - a; e \in E\}$. Analogously to (J6) one has the space $\mathcal{M}' = \mathcal{M}_{E'}$. An operator $\partial^a : \mathcal{M} \rightarrow \mathcal{M}'$ is called a **derivative of order a** , iff it satisfies the properties:

- i) $\partial^a (\lambda f(r) + \mu g(r)) = \lambda \partial^a f(r) + \mu \partial^a g(r) \quad \forall \lambda, \mu \in \mathbb{R} \text{ and all } f, g \in \mathcal{M}$
- ii) $\partial^a r^e = \frac{\Gamma(e+1)}{\Gamma(e-a+1)} r^{e-a} + \sum_{n=0}^{\infty} \delta_{a,e+n+1} \Gamma(e+1) \delta^{(n)}(r) \quad (\text{J7})$
- iii) $\partial^a \delta^{(n)}(r) = \frac{r^{-1-n-a}}{\Gamma(-a-n)} + \sum_{m=0}^{\infty} \delta_{a,m} \delta^{(n+m)}(r)$

In particular, it follows from (J7) that the prefactor for any monomial r^{-n-1} with $n \in \mathbb{N}$ indeed vanishes. For our purposes, we may consider the set E' therefore also as an E-set and $\mathcal{M}'$ as an $\mathcal{M}$ -space. In particular, ∂^a can be

⁴ This last point may resolved by considering the Caputo fractional derivative, but which still suffers from the other two problems [601].

applied term-by-term to any function $f \in \mathcal{M}$. Often, we shall also write $\partial^a = \partial_r^a$ if we want to specify explicitly the variable r on which ∂^a is supposed to act. As the Riemann-Liouville derivative, ∂^a is not defined on negative integer powers r^{-n-1} with $n \geq 0$. It is also possible to introduce this derivative for generalised functions concentrated on the half-line $r > 0$ by setting [274, p. 115]

$$\partial^a f(r) := f(r) * \frac{r_+^{-a-1}}{\Gamma(-a)} = \frac{1}{\Gamma(-a)} \int_0^r d\rho f(\rho) (r - \rho)^{-a-1} \quad (\text{J8})$$

where the integral must be regularised [274]. The main calculational rules are as follows.

Lemma 1: *For an E-set and the associated $\mathcal{M}$ -space, $f \in \mathcal{M}$ such that all fractional derivatives $\partial^a f$ of order $a \in \mathbb{R}$ considered below exist, we have on $\mathcal{M}$*

$$\partial^{a+b} f(r) = \partial^a \partial^b f(r) = \partial^b \partial^a f(r) \quad (\text{J9})$$

$$[\partial^a, r] f(r) = a \partial^{a-1} f(r) \quad (\text{J10})$$

$$\partial_r^a f(\lambda r) = \lambda^a \partial_{\lambda r}^a f(\lambda r) \quad (\text{J11})$$

$$\partial_{\lambda r}^a f(r) = \lambda^{-a} \partial_r^a f(r) \quad (\text{J12})$$

where $\lambda > 0$ is a real constant. If f is analytic and non-singular and $g \in \mathcal{M}$, we have a generalised Leibniz formula

$$\partial^a (f(r)g(r)) = \sum_{\ell=0}^{\infty} \binom{a}{\ell} \frac{d^\ell f(r)}{dr^\ell} \partial^{a-\ell} g(r) \quad (\text{J13})$$

where d^ℓ/dr^ℓ are ordinary derivatives of integer order ℓ . If $f \in \mathcal{M}$ and non-singular and $n \in \mathbb{N}$, one has with the ordinary derivative $f^{(\ell)}(x) = d^\ell f(x)/dx^\ell$ of integer order ℓ

$$[f(\partial_r), r^n] = \sum_{\ell=1}^n \binom{n}{\ell} r^{n-\ell} f^{(\ell)}(\partial_r). \quad (\text{J14})$$

While these definitions are to be used for $r > 0$ as they stand, we may use (J11) with $\lambda < 0$ to formally extend this to all $r \neq 0$.

Let us note two corollaries from (J13). First, we have

$$[\partial_t^a, t^n] = \sum_{k=1}^n \binom{a}{k} \binom{n}{k} k! t^{n-k} \partial_t^{a-k}. \quad (\text{J15})$$

Second, we set $g(r) = 1$. if $f(r)$ is analytic, one has

$$\partial_r^a f(r)|_{\text{reg}} = r^{-a} \sum_{\ell=0}^{\infty} \frac{\Gamma(a+1)}{\ell!} \frac{\sin(\pi(a-\ell))}{\pi(a-\ell)} r^\ell \frac{d^\ell f(r)}{dr^\ell} \quad (\text{J16})$$

which expresses the regular part of $\partial^a f$ in terms of ordinary derivatives and for $a \rightarrow n \in \mathbb{N}$, one recovers the ordinary n^{th} derivative. For illustration, we reconsider the function $\mathfrak{e}(r)$ defined above in eq. (J4). We find $\partial_r^{1/N} \mathfrak{e}(r) = \alpha \mathfrak{e}(r) + \delta(r)$ and furthermore obtain

$$\partial_r^{2/N} \mathfrak{e}(r) = \alpha^2 \mathfrak{e}(r) + \frac{r^{-1-1/N}}{\Gamma(-1/N)} + \alpha \delta(r) = \partial_r^{1/N} \left(\partial_r^{1/N} \mathfrak{e}(r) \right). \quad (\text{J17})$$

The singular terms are essential for the composition property (J9).

It remains to discuss the convergence of the series involved. To do so, consider an E-set E which is countable and ordered, and its elements can be labelled e_n with $n \in \mathbb{N}$. Let $\nu_n := e_{n+1} - e_n > 0$. Call such an E-set E **well-separated with separation constant** ϵ , if there is an $\epsilon > 0$ such that $\nu_n \geq \epsilon$. For the regular part of a function $f \in \mathcal{M}_E$, we have

$$f(r)|_{\text{reg}} = \sum_{e \in E} f_e r^e = \sum_{n=0}^{\infty} f_n r^{e_n} \quad ; \quad f_n := f_{e_n}. \quad (\text{J18})$$

Questions of existence and of convergence of $\partial_r^a f(r)$ are treated in the following

Lemma 2: *Let E be a well-separated E-set with separation constant ϵ , $f \in \mathcal{M}_E$ with coefficients as in (J18), $e_n > 0$, $e_n - a > 0$ and*

$$\rho^{-1} := \limsup_{n \rightarrow \infty} |f_n|^{1/e_n} \geq 0. \quad (\text{J19})$$

Then: (i) $f(r)$ converges absolutely for $|r| < \rho$.

(ii) If $\nu_n/e_n < B$ for some constant B , $\partial_r^a f(r)$ converges absolutely, provided $|r| < \rho \min(1, (1+B)^{-a/\epsilon})$.

(iii) If $f : I \rightarrow \mathbb{R}$ is analytic with a radius of convergence $\rho > 0$ around $r = 0$, then the series (J16) for $\partial_r^a f(r)$ converges absolutely for $|r| < \rho/2$.

The first property is well-known for ψ -series [363]. In particular, the admissible functions include $r^\lambda f(r^\mu)$ with $f(r)$ analytic, $\mu > 0$ and $\lambda \neq -\mu m - n$ which is enough for the purposes of this book. Series of this kind have effectively $B = 0$.

J.2 Fractional Laplacians

The construction of the infinitesimal generators with generalised ‘mass terms’ in Chap. 5 can be carried out using a definition of a fractional derivative as a pseudo-differential operator. We follow here [44, 48].

Definition: *The action of the operator $\nabla_{\mathbf{r}}^\alpha$ with $\alpha \in \mathbb{R}$ on the function $f(\mathbf{r})$ is defined as*

$$\nabla_{\mathbf{r}}^\alpha f(\mathbf{r}) := i^\alpha \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^\alpha e^{i\mathbf{r} \cdot \mathbf{k}} \widehat{f}(\mathbf{k}) \quad (\text{J20})$$

where the right-hand side of (J20) is to be understood in a distributional sense [274] and $\widehat{f}(\mathbf{k})$ denotes the Fourier transform.

A simple example is $\nabla_{\mathbf{r}}^{\alpha} \exp(i\mathbf{q} \cdot \mathbf{r}) = i^{\alpha} |\mathbf{q}|^{\alpha} \exp(i\mathbf{q} \cdot \mathbf{r})$. Since also $\cos(\mathbf{q} \cdot \mathbf{r})$ and $\sin(\mathbf{q} \cdot \mathbf{r})$ are eigenfunctions of $\nabla_{\mathbf{r}}^{\alpha}$, the derivative (J20) is probably best viewed as a generalised Laplace operator, viz. $\nabla_{\mathbf{r}}^{\alpha} = \Delta_{\mathbf{r}}^{\alpha/2}$, to which it reduces for $\alpha = 2$. The main calculational rules are as follows.

Lemma 3: *The linear operator $\nabla_{\mathbf{r}}^{\alpha}$ has the following properties*

$$\begin{aligned}
 \text{i.) } & \nabla_{\mathbf{r}}^{\alpha} \nabla_{\mathbf{r}}^{\beta} = \nabla_{\mathbf{r}}^{\alpha+\beta} \\
 \text{ii.) } & \sum_{i=1}^d \partial_{r_i}^2 = \nabla_{\mathbf{r}}^2 = \Delta_{\mathbf{r}} \\
 \text{iii.) } & [\nabla_{\mathbf{r}}^{\alpha}, r_i] = \alpha \partial_{r_i} \nabla_{\mathbf{r}}^{\alpha-2} \\
 \text{iv.) } & [\nabla_{\mathbf{r}}^{\alpha}, \mathbf{r}^2] = 2\alpha(\mathbf{r} \cdot \partial_{\mathbf{r}}) \nabla_{\mathbf{r}}^{\alpha-2} + \alpha(\alpha - 2 + d) \nabla_{\mathbf{r}}^{\alpha-2} \\
 \text{v.) } & \nabla_{\mu\mathbf{r}}^{\alpha} f(\mu\mathbf{r}) = |\mu|^{-\alpha} \nabla_{\mathbf{r}}^{\alpha} f(\mu\mathbf{r}) \\
 \text{vi.) } & \nabla_{\mu\mathbf{r}}^{\alpha} f(\mathbf{r}) = |\mu|^{-\alpha} \nabla_{\mathbf{r}}^{\alpha} f(\mathbf{r})
 \end{aligned} \tag{J21}$$

where $\Delta_{\mathbf{r}}$ is the Laplacian with respect to $\mathbf{r}$. We use the shorthand $\mathbf{r} \cdot \partial_{\mathbf{r}} := \sum_{i=1}^d r_i \partial_{r_i}$ and $\partial_{r_i} = \partial/\partial r_i$ denotes the usual partial derivative into the i^{th} direction.

Here, the commutativity property in (J1) is modified. In contrast to the definition of $\partial_{\mathbf{r}}^{\alpha}$, there is no simple way to define the action of $\nabla_{\mathbf{r}}^{\alpha}$ on monomials.

Lemma 4: *For $d = 1$ and $\alpha \notin -\mathbb{N}$, one has for a complex analytic function $f(z)$ a generalised Cauchy formula*

$$\nabla_{\mathbf{r}}^{\alpha} f(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \oint_{C_z} dw \frac{f(w)}{(w - z)^{\alpha+1}} \tag{J22}$$

where C_z is a closed smooth contour which goes around z once.

For the resolution of fractional differential equations, the ‘singular’ definition of $\partial_{\mathbf{r}}^{\alpha}$ (J7) calls for series expansion methods, while the ‘Laplacian’ definition $\nabla_{\mathbf{r}}^{\alpha}$ (J20) works best with Fourier/Laplace transforms.

K. Conformally Invariant Interacting Fields

In Chap. 2 in Volume 1, we discussed the conformal invariance of the Laplace equation. However, the technique applied there cannot be extended straightforwardly to semi-linear generalisations $\Delta_{\text{L}} \phi = \frac{n}{2} g_n \phi^{n-1}$, where g_n is a coupling constant. Furthermore, it is a well-established mathematical result, see e.g. [267, 665], that conformal invariance of the above equation only holds at the peculiar dimension $d^{(c)} = 2n/(n-2)$. Since from a physicist’s perspective,

symmetries of a theory are better discussed in terms of the energy-momentum tensor, we give a simple argument [218] of how the usual construction of an ‘improved’ energy-momentum tensor (the **Belinfante tensor** [191]), at least for classical field-theories, can be understood in a simple way by including the transformation of the dimensionful constant g_n in the conformal generators.⁵

K.1 Conformal Invariance and Coupling Constants

Consider a scalar field $\phi_0(\mathbf{r})$, $\mathbf{r} \in \mathbb{R}^d$, which transforms under an infinitesimal diffeomorphism $\mathbf{r} \mapsto \mathbf{r}' = \mathbf{r} + \boldsymbol{\epsilon}(\mathbf{r})$, as

$$\phi_0(\mathbf{r}) \mapsto \phi'_0(\mathbf{r}') = \left(1 - \frac{x_0}{d} \boldsymbol{\partial} \cdot \boldsymbol{\epsilon}(\mathbf{r})\right) \phi_0(\mathbf{r}), \quad (\text{K1})$$

where x_0 is the scaling dimension of the field. The variation of the free-field theory action $S_0 = \int_{\mathbf{r}} (\partial \phi_0(\mathbf{r}))^2$, with the short-hand $\int_{\mathbf{r}} := \int d^d \mathbf{r}$ is given by

$$\begin{aligned} \delta S_0 &= \int_{\mathbf{r}} (\partial \phi'_0(\mathbf{r}))^2 - \int_{\mathbf{r}} (\partial \phi_0(\mathbf{r}))^2 \\ &= \int_{\mathbf{r}} \left[\left(1 - \frac{2x_0}{d}\right) (\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) (\partial \phi_0)^2 - 2\partial^\mu \phi_0 (\partial_\mu \epsilon_\nu) \partial_\nu \phi_0 - \frac{2x_0}{d} \partial^\mu \phi_0 (\partial_\mu \boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) \phi_0 \right] \\ &= \int_{\mathbf{r}} \left[(\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) \left(\frac{1}{2} - \frac{1+x_0}{d}\right) (\partial \phi_0)^2 - \frac{x_0}{d} (\partial_\mu (\boldsymbol{\partial} \cdot \boldsymbol{\epsilon})) \partial^\mu \phi_0^2 \right] \end{aligned} \quad (\text{K2})$$

where in the last line we specialised to conformal transformations $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (2/d) \delta_{\mu,\nu} \boldsymbol{\partial} \cdot \boldsymbol{\epsilon}$. We observe: (i) $\partial_\mu (\boldsymbol{\partial} \cdot \boldsymbol{\epsilon}) = b_\mu$ is a constant for a special conformal transformation and vanishes for the other conformal transformations. Then $\delta_{\text{conf}} S_0 = 0$, up to a surface term, iff the scaling dimension $x_0 = d/2 - 1$ has its canonical value. (ii) For an interacting classical field, conformal invariance in general is lost, unless $d = d^{(c)}$ [191, 759]. For illustration, consider a scalar ϕ_0^n theory with the action $S_1 = \int_{\mathbf{r}} [(\partial \phi_0(\mathbf{r}))^2 + g_n \phi_0^n(\mathbf{r})]$. This action becomes conformally invariant in any dimension d if we also transform the coupling $g \rightarrow g'$, such that [686]

$$g' = \left(1 + \frac{y}{d} \boldsymbol{\partial} \cdot \boldsymbol{\epsilon}(\mathbf{r})\right) g, \quad (\text{K3})$$

where $y = n(d/2 - 1) - d$. In other words, the spatial symmetries of the action are dictated by the properties of the kinetic term, while the transformation properties of the coupling g is chosen so as to obtain the invariance of the full action under a specific representation of the corresponding group of spatial transformations. Next, construct a quasi-primary field $\phi(\mathbf{r})$ with an arbitrary scaling dimension x , as $\phi(\mathbf{r}) := g^\alpha \phi_0(\mathbf{r})$ where $\alpha = (x_0 - x)/y$. Then

⁵ MH thanks S.B. Dutta for intensive discussions on this topic.

$$S_1 = \int_{\mathbf{r}} \left[g^{-2\alpha} (\partial\phi(\mathbf{r}))^2 + g_n^{1-n\alpha} \phi^n(\mathbf{r}) \right]. \quad (\text{K4})$$

This extends to any polynomial potential with an arbitrary number of couplings.

K.2 Conformally Conserved Currents

Consider actions of the form

$$S = \int_{\mathbf{r}} \mathcal{L} [\phi(\mathbf{r}), \partial_\mu \phi(\mathbf{r}); g], \quad (\text{K5})$$

which generalises the special case (K4). We look for the conserved currents associated with the conformal transformations $\mathbf{r} \mapsto \mathbf{r}'$, collectively parametrised as $\{\omega_a\}$, and which include d translations $\{a^\mu\}$, $d(d-1)/2$ rotations $\{\omega^{\mu\nu}\}$, one scale $\{\lambda\}$, and d special conformal transformations $\{b^\mu\}$. Here $\phi(\mathbf{r})$ is a quasi-primary scaling operator with scaling dimension x , and g transforms as in (K3). Consider the simultaneous transformations

$$\phi'(\mathbf{r}') := \phi(\mathbf{r}) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(\mathbf{r}), \quad g' := g + \omega_a \frac{\delta g}{\delta \omega_a}, \quad \mathbf{r}' := \mathbf{r} + \omega_a \frac{\delta \mathbf{r}}{\delta \omega_a}. \quad (\text{K6})$$

Now modulate the symmetry parameters $\{\omega_a\} \rightarrow \{\omega_a(\mathbf{r})\}$ and write down the variation of the action. It is useful to independently vary g and transform the fields as follows,

$$g' = g + \overline{\delta g}(\mathbf{r}); \quad \phi'(\mathbf{r}) = \phi(\mathbf{r}) + \overline{\delta g}(\mathbf{r}) \partial_g \phi(\mathbf{r}), \quad (\text{K7})$$

where $\partial_g \phi(\mathbf{r}) = (\alpha/g)\phi(\mathbf{r})$. The variation of the action is then given by

$$\begin{aligned} \delta_{\omega, \overline{\delta g}} S &= \int_{\mathbf{r}} \mathcal{L} [\phi'(\mathbf{r}), \partial_\mu \phi'(\mathbf{r}); g'] - \int_{\mathbf{r}} \mathcal{L} [\phi(\mathbf{r}), \partial_\mu \phi(\mathbf{r}); g] \\ &= - \int_{\mathbf{r}} [(\partial_\mu \omega_a(\mathbf{r})) J_a^\mu(\mathbf{r}) + (\partial_\mu \overline{\delta g}) G^\mu(\mathbf{r}) + \overline{\delta g} R(\mathbf{r})], \end{aligned} \quad (\text{K8})$$

where

$$J_a^\mu(\mathbf{r}) = \overbrace{\left(\partial_\nu \phi \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} - \delta_\nu^\mu \mathcal{L} \right)}^{\tilde{T}_\nu^\mu(\mathbf{r})} \frac{\delta r^\nu}{\delta \omega_a} - \frac{\delta \mathcal{F}}{\delta \omega_a} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}, \quad (\text{K9})$$

$$G^\mu(\mathbf{r}) = -\partial_g \phi \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}, \quad (\text{K10})$$

$$R(\mathbf{r}) = - \left[\partial_g \phi \frac{\partial \mathcal{L}}{\partial \phi} + (\partial_\mu \partial_g \phi) \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} + \frac{\partial \mathcal{L}}{\partial g} \right] \quad (\text{K11})$$

and where $\tilde{T}_\nu^\mu(\mathbf{r})$ is the canonical energy-momentum tensor. Since explicitly $\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(\mathbf{r}) = -\frac{x}{d} \omega_a \left(\partial_\nu \frac{\delta r^\nu}{\delta \omega_a} \right) \phi(\mathbf{r})$, the conformal current (K9) becomes

$$J_a^\mu(\mathbf{r}) = \tilde{T}_\nu^\mu(\mathbf{r}) \frac{\delta r^\nu}{\delta \omega_a} - \frac{gx}{\alpha d} \left(\partial_\nu \frac{\delta r^\nu}{\delta \omega_a} \right) G^\mu(\mathbf{r}). \quad (\text{K12})$$

Specialising to scale-invariance, $\omega_a \mapsto \lambda$, the dilatation current J_D^μ is obtained by substituting the scale transformation $\delta r^\nu / \delta \lambda = r^\nu$ in the above expression, and is given by

$$J_D^\mu(\mathbf{r}) = r^\nu \tilde{T}_\nu^\mu(\mathbf{r}) - \frac{gx}{\alpha} G^\mu(\mathbf{r}). \quad (\text{K13})$$

This form of the dilatation current is identical to the one quoted in [191], where it is obtained by an *ad hoc* improvement of the canonical scale current. Dilatation-invariance of the action, $\delta_{\omega,0} S = \int_{\mathbf{r}} \omega_a \partial_\mu J_a^\mu = 0$ implies

$$\partial_\mu J_D^\mu(\mathbf{r}) = \tilde{T}_\mu^\mu - \frac{x}{\alpha} \partial_\mu (gG^\mu) = \partial_\mu K^\mu(\mathbf{r}), \quad (\text{K14})$$

where $K^\mu(\mathbf{r})$ is some vector field. Equation (K14) is the familiar statement that scale-invariance holds if the trace of the energy-momentum tensor can be written as divergence of a current, $\tilde{T}_\mu^\mu = -\partial_\mu \tilde{K}^\mu$ [602].

Special conformal transformations, with $\delta r^\nu / \delta b_\alpha = 2r^\nu r^\alpha - \delta^{\nu\alpha} \mathbf{r} \cdot \mathbf{r}$, are treated analogously. From eq. (K12) we find the special conformal current, using (K13) $J_C^{\mu\alpha}(\mathbf{r}) = 2r^\alpha J_D^\mu(\mathbf{r}) - \mathbf{r} \cdot \mathbf{r} \tilde{T}^{\mu\alpha}(\mathbf{r})$. Special conformal invariance of S gives $\int_{\mathbf{r}} \partial_\mu J_C^{\mu\alpha} = 0$, which upon using $J_C^{\mu\alpha}$, eq. (K13) and $\partial_\mu \tilde{T}_\nu^\mu = 0$ leads to

$$\frac{x}{\alpha} gG^\mu + K^\mu = \partial_\nu L^{\mu\nu}. \quad (\text{K15})$$

This is equivalent to the condition of special conformal invariance, $\tilde{T}_\mu^\mu = \partial_\mu \partial_\nu L^{\mu\nu}$ as formulated in [602] (or $\tilde{T}_\mu^\mu = \partial^2 L(\mathbf{r})$ if $d = 2$). If $L^{\mu\nu}$ is explicitly known, one may construct a traceless improved energy-momentum tensor. For example, in $2D$ the improved energy-momentum tensor

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + \partial_\mu \partial_\nu L(\mathbf{r}) - \delta_{\mu\nu} \partial^2 L(\mathbf{r}) \quad (\text{K16})$$

is symmetric, conserved and traceless. Similar, but more lengthy formulæ exist for $d > 2$. The construction presented here is equivalent to, but may appear more natural than, the *ad hoc* procedures found in the literature, and thus might be more easy to generalise.

We stress that the two conditions $d = d^{(c)}$ and $x = \frac{d}{2} - 1$ are both no longer required [218].

Example : For the classical ϕ^n -theory, one easily finds from the action S_1

$$\frac{x}{\alpha} gG^\mu = -x\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -2xg^{-2\alpha} \phi \partial_\mu \phi. \quad (\text{K17})$$

As expected, this confirms scale-invariance, because of eq. (K14), and also conformal invariance, since G^μ is of a divergence form, viz. $\frac{x}{\alpha}gG^\mu = -\partial_\nu L^{\mu\nu}$, where $L_{\mu\nu} = \delta_{\mu\nu}xg^{-2\alpha}\phi^2$. Hence, making the ‘minimal’ choice $K^\mu = 0$, one has $\tilde{T}_\mu^\mu = \partial_\mu\partial_\nu L^{\mu\nu}$. Thus, in this example at least, scale-invariance is indeed sufficient to imply (special) conformal invariance.⁶

L. Lie Groups and Lie Algebras: a Reminder

To make this book a little more self-contained, we collect some basic definitions and facts from the theory of finite groups and of Lie groups and their Lie algebras. This appendix is intended as a reminder of a few well-known facts which are routinely used in this book, for a reader who will have met Lie algebras in the context of the quantum-mechanical theory of angular momentum. It is not meant to be a systematic introduction and we refer for this to several excellent texts, e.g. [309, 279, 387]. Finite- and infinite-dimensional Lie algebras and their relationship with conformal field-theories are superbly exposed in [191]. For the relationship with symmetries of differential equations, see [231, 267]. More mathematically oriented texts include [446, 184, 303, 415, 624].

L.1 Finite groups

We begin with the definition of a (finite) group:

Definition: A set $G = \{g_1, \dots, g_n\}$ of n elements is called a **finite group** if there is a map $\cdot : G \times G \rightarrow G$ which associates to each pair (g_1, g_2) of elements $g_{1,2} \in G$ another element $g_3 \in G$, written as $(g_1, g_2) \mapsto g_3 := g_1 \cdot g_2$ such that the following holds true:

1. associativity: $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 \quad \forall g_1, g_2, g_3 \in G$.
2. neutral element: there is a unique element $\mathbf{1} \in G$ such that $g \cdot \mathbf{1} = \mathbf{1} \cdot g = g \quad \forall g \in G$. $\mathbf{1}$ is also called the **unit element**.
3. inverse elements: for each $g \in G$, there is a unique element $h_g \in G$ such that $g \cdot h_g = h_g \cdot g = \mathbf{1}$. One writes $g^{-1} := h_g$.

The number $n = |G|$ of elements of G is called the **order** of the finite group G . If furthermore, one has commutativity: $g_1 \cdot g_2 = g_2 \cdot g_1 \quad \forall g_1, g_2 \in G$, G is called an **abelian group**. If a subset $H \subset G$ of the finite group G , which is closed under the group product, that is $h_3 := h_1 \cdot h_2 \in H \quad \forall h_{1,2} \in H$, then H is called a finite **subgroup** of G .⁷

⁶ Well-known examples of scale-invariant, but not conformally invariant systems include randomly branched polymers [519, 380] or two-dimensional elasticity [621].

⁷ In practise, one often drops the group product and writes concisely $g_1 g_2 := g_1 \cdot g_2$.

Examples: we recall the definition of three elementary finite groups.

1. Consider n distinct objects $1, \dots, n$. The **permutation group** $\mathbb{S}_n$ is the finite group of all permutations $(1, \dots, n) \mapsto (\sigma(1), \dots, \sigma(n))$ of these objects. There are $|\mathbb{S}_n| = n!$ permutations.

The most simple permutations merely exchange two of the n objects and leave the other ones unchanged. Such a permutation is a **transposition**. Those elements of $\mathbb{S}_n$ which can be written as the product of an *even* number of transpositions, form the **alternating group** $\mathbb{A}_n \subset \mathbb{S}_n$, of order $|\mathbb{A}_n| = \frac{1}{2} n!$.

2. Consider n particles arranged on a circle, with angles $\theta_j = \frac{2\pi}{n} j$, where $j = 0, 1, \dots, n-1$. The **cyclic group** $\mathbb{Z}_n$ is the finite group of rigid rotations, where all particles are rotated by an angle $\frac{2\pi}{n} k$ where k is an integer between 0 and $n-1$. There are $|\mathbb{Z}_n| = n$ such rotations. The cyclic group $\mathbb{Z}_n \subset \mathbb{S}_n$ is a finite abelian subgroup of the permutation group.
3. The **dihedral group** $\mathbb{D}_n$, of order $|\mathbb{D}_n| = 2n$, is the symmetry group of regular polygons with $n \geq 3$ sides in the plane. The group product for the n rotations r_i and the n reflections s_i with $i = 0, 1, \dots, n-1$ is given by

$$r_i \cdot r_j = r_{i+j} \quad , \quad r_i \cdot s_j = s_{i+j} \quad , \quad s_i \cdot r_j = s_{i-j} \quad , \quad s_i \cdot s_j = r_{i-j} \quad (\text{L1})$$

and the addition of the indices is *modulo* n . Evidently, $\mathbb{Z}_n \subset \mathbb{D}_n$. For $n \geq 3$, one has the semi-direct product $\mathbb{D}_n \cong \mathbb{Z}_n \circ \mathbb{Z}_2 \subset \mathbb{S}_n$.

We leave it as an exercise to the reader to identify the group products and to check that the group properties are indeed satisfied. **Cayley's theorem** states that each finite group of order n is a subgroup of $\mathbb{S}_n$.

In appendix A, we have given examples for spin systems which have these finite groups as global symmetry groups of their Hamiltonian $\mathcal{H}$. For example, the Ising model has $\mathbb{Z}_2$ as the global symmetry group, where the two elements correspond to $(\sigma_1, \dots, \sigma_{\mathcal{N}}) \mapsto (\sigma_1, \dots, \sigma_{\mathcal{N}})$ and $(\sigma_1, \dots, \sigma_{\mathcal{N}}) \mapsto (-\sigma_1, \dots, -\sigma_{\mathcal{N}})$. The action of these group elements leaves the Ising model Hamiltonian $\mathcal{H} = -\sum_{1 \leq i, j \leq \mathcal{N}} J_{i,j} \sigma_i \sigma_j$ invariant, where $\mathcal{N}$ is the number of lattice sites. The group of global symmetries of the q -states Potts model is $\mathbb{S}_q$ and the symmetry group of the p -states clock model is $\mathbb{D}_p$. The reader should work this out for himself.

Two finite groups G and G' , of equal order $|G| = |G'|$, are **isomorphic** if there is an invertible and structure-preserving map $f : G \rightarrow G'$ such that $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2) \quad \forall g_{1,2} \in G$ and that the inverse map $f^{-1} : G' \rightarrow G$ is structure-preserving as well. One then writes $G \cong G'$. For example, one has the group isomorphism $\mathbb{S}_2 \cong \mathbb{Z}_2$.

From a pair G, G' of finite groups, one can form the **direct product**, denoted by $G \otimes G'$ with the group product $(g_1, g'_1) \cdot (g_2, g'_2) := (g_1 \cdot g_2, g'_1 \cdot g'_2)$. Clearly, both G and G' can be considered as finite subgroups of the finite

group $G \otimes G'$. As an example, note that the two finite groups of the same order $|\mathbb{Z}_4| = |\mathbb{Z}_2 \otimes \mathbb{Z}_2| = 4$ are not isomorphic, $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

The **semi-direct product** of two groups G and H is defined as follows.⁸ Find for each $g \in G$ an **automorphism** $\tau_g : H \rightarrow H$ such that each $h \in H$ is mapped onto a $\tau_g(h) \in H$ in a structure-preserving way, i.e. $\tau_g \tau_{g'} = \tau_{gg'}$. The semi-direct product group is denoted in this book by $G \circ H$ and is given by the group product $(g, h) \cdot (g', h') := (gg', h\tau_g(h'))$. For example, $\mathbb{D}_n \cong \mathbb{Z}_n \circ \mathbb{Z}_2$ if $n \geq 3$, but $\mathbb{D}_2 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

An overview over the finite groups can be obtained as follows. First, fix some $g \in G$ and define a **conjugation map** $I_g : G \rightarrow G$, $x \mapsto gxg^{-1} \quad \forall x \in G$. Next, if $H \subset G$ is a subgroup of G , H is called an **invariant subgroup** if $I_g H = H \quad \forall g \in G$. Finally, a group G is called **simple** if it has *no* invariant subgroups. Simple finite groups may be considered as the building blocks from which all finite groups can be constructed (**Jordan-Hölder theorem**). For example, direct or semi-direct products of simple groups are not simple. It is thought that there is a classification of all simple finite groups:⁹ *every simple finite group G is either one of the 26 so-called ‘sporadic groups’ or else belongs to one of three infinite series, namely (i) $\mathbb{Z}_p$ with p prime, (ii) $\mathbb{A}_n$ with $n \geq 5$ or (iii) one of 16 families which arise from ‘discretising the simple Lie groups’.* See [628, 735] and references therein for further information.

L.2 Continuous Groups and Lie Groups

In many physical situations, one encounters groups whose elements depend continuously on one or several parameters such that the formal group properties defined above remain valid. A simple example is given by the continuous group $SO(2)$, whose elements are 2×2 matrices which describe a planar rotation by an angle $\theta \in [0, 2\pi)$

$$g_\theta \rightarrow r(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (\text{L2})$$

In this example, group multiplication corresponds to the matrix multiplication of two rotation matrices, parametrised by the angles $\theta_{1,2}$ such that

$$g_{\theta_1} \cdot g_{\theta_2} \rightarrow r(\theta_1)r(\theta_2) = r(\theta_1 + \theta_2) \rightarrow g_{\theta_1 + \theta_2}.$$

Of course, the addition of the angles has to be done *modulo* 2π .

A **Lie group** is a continuous group where furthermore the group elements depend analytically on the group parameters. For example, the continuous group $SO(2)$ is a Lie group. The number $n = \dim G$ of independent

⁸ In distinction to the direct product, the order of the factors is essential for the semi-direct product.

⁹ Informally referred to as the **enormous theorem** because of the length of the proof required. A first version of the proof was thought to be complete in 1981, after about 120 years of effort.

group parameters is called the **dimension** of the Lie group G . For example, $\dim SO(2) = 1$. The concepts of **Lie subgroups**, **abelian Lie groups**, **simple Lie groups** and of **Lie isomorphism** between two Lie groups of equal dimensions are defined in a way analogous to finite groups. For example, since the group product in $SO(2)$ is of the form $g_\theta \cdot g_{\theta'} = g_{\theta+\theta'}$, there is a Lie group isomorphism $SO(2) \cong U(1)$, where $U(1)$ is the abelian Lie group where the elements are real numbers $\theta \in [0, 2\pi)$ and the group product corresponds to addition *modulo* 2π .

The XY model defined in appendix A has the Lie group $SO(2)$ as a global symmetry group.

Lie groups also arise as **Lie transformation groups** when acting on some external, i.e. geometric, object. If G is a Lie group and M a geometric object (technically, a differentiable manifold), a *Lie transformation group* is a map $\varphi : G \times M \rightarrow M$ with $\varphi(g, x) = gx$ such that (i) $(g_1 \cdot g_2)x = g_1(g_2x) \quad \forall g_{1,2} \in G$ and $\forall x \in M$ and (ii) $1x = x \quad \forall x \in M$. φ is called the **group action** on x .

Example: the set of projective conformal (or Möbius) transformations of the complex variable $z \in \mathbb{C}$, parametrised by the real constants $\alpha, \beta, \gamma, \delta$

$$z \mapsto z' = f(z) := \frac{\alpha z + \beta}{\gamma z + \delta} ; \quad \alpha\delta - \beta\gamma = 1 \quad (\text{L3})$$

is a non-abelian Lie group. Here, the group product $g_1 \cdot g_2$ corresponds to function composition $f_1(f_2(z))$. This Lie group is isomorphic to the Lie group $Sl(2, \mathbb{R})$ generated by the 2×2 matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha\delta - \beta\gamma = 1$ and where the group product is given by matrix multiplication. Clearly, $\dim Sl(2, \mathbb{R}) = 3$.

Direct and semi-direct products of Lie groups are defined analogously with respect to finite groups. We illustrate this with several examples involving Lie transformation groups which are met frequently in this book.

1. The **translation group** $T(d)$ is defined by the group action on a vector $\mathbf{r} \in \mathbb{R}^d$:

$$\mathbf{r} \mapsto \mathbf{r}' := \mathbf{r} + \mathbf{a}, \quad (\text{L4})$$

where $\mathbf{a} \in \mathbb{R}^d$. One has $T(d) = \underbrace{T(1) \otimes \cdots \otimes T(1)}_{d \text{ times}}$. This is an abelian Lie group and $\dim T(d) = d$.

2. The **Euclidean group** $Eucl(d)$ is defined by its group action:

$$\mathbf{r} \mapsto \mathbf{r}' = \mathcal{D}\mathbf{r} + \mathbf{a} \quad (\text{L5})$$

with the group parameters $\mathbf{a} \in \mathbb{R}^d$ and $\mathcal{D} \in SO(d)$ is the matrix for a spatial rotation in d dimensions. It can be obtained as the semi-direct product $Eucl(d) = T(d) \circledcirc SO(d)$. Clearly, $\dim Eucl(d) = \dim T(d) + \dim SO(d) = d + \frac{1}{2}d(d-1)$.

3. The **Galilei group** is defined by its group action

$$(t, \mathbf{r}) \mapsto (t', \mathbf{r}') = (t + \beta, \mathcal{D}\mathbf{r} + \mathbf{a} + \mathbf{v}t) \quad (\text{L6})$$

with the additional group parameter $\mathbf{v} \in \mathbb{R}^d$. It is a semi-direct product $\text{Gal}^{(0)}(d) \cong (T(d) \otimes T(d)) \otimes (SO(d) \otimes T(1))$. This is reflected in the group action in that the transformation of the space coordinates is time-dependent, whereas no space-dependence enters into the transformation of the time coordinate.¹⁰ One has $\dim \text{Gal}^{(0)}(d) = \frac{1}{2}d(d+3) + 1$.

4. The **Schrödinger group** defined in Chap. 4 has the group action

$$(t, \mathbf{r}) \mapsto (t', \mathbf{r}') = \left(\frac{\alpha t + \beta}{\gamma t + \delta}, \frac{\mathcal{D}\mathbf{r} + \mathbf{a} + \mathbf{v}t}{\gamma t + \delta} \right) ; \quad \alpha\delta - \beta\gamma = 1. \quad (\text{L7})$$

Extending the previous example, one has the semi-direct product $\text{Sch}^{(0)}(d) \cong (T(d) \otimes T(d)) \otimes (SO(d) \otimes \text{Sl}(2, \mathbb{R}))$ and $\dim \text{Sch}^{(0)}(d) = \frac{1}{2}d(d+3) + 3$.¹¹

Since the parameter space of a Lie group can be considered as a geometric object (a differentiable manifold), any Lie group is a Lie transformation group acting on itself.

L.3 From Lie Groups to Lie Algebras and Back

Since the explicit form of Lie groups becomes rapidly quite complicated (recall for example the explicit representation of rotation matrices in $d > 2$ dimensions in terms of Euler angles) it is often helpful to concentrate on *infinitesimal* transformations in the vicinity of the group identity $\mathbf{1}$.

To see how this comes about, consider a Lie transformation group G . Any point $p \in M$ may be expressed in local coordinate systems, e.g. the system S with coordinates $\mathbf{r}(p)$ or the system S' with coordinates $\mathbf{r}'(p)$. Furthermore, consider a function $F : M \rightarrow \mathbb{R}$, which in local coordinates becomes

$$F(p) = F^{(S)}(\mathbf{r}(p)) = F^{(S')}(\mathbf{r}'(p)).$$

The relation between the coordinate systems is given by an element $g \in G$: $\mathbf{r}'(p) = \varphi(\alpha, \mathbf{r}(p))$, where α stands for the group parameters which characterise $g \in G$. One now asks for the relationship between $F^{(S')}$ in terms of $F^{(S)}$. Formally, $F^{(S')}(\mathbf{r}'(p)) = F^{(S)}(\varphi(\alpha^{-1}, \mathbf{r}'(p)))$, where now α^{-1} parametrises

¹⁰ The subscript (0) indicates that the definitions considered in this section refer to the case of ‘vanishing mass’. Throughout this book, we reserve the names of Galilei and Schrödinger groups and algebras to the case with a non-vanishing mass, to be defined below on p. 417 via central extensions.

¹¹ In expanding/contracting space-times, the dynamical group is sometimes referred to as **Newton-Hooke group**. It is isomorphic to the Schrödinger group [545].

the inverse $g^{-1} \in G$. This result simplifies considerably if one considers infinitesimal transformations in the vicinity of the group identity $\mathbf{1}$. Then¹² $\delta\alpha^{-1} \simeq -\delta\alpha$ and, to leading order $\mathbf{r}(p) \simeq \mathbf{r}'(p) - \delta\alpha^\mu \frac{\partial \varphi(\beta, \mathbf{r}'(p))}{\partial \beta^\mu} \Big|_{\beta=0}$. Finally, $F^{(S')}(\mathbf{r}'(p)) - F^{(S)}(\mathbf{r}(p)) = \delta\alpha^\mu X_\mu(\mathbf{r}') F^{(S)}(\mathbf{r}'(p))$ with the **infinitesimal generators**

$$X_\mu(\mathbf{r}') := - \frac{\partial \varphi^j(\beta, \mathbf{r}')}{\partial \beta^\mu} \Big|_{\beta=0} \frac{\partial}{\partial r'^j} = \sum_{j=1}^m \xi_j(\mathbf{r}') \frac{\partial}{\partial r'^j}. \quad (\text{L8})$$

This construction can be immediately translated to Lie groups, acting on themselves as Lie transformation groups. It is particularly straightforward for Lie groups realised as matrix groups, since it is enough to calculate the derivatives with respect to the group parameters.

Examples: the abstract procedure becomes more clear by explicit calculations:

1. Consider the matrix group $SO(2)$, realised by (L2). The infinitesimal generator is

$$X = \frac{\partial r(\theta)}{\partial \theta} \Big|_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{L9})$$

2. Consider the Lie transformation group $Sl(2, \mathbb{R})$, realised by (L3). The three independent transformations may be taken as follows: $\varphi_{-1}(\beta, z) = z + \beta$, $\varphi_0(\epsilon, z) = (1 + \epsilon)z$ and $\varphi_1(\gamma, z) = z/(1 + \gamma z)$, respectively. Hence the three infinitesimal generators become

$$\begin{aligned} X_{-1} &= - \frac{\partial \varphi_{-1}(\beta, z)}{\partial \beta} \Big|_{\beta=0} \partial_z = -\partial_z \\ X_0 &= - \frac{\partial \varphi_0(\epsilon, z)}{\partial \epsilon} \Big|_{\epsilon=0} \partial_z = -z \partial_z \\ X_1 &= - \frac{\partial \varphi_1(\gamma, z)}{\partial \gamma} \Big|_{\gamma=0} \partial_z = -z^2 \partial_z \end{aligned} \quad (\text{L10})$$

The infinitesimal generators satisfy an algebraic structure, to which we turn now. Let $a, b \in G$. The Lie group element $c := (ab)(ba)^{-1}$ measures the degree of non-commutativity in G . Close to the identity, one expands

$$\begin{aligned} a &\simeq \mathbf{1} + \delta\alpha^\mu X_\mu + \frac{1}{2} \delta\alpha^\mu X_\mu \delta\alpha^\nu X_\nu \\ b &\simeq \mathbf{1} + \delta\beta^\mu X_\mu + \frac{1}{2} \delta\beta^\mu X_\mu \delta\beta^\nu X_\nu \end{aligned}$$

such that

¹² We frequently use Einstein's summation convention: sum over repeated indices.

$$(ab)(ba)^{-1} \simeq \mathbf{1} + \delta\alpha^\mu \delta\beta^\nu (X_\mu X_\nu - X_\nu X_\mu) =: \mathbf{1} + \delta\alpha^\mu \delta\beta^\nu [X_\mu, X_\nu] \quad (\text{L11})$$

which defines the **commutator** of the two generators X_μ, X_ν . Since $c \in G$, the commutator can be rewritten in terms of the generators X_μ :

$$[X_\mu, X_\nu] = C_{\mu\nu}^\lambda X_\lambda. \quad (\text{L12})$$

Hence the generators X_μ form a **Lie algebra** $\mathfrak{g}$ associated to the Lie group G the structure of which is given by the **structure constants** $C_{\mu\nu}^\lambda$. The number of independent generators in $\mathfrak{g}$ is the same as the number of independent parameters in the Lie group G , hence $\dim \mathfrak{g} = \dim G$.

Abstractly: a **Lie algebra** $\mathfrak{g}$ is a linear vector space, together with a bilinear Lie bracket (commutator) $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\forall X, Y, Z \in \mathfrak{g}$

1. antisymmetry: $[X, Y] = -[Y, X]$.
2. Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

The most direct characterisation of Lie algebras in terms of their structure constants depends on the choice of basis for the generators X_μ . Considerable mathematical effort has been spent on finding basis-independent characteristic properties of Lie algebras. A **Lie sub-algebra** $\mathfrak{h} \subset \mathfrak{g}$ is defined by $[h_1, h_2] \in \mathfrak{h} \forall h_{1,2} \in \mathfrak{h}$ and one writes formally $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ of equal dimension are **isomorphic** if there is a vector space isomorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ which is structure-preserving, that is $f([g_1, g_2]) = [f(g_1), f(g_2)] \forall g_{1,2} \in \mathfrak{g}$. A **direct sum** of two Lie algebras $\mathfrak{g}, \mathfrak{g}'$, written as $\mathfrak{g} \oplus \mathfrak{g}'$, is given by the direct sum of the vector spaces such that $[g, g'] = 0 \forall g \in \mathfrak{g}$ and $\forall g' \in \mathfrak{g}'$. A **semi-direct sum** $\mathfrak{g}' \ltimes \mathfrak{g}''$ of two Lie sub-algebras $\mathfrak{g}', \mathfrak{g}''$ is the sum of the vector spaces such that $[\mathfrak{g}', \mathfrak{g}''] \subset \mathfrak{g}''$. Direct and semi-direct products of Lie groups lead to direct and semi-direct sums, respectively, of their associated Lie algebras.¹³ In Table L1 we list the definitions of several Lie groups and Lie algebras, in the (non-standard) notation which will be used in this book.

Example: the generators (L10) of $Sl(2, \mathbb{R})$ obtained above satisfy the commutator relations $[X_n, X_m] = (n - m)X_{n+m}$, with $n, m \in \{\pm 1, 0\}$, and form the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. One has the isomorphism $\mathfrak{conf}(2) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

Example: the Euclidean Lie algebra $\mathfrak{eucl}(3)$ has the sub-algebras $\mathfrak{so}(3)$, spanned by the generators J_μ (rotations), and $\mathfrak{t}(3)$, spanned by the generators Y_μ (translations), with $\mu = 1, 2, 3$. The non-vanishing commutators are

$$[J_\mu, J_\nu] = i\varepsilon_{\mu\nu}^\lambda J_\lambda, \quad [Y_\mu, J_\nu] = i\varepsilon_{\mu\nu}^\lambda Y_\lambda \quad (\text{L13})$$

where $\varepsilon_{\mu\nu}^\lambda$ is the totally antisymmetric tensor and $\varepsilon_{12}^3 = 1$. Hence $\mathfrak{t}(3)$ is an invariant sub-algebra, $[\mathfrak{so}(3), \mathfrak{t}(3)] \subset \mathfrak{t}(3)$ and one has the semi-direct sum $\mathfrak{eucl}(3) \cong \mathfrak{so}(3) \ltimes \mathfrak{t}(3)$.

¹³ We follow the habit of the mathematical literature and denote the Lie algebra $\mathfrak{g}$ of the Lie group G by the corresponding german letter(s).

name	group	algebra	dimension
translational	$T(d)$	$\mathfrak{t}(d) \cong \bigoplus_{i=1}^d \mathfrak{t}(1)$	d
Euclidean	$Eucl(d)$	$\mathfrak{eucl}(d) \cong \mathfrak{so}(d) \ltimes \mathfrak{t}(d)$	$\frac{1}{2}d(d+1)$
Galilean	$Gal^{(0)}(d)$	$\mathfrak{gal}^{(0)}(d) \cong \left(\mathfrak{t}(1) \oplus \mathfrak{so}(d) \right) \ltimes \mathfrak{t}(2d)$ $\cong \mathfrak{t}(1) \ltimes \left(\mathfrak{so}(d) \ltimes \mathfrak{t}(2d) \right)$	$\frac{1}{2}d(d+3) + 1$
Schrödingerian	$Sch^{(0)}(d)$	$\mathfrak{sch}^{(0)}(d) \cong \left(\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(d) \right) \ltimes \mathfrak{t}(2d)$ $\cong \mathfrak{sl}(2, \mathbb{R}) \ltimes \left(\mathfrak{so}(d) \ltimes \mathfrak{t}(2d) \right)$	$\frac{1}{2}d(d+3) + 3$
conformal	$Alt(d)$	$\mathfrak{alt}(d) \cong \left(\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(d) \right) \ltimes \mathfrak{t}(3d)$	$\frac{1}{2}d(d+5) + 3$
Galilean/altern		$\cong \mathfrak{sl}(2, \mathbb{R}) \ltimes \left(\mathfrak{so}(d) \ltimes \mathfrak{t}(3d) \right)$	
conformal	$Conf(d)$	$\mathfrak{conf}(d) \cong \mathfrak{so}(d+2)$	$\frac{1}{2}(d+1)(d+2)$

^(a) The name of $Alt(d)$ was taken from the german for ‘ageing’ [328, 357]. The more common name is **conformal Galilean group** [322, 533]. Its Lie algebra is the **conformal Galilean algebra**, denoted by $CGA(d) \equiv \mathfrak{alt}(d)$.

Table L1. Some Lie groups G and their (complexified) Lie algebras $\mathfrak{g}$ of space-time transformations, with some of their (semi-)direct decompositions.

Theorem: (Lie) *The first-order differential equation $\frac{d\mathbf{r}'}{d\tau} = \boldsymbol{\xi}(\mathbf{r}')$ with $\mathbf{r}'(0) = \mathbf{r}$ is equivalent to a one-parameter Lie transformation group. The infinitesimal parameter ϵ is related to $\tau = \tau(\epsilon)$ by $\tau(\epsilon) = \int_0^\epsilon d\epsilon' \Gamma(\epsilon')$, with $\Gamma(\epsilon) = \left. \frac{\partial \phi(a, b)}{\partial b} \right|_{(a=\epsilon^{-1}, b=\epsilon)}$ where $\phi(a, b)$ stands for the Lie group transformation law parametrised by a and b , ϵ^{-1} is the inverse parameter to ϵ and $\Gamma(0) = 1$. In terms of ϵ , the one-parameter Lie group is given by the solutions of the initial-value problem*

$$\frac{d\mathbf{r}'(\epsilon)}{d\epsilon} = \Gamma(\epsilon)\boldsymbol{\xi}(\mathbf{r}') \quad (\text{L14})$$

and $\boldsymbol{\xi}(\mathbf{r})$ gives the infinitesimal transformation $\mathbf{r}' = \mathbf{r} + \epsilon\boldsymbol{\xi}(\mathbf{r})$. Generalising this to n coupled partial differential equations, Lie showed that these equations can be decoupled into a product of two matrices, where the first one only depends on the group parameters and the second one only on the initial values.

For illustration, we collect in Table L2 the definitions of the classical examples, namely the general and special linear, unitary, orthogonal and symplectic Lie groups and their associated Lie algebras. They are defined in terms of matrix groups/algebras such that the Lie groups preserve the metric $\mathbf{x}^\dagger M \mathbf{y}$ for the unitary and symplectic cases and $\mathbf{x}^T M \mathbf{y}$ for the orthogonal case, respectively. Similarly, the matrices $A \in G$ and $X \in \mathfrak{g}$ satisfy the conditions $A^\dagger M A = M$ and $X^\dagger M = -M X$, respectively (for the orthogonal case, replace the hermitian conjugate $A^\dagger$ by the transpose A^T). The matrix M is listed in Table L2, where $\mathbf{1}$ (or $\mathbf{1}_n$) is the $n \times n$ unit matrix and

label	name	Lie group		Lie algebra		M dim	compact
	general	$Gl(n, \mathbb{C})$		$\mathfrak{gl}(n, \mathbb{C})$		$-2n^2$	no
		$Gl(n, \mathbb{R})$		$\mathfrak{gl}(n, \mathbb{R})$		$-n^2$	no
	special	$Sl(n, \mathbb{C})$	$\det A = 1$	$\mathfrak{sl}(n, \mathbb{C})$	$\text{tr } X = 0$	$-2(n^2 - 1)$	no
		$Sl(n, \mathbb{R})$	$\det A = 1$	$\mathfrak{sl}(n, \mathbb{R})$	$\text{tr } X = 0$	$-n^2 - 1$	no
A_{n-1}	unitary	$U(n)$		$\mathfrak{u}(n)$		n^2	yes
		$U(p, n-p)$		$\mathfrak{u}(p, n-p)$		$L n^2$	no
		$SU(n)$	$\det A = 1$	$\mathfrak{su}(n)$	$\text{tr } X = 0$	$n^2 - 1$	yes
		$SU(p, n-p)$	$\det A = 1$	$\mathfrak{su}(p, n-p)$	$\text{tr } X = 0$	$L n^2 - 1$	no
B_n	ortho- gonal	$O(n)$		$\mathfrak{o}(n)$		$n(n-1)/2$	yes
		$O(p, n-p)$		$\mathfrak{o}(p, n-p)$		$L n(n-1)/2$	no
		$SO(2n+1)$	$\det A = 1$	$\mathfrak{so}(2n+1)$	$\text{tr } X = 0$	$n(2n+1)$	yes
		$SO(p, q)$	$\det A = 1$	$\mathfrak{so}(p, q)$	$\text{tr } X = 0$	$L n(2n+1)$	no
D_n		$SO(2n)$	$\det A = 1$	$\mathfrak{so}(2n)$	$\text{tr } X = 0$	$n(2n-1)$	yes
		$SO(p, q)$	$\det A = 1$	$\mathfrak{so}(p, q)$	$\text{tr } X = 0$	$L n(2n-1)$	no
C_n	sym- plectic	$Sp(2n, \mathbb{C})$		$\mathfrak{sp}(2n, \mathbb{C})$		$2n(2n+1)$	no
		$Sp(2n, \mathbb{R})$		$\mathfrak{sp}(2n, \mathbb{R})$		$J n(2n+1)$	no
		$SSp(2n, \mathbb{C})$	$\det A = 1$	$\mathfrak{ssp}(2n, \mathbb{C})$	$\text{tr } X = 0$	$J 2n(2n+1) - 1$	no
		$SSp(2n, \mathbb{R})$	$\det A = 1$	$\mathfrak{ssp}(2n, \mathbb{R})$	$\text{tr } X = 0$	$J n(2n+1)$	no

Table L2. Classical finite-dimensional Lie groups and Lie algebras, identified either by their name or their label from the Cartan classification of simple Lie algebras, see text for details. For the algebras B_n, D_n , one has $p+q = 2n+1, 2n$, respectively.

$$L = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \quad (\text{L15})$$

define the Lorentzian (either with $q = n-p$ or as stated in Table L2) and symplectic metrics, where we also indicate the size of the subspaces. It is also stated whether the group manifold G is compact (that is, closed and bounded; like a sphere and in contrast to a hyperboloid). The commutation relations of the classical Lie algebras may be written as $[X_{ij}, X_{jk}] = X_{ik}$, for $i \leq j \leq k$ with the following additional conditions on the generators [278]: (i) unitary $X_{ji} = X_{ij}^\dagger$, (ii) orthogonal $X_{ji} = -X_{ij}$ and (iii) symplectic: the indices have the values $i = -n, \dots, -1, 1, \dots, n$. Relabel the indices $X_{i'j'} \mapsto X_{ij}$ such that $i = 2i'$ if $i' > 0$ and $i = -2i' - 1$ if $i' < 0$. Then $X_{-j', -i'} = -\sigma(i')\sigma(j')X_{i'j'}$, where $\sigma(i')$ is the sign of i' .

Having seen how to find the Lie algebra from a given Lie group, we now discuss how the opposite task can be performed. Indeed, if $X(\mathbf{r})$ is the generator of a one-parameter sub-algebra of a Lie algebra $\mathfrak{g}$, the corresponding one-parameter Lie transformation group is given by the **Lie series**

$$\mathbf{r}' = \exp(\epsilon X(\mathbf{r}))\mathbf{r} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} X^k(\mathbf{r})\mathbf{r}. \quad (\text{L16})$$

Similarly, for any analytic function, one has a Lie series in the form $f(\mathbf{r}') = e^{\epsilon X} f(\mathbf{r})$.

Examples: we reconsider the two Lie algebras discussed previously.

1. For matrix groups, it may be easiest to carry out the exponentiation (L16) directly. For the Lie algebra $\mathfrak{so}(2)$, one has the generator $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Using the property $X^2 = -\mathbf{1}$, one finds

$$\begin{aligned} \exp(\epsilon X) &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} (-1)^{2k} \frac{\epsilon^{2k}}{(2k)!} & \sum_{k=0}^{\infty} (-1)^{2k+1} \frac{\epsilon^{2k+1}}{(2k+1)!} \\ \sum_{k=0}^{\infty} (-1)^{2k} \frac{\epsilon^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty} (-1)^{2k} \frac{\epsilon^{2k}}{(2k)!} \end{pmatrix} = \begin{pmatrix} \cos \epsilon & -\sin \epsilon \\ \sin \epsilon & \cos \epsilon \end{pmatrix}. \end{aligned}$$

As expected, the matrix (L2) is recovered.

2. Cartan introduced the following infinite-dimensional extension of $\mathfrak{sl}(2, \mathbb{R})$, given by the generators

$$X_n = -z^{n+1} \partial_z - \Delta(n+1) z^n \quad (\text{L17})$$

with $n \in \mathbb{Z}$ and where Δ is a real constant. The commutation relations are $[X_n, X_m] = (n-m)X_{n+m}$ for $n, m \in \mathbb{Z}$. Here, it may be better to define formally the finite transformation $f(\epsilon, z) = e^{\epsilon X} f(0, z)$, which satisfies the differential equation

$$\frac{\partial f(\epsilon, z)}{\partial \epsilon} = X f(\epsilon, z). \quad (\text{L18})$$

If one is interested in the transformation of the coordinate z , one will complete eq. (L18) by the initial condition $f(0, z) = z$. Since the term parametrised by Δ describes the transformation of scaling operators (see below), we set $\Delta = 0$ and then have the initial value problem

$$\left(\frac{\partial}{\partial \epsilon} + z^{n+1} \frac{\partial}{\partial z} \right) f(\epsilon, z) = 0, \quad f(0, z) = z. \quad (\text{L19})$$

This may be solved by the method of characteristics [421] or more simply by the lemma given in the solution of exercise 1 in Volume 1. For $n \neq 0$, we have $f(\epsilon, z) = F(\epsilon + (nz^n)^{-1})$ where the function $F = F(\alpha) = (n\alpha)^{-1/n}$ is found from the initial condition, see again exercise 1 in Volume 1. Hence

$$z' = \frac{z}{(1 + \epsilon n z^n)^{1/n}}, \quad n \neq 0 \quad (\text{L20})$$

and it is left to the reader to add the case $n = 0$. For $n = \pm 1$ we recover the projective conformal transformations discussed earlier.

One may extend this and discuss the transformation of a scaling operator $\phi(z)$. Then we have the initial value problem

$$\left(\frac{\partial}{\partial \epsilon} + z^{n+1} \frac{\partial}{\partial z} + \Delta(n+1)z^n \right) f(\epsilon, z) = 0 \quad , \quad f(0, z) = \phi(z). \quad (\text{L21})$$

Solutions to this type of equation are either looked up in [421] or may be found directly from the following

Lemma: [421] *Consider the solutions $f = f(x, y)$ of the partial differential equation*

$$(a(x, y)\partial_x + b(x, y)\partial_y + c(x, y))f(x, y) = d(x, y). \quad (\text{L22})$$

Take $g = g(x, y)$ to be a solution of the associated homogeneous equation $(a\partial_x + b\partial_y)g = 0$. Then the solutions to (L22) are found by the change of variables $F(g, y) := f(x, y)$ where F satisfies the ordinary differential equation

$$b \frac{\partial F}{\partial y} + cF = d \quad (\text{L23})$$

and where $x = x(g, y)$ has been inserted throughout.

We apply this to (L21), again for $n \neq 0$. A solution to the homogeneous equation (L19) was $y = \epsilon + (nz^n)^{-1}$. Using y as a new variable, we set $f(\epsilon, z) = F(y, z)$ and find from the lemma the ordinary differential equation $(z\partial_z + \Delta(n+1))F(y, z) = 0$. Hence $F(y, z) = z^{-\Delta(n+1)}F_0(y) = z^{-\Delta(n+1)}F_0(\epsilon + (nz^n)^{-1}) = f(\epsilon, z)$. The remaining function F_0 is found from the initial condition and reads $F_0(\alpha) = (n\alpha)^{-\Delta(n+1)/n}\phi((n\alpha)^{-1/n})$ such that finally

$$\begin{aligned} \phi'(z') &= f(\epsilon, z) = (1 + \epsilon nz^n)^{-\Delta(n+1)/n} \phi \left(\frac{z}{(1 + \epsilon nz^n)^{1/n}} \right) \\ &= \left(\frac{d\beta(z)}{dz} \right)^{-\Delta} \phi(\beta(z)) \end{aligned} \quad (\text{L24})$$

where we introduced the transformation $\beta(z) = z(1 + \epsilon nz^n)^{-1/n}$. The last form also includes the case $n = 0$, leading to $\beta(z) = \lambda z$ (we leave this to the reader).

For a compact Lie group G , it is known that every $g \in G$ lies on a one-parameter subgroup of G and can be obtained by exponentiating an element of the Lie algebra $\mathfrak{g}$. Two Lie groups with isomorphic Lie algebras are either globally isomorphic or else are images of a third Lie group, called the **universal covering group** with a simply connected group manifold. For example, recall the well-known fact that $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, but $O(3) \cong SO(3) \otimes \mathbb{Z}_2$ is not connected, since there are orthogonal transformations with $\det A = \pm 1$. The universal covering group of $SO(3)$ is $SU(2)$. We give two more examples of Lie algebras (and leave the verifications again to you, dear reader).

1. Consider the real vector space $\mathbb{R}^3$ and define the Lie bracket $[\mathbf{x}, \mathbf{y}] := \mathbf{x} \wedge \mathbf{y}$ through the vector product. The Lie algebra is isomorphic to $\mathfrak{so}(3)$.
2. Consider continuously differentiable functions $f = f(p, q)$ of two variables. The space $C^1(\mathbb{R}^2)$ of these functions becomes a Lie algebra, where the Lie bracket is the **Poisson bracket**

$$\{f, g\} := \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}. \quad (\text{L25})$$

The compatibility with the (commutative) ordinary product of functions fg is expressed by the *Leibniz identity* $\{fg, h\} = f\{g, h\} + \{f, h\}g$. In this way, one obtains a **Poisson algebra**. Practical investigations are often more easy and rapid to carry out in Poisson algebras.

Example: an isomorphism of $\mathfrak{sl}(2, \mathbb{R})$ onto the Poisson algebra of polynomials of order two is given by $X_{-1} \leftrightarrow \frac{1}{2}q^2$, $X_0 \leftrightarrow -\frac{1}{2}qp$ and $X_1 \leftrightarrow \frac{1}{2}p^2$.

L.4 Matrix Representations and the Cartan-Weyl Basis

Definition: A **matrix representation of a Lie group** G is a structure preserving map $A : G \rightarrow \text{Gl}(V)$ with $g \mapsto A(g)$ and where V is a vector space such that $A(g)\mathbf{r}$ is linear $\forall \mathbf{r} \in V$ and $A(\mathbf{1}) = \mathbf{1}$ and $A(g_1)A(g_2) = A(g_1g_2) \forall g_{1,2} \in G$. Similarly, a **matrix representation of a Lie algebra** $\mathfrak{g}$ is a structure-preserving linear map $X : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. The representation is **unitary** if $A(g)$ resp. $X(g)$ are unitary matrices. A representation is **irreducible** if there are no non-trivial subspaces of V . It is **completely reducible** if $V = V_1 \oplus V_2 \oplus \dots \oplus V_m$ such that each subspace V_i has an irreducible representation. Then the matrix $A(g) = A_1(g) \oplus A_2(g) \oplus \dots \oplus A_m(g)$ has a block diagonal form (and similarly for $X(g)$).

One of the motives for being interested in unitary representations comes from the following

Theorem: Every unitary representation is completely reducible.

For example, one may consider the matrices in (L2) as a unitary (more precisely orthogonal) irreducible representation of $SO(2)$. An important representation for Lie algebras is the **adjoint representation** which is defined by fixing a generator $X \in \mathfrak{g}$. Then one has a map $\text{ad } X : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$Y \mapsto (\text{ad } X)(Y) := [X, Y]. \quad (\text{L26})$$

If the set $\langle X_i \rangle_{i=1, \dots, n}$ forms a basis for the Lie algebra, one has $(\text{ad } X_i)(X_j) = \sum_{k=1}^n C_{ij}^k X_k$ such that the matrix associated with $\text{ad } X_i$ is $(M_i)_{jk} = C_{ik}^j$.

Example: For $\mathfrak{sl}(2, \mathbb{R})$, we had the generators $X_n = -z^{n+1}\partial_z$ with $[X_n, X_m] = (n-m)X_{n+m}$ and $n, m = \pm 1, 0$. From the commutators, $\text{ad } X_n$ is rapidly worked out. Using a basis for the vector space such that $X_{-1} \rightarrow (1, 0, 0)^T$, $X_0 \rightarrow (0, 1, 0)^T$ and $X_1 \rightarrow (0, 0, 1)^T$, the adjoint representation is

$$\text{ad } X_{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{ad } X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{ad } X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (\text{L27})$$

An important characterisation of Lie algebras may be read from the adjoint representation. *If ad is irreducible, $\mathfrak{g}$ is simple. If ad is fully reducible, $\mathfrak{g}$ is semi-simple.* For example, the adjoint representation (L27) of $\mathfrak{sl}(2, \mathbb{R})$ is irreducible, hence $\mathfrak{sl}(2, \mathbb{R})$ is simple. The **Killing form** of a Lie algebra is the symmetric bilinear form

$$K(X, Y) := \text{tr} (\text{ad } X \text{ ad } Y). \quad (\text{L28})$$

Its properties are independent of the choice of the basis. For a given basis of $\mathfrak{g}$, one has the **Killing metric**

$$g_{ij} := K(X_i, X_j) = \sum_{r,s=1}^n C_{is}^r C_{jr}^s. \quad (\text{L29})$$

Theorem: (Cartan) (i) A Lie algebra is semi-simple if and only if the Killing metric is non-singular, i.e. $\det g \neq 0$. (ii) A Lie algebra is compact if and only if the Killing metric tensor is negative definite.

Example: for $\mathfrak{sl}(2, \mathbb{R})$ in the chosen basis (L27), the Killing metric tensor is $g = \begin{pmatrix} 0 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 0 \end{pmatrix}$. Since $\det g = -32$, the algebra is semi-simple. Since the eigenvalues of g are $2, \pm 4$, the algebra is not compact. However, since one eigenvalue of g is negative, there is a one-dimensional sub-algebra which can be exponentiated to the maximal compact subgroup.

Semi-simple algebras are found as direct sums of simple algebras. More generally, one has the

Theorem: (Levi decomposition) Any Lie algebra can be decomposed as $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$, where $\mathfrak{s}$ is a semi-simple sub-algebra and the **radical** $\mathfrak{r}$ is an invariant sub-algebra such that $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{r}$ and the derived radical $\mathfrak{r}^{(1)} := [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}$. $\mathfrak{r}$ necessarily contains a **nilpotent** sub-algebra $\mathfrak{n}$.¹⁴

For example, the abelian algebra $\mathfrak{t}(d)$ is the radical of $\mathfrak{euct}(d)$. The Killing metric vanishes on $\mathfrak{n}$, but is non-singular otherwise.

For semi-simple Lie algebras, the Killing metric tensor has an inverse $g^{ij} g_{jk} = \delta_k^i$. The quadratic **Casimir operator** is

$$C := \sum_{i,j=1}^n g^{ij} X_i X_j \quad (\text{L30})$$

and commutes with the entire Lie algebra $\mathfrak{g}$. Recall the

¹⁴ A Lie algebra $\mathfrak{g}$ is nilpotent if the sequence of derived algebras $\mathfrak{g}^{(n+1)} := [\mathfrak{g}, \mathfrak{g}^{(n)}]$, with $\mathfrak{g}^{(0)} = \mathfrak{g}$, terminates in zero for some finite n .

Theorem: (Schur) *If $A(g)$ is an irreducible representation and if $[C, A(g)] = 0$ for all $g \in \mathfrak{g}$, then $C = c\mathbf{1}$ with $c \in \mathbb{C}$.*

Hence the values of the Casimir operator(s) can be used to characterise irreducible representations.

Example: For the adjoint representation of $\mathfrak{sl}(2, \mathbb{R})$ in the basis (L27), the inverse Killing metric tensor is $g^{-1} = \frac{1}{4} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Hence the quadratic Casimir operator (L30) reads

$$C = \frac{1}{2}X_0^2 - \frac{1}{4}(X_1X_{-1} + X_{-1}X_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1}. \quad (\text{L31})$$

Example: consider the Galilean algebra $\mathfrak{gal}^{(0)}(3)$. Using the notation of the example on p. 408, it is spanned by three vectors $\mathbf{J}$ (angular momentum), $\mathbf{v}$ (Galilean boost) and $\mathbf{p}$ (space translation) and a scalar generator T . We write $\mathbf{J}_i = J_i \mathbf{e}_i$ etc., with the unit vectors $\mathbf{e}_i$. The non-vanishing commutators are $\mathbf{J}_i \wedge \mathbf{J}_j$, $\mathbf{J}_i \wedge \mathbf{v}_j$, $\mathbf{J}_i \wedge \mathbf{p}_j$ and $[T, \mathbf{v}] = \mathbf{p}$ with $i, j = 1, 2, 3$. A general element may be parametrised by $\boldsymbol{\theta} \cdot \mathbf{J} + \mathbf{a} \cdot \mathbf{v} + tT + \mathbf{b} \cdot \mathbf{p}$. Following the same order, the adjoint representation has the following block structure

$$\text{ad} \rightarrow \begin{pmatrix} -\boldsymbol{\theta} \cdot \boldsymbol{\Sigma} & -\mathbf{a} \cdot \boldsymbol{\Sigma} & -\mathbf{b} \cdot \boldsymbol{\Sigma} \\ & -\boldsymbol{\theta} \cdot \boldsymbol{\Sigma} & t\mathbf{1}_3 \\ & & 0 & -\mathbf{a}^T \\ & & & -\boldsymbol{\theta} \cdot \boldsymbol{\Sigma} \end{pmatrix} \quad (\text{L32})$$

with the abbreviation $\mathbf{A} \cdot \boldsymbol{\Sigma} = \begin{pmatrix} 0 & A_3 - A_2 \\ -A_3 & 0 & A_1 \\ A_2 - A_1 & 0 & 0 \end{pmatrix}$ and empty blocks are zero.

Several sub-algebras are recognised and an invariant sub-algebra is spanned by $\langle \mathbf{v}, T, \mathbf{p} \rangle$. Since ad in (L32) cannot be brought to block-diagonal form, it follows that $\mathfrak{gal}^{(0)}(3)$ is not semi-simple. The Killing metric tensor becomes

$$g = \begin{pmatrix} -6\mathbf{1}_3 & & & \\ & \mathbf{0}_3 & & \\ & & 0 & \\ & & & \mathbf{0}_3 \end{pmatrix} \quad (\text{L33})$$

from which one recognises the Levi decomposition $\mathfrak{gal}^{(0)}(3) \cong \mathfrak{so}(3) \ltimes \mathfrak{n}$ into a semi-simple sub-algebra on which g is non-singular (in this case the compact $\mathfrak{so}(3)$) and an invariant, nilpotent, sub-algebra $\mathfrak{n}$ on which g vanishes.

In order to find a canonical form for the commutation relations of a Lie algebra, consider a general element $r^i X_i \in \mathfrak{g}$ and look for the eigenvalues in the adjoint representation: $\det(\text{ad}(r^i X_i) - \lambda \mathbf{1}) = 0$. This is a polynomial

$\mathfrak{g}$	non-zero roots	$\mathcal{N}_r$	$\dim \mathfrak{g}$	$\mathfrak{g}$
$A_r \ r \geq 1$	$\mathbf{e}_i - \mathbf{e}_j \quad 1 \leq i \neq j \leq r+1$	$r(r+1)$	$r(r+2)$	$r+1$
$B_r \ r \geq 2$	$\pm \mathbf{e}_i \pm \mathbf{e}_j, \pm \mathbf{e}_i \quad 1 \leq i \neq j \leq r$	$2r(r-1) + 2r$	$r(2r+1)$	$2r-1$
$C_r \ r \geq 2$	$\pm \mathbf{e}_i \pm \mathbf{e}_j, \pm 2\mathbf{e}_i \quad 1 \leq i \neq j \leq r$	$2r(r-1) + 2r$	$r(2r+1)$	$r+1$
$D_r \ r \geq 4$	$\pm \mathbf{e}_i \pm \mathbf{e}_j \quad 1 \leq i \neq j \leq r$	$2r(r-1)$	$r(2r-1)$	$2r-2$

Table L3. Cartan's classification of the infinite series of complex root spaces A_r, B_r, C_r, D_r , of rank r , of simple finite-dimensional Lie algebras $\mathfrak{g}$, in terms of the usual unit vectors $\mathbf{e}_i$. $\mathcal{N}_r$ is the number of non-zero roots and $\mathfrak{g}$ is the dual Coxeter number.

equation in λ of some order r , which is called the **rank** of the Lie algebra. If $\dim \mathfrak{g}$ is finite, a detailed study shows that in the so-called **Cartan-Weyl basis** the non-vanishing commutators for a semi-simple (complex) Lie algebra take the form (with $i, j = 1, \dots, r$)

$$\begin{aligned}
[H_i, H_j] &= 0 \\
[H_i, E_\alpha] &= \alpha_i E_\alpha \\
[E_\alpha, E_\beta] &= \begin{cases} N_{\alpha, \beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\ \frac{2}{|\alpha|^2} \alpha \cdot \mathbf{H} & \text{if } \alpha + \beta = \mathbf{0} \end{cases} .
\end{aligned} \tag{L34}$$

Here, the r generators H_i form an abelian sub-algebra, the **Cartan sub-algebra**. The components of the **root vectors** α can be obtained as eigenvalues of the H_i : $\text{ad}(H_i)E_\alpha = \alpha_i E_\alpha$.¹⁵ Δ is the set of all roots and $N_{\alpha, \beta}$ are constants.¹⁶ The quadratic Casimir operator reads in this basis¹⁷

$$C = \sum_{i=1}^r H_i H_i + \sum_{\alpha > 0} \frac{1}{2} |\alpha|^2 (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha). \tag{L35}$$

A simple Lie algebra of rank r has r Casimir invariants, which commute with the entire algebra. For complex simple finite-dimensional Lie algebras, one has the following celebrated result.

Theorem: (Cartan) *The root spaces of a complex simple, finite-dimensional Lie algebra $\mathfrak{g}$ either belong to one of the four infinite series listed in Table L3, or else are one of the five ‘exceptional’ cases G_2, F_4, E_6, E_7, E_8 , where the index indicates the rank.*

The correspondence with the classical Lie algebras is indicated in Table L2. Motivated by the applications to statistical physics, we mainly use here an

¹⁵ Analogously to quantum mechanics, one may introduce states $|\alpha\rangle = |E_\alpha\rangle$ by $H_i |\alpha\rangle = \alpha_i |\alpha\rangle$ on which the E_α act as ladder operators.

¹⁶ An even more economical presentation is given by the **Chevalley basis**, which involves the *simple roots* only.

¹⁷ A root $\alpha = \sum_{i=1}^r \alpha_i \mathbf{e}_i$, where $\mathbf{e}_i$ are the usual unit vectors, is called **positive** and written $\alpha > 0$ if the first non-vanishing component is positive.



Fig. L1. Root diagram of the simple Lie algebra $A_1 \cong \mathfrak{su}(2) \cong \mathfrak{sl}(2, \mathbb{R})$.

‘Euclidean’ form. At the present state of knowledge, there is no apparent need for us to consider the well-known *real forms* of these complex root spaces.

Example: reconsider $\mathfrak{sl}(2, \mathbb{R}) = \langle X_{\pm 1,0} \rangle$ and the commutators $[X_n, X_m] = (n - m)X_{n+m}$. We compare this with the Lie algebra $A_1 = \langle E_{-\alpha}, H, E_{\alpha} \rangle$, of rank 1. Its root space is shown in Fig. L1 and we have the isomorphism

$$X_{-1} \mapsto E_{-\alpha} \mapsto -e_1, \quad X_0 \mapsto H \mapsto \mathbf{0}, \quad X_1 \mapsto E_{\alpha} \mapsto +e_1 \quad (\text{L36})$$

which relates each generator to a vector in the root diagram. Because of the canonical commutation relations (L34), this relation is such that forming the commutator $[X_n, X_m]$ corresponds to taking the vector sum of the associated vectors. If that sum hits a point in the diagram, that point gives the commutator, up to a multiplicative factor. Otherwise, the commutator vanishes.

Note that the entire Cartan sub-algebra $\mathfrak{h}$ is mapped onto the zero vector $\mathbf{0}$. This can lead to ambiguities when one tries to identify the generators of a given Lie algebra with root vectors.

Similarly, we show in Fig. L2 the root diagrams of all complex simple Lie algebras of rank 2 ($D_2 \cong A_2 \oplus A_2$ is not simple). The root spaces A_2 and G_2 are projected into the $2D$ plane, but have a simpler presentation in $3D$, see also Table L3. For the interpretation, one identifies each root E_{α} with a vector α , pointing from the origin to one of the dots in the diagram. By the two simple rules, which follow from (L34), the commutator $[E_{\alpha}, E_{\beta}]$ is related to the vector sum $\alpha + \beta$ as follows:

1. if $\alpha + \beta$ corresponds to a point in the diagram ($\alpha + \beta \in \Delta$), the commutator either is proportional to $E_{\alpha+\beta}$ if $\alpha + \beta \neq \mathbf{0}$, or else falls into $\mathfrak{h}$ if $\alpha + \beta = \mathbf{0}$.
2. if $\alpha + \beta$ falls outside the diagram ($\alpha + \beta \notin \Delta$), the commutator vanishes.

In particular, sub-algebras are readily identified, since they correspond to *convex sets* in the root diagram. Applications to the Schrödinger group and ageing are discussed in Chap. 4.

L.5 Function-space Representations

Having looked at length so far at the matrix representations, which involve finite-dimensional vector spaces, we now consider representations on function spaces, which are infinite-dimensional.

Definition: A **function-space representation** of a Lie group G is a structure-preserving map $T : G \rightarrow L(\mathcal{F}, \mathcal{F})$ with $g \mapsto T_g$, where T_g is a linear

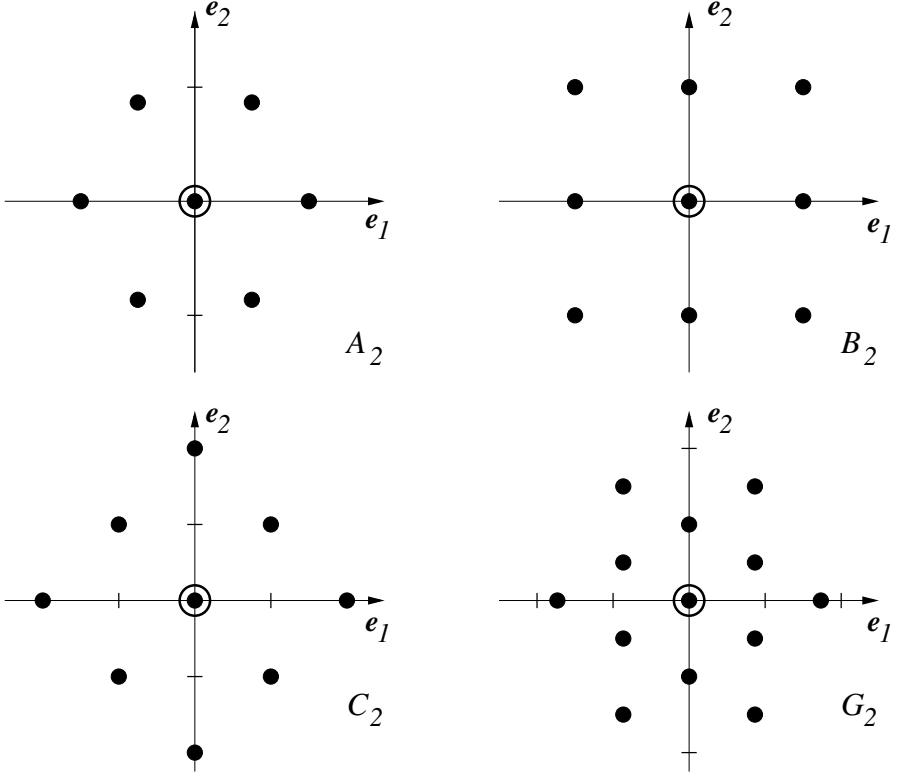


Fig. L2. Root diagrams of the complex simple Lie algebras A_2 , B_2 , C_2 and G_2 of rank 2. The units on the root lattice are indicated by the short ticks on the axes, if possible. The double circle in the centre indicates the (two-dimensional) Cartan sub-algebra.

operator acting on some space $\mathcal{F}$ of sufficiently smooth functions such that $T_g \mathbf{1} = \mathbf{1}$ and $T_{g_1} T_{g_2} = T_{g_1 g_2} \quad \forall g_{1,2} \in G$. A function-space representation of a Lie algebra is defined analogously. In the same way as for matrix representations, one can introduce unitary or irreducible representations, and so on. If G acts as a Lie transformation group on some geometric object with local coordinates $\mathbf{r} \mapsto \mathbf{r}' = g\mathbf{r}$, the action of the function-space representation of G may be written as $\phi(\mathbf{r}) \mapsto T_g(\mathbf{r}) = \phi'(g\mathbf{r})$. In this way, the change brought onto the function ϕ by the action of $g \in G$ is (i) by the change of $\mathbf{r} \mapsto \mathbf{r}'$ (called a **passive transformation**) and (ii) by the functional change $\phi \mapsto \phi'$ of ϕ itself (called an **active transformation**). Therefore, one often writes the action of a representation on a function space as follows

$$\phi(\mathbf{r}) \mapsto (T_g \phi)(\mathbf{r}) = \phi(g^{-1} \mathbf{r}) \quad , \quad T_{g_1} T_{g_2} = T_{g_1 g_2} \quad . \quad (\text{L37})$$

The concept analogous to the isomorphism of Lie groups is the equivalence of representations: *Two representations T and T' of a Lie group G are **equiv-***

alent, if there is a fixed linear map $P : T \rightarrow T'$ such that $T'_g = P^{-1}T_gP$ $\forall g \in G$. P is called an **intertwiner**. For Lie algebras, everything is set up in complete analogy.

Examples: Function-space representations arise in quantum mechanics:

1. The elements of the translation group $T(d)$ are parametrised by a vector $\mathbf{a} \in \mathbb{R}$ such that $\mathbf{r} \mapsto \mathbf{r}' = \mathbf{r} + \mathbf{a}$. A representation on a space of functions (think of the wave functions in quantum mechanics) is

$$\phi(\mathbf{r}) \mapsto (T_{\mathbf{a}}\phi)(\mathbf{r}) = \phi(\mathbf{r} - \mathbf{a}) = \exp(-i\mathbf{a} \cdot \mathbf{p}) \phi(\mathbf{r}), \quad (\text{L38})$$

where $\mathbf{p} = -i\nabla_{\mathbf{r}}$ are the generators of the Lie algebra $\mathfrak{t}(d)$.

2. The elements of the rotation group $O(3)$ can be parametrised in terms of the rotation (Euler) angles θ_i with $i = 1, 2, 3$. The action on scalar (wave) functions reads

$$\phi(\mathbf{r}) \mapsto (T_{\boldsymbol{\theta}}\phi)(\mathbf{r}) = \exp(-i\boldsymbol{\theta} \cdot \mathbf{J}) \phi(\mathbf{r}), \quad (\text{L39})$$

where $\mathbf{J}$ stands for the generators of $\mathfrak{so}(3)$.

3. The elements of the Lie group $Sl(2, \mathbb{R})$ are matrices $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha\delta - \beta\gamma = 1$. A function-space representation, on the smooth functions of a single variable, was constructed in (L24) and is given by

$$\phi(z) \mapsto (T_g\phi)(z) = \left(\frac{1}{\gamma z + \delta} \right)^{2\Delta} \phi\left(\frac{\alpha z + \beta}{\gamma z + \delta} \right). \quad (\text{L40})$$

The infinitesimal generators are listed in (L17) and span the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) = \langle X_{\pm 1, 0} \rangle$. The quadratic Casimir operator reads explicitly in this representation

$$C = \frac{1}{2}X_0^2 - \frac{1}{4}(X_1X_{-1} + X_{-1}X_1) = \frac{1}{2}\Delta(\Delta - 1). \quad (\text{L41})$$

We see that the representations of this non-compact Lie algebra are labelled by the real number Δ , in contrast to what happens for the compact Lie algebra $\mathfrak{so}(3)$ (familiar from the quantum-mechanical theory of angular momentum) where the spectrum of the quadratic Casimir is *discrete*.

When going beyond the semi-simple Lie groups/algebras considered so far, it turns out that the notion of representation has to be extended: A *function-space representation is projective*, if its action has the form

$$\phi(\mathbf{r}) \mapsto (T_g\phi)(\mathbf{r}) = f_g(g^{-1}\mathbf{r})\phi(g^{-1}\mathbf{r}) \quad , \quad T_{g_1}T_{g_2} = \omega(g_1, g_2)T_{g_1g_2} \quad (\text{L42})$$

where ω is called a **cocycle** and f a **companion function**. This definition is physically well-motivated, since the axioms of quantum mechanics merely guarantee the existence of projective representations. However, one has [37]:

Theorem: (Bargman) *The representations of semi-simple Lie groups or Lie algebras are non-projective, that is $\omega(g_1, g_2) = 1$.*

The classical example for a physically relevant projective representation comes from the Galilei-transformations of space-time $(t, \mathbf{r}) \mapsto (t', \mathbf{r}') = (t, \mathbf{r} + \mathbf{v}t)$. The action of this element of $Gal^{(0)}(d)$ on a (wave) function is

$$\phi(t, \mathbf{r}) \mapsto (T_{\mathbf{v}}\phi)(t, \mathbf{r}) = \exp \left[-im \left(\mathbf{v} \cdot \mathbf{r} - \frac{1}{2} \mathbf{v}^2 t \right) \right] \phi(t, \mathbf{r} - \mathbf{v}t). \quad (\text{L43})$$

The companion function is parametrised by the mass m of the particle. This representation is projective, with a non-trivial cocycle $\omega(\mathbf{v}_1, \mathbf{v}_2) = e^{im\mathbf{v}_1 \cdot \mathbf{v}_2 t}$. It can be directly checked that the free Schrödinger equation $(2mi\partial_t + \Delta_L)\phi(t, \mathbf{r}) = 0$ is invariant under the transformation (L43).

Applied to quantum mechanics, Galilei-invariance leads to the conceptually important **Bargman superselection rules** [37]. To see how this comes about, consider the sequence of transformations, with t being kept fixed

$$\mathbf{r} \mapsto \mathbf{r} - \mathbf{v}t \mapsto \mathbf{r} - \mathbf{v}t - \mathbf{a} \mapsto \mathbf{r} - \mathbf{a} \mapsto \mathbf{r}$$

which amounts to the group identity $\mathbf{r} \mapsto \mathbf{r}$ and $t \mapsto t$. On the other hand, when this sequence of transformation is applied to a wave function $\phi(t, \mathbf{r})$, one rather obtains the transformation $\phi(t, \mathbf{r}) \mapsto e^{im\mathbf{a} \cdot \mathbf{v}} \phi(t, \mathbf{r})$. This is certainly in agreement with the postulates of quantum mechanics which associate two wave functions which merely differ by an overall phase to the same physical state. On the other hand, the space of physical states has to be decomposed into a direct sum of spaces, each of which is characterised by a different value of m , since any term in the linear combination of states $|\Phi\rangle = \sum_{\alpha} a_{\alpha} |\phi_{\alpha}\rangle$ with different masses m_{α} will under the above operation pick up a different phase such that $|\Phi\rangle$ cannot be Galilei-invariant. It follows that *any matrix elements of observables between states of different mass have to vanish*. In the main text, we shall encounter Bargman's superselection rules in this form.

L.6 Central Extensions

The consideration of the peculiarities of the representations of non-semi-simple Lie algebras motivates us to look for so-called central extensions of the Lie algebras. A **central extension of a Lie algebra $\mathfrak{g}$** is characterised by an extension of the commutation relations

$$[X_{\mu}, X_{\nu}] = C_{\mu\nu}^{\lambda} X_{\lambda} + c_{\mu,\nu} \quad (\text{L44})$$

such that $[c_{\mu,\nu}, X_{\lambda}] = 0 \quad \forall X_{\lambda} \in \mathfrak{g}$.

Theorem: (Whitehead) *A semi-simple and finite-dimensional Lie algebra $\mathfrak{g}$ has no central extensions.*

Indeed, if one tries to introduce central extensions for semi-simple, finite-dimensional Lie algebras, one finds that they can be re-absorbed into the

generators. Consequently, central extensions can be found only for (i) non-semi-simple Lie algebras or (ii) infinite-dimensional Lie algebras. We shall show first how to complete the definition of Galilei- and Schrödinger-algebras before considering Virasoro and Kac-Moody algebras.

Example: For the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, one might try to find a central extension and write $[X_n, X_m] = (n - m)X_{n+m} + c(n, m)$. Obviously, $c(n, m) = -c(m, n)$ and the independent elements are $c(1, -1)$ and $c(\pm 1, 0)$. However, these can be absorbed into redefined generators X'_n , since

$$\begin{aligned} [X_{\pm 1}, X_0] &= \pm 1 X_{\pm 1} + c(\pm 1, 0) = \pm 1 (X_{\pm 1} \pm 1 c(\pm 1, 0)) =: \pm 1 X'_{\pm 1} \\ [X_1, X_{-1}] &= 2X_0 + c(1, -1) = 2 \left(X_0 + \frac{1}{2} c(1, -1) \right) =: 2X'_0 \end{aligned}$$

such that $[X'_n, X'_m] = (n - m)X'_{n+m}$ without a central extension is recovered for all $n, m = \pm 1, 0$.

A physically interesting example of a central extension is obtained from the projective representation (L43) of the Galilei group which describes how the wave function of a free particle with mass m changes if one goes from one inertial system to another. Expanding in $\mathbf{v}$, one obtains the infinitesimal generators of the Galilei boosts as ($j = 1, \dots, d$)

$$Y_{1/2}^{(j)} = -t \frac{\partial}{\partial r_j} - imr_j \quad (\text{L45})$$

and we see that the mass m now explicitly appears in the generator. In consequence, these generators no longer commute with the generators $Y_{-1/2}^{(j)} = -\partial_{r_j}$ of spatial translations, but rather

$$[Y_{1/2}^{(j)}, Y_{-1/2}^{(j')}] = -im \delta^{j,j'} =: \delta^{j,j'} M_0. \quad (\text{L46})$$

Formally, we have obtained a central extension of the Lie algebra $\mathfrak{t}(d) \oplus \mathfrak{t}(d)$ to the so-called **Heisenberg algebra** $\mathfrak{hei}(d) = \langle Y_{\pm 1/2}^{(j)}, M_0 \rangle_{j=1, \dots, d}$. Clearly, $\dim \mathfrak{hei}(d) = 2d + 1$.

Having introduced the new generator M_0 , it is natural to ask on which variable it acts. A convenient way to see this is to consider m no longer as some parameter, but rather as a new physical coordinate [280] and to go over to a ‘dual’ variable ζ via a Fourier transformation

$$\phi(t, \mathbf{r}) = \phi_m(t, \mathbf{r}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\zeta e^{-im\zeta} \psi(\zeta, t, \mathbf{r}) \quad (\text{L47})$$

such that the generators of Galilei boosts, when acting on the new wave function $\psi = \psi(\zeta, t, \mathbf{r})$, take the form $Y_{1/2}^{(j)} = -t\partial_{r_j} - r_j\partial_{\zeta}$. This implies $M_0 = -\partial_{\zeta}$. When integrating the generators $Y_{1/2}^{(j)}$, one finds the transformation

$$\zeta \mapsto \zeta - \mathbf{r} \cdot \mathbf{v} + \frac{1}{2} \mathbf{v}^2 t, \quad t \mapsto t, \quad \mathbf{r} \mapsto \mathbf{r} - \mathbf{v} t \quad (\text{L48})$$

and one has traded a projective representation for an ordinary representation on a larger coordinate space.

In this way, one can now complete the definition of the **Galilei algebra**, by centrally extending the translations $\mathfrak{t}(2d) \rightarrow \mathfrak{hei}(d)$ and one has from Table L1 the central extension of $\mathfrak{gal}^{(0)}(d)$ in the form

$$\mathfrak{gal}(d) \cong \mathfrak{t}(1) \ltimes (\mathfrak{so}(d) \ltimes \mathfrak{hei}(d)) \quad , \quad \dim \mathfrak{gal}(d) = \frac{1}{2}d(d+3) + 2. \quad (\text{L49})$$

Notice that the Schrödinger operator, with $X_{-1} = -\partial_t$, $\mathcal{S} = 2mi\partial_t - \Delta_L = 2M_0X_{-1} - \mathbf{Y}_{-1/2} \cdot \mathbf{Y}_{-1/2}$ is a Casimir operator of $\mathfrak{gal}(d)$. Analogously, by a central extension of $\mathfrak{sch}^{(0)}(d)$, the complete **Schrödinger algebra** becomes

$$\mathfrak{sch}(d) \cong \mathfrak{sl}(2, \mathbb{R}) \ltimes (\mathfrak{so}(d) \ltimes \mathfrak{hei}(d)) \quad , \quad \dim \mathfrak{sch}(d) = \frac{1}{2}d(d+3) + 4 \quad (\text{L50})$$

which includes $\mathfrak{gal}(d)$ as a subalgebra and we refer to Table L1 for other possible decompositions. Explicit function-space representations of $\mathfrak{sch}(d)$ are given in Chap. 4. Since the projective representations of these algebras are, by the above procedure, equivalent to ordinary representations, one may inquire whether these algebras might be considered as subalgebras of a larger algebra where M_0 is no longer central. One example of this is the embedding $\mathfrak{sch}(d) \subset \mathfrak{conf}(d+2)$ which we study, including its physical consequences, in Chap. 4.

Another example of the same kind is provided by the recently discovered so-called **exotic conformal Galilean algebra** ECGA in $d = 2$ space dimensions [483, 484], see Chap. 5.

We now consider central extensions of *infinite-dimensional* Lie algebras.

Example: A different kind of central extension is met when considering the infinite-dimensional extension introduced on p. 406 above. The centrally extended Lie algebra will have the commutators $[X_n, X_m] = (n-m)X_{n+m} + c(n, m)$ with $c(n, m) = -c(m, n)$ and $n, m \in \mathbb{Z}$. For $n \neq 0$ one can re-define, as for $\mathfrak{sl}(2, \mathbb{R})$, $X'_n = X_n + c(n, 0)/n$. Briefly, one captures this by saying that one can always arrange for $c(n, 0) = 0$. Next, one uses the Jacobi identity for $X_{0,n,m}$ which reads $[X_0, [X_n, X_m]] = -(n+m)[X_n, X_m]$ and from which it follows that $(n+m)c(n, m) = 0$. This constraint is solved by $c(n, m) = \delta_{n+m,0}c(m)$ and $c(m) = -c(-m)$. Now, write down the Jacobi identity for $X_{n,m,\ell}$ with $n+m+\ell=0$ which leads to

$$(m-n)c(n+m) - (2n+m)c(m) + (n+2m)c(n) = 0$$

and setting in there $n = 1$, one has the recursion $(m-1)c(m+1) = (m+2)c(m) - (2m+1)c(1)$, for all $m \geq 1$. This linear recursion can have two independent solutions. It is readily checked that $c(m) = m$ and $c(m) = m^3$ are indeed solutions. Therefore, the most general form of the central extension

is $c(n, m) = \delta_{n+m,0}(c_1 m^3 + c_2 m)$. Conventionally, one redefines X_0 to arrange for $c_2 = -c_1$ and sets $c_1 = \frac{c}{12}$ which gives the commutator of the **Virasoro algebra** $\mathfrak{vir}$

$$[X_n, X_m] = (n - m)X_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \quad (\text{L51})$$

and where c is called the **central charge**. In this way, the inclusion $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{vir}$ is made obvious.

When extending the function-space representation (L17) of $\mathfrak{sl}(2, \mathbb{R})$ to $\mathfrak{vir}$, a celebrated result clarifies when these representations are unitary (on a suitably defined Hilbert space).¹⁸

Theorem: [262, 282] *The function-space representations of $\mathfrak{vir}$, characterised by the conformal weight Δ , are unitary if and only if one of the following conditions hold true:*

- (i) $c \geq 1$ and $\Delta \geq 0$,
- (ii) $c = c_m$ and $\Delta = \Delta_{r,s}(m)$, where $m = 2, 3, 4, 5 \dots$, r, s are integers such that $1 \leq r \leq m - 1$, $1 \leq s \leq m$ and

$$c_m = 1 - \frac{6}{m(m+1)}, \quad \Delta_{r,s}(m) = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}. \quad (\text{L52})$$

The last example of centrally extended Lie algebras we consider here are the **Kac-Moody algebras**. Consider a finite-dimensional, semi-simple Lie algebra $\mathfrak{g}$ with generators J^a , $a = 1, \dots, \dim \mathfrak{g}$ and commutator relations $[J^a, J^b] = if^{ab}_c J^c$, where f^{ab}_c are the structure constants and the factor i is inserted in order to have hermitian representations on function spaces. One now adds an extra variable t and considers the generators $J_n^a := J^a t^n$ with $n \in \mathbb{Z}$. The commutators of this infinite-dimensional Lie algebra $\mathfrak{g} \oplus \mathbb{C}[t, t^{-1}]$ (referred to as **loop algebra**) read $[J_n^a, J_m^b] = [J^a t^n, J^b t^m] = if^{ab}_c J^c t^{n+m} = if^{ab}_c J_{n+m}^c$. This allows for the following central extension

$$[J_n^a, J_m^b] = if^{ab}_c J_{n+m}^c + kn\delta_{n+m,0}\delta^{ab} \quad (\text{L53})$$

where the constraints coming from the Jacobi identities are already taken into account.¹⁹ The constant k is the **central charge** and commutes with the entire algebra. Algebras of this kind, in a slightly different dressing which we shall not describe here, were first discussed in elementary particle physics under the name of **current algebras**, where the central extensions arose from quantum effects, before their mathematical structure has been profoundly studied in its own right. In the Cartan-Weyl basis, the non-vanishing commutators read

¹⁸ Very frequently, the conformal weights of the representations of $\mathfrak{vir}$ are denoted by h instead of Δ , as in this book.

¹⁹ Implicitly, this is written here in a basis of generators such that the Killing metric tensor $K(J_0^a, J_0^b) = \delta^{ab}$ is reduced to the identity matrix, which is possible whenever $\mathfrak{g}$ is semi-simple. For a more general choice of basis, the Killing metric must be introduced in (L53).

$$\begin{aligned}
 [H_n^i, H_m^j] &= kn\delta_{n+m,0}\delta^{ij} \\
 [H_n^i, E_m^\alpha] &= \alpha_i E_{n+m}^\alpha \\
 [E_n^\alpha, E_m^\beta] &= \begin{cases} N_{\alpha,\beta} E_{n+m}^{\alpha+\beta} & ; \text{ if } \alpha + \beta \in \Delta \\ \frac{2}{|\alpha|^2} \alpha \cdot \mathbf{H}_{n+m} + kn\delta_{n+m} & ; \text{ if } \alpha + \beta = \mathbf{0} \end{cases}
 \end{aligned} \tag{L54}$$

in a notation which we hope is obvious. The Cartan subalgebra is $\mathfrak{h} = \langle H_0^1, \dots, H_0^r, k, L_0 \rangle$, where r is the rank of $\mathfrak{g}$ and $L_0 = -t \frac{d}{dt}$, with the action $[L_0, J_n^a] = -nJ_n^a$. The resulting Lie algebra $\widehat{\mathfrak{g}}_k = (\mathfrak{g} \oplus \mathbb{C}[t, t^{-1}]) \ltimes (\mathbb{C}k \oplus \mathbb{C}L_0)$ is called an **affine Lie algebra**²⁰ or a **Kac-Moody algebra** and k is called the **level** of that Lie algebra.

Example: For the affine Lie algebra $\widehat{\mathfrak{su}}(2)_k$, one can extend the root space of $\mathfrak{su}(2)$, see Fig. L1, to an infinite straight line in both directions.

If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, the **Sugawara construction** defines canonically an extension of the associated affine Lie algebra $\widehat{\mathfrak{g}}_k$ as a semi-direct product with a Virasoro algebra such that the commutators become

$$\begin{aligned}
 [X_n, X_m] &= (n-m)X_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0} \\
 [X_n, J_m^a] &= -mJ_{n+m}^a \\
 [J_n^a, J_m^b] &= if^{ab} J_{n+m}^c + kn\delta_{n+m,0}\delta^{ab}.
 \end{aligned} \tag{L55}$$

Consequently, the Virasoro central charge reads explicitly

$$c = \frac{k \dim \mathfrak{g}}{k + \mathfrak{g}} \tag{L56}$$

where $\mathfrak{g}$ is the **dual Coxeter number** of the Lie algebra $\mathfrak{g}$. The values of $\mathfrak{g}$ for the classical Lie algebras are listed in Table L3. If $\mathfrak{g}$ is semi-simple, the analogous formula sums the contributions to c from the simple components. Compactly, this may be written in terms of fields $J^a(z) = \sum_{n \in \mathbb{N}} J_n^a z^{-1+n}$ and $X(z) = \sum_{n \in \mathbb{N}} X_n z^{-2+n}$. In the Cartan-Weyl basis, the Virasoro field $X(z)$ (related to the energy-momentum tensor) becomes

$$\begin{aligned}
 X(z) &= \frac{1}{2(k + \mathfrak{g})} \left\{ \sum_{i=1}^r : H^i H^i : (z) \right. \\
 &\quad \left. + \sum_{\alpha > 0} \frac{1}{2} |\alpha|^2 [: E^\alpha E^{-\alpha} : (z) + : E^{-\alpha} E^\alpha : (z)] \right\}. \tag{L57}
 \end{aligned}$$

Therefore, whenever one encounters an affine Lie algebra, the Sugawara construction is automatically performed in order to obtain the Virasoro generators and this is not always mentioned explicitly.

²⁰ In the mathematical literature, these are referred to as ‘non-twisted’ affine Lie algebras.

Example: Almost like M. Jourdain in Molière's *Bourgeois gentilhomme*, the reader will recognise the following prosaic example: consider a free boson, with the Hamiltonian $H = \sum_k \omega(k) (b(k)^\dagger b(k) + \frac{1}{2})$ and the canonical non-vanishing commutator $[b(k), b(k')^\dagger] = \delta_{k,k'}$. In the extremely relativistic limit,²¹ where one has $\omega(k) = |k|$, it is of interest to redefine $b_k := -i\sqrt{k} b(k)$ and $b_{-k} := i\sqrt{k} b(k)^\dagger$ for $k > 0$ such that the commutator now becomes

$$[b_k, b_{k'}] = k\delta_{k+k',0} \quad (\text{L58})$$

and the Hamiltonian reads $H = \sum_{k>0} (b_{-k} b_k + \frac{1}{2})$. In this way, one has obtained a $\widehat{\mathfrak{u}}(1)$ Kac-Moody algebra. In order to carry out the Sugawara construction, one first introduces the normal ordering prescription

$$b_k b_{k'} = : b_k b_{k'} : + k\delta_{k+k',0} \Theta(k - k') \quad (\text{L59})$$

and then one can write the Virasoro generators as follows:

$$X_n = \frac{1}{2} \sum_{n \in \mathbb{Z}} : b_{n-p} b_p : \quad (\text{L60})$$

The central charge of this Virasoro algebra is $c = 1$, but we warn the reader that the calculation does require a careful treatment of the normal ordering.²²

Example: It is possible to extend the Schrödinger algebra $\mathfrak{sch}(d)$ to an infinite-dimensional Lie algebra, called $\mathfrak{sv}(d)$ in this book, quite in analogy to the introduction of the loop algebras described above [327], see Chap. 4. When considering the possible central extensions, we observe that $\mathfrak{sv}(d)$ contains the Lie algebra $\mathfrak{vir} \ltimes \widehat{\mathfrak{so}}(d)_k$ as a subalgebra. Hence the central charge of the Virasoro generators X_n is given for $d \geq 3$ by the Sugawara formula (L56).²³

M. On the Central Limit Theorem

We recall some background knowledge of random variables, stochastic process and in particular the central limit theorem. For further information, consult e.g. [679, 710, 74, 216, 584, 238, 91]. As formulated by Kolmogorov, probabilistic events are mathematically described by a **probability space** $(\Omega, \mathcal{F}, \mu)$,

²¹ This situation also arises in condensed-matter systems such as quantum antiferromagnets, where the low-energy dispersion $\omega(k)$ relation may become linear in certain cases.

²² For an abelian Lie algebra, $\mathfrak{g} = 0$. Hence $c = \dim \mathfrak{g}$ is independent of the level k .

²³ Here, one uses the following elementary fact: consider two Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ with central extensions c_a and c_b . Then in the semi-direct product $\mathfrak{a} \ltimes \mathfrak{b}$, the central extension of the generators $A_a \in \mathfrak{a}$ remains unchanged, whereas the central extension of the generators $B_b \in \mathfrak{b}$ depends on the way $\mathfrak{a}$ acts on $\mathfrak{b}$.

where Ω is a set, $\mathcal{F}$ is a **σ -algebra** of subsets of Ω and $\mu : \mathcal{F} \rightarrow [0, 1]$ is a **measure** and by definition satisfies the property

$$\text{if } A_i \cap A_j = \emptyset \text{ for } i \neq j, \text{ then } \mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i).$$

An **event** corresponds to a set $A \in \mathcal{F}$ and occurs with probability $\mu(A)$. Clearly, one requires $\mu(\Omega) = 1$. A **σ -algebra** $\mathcal{F}$ is defined by the following properties: (i) $\Omega \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$, (ii) if $A \in \mathcal{F}$, then its complement $A^c := \Omega - A \in \mathcal{F}$, (iii) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ and (iv) if $A_i \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Example: An almost trivial example of a probability space is given by the σ -algebra $\mathcal{F}_1 := (\emptyset, A, A^c, \Omega)$, where $A \subset \Omega$, and the measure $\mu(\emptyset) = 0$, $\mu(A) = \mu_a$, $\mu(A^c) = 1 - \mu_a$ and $\mu(\Omega) = 1$.

Example: Much more often used is the **Lebesgue integral** over the real line $\mathbb{R}$. The measure space is $(\mathbb{R}, \mathcal{B}, \mu)$, where the **Borel algebra** $\mathcal{B}$ is the σ -algebra generated over the open intervals $A = (a, b) \subset \mathbb{R}$. The Lebesgue measure is $\mu(A) := b - a$ (since $\mu(\mathbb{R}) = \infty$, this is not a conventional probability space).

A **random variable** X is an integrable function on the probability space $(\Omega, \mathcal{F}, \mu)$ and has the values $X(A) = \int_A d\mu(\omega) X(\omega)$. The **expectation value** of a random variable X is

$$\langle X \rangle = \mathbb{E}[X] := \int_{\Omega} d\mu(\omega) X(\omega). \quad (\text{M1})$$

For real probability measures, one often considers the **cumulative probability distribution**

$$\Pr\{x \leq X(\omega) \leq x + dx\} = \mu(\omega \in \Omega | x \leq X(\omega) \leq x + dx) =: p(x)dx$$

which allows us to re-write the expectation value in terms of the **probability density** $p(x)$ in the form $\mathbb{E}[X] = \int_{\mathbb{R}} dx xp(x)$, more appealing to physicists. For a vector $\mathbf{X} \in \mathbb{R}^n$ of random variables, the **characteristic function** is defined from the joint probability density $p(\mathbf{x})$ by

$$G(\mathbf{k}) := \int_{\mathbb{R}^n} d\mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} p(\mathbf{x}). \quad (\text{M2})$$

Expanding $G(\mathbf{k})$ around $\mathbf{k} = \mathbf{0}$ will produce the *correlations* of the random variables X_i , whereas an analogous expansion of $\ln G(\mathbf{k})$ generates the *connected correlations* or *Ursell functions* or **cumulants** $\langle\langle X_1 \cdots X_m \rangle\rangle$. Cumulants are related to expectation values, for example

$$\begin{aligned}
\langle\langle X_1 X_2 \rangle\rangle &= \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle \\
\langle\langle X_1 X_2 X_3 \rangle\rangle &= \langle X_1 X_2 X_3 \rangle - \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_1 \rangle \langle X_2 X_3 \rangle \\
&\quad + 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle.
\end{aligned} \tag{M3}$$

Two random variables $X_{1,2}$ are **uncorrelated**, iff $\langle\langle X_1 X_2 \rangle\rangle = 0$. In contrast, two random variables $X_{1,2}$ are **independent**, iff $G(k_1, k_2) = G(k_1)G(k_2)$. Independence is equivalent to any of the following three conditions

$$\begin{cases} p(x_1, x_2) = p(x_1)p(x_2) \\ \langle X_1^{m_1} X_2^{m_2} \rangle = \langle X_1^{m_1} \rangle \langle X_2^{m_2} \rangle & ; \text{ for all } m_1, m_2 \\ \langle\langle X_1^{m_1} X_2^{m_2} \rangle\rangle = 0 & ; \text{ for all } m_1 \neq 0, m_2 \neq 0 \end{cases} \tag{M4}$$

Finally, a **stochastic process** is an indexed set $\{X_t\}_{t \in I}$ of random variables.

After these preparations, we can now formulate the **central limit theorem**. First, we give a simplified version of this result, which is spectacularly easy to prove.

Lemma: *Let $\{X_1, \dots, X_n\}$ be a set of independent and identically distributed random variables, with vanishing mean $\langle X_1 \rangle = \dots = \langle X_n \rangle = 0$ and variance $\langle X_1^2 \rangle = \dots = \langle X_n^2 \rangle = \sigma^2$. Then the probability density $P_n(x)$ of the normalised sum*

$$S_n := \frac{1}{\sqrt{n}} \sum_{j=1}^n X_j \tag{M5}$$

tends in the limit $n \rightarrow \infty$ to a Gaussian

$$\lim_{n \rightarrow \infty} P_n(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{z^2}{2\sigma^2} \right]. \tag{M6}$$

Proof: In order to find $P_n(z) = \int_{\mathbb{R}^n} d\mathbf{x} \, \delta(z - S_n) p(\mathbf{x})$, one considers the characteristic function

$$\begin{aligned} G_n(k) &= \int_{\mathbb{R}} dz \, e^{ikz} P_n(z) = \int_{\mathbb{R}^n} d\mathbf{x} \, p(\mathbf{x}) \int_{\mathbb{R}} dz \, e^{ikz} \delta(z - S_n) \\ &= \left[\int_{\mathbb{R}} dx \, p(x) e^{ikx/\sqrt{n}} \right]^n = \left[G \left(\frac{k}{\sqrt{n}} \right) \right]^n, \end{aligned}$$

where in the second line the factorisation $p(\mathbf{x}) = p(x_1) \cdots p(x_n)$, valid for independent random variables, was used and G is the characteristic function for a single random variable. From the definition of S_n , one has directly $\langle S_n \rangle = 0$ and $\langle S_n^2 \rangle = \sigma^2$. Expanding the characteristic function G , one obtains

$$G_n(k) = \left[1 - \frac{1}{n} \frac{(k\sigma)^2}{2} + O(k^3 n^{-3/2}) \right]^n \xrightarrow{n \rightarrow \infty} \exp \left[-\frac{(k\sigma)^2}{2} \right]$$

which proves the assertion, after an inverse Fourier transformation. q.e.d.

This result goes a long way to explain the ever so frequent Gaussian distributions in almost all kinds of statistical phenomena. Provided only that the second moment of the single-variable probability density exists,²⁴ this conclusion is independent of the form of the probability density $p(x)$. Furthermore, the conclusion eq. (M6) even remains valid for sums of independent random variables with distinct single-variable distributions, under some weak conditions. A simple sufficient criterion was found by Lyapounov: *if X_i are independent random variables with $\langle X_j \rangle = 0$ and $\langle X_j^2 \rangle = \sigma_j^2$, $s_n^2 := \sum_{j=1}^n \sigma_j^2$ is finite and there exists a constant $\delta > 0$ such that the Lyapounov condition*

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{s_n^{2+\delta}} \sum_{j=1}^n \mathbb{E}[|X_j|^{2+\delta}] \right\} = 0 \quad (\text{M7})$$

holds true, then the distribution of $s_n^{-1} \sum_{j=1}^n X_j$ converges to a Gaussian.

Precise information on the speed of convergence is also available. The Berry-Esséen theorem states that *if the third moment $\langle |X_j|^3 \rangle$ is finite, the convergence towards the Gaussian is uniform and at least of the order $O(n^{-1/2})$.*

Finally, the condition of independence can be slightly relaxed. Some further terminology is needed. Consider a stochastic process $\{X_t\}_{t \in I}$ with $I \subset \mathbb{R}$ and let $\mathcal{X}_a^b := \{X_a, X_{a+1}, \dots, X_b\}$ be the σ -algebra generated by the random variables X_t with $a \leq t \leq b$. Then define the **strong mixing coefficient**

$$\alpha(s) := \sup_t \left\{ \left| \Pr(A \cap B) - \Pr(A)\Pr(B) \right| \mid t \in \mathbb{R}, A \in \mathcal{X}_{-\infty}^t, B \in \mathcal{X}_{t+s}^\infty \right\}$$

A random process is **strongly mixing** iff $\lim_{s \rightarrow \infty} \alpha(s) = 0$ [91]. Then an extension of the central limit theorem to the case of weakly dependent random variables X_j is possible if expectation values of the X_j of a sufficiently high order exist and furthermore, $\alpha(s) \rightarrow 0$ sufficiently fast for $s \rightarrow \infty$. A mathematically precise form of this statement follows.

Theorem: [216] *If the stochastic process $\{X_t\}_{t \in I}$ with $\mathbb{N} \subset I \subset \mathbb{R}$ is strongly mixing, centred $\mathbb{E}[X_n] = 0$ and if there is a constant $\delta > 0$ such that*

$$\mathbb{E}[|X_n|^{2+\delta}] < \infty, \quad \sum_{n \in \mathbb{N}} \alpha(n)^{\delta/(4+2\delta)} < \infty. \quad (\text{M8})$$

Then $\sigma^2 = \lim_{n \rightarrow \infty} \mathbb{E}[(X_1 + \dots + X_n)/\sqrt{n}]^2$ exists and the distribution of $(X_1 + \dots + X_n)/\sqrt{n}$ converges towards the Gaussian (M6) for $n \rightarrow \infty$.

In [74], the same conclusion is reached under the conditions $\mathbb{E}[X_n^{12}]$ finite and $\alpha(n) = O(n^{-5})$ for $n \rightarrow \infty$.

²⁴ The Gnedenko-Kolmogorov theorem allows to relax this condition a little further.

Q. Lexique/Lexikon

Les rapports de stages M1/M2 et les thèses universitaires s'écrivent toujours en français. Comme la littérature spécialisée apparaît aujourd'hui presque exclusivement en anglais, la traduction en français des termes techniques est parfois difficile. Cette liste brève, dans un ordre pas trop systématique, pourra aider (le genre grammatical *m, f* est indiqué).

Es soll vorkommen, daß Abschlußarbeiten L3/M2 und Dissertationen an deutschsprachigen Universitäten immer noch auf Deutsch geschrieben werden. Da die Fachliteratur heute fast ausschließlich auf Englisch erscheint, ist die Übertragung spezifischer Terminologie ins Deutsche manchmal mühsam. Dieses kurze Verzeichnis kann dabei helfen (das grammatikalische Geschlecht *m, f, n* ist angegeben).

This is a short list of specialised terms, translated into German and French.

Deutsch	English	Français
Primäroperator, <i>m</i>	primary operator	opérateur primaire, <i>m</i>
Sekundäroperator, <i>m</i>	secondary operator	opérateur secondaire, <i>m</i>
Operatorinhalt, <i>m</i>	operator content	contenu opératoire, <i>m</i>
Liegruppe, <i>f</i>	Lie group	groupe de Lie, <i>m</i>
Darstellung, <i>f</i>	representation	représentation, <i>f</i>
Abbildung, <i>f</i>	map	application, <i>f</i>
Anwendung, <i>f</i>	application	application, <i>f</i>
relevant	relevant	pertinent
irrelevant	irrelevant	non pertinent
Eigenwert, <i>m</i>	eigenvalue	valeur propre, <i>f</i>
Eigenvektor, <i>m</i>	eigenvector	vecteur propre, <i>m</i>
Entartung, <i>f</i>	degeneracy	dégénérescence, <i>f</i>
Energielücke, <i>f</i>	energy gap	lacune d'énergie, <i>f</i>
Renormierungsgruppe, <i>f</i>	renormalization group	groupe de renormalisation, <i>m</i>
Skalenverhalten, <i>n</i>	scaling (behaviour)	comportement d'échelle, <i>m</i>
Skaleninvarianz, <i>f</i>	scaling (invariance)	invariance d'échelle, <i>f</i>
Hyperskalengesetz, <i>n</i>	hyperscaling (law)	loi de hyperéchelle, <i>f</i>
Endlichkeits- skalenverhalten, <i>n</i>	finite-size scaling	comportement d'échelle de taille finie, <i>m</i>
Endlichkeits- skalenregion, <i>f</i>	finite-size scaling region	région du comportement d'échelle sur réseau, <i>f</i>
Gitterkorrektur, <i>f</i>	finite-size correction	correction de taille finie, <i>f</i>
Gitterpunkt, -platz, <i>m</i>	site	site, <i>m</i>
Gitter, <i>n</i>	lattice	réseau, <i>m</i>
Kante, <i>f</i>	edge	lisière, <i>f</i> ; arête, <i>f</i>
Schicht, <i>f</i>	layer	couche, <i>f</i>

Abrundungstemperatur, <i>f</i>	rounding temperature	température d'arondissement, <i>f</i>
Überkeuzen, <i>n</i>	cross-over	changement de comportement, <i>m</i>
Kreuzexponent, <i>m</i>	crossing exponent	exposant de changement de comportement, <i>m</i>
Verschiebungsexponent, <i>m</i>	shift exponent	exposant de déplacement, <i>m</i>
Wanderexponent, <i>m</i>	wandering exponent	exposant de divagation, <i>m</i>
Volumenexponent, <i>m</i>	bulk exponent	exposant de volume, <i>m</i>
dynamischer Exponent, <i>m</i>	dynamical exponent	exposant dynamique, <i>m</i>
Übergangsexponent, <i>m</i>	passage exponent	exposant de passage, <i>m</i>
Alterungsexponent, <i>m</i>	ageing exponent	exposant de vieillissement, <i>m</i>
Oberflächenexponent, <i>m</i>	surface exponent	exposant de surface, <i>m</i>
Eckenexponent, <i>m</i>	corner exponent	exposant de coin, <i>m</i>
Eckentransfermatrix, <i>f</i>	corner transfer matrix	matrice (de transfert) de coin, <i>f</i>
Eckensingularität, <i>f</i>	edge singularity	singularité de seuil, <i>f</i>
Schleife, <i>f</i>	loop	boucle, <i>f</i> , lacet, <i>m</i>
Zopf, <i>m</i>	braid	tresse, <i>f</i>
Abbildung, <i>f</i>	mapping	application, <i>f</i>
Menge, <i>f</i>	set	ensemble, <i>m</i>
Teilmenge, <i>f</i>	subset	sous-ensemble, <i>m</i>
Gruppe, <i>f</i>	group	groupe, <i>m</i>
selbstdual	self-dual	auto-dual
Dualität, <i>f</i>	duality	dualité, <i>f</i>
Saite, <i>f</i>	string	corde, <i>f</i>
Saitentheorie, <i>f</i>	string theory	théorie de cordes, <i>f</i>
Urknall, <i>m</i>	big bang	'big bang', <i>m</i> , (gros boum, <i>m</i>)
Tröpfchenbild, <i>n</i>	droplet picture	image de gouttes, <i>m</i>
Auswahlregel, <i>f</i>	selection rule	règle de sélection, <i>f</i>
Zustandssumme, <i>f</i>	partition function	fonction de partition, <i>f</i>
Hamiltonoperator, <i>m</i>	Hamiltonian	l'hamiltonien, <i>m</i>
Laplaceoperator, <i>m</i>	Laplacian	le laplacien, <i>m</i>
Mastergleichung, <i>f</i>	master equation	équation maîtresse, <i>f</i>
Transfermatrix, <i>f</i>	transfer matrix	matrice de transfert, <i>f</i>
Transversalfeld, <i>n</i>	transverse field	champ transverse, <i>m</i>
Isingmodell, <i>n</i>	Ising model	modèle d'Ising, <i>m</i>
Pottsmodell, <i>n</i>	Potts model	modèle de Potts, <i>m</i>
Uhrmodell, <i>n</i>	clock model	modèle d'horloge, <i>m</i>
Wählermodell, <i>n</i>	voter model	modèle d'électeur, <i>m</i>
Starkbindungsmodell, <i>n</i>	tight-binding model	modèle de liaisons fortes, <i>m</i>
Spinwelle, <i>f</i>	spin wave	onde de spins, <i>f</i>
Punktperkolation, <i>f</i>	site percolation	percolation de sites, <i>f</i>
Kantenperkolation, <i>f</i>	bond percolation	percolation de liens, <i>f</i>
gerichtete Perkolation, <i>f</i>	directed percolation	percolation dirigée (orientée), <i>f</i>
Spannungstensor, <i>m</i>	stress-energy tensor	tenseur d'énergie- contrainte, <i>m</i>
Energieimpulstensor, <i>m</i>	energy-momentum tensor	tenseur impulsion- énergie, <i>m</i>

selbstkonsistent	self-consistent	auto-cohérent
Betheansatz, <i>m</i>	Bethe ansatz	ansatz de Bethe, <i>m</i>
Abschneidemethode, <i>f</i>	truncation method	méthode de troncation, <i>f</i>
Anfangsbedingung, <i>f</i>	initial condition	condition initiale, <i>f</i>
Randbedingung, <i>f</i>	boundary condition	condition de bord, <i>f</i>
Dauerstrom, <i>m</i>	persistent current	courant permanent, <i>m</i>
Magnet, <i>m</i>	magnet	aimant, <i>m</i>
Magnetisierung, <i>f</i>	magnetisation	aimantation, <i>f</i>
Magnetismus, <i>m</i>	magnetism	magnétisme, <i>m</i>
magnetisch	magnetic	magnétique (aimanté)
Glas, <i>n</i>	glass	verre, <i>m</i>
Abschrecken, <i>n</i>	quench	trempe, <i>f</i>
Altern, <i>n</i>	ageing	vieillissement, <i>m</i>
Unter-(Über-)altern, <i>n</i>	sub-(super-)ageing	sous-(sur-)vieillissement, <i>m</i>
alternd	ageing	vieillissant
abschrecken	to quench	tremper
ausglühen	to anneal	recuire
reifen	to ripen	mûrir
Reifung, <i>f</i>	ripening	mûrissement, <i>m</i>
Glätten, <i>n</i> ; Glättung, <i>f</i>	coarse-graining	lissage, <i>m</i>
Benetzung, <i>f</i>	wetting	mouillage, <i>m</i>
Scherung, <i>f</i>	shear	cisaillement, <i>m</i>
Steifigkeit, <i>f</i>	stiffness	rigidité, <i>f</i>
Rauhigkeit, <i>f</i>	roughness	rugosité, <i>f</i>
Zinnpest, <i>f</i>	tin pest (disease)	peste de l'étain, <i>f</i>
Phasenordnung, <i>f</i>	phase-ordering	mise en ordre de phases, <i>f</i>
Phasentrennung, <i>f</i>	phase-separation	séparation de phases, <i>f</i>
Antwort, <i>f</i>	response	réponse, <i>f</i>
Selbstantwort, <i>f</i>	autoresponse	autoréponse, <i>f</i>
Diffusion, <i>f</i>	diffusion	diffusion, <i>f</i>
Streuung, <i>f</i>	scattering	diffusion, <i>f</i>
Irrfahrt, <i>f</i> , Zufallsweg, <i>m</i>	random walk	marche aléatoire, <i>f</i>
Lévyflug, <i>m</i>	Levy flight	vol de Lévy, <i>m</i>
Haufen, <i>m</i>	cluster	amas, <i>m</i>
verzweigtes Polymer, <i>n</i>	lattice animal	animal sur réseau, <i>m</i>
selbstvermeidend	self-avoiding	auto-évitant
selbstmittelnd	self-averaging	auto-moyennant
Zufallszahl, <i>f</i>	random number	nombre aléatoire, <i>f</i>
Rechner, <i>m</i>	computer	ordinateur, <i>m</i>
Datei, <i>f</i>	file	fichier, <i>m</i>
Dateienverzeichnis, <i>n</i>	directory	répertoire, <i>m</i>
Internetz, <i>n</i>	internet	réseau inter, <i>m</i> ; toile, <i>f</i>
Elektropost, <i>f</i>	e-mail	courriel (c. électronique), <i>m</i>
Register, <i>n</i>	index	index, <i>m</i>
Anhang, <i>m</i>	appendix	annexe, <i>m</i>
Urheberrecht, <i>n</i>	copyright	droit d'auteur, <i>m</i>
aufzählen	to list	énumérer
(nach-, über-) prüfen	to test	vérifier, contrôler
verschieben	to shift	déplacer, glisser

Solutions

Problems of Chapter 1

1.1 With the initial condition $v(0) = v_0$, the solution of (1.135) is for $F = 0$

$$v(t) = v_0 e^{-\gamma t} + e^{-\gamma t} \int_0^t dt' e^{\gamma t'} \eta(t') .$$

Squaring and averaging leads for large times to $\langle v^2(\infty) \rangle = B/\gamma$ and the Einstein relation follows from the stated equipartition condition, valid at equilibrium. Next, for the calculation of the equilibrium response function, recall that $\langle v(t) \rangle = \int_{\mathbb{R}} dt' R(t - t') F(t')$ is a convolution. Hence, introducing the Fourier transform $\hat{f}(\omega) = \int_{\mathbb{R}} dt e^{i\omega t} f(t)$, one has $\langle \hat{v}(\omega) \rangle = \hat{R}(\omega) \hat{F}(\omega)$. Averaging over the noise in (1.135) then gives

$$\hat{R}(\omega) = \frac{1}{m} \frac{1}{\gamma - i\omega} .$$

For $F = 0$, the correlation function $C(t) = C(-t) = (T/m)e^{-\gamma|t|}$ is symmetric in time. Its Fourier transform is $\hat{C}(\omega) = (2T/m)\gamma/(\gamma^2 + \omega^2)$ and comparison with $\hat{R}(\omega)$ gives the *fluctuation-dissipation theorem*

$$\text{Im } \hat{R}(\omega) = \frac{\omega}{2\gamma T} \hat{C}(\omega) .$$

In order to see the equivalence of this with the form (1.8), one notes that

$$2 \text{Im } \hat{R}(\omega) = \frac{1}{i} \int_0^\infty dt (e^{i\omega t} - e^{-i\omega t}) R(t)$$

and where the lower integration limit follows from the *causality condition*: $R(t - t') = 0$ for $t < t'$. In the second term, one now changes $t \mapsto -t$ and

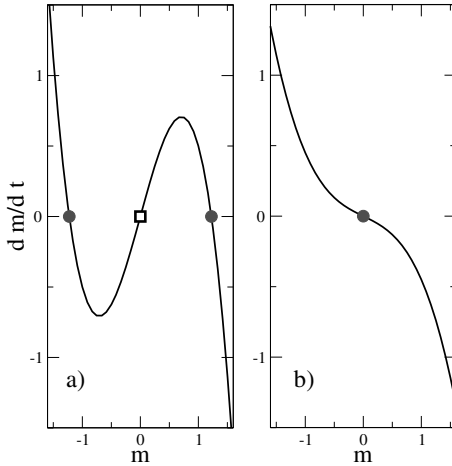


Fig. S.1. Vector field $(m(t), \dot{m}(t))$ for the relaxation of a simple magnet with a non-conserved order-parameter for (a) ordered phase with $\lambda^2 > 0$ ($T < T_c$) and (b) disordered phase with $\lambda^2 < 0$ ($T > T_c$). The fixed points m_∞ are shown, where filled points are stable and open points are unstable.

defines the *analytic continuation* $R(-t) \mapsto R(t)$ for $t > 0$. On the other hand, rewriting $\omega \hat{C}(\omega) = i \int_{\mathbb{R}} dt e^{i\omega t} \partial_t C(t)$ and comparing with the above form of the FDT leads back to (1.8), if one further scales $\gamma k_B \stackrel{!}{=} 1$.

Lastly, one can arrange for $x(0) = 0$ and $x(t) = \int_0^t dt' v(t')$ gives for the diffusion constant

$$D = \lim_{t \rightarrow \infty} \frac{\langle x^2(t) \rangle}{2t} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t dt' \int_0^{t'} d\tau \langle v(\tau) v(0) \rangle = \int_0^\infty d\tau \langle v(\tau) v(0) \rangle = \hat{C}(0)$$

where the limit $t \rightarrow \infty$ of the double integral was estimated by using l'Hôpital's rule. From this Einstein's expression of the diffusion constant is recovered.

1.2 The stationary solutions are $m_\infty = 0, \pm\sqrt{3}\lambda$. In order to analyse the stability of these, consider the vector field $(m(t), \dot{m}(t))$ associated to the mean-field equation of motion, illustrated in Fig. S.1. Letting $\delta m(t) = m(t) - m_\infty$, a fixed point of the equation $\frac{dm}{dt} = f(m)$ is *stable*, if $f'(m_\infty) < 0$ and is *unstable*, if $f'(m_\infty) > 0$. Clearly, for $\lambda^2 > 0$ the fixed points $m_\infty = \pm\sqrt{3}\lambda$ are stable, and the sign of m_0 decides to which fixed point the system evolves. On the other hand, for $\lambda^2 < 0$ the only stable fixed point is $m_\infty = 0$. Unless $m_0 = 0$, there is for large times an exponential relaxation $m(t) - m_\infty \sim e^{-t/\tau_{\text{rel}}}$, with the finite relaxation time

$$\frac{1}{\tau_{\text{rel}}} = \begin{cases} 3|\lambda^2| & ; \text{ if } \lambda^2 < 0 \\ 6\lambda^2 & ; \text{ if } \lambda^2 > 0 \end{cases}.$$

Studies of ageing consider what happens for $\lambda^2 > 0$ at the unstable fixed point $m_\infty = 0$, realised by the initial condition $m_0 = 0$.

1.3 From the Langevin equation, one has

$$m(t) = m(0) - \int_0^t dt' F(m(t')) + \int_0^t dt' (h(t') + \eta(t')) .$$

On one hand, one can set $h = 0$ and expand around the noiseless solution $\mu(t)$ which would be obtained for $\eta = 0$, where $t > s$ is assumed:

$$\begin{aligned} \langle m(t)\eta(s) \rangle &= \langle m(0)\eta(s) \rangle - \int_0^t dt' \left\langle \left[F(\mu(t')) + \frac{\partial F}{\partial m} \Big|_{m=\mu(t')} \eta(t') + \dots \right] \eta(s) \right\rangle \\ &\quad + \int_0^t dt' \langle \eta(t')\eta(s) \rangle \\ &= - \left\langle \frac{\partial F}{\partial m} \Big|_{m=\mu(s)} \right\rangle 2T + 2T + \dots \end{aligned}$$

where the neglected terms correspond to higher orders in η and one uses that $\mu(t)$ and $m(0)$ are by definition independent of the noise. On the other hand, one now expands in h , again for $t > s$:

$$\begin{aligned} \frac{\delta \langle m(t) \rangle}{\delta h(s)} &= - \frac{\delta}{\delta h(s)} \int_0^t dt' \left\langle \left[F(m(t')) + \frac{\partial F}{\partial m} \Big|_{m=m(t')} h(t') + \dots \right] \right\rangle \\ &\quad + \int_0^t dt' \delta(t' - s) \\ &= 1 - \left\langle \frac{\partial F}{\partial m} \Big|_{m=m(s)} \right\rangle + \dots \end{aligned}$$

where the neglected terms are at least of order $O(h^2)$. If one now lets $h \rightarrow 0$, then $m(s) \rightarrow \mu(s)$. The comparison of the two expansions gives the assertion.

1.4 Use (1.12) at time s and multiply with $m(t)$. Then use that the cumulant $\langle\langle m(t)m(s)^3 \rangle\rangle = \langle m(t)m(s)^3 \rangle - 3\langle m(t)m(s) \rangle \langle m(s)^2 \rangle \stackrel{!}{=} 0$ in mean-field theory in order to carry out the average.

1.5 Defining the variance $v(t) := \langle m(t, \mathbf{r})^2 \rangle$ and using $m^3 \simeq 3v(t)m$, the mean-field approximation to the equation of motion (1.136) becomes $\partial_t m = \Delta m + 3(\lambda^2 - v(t))m + \eta$. Multiplying with $\eta(s, \mathbf{r}')$ and similarly with m leads to (in the equation for C rotation-invariance was also used)

$$\begin{aligned} \partial_t R(t, s; \mathbf{r}) &= [\Delta_{\mathbf{r}} + 3(\lambda^2 - v(t))] R(t, s; \mathbf{r}) + \delta(t - s)\delta(\mathbf{r}) \\ \partial_s C(t, s; \mathbf{r}) &= [\Delta_{\mathbf{r}} + 3(\lambda^2 - v(s))] C(t, s; \mathbf{r}) + 2TR(t, s; \mathbf{r}) \end{aligned}$$

If one takes first $\mathbf{r} = \mathbf{r}'$ and then lets $t \rightarrow s$, one obtains the equation of motion $\dot{v}(t) = 6(\lambda^2 - v(t))v(t)$ for the variance. Its time-dependence determines the behaviour of the observables R and C and since $\lambda^2 - v(t) \rightarrow 0$ for $\lambda^2 \geq 0$ while $\lambda^2 - v(t) \rightarrow -|\lambda^2| < 0$ for $\lambda^2 < 0$, one understands where the slow dynamics for $T \leq T_c$ comes from. Explicitly, if $\lambda^2 \neq 0$

$$v(t) = \lambda^2 \left[1 - \left(1 - \frac{\lambda^2}{v(0)} \right) e^{-6\lambda^2 t} \right]^{-1} \simeq \begin{cases} \lambda^2 + O\left(e^{-6\lambda^2 t}\right) & ; \text{ if } \lambda^2 > 0 \\ \frac{|\lambda^2|}{1 + |\lambda^2|/v(0)} e^{-6|\lambda^2|t} & ; \text{ if } \lambda^2 < 0 \end{cases}$$

together with the leading long-time behaviour. In the critical case $\lambda^2 = 0$, we have $v(t) = v(0)/(1 + 6v(0)t) \sim (6t)^{-1}$. The above equations of motion are best solved in Fourier space, where

$$\begin{aligned} \partial_t \hat{R}_{\mathbf{q}}(t, s) &= [-q^2 + 3(\lambda^2 - v(t))] \hat{R}_{\mathbf{q}}(t, s) + \delta(t - s) \\ \partial_s \hat{C}_{\mathbf{q}}(t, s) &= [-q^2 + 3(\lambda^2 - v(s))] \hat{C}_{\mathbf{q}}(t, s) + 2T \hat{R}_{\mathbf{q}}(t, s) \end{aligned} \quad (\text{S.29})$$

The case treated in the text is now recovered in the long-range limit $q \rightarrow 0$.

The response function must respect causality. It will therefore be sought in the form $\hat{R}_{\mathbf{q}}(t, s) = \Theta(t - s) r_{\mathbf{q}}(t, s)$ which implies the differential equation

$$\partial_t r_{\mathbf{q}}(t, s) = [-q^2 + 3(\lambda^2 - v(t))] r_{\mathbf{q}}(t, s)$$

such that the singular terms are cancelled by the initial condition $r_{\mathbf{q}}(t, t) = 1$. The solution reads, after a straightforward calculation

$$\hat{R}_{\mathbf{q}}(t, s) = \Theta(t - s) e^{-q^2(t-s)} \begin{cases} \left(\frac{1 - \alpha \exp(-6\lambda^2 s)}{1 - \alpha \exp(-6\lambda^2 t)} \right)^{1/2} & ; \text{ if } \lambda^2 \neq 0 \\ \left(\frac{1 + 6v_0 s}{1 + 6v_0 t} \right)^{1/2} & ; \text{ if } \lambda^2 = 0 \end{cases}$$

where $\alpha = 1 - \lambda^2/v_0$ and $v_0 = v(0)$. In particular, one recovers (1.16) in the $q \rightarrow 0$ limit and for large enough times (where instead of correctly $\hat{R}_{\mathbf{0}}(t, s)$, we sloppily wrote $R(t, s)$).

Next, we consider the correlator $\hat{C}_{\mathbf{q}}(t, s)$. For clarity, we consider first the case $\mathbf{q} = \mathbf{0}$ and treat the case $\lambda^2 \neq 0$. The equation of motion (S.29) without the right term has in the limit $\mathbf{q} \rightarrow \mathbf{0}$ the ‘homogeneous’ solution $\hat{C}_{\mathbf{h},0}(t, s) = \gamma(t) [(1 - \alpha)/(1 - \alpha \exp(-6\lambda^2 s))]^{1/2}$, where $\alpha = 1 - \lambda^2/v_0$ and the function $\gamma(t)$ remains to be determined. Performing a variation of the constant, one considers $\gamma(t) \mapsto \gamma(t, s)$ for which one easily derives an equation from (S.29). Inserting the result into the full correlator, we have

$$\begin{aligned} \hat{C}_{\mathbf{0}}(t, s) &= \frac{\gamma(t)(1 - \alpha)^{1/2}}{(1 - \alpha e^{-6\lambda^2 s})^{1/2}} \\ &\quad + 2T \frac{\min(t, s) - \alpha/(6\lambda^2) [\exp(-6\lambda^2 \min(t, s)) - 1]}{[(1 - \alpha e^{-6\lambda^2 t})(1 - \alpha e^{-6\lambda^2 s})]^{1/2}}. \end{aligned}$$

Herein, the remaining function $\gamma(t)$ is found from the condition that $\hat{C}_{\mathbf{0}}(t, s) \stackrel{!}{=} \hat{C}_{\mathbf{0}}(s, t)$ must be symmetric in t and s . The most simple way to use this is to set $s = 0$, which leads to

$$\gamma(t) = \widehat{C}_0(t, 0) = \widehat{C}_0(0, t) = \gamma(0) [(1 - \alpha)/(1 - \alpha \exp(-6\lambda^2 t))]^{1/2}.$$

Identifying $\gamma(0) = \widehat{C}_0(0)$, the final result is, for $\lambda^2 \neq 0$

$$\begin{aligned} \widehat{C}_0(t, s) = & \frac{\widehat{C}_0(0)(1 - \alpha)}{(1 - \alpha e^{-6\lambda^2 t})^{1/2}(1 - \alpha e^{-6\lambda^2 s})^{1/2}} \\ & + 2T \frac{\min(t, s) - \alpha/(6\lambda^2) [\exp(-6\lambda^2 \min(t, s)) - 1]}{[(1 - \alpha e^{-6\lambda^2 t})(1 - \alpha e^{-6\lambda^2 s})]^{1/2}}. \end{aligned}$$

For $\lambda^2 = 0$, an analogous calculation leads to

$$\widehat{C}_0(t, s) = \frac{\widehat{C}_0(0) + 2T_c [\min(t, s) + 3v_0 \min(t, s)^2]}{[(1 + 6v_0 t)(1 + 6v_0 s)]^{1/2}}.$$

From these expressions, the limit $t, s \gg 1$ with $t > s$ leads to the scaling forms quoted in the text (writing sloppily $\widehat{C}(t, s)$ instead of correctly $\widehat{C}_0(t, s)$).

The non-global correlator $\widehat{C}_q(t, s)$ with $q \neq 0$ is found by an analogous, if just a little more tedious, calculation. Taking the symmetry condition $\widehat{C}_q(t, s) \stackrel{!}{=} \widehat{C}_q(s, t)$ into account, we find for $\lambda^2 \neq 0$

$$\begin{aligned} \widehat{C}_q(t, s) = & \frac{\widehat{C}_q(0)(1 - \alpha) \exp(-q^2(t + s))}{[(1 - \alpha e^{-6\lambda^2 t})(1 - \alpha e^{-6\lambda^2 s})]^{1/2}} \\ & + \frac{2T}{[(1 - \alpha e^{-6\lambda^2 t})(1 - \alpha e^{-6\lambda^2 s})]^{1/2}} \left[\frac{1}{2q^2} \left(e^{-q^2|t-s|} - e^{-q^2(t+s)} \right) \right. \\ & \left. - \frac{\alpha}{2q^2 - 6\lambda^2} \left(e^{-q^2|t-s| - 6\lambda^2 \min(t, s)} - e^{-q^2(t+s)} \right) \right] \end{aligned}$$

and similarly for $\lambda^2 = 0$

$$\begin{aligned} \widehat{C}_q(t, s) = & \frac{\widehat{C}_q(0)(1 - \alpha) \exp(-q^2(t + s))}{[(1 + 6v_0 t)(1 + 6v_0 s)]^{1/2}} \\ & + \frac{2T_c}{[(1 + 6v_0 t)(1 + 6v_0 s)]^{1/2}} \frac{1}{2q^2} \left[e^{-q^2|t-s|} (1 + 6v_0 \min(t, s)) \right. \\ & \left. - e^{-q^2(t+s)} - \frac{3v_0}{q^2} \left(e^{-q^2|t-s|} - e^{-q^2(t+s)} \right) \right]. \end{aligned}$$

The leading critical long-time behaviour is obtained in the $q \rightarrow 0$ limit and leads to (1.17). From these, the mean-field fluctuation-dissipation ratio quoted in the text is readily found.

1.6 Inserting $\mathcal{V}(\phi) = (1 - \phi^2)^2$ into (1.37) readily gives, using the initial condition $\phi(0) = 0$, the profile $\phi(g) = \tanh(\sqrt{2}g)$. Similarly, one finds the surface tension $\sigma = 4\sqrt{2}/3$.

For a discussion of the BW and GL potentials, begin by tracing these as a function of ϕ and discuss their qualitative and quantitative properties. Make

similar plots for the profiles. If necessary, perform the integrals numerically and discuss the dependence on T .

1.7 In both cases, one has $\rho(\mathbf{x}) = Z^{-1} \exp(-\mathcal{H}[\mathbf{x}]/T)$, where Z is the canonical partition function, hence $s_j = T^{-1} \partial \mathcal{H}[\mathbf{x}] / \partial x_j$. If the dynamics is given by a standard Langevin equation, one has after a suitable choice of units $s_j = -T^{-1} \dot{x}_j$, hence $R_{x_i x_j}(t) = T^{-1} \langle x_i(t) \dot{x}_j(0) \rangle$ and the assertion follows.

1.8 Following [215], we admit that for critical quenches, the ageing part of the two-time correlator has the scaling form $C_{\text{age}}(t, s) = s^{-b} \mathcal{C}(h(t)/h(s))$. The form of the function $h = h(s)$ is found by matching this with the form which describes the passage between the quasi-static and the ageing regime

$$C_{\text{age}}(t, s) = s^{-b} \mathcal{C} \left(\frac{h(t)}{h(s)} \right) \stackrel{!}{=} s^{-\alpha} c_1 \left(\frac{t-s}{s^\zeta} \right) \quad (\text{S.30})$$

where α is some exponent and c_1 is a scaling function. In the scaling limit where both s and $t-s$ are large, but such that the scaling variable x in $t-s = xs^\zeta$ remains finite, a first-order expansion gives

$$C_{\text{age}}(t, s) \simeq s^{-b} \mathcal{C} \left(1 + xs^\zeta \frac{d \ln h(s)}{ds} \right) \simeq s^{-b} \left(A xs^\zeta \frac{d \ln h(s)}{ds} \right)^\beta.$$

In the second step a power-law behaviour $\mathcal{C}(1+x) \sim x^\beta$ for $x \rightarrow 0$ was assumed and A is a constant. Equation (S.30) leads to the differential equation

$$\frac{d \ln h(s)}{ds} = A^{-1} s^{-\mu}, \quad \mu := \zeta - (b - \alpha)/\beta$$

which upon integration reproduces (1.82).

1.9 The assertion follows from a straightforward generalisation of the derivation of the YRD inequality and is indeed reproduced in the spin-wave approximation of the 2D XY model, see Table 2.10 and [633].

1.10 See Fig. 1.16 for the experimental motivation of this exercise. Through a change of variables, the correlator reads

$$C(t, \mathbf{r}) = e^{i\mathbf{k}_0 \cdot \mathbf{r}} \int_{\mathbb{R}^d} d\mathbf{q} \frac{\exp(i\mathbf{q} \cdot \mathbf{r})}{(1 + q^2/\Gamma^2)^2} =: e^{i\mathbf{k}_0 \cdot \mathbf{r}} I(\mathbf{r}).$$

The global phase factor is unobservable and the explicit calculation gives

$$C(t, \mathbf{r}) = C(\mathbf{0}) \times \begin{cases} e^{-r\Gamma}(1 + r\Gamma) & ; \text{ if } d = 1 \\ r\Gamma K_1(r\Gamma) & ; \text{ if } d = 2 \\ e^{-r\Gamma} & ; \text{ if } d = 3 \end{cases} \quad (\text{S.31})$$

where K_1 is a modified Bessel function [4].

The calculation of $I(\mathbf{r})$ in $d = 1, 3$ is simplified through the following

Lemma 1: *If $P(z)$ and $Q(z)$ are entire functions of the complex variable z and furthermore $P(z_0) \neq 0$, $Q(z_0) = Q'(z_0) = 0$ and $Q''(z_0) \neq 0$, then the complex function $f(z) = P(z)/Q(z)$ has a pole of second order at $z = z_0$ with the residue*

$$\text{Res} \left(\frac{P(z)}{Q(z)}, z_0 \right) = \frac{2P'(z_0)}{Q''(z_0)} - \frac{2}{3} \frac{P(z_0)Q'''(z_0)}{(Q''(z_0))^2}. \quad (\text{S.32})$$

Proof: by direct expansion of $P(z)$ and $Q(z)$ around $z = z_0$. q.e.d.

Now, for $d = 1$ the announced result directly follows from (S.32) and the residue theorem. For $d = 3$, one introduces spherical coordinates and finds

$$I_{(d=3)}(\mathbf{r}) = \frac{4\pi}{r} \int_0^\infty dq \frac{q \sin rq}{(1 + q^2/\Gamma^2)^2} = \frac{2\pi}{r} \Im \int_{\mathbb{R}} dq \frac{q \exp(irq)}{(1 + q^2/\Gamma^2)^2}$$

since the integrand is even in q . This is now easily evaluated from the residue calculus, using again (S.32). Finally, for $d = 2$ one uses polar coordinates

$$I_{(d=2)}(\mathbf{r}) = \int_0^\infty dq q \int_0^{2\pi} d\theta \frac{\exp(iqr \cos \theta)}{(1 + q^2/\Gamma^2)^2} = 2\pi \int_0^\infty dq \frac{q J_0(qr)}{(1 + q^2/\Gamma^2)^2}$$

where J_0 is a Bessel function [4]. The calculation is now completed by the identity [293, (6.565.4)]

$$\int_0^\infty dx J_0(bx) x (x^2 + a^2)^{-2} = \frac{b}{2a} K_1(ab).$$

For $d = 3$, a simple Lorentzian does not satisfy Porod's law, but it does so for $d = 1$. On the other hand, a Lorentzian-squared structure factor does not reproduce Porod's law for $d \neq 3$ and $C(\mathbf{r})$ no longer has a cusp at $r = 0$. The calculation of $C(t, \mathbf{r})$ is analogous to the case above, and is facilitated by

Lemma 2: *If $P(z)$ and $Q(z)$ are entire functions of the complex variable z and furthermore $P(z_0) \neq 0$, $Q(z_0) = 0$ and $Q'(z_0) \neq 0$, then the complex function $f(z) = P(z)/Q(z)$ has a pole of first order at $z = z_0$ with the residue*

$$\text{Res} \left(\frac{P(z)}{Q(z)}, z_0 \right) = \frac{P(z_0)}{Q'(z_0)}. \quad (\text{S.33})$$

The proof of (S.33) is left to the reader.

1.11 In the limit $s \rightarrow \infty$, with $y = t/s > 1$, one is in the ageing regime and therefore one may write

$$\begin{aligned} f_{\text{Int}}(y) &= \lim_{s \rightarrow \infty} s^a \int_{s/2}^s du u^{-1-a} f_R \left(\frac{ys}{u} \right) \\ &= \int_{1/2}^1 dv v^{-1-a} f_R \left(\frac{y}{v} \right) \sim y^{-\lambda_R/z} \underbrace{\int_{1/2}^1 dv v^{\lambda_R/z-1-a}}_{= \text{constant}}. \end{aligned}$$

where in the last step, the asymptotic behaviour of $f_R(y) \sim y^{-\lambda_R/z}$ for $y \rightarrow \infty$ was used. Hence $f_{\text{Int}}(y) \sim y^{-\lambda_R/z}$. Analogous results hold for the scaling functions $f_M(y)$ and $f_\chi(y)$ of the other integrated susceptibilities.

1.12 The global correlators and responses are obtained as the $\mathbf{q} \rightarrow \mathbf{0}$ limit of the Fourier transforms of the real-space correlators and responses

$$\begin{aligned}\widehat{C}_{\mathbf{q}}(t, s) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} C(t, s; \mathbf{r}) \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} L(s)^{-bz} F_C \left(\frac{L(t)}{L(s)}, \frac{\mathbf{r}}{L(t-s)} \right) \\ &= L(s)^{-bz} L(t-s)^d \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} d\mathbf{u} e^{-i\mathbf{q} \cdot \mathbf{u} L(t-s)} F_C \left(\frac{L(t)}{L(s)}, \mathbf{u} \right)\end{aligned}$$

Hence, as $\mathbf{q} \rightarrow \mathbf{0}$, one obtains for the global correlator the scaling form $\widehat{C}_{\mathbf{0}}(t, s) = L(s)^{-bz} L(t-s)^d \widehat{f}_C(t/s)$, distinct from the form (1.74). A similar result holds for the global response function $\widehat{R}_{\mathbf{0}}(t, s)$, with b replaced by $1+a$.

We warn the reader that distinction between *autocorrelators/autoreponses* on one hand and *global correlators/responses* on the other hand is often disrespected in the literature, which can easily lead to confusions.

1.13 The proof outlined here uses times, rather than frequencies as in the original formulation [427]. Recall from exercise 1.3 how to rewrite the response function as a correlator of the order-parameter $\phi(t)$ with the noise $\eta(t')$. Using this relationship in Fourier space, one has

$$\begin{aligned}\left| 2T \widehat{R}_{\mathbf{k}}(t, t) \right| &\leq \left\langle \left| \widehat{\phi}_{\mathbf{k}}(t) \widehat{\eta}_{\mathbf{k}}(t) \right| \right\rangle \\ &\leq \sqrt{\left\langle \left| \widehat{\phi}_{\mathbf{k}}(t) \right|^2 \right\rangle \left\langle \left| \widehat{\eta}_{\mathbf{k}}(t) \right|^2 \right\rangle} = \sqrt{\widehat{C}_{\mathbf{k}}(t, t) 2T},\end{aligned}$$

where the average was considered as a scalar product and the Cauchy-Schwarz inequality was applied. Squaring leads to the assertion. It does provide a non-trivial constraint in the study of dynamical systems.

If one further speculates that this statement might remain valid in the scaling regime, one might reconsider the scaling forms $\widehat{C}_{\mathbf{0}}(s, s) \sim s^{d/z-b}$ and $\widehat{R}_{\mathbf{0}}(s, s) \sim s^{d/z-1-a}$ from exercise 1.12. If one has $1+a \leq d/z$, then the Katzav-Schwartz inequality would lead to $b-2a \leq 2-d/z$, or equivalently

$$b-a \leq \underbrace{1+a-d/z}_{\leq 0} + 1 \leq 1.$$

1.14 Expanding the initial probability distribution up to terms linear in m_0 gives

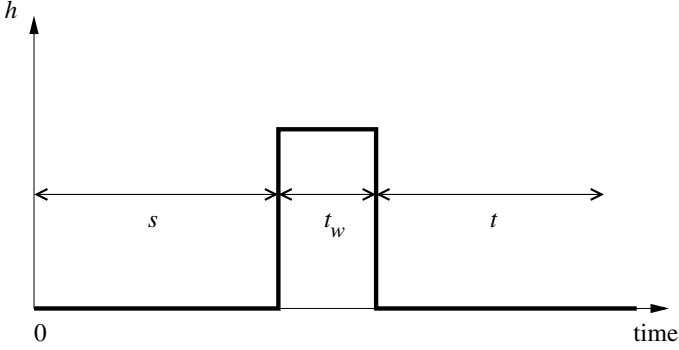


Fig. S.2. Schematic measurement protocol of an integrated response in equilibrium critical dynamics. The initial instant is defined by the quench to $T = T_c$, the time s is the equilibration time, after which an external field is applied for a waiting time t_w . The measurement time t is here defined from the instant the external field was turned off.

$$P(\{\sigma\}; 0) \simeq 2^{-\mathcal{N}} \left(1 + m_0 \sum_{j \in \Lambda} \sigma_j \right)$$

and hence the magnetisation is, up to terms linear in m_0 and, where $\mathcal{T} = \exp(tW)$ is the formal time-evolution operator,

$$M(t) = 2^{-\mathcal{N}} \sum_{\sigma, \sigma'} \overbrace{\left(\sum_{j \in \Lambda} \sigma_j \right)}^{=0} \mathcal{T}(\{\sigma, \sigma'\}; t) + \frac{m_0}{2^{\mathcal{N}}} \sum_{\sigma, \sigma'} \left(\sum_{i \in \Lambda} \sigma_i \right) \mathcal{T}(\{\sigma, \sigma'\}; t) \left(\sum_{j \in \Lambda} \sigma'_j \right)$$

and the term in the first line vanishes because of the up-down symmetry. Since $P(\{\sigma\}; 0) = 2^{-\mathcal{N}}$ is the probability of the initial state with a vanishing correlation length and vanishing magnetisation, the first part of the assertion (1.140) follows. On the other hand, one expects $M(t) \sim m_0 t^\Theta$ in the critical short-time regime.

1.15 This protocol, with the initial long equilibration time s (see Fig. S.2), rather brings the system back to equilibrium, after the long equilibration time s has passed. Rather than studying ageing, which usually occurs far from equilibrium, this kind of experiment looks at *equilibrium critical dynamics*. The presence of some kind of critical dynamics is indeed suggested by the empirical observation of t_w/t dynamical scaling.

Assuming that the measured conductance is still in the linear response regime, one is led to consider the following phenomenological form of a two-

time response function $R(t, u) = r_0 [(t-u)/t_0]^{-1-a}$, where the assumed power-law scaling is taken into account. Here t_0 is a microscopic reference time and r_0 is a dimensionless normalisation constant. For $s \rightarrow \infty$, one relaxes back to equilibrium, where one expects from the FDT that $a = (d - 2 + \eta)/z$. Furthermore, the exponent a should be rather small, since the disorder in samples which may be considered as Anderson insulators should lead to a large value of the dynamical exponent z and the equilibrium critical exponent η is in general small as well.

We therefore consider the following integrated response, in the limit of a large equilibration time $s \rightarrow \infty$,

$$\begin{aligned}\chi(t, t_w) &= \int_s^{s+t_w} du R(s + t_w + t, u) \\ &= r_0 \int_s^{s+t_w} du \left(\frac{s + t_w + t - u}{t_0} \right)^{-1-a} \\ &= \frac{\chi_0}{a} \left[\left(e^{-a \ln(t/t_0)} - 1 \right) - \left(e^{-a \ln((t_w+t)/t_0)} - 1 \right) \right] \\ &\simeq \chi_0 \ln \left(1 + \frac{t_w}{t} \right) \left[1 - a \left(\ln \frac{t}{t_0} + \frac{1}{2} \ln \left(1 + \frac{t_w}{t} \right) \right) + O(a^2) \right]\end{aligned}$$

with $\chi_0 := r_0 t_0$. In the second line the phenomenological form of $R(t, u)$ was used, together with a change of variable and in the last line an expansion in a was performed. In the limit $a \rightarrow 0$, one recovers the observed data collapse and the scaling function $g(y) \sim \ln(1 + y)$.

Very similar scaling forms have been discussed in the literature [15]. On the other hand, the real experimental data on 5nm thick films of the glassy Anderson insulator $\text{In}_2\text{O}_{3-x}$ rather produce a stretched exponential scaling form $\Delta G(t, t_w) \sim \exp(-b(t/t_w)^\alpha)$, with $\alpha \approx 0.21$ and b a constant [709].

1.16 Using the scaling forms (1.74, 1.75), one has

$$X(t, s) = -s^{b-a} \frac{T f_R(t/s)}{b f_C(t/s) + (t/s) f'_C(t/s)}$$

where the prime denotes the derivative. This has a finite limit for $s \rightarrow \infty$ only if $a = b$ (for quenches to $T > T_c$, $X(t, s) \rightarrow 0$) and then $X(t, s) = f_X(t/s)$. With the asymptotic form $f_{C,R}(y) \simeq f_{C,R}^{(\infty)} y^{-\lambda_{C,R}/z}$, one has for $y = t/s \gg 1$

$$f_X(y) \simeq T \frac{f_R^{(\infty)}}{f_C^{(\infty)}} \left(\frac{\lambda_C}{z} - b \right)^{-1} y^{(\lambda_C - \lambda_R)/z}$$

which has a finite limit $X_\infty := \lim_{y \rightarrow \infty} f_X(y)$ only if furthermore $\lambda_C = \lambda_R$. For non-equilibrium critical dynamics, the two conditions on the exponents are satisfied and $X_\infty > 0$ follows from the YRD inequality (1.91). If $X_\infty \neq 1$, the system never reaches equilibrium.

The universality of X_∞ , initially conjectured by Godrèche and Luck [285, 286], has been confirmed by detailed field-theoretical renormalisation-group studies [118, 116, 119, 121] for quenches to $T = T_c$.

1.17 The quantity $\overline{X}_\infty = 1$. To understand this, consider the second limit (1.142). Since the passage time between the quasi-stationary regime and the ageing regime is of the order $t_p(s) \sim t^\zeta$ with $0 < \zeta < 1$, one arrives for s sufficiently large always in the regime $\tau \ll t_p(s)$, where $X(\tau + s, s) \xrightarrow{s \rightarrow \infty} 1$.

The reader should feel encouraged to check this statement on the explicit results of the exactly solved systems of Chap. 2, see also [285, 286] for a detailed discussion of the Ising and spherical models.

1.18 Following [144], one can always arrange for $\langle X(t) \rangle = 0$. Let $\sigma^2(t) := \langle X(t)X(t) \rangle$ and $N(t_1, t_2) := \langle X(t_1)X(t_2) \rangle / (\sigma(t_1)\sigma(t_2))$. Then the conditional probability for a Gaussian process to be at x_1 at time t_1 , provided that it was at x_2 at an earlier time $t_2 < t_1$, reads explicitly

$$P(t_1, x_1 | t_2, x_2) = (2\pi\sigma^2(t_1) (1 - N^2(t_1, t_2)))^{-1} \\ \times \exp \left[-\frac{1}{2} \frac{1}{\sigma^2(t_1) (1 - N^2(t_1, t_2))} \left(x_1 - N(t_1, t_2) \frac{\sigma(t_1)}{\sigma(t_2)} \right)^2 \right]. \quad (\text{S.34})$$

Consider the conditional average of $X(t)$ at time t_1 , given that it had the value x_3 at an earlier time t_3 and calculate this in two different ways. First, one uses (1.146) and from the explicit form (S.34) one reads off

$$\langle X(t_1) \rangle |_{X(t_3)=x_3} = \int dx_1 x_1 P(t_1, x_1 | t_3, x_3) = \frac{\sigma(t_1)}{\sigma(t_3)} N(t_1, t_3) x_3.$$

On the other hand, one uses (1.145) to insert an integration over an intermediate position x_2 at time $t_1 > t_2 > t_3$

$$\begin{aligned} \langle X(t_1) \rangle |_{X(t_3)=x_3} &= \int dx_1 dx_2 x_1 P(t_1, x_1 | t_2, x_2) P(t_2, x_2 | t_3, x_3) \\ &= \frac{\sigma(t_1)}{\sigma(t_2)} N(t_1, t_2) \int dx_2 x_2 P(t_2, x_2 | t_3, x_3) \\ &= \frac{\sigma(t_1)}{\sigma(t_2)} \frac{\sigma(t_2)}{\sigma(t_3)} N(t_1, t_2) N(t_2, t_3) x_3 \end{aligned}$$

and comparison with the first form leads to $N(t_1, t_3) = N(t_1, t_2)N(t_2, t_3)$. For a stationary process, (1.144) implies $N(t_1, t_2) = n(t_1 - t_2)$, hence $n(t_1 - t_3) = n(t_1 - t_2)n(t_2 - t_3)$. This is the functional equation of the exponential, hence $n(t) = e^{-\mu t}$ and the first part of the assertion (1.147) follows.

For the converse, recall that the Ornstein-Uhlenbeck process satisfies the condition on the autocorrelator and is explicitly Markovian. Since any Gaus-

sian process is uniquely characterised by its first two moments, the proof is complete.

1.19 For sufficiently large blocks, i.e. $\ell \gg L(t)$, the block persistence $P_b(t, \ell) \sim t^{-\theta_g}$ behaves essentially as the global persistence while for small blocks with $\ell \ll L(t)$ one finds the behaviour $P_b(t, \ell) \sim t^{-\theta_\ell}$ of the local persistence. For large ℓ with the ratio $\ell/L(t)$ being kept fixed, one expects the scaling behaviour

$$P_b(t, \ell) = \ell^{-z\theta_g} F(L(t)/\ell) = \ell^{-z\theta_g} f(t/\ell^z)$$

where $f(x) \sim x^{-\theta_g}$ for $x \rightarrow 0$ and $f(x) \sim x^{-\theta_\ell}$ for $x \rightarrow \infty$. The exponent in the above scaling form is fixed from the requirement that for t fixed and $\ell \rightarrow \infty$, one must recover the global persistence.

This cross-over has been confirmed in detail, both for $T < T_c$ and for $T = T_c$, in several variants of kinetic Ising models and the TDGL, without and with a conserved order-parameter [165].

Problems of Chapter 2

2.1 From the heat-bath rule (2.1) one has directly

$$\frac{dM_n(t)}{dt} = -M_n(t) + \langle \tanh(h_n(t)/T) \rangle$$

with three possible values $h_n(t) = 0, \pm 2J$ for the local magnetic field. Hence $\tanh(h_n(t)/T) = (2J)^{-1} \tanh(2J/T) h_n(t)$ and the assertion (2.174) follows.

An important special case is obtained for the initial condition $M_n(0) = \delta_{n,0}$. The corresponding solution is the **Green's function** and will be denoted here and in the main text by $G_n(t)$. A simple way to solve (2.174) uses a generating function $F(z, t) := \sum_{n \in \mathbb{Z}} z^n M_n(t)$ which satisfies the differential equation

$$\frac{\partial F(z, t)}{\partial t} = \sum_{n \in \mathbb{Z}} z^n \frac{dM_n(t)}{dt} = \left(-1 + \frac{\gamma}{2} (z + z^{-1}) \right) F(z, t).$$

Hence

$$F(z, t) = F(z, 0) \exp \left(-t + \frac{\gamma t}{2} (z + z^{-1}) \right).$$

If one lets $z = 1$, one obtains from $F(1, t) = \sum_{n \in \mathbb{Z}} M_n(t) = M(t)$ the *global magnetisation* $M(t)$, which reads explicitly $M(t) = M(0)e^{-(1-\gamma)t}$, such that the relaxation time becomes $\tau_{\text{rel}}^{-1} = 1 - \gamma \approx 2e^{-4J/T}$ as $T \rightarrow 0$.

In order to find the Green's function $G_n(t)$, one now uses $F(z, 0) = \sum_{n \in \mathbb{Z}} z^n G_n(0) = 1$ and from the well-known generating function of the modified Bessel functions $I_n(t)$ [4], the final result is

$$G_n(t) = e^{-t} I_n(\gamma t) . \quad (\text{S.35})$$

The general solution for the time-dependent magnetisation on an infinite chain then becomes a convolution with $G_n(t)$

$$M_n(t) = (M_n * G_n)(t) = e^{-t} \sum_{\ell=-\infty}^{\infty} M_\ell(0) I_{n-\ell}(\gamma t) .$$

We observe in passing that from the long-time asymptotic formula $I_n(t) \simeq (2\pi t)^{-1/2} e^t (1 + O(1/t))$ [4] one has $M_n(t) \simeq M(0)(2\pi t)^{-1/2} \exp(-(1 - \gamma)t)$ which only depends on the initial global magnetisation $M(0)$.

Alternatively, one may as well use the discrete spatial Fourier transform

$$\widehat{f}(t, q) = \sum_{n \in \mathbb{Z}} f_n(t) e^{-inq} , \quad f_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \widehat{f}(t, q) e^{inq} .$$

The equation of motion for the magnetisation becomes in Fourier space $d\widehat{M}(t, q)/dt = (\gamma \cos q - 1)\widehat{M}(t, q)$. For the Green's function, the specific initial condition $\widehat{G}(0, q) = 1$ gives $\widehat{G}(t, q) = \exp((\gamma \cos q - 1)t)$ which in real space leads back to (S.35).

The equation of motion for the single-time correlation function $C_{n,m}(t) = \langle \sigma_n(t) \sigma_m(t) \rangle$ can be derived analogously. A straightforward calculation leads to

$$\frac{dC_{n,m}(t)}{dt} = -2C_{n,m}(t) + \frac{\gamma}{2} [C_{n,m-1}(t) + C_{n,m+1}(t) + C_{n-1,m}(t) + C_{n+1,m}(t)] \quad (\text{S.36})$$

If the initial conditions are such that $C_{n,m}(0) = C_{n-m}(0)$, then (S.36) conserves spatial translation-invariance for all times and $C_{n-m}(t) = C_{n,m}(t)$. We leave it to the reader to verify that for $n \neq 0$, $C_n(t)$ satisfies almost the same equation of motion (2.174) with M_n replaced by C_n but where the r.h.s. is multiplied by an extra factor 2. However, in order to take the boundary condition $C_0(t) = 1$ into account, it is easier to rewrite it as

$$\frac{dC_n(t)}{dt} = -2C_n(t) + \gamma (C_{n-1}(t) + C_{n+1}(t)) + v(t) \delta_{n,0}$$

for all $n \in \mathbb{Z}$ and where $v(t)$ is chosen such that $C_0(t) = 1$ is satisfied [285]. Equation (2.2) follows by exact replication of the derivation of (2.174) for the magnetisation, with the compatibility condition $C_n(t, t) = C_n(t)$.

The Green's function $G_n(t)$ derived above may be used to express the two-time correlator in terms of the single-time correlator via a convolution

$$C_n(s + \tau, s) = C_n(s) * G_n(\tau) = \sum_{m \in \mathbb{Z}} C_m(s) G_{n-m}(\tau)$$

with the time difference $\tau = t - s$. This is (2.17) in the main text.

2.2 The detailed balance condition (G3) relates the transition rates with the equilibrium probability distribution $P_\infty(\{\sigma\}) = Z^{-1} \exp(-\mathcal{H}[\{\sigma\}]/T)$. From the Ising Hamiltonian, one has (concentrating on the parts which change by the flip)

$$\begin{aligned} \frac{P_\infty(-\sigma_j)}{P_\infty(\sigma_j)} &= \frac{\exp[-(J/T) \sigma_j (\sigma_{j-1} + \sigma_{j+1})]}{\exp[(J/T) \sigma_j (\sigma_{j-1} + \sigma_{j+1})]} \\ &\stackrel{!}{=} \frac{w_j(\sigma_j)}{w_j(-\sigma_j)} = \frac{1 - \frac{1}{2}\gamma \sigma_j (\sigma_{j-1} + \sigma_{j+1})}{1 + \frac{1}{2}\gamma \sigma_j (\sigma_{j-1} + \sigma_{j+1})} \end{aligned}$$

which only depends on the values of the three spins $\sigma_{j,j\pm 1}$. Because of global spin-reversal symmetry, it is enough to consider the case $\sigma_j = +1$. Then, if $\sigma_{j-1} + \sigma_{j+1} = \pm 2$, the above condition leads to $\exp(-4J/T) = \frac{1-\gamma}{1+\gamma}$. Solving for γ , one obtains (2.5) in the main text. The other possibility $\sigma_{j-1} + \sigma_{j+1} = 0$ merely gives the trivial relation $1 = 1$.

2.3 Rewrite the master equation as a single sum, using an auxiliary variable σ' [281]

$$\partial_t P(\sigma_1, \dots, \sigma_N; t) = - \sum_n \sigma_n \left(\sum_{\sigma'=\pm 1} \sigma' \right) w_n(\sigma') P(\sigma_1, \dots, \underbrace{\sigma'}_{\text{site } n}, \dots, \sigma_N; t) \quad (\text{S.37})$$

In what follows, we use the abbreviation $\sum_{\{\sigma\}} := \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_N=\pm 1}$. For notational simplicity, consider first a one-point function

$$\begin{aligned} \frac{d\langle \sigma_a \rangle}{dt} &= \sum_{\{\sigma\}} \sigma_a \partial_t P(\sigma_1, \dots, \sigma_N; t) \\ &= - \sum_{\{\sigma\}} \sum_n \sigma_a \sigma_n \sum_{\sigma'=\pm 1} \sigma' w_n(\sigma') P(\sigma_1, \dots, \underbrace{\sigma'}_{\text{site } n}, \dots, \sigma_N; t) \\ &= - \sum_{\{\sigma\}} \sum_{\sigma'=\pm 1} \sigma' w_a(\sigma') P(\sigma_1, \dots, \underbrace{\sigma'}_{\text{site } a}, \dots, \sigma_N; t) \\ &= -2 \sum_{\sigma_a \notin \{\sigma\}} \sum_{\sigma'=\pm 1} \sigma' w_a(\sigma') P(\sigma_1, \dots, \underbrace{\sigma'}_{\text{site } a}, \dots, \sigma_N; t) \\ &= -2 \sum_{\{\sigma\}} \sigma_a w_a(\sigma_a) P(\sigma_1, \dots, \sigma_a, \dots, \sigma_N; t) \end{aligned}$$

which is (2.6) for the case $N = 1$. In the second line, one uses (S.37) and since $\sum_{\{\sigma\}} \sigma_n = \sum_{\sigma_n \notin \{\sigma\}} \sum_{\sigma_n=\pm 1} \sigma_n = 0$, it follows that all terms with $n \neq a$ vanish. In the third line, one isolates the variable σ_a , which may be summed over. In the last step, the auxiliary variable $\sigma' \mapsto \sigma_a$ is relabelled in a natural way.

The same method goes through for $N > 1$, but in the first step, one has the possibilities $n = n_1, \dots, n_N$, such that now N terms need to be retained.

2.4 Writing $M_n(t) := \langle \sigma_n \rangle(t)$, one has from (2.6) with $N = 1$ and with Glauber's transition rates (with $\alpha = 1$)

$$\begin{aligned}\dot{M}_n(t) &= - \left\langle \sigma_n - \frac{\gamma}{2} \sigma_n^2 (\sigma_{n-1} + \sigma_{n+1}) \right\rangle \\ &= -M_n(t) + \frac{\gamma}{2} (M_{n-1}(t) + M_{n+1}(t))\end{aligned}$$

since $\sigma_n^2 = 1$. This is indeed (2.174).

The extension to the correlators is completely analogous and (S.36) and (2.2) are readily recovered. As a simple illustration, consider the *global* magnetisation

$$\widehat{M}(t) = L^{-1} \sum_n M_n(t) = \widehat{M}(0) e^{-(1-\gamma)t}.$$

This decays monotonously towards zero, with a characteristic time-scale $\tau^{-1} = 1 - \gamma$.

2.5 The equations of motion obtained from exercises 2.1 or 2.4 have to be generalised as follows [281]

$$\begin{aligned}\dot{M}_0(t) &= 0 && \text{(recall the constraint } M_0(t) = 1) \\ \dot{M}_{\pm 1}(t) &= -M_{\pm 1}(t) + \frac{\gamma}{2} (M_0(t) + M_{\pm 2}(t)) \\ \dot{M}_n(t) &= -M_n(t) + \frac{\gamma}{2} (M_{n-1}(t) + M_{n+1}(t)) \quad ; \quad \text{if } |n| \geq 2\end{aligned}$$

Since the parts with $n > 0$ and $n < 0$ decouple from each other, it is enough to consider a semi-infinite system with $n = 0, 1, \dots$. The general solution of this kind of coupled linear equation is the sum of (i) a special solution with the boundary and (ii) the general solution of the infinite system without boundary, but here such that $M_{-\ell}(0) = -M_\ell(0)$.

A special solution is found by considering the stationary equation

$$M_n = \frac{\gamma}{2} (M_{n-1} + M_{n+1}) .$$

This may be solved by the ansatz $M_n = M^{(0)} \eta^{|n|}$, which leads to $\eta^2 - \frac{2}{\gamma} \eta + 1 = 0$. Hence, with the use of (2.5)

$$\eta = \frac{1}{\gamma} \left(1 - \sqrt{1 - \gamma^2} \right) = \tanh \frac{J}{T} . \quad (\text{S.38})$$

A basis for the solution of the system without boundary was shown in exercise 2.1 to be given by $M_n(0) e^{-t} I_n(\gamma t)$. The required boundary condition is taken into account by adapting the method of images, e.g. from electrostatics.

The required solution is then, with η given by (S.38)

$$M_n(t) = \eta^{|n|} + e^{-t} \sum_{\ell=1}^{\infty} [M_\ell(0) - \eta^\ell] [I_{|n|-\ell}(\gamma t) - I_{|n|+\ell}(\gamma t)] . \quad (\text{S.39})$$

We leave it to the reader to check that this indeed solves the above system of equations of motion and the constraint $M_0(t) = 1$, for all temperatures $T < \infty$.

2.6 Since $C_n(t) = C_{n,0}(t) = \langle \sigma_n(t) \sigma_0(t) \rangle$, $C_n(t)$ may be interpreted as the time-dependent conditional probability that $\sigma_n = 1$, provided that $\sigma_0 = 1$ is given. From (S.36) and the constraint $C_0(t) = 1$ it follows that one is back to the system treated in the previous exercise 2.5, but with the replacement $t \mapsto 2t$. Hence, from (S.39)

$$C_n(t) = \eta^{|n|} + e^{-2t} \sum_{\ell=1}^{\infty} [C_\ell(0) - \eta^\ell] [I_{|n|-\ell}(2\gamma t) - I_{|n|+\ell}(2\gamma t)] \quad (\text{S.40})$$

where again η is given by (S.38).

In order to describe the relaxation towards equilibrium, observe that from the equilibrium correlator $C_{n,\text{eq}} = C_n(\infty) = \eta^{|n|} = \exp(-|n| \ln \tanh J/T)$, one may read off the well-known correlation length (which is also easily found from the equilibrium correlator, see e.g. [743])

$$\xi := |\ln \tanh J/T|^{-1} \simeq \frac{1}{2} e^{2J/T} \quad ; \quad \text{as } T \rightarrow 0.$$

Next, using the asymptotic behaviour of the modified Bessel functions I_n for $t \rightarrow \infty$ [4], the approach towards equilibrium is of the order $C_n(t) - C_{n,\text{eq}} \sim t^{-1/2} e^{-2(1-\gamma)t}$ for large times t . Hence the relaxation time $\tau_{\text{eq}} = (2(1-\gamma))^{-1} \simeq e^{4J/T}$ as $T \rightarrow 0$. Therefore, for sufficiently small temperatures T , one has indeed an algebraic relation

$$\tau_{\text{eq}} \sim \xi^z, \quad \text{where } z = 2.$$

If we consider a quench from a fully disordered initial state with $C_n(0) = \delta_{n,0}$ to zero temperature final state, such that $\gamma = \eta = 1$, it can be shown that (S.40) can be rewritten in the form (2.18), after some manipulation of Bessel function identities [4]. We leave the details to the reader.

2.7 The two-time correlation function is given by (2.17), whereas the single-time correlator is read off from (S.40). An initial state at thermal equilibrium means that $C_n(0) = \eta^{|n|}$ for all $n \in \mathbb{Z}$. Hence, with the explicit form of the Green's function $G_n(\tau)$, one has

$$C_{n-m}(s+\tau, s) = \langle \sigma_n(s+\tau) \sigma_m(s) \rangle = e^{-\tau} \sum_{\ell=-\infty}^{\infty} \eta^{|n-m+\ell|} I_\ell(\gamma\tau)$$

which indeed *only* depends on the time difference $\tau = t - s$.

2.8 The Hamiltonian now reads $\mathcal{H} = -J \sum_n \sigma_n \sigma_{n+1} - \sum_n H_n \sigma_n$. For rates which allow a single flip at site n , one considers the ratio of the equilibrium probability distribution

$$\begin{aligned}
\frac{P_\infty(-\sigma_n)}{P_\infty(\sigma_n)} &= \frac{\exp\left[-\frac{J}{T}\sigma_n(\sigma_{n-1} + \sigma_{n+1}) - \frac{H_n}{T}\sigma_n\right]}{\exp\left[\frac{J}{T}\sigma_n(\sigma_{n-1} + \sigma_{n+1}) + \frac{H_n}{T}\sigma_n\right]} \\
&= \frac{w_n^{(G)}(\sigma_n)}{w_n^{(G)}(-\sigma_n)} \frac{\exp\left[-\frac{H_n}{T}\sigma_n\right]}{\exp\left[\frac{H_n}{T}\sigma_n\right]} = \frac{w_n^{(G)}(\sigma_n) \cdot [1 - \sigma_n \tanh(H_n/T)]}{w_n^{(G)}(-\sigma_n) \cdot [1 + \sigma_n \tanh(H_n/T)]}.
\end{aligned}$$

where $w_n^{(G)}(\sigma_n)$ are Glauber's transition rates (2.4). A field-dependent choice for the reaction rates consistent with detailed balance is therefore

$$w_n(\sigma_n) = w_n^{(G)}(\sigma_n) [1 - \sigma_n \tanh(H_n/T)].$$

We point out that different choices are possible. Inserting this into (2.6) and carrying out the functional derivative leads to the equation of motion

$$\partial_t R_{n,m}(t, s) = -R_{n,m}(t, s) + \frac{\gamma}{2} (R_{n-1,m}(t, s) + R_{n+1,m}(t, s)),$$

which is of the same form as (2.2), together with the equal-time boundary condition $R_{n,m}(s, s) = \delta_{n,m} [1 - \frac{\gamma}{2} (C_{n-1,n}(s) + C_{n,n+1}(s))]$ which involves a single-time correlation function. While these equations are different from those derived in the main text and will result in a different short-time behaviour, the resulting long-time scaling behaviour is the same.

2.8 Following [165], consider the equation of motion (2.2) for the two-time correlator $C_n(t, s)$, where one may also assume that $t > s$. Summing over n , one finds that $\partial_t \langle \widehat{M}(t) \widehat{M}(s) \rangle = 0$, where $\widehat{M}(t)$ denotes the time-dependent global magnetisation. Hence $\langle \widehat{M}(t) \widehat{M}(s) \rangle = \langle \widehat{M}^2(\min(t, s)) \rangle$. From the previous exercises, one readily finds $\langle \widehat{M}^2(t) \rangle \sim \sqrt{t}$ in the large-time limit which gives the normalised autocorrelator $\widehat{N}(t, s) = (s/t)^{1/4}$ (with $t > s$). By Slepian's formula, one reads off $\theta_g = \frac{1}{4}$.

2.10 One uses (2.6) derived in exercise 2.3 together with the explicit rates. The result is

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \sigma_n \rangle &= -\langle \sigma_n \rangle + \frac{\gamma}{2} \langle \sigma_{n-1} + \sigma_{n+1} \rangle - \delta \langle q_n \rangle \\
\frac{\partial}{\partial t} \langle q_n \rangle &= -3 \langle q_n \rangle + \gamma \langle \sigma_{n-1} + \sigma_{n+1} \rangle - \delta \langle \sigma_n \rangle + \delta \langle A_n \rangle
\end{aligned} \tag{S.41}$$

where

$$\begin{aligned}
A_n &:= \left[\frac{\gamma}{2\delta} \sigma_{n-2} \sigma_n \sigma_{n+1} - \sigma_{n-1} \sigma_{n+1} \sigma_{n+2} \right] \\
&\quad + \left[\frac{\gamma}{2\delta} \sigma_{n-1} \sigma_n \sigma_{n+2} - \sigma_{n-2} \sigma_{n-1} \sigma_{n+1} \right].
\end{aligned} \tag{S.42}$$

If the three-point functions arising in A_n are translationally invariant, then $\langle A_n \rangle = 0$ for the case of KDH dynamics and the system (S.41) closes [441, 188].

2.11 Using the same methods as in exercise 2.10, we find for $|n - m| > 1$

$$\begin{aligned}\frac{\partial}{\partial t} \langle \sigma_m \sigma_n \rangle &= -2 \langle \sigma_m \sigma_n \rangle + \frac{\gamma}{2} \langle \sigma_m (\sigma_{n-1} + \sigma_{n+1}) + \sigma_n (\sigma_{m-1} + \sigma_{m+1}) \rangle \\ &\quad - \delta \langle \sigma_m q_n + \sigma_n q_m \rangle \\ \frac{\partial}{\partial t} \langle \sigma_m q_n \rangle &= -4 \langle \sigma_m q_n \rangle + \frac{\gamma}{2} \langle 2 \sigma_m (\sigma_{n-1} + \sigma_{n+1}) + q_n (\sigma_{m-1} + \sigma_{m+1}) \rangle \\ &\quad - \delta \langle \sigma_m \sigma_n + q_m q_n \rangle + \delta \langle \sigma_m A_n \rangle\end{aligned}$$

where A_n is given in (S.42). Furthermore, $\langle \sigma_n q_n \rangle = \langle \sigma_{n-1} \sigma_{n+1} \rangle$ and $\langle \sigma_{n \pm 1} q_n \rangle = \langle \sigma_n \sigma_{n \mp 1} \rangle$. Finally, for $|n - m| > 2$, we have

$$\begin{aligned}\frac{\partial}{\partial t} \langle q_m q_n \rangle &= -6 \langle q_m q_n \rangle + \gamma \langle q_m (\sigma_{n-1} + \sigma_{n+1}) + q_n (\sigma_{m-1} + \sigma_{m+1}) \rangle \\ &\quad - \delta \langle q_m \sigma_n + q_n \sigma_m \rangle + \delta \langle q_m A_n + q_n A_m \rangle.\end{aligned}$$

In addition, $\langle q_n q_{n+1} \rangle = \langle \sigma_{n-1} \sigma_{n+2} \rangle$ is already known. The next-nearest correlator $\langle q_{n-1} q_{n+1} \rangle = \langle \sigma_{n-2} \sigma_{n-1} \sigma_{n+1} \sigma_{n+2} \rangle$, is found from

$$\begin{aligned}\frac{\partial}{\partial t} \langle q_{n-1} q_{n+1} \rangle &= -4 \langle q_{n-1} q_{n+1} \rangle + \gamma \langle \sigma_{n+1} \sigma_{n+2} + \sigma_{n-2} \sigma_{n-1} \rangle + 2\delta \langle \sigma_{n-2} \sigma_{n+2} \rangle \\ &\quad - \delta \langle \sigma_{n-1} q_{n+1} + q_{n-1} \sigma_{n+1} \rangle + \delta \langle q_{n-1} A_{n+1} + q_{n+1} A_{n-1} \rangle.\end{aligned}$$

These equations only close when the terms containing A_n can be made to disappear.

2.12 We begin by writing down the equations of motion for $C_r^{\sigma\sigma}(t)$ for $r \neq 0$,

$$\frac{d}{dt} C_r^{\sigma\sigma}(t) = -2C_r^{\sigma\sigma}(t) + 2\delta (C_{r-1}^{\sigma\sigma}(t) + C_{r+1}^{\sigma\sigma}(t)) - \delta (C_r^{\sigma q}(t) + C_r^{q\sigma}(t)).$$

and $C_0^{\sigma\sigma}(t) = 1$. Next, for $r \neq 0, \pm 1$, we have

$$\begin{aligned}\frac{d}{dt} C_r^{\sigma q}(t) &= -4C_r^{\sigma q}(t) + 2\delta (C_{r-1}^{\sigma\sigma}(t) + C_{r+1}^{\sigma\sigma}(t)) - \delta (C_r^{\sigma\sigma}(t) + C_r^{qq}(t)) \\ &\quad + \delta (C_{r-1}^{\sigma q}(t) + C_{r+1}^{\sigma q}(t)) + \delta C_r^{\sigma A}(t),\end{aligned}$$

where $C_r^{\sigma A}(t) := \sum_n \langle \sigma_n A_{n+r} \rangle / L$. Furthermore $C_0^{\sigma q}(t) = C_2^{\sigma\sigma}(t)$ and $C_{\pm 1}^{\sigma q}(t) = C_1^{\sigma\sigma}(t)$. Finally, for $r \neq 0, \pm 1, \pm 2$, we have

$$\begin{aligned}\frac{d}{dt} C_r^{qq}(t) &= -6C_r^{qq}(t) + 2\delta (C_{r-1}^{q\sigma}(t) + C_{r-1}^{\sigma q}(t)) + 2\delta (C_{r+1}^{q\sigma}(t) + C_{r+1}^{\sigma q}(t)) \\ &\quad - \delta (C_r^{q\sigma}(t) + C_r^{\sigma q}(t)) + \delta (C_r^{qA}(t) + C_r^{Aq}(t)), \\ \frac{d}{dt} C_2^{qq}(t) &= -4C_2^{qq}(t) + 4\delta C_1^{\sigma\sigma}(t) + 2\delta C_4^{\sigma\sigma}(t) - 2\delta C_2^{\sigma q}(t) \\ &\quad + \delta (C_2^{qA}(t) + C_2^{Aq}(t)),\end{aligned}$$

with $C_r^{qA}(t) := \sum_n \langle q_n A_{n+r} \rangle / L$. These are completed by $C_0^{qq}(t) = 1$ and $C_1^{qq}(t) = C_3^{\sigma\sigma}(t)$. In order to obtain a closed system of equations, we now use the expansion

$$\begin{aligned} \frac{d}{dt} \hat{C}^{\sigma\sigma}(t) &= \frac{1}{L^2} \sum_n \sum_{m \neq n} \frac{d}{dt} \langle \sigma_n \sigma_m \rangle \\ &= (4\delta - 2) \hat{C}^{\sigma\sigma}(t) - 2\delta \hat{C}^{q\sigma}(t) + \frac{2}{L} \left(1 - 2\delta C_1^{\sigma\sigma}(t) + \delta C_2^{\sigma\sigma}(t) \right) \end{aligned}$$

and similarly

$$\begin{aligned} \frac{d}{dt} \hat{C}^{\sigma q}(t) &= \frac{1}{L^2} \sum_n \sum_{m \neq n, n \pm 1} \frac{d}{dt} \langle q_m \sigma_n \rangle + \frac{1}{L} \left(\frac{d}{dt} C_0^{\sigma q}(t) + 2 \frac{d}{dt} C_1^{\sigma q}(t) \right) \\ &= 3\delta \hat{C}^{\sigma\sigma}(t) + (2\delta - 4) \hat{C}^{\sigma q}(t) - \delta \hat{C}^{qq}(t) \\ &\quad + \frac{2}{L} \left(\delta + 2(1 - \delta) C_1^{\sigma\sigma}(t) + (1 - 2\delta) C_2^{\sigma\sigma}(t) + 2\delta C_3^{\sigma\sigma}(t) - 2\delta C_2^{\sigma q}(t) \right). \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \hat{C}^{qq}(t) &= \frac{1}{L^2} \sum_n \sum_{m \neq n, n \pm 1, n \pm 2} \frac{d}{dt} \langle q_m q_n \rangle + \frac{2}{L} \left(\frac{d}{dt} C_1^{qq}(t) + \frac{d}{dt} C_2^{qq}(t) \right) \\ &= 6\delta \hat{C}^{\sigma q}(t) - 6\hat{C}^{qq}(t) \\ &\quad + \frac{2}{L} \left(3 - 2\delta C_1^{\sigma\sigma}(t) - \delta C_2^{\sigma\sigma}(t) + 4C_3^{\sigma\sigma}(t) + 2\delta C_4^{\sigma\sigma}(t) \right. \\ &\quad \left. - 4\delta C_2^{\sigma q}(t) - 2\delta (C_3^{\sigma q}(t) + C_3^{q\sigma}(t)) + 2C_2^{qq}(t) \right). \end{aligned}$$

From these expansions, it becomes clear that if the global correlators $\hat{C}^{gf}(0) = O(1)$ are large with respect to the non-global ones $C_n^{gf}(0)$, those terms which would destroy the closed system (2.68) become negligible in the $L \rightarrow \infty$ limit.

This means that for example a fully disordered initial state cannot be treated with the present method.

2.13 The admissible reactions merely exchange the spins σ_n and σ_{n+1} and their effect is illustrated in the following Table (the leftmost spin is arbitrarily fixed to be $\uparrow$).

process	rate
$\uparrow\uparrow\downarrow\uparrow \longleftrightarrow \uparrow\downarrow\uparrow\uparrow$	1
$\uparrow\uparrow\downarrow\downarrow \longrightarrow \uparrow\downarrow\uparrow\downarrow$	$1 - \gamma$
$\uparrow\downarrow\uparrow\downarrow \longrightarrow \uparrow\uparrow\downarrow\downarrow$	$1 + \gamma$

Therefore, the system evolves for $\gamma > 0$ towards a phase-separated state $\dots \uparrow\uparrow\uparrow\downarrow\downarrow\downarrow \dots$, while for $\gamma < 0$ a homogeneous state $\dots \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \dots$ is reached.

The relaxation time, or rather the characteristic frequency, is found by considering the linear response to an external perturbation $\delta\mathcal{H} = -\sum_n h_n(t)\sigma_n$.

Via the master equation, this generates the extra spin exchange term $\delta w_n(t) = -w_n(h_n(t)\sigma_n + h_{n+1}(t)\sigma_{n+1})$. Since the total magnetisation is conserved, one must have a continuity equation $\partial_t m_n = j_n - j_{n+1}$, where j_{n+1} is the magnetisation current which describes the flow of magnetisation from the site n to the site $n+1$. Hence j_{n+1} is related to the rate of exchange of the pair (σ_n, σ_{n+1}) and is given by

$$j_{n+1} = \left\langle \frac{1}{2}(1 - \sigma_n \sigma_{n+1})(w_n + \delta w_n(t))\sigma_n \right\rangle = \frac{1}{2}\langle w_n \rangle(h_n - h_{n+1})$$

where the average $\langle w_n \rangle$ can be found as an equilibrium average [762]. One now goes over to Fourier space $\hat{m}_q = \mathcal{N}^{-1/2} \sum_n e^{iqn} \langle \sigma_n \rangle = \hat{\chi}_{\text{eq}}(q) \hat{h}_q$, where $\mathcal{N}$ is the number of sites, and uses a ‘linear law’ for the magnetisation. The equilibrium susceptibility is then readily calculated for $\mathcal{N} \rightarrow \infty$ from the equilibrium fluctuation-dissipation theorem

$$\hat{\chi}_{\text{eq}}(q) = \frac{1}{4} \sum_{n=-\infty}^{\infty} \langle \sigma_n \sigma_0 \rangle_{\text{eq}} e^{iqn} = \frac{1}{4} \frac{1 - \eta^2}{1 - 2\eta \cos q + \eta^2}, \quad \eta := \tanh J/T$$

and the equilibrium correlator $\langle \sigma_n \sigma_0 \rangle_{\text{eq}} = \eta^{|n|}$. Hence the magnetisation satisfies an effective diffusion equation

$$\partial_t \hat{m}_q = -\langle w \rangle \frac{1 - \cos q}{\hat{\chi}_{\text{eq}}(q)} \hat{m}_q =: -D(q) q^2 \hat{m}_q$$

and the average $\langle w \rangle = \langle w_n \rangle = \alpha(1 - (1 + \gamma)\eta + \gamma\eta^2) \rightarrow \alpha(1 - \eta)^2$ in the $T \rightarrow 0$ limit, where $\gamma \rightarrow 1$. For small temperatures, one has $1 - \eta = \xi^{-1}$, where ξ is the equilibrium correlation length. From the diffusion equation, one reads off the critical frequency

$$\omega_c(q) = D(q) q^2 = \langle w \rangle \frac{1 - \cos q}{\hat{\chi}(q)} \simeq \alpha \xi^{-5} (q\xi)^2 (1 + (q\xi)^2) \stackrel{!}{=} \xi^{-z} \Omega(q\xi)$$

where a simultaneous expansion $q \rightarrow 0$ and $\xi \rightarrow \infty$ was performed and the end result is compared to the expected dynamical scaling form. Comparison gives the dynamical exponent $z = 5$ [762].

This result disagrees with the standard renormalisation-group expectation $z = 4 - \eta$ [759], which would have led to $z = 3$. This comes about since in 1D one has the anomalous behaviour $\langle w \rangle \sim \xi^{-2}$, see [161] for a review.

Variational ansätze have been used by Haake and Thol [305] to derive rigorous upper and lower bounds on the dynamical exponents of the 1D kinetic Ising model. For a conserved order-parameter, they find $z \geq 5$.

A different way to understand the value of z is through a consideration of the domain wall movements which may lead to a relaxation towards equilibrium. This should lead to upper bounds on z [160] and this kind of argument can also be extended to 1D Potts- q models [208].

2.14 This kind of calculation is already needed at equilibrium and can be performed at different levels of rigour. Here we give a simple heuristic way to arrive at the correct result. Decompose

$$\bar{f}(p) = \int_0^\infty du e^{-(p+4d)u} I_0(4u)^d = \bar{f}_{\text{reg}}(p) + \bar{f}_{\text{sing}}(p)$$

by setting $\int_0^\infty = \int_0^\eta + \int_\eta^\infty$ which will be evaluated in the limit $\eta \gg 1$, $p \ll 1$ and $\eta p = O(1)$. The end result should be independent of η , of course. Then, for the regular term $\bar{f}_{\text{reg}}(p) \rightarrow (2T_c(d))^{-1} + O(p)$. Any singular behaviour in p should come from the upper limit of the integration. One may use the asymptotic form $I_0(u) \simeq e^u/\sqrt{2\pi u}$ for $u \rightarrow \infty$ [4] and have for $d \neq 4$

$$\begin{aligned} \int_\eta^\infty du e^{-pu} (8\pi)^{-d/2} u^{-d/2} &= (8\pi)^{-d/2} p^{d/2-1} \int_{\eta p}^\infty dv v^{-d/2} e^{-v} \\ &\simeq (8\pi)^{-d/2} \Gamma(1-d/2) p^{d/2-1} \end{aligned} \quad (\text{S.43})$$

which gives the leading singular term as $\eta p \rightarrow 0$. For $d = 4$, we use the identities (3.351(4)) and (8.214(1)) from [293] for the exponential integral Ei and find

$$\begin{aligned} \bar{f}_{\text{sing}}(p) &= \int_\eta^\infty du e^{-(p+16)u} I_0(4u)^4 \\ &= (8\pi)^{-2} \left[p \text{Ei}(-p\eta) + \frac{e^{-p\eta}}{\eta} \right] \\ &\simeq (8\pi)^{-2} [pC_E + p \ln p] \end{aligned}$$

where we dropped an η -dependent term $\sim p \ln \eta$ which should cancel in any case against a corresponding correction coming from the regular term $\bar{f}_{\text{reg}}(p)$.

2.15 From (2.97) we get for the correlation function in the scaling region

$$C(t, s) = M_{\text{eq}}^4 \frac{t^{d/4} s^{d/4}}{\left(\frac{t+s}{2}\right)^{d/2}} + 2T M_{\text{eq}}^4 (ts)^{d/4} \int_0^s dt_1 f\left(\frac{t+s}{2} - t_1\right) g(t_1).$$

We estimate the value of the integral by developing the auxiliary function f at first order

$$\begin{aligned} &\int_0^s dt_1 f\left(\frac{t+s}{2} - t_1\right) g(t_1) \\ &\approx f\left(\frac{t+s}{2}\right) \int_0^s dt_1 g(t_1) - f'\left(\frac{t+s}{2}\right) \int_0^s dt_1 t_1 g(t_1) \\ &\approx \left(\frac{t+s}{2}\right)^{-d/2} \frac{1}{2T_c M_{\text{eq}}^2} + d \left(\frac{t+s}{2}\right)^{-d/2} \frac{1}{t+s} \frac{(8\pi)^{-d/2}}{M_{\text{eq}}^4} \int_0^s dt_1 t_1^{-d/2+1} \\ &= \left(\frac{t+s}{2}\right)^{-d/2} \frac{1}{2T_c M_{\text{eq}}^2} + \text{constant} \cdot (t+s)^{-d/2-1} s^{2-d/2}. \end{aligned}$$

The sum rule $\int_0^\infty du g(u) = (2T_c M_{\text{eq}}^2)^{-1}$ [286, (2.39)], which follows from the first singular term $\sim \delta(t)$ in (2.105) was used in the first integral in the second line. It follows from the explicit form of $g(t)$ that the regular long-time approximation for $g(t)$ may be used in the second integral in the second line. In addition, we replace the derivative of the function f by its asymptotic value. In the last line, we see that the second term is negligible in the $s \rightarrow \infty$ limit.

Therefore, using the exact expression $M_{\text{eq}}^2 = 1 - T/T_c$ [67], we find for any dimension d

$$C(t, s) \approx \left(\frac{4ts}{(t+s)^2} \right)^{d/4} \left(M_{\text{eq}}^4 + \frac{T}{T_c} M_{\text{eq}}^2 \right) = M_{\text{eq}}^2 \left(\frac{4ts}{(t+s)^2} \right)^{d/4}$$

from which the scaling limit, with $t, s \rightarrow \infty$ and $y = t/s$ fixed, is obtained.

2.16 We first observe that $L(t)$ can be calculated from

$$L^2(t) = 4d \frac{t + 2T \int_0^t dt' (t - t') g(t')}{1 + 2T \int_0^t dt' g(t')} = 4d \frac{\mathcal{L}^{-1}(p^{-2} + 2Tp^{-2}\bar{g}(p))(t)}{\mathcal{L}^{-1}(p^{-1} + 2Tp^{-1}\bar{g}(p))(t)}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Since for $T < T_c$, one knows that $\bar{g}(p) = (2T_c)^{-1}(1 - T/T_c)^{-1}(1 + o(p))$, it follows that $L^2(t) = 4dt(1 + O(1/t))$ for all $d > 2$. Indeed, for $T = 0$ this might have been read off directly from the above equation.

Consider now the case $T = T_c$. For $d > 4$, we have $\bar{g}(p) = (A_2 p)^{-1}$. Then

$$L^2(t) = 4d \frac{\mathcal{L}^{-1}(p^{-2} + (2T_c/A_2)p^{-3})(t)}{\mathcal{L}^{-1}(p^{-1} + (2T_c/A_2)p^{-2})(t)} \stackrel{t \rightarrow \infty}{\simeq} 4d \frac{1}{2} \frac{(2T_c/A_2) \cdot t^2}{(2T_c/A_2) \cdot t} = 2dt$$

since the second term in both the numerator and the denominator is the dominant one for $p \rightarrow 0$. Similarly, for $2 < d < 4$ we have

$$L^2(t) = 4d \frac{\mathcal{L}^{-1}(p^{-2} + (2T_c/A_1)p^{-1-d/2})(t)}{\mathcal{L}^{-1}(p^{-1} + (2T_c/A_1)p^{-d/2})(t)} \stackrel{t \rightarrow \infty}{\simeq} 4d \frac{1/\Gamma(d/2 + 1)}{1/\Gamma(d/2)} t = 8t.$$

Finally, for $d = 4$, we must find the inverse Laplace transforms of $h_k(p) := p^{-k}(C_E + \ln p)^{-1}$, with $k = 1, 2, 3$. We illustrate the calculation for the most simple case $k = 1$ which, again, should be considered as an approximate heuristic device. Recall the relation $\ln p = \lim_{n \rightarrow 0} (p^n - 1)/n$ and write, for p small enough

$$\begin{aligned} \frac{1}{C_E p + p \ln p} &= \lim_{n \rightarrow 0} \frac{n}{nC_E - 1} \frac{p^{-1}}{(1 + p^n/(nC_E - 1))} \\ &\simeq \lim_{n \rightarrow 0} \frac{n}{nC_E - 1} \left(\frac{1}{p} - \frac{p^{n-1}}{nC_E - 1} + \dots \right). \end{aligned}$$

Carrying out the inverse Laplace transform, we find the large-time asymptotics

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{C_E p + p \ln p}\right)(t) &\simeq \lim_{n \rightarrow 0} \frac{n}{nC_E - 1} \left(1 - \frac{t^{-n}}{(nC_E - 1)\Gamma(1-n)} + \dots\right) \\
&\simeq \lim_{n \rightarrow 0} \frac{n}{nC_E - 1} \left(1 + \frac{e^{-n \ln t}}{(nC_E - 1)\Gamma(1-n)} + \dots\right)^{-1} \\
&= -\frac{1}{\ln t}.
\end{aligned}$$

The long-time behaviour for $k = 2, 3$ is found similarly [224]:

$$\mathcal{L}^{-1}(h_2(p))(t) \simeq -\frac{t}{\ln t - 1}, \quad \mathcal{L}^{-1}(h_3(p))(t) \simeq -\frac{1}{2} \frac{t^2}{\ln t - 3/2}$$

and this gives, up to *additive* logarithmic corrections when $d = 4$,

$$L^2(t) \simeq t \times \begin{cases} 8 & ; \text{ if } 2 < d \leq 4 \\ 2d & ; \text{ if } d > 4 \end{cases}$$

That this heuristic way of calculation gives indeed accurate results can be checked by solving the Volterra integral equation numerically [224].

2.17 For the global order-parameter, one considers the normalised autocorrelator

$$\hat{N}(t, s) = \frac{\hat{C}_0(t, s)}{\sqrt{\hat{C}_0(t)\hat{C}_0(s)}} = \sqrt{\frac{\hat{C}_0(s)g(s)}{\hat{C}_0(t)g(t)}} = \sqrt{\frac{\hat{C}_0(0) + 2T \int_0^s d\tau g(\tau)}{\hat{C}_0(0) + 2T \int_0^t d\tau g(\tau)}}$$

where the initial correlator $\hat{C}_0(0) = 1$. Solving the Volterra integral equation for $g(t)$ and integrating, we find in the scaling limit $t, s \rightarrow \infty$ with $y = t/s > 1$ fixed that

$$\hat{N}(t, s) = (s/t)^\mu, \quad \mu = \begin{cases} 0 & ; \text{ if } T < T_c \\ (d-2)/4 & ; \text{ if } T = T_c \text{ and } 2 < d < 4 \\ 1/2 & ; \text{ if } T = T_c \text{ and } d > 4 \end{cases}.$$

Going over to logarithmic time, the conditions for Slepian's formula are satisfied and θ_g can be identified with the values of μ just calculated. This reproduces the corresponding entries in Tables 1.9 and 1.10.

For quenches to $T > T_c$, $\hat{N}(t, s)$ decays exponentially with $t - s$ and consequently, $P_g(t)$ also decays exponentially.

It is left to the reader to generalise this further to the *long-ranged* spherical model. The only non-trivial result is found for $\sigma < d < 2\sigma$ at $T = T_c$, when $\hat{N}(t, s) = (s/t)^\mu$, with $\mu = (d - \sigma)/(2\sigma)$, hence $\theta_g = \mu$ as it should be for a Markov process.

2.18 Using the same notations as in the main text, one has

$$\chi_{\text{FC}}(t) = \int_0^t du R(t, u) = \int_0^t du f\left(\frac{t-u}{2}\right) \sqrt{\frac{g(u)}{g(t)}}.$$

This may be rewritten in terms of a correlator as follows

$$\begin{aligned} \frac{1}{|A|} \overline{\left\langle \sum_{\mathbf{x} \in A} S_{\mathbf{x}}(t) h_{\mathbf{x}} \right\rangle} &= \frac{(2\pi)^{-2d}}{|A|} \sum_{\mathbf{x} \in A} \int_{\mathcal{B}^2} d\mathbf{q} d\mathbf{q}' e^{i(\mathbf{q}+\mathbf{q}') \cdot \mathbf{x}} \overline{\left\langle \widehat{S}(\mathbf{q}, t) \widehat{h}(\mathbf{q}') \right\rangle} \\ &= \frac{(2\pi)^{-2d}}{|A|} \sum_{\mathbf{x} \in A} \int_{\mathcal{B}^2} d\mathbf{q} d\mathbf{q}' e^{i(\mathbf{q}+\mathbf{q}') \cdot \mathbf{x}} \int_0^t dt' e^{\omega(\mathbf{q})(t-t')} \sqrt{\frac{g(t')}{g(t)}} \cdot \overline{\widehat{h}(\mathbf{q}) \widehat{h}(\mathbf{q}')} \\ &= (2\pi)^{-d} 2\Gamma \int_0^t dt' \int_{\mathcal{B}} d\mathbf{q} e^{-\omega(\mathbf{q})(t-t')} \sqrt{\frac{g(t')}{g(t)}} \\ &= 2\Gamma \chi_{\text{FC}}(t). \end{aligned}$$

In the second line, the explicit solution of the equation of motion for $\widehat{S}(\mathbf{q}, t)$ was used whereas the averages over the magnetic field are calculated from the field correlator.

2.19 Adapting the formalism of the spherical model as developed in this chapter, the average magnetisation is for a space-independent external field $H(t)$ given by

$$m(t) = \frac{1}{\sqrt{g(t)}} \left[m_0 + \int_0^t dt' H(t') \sqrt{g(t')} \right]$$

with the initial value m_0 and where $g(t)$ is found from the integral equation

$$\begin{aligned} g(t) &= A(t) + 2T \int_0^t dt' f(t-t') g(t') \\ &\quad + 2m_0 \int_0^t dt' B_{\mathbf{r}}(t') \sqrt{g(t')} + \left(\int_0^t dt' B_{\mathbf{r}}(t') \sqrt{g(t')} \right)^2 \end{aligned}$$

where $B_{\mathbf{r}}(t) = (2\pi)^{-d} \int_{\mathcal{B}} d\mathbf{q} e^{\omega(\mathbf{q})t + i\mathbf{q} \cdot \mathbf{r}} \widehat{H}(\mathbf{q}, t)$ and $\omega(\mathbf{q})$, $f(t)$ and $A(t)$ are defined in the text [168, 570].

The solution $g(t)$ can be found numerically [604, 570] and the result is shown in Fig. S.3. A non-monotonous behaviour of $m(t)$ should have been expected, since the individual spins will tend to align with their neighbours and because $m_0 = m_{\text{eq}} > 0$, one orientation is preferred. Hence $m(t)$ should initially increase with respect to its initial value $m_0 = m_{\text{eq}}$. Only when the domains have become sufficiently large, this cooperative effect will decrease and rapid relaxation of $m(t)$ towards equilibrium follows. As in the original Kovacs effect [451], this example shows that it is not sufficient to fix the value of a single observable to its equilibrium value in order to bring the full system to its equilibrium state.

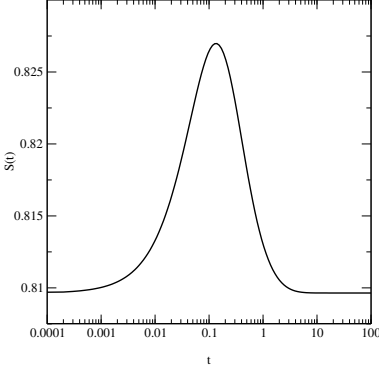


Fig. S.3. Time-dependent magnetisation $m(t)$ in the mean spherical model for $d = 3.5$ dimensions, $T = 2 < T_c$, magnetic field $H = 0.2$, and $m_0 \approx 0.810$. Reprinted from [570]. Copyright (2003) Institute of Physics Publishing.

2.20 From the explicit expressions (2.155) and (2.148) for the angular response and correlator, it is clear that the first one is of order $R = O(T_i^0, T_f^0) = O(1)$ with respect to initial and final temperatures while $C = O(T_i, T_f)$. Next, a first-order perturbative expansion in g_4 shows that the corresponding correction term to the correlator is

$$\delta C = \int du d\mathbf{R} \left\langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \tilde{\phi}(u, \mathbf{R}) \nabla (\nabla \phi(u, \mathbf{R}))^3 \right\rangle.$$

Expanding this six-point function in terms of two-point functions by using Wick's theorem and applying the above order estimates leads to the assertion.

2.21 Because of exercise 2.20, the spin-wave expansion as a function of temperature is reliable to first order. It is sufficient to expand the given expressions and check that they agree with (2.155) and (2.154), up to terms $O(T_i^2, T_f^2)$ [589].

2.22 The Langevin equation of motion for the angular variable $\phi(t, r)$ is at $T = 0$ a diffusion equation $\partial_t \phi = \partial_r^2 \phi$. In order to characterise the initial conditions, one assumes $\phi(0, r)$ to be Gaussian, with the distribution [537, 93]

$$\mathcal{P}[\hat{\phi}_k(0)] \sim \exp \left[-\frac{1}{2} \sum_k \beta_k \hat{\phi}_k(0) \hat{\phi}_{-k}(0) \right]$$

and the initial single-time correlator becomes

$$\begin{aligned} C_0(r) &= \langle \cos [\phi(0, r) - \phi(0, 0)] \rangle \\ &= \exp \left[-\frac{1}{2} \left\langle [\phi(0, r) - \phi(0, 0)]^2 \right\rangle \right] = \exp \left[-\sum_k \beta_k^{-1} [1 - \cos kr] \right] \end{aligned}$$

such that the choice $\beta_k = \frac{1}{2} \xi k^2$ gives $C_0(r) = e^{-|r|/\xi}$, which describes a disordered state with a finite correlation length ξ . Since the solution of the

diffusion equation reads in Fourier space $\hat{\phi}_k(t) = \hat{\phi}_k(0)e^{-k^2 t}$, one has with the above initial correlator for the two-time correlation function

$$\begin{aligned} C(t, s; r) &= \exp \left[-\frac{1}{2} \left\langle [\phi(t, r) - \phi(s, 0)]^2 \right\rangle \right] \\ &= \exp \left[-\frac{1}{\xi} \sum_k k^{-2} \left([e^{-k^2 t} - e^{-k^2 s}]^2 + 2[1 - \cos kr] e^{-k^2(t+s)} \right) \right] \\ &\simeq \exp \left[-\frac{1}{\xi \sqrt{\pi}} \left(\frac{r^2}{2\sqrt{t+s}} + 2\sqrt{t+s} - \sqrt{2t} - \sqrt{2s} \right) \right]. \end{aligned}$$

where in the last line the scaling limit $kr \ll 1$ was taken [537, 93]. The single-time correlator $C_t(r) = C(t, t; r) = \exp(-(8\pi\xi^2)^{-1/2} \cdot r^2/t^{1/2})$ satisfies the usual scaling form, but with the unhabitual growth law $L(t) \sim t^{1/4}$. The resulting dynamical exponent $z = 4$ does not agree with the prediction of the Bray-Rutenberg theory. On the other hand, dynamical scaling does *not* hold for the two-time correlator, which renders the Bray-Rutenberg theory inapplicable.

Since the initial correlation length ξ enters into the scaling function, this suggests an unusual dependence on the initial conditions. For example, the choice $\beta_k = |k|/\gamma$ generates a long-range power-law initial correlator $C_0(r) \sim r^{-\gamma/\pi}$. Then one recovers the usual growth law $L(t) \sim t^{1/2}$ and the two-time autocorrelator $C(t, s; 0) \sim (4ts/(t+s)^2)^{\gamma/4\pi}$ and space-time correlator $C(t, s; r) = C(t, s; 0)f_c(r/\sqrt{t+s})$ with $f_c(u) \sim u^{\gamma/\pi}$ do satisfy the usual dynamical scaling for simple ageing, with $b = 0$ and $\lambda_C = \gamma/2\pi$ [93].

See [165] for an application of this result to the calculation of the global persistence exponent θ_g .

2.23 This is a much-studied model with many exact results available, much beyond our scope. In $1D$, this reduces to the Glauber-Ising model, see (2.4). For $d \geq 2$, detailed balance cannot be satisfied. To see this in $2D$, one may consider the cyclic sequence of configurations

$$\begin{array}{ccccccccc} + & + & & + & + & & + & + & & + & + \\ + & + & + & - & \rightarrow & + & + & - & - & \rightarrow & + & - & - & - & \rightarrow & + & - & + & - & \rightarrow & + & + & + & - \\ + & + & & + & + & & + & + & & + & + & & + & + & & + & + & & + & + \end{array}$$

with a transition probability proportional to $\mathcal{P}_\rightarrow = (1 - \gamma/2)(1 + \gamma)$ and the inverse cyclic sequence

$$\begin{array}{ccccccccc} + & + & & + & + & & + & + & & + & + \\ + & + & + & - & \rightarrow & + & - & + & - & \rightarrow & + & - & - & - & \rightarrow & + & + & - & - & \rightarrow & + & + & + & - \\ + & + & & + & + & & + & + & & + & + & & + & + & & + & + & & + & + \end{array}$$

with a transition probability proportional to $\mathcal{P}_\leftarrow = (1 + \gamma/2)(1 - \gamma) \neq \mathcal{P}_\rightarrow$. Hence stationary probability currents are possible which is incompatible with detailed balance.

The derivation of the equations of motion uses (2.6). For example, the average spin satisfies

$$\partial_t \langle \sigma_{\mathbf{r}} \rangle = \frac{\gamma}{2d} \sum_{j=1}^d \langle \sigma_{\mathbf{r}+\mathbf{e}_j} + \sigma_{\mathbf{r}-\mathbf{e}_j} - 2\sigma_{\mathbf{r}} \rangle + (\gamma-1) \langle \sigma_{\mathbf{r}} \rangle = \frac{\gamma}{2d} \Delta_{\mathbf{L}} \langle \sigma_{\mathbf{r}} \rangle + (\gamma-1) \langle \sigma_{\mathbf{r}} \rangle$$

which shows that the characteristic length scale $\sim 1/|\gamma-1|$ diverges as $\gamma \rightarrow \gamma_c = 1$. The critical voter model has two *distinct* stationary states, namely $\dots + + + \dots$ and $\dots - - - \dots$, another fact incompatible with detailed balance (see appendix G in volume 1). Similar equations, including for the higher-order correlators and responses, will be derived in Chap. 3 for the bosonic contact process and the Edwards-Wilkinson model, whose ageing behaviour is in the same universality class as the voter model.

A recent extension to $q \geq 2$ states with global $\mathbb{S}_q$ -symmetry has non-equilibrium critical dynamics in d dimensions, with $z = 2$, $\lambda_{C'} = d$, $\Theta = 0$ and $X_{\infty} = \frac{1}{2}$ [316].

Problems of Chapter 3

3.1 One considers the average $m(t) = \langle \phi(t, \mathbf{r}) \rangle$ of the order-parameter and the fluctuation $\psi(t, \mathbf{r}) = \phi(t, \mathbf{r}) - m(t)$. From the habitual Langevin equation, one has the mean-field equations [579]

$$\partial_t \psi(t, \mathbf{r}) = (\Delta_{\mathbf{L}} - m(t)^2) \psi(t, \mathbf{r}) + \eta(t, \mathbf{r}) \quad , \quad \partial_t m(t) = -\frac{1}{3} m(t)^3$$

where all non-linearities in ψ are neglected, $\Delta_{\mathbf{L}}$ is the spatial Laplacian and η is a centred Gaussian noise with variance $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2T \delta(t-t') \delta(\mathbf{r}-\mathbf{r}')$. From this, one has $m(t)^2 = m_0^2 / (1 + 2m_0^2 t/3)$. In the case at hand, the following variable will in the infinite-size limit $1 \ll |\Omega| \ll V$ be described by a Gaussian process, because of the central limit theorem

$$\hat{\psi}_{\mathbf{0}}(t) = \frac{1}{\sqrt{|\Omega|}} \int_{\Omega} d\mathbf{r} \psi(t, \mathbf{r}).$$

Fourier-transforming the above equation of motion gives a linear equation for $\hat{\psi}_{\mathbf{0}}$. Solving it explicitly, one obtains the normalised autocorrelator

$$\hat{N}(t, s) = \frac{\langle \hat{\psi}_{\mathbf{0}}(t) \hat{\psi}_{\mathbf{0}}(s) \rangle}{\sqrt{\langle \hat{\psi}_{\mathbf{0}}(t) \hat{\psi}_{\mathbf{0}}(t) \rangle \langle \hat{\psi}_{\mathbf{0}}(s) \hat{\psi}_{\mathbf{0}}(s) \rangle}} = \frac{L(t/\tau_0)}{L(s/\tau_0)}$$

where $L(t) = \sqrt{(t+1)^4 - 1}$ and $\tau_0 = 3/(2m_0^2)$. Going over to a logarithmic time $T = \ln t$, the conditions for the application of Doob's lemma are satisfied

and consequently, one may use Slepian's formula to find the global persistence, for sufficiently large times

$$P_g(t; m_0) \sim \frac{L(t_{\text{micro}}/\tau_0)}{L(t/\tau_0)} \simeq \frac{(t_{\text{micro}}/\tau_0)^{1/2}}{\sqrt{(1+t/\tau_0)^4 - 1}} \sim \begin{cases} t^{-1/2} & ; \text{ if } t \ll \tau_0 \\ t^{-2} & ; \text{ if } t \gg \tau_0 \end{cases}$$

where t_{micro} is a microscopic reference-time. One identifies the global persistence exponents $\theta_{g,0} = \frac{1}{2}$ and $\theta_{g,\infty} = 2$. The cross-over between these regimes is also evident.

3.2 Integrating (3.52) and using the initial condition $L(0) = 0$ leads to

$$\int_0^L dL L \exp\left(\frac{\epsilon}{T} (\ln L)^\omega\right) = D_0 \int_0^t dt.$$

The left-hand side may be evaluated by using the mean-value theorem of integral calculus, which states that for *any continuous function* $f : [0, L] \rightarrow \mathbb{R}$, *there exists* $L^* \in (0, L)$ *such that* $\int_0^L dL f(L) = Lf(L^*)$. For L large enough, the exponential term dominates such that $L^* \simeq L$. This gives

$$L^2 \exp\left(\frac{\epsilon}{T} (\ln L)^\omega\right) \simeq D_0 t$$

If one tries to parametrise $L = L(t)$ in terms of an effective dynamic exponent z_{eff} via $L^{z_{\text{eff}}} = t$, one finds $z_{\text{eff}} = 2 + \frac{\epsilon\omega}{T} (\ln L)^{\omega-1}$. For $\omega = 1$ one arrives at (3.54), while $z_{\text{eff}} \rightarrow 2$ for $\omega < 1$. For $\omega > 1$ the growth law is not described by a simple algebraic form.

3.3 The equilibrium scaling form is independent of the initial magnetisation m_0 and reads $M_{\text{eq}}(t, r_\perp; \tau) = b^{-x} M_{\text{eq}}(tb^{-z}, r_\perp/b; \tau b^{1/\nu})$. In the static case, one may set $t = 0$ and obtain $M_{\text{eq}}(0, r_\perp; \tau) = \tau^{x\nu} \bar{M}_{\text{eq}}(r_\perp \tau^\nu)$. Since in the bulk, $M_{\text{eq}, \text{bulk}}(\tau) \sim \tau^\beta$, one must have $\lim_{r \rightarrow \infty} \bar{M}_{\text{eq}}(r) = \text{constant}$. On the other hand, near to the surface one has $M_{\text{eq}}(0; \tau) \sim \tau^{\beta_1}$ which implies that $\bar{M}_{\text{eq}}(r) \sim r^{x_1 - x}$ for $r \rightarrow 0$.

Returning to the time-dependent case *at* criticality ($\tau = 0$), one expects the behaviour $M_{\text{eq}}(t, r_\perp; 0) \sim t^{-x/z} (r_\perp t^{-1/z})^{x_1 - x}$ in the limit $r_\perp t^{-1/z} \rightarrow 0$.

3.4 Deep inside the bulk, one may simply set $r_\perp \rightarrow \infty$. For large times, the scaling should be independent of the initial value m_0 which means that $\bar{M}(\mathfrak{z}) = \text{constant}$ for $\mathfrak{z} \rightarrow \infty$. On the other hand, for sufficiently small times, the magnetisation should be proportional to m_0 , hence $\bar{M}(\mathfrak{z}) \sim \mathfrak{z}$.

3.6 Combine the scaling relation (3.99) with the hyperscaling relation (4.89) from Volume 1, which gives

$$\lambda_C = z + \frac{3}{2}d - \frac{1}{2}\Theta.$$

3.7 The derivation of the equation of motion is analogous to the procedure outlined in the text. Since the density evolves independently for each site, one can drop the dependence on $\mathbf{r}$. From the first equation, the density $\langle a(t) \rangle = \langle a(0) \rangle$ is a constant. However, the variance increases as a function of time, namely (i) linearly for the BCP with $m = 1$ and (ii) exponentially for the BPCP with $m = 2$.

3.8 To shorten the notation, any time-dependence will be suppressed unless needed. Using the property $\langle s | a^\dagger(\mathbf{r}) = \langle s |$, one easily verifies that on the critical line (here merely for the $1D$ case for brevity)

$$\partial_t \langle s | a(x) = \langle s | [a(x), \mathcal{L}] = D \langle s | (a(x+1) + a(x-1) - 2a(x)) .$$

Introducing the Fourier transform $\langle s | \hat{a}(\mathbf{k}) = \sum_{\mathbf{r}} e^{-i\mathbf{k} \cdot \mathbf{r}} \langle s | a(\mathbf{r})$, and rescaling all times according to $t \mapsto t/(2D)$, one has for all times $s \leq t$ the representation

$$\langle s | \hat{a}(t, \mathbf{k}) = \exp \left(-\frac{1}{2} \omega(\mathbf{k})(t-s) \right) \langle s | \hat{a}(s, \mathbf{k})$$

where $\omega(\mathbf{k})$ is the dispersion relation. Using Fourier's theorem $\langle s | a(\mathbf{x}) = (2\pi)^{-d} \int_{\mathcal{B}} d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} \langle s | \hat{a}(\mathbf{k})$ where $\mathcal{B}$ is the first Brillouin zone, we have for all times $s \leq t$

$$\begin{aligned} \langle s | a(t, \mathbf{x}) &= \sum_{\mathbf{y}} \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \exp \left(-\frac{1}{2} \omega(\mathbf{k})(t-s) \right) \langle s | a(s, \mathbf{y}) \\ &= \sum_{\mathbf{y}} \prod_{i=1}^d \left(e^{-(t-s)} I_{x_i - y_i}(t-s) \right) \langle s | a(s, \mathbf{y}) \\ &= \sum_{\mathbf{y}} R(t-s; \mathbf{x} - \mathbf{y}) \langle s | a(s, \mathbf{y}) \end{aligned} \quad (\text{S.44})$$

where I_n is a modified Bessel function [4].

Now we apply (S.44) to the two-point correlation and find

$$F(t, s; \mathbf{r}) = \langle a(t, \mathbf{x}) a(s, \mathbf{x} + \mathbf{r}) \rangle = \sum_{\mathbf{y}} R(t-s; \mathbf{x} - \mathbf{y}) F(s, \mathbf{r} + \mathbf{x} - \mathbf{y}) .$$

On the other hand, the density-density correlator is

$$\begin{aligned} \langle n(t, \mathbf{x}) n(s, \mathbf{x} + \mathbf{r}) \rangle &= \langle a^\dagger(t, \mathbf{x}) a(t, \mathbf{x}) a^\dagger(s, \mathbf{x} + \mathbf{r}) a(s, \mathbf{x} + \mathbf{r}) \rangle \\ &= \sum_{\mathbf{y}} R(t-s; \mathbf{x} - \mathbf{y}) \langle a(s, \mathbf{x}) a^\dagger(s, \mathbf{x} + \mathbf{r}) a(s, \mathbf{x} + \mathbf{r}) \rangle \\ &= \sum_{\mathbf{y}} R(t-s; \mathbf{x} - \mathbf{y}) \{ \langle a^\dagger(s, \mathbf{x} + \mathbf{r}) a(s, \mathbf{x}) a(s, \mathbf{x} + \mathbf{r}) \rangle \\ &\quad + \delta(\mathbf{y} - \mathbf{x} - \mathbf{r}) \langle a(s, \mathbf{x} + \mathbf{r}) \rangle \} \\ &= \sum_{\mathbf{y}} R(t-s; \mathbf{x} - \mathbf{y}) F(s, \mathbf{r} + \mathbf{x} - \mathbf{y}) + R(t-s; \mathbf{r}) \rho_0 \\ &= F(t, s; \mathbf{r}) + R(t-s; \mathbf{r}) \rho_0 \end{aligned}$$

where (S.44) was applied again and in going from the second to the third line, the equal-time commutator of a and $a^\dagger$ was used. Comparison with the above expression for $C(t, s; \mathbf{r}) = F(t, s; \mathbf{r}) - \rho_0^2$ gives the assertion.

We remark that the argument is readily extended to the case of long-range jumps generated by Lévy flights of single particles [215].

Problems of Chapter 4

4.1 The correspondence is readily checked for translations $z \mapsto z + a$, dilations $z \mapsto bz$ with $b \in \mathbb{R}$ and rotations $z \mapsto e^{i\alpha}z$ with $\alpha \in [0, 2\pi]$. For the special transformations (2.147) in Volume 1, let $a = a_1 - ia_2$. Then $z \mapsto z'$ where

$$z' = \frac{z + \bar{a}z\bar{z}}{1 + \bar{a}\bar{z} + az + a\bar{a}z\bar{z}} = \frac{z(1 + \bar{a}\bar{z})}{(1 + az)(1 + \bar{a}\bar{z})} = \frac{z}{1 + az}.$$

4.2 Use complex coordinates $z = r_1 + ir_2$, $\bar{z} = r_1 - ir_2$, when the Laplacian becomes $\Delta_L = 4\partial_z\partial_{\bar{z}}$. The infinitesimal conformal generators read $\ell_n = -z^{n+1}\partial_z - (n+1)z^n\Delta$ with $n \in \mathbb{Z}$ and satisfy $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$; and similarly $\bar{\ell}_n$ with z replaced by $\bar{z}$ and Δ by $\bar{\Delta}$. Here, $(\Delta, \bar{\Delta})$ are the conformal weights such that $x = \Delta + \bar{\Delta}$ is the scaling dimension of the field ϕ and $s = \Delta - \bar{\Delta}$ is its spin. The conformal invariance of the Laplace equation is expressed by the commutator

$$[\Delta_L, \ell_n] = -(n+1)z^n\Delta_L - 4(n+1)n z^{n-1} \frac{\partial}{\partial z} \cdot \Delta$$

and similarly for $\bar{\ell}_n$. Since for scalars, $s = 0$, one has a dynamical invariance iff $\Delta = \bar{\Delta} = 0$, then the transformation $\ell_n\phi$ of a solution of $\Delta_L\phi = 0$ also solves the Laplace equation.

4.3 The exponentiation of a Lie algebra generator follows the procedure outlined in appendix L, p. 406, and amounts to solve differential equations of the type (L18). We shall limit ourselves to the 1D case $\mathfrak{sv}(1)$. The presence of the spatial variables requires that several invariant variables must be identified, which can be done by inspection and it is enough to quote the results [357]. From the generators X_n , one finds the transformations $t \mapsto t'$, $r \mapsto r'$ with $t = \beta(t')$ and $r = r'\sqrt{\dot{\beta}(t')}$ where $\beta(t')$ is a non-decreasing function of t' . The scaling operator transforms

$$\phi(t, r) = \dot{\beta}(t')^{-x/2} \exp \left[-\frac{\mathcal{M}}{4} \frac{\ddot{\beta}(t')}{\dot{\beta}(t')} r'^2 \right] \phi'(t', r')$$

such that for $\beta(t) = t - \epsilon t^{n+1}$ the generator X_n is recovered in the limit $\epsilon \rightarrow 0$. From the generators Y_m , one has the time-dependent spatial translations $t = t'$, $r = r' - \alpha(t)$ and the scaling operator transforms as

$$\phi(t, r) = \exp \left[\mathcal{M} \left(\frac{1}{2} \alpha(t') \dot{\alpha}(t') - r' \dot{\alpha}(t') \right) \right] \phi'(t', r')$$

such that for $\alpha(t) = -\epsilon t^{m+1/2}$ the generator Y_m is recovered in the limit $\epsilon \rightarrow 0$. Exponentiation of M_n keeps the coordinates unchanged, $t = t', r = r'$ and changes the phase of the scaling operator

$$\phi(t, r) = \exp [\mathcal{M} \gamma(t')] \phi'(t', r')$$

such that for $\gamma(t) = -\epsilon t^n$ the generator M_n is recovered in the limit $\epsilon \rightarrow 0$. Finally, if $d > 1$, the $R_n^{(jk)}$ lead to spatial rotations $t = t', \mathbf{r} = \mathfrak{D}^{-1} \mathbf{r}'$ with time-dependent rotation angles.

A different, more formal, route is followed by [623, 624].

4.4 (a) The confirmation that $\mathcal{S}$ commutes with all generators of $\mathfrak{gal}(d)$ is straightforwardly seen by working out the commutators. (b) Similarly, the Schrödinger-invariance of $\mathcal{S}\psi = 0$ is an easy generalisation of (4.11) and consideration of $[\mathcal{S}, X_1]$ leads to $x = d/2$.

4.5 Following [546], denote the transformed coordinates by $(t', \mathbf{r}') = g(t, \mathbf{r})$. Define the abbreviations

$$e^2 := \frac{\partial t}{\partial t'} \quad , \quad c_i := \frac{\partial t}{\partial r'_i} \quad , \quad b_i := \frac{\partial r_i}{\partial t'} \quad , \quad d_{ij} := \frac{\partial r_i}{\partial r'_j}. \quad (\text{S.45})$$

From the requirement of co-variance of the Schrödinger equation (4.171) one may reduce the transformed equations back to the variables $t, \mathbf{r}$ via the abbreviations defined above. Comparing the different orders of space-time derivatives of ψ , one finds, suppressing the arguments [546],

$$\begin{aligned} c_i &= 0 \\ d_{il} d_{jl} &= e^2 \delta_{ij} \\ 2e^2 \partial_i f_g + (d_{nm} \partial_n d_{im} + 2im b_i) f_g &= 0 \\ \left[e^2 \Delta_L + (d_{nm} \partial_n d_{im} + 2im b_i) + 2ime^2 + 2m(e^2 V(t, \mathbf{r}) - V(t', \mathbf{r}')) \right] f_g &= 0 \end{aligned}$$

Express d_{ij} as a rotation $d_{ij} = e \mathfrak{D}_{ji}$. Then the above derivatives (S.45) can be inverted

$$\frac{\partial t'}{\partial t} = e^{-2} \quad , \quad \frac{\partial t'}{\partial r_i} = 0 \quad , \quad \frac{\partial r'_i}{\partial t} = -e^{-3} \mathfrak{D}_{in} b_n \quad , \quad \frac{\partial r'_i}{\partial r_j} = e^{-1} \mathfrak{D}_{ij} \quad ,$$

which leads to the following integrability conditions $e = e(t)$, $\mathfrak{D}_{ij} = \mathfrak{D}_{ij}(t)$ and $\partial_i b_j = e \dot{e} \delta_{ij} - e^2 \mathfrak{D}_{nj} \dot{\mathfrak{D}}_{ni}$. Then the above equations for the companion function f_g become

$$\begin{aligned} \partial_i \ln f_g &= -ime^{-2} b_i \quad , \\ \partial_t \ln f_g &= \frac{im}{2} e^{-4} b_j b_j + \frac{1}{2} d^{-2} \partial_j b_j + i[V(t, \mathbf{r}) - e^{-2} V(t', \mathbf{r}')] \quad . \end{aligned}$$

The integrability condition $\partial_i b_j = \partial_j b_i$ for the first of those gives $\dot{D}_{ij} = 0$ and $\partial_i b_j = e\dot{e}\delta_{ij}$. Defining the vector $\mathbf{s}(t)$ via

$$\mathbf{b} = e\dot{e}(\mathbf{r} + \mathfrak{D}^{-1}\mathbf{s}) - e^2\mathfrak{D}^{-1}\dot{\mathbf{s}}$$

the explicit transformation can now be found and the companion function becomes

$$f_g(t, \mathbf{r}) = e^{d/2} \exp \left\{ -\frac{im}{2} \left[\frac{\dot{e}\mathbf{r}^2}{e} + 2(\mathfrak{D}\mathbf{r}) \cdot \left(\frac{\dot{e}}{e}\mathbf{s} - \dot{\mathbf{s}} \right) + \frac{\dot{e}}{e}\mathbf{s}^2 - \dot{\mathbf{s}} \cdot \mathbf{s} + \ell(t) \right] \right\}$$

where $\ell(t)$ is a scalar. The condition (4.173) now also follows.

The maximal dimension of the symmetry group follows, since e and $\mathbf{s}$ are determined by second-order equations. Adding the integration constants and taking the rotations into account, the maximal dimension is $3 + 2d + \frac{1}{2}d(d-1) = \frac{1}{2}(d+1)(d+2) + 2$. The Schrödinger group is an example where this maximal dimensionality is realised.

4.6 For scalar quasi-primary scaling operators, the three space points $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ can be brought to any pre-determined plane. It is then straightforward to check by direct application of the generators of $\mathfrak{sch}(2)$ that the given expression is indeed covariant. A constructive proof is in [327].

We remark that there exists a second form of a Schrödinger-covariant three-point function, namely

$$\begin{aligned} \langle \phi_1(t_1, \mathbf{r}_1) \phi_2(t_2, \mathbf{r}_2) \phi_3(t_3, \mathbf{r}_3) \rangle &= \delta_{\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3, 0} \exp \left[-\frac{\mathcal{M}_2}{2} \frac{\mathbf{r}_{12}^2}{t_{12}} - \frac{\mathcal{M}_3}{2} \frac{\mathbf{r}_{13}^2}{t_{13}} \right] \\ &\times t_{13}^{-x_{13,2}/2} t_{23}^{-x_{23,1}/2} t_{12}^{-x_{12,1}/2} \Psi_{1,23} \left(\frac{(\mathbf{r}_{13}t_{12} - \mathbf{r}_{12}t_{13})^2}{t_{12}t_{13}t_{23}} \right). \end{aligned}$$

4.7 The form of the autoresponse $R(t, s) = \langle \phi(t) \tilde{\phi}(s) \rangle$ is determined from the covariance under the two generators $X_0 = -t\partial_t - x/2$ and $X_1 = -t^2\partial_t - (x + \xi)t$. This leads to

$$\begin{aligned} X_0 R(t, s) &= \left(-t\partial_t - s\partial_s - \frac{1}{2}(x + \tilde{x}) \right) R \stackrel{!}{=} 0 \\ X_1 R(t, s) &= \left(-t^2\partial_t - s^2\partial_s - (x + \xi)t - (\tilde{x} + \tilde{\xi})s \right) R \stackrel{!}{=} 0. \end{aligned}$$

These can be decoupled by going over to the variables $u = t - s$ and $v = t/s$ [352, 346]. With $R(t, s) = \bar{R}(u, v)$, this gives

$$\begin{aligned} \left(u\partial_u + \frac{1}{2}(x + \tilde{x}) \right) \bar{R}(u, v) &= 0 \\ u \left(v\partial_v + \frac{v}{v-1} \frac{x - \tilde{x} + 2\xi}{2} + \frac{1}{v-1} \frac{\tilde{x} - x + 2\tilde{\xi}}{2} \right) \bar{R}(u, v) &= 0 \end{aligned}$$

and the solution is found in a factorised form $\bar{R}(u, v) = f(u)g(v)$. This leads to the form

$$R(t, s) = r_0 s^{-1-a} \left(\frac{t}{s} \right)^{1+a'-\lambda_R/2} \left(\frac{t}{s} - 1 \right)^{-1-a'}$$

and

$$1+a = \frac{1}{2}(x+\tilde{x}) \quad , \quad a'-a = \xi + \tilde{\xi} \quad , \quad \lambda_R/2 = x + \xi.$$

4.8 Denote the Schrödinger-invariant bulk two-point function by $R_b(\tau, \mathbf{r}) = R_b(\tau, \mathbf{r}_{\parallel}, r_{\perp})$ and similarly the two-point function in the semi-infinite system by $R_s(\tau, \mathbf{r}, \mathbf{r}')$, with the decomposition $\mathbf{r} = (\mathbf{r}_{\parallel}, r_{\perp})$. Then, by the familiar method of images, one has for Dirichlet boundary conditions on the surface

$$\begin{aligned} R_s(\tau, \mathbf{r}, \mathbf{r}') &= R_b(\tau, \mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}, r_{\perp} - r'_{\perp}) - R_b(\tau, \mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel}, r_{\perp} + r'_{\perp}) \\ &= r_0 \tau^{-x} \exp \left[-\frac{\mathcal{M}_1}{2} \frac{(\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel})^2}{\tau} \right] \\ &\quad \times \left(\exp \left[-\frac{\mathcal{M}_1}{2} \frac{(r_{\perp} - r'_{\perp})^2}{\tau} \right] - \exp \left[-\frac{\mathcal{M}_1}{2} \frac{(r_{\perp} + r'_{\perp})^2}{\tau} \right] \right) \\ &= 2r_0 \tau^{-x} \sinh \left(\mathcal{M}_1 \frac{r_{\perp} r'_{\perp}}{\tau} \right) \exp \left[-\frac{\mathcal{M}_1}{2} \frac{r_{\perp}^2 + r'^2_{\perp}}{\tau} \right] \exp \left[-\frac{\mathcal{M}_1}{2} \frac{(\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel})^2}{\tau} \right] \end{aligned}$$

and comparison with the form (4.33) gives $\chi_{12}(u) = 2 \sinh(\mathcal{M}_1 u)$.

Extension systems which in the bulk are only $\mathfrak{age}(d)$ -invariant leads to

$$\begin{aligned} R_s(t, s; \mathbf{r}, \mathbf{r}') &= R(t, s) \exp \left[-\frac{\mathcal{M}_1}{2} \frac{(\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel})^2}{\tau} \right] \\ &\quad \times 2 \sinh \left(\mathcal{M}_1 \frac{r_{\perp} r'_{\perp}}{\tau} \right) \exp \left[-\frac{\mathcal{M}_1}{2} \frac{r_{\perp}^2 + r'^2_{\perp}}{\tau} \right] \end{aligned} \quad (\text{S.46})$$

where $R(t, s)$ is the bulk autoresponse function.

Similarly, one may treat Neumann boundary conditions. Then the two terms add and in (S.46) $\sinh$ is replaced by $\cosh$.

4.9 Consider a straight line going through the origin of the root diagram of B_2 , and with a slope ≥ 1 and $\leq \infty$. If the slope is > 1 and $< \infty$, then the generator X_{-1} is to the left of the line and does not belong to the parabolic subalgebra $\mathfrak{p}$. Neither is the generator V_+ contained, whereas the generators X_1 and $Y_{-\frac{1}{2}}$ are to the right of the line and belong to $\mathfrak{p}$. Then $\mathfrak{p} \cong \widetilde{\mathfrak{age}}(1)$. There are two extreme limit cases: first, if the slope is exactly one, then both $X_{\pm 1} \in \mathfrak{p} \cong \mathfrak{sch}(1)$. Second, if the slope is ∞ , then both $Y_{-\frac{1}{2}}, V_+ \in \mathfrak{p} \cong \widetilde{\mathfrak{alt}}(1)$.

From the Weyl symmetries it is clear that no other non-isomorphic maximal subalgebras can exist.

A formal proof of the classification is given in appendix C of [357].

4.10 The calculations proceed in a manner analogous to exercise 4.3 and we quote only the result. The coordinate changes are $(\zeta, t, r) \mapsto (\zeta', t', r')$ where

$$\begin{aligned} X_n : \quad \zeta &= \zeta' - \frac{i}{4} \frac{\ddot{\beta}(t')}{\dot{\beta}(t')} r'^2, \quad t = \beta(t'), \quad r = r' \sqrt{\dot{\beta}(t')} \\ Y_M : \quad \zeta &= \zeta' + i r' \dot{\alpha}(t') - \frac{i}{2} \alpha(t') \dot{\alpha}(t'), \quad t = t', \quad r = r' - \alpha(t') \\ M_n : \quad \zeta &= \zeta' + i \gamma(t), \quad t = t', \quad r = r' \end{aligned} \quad (\text{S.47})$$

along with the trivial transformation $\psi(\zeta, t, r) = \psi'(\zeta', t', r')$. Therefore, the non-local change of variables (4.43) transforms a projective representation of $\mathfrak{su}(1)$ into a usual representation.

4.11 We derive the causal representation (4.63) for the two-point function [357]. Because of rotation-invariance and since the Ψ_a are scalars, it is enough to consider the case $d = 1$ explicitly. Introduce centre-of-mass coordinates $\eta = \zeta_1 + \zeta_2$ and relative coordinates $\zeta = \zeta_1 - \zeta_2$. Then from (4.62)

$$\begin{aligned} \langle \phi_1 \phi_2^* \rangle &= \frac{1}{4\pi} \int_{\mathbb{R}^2} d\zeta d\eta e^{-\frac{i}{2}(\mathcal{M}_1 - \mathcal{M}_2)\eta} e^{-\frac{i}{2}(\mathcal{M}_1 + \mathcal{M}_2)\zeta} \langle \psi_1 \psi_2 \rangle \\ &= \delta(\mathcal{M}_1 - \mathcal{M}_2) \int_{\mathbb{R}} d\zeta e^{-i\mathcal{M}_1 \zeta} \langle \psi_1 \psi_2 \rangle \\ &= \delta(\mathcal{M}_1 - \mathcal{M}_2) \delta_{x_1, x_2} \psi_0 t^{-x_1} \mathcal{M}_1^{x_1-1} \int_{\mathbb{R}} d\zeta e^{-i\zeta} \left(\zeta + \frac{i\mathcal{M}_1}{2} \frac{r^2}{t} \right)^{-x_1} \\ &= \delta(\mathcal{M}_1 - \mathcal{M}_2) \delta_{x_1, x_2} \psi_0 t^{-x_1} \mathcal{M}_1^{x_1-1} \exp\left(-\frac{\mathcal{M}_1}{2} \frac{r^2}{t}\right) I \end{aligned}$$

where

$$I = \int_{\mathbb{R} + i\frac{\mathcal{M}_1}{2} \frac{r^2}{t}} du u^{-x_1} e^{-iu}$$

and $r = r_1 - r_2$, $t = t_1 - t_2$ and $\psi_0 = 4^{-x_1} \Psi_0$. In order to evaluate this, we consider the contour integral

$$J = \oint_{C_{\pm}} du u^{-x_1} e^{-iu}$$

where the contours $C_{\pm}$ apply to $t > 0$ and $t < 0$ and are indicated in Fig. S.4 where of course the radius of the upper (lower) semicircle has to be sent to infinity. The only singularity of the integrand is the cut along the negative real axis, hence $J = 0$ for both $t > 0$ and $t < 0$. Provided the negative real axis is not crossed, the integration contour can be arbitrarily shifted and therefore $I = I(x_1)$ depends *only* on the sign of $\frac{\mathcal{M}_1}{2} \frac{r^2}{t}$.

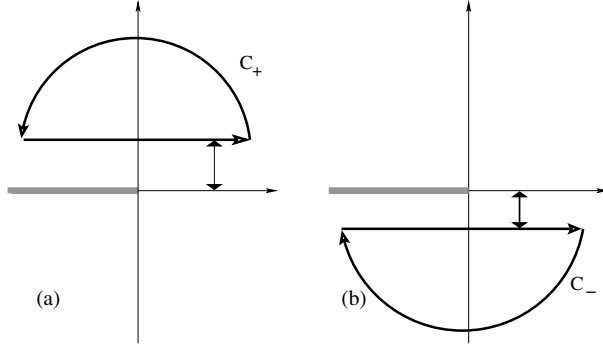


Fig. S.4. Integration contours $C_{\pm}$ in the complex u -plane for (a) $t > 0$ and (b) $t < 0$ for the proof of the causality of the two-point function. The thick grey lines indicate the cut along the negative real axis and the distance of the horizontal part of $C_{\pm}$ from the x -axis is $\frac{\mathcal{M}}{2} \frac{r^2}{t}$.

Consider the case $\frac{\mathcal{M}_1}{2} \frac{r^2}{t} < 0$, which implies $t < 0$ because of the physical convention $\mathcal{M}_1 \geq 0$. Then the contour of integration C_- may be taken to consist of the horizontal part $\mathbb{R} - i\epsilon$ with $\epsilon > 0$ arbitrarily small, and can be closed by a semicircle in the lower half-plane, see Fig. S.4b, and we have $J = I(x_1) + J_{\text{inf}}$. Using polar coordinates $u = Re^{-i\theta}$, the contribution J_{inf} of the lower semicircle of radius R can be estimated in a standard fashion

$$|J_{\text{inf}}| \leq R^{1-x_1} \int_0^\pi d\theta e^{-R \sin \theta} \leq 2R^{1-x_1} \int_0^{\pi/2} d\theta e^{-(2R/\pi)\theta} \leq \pi R^{-x_1}$$

and therefore vanishes as $R \rightarrow \infty$, provided $x_1 > 0$. It follows that $I = J = 0$ for $t < 0$. On the other hand, if $t > 0$ and thus $\frac{\mathcal{M}_1}{2} \frac{r^2}{t} > 0$, the contribution J_{sup} of the upper semicircle, see Fig. S.4a, cannot be bounded. It follows that $0 = J = I(x_1) + J_{\text{sup}}$. This proves (4.63).

For the three-point function, an analogous argument works [357, app. B].

4.12 Using (4.82), one has for the changes of the action

$$\begin{aligned} \delta_{X_1} \mathcal{J}_{(b)} &= -\varepsilon \int d\zeta dt dr \left[(2T_0^0 + T_1^1)t + (T_0^1 - iT_1^{-1})r \right] = 0 \\ \delta_W \mathcal{J}_{(b)} &= -\varepsilon \int d\zeta dt dr \left[(2T_{-1}^{-1} + T_1^1)\zeta + (T_{-1}^1 - iT_1^0)r \right] = 0 \\ \delta_{V_+} \mathcal{J}_{(b)} &= -\varepsilon \int d\zeta dt dr \left[(2T_{-1}^{-1} + T_0^0 + T_1^1)r \right. \\ &\quad \left. + (T_0^1 - iT_1^{-1})i\zeta + (T_{-1}^1 - iT_1^0)it \right] = 0 \end{aligned}$$

where the conditions (4.83) were used.

In general, the transformation under the generators of the families X and Y are (once more, $\{\beta(t), t\}$ denotes the Schwarzian derivative)

$$\delta_X \mathcal{J}_{(b)} = \int d\zeta dt dr \frac{1}{2} r^2 \{\beta(t), t\} \left(\frac{\partial \psi}{\partial \zeta} \right)^2$$

$$\delta_Y \mathcal{J}_{(b)} = \int d\zeta dt dr (\alpha(t) - 2r) \ddot{\alpha}(t) \left(\frac{\partial \psi}{\partial \zeta} \right)^2$$

and comparison with the general infinitesimal transformation (4.82) leads to the identification $T_0^{-1} = 2i(\partial_\zeta \psi)^2$. This is the analogue of (4.75).

4.13 For the action (4.80), the energy-momentum tensor is found as follows [357]. The canonical energy-momentum tensor is given by

$$\tilde{T}_\mu^\nu = -\delta_\mu^\nu \mathcal{L}_b + \frac{\partial \mathcal{L}_b}{\partial(\partial^\mu \psi)} \partial^\nu \psi.$$

and may be written in a matrix form (here for the $d = 1$ case, where ν labels the columns and we write $\psi_\zeta = \partial\psi/\partial\zeta, \dots$)

$$\tilde{T}_\mu^\nu = \begin{pmatrix} -\psi_r^2 & 2i\psi_t^2 & 2i\psi_r\psi_t \\ 2i\psi_\zeta^2 & -\psi_r^2 & 2i\psi_\zeta\psi_r \\ 2\psi_r\psi_\zeta & 2\psi_r\psi_t & \psi_r^2 - 2i\psi_\zeta\psi_t \end{pmatrix}.$$

It is classically conserved and satisfies all Ward identities (4.83) coming from the 3D conformal group, but not scaling identity. To correct this, define the improved (Belinfante) tensor [124]

$$T_\mu^\nu = \tilde{T}_\mu^\nu + \partial^\rho B_{\rho\mu}^\nu,$$

with $B_{\rho\mu}^\nu$ antisymmetric in ρ in μ , which has the same divergence as $\tilde{T}$. Choosing

$$B_{-10}^\nu = \begin{pmatrix} -\frac{1}{2}\psi\psi_\zeta \\ \frac{1}{2}\psi\psi_t \\ 0 \end{pmatrix}, B_{10}^\nu = \begin{pmatrix} 0 \\ \frac{1}{2}\psi\psi_r \\ -\frac{1}{2}\psi\psi_\zeta \end{pmatrix}, B_{1-1}^\nu = \begin{pmatrix} \frac{1}{2}\psi\psi_r \\ 0 \\ -\frac{1}{2}\psi\psi_t \end{pmatrix},$$

the improved energy-momentum tensor reads (where the equations of motion were used)

$$T_\mu^\nu = \frac{1}{2} \begin{pmatrix} i\psi_\zeta\psi_t - i\psi\psi_{\zeta t} - \psi_r^2 & 3i\psi_t^2 - i\psi\psi_{tt} & i[3\psi_t\psi_r - \psi\psi_{tr}] \\ 3i\psi_\zeta^2 - i\psi\psi_{\zeta\zeta} & i\psi_\zeta\psi_t - i\psi\psi_{\zeta t} - \psi_r^2 & i[3\psi_\zeta\psi_r - \psi\psi_{\zeta r}] \\ 3\psi_\zeta\psi_r - \psi\psi_{\zeta r} & 3\psi_t\psi_r - \psi\psi_{tr} & -2[i\psi_\zeta\psi_t - i\psi\psi_{\zeta t} - \psi_r^2] \end{pmatrix}.$$

This is a particular case of the well-known construction of the **Belinfante tensor** in 3D conformal field-theory [124, 191].

The consideration of the improved energy-momentum tensor in Poincaré-invariant interacting theories is of particular interest, since satisfying the conformal Ward identities implies that the elements of T_μ^ν remain finite in the limit of a large cut-off for renormalisable interactions [124].

4.14 Combining Galilei- and spatial translation-invariance, one has the Bargman condition (4.92). Since all $\mathcal{M}_I > 0$ by assumption, the co-variant n -point correlator must vanish. This implies that in ageing-invariant theories, only responses, but not correlators, can be found directly from the requirement of covariance under the generators of the algebra.

4.15 In d space dimensions, form a first-order linear differential operator

$$\mathcal{R}\Psi := \left(A \frac{\partial}{\partial t} + \sum_{i=1}^d B_i \frac{\partial}{\partial r^i} + \mathcal{M}C \right) \Psi = 0$$

where A, B_i and C are constant matrices to be determined from $\mathcal{R}^2 \stackrel{!}{=} \mathcal{S}$. It is easy to check [468] that the matrices A, B_i, C give a representation of a **Clifford algebra**²⁵ (with an unconventional metric) in $d + 2$ dimensions. Namely, if we set

$$\begin{aligned} \mathfrak{B}_j &:= i\sqrt{2} B_j \quad ; \quad j = 1, \dots, d \\ \mathfrak{B}_{d+1} &:= A + \frac{1}{2}C \quad , \quad \mathfrak{B}_{d+2} := i \left(A - \frac{1}{2}C \right) \end{aligned}$$

then the condition on $\mathcal{R}$ is equivalent to the canonical form $\{\mathfrak{B}_j, \mathfrak{B}_k\} = 2\delta_{j,k}$ for $j, k = 1, \dots, d+2$ and where $\{A, B\} := AB + BA$ is the anti-commutator.

In the $1D$ case, the Clifford algebra generated by $\mathfrak{B}_j$, $j = 1, 2, 3$, has exactly one irreducible representation, up to equivalence. One may for example express the $\mathfrak{B}_i$ in terms of Pauli matrices σ^μ [358]

$$\mathfrak{B}_1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \mathfrak{B}_2 = \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad , \quad \mathfrak{B}_3 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Rescaling $r \mapsto \sqrt{2}r$, the wave equation $\mathcal{R}\Psi = \mathcal{R} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = 0$ becomes

$$\partial_t \psi = \partial_r \phi \quad , \quad 2\mathcal{M}\phi = \partial_r \psi . \tag{S.48}$$

These are called the *Dirac-Lévy-Leblond equations*, in one space dimension.²⁶

4.16 Since the masses $\mathcal{M} > 0$ by physical convention, define their dual ζ through a Laplace transform

²⁵ A Clifford algebra is defined by a set of generators which close with respect to the anti-commutator $\{a_i, a_j\}$. Representations of Clifford algebras only exist on even-dimensional spaces.

²⁶ A similar generalisation to spin-1 fields gives a Galilei-invariant analogue of Maxwell's equations. Compared to the relativistic case, only the terms describing the displacement current are absent – in particular there are no electromagnetic waves and no ‘Galilean’ radio ! [468].

$$\psi(\zeta, t, r) = \int_0^\infty d\mathcal{M} e^{-2\mathcal{M}\zeta} \tilde{\psi}(\mathcal{M}; t, r) \quad (\text{S.49})$$

and similarly for ϕ , where the tilde refers to solutions of the Dirac-Lévy-Leblond equations (S.48). Then, for $d = 1$, (S.48) become

$$\partial_t \psi = \partial_r \phi, \quad \partial_\zeta \phi = -\partial_r \psi. \quad (\text{S.50})$$

These are equivalent to the three-dimensional massless free Dirac equation $\gamma^\mu \partial_\mu \Phi = 0$, with coordinates ξ^μ , $\mu = 1, 2, 3$, $\partial_\mu = \partial/\partial \xi^\mu$ and the γ^μ form a Clifford algebra. To recover (S.50) from the Dirac equation, set $t = \frac{1}{2}(\xi^1 + i\xi^2)$, $\zeta = \frac{1}{2}(\xi^1 - i\xi^2)$, $r = \xi^3$, and choose the representation $\gamma^\mu = \sigma^\mu$.

Pauli (1940) showed that the dynamical symmetry of the free massless Dirac equation is the conformal group. Hence the known Schrödinger-invariance of a free non-relativistic particle of spin S , in particular $S = 0$ and $S = \frac{1}{2}$ [307], extends to conformal invariance. An explicit representation of the generators is given in 1D by $X_{-1} = -\partial_t$, $Y_{-\frac{1}{2}} = -\partial_r$, $M_0 = \frac{1}{2}\partial_\zeta$ and [358]

$$\begin{aligned} V_- &= -r\partial_t + 2\zeta\partial_r - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N = -t\partial_t + \zeta\partial_\zeta + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ Y_{\frac{1}{2}} &= -t\partial_r + \frac{1}{2}r\partial_\zeta - \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X_0 = -t\partial_t - \frac{1}{2}r\partial_r - \frac{1}{2} \begin{pmatrix} x & 0 \\ 0 & x+1 \end{pmatrix} \\ W &= -\frac{1}{2}r^2\partial_t + 2\zeta^2\partial_\zeta + 2\zeta r\partial_r + \begin{pmatrix} 2(x+1)\zeta & -r \\ 0 & 2x\zeta \end{pmatrix} \\ V_+ &= -tr\partial_t - \zeta r\partial_\zeta - \frac{1}{2}(r^2 - 4\zeta t)\partial_r - \frac{1}{2} \begin{pmatrix} (2x+1)r & 2t \\ -2\zeta & (2x+1)r \end{pmatrix} \\ X_1 &= -t^2\partial_t - tr\partial_r + \frac{1}{4}r^2\partial_\zeta - \begin{pmatrix} xt & 0 \\ r/2 & (x+1)t \end{pmatrix} \end{aligned}$$

and their commutators are again encoded in Fig. 4.2. Write the Dirac operator as

$$\mathcal{D} = \frac{1}{i}\mathcal{R} = \begin{pmatrix} \partial_r & \partial_\zeta \\ \partial_t & -\partial_r \end{pmatrix}.$$

Then the conformal invariance (*a fortiori* the Schrödinger-invariance) of the 1D Lévy-Leblond equation $\mathcal{D}\Psi = 0$ follows from the commutators

$$\begin{aligned} [\mathcal{D}, X_1] &= -t\mathcal{D} - \left(x - \frac{1}{2}\right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ [\mathcal{D}, X_{-1}] &= [\mathcal{D}, Y_{-\frac{1}{2}}] = [\mathcal{D}, V_-] = 0 \end{aligned}$$

if only $x = \frac{1}{2}$. Since $X_1, X_{-1}, Y_{-\frac{1}{2}}, V_-$ span $\mathfrak{conf}(3)$, see Fig. 4.2, the invariance under the remaining generators follows from the Jacobi identities.

It is sometimes useful to work with the generator $D := 2X_0 - N$ whose differential part is the Euler operator $-t\partial_t - r\partial_r - \zeta\partial_\zeta$.

We also see that the individual components ψ, ϕ of the spinor Ψ have scaling dimensions $x_\psi = x$ and $x_\phi = x + 1$, respectively. The generalisation to $d > 1$ should be straightforward and is left to the reader.

4.17 Given the obvious invariance under the three translations, consider first the invariance under the special transformation X_1 . From the explicit form given in exercise 4.16 and by using X_0 and $Y_{\frac{1}{2}}$, this simplifies to the condition $(x_1 - x_2)t\langle\psi_1\psi_2\rangle = 0$ and furthermore

$$\begin{aligned} 2(x_1 - x_2 - 1)t\langle\psi_1\phi_2\rangle - r\langle\psi_1\psi_2\rangle &= 0, \\ 2(x_1 - x_2 + 1)t\langle\phi_1\psi_2\rangle + r\langle\psi_1\psi_2\rangle &= 0, \\ 2(x_1 - x_2)t\langle\phi_1\phi_2\rangle + r\langle\psi_1\phi_2\rangle - r\langle\phi_1\psi_2\rangle &= 0 \end{aligned}$$

where for brevity here and below the arguments of the two-point functions are suppressed. From the first of these conditions, we distinguish two cases.

A: $x_1 = x_2$. We find, from the remaining three equations

$$\langle\psi_1\phi_2\rangle = \langle\phi_1\psi_2\rangle = -\frac{r}{2t}\langle\psi_1\psi_2\rangle.$$

Covariance under $Y_{\frac{1}{2}}$, N and X_0 , respectively, leads to the conditions

$$\begin{aligned} \left(-t\frac{\partial}{\partial r} + \frac{r}{2}\frac{\partial}{\partial \zeta}\right)\langle\psi_1\psi_2\rangle &= 0, \quad \left(-t\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial \zeta} + 1\right)\langle\psi_1\psi_2\rangle = 0, \\ \left(-t\frac{\partial}{\partial t} - \frac{r}{2}\frac{\partial}{\partial r} - x_1\right)\langle\psi_1\psi_2\rangle &= 0 \end{aligned}$$

and a similar system is found for $\langle\phi_1\phi_2\rangle$. The solution is unique up to a normalisation constant and reads

$$\begin{aligned} \langle\psi_1\psi_2\rangle &= \psi_0 t (4\zeta t + r^2)^{-x_1-1} \\ \langle\psi_1\phi_2\rangle &= \langle\phi_1\psi_2\rangle = -\frac{1}{2}\psi_0 r (4\zeta t + r^2)^{-x_1-1} \\ \langle\phi_1\phi_2\rangle &= \frac{\psi_0}{4}\frac{r^2}{t} (4\zeta t + r^2)^{-x_1-1} + \phi_0 \frac{1}{t} (4\zeta t + r^2)^{-x_1} \end{aligned} \tag{S.51}$$

In contrast to standard conformal invariance or Schrödinger-invariance for *scalars*, we see that certain two-point functions can be non-vanishing, even if they contain scaling operators with different scaling dimensions.

B: $\langle\psi_1\psi_2\rangle = 0$. The remaining conditions coming from X_1 are

$$\begin{aligned} (x_1 - x_2 - 1)t\langle\psi_1\phi_2\rangle &= 0, \quad (x_1 - x_2 + 1)t\langle\phi_1\psi_2\rangle = 0, \\ (x_1 - x_2)t\langle\phi_1\phi_2\rangle + \frac{r}{2}(\langle\psi_1\phi_2\rangle - \langle\phi_1\psi_2\rangle) &= 0. \end{aligned}$$

One of the conditions $x_1 = x_2 \pm 1$ must hold true. Supposing that $x_1 = x_2 + 1$, we get $\langle\phi_1\phi_2\rangle = -\frac{1}{2}(r/t)\langle\psi_1\phi_2\rangle$ and an analogous relation holds (with the

first and second field exchanged) in the other case. Again, covariance under $Y_{\frac{1}{2}}, N, X_0$ leads to a system of three linear equations for $\langle \psi_1 \phi_2 \rangle$, with the solution $\langle \psi_1 \psi_2 \rangle = \langle \phi_1 \psi_2 \rangle = 0$ and

$$\langle \psi_1 \phi_2 \rangle = \psi_0 (4\zeta t + r^2)^{-x_1} \quad , \quad \langle \phi_1 \phi_2 \rangle = -\frac{\psi_0}{2} \frac{r}{t} (4\zeta t + r^2)^{-x_1} \quad . \quad (\text{S.52})$$

The case $x_1 = x_2 - 1$ is obtained by exchanging the spinor Ψ_1 with Ψ_2 .

Extending this result to full conformal invariance, it is enough to check the co-variance under the further generator V_- . For case A, this requires $\psi_0 = -4\phi_0$. For case B, one has only the trivial solution $\psi_0 = 0$.

4.18 Returning to the scaling operators $\tilde{\psi}, \tilde{\phi}$ with $\mathcal{M} > 0$ fixed, by inverting the Laplace transform (S.49), the $\widehat{\mathfrak{sch}}(1)$ -covariant two-point functions of (S.51) take the form

$$\begin{aligned} \langle \tilde{\psi}_1 \tilde{\psi}_2 \rangle &= \psi'_0 \left(\frac{\mathcal{M}}{t} \right)^{x_1} \exp \left(-\frac{\mathcal{M}}{2} \frac{r^2}{t} \right) \\ \langle \tilde{\psi}_1 \tilde{\phi}_2 \rangle &= \langle \tilde{\phi}_1 \tilde{\psi}_2 \rangle = -\psi'_0 \frac{r}{2t} \left(\frac{\mathcal{M}}{t} \right)^{x_1} \exp \left(-\frac{\mathcal{M}}{2} \frac{r^2}{t} \right) \\ \langle \tilde{\phi}_1 \tilde{\phi}_2 \rangle &= \frac{\psi'_0}{4} \frac{r^2}{t} \left(\frac{\mathcal{M}}{t} \right)^{x_1} \exp \left(-\frac{\mathcal{M}}{2} \frac{r^2}{t} \right) + \phi'_0 \left(\frac{\mathcal{M}}{t} \right)^{x_1-1} \exp \left(-\frac{\mathcal{M}}{2} \frac{r^2}{t} \right) \end{aligned} \quad (\text{S.53})$$

Here, $\psi'_0 = \psi_0 / (\Gamma(x_1 + 1) 2^{x_1+1})$ and $\phi'_0 = \phi_0 / (\Gamma(x) 2^{x_1})$. Similarly to the treatment of exercise 4.11, a causality condition $t > 0$ via a Theta-function prefactor $\Theta(t)$ is obtained for $x_1 = x_2 > 0$, so that the covariant two-point functions obtained here must be interpreted as response functions.

For conformal covariance, one has the additional condition $\psi_0 = -4\phi_0$.

The case (S.52) is treated similarly.

4.19 The generator $\mathcal{X}$ is related to the associated group transformation via $T_g = \mathbf{1} + \varepsilon \mathcal{X} + \dots$. Therefore, the transformed Schrödinger equation becomes

$$\mathcal{S}[T_g \phi](t, \mathbf{r}) = F \left(t, \mathbf{r}, [T_g \phi](t, \mathbf{r}), [T_g \phi]^*(t, \mathbf{r}) \right)$$

after a relabelling of the coordinates. Upon expansion, this gives $\mathcal{S}(\mathcal{X}\phi) = (\mathcal{X}\phi)\partial_\phi F + (\mathcal{X}\phi)^*\partial_\phi F$. Inserting the explicit form of $\mathcal{X}$ gives the assertion (4.174). We emphasise that projective representations of the form (4.4) are admissible and that the phase $c(t, \mathbf{r}) \in \mathbb{C}$. When treating formally real equations such as an diffusion equation, the response field $\tilde{\phi}$ formally acts as a complex conjugate.

4.20 We discuss explicitly only the case $d = 1$ and leave the obvious extensions to $d > 1$ to the reader. The Galilei-algebra $\mathfrak{gal}(1) = \langle X_{-1}, Y_{\pm \frac{1}{2}}, M_0 \rangle$.

First, consider the case of fixed masses and use the representation (4.7). We apply (4.174) to each of the generators and find $\partial_t F = \partial_r F = 0$ from the translations $X_{-1}, Y_{-\frac{1}{2}}$, while invariance under either of the generators $Y_{\frac{1}{2}}$ or M_0 leads to

$$\mathcal{M}[-\phi\partial_\phi + \phi^*\partial_{\phi^*} + 1]F = 0 \implies F = \phi f(\phi\phi^*)$$

where the differentiable function f is not determined. Second, we use the dual variable ζ and the representation (4.44). The conclusions of temporal and spatial translation-invariance remain unchanged, whereas invariance under the generators $Y_{\frac{1}{2}}$ and M_0 now merely leads to $\partial_\zeta F = 0$. Hence the form of $F = F(\psi, \psi^*)$ is still left unconstrained.

4.21 Consider first the case of Schrödinger-invariance with fixed masses, for $d = 1$ for simplicity. Dilatation-invariance (X_0) gives the condition

$$\left[t\partial_t + \frac{1}{2}r\partial_r - \frac{x}{2}(\phi\partial_\phi + \phi^*\partial_{\phi^*}) + \left(\frac{x}{2} + 1\right) \right] F = 0.$$

Inserting the result from exercise 4.20, one obtains $-xuf'(u) + f(u) = 0$, hence $F = f_0 \phi(\phi\phi^*)^{1/x}$, where f_0 is a constant. Special Schrödinger-invariance fixes $x = d/2$ and we have re-derived a classical result.

We now turn to the case where the dual variable ζ is used and restrict ourselves to $d = 1$. Now also the conformal generators (4.49) not included in $\mathfrak{sch}(1)$ are first-order differential operators such that one can apply (4.174). Then we have to distinguish between the three (almost) parabolic subalgebras [686].

Case $\mathfrak{sch}(1)$: Both dilatation- as well as special Schrödinger-invariance ($X_{0,1}$) lead to the condition

$$\left[-\frac{x}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) + \left(\frac{x}{2} + 1\right) \right] F = 0$$

and where furthermore $x = \frac{1}{2}$. This gives

$$F = \psi^5 f(\psi/\psi^*) \tag{S.54}$$

where f denotes an arbitrary differentiable function. One can extend this to the parabolic subalgebra $\mathfrak{sch}(1)$ by adding the generator N , which does not modify the result for F .

Case $\mathfrak{age}(1)$: With respect to (4.44, 4.49), the generators X_1, V_+ take a slightly more general form

$$\begin{aligned} X_1 &= -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - (x + \xi)t \\ V_+ &= -2tr\partial_t - 2\zeta r\partial_\zeta - (r^2 + 2\zeta t)\partial_r - 2(x + \xi)r \end{aligned}$$

where the assumption of ageing-invariance of the kinetic term gives $x + \xi = \frac{1}{2}$ in 1D, see Sect. 4.3. Applying (4.174), the non-linearity now depends explicitly on time and after a straightforward calculation, we find

$$F = t^{-4\xi/(2\xi+1)} \psi^{(2\xi+5)/(2\xi+1)} f(\psi/\psi^*)$$

and f denotes an arbitrary differentiable function. If we extend the symmetry to $\widetilde{\mathfrak{age}}(1)$ by adding the generator N , consistency requires that $\xi = 0$ and we are back to (S.54).

Case $\mathfrak{alt}(1)$: In this case, the dilatation generator X_0 must be replaced by $\overline{D} := 2X_0 - N = -\zeta\partial_\zeta - t\partial_t - r\partial_r - x$. Applying (4.174), there are two solutions: (i) if $\xi = 0$, one recovers (S.54) and (ii) if $\xi \neq 0$, we obtain

$$F = t^{-2} \psi f(\psi/\psi^*) .$$

This second solution disappears if one extends the symmetry to $\widetilde{\mathfrak{alt}}(1)$ by adding the generator N .

4.22 Technically, it is simpler to work with the ‘dual’ coordinate ζ . We look for new representations of $\mathfrak{age}(1)$, whose generators may contain extra terms with respect to the coupling g , but we require that the spatial translation generator $Y_{-1/2} = -\partial_r$ should remain unchanged. For example, the mass generator may be written as $M_0 = -\partial_\zeta - L(\zeta, t, r, g)\partial_g$. Similarly, one has to modify the other generators and the extra functions will be determined from the commutation relations of $\mathfrak{age}(1)$.

It turns out [686] that one must distinguish two classes of solutions:

a) *Non-modified mass generator* $M_0 = -\partial_\zeta$. It is sufficient to give explicitly the ‘special’ generator

$$X_1 = -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta - t^{y+1}m(t^y/g)\partial_g - xt,$$

where the function $m(v)$ remains arbitrary. The remaining generators can be found from the commutation relations of $\mathfrak{age}(1)$, see also Fig. 4.3.

b) *Modified mass generator* $M_0 = -\partial_\zeta + 2y\zeta^{-1}g\partial_g$. Now, we find

$$\begin{aligned} X_1 = & -t^2\partial_t - tr\partial_r - \frac{1}{2}r^2\partial_\zeta \\ & - \frac{2y}{\zeta} \left(\frac{r^2g}{2} + h_0 tr g^{(2y-1)/(2y)} + h_1 t^{3/2} g^{(2y-1)/(2y)} \right) \partial_g - xt, \end{aligned}$$

where $h_{0,1}$ are arbitrary constants.

This approach can be extended to the other algebras discussed in the text. The resulting representations and the corresponding invariant equations have been classified [686]. Extensions to more than a single coupling are briefly summarised in [685].

If instead, one prefers to work with fixed masses $\mathcal{M}$, a similar analysis shows [52] that only the representation with a non-modified mass generator is possible and

$$X_1 = -t^2\partial_t - tr\partial_r - t^{y+1}p_0m(t^y/g)\partial_g - \frac{\mathcal{M}r^2}{2} - xt, \quad (\text{S.55})$$

where $p_0 = p_0(\mathcal{M})$ is a mass-dependent constant. The function $m(v)$ is found as follows [52]. Co-variance of the Schrödinger equation $\mathcal{S}\phi = 0$ requires us to consider the commutator

$$[\mathcal{S}, X_1] = -4M_0X_0 + (Y_{1/2}Y_{-1/2} + Y_{-1/2}Y_{1/2}) = -2t\mathcal{S} + \mathcal{M}\mathcal{Q}$$

with $\mathcal{Q} := 1 - x - 4yg\partial_g$. On the other hand, the explicit form (S.55) leads to

$$[\mathcal{S}, X_1] = -2t\mathcal{S} + \mathcal{M}(1 - 2x) - 2\mathcal{M}p_0t^y((y+1)vm(v) + ym'(v))\partial_g.$$

Comparison with the previous form gives a differential equation for $m(v)$, with the solution $m(v) = (2y/p_0)v^{-1} + (m_0/p_0)v^{-1-1/y}$. The final form of the generator X_1 is

$$X_1 = -t^2\partial_t - tr\partial_r - 2ytg\partial_g - m_0g^{1+1/y}\partial_g - \frac{\mathcal{M}r^2}{2} - xt.$$

In order to make this a dynamical symmetry of the Schrödinger equation $\mathcal{S}\phi_g = 0$, we must impose the auxiliary condition $\mathcal{Q}\phi_g = 0$, which leads to $\phi_g(t, r) = g^{(1-2x)/4y}\phi(t, r)$.

At last, we can find invariant semi-linear equations $\mathcal{S}\phi = F(t, r, g; \phi, \phi^*)$, using the same method as in the previous exercise 4.19. We leave the details to the motivated reader and merely quote the result [52]

$$F = \phi(\phi\phi^*)^{1/x}f\left((\phi\phi^*)^y\left[g^{1/y} - \frac{m_0}{y}\frac{1}{t}\right]\right)$$

with a scaling function f . In particular, this form of an invariant deterministic equation contains the bosonic pair-contact process as a special case.

For the Schrödinger algebra, the same analysis applies, but with $m_0 = 0$. This case is also recovered for the more general ageing algebra in the limit $t \rightarrow \infty$.

4.23 Use the autoresponse function (4.132), or (5.41) when $z \neq 2$. Then

$$\begin{aligned} \chi_{\text{TRM}}(t, s) &= \int_0^s du R(t, u) = s^{-a} \int_0^1 dv v^{-1-a} f_R\left(\frac{t}{s} \frac{1}{v}\right) \\ &= s^{-a} r_0 \left(\frac{t}{s}\right)^{-\lambda_R/z} \int_0^1 dv v^{-1-a+\lambda_R/z} \left(1 - \frac{s}{t}v\right)^{-1-a'} \end{aligned}$$

and rewrite the integral as a hypergeometric function [4, (15.3.1)].

4.24 Variation with respect to $\tilde{\phi}$ and ϕ , respectively, gives the equations of motion

$$\begin{aligned}\mathcal{S}\phi &:= (2\mathcal{M}\partial_t - \Delta_L)\phi + v(t)\phi = 0 \\ \tilde{\mathcal{S}}\tilde{\phi} &:= (-2\mathcal{M}\partial_t - \Delta_L)\tilde{\phi} + v(t)\tilde{\phi} = 0\end{aligned}$$

Using the representation of $\mathfrak{age}(1)$ defined in the text and (4.11), we have

$$[\mathcal{S}, X_0] = -2\mathcal{M}\partial_t + \partial_r^2 + t\dot{v} \stackrel{!}{=} -\mathcal{S}$$

which implies $t\dot{v} = -v$, hence $v(t) = v_0 t^{-1}$. Similarly, $[\tilde{\mathcal{S}}, X_0] = -\tilde{\mathcal{S}}$. Next, the condition $[\mathcal{S}, X_1] \stackrel{!}{=} -2t\mathcal{S}$ leads to $v_0 = \mathcal{M}(2x + 2\xi - 1)$ and similarly, $[\tilde{\mathcal{S}}, X_1] \stackrel{!}{=} -2t\tilde{\mathcal{S}}$ leads to $v_0 = \mathcal{M}(1 - 2\tilde{x} - 2\tilde{\xi})$. Consistency gives the constraint $x + \xi + \tilde{x} + \tilde{\xi} = 1$. The straightforward generalisation to $d \geq 1$ is left to the reader.

Problems of Chapter 5

5.1 Introduce the additional generators

$$Z_n^{(2)} = -nt^{n-1}r^2, \quad Z_m^{(1)} = -2t^{m-1/2}r, \quad Z_n^{(0)} = -2t^n.$$

Then the commutation relations (5.7) are completed as follows

$$\begin{aligned}[Y_m, Y_{m'}] &= (m - m') \left(4B_{20} Z_{m+m'}^{(2)} + B_{10} Z_{m+m'}^{(0)} \right) \\ [X_n, Z_{n'}^{(2)}] &= -n' Z_{n+n'}^{(2)}, \quad [Y_m, Z_n^{(2)}] = -n Z_{n+m}^{(1)} \\ [X_n, Z_m^{(1)}] &= -\left(\frac{n}{2} + m\right) Z_{n+m}^{(1)}, \quad [Y_m, Z_{m'}^{(1)}] = -Z_{m+m'}^{(0)} \\ [X_n, Z_{n'}^{(0)}] &= -n' Z_{n+n'}^{(0)}\end{aligned}$$

and all other commutators vanish. This is the maximal extension of $\mathfrak{sv}(1)$ through first-order differential operators such that the time- and space- translations $X_{-1}, Y_{-1/2}$ and the dilatation X_0 are given by their standard form (4.7) [330]. For $B_{20} = 0$, we recover $\mathfrak{sv}(1)$.

5.2 Let $\ell_n = -z^{n+1}\partial_z$ and $\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$ be the generators of the infinitesimal $2D$ conformal transformations, see Chap. 2 in Volume 1. The generators of (5.83) are obtained by setting [330]

$$X_n = \ell_n + \bar{\ell}_n, \quad Y_n = A_{10}\bar{\ell}_n.$$

In conformal field-theory, one introduces via $\ell_n \mapsto L_n$, $\bar{\ell}_n \mapsto \bar{L}_n$ a pair of commuting Virasoro algebras with the same central charge c . Generalising the

above correspondence, and choosing units such that the inverse characteristic velocity $A_{10} \stackrel{!}{=} 1$, three central extensions parametrised by a single constant c are found.

5.3 Direct computations give for $A_{10} \neq 0$ the commutators

$$[\mathcal{S}, X_n] = -(n+1)t^n \mathcal{S} + n(n+1)A_{10}t^{n-1} \left(x - \frac{B_{10}}{2A_{10}} \right), \quad [\mathcal{S}, Y_n] = 0$$

and one obtains a dynamical symmetry for the choice $x = B_{10}/2A_{10}$.

5.4 1. We begin with the two-point function, following [330]. We set $F(t_1, t_2; r_1, r_2) := \langle \phi_1(t_1, r_1) \phi_2(t_2, r_2) \rangle$, where each field ϕ_i , $i = 1, 2$ is characterised by the set of constants (x_i, μ_i, γ_i) . Time- and space-translation-invariance imply that $F = F(t, r)$. While the invariance under Y_1 merely gives the generalised Bargman superselection rules $\mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$, the form of the function $F(t, r)$ follows from

$$\begin{aligned} X_0 F &= (-t\partial_t - r\partial_r - x)F(t, r) = 0 \\ Y_0 F &= (-t\partial_r - \mu_1 r\partial_r - 2\gamma_1)F(t, r) = 0 \end{aligned}$$

while the last condition $X_1 F = 0$ reduces to $(x_1 - x_2)tF(t, r) = 0$ as in the text, hence $x_1 = x_2$. The solution of these equations reads for $\mu_1 \neq 0$

$$F(t, r) = t^{-2x_1} f_0 \left(1 + \frac{\gamma_1}{\mu_1} \frac{r}{t} \right)^{-2\gamma_1/\mu_1} \quad (\text{S.56})$$

and where the Bargman conditions are implied.

2. Turning to the three-point function, we now set $F(t_1, t_2, t_3; r_1, r_2, r_3) := \langle \phi_1(t_1, r_1) \phi_2(t_2, r_2) \phi_3(t_3, r_3) \rangle$. From time- and space-translation-invariance, we have

$$F = F(\tau, \sigma; r, s) \quad ; \quad \tau := t_1 - t_3, \quad \sigma := t_2 - t_3, \quad r := r_1 - r_3, \quad s := r_2 - r_3.$$

We now list the co-variance conditions for F . First, $X_0 F = 0$ leads to

$$[-\tau\partial_\tau - \sigma\partial_\sigma - r\partial_r - s\partial_s - x_1 - x_2 - x_3]F = 0 \quad (\text{S.57})$$

Next, the condition $Y_0 F = 0$ gives the two equations, since no explicit dependence on r_3 may remain

$$\begin{aligned} [-\tau\partial_r - \sigma\partial_s - \mu_1 r\partial_r - \mu_2 s\partial_s - \gamma_1 - \gamma_2 - \gamma_3]F &= 0 \\ [(\mu_1 - \mu_3)\partial_r + (\mu_2 - \mu_3)\partial_s]F &= 0 \end{aligned} \quad (\text{S.58})$$

Next, the condition $Y_1 F = 0$ leads to the following, whereby terms depending explicitly on t_3 can be eliminated from the previous equations, and terms explicitly containing r_3 give additional explicit conditions

$$\begin{aligned}
& [-(\tau + \mu_1 r)^2 \partial_r - (\sigma + \mu_2 s)^2 \partial_s - 2\gamma_1 \tau - 2\gamma_2 \sigma - 2\gamma_1 \mu_1 r - 2\gamma_2 \mu_2 s] F = 0 \\
& [-\mu_1 (\tau + \mu_1 r) \partial_r - \mu_2 (\sigma + \mu_2 s) \partial_s - \mu_1 \gamma_1 - \mu_2 \gamma_2 - \mu_3 \gamma_3] F = 0 \\
& [(\mu_1 + \mu_3)(\mu_1 - \mu_3) \partial_r + (\mu_2 + \mu_3)(\mu_2 - \mu_3) \partial_s] F = 0
\end{aligned} \tag{S.59}$$

At this point, a first important simplification can be made. Multiply the second equation of (S.58) with $\mu_1 + \mu_3$ and subtract from the third equation of (S.59), which gives

$$(\mu_2 - \mu_1)(\mu_2 - \mu_3) \partial_s F = 0.$$

Unless F is independent of s (and then a rather trivial result would follow), we must have either $\mu_1 = \mu_2$ or $\mu_2 = \mu_3$. In the first case, the second equation of (S.59) can be rewritten as μ_1 times the first equation of (S.58), provided $\mu_1 = \mu_3$. On the other hand, if the second case $\mu_2 = \mu_3$ applies, the second equation of (S.58) becomes $(\mu_1 - \mu_3) \partial_r F = 0$, hence $\mu_1 = \mu_3$. We have thus obtained a generalised Bargman superselection rule

$$\mu_1 = \mu_2 = \mu_3 =: \mu$$

and μ will be a *universal* constant with the dimension of an inverse velocity, but independent of the scaling operators under consideration.

Now, the equations determining the form of F take the simpler form

$$\begin{aligned}
& [-\tau \partial_r - \sigma \partial_s - \mu(r \partial_r + s \partial_s) - \gamma_1 - \gamma_2 - \gamma_3] F = 0 \\
& [-\tau \partial_r - \sigma \partial_s - r \partial_r - s \partial_s - x_1 - x_2 - x_3] F = 0 \\
& [-(\tau + \mu r)^2 \partial_r - (\sigma + \mu s)^2 \partial_s - 2\gamma_1 \tau - 2\gamma_2 \sigma - 2\mu \gamma_1 r - 2\mu \gamma_2 s] F = 0
\end{aligned}$$

A solution to the first of these is $F = F_0 \cdot (\tau + \mu r)^{-\sum_i \gamma_i / \mu}$, where $F_0 = F_0(\tau, \sigma, (\tau + \mu r)/(\sigma + \mu s))$. Using now the last of the above conditions, this reduces to

$$F = f_0 \cdot (\tau + \mu r)^{-\gamma_{31,2}/\mu} (\sigma + \mu s)^{-\gamma_{23,1}/\mu} (\tau - \sigma + \mu(r - s))^{-\gamma_{12,3}/\mu}$$

where $f_0 = f_0(\tau, \sigma)$ and $\gamma_{ij,k} := \gamma_i + \gamma_j - \gamma_k$.

At last, we use the condition $X_1 F = 0$ to find the form of f_0 . Using the results obtained so far, the co-variance condition becomes

$$[-\tau^2 \partial_r - \sigma^2 \partial_s - 2\tau r \partial_r - 2\sigma s \partial_s - \mu(r^2 \partial_r + s^2 \partial_s) - 2x_1 \tau - 2x_2 \sigma - 2\gamma_1 r - 2\gamma_2 s] F = 0$$

Together with the condition (S.57), derived above from global scale-invariance $X_0 F = 0$, we find the following system

$$\begin{aligned}
& \left[-\tau \partial_r - \sigma \partial_s - (x_1 + x_2 + x_3) + \frac{\gamma_1 + \gamma_2 + \gamma_3}{\mu} \right] f_0 = 0 \\
& \left[-\tau^2 \partial_r - \sigma^2 \partial_s - 2x_1 \tau - 2x_2 \sigma + \frac{2\gamma_1}{\mu} \tau + \frac{2\gamma_2}{\mu} \sigma \right] f_0 = 0
\end{aligned}$$

Indeed, if we now make the substitution $x_i - \gamma_i/\mu \mapsto x_i$, we would recover exactly the conditions for a three-point function co-variant under conventional conformal transformations, see Chap. 2 in Volume 1. We then have the final result, with the abbreviation $x_{ij,k} := x_i + x_j - x_k$

$$F(\tau, \sigma; r, s) = F_{123} \tau^{-x_{13,2}} \sigma^{-x_{23,1}} (\tau - \sigma)^{-x_{12,3}} \quad (\text{S.60})$$

$$\times \left[1 + \mu \frac{r}{\tau}\right]^{-\gamma_{13,2}/\mu} \left[1 + \mu \frac{s}{\sigma}\right]^{-\gamma_{23,1}/\mu} \left[1 + \mu \frac{r-s}{\tau-\sigma}\right]^{-\gamma_{12,3}/\mu}$$

and where F_{123} is an undetermined constant. There is no selection rule on the parameters γ_i , which rather appear to be a second kind of exponent.

Although this Lie algebra is isomorphic to $\mathfrak{conf}(2)$, the forms of the co-variant n -point functions are distinct.

5.5 Dimensional analysis of the terms in the generators X_n, Y_n , from Table 5.1 case (iii), shows that $c := 1/A_{10}$ has the dimension of a velocity such that the group contraction achieved by taking the $c = A_{10}^{-1} \rightarrow \infty$ may physically be interpreted as a non-relativistic limit (this is a contraction since the dimension of the algebra does not change). The isomorphism of the finite-dimensional subalgebra $\langle X_{\pm 1,0}, Y_{\pm 1,0} \rangle \cong \mathfrak{alt}(1)$ can be seen from the root diagram Fig. 4.3c, where the three generators X_{-1}, X_0, X_{+1} correspond to the three roots on the e_2 -axis, from bottom to top. The three generators Y_{-1}, Y_0, Y_{+1} , from bottom to top, correspond to the three roots on the straight line to the right of the e_2 -axis. Performing the limit $A_{10} \rightarrow 0$ in the generators as listed in Table 5.1 case (iii), we find with $\gamma := B_{10}$ and $n \in \mathbb{Z}$ [354, 355]

$$X_n = -t^{n+1} \partial_t - (n+1)t^n r \partial_r - (n+1)xt^n - n(n+1)\gamma t^{n-1}r$$

$$Y_n = -t^{n+1} \partial_r - (n+1)\gamma t^n. \quad (\text{S.61})$$

This function-space representation of $\mathfrak{altv}(1)$, with vanishing central charges, is characterised by the pair of constants (x, γ) .

We can write the generators of the centre-less **altern-Virasoro algebra** $\mathfrak{altv}(d)$ in d spatial dimensions, with $j, k = 1, \dots, d$ as follows [146]

$$X_n = -t^{n+1} \partial_t - (n+1)t^n \mathbf{r} \cdot \nabla_{\mathbf{r}} - (n+1)xt^n - n(n+1)t^{n-1} \boldsymbol{\gamma} \cdot \mathbf{r}$$

$$Y_n^{(j)} = -t^{n+1} \frac{\partial}{\partial r_j} - (n+1)\gamma_j t^n \quad (\text{S.62})$$

$$R_n^{(jk)} = -t^n \left(r_j \frac{\partial}{\partial r_k} - r_k \frac{\partial}{\partial r_j} \right) - t^n \left(\gamma_j \frac{\partial}{\partial \gamma_k} - \gamma_k \frac{\partial}{\partial \gamma_j} \right).$$

and we introduced a vector $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$ of constants. A quasi-primary operator under this representation will be characterised by the constants $(x, \boldsymbol{\gamma})$.

Central extensions: a glance at the root diagram Fig. 4.3c shows that no further generators, commuting with all $X \in \mathfrak{alt}(1)$, can be introduced for the

finite-dimensional subalgebra, although it is not semi-simple [354].²⁷ However, there does exist a central extension of $\mathfrak{alt}(2)$, which is commonly called **exotic** [483, 484]. For the infinite-dimensional algebra $\mathfrak{altv}(1)$, one may either follow the procedure outlined in appendix L or else apply the powerful mathematical techniques of cohomology [569, 303, 354], with the result

$$\begin{aligned}[X_n, X_{n'}] &= (n - n')X_{n+n'} + \frac{c_X}{12} (n^3 - n) \delta_{n+n',0} \\ [X_n, Y_{n'}] &= (n - n')Y_{n+n'} + \frac{c_Y}{12} (n^3 - n) \delta_{n+n',0}.\end{aligned}\quad (\text{S.63})$$

The two central charges $c_{X,Y}$ are independent. We illustrate this by an example [355]. Consider the generators L_n and L'_n ($n \in \mathbb{Z}$) of two commuting Virasoro algebras with central charges c and c' . Then set

$$X_n := \begin{pmatrix} L_n + L'_n & 0 \\ 0 & L_n + L'_n \end{pmatrix}, Y_n := \begin{pmatrix} 0 & L_n \\ 0 & 0 \end{pmatrix}, K_X := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K_Y := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This reproduces the commutators (S.63) and $K_{X,Y}$ are central. The values of the central charges $c_X = (c + c')K_X$ and $c_Y = cK_Y$ can be chosen independently.

5.6 The two- and three-point functions can of course be found by writing down the co-variance conditions, using the explicit representation (S.62), and solving them, as has been done in [330, 355, 32]. Here, we rather obtain their form from the solution of exercise 5.4. In the explicit expressions (S.56,S.60), we take the limit $\mu \rightarrow 0$ and use the relation

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{\mu \rightarrow 0} (1 + \mu x)^{1/\mu}.$$

We then find for the two-point function in $d = 1$

$$\langle \phi_1(t_1, r_1) \phi_2(t_2, r_2) \rangle = \phi_0 \delta_{x_1, x_2} \delta_{\gamma_1, \gamma_2} (t_1 - t_2)^{-2x_1} \exp \left[-2\gamma_1 \frac{r_1 - r_2}{t_1 - t_2} \right]$$

and for the three-point function in $d = 1$

$$\begin{aligned}\langle \phi_1(t_1, r_1) \phi_2(t_2, r_2) \phi_3(t_3, r_3) \rangle &= \\ \phi_{123} (t_1 - t_3)^{-x_{13,2}} (t_2 - t_3)^{-x_{23,1}} (t_1 - t_2)^{-x_{12,3}} \\ &\times \exp \left[-\gamma_{13,2} \frac{r_1 - r_3}{t_1 - t_3} - \gamma_{23,1} \frac{r_2 - r_3}{t_2 - t_3} - \gamma_{12,3} \frac{r_1 - r_2}{t_1 - t_2} \right]\end{aligned}$$

where ϕ_0 and ϕ_{123} are undetermined constants and $x_{ij,k} = x_i + x_j - x_k$ and $\gamma_{ij,k} = \gamma_i + \gamma_j - \gamma_k$.

²⁷ On the contrary, if one considers the massless Schrödinger algebra $\mathfrak{sch}^{(0)}(1)$, the root diagram Fig. 4.3a indicates that one can construct a central extension $\mathfrak{sch}(1) = \mathfrak{sch}^{(0)}(1) \oplus \mathbb{C}M_0$, where the generator M_0 corresponds to the root in the lower right corner and indeed $[\mathfrak{sch}^{(0)}(1), M_0] = 0$.

Generalisations to higher dimensions $d > 1$ of two- and three-point functions transforming co-variantly under the representation (S.62) are obtained by replacing $\gamma r \mapsto \boldsymbol{\gamma} \cdot \mathbf{r}$ in the above results. Then the rotation-invariance of the resulting expressions is manifest.

5.7 In the first case, when the generators are given by (4.44,4.49), the two-point function reads [357]

$$\begin{aligned} \langle \psi_1(\zeta_1, t_1, r_1) \psi_2(\zeta_2, t_2, r_2) \rangle &= (t_1 - t_2)^{-(x_1+x_2)/2} \left(\frac{t_1}{t_2} \right)^{(x_2-x_1)/2} \\ &\times f \left(\zeta_1 - \zeta_2 + \frac{i}{2} \frac{(r_1 - r_2)^2}{t_1 - t_2} \right) \end{aligned} \quad (\text{S.64})$$

where f is an arbitrary function. Similarly, in the second case, imposing co-variance under the representation (5.86) leads to [355]

$$\langle \psi_1(\zeta_1, t_1, r_1) \psi_2(\zeta_2, t_2, r_2) \rangle = \delta_{x_1, x_2} (t_1 - t_2)^{-x_1} f \left(\zeta_1 + \zeta_2 + \frac{i}{2} \frac{r_1^2 - r_2^2}{t_1 - t_2} \right) \quad (\text{S.65})$$

where again f is an arbitrary function. It is evident that these two representations will describe quite distinct physical systems.

A more habitual form can be obtained if we trade the dual coordinate ζ for a fixed mass $\mathcal{M}$ via the transformation (4.42). Analogously to the calculations in exercise 4.11, we find from the first representation and (S.64) [357]

$$\begin{aligned} \langle \phi_1(t_1, r_1) \phi_2^*(t_2, r_2) \rangle &= \phi_0 \delta(\mathcal{M}_1 - \mathcal{M}_2) \Theta(t_1 - t_2) \\ &\times \left(\frac{t_1}{t_2} \right)^{(x_2-x_1)/2} (t_1 - t_2)^{-(x_1+x_2)/2} \exp \left[-\frac{\mathcal{M}_1}{2} \frac{(r_1 - r_2)^2}{t_1 - t_2} \right] \end{aligned}$$

where $\Theta(t)$ is the Heaviside function and ϕ_0 a normalisation constant. From the second representation and (S.65) we find [355]

$$\langle \phi_1(t_1, r_1) \phi_2(t_2, r_2) \rangle = \phi_0 \delta_{x_1, x_2} \delta(\mathcal{M}_1 - \mathcal{M}_2) (t_1 - t_2)^{-x_1} \exp \left[-\frac{\mathcal{M}_1}{2} \frac{r_1^2 - r_2^2}{t_1 - t_2} \right].$$

In contrast to most cases studied in this volume, we obtain here a simple Bargman-type rule between two scaling operators and need not invoke either a ‘complex conjugate’ or a conjugate response operator.

The two forms obtained here have the same dynamic exponent $z = 2$, whereas the representation (S.62) used in exercise 5.6, which gave $z = 1$. Still, these examples clearly illustrate that the knowledge of the value of z does not yet fix the form of the n -point scaling functions, but rather admit quite distinct space-time behaviours.

5.8 Recall from exercise 5.3 that the invariant equation found there becomes trivial in the limit $A_{10} \rightarrow 0$ which is required for obtaining the algebra $\text{CGA}(2)$.

Because of the ‘exotic’ central extensions, a non-trivial equation is now identified, because with (5.88) the only non-vanishing commutators with $\mathcal{S}$ are

$$[\mathcal{S}, X_0] = -S \quad , \quad [\mathcal{S}, X_1] = -2t\mathcal{S} + 2\theta(x-3)$$

and we read off $x = 3$.

One may obtain ECGA from a contraction of the conformal algebra $\mathfrak{so}(3, 2)$ and derive (5.89) by the same contraction from the Dirac equation [503].

5.9 Following [503], form the two-point function

$$F = F(t_1, t_2; \mathbf{r}_1, \mathbf{r}_2; \mathbf{v}_1, \mathbf{v}_2) = \langle \phi_1(t_1, \mathbf{r}_1, \mathbf{v}_1) \phi_2(t_2, \mathbf{r}_2, \mathbf{v}_2) \rangle.$$

In what follows, we specialise to $\gamma = \mathbf{0}$, so that a quasi-primary scaling operator ϕ is characterised by a pair (x, θ) . Because of translation-invariance in t , $\mathbf{r}$, one sets $t := t_1 - t_2$, $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{w} := \mathbf{v}_1 + \mathbf{v}_2$ and adding the effect of dilatation-invariance, one has

$$F = t^{-x_1-x_2} f(\mathbf{u}, \mathbf{v}, \mathbf{w}) \quad , \quad \mathbf{u} := \mathbf{r}/t.$$

Invariance under $Y_0^{(j)}$ gives

$$\left(\frac{\partial}{\partial u_j} - 2 \frac{\partial}{\partial w_j} + \frac{1}{4} \epsilon_{jk} (w_k \theta_+ + v_k \theta_-) \right) f = 0$$

where $\theta_{\pm} := \theta_1 \pm \theta_2$. From the effect of the $Y_1^{(j)}$ it is now easily seen that one obtains a *generalised Bargman superselection rule* $\theta_+ = \theta_1 + \theta_2 = 0$. Simplifying the above relation and writing the Ward identity which follows from $Y_1^{(j)} F = 0$, one has the system

$$\left(\frac{\partial}{\partial u_j} - 2 \frac{\partial}{\partial w_j} + \frac{1}{2} \theta_1 \epsilon_{jk} v_k \right) f = \left(\frac{\partial}{\partial v_j} - \theta_1 \epsilon_{jk} \left(u_k + \frac{1}{4} w_k \right) \right) f = 0.$$

Solving these equations, and also taking into account that the condition $X_1 F = 0$ merely leads to $x_1 = x_2$, the final result is

$$F = \delta_{x_1, x_2} \delta_{\theta_1 + \theta_2, 0} t^{-2x_1} f_0 \left(\frac{\mathbf{r}}{t} + \frac{\mathbf{w}}{2} \right) \exp(\theta_1 \mathbf{v} \wedge (\mathbf{r}/t + \mathbf{w}/4))$$

and where the function f_0 remains undetermined.

5.10 Following [330, app. C], first verify the commutators $[X_0, X_{-1}] = X_{-1}$, $[X_{-1}, Y_{-1/2}] = 0$ and $[X_0, Y_{-1/2}] = \frac{1}{z} Y_{-1/2}$. Using (J14), we find

$$[X_0, Y_{1/2}] = \frac{z-1}{z} t \partial_r - \frac{1}{z} \mathcal{M}'(\partial_r) \partial_r r + \frac{1}{z} \mathcal{M}(\partial_r) r \stackrel{!}{=} -\frac{z-1}{z} Y_{1/2}$$

where the last equation is motivated from the first term in the generator $Y_{1/2}$ and our construction principle. If we let $x := \partial_r$, we find $(z-2)\mathcal{M}(x) =$

$-\mathcal{M}'(x)x$. The solution is $\mathcal{M}(x) = \mathcal{M}_0 x^{2-z}$, where $\mathcal{M}_0$ is a constant. Next, we calculate

$$[X_{-1}, Y_{1/2}] = -Y_{-1/2} \quad , \quad [Y_{1/2}, Y_{-1/2}] = -\mathcal{M}_0 \partial_r^{2-z} =: M_0$$

with the new generator M_0 . For $z = 2$, we recover the Galilei algebra and M_0 is central. Otherwise, the non-vanishing commutators with M_0 read

$$[X_0, M_0] = -\frac{2-z}{z} M_0 \quad , \quad [Y_{1/2}, M_0] = -(2-z) \mathcal{M}_0^2 \partial_r^{3-2z}$$

and we see that we must define a new generator N . In the special case $z = 3/2$, we have $N := -\frac{1}{2} \mathcal{M}_0^2$. Then $[Y_{1/2}, M_0] = N$ and N is central. Therefore, if $z = 3/2$, the set

$$\mathfrak{a} := \langle X_{-1}, X_0, Y_{-1/2}, Y_{1/2}, M_0, N \rangle$$

closes as a Lie algebra and satisfies the required counting condition on X_0 .

5.11 Recall the generators constructed in exercise 5.10

$$Y_{1/2} = -t\partial_r - \mathfrak{M} \partial_r^{1/2} r \quad , \quad M_0 = -\mathfrak{M} \partial_r^{1/2} \quad , \quad N = -\frac{1}{2} \mathfrak{M}^2 \quad .$$

Co-variance implies that the action of all generators of $\mathfrak{a}$ on the two-point function $F = F^{(\alpha, \beta)}(t_1, t_2; r_1, r_2) = \langle \phi_1^{(\alpha)}(t_1, r_1) \phi_2^{(\beta)}(t_2, r_2) \rangle$ vanishes, where $x = x_1 + x_2$ is the total scale-dimension. Spatio-temporal translation invariance yields $F = F(t, r)$, where $t = t_1 - t_2$ and $r = r_1 - r_2$. Invariance under the action of N leads to the condition $\mathcal{M}_2^2 = (-1)^{\beta-\alpha+1} \mathcal{M}_1^2$ or alternatively

$$\mathcal{M}_2 = -i^{\beta-\alpha+1} \mathcal{M}_1 \quad , \quad \text{where } \beta = \alpha + 1 \bmod 2 \quad . \quad (\text{S.66})$$

Then the action of M_0 on F is, using translation invariance and (J11), $M_0 F(t, r) = -i^\alpha \mathcal{M}_1 (1 - (-1)^{\beta-\alpha+1}) \partial_r^{1/2} F(t, r) = 0$ because of (S.66). Next, recall from the Leibniz rule (J13) that for $f \in \mathcal{M}_E$ one must have $\partial_r^{1/2}(rf(r)) = r\partial_r^{1/2}f(r) + \frac{1}{2}\partial_r^{-1/2}f(r)$, hence using again (J11),

$$Y_{1/2}F(t, r) = -\left(t\partial_r + \mathfrak{M}\left(r\partial_r^{1/2} + \partial_r^{-1/2}\right)\right)F(t, r)$$

where we have set $\mathfrak{M} := \mathcal{M}_1 i^\alpha$. Finally, $X_0 F = 0$ is solved by the scaling ansatz $F(t, r) = t^{-2x/3} \Psi(u)$ with $u = rt^{-2/3}$. This leads to a fractional differential equation for the scaling function $\Psi(u)$

$$\left(\partial_u + \mathfrak{M} u \partial_u^{1/2} + \mathfrak{M} \partial_u^{-1/2}\right) \Psi(u) = 0 \quad .$$

This may be solved by the series ansatz $\Psi(u) = \sum_{n=0}^{\infty} \Psi_n u^{s+3n/2}$ with $\Psi_0 \neq 0$, gives the recursion $\Psi_{n+2} = \Psi_n (2\mathfrak{M}^2/3)(n+1)((n+2/3)(n+4/3))^{-1}$ and finally the scaling function

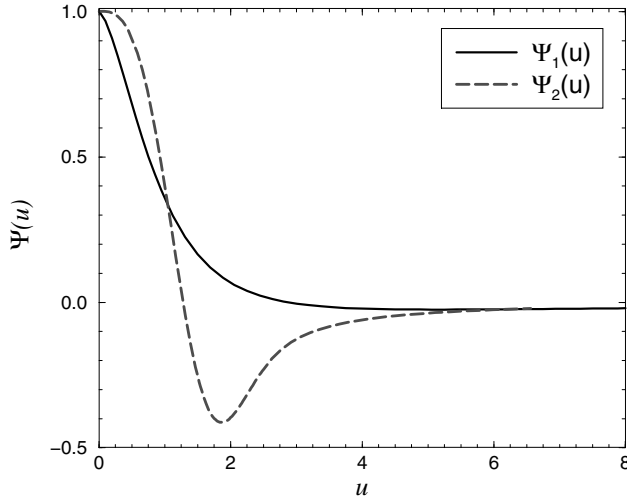


Fig. S.5. Covariant scaling functions $\Psi_{1,2}(u)$ for the two-point functions $F_i(t, r) = t^{-2x/3}\Psi_i(u)$, for $\mathcal{M} = 1$ and $\Psi_0 = 1$, of the $z = \frac{3}{2}$ algebra of exercises 5.10, 5.11. Adapted from [330].

$$\Psi(u) = \Psi_0 \left[{}_2F_2 \left(1, \frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{\mathfrak{M}^2}{3} u^3 \right) - \sqrt{\frac{4}{\pi} \mathfrak{M}^2 u^3} {}_2F_2 \left(1, 1; \frac{5}{6}, \frac{7}{6}; \frac{\mathfrak{M}^2}{3} u^3 \right) \right]$$

where ${}_2F_2$ is a generalised hypergeometric function and $\Psi_0 = \Psi(0)$ is the initial value.

For a physical interpretation, recall that the four scaling operators $\phi^{(\alpha)}$ had the ‘masses’ $\mathfrak{M} = i^\alpha \mathcal{M}_1$, where $\mathcal{M}_1 > 0$ and $\alpha = 0, 1, 2, 3$. In addition, considering global scale invariance with either $t = 0$ or $r = 0$, it follows that for $u \rightarrow \infty$, the boundary condition $\Psi(u) \rightarrow 0$ must be satisfied. That is possible in two cases: (i) when $\mathfrak{M} > 0$ and (ii) when $\mathfrak{M}$ is imaginary and the real part of the two-point function is retained. In these cases, the two-point functions read

$$\begin{aligned} F_1 &= F^{(0,1)}(t, r) = t^{-2x/3} \Psi(u) \quad ; \quad \text{where } \mathfrak{M} = \mathcal{M}_1 = \mathcal{M}_2 > 0 \\ F_2 &= \frac{1}{2} \left(F^{(1,2)}(t, r) + F^{(3,0)}(t, r) \right) = t^{-2x/3} \Psi_0 {}_2F_2 \left(1, \frac{1}{2}; \frac{1}{3}, \frac{2}{3}; -\frac{\mathcal{M}_1^2}{3} u^3 \right) \\ &\quad ; \quad \text{where } \mathfrak{M}^2 = -\mathcal{M}_1^2, \mathcal{M}_1 = \mathcal{M}_2 > 0 \end{aligned}$$

The scaling functions $\Psi_{1,2}(u)$ so obtained are shown in Fig. S.5 [330].

5.12 For $a \neq e + n + 1$ with $n \in \mathbb{N}$, one applies the definition to $f(r) = r^e$

$$\frac{1}{\Gamma(-a)} \int_0^r d\rho \frac{\rho^e}{(r-\rho)^{a+1}} = \frac{r^{1+e-a-1}}{\Gamma(-a)} \int_0^1 du \frac{u^e}{(1-u)^{a+1}} = \frac{r^{1+e-a-1} \Gamma(e+1)}{\Gamma(-a+e+1)}$$

whereas the case $a = e + n + 1$ is established from the regularisation of the integral [274]. On the other hand, if $f(r) = \delta^{(n)}(r)$ and $a \neq n$, one finds from n -fold partial integration

$$\frac{1}{\Gamma(-a)} \int_0^r d\rho \delta^{(n)}(\rho) (r - \rho)^{-a-1} = \frac{(-a-1) \cdots (-a-1-n) r^{-a-1-n}}{(-a-1) \cdots (-a-1-n) \Gamma(-a-n)}$$

and finally, the case $a = n$ is obvious.

5.13 One may decompose the Janssen-de Dominics functional (D16) as $\mathcal{J}_{\text{eff}}[\phi, \tilde{\phi}] = \mathcal{J}[\phi, \tilde{\phi}] + \mathcal{J}_{\text{ini}}[\tilde{\phi}]$ where the contributions from the initial state are given by $\mathcal{J}_{\text{ini}}$ in (D15). Using again the notion of the deterministic average $\langle \cdot \rangle_0$, we have

$$m(t) = \langle \phi(t, \mathbf{0}) \rangle = \left\langle \phi(t, \mathbf{0}) e^{-\mathcal{J}_{\text{ini}}[\tilde{\phi}]} \right\rangle_0 = m_0 \int_{\mathbb{R}^d} d\mathbf{r} \langle \phi(t, \mathbf{0}) \tilde{\phi}(0, \mathbf{r}) \rangle_0$$

where spatial translation-invariance was used and the contribution coming from the initial state was expanded in m_0 before the generalised Bargman superselection rules were applied. For the calculation of the response function arising herein, we see in the same way that it is independent of the initial states. Now, for $t \ll t_M$ (this may be realised by considering $m_0 \ll 1$), the response function is scale-invariant and one may use the usual procedure of local scale-invariance to find its form. Generalising (1.79), we expect for t not too small $R(t, 0; \mathbf{r}) = \langle \phi(t, \mathbf{0}) \tilde{\phi}(0, \mathbf{r}) \rangle_0 = t^{-\lambda_{R'}/z} \mathcal{F}^{(\alpha, \beta)}(\mathbf{r} t^{-1/z})$ and where the scaling function $\mathcal{F}^{(\alpha, \beta)}(\mathbf{u})$ should be given by (5.36). Performing a change of variables, we find

$$m(t) = t^{(d-\lambda_{R'})/z} m_0 \int_{\mathbb{R}^d} d\mathbf{u} \mathcal{F}^{\alpha, \beta}(\mathbf{u}) \sim t^\Theta$$

where the identification with the slip exponent Θ is justified because only the term linear in m_0 survived. Comparison leads to $\lambda_{R'} = d - \Theta z$. This reproduces (1.112), provided the identifications $\lambda_{R'} = \lambda_R$ and $\lambda_R = \lambda_C$ are admissible.

The main point of this calculation is to show that the initial conditions lead to $m(t) \neq 0$. The cross-over between this early-time regime and the relaxation regime where $t \gg t_M$, which again obeys a simple scale-invariance, is not accessible by these methods. In Chap. 2, an explicit calculation was described for the spherical model.

5.14 An uncorrelated initial state is described by the initial correlator $\hat{a}(\mathbf{k}) = a_0 \text{constant}$ in Fourier space. Before starting to insert this into (5.48), a conceptual remark is in order [48]. Since the form of the initial correlator means that we have to bring together two scaling operators $\tilde{\phi}$, one should consider that these will rather form a *composite* operator $\tilde{\phi}^2$ and characterised by the scaling exponents $(\tilde{x}_2, \tilde{\xi}_2)$, which become $(2\tilde{x}, 2\tilde{\xi})$ only in the case of free

fields. When using (5.48), the replacements $\tilde{x} \mapsto \tilde{x}_2$ and $\tilde{\xi} \mapsto \tilde{\xi}_2$ should hence be made. Then the Fourier integral is readily calculated and comparison with the expected scaling form leads to

$$b = \frac{2}{z}(x + \tilde{x}_2 + 2\tilde{\xi}_2) - \frac{d}{z} - 2F \quad , \quad \lambda_C = \lambda_R + \beta$$

and the scaling function reads

$$f_C(y) = f_0 y^{-b+\lambda_C/z} (y-1)^{b+(d-2\lambda_C+2\beta)/z} (y+1)^{-(d+2\beta)/z}.$$

In contrast to the case $z = 2$, the autocorrelation and autoresponse exponents have the same value only if $\beta = 0$, which by (5.30) occurs either for $2\gamma = \mu_1$ or for $z = 2$.

5.15 Following [48], we use (5.51) with $z = 2$, as appropriate for phase-ordering. Using the given equal-time correlator, we have in Fourier space

$$\begin{aligned} \hat{a}(t, \mathbf{k}) &= \int_{\mathbb{R}^2} d\mathbf{r} \arcsin \left[\exp \left(-\nu \frac{r^2}{t} \right) \right] e^{-i\mathbf{r} \cdot \mathbf{k}} \\ &\sim \int_0^\infty dr \, r \arcsin \left[\exp \left(-\nu \frac{r^2}{t} \right) \right] J_0(kr) \end{aligned}$$

where the angular integration has been carried out and an unimportant amplitude was dropped. Since we have $z = 2$, it also follows that $\beta = 0$. The remainder of the calculation is immediate.

Problems of Chapter 6

6.1 Assume for the two-point correlation functions the scaling form

$$G_i(r_\perp, r_\parallel; \tau, h) = b^{-2x_i} G_i \left(\frac{r_\perp}{b}, \frac{r_\parallel}{b^\theta}; \tau b^{y_t}, h b^{y_h} \right) \quad (\text{S.67})$$

where b is the rescaling factor and y_t, y_h, x_i are scaling exponents. The index i refers to different physical quantities of which the two-point function is formed, e.g. $i = \sigma$ for the spin operator or $i = \varepsilon$ for the energy density. At the Lifshitz point, $\tau = h = 0$, and one has $G_i(r_\perp, 0) \sim r_\perp^{-2x_i}$ and $G_i(0, r_\parallel) \sim r_\parallel^{-2x_i/\theta}$. We use the equilibrium fluctuation-dissipation theorem

$$\chi = \int dr_\parallel d^d r_\perp G_\sigma(r_\perp, r_\parallel) \quad , \quad C = \int dr_\parallel d^d r_\perp G_\varepsilon(r_\perp, r_\parallel), \quad (\text{S.68})$$

where d is the number of ‘spatial’ dimensions and χ, C are the susceptibility and specific heat. For simplicity, we shall restrict ourselves to $m = 1$ directions with modulation in what follows and leave the further extensions to the reader.

Units are such that the critical temperature $T_c = 1$. Integrating (S.68) and comparing with the scaling of χ directly gives the expected scaling relation for γ and we now concentrate on the derivation of hyperscaling [353].

From (S.67) and integrating, one gets immediately the scaling forms for $\chi = \chi(\tau, h) = -\partial^2 g^{(\text{sing})}/\partial h^2$ and for $C = C(\tau, h) = -\partial^2 g^{(\text{sing})}/\partial \tau^2$, where $g^{(\text{sing})}$ is the singular part of the density of the Gibbs functional. Hence

$$\begin{aligned} g^{(\text{sing})}(\tau, h) &= b^{\theta+d-2x_\sigma-2y_h} g^{(\text{sing})}(\tau b^{y_t}, h b^{y_h}) \\ &= b^{\theta+d-2x_\varepsilon-2y_t} g^{(\text{sing})}(\tau b^{y_t}, h b^{y_h}). \end{aligned}$$

These two forms are consistent, if $x_\sigma + y_h = x_\varepsilon + y_t = w$. Repeating the usual scaling analysis, one has $\nu_\perp = 1/y_t$, $\theta = \nu_\parallel/\nu_\perp$ and $\beta + \gamma = y_h/y_t$. From dimensional counting, we expect $w = d + \theta$ and the Gibbs functional density should then scale as $g^{(\text{sing})}(\tau, h) = b^{-d-\theta} g^{(\text{sing})}(\tau b^{y_t}, h b^{y_h})$. We explicitly exclude dangerously irrelevant scaling fields and recover the hyperscaling relation $2 - \alpha = \nu_\parallel + d\nu_\perp$. Scaling out b gives the usual thermodynamic scaling form

$$g^{(\text{sing})}(\tau, h) = A_1 |\tau|^{2-\alpha} W^\pm (A_2 h |\tau|^{-\beta-\gamma}), \quad (\text{S.69})$$

where $W^\pm$ are the universal scaling functions which are obtained for $\tau > 0$ and $\tau < 0$, respectively, and $A_{1,2}$ are non-universal metric constants.

6.2 Equation (S.69) is the starting point for our discussion of finite-size scaling [353]. Consider a situation where the ‘spatial’ directions are of finite extent L whereas the ‘temporal’ direction remains infinite. The habitual assumption of finite-size scaling now reads

$$g^{(\text{sing})}(\tau, h; L) = A_1 |\tau|^{2-\alpha} W^\pm (A_2 h |\tau|^{-\beta-\gamma}; L \xi_\perp^{-1}), \quad (\text{S.70})$$

where the *bulk* ‘spatial’ correlation length $\xi_\perp = \xi_0 t^{-\nu_\perp}$. There is *no* extra metric factor in the second argument of $W^\pm$, whereas ξ_0 is non-universal.²⁸

Analogously to Privman and Fisher, one traces the non-universal constants, which for simplicity is done first for the infinite system. We expect for the connected spin-spin correlator

$$G_\sigma(r_\perp, r_\parallel; \tau, h) = D_0 D_1 r_\perp^{-2x_\sigma} X^\pm (r_\perp/\xi_\perp, D_0 r_\parallel/\xi_\perp^\theta; D_2 h |\tau|^{-\beta-\gamma})$$

where $X^\pm$ is a universal scaling function and $D_{0,1,2}$ are non-universal metric factors. From the fluctuation-dissipation theorem (S.68) one has

$$\chi(\tau, h) = D_1 \xi_\perp^{\gamma/\nu_\perp} \tilde{X}^\pm (D_2 h |\tau|^{-\beta-\gamma}), \quad (\text{S.71})$$

where $\tilde{X}^\pm$ is a new scaling function obtained from $X^\pm$. For the *non-connected* spin-spin correlator

²⁸ To simplify the notation, we assume that there is no phase transition for L finite. The ‘temporal’ direction remains infinite, otherwise one must deal with two *distinct* finite length scales.

$$\Gamma_\sigma(r_\perp, r_\parallel; \tau, h) = D_0 D_1 r_\perp^{-2x_\sigma} Z^\pm(r_\perp/\xi_\perp, D_0 r_\parallel/\xi_\perp^\theta; D_2 h|\tau|^{-\beta-\gamma})$$

with another universal scaling functions $Z^\pm$. We assume translation-invariance with respect to $r_\perp$ and $r_\parallel$. Therefore, there should exist a mean magnetisation m which is independent of $r_\perp$ and $r_\parallel$ and which can be found considering Γ_σ at large separations $r_\perp, r_\parallel$:

$$\begin{aligned} \Gamma_\sigma(r_\perp, r_\parallel; \tau, h) &= D_0 D_1 \xi_\perp^{-2x_\sigma} Z^\pm(1, D_0 r_\parallel/r_\perp^\theta; D_2 h|\tau|^{-\beta-\gamma}) \\ &\quad \downarrow \qquad \qquad \downarrow \\ m^2(\tau, h) &= D_0 D_1 \xi_\perp^{-2x_\sigma} \tilde{Z}^\pm(D_2 h|\tau|^{-\beta-\gamma}), \end{aligned} \quad (\text{S.72})$$

where the arrow indicates taking the limit of large ‘spatio-temporal’ separations. Because of translation invariance, m^2 should become independent of D_0 . On the other hand, applying standard thermodynamics to the free energy (S.69) yields

$$m(\tau, h) = A_1 A_2 |\tau|^\beta W_1^\pm(A_2 h|\tau|^{-\beta-\gamma}), \quad (\text{S.73})$$

$$\chi(\tau, h) = A_1 A_2^2 |\tau|^{-\gamma} W_2^\pm(A_2 h|\tau|^{-\beta-\gamma}), \quad (\text{S.74})$$

where $W_n^\pm(x) = d^n W^\pm(x)/dx^n$.

We now compare (S.71) and (S.74). Letting first $h = 0$, we find $D_1 \xi_0^{\gamma/\nu_\perp} = A_1 A_2^2 U_1$. Comparing the arguments of the scaling functions, we have $D_2 = A_2 U_2$. Comparison of (S.72) and (S.73) gives for $h = 0$ that $D_0 D_1 \xi_0^{-2\beta/\nu_\perp} = A_1^2 A_2^2 U_3$ (since $x_\sigma = \beta/\nu_\perp$). Here, $U_{1,2,3}$ are *universal* constants whose universality follows from the universality of the scaling functions considered. Using the hyperscaling relation $\gamma + 2\beta = (d + \theta)\nu_\perp$, we find that

$$\begin{aligned} A_1 \xi_0^{d+\theta} D_0^{-1} &= Q_1 = U_1/U_3 \\ D_2 A_2^{-1} &= Q_2 = U_2 \\ D_0^{\gamma/(\nu_\perp(d+\theta))} D_1 A_1^{-1-\gamma/(\nu_\perp(d+\theta))} A_2^{-2} &= Q_3 = U_1^{1-\gamma/(\nu_\perp(d+\theta))} U_3^{\gamma/(\nu_\perp(d+\theta))} \end{aligned} \quad (\text{S.75})$$

and the $Q_{1,2,3}$ are universal constants.

Finally, we come back to the finite-size scaling behaviour. In (S.70), we replace ξ_0 by A_1 using (S.75). Scaling out L and using again hyperscaling, we find

$$g^{(\text{sing})}(\tau, h; L) = L^{-d-\theta} D_0 Y\left(C_1 \tau L^{1/\nu_\perp}, C_2 h L^{(\beta+\gamma)/\nu_\perp}\right), \quad (\text{S.76})$$

where Y is a universal scaling function and $C_{1,2}$ are non-universal metric factors related to $A_{1,2}$. In contrast with the isotropic situation, the finite-size scaling amplitude of the Gibbs functional is no longer universal. Furthermore, since $\xi_\parallel = \xi_\perp^\theta/D_0$, we have

$$\begin{aligned} g^{(\text{sing})}(\tau, 0; L) \xi_\perp^d(\tau, 0; L) \xi_\parallel(\tau, 0; L) &= D_0^{-1} g^{(\text{sing})}(\tau, 0; L) \xi_\perp^{d+\theta}(t, 0; L) \\ &\rightarrow_{\tau \rightarrow 0} \text{univ. constant}, \end{aligned}$$

and the ‘spatial’ correlation length becomes

$$\xi_{\perp}^{-1} = L^{-1} S \left(C_1 \tau L^{1/\nu_{\perp}}, C_2 h L^{(\beta+\gamma)/\nu_{\perp}} \right)$$

with a universal scaling function S and the same metric factors $C_{1,2}$ as in (S.76). The scaling functions Y and S should depend on the boundary conditions and, if $d \geq 2$, on the shape of the finite ‘spatial’ domain. Similarly, for the ‘temporal’ correlation length

$$\xi_{\parallel}^{-1} = L^{-\theta} D_0 R \left(C_1 \tau L^{1/\nu_{\perp}}, C_2 h L^{(\beta+\gamma)/\nu_{\perp}} \right)$$

with a universal scaling function R and the same metric factors $C_{1,2}$.

6.3 We closely follow the discussion in [746] of the ANNNI model in mean-field approximation.

With the choice (6.75) for the trial Hamiltonian $\mathcal{H}_0$, the right-hand side of (6.74) is readily evaluated, yielding the expression

$$\begin{aligned} \Phi = & -T\mathcal{N}^2 \sum_z \ln(2 \cosh(\eta_z/T)) - \frac{1}{2}\mathcal{N}^2 \sum_z [4J_0 m_z^2 + J_1 m_z(m_{z-1} + m_{z+1}) \\ & + J_2 m_z(m_{z-2} + m_{z+2})] + \mathcal{N}^2 \sum_z \eta_z m_z . \end{aligned}$$

The average magnetisation per spin in the layer z is thereby given by $m_z = \tanh(\eta_z/T)$. Minimising this expression with respect to the variational parameters η_z gives

$$\eta_z = 4J_0 m_z + J_1(m_{z-1} + m_{z+1}) + J_2(m_{z-2} + m_{z+2}) ,$$

which finally yields the mean-field free energy of the ANNNI model,

$$\begin{aligned} \mathcal{N}^{-3} F_{MF}(T, \mathcal{N}) = & -T \ln 2 \\ & + \frac{T}{2\mathcal{N}} \sum_z [(1 + m_z) \ln(1 + m_z) + (1 - m_z) \ln(1 - m_z)] \\ & - \frac{1}{2\mathcal{N}} \sum_z [4J_0 m_z^2 + J_1 m_z(m_{z-1} + m_{z+1}) + J_2 m_z(m_{z-2} + m_{z+2})] \end{aligned}$$

as well as the mean-field equations for the layer magnetisations:

$$m_z = \tanh \left[(4J_0 m_z + J_1(m_{z-1} + m_{z+1}) + J_2(m_{z-2} + m_{z+2}))/T \right] . \quad (\text{S.77})$$

The mean-field phase diagramme shown in Fig. 6.3 is obtained by solving numerically this set of $\mathcal{N}$ coupled equations for various values of the interactions J_0 , J_1 , and J_2 [657].

In order to discuss the transition lines between the disordered high-temperature phase (where $m_z = 0$) and the ordered low-temperature phases (where $m_z \neq 0$), we consider the equations (S.77) for small m_z :

$$m_z \approx (4J_0 m_z + J_1(m_{z-1} + m_{z+1}) + J_2(m_{z-2} + m_{z+2}))/T ,$$

which yields for the Fourier components of m_z the expressions

$$\hat{m}_{\mathbf{q}} \approx J(q) \hat{m}_{\mathbf{q}}/T \quad (\text{S.78})$$

with

$$J(q) = 4J_0 + 2J_1 \cos q + 2J_2 \cos 2q .$$

The transition temperature is obtained by observing that, when decreasing the temperature, non-vanishing solutions of (S.78) appear for a critical value of q that maximises $J(q)$:

$$T_c = \max_{\mathbf{q}} [J(q)] = J(q_c) .$$

From this, we immediately obtain that the modulated phase near the transition temperature is characterised by $\cos q_c = -J_1/4J_2$, whereas $q_c = 0$ for the ferromagnetic phase. It then follows that the ordered phase is modulated for $\kappa > 1/4$ and ferromagnetic for $\kappa < 1/4$. The transition temperature for the transition to the modulated phase is given by

$$T_q = J(q_c) = 4J_0 - 2J_2 - \frac{J_1^2}{4J_2}$$

and that for the transition to the ferromagnetic phase is

$$T_0 = J(0) = 4J_0 + 2J_1 + 2J_2 .$$

Both lines meet at the Lifshitz point $\kappa_L = 1/4$ and $T_L = 4J_0 + \frac{3}{2}J_1$.

6.4 Recall from the text the commutators (6.19) and (6.20). Then, since $[X_1, Y_{-N/2+k}^{(i)}] = (N-k)Y_{-N/2+k+1}^{(i)}$, use the Jacobi identities to arrive at

$$\begin{aligned} & \left[\mathcal{S}, Y_{-N/2+k+1}^{(i)} \right] \psi = \\ & (N-k)^{-1} \left(\left[X_1, \left[\mathcal{S}, Y_{-N/2+k}^{(i)} \right] \right] - \left[Y_{-N/2+k}^{(i)}, [\mathcal{S}, X_1] \right] \right) \psi = 0 \end{aligned}$$

For solutions of $\mathcal{S}\psi = 0$, the scaling dimension is chosen such that $[\mathcal{S}, X_1]\psi = 0$. Since from the above $[\mathcal{S}, Y_{-N/2}^{(i)}]\psi = 0$, the assertion follows for all $k \neq N$ by induction over k .

6.5 Re-summation of the series gives $\Omega_{N=1}(v) = b_0 v^{-\zeta} e^{-\alpha_1/v}$, where $v = tr^{-2}$ and $\zeta = x_1$. This fully agrees with the form of the Schrödinger-invariant two-point function $\langle \phi \phi^* \rangle$, as expected.

6.6 For an m -axial Lifshitz point the Fourier transform of the exchange integral is given by

$$\widehat{J}(\phi) = 2J \left(\sum_{j=1}^d \cos \phi_j - \sum_{j=1}^m \kappa_1 \cos(2\phi_j) \right)$$

which can be expanded into

$$\widehat{J}(\phi) = 2Jd + m\kappa_1 - \frac{1}{2} \sum_{j=m+1}^d \phi_j^2 - c \sum_{j=1}^m \phi_j^4 + \dots$$

where $c = -(1 - 16\kappa_1)/24$.

For $|\mathbf{r}|$ large enough, the principal contribution to the integrals comes from $|\phi| \sim 0$, and the leading singular behaviour of $C(\mathbf{r})$ is contained in

$$C(\mathbf{r}) = \frac{T_c}{2J(2\pi)^d} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} d\phi_1 \dots d\phi_d \frac{\cos(\mathbf{r} \cdot \phi)}{\sum_{j=m+1}^d \frac{1}{2} \phi_j^2 + c \sum_{j=1}^m \phi_j^4}.$$

From the identity $\frac{1}{a} = \int_0^\infty du e^{-au}$, we have $C(\mathbf{r}) = T_c/(2J(2\pi)^d) \int_0^\infty du F(u)$ with

$$F(u) = \Re \left\{ \prod_{j=1}^m \left[\int_{-\pi}^{\pi} \exp(i\phi_j r_j - c\phi_j^4 u) d\phi_j \right] \right. \\ \left. \times \prod_{j=m+1}^d \left[\int_{-\pi}^{\pi} \exp\left(i\phi_j r_j - \frac{1}{2}\phi_j^2 u\right) d\phi_j \right] \right\}$$

where $\Re$ denotes the real part. An Abelian theorem for the Laplace transform [525] states that if $F(u)$ is analytic, the behaviour of $C(\mathbf{r})$ is determined by the behaviour of $F(u)$ at $u \rightarrow \infty$. For $u \gg 1$ we can therefore replace the range of integration $(-\pi, \pi)$ on the ϕ_j integrals by $(-\infty, \infty)$.

The correlation function now reads

$$C(\mathbf{r}) = \frac{T_c}{2J(2\pi)^d} \int_0^\infty du \left(\Re \left\{ \prod_{j=1}^m \left[\int_{-\infty}^{\infty} \exp(i\phi_j r_j - c\phi_j^4 u) d\phi_j \right] \right\} \right. \\ \left. \times \left(\frac{2\pi}{u} \right)^{\frac{d-m}{2}} \exp \left(-\frac{1}{2u} \sum_{j=m+1}^d r_j^2 \right) \right)$$

Expanding the cosine and integrating term by term we get

$$\int_{-\infty}^{\infty} e^{-c\phi_j^4 u} \cos(r_j \phi_j) d\phi_j = \frac{1}{2} \left(\frac{1}{cu} \right)^{\frac{1}{4}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \Gamma \left(\frac{k}{2} + \frac{1}{4} \right) \left(\frac{r_j^2}{(cu)^{\frac{1}{2}}} \right)^k$$

In what follows, we specialise to the choice

$$r_{\parallel} = r_1 \neq 0 \quad , \quad r_i = 0 \quad (\text{for } i = 2, \dots, m) \quad , \quad r_{\perp}^2 = \sum_{j=m+1}^d r_j^2$$

and the correlation function becomes

$$C(r_{\parallel}, r_{\perp}) = \frac{T_c}{2J(2\pi)^{\frac{d+m}{2}}} \left(\frac{1}{2}\right)^m \Gamma\left(\frac{1}{4}\right)^{m-1} \left(\frac{1}{c}\right)^{\frac{m}{4}} \\ \times \int_0^{\infty} du \left\{ u^{\frac{m}{4} - \frac{d}{2}} \exp\left(-\frac{r_{\perp}^2}{2u}\right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \Gamma\left(\frac{k}{2} + \frac{1}{4}\right) \left(\frac{r_{\parallel}^2}{(cu)^{\frac{1}{2}}}\right)^k \right) \right\}$$

The integral converges if $d > 2 + \frac{m}{2} =: d_*$ and the sum is absolutely convergent. Exchanging the sum with the integral, we thus obtain the exact result

$$C(r_{\parallel}, r_{\perp}) = \frac{T_c}{2J(2\pi)^{\frac{d+m}{2}}} \left(\frac{1}{2}\right)^m \Gamma\left(\frac{1}{4}\right)^{m-1} \left(\frac{1}{c}\right)^{\frac{m}{4}} \left(\frac{r_{\perp}^2}{2}\right)^{-\frac{d-d_*}{2}} \\ \times \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \Gamma\left(\frac{k}{2} + \frac{1}{4}\right) \Gamma\left(\frac{k}{2} + \frac{d-d_*}{2}\right) \left(\left(\frac{2}{c}\right)^{1/2} \frac{r_{\parallel}^2}{r_{\perp}}\right)^k$$

We remark that the series is absolutely convergent on the whole real axis.

If we now restrict ourselves to the uniaxial case $m = 1$ and use the identity

$$(2k)! = \Gamma(2k+1) = (2\pi)^{-1} 2^{2k+\frac{1}{2}} k! \Gamma\left(k + \frac{1}{2}\right)$$

we recover the expressions (6.64) and (6.65) for the correlator at the Lifshitz point of the d -dimensional ANNS model where the Lifshitz point value $\kappa_1 = 1/4$ has to be plugged in.

List of frequently used symbols

$\lg, \ln$	logarithms at base 10 and e
$\Theta(x)$	Heaviside function ($= 1$ for $x > 0$ and $= 0$ for $x \leq 0$)
Δ_L	spatial Laplacian
ϕ	order-parameter (or magnetisation m)
$\tilde{\phi}$	response field
h	field conjugated to the order parameter
T_c	critical temperature
$\mathcal{N}$	total number of sites
χ	susceptibility
α	equilibrium critical exponent related to specific heat
η	equilibrium critical exponent related to correlation functions
$\mathcal{H}$	Hamiltonian of an equilibrium model
$\langle \dots \rangle$	average over independent thermal realisations
$[\dots]_{\text{av}}$	average over independent disorder (sample) realisations
$\nu_{\perp}, \nu_{\parallel}$	exponents related to spatial and temporal correlations
$L = L(t)$	time-dependent linear cluster size
t, s	ageing: observation time and waiting time
τ	time difference $\tau = t - s$
z, θ	dynamical exponent or anisotropy exponent $\theta = \nu_{\parallel}/\nu_{\perp}$
a, a', b	ageing exponents of response and correlation
λ_C, λ_R	autocorrelation and autoresponse exponents
Θ	critical initial slip exponent
θ_g	global persistence exponent
μ	subageing exponent
ζ	passage exponent/dual coordinate to mass
$\mathcal{M}$	‘mass’ parameter
X_{∞}	limit fluctuation-dissipation ratio
d_c	upper critical dimension
$\langle s $	sum state over all configurations
$\mathcal{L}$	Liouville operator for random sequential updates
$\mathcal{T}$	transfer matrix for parallel updates
MF	subscript marking mean-field results
$\mathbb{Z}_n$	cyclic group with n elements
$\mathbb{D}_n$	dihedral group with $2n$ elements
$\mathbb{S}_n$	permutation group of n objects
$O(n)$	orthogonal group (rotations) in n dimensions
$\otimes, \oplus$	direct product/sum of groups/Lie algebras
$\oslash, \ltimes$	semi-direct product/sum of groups/Lie algebras
S_d	surface of unit sphere in d dimensions
$\hat{f}(k)$	Fourier transform of $f(r)$
$\bar{g}(p)$	Laplace transform ($\mathcal{L}g$)(p) of $g(r)$

List of commonly used abbreviations

ANNNI	axial next-nearest-neighbour Ising
ANNNO(n)	axial next-nearest-neighbour $O(n)$
ANNNS	axial next-nearest-neighbour spherical
BCPD	bosonic contact process (with diffusion)
BPCPD	bosonic pair-contact process (with diffusion)
BCPL	bosonic contact process (with Lévy flight)
BPCPL	bosonic pair-contact process (with Lévy flight)
BPT	Bray-Puri-Toyoki
CP	contact process
DMRG	density-matrix renormalisation group
DP	directed percolation
EA	Edwards-Anderson
EW	Edwards-Wilkinson
FDT	fluctuation-dissipation theorem
FDR	fluctuation-dissipation ratio
FSS	finite-size scaling
FT	field-theory
KPZ	Kardar-Parisi-Zhang
KT	Kosterlitz-Thouless
LCTMRG	light-cone transfer matrix renormalisation group
LP	Lifshitz point
LSI	local scale-invariance
MH	Mullings-Herring
NEKIM	non-equilibrium kinetic Ising model
OJK	Ohta-Jasnow-Kawasaki
PCP	pair-contact process
PCPD	pair-contact process with diffusion
RG	renormalisation group
TDGL	time-dependent Ginzburg-Landau
TMRG	transfer matrix renormalisation group
TRM	thermoremanent
SOS	solid-on-solid model
YRD	Yeung-Rao-Desai
ZFC	zero-field-cooled
$2D, 3D$	two-dimensional, three-dimensional

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