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Non-Linear Dynamics Near and Far from Equilibrium



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Preface

Non linear dynamics has been almost synonymous with the dynamics of a few degrees of freedom. This is somewhat extra-ordinary given the fact that natural phenomena are most often described by partial differential equations which have an infinite number of degrees of freedom and more often than not the relevant partial differential equations are nonlinear. We thought that, for a change, it would be worthwhile to focus on the continuous systems and describe some of the nonlinear phenomena that have captured the attention of researchers over the last fifty years. Interestingly enough for these systems with infinite degrees of freedom, it is real life experiments (as opposed to computer simulations) that are more common and this allows one to confront theoretical findings with laboratory experiments. Calculations use idealised phenomenological models which have a fair amount of approximation and try their best to describe the complex phenomena of the real world and it is quite exhilarating when the calculations and observations match. The theoretical analysis can proceed along two different routes-analytical and computational. In this book, we have tried to bring together some of the analytical tools which have been successful in producing results for systems which are close to equilibrium as well as for those that are far from it. We hope the beginning graduate students who want to take up the study of statistical physics will find it useful.

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Introduction

We will be concerned mainly with systems with infinite degrees of freedom which can however, be described by a few variables. These variables must necessarily be fields i.e. functions of space and time. A typical example would be to try to describe the flow of air around us. The variables that would be necessary to describe the state of air would certainly be its density, its temperature and its velocity. All these variables (density, temperature and velocity) are, in general, functions of space and time. They are mesoscopic variables. They do not reflect the variations occurring at the molecular level. To define a density, it should be recalled, we take a small volume (small compared to the total system size, yet large compared to atomic dimensions) and consider the mass of this small volume. The ratio of mass to volume remains constant for a reasonably large variation in the size of the volume chosen and defines the density of the system. It fails to be a constant if the volume becomes so small that it contains only a few molecules. In that case our description in terms of a density fails. All the systems that we will talk about can be described in terms of a coarse grained field like the density. Because of the smallness (at the macroscopic level) of the volume used in defining density it can be considered a local variable. This is what makes it a field. Similarly we can talk about the local temperature and local velocity. The local velocity is not the velocity of an individual molecule but the velocity associated with a macroscopically small, yet microscopically large volume of air. These fields evolve in time and space according to some definite laws. Every separate system that we discuss will have its own rule for evolution. All these systems have attracted a vast amount of attention for several decades. A lot of spectacular progress has been made and a lot more remains to be understood. Our aim in this book will be to describe a few analytic techniques which have been very effective in providing some sort of common platform for widely varying problems of dynamics. While often the eventual detailed understanding

of these systems require extensive numerical work, some of the essential features can often be captured by these analytic techniques. The confrontation between calculations and real life experiments in such cases provides a good measure of the worth of the theory. In this introductory chapter, our focus will be on providing a glimpse of some of the more interesting comparisons of theory and experiment in one of the systems we wish to study - the second order phase transitions.

The thing to stress, as the theme of this book, is the common platform of theoretical techniques suited for diverse problems - ‘universal’ techniques to borrow a much used word. Universal was the key word that triggered a tremendous amount of interest in critical phenomena in the nineteen sixties and seventies. Critical phenomena exhibited universality and hence it was worth the effort spent to get to the bottom of it. We will begin by explaining what universality meant in that context.

The two most common second order phase transitions or critical phenomena are the paramagnet to ferromagnet transition at the Curie point and the gas liquid transition at the critical point. These are second order transitions since they require no latent heat.

If a material like iron is taken and kept at a temperature significantly higher than the room temperature (but lower than its melting point, needless to say) then it will not show any spontaneous magnetization. Spontaneous magnetization is the magnetization that one obtains in the absence of any external field. Experimentally it is determined by switching on a small external field, finding the magnetization at different values of the field and taking the limit of the external field going to zero. If we cool the piece of iron, then at a certain temperature T_c , the critical temperature or the Curie temperature, the system undergoes a phase transition. For temperatures above the critical temperature T_c the spontaneous magnetization is zero, while below T_c the iron exhibits non-zero spontaneous magnetization. The magnitude of the magnetization rises automatically from its value of zero at T_c as the temperature is lowered. A typical measurement yields the kind of behaviour shown in Fig. 1.1.

It is the region very close to T_c which is interesting. Here the magnetization has the behaviour

$$M \propto (T_c - T)^\beta \quad \text{for} \quad \frac{T_c - T}{T_c} \ll 1 \quad (1.0.1)$$

The quantity β is called a critical exponent and for this magnetic transition, β is independent of what material one is using for the experiments. It could be a transition metal like iron, nickel or manganese, or it could be a rare earth compound like EuO or EuS. For different materials the actual values of magnetization are different but the exponent β remains the same. This is universality.

Large fluctuations characterize the critical point. This can be seen from a study of the linear response functions. In the case of the magnet the response function is the zero field susceptibility. One applies an infinitesimal field H and determines the corresponding magnetization M . The isothermal susceptibility is defined as $\chi_T = \left(\frac{\partial M}{\partial H}\right)_T$. It is the zero external field that concerns us. Whether for $T > T_c$ or $T < T_c$,

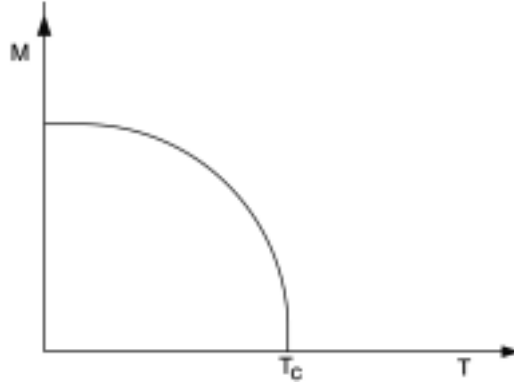


Figure 1.1. Spontaneous magnetisation as a function of temperature

the susceptibility becomes very big as one approaches T_c , and this behaviour, which culminates in the susceptibility diverging at $T = T_c$, can be represented as

$$\chi_T \propto |T - T_c|^{-\gamma} \quad (1.0.2)$$

Once again, the exponent γ is independent of what material was used to study the phase transition. The strong response of the system comes from the fact that near T_c , the fluctuations are very big. An infinitesimal change in temperature around T_c changes the system from a disordered to an ordered state. As the critical temperature is approached blobs of ordered phase start appearing in the disordered region and gradually these blobs become correlated as the temperature is further lowered to the critical point. This makes the system very susceptible to small perturbations. At a temperature infinitesimally close to T_c , switching on an infinitesimal field can produce a finite amount of magnetization in the system. This is why the susceptibility (isothermal) diverges at the critical point.

It should be clear from the above discussion that the description of the phase transition should be couched in terms of a local magnetization, the magnetization of the blob of the ordered phase that appears in the disordered system. This blob is like the ‘small’ volume that is used in defining density. It is small in comparison to the system size but it is large enough to wash out the individual effects of the atomic magnetic moments. This local magnetization is a function of space (and also time in general) and can be described by a field $\phi(\vec{r}, t)$. The field is usually a vector field in the case of magnetization. However, there can be uniaxial magnets where the magnetization can only be along a definite direction in space. In such situations the field $\phi(\vec{r}, t)$ is a scalar field. In certain planar magnets, the magnetization is confined to lie in a plane and then the field $\phi(\vec{r}, t)$ is a two-dimensional vector field. This brings in an even wider instance of universality.

We focus now on yet another critical point - the one discovered by Andrews in his experiment on $C O_2$ in 1869. The experiment measured the P-V curve for a fixed

mass of CO_2 at different temperatures. At the lower temperatures (T_1, T_2, T_3), as the pressure is increased, the volume decreases and over a range of volume when the vapour changes to liquid the pressure remains constant. When all the vapour is converted to liquid, the volume decreases slightly as the pressure is further increased.

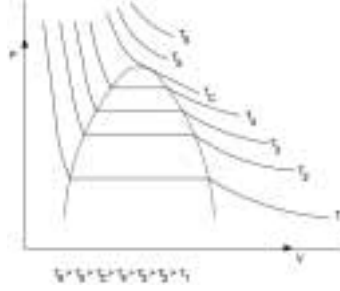


Figure 1.2. Isotherms for a liquid gas transition

The conversion from gas to liquid occurs with the emission of latent heat. As the temperature was increased, it was seen that the flat part of the curve corresponding to the phase transformation and eventually at $T = T_c$, the flat part disappeared which meant that the density of the liquid and vapour had become equal at this temperature. For $T > T_c$, there is only a single-phase region. The transition from a two-phase region below T_c to a single-phase region above T_c is a second order phase transition which occurs without any latent heat. If we consider the density difference $\rho_L - \rho_g$, the difference between densities of liquid and vapour (gas) phase, then it vanishes for $T > T_c$ (single-phase region) and increases with temperature for $T \leq T_c$. The graph of $\rho_L - \rho_g$ vs. T is shown in Fig. 1.3 and is remarkably similar to the corresponding figure for magnetization shown in Fig. 1.1. For T sufficiently close to T_c , once again

$$\rho_L - \rho_g \propto (T_c - T)^\beta \quad (1.0.3)$$

If one looks at the isothermal compressibility, which is the change of density for an infinitesimal change in pressure, then it is found that the compressibility diverges at the critical point. Once again this is because of the large scale fluctuations in the system. Infinitesimally above T_c , it is a single-phase region, while infinitesimally below it is a two-phase region. In this range, a small change in an external parameter like the pressure can cause large changes in the density and that is what causes the singular response

$$\chi_T = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial P} \right)_T \propto |T - T_c|^{-\gamma} \quad (1.0.4)$$

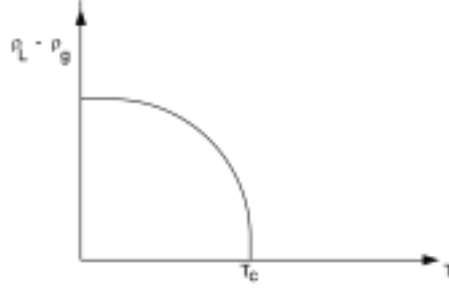


Figure 1.3. Density difference between the gas and liquid phases as a function of temperature

The exponents β and γ are independent of what substance is used to carry out the experiment. A few typical results are shown below.

Substance	β	γ
Xe	0.0321	1.236
SF ₆	0.0327 ± 0.006	1.251
CO ₂	0.329	1.280
He ³	0.321 ± 0.006	

Table 1.1 Univesality of critical exponents for the liquid gas transition.

As T approaches T_c from above local blobs are formed where $\rho_L - \rho_g$ acquires nonzero value. These blobs get more and more correlated as T gets closer and closer to T_c and eventually one ends up with a globally non-zero value of $\rho_L - \rho_g$. If we denote the local value of $\rho_L - \rho_g$ by $\phi(\vec{r}, t)$, then $\phi(\vec{r}, t)$ is a scalar function very much like the uniaxial magnet and sure enough if one looks at the experiments involving phase transitions in uniaxial magnets, then

$$\beta = 0.32 \quad \gamma \simeq 1.24. \quad (1.0.5)$$

A similar phase transition without latent heat occurs in the critical mixing of some liquids. If we take two well chosen liquids of densities ρ_1 and ρ_2 and consider a mixture of these at concentrations C_1 and $C_2 = 100 - C_1$, then it is possible that at high temperatures these liquids are completely miscible but at low temperatures they become immiscible and a meniscus develops. The separation begins at a temperature of T_c and becomes more and more pronounced as the temperature is lowered. If we consider the variable $C = X_1(T) - C_1$, where $X_1(T)$ is the relative concentration of liquid number one at temperature T , then for all $T > T_c$, $C = 0$, while for $T < T_c$, $C \neq 0$ and increases as T is lowered. The transition at T_c requires no latent heat and is a second order transition. If one now thinks of the concentration $C(\vec{r}, t)$ being the field ϕ , then ϕ is once again a scalar field. From our previous discussion it would appear that once again β should be near 0.32 and γ near 1.25.

Substance	β	γ
Isobutyric acid+Water	0.324 ± 0.012	1.23 ± 0.02
C S ₂ +Nitromethane	0.316 ± 0.008	
Polystyrene+Cyclohexane	0.327 ± 0.008	
3-methylpentane+Nitroethane		1.240 ± 0.017

Table 1.2 Universality of critical exponents for the mixing transition in binary liquids.

The underlying message is very clear. Critical phenomena are universal to the point that fluids, magnets, binary mixtures all exhibit the same behaviour. Numbers like β and γ are called critical exponents. The field $\phi(\vec{r}, t)$, which is identically zero above the transition point and has a global non-zero value below T_c , is called an order parameter. The order parameter is characterized by the dimension of the field $\phi(\vec{r}, t)$ which can be a three-dimensional vector, a planar vector or a scalar. In general it can be thought of as a n -component field $\phi_i(\vec{r}, t)$ $i = 1, 2, 3, \dots, n$ with $n = 1$ for the fluid, the binary liquid and the uniaxial magnet. Universality, means that in three-dimensional space critical exponents are determined by the value of n alone. If the dimension of the space is D , then critical exponents depend solely on n and D . The method of tackling this problem of determining the critical exponents has to be a general technique independent of the specific nature of the system. It is because of this that critical phenomena had attracted a great deal of attention in the last century.

We start by pointing out the different ways of tackling the individual problems. We begin with the magnet. The local magnetization field is $m(\vec{r})$ [we are going to consider statics now and will take the fields to be time independent]. The magnetization at two different points $\vec{r}$ and $\vec{r}'$ will interact and produce an interaction energy

$$E_{int} = - \int J(\vec{r} - \vec{r}') m(\vec{r}) m(\vec{r}') d^D r d^D r' \quad (1.0.6)$$

For convenience we take $m(\vec{r})$ to be a scalar function (uniaxial magnet). The energy is lowered parallel moments if $J > 0$. If there is an external field H , then there is an additional term $-H \int m(\vec{r}) d^D r$ and we have

$$E = - \int J(\vec{r} - \vec{r}') m(\vec{r}) m(\vec{r}') d^D r d^D r' - H \int m(\vec{r}) d^D r \quad (1.0.7)$$

The strategy is to find the statistical mechanical partition function

$$Z = \sum_{m(\vec{r})} e^{-\frac{E}{k_B T}} \quad (1.0.8)$$

and thence determine all the thermodynamic quantities like M , χ_T etc. from the free energy $F = -k_B T \ln Z$. The problem is simplified by writing

$$H_{int}(\vec{r}) = \int J(\vec{r} - \vec{r}') m(\vec{r}') d^D r' \quad (1.0.9)$$

and then assuming that the internal field $H_{int}(\vec{r})$ is independent of the position $\vec{r}$. This is the so called mean field approximation. This uniform H_{int} can only be caused by the net magnetization and can be taken to be proportional to it. Thus, $H_{int} = J M$, where J is a constant and we have

$$E = -(J M + H) \int m(\vec{r}) d^D r \quad (1.0.10)$$

At every spatial point $\vec{r}$, the magnetization can be $\pm\mu$ in the simplest situation that one may consider (Ising model) and the partition function in Eq.(1.0.8), now yields

$$Z = \left[2 \cosh \frac{\mu(JM + H)}{k_B T} \right]^N \quad (1.0.11)$$

where N is the number of spatial sites in the volume under consideration. The free energy is

$$F = -k_B T \ln Z = -N k_B T \ln \left[2 \cosh \frac{\mu(JM + H)}{k_B T} \right] \quad (1.0.12)$$

The magnetization is found from $-\left(\frac{\partial F}{\partial H}\right)_T$ as

$$M = \mu N \tanh \left[\frac{\mu(JM + H)}{k_B T} \right] \quad (1.0.13)$$

We note that M is extensive, while H_{int} is intensive and hence J must have the form $\frac{J_0}{N}$, which we substitute in Eq.(1.0.13) and write as

$$\frac{M}{\mu N} = \tanh \left[\frac{\mu^2}{k_B T} \left(\frac{J_0 M}{\mu N} + \frac{H}{\mu} \right) \right]. \quad (1.0.14)$$

The zero field magnetization (spontaneous magnetization) is obtained as

$$\frac{M}{\mu N} = \tanh \left[\frac{\mu^2 J_0}{k_B T} \frac{M}{\mu N} \right]. \quad (1.0.15)$$

The solution can be obtained graphically as shown in Fig. 1.4

Clearly, for high temperatures, $\frac{\mu^2 J_0}{k_B T} < 1$, there is no magnetization. The only solution is $M = 0$. For $\frac{\mu^2 J_0}{k_B T} > 1$, there are two solutions $M = 0$ and a finite value of M . It is the finite value of M which gives a lower free energy and hence there is a critical temperature T_c defined by

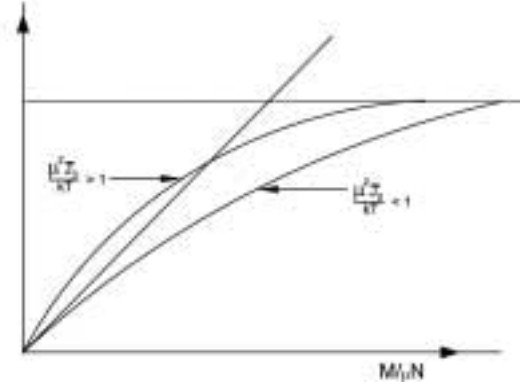


Figure 1.4. The graphical solution of Eq.(1.15)

$$\frac{\mu^2 J_0}{k_B T_c} = 1, \quad (1.0.16)$$

which separates a phase of zero spontaneous magnetization (paramagnetic) from a phase of non-zero spontaneous magnetization (ferromagnetic).

To find how big $\frac{M}{\mu N}$ is in the ordered phase, we expand the right hand side of Eq.(1.0.15), assuming T is such that the argument of the hyperbolic tangent is still small. Then,

$$\frac{M}{\mu N} = \frac{\mu^2 J_0}{k_B T} \frac{M}{\mu N} - \left(\frac{1}{3} \frac{\mu^2 J_0}{k_B T} \right)^3 \left(\frac{M}{\mu N} \right)^3 + \dots \quad (1.0.17)$$

In the second term on the right hand side, we can replace T by T_c since we are interested in the region $T \simeq T_c$. Using Eq.(1.0.16), and the fact that $M/\mu N \neq 0$, we find

$$\left(\frac{M}{\mu N} \right)^2 \simeq 3 \left(\frac{\mu^2 J_0}{k_B T_c^2} \right) (T_c - T) = 3 \frac{(T_c - T)}{T_c} \quad (1.0.18)$$

or $M \propto (T_c - T)^{1/2}$

This leads to $\beta = \frac{1}{2}$ when we compare with Eq.(1.0.1), independent of any material dependent constant J_0 and T_c .

In the presence of a small external field H , the Taylor expansion of Eq.(1.0.14) for $T > T_c$ leads to

$$\begin{aligned} \frac{M}{\mu N} &\simeq \frac{\mu^2 J_0}{k_B T} \frac{M}{\mu N} + \frac{\mu}{k_B T} H \\ \text{or} \quad \frac{\mu^2 J_0}{k_B} \frac{T - T_c}{T_c^2} \frac{M}{\mu N} &\simeq \frac{\mu}{k_B T_c} H \\ \chi_T = \left(\frac{\partial M}{\partial H} \right)_T &\simeq \frac{N T_c}{J_0} \frac{1}{T - T_c} \end{aligned} \quad (1.0.19)$$

Comparing with Eq.(1.0.19), we find $\gamma = 1$, once more independent of the material considered.

If $T < T_c$, then the expansion needs to be done with care. Taking the derivative of Eq.(1.0.14) with respect to H ,

$$\left. \frac{1}{\mu N} \frac{\partial M}{\partial H} \right|_T = \text{sech}^2 \left(\frac{\mu^2}{k_B T} \left(J_0 \frac{M}{\mu N} + \frac{H}{\mu} \right) \right) \left[\frac{\mu^2 J_0}{k_B T} \frac{1}{\mu N} \frac{\partial M}{\partial H} \right]_T + \frac{\mu}{k_B T} \quad (1.0.20)$$

In the limit of $H \rightarrow 0$

$$\left. \frac{1}{\mu N} \frac{\partial M}{\partial H} \right|_T \frac{\mu^2 J_0}{k_B T} \left[\frac{T - T_c}{T_c} + \tanh^2 \frac{\mu^2 J_0}{k_B T} \frac{M}{\mu N} \right] = \frac{\mu}{k_B T} \text{sech}^2 \left(\frac{\mu^2 J_0}{k_B T} \frac{M}{\mu N} \right). \quad (1.0.21)$$

Using Eqns.(1.0.16),(1.0.18) and remembering $T \simeq T_c$, i.e.(M very small), we arrive at

$$\begin{aligned} \frac{1}{\mu N} \chi_T \left[\frac{T - T_c}{T_c} + \left(\frac{M}{\mu N} \right)^2 \right] &\simeq \frac{\mu}{K_B T_c} \\ \frac{1}{\mu N} \chi_T \frac{2(T - T_c)}{T_c} &\simeq \frac{\mu}{K_B T_c} \\ \text{or} \quad \chi_T &= \frac{N T_c}{2 J_0} \frac{1}{T_c - T} \end{aligned} \quad (1.0.22)$$

For $T > T_c$, we have $\gamma = 1$ exactly as above but the amplitude of $(T - T_c)^{-1}$ is different now from what it was above T_c . The amplitude contains material dependent quantities like T_c and J_0 and hence is non universal. However, if we write

$$\begin{aligned} \chi_T &\sim A^> (T - T_c)^{-\gamma} \quad \text{for } T > T_c \\ \text{and} \quad &\sim A^< (T - T_c)^{-\gamma} \quad \text{for } T < T_c \end{aligned} \quad (1.0.23)$$

then

$$\frac{A^>}{A^<} = 2 \quad (1.0.24)$$

which is universal once again!

Leaving the magnets, we turn to the gases and try to draw the isotherms of a typical non-ideal gas. The best empirical equation of state for a non-ideal gas is that due to Van der Waals which gives (for one mole)

$$P = \frac{RT}{V - b} - \frac{a}{V^2} \quad (1.0.25)$$

where a and b are constants that depend on the particular gas chosen. The typical isotherms are shown in Fig. 1.5.

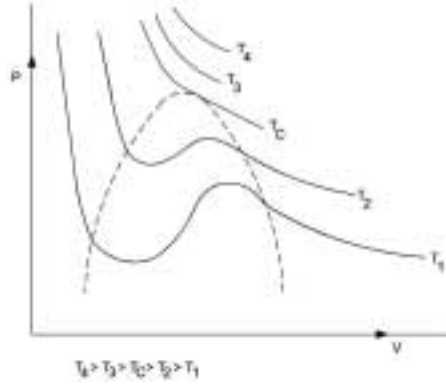


Figure 1.5. Isotherms for the Van der Waal equation of State

Below the isotherm at $T = T_c$, there is maximum and minimum in each isotherm obtained from the condition $\left(\frac{\partial P}{\partial V}\right)_T = 0$. Between these extrema P is an increasing function of V which is unphysical. We can interpret this unphysical region as the coexistence curve. To the right is the pure vapour phase and to the left is the liquid phase. The extrema come closer to each other as T_c is approached and above T_c , the isotherms are of the pure gas variety. At T_c , there is a transition from a two phase to a single phase region which makes T_c the critical temperature. The point (value of P and V) at which the two extrema merge in the critical isotherm is the critical point characterized by a critical value V_c and a critical pressure P_c . The coexistence volume for one molar mass is found from

$$\left(\frac{\partial P}{\partial V}\right)_T = 0 = \frac{2a}{V^3} - \frac{RT}{(V-b)^2}$$

or $V^3 - \frac{2a}{RT}(V-b)^2 = 0$ (1.0.26)

The point where the extrema merge is clearly a point of inflection where the second derivative of P vanishes i.e.

$$\left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0 = \frac{3a}{V_c^4} - \frac{RT_c}{(V_c-b)^3} \quad (1.0.27)$$

Using Eqns.(1.0.27) and (1.0.26) at $T = T_c$ and $V = V_c$, we have

$$V_c = 3b. \quad (1.0.28)$$

It follows that

$$RT_c = \frac{8a}{27b} \quad (1.0.29)$$

Having located the critical point, we can find the the susceptibility at a temperature T slightly above T_c . We work along the critical isotherm where $V = V_c = 3b$

$$\begin{aligned} -\left(\frac{\partial P}{\partial V}\right)_T &= \frac{RT}{(V_c - b)^2} - \frac{2a}{V_c^3} = \frac{RT}{4b^2} - \frac{2a}{27b^3} \\ &= \frac{1}{4b^2} \left[RT - \frac{8a}{27b} \right] = \frac{R}{4b^2} (T - T_c) \\ \text{leading to } \chi_T &= -\frac{1}{V_c} \left(\frac{\partial V}{\partial P} \right)_T = \frac{4b}{3R} \frac{1}{T - T_c} \end{aligned} \quad (1.0.30)$$

This shows that $\gamma = 1$ on comparing with Eq.(1.0.4). The magnitude of χ_T is dependent on material constants (through b) but the exponent is independent of what gas we are dealing with.

Determination of the critical exponent β proceeds by determining the position of the extrema for $T < T_c$ but very close to it. The two roots of V in the vicinity of $V_c = 3b$ give the two densities of the liquid and gaseous phase. In Eq.(1.0.26) we set $T = T_c(1 - \epsilon)$, $0 < \epsilon \ll 1$ and $V = 3b(1 + v)$ with $v \ll 1$, to obtain

$$\begin{aligned} 27b^3(1 + v)^3 &= \left(1 + \frac{3v}{2}\right)^2 (1 + \epsilon + \dots) \\ \text{or } \frac{(1 + v)^3}{(1 + \frac{3v}{2})^2} - 1 &= \epsilon \end{aligned}$$

leading to

$$\frac{3}{4}v^2 \simeq \epsilon \quad \text{or} \quad v = \frac{2}{\sqrt{3}} \epsilon^{1/2}. \quad (1.0.31)$$

The specific volumes of the two phases are then $3b(1 \pm \frac{2}{\sqrt{3}}\epsilon^{1/2})$, so that the density difference

$$\Delta\rho = M \left(\frac{1}{V_l} - \frac{1}{V_g} \right) \simeq \frac{M\delta V}{V_c^2} = \frac{4M}{9b} \epsilon^{1/2}. \quad (1.0.32)$$

This shows that $\beta = 1/2$, once again independent of the material while the amplitude does depend on the gas considered, through the molar mass M and the constant b .

The exponents β and γ have turned out identical for the magnet and the fluid - this is the greater universality that the experiments had indicated. The methods of calculation have been very different for the magnet and the fluid. The fact that the exponents are the same in the two cases indicate that very close to the critical point, the method of calculation should be the same if one recognizes the essential ingredients. This was the approach of Landau, who said that there would be an additional contribution to the free energy coming from the local order

parameter field that is set up for temperatures very close to T_c . This contribution to the free energy from an order parameter field $\phi(\vec{r})$ can be written from general considerations. First, one does not expect the free energy to depend on the sign of $\phi(\vec{r})$. Second, if the function $\phi(\vec{r})$ is a vector function with components $\phi_i(\vec{r})$, $\{i = 1, 2, \dots, n\}$ then one expects rotational invariance (isotropy) in the space of $\phi_i(\vec{r})$, so that the free energy is determined by the magnitude $\phi_i(\vec{r})\phi_i(\vec{r})$ [repeated index is assumed to be summed over]. But for models that are not isotropic we need to relax this requirement. With this in mind we can write what is known as the Ginzburg Landau free energy functional

$$F = \int d^D r \left[\frac{m^2}{2} \phi_i \phi_i + \frac{1}{2} (\partial_j \phi_i) \cdot (\partial_j \phi_i) + \frac{\lambda}{4} (\phi_i \phi_i)^2 + \dots \right]. \quad (1.0.33)$$

It will cost some free energy to have $\phi_i(\vec{r})$ vary spatially and that comes through the gradient term. We talked about long range correlations developing near the critical point and so our interest will primarily be on the large distance behaviour. This means only lowest order spatial variation would suffice. Invariance under spatial reflection requires the two derivatives shown in Eq.(1.0.33). The problem is that we do not know a-priori the function $\phi_i(\vec{r})$. In fact it is an arbitrary function and hence one would have to use statistical mechanics to get at the thermodynamic free energy. To do a statistical mechanical calculation, we need the partition function which is obtained as,

$$Z = \sum_{\text{all states}} e^{-\frac{E}{k_B T}} \quad (1.0.34)$$

“All states” in this case means all possible functions $\phi_i(\vec{r})$. In the sum over states there would also be the degeneracy factor $g(\phi_i(\vec{r}))$ for each $\phi_i(\vec{r})$ - this is the number of microscopic states corresponding to the macrostate $\phi_i(\vec{r})$. We rewrite Eq.(1.0.34) as

$$\begin{aligned} Z &= \sum_{\text{all } \phi_i(\vec{r})} g(\phi_i(\vec{r})) e^{-\frac{E(\phi_i(\vec{r}))}{k_B T}} \\ &= \sum_{\text{all } \phi_i(\vec{r})} e^{\frac{TS(\phi_i(\vec{r}))}{k_B T}} e^{-\frac{E(\phi_i(\vec{r}))}{k_B T}} \\ &= \sum_{\text{all } \phi_i(\vec{r})} e^{-\frac{F(\phi_i(\vec{r}))}{k_B T}} \end{aligned} \quad (1.0.35)$$

The sum over all $\phi_i(\vec{r})$ in Eq.(1.0.35) is a functional integration and hence the basic problem in critical phenomena is the computation of

$$Z = \int \mathcal{D}[\phi] e^{-\int d^D x \left[\frac{m^2}{2} \phi_i \phi_i + \frac{1}{2} (\partial_j \phi_i) \cdot (\partial_j \phi_i) + \frac{\lambda}{4} (\phi_i \phi_i)^2 + \dots \right] / k_B T_c} \quad (1.0.36)$$

Since this form of F is valid very near T_c , the temperature dependence is through m^2 alone. The scale of F is set by $k_B T_c$.

Before worrying about how to do this functional integration (see Appendix 1), we return to Eq.(1.0.33) and follow the path that Landau had taken. Landau assumed that $\phi_i(\vec{r})$ could be written as M - a constant which would minimize the free energy as required for thermodynamic stability. We immediately find the thermodynamic free energy by minimizing

$$F = \frac{m^2}{2} M^2 + \frac{\lambda}{4} M^4 \quad (1.0.37)$$

for $m^2 > 0$ there is only one minimum, namely $M = 0$ for which $F = 0$. If $m^2 < 0$, there are three minima $M = 0$ and $M \pm \sqrt{\frac{-m^2}{\lambda}}$. For $M = 0$ the free energy $F = 0$ while for $M \pm \sqrt{\frac{-m^2}{\lambda}}$ $F = -\frac{m^4}{4\lambda}$ which is lower than $F = 0$. Consequently, for $m^2 > 0$, we have $M = 0$, while for $m^2 < 0$ $M^2 = -\frac{m^2}{\lambda}$. This will match the critical point phenomenology exactly if we choose for $T \simeq T_c$, $m^2 = a_0(T - T_c)$ with $a_0 > 0$ and $\lambda = \text{constant}$. Then for $T > T_c$, $M = 0$, while for $T < T_c$, we have a finite value of M with

$$M = \pm \sqrt{\frac{a_0}{\lambda}} (T_c - T)^{1/2} \quad (1.0.38)$$

leading to $\beta = \frac{1}{2}$. If ϕ couples to an external field H , then there is an additional term $-\int d^D r \phi_i(\vec{r}) H_i(\vec{r})$. In the Landau approximation, Eq.(1.0.37) would become

$$F = \frac{m^2}{2} M^2 + \frac{\lambda}{4} M^4 - MH \quad (1.0.39)$$

The minimization condition is

$$m^2 M + \lambda M^3 - H = 0 \quad (1.0.40)$$

Derivative w.r.t H yields

$$(m^2 + 3\lambda M^2) \frac{\partial M}{\partial H} \bigg|_T = 1 \quad (1.0.41)$$

If $M = 0$ (i.e. $T > T_c$)

$$\chi_T = \frac{\partial M}{\partial H} \bigg|_T = \frac{1}{a_0(T - T_c)} \quad (1.0.42)$$

while for $M \neq 0$ (i.e. $T < T_c$)

$$\chi_T = \frac{\partial M}{\partial H} \bigg|_T = \frac{1}{2 a_0(T - T_c)}. \quad (1.0.43)$$

Clearly, $\gamma = 1$ and the ratio of χ_T above T_c to χ_T below T_c is 2.

Thus Landau theory is a completely general description of phase transitions which is independent of the exact nature of the material undergoing the transition. It is applicable to magnets, fluids, superfluids and superconductors alike. Let us point out another feature of Eq.(1.0.40). If we were to sit exactly at the critical point, i.e. set $m^2 = 0$, then

$$M \propto H^{1/3}. \quad (1.0.44)$$

In general this defines a critical exponent δ in the form

$$M \propto H^{1/\delta} \quad \text{at } T = T_c \quad (1.0.45)$$

In this case $\delta = 3$. If we solve for M from Eq.(1.0.40), then we would find M as a function of both $T_c - T$ and H . We write down the function in terms of the reduced temperature $t = \frac{T_c - T}{T}$ as,

$$M = \frac{M_0 t^{1/2}}{2^{1/3} \lambda^{1/3}} \times \begin{cases} \left[\frac{H}{M_0^3 t^{3/2}} + \sqrt{\frac{H^2}{M_0^6 t^3} - 1} \right]^{1/3} + \left[\frac{H}{M_0^3 t^{3/2}} - \sqrt{\frac{H^2}{M_0^6 t^3} - 1} \right]^{1/3} & \text{if } H > M_0^3 t^{3/2} \\ 2 \cos \left[\frac{1}{3} \tan^{-1} \left(\sqrt{\frac{M_0^6 t^3}{H^2} - 1} \right) \right] & \text{if } H < M_0^3 t^{3/2} \end{cases} \quad (1.0.46)$$

where

$$M_0^6 = \frac{4}{27} \frac{a_0^3 T_c^3}{\lambda} \quad (1.0.47)$$

If $t = 0$ (i.e. $a = 0$), then Eq.(1.0.46) shows $M = (\frac{H}{\lambda})^{1/3}$, as expected from Eq.(1.0.40). On the other hand, when $H = 0$, $M = [a_0(T_c - T)/\lambda]^{1/2}$, once again as expected from Eq.(1.0.40). The function in Eq.(1.0.46) connects the two extremal behaviours - the one at $H = 0$ and the other at $t = 0$ and is called a **scaling function**.

We can express Eq.(1.0.46) as

$$M = \frac{M_0 t^{1/2}}{(2\lambda)^{1/3}} \mathbf{f} \left(\frac{H}{(M_0 t^{1/2})^3} \right) \quad (1.0.48)$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \left[x + \sqrt{x^2 - 1} \right]^{1/3} + \left[x - \sqrt{x^2 - 1} \right]^{1/3} & \text{if } x \gg 1 \\ 2 \cos \left[\frac{1}{3} \tan^{-1} \left(\sqrt{\frac{1}{x^2} - 1} \right) \right] & \text{if } x \ll 1 \end{cases} \quad (1.0.49)$$

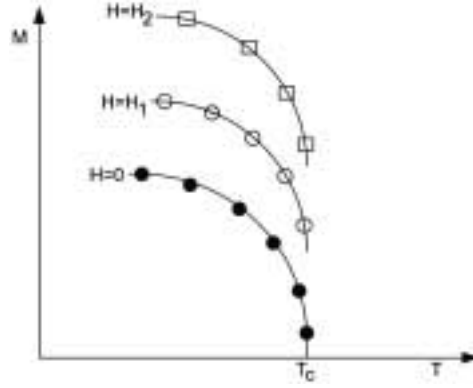


Figure 1.6. Magnetisation as a function of temperature for different external magnetic fields

In terms of the exponents β and δ , we see that

$$M = Ct^\beta f\left(\frac{H}{t^{\beta\delta}}\right) \quad (1.0.50)$$

To obtain the two extremal behaviours, we note that $f(x) \rightarrow \text{constant}$ for $x \rightarrow 0$ and $f(x) \propto x^{1/\delta}$ for $x \rightarrow \infty$. We can easily check that the $f(x)$ of Eq.(1.0.49) has these properties. A relation of the type shown in Eq.(1.0.50) is called a **scaling relation**. The order parameter is a homogeneous function of H and t . A relation of the form of Eq.(1.0.46) can be tested experimentally in a spectacular manner. For the paramagnet-ferromagnet transition, the magnetization is measured as a function of temperature T and the external magnetic field H . The data is generally plotted showing the magnetization M as a function of T for different values of H .

The result is a series of curves as shown in Fig. 1.6. If we now define $t = \frac{T_c - T}{T_c}$ and plot $\frac{M}{t^\beta}$ against $\frac{H}{t^{\beta\delta}}$, the result is a single curve shown in Fig. 1.7.

This spectacular data collapse seen in several experiments is a convincing demonstration of Eq.(1.50). The thing to note is that $\beta \simeq 0.32$ and $\delta \simeq 4.99$ instead of $\beta = 1/2$ and $\delta = 3$ that the Landau approximation and our earlier calculations yielded. Before returning to this discrepancy, we would like to look at the correlations in the field $\phi(\vec{r})$. To find out whether the correlations of the fields at two different spatial points $\vec{r}_1$ and $\vec{r}_2$ are indeed long ranged, we return to Eq.(1.36) and use it to calculate the correlation function defined as

$$C(\vec{r}_1, \vec{r}_2) = \frac{\int \mathcal{D}[\phi(\vec{r})] \phi(\vec{r}_1) \phi(\vec{r}_2) e^{-\int d^D r [\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4]} \int \mathcal{D} \phi(\vec{r}) e^{-\int d^D r [\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4]} \quad (1.0.51)$$

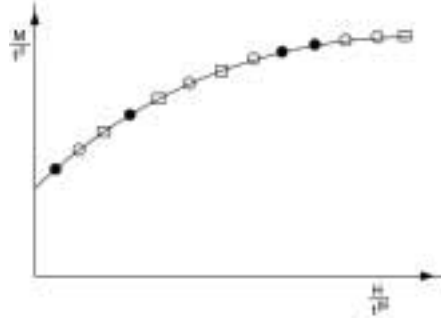


Figure 1.7. Scaling plot of the magnetisation as a temperature and external magnetic field

As explained in Appendix 1, the calculation can be done exactly only for $\lambda = 0$. The result for $D = 3$ and $T \geq T_c$ ($a \geq 0$) is

$$C(\vec{r}_1, \vec{r}_2) = \frac{e^{-m r_{12}}}{r_{12}} \quad (1.0.52)$$

The dependence of C on r_{12} alone is a reflection of translational invariance where $r_{12} = \vec{r}_1 - \vec{r}_2$. In an arbitrary spatial dimension D

$$C(\vec{r}_1, \vec{r}_2) = \frac{e^{-m r_{12}}}{r_{12}^{D-2}}$$

for large r_{12} where $\xi = \frac{1}{m}$ is the correlation length. Since m vanishes as $T \rightarrow T_c$, ξ diverges as the critical point is approached. The characteristic behaviour is $\xi \propto (T - T_c)^{1/2}$. It is this diverging correlation length which is responsible for the strong responses to a perturbation when the system is very near the critical point. At $T = T_c$, the correlation function has no scale and shows a power law behaviour $C(\vec{r}_1, \vec{r}_2) \sim r^{-(D-2)}$. The corresponding behaviour in momentum space is found from the Fourier transform

$$\begin{aligned} C(k) &= \frac{1}{(2\pi)^D} \int d^D r_{12} e^{\vec{k} \cdot \vec{r}_{12}} C(r_{12}) \\ &\propto \frac{1}{(2\pi)^D} \int d^D r_{12} e^{\vec{k} \cdot \vec{r}_{12}} \frac{1}{r_{12}^{D-2}} \\ &\propto \frac{1}{k^2} \end{aligned} \quad (1.0.53)$$

The correlation function (also known as the structure factor) can be measured very accurately by neutron scattering for magnets and light scattering for fluids. In momentum space, Eq.(1.0.52) reads

$$C(k) = \frac{1}{k^2 + m^2} = \frac{1}{k^2 + \frac{1}{\xi^2}} \quad (1.0.54)$$

At $T = T_c$, ξ is definitely big and $C(k)$ goes over to the form shown in Eq.(1.0.53). The experiments are carried out at fixed temperature for different wave vectors by carrying out the detection of the scattered beam at different angles. The intensity of scattering is proportional to the correlation function. The result is shown in Fig. 1.8.

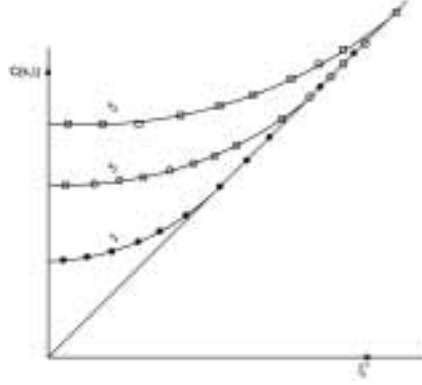


Figure 1.8. Correlation function as a function of the correlation length at different wave vectors

Two things are to be noted from the above data. To make a plot of the structure factor against ξ^{-2} and get a linear plot for $\xi^2 \ll k^2$ we need $\xi \propto (T - T_c)^{-0.63}$ rather than $(T - T_c)^{-0.5}$. Thus the correlation length diverges at $T = T_c$ alright, but the exponent is not $\frac{1}{2}$. We define the correlation length exponent ν by

$$\xi \propto (T - T_c)^{-\nu} \quad (1.0.55)$$

A second point which is small but vital is that if one analyses the data as $\xi^{-2} \rightarrow 0$, i.e., looks at the intercept on the $C(k, \xi)$ axis as a function of k , then $C(k)$ is not proportional to k^{-2} but rather to $k^{-2+\eta}$, where η is very small - of the order of 0.04! The small but non-zero η is a challenge to theorists and experimentalists. In coordinate space the correlation function falls off as $r^{-(D-2+\eta)}$.

We would like to point out how one goes about the calculation of thermodynamic quantities from Eq.(1.36). To do that, we need to have an external field H and write the partition function in the presence of the field as

$$Z(T, H) = \int \mathcal{D}[\phi] e^{-\int d^D r [\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4 - H \phi] / k_B T_c} \quad (1.0.56)$$

The magnetization M (which is the typical order parameter) is obtained as the derivative

$$M = \left. \frac{\partial F_{th}}{\partial H} \right|_{H \rightarrow 0},$$

where F_{th} is the thermodynamic free energy, expressed in terms of Z by $F_{th} = -k_B T_c \ln Z$. Therefore,

$$\begin{aligned} M &= k_B T_c \frac{\partial}{\partial H} (\ln Z) \Big|_{H \rightarrow 0} = k_B T_c \frac{1}{Z} \frac{\partial Z}{\partial H} \Big|_{H \rightarrow 0} \\ &= k_B T_c \frac{1}{Z} \int \mathcal{D}[\phi] \frac{1}{k_B T_c} \int d^D r \phi e^{-\int d^D r [\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} (\phi^2)^2] / k_B T_c} \\ &= \int \langle \phi \rangle d^D r \end{aligned} \quad (1.0.57)$$

The susceptibility is obtained as the derivative of the magnetization and from the last but one step of Eq.(1.0.57), we have

$$\begin{aligned} \chi_T &= \frac{\partial M}{\partial H} \Big|_{H \rightarrow 0} \\ &= \frac{1}{k_B T_c} \left[\frac{1}{Z} \int \mathcal{D}[\phi] \int d^D r_1 \phi(\vec{r}_1) \int d^D r_2 \phi(\vec{r}_2) e^{-\int d^D r [\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4] / k_B T_c} \right. \\ &\quad \left. - \frac{1}{Z^2} \int \mathcal{D}[\phi] \int d^D r_1 \phi(\vec{r}_1) e^{-\int d^D r [\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4] / k_B T_c} \right] \\ \text{or } k_B T_c \chi_T &= \int d^D r_1 d^D r_2 \langle \phi(\vec{r}_1) \phi(\vec{r}_2) \rangle - \left[\int d^D r_1 \langle \phi(\vec{r}_1) \rangle \right]^2 \\ &= V \int C(r_{12}) d^D r_{12} - \left[\int d^D r_1 \langle \phi(\vec{r}_1) \rangle \right]^2 \end{aligned} \quad (1.0.58)$$

where we have taken note of the fact that the correlation function $\langle \phi(\vec{r}_1) \phi(\vec{r}_2) \rangle$ is a function of r_{12} alone. For $T > T_c$, $\langle \phi(\vec{r}) \rangle = 0$ and

$$k_B T_c \chi_T = V \int C(r_{12}) d^D r_{12} \quad (1.0.59)$$

For $T < T_c$, $\langle \phi(\vec{r}) \rangle \neq 0$ and recognizing that $M = \int d^D r_1 \langle \phi(\vec{r}) \rangle$, then $C(r_{12})$ needs to be defined as $C(r_{12}) = \langle (\phi(\vec{r}_1) - M)(\phi(\vec{r}_2) - M) \rangle$ and Eq.(1.0.55) will again be valid. With $C(r_{12}) \sim e^{-\frac{r_{12}}{\xi}} / r^{D-2+\eta}$

$$\chi_T \propto \xi^{2-\eta} \quad (1.0.60)$$

This is what we have been driving at. The divergence of χ_T is caused by the divergence of ξ . With $\chi_T \propto (T - T_c)^{-\gamma}$ and $\xi \propto (T - T_c)^{-\nu}$, Eq.(1.0.60) also yields the relation

$$\gamma = (2 - \eta)\nu \quad (1.0.61)$$

We round off our discussion of response functions by looking at the entropy which is the response to a temperature fluctuation. The entropy is defined as $S = -\frac{\partial F}{\partial T}$ with the external field equal to zero. With $m^2 = a_0(T - T_c)$, we can write $S = -a_0 \frac{\partial F}{\partial m^2} = a_0 \int \langle \phi^2(\vec{r}) \rangle d^D r$. A further derivative yields the specific heat as $C = T \frac{\partial S}{\partial T} = T \frac{\partial S}{\partial m^2}$. This leads to

$$\begin{aligned}
 C &= a_0^2 T \left[\frac{1}{Z} \int d^D r_1 d^D r_2 \phi^2(\vec{r}_1) \phi^2(\vec{r}_2) e^{-\int d^D r [\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4] / k_B T_c} \right. \\
 &\quad \left. - \frac{1}{Z^2} \left(\int d^D r_1 \phi^2(\vec{r}_1) e^{-\int d^D r [\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4] / k_B T_c} \right)^2 \right] \\
 &= a_0^2 T \int \left\langle \phi^2(\vec{r}_1) \phi^2(\vec{r}_2) \right\rangle_c d^D r_1 d^D r_2 \\
 &\simeq a_0^2 T_c \int \left\langle \phi^2(\vec{r}_1) \phi^2(\vec{r}_2) \right\rangle_c d^D r_1 d^D r_2
 \end{aligned} \tag{1.0.62}$$

where the subscript c denotes the *connected part*. To get a feel for how Eq.(1.0.62) works, we can approximate the connected correlation (i.e. $\vec{r}_1$ and $\vec{r}_2$ are connected) as

$$\left\langle \phi^2(\vec{r}_1) \phi^2(\vec{r}_2) \right\rangle_c \simeq \left\langle \phi(\vec{r}_1) \phi(\vec{r}_2) \right\rangle^2,$$

and use the asymptotic form of

$$\langle \phi(\vec{r}_1) \phi(\vec{r}_2) \rangle \simeq \frac{e^{-r_{12}/\xi}}{r_{12}^{D-2}}$$

to obtain

$$C \propto V \int d^D r_{12} \frac{e^{-2r_{12}/\xi}}{r_{12}^{2D-4}} \propto \xi^{4-D} \propto (T - T_c)^{\frac{4-D}{2}} \tag{1.0.63}$$

Here is another response function which diverges as the correlation length becomes infinitely big. The relevant exponent (generally denoted by α)

$$C = C_0 |T - T_c|^{-\alpha} \tag{1.0.64}$$

is in the approximation $\alpha = \frac{4-D}{2}$. The divergent specific heat at the liquid-gas critical point is shown in Fig. 1.9a. It also diverges near the superfluid transition in liquid He⁴ Fig. 1.9b. The exponents however are different from $\frac{4-D}{2}$, for the liquid-gas system $\alpha \simeq 0.11$ while for the superfluid transition $\alpha \simeq 0$.



Figure 1.9. Constant volume specific heat near the liquid gas critical point.

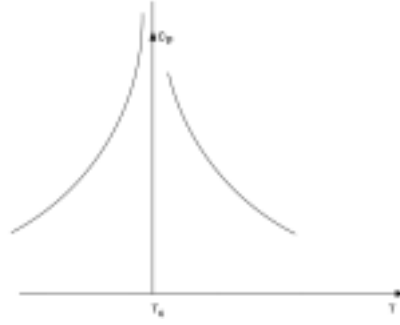


Figure 1.9. Constant pressure specific heat near the superfluid transition.

We can also find the specific heat from the approximation of Eq.(1.0.37). In this case

$$\begin{aligned} F &= 0 & \text{for} & \quad T > T_c \\ \text{and} \quad F &= -\frac{a_0^2}{2\lambda} & \text{for} & \quad T < T_c \end{aligned} \quad (1.0.65)$$

This implies a discontinuity in the measured specific heat which will be a combination of the critical and non-critical parts (see Fig. 1.10).

If we compare this with the specific heat near the superconducting transition, Fig. 1.11, then the similarity is striking. We now have a situation which may appear contradictory. The liquid-gas transition, the superfluid transition and the superconducting transition are all second order transitions. In the way we handled the Landau-Ginzburg free energy, the specific heat diverged at the critical point while in the mean-field approximation, the specific heat has a discontinuity at $T = T_c$. The specific heat at the liquid-gas and superfluid transitions diverge at the critical point but remains finite with a discontinuity at the superconducting transition. To understand the differing behaviours, we need to understand the role of fluctuations. In the mean field approach, where the fluctuating field $\phi(\vec{r})$ is

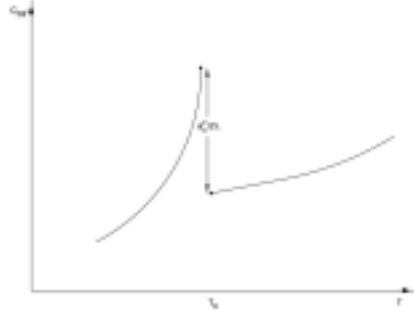


Figure 1.10. Jump in the mean field specific heat

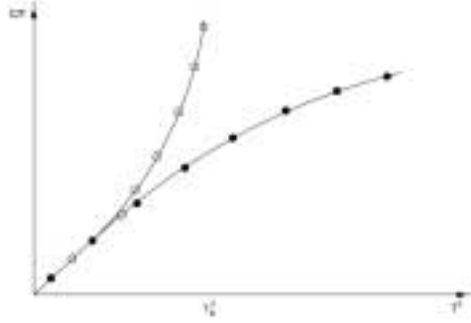


Figure 1.11. Experimental specific heat near the superconducting transition

replaced by the spatially uniform m , the role of fluctuations is minimal. The role of fluctuations is characterized by the correlation length $\xi = \frac{1}{|a|^{1/2}} = \frac{1}{a_0^{1/2}|T-T_c|^{1/2}}$. There is another length scale in the problem which is characterized by the coupling constant λ . Since $\int d^D r [\frac{m^2}{2}\phi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\lambda}{4}\phi^4]$ appears in the exponent of an exponential, every term in it must be dimensionless (in units where $K_B T = 1$) we find that ϕ has the dimension $L^{1-\frac{D}{2}}$ where L is a length (from the middle term). The last term now shows that λ has the dimension L^{D-4} giving a length scale $\lambda^{\frac{1}{D-4}}$.

At a particular temperature T_m the correlation length becomes the same as the other length scale and for smaller values of $T - T_c$, the fluctuations break up the order. The mean field theory is valid if

$$T - T_c \geq \frac{1}{a_0} \lambda^{\frac{2}{4-D}} = \frac{1}{a_0 \xi_0^2} \quad (1.0.66)$$

where ξ_0 is a non-critical characteristic length. If ξ_0 is very big, then the condition of Eq.(1.0.67) is satisfied for almost all T and the system shows the mean field exponents. This is true for the superconducting transition where the pair coherence length ξ_0 is the non-critical length scale and is extremely big. This is the reason

behind the excellent agreement between the theory and experiment in Figs.1.10 and 1.11. For other systems which have a much smaller value of ξ_0 , there will be a temperature T_m where the equality of Eq.(1.0.67) holds and if one is further away from T_c than T_m , then the mean field exponents will be observed. Closer to T_c than T_m , the exponents will change and the phenomenon is called *crossover*. Writing Eq.(1.0.67) as

$$[a_0(T - T_c)]^{4-D} \geq \lambda^2 \quad (1.0.67)$$

we see that for $T \simeq T_c$, this relation will always be satisfied for $D > 4$. Thus mean field results are always true for $D > 4$. It is for $D < 4$, that the crossover occurs from mean field to nontrivial exponents. If we are measuring the compressibility of a fluid or susceptibility of a uniaxial magnet, then for temperatures very close to T_c , $\gamma \simeq 1.24$ is observed while further away one finds $\gamma = 1$. Similarly, for the magnetization it is $\beta \simeq 0.32$ very close to T_c and $\beta = 0.5$ somewhat further away. For the exponent δ , it is $\delta \simeq 5$ for very small magnetic fields and $\delta \simeq 3$ for large fields. This brings up the final question : How does one calculate the nontrivial exponents $\beta \simeq 0.32$, $\gamma \simeq 1.24$ and $\delta \simeq 5$? Two exponents α and η are identically zero in the mean field theory while experiments very closely show that $\alpha \simeq 0.11$ for the fluid and $\eta \simeq 0.04$. These two small exponents are consequently crucial to the theory. In **Chapter 3**, we will describe the technique which allows us to arrive at non-trivial values for the critical exponents. The success in setting up a realistic theory for second order phase transitions is the inspiration behind bringing together various topics in dynamics of such macroscopic systems - both near equilibrium and out-of-equilibrium - in the subsequent chapter.

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Models of Dynamics

2.1 Introduction

Our concern will mainly be with mesoscopic systems. These are systems whose length and time scales are significantly larger (a few orders of magnitude) than atomic scales but still small compared to macroscopic scales (system size etc.). A typical example is provided by a system near a second order phase transition point, e.g. a magnet being cooled towards the Curie point. For temperatures far above the Curie point the individual magnetic moments inside the magnet are moving around randomly and the overall magnetization of the sample is zero. As the temperature is lowered and the Curie point is approached, the individual magnetic moments become more correlated. The energy is lowered if the moments are aligned and as the temperature decreases, the entropy effects become smaller and the gradually dominating energy part of the thermodynamic free energy causes the correlations to build up. The distance over which the correlations exist is called the correlation length ξ . As the temperature reaches the Curie point, the correlation length becomes infinitely big, leading to correlation functions which become infinitely long ranged. For temperatures very close to the critical point (say a millikelvin away from the critical point), the correlation length is of the order of a few microns which is about three or four orders of magnitude larger than the atomic scale which is of the order of a few angstroms. This makes for an ideal mesoscopic system. If we are to describe the statics or dynamics of such a system, the task would be quite difficult if it were to be in terms of individual atoms or molecules. Consequently, one uses a coarse grained description. One talks about the averaged magnetic moment at any point of space. The averaging at any point of space is done over several atomic dimensions and the result is a magnetization field $\vec{m}(\vec{r})$, in terms of which the critical phenomena is described. The generic name for the field $\vec{m}(\vec{r})$ is the order parameter field - $\phi(\vec{r})$. It could be the density fluctuation field for the liquid-vapour

transition (or a binary fluid near its consolute point) or the superfluid fluctuations (amplitude and phase) near the lambda point or the fluctuations in the staggered magnetization near the antiferromagnetic transition.

Our interest, as we emphasized in the last chapter is in systems which are described in terms of macroscopic fields. In this chapter, we will describe the kind of dynamics that one expects for these systems. We begin with a system near the critical point that we have discussed in the last chapter. If we are to describe dynamics of this system, then the equation of motion is not so obvious. For one thing, those degrees of freedom which have been averaged over, are going to influence the dynamics and this cannot be in a deterministic fashion. Consequently, the dynamics will be described in terms of stochastic rather than deterministic differential equations. The dynamics of the fluctuations is generally expected to be relaxational as we expect a small fluctuation from the equilibrium state to disappear in time. This relaxation generally is the linear part of the equation of motion - the expectation is that the equation of motion will be first order in time since specification of only the initial value of the order parameter should be enough to specify its future dynamics. Thus the expected dynamics is of Langevin variety.

A different example is provided by crystal growth by ballistic deposition of atoms. The depositing atoms diffuse across the surface to settle down at places where the local energy is a minimum. This smooths out the growing surface. On the other hand, there is intrinsic noise in the deposition process and this causes the surface to be rough. The natural variable to describe the growth of this surface is the local height variable $h(\vec{r}, t)$. Here $\vec{r}$ is the coordinate in the substrate. The dynamics of the field $h(\vec{r}, t)$ has two clear cut parts:

- i) a surface diffusion which helps smooth out any fluctuation in h and
- ii) a noise part corresponding to the fluctuation in the deposition process

Thus, once more the dynamics is of the Langevin variety.

Yet another example of a mesoscopic system is the dynamics of polymer chains. Consider a polymer chain put in a solvent. If the polymer is hydrophobic, then to prevent contact with the water molecules the polymer tends to fold into a ball. The entropy term in the thermodynamic free energy would like the polymer to spread out and hence there is a competition between the energy and the entropy effects. As the temperature is lowered, the chain is expected to undergo a transition into a compact structure. The compact structure is typically of the order of microns (examples of these compact structures are the enzyme like proteins) and constitute another example of a mesoscopic system. The dynamics of a polymer chain is governed again by a Langevin equation. For a randomly hydrophobic and hydrophilic chain, thus the dynamics can be of some relevance to the interesting problem of protein folding.

Finally, we mention the problem of turbulence. Here the randomness is generated by the non-linear term in the Navier-Stokes' equation. However, to maintain the turbulence we need to have a maintained mean flow and energy transfer has to occur from the mean flow to the fluctuations. There would also be the question

of boundary conditions on the various bounding surfaces, however far away. It is the information about boundaries and maintained mean flows that we can average over and cast the Navier-Stokes' equation as a Langevin equation with a fluctuating force. We begin our discussion of Langevin equations by an explicit construction. A single particle interacts with a set of particles which constitute a bath. It will be demonstrated how averaging over the bath variables leads to the fluctuating force. The derivation that follows is due to Zwanzig.

2.2 Langevin Picture

Without any loss of generality, we will confine ourselves to one dimensional dynamics. The bath will be a set of harmonic oscillators, each characterized by a coordinate q_i and a conjugate momentum p_i . The particle whose equation of motion is our concern has a coordinate X , moves in a potential $V(X)$ and interacts with the bath via a quadratic coupling, i.e. a coupling of the form Xq_i for all i . The Hamiltonian describing the system is

$$H = \frac{P^2}{2} + V(X) + \sum_i \frac{1}{2} \omega_i^2 \left(q_i - \frac{\gamma_i}{\omega_i^2} X \right)^2 + \sum_i \frac{p_i^2}{2} \quad (2.2.1)$$

where all masses have been set to unity. Now the equation of motion for P is

$$\dot{P} = -\frac{\partial V}{\partial X} + \sum_i \gamma_i \left(q_i - \frac{\gamma_i}{\omega_i^2} X \right) \quad (2.2.2)$$

and that for q_i is

$$\ddot{q}_i + \omega_i^2 q_i = \gamma_i X \quad (2.2.3)$$

The solution for $q_i(t)$ can be written down as

$$\begin{aligned} q_i(t) &= q_i(0) \cos \omega_i t + \frac{p_i(0)}{\omega_i} \sin \omega_i t + \int_0^t \gamma_i \frac{\sin \omega_i(t-s)}{\omega_i} X(s) ds \\ &= q_i(0) \cos \omega_i t + \frac{p_i(0)}{\omega_i} \sin \omega_i t + \frac{\gamma_i}{\omega_i^2} X(t) - \frac{\gamma_i}{\omega_i^2} X(0) \cos \omega_i t \\ &\quad - \int_0^t \frac{\gamma_i}{\omega_i^2} \cos \omega_i(t-s) P(s) ds \end{aligned}$$

leading to,

$$\begin{aligned} q_i(t) - \frac{\gamma_i}{\omega_i^2} X(t) &= \left[q_i(0) - \frac{\gamma_i}{\omega_i^2} X(0) \right] \cos \omega_i t + \frac{p_i(0)}{\omega_i} \sin \omega_i t \\ &\quad - \int_0^t \frac{\gamma_i}{\omega_i^2} \cos \omega_i(t-s) P(s) ds \end{aligned} \quad (2.2.4)$$

(note that $\dot{X} = P$ since $m = 1$).

Inserting this solution in Eq.(2.2.2), we have

$$\begin{aligned}\dot{P} &= -\frac{\partial V}{\partial X} - \sum_i \int_0^t \frac{\gamma_i^2}{\omega_i^2} \cos \omega_i(t-s) P(s) ds \\ &\quad + \sum_i \gamma_i \left(q_i(0) - \frac{\gamma_i}{\omega_i^2} X(0) \right) \cos \omega_i t + \sum_i \frac{\gamma_i}{\omega_i} p_i(0) \sin \omega_i t \\ &= -\frac{\partial V}{\partial X} - \int_0^t K(t-s) P(s) ds + f(t)\end{aligned}\quad (2.2.5)$$

where

$$\begin{aligned}K(t-s) &= \frac{\gamma_i^2}{\omega_i^2} \cos \omega_i(t-s) \\ \text{and } f(t) &= \sum_i \gamma_i \left(q_i(0) - \frac{\gamma_i}{\omega_i^2} X(0) \right) \cos \omega_i t + \sum_i \frac{\gamma_i}{\omega_i} p_i(0) \sin \omega_i t\end{aligned}\quad (2.2.6)$$

As expected, at this point the equation of motion for $P(t)$ is completely deterministic. Note that the force $f(t)$ is dependent on the initial values of all the bath variables which would be of the order of the Avogadro number. Hence, practical specification of the force is impossible. Consequently, it pays to go to a statistical description. It is more convenient to think of $f(t)$ as a stochastic force with only its moments specified. To specify the moments, we require a distribution. In this case, the distribution has to do with the distribution of the initial values of the coordinates and momenta of the heat bath. If we assume that the system is characterized by a temperature T , then the distribution is of Maxwell-Boltzmann variety (i.e. Gaussian) and we have

$$\begin{aligned}P\left(\left\{q_i - \frac{\gamma_i}{\omega_i^2} X\right\}, \{p_i\}\right) \\ = \left(\frac{\omega}{\pi K_B T}\right)^N \prod_{i=1}^N \exp\left[-\left\{\frac{1}{2} p_i^2 + \frac{1}{2} \omega_i^2 \left(q_i - \frac{\gamma_i}{\omega_i^2} X\right)^2\right\} / K_B T\right]\end{aligned}\quad (2.2.7)$$

leading to the moments

$$\begin{aligned}\langle p_i(0) \rangle &= \langle q_i(0) - \frac{\gamma_i}{\omega_i^2} X(0) \rangle = 0 \\ \langle p_i(0) p_j(0) \rangle &= K_B T \delta_{ij} \\ \langle (q_i(0) - \frac{\gamma_i}{\omega_i^2} X(0)) \omega_i^2 (q_j(0) - \frac{\gamma_j}{\omega_j^2} X(0)) \rangle &= K_B T \delta_{ij} \\ \langle p_i(0) (q_j(0) - \frac{\gamma_j}{\omega_j^2} X(0)) \rangle &= 0 \quad \text{for all } i \text{ and } j\end{aligned}\quad (2.2.8)$$

We can now calculate the moments of the random force $f(t)$. Clearly,

$$\langle f(t) \rangle = 0 \quad (2.2.9)$$

As for the second moment

$$\begin{aligned} \langle f(t)f(s) \rangle &= \left\langle \sum_i \left[\gamma_i \left(q_i(0) - \frac{\gamma_i}{\omega_i^2} X(0) \right) \cos \omega_i t + \frac{\gamma_i}{\omega_i} P_i(0) \sin \omega_i t \right] \times \right. \\ &\quad \left. \left[\gamma_j \left(q_j(0) - \frac{\gamma_j}{\omega_j^2} X(0) \right) \cos \omega_j s + \frac{\gamma_j}{\omega_j} P_j(0) \sin \omega_j s \right] \right\rangle \\ &= \sum_{i,j} \gamma_i \gamma_j \left\langle q_i(0) - \frac{\gamma_i}{\omega_i^2} X(0) \right\rangle \left\langle q_j(0) - \frac{\gamma_j}{\omega_j^2} X(0) \right\rangle \times \\ &\quad \cos \omega_i t \cos \omega_j s + \sum_{i,j} \frac{\gamma_i \gamma_j}{\omega_i \omega_j} \langle p_i(0) p_j(0) \rangle \sin \omega_i t \sin \omega_j s \\ &= K_B T \sum_j \frac{\gamma_j^2}{\omega_j^2} \cos \omega_j (t-s) \\ &= K_B T K(t-s) \end{aligned} \quad (2.2.10)$$

Let us return to Eq.(2.2.5) and examine the different forces on the R.H.S. The first term, i.e. $\frac{\partial V}{\partial X}$, comes from an externally imposed potential while the second and third terms come from the interaction with the environment. We will make the assumption of being in the limit of large coupling between the particle and the bath variables and drop the potential term $\frac{\partial V}{\partial X}$. The R.H.S of Eq.(2.2.5) now contains two terms

- i) a damping term $-\int_0^t K(t-s)p(s)ds$ and
- ii) a random term $f(t)$, such that $\langle f(t) \rangle$ and $\langle f(t)f(s) \rangle = K_B T K(t-s)$

An equation of this sort is called a Langevin equation - it has resulted entirely from the interaction with the environment (we will see later that there can be additional terms).

$$\dot{p} = - \int_0^t K(t-s)p(s)ds + f(t) \quad (2.2.11)$$

In the particular example, we have chosen

$$\langle f(t)f(s) \rangle = K_B T K(t-s) \quad (2.2.12)$$

relating the stochastic term to the dissipative term. This may not always happen.

We need to discuss the classification of noise. For a correlation of the kind shown in Eq.(2.2.12), the noise at time t is correlated with the noise at time s . It thus retains memory and is called a coloured noise. Returning to the expression

for $K(t-s)$, if the couplings γ_i and the frequencies ω_i are such that $g_i \frac{\gamma_i}{\omega_i^2}$ (where g_i is the weight factor) is independent of i , in that case,

$$\begin{aligned}
K(t-s) &= \sum_i \frac{\gamma_i^2}{\omega_i^2} \cos \omega_i(t-s) \\
&= \lim_{\omega_{\max} \rightarrow \infty} \int_{-\omega_{\max}}^{\omega_{\max}} \frac{\gamma^2}{\omega^2} P(\omega) \cos \omega(t-s) d\omega \\
&= \Gamma \lim_{\omega_{\max} \rightarrow \infty} \int_0^{\omega_{\max}} \cos \omega(t-s) d\omega \quad (\text{using } P(\omega) = \frac{\Gamma \omega^2}{2\gamma^2}) \\
&= \Gamma \lim_{\omega_{\max} \rightarrow \infty} \frac{\sin \omega_{\max}(t-s)}{t-s} \\
&= \Gamma \delta(t-s)
\end{aligned} \tag{2.2.13}$$

In this limit, the noise at time t is not correlated with the noise at a different time s . Such a noise is called a Markovian or a white noise. Using the δ -function form of $K(t-s)$, we now have the Langevin equation in the form

$$\dot{p}(t) = -\Gamma p(t) + f(t) \tag{2.2.14}$$

where

$$\langle f(t)f(s) \rangle = \Gamma K_B T K(t-s) \tag{2.2.15}$$

It is worth noting that the appearance of Γ in the relaxation term in Eq.(2.2.14) and the correlation function in Eq.(2.2.15) is not accidental and is related to the existence of an equilibrium distribution.

We now turn to Eq.(2.2.14) and discuss the solution. As is to be expected, the solution for $p(t)$ is stochastic and hence only quantities which are sensible are the various moments of $p(t)$. The solution of Eq.(2.2.14) is straightforward to write down and we have

$$p(t) = e^{-\Gamma t} \int_0^t e^{\Gamma s} f(s) ds + p(0) e^{-\Gamma t} \tag{2.2.16}$$

Clearly, $\langle p(t) \rangle = \langle p(0) \rangle e^{-\Gamma t}$. For the second moment

$$\begin{aligned}
\langle p(t_1)p(t_2) \rangle &= e^{-\Gamma(t_1+t_2)} \int_0^{t_1} e^{\Gamma s_1} \left\langle f(s_1) ds_1 \int_0^{t_2} e^{\Gamma s_2} f(s_2) ds_2 \right\rangle \\
&\quad + \langle p^2(0) \rangle e^{-\Gamma(t_1+t_2)} \\
&= e^{-\Gamma(t_1+t_2)} \times \Gamma K_B T \int_0^{t_1} e^{2\Gamma s_1} ds_1 + \langle p^2(0) \rangle e^{-\Gamma(t_1+t_2)} \quad (t_2 > t_1) \\
&= e^{-\Gamma(t_1+t_2)} \frac{K_B T}{2} (e^{2\Gamma t_1} - 1) + \langle p^2(0) \rangle e^{-\Gamma(t_1+t_2)} \\
&= \frac{K_B T}{2} e^{-\Gamma(t_2-t_1)} + \left(\langle p^2(0) \rangle - \frac{K_B T}{2} \right) e^{-\Gamma(t_1+t_2)}
\end{aligned} \tag{2.2.17}$$

Considering the case $t_1 > t_2$, we have in general

$$\langle p(t_1)p(t_2) \rangle = \frac{K_B T}{2} e^{-\Gamma|t_1-t_2|} + \left(\langle p^2(0) \rangle - \frac{K_B T}{2} \right) e^{-\Gamma(t_1+t_2)}.$$

If the initial distribution is such that

$$\langle p^2(0) \rangle = \frac{K_B T}{2},$$

then the fluctuation correlations decay as

$$\langle p(t_1)p(t_2) \rangle = \langle p^2(0) \rangle e^{-\Gamma|t_1-t_2|}.$$

For equal time correlation on the other hand

$$\langle p^2(t) \rangle = \frac{k_B T}{2} + \left(\langle p^2(0) \rangle - \frac{k_B T}{2} \right) e^{-2\Gamma t}.$$

As $t \rightarrow \infty$, $\langle p^2(t) \rangle \rightarrow \frac{k_B T}{2}$. We can construct the n^{th} order correlation function from the solution written down in Eq.(2.2.16) in a straightforward manner and one finds for $t \rightarrow \infty$,

$$\begin{aligned} \langle p^{2n+1}(t) \rangle &= 0 \\ \langle p^{2n}(t) \rangle &= \frac{(2n)!}{n! 2^n} \left(\frac{k_B T}{2} \right)^n \end{aligned} \quad (2.2.18)$$

With all the correlation functions known we can write down what the distribution is for $t \rightarrow \infty$. The moments of Eq.(2.2.18) indicate a Gaussian distribution with,

$$P(p) = \frac{1}{\sqrt{\pi K_B T}} e^{-\frac{p^2}{K_B T}} \quad (2.2.19)$$

Writing $F(p) = \frac{p^2}{2}$ as a kind of *energy expression*, we can write the Langevin equation that we started out with as

$$\begin{aligned} \dot{p} &= -\Gamma \frac{\partial F}{\partial p} + f \\ \langle f(t) f(s) \rangle &= 2\Gamma \frac{k_B T}{2} \delta(t-s) \\ &= 2\Gamma D \delta(t-s) \end{aligned} \quad (2.2.20)$$

where $D = \frac{k_B T}{2}$. With the Langevin equation appearing in the form of Eq.(2.2.20), one is guaranteed the existence of an equilibrium distribution given by

$$P(p) \propto e^{-\frac{F(p)}{D}}.$$

In our specific example $F(p)$ was a quadratic function of p . As we will show in the next section, this restriction is not necessary. Any Langevin equation with the structure

$$\begin{aligned}\dot{p} &= -\Gamma \frac{\partial F}{\partial p} + f \\ \langle f(t) f(s) \rangle &= 2\Gamma D \delta(t-s)\end{aligned}\quad (2.2.21)$$

has the equilibrium distribution $e^{-\frac{F}{D}}$.

To end this section, we ask what is the probability distribution associated with the noise $f(t)$. It is a straightforward matter to carry out the calculation of the higher moments since all the moments of the initial values $q_i(0)$ and $p_i(0)$ are known. One finds,

$$\begin{aligned}\langle f(t_1) f(t_2) \dots f(t_{2n+1}) \rangle &= 0 \\ \text{while } \langle f(t_1) f(t_2) \dots f(t_{2n+1}) \rangle &= (2\Gamma D)^n \{ \delta(t_1 - t_2) \delta(t_3 - t_4) \\ &\quad \dots \delta(t_{2n-1} - t_{2n}) + \text{all permutations} \\ &\quad \text{of the time intervals} \}\end{aligned}\quad (2.2.22)$$

Given the above correlation functions, the probability of $f(t)$ being a given function $f(t)$ is

$$P[f] = N \exp \left[-\frac{1}{4\Gamma D} \int_{-\infty}^{\infty} |f(s)|^2 ds \right]$$

where the normalization is given by the functional integral

$$N^{-1} = \int \mathcal{D}[f] \exp \left[-\frac{1}{4\Gamma D} \int_{-\infty}^{\infty} |f(s)|^2 ds \right].$$

It will be an useful exercise for the reader to verify that the above distribution does lead to the correlation function in Eq.(2.2.22).

2.3 Fokker Planck Description

In this section, we discuss a different kind of equation of motion - an equation of motion for the probability P itself. The question is what is the probability $P(p(t), t)$ of the variable p acquiring the value $p(t)$ at time t ? The equation of motion for the variable p is that given in Eq.(2.2.21). It is easiest to proceed by considering time as a discrete variable and taking a sequence $t_0, t_1, t_2, \dots, t_n, t_{n+1}, \dots$ such

that $t_{n+1} - t_n = \delta t$ is an infinitesimal. Writing $p(t_n) = p_n$, the discretized form of Eq.(2.2.21) is

$$p_{n+1} = p_n - g \quad (2.3.1)$$

$$\text{where } g = \Gamma \frac{\partial F}{\partial p} \delta t - \bar{f}(t) (\delta t)^{1/2} \quad (2.3.2)$$

with the definition

$$\bar{f}(t) = f(t)(\delta t)^{1/2} \quad (2.3.3)$$

so that the correlation function of $\bar{f}(t)$ is

$$\langle \bar{f}(t) \bar{f}(t') \rangle = 2\Gamma D \delta_{tt'} \quad (2.3.4)$$

$\delta_{tt'}$ being the Kronecker delta. Our aim is to go from $P(p, t)$ to $P(p(t + \delta t), t + \delta t)$, i.e. the probability of finding the value $p(t + \delta t)$ of the variable p at time $t + \delta t$. In terms of the discretized time variable, we need to go from $P(p_n, t_n)$ to $P(p_{n+1}, t_{n+1})$. Obviously, this is achieved by using the equation of motion i.e.

$$P(p_{n+1}, t_{n+1}) = \left\langle \int P(p_n, t_n) \delta(p_{n+1} - p_n + g) dp_n \right\rangle \quad (2.3.5)$$

The averaging has to be done over all possible realizations of the noise $f(t)$ that occurs in g i.e.

$$\langle \dots \rangle = N \int \mathcal{D}[f](\dots) \exp \left[-\frac{1}{4\Gamma D} \int_{-\infty}^{\infty} |f(s)|^2 ds \right] \quad (2.3.6)$$

We now make use of the fact that g is small ($O(\delta t)^{1/2}$) and expand the δ -function as

$$\begin{aligned} \delta(p_{n+1} - p_n + g) &= \delta(p_{n+1} - p_n) + g \frac{\partial}{\partial p_n} \delta(p_{n+1} - p_n) \\ &+ \frac{g^2}{2} \frac{\partial^2}{\partial p_n^2} \delta(p_{n+1} - p_n) + \dots \end{aligned} \quad (2.3.7)$$

Inserting this expansion in Eq.(2.3.5), we note the following

$$\left\langle \int P(p_n, t_n) \delta(p_{n+1} - p_n) dp_n \right\rangle = P(p_{n+1}, t_n) \quad (2.3.8)$$

$$\begin{aligned}
& \left\langle \int P(p_n, t_n) g \frac{\partial}{\partial p_{n+1}} \delta(p_{n+1} - p_n) dp_n \right\rangle \\
&= - \left\langle \int P(p_n, t_n) g \frac{\partial}{\partial p_n} \delta(p_{n+1} - p_n) dp_n \right\rangle \\
&= \left\langle -P(p_n, t_n) g \delta(p_{n+1} - p_n) \right\rangle \Big|_{p_n=-\infty}^{p_n=\infty} \\
&\quad + \left\langle \int \delta(p_{n+1} - p_n) \frac{\partial}{\partial p_n} \left(g P(p_n, t_n) \right) dp_n \right\rangle \\
&= \left\langle \frac{\partial}{\partial p_{n+1}} \left(g P(p_{n+1}, t_n) \right) \right\rangle
\end{aligned} \tag{2.3.9}$$

and

$$\begin{aligned}
& \left\langle \int P(p_n, t_n) \frac{g^2}{2} \frac{\partial^2}{\partial p_{n+1}^2} \delta(p_{n+1} - p_n) dp_n \right\rangle \\
&= \left\langle \int P(p_n, t_n) \frac{g^2}{2} \frac{\partial^2}{\partial p_n^2} \delta(p_{n+1} - p_n) \right\rangle \\
&= \left\langle P(p_n, t_n) \frac{g^2}{2} \frac{\partial}{\partial p_n} \delta(p_{n+1} - p_n) \right\rangle \Big|_{p_n=-\infty}^{p_n=\infty} \\
&\quad - \left\langle \int \frac{\partial}{\partial p_n} [\delta(p_{n+1} - p_n)] \times \frac{\partial}{\partial p_n} \left[\frac{g^2}{2} P(p_n, t_n) \right] dp_n \right\rangle \\
&= \left\langle \int \frac{\partial^2}{\partial p_n^2} \left[\frac{g^2}{2} P(p_n, t_n) \right] \delta(p_{n+1} - p_n) dp_n \right\rangle \\
&= \left\langle \frac{\partial^2}{\partial p_{n+1}^2} \left[\frac{g^2}{2} P(p_{n+1}, t_n) \right] \right\rangle
\end{aligned} \tag{2.3.10}$$

In the above derivations, we have always assumed that $P(p_n) \rightarrow 0$ as p_n becomes large. Inserting the expression for g from Eq.(2.3.2) in Eq.(2.3.9), we find on implementing the average that

$$\left\langle \int P(p_n, t_n) g \frac{\partial}{\partial p_{n+1}} \delta(p_{n+1} - p_n) dp_n \right\rangle = \frac{d}{dp_{n+1}} \left(\Gamma P(p_{n+1}, t_n) \frac{\partial F}{\partial p_{n+1}} \right) \delta t \tag{2.3.11}$$

Doing the same in Eq.(2.3.10) and retaining the terms only up to $O(\delta t)$, we have

$$\left\langle \int P(p_n, t_n) \frac{g^2}{2} \frac{\partial^2}{\partial p_{n+1}^2} \delta(p_{n+1} - p_n) dp_n \right\rangle = \frac{d^2}{dp_{n+1}^2} \left[\frac{1}{2} 2\Gamma D P(p_{n+1}, t_n) \right] \delta t \tag{2.3.12}$$

Making use of Eqns.(2.3.8), (2.3.11) and (2.3.12) the evolution of Eq.(2.3.5) reads

$$P(p_{n+1}, t_{n+1}) = P(p_{n+1}, t_n) + \frac{d}{dp_{n+1}} \left(\Gamma P(p_{n+1}, t_n) \frac{\partial F}{\partial p_{n+1}} \right) \delta t + \frac{d^2}{dp_{n+1}^2} \left(\Gamma D P(p_{n+1}, t_n) \right) \delta t \quad (2.3.13)$$

Absorbing the Γ to simplify redefine the time scale, the differential equation for $P(p, t)$ reads,

$$\frac{\partial P(p, t)}{\partial t} = \frac{\partial}{\partial p} \left[\frac{\partial F}{\partial p} P \right] + D \frac{\partial^2}{\partial p^2} P \quad (2.3.14)$$

This is the Fokker-Planck equation that corresponds to the Langevin equation of Eq.(2.2.21). What about the equilibrium distribution P_{eq} ? This is obtained by setting

$$\frac{\partial}{\partial t} P_{eq} = 0$$

and immediately leads to

$$P_{eq} \sim e^{-F/D} \quad (2.3.15)$$

This establishes what we mentioned at the end of the last section. The Langevin equation given in Eq.(2.2.21) has associated with it an equilibrium distribution.

We have thus far discussed Langevin and Fokker-Planck equations for a single variable. As discussed in the introduction, mesoscopic systems which will be our concern are described by fields $\phi(\vec{r}, t)$ rather than a single variable. Instead of a function F of a single variable for the equilibrium distribution, as we had before, we would have to consider a functional $F(\phi(\vec{r}, t))$. If we are discussing critical phenomena, this functional would be the *Ginzburg-Landau* free energy functional (for a scalar field ϕ)

$$F(\phi(\vec{r}, t)) = \int d^D r \left[\frac{m^2}{2} \phi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{\lambda}{4} \phi^4 \right] \quad (2.3.16)$$

In the above λ is a coupling constant and

$$m^2 \propto (T - T_c),$$

where T_c is the critical temperature. The Langevin equation corresponding to Eq.(2.2.21) is now given by

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\delta F}{\delta \phi} + N$$

$$\langle N(\vec{r}_1, t_1) N(\vec{r}_2, t_2) \rangle = 2\Gamma D \delta(t_1 - t_2) \delta(\vec{r}_2 - \vec{r}_1) \quad (2.3.17)$$

Notice that instead of the ordinary derivative, we have a functional derivative in Eq.(2.3.17), the Markovian property extends to space as well - the noise is not correlated in time or space. For completeness, we provide the definition of the functional derivative as

$$\frac{\delta F}{\delta \phi} = \frac{\partial F}{\partial \phi} - \sum_i \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial (\frac{\partial \phi}{\partial x_i})} \right) + \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{\partial F}{\partial (\frac{\partial^2 \phi}{\partial x_i \partial x_j})} \right) + \dots \quad (2.3.18)$$

Instead of working with the field $\phi(\vec{r}, t)$, we can work with its Fourier components $\phi_k(t)$

$$\phi(\vec{r}, t) = \frac{1}{(2\pi)^{D/2}} \int d^D k e^{i\vec{k}\cdot\vec{r}} \phi_{\vec{k}}(t) \quad (2.3.19)$$

The Langevin equation for $\phi_k(t)$ now reads

$$\begin{aligned} \frac{\partial \phi_{\vec{k}}}{\partial t} &= -\Gamma \frac{\partial F}{\partial \phi_{-\vec{k}}} + N_{\vec{k}} \\ \text{with} \quad \langle N_{\vec{k}_1}(t_1) N_{\vec{k}_2}(t_2) \rangle &= 2D\Gamma \delta(\vec{k}_1 + \vec{k}_2) \delta(t_1 - t_2) \end{aligned} \quad (2.3.20)$$

Associated with the Langevin equations of Eq.(2.3.17) or Eq.(2.3.20) is the probability distribution $e^{-F/D}$. We now ask what is the associated Fokker-Planck equation. This is easiest to answer when the Langevin equations are cast in the form of Eq.(2.3.20). Instead of a single variable Langevin equation, we now have Langevin equations for an infinite number of variables ϕ_k . The Fokker-Planck equation is that for $P(\{\phi_k\}, t)$, where P is the probability of each of the variables ϕ_k acquiring the value ϕ_k at time t . The derivation of the evolution equation parallels the one we have given already and the final result is

$$\frac{\partial P(\{\phi_k\}, t)}{\partial t} = \sum_{\vec{k}} \frac{\partial}{\partial \phi_{\vec{k}}} \left[P \frac{\partial F}{\partial \phi_{-\vec{k}}} \right] + D \sum_{\vec{k}} \frac{\partial^2 P}{\partial \phi_{\vec{k}} \partial \phi_{-\vec{k}}} \quad (2.3.21)$$

As expected, this Fokker-Planck equation does support the equilibrium distribution $e^{-F/D}$.

We can write down Langevin equations for fields which cannot be cast in the form of Eq.(2.3.17), i.e. the drive on the R.H.S does not have the structure $\frac{\delta F}{\delta \phi}$. In this case the existence of an equilibrium state is not assured. A non-equilibrium steady state can, however, exist.

2.4 Dynamics of a Magnet near Its Critical Point

In this section we will be setting up the equation of motion that is appropriate for describing the dynamics of a Heisenberg ferromagnet. At first, we look at the statics. The order parameter for the transition is the three dimensional magnetization

vector $\phi(\vec{r}, t)$ with the components ϕ_1, ϕ_2 and ϕ_3 and the free energy functional determining the probability distribution is

$$F = \int d^D r \left[\frac{m^2}{2} \phi_i \phi_i + \frac{1}{2} (\vec{\nabla} \phi_i) \cdot (\vec{\nabla} \phi_i) + \frac{\lambda}{4} (\phi_i \phi_i)^2 \right] \quad (2.4.1)$$

The Langevin equation is, as recommended in Eq.(2.3.17)

$$\dot{\phi}_i = -\Gamma \frac{\delta F}{\delta \phi_i} + N_i \quad (2.4.2)$$

$$\langle N_i(\vec{r}_1, t_1) N_j(\vec{r}_2, t_2) \rangle = 2\Gamma (K_B T) \delta_{ij} \delta(t_1 - t_2) \delta(\vec{r}_1 - \vec{r}_2)$$

The noise comes from an averaging over very short wavelength fluctuations. The equations of motion, as written down in Eq.(2.4.2), have a relaxational part and a stochastic part. We now have to worry about a conservation law associated with the ferromagnet. If we consider the total magnetization $\int d^D r \phi$, then that has to be conserved in time. The dynamics as written down in Eq.(2.4.2) does not conserve the order parameter. For the conserved order parameter,

$$\dot{\phi}_i = \Gamma \nabla^2 \frac{\delta F}{\delta \phi_i} + N_i \quad (2.4.3)$$

$$\text{and } \langle N_i(\vec{r}_1, t_1) N_j(\vec{r}_2, t_2) \rangle = -2\Gamma \nabla^2 (K_B T) \delta_{ij} \delta(t_1 - t_2) \delta(\vec{r}_1 - \vec{r}_2)$$

The term ∇^2 acts like the divergence of a current and hence provides the conservation law. For both Eqns.(2.4.2) and (2.4.3), we have the equilibrium distribution given by $e^{-F/K_B T}$, the distribution which correctly gives the statics. The relations between the stochastic and dissipative parts in Eqns.(2.4.2) and (2.4.3) are called fluctuation dissipation relations (FDR). In dealing with the statics, we will assume that the static effects have been taken into account already and accordingly the mass m^2 will be identified with the inverse correlation length and assigned the correct critical behaviour, i.e.

$$m^2 = \frac{1}{\xi^2} = \xi_0^{-2} t^{2\nu} = \xi_0^{-2} \left(\frac{T - T_c}{T_c} \right)^{2\nu} \quad (2.4.4)$$

with the exponent ν given by

$$\nu = \frac{1}{2} + \frac{1}{4} \frac{n+2}{n+8} \epsilon + O(\epsilon^2) \quad (2.4.5)$$

where n is the number of components of the order parameter field ($n=3$ for our ferromagnet) and $\epsilon = 4 - D$, where D is the dimensionality of space. The susceptibility at $T = T_c$, will be given by

$$\chi^{-1} = k^{2-\eta} \quad (2.4.6)$$

$$\text{with } \eta = \frac{1}{2} \frac{n+2}{(n+8)^2} \epsilon^2 + O(\epsilon^3) \quad (2.4.7)$$

Our Eq.(2.4.2) constitutes what *Halperin and Hohenberg* describes as the *model A* of critical dynamics, while our Eq.(2.4.3) constitutes their *model B*. These models comprise a relaxational and a stochastic part. However, physical considerations dictate another term in the equation of motion. The magnetic moment experiences a torque due to the local field $\vec{h}(\vec{r})$ and that causes the magnetization $\vec{\phi}_i$ to change as

$$\dot{\vec{\phi}} = \tilde{g}\vec{\phi} \times \vec{h} \quad (2.4.8)$$

The local field $\vec{h}$ is produced by the local magnetization $\vec{\phi}$ and we have the expansion

$$\vec{h} = a_1\vec{\phi} + a_2\nabla^2\vec{\phi} + \dots \quad (2.4.9)$$

Inserting in Eq.(2.4.8)

$$\begin{aligned} \dot{\vec{\phi}} &= a_2\tilde{g}\vec{\phi} \times \nabla^2\vec{\phi} \\ &= g\vec{\phi} \times \nabla^2\vec{\phi} \end{aligned} \quad (2.4.10)$$

It should be noted that all other terms in the expansion of Eq.(2.4.9) would give terms which would be irrelevant for the long wavelength behaviour. In the above g is a coupling constant.

Knowing that the relaxational part of the equation of motion has to preserve the total magnetization, we write down the full equation of motion for the ferromagnet as

$$\dot{\vec{\phi}} = g\vec{\phi} \times \nabla^2\vec{\phi} + \Gamma\nabla^2[(m^2 - \nabla^2)^2 + \lambda\phi^2]\vec{\phi} + \vec{N} \quad (2.4.11)$$

The first term on the R.H.S. is known as the streaming term or reversible term in the equation of motion. The obvious question is what does the streaming term do to the equilibrium distribution. The answer is that the equilibrium distribution is not affected by the streaming term. This is easy to see on constructing $\frac{dF}{dt}$ using Eqns.(2.4.1) and (2.4.10). We find

$$\frac{dF}{dt} = 0$$

since

$$\vec{\phi} \cdot \dot{\vec{\phi}} = g\vec{\phi} \cdot (\vec{\phi} \times \nabla^2\vec{\phi})$$

is trivially zero. Thus the streaming term does not disturb the equilibrium distribution. The model we have just described is model *J* in the notation of *Halperin and Hohenberg*.

We now turn to a different critical system where we will find that we need two Langevin equations. This is the liquid-gas critical point or the consolute point in a binary liquid. The liquid-gas critical point is the usual Van der Waals critical point

which is very familiar. The transition at the consolute point of a binary liquid is similar to the liquid-gas critical point. Above the consolute point, the two liquids in the binary mixture are completely miscible while below the critical temperature they separate and a meniscus is formed. The order parameter for the transition is the scalar field $\phi(\vec{r}, t)$ which is the fluctuation of the local concentration of one of the binary liquid components from the concentration required for complete miscibility. As in the case of the ferromagnet, the statics is described by the free energy,

$$F = \int d^D r \left[\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} (\phi)^4 \right] \quad (2.4.12)$$

There is a conservation law for concentration and hence the equation of motion for the order parameter is

$$\dot{\phi} = \Gamma \nabla^2 \frac{\delta F}{\delta \phi} + N \quad (2.4.13)$$

Now comes the question whether the above equation is complete. One is discussing fluids and fluctuations, so there would be fluctuations in the velocity - although there is no overall flow there will always be a fluctuating velocity field and the velocity will carry the concentration field, so that the above equation will become

$$\dot{\phi} = -(\vec{v} \cdot \vec{\nabla}) \phi + \Gamma \nabla^2 \frac{\delta F}{\delta \phi} + N \quad (2.4.14)$$

But now we have coupled to an additional field in the streaming term $(\vec{v} \cdot \vec{\nabla}) \phi$ [this additional term did not come in the ferromagnet] the free energy expression of Eq.(2.4.12) has to be modified to include the kinetic energy $\frac{1}{2} \rho v^2$ and we have

$$\tilde{F} = \int d^D r \left[\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} (\phi)^4 + \frac{1}{2} \rho v^2 \right] \quad (2.4.15)$$

The equation of motion in Eq.(2.4.14) simply becomes

$$\dot{\phi} = -(\vec{v} \cdot \vec{\nabla}) \phi + \Gamma \nabla^2 \frac{\delta \tilde{F}}{\delta \phi} + N \quad (2.4.16)$$

How about the equation of motion for v ? We expect

$$\dot{v}_i = \text{Streaming term} + \nu \nabla^2 \frac{\delta \tilde{F}}{\delta v_i} + N_i^v \quad (2.4.17)$$

What determines the streaming term? The streaming term has to be determined so that

$$\frac{d\tilde{F}}{dt} = 0$$

and this gives

$$\dot{v}_i = (m^2 - \nabla^2)\phi \frac{\partial \phi}{\partial x_i} + v \nabla^2 \frac{\delta \tilde{F}}{\delta v_i} + N_i^v \quad (2.4.18)$$

The critical dynamics of a liquid-gas system or a binary liquid is governed by Eqns.(2.4.16) and (2.4.18). They constitute the *model H* of Hohenberg and Halperin.

To end this section, we discuss the linearized form of *models A and B*. These have the form

$$\dot{\phi}_i = -\Gamma(m^2 - \nabla^2)\phi_i + N_i \quad (2.4.19)$$

and

$$\dot{\phi}_i = \Gamma \nabla^2(m^2 - \nabla^2)\phi_i + N_i \quad (2.4.20)$$

These equations are best handled in momentum space, where they read

$$\dot{\phi}_i(k) = -\Gamma(m^2 + k^2)\phi_i(k) + N_i(k) \quad (2.4.21)$$

and

$$\dot{\phi}_i(k) = -\Gamma k^2(m^2 + k^2)\phi_i(k) + N_i(k) \quad (2.4.22)$$

Straightforward calculation shows that for a system in thermal equilibrium at $t = 0$, correlations decay in the two cases as

$$\langle \phi_i(k, t_2) \phi_i(-k, t_1) \rangle = \frac{1}{k^2 + m^2} e^{-\Gamma(k^2 + m^2)|t_2 - t_1|} \quad (2.4.23)$$

$$\langle \phi_i(k, t_2) \phi_i(-k, t_1) \rangle = \frac{1}{k^2 + m^2} e^{-\Gamma k^2(k^2 + m^2)|t_2 - t_1|} \quad (2.4.24)$$

The characteristic relaxation frequency for the two cases are

$$\omega_A = \Gamma(k^2 + m^2) \quad (2.4.25)$$

$$\omega_B = \Gamma k^2(k^2 + m^2) \quad (2.4.26)$$

At the critical point (i.e. $m = 0$), the relaxation rate is determined by the wavenumber - the longer the wavelength, the longer the relaxation time - this is known as critical slowing down. The critical slowing down is determined by the dynamic critical exponent

$$\omega \propto k^z \quad (2.4.27)$$

for linearized *model A* (see Eq.(2.4.19))

$$z = 2 \quad (2.4.28)$$

and for the linearized *model B* (Eq.(2.4.20))

$$z = 4 \quad (2.4.29)$$

The nonlinear contributions to the equation of motion for models A and B will change the dynamic scaling exponents. We will discuss this in the next chapter. The dynamics with the streaming terms included will further change the dynamics. Notice that away from the critical point ($m \neq 0$), the relaxation frequency can be written as

$$\omega = k^z f(k\xi) \quad (2.4.30)$$

where $\xi = m^{-1}$ for the non-conserved case Eq.(2.4.28). This is the dynamic scaling hypothesis, which holds in general. For the conserved case,

$$\omega = k^2 k^{z-2} g(k\xi), \quad (2.4.31)$$

where, the k^2 is a consequence of the conservation law.

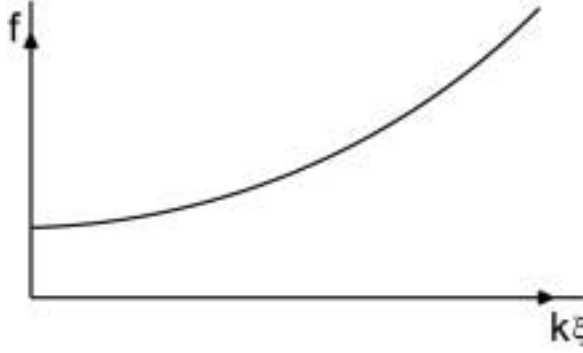


Figure 2.1. Scaling plot for the temperature and wave number dependent relaxation rate of the density fluctuations near the liquid gas critical point (Phys Rev A28 2486 (1973)).

The function $f(x)$ has the behaviour

$$f(x) \rightarrow \text{constant} \quad \text{for} \quad x \rightarrow \infty$$

while

$$f(x) \sim x^{-z} \quad \text{for} \quad x \ll 1.$$

The function $g(x)$ has the behaviour

$$g(x) \rightarrow \text{constant} \quad \text{for} \quad x \rightarrow \infty$$

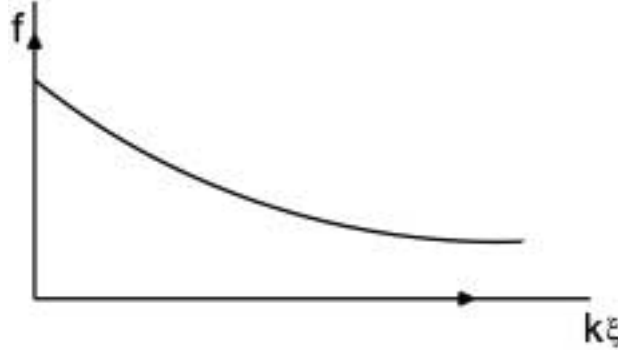


Figure 2.2. Scaling plot for the relaxation rate of magnetic fluctuations near the Curie point (Phys Rev Lett 24 514 (1970)).

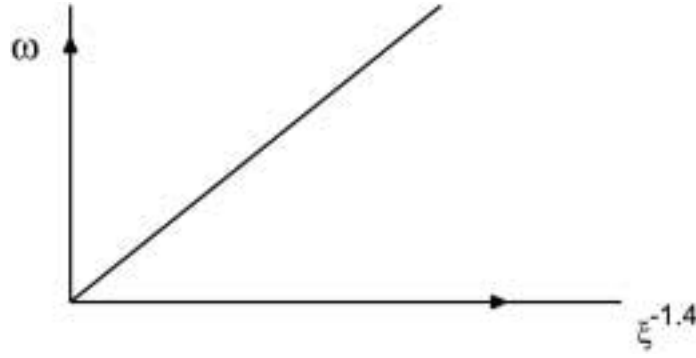


Figure 2.3. Relaxation rate of the order parameter fluctuations as a function of the wave number at the Neel point (Phys Rev B4 3204 (1971))

while

$$g(x) \rightarrow x^{z-2} \quad \text{as} \quad x \rightarrow 0$$

In Eq.(2.4.23) $f(x) = 1 + \frac{1}{x^2}$ with $z = 2$, and in Eq.(2.4.24) $g(x) = 1 + \frac{1}{x^2}$ with $z = 4$. The most significant change in z comes from the inclusion of the streaming terms (also known as the mode coupling terms). For the isotropic ferromagnet (conserved order parameter) z changes from $z = 4$ to $z = 2.5$ while for the liquid-gas system (once again conserved order parameter) z changes from $z = 4$ to $z = 3.068$. The confrontation between theory and experiment is less than one part in thousand. The technique of calculating the exponents will be explained in **Chapters 3** and **4**. For an antiferromagnet (where the order parameter is not conserved), z changes from $z = 2$ to $z = 1.5$. To end this section, we see the spectacular data collapse of scaling for the liquid-gas system in Fig. 2.1, the scaling function for the magnet in Fig. 2.2 and the determination of z for the antiferromagnet in Fig. 2.3.

In **Chapters 4** and **5**, we take up the detailed study of the dynamics of the systems discussed here.

2.5 Systems not in Equilibrium

In this section, we discuss systems derived from those written down in section (2.4), but where fluctuation dissipation relations do not hold. We begin by writing down model *B* for a scalar field ϕ as

$$\frac{\partial \phi(\vec{r}, t)}{\partial t} = \Gamma \{ (m^2 - \nabla^2) \nabla^2 \phi + \lambda \nabla^2 \phi^3 \} + N \quad (2.5.1)$$

$$\text{with } \langle N(\vec{r}_1, t_1) N(\vec{r}_2, t_2) \rangle = -2\Gamma(k_B T) \nabla^2 \delta(\vec{r}_1 - \vec{r}_2) \delta(t_1 - t_2) \quad (2.5.2)$$

The above equation is explicitly in the form of a conservation law

$$\frac{\partial \phi}{\partial t} = -\vec{\nabla} \cdot \vec{j} \quad (2.5.3)$$

with

$$\vec{j} = -\vec{\nabla} \frac{\delta F}{\delta \phi} + \vec{\xi} \quad (2.5.4)$$

To take the system away from equilibrium, the simplest thing to do is to add a driving field. This is the model of Katz, Lebowitz and Spohn. The driving field has two main characteristics

- *i) adding to the current a term proportional to the applied field and*
- *ii) breaking the symmetry among the different spatial directions - the magnitude of the applied field is different in the directions transverse to the field*

Further, it would be very difficult for a fluctuation which is strongly correlated to its surrounding to feel the effect of the driving force. Consequently, if $\phi(\vec{r}) = \pm 1$, we want the current to vanish and thus the assumed form of the current is

$$\vec{j}_\epsilon = (1 - \phi^2) \vec{\epsilon} \quad (2.5.5)$$

We consider the total current as $\vec{j} + \vec{j}_\epsilon$ and use the derivative in Eq.(2.5.2) remembering that there has to be anisotropy in space. The derivative parallel to $\vec{\epsilon}$ will be denoted by ∂ and the derivative in the transverse direction is $\nabla_\perp$ and the equation of motion is

$$\begin{aligned} \frac{\partial \phi}{\partial t} = & \Gamma \{ (m_\perp^2 - \alpha_\perp \nabla_\perp^2) \nabla_\perp^2 \phi + (m_\parallel^2 - \alpha_\parallel \partial^2) \partial^2 \phi \\ & - 2\beta \partial^2 \nabla_\perp^2 \phi + \lambda (\nabla_\perp^2 \phi^3 + \partial^2 \phi^3) + \epsilon \partial \phi^2 \} \\ & - (\vec{\nabla}_\perp \cdot \vec{\xi}_\perp + \partial \xi_\parallel) \end{aligned} \quad (2.5.6)$$

with

$$\begin{aligned} \langle (\vec{\nabla}_\perp \cdot \vec{\xi}_\perp) (\vec{\nabla}'_\perp \cdot \vec{\xi}_\perp) \rangle &= N_\perp (-\nabla_\perp^2) \delta(\vec{x} - \vec{x}') \delta(t - t') \\ \langle \partial \xi_\parallel \partial \xi'_\parallel \rangle &= N_\parallel (-\partial^2) \delta(\vec{x} - \vec{x}') \delta(t - t') \end{aligned} \quad (2.5.7)$$

It is possible that Eqns.(2.5.6) and (2.5.7) might describe an anisotropic equilibrium system but in that case FDR must be valid. In this case at least $\frac{N_\perp}{N_\parallel} = \frac{m_\perp^2}{m_\parallel^2}$ must hold.

In general, however, we will have

$$\frac{N_\perp}{N_\parallel} \neq \frac{m_\perp^2}{m_\parallel^2} \quad (2.5.8)$$

This inequality can be regarded as a signal for violation of FDR. The consequences of FDR violation will be further pursued in chapter V. One of the interesting puzzles of the driven system is which mass ($m_\perp^2$ or $m_\parallel^2$) approaches zero at the phase transition. One could have

- i) $m_\perp^2 \rightarrow 0$, $m_\parallel^2 > 0$
- ii) $m_\perp^2 > 0$, $m_\parallel^2 \rightarrow 0$
- iii) $m_\perp^2 \rightarrow 0$, $m_\parallel^2 \rightarrow 0$

We now turn to a different out of equilibrium problem that involves a sudden temperature change that takes a system from a homogeneous phase to a temperature appropriate to a phase separated state. To focus one's ideas it helps to consider a ferromagnetic Ising model. The Ising spins can only point up or down and at $t = 0$ the system is at a temperature $T > T_c$, where it is totally disordered with as many spins pointing up as are down. Suddenly at $t = 0$, the temperature is lowered to $T_f < T_c$, where there can be two possible equilibrium phases with magnetization $\pm M_0$. Immediately after the quench, the system is in an unstable disordered state and as time goes on it has to evolve towards the final equilibrium state. The problem is interesting because the largest relaxation time diverges with the system size in the ordered phase and in the thermodynamic limit the equilibrium state is never reached. A network of domains of the equilibrium phases appear and the typical domain size increases with time. There is scaling in the sense that the domain patterns at later times are statistically similar to the domain patterns at earlier times. A related phenomenon, studied by metallurgists, is the spinodal decomposition of binary alloys. A binary alloy AB can be thought of as Ising spins with the difference that spin-flip which is allowed in the Ising case is not allowed for the binary alloy in which flipping would change A to B or vice-versa but there is a conservation law for each of the species. So while the Ising system corresponds to model A, the binary alloy corresponds to model B.

We start with the *Ginzburg-Landau* free energy which in the ordered phase can be written as

$$F = \int d^D r \left[\frac{1}{2} (\nabla \phi)^2 + V(\phi) \right] \quad (2.5.9)$$

The potential $V(\phi)$ has double well structure and will be taken to be

$$V(\phi) = (1 - \phi^2)^2 \quad (2.5.10)$$

For the non-conserved ϕ -field

$$\dot{\phi} = -\Gamma \frac{\delta F}{\delta \phi} = \Gamma \left[\nabla^2 \phi - V'(\phi) \right] \quad (2.5.11)$$

while for the conserved ϕ -field

$$\dot{\phi} = -\Gamma \nabla^2 \left[\nabla^2 \phi - V'(\phi) \right] \quad (2.5.12)$$

The absence of the noise means that we are effectively working at $T = 0$. This means that the final temperature T_f of the quench is an irrelevant variable and the system behaves as if it were at $T = 0$. By the same token the initial temperature can be taken to be $T = \infty$ i.e. the system at $t = 0$ is completely disordered. So the problem that we need to solve is to find the solution of Eqns.(2.5.11) or (2.5.12) under the initial condition

$$\langle \phi(\vec{x}_1, 0) \phi(\vec{x}_2, 0) \rangle = \delta(\vec{x}_1 - \vec{x}_2) \quad (2.5.13)$$

If the order parameter field ϕ is a vector, then additional complications can occur. In the ordered phase, the vector magnetization can point in different directions in different regions of space and hence singular lines (vortex lines) can be formed where the direction is not well defined. These are the topological defects. For the scalar field that we have talked about the defect is the domain wall - the boundary between $+M_0$ and $-M_0$.

The scaling hypothesis for the problem is the existence of a characteristic length $L(t)$ such that the domain structure (in a statistical sense) is independent of time when scaled by $L(t)$. Two commonly calculated correlation functions are

$$C(\vec{r}, t) = \langle \phi(\vec{x} + \vec{r}, t) \phi(\vec{x}, t) \rangle \quad (2.5.14)$$

and its Fourier transform

$$S(\vec{k}, t) = \langle \phi_k(t) \phi_{-k}(t) \rangle \quad (2.5.15)$$

The existence of a characteristic length $L(t)$ implies the scaling forms

$$C(\vec{r}, t) = M^2 f(r/L) \quad (2.5.16)$$

and

$$S(k, t) = M^2 L^D g(kL) \quad (2.5.17)$$

In the above, M is the equilibrium magnetization. The structure factor can be probed by scattering measurements. There is another structure factor

$$C(\vec{r}, t, t') = \langle \phi(\vec{x} + \vec{r}, t) \phi(\vec{x}, t') \rangle \quad (2.5.18)$$

The scaling form for this correlation function would be

$$C(\vec{r}, t, t') = M^2 f\left(\frac{r}{L}, \frac{r}{L'}\right) \quad (2.5.19)$$

where $L = L(t)$ and $L' = L(t')$. In the limit $L \gg L'$, the above correlation function becomes

$$C(\vec{r}, t, t') \sim M^2 \left(\frac{L'}{L}\right)^{\bar{\lambda}} h\left(\frac{r}{L}\right) \quad (2.5.20)$$

For $\vec{r} = 0$, the autocorrelation function behaves as $(\frac{L'}{L})^{\bar{\lambda}}$. The exponent $\bar{\lambda}$ is a non-trivial exponent in phase ordering kinetics.

To end this section, we will discuss the role of the domain wall in this phase ordering dynamics. To do so, we consider the simplest situation - a flat wall as shown in Fig. 2.1. The wall is at $z = 0$. The field $\phi(\vec{r}) = +1$ for $z \rightarrow \infty$ and $\phi(\vec{r}) = -1$ for $z \rightarrow -\infty$. The field is zero on the wall. The profile of the order parameter field is obtained from Eq.(2.5.11) as

$$\begin{aligned} \frac{d^2 \phi}{dz^2} &= V'(\phi) \\ \text{or } \frac{1}{2} \left(\frac{d\phi}{dz} \right)^2 &= V(\phi) \end{aligned} \quad (2.5.21)$$

The energy per unit area of the wall is the surface tension σ and is given by

$$\sigma = \int_{-\infty}^{\infty} dz \left(\frac{d\phi}{dz} \right)^2 = 2 \int_{-\infty}^{\infty} [V(\phi)] dz \quad (2.5.22)$$

For $\phi^2 \sim 1$, the solution can be written down as

$$1 \pm \phi = e^{-[V''(\pm 1)]^{1/2} |z|} \quad (2.5.23)$$

as $|z| \rightarrow \infty$. The order parameter saturates exponentially away from the wall. The existence of a surface tension implies a force proportional to the local curvature at every point of the wall. Consider a three dimensional spherical bubble of radius R . If the force per unit area is F , then the work done by the force in decreasing the surface area is $4\pi R^2 F dR$. The decrease in surface energy is $8\pi R \sigma dR$. Equating

the two, the force F is $2\sigma/R$. For model A, this force causes the walls to move with a velocity proportional to the local curvature. Thus

$$\eta \frac{dR}{dt} = -2 \frac{\sigma}{R} \quad (2.5.24)$$

For arbitrary dimension D , 2 gets replaced by $D - 1$. This picture allows us to give an intuitive picture of the growth law for $L(t)$. If there is a single characteristic length scale L , then the R.H.S of Eq.(2.5.24) goes as $\frac{dL}{dt}$. The L.H.S goes as L^{-1} . Equating and integrating $L \sim t^{1/2}$.

For the conserved order parameter, the arguments are somewhat more complicated. We will return to it in **Chapter 5**.

2.6 Models of Growth

In this section, we will discuss some models of growth by deposition which have, in the last few years transcended the purpose they were invented for and became important prototypes for problems in different areas in physics and mathematics. The simplest problem in this genre is the problem of crystal growth by depositing atoms in the form of an atomic beam on a substrate. The substrate is D -dimensional and growth occurs in the z -direction. At any time t , the growth is characterized by the scalar field $h(\vec{r}, t)$ where h is the height at the point $\vec{r}$ on the substrate. One of the interesting questions is whether the surface is rough or smooth. To answer that, one looks at the correlation function

$$C(\vec{r}, t) = \langle h(\vec{x} + \vec{r}, t) h(\vec{x}, t) \rangle \quad (2.6.1)$$

This correlation function has the scaling form

$$C(\vec{r}, t) = r^{2\alpha} f\left(\frac{r}{2\xi(t)}\right) \quad (2.6.2)$$

For $\alpha > 0$, this correlation diverges as the separation increases and the surface is deemed to be rough.

It is the random fluctuations f in the beam intensity $I(\vec{r}, t)$ (beam comes in the z -direction) that cause the surface to become rough. The fluctuations $f(\vec{r}, t)$ are uncorrelated in space and time. Thus,

$$\langle f(\vec{r}, t) f(\vec{r}', t') \rangle = D_0 \delta(\vec{r} - \vec{r}') \delta(t - t') \quad (2.6.3)$$

where D_0 is approximately the square of the average beam intensity. In the absence of atomic movement on the surface, the surface would be rough. But there can be

- *i) evaporation or deposition from the surface*
- *ii) surface diffusion*

and these effects tend to smooth out the surface. Thus one has a competition between and randomizing effects.

It is important to show that the effects one is discussing are significant. To do so, we calculate the effect of fluctuations in the beam intensity for a beam of thickness h on a substrate $R \times R$. The number of atoms deposited is $R^2 h$. The fluctuation in the number is $\sqrt{R^2 h}$. The fluctuation in the height is thus approximately

$$\delta h = \frac{(R^2 h)^{1/2}}{R^2} = \frac{h^{1/2}}{R} \quad (2.6.4)$$

For $h = 100$ on a base size of 100×100 atoms, fluctuation is 0.1 atomic unit. This is sizable.

We now turn to the smoothening dynamics.

i) Evaporation Dynamics

In this case it is the chemical potential difference between the local value on the surface and the average value of the ambient vapour that determines the evaporation rate. The rate at which the local height changes,

$$\dot{h}(\vec{r}, t) = f(\mu(\vec{r}, t) - \bar{\mu}) \quad (2.6.5)$$

The local chemical potential $\mu(\vec{r}, t)$ is determined by the local shape of the surface. The simplest mathematical expression of this is that $\delta\mu = \mu(\vec{r}, t) - \bar{\mu}$ will be determined by the different derivatives $\frac{\partial^n h}{\partial x_i^n}$ and the function $f(\delta\mu)$ can be expanded in the powers of the various derivatives. To fix which terms would be appropriate at the linear order, we note that the linear derivative $\frac{\partial h}{\partial x_i}$ cannot occur because that would imply a possible instability caused simply by changing the orientation of the plane. So the first term that is allowed is $\frac{\partial^2 h}{\partial x_i \partial x_i}$ and we have the linear equation as a diffusion equation

$$\frac{\partial h}{\partial t} = \Gamma \nabla^2 h \quad (2.6.6)$$

To the smoothening dynamics of the above equation, if we simply add a random noise component, we have the simple possible dynamics determining the competition between randomizing and smoothening effects. Thus,

$$\begin{aligned} \frac{\partial h}{\partial t} &= \Gamma \nabla^2 h + f \\ \langle f(\vec{r}, t) f(\vec{r}', t) \rangle &= 2D \delta(\vec{r} - \vec{r}') \delta(t - t') \end{aligned} \quad (2.6.7)$$

This is the Edwards-Wilkinson model for surface growth.

ii) Surface Diffusion

In this case, there is a conservation law in the sense that all the atoms that we

deposited have to be accounted for. This means that we should write the dynamics in the form

$$\frac{\partial h}{\partial t} = -\vec{\nabla} \cdot \vec{j} \quad (2.6.8)$$

where $\vec{j}$ is a current. An equation of this sort has the conservation law built in. The current is determined, as always, by the gradient of the chemical potential

$$\vec{j} = -\vec{\nabla} \mu \quad (2.6.9)$$

With μ determined by $\nabla^2 h$ as discussed in the previous subsection, we can write Eq.(2.6.8) as

$$\frac{\partial h}{\partial t} = \Gamma \nabla^4 h \quad (2.6.10)$$

Once again to get the randomizing effects the simplest thing to do is to have a random part ξ in the current $\vec{j}$, i.e. $\vec{j} = -\vec{\nabla} \mu + \vec{\xi}$ and Eq.(2.6.10) becomes

$$\frac{\partial h}{\partial t} = \Gamma \nabla^4 h + \tilde{f} \quad (2.6.11)$$

$$\text{with } \langle \tilde{f}(\vec{r}, t) \tilde{f}(\vec{r}', t') \rangle = -2D_0 \nabla^2 \delta(\vec{r} - \vec{r}') \delta(t - t')$$

This sets up the Mullins-Sekerka equation.

We will discuss the solutions of these equations in chapter 6 showing the importance of the substrate dimension on whether the surface is rough or smooth. What would be the first nonlinearity in the above equations? Since long distance effects are of importance, we will keep the lowest gradient terms and clearly the most relevant term in $\delta\mu$ should be the term $(\vec{\nabla} h)^2$. This will enlarge the scope of Eq(2.6.7) to

$$\frac{\partial h}{\partial t} = \Gamma \nabla^2 h + \lambda (\nabla h)^2 + f \quad (2.6.12)$$

with the noise correlation remaining the same as before. Similarly, Eq.(2.6.11) will be augmented to

$$\frac{\partial h}{\partial t} = \Gamma \nabla^4 h + \lambda \nabla^2 (\nabla h)^2 + \tilde{f} \quad (2.6.13)$$

with the $\tilde{f}$ correlations same as in Eq.(2.6.11). The non-linear equation shown in Eq.(2.6.12) is the Kardar-Parisi-Zhang (KPZ) equation which has been studied very extensively in the last decade. The model of Eq.(2.6.13) is the continuum version of a discrete model introduced by Wolf and Villain and Das Sarma and Tamborenea.

2.7 Turbulence

In this section, we discuss the classic problem in nonlinear dynamics - turbulence in fluids. The velocity field of a fluid satisfies Navier-Stokes equation which reads

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} P}{\rho} + \nu \nabla^2 \vec{v} \quad (2.7.1)$$

with

$$\vec{\nabla} \cdot \vec{v} = 0 \quad (2.7.2)$$

for an incompressible flow. The above equation needs boundary conditions and initial conditions. If $\vec{v} = 0$ at infinite distances or bounding surface, then Eqns.(2.7.1) and (2.7.2) lead to

$$\frac{\partial}{\partial t} \int \frac{1}{2} v^2 d^3 r = -\nu \int \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} d^3 r \leq 0 \quad (2.7.3)$$

Thus, unless there is an external force the motion ultimately ceases. For maintained turbulence there must exist an external force such that

$$\int \vec{f} \cdot \vec{v} d^3 r = \nu \int \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} d^3 r \quad (2.7.4)$$

This makes possible a steady state. We will assume that the external force puts energy per unit mass into the system at the rate

$$\epsilon = \int \vec{f} \cdot \vec{v} d^3 r.$$

We now turn to a complication exhibited by Eq.(2.7.1). If the nonlinear term dominates (which it does if $R_e = \frac{vL}{\nu} \gg 1$, where v is a characteristic velocity and L a characteristic length), the solutions of Eq.(2.7.1) are chaotic. This means that they are sensitive to initial conditions and that, in turn, means that talking about $v(\vec{r}, t)$ is not sensible. One has to talk about average values. One has to prescribe how to take the averages. One way would be to consider an ensemble where different realizations correspond to different initial conditions and averaging over the realizations gives $\langle v(\vec{r}, t) \rangle$. The specification of the velocity field now requires the knowledge of all the correlation functions - the two point correlation function $\langle v(\vec{x} + \vec{r}, t) v(\vec{x}, t) \rangle$ and the correlation of all the composite operators e.g $\langle v^2(\vec{x} + \vec{r}, t) v^2(\vec{x}, t) \rangle$ etc. The problem of turbulence thus becomes a problem in statistical mechanics. Calculating correlation functions is facilitated by having a stochastic differential equation, as we have had in all our previous examples. To set up a Langevin equation in this case, we split the velocity field into a mean part and a fluctuation

$$\vec{v}(\vec{r}, t) = \langle \vec{v}(\vec{r}, t) \rangle + \vec{u}(\vec{r}, t) = \vec{V}(\vec{r}, t) + \vec{u}(\vec{r}, t) \quad (2.7.5)$$

Substituting Eq.(2.7.5) into Eq.(2.7.1)

$$\frac{\partial \vec{V}}{\partial t} + \frac{\partial \vec{u}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V} + (\vec{V} \cdot \vec{\nabla}) \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{V} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\frac{\vec{\nabla} P}{\rho} + \nu \nabla^2 \vec{V} + \nu \nabla^2 \vec{u} \quad (2.7.6)$$

If we now consider ensemble averages, then

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V} + \langle (\vec{u} \cdot \vec{\nabla}) \vec{u} \rangle = -\left\langle \frac{\vec{\nabla} P}{\rho} \right\rangle + \nu \nabla^2 \vec{V} \quad (2.7.7)$$

This transforms Eq.(2.7.6) to

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\frac{\vec{\nabla} P}{\rho} + \nu \nabla^2 \vec{u} + \vec{T} \quad (2.7.8)$$

where

$$\vec{T} = -(\vec{V} \cdot \vec{\nabla}) \vec{u} - (\vec{u} \cdot \vec{\nabla}) \vec{V} - \langle (\vec{u} \cdot \vec{\nabla}) \vec{u} \rangle + \left\langle \frac{\vec{\nabla} P}{\rho} \right\rangle \quad (2.7.9)$$

The force $\vec{T}$ includes the interaction between the mean flow and the fluctuation and thus contains the mechanism for transferring energy into the fluctuating field. In Eq.(2.7.8) for the fluctuating field, we will replace the force $\vec{T}$ by a fluctuating force $\vec{f}$ which is operative mainly at the large distance scales. Thus,

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\frac{\vec{\nabla} P}{\rho} + \nu \nabla^2 \vec{u} + \vec{f} \quad (2.7.10)$$

with $\vec{\nabla} \cdot \vec{v} = 0$

The random force f will exhibit spatial correlation since it will be operative mainly at the large distance scales and we take

$$\langle f_\alpha(\vec{r}_1, t_1) f_\beta(\vec{r}_2, t_2) \rangle = D_0 P_{\alpha\beta} |\vec{r}_1 - \vec{r}_2|^{y-4} \delta(t_2 - t_1) \quad (2.7.11)$$

where $P_{\alpha\beta}$ is the projection operator

$$P_{\alpha\beta} = \delta_{\alpha\beta} - \frac{\partial_\alpha \partial_\beta}{\nabla^2} \quad (2.7.12)$$

It is more usual to write the above in momentum space, where for the Fourier component $\vec{u}(\vec{k}, t)$, we have the equation of motion in the form

$$\frac{\partial}{\partial t} u_\alpha(\vec{k}, t) = \sum_{\vec{p}} M_{\alpha\beta\gamma}(k) u_\beta(\vec{p}) u_\gamma(\vec{k} - \vec{p}) - \nu k^2 u_\alpha + f_\alpha \quad (2.7.13)$$

where

$$M_{\alpha\beta\gamma}(k) = i[k_\beta P_{\alpha\gamma}(k) + k_\gamma P_{\alpha\beta}(k)] \quad (2.7.14)$$

with the projection operator

$$P_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \quad (2.7.15)$$

and the noise correlation

$$\langle f_\alpha(\vec{k}_1, t_1) f_\beta(\vec{k}_2, t_2) \rangle = \frac{D_0}{k^{D-4+y}} P_{\alpha\beta}(k) \delta(\vec{k}_1 + \vec{k}_2) \delta(t_2 - t_1) \quad (2.7.16)$$

It is the properties of Eqns.(2.7.13) - (2.7.16), that we will be investigating in **Chapter 8**. Here we discuss what is the phenomenology that one would like to reproduce.

The phenomenology of homogeneous isotropic turbulence was formulated by Kolmogorov more than fifty years ago. The important thing to do first is to recognize the existence of three different regimes:

- i) *a long wavelength regime, where the energy needed to maintain the turbulence is inserted. The wave vectors in this regime are of the order of L^{-1} , where L is the characteristic size of the system.*
- ii) *a short wave length regime, where the energy inserted by the stirring forces is dissipated by molecular forces. This length scale is formed out of the energy input rate ϵ and the molecular viscosity ν and is given by k_D^{-1} , where $k_D = (\epsilon/\nu^3)^{1/4}$. The short wavelength region corresponds to wavenumbers $k \geq O(k_D)$.*
- iii) *an inertial range, corresponding to wavenumbers k such that $\frac{1}{L} \ll k \ll k_D$. In this range of wavelengths, the boundaries and the energy input mechanism or the dissipative mechanisms and molecular viscosity do not affect the dynamics which is consequently expected to be universal. In this inertial range the energy cascades from one scale to another at the constant rate ϵ . We note that $k_D L = (\epsilon/\nu^3)^{1/4} L \sim (\frac{\nu L}{\epsilon})^{3/4}$, implying a large inertial range for $Re \gg 1$.*

The Kolmogorov phenomenology deals with the supposedly universal inertial range, where it predicts the correlation functions by a dimensional analysis. The two point structure factor,

$$C(k) = \langle v(\vec{k}, t) v(-\vec{k}, t) \rangle \quad (2.7.17)$$

is the most commonly investigated one. From $C(k)$, we construct the energy spectrum by the relation

$$E = \int C(k) d^3k = \int E(k) dk \quad (2.7.18)$$

In the above E is the total energy per unit mass with dimensions L^2/T^2 . The quantity $E(k)$ is called the energy spectrum and has the dimension $E(k) \sim L^3/T^2$. The universality of the inertial range requires that $E(k)$ needs to be determined by ϵ and k alone and dimensional analysis immediately yields

$$E(k) = C_k \epsilon^{2/3} k^{-5/3} \quad (2.7.19)$$

where C_k is a universal dimensionless constant. In coordinate space, this implies that the two-point function has the scaling behaviour

$$\langle [v(\vec{x} + \vec{r}) - v(\vec{x})]^2 \rangle = C_2 \epsilon^{2/3} r^{2/3} \quad (2.7.20)$$

Can one actually probe the correlation function of Eq.(2.7.20) or equivalently the energy spectrum of Eq.(2.7.19)? Unlike the accurate measurements that can be carried out in critical phenomena, the early measurements in turbulence that were carried out as late as the nineteen seventies, have a *certain real world charm*. The first requirement is high Reynold's number which we have already noticed is VL/ν . It is difficult to raise the Reynold's number by increasing the speed. So one depends on L . This is why the experiments are carried out in large water bodies like rivers or sea inlets. The classic study of **Grant, Stuart and Moillet** was set near Vancouver Island in Canada and the data were taken at different points of the Campbell river. The data at different Reynolds number show the same slope for $E(k)$ and the slope is very close to $-5/3$.

For the higher order structure factors, this dimensional analysis leads to

$$\langle |v(\vec{x} + \vec{r}) - v(\vec{x})|^p \rangle = C_p \epsilon^{p/3} r^{p/3} \quad (2.7.21)$$

For the third order structure factor

$$\langle |v(\vec{x} + \vec{r}) - v(\vec{x})|^3 \rangle = -\frac{4}{5} \epsilon r \quad (2.7.22)$$

The Kolmogorov spectrum was considered correct when confronted with early experimental data, but later more accurate determination of the higher order structure factors showed clear cut departures from Kolmogorov scaling. The dissipation rate which was assumed to be constant in the Kolmogorov picture, showed strong intermittent fluctuations when looked at closely. This phenomenon is called *intermittency*. It shows up the departure from Kolmogorov scaling in a particularly transparent matter. The energy dissipation rate ϵ can be written as $v^2/t \sim v^3/L$. If we define a local dissipation rate $\epsilon(\vec{x})$ by

$$\epsilon(\vec{x}) = \nu \int_{\text{over a ball surrounding } \vec{x}} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 d^3r \quad (2.7.23)$$

then the dissipation correlation function $k_{\epsilon\epsilon}(r)$ is defined as

$$k_{\epsilon\epsilon}(r) = \langle \epsilon(\vec{x} + \vec{r}) \epsilon(\vec{x}) \rangle \quad (2.7.24)$$

With $\epsilon \sim v^3/L$, the Kolmogorov scaling of Eq.(2.7.21) would imply $k_{\epsilon\epsilon}(r)$ is independent of r . Consequently with

$$k_{\epsilon\epsilon}(r) \sim r^{-\mu} \quad (2.7.25)$$

where μ is called the *intermittency exponent*, Kolmogorov scaling gives $\mu = 0$. A non-zero value of μ is a very clear indication of departure from Kolmogorov scaling. In general, taking the intermittency phenomenology into account, we have

$$\langle |v(\vec{x} + \vec{r}) - v(\vec{x})|^p \rangle \sim r^{\zeta_p} \quad (2.7.26)$$

where $\zeta_p \neq p/3$. The departure of ζ_p from $p/3$ is linked to the existence of coherent structures.

The coherent structures have of late brought interest back into the turbulence exhibited by the forced Burgers equation. Burgers equation, often studied in one dimension, gives the equation of motion for the velocity as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad (2.7.27)$$

The coherent structures in this equation are shocks and have been well studied. The forced Burgers equation acquires the form

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} &= \nu \frac{\partial^2 u}{\partial x^2} + f \\ \text{with } \langle f(x_1, t_1) f(x_2, t_2) \rangle &= -2D_0 \nabla^2 \delta(x_1 - x_2) \delta(t_1 - t_2) \end{aligned} \quad (2.7.28)$$

Interestingly enough, the KPZ equation that we discussed before can be cast in the Burgers equation by the transformation $u = \frac{\partial h}{\partial x}$. This can be seen by taking the gradient in Eq(2.7.27) and setting $\vec{u} = \vec{\nabla} h$. We get

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} &= \Gamma \nabla^2 \vec{u} + \lambda \vec{\nabla} u^2 + \vec{\nabla} f \\ &= \Gamma \nabla^2 \vec{u} + \lambda \vec{\nabla} u^2 + \vec{\xi} \end{aligned} \quad (2.7.29)$$

where

$$\langle \xi_\alpha(\vec{x}_1, t_1) \xi_\beta(\vec{x}_2, t_2) \rangle = -2D_0 \nabla^2 \delta(\vec{x}_1 - \vec{x}_2) \delta(t_1 - t_2) \delta_{\alpha\beta} \quad (2.7.30)$$

In $D = 1$, this is the same as Eq.(2.7.28).

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The Renormalization Group

3.1 Introduction

Critical phenomena had posed a challenge for a long time since it is an intrinsically strong coupling problem where traditional perturbation techniques did not work. The difficulty is enshrined in the infinite correlation length at the critical point. The renormalization group (RG) is a technique that is specifically designed to lead to an infinite correlation length. The idea is best set out using the Kadanoff construction in the two dimensional Ising model shown in Fig. 3.1.

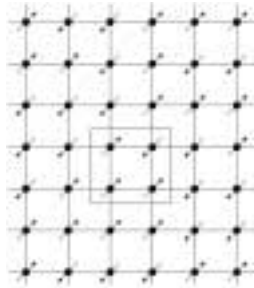


Figure 3.1. Making blocks of Ising spins

The Hamiltonian for the two dimensional Ising model is given by

$$H = -J \sum S_{i,j} S_{i+1,j+1} \quad (3.1.1)$$

where $S_{i,j}$ is the spin at the lattice site (i, j) . The idea is to bunch together some spins and form a new spin (the spins shown inside the dotted square in Fig. 3.1 and

form an effective lattice, where the lattice spacing is double the previous lattice spacing. As the spins change to an effective spin $S'_{i,j}$, the coupling constant changes to a new J' , such that

$$J' = f(J) \quad (3.1.2)$$

The correlation length, which is $\xi(J)$ in terms of the lattice spacing undergoes a transformation when the spin regrouping occurs. The new correlation length $\xi(J')$ will clearly be half of the previous correlation length (since the lattice spacing is doubled) and we have

$$\xi(J') = \frac{1}{2}\xi(J) \quad (3.1.3)$$

One can repeat this transformation and the important question is what will happen as the transformation is repeated again and again. There is the possibility that ultimately we will reach a fixed point J^* , where

$$J^* = f(J^*) \quad (3.1.4)$$

and further regrouping does not change J anymore. If this occurs, then Eq.(3.1.3) tells us that $\xi = \infty$ or 0. Thus this transformation is designed to lead to an infinite correlation length, if the fixed point is arrived at and this corresponds to the critical point. If, we examine Eq.(3.1.2) near the fixed point J^* , we find

$$\begin{aligned} J' &= J^* + \left. \frac{\partial f}{\partial J} \right|_{J=J^*} (J - J^*) + \dots \\ \text{or } \delta J' &= \left. \frac{\partial f}{\partial J} \right|_{J=J^*} \delta J = \lambda_1 \delta J \end{aligned} \quad (3.1.5)$$

If $|\lambda_1| > 1$, then δJ grows on iteration and to reach criticality, we have to hold $\delta J = 0$. Such a variable is called a relevant variable - the fixed point is unstable in the direction of the relevant variable. Turning to Eq.(3.1.3), one can now write

$$\begin{aligned} \xi(\delta J) &= 2\xi(\delta J') \\ &= 2\xi(\lambda_1 \delta J) \\ &= 2^2 \xi(\lambda_1^2 \delta J) \\ &\dots\dots\dots \\ &= 2^n \xi(\lambda_1^n \delta J) \end{aligned} \quad (3.1.6)$$

Since λ_1 is greater than unity, we can choose n large enough so that however small δJ may be (δJ is very small close to the critical point) $\lambda_1^n \delta J \sim 1$. For this choice, we have

$$n = \frac{\ln(\frac{1}{\delta J})}{\ln(\lambda_1)} \quad (3.1.7)$$

with this value of n Eq.(3.1.6) becomes

$$\xi(\delta J) = 2^{[ln(\frac{1}{\delta J})/ln(\lambda_1)]} \xi(1) \quad (3.1.8)$$

Now, $\xi(1)$ is an ordinary number say ξ_0 and we have

$$\xi(\delta J) = \xi_0 (\delta J)^{-\frac{ln 2}{ln \lambda_1}} = \xi_0 (\delta J)^{-\nu} \quad (3.1.9)$$

Since δJ is a relevant variable, which has to be zero at criticality, we identify δJ with $\delta T = \frac{T-T_c}{T_c}$, and one arrives at

$$\xi(T - T_c) = \xi_0 \left[\frac{T - T_c}{T_c} \right]^{-\nu}$$

where $\nu = \frac{ln 2}{ln \lambda_1}$ (3.1.10)

To get an actual value of ν , one must have a calculational scheme that will give us the function f of Eq.(3.1.1). The simple steps that we have gone through have yielded the scaling law of the first of Eq.(3.1.10) and illustrates the basic working of the renormalization group.

In general, we would need to start out with a larger Hamiltonian, e.g. one that would include four spin interaction, two spin interaction with next nearest neighbour and so on. We denote the coupling constants of these various interactions by the set $\{K\} = (K_1, K_2, \dots)$. Under the scaling transformations described above all the K_i 's reach the fixed point values K_i^{*} - that is the entire Hamiltonian reaches a fixed point. The transformation that we describe takes us from $J, \{K\} \rightarrow J', \{K'\}$ and in general $J = f(J, \{K\})$ and similarly for the set $\{K\}$. From the set $J, \{K\}$ we can form linear combinations $\{u_i\}$, called scaling fields, such that when linearized about the fixed point

$$\delta u'_i = \lambda_i \delta u_i \quad (3.1.11)$$

for each i . For $|\lambda_i| > 1$, we have a relevant variable that we have discussed before, while for $|\lambda_i| < 1$, the variables are irrelevant. Marginal variables correspond to $|\lambda_i| = 1$. The existence of the irrelevant variables leads to universality. If one starts out with a general Hamiltonian, then under the iterations described above, the irrelevant variables die out and one ends up with the critical Hamiltonian where only relevant variables matter. Thus a huge number of Hamiltonians end up giving the same critical behaviour thus leading to universality.

We summarize the above discussion by repeating that a convenient way of describing critical phenomena is to set up a transformation which removes degrees of freedom on the small scale (the lattice spacing is doubled in the transformation) and focusses on the change in the coupling constants that this entails. The transformation is called a renormalization group transformation (RGT) - the Hamiltonian after the transformation has the same structure as before (notice that the spins have

to be rescaled as well so that they are always ± 1) but with “renormalized” coupling constants. In the next section, we will see how one implements this procedure for continuous fields - our primary concern in this book.

3.2 Renormalization Group: General Framework

In this section, we show how to implement the RGTs for a continuum model. The order parameter field $\phi(x)$ (we will restrict ourselves to a scalar field for the present) determines the free energy density which in turn determines the probability distribution of the field $\phi(x)$. The free energy density is given, as is well known, by the Ginzburg - Landau free energy density of Eq.(2.3.16). We will drop the quartic term to begin with and consider the Gaussian model with the free energy

$$F = \int d^D x \left[\frac{1}{2} m^2 \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 \right] \quad (3.2.1)$$

which can be exactly solved.

We will first discuss the method of implementing the transformation procedure. This discussion will be general and independent of the model. To appreciate the existence of all scales in the field $\phi(x)$, we carry out the Fourier decomposition

$$\phi(\vec{x}) = \frac{1}{(2\pi)^{D/2}} \int^\Lambda \phi(\vec{k}) e^{i\vec{k} \cdot \vec{x}} d^D k \quad (3.2.2)$$

The momentum integration is essentially over the entire momentum range - the cutoff Λ corresponds to inverse lattice spacing. Thus $\phi(x)$ is composed of Fourier components $\phi(k)$, where k can range from 0 to Λ . This shows the existence of all length scales (k^{-1} corresponds to length). The Fourier components with low values of k are called the *slow variables* while the components with high values of k are called the *fast variables*. Quantitatively we call all components with $0 < k < \frac{\Lambda}{b}$ ($b > 1$) the slow variables and the components with $\frac{\Lambda}{b} < k < \Lambda$ are the fast variables. The slow variables will be denoted by $\phi_<(k)$ and the fast variables by $\phi_>(k)$. The first step of the RGT is now clear - integrate out the fast variables and write the free energy as a function of $\phi_<(k)$. But one is not ready to compare coupling constants yet - the new Hamiltonian (a term which will be used interchangeably with free energy) has $\phi(k)$ with k ranging only up to $\frac{\Lambda}{b}$. The previous Hamiltonian had $\phi(k)$ with k ranging up to Λ . To be able to compare the two Hamiltonians, the momentum after the elimination of the degrees of freedom has to go up to Λ . This requires scaling all momenta by a factor b , i.e. $k \rightarrow k' = bk$. In terms of k' , the range is now 0 to Λ . We now need to rescale the field $\phi(x)$, so that the coefficient of $(\vec{\nabla} \phi)^2$ remains fixed at $1/2$. This ensures that the new Hamiltonian does not become the old Hamiltonian by a mere rescaling of ϕ . At this point, one is ready to compare the coupling constants. Thus the three steps of the RGT are

- i) *elimination of the fast variables, $\phi(k)$ with $\frac{\Lambda}{b} < k < \Lambda$*
- ii) *rescaling of k to $k' = bk$*
- iii) *rescaling of ϕ to $\phi' = \phi/\zeta$*

We now implement the steps for the Gaussian model. To do so it helps to write the free energy in the Fourier space. This gives

$$F = \int \frac{1}{2} (m^2 + k^2) \phi(k) \phi(-k) d^D k \quad (3.2.3)$$

and the partition function is

$$\begin{aligned} Z &= \int \Pi d\phi(k) e^{-\int \frac{1}{2} (m^2 + k^2) \phi(k) \phi(-k) d^D k} \\ &= \int \Pi d\phi(k) e^{-\frac{1}{2} \int_{k=0}^{\Lambda/b} (m^2 + k^2) \phi_{<}(k) \phi_{<}(-k) d^D k} \\ &\quad \times e^{-\frac{1}{2} \int_{k=\Lambda/b}^{\Lambda} (m^2 + k^2) \phi_{>}(k) \phi_{>}(-k) d^D k} \\ &= \int \Pi d\phi_{<}(k) e^{-\frac{1}{2} \int_{k=0}^{\Lambda/b} (m^2 + k^2) \phi_{<}(k) \phi_{<}(-k) d^D k} \\ &\quad \times \int \Pi d\phi_{>}(k) e^{-\frac{1}{2} \int_{k=\Lambda/b}^{\Lambda} (m^2 + k^2) \phi_{>}(k) \phi_{>}(-k) d^D k} \\ &= C_0 \int \Pi d\phi_{<}(k) e^{-\frac{1}{2} \int_{k=0}^{\Lambda/b} (m^2 + k^2) \phi_{<}(k) \phi_{<}(-k) d^D k} \\ &= C_0 \int \Pi d\phi_{<}(k') e^{-\frac{1}{2} \int_{k'=0}^{\Lambda} (m^2 + \frac{k'^2}{b^2}) \phi_{<}(\frac{k'}{b}) \phi_{<}(-\frac{k'}{b}) \frac{d^D k'}{b^D}} \\ &= C_0 \int \Pi d\phi'_{<}(k') e^{-\frac{1}{2} \int_{k'=0}^{\Lambda} (m^2 + \frac{k'^2}{b^2}) \zeta^2 \phi'_{<}(k') \phi'_{<}(-k') \frac{d^D k'}{b^D}} \\ &= C_0 \int \Pi d\phi'_{<}(k') e^{-\frac{1}{2} \int_{k'=0}^{\Lambda} (m'^2 + \frac{k'^2 \zeta^2}{b^2 + D}) \phi'_{<}(k') \phi'_{<}(-k') d^D k'} \end{aligned} \quad (3.2.4)$$

with the definition

$$\phi(k) = \phi_{<}(k) + \phi_{>}(k)$$

where

$$\begin{aligned} \phi_{<}(k) &= \phi(k) \quad \text{for} \quad 0 \leq k < \frac{\Lambda}{b} \\ &= 0 \quad \text{for} \quad \frac{\Lambda}{b} \leq k < \Lambda \end{aligned}$$

and

$$\begin{aligned} \phi_{>}(k) &= 0 \quad \text{for} \quad 0 \leq k < \frac{\Lambda}{b} \\ &= \phi(k) \quad \text{for} \quad \frac{\Lambda}{b} \leq k < \Lambda. \end{aligned}$$

Now

$$m'^2 = \frac{\zeta^2 m^2}{b^D} \quad (3.2.5)$$

and

$$\zeta^2 = b^{2+D} \quad (3.2.6)$$

to keep the coefficient of $k^2 \phi(k) \phi(-k)$ at 1/2. Thus the recursion relation for m becomes

$$m'^2 = b^2 m^2 \quad (3.2.7)$$

Clearly, m^2 is relevant, since $b^2 > 1$ and for criticality $m^2 = 0$ - this is consistent with $m^2 \sim (T - T_c)$. The rescaling of momentum scale by b , means that

$$\xi(m') = \xi(m)/b \quad (3.2.8)$$

or

$$\xi(m) = b \xi(bm) = b^2 \xi(b^2 m) = \dots = b^n \xi(b^n m) \quad (3.2.9)$$

Choosing $b^n m = 1$

$$\begin{aligned} \xi(m) &= \frac{1}{m} \xi(1) \\ &= \xi_0 \left(\frac{T - T_c}{T_c} \right)^{-1/2} \end{aligned} \quad (3.2.10)$$

giving the critical exponent $\nu = 1/2$.

Are terms like $(\nabla^2 \phi)^2$, $[\nabla^2 (\vec{\nabla} \phi)]^2$ etc. in the free energy relevant? In momentum space these terms contribute $k^{2n} \phi(k) \phi(-k)$ to the free energy density, with $n = 2, 3, \dots$. For a term $K_n k^{2n} \phi(k) \phi(-k)$ in Eq.(3.2.3) we get $K_n \zeta^2 / b^{2n+D}$ as the coefficient of $k'^{2n} \phi_<(k') \phi_<(-k')$ after the steps of the RGT and hence

$$K'_n = K_n \zeta^2 / b^{2n+D} = K_n b^{2(1-n)} \quad (3.2.11)$$

Clearly, for $n \geq 2$, all couplings K_n are irrelevant.

We now look at the couplings of the form $u_n \int \phi^{2n} d^D x$ in F for $n \geq 2$. In momentum space this term in F has the form

$$u_n \int \phi(p_1) \phi(p_2) \dots \phi(-p_1 - p_2 \dots - p_{2n-1}) d^D p_1 d^D p_2 \dots d^D p_{2n-1}.$$

Under the transformations described above this term becomes

$$\begin{aligned} &\frac{u_n \zeta^{2n}}{b^{(2n-1)D}} \int \phi_<(p'_1) \phi_<(p'_2) \dots \phi_<(-p'_1 - p'_2 \dots - p'_{2n-1}) d^D p'_1 \\ &d^D p'_2 \dots d^D p'_{2n-1}. \end{aligned}$$

Thus the coupling constant u_n goes to u'_n , where

$$\begin{aligned} u'_n &= \frac{\zeta^{2n}}{b^{(2n-1)D}} u_n = \frac{b^{2n+nD}}{b^{(2n-1)D}} u_n \\ &= b^{2n+(1-n)D} u_n \end{aligned} \quad (3.2.12)$$

The exponent is positive if (this makes u_n relevant)

$$D < \frac{2n}{n-1} \quad (3.2.13)$$

For $n=2$, u_2 is relevant for $D < 4$, while for $n=3$, u_3 is relevant for $D < 3$ and so on. Consequently, if we are interested in critical phenomena in $D=3$, the only relevant perturbation to the Gaussian model of Eq.(3.2.1) is a term $\int \phi^4 d^D x$. (it is easy to check that terms of the form $\int (\vec{\nabla} \phi^2)^2 d^D x$ etc. where derivatives are included will be irrelevant). This leads to the standard model for treating critical phenomena

$$F = \int d^D x \left[\frac{1}{2} m^2 \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4 \right] \quad (3.2.14)$$

The performance of the RGT on the Hamiltonian F , leads to renormalized m and λ . For dimensions $D > 4$, λ is irrelevant and the Gaussian model holds. For dimensions $D < 4$, the ϕ^4 term is relevant and a new fixed point is reached, where a first order in λ calculation yields

$$\nu = \frac{1}{2} + \frac{\epsilon}{12} \quad (3.2.15)$$

We will not provide a full derivation of Eq.(3.2.15) which can be found in a large number of texts, but will content ourselves with outlining how the mode elimination is done. This is done perturbatively by writing

$$\phi = \phi_{<} + \phi_{>} \quad (3.2.16)$$

Where $\phi_{<}$ contains wavenumbers from 0 to Λ/b and $\phi_{>}$ contains wavenumbers from Λ/b to Λ

$$\begin{aligned} Z &= \int D[\phi_{<}] D[\phi_{>}] e^{-\int d^D x \left[\frac{m^2}{2} (\phi_{<} + \phi_{>})^2 + \frac{1}{2} (\vec{\nabla} \phi_{<} + \vec{\nabla} \phi_{>})^2 + \frac{\lambda}{4} (\phi_{<} + \phi_{>})^4 \right]} \\ &= \int \Pi d\phi_{<}(k) \Pi d\phi_{>}(k) \exp \left[-\frac{1}{2} \int d^D k (k^2 + m^2) \phi_{<}(k) \phi_{<}(-k) \right] \\ &\quad \times \exp \left[-\frac{\lambda}{4} \int d^D k_1 d^D k_2 d^D k_3 \phi_{<}(k_1) \phi_{<}(k_2) \phi_{<}(k_3) \phi_{<}(-k_1 - k_2 - k_3) \right] \\ &\quad \times \exp \left[-\frac{1}{2} \int d^D k (k^2 + m^2) \phi_{>}(k) \phi_{>}(-k) \right] \times \exp \left[-\frac{\lambda}{4} \int d^D k_1 d^D k_2 d^D k_3 \right. \end{aligned}$$

$$\begin{aligned}
& \times \{4\phi_{<}(k_1)\phi_{<}(k_2)\phi_{<}(k_3)\phi_{>}(-k_1-k_2-k_3) + 4\phi_{<}(k_1)\phi_{>}(k_2) \\
& \times \phi_{>}(k_3)\phi_{>}(-k_1-k_2-k_3) + 6\phi_{<}(k_1)\phi_{<}(k_2)\phi_{>}(k_3) \\
& \times \phi_{>}(-k_1-k_2-k_3) + \phi_{>}(k_1)\phi_{>}(k_2)\phi_{>}(k_3)\phi_{<}(-k_1-k_2-k_3)\} \Big] \\
& = \int \Pi d\phi_{<}(k) e^{-F_{<}} \int \Pi d\phi_{>}(k) e^{-F_{>}^0} e^{-\lambda F_I(\phi_{<}, \phi_{>})} \quad (3.2.17)
\end{aligned}$$

The strategy is to separate the Hamiltonian $F_{<}$ which contains only the fields $\phi_{<}$ from the rest where $\phi_{>}$ appears. In the part which contains $\phi_{>}$, we split the Hamiltonian into the Gaussian part $F_{>}^0$ and a part $\lambda F_I(\phi_{<}, \phi_{>})$, which involves both $\phi_{<}$ and $\phi_{>}$ or $\phi_{>}$ alone in a non-quadratic form. The perturbative calculation proceeds by expanding

$$e^{-\lambda F_I} = 1 - \lambda F_I + \frac{\lambda^2}{2} F_I^2 + \dots \quad (3.2.18)$$

One now performs the integration over the fields $\phi_{>}$, and exponentiating the result ends up with a contribution of the form $\exp(-\tilde{F}_{<}(\phi_{<}))$. This casts Eq.(3.2.18) in the form

$$Z = \int D[\phi_{<}] \exp \left[- (F_{<} + \tilde{F}_{<}) \right]$$

Rescaling of k and ϕ paves the way to the recursion relations.

3.3 Dynamics of Model A

When dealing with dynamics there is the additional variable which is time and when differentiating between fast and slow components of the Fourier field $\phi(\vec{x}, t)$, we need to think about the double Fourier transform

$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^{D/2}} \frac{1}{\sqrt{2\pi}} \int \phi(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{x} - \omega t)} d^D k d\omega \quad (3.3.1)$$

The fast variables are now the high momentum, high frequency components and the slow variables are the low frequency, low momentum components. For the momentum variable, as before, we introduce the quantity $b(> 1)$ in terms of which, we define the fast range as $\frac{\Lambda}{b} < k < \Lambda$ and the slow range is $0 < k < \frac{\Lambda}{b}$. For the frequency variable, we need to define a similar variable ω_b . If the full frequency range is 0 to Ω , then the slow variable range is $0 < \omega < \frac{\Omega}{\omega_b}$ and the fast variable range is $\frac{\Omega}{\omega_b} < \omega < \Omega$. The parameter ω_b is the additional one for the dynamics and its dependence on b is specified as

$$\omega_b \propto b^z \quad (3.3.2)$$

where z is called the dynamic scaling exponent which was defined from a slightly different standpoint in Eq.(2.4.27). A considerable part of the effort in dynamic critical phenomena revolves around the calculation of the exponent z . To explain how the RG works in this context, we consider the trivial problem of the linear part of model A (Eq.(2.4.19)) where by directly integrating the linear equation, we already saw in **Chapter 2**, that $z=2$. The equation of motion for ϕ in the k, ω space (Fourier transform of Eq.(2.4.19)) is

$$\left[-\frac{i\omega}{\Gamma} + k^2 + m^2 \right] \phi(\vec{k}, \omega) = \frac{N(\vec{k}, \omega)}{\Gamma} \quad (3.3.3)$$

The first step is equivalent to the first step in the statics (previous section)- we need to find the effective equation of motion for the slow variable $\phi_{<}(\vec{k}, \omega)$. For the linear equation of motion in Eq.(3.3.3), this is trivially done - the equation for $\phi_{<}$ is

$$\left[-\frac{i\omega}{\Gamma/\chi} + 1 \right] \phi_{<}(\vec{k}, \omega) = \frac{N_{<}}{\Gamma/\chi} \quad (3.3.4)$$

where $\chi = (k^2 + m^2)^{-1}$ is the static susceptibility. The next few steps are,

- i) rescale momentum $k \rightarrow k' = bk$, so that the range of k' is back to $0 \rightarrow \lambda$
- ii) rescale frequency $\omega \rightarrow \omega' = b^z \omega$, so that the range of ω' is back to $0 \rightarrow \Omega$
- iii) rescale field $\phi_{<}(k, \omega)$ to $\phi' = \phi/\zeta$ with ζ such that it is consistent with the statics.

This gives the transformation law for Γ/χ . The above rescalings lead to

$$\left[-\frac{i\omega'}{\Gamma b^z/\chi} + 1 \right] \phi'_{<} = \frac{N_{<}/\zeta}{\Gamma/\chi} \quad (3.3.5)$$

To keep the scale factor of ϕ consistent with statics,

$$\zeta = b^{1+\frac{D}{2}+z} \quad (3.3.6)$$

$$\chi' = \chi b^{-2} \quad (3.3.7)$$

The above equation of motion becomes

$$\left[-\frac{i\omega'}{\Gamma'/\chi'} + 1 \right] \phi'_{<} = \frac{N'_{<}}{\Gamma'/\chi'} \quad (3.3.8)$$

where

$$\Gamma' = \frac{\Gamma b^z}{b^2} = \Gamma b^{z-2} \quad (3.3.9)$$

$$N' = N b^z / \zeta = N b^{z-2} \quad (3.3.10)$$

The relaxation rate renormalizes as shown in Eq.(3.3.9) and the fixed point corresponds to

$$z = 2. \quad (3.3.11)$$

as expected.

We now turn to the complete model A, when the equation of motion is defined by Eq.(2.4.2) with F given by Eq.(2.4.1). We work with $n = 1$ i.e. a scalar order parameter. In Fourier space, the equation of motion reads

$$\begin{aligned} \left[-\frac{i\omega}{\Gamma} + k^2 + m^2 \right] \phi(k, \omega) = & -\lambda \sum \phi(\vec{p}_1, \omega_1) \phi(\vec{p}_2, \omega_2) \\ & \phi(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) + \frac{N(\vec{k}, \omega)}{\Gamma} \end{aligned} \quad (3.3.12)$$

The slow variable $k < \frac{\Lambda}{b}$, $\omega < \frac{\Omega}{b^z}$ satisfies

$$\begin{aligned} \left[-\frac{i\omega}{\Gamma} + k^2 + m^2 \right] \phi_{<}(k, \omega) = & -\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\ & \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\ & - 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{>}(\vec{p}_2, \omega_2) \\ & \times \phi_{>}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\ & - 3\lambda \sum \phi_{>}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\ & \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\ & - \lambda \sum \phi_{>}(\vec{p}_1, \omega_1) \phi_{>}(\vec{p}_2, \omega_2) \\ & \times \phi_{>}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) + \frac{N_{<}}{\Gamma} \end{aligned} \quad (3.3.13)$$

The fast variable $\phi_{>}$ satisfies a similar equation with the understanding that $\frac{\Lambda}{b} < k < \Lambda$ and $\frac{\Omega}{b^z} < \omega < \Omega$. We have

$$\begin{aligned} \left[-\frac{i\omega}{\Gamma} + k^2 + m^2 \right] \phi_{>}(k, \omega) = & -\lambda \sum \phi_{>}(\vec{p}_1, \omega_1) \phi_{>}(\vec{p}_2, \omega_2) \\ & \times \phi_{>}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\ & - 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{>}(\vec{p}_2, \omega_2) \\ & \times \phi_{>}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\ & - 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\ & \times \phi_{>}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \end{aligned}$$

$$\begin{aligned}
& -\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\
& \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) + \frac{N_{>}}{\Gamma}
\end{aligned} \tag{3.3.14}$$

The strategy is to solve for $\phi_{>}$ from Eq.(3.3.14) in terms of $\phi_{<}$ and substitute into Eq.(3.3.13) to get the equation of motion for $\phi_{<}$. The way to solve for $\phi_{>}$ from Eq.(3.3.14) is by perturbation theory. So one expands,

$$\phi_{>} = \phi_{>}^{(0)} + \lambda \phi_{>}^{(1)} + \lambda^2 \phi_{>}^{(2)} + \dots \tag{3.3.15}$$

and at the different orders

$$\left[-\frac{i\omega}{\Gamma} + k^2 + m^2 \right] \phi_{>}^{(0)}(k, \omega) = \frac{N_{>}(k, \omega)}{\Gamma} \tag{3.3.16}$$

$$\begin{aligned}
\left[-\frac{i\omega}{\Gamma} + k^2 + m^2 \right] \phi_{>}^{(1)}(k, \omega) = & -3 \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - 3 \sum \phi_{>}^{(0)}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\
& \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - \sum \phi_{>}^{(0)}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\
& \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2)
\end{aligned} \tag{3.3.17}$$

Inserting the expansion of Eq.(3.3.15) into Eq.(3.3.13), we have

$$\begin{aligned}
\left[-\frac{i\omega}{\Gamma} + k^2 + m^2 \right] \phi_{<}(\vec{k}, \omega) = & -\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\
& \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2)
\end{aligned}$$

$$\begin{aligned}
& -\lambda \sum \phi_{>}^{(0)}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - 3\lambda^2 \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(1)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - 3\lambda^2 \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{>}^{(1)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - 3\lambda^2 \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(1)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - \lambda^2 \sum \phi_{>}^{(1)}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - \lambda^2 \sum \phi_{>}^{(0)}(\vec{p}_1, \omega_1) \phi_{>}^{(1)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& - \lambda^2 \sum \phi_{>}^{(0)}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(1)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& + O(\lambda^3) + \frac{N_{<}}{\Gamma}
\end{aligned} \tag{3.3.18}$$

The solution to Eq.(3.3.16) is

$$\phi_{>}^{(0)} = GN_{>}/\Gamma \tag{3.3.19}$$

where

$$G^{-1} = -\frac{i\omega}{\Gamma} + k^2 + m^2 \tag{3.3.20}$$

The solution to Eq.(3.3.17) is

$$\begin{aligned}
\phi_{>}^{(1)}(\vec{k}, \omega) = & -G(\vec{k}, \omega) \left[3 \sum \phi_{>}^{(0)}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \right. \\
& \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& + 3 \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& + \sum \phi_{>}^{(0)}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\
& \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\
& + \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \\
& \left. \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \right]
\end{aligned} \tag{3.3.21}$$

The thing to note is that $\phi_{>}^{(0)}$ and $\phi_{>}^{(1)}$ are determined by the noise (see Eq.(3.3.19)) and hence to determine the equation of motion satisfied by $\phi_{<}$, we will have to average over the noise which can be done by using Eq.(2.4.2) in Fourier space

$$\langle N_{>}(\vec{k}_1, \omega_1) N_{>}(\vec{k}_2, \omega_2) \rangle = 2\Gamma \delta(\vec{k}_1 + \vec{k}_2) \delta(\omega_1 + \omega_2)$$

We will be looking at the $O(\lambda)$ term in Eq.(3.3.18) in some detail to establish the procedure,

$$\begin{aligned} &= \left\langle 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \right\rangle \\ &= \left\langle 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \langle G(\vec{p}_2, \omega_2) N(\vec{p}_2, \omega_2) G(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \rangle \right. \\ &\quad \times \left. N(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \right\rangle \Gamma^{-2} \\ &= 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) G(\vec{p}_2, \omega_2) G(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\ &\quad \times \langle N(\vec{p}_2, \omega_2) N(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \rangle \Gamma^{-2} \\ &= 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) G(\vec{p}_2, \omega_2) G(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\ &\quad \times 2\Gamma \delta(\vec{k} - \vec{p}_1) \delta(\omega - \omega_1) / \Gamma^2 \\ &= 6 \left(\frac{\lambda}{\Gamma} \right) \phi_{<}(\vec{k}, \omega) \sum_{\vec{p}, \omega_1} \frac{1}{[-\frac{i\omega_1}{\Gamma} + p^2 + m^2]} \frac{1}{[\frac{i\omega_1}{\Gamma} + p^2 + m^2]} \\ &= 6 \left(\frac{\lambda}{\Gamma} \right) \phi_{<}(\vec{k}, \omega) \int \frac{d^D p}{(2\pi)^D} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{1}{(p^2 + m^2)^2 + \frac{\omega_1^2}{\Gamma^2}} \\ &= 3\lambda \phi_{<}(\vec{k}, \omega) \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2} \end{aligned} \quad (3.3.22)$$

Similarly, the third term on the R.H.S of Eq.(3.3.18) is

$$\begin{aligned} &= 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \\ &= 3\lambda \sum \phi_{<}(\vec{p}_1, \omega_1) \phi_{<}(\vec{p}_2, \omega_2) G(N_{>}) \Gamma^{-1} \\ &= 0 \end{aligned} \quad (3.3.23)$$

The fourth term on the R.H.S of Eq.(3.3.18) does not contribute either. Correct to $O(\lambda)$ in Eq.(3.3.18), we have, on using the results of Eqns.(3.3.22) and (3.3.23)

$$\left[-\frac{i\omega}{\Gamma} + k^2 + m'^2 \right] \phi_{<} = -\lambda \phi_{<} \phi_{<} \phi_{<}$$

where

$$m'^2 = m^2 + 3\lambda \phi_{<}(\vec{k}, \omega) \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2} \quad (3.3.24)$$

The dynamics is not changed. It is the statics that gets modified and Eq.(3.3.24) is exactly what a one-loop statics calculation would give if carried out in detail in §(3.2). Similarly, the fourth, fifth and sixth terms on the R.H.S of Eq.(3.3.18) contribute to the statics. These terms renormalize the coupling constant λ . (The sixth is not important as it gives a reducible contribution).

To find out the effect on the dynamics, we need to find a contribution to the equation of motion that can renormalize the coefficient Γ^{-1} of the $-i\omega$ term on the L.H.S of Eq.(3.3.18). To do so, we need to consider the last three terms on the R.H.S and pick out the terms that are linear in $\phi_{<}$. Further, we are interested in terms which have the structure $\Sigma(k, \omega)\phi_{<}(k, \omega)$, since the coefficient $\Sigma(k, \omega)$ can then modify the relaxation rate Γ . Let us now look at the eighth term on the R.H.S of Eq.(3.3.18) and substitute for $\phi_{>}^{(1)}$ by the first term on the R.H.S of Eq.(3.3.21). The term becomes (after the averaging)

$$\begin{aligned} & 3\lambda^2 \sum G(\vec{p}_1, \omega_1) \phi_{<}(\vec{q}_1, \omega'_1) \langle \phi_{>}^{(0)}(\vec{q}_2, \omega'_2) \\ & \times \phi_{>}^{(0)}(\vec{p}_1 - \vec{q}_1 - \vec{q}_2, \omega_1 - \omega'_1 - \omega'_2) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \\ & \times \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \rangle \end{aligned}$$

In factoring the above average to get the contribution that has the above structure $\Sigma(k, \omega)\phi_{<}(k, \omega)$ requires the $\phi_{>}^{(0)}$ coming from Eq.(3.3.21), to be correlated with $\phi_{>}^{(0)}$ coming from Eq.(3.3.18). This takes the above term to

$$\begin{aligned} & 6\lambda^2 \sum G(\vec{p}_1, \omega_1) \phi_{<}(\vec{q}_1, \omega'_1) \langle \phi_{>}^{(0)}(\vec{q}_2, \omega'_2) \phi_{>}^{(0)}(\vec{p}_2, \omega_2) \rangle \\ & \times \langle \phi_{>}^{(0)}(\vec{p}_1 - \vec{q}_1 - \vec{q}_2, \omega - \omega'_1 - \omega'_2) \phi_{>}^{(0)}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \rangle \\ & = 6\lambda^2 \sum G(\vec{p}_1, \omega_1) \phi_{<}(\vec{q}_1, \omega'_1) \left(\frac{2}{\Gamma} \right) \frac{1}{\frac{\omega_2^2}{\Gamma^2} + (p_2^2 + m^2)^2} \\ & \times \left(\frac{2}{\Gamma} \right) \frac{1}{\frac{(\omega - \omega_1 - \omega_2)^2}{\Gamma^2} + [(\vec{k} - \vec{p}_1 - \vec{p}_2)^2 + m^2]^2} \delta(\vec{q}_1 - \vec{k}) \delta(\omega'_1 - \omega) \end{aligned}$$

Noting that

$$\left(\frac{2}{\Gamma} \right) \int \frac{d\omega}{2\pi} \frac{1}{\frac{\omega^2}{\Gamma^2} + \beta^2} = \frac{1}{\beta}$$

and that

$$\left(\frac{2}{\Gamma} \right) \int \frac{d\omega'}{2\pi} \frac{1}{[-\frac{i\omega'}{\Gamma} + \beta_1]} \frac{1}{[-\frac{i(\omega - \omega')}{\Gamma} + \beta_2]} = \frac{1}{[-\frac{i\omega}{\Gamma} + \beta_1 + \beta_2]}$$

We can write the above term as

$$6\lambda^2 \phi_{<}(\vec{k}, \omega) \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \frac{1}{p_2^2 + m^2} \frac{1}{(\vec{k} - \vec{p}_1 - \vec{p}_2)^2 + m^2} \\ \times \frac{1}{\frac{-i\omega}{\Gamma} + p_1^2 + p_2^2 + (\vec{k} - \vec{p}_1 - \vec{p}_2)^2 + 3m^2}$$

The eighth and ninth terms on the R.H.S of Eq.(3.3.18), when treated in the same manner yield terms similar to the above with p exchanging places with p_1 in one and $\vec{k} - \vec{p}_1 - \vec{p}_2$ in the other. We can now write the effective equation of motion for $\phi_{<}$ as

$$[\frac{-i\omega}{\Gamma} + k^2 + m^2 - \Sigma(k, \omega)]\phi_{<}(k, \omega) = -\lambda \sum \phi_{<}(\vec{p}_1, \omega_1)\phi_{<}(\vec{p}_2, \omega_2) \\ \times \phi_{<}(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) + N_{<} \quad (3.3.25)$$

where the “self-energy” $\Sigma(k, \omega)$ is given by

$$\Sigma(k, \omega) = 6\lambda^2 \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \frac{p_1^2 + p_2^2 + (\vec{k} - \vec{p}_1 - \vec{p}_2)^2 + 3m^2}{(p_1^2 + m^2)(p_2^2 + m^2)[(\vec{k} - \vec{p}_1 - \vec{p}_2)^2 + m^2]} \\ \times \frac{1}{\frac{-i\omega}{\Gamma} + p_1^2 + p_2^2 + (\vec{k} - \vec{p}_1 - \vec{p}_2)^2 + 3m^2} \quad (3.3.26)$$

We can expand the above self-energy as

$$\Sigma(k, \omega) = \Sigma(k, 0) + (-i\omega) \left. \frac{\partial \Sigma(k, \omega)}{\partial (-i\omega)} \right|_0 + \dots \quad (3.3.27)$$

The first term gives a contribution to the statics. The statics with the quartic interaction showed us in §3.2, that the quartic interaction is relevant for $D < 4$ and thus inspires a perturbation theory in $\epsilon = 4 - D$ and to the lowest order in ϵ , this means the integrals like the one shown in Eq.(3.3.26) need to be evaluated at $D = 4$. Keeping in mind that the momentum integration runs between $\frac{\Lambda}{b}$ and Λ , one gets $\Sigma(k, 0) = -\eta k^2 \ln b$, where the anomalous dimension exponent

$$\eta = \frac{\epsilon^2}{54} \quad (3.3.28)$$

for a general value of n , the this answer is

$$\eta = \frac{\epsilon^2}{2} \frac{n+2}{(n+8)^2}.$$

In deriving $\eta = \frac{\epsilon^2}{54}$, we need the fixed point value of λ . This is obtained from statics. The renormalization of ‘ m ’ and ‘ λ ’ that come from the purely static part

give transformation equations for these couplings which for $D < 4$ support a non-Gaussian fixed point. The appropriate fixed point value λ^* of λ , when used in Eq.(3.3.26) gives

$$\left. \frac{\partial \Sigma(k, \omega)}{\partial k^2} \right|_{\omega=k=0} = -\eta \ln b \quad (3.3.29)$$

with η prescribed by Eq.(3.3.28). The equation of motion for $\phi_<$ now reads

$$\begin{aligned} \left[-i\omega \left(\frac{1}{\Gamma} - \frac{\partial \Sigma}{\partial(-i\omega)} \right) \right]_{\omega=k=0} + k^2(1 + \eta \ln b) + \tilde{m}^2 \Big] \phi_< \\ = -\lambda \sum \phi_<(\vec{p}_1) \phi_<(\vec{p}_2) \phi_<(\vec{k} - \vec{p}_1 - \vec{p}_2) \end{aligned} \quad (3.3.30)$$

We now carry out the nest two steps in the RGT

- i) the momenta are scaled to $k' = bk$, and frequency to $\omega' = b^z \omega$ to restore the cutoff to Λ and Ω and
- ii) The field $\phi_<$ is rescaled to $\phi'_< \zeta$, so that the coefficient of k^2 is unchanged.

This now gives us the equation of motion as

$$\begin{aligned} \left[-\frac{i\omega'}{b^z} \left(\frac{1}{\Gamma} - \frac{\partial \Sigma}{\partial(-i\omega')} \right) \right]_{\omega'=k=0} + \\ \frac{k'^2 \zeta}{b^2} (1 + \eta \ln b) + \tilde{m}^2 \zeta \Big] \phi_< = -\lambda \sum \zeta^3 \phi_< \phi_< \phi_< + \frac{N_<}{\Gamma} \end{aligned} \quad (3.3.31)$$

The requirement of the coefficient of k'^2 be unity fixes $\zeta = b^{2-\eta}$. The relaxation rate is seen to be renormalized to

$$\begin{aligned} \frac{1}{\Gamma} &= b^{2-\eta-z} \left(\frac{1}{\Gamma} - \frac{\partial \Sigma}{\partial(-i\omega')} \right)_{\omega'=k=0} \\ &= b^{2-\eta-z} \left(\frac{1}{\Gamma} + 6 \frac{\lambda^{*2} b^z}{\Gamma b^z} \right. \\ &\quad \times \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \frac{1}{(p_1^2 + m^2)} \frac{1}{(p_2^2 + m^2)} \frac{1}{(\vec{p}_1 + \vec{p}_2)^2 + m^2} \\ &\quad \left. \times \frac{1}{\vec{p}_1^2 + \vec{p}_2^2 + (\vec{p}_1 + \vec{p}_2)^2 + 3m^2} \right) \end{aligned} \quad (3.3.32)$$

The integral needs to be evaluated at $D=4$, since λ^* is already $O(\epsilon)$ and standard techniques of doing such integrals (note that the momentum integrations are between $\frac{\lambda}{b}$ and λ and in this high momentum range m^2 can be dropped) leads to (power counting shows a logarithmic divergence)

$$\begin{aligned}
\frac{1}{\Gamma'} &= \frac{b^{2-\eta-z}}{\Gamma} \left[1 + 6\eta \ln \frac{4}{3} \ln b \right] \\
&= \frac{1}{\Gamma} \left[1 + \left\{ \left(6 \ln \frac{4}{3} \right) \eta + (2 - \eta - z) \right\} \ln b + \dots \right] \quad (3.3.33)
\end{aligned}$$

The fixed point of the relaxation rate leads to

$$z = 2 + \left(6 \ln \frac{4}{3} - 1 \right) \eta + O(\epsilon^3) \quad (3.3.34)$$

Thus in model A, the nontrivial dynamic exponent is determined to $O(\epsilon^2)$. We have demonstrated this calculation in great detail because this is the way all the dynamic renormalization group (DRG) calculations can be done.

For model B, an identical calculation can be carried out. However, as should be explicitly verified by the reader, the integral

$$\left. \frac{\partial \Sigma}{\partial(-i\omega)} \right|_{k, \omega=0}$$

in this case does not show the logarithmic divergence shown in the first step of Eq.(3.3.33) and consequently, for the model with the conservation law $z = 4 - \eta$.

3.4 Inclusion of Reversible Terms

This section will be devoted to studying the operation of RG on the reversible nonlinear part in the equation of motion. We will take as our example the Heisenberg ferromagnet, where the equation of motion is given in Eq.(2.4.11), which we now write in momentum space,

$$\begin{aligned}
\dot{\phi}_i(k, t) &= g \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \epsilon_{ilm} (p^2 - p'^2) \phi_l(\vec{p}) \phi_m(\vec{p}') - \Gamma k^2 (k^2 + m^2) \phi_i(k) \\
&\quad - \lambda \Gamma k^2 \sum_{\vec{p}_1, \vec{p}_2} [\phi_j(\vec{p}_1) \phi_j(\vec{p}_2)] \phi_i(\vec{k} - \vec{p}_1 - \vec{p}_2) + N_i \quad (3.4.1)
\end{aligned}$$

In the first term ϵ_{ilm} is the Levi Civita symbol which is zero if any of the two indices are equal and $\epsilon_{123} = 1$. One permutation of the subscripts changes the sign. It should be noted that there are two kinds of nonlinear terms in the above equation.

- i) the dissipative kind with coupling constant λ which we have already dealt with
- ii) the reversible term with coupling constant g which ensures

$$\frac{d}{dt} [\phi_i(k) \phi_i(-k)] = 0$$

From our previous discussion, we know that the coupling constant λ is relevant for $D < 4$. The first issue that we need to determine is when is the coupling constant g relevant? For this discussion, we need not consider the λ term and we have the equation of motion

$$\left[-\frac{i\omega}{\Gamma/\chi} + k^2 \right] \phi_1(k, \omega) = \frac{g}{\Gamma/\chi} \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}}^{\omega_1 + \omega_2 = \omega} \phi_2(\vec{p}_1, \omega_1) \phi_3(\vec{p}_2, \omega_2) (p_1^2 - p_2^2) + \frac{N}{\Gamma/\chi} \quad (3.4.2)$$

As the detailed calculations of the previous section show, removing the fast variables produces an effective equation of motion for the slow variables. We are not interested in all the different terms, right now we want to concentrate on one term which is guaranteed to be there, which makes the effective equation of motion

$$\left[-\frac{i\omega}{\Gamma/\chi} + k^2 \right] \phi_1^<(k, \omega) = \frac{g}{\Gamma/\chi} \sum \phi_2^<(\vec{p}_1, \omega_1) \phi_3^<(\vec{p}_2, \omega_2) (p_1^2 - p_2^2) + \frac{N^<}{\Gamma/\chi} \quad (3.4.3)$$

Doing the rescalings $k' = bk$, $\omega' = b^z \omega$ and $\phi^< = \zeta \phi^<$, we have

$$\left[-\frac{i\omega'}{b^{z-2}k'^2\Gamma/\chi} + 1 \right] \phi_1^<(k, \omega) = \left[\frac{g}{\Gamma/\chi} \right] \left[\frac{\zeta}{k'^2} \right] \left[\frac{1}{b^{D+z}} \right] \sum (p_1'^2 - p_2'^2) \times \phi_2^<(\vec{p}'_1, \omega'_1) \phi_3^<(\vec{p}'_2, \omega'_2) + \frac{N^</\zeta}{\Gamma/\chi} \quad (3.4.4)$$

We need to choose ζ to be consistent with the states and hence with $\chi = \chi' b^2$

$$\zeta = b^{1+\frac{D}{2}+z} \quad (3.4.5)$$

Rewriting Eq.(3.4.4) in the form

$$\begin{aligned} \left[-\frac{i\omega'}{k'^2\Gamma'/\chi'} + 1 \right] \phi_1^< &= \frac{g}{k'^2\Gamma'/\chi'} b^{z-4} b^{3-\frac{D}{2}} \sum (p_1'^2 - p_2'^2) \\ &\times \phi_2^<\phi_3^< + \frac{N'^<}{\Gamma'} \\ &= \frac{g'}{\Gamma'/\chi'} \sum (p_1'^2 - p_2'^2) \phi_2^<(p_1) \phi_3^<(p_2) + \frac{N'}{\Gamma'} \end{aligned}$$

where

$$\Gamma' = \Gamma b^{z-4} \quad (3.4.6)$$

$$g' = g b^{z-1-\frac{D}{2}} \quad (3.4.7)$$

At the fixed point for Γ , $z = 4$ and we have

$$g' = gb^{6-D} \quad (3.4.8)$$

The reversible term is relevant for $D < 6$ and irrelevant for $D > 6$. Consequently, for $D > 6$, the reversible term has no effect on the dynamics i.e. $z = 4$ will be exact. For $D < 6$, the reversible will change the value of z and this is what we now turn to.

We now proceed to carry out the RGT as done in the previous section for the dissipative model and write (i, j and k are always cyclic in 1, 2, 3)

$$\begin{aligned} \left[-\frac{i\omega}{\Gamma} + k^2(k^2 + m^2) \right] \phi_i^<(k, \omega) = & \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_j^<(\vec{p}_1) \phi_k^<(\vec{p}_2) \\ & + \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_j^<(\vec{p}_1) \phi_k^>(\vec{p}_2) \\ & + \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_j^>(\vec{p}_1) \phi_k^<(\vec{p}_2) \\ & + \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_j^>(\vec{p}_1) \phi_k^>(\vec{p}_2) + \frac{N^<}{\Gamma} \end{aligned} \quad (3.4.9)$$

and

$$\begin{aligned} \left[-\frac{i\omega}{\Gamma} + k^2(k^2 + m^2) \right] \phi_1^>(k, \omega) = & \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_2^<(\vec{p}_1) \phi_3^<(\vec{p}_2) \\ & + \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_2^>(\vec{p}_1) \phi_3^<(\vec{p}_2) \\ & + \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_2^<(\vec{p}_1) \phi_3^>(\vec{p}_2) \\ & + \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_2^>(\vec{p}_1) \phi_3^>(\vec{p}_2) + \frac{N^>}{\Gamma} \end{aligned} \quad (3.4.10)$$

As before, one is expected to solve for $\phi^>$ and insert the result in Eq.(3.4.9) to obtain the effective equation of motion for $\phi^<$. To solve for $\phi^>$ perturbatively we expand

$$\phi_i^> = \phi_i^{>(0)} + \frac{g}{\Gamma} \phi_i^{>(1)} + ..$$

with

$$\left[-\frac{i\omega}{\Gamma} + k^2(k^2 + m^2) \right] \phi_1^{>(0)} = \frac{N^>}{\Gamma} \quad (3.4.11)$$

$$\begin{aligned}
[-\frac{i\omega}{\Gamma} + k^2(k^2 + m^2)]\phi_1^{>(1)} &= \sum (p_1^2 - p_2^2)\phi_2^{<}(\vec{p}_1)\phi_3^{<}(\vec{p}_2) \\
&+ \sum (p_1^2 - p_2^2)\phi_2^{>(0)}(\vec{p}_1)\phi_3^{<}(\vec{p}_2) \\
&+ \sum (p_1^2 - p_2^2)\phi_2^{<}(\vec{p}_1)\phi_3^{>(0)}(\vec{p}_2) \\
&+ \sum (p_1^2 - p_2^2)\phi_2^{>(0)}(\vec{p}_1)\phi_3^{>(0)}(\vec{p}_2) \quad (3.4.12)
\end{aligned}$$

and

$$\begin{aligned}
[-\frac{i\omega}{\Gamma} + k^2(k^2 + m^2)]\phi_1^{>(2)} &= \sum (p_1^2 - p_2^2)\phi_2^{>(1)}(\vec{p}_1)\phi_3^{<}(\vec{p}_2) \\
&+ \sum (p_1^2 - p_2^2)\phi_2^{<}(\vec{p}_1)\phi_3^{>(1)}(\vec{p}_2) \\
&+ \sum (p_1^2 - p_2^2)\phi_2^{>(0)}(\vec{p}_1)\phi_3^{>(1)}(\vec{p}_2) \\
&+ \sum (p_1^2 - p_2^2)\phi_2^{>(1)}(\vec{p}_1)\phi_3^{>(0)}(\vec{p}_2) \quad (3.4.13)
\end{aligned}$$

Inserting the power series expansion of $\phi^{>}$ in Eq.(3.4.9), we get

$$\begin{aligned}
[-\frac{i\omega}{\Gamma} + k^2(k^2 + m^2)]\phi_1^{<}(k, \omega) &= \frac{g}{\Gamma} \sum (p_1^2 - p_2^2)\phi_2^{<}(\vec{p}_1, \omega_1)\phi_3^{<}(\vec{p}_2, \omega_2) \\
&+ \left[\frac{g}{\Gamma} \sum (p_1^2 - p_2^2)\phi_2^{<}(\vec{p}_1, \omega_1)\phi_3^{>(0)}(\vec{p}_2, \omega_2) \right. \\
&+ \frac{g^2}{\Gamma^2} \sum (p_1^2 - p_2^2)\phi_2^{<}(\vec{p}_1, \omega_1)\phi_3^{>(1)}(\vec{p}_2, \omega_2) \\
&+ \frac{g^3}{\Gamma^3} \sum (p_1^2 - p_2^2)\phi_2^{<}(\vec{p}_1, \omega_1)\phi_3^{>(2)}(\vec{p}_2, \omega_2) \\
&+ \text{a similar term with subscripts 2 and 3} \\
&\quad \left. \text{on } \phi \text{ interchanged} \right] \\
&+ \frac{g}{\Gamma} \sum (p_1^2 - p_2^2)\phi_2^{>(0)}(\vec{p}_1, \omega_1)\phi_3^{>(0)}(\vec{p}_2, \omega_2) \\
&+ \frac{g^2}{\Gamma^2} \sum (p_1^2 - p_2^2) \left(\phi_2^{>(1)}\phi_3^{>(0)} + \phi_3^{>(1)}\phi_2^{>(0)} \right) \\
&+ \frac{g^3}{\Gamma^3} \sum (p_1^2 - p_2^2) \left(\phi_2^{>(2)}\phi_3^{>(0)} + \phi_3^{>(2)}\phi_2^{>(0)} \right. \\
&\quad \left. + \phi_2^{>(1)}\phi_3^{>(1)} \right) + \dots \quad (3.4.14)
\end{aligned}$$

The solution for $\phi_1^{>(i)}$ can be written as

$$\begin{aligned}
\phi_1^{>(0)} &= GN^{>}/\Gamma \\
\text{with} \quad G^{-1} &= -\frac{i\omega}{\Gamma} + k^2(k^2 + m^2) \quad (3.4.15)
\end{aligned}$$

(Note that G is independent of the component ϕ_i).

$$\begin{aligned} \phi_1^{>(1)}(k, \omega) = G(k, \omega) \Big\{ & \sum (p_1^2 - p_2^2) \phi_2^<(\vec{p}_1, \omega_1) \phi_3^<(\vec{p}_2, \omega_2) \\ & + \sum (p_1^2 - p_2^2) \left[\phi_2^{>(0)}(\vec{p}_1, \omega_1) \phi_3^<(\vec{p}_2, \omega_2) \right. \\ & \left. + \phi_2^<(\vec{p}_1, \omega_1) \phi_3^{>(0)}(\vec{p}_2, \omega_2) + \phi_2^{>(0)}(\vec{p}_1, \omega_1) \phi_3^{>(0)}(\vec{p}_2, \omega_2) \right] \Big\} \end{aligned} \quad (3.4.16)$$

From the R.H.S of Eq.(3.4.14) we are supposed to find

- i) a term linear in $\phi_1^<(k, \omega)$ that will help renormalize Γ^{-1} and
- ii) a term of the form $\phi_2^<\phi_3^<$ that will help renormalize the coupling constant g

To find the renormalization of Γ , the correct terms on the R.H.S of Eq.(3.4.14) are

$$\left(\frac{g}{\Gamma}\right)^2 \sum (p_1^2 - p_2^2) \left(\phi_2^{>(1)}(p_1) \phi_3^{>(0)}(p_2) + \phi^{>(0)}(p_1) \phi_3^{>(1)}(p_2) \right)$$

Looking at the first term of the above expression, we have

$$\begin{aligned} \left(\frac{g}{\Gamma}\right)^2 \sum (p_1^2 - p_2^2) \phi_2^{>(1)}(p_1) \phi_3^{>(0)}(p_2) &= \left(\frac{g}{\Gamma}\right)^2 \sum (p_1^2 - p_2^2) \sum G(p_1, \omega_1) \\ &\times [q^2 - (\vec{p}_1 - \vec{q})^2] \phi_1^<(p_1 - q, \omega_1 - \omega') \\ &\times \phi_3^{>(0)}(q, \omega') \phi_3^{>(0)}(p_2, \omega_2) \end{aligned}$$

Doing the averaging over the noise, it becomes

$$\begin{aligned} &\left(\frac{g}{\Gamma}\right)^2 \sum (p_1^2 - p_2^2) [-q^2 + (\vec{p}_1 - \vec{q})^2] G(p_1, \omega_1) \phi_1^<(q, \omega') \\ &\times \left\langle \phi_3^>(p_2, \omega_2) \phi_3^>(-p_2, -\omega_2) \right\rangle \delta(\vec{k} - \vec{q}) \delta(\omega - \omega') \\ &= \phi_1^<(k', \omega) \left(\frac{g}{\Gamma}\right)^2 \sum (p_1^2 - p_2^2) (-k^2 + p_2^2) \\ &\times \frac{2}{\Gamma} \frac{p_2^2}{[\frac{\omega_2^2}{\Gamma^2} + p_2^4(p_2^2 + m^2)^2]} \frac{1}{[-\frac{i\omega_1}{\Gamma} + p_1^2(p_1^2 + m^2)]} \\ &= \phi_1^<(k', \omega) \left(\frac{g}{\Gamma}\right)^2 \sum \frac{(p_1^2 - p_2^2)(k^2 - p_2^2)}{(p_2^2 + m^2)} \frac{1}{-\frac{i\omega}{\Gamma} + p_1^2(p_1^2 + m^2) + p_2^2(p_2^2 + m^2)} \end{aligned}$$

The two terms together give

$$\left[-\frac{i\omega}{\Gamma} + k^2(k^2 + m^2) + \Sigma(k, \omega) \right] \phi_1^<(k, \omega) \quad (3.4.17)$$

$$= \frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_2^<(p_1, \omega_1) \phi_2^<(p_1, \omega_1) + \frac{N^<}{\Gamma} \quad (3.4.18)$$

where

$$\begin{aligned} \Sigma(k, \omega) &= \left(\frac{g}{\Gamma} \right)^2 (k^2 + m^2) \int \frac{d^D p_1}{(2\pi)^D} \frac{[p_1^2 - (\vec{k} - \vec{p}_1)^2]^2}{(p_1^2 + m^2)[(\vec{k} - \vec{p}_1)^2 + m^2]} \\ &\quad \times \frac{1}{[-\frac{i\omega}{\Gamma} + p_1^2(p_1^2 + m^2) + (\vec{k} - \vec{p}_1)^2\{(\vec{k} - \vec{p}_1)^2 + m^2\}]} \end{aligned} \quad (3.4.19)$$

Note that $\Sigma(k, \omega)$ is dynamic in origin and unlike the dissipative case, does not have any static component which has to be separated. This self-energy $\Sigma(k, \omega)$ renormalizes Γ and we can carry out the momentum integration over p such that $\frac{\Lambda}{b} < p < \Lambda$. In this high momentum region, the integral has a logarithmic divergence at $D = 6$, which is the upper critical dimension, i.e. where g becomes relevant. We evaluate the integral in $D = 6$, to find (one needs to remember that p is large and appropriate approximations are necessary)

$$\begin{aligned} \Sigma(k, \omega) &= \left(\frac{g}{\Gamma} \right)^2 (k^2 + m^2) k^2 \frac{C_6}{(2\pi)^6} \frac{4}{D} \int \frac{p^5 dp}{(p^2 + m^2)^2} \frac{p^2}{2p^2(p^2 + m^2)} \\ &= k^2(k^2 + m^2) \left(\frac{g}{\Gamma} \right)^2 \frac{2}{D} K_6 \int \frac{p^5 dp}{(p^2 + m^2)^3} \\ &= k^2(k^2 + m^2) \left(\frac{g}{\Gamma} \right)^2 \frac{2}{D} K_6 \ln b \end{aligned} \quad (3.4.20)$$

Looking back at Eq.(3.4.17) and multiplying by Γ we can identify

$$\Delta\Gamma = \frac{g^2}{\Gamma} \frac{2}{D} K_6 \ln b \quad (3.4.21)$$

To find the renormalization of the coupling constant g , we need to look at the last group of terms (involving g^3/Γ^3) in the R.H.S of Eq.(3.4.14) and we use $\phi_i^{>(2)}$ from the first two terms of the R.H.S of Eq.(3.4.13), with $\phi_i^{>(1)}$ coming from the second and third terms on the R.H.S of Eq.(3.4.16). We then generate all the non-zero terms (after averaging over $\phi_i^{>(0)}$), of the form

$$\frac{g}{\Gamma} \sum (p_1^2 - p_2^2) \phi_j^<(\vec{p}_1) \phi_k^<(\vec{p}_2) + \frac{N^<}{\Gamma}$$

and the sum of the three contributing terms add upto zero. Hence, to this order, g is not renormalized.

Returning to Eq.(3.4.7) the fact that g is not normalized leads to the relation

$$z = 1 + \frac{D}{2} \quad (3.4.22)$$

For $D < 6$, $z < 4$ and that implies the Onsager coefficient Γ has to diverge for $D < 6$ which is represented by the scaling of Eq.(3.4.6) and we have

$$\begin{aligned} \Gamma' &= \Gamma b^{z-4} = \Gamma b^{\frac{D}{2}-3} = \Gamma b^{-\frac{\epsilon}{2}} \\ &= \Gamma \left(1 - \frac{\epsilon}{2} \ln b + O(\epsilon^2) \right) \end{aligned} \quad (3.4.23)$$

Adding in the contribution from Eq.(3.4.20) we find

$$\Gamma' = \Gamma + \Gamma \left(-\frac{\epsilon}{2} + \frac{2}{6} \frac{g^2}{\Gamma^2} K_6 \right) \ln b \quad (3.4.24)$$

The fixed point is reached for

$$\frac{g^2}{\Gamma^2} = \left(\frac{3}{2 K_6} \right) \epsilon = 96\pi^3 \epsilon \quad (3.4.25)$$

which is a universal number (dependent only on dimensionality of space). The Onsager coefficient Γ thus diverges as $k^{-\epsilon/2}$ and $\epsilon > 0$ i.e. $D < 6$, and has an amplitude Γ_0 such that the number g^2/Γ_0^2 is an universal number. Thus, in the dynamics, we can have universal exponents and universal amplitude combinations.

In the above discussion, calculations have been carried out to $O(\epsilon)$. The non-renormalizability of the coupling constant ' g ' is however correct to all orders in ϵ and thus the conclusion $z = 1 + \frac{D}{2}$ is exact. However, in getting this answer, η has been ignored. For $6 > D > 4$, this does not matter since in that range $\eta = 0$. However, for $D < 4$, there will be $O(\eta)$ correction to the dynamic scaling exponent. By going through the detailed calculation for the dissipative coupling and the reversible coupling, we have covered all possible ways in RG techniques that one uses in dynamics.

3.5 Field Theoretic Form

In this section, we give a very sketchy indication of how the dynamics problem can be made to look very much like a problem in statics and the field theoretic methods of statics can be brought to bear to solve it. We do this first for the simplest system - linearized version of model A. The equation of motion is,

$$\frac{\partial \phi(\vec{k})}{\partial t} = -\Gamma(k^2 + m^2)\phi(k) + N(k, t) \quad (3.5.1)$$

with

$$\langle N(k_1, t_1) N(k_2, t) \rangle = 2\Gamma \delta(\vec{k}_1 + \vec{k}_2) \delta(t_1 - t) \quad (3.5.2)$$

In frequency space,

$$\begin{aligned} [-i\omega + \Gamma(k^2 + m^2)]\phi(k, \omega) &= N(k, \omega) \\ \langle N(k_1, \omega_1) N(k_2, \omega_2) \rangle &= 2\Gamma \delta(\vec{k}_1 + \vec{k}_2) \delta(\omega_1 + \omega_2) \end{aligned} \quad (3.5.3)$$

The solution to the problem implies complete specification of the Green's function $G(k, \omega)$ and the correlation function $C(k, \omega)$. The results for the model of Eq.(3.5.3) are

$$\begin{aligned} G(k, \omega) &= \Gamma \left\langle \frac{\delta\phi(k_1, \omega_1)}{\delta N(k_2, \omega_2)} \right\rangle \frac{1}{\delta(\vec{k}_1 + \vec{k}_2) \delta(\omega_1 + \omega_2)} \\ &= \left(-\frac{i\omega}{\Gamma} + m^2 + k^2 \right)^{-1} \end{aligned} \quad (3.5.4)$$

$$\begin{aligned} C(k, \omega) &= \langle \phi(k_1, \omega_1) \phi(k_2, \omega_2) \rangle \frac{1}{\delta(\vec{k}_1 + \vec{k}_2) \delta(\omega_1 + \omega_2)} \\ &= \frac{1}{\frac{\omega^2}{\Gamma^2} + (k^2 + m^2)^2} \end{aligned} \quad (3.5.5)$$

The probability distribution of the noise N is given by a Gaussian distribution

$$p(N) \propto \exp\left[-\frac{1}{4\Gamma} \int d^D k dt |N(k, t)|^2\right]$$

Instead of working with $N(k, t)$, an equivalent description would be in terms of $\phi(k, t)$ and $\dot{\phi}(k, t)$ obtained by expressing $N(k, t)$ in terms of them through Eq.(3.5.1). It is even more convenient to introduce a set of auxiliary fields $\tilde{\phi}$, noting the result

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\tilde{x} e^{-\tilde{x}^2 - 2ix\tilde{x}} \quad (3.5.6)$$

and write the probability distribution as a function of ϕ and $\tilde{\phi}$ in the form,

$$\mathcal{Z}(\phi, \tilde{\phi}) \sim \int \mathcal{D}[\phi] \mathcal{D}[\tilde{\phi}] e^{-S(\phi, \tilde{\phi})} \quad (3.5.7)$$

where apart from a Jacobian $S(\phi, \tilde{\phi})$ is given in coordinate space by the expression

$$S(\phi, \tilde{\phi}) = \int d^D x dt [-\tilde{\phi} \Gamma \tilde{\phi} + \tilde{\phi} \{\dot{\phi} + \Gamma(m^2 - \nabla^2)\phi\}] \quad (3.5.8)$$

The problem cast in the form of Eqns.(3.5.7) and (3.5.8) looks very much like the problem in statics - set up a free energy functional (statics) and then evaluate the partition function. The action $S(\phi, \tilde{\phi})$ of Eq(2.5.8) is the analogue of the free energy functional and the $Z(\phi, \tilde{\phi})$ of Eq.(3.5.7) is the 'partition function'.

The physical meaning of the field $\tilde{\phi}$ can be seen from the Eqns(3.5.7) and (3.5.8), if we add an external force $\vec{F}$ to the equation of motion in Eq.(3.5.1). The response function χ is then given by

$$\chi = \frac{\delta \langle \phi(\vec{r}_1, t_1) \rangle}{\delta F(\vec{r}_2, t_2)} \Big|_{f \rightarrow 0} = \langle \phi(\vec{r}_1, t_1) \tilde{\phi}(\vec{r}_2, t_2) \rangle \quad (3.5.9)$$

Thus the response function is cast as a correlation function and we can now write down the response and correlation function by an inspection of the action in momentum and frequency space.

$$S = \int \frac{d^D k}{(2\pi)^D} \frac{d\omega}{2\pi} [-\Gamma \tilde{\phi}(k, \omega) \tilde{\phi}(-k, -\omega) + \tilde{\phi}(-k, -\omega) \{-i\omega + \Gamma(k^2 + m^2)\} \tilde{\phi}(k, \omega)] \quad (3.5.10)$$

Evaluation of the Gaussian functional integral to obtain the correlation function $\langle \tilde{\phi}(k, \omega) \tilde{\phi}(-k, -\omega) \rangle$ and $\langle \phi(k, \omega) \phi(-k, -\omega) \rangle$ leads to the answers written down in Eqns.(3.5.4) and (3.5.5).

In general the action will not be quadratic. If we consider the full model A with the equation of motion given by Eq.(2.4.2) with the free energy given by Eq.(2.4.1), the action will be

$$S = \int \frac{d^D k}{(2\pi)^D} \frac{d\omega}{2\pi} \left[-\Gamma \tilde{\phi}(k, \omega) \tilde{\phi}(-k, -\omega) + \tilde{\phi}(-k, -\omega) \{-i\omega + \Gamma(k^2 + m^2)\} \right. \\ \times \tilde{\phi}(k, \omega) + \Gamma u \tilde{\phi}(-k, -\omega) \sum \phi(\vec{k} - \vec{p}_1 - \vec{p}_2, \omega - \omega_1 - \omega_2) \Big] \\ \times \phi(\vec{p}_1, \omega_1) \phi(\vec{p}_2, \omega_2) \quad (3.5.11)$$

The kinetic coefficient Γ now needs a renormalization constant Z to render the dynamic correlations finite because of the quartic term in the action. For the Heisenberg model of Eq.(3.4.2), the action will be

$$S = \int \frac{d^D k}{(2\pi)^D} \frac{d\omega}{2\pi} \left[-\Gamma \tilde{\phi}_\alpha(k, \omega) \tilde{\phi}_\alpha(-k, -\omega) + \tilde{\phi}_\alpha(-k, -\omega) \right. \\ \{-i\omega + \Gamma(k^2 + m^2)\} \\ \times \tilde{\phi}_\alpha(k, \omega) - g \sum_{p, \omega'} \tilde{\phi}_\alpha(-k, -\omega) \tilde{\phi}_\beta(p, \omega') \tilde{\phi}_\gamma(\vec{k} - \vec{p}, \omega - \omega') \\ \left. [p^2 - (\vec{k} - \vec{p})^2] \right] \quad (3.5.12)$$

Once again the non-quadratic term will renormalize the kinetic coefficient Γ and the calculation will parallel the field theoretic treatment in statics.

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Mode Coupling Theories

4.1 Introduction

In this chapter, we will deal with two techniques which are non-perturbative in nature and hence can be very effective in features that may elude perturbation theory. The implementation of the RG that we discussed in the last chapter often requires perturbation theory. For a given problem it is consequently a good idea to try both the RG and the techniques that we will discuss here. We will explain the method by using the same ferromagnetic system that we used in §3.4. The equation of motion is

$$\begin{aligned} \dot{\phi}_i(\vec{k}) = & -\Gamma_0 k^2 (k^2 + m^2) \phi_i(k) + N_i(k) \\ & + g \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \epsilon_{ijk} (p_1^2 - p_2^2) \phi_j(\vec{p}_1) \phi_k(\vec{p}_2) \end{aligned} \quad (4.1.1)$$

with

$$\langle N_i(\vec{k}_1, \omega_1) N_j(\vec{k}_2, \omega_2) \rangle = -2\Gamma_0 k_1^2 \delta_{ij} \delta(\vec{k}_1 + \vec{k}_2) \delta(\omega_1 + \omega_2) \quad (4.1.2)$$

By the word mode, one refers to the Fourier mode $\phi(\vec{k})$. The non-linear term in the equation of motion couples the different modes. The second term on the R.H.S of Eq.(4.1.2) is the non-linear term and as is obvious the term couples the modes $\phi(\vec{p}_1)$ and $\phi(\vec{p}_2)$ with the constraint $\vec{p}_1 + \vec{p}_2 = \vec{k}$. The linear term, the first term on the R.H.S of Eq.(4.1.2), on the other hand, does not couple modes and consequently the linear equation of motion is trivially solvable. Before discussing how to handle the non-linear terms we need to discuss some general issues. The response or Green's function is defined as

$$G_{ij}(\vec{k}, \omega) = \left\langle \frac{\delta \phi_i(k, \omega)}{\delta N_j(k', \omega)} \right\rangle \frac{1}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')} \quad (4.1.3)$$

and the correlation function by

$$C_{ij}(\vec{k}, \omega) = \langle \phi_i(\vec{k}, \omega) \phi_j(\vec{k}', \omega') \rangle \frac{1}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')} \quad (4.1.4)$$

For an isotropic system (both in order parameter space and in actual space time) and the noise uncorrelated in different directions, we have

$$G_{ij}(\vec{k}, \omega) = G(k, \omega) \delta_{ij}$$

and similarly,

$$C_{ij}(\vec{k}, \omega) = C(k, \omega) \delta_{ij}.$$

In this chapter, we will always assume that the isotropy holds.

The calculation of G becomes straight forward if it can be expressed as a correlation function. Examination of Eq.(4.1.1) shows us that the required correlation function is

$$G(k, \omega) = (2\Gamma_0 k^2)^{-1} \langle \phi_i(\vec{k}, \omega) N(\vec{k}', \omega') \rangle \frac{1}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')} \quad (4.1.5)$$

If we work with the linear part alone, then Eq.(4.1.1) shows that in Fourier space

$$[-i\omega + \Gamma_0 k^2 (k^2 + m^2)] \phi_i(k, \omega) = N_i(k, \omega)$$

leading to, according to Eq.(4.1.3)

$$G(k, \omega) = [-i\omega + \Gamma_0 k^2 (k^2 + m^2)]^{-1} \quad (4.1.6)$$

If we use Eq.(4.1.5)

$$\begin{aligned} G(k, \omega) &= \frac{1}{2\Gamma_0 k^2} \langle \phi_i(\vec{k}, \omega) N(\vec{k}', \omega') \rangle \frac{1}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')} \\ &= \frac{1}{2\Gamma_0 k^2} \langle [-i\omega + \Gamma_0 k^2 (k^2 + m^2)]^{-1} N(\vec{k}, \omega) N(\vec{k}', \omega') \rangle \\ &\quad \times \frac{1}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')} \\ &= [-i\omega + \Gamma_0 k^2 (k^2 + m^2)]^{-1} \end{aligned}$$

the same as that obtained in Eq.(4.1.6).

The static response function of the system is

$$\chi = \frac{1}{k^2 + m^2}$$

and it is useful to normalize the dynamic response such that the zero frequency limit correctly reproduces the statics. We can do this by recognizing that the relaxation rate

$$\Gamma = \Gamma_0 k^2 (k^2 + m^2) = \Gamma_0 k^2 / \chi$$

and thus the desired normalization is obtained if

$$G(k, \omega) = \Gamma \chi \left\langle \frac{\delta \phi}{\delta N} \right\rangle = \frac{1}{2} \langle \phi N \rangle \quad (4.1.7)$$

Thus, instead of using the definitions given in Eqns.(4.1.3) and (4.1.5), we will use the prescription in Eq.(4.1.7) which correctly reproduces the static limit. The normalization of $G(k, \omega)$ to the static response is vital for the following discussion. In statics, the response function is also the two-point correlation function. With this normalization we will be able to generalize this result to dynamics. Since $G(k, \omega)$ is a response function, it will be causal and in general will satisfy all the conditions for satisfying the Kramers-Kronig relation and we have

$$G(k, \omega) = \frac{1}{\pi} \int \frac{\text{Im } G(k, \omega')}{\omega' - \omega} d\omega'$$

For the static response,

$$G(k, 0) = \frac{1}{\pi} \int \frac{\text{Im } G(k, \omega')}{\omega'} d\omega'. \quad (4.1.8)$$

Turning to the correlation function $C(k, \omega)$, the equal time correlation function $C(k, t_{12} = 0)$ is obtained from the integral

$$C(k, t_{12} = 0) = \int \frac{d\omega'}{2\pi} C(k, \omega') \quad (4.1.9)$$

The equal time correlation is the static response function when we have an equilibrium distribution function. In that case, the equality of the L.H.S of Eqns.(4.1.8) and (4.1.9) lead to the *fluctuation dissipation theorem* (FDT).

$$C(k, \omega) = \frac{1}{2\omega} \text{Im } G(k, \omega) \quad (4.1.10)$$

For our linearized system, the correlation function $C(k, \omega)$ according to Eq.(4.1.4) is

$$C(k, \omega) = \frac{2\Gamma_0 k^2}{\omega^2 + [\Gamma_0 k^2 (k^2 + m^2)]^2} = \frac{1}{\Gamma_0 k^2} \frac{2}{(\frac{\omega}{\Gamma_0 k^2})^2 + (k^2 + m^2)^2} \quad (4.1.11)$$

With $G(k, \omega)$ given by Eq.(4.1.7) as

$$G(k, \omega) = \left[-\frac{i\omega}{\Gamma_0 k^2} + (k^2 + m^2) \right]^{-1} \quad (4.1.12)$$

We find that Eq.(4.1.10) is satisfied as expected.

If the FDT is satisfied, then we do not need to consider the equations for $G(k, \omega)$ and $C(k, \omega)$ separately, since the information obtained automatically leads to the information on the other. For systems without an equal time (equilibrium) distribution function, however, the FDT does not hold and $G(k, \omega)$ and $C(k, \omega)$ must be separately considered. For the models of dynamic critical phenomena discussed in **Chapter 1**, FDT holds and one simply needs to consider the response function (or correlation function), while for the driven diffusive lattice gas (§2.5) and the growth models (§2.6), there is no equilibrium distribution and there is no FDT. In the detailed example that we consider in this chapter - the critical dynamics of the Heisenberg ferromagnet as given in Eq.(4.1.1) there is an equilibrium distribution, and the FDT holds. This simplifies our work.

4.2 Self Consistent Mode Coupling

We will now try to find the response function for the dynamics of the Heisenberg model as represented by Eq.(4.1.1). We begin by working with perturbation theory by expanding

$$\phi(\vec{k}, t) = \phi_i^{(0)}(\vec{k}, t) + g\phi_i^{(1)}(\vec{k}, t) + g^2\phi_i^{(2)}(\vec{k}, t) + \dots \quad (4.2.1)$$

Inserting the expansion in Eq.(4.1.1) and equating equal powers of g on either side of the equation, we have

$$\dot{\phi}_i^{(0)} = -\Gamma_0 k^2 (k^2 + m^2) \phi_i^{(0)} + N_i \quad (4.2.2)$$

$$\dot{\phi}_i^{(1)} = \sum \epsilon_{ijk} (p_1^2 - p_2^2) \phi_j^{(0)}(\vec{p}_1) \phi_k^{(0)}(\vec{p}_2) - \Gamma_0 k^2 (k^2 + m^2) \phi_i^{(1)} \quad (4.2.3)$$

$$\begin{aligned} \dot{\phi}_i^{(2)} = & -\Gamma_0 k^2 (k^2 + m^2) \phi_i^{(2)} + \sum \epsilon_{ijk} (p_1^2 - p_2^2) [\phi_j^{(0)}(\vec{p}_1) \phi_k^{(1)}(\vec{p}_2) \\ & + \phi_j^{(1)}(\vec{p}_1) \phi_k^{(0)}(\vec{p}_2)] \end{aligned} \quad (4.2.4)$$

The Green's function has the expansion

$$\begin{aligned} G &= G^{(0)} + gG^{(1)} + g^2G^{(2)} + \dots \\ &= \frac{1}{2} [\langle \phi^{(0)}(k, \omega) N(-k, -\omega) \rangle + g \langle \phi^{(1)}(k, \omega) N(-k, -\omega) \rangle \\ &\quad + g^2 \langle \phi^{(2)}(k, \omega) N(-k, -\omega) \rangle + \dots] \end{aligned} \quad (4.2.5)$$

The solution for $\phi_i^{(0)}$ from Eq.(4.2.2) is

$$\dot{\phi}_i^{(0)}(-\vec{k}, t) = e^{-\Gamma_0 k^2 (k^2 + m^2)t} \int_{-\infty}^t dt' e^{\Gamma_0 k^2 (k^2 + m^2)t'} N(k, t')$$

This leads to the zeroth order response as

$$\begin{aligned} G^{(0)} &= \frac{1}{2} \langle \phi_i^{(0)}(\vec{k}, t_1) N_i(-\vec{k}, t_2) \rangle \\ &= \frac{1}{2} \langle e^{-\Gamma_0 k^2 (k^2 + m^2)t_1} \int_{-\infty}^{t_1} dt' e^{\Gamma_0 k^2 (k^2 + m^2)t'} N_i(k, t') N_i(-k, t_2) \rangle \\ &= \frac{1}{2} e^{-\Gamma_0 k^2 (k^2 + m^2)t_1} 2\Gamma_0 k^2 \int_{-\infty}^{t_1} dt' e^{\Gamma_0 k^2 (k^2 + m^2)t'} \delta(t' - t_2) \\ &= \Gamma_0 k^2 e^{-\Gamma_0 k^2 (k^2 + m^2)(t_1 - t_2)} \theta(t_1 - t_2) \end{aligned}$$

The Fourier transform

$$G^{(0)}(k, \omega) = \int_{-\infty}^{\infty} dt_{12} G(k, t_{12}) e^{i\omega t_{12}}$$

immediately leads us to the response function given by Eq.(4.1.12)

$$G^{(0)}(k, \omega) = \left[-\frac{i\omega}{\Gamma_0 k^2} + (k^2 + m^2) \right]^{-1} \quad (4.2.6)$$

The solution at $O(g)$ (Eq.(4.2.3)) is

$$\begin{aligned} \phi_i^{(1)}(t_1) &= e^{-\Gamma_0 k^2 (k^2 + m^2)t_1} \int_{-\infty}^{t_1} dt' e^{\Gamma_0 k^2 (k^2 + m^2)t'} \left[\sum \epsilon_{ijk} (p_1^2 - p_2^2) \right. \\ &\quad \times \phi_j^{(0)}(\vec{p}_1, t') \phi_k^{(0)}(\vec{p}_2, t') \big] \\ &= e^{-\Gamma_0 k^2 (k^2 + m^2)t_1} \int_{-\infty}^{t_1} dt' e^{\Gamma_0 k^2 (k^2 + m^2)t'} \left[\sum \epsilon_{ijk} (p_1^2 - p_2^2) \right. \\ &\quad \times \int e^{-\Gamma_0 p_1^2 (p_1^2 + m^2)(t' - t'')} N_j(\vec{p}_1, t'') dt'' \\ &\quad \times \left. \int e^{-\Gamma_0 p_2^2 (p_2^2 + m^2)(t' - t''')} N_j(\vec{p}_2, t''') dt''' \right] \end{aligned} \quad (4.2.7)$$

For the response function at $O(g)$,

$$\begin{aligned} G^{(1)} &= \frac{1}{2} \langle \phi_i^{(1)} N_i \rangle \\ &= 0 \end{aligned} \quad (4.2.8)$$

The above solution for $\phi_i^{(1)}$ can be expressed graphically as shown in Fig. 4.1. The cross indicates the noise. With

$$\frac{G^{(0)}(t_{12})}{\Gamma_0 k^2} = e^{-\Gamma_0 k^2 (k^2 + m^2)(t_{12})} \theta(t_{12})$$

the lines in the figure corresponding to $G^{(0)}(t_{12})/\Gamma_0 k^2$ with the arrow in the direction of increasing time and the solid circle corresponds to an interaction.

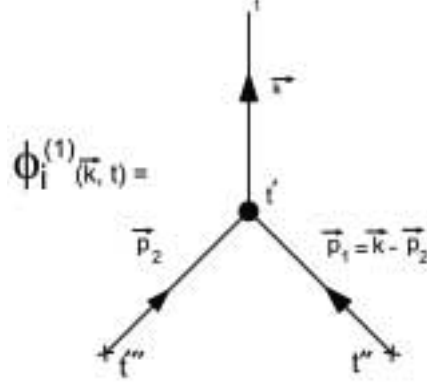


Figure 4.1. Three point vertex where a magnetic fluctuation at a given wave Vector couples to two other fluctuations with different wave vectors

Inserting Eq.(4.2.2) and using $\phi_i^{(1)}$ from Eq.(4.2.5), we have

$$\begin{aligned} \phi^{(2)}(k, t_1) &= \int e^{-\Gamma_0 k^2 (k^2 + m^2)(t_1 - t')} \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \epsilon_{ijk} (p_1^2 - p_2^2) [\phi_j^{(1)}(\vec{p}_1, t') \\ &\quad \times \phi_k^{(0)}(\vec{p}_1, t') + \phi_j^{(0)}(\vec{p}_1, t') \phi_k^{(1)}(\vec{p}_1, t')] dt' \\ &= \int dt' e^{-\Gamma_0 k^2 (k^2 + m^2)(t_1 - t')} \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \epsilon_{ijk} (p_1^2 - p_2^2) \left\{ \int dt'' dt''' \right. \\ &\quad \times dt'''' \sum_{\vec{q}_1 + \vec{q}_2 = \vec{p}_1} e^{-\Gamma_0 p_1^2 (p_1^2 + m^2)(t' - t'')} e^{-\Gamma_0 q_1^2 (q_1^2 + m^2)(t'' - t''')} \\ &\quad \times e^{-\Gamma_0 q_2^2 (q_2^2 + m^2)(t''' - t''')} N_k(q_1, t''') N_i(q_2, t''') \int dt'''' \\ &\quad \times e^{-\Gamma_0 p_2^2 (p_2^2 + m^2)(t' - t''')} N_k(p_2, t''') \\ &\quad \left. + \text{an identical term with } \vec{p}_1 \leftrightarrow \vec{p}_2. \right\} \end{aligned} \quad (4.2.9)$$

The diagrammatic representation of the two terms on the R.H.S of Eq.(4.2.7) would be,

Forming the correlation length with $N(-k, t_2)$ gives the second order response function as

$$\begin{aligned}
G^{(2)}(k, t_{12}) &= \frac{1}{2} \int dt' e^{-\Gamma_0 k^2 (k^2 + m^2)(t_1 - t')} \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} (p_1^2 - p_2^2)(p_2^2 - k^2) \\
&\quad \times \int dt'' dt''' e^{-\Gamma_0 p_1^2 (p_1^2 + m^2)(t' - t'')} 2\Gamma_0 k^2 e^{-\Gamma_0 k^2 (k^2 + m^2)(t'' - t_2)} 2\Gamma_0 p_2^2 \\
&\quad \times e^{-\Gamma_0 p_2^2 (p_2^2 + m^2)(t'' + t' - 2t''')} + \text{an identical term with } \vec{p}_1 \leftrightarrow \vec{p}_2 \\
&= \left\{ \Gamma_0 k^2 \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \int dt' e^{-\Gamma_0 k^2 (k^2 + m^2)(t_1 - t')} \frac{(p_1^2 - p_2^2)(p_2^2 - k^2)}{p_2^2 + m^2} \right. \\
&\quad \times dt'' e^{-[\Gamma_0 p_1^2 (p_1^2 + m^2) + \Gamma_0 p_2^2 (p_2^2 + m^2)](t' - t'')} e^{-\Gamma_0 k^2 (k^2 + m^2)(t'' - t_2)} \\
&\quad \left. + \text{an identical term with } \vec{p}_1 \leftrightarrow \vec{p}_2 \right\} \\
&= -\Gamma_0 k^2 \int dt' dt'' \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \frac{(p_1^2 - p_2^2)^2 (k^2 + m^2)}{(p_1^2 + m^2)(p_2^2 + m^2)} \\
&\quad \times e^{-\Gamma_0 k^2 (k^2 + m^2)(t_1 - t')} \times e^{-[\Gamma_0 p_1^2 (p_1^2 + m^2) + \Gamma_0 p_2^2 (p_2^2 + m^2)](t' - t'')} \\
&\quad \times e^{-\Gamma_0 k^2 (k^2 + m^2)(t'' - t_2)} \tag{4.2.10}
\end{aligned}$$

In Fourier space

$$\begin{aligned}
G^{(2)}(k, \omega) &= \int_{-\infty}^{\infty} G^{(2)}(k, t_{12}) e^{i\omega t_{12}} dt_{12} \\
&= -\Gamma_0 k^2 \frac{(k^2 + m^2)}{[-i\omega + \Gamma_0 k^2 (k^2 + m^2)]^2} \\
&\quad \times \int \frac{d^D p}{(2\pi)^D} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{(p^2 + m^2)[(\vec{k} - \vec{p})^2 + m^2]} \\
&\quad \times \frac{1}{[-i\omega + \Gamma_0 p^2 (p^2 + m^2) + \Gamma_0 (\vec{k} - \vec{p})^2 \{(\vec{k} - \vec{p})^2 + m^2\}]} \\
&= -G_0 \tilde{\Sigma}(k, m, \omega) G_0 \tag{4.2.11}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\Sigma}(k, m, \omega) &= \frac{(k^2 + m^2)}{\Gamma_0 k^2} \int_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \frac{d^D p_1}{(2\pi)^D} \frac{[p_1^2 - p_2^2]^2}{(p_1^2 + m^2)(p_2^2 + m^2)} \\
&\quad \times \frac{1}{-i\omega + \Gamma_0 p_1^2 (p_1^2 + m^2) + \Gamma_0 p_2^2 (p_2^2 + m^2)} \tag{4.2.12}
\end{aligned}$$

The series for $G(k, \omega)$ is thus given by

$$G(k, \omega) = G^{(0)}(k, \omega) - g^2 G_0 \tilde{\Sigma} G_0 + O(g^4)$$

The diagrammatic representation for the response function can be found from Fig. 4.2 by taking the product with $N_i(-k, t_2)$ and averaging. The process of averaging two noise terms brings two time arguments together and one momentum is the negative of the other. The resulting diagrams are shown in Fig. 4.3,

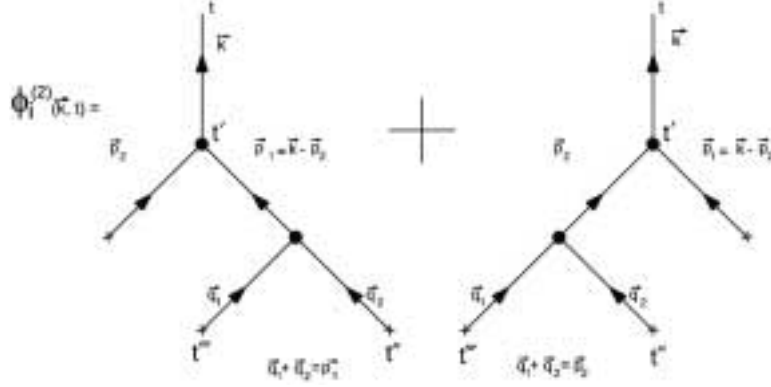


Figure 4.2. How a field at present is affected by the past

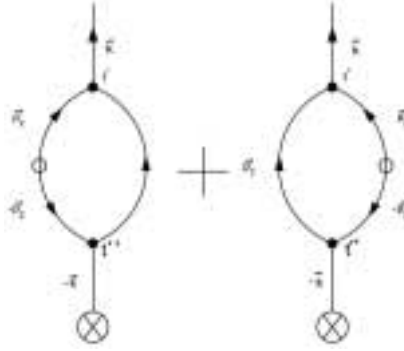


Figure 4.3. The one loop response function

where the line with momentum p carrying the symbol $\otimes$ in the middle has the static correlation factor $(2\Gamma_0 p^2)$ attached to it, while a line of momentum p carrying the symbol $\otimes$ at the end carries only an additional factor of $2\Gamma_0 p^2$. If we imagine working to higher order in the coupling constant in g , then new diagrams

will be generated, but one particular set of diagrams will be simply an iteration of the lower order one, i.e. graphs of the structure shown in Fig. 4.4



Figure 4.4. A two loop disconnected response function

With these structures repeating at every order

$$\begin{aligned} G(k, \omega) &= G^{(0)}(k, \omega) - g^2 G^{(0)}(k, \omega) \tilde{\Sigma}(k, \omega) G^{(0)}(k, \omega) \\ &\quad + g^4 G^{(0)}(k, \omega) \tilde{\Sigma}(k, \omega) G^{(0)}(k, \omega) \tilde{\Sigma}(k, \omega) G^{(0)}(k, \omega) + \dots \\ &= G^{(0)}(k, \omega) - g^2 \tilde{\Sigma}(k, \omega) G(k, \omega) \end{aligned}$$

leading to

$$G(k, \omega) = \frac{G^{(0)}(k, \omega)}{1 + g^2 \tilde{\Sigma}(k, \omega) G^{(0)}(k, \omega)}$$

or

$$G^{-1}(k, \omega) = G^{(0)-1}(k, \omega) + g^2 \tilde{\Sigma}(k, \omega) \quad (4.2.13)$$

which is just the Dyson's equation for the problem. It is convenient to write Eq.(4.2.13) as

$$\begin{aligned} G^{-1}(k, \omega) &= -\frac{i\omega}{\Gamma_0 k^2} + k^2 + m^2 + g^2 \tilde{\Sigma}(k, \omega) \\ &= (\Gamma_0 k^2)^{-1} [-i\omega + \Gamma_0(k^2 + m^2) + \Sigma(k, \omega)] \end{aligned} \quad (4.2.14)$$

where

$$\begin{aligned} \Sigma(k, m, \omega) &= g^2(k^2 + m^2) \int \frac{d^D p_1}{(2\pi)^D} \frac{(p_1^2 - p_2^2)^2}{(p_1^2 + m^2)(p_2^2 + m^2)} \\ &\quad \times \frac{1}{-i\omega + \Gamma_0 p_1^2(p_1^2 + m^2) + \Gamma_0 p_2^2(p_2^2 + m^2)} + O(g^4) \end{aligned} \quad (4.2.15)$$

It is convenient to write this answer as an integral over the correlation function as

$$\begin{aligned} \Sigma(k, m, \omega) = g^2(k^2 + m^2) \int_{\vec{p}_1 + \vec{p}_2 = \vec{k}}^{\omega_1 + \omega_2 = \omega} \frac{d^D p_1}{(2\pi)^D} \frac{d\omega_1}{2\pi} (p_1^2 - p_2^2)^2 \\ \times C^{(0)}(\vec{p}_1, \omega_1) C^{(0)}(\vec{p}_2, \omega_2) + O(g^4) \end{aligned} \quad (4.2.16)$$

It should be noted, however, that the possibility of writing the self-energy $\Sigma(k, \omega)$ in the form of Eq.(4.2.16) as an integral over correlation function is a consequence of the FDT. The general structure is that shown in Fig. 4.3. The self-energy is extracted from the graphs by removing the external lines and in Fig. 4.4, we show the generally true form of the self-energy at this order

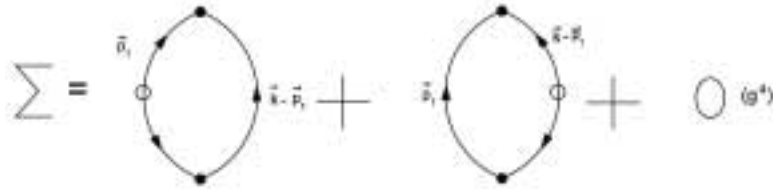


Figure 4.5. The one loop self energy

It should be noted that in writing Eq.(4.2.13) one has included graphs of the form chosen in Fig. 4.4 (technically called reducible - because a cut across the connecting propagator would divide the graph into two) but has left out graphs of the form shown in Fig. 4.6.

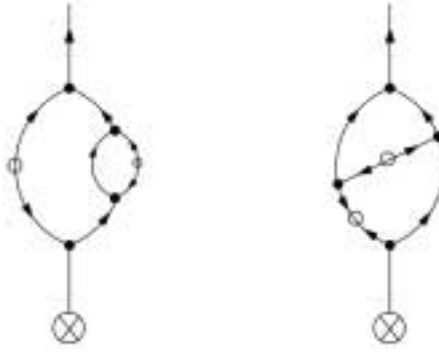


Figure 4.6. The two loop response function

Graphs for G such as shown in Fig. 4.6 would contribute at $O(g^4)$ to $\Sigma(k, m, \omega)$ and the corresponding contributions would be as shown in Fig. 4.7

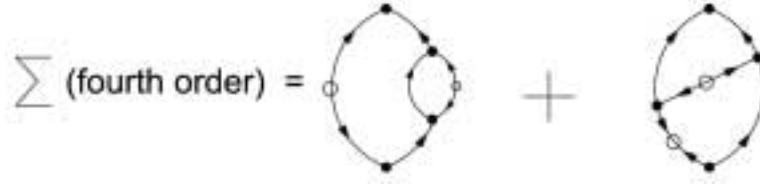


Figure 4.7. The two loop contributions to the self energy

We have not shown all the possibilities at the fourth order.

Because of the FDT, we do not need to set up separately the perturbation theory for the correlation function. When there is no FDT, this needs to be done and the discussion that we have had so far, when carried out for $C(k, \omega)$ to the expansion shown in Fig. 4.8

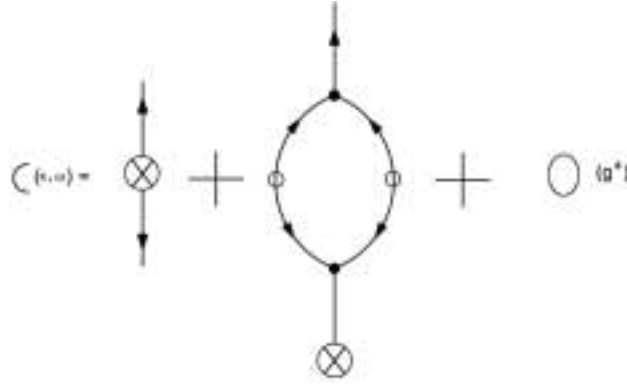


Figure 4.8. The one loop correlation function

We now turn to the physical significance of the self-energy $\Sigma(k, \omega)$. This is best done by referring to Eq.(4.2.14). The self-energy, as is evident from that equation is an addition to $\Gamma_0 k^2(k^2 + m^2)$ the relaxation rate in the linearized theory. Thus $\Sigma(k, \omega)$ is the contribution to the relaxation rate coming from the nonlinear terms. We now turn to Eq.(4.2.15) to estimate how big the contribution is. We work at $m = 0$ (critical point), $\omega = 0$ (long time) and consider the long wavelength limit ($k \rightarrow 0$). The self-energy becomes

$$\Sigma(k) \simeq g^2 k^4 \frac{4}{2D\Gamma_0} \int \frac{C_D}{(2\pi)^D} \frac{p^{1+D}}{p^8} dp \quad (4.2.17)$$

In the above the limit $k \rightarrow 0$, makes

$$p_1 \simeq p_2 = p$$

and

$$p_1^2 - p_2^2 = 2\vec{k} \cdot \vec{p}.$$

also

$$\langle \cos^2 \theta \rangle = \frac{1}{D}$$

The integral converges at the lower limit for $D > 6$ but for $D < 6$, has to be cut off at $p \simeq k$ to prevent the divergence. So $\Sigma(k)$ has a long wavelength divergence for $D < 6$ and in this limit the non-linear contribution to the relaxation rate completely dominates the linear contribution $\Gamma_0 k^4$. When the nonlinear terms dominate, we have (keeping the normalization $G^{-1}(k, 0) = \chi^{-1}$)

$$G^{-1}(k, \omega) = [-i\omega + \Sigma(k, \omega)] \frac{\chi^{-1}}{\Sigma(k, 0)} \quad (4.2.18)$$

For the ferromagnet this happens at $D < 6$. For $D > 6$, the nonlinear terms do not affect the momentum dependence of the relaxation rate as obtained from the linearized model and the nonlinear terms are irrelevant. Thus $D = 6$ is the upper critical dimension. This is exactly the same as we found from the RG treatment in **Chapter 3**.

We are now ready to convert the perturbative result of Eq.(4.2.16) into a non perturbative one. We assume that the perturbative Σ on the L.H.S of Eq.(4.2.16) will become the full Σ , if we replace the bare correlation functions on the R.H.S by the full correlation function $C(k, \omega)$. Because of the FDT, the full $C(k, \omega)$ is related to the full $G(k, \omega)$ and hence to the $\Sigma(k, \omega)$. Thus we have,

$$\begin{aligned} \Sigma(k, \omega) &= (k^2 + m^2)g^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} C(\vec{p}, \omega') \\ &\quad C(\vec{k} - \vec{p}, \omega - \omega') [p^2 - (\vec{k} - \vec{p})^2] \\ C(k, \omega) &= \frac{1}{\omega} \text{Im } G(k, \omega) \\ \text{and } G^{-1}(k, \omega) &= [-i\omega + \Sigma(k, m, \omega)] \frac{\chi^{-1}(k, m)}{\Sigma(k, m)} \end{aligned} \quad (4.2.19)$$

The above set of equations determine the self energy in a self consistent manner. The diagrammatic representation of the above equation is shown in Fig. 4.9. This technique of finding $\Sigma(k, \omega)$ is called the self consistent mode coupling approximation.

(When FDT is not valid, the self consistent mode coupling would involve the lines in the self energy part of Fig. 4.3 represent the full Green's function and the full correlation function.)

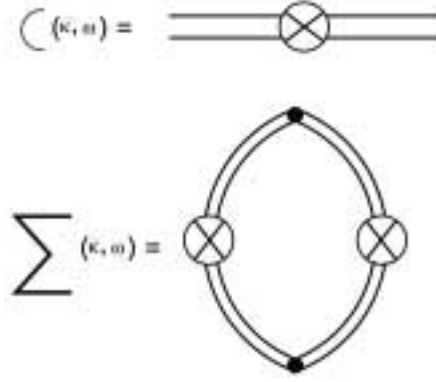


Figure 4.9. The one loop dressed correlation function

We now ask whether Eq.(4.2.19) has a scaling solution. This involves writing

$$\Sigma(k, \omega) = k^z f\left(\frac{m}{k}, \frac{\omega}{k^z}\right) \quad (4.2.20)$$

with the constraint

$$\int C(k, \omega) \frac{d\omega}{2\pi} = \chi(k) = k^{-2} \quad \text{for} \quad m = 0$$

the dimension of $C(k, \omega)$ is k^{-2-z} (the frequency corresponds to k^z) i.e. the scaling form of $C(k, \omega)$

$$C(k, \omega) = \frac{1}{k^{2+z}} g\left(\frac{m}{k}, \frac{\omega}{k^z}\right) \quad (4.2.21)$$

The power count of the integral on the R.H.S in Eq.(4.2.20) gives k^{2+D-z} . The L.H.S is k^z and matching immediately leads to

$$z = 1 + \frac{D}{2} \quad (4.2.22)$$

for $D < 6$, exactly as in the RG treatment of **Chapter 3**.

We now ask how much can one infer about the function $f(\frac{\omega}{k^z})$ in Eq.(4.2.20). At the critical point and at $\omega = 0$, the function is a constant Γ , i.e. $\Sigma(k, 0) = \Gamma k^z$ and one needs to find the value of Γ . In general the determination of the function f has to be done numerically from Eq.(4.2.19). However, we show here that for $D \simeq 6$, there is an analytic solution. For $D = 6$, referring to Eq.(4.2.17) the integral on the R.H.S will be dominated by the high momenta and the self energies on the R.H.S of the integral in Eq.(4.2.19) are effectively at zero frequency and we can write

$$C(k, \omega) = \frac{\chi(k)}{2\Sigma(k)} \left[\frac{1}{-i\omega + \Sigma(k)} + \frac{1}{i\omega + \Sigma(k)} \right] \quad (4.2.23)$$

This allows the frequency integration in Eq.(4.2.19) to be done and leads to

$$\begin{aligned} \Sigma(k, \omega) &= g^2 \chi^{-1}(k) \int \frac{d^D p}{(2\pi)^D} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{p^2 (\vec{k} - \vec{p})^2} \\ &\times \frac{1}{-i\omega + \Sigma(\vec{p}) + \Sigma(\vec{k} - \vec{p})} \end{aligned} \quad (4.2.24)$$

Since the self consistency of the above approach is for high values of the internal momentum p , we can carry out the high momentum approximation in the integrand of the above equation. For $\omega = 0$, and remembering that $D \simeq 6$,

$$\begin{aligned} \Gamma k^{1+\frac{D}{2}} &= g^2 k^4 \frac{C_6}{(2\pi)^6} \frac{4}{D} \frac{1}{2\Gamma} \int_k^\infty \frac{p^{D-1} p^2}{p^{1+\frac{D}{2}} p^4} dp \\ &= g^2 k^{1+\frac{D}{2}} \frac{C_6}{(2\pi)^6} \frac{4}{D} \frac{1}{2\Gamma} \int_1^\infty \frac{1}{x^{4-\frac{D}{2}}} dx \\ &= \frac{g^2}{\Gamma} k^{1+\frac{D}{2}} \frac{C_6}{(2\pi)^6} \frac{4}{6} \frac{1}{2} \frac{2}{\epsilon} \end{aligned} \quad (4.2.25)$$

where $\epsilon = 6 - D$. Thus

$$\frac{\Gamma^2}{g^2} = \frac{2}{3} \frac{C_6}{(2\pi)^6} \frac{1}{\epsilon} \quad (4.2.26)$$

for $\epsilon \ll 1$, which is exactly identical to Eq.(4.2.24) obtained from the RG technique. Thus, if the self consistent mode coupling equations are solved by iterating around the upper critical dimension, one obtains a perturbation series which matches exactly the perturbative RG result.

Before ending this section, we note that making the Ansatz of Eq.(4.2.23) for the correlations functions on the R.H.S. of Eq.(4.2.19) is a first step in an iterative process of situation and is in general a very effective technique for obtaining a solution. With this approximation for $C(k, \omega)$, the self-energy $\Sigma(k, \omega)$ is given in general by,

$$\begin{aligned} \Sigma(k, m, \omega) &= g^2 \chi^{-1}(k, m) \int \frac{d^D p}{(2\pi)^D} \frac{1}{p_1^2 + m^2} \frac{1}{p_2^2 + m^2} \\ &\times \frac{(p_1^2 - p_2^2)^2}{-i\omega + \Sigma(p_1, m) + \Sigma(p_2, m)} \end{aligned} \quad (4.2.27)$$

As we have already seen, for $m = 0$ and $\omega = 0$,

$$\Sigma(k, \omega) = \Gamma k^{1+\frac{D}{2}}.$$

Now for $m \gg k$, we need to find out $\Sigma(k, m)$. Inspection of Eq.(4.2.27), shows that the integral is proportional to k^2 and we anticipate

$$\Sigma(k, m) \sim k^2 m^y \quad \text{for} \quad m \gg k.$$

Self consistency of Eq.(4.2.27) requires

$$\begin{aligned} \Sigma(k, m) &= g^2 m^2 \frac{4}{D} k^2 \frac{C_6}{(2\pi)^6} \int \frac{p^{D-1} p^2}{(p^2 + m^2)^2 2\Sigma(p, m)} dp \\ &= g^2 \frac{2}{D} \frac{C_6}{(2\pi)^6} k^2 m^2 \frac{p^{D-1} p^2}{(p^2 + m^2)^2} dp \\ &\propto m^{D-2-y}, \end{aligned} \quad (4.2.28)$$

leading to

$$y = \frac{D}{2} - 1 \quad (4.2.29)$$

We now investigate another limit, namely the case of very high frequency, i.e.

$$\begin{aligned} \omega &\gg \Gamma k^{1+\frac{D}{2}} \\ \text{or} \quad \omega &\gg \tilde{\Gamma} k^2 m^{\frac{D}{2}-1}. \end{aligned}$$

In this limit, k and m both tend to zero and appropriate approximations in Eq.(4.2.27), lead to

$$\begin{aligned} \Sigma(k, \omega) &= g^2 (k^2 + m^2) k^2 \frac{4}{D} \frac{C_D}{(2\pi)^D} \int \frac{p^{D-1} p^2}{p^2 p^2 [-i\omega + 2\Sigma(p)]} dp \\ &= g^2 (k^2 + m^2) k^2 \frac{2}{D} \frac{C_D}{(2\pi)^D} \int \frac{p^{D-3}}{-i\frac{\omega}{2} + p^{1+\frac{D}{2}}} dp \\ &= g^2 (k^2 + m^2) k^2 \left(\frac{2}{-i\omega} \right)^{\frac{3-D/2}{1+D/2}} \frac{1}{D} \frac{C_D}{(2\pi)^D} \beta \left(\frac{7D-4}{2(2+D)}, \frac{12-5D}{2(2+D)} \right) \end{aligned} \quad (4.2.30)$$

With the above limiting forms for $\Sigma(k, m, \omega)$, we have the following results:

$$\Sigma(k, m, \omega) = k^2 (k^2 + m^2) L(k, m, \omega) \quad (4.2.31)$$

where

$$\begin{aligned} L(k, m, \omega) &\propto k^{-\epsilon/2} && \text{for} \quad m = \omega = 0 \\ &\propto m^{-\epsilon/2} && \text{for} \quad k = \omega = 0 \\ &\propto \left(\frac{2}{-i\omega} \right)^{-\frac{\epsilon}{2+D}} && \text{for} \quad k = m = 0 \end{aligned} \quad (4.2.32)$$

where $\epsilon = 6 - D$

4.3 Spherical Limit

In this section, we discuss an approximation that was invented forty years ago for problems in non-linear dynamics but has attracted attention only recently. This is the spherical limit. It has been a popular approximation in static critical phenomena for more than two decades, but its place in dynamics was not appreciated until recently. There are some contemporary ways of introducing this technique, but we will use the very general framework that Kraichnan used in his pioneering work, and cast it in a language appropriate to the discussion of the Heisenberg model with which we have demonstrated both the RG and self consistent mode coupling techniques. Our equation of motion is, in momentum space

$$\begin{aligned}\dot{\phi}_i(k) &= -\Gamma k^2(k^2 + m^2)\phi_i \\ &\quad + g \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \epsilon_{ijl}(p_1^2 - p_2^2) \phi_j(\vec{p}_1) \phi_l(\vec{p}_2) + N_i \\ \langle N_i(k, \omega) N_j(k', \omega') \rangle &= 2\Gamma k^2 \delta_{ij} \delta^D(\vec{k} + \vec{k}') \delta(\omega + \omega')\end{aligned}\quad (4.3.1)$$

To set up a spherical limit, one needs to introduce a M -component generalization of the above dynamical system. Kraichnan's method is to introduce M -copies of the system with the magnetization vector for the m^{th} copy being denoted by $\vec{\phi}_{[m]}$. A collective coordinate $\vec{\phi}^\alpha$ is now defined as

$$\vec{\phi}^\alpha = \frac{1}{\sqrt{M}} \sum_{m=1}^M e^{2\pi i \alpha m / M} \vec{\phi}_{[m]} \quad (4.3.2)$$

This index α runs from $-s$ to s in steps of unity with $2s + 1 = M$. One now needs to introduce an equation of motion $\dot{\phi}^\alpha$, which will reduce to Eq.(4.3.1) if $M = 1$. The required equation of motion is

$$\begin{aligned}\dot{\phi}^\alpha(\vec{k}) &= -\Gamma k^2(k^2 + m^2)\phi^{(\alpha)}(k) + \frac{g}{\sqrt{M}} \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} \epsilon_{ijl} \\ &\quad \times \sum_{\beta} \lambda_{\alpha, \beta, \alpha - \beta} (p_1^2 - p_2^2) \phi_j^{(\beta)}(\vec{p}_1) \phi_l^{(\alpha - \beta)}(\vec{p}_2) + N_i^\alpha \\ \langle N_i^{(\alpha)}(k, \omega) N_j^{(\beta)}(k', \omega') \rangle &= 2\Gamma k^2 \delta_{ij} \delta^D(\vec{k} + \vec{k}') \delta(\omega + \omega') \delta_{\alpha\beta}\end{aligned}\quad (4.3.3)$$

The coupling $\lambda_{\alpha, \beta, \alpha - \beta}$ are endowed with the following properties

- i) $\lambda_{\alpha, \beta, \alpha - \beta} = \lambda_{\alpha, \alpha - \beta, \beta}$
- ii) $\lambda_{\alpha, \beta, \alpha - \beta} = 1$ *if anyone of the subscripts is zero*
- iii) $\lambda_{\alpha, \beta, \alpha - \beta} = \lambda_{-\alpha, -\beta, -(\alpha - \beta)}^*$
- iv) $\lambda_{\alpha, \beta, \alpha - \beta}^* = \lambda_{\alpha - \beta, -\beta, \alpha} = \lambda_{\beta, \alpha, -(\alpha - \beta)}$

The second condition ensures that Eq.(4.3.4) reduces to Eq(4.3.1) for $M = 1$. Thus Eqns.(4.3.3) are a proper generalization of Eqns.(4.3.1). The first constraint written down above expresses a natural symmetry of the system, namely in the coupling of modes on the R.H.S of Eq.(4.3.3) it does not matter whether the mode with momentum $\vec{p}_1$ belongs to the β^{th} or the $(\alpha - \beta)^{th}$ replica of the system. The third constraint ensures that in the course of evolution, the field $\vec{\phi}_m$ remains real. This is done by demanding that the equation of motion for $\vec{\phi}_m$, corresponding to Eq.(4.3.3), the non-linear term remains real. The fourth constraint above is obtained from the fact there must exist an equilibrium probability distribution at all times for doing averages. For the replicated system, this distribution corresponds to the action (“free energy”)

$$F = \sum \phi_i^\alpha(k) \phi_i^\alpha(-k)(k^2 + m^2)$$

and the equation of motion must remain

$$\frac{dF}{dt} = 0.$$

For this to hold under Eq.(4.3.3), we must have the fourth constraint.

It should be noted that the procedure carried out so far is absolutely general and can be carried through any dynamical system. The idea is to introduce M replicas and coupling between the replicas in the non-linear term in a manner such that the correct equation of motion is obtained for $M = 1$. We now introduce the vital step that will allow the model to be exactly solvable for $M \rightarrow \infty$ and it should be noted that this step too is very general and can be carried out for any system. This step constitutes writing

$$\lambda_{\alpha,\beta,\alpha-\beta} = e^{i\theta_{\alpha,\beta,\alpha-\beta}} \quad (4.3.4)$$

where the phase factors $\theta_{\alpha,\beta,\alpha-\beta}$ take a value randomly between 0 and 2π for every assigned value of α and β .

The task of the theory is to calculate the self-energy $G(k, \omega)$, which we learnt in the previous section can be written as

$$G^{-1}(k, \omega) = -i\omega + \Gamma k^2(k^2 + m^2) + \Sigma(k, \omega)$$

where $\Sigma(k, \omega)$ has an expansion in powers of g . The complete terms of $O(g^2)$ are shown in Fig. 4.5 and the typical fourth order term (i.e. $O(g^4)$) terms are shown in Fig. 4.7. In Fig. 4.9, we show the same diagrams with two contributions of Fig. 4.5 combined into one and some similar combinations carried out in Fig. 4.7.

Imagine constructing $\Sigma^{\alpha\alpha}$ where α is a fixed replica index. We first consider the $O(g^2)$ calculation corresponding to graph (a) in Fig. 4.10. The coupling constant brings in g^2/M . Now the mode α connects to modes β and $\alpha - \beta$. Since the noise correlation involves the same replica (see Eq.(4.3.3)), if the λ factor at one vertex

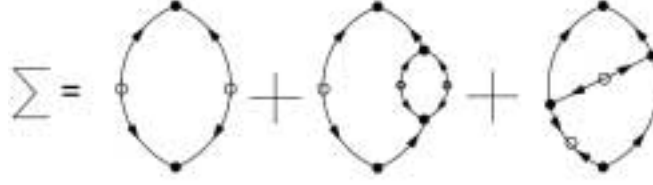


Figure 4.10. The two lowest orders for the self energy

is $\lambda_{\alpha, \beta, \alpha-\beta}$, then at the other it has to be $\lambda_{\beta, \beta-\alpha, \alpha}$. Thus, the replica part of the contribution to Σ is

$$\begin{aligned}
 \frac{1}{M} \sum_{\beta=1}^M \lambda_{\alpha, \beta, \alpha-\beta} \lambda_{\beta, \beta-\alpha, \alpha} &= \frac{1}{M} \sum_{\beta=1}^M \lambda_{\alpha, \beta, \alpha-\beta} \sum_{\beta=1}^M \lambda_{\alpha, \beta, \alpha-\beta}^* \\
 &= \frac{1}{M} \sum_{\beta=1}^M |\lambda_{\alpha, \beta, \alpha-\beta}|^2 \quad (\text{Using (iv) and (i)}) \\
 &= 1
 \end{aligned}$$

where we have used Eq.(4.3.4) in arriving at the last line. This the contributions of Fig. 4.10a to $\Sigma(k, \omega)$ is $O(1)$.

Turning to the contribution of graph(b) of Fig. 4.10, we note that these are **two independent free replica indices** left to sum over after the preliminary algebra. These correspond to the line on the left to the graph(b) and one of the lines in the sub graph on the right hand line Fig 4.10b. These two sums provide a factor M^2 , while the four ' g ' vertices each come with a factor of $M^{-1/2}$ and thus produce M^{-2} . Consequently, this contribution is $O(1)$.

Now, consider the contribution from Fig. 4.10c. The four vertices produces factors which can be written as the string

$$\sum_{\beta, \gamma} \lambda_{\alpha, \beta, \alpha-\beta} \lambda_{\beta, \gamma, \beta-\gamma} \lambda_{\gamma, \beta-\alpha, \gamma+\alpha-\beta} \lambda_{\beta-\gamma, \beta-\gamma-\alpha, \alpha}$$

Note that every time, we choose a value of β , the γ -value has to be fixed at $\gamma = 2\beta - \alpha$, so that the λ factors can combine to produce $|\lambda|^4 = 1$. If it did not do this, then the sum over indices would produce zero due to the random phase averaging introduced following Eq.(4.3.4). Thus β and γ are not independent. Hence the double sum in the string produces a factor of M only and the contribution

of Fig. 4.10c to Σ is $O(\frac{1}{M^2}M) = O(\frac{1}{M})$. If we take $M \rightarrow \infty$ limit, this contribution can be dropped.

Thus, we notice a great simplification in the limit $M \rightarrow \infty$ which is dubbed the *spherical limit*. The contribution to $\Sigma(k, \omega)$ is given by graph(a) in Fig. 4.10 and all other graphs where the lines of graph(a) are dressed by a graph of the form shown in graph(a), e.g. the contribution from graph(b). Thus, in the spherical limit, the graphical expansion of Σ is as given in Fig. 4.11.

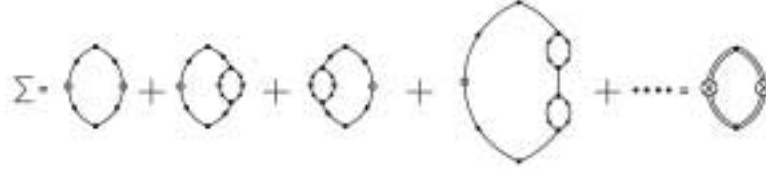


Figure 4.11. Summing the self energy in the spherical limit

which is simply the self consistent mode coupling form of Eq.(4.2.19). If one is interested in the scaling limit, where the contribution coming from the linear term in the equation of motion can be dropped,

$$G^{-1}(k, \omega) = -i\omega + \Sigma(k, \omega) \quad (4.3.5)$$

corresponding to Fig. 4.11

$$\begin{aligned} \Sigma(k, \omega) = & g^2(k^2 + m^2) \int_{\vec{p}_1 + \vec{p}_2 = \vec{k}}^{\omega_1 + \omega_2 = \omega} \frac{d^D p_1}{(2\pi)^D} \frac{d\omega_1}{2\pi} (p_1^2 - p_2^2)^2 \\ & \times C(p_1, \omega_1) C(p_2, \omega_2) \end{aligned} \quad (4.3.6)$$

and from the FDT

$$C(k, \omega) = \frac{1}{\omega} \text{Im } G(k, \omega) \quad (4.3.7)$$

In the spherical limit and in the scaling regime, Eqns.(4.3.5)-(4.3.7) constitute an exact solution to the problem of critical dynamics in the Heisenberg model.

We end this section by a discussion of whether the spherical limit and self consistent mode coupling are identical. The answer is no. Self consistent mode coupling can be carried beyond the single loop approximation shown in Fig. 3.9 and indeed it is often necessary to do that in the field of dynamic critical phenomena where the experiments are precise and better accuracy than that provided by Fig. 3.9 is necessary. The spherical limit on the other hand is exact. There are no $O(g^4)$ in this limit. The self consistent mode coupling approximation of Fig. 3.9 gives an exact answer to a generalized problem in the $M \rightarrow \infty$ limit.

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Critical Dynamics in Fluids

5.1 Introduction

We have already encountered the phenomenon of critical slowing down in our discussion of ferromagnetic transition in chapters 2 and 3. In the study of static response in second order phase transitions, there is a very strong universality which tells us that it is only the dimensionality of the order parameter which determines universality classes. Consequently, the best known critical phenomenon which is the critical point of a liquid-vapour system (well studied both theoretically and experimentally for about one hundred and twenty five years) has the same behaviour as the Ising model which was introduced to study the magnetic transition. The order parameter (the quantity whose expectation value is zero above the transition and non-zero below) of the liquid vapour system is the difference between the densities in the liquid and vapour state. This field is a scalar - capable of only being only positive or negative. The Ising model consists of spins which can point up or down and thus can be positive or negative and hence in the continuum limit is represented by a scalar order parameter field. The static critical behaviour of the two systems are consequently identical. For the dynamic responses, this broad universality does not hold. The liquid-vapour system has nothing to do with the Ising model.

We have already seen evidence of this in our discussion of the ferromagnetic transition. The *Ginzburg-Landau* model for a n -component order parameter is adequate for discussing the static responses near the paramagnet-ferromagnet transition but is totally irrelevant for studying the dynamic responses. The dynamic response is dictated by reversible terms in the equation of motion for the order parameter. The reversible terms are so called because they keep the equilibrium distribution unchanged in time and are simply the physical terms which determine the dynamics. In our ferromagnet, this simply meant recognizing that the magnetization which is the order parameter responds to the torque (magnetization comes from the 'angular momentum') and writing down equations of motion based on that ensures a correct handling of the dynamics. This is true for all systems showing

dynamic critical behaviour. In this chapter, we discuss the most well-known system exhibiting critical behaviour from this point of view and then discuss another well known system - the superfluid transition - of liquid ^4He which gave rise to the concept of dynamic scaling.

To start our discussion we focus on a system which is in the same universality class as the liquid-vapour system - this is the symmetric binary mixture. This is a 50 – 50 mixture of atoms A and atoms B , which is completely miscible above a critical temperature T_c . Below T_c , there is phase separation. If C_A is the concentration of the A atoms and C_B the concentration of B atoms, The order parameter is $\delta C = C_A - C_B$. For $T > T_c$, $\langle \delta C \rangle = 0$, where the angular brackets denote statistical averaging. When phase separation occurs for $T < T_c$, $\langle \delta C \rangle$ is clearly non-zero. Thus $\langle \delta C \rangle$ is clearly the order parameter ϕ of the transition. This clear cut definition of the order parameter makes us choose this as a better example - for the liquid vapour system the analogue of $\delta C = C_A - C_B$ is $\delta \rho = \rho_L - \rho_g$, ρ_L and ρ_g being the densities in the liquid and gaseous phase. However, experiments show that the coexistence curve is not symmetric and the order parameter is really a mixture of density and energy fluctuations. In fact the relevant combination of density and energy is very much like the entropy.

To discuss the dynamic response in the binary mixture, we imagine the system to be in equilibrium and then we imagine applying an external force. The force is $\vec{F}$ per unit mass on every atom A and $-\vec{F}$ on every atom B . The system now settles in to a steady state where there will be a finite current corresponding to the concentration difference ψ . The A -current and B -current are equal and opposite and the current $\vec{j}^\psi$ corresponding to the concentration difference ψ is

$$\vec{j}^\psi = \vec{j}^A - \vec{j}^B = 2\vec{j}^A \quad (5.1.1)$$

For small values of the force $\vec{F}$, the current is proportional to the force and we can write

$$\vec{j}^\psi = \lambda \vec{F} \quad (5.1.2)$$

the constant of proportionality defining the transport coefficient for the concentration difference. The same current can be generated if instead of an external force, we have a concentration gradient in the system. In this case, the current is proportional to the gradient of chemical potential and we have

$$\vec{j}^\psi = -\lambda \vec{\nabla} \mu = -\frac{\lambda}{\chi_\psi} \vec{\nabla} \psi \quad (5.1.3)$$

where

$$\chi_\psi = \left. \frac{\partial \mu}{\partial \psi} \right|_{P,T} \quad (5.1.4)$$

is the isothermal susceptibility of the system. The equation of motion for ψ can be found from the conservation law

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= -\vec{\nabla} \cdot \vec{j} \psi = \frac{\lambda}{\chi_\psi} \nabla^2 \psi \\ &= D_\psi \nabla^2 \psi\end{aligned}\quad (5.1.5)$$

where D_ψ is the familiar concentration diffusion coefficient. In terms of the transport coefficient λ , we find

$$D_\psi = \frac{\lambda}{\chi_\psi}. \quad (5.1.6)$$

The relaxation rate for the Fourier component $\psi(k)$ is clearly $D_\psi k^2$.

If instead of a concentration current, we talk about a heat current, then the transport coefficient is the thermal conductivity λ_T , the corresponding susceptibility is the constant pressure specific heat and for the thermal diffusion coefficient is

$$D_T = \frac{\lambda_T}{C_P}. \quad (5.1.7)$$

For the momentum current, the transport coefficient is the shear viscosity η and the corresponding susceptibility is simply the density of the system. The momentum diffusion coefficient is the kinematic viscosity ν and we have

$$\nu = \frac{\eta}{\rho}. \quad (5.1.8)$$

We now need to discuss what happens to the transport coefficient near the critical point. Close to the critical point, regions where fluctuations are correlated become very large and the correlation length ξ which characterizes the system becomes infinite at $T = T_c$. Close to $T = T_c$, $\xi \sim t^{-\nu}$ where

$$t = \frac{T - T_c}{T_c}.$$

Over a region of size ξ in a D dimensional space, the applied force $\vec{f}$ is given by

$$\vec{f} = \xi^D \psi \vec{F} \quad (5.1.9)$$

where $\vec{F}$ is the applied force per unit mass. In the steady state, where there is a uniform velocity $\vec{v}$, there is an equal and opposite resistive Stokes' law force which is given by

$$\vec{f}_r \propto \vec{v} \eta \xi^{D-2} \quad (5.1.10)$$

The current $\vec{j}$ is given by

$$\begin{aligned}\vec{j} &= \langle \psi \vec{v} \rangle = \left\langle \frac{\psi}{\eta} \xi^{2-D} \vec{f}_r \right\rangle \\ &= \left\langle \frac{\psi}{\eta} \xi^{2-D} \vec{f} \right\rangle \\ &= \left\langle \xi^2 \frac{\psi^2}{\eta} \right\rangle \vec{F}\end{aligned}\quad (5.1.11)$$

leading to the transport coefficient

$$\lambda = \frac{\xi^2}{\eta} \langle \psi^2 \rangle. \quad (5.1.12)$$

The expectation value $\langle \psi^2 \rangle$ is proportional to the susceptibility per unit volume and thus

$$\langle \psi^2 \rangle \propto \frac{\chi_\psi}{\xi^D} \propto \xi^{2-\eta_0-D}. \quad (5.1.13)$$

From static critical phenomenon

$$\chi_\psi \propto \xi^{2-\eta_0},$$

where η_0 is the anomalous dimension index, the subscript put to differentiate it from the shear viscosity η . The transport coefficient is given by

$$\lambda = \frac{\xi^2}{\eta} \langle \psi^2 \rangle \propto \frac{\xi^{4-D-\eta_0}}{\eta} \quad (5.1.14)$$

Assuming that the shear viscosity has no critical variation (or a very weak one even if it has one, since it is not the response function directly associated with the order parameter), we see that the transport coefficient diverges for $D < 4 - \eta_0 \simeq 4(\eta_0 \simeq O(10^{-2}))$.

The above discussion shows two things

- i) *the transport of order parameter is strongly affected by critical fluctuations and the transport coefficient diverges at the critical point.*
- ii) *The velocity fluctuations are involved in the process since the current is provided by the velocity. The velocity fluctuations are not critical but couple to the order parameter. Without detailed calculations, it is not possible to say how the transport of the momentum fluctuations is affected, and what would be if any, the critical behaviour of the shear viscosity.*

5.2 Equations for Transport Coefficients

We thus see that to proceed further and get a quantitative estimate of the different transport coefficients, the coefficients associated with concentration flow, the heat flow, the momentum flow, and also the density fluctuations determining the sound propagation - equations of motion need to be established. For the concentration difference ψ for the binary liquid, the equation of motion in Eq.(5.1.5) needs to be augmented by including the coupling to the velocity field. This is a very straightforward modification of the time derivative in Eq.(5.1.5). Instead of the partial time derivative at a given point in space, we need the total time derivative which includes the change brought about by the transport by the velocity field. This leads to the equation of motion

$$\frac{\partial \psi}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \psi = D \nabla^2 \psi = \frac{\lambda}{\chi} \nabla^2 \psi \quad (5.2.1)$$

We now need to write down the equation of motion for the velocity field. This is the Navier-Stokes equation. For the time being we will ignore the sound wave and assume incompressible flow i.e.

$$\vec{\nabla} \cdot \vec{v} = 0 \quad (5.2.2)$$

The velocity fluctuations are small and all flows are in the very low Reynold's number regime. So the usual non-linearity $(\vec{v} \cdot \vec{\nabla}) \vec{v}$ [and also $\vec{\nabla} P$, P being the pressure, which for an incompressible flow is proportional to $(\vec{v} \cdot \vec{\nabla}) \vec{v}$] is negligible and for the i^{th} component of the velocity vector

$$\frac{\partial v_i}{\partial t} = T_{ij,j} \quad (5.2.3)$$

where T_{ij} is the stress tensor. The usual contribution to the stress tensor is

$$\eta \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right].$$

What we need to know is how does the concentration fluctuation affect the stress tensor, because it is through the coupling to the order parameter field that we expect behaviour to arise. To this end, we need to know the equilibrium free energy. This is the *Ginzburg-Landau* free energy augmented by a quadratic term in velocity to account for the probabilistic distribution of the velocity fluctuations. We will drop the quadratic term in the *Ginzburg-Landau* free energy since it is not particularly relevant directly to the dynamics. It's effect on the statics will be taken care of by assuming that the mass term in the free energy has the correct critical behaviour. The free energy F is

$$F = \int d^D x \left[\frac{\kappa}{2} \psi^2 + \frac{1}{2} (\nabla \psi)^2 + \frac{\rho}{2} v^2 \right] \quad (5.2.4)$$

As stated above the term ψ^4 is missing in F , but its effect on the statics is included by stipulating that the mass term $\kappa \propto t^\nu$ where ν is the correct critical exponent ($\nu \simeq 0.62$ for $n = 1$, i.e. scalar order parameter) and t is the reduced temperature

$$t = \frac{T - T_c}{T_c},$$

T_c being the critical point temperature. To find the stress tensor, we imagine the fluid away from the critical point and apply an external potential $V(\vec{x})$, to which the system responds by setting up a spatially dependent $\psi(\vec{x})$, such that the contribution to the energy is $\psi(\vec{x})V\vec{x}$. The free energy F in the presence of $V(\vec{x})$ is

$$F(\psi, \vec{\nabla}\psi, V(\vec{x})) = F(\psi, \vec{\nabla}\psi, V=0) + \psi V \quad (5.2.5)$$

The force $\vec{K}$ is obtained from the gradient of F and

$$K_i = -\frac{\partial F}{\partial x_i} = -\frac{\partial F}{\partial \psi} \frac{\partial \psi}{\partial x_i} - \sum_j \frac{\partial F}{\partial \left(\frac{\partial \psi}{\partial x_j}\right)} \frac{\partial}{\partial x_i} \left(\frac{\partial \psi}{\partial x_j}\right) - \psi \frac{\partial V}{\partial x_i} \quad (5.2.6)$$

In the presence of $V(\vec{x})$, the system will have to adjust itself to minimize

$$\int F(\psi, \vec{\nabla}\psi, V(\vec{x})) d^D x$$

and that leads to the canonical Euler-Lagrange equation

$$\frac{\partial F}{\partial \psi} = \sum_i \frac{\partial}{\partial x_i} \left(\frac{\partial F}{\partial \left(\frac{\partial \psi}{\partial x_i}\right)} \right) \quad (5.2.7)$$

Using this condition in Eq.(5.2.6)

$$\begin{aligned} K_i &= -\sum_j \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial \left(\frac{\partial \psi}{\partial x_j}\right)} \right) \frac{\partial \psi}{\partial x_i} - \sum_j \left(\frac{\partial F}{\partial \left(\frac{\partial \psi}{\partial x_j}\right)} \right) \frac{\partial}{\partial x_j} \left(\frac{\partial \psi}{\partial x_i} \right) - \psi \frac{\partial V}{\partial x_i} \\ &= -\sum_j \frac{\partial}{\partial x_j} \left[\left(\frac{\partial F}{\partial \left(\frac{\partial \psi}{\partial x_j}\right)} \right) \frac{\partial \psi}{\partial x_i} \right] - \psi \frac{\partial V}{\partial x_i} \\ &= -\sum_j \frac{\partial T_{ij}}{\partial x_j} - \psi \frac{\partial V}{\partial x_i} \end{aligned} \quad (5.2.8)$$

where

$$T_{ij} = \left(\frac{\partial F}{\partial \left(\frac{\partial \psi}{\partial x_j}\right)} \right) \frac{\partial \psi}{\partial x_i} \quad (5.2.9)$$

For a more general way of finding the non-linear contribution to the velocity equation due to concentration fluctuations we need to ensure that the free energy F is

unaffected by the non-linear terms. With $\dot{\psi}$ given in Eq.(5.2.1) we have (*keeping only the non-linear terms in the equation of motion*)

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \int F d^D x = \int \left[\rho v_i \dot{v}_i + \kappa^2 \psi \dot{\psi} + \frac{\partial \psi}{\partial x_j} \frac{\partial \dot{\psi}}{\partial x_j} \right] d^D x \\ &= \int \left[\rho v_i \dot{v}_i - \kappa^2 \psi v_i \frac{\partial \psi}{\partial x_i} - \frac{\partial \psi}{\partial x_j} \frac{\partial}{\partial x_j} \left\{ v_i \frac{\partial \psi}{\partial x_i} \right\} \right] d^D x \\ &= \int \left[\rho v_i \dot{v}_i - \kappa^2 \psi v_i \frac{\partial \psi}{\partial x_i} + v_i \frac{\partial \psi}{\partial x_i} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right] d^D x \quad (5.2.10) \end{aligned}$$

keeping in mind that $\vec{v}$ is compressible thus gives the equation of motion (so far as the non-linear terms go)

$$\rho \dot{v}_i = P_{il} \left[-\frac{\partial \psi}{\partial x_l} \frac{\partial^2 \psi}{\partial x_j \partial x_j} + \kappa^2 \psi \frac{\partial \psi}{\partial x_l} \right] \quad (5.2.11)$$

The missing linear term in the above equation comes from the shear contribution to the stress tensor and

$$\dot{v}_i = \nu \nabla^2 v_i + P_{il} \left[\kappa^2 \psi \frac{\partial \psi}{\partial x_l} - \frac{\partial \psi}{\partial x_l} \nabla^2 \psi \right] \quad (5.2.12)$$

The deterministic equations shown in Eqns.(5.2.11) and (5.2.12) cannot maintain the equilibrium distribution because the linear terms are dissipative. These are hydrodynamic terms and are associated with the long wavelength behaviour. The short wavelength fluctuations give rise to noise terms in the equations of motion for ψ and $\vec{v}$. These noise terms with appropriate correlations to ensure fluctuation dissipation theorem maintain the equilibrium. The final equations of motion, consequently, are

$$\dot{\psi} = \frac{\lambda}{\chi} \nabla^2 \psi - (\vec{v} \cdot \vec{\nabla}) \psi + N_\psi \quad (5.2.13)$$

$$\dot{v}_i = \nu \nabla^2 v_i + P_{il} \left[\kappa^2 \psi \frac{\partial \psi}{\partial x_l} - \frac{\partial \psi}{\partial x_l} \nabla^2 \psi \right] + N_i^v \quad (5.2.14)$$

The correlations are

$$\begin{aligned} \langle N_\psi(\vec{x}, t) N_\psi(\vec{x}', t') \rangle &= -2\lambda \nabla^2 \delta(\vec{x} - \vec{x}') \delta(t - t') \\ \langle N_i^v(\vec{x}, t) N_j^v(\vec{x}', t') \rangle &= -2\nu \nabla^2 \delta(\vec{x} - \vec{x}') \delta(t - t') P_{ij} \end{aligned} \quad (5.2.15)$$

where P_{ij} is the projection operator

$$P_{ij} = \delta_{ij} - \frac{\partial^2}{\partial x_i \partial x_j}$$

which is necessary to maintain the transverse nature of the velocity field.

In momentum space, the equations of motion are

$$\begin{aligned}\dot{\psi}(\vec{k}) &= -\frac{\lambda}{\chi} k^2 \psi(\vec{k}) - i \sum_{\vec{p}} v_j(\vec{k} - \vec{p}) p_j \psi(\vec{p}) + N_\psi(\vec{k}) \\ &= -\lambda(k^2 + \kappa^2) k^2 \psi(\vec{k}) - i \sum_{\vec{p}} k_j v_j(\vec{k} - \vec{p}) \psi(\vec{p}) + N_\psi(\vec{k})\end{aligned}\quad (5.2.16)$$

$$\begin{aligned}\dot{v}_j(\vec{k}) &= -\nu k^2 v_j(\vec{k}) + i\kappa^2 \sum_{\vec{p}} p_j \psi(\vec{k} - \vec{p}) \psi(\vec{p}) \\ &\quad + \frac{i}{2} P_{jl}(k) \sum_{\vec{p}} p_l [(\vec{k} - \vec{p})^2 - p^2] \psi(\vec{k} - \vec{p}) \psi(\vec{p}) + N_j^v(\vec{k})\end{aligned}\quad (5.2.17)$$

In the above, we have used

$$\chi^{-1} = k^2 + \kappa^2.$$

If the non-linear terms are absent, then the Green's functions and correlation functions can be written down as

$$\begin{aligned}G_\psi(k, \kappa, \omega) &= -i\omega + \lambda(k^2 + \kappa^2)k^2 \\ C_\psi^{(0)}(k, \kappa, \omega) &= \frac{2\lambda k^2}{\omega^2 + \lambda^2 k^4 (k^2 + \kappa^2)^2} \\ [G_v^{(0)-1}]_{ij} &= (-i\omega + \eta k^2) P_{ij}(k) \\ \text{and} \quad [C_v^{(0)}]_{ij} &= \frac{2\eta k^2}{\omega^2 + \eta^2 k^4} P_{ij}(k)\end{aligned}\quad (5.2.18)$$

where the momentum space projection operator is

$$P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2}$$

The dynamic critical exponents for the order parameter and velocity fields are different at this zeroth level. At the critical point, $\kappa = 0$, the relaxation rate for the ψ field is λk^4 , giving $z_\psi = 4$. For the velocity field relaxation rate is ηk^2 , giving $z_v = 2$. The non-linear terms are likely to affect λ and η , and we use the self-consistent technique developed in **Chapter 4** to find the answer in the next section.

5.3 One-Loop Perturbation Theory

In this section, we will investigate the single loop perturbation theory contribution to the transport coefficients λ and η . To do so, we first look at the lowest order

perturbation theory for the transport coefficient λ . What we need to calculate is the lowest order contribution to the self-energy for ψ and $\vec{v}$, in terms of which, we write the response function as,

$$G_{\psi}^{-1}(k, \kappa, \omega) = G_{\psi}^{(0)-1} + \Sigma_{\psi}(k, \kappa, \omega) \quad (5.3.1)$$

$$G_v^{-1}(k, \kappa, \omega) = G_v^{(0)-1} + \Sigma_v(k, \kappa, \omega) \quad (5.3.2)$$

A practical way of calculating the response function like

$$G_{\psi} = \left\langle \frac{\partial \psi}{\partial N_{\psi}} \right\rangle$$

is to calculate the correlation function $\langle \psi(k) N_{\psi}(-k) \rangle$. If we write Eqns.(5.2.13) and (5.2.14) in frequency space,

$$(-i\omega + \frac{\lambda}{\chi} k^2) \psi(\vec{k}, \omega) = -i \sum_{p, \omega'} k_j v_j(\vec{k} - \vec{p}, \omega - \omega') \psi(\vec{p}, \omega') + N_{\psi}(\vec{k}, \omega) \quad (5.3.3)$$

$$\begin{aligned} (-i\omega + v k^2) v_j(\vec{k}, \omega) &= \frac{i}{2} P_{jl}(k) \sum_{p, \omega'} [p^2 + (\vec{k} - \vec{p})^2] p_l \psi(\vec{k} - \vec{p}, \omega - \omega') \\ &\quad \times \psi(\vec{p}, \omega') + N_j^v(\vec{k}, \omega) \end{aligned} \quad (5.3.4)$$

then the zeroth order solutions (ignores the non-linear terms) are

$$\psi^{(0)}(\vec{k}, \omega) = G_{\psi}^{(0)}(\vec{k}, \omega) N_{\psi}(\vec{k}, \omega) \quad (5.3.5)$$

$$v_j^{(0)}(\vec{k}, \omega) = G_v^{(0)}(\vec{k}, \omega) N_j^v(\vec{k}, \omega) \quad (5.3.6)$$

If we expand the fields as

$$\psi(\vec{k}, \omega) = \psi^{(0)}(\vec{k}, \omega) + \psi^{(1)}(\vec{k}, \omega) + \psi^{(2)}(\vec{k}, \omega) + \dots \quad (5.3.7)$$

$$v_j(\vec{k}, \omega) = v_j^{(0)}(\vec{k}, \omega) + v_j^{(1)}(\vec{k}, \omega) + v_j^{(2)}(\vec{k}, \omega) + \dots \quad (5.3.8)$$

where $\psi^{(1)}(\vec{k}, \omega)$ and $v_j^{(1)}(\vec{k}, \omega)$ are the first iteration contribution coming from Eqns.(5.3.3) and (5.3.4). Explicitly,

$$\psi^{(1)}(\vec{k}, \omega) = -i G_{\psi}^{(0)}(\vec{k}, \omega) \sum_{p, \omega'} k_j v_j^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(0)}(\vec{p}, \omega') \quad (5.3.9)$$

and

$$v_j^{(1)}(\vec{k}, \omega) = \frac{i}{2} P_{jl}(k) G_v^{(0)}(\vec{k}, \omega) \sum_{p, \omega'} [-p^2 + (\vec{k} - \vec{p})^2] \times p_l \psi^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(0)}(\vec{p}, \omega') \quad (5.3.10)$$

(the projection operator P_{jl} ensures the incompressibility condition). Going to the next order

$$\psi^{(2)}(\vec{k}, \omega) = -i G_\psi^{(0)}(\vec{k}, \omega) \sum_{\vec{p}, \omega'} k_j [v_j^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(1)}(\vec{p}, \omega') + v_j^{(1)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(0)}(\vec{p}, \omega')] \quad (5.3.11)$$

$$v_j^{(2)}(\vec{k}, \omega) = \frac{i}{2} G_v^{(0)}(\vec{k}, \omega) \sum_{\vec{p}, \omega'} [(\vec{k} - \vec{p})^2 - p^2] p_j \{ \psi^{(1)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(0)}(\vec{p}, \omega') + \psi^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(1)}(\vec{p}, \omega') \} \quad (5.3.12)$$

To construct the response function

$$G_\psi(\vec{k}, \omega) = \langle \psi(\vec{k}, \omega) N(-\vec{k}, -\omega) \rangle = \langle \psi^{(0)}(\vec{k}, \omega) N(-\vec{k}, -\omega) \rangle + \langle \psi^{(1)}(\vec{k}, \omega) N(-\vec{k}, -\omega) \rangle + \langle \psi^{(2)}(\vec{k}, \omega) N(-\vec{k}, -\omega) \rangle + \dots \quad (5.3.13)$$

The first term is our $G_\psi^{(0)}(k, \omega)$. The second term is zero, when we use Eq.(5.3.9) and find that $v_j^{(0)}$ cannot be paired. The first non-zero contribution has to come from the third term.

$$\langle \psi^{(1)} N_\psi \rangle = -i G_\psi^{(0)}(k, \omega) \sum [v_j^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(1)}(p, \omega') + v_j^{(1)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(0)}(p, \omega')] k_j N_\psi(-k, -\omega) \quad (5.3.14)$$

We will handle the two terms separately. First

$$\begin{aligned} & \sum_{p, \omega'} \langle v_j^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(1)}(p, \omega') k_j N_\psi(-k, -\omega) \rangle \\ &= -i \sum_{p, \omega'} \langle v_j^{(0)}(\vec{k} - \vec{p}, \omega - \omega') k_j G_\psi^{(0)}(p, \omega') \\ & \times \sum_{q, \omega''} p_l v_l^{(0)}(\vec{k} - \vec{p}, \omega' - \omega'') \psi^{(0)}(q, \omega'') N_\psi(-k, \omega) \rangle \end{aligned}$$

$$\begin{aligned}
& \sum_{p, \omega'} \langle v_j^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(1)}(p, \omega') k_j N_\psi(-k, -\omega) \rangle \\
&= -i \sum_{p, \omega'} k_j p_l \sum_{q, \omega''} \langle v_j^{(0)}(\vec{k} - \vec{p}, \omega - \omega') v_l^{(0)}(\vec{p} - \vec{q}, \omega' - \omega'') \rangle \\
&\quad \times G_\psi^{(0)}(p, \omega') \langle \psi^{(0)}(q, \omega'') N_\psi(-k, -\omega) \rangle \\
&= -i \sum_{p, \omega'} k_j p_l \sum_{q, \omega''} C_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') P_{jl}(\vec{k} - \vec{p}) \\
&\quad \times G_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k, \omega) \delta(\vec{q} - \vec{k}) \delta(\omega - \omega'') \\
&= -i \sum_{p, \omega'} k_j p_l P_{jl}(\vec{k} - \vec{p}) C_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') G_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k, \omega)
\end{aligned} \tag{5.3.15}$$

where we have used Eq.(5.3.9) for $\psi^{(1)}(p, \omega')$ and the fact that in the zeroth order, a complicated correlation function factors into products of correlations of two fields at a time, which is Wick's theorem for problems of this sort. Very often there will be more than one way of doing the factoring and the final answer is the sum of all possible factorings. For the second contribution in Eq.(5.3.14)

$$\begin{aligned}
& \sum_{p, \omega'} k_j \langle v_j^{(1)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(0)}(p, \omega') N(-k, -\omega) \rangle \\
&= i \sum_{p, \omega} k_j G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') P_{jl}(\vec{k} - \vec{p}) \sum_{q, \omega''} [\kappa^2 + (\vec{k} - \vec{p} - \vec{q})^2] q_l \\
&\quad \times \langle \psi^{(0)}(\vec{k} - \vec{p} - \vec{q}, \omega - \omega' - \omega'') \psi^{(0)}(q, \omega'') \psi^{(0)}(p, \omega') N(-k, -\omega) \rangle \\
&= i \sum_{p, \omega'} \sum_{q, \omega''} k_j q_l P_{jl}(\vec{k} - \vec{p}) G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') [\kappa^2 + (\vec{k} - \vec{p} - \vec{q})^2] \\
&\quad \times \{ \langle \psi^{(0)}(\vec{k} - \vec{p} - \vec{q}, \omega - \omega' - \omega'') \psi^{(0)}(p, \omega') \rangle \langle \psi^{(0)}(q, \omega'') N(-k, -\omega) \rangle \\
&\quad + \langle \psi^{(0)}(q, \omega'') \psi^{(0)}(p, \omega') \rangle \langle \psi^{(0)}(\vec{k} - \vec{p} - \vec{q}, \omega - \omega' - \omega'') N(-k, -\omega) \rangle \} \\
&= i \sum_{p, \omega'} \sum_{q, \omega''} k_j q_l P_{jl}(\vec{k} - \vec{p}) G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') [\kappa^2 + (\vec{k} - \vec{p} - \vec{q})^2] \\
&\quad \times \{ C_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k, \omega) \delta(\vec{q} - \vec{k}) \delta(\omega - \omega'') \\
&\quad + C_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k, \omega) \delta(\vec{q} + \vec{p}) \delta(\omega' + \omega'') \} \\
&= i \sum_{p, \omega'} k_j k_l P_{jl}(\vec{k} - \vec{p}) G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') (p^2 + \kappa^2) C_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k, \omega) \\
&\quad + i \sum_{p, \omega'} (-k_j p_l) P_{jl}(\vec{k} - \vec{p}) G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \\
&\quad \times (k^2 + \kappa^2) C_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k, \omega)
\end{aligned} \tag{5.3.16}$$

We now use the FDT to write

$$\begin{aligned} C_{\psi}^{(0)}(k, \omega) &= \frac{1}{k^2 + \kappa^2} [G_{\psi}^{(0)} + G_{\psi}^{(0)*}] \\ C_v^{(0)} &= G_v^{(0)} + G_v^{(0)*} \end{aligned} \quad (5.3.17)$$

The following identities are used to simplify the various expressions

$$\begin{aligned} k_j p_l P_{jl}(\vec{k} - \vec{p}) &= k_j p_l \left[\delta_{jl} - \frac{(\vec{k} - \vec{p})_j (\vec{k} - \vec{p})_l}{(\vec{k} - \vec{p})^2} \right] \\ &= \vec{k} \cdot \vec{p} - \frac{(k^2 - \vec{k} \cdot \vec{p})(\vec{k} \cdot \vec{p} - p^2)}{(\vec{k} - \vec{p})^2} \\ &= \frac{(\vec{k} \cdot \vec{p})[k^2 + p^2 - 2(\vec{k} \cdot \vec{p})] - (p^2 + k^2)(\vec{k} \cdot \vec{p}) + p^2 k^2 + (\vec{k} \cdot \vec{p})^2}{(\vec{k} - \vec{p})^2} \\ &= \frac{p^2 k^2 - (\vec{k} \cdot \vec{p})^2}{(\vec{k} - \vec{p})^2} \end{aligned} \quad (5.3.18)$$

$$k_j k_l P_{jl}(\vec{k} - \vec{p}) = k^2 - \frac{(k^2 - \vec{k} \cdot \vec{p})^2}{(\vec{k} - \vec{p})^2} = \frac{p^2 k^2 - (\vec{k} \cdot \vec{p})^2}{(\vec{k} - \vec{p})^2} \quad (5.3.19)$$

We now put together Eqns.(5.3.14), (5.3.15) and (5.3.16) to write

$$\begin{aligned} \langle \psi^{(2)} N_{\psi} \rangle &= -G_{\psi}^{(0)}(k, \omega) \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega'}{2\pi} \left\{ \frac{p^2 k^2 - (\vec{k} \cdot \vec{p})^2}{(\vec{k} - \vec{p})^2} C_v^{(0)}(k - p, \omega - \omega') \right. \\ &\quad \times G_{\psi}^{(0)}(p, \omega') + \frac{p^2 k^2 - (\vec{k} \cdot \vec{p})^2}{(\vec{k} - \vec{p})^2} G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') C_{\psi}^{(0)}(p, \omega') \\ &\quad \times (k^2 + \kappa^2) - \frac{p^2 k^2 - (\vec{k} \cdot \vec{p})^2}{(\vec{k} - \vec{p})^2} G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \\ &\quad \left. \times C_{\psi}^{(0)}(p, \omega')(p^2 + \kappa^2) \right\} G_{\psi}^{(0)}(k, \omega) \end{aligned} \quad (5.3.20)$$

We now make use of the FDT. This expresses the correlation function in terms of the response functions as shown in Eq.(5.3.17). The response function has the advantage that it is causal and so in the frequency space all the poles in $G(k, \omega)$ have to occur in the lower half plane. Consequently, we have the frequency integral

$$\begin{aligned} C_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') G_{\psi}(\vec{p}, \omega') d\omega' &= \int d\omega' \left[G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \right. \\ &\quad \left. + G_v^{(0)*}(\vec{k} - \vec{p}, \omega - \omega') \right] G_{\psi}^{(0)}(\vec{p}, \omega') \\ &= \int G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') G_{\psi}^{(0)}(\vec{p}, \omega') d\omega' \end{aligned} \quad (5.3.21)$$

The last step follows from the fact that $G_{\psi}^{(0)}$ has poles only in the lower half of the complex ω' -plane. Similarly, $G_v^{(0)}(\omega - \omega')$ has poles only in the lower half of the complex $\omega - \omega'$ plane and hence in the upper half of the complex ω' -plane. This puts the ω' -poles of $G_v^{(0)*}(\omega - \omega')$ in the lower half of the complex ω' -plane. Since $G_v^{(0)*}(\omega - \omega')$ and $G_{\psi}^{(0)}$ have no poles in the upper half plane the integral

$$\int G_v^{(0)*}(\omega - \omega') G_{\psi}^{(0)}(\omega') d\omega' = 0.$$

In each of the three terms on the R.H.S of Eq.(5.3.20) an identical argument when carried through gives

$$\begin{aligned} \langle \psi^{(2)} N_{\psi} \rangle &= -[G_{\psi}^{(0)}(k, \omega)]^2 \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega'}{2\pi} \frac{p^2 k^2 - (\vec{k} \cdot \vec{p})^2}{(\vec{k} - \vec{p})^2} \\ &\quad \times \frac{k^2 + \kappa^2}{p^2 + \kappa^2} G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') G_{\psi}^{(0)}(p, \omega') \end{aligned} \quad (5.3.22)$$

Noting that

$$\begin{aligned} (\vec{k} - \vec{p})^2 - [(\vec{k} - \vec{p}) \cdot \vec{k}]^2 &= k^4 + p^2 k^2 - 2k^2(\vec{k} \cdot \vec{p}) - [k^2 - \vec{p} \cdot \vec{k}]^2 \\ &= p^2 k^2 - (\vec{p} \cdot \vec{k})^2 \end{aligned}$$

We can now cast Eq.(5.3.22) in the form

$$\begin{aligned} \langle \psi^{(2)} N_{\psi} \rangle &= -[G_{\psi}^{(0)}(\vec{k}, \omega)]^2 \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega'}{2\pi} \frac{k^2 + \kappa^2}{p^2 + \kappa^2} \\ &\quad \times k^2 \sin^2 \theta G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') G_{\psi}^{(0)}(\vec{p}, \omega') \\ &= -k^2(k^2 + \kappa^2)[G_{\psi}^{(0)}(\vec{k}, \omega)]^2 \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega'}{2\pi} \frac{\sin^2 \theta}{p^2 + \kappa^2} \\ &\quad \times G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') G_{\psi}^{(0)}(\vec{p}, \omega') \end{aligned} \quad (5.3.23)$$

where θ is the angle between $\vec{k}$ and $\vec{q} = \vec{k} - \vec{p}$. From the contribution to the response function in the second order perturbation as shown in Eq.(5.3.23), the self-energy Σ_{ψ} is (as should be clear from our discussion in Chapter 2),

$$\begin{aligned} \Sigma_{\psi}(k, \kappa, \omega) &= k^2(k^2 + \kappa^2) \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega'}{2\pi} \frac{\sin^2 \theta}{p^2 + \kappa^2} \\ &\quad \times G_v^{(0)}(\vec{k} - \vec{p}, \omega - \omega') G_{\psi}^{(0)}(\vec{p}, \omega') \end{aligned} \quad (5.3.24)$$

We now turn to the velocity response and write

$$\begin{aligned} \langle v_j(\vec{k}, \omega) N_j^{(v)}(-\vec{k}, -\omega) \rangle &= \langle v_j^{(0)} N_j^{(v)} \rangle + \langle v_j^{(1)} N_j^{(v)} \rangle + \langle v_j^{(2)} N_j^{(v)} \rangle + \dots \\ &= G^{(0)}(\vec{k}, \omega)(D - 1) + \langle v_j^{(1)} N_j^{(v)} \rangle + \langle v_j^{(2)} N_j^{(v)} \rangle + \dots \end{aligned} \quad (5.3.25)$$

The factor $D - 1$ comes from the sum over j in the projection operator in a D -dimensional space. With $v_j^{(1)}$ from Eq.(5.3.10), we find easily that

$$\langle v_j^{(1)} N_j^{(1)} \rangle = 0.$$

The first nontrivial contribution comes from the third term in Eq.(5.3.25). Using Eq.(5.3.12)

$$\begin{aligned} \langle v_j^{(2)} N_j^{(v)} \rangle &= \frac{i}{2} G_v^{(0)}(\vec{k}, \omega) P_{jl}(\vec{k}) \left\langle \sum_{\vec{p}, \omega'} [(\vec{k} - \vec{p})^2 - p^2] p_l \left\{ \psi^{(0)}(\vec{p}, \omega') \right. \right. \\ &\quad \times \psi^{(1)}(\vec{k} - \vec{p}, \omega - \omega') + \psi^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(1)}(\vec{p}, \omega') \Big\} \\ &\quad \times N_j^v(\vec{k}, \omega) \Big\rangle \\ &= \frac{1}{2} G_v^{(0)}(\vec{k}, \omega) P_{jl}(\vec{k}) \left\langle \sum_{\vec{p}, \omega'} [(\vec{k} - \vec{p})^2 - p^2] p_l \sum_{\vec{q}, \omega''} \left\{ (\vec{k} - \vec{p})_n \right. \right. \\ &\quad \times G_\psi^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \psi^{(0)}(\vec{q}, \omega'') v_n^{(0)}(\vec{k} - \vec{p} - \vec{q}, \omega - \omega' - \omega'') \\ &\quad \times \psi^{(0)}(\vec{p}, \omega') + p_n G_\psi^{(0)}(\vec{p}, \omega') \psi^{(0)}(\vec{q}, \omega'') v_n^{(0)}(\vec{p} - \vec{q}, \omega' - \omega'') \\ &\quad \times \psi^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \Big\} \Big\rangle \\ &= \frac{1}{2} G_v^{(0)}(\vec{k}, \omega) P_{jl}(\vec{k}) \sum_{\vec{p}, \vec{q}}^{\omega', \omega''} [(\vec{p} - \vec{k})^2 - p^2] \left\{ p_l (\vec{k} - \vec{p})_n \right. \\ &\quad \times G_\psi^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \langle v_n^{(0)}(\vec{k} - \vec{p} - \vec{q}, \omega - \omega' - \omega'') N_j^v(\vec{k}, \omega) \rangle \\ &\quad \times \langle \psi^{(0)}(\vec{q}, \omega'') \psi^{(0)}(\vec{p}, \omega') \rangle + p_l p_n G_\psi^{(0)}(\vec{p}, \omega') \langle \psi^{(0)}(\vec{q}, \omega'') \\ &\quad \times \psi^{(0)}(\vec{k} - \vec{p}, \omega - \omega') \rangle \langle v_n^{(0)}(\vec{p} - \vec{q}, \omega' - \omega'') N_j^v(\vec{k}, \omega) \rangle \Big\} \\ &= \frac{1}{2} G_v^{(0)}(\vec{k}, \omega) P_{jl}(\vec{k}) \sum_{\vec{p}, \omega'} [(\vec{p} - \vec{k})^2 - p^2] \left\{ p_l (\vec{k} - \vec{p})_n C_\psi^{(0)}(p, \omega') \right. \\ &\quad \times G_\psi^{(0)}(\vec{k} - p, \omega - \omega') G_v^{(0)}(\vec{k}, \omega) P_{nj}(\vec{k}) + p_l p_n G_\psi^{(0)}(p, \omega') \\ &\quad \times C_\psi^{(0)}(\vec{k} - p, \omega - \omega') G_v^{(0)}(\vec{k}, \omega) P_{nj}(\vec{k}) \Big\} \quad (5.3.26) \end{aligned}$$

We now use the FDT and the identities

$$P_{jl}(k) p_l (\vec{k} - \vec{p})_n P_{nj}(k) = -P_{jl}(k) p_l p_n P_{nj}(k) = -p^2 + (\vec{p} \cdot \vec{k})^2 / k^2,$$

to simplify Eq.(5.3.26) and write it as

$$\begin{aligned} \langle v_j^{(2)} N_j^{(v)} \rangle &= -\frac{1}{2} [G_v^{(0)}(k, \omega)]^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} G_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k-p, \omega-\omega') \\ &\quad \times \frac{[p^2 - (\vec{k} - \vec{p})^2]^2 [p^2 - \frac{(\vec{p} \cdot \vec{k})^2}{k^2}]}{(p^2 + \kappa^2) [\kappa^2 + (\vec{p} - \vec{k})^2]} \end{aligned} \quad (5.3.27)$$

Comparing with Eq.(5.3.25) and remembering the construction of the self energy from the perturbation series for G , we have for the second order perturbation theory

$$\begin{aligned} \Sigma_v &= \frac{1}{2(D-1)} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} G_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k-p, \omega-\omega') \\ &\quad \times \frac{p^2 \sin^2 \theta [p^2 - (\vec{k} - \vec{p})^2]^2}{(p^2 + \kappa^2) [\kappa^2 + (\vec{p} - \vec{k})^2]} \end{aligned} \quad (5.3.28)$$

In Eqns.(5.3.24) and (5.3.28), we have constructed the lowest order (second order in perturbation theory) self energies for the order parameter and velocity fields. These are the contributions to the relaxation rate for these and we will analyse them in what follows to see what they imply for the contribution of the critical fluctuations.

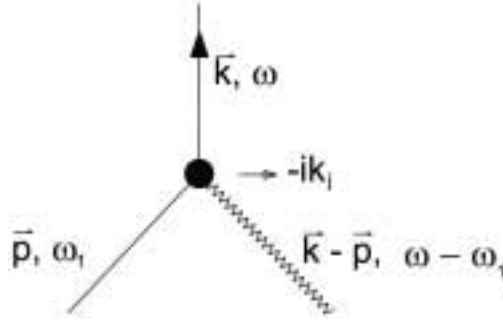


Figure 5.1. Vertex for the order parameter fluctuations

5.4 Diagrammatic Perturbation Theory

In this section, we show the perturbation theory in diagrammatic terms. We have two kinds of fields - straight lines which denote the order parameter field and wavy lines which denote the velocity field. There are two kinds of vertices shown in Figs. 5.1a and 5.1b

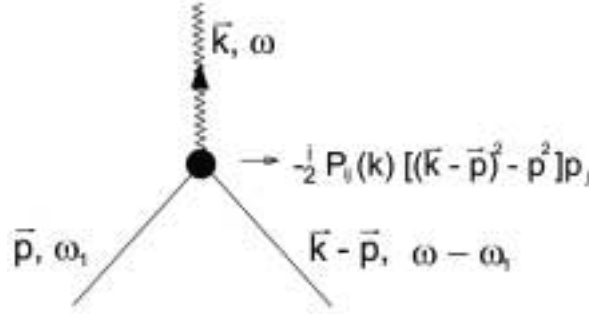


Figure 5.1. Vertex for the velocity fluctuations

- i) the order parameter line breaking into a order parameter and a velocity field
- ii) the velocity line breaking into two order parameter lines

The momentum factors occurring with the vertices are shown in the figure. At each vertex, momentum and frequency are conserved. We now make the convention that a line with an arrow is a propagator and a line with a circle is a correlator. The noise for the concentration field is shown by a dot within the circle and that for the velocity field is shown by a cross within a circle. The response function being a correlation of the field and noise, the diagrammatic representation is shown in Fig. 5.2

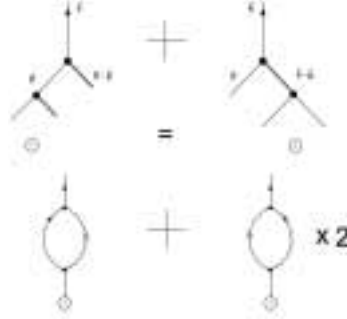


Figure 5.2. Order parameter response function at one loop order

The factor of 2 shown in one of the diagrams is = in? Fig. 5.2a comes from the combinatorics - either of the two lines emanating from the velocity line at momentum $\vec{k} - \vec{p}$ can combine with the noise term. Reading off the contribution from Fig. 5.2a according to the rules, we have the second order contribution $G_{\psi}^{(2)}$ to G_{ψ} work out as

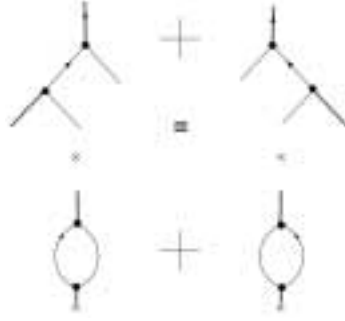


Figure 5.2. Velocity response function at one loop order

$$\begin{aligned}
 G_{\psi}^{(2)} &= [G_{\psi}^{(0)}]^2 \sum_{p, \omega'} \{(-ik_j)(-ip_l)P_{jl}(k-p)G_{\psi}^{(0)}(p, \omega') \\
 &\quad \times C^{(0)}(k-p, \omega-\omega') + 2 \times (-ik_j)\frac{i}{2}P_{jl}(k-p)p_l[k^2-p^2] \\
 &\quad \times C_{\psi}^{(0)}(p, \omega')G_v^{(0)}(k-p, \omega-\omega')\} \quad (5.4.1)
 \end{aligned}$$

The reader should verify that this is exactly the same as obtained in Eq.(5.3.23). The contribution $G_v^{(2)}$ from Fig. 5.2b can be similarly worked out. If we now go to the next two orders then the proliferations are shown in Fig. 5.3 and (5.4).

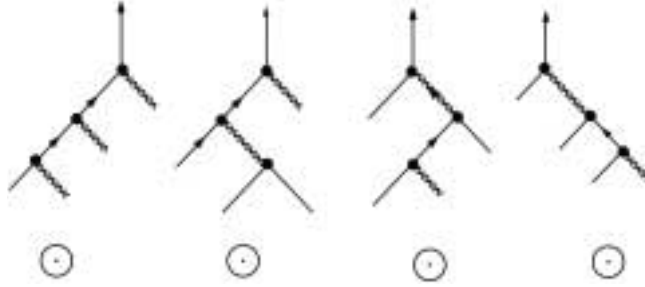


Figure 5.3. Tracing back the order parameter field to the noise to find the response

With the graphs shown in Figs. 5.3 and 5.4, we now try to form correlation functions. The graph of Fig. 5.3a gives zero because there is always an uncorrelated velocity field left. The non-zero contribution comes from Fig. 5.3b. After forming correlators, the resulting diagrams are shown in Fig. 5.5.

The diagrams in the first row of Fig. 5.5 dress the external $G_{\psi}(k)$. The diagrams in the second row of Fig. 5.5 are self energy insertions - so called because they consist of dressing the order parameter and velocity lines in the loop of the second order diagrams shown in Fig. 5.2. These diagrams can in principle

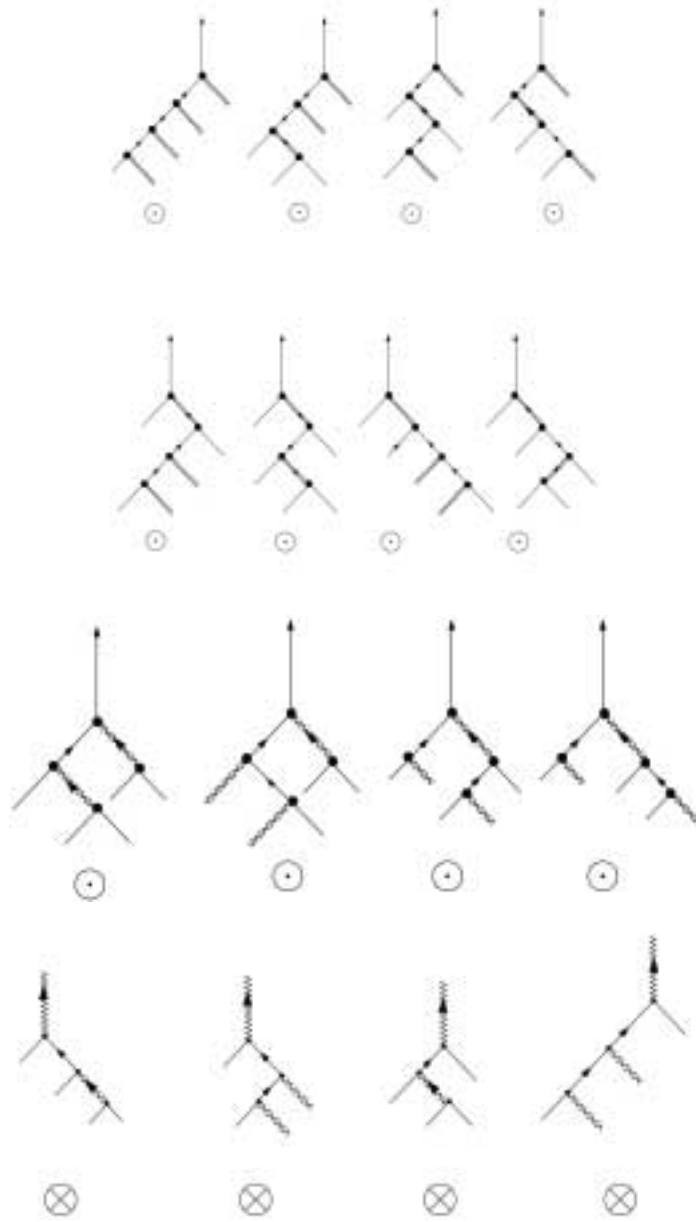
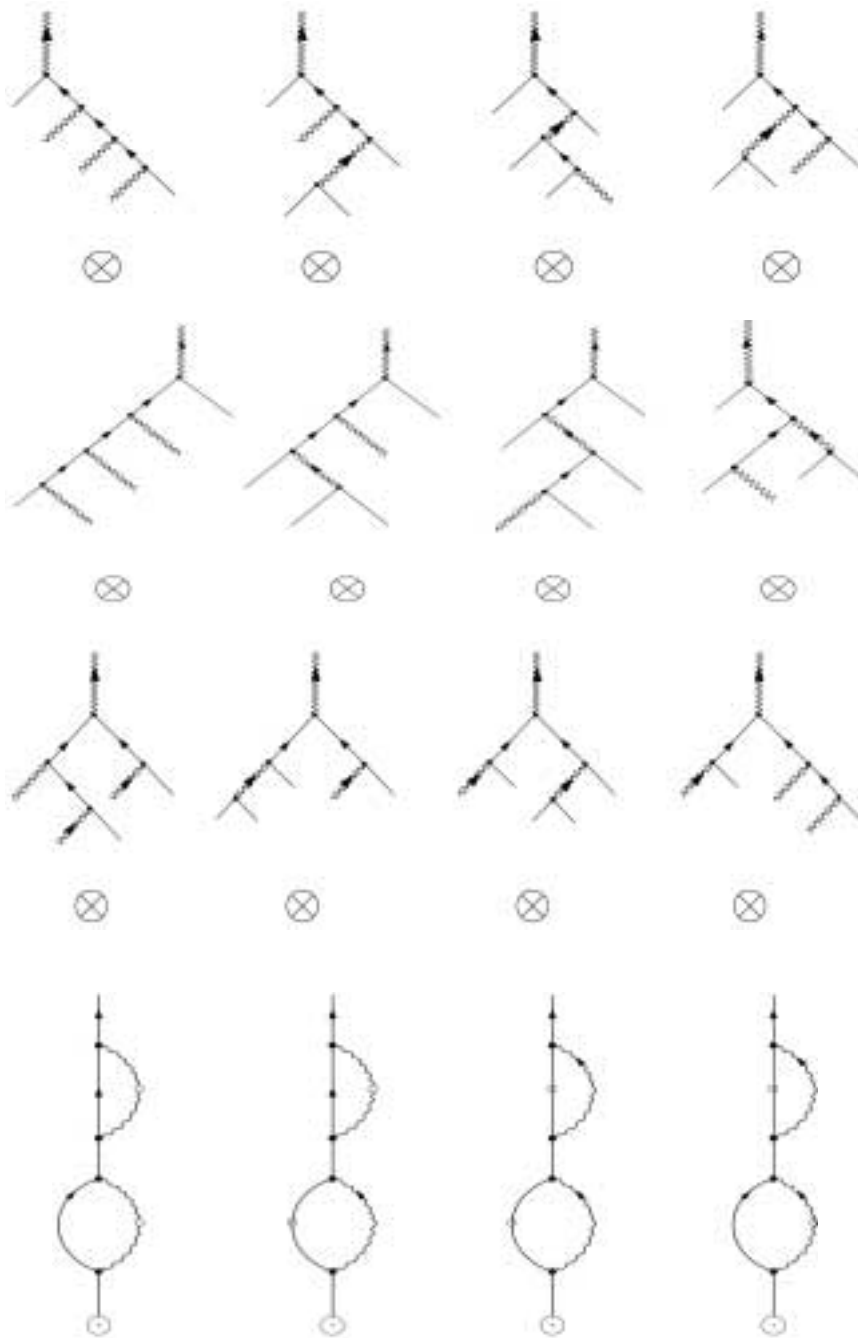


Figure 5.4. Tracing back the velocity field to the noise to find the response function (two loops)



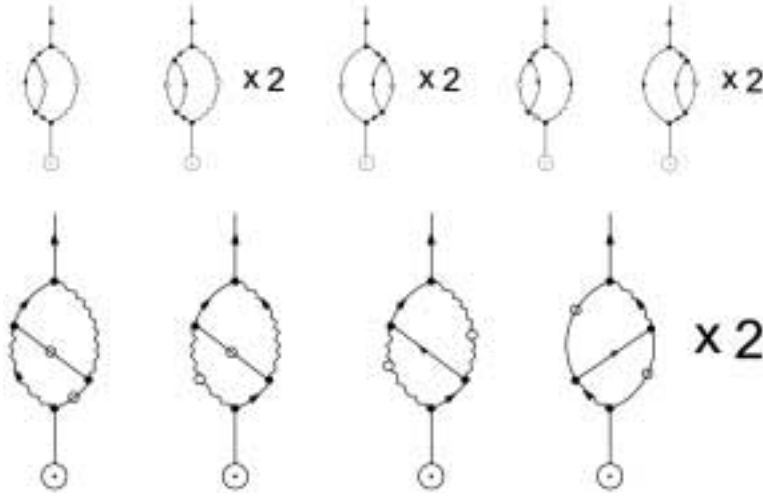


Figure 5.5. The two loop response function for the order parameter

be handled by dressing the lines involved in the single loop graphs. The diagrams in third row of Fig. 5.5 constitute the vertex correction diagrams and are in some sense the genuine two-loop diagrams. The vertex correction contributions to the fourth order velocity response function are shown in Fig. 5.6. The reader should write down the algebraic expressions corresponding to the figures.

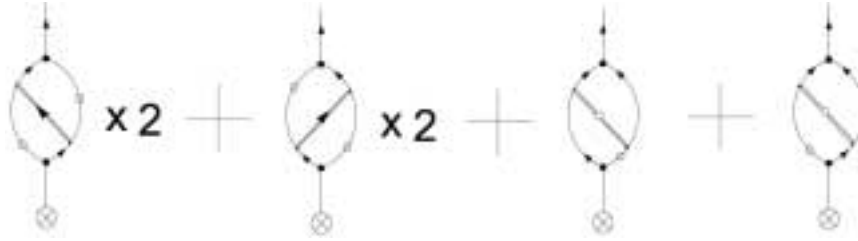


Figure 5.6. The two loop response function for the velocity

5.5 Self-Consistent Perturbation Theory

We now use the self consistent perturbation theory to determine the critical behaviour. To do the self consistent calculation with Eqns.(5.3.24) and (5.3.28), we replace the bare response and correlation functions by the full response and correlations functions. Further we assume that the nonlinear contributions Σ_ψ and Σ_v

to the relaxation rates dominate the bare relaxation rates $\lambda(k^2 + \kappa^2)k^2$ and ηk^2 . Thus

$$G_\psi^{-1} = -i\omega + \Sigma_\psi(k, \kappa, \omega) \quad (5.5.1)$$

$$G_v^{-1} = -i\omega + \Sigma_v(k, \kappa, \omega) \quad (5.5.2)$$

$$\begin{aligned} \Sigma_\psi &= k^2(k^2 + \kappa^2) \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} \frac{\sin^2 \theta}{k^2 + \kappa^2} G_v(k - p, \omega - \omega') \\ &\quad \times G_\psi(p, \omega') \end{aligned} \quad (5.5.3)$$

$$\begin{aligned} \Sigma_v(k, \kappa, \omega) &= \frac{1}{2(D-1)} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} \frac{\sin^2 \theta}{k^2 + \kappa^2} \frac{[p^2 - (\vec{p} - \vec{k})^2]}{[\kappa^2 + (\vec{k} - \vec{p})^2]} \\ &\quad \times G_\psi(p, \omega') G_\psi(k - p, \omega - \omega') \end{aligned} \quad (5.5.4)$$

The self consistent set of equations for the problem are provided by Eqns.(5.5.1)-(5.5.4). With the scaling Ansatz

$$\Sigma_\psi = k^2(k^2 + \kappa^2)k^{-X_\psi} f\left(\frac{k}{\kappa}, \frac{\omega}{k^{4-X_\psi}}\right) \quad (5.5.5)$$

$$\Sigma_v = k^2 k^{-X_\eta} g\left(\frac{k}{\kappa}, \frac{\omega}{k^{4-X_\psi}}\right) \quad (5.5.6)$$

a simple power counting of Eq.(5.5.4) gives

$$X_\psi + X_\eta = 4 - D = \epsilon \quad (5.5.7)$$

It should be apparent from Eqns.(5.5.5) and (5.5.6) that our nonlinear contributions Σ_ψ and Σ_v to the relaxation rates will dominate the non-critical contributions $\lambda(k^2 + \kappa^2)k^2$ and ηk^2 if X_ψ and X_η are positive. This implies from Eq.(5.5.7), that the critical fluctuations will dominate for $\epsilon > 0$ i.e. $D < 4$ i.e. in spatial dimensions less than four. For $D > 4$, the nonlinear contributions are irrelevant in the non-longwavelength limit and the mean field results are valid.

For the purpose of doing the frequency integration in Eqns.(5.5.3) and (5.5.4), we assume the response functions to be Lorentzian i.e.

$$G(k, \kappa, \omega) = -i\omega + \Sigma(k, \kappa)$$

and that leads to

$$\Sigma_\psi(k, \kappa, \omega) = k^2(k^2 + \kappa^2) \int \frac{d^D p}{(2\pi)^D} \frac{\sin^2 \theta}{k^2 + \kappa^2} \frac{1}{-i\omega + \Sigma_\psi(p, \kappa) + \Sigma_v(k-p, \kappa)} \quad (5.5.8)$$

and

$$\Sigma_v(k, \kappa, \omega) = \frac{1}{2(D-1)} \int \frac{d^D p}{(2\pi)^D} \frac{p^2 \sin^2 \theta}{k^2 + \kappa^2} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{[(\vec{p} - \vec{k})^2 + \kappa^2]} \times \frac{1}{-i\omega + \Sigma_\psi(p, \kappa) + \Sigma_\psi(k-p, \kappa)} \quad (5.5.9)$$

We are now in a position to get the self consistent value of X_ψ and X_η . To do this, we can afford to work at the critical point i.e. $\kappa = 0$. In this limit, $\Sigma_\psi \sim k^{4-X_\psi}$ and $\Sigma_v \sim k^{2-X_\eta}$ and clearly for low k (the long wavelength limit, in which we are interested), $\Sigma_v \gg \Sigma_\psi$ (for all $D > 2$). It should also be pointed out here that when the dependence on the external frequency ω is considered, these frequencies are of the order of the relaxation rate of the order parameter and we will always have $\omega \ll \Sigma_v$. Writing Eqns.(5.5.8) and (5.5.9) in the limit $\kappa \rightarrow 0$, $\omega \rightarrow 0$,

$$\Sigma_\psi(k) = k^4 \int \frac{d^D p}{(2\pi)^D} \frac{\sin^2 \theta}{p^2} \frac{1}{\Sigma_v(k-p)} \quad (5.5.10)$$

$$\Sigma_v(k) = \frac{1}{2(D-1)} \int \frac{d^D p}{(2\pi)^D} \frac{p^2 \sin^2 \theta}{p^2 (\vec{p} - \vec{k})^2} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{[\Sigma_\psi(p) + \Sigma_\psi(k-p)]} \quad (5.5.11)$$

Writing

$$\Sigma_\psi = \Gamma_\psi k^{4-X_\psi} \quad (5.5.12)$$

$$\Sigma_v = \Gamma_v k^{2-X_\eta} \quad (5.5.13)$$

Eq.(5.5.10) yields

$$\begin{aligned} \Gamma_\psi \Gamma_v &= k^{X_\psi} \int \frac{d^D p}{(2\pi)^D} \frac{\sin^2 \theta}{p^2} \frac{1}{(\vec{k} - \vec{p})^{2-X_\eta}} \\ &= \int \frac{d^D p}{(2\pi)^D} \frac{\sin^2 \theta}{p^2} \frac{1}{(\vec{1} - \vec{p})^{2-X_\eta}} = I(X_\eta) \end{aligned} \quad (5.5.14)$$

while Eq.(5.5.11) yields

$$\begin{aligned}
 \Gamma_\psi \Gamma_v &= \frac{k^{X_\eta-2}}{2(D-1)} \int \frac{d^D p}{(2\pi)^D} \frac{p^2 \sin^2 \theta}{p^2 (\vec{p} - \vec{k})^2} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{[p^{4-X_\eta} + |\vec{k} - \vec{p}|^{4-X_\eta}]} \\
 &= \frac{1}{2(D-1)} \int \frac{d^D p}{(2\pi)^D} \frac{p^2 \sin^2 \theta}{p^2 (\vec{1} - \vec{p})^2} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{[p^{4-X_\eta} + |\vec{1} - \vec{p}|^{4-X_\eta}]} \\
 &= J(X_\eta)
 \end{aligned} \tag{5.5.15}$$

where all momenta have been scaled by k . The equality,

$$I(X_\eta) = J(X_\eta) \tag{5.5.16}$$

following from Eqns.(5.5.14) and (5.5.15) yields the exponent X_η and from Eq.(5.5.7) the exponent X_ψ .

At this point we will carry out an approximation. If $D \simeq 4$, i.e. $\epsilon \simeq 0$, then both X_η and X_ψ are small, and assuming that both exponents are $O(\epsilon)$, we note that both the integrals I and J have poles at $\epsilon = 0$. The contribution to the pole comes from the momenta $p \gg 1$ and we have

$$I(X_\eta) \simeq \frac{C_D}{(2\pi)^D} \int_1^\Lambda \frac{d^D p}{C_D} \frac{\sin^2 \theta}{p^{4-X_\eta}}$$

where

$$C_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

is the surface area of the D-dimensional sphere. Noting that the average of $\cos^2 \theta$ in D-dimensional space is D^{-1} , we have

$$\begin{aligned}
 I(X_\eta) &\simeq \frac{C_D}{(2\pi)^D} \left(1 - \frac{1}{D}\right) \int_1^\Lambda \frac{dp}{p^{1+\epsilon-X_\eta}} \\
 &= \frac{C_4}{(2\pi)^4} \left(1 - \frac{1}{4}\right) \frac{1}{\epsilon - X_\eta}
 \end{aligned} \tag{5.5.17}$$

In the same limit

$$\begin{aligned}
 J(X_\eta) &\simeq \frac{1}{2(D-1)} \frac{C_D}{(2\pi)^D} \int \frac{d^D p}{C_D} \frac{4p^4 \sin^2 \theta \cos^2 \theta}{2p^4 p^{4-X_\eta}} \\
 &= \frac{C_D}{(2\pi)^D} \frac{1}{D-1} \frac{1}{D+2} \left(1 - \frac{1}{D}\right) \frac{1}{\epsilon - X_\eta} \\
 &\simeq \frac{C_4}{(2\pi)^4} \frac{1}{18} \left(1 - \frac{1}{4}\right) \frac{1}{X_\eta}
 \end{aligned} \tag{5.5.18}$$

Equating, according to Eq.(5.5.16)

$$X_\eta = \frac{\epsilon}{19} + O(\epsilon^2) \quad (5.5.19)$$

$$X_\psi = \frac{18\epsilon}{19} + O(\epsilon^2) \quad (5.5.20)$$

Hence the interesting physical result that for $D < 4$, the transport coefficient λ and the shear viscosity η diverge as the critical point is approached. scaling implies that if we are in the long wavelength limit (i.e. $k \simeq 0$) and approach the critical point (i.e. start from finite κ and approach $\kappa = 0$), then the excess transport coefficient $\Delta\lambda$ and the excess shear viscosity $\Delta\eta$ coming from the critical fluctuations behave as

$$\Delta\lambda \sim \xi^{X_\psi} \quad (5.5.21)$$

$$\Delta\eta \sim \xi^{X_\eta} \quad (5.5.22)$$

where ξ is the correlation length.

It should be noted that the divergence of the shear viscosity is weak ($X_\eta = \frac{\epsilon}{19}$ for $\epsilon \ll 1$) and often it is a good approximation to set $X_\eta = 0$ to find the concentration transport coefficient $\Delta\lambda$. If this is done then Eq.(5.5.8) yields in the physical dimension $D = 3$

$$\begin{aligned} \Delta\lambda(k, \kappa) &= \int \frac{d^3 p}{(2\pi)^3} \frac{\sin^2 \theta}{p^2 + \kappa^2} \frac{1}{\Sigma_v(k, \kappa)} \\ &= \frac{1}{\eta} \int \frac{d^3 p}{(2\pi)^3} \frac{\sin^2 \theta}{p^2 + \kappa^2} \frac{1}{(\vec{k} - \vec{p})^2} \end{aligned} \quad (5.5.23)$$

In the limit $\kappa = 0$

$$\begin{aligned} \Delta\lambda &= \frac{1}{\eta k} \int \frac{d^3 p}{(2\pi)^3} \frac{\sin^2 \theta}{p^2 (1 - \vec{p})^2} \\ &= \frac{1}{\eta k} \frac{4\pi}{(2\pi)^3} \frac{\pi^2}{8} \end{aligned} \quad (5.5.24)$$

while in the limit $k = 0$,

$$\begin{aligned} \Delta\lambda &= \frac{1}{\eta} \frac{1}{\kappa} \frac{4\pi}{(2\pi)^3} \int \frac{d^3 p}{4\pi} \frac{\sin^2 \theta}{1 + p^2} \frac{1}{p^2} \\ &= \frac{1}{\eta} \frac{1}{\kappa} \frac{4\pi}{(2\pi)^3} \frac{\pi}{3} = \frac{1}{6\pi \eta \kappa} \end{aligned} \quad (5.5.25)$$

For arbitrary k and κ , the integral in Eq.(5.5.23) can be done exactly and we have

$$\Delta\lambda(k, \kappa) = \frac{1}{6\pi \eta \kappa} K(k\xi) \quad (5.5.26)$$

where

$$K(x) = \frac{3}{4x^2} \left[1 + x^2 + \left(x^3 - \frac{1}{x} \right) \tan^{-1} x \right] \quad (5.5.27)$$

is generally known as the Kawasaki function.

What we have thus far described is the dynamic critical behaviour of the concentration fluctuations and velocity fluctuations in binary liquids. The ordinary fluid near the liquid-vapour critical point is in the same universality class but as we have discussed before the order parameter for the liquid-vapour critical point is not a clear density fluctuation but a mixture of density and energy fluctuations - in fact it is very close to the entropy fluctuations. Consequently, in Eq.(5.2.1), ψ stands for the entropy fluctuations and λ is the thermal conductivity and χ denotes the specific heat at constant pressure. The coupling to the velocity fluctuations and the nonlinearity in the velocity equation remain unaltered and Eqns.(5.5.21) and (5.5.22) imply that the thermal conductivity and shear viscosity diverge near the liquid-vapour critical point.

5.6 Sound Propagation

The three basic dissipative processes in a fluid have to do with thermal conductivity, shear viscosity and bulk viscosity. The first two have been dealt with - the third has to do with sound propagation. Sound waves are the propagation of density fluctuations in the system. The density fluctuations are directly related to the order parameter fluctuations for the liquid vapour critical point and only indirectly related for the binary liquid critical point. As the critical point is approached the order parameter fluctuations become extremely prominent and absorb energy from wave - this makes the propagation of low frequency sound waves impossible near the critical point. To see this, let us write down the hydrodynamic equations giving the propagation of sound waves. The hydrodynamic equations are,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0 \quad (5.6.1)$$

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (\rho v_i v_j) = - \frac{\partial P}{\partial x_i} \quad (5.6.2)$$

$$\frac{\partial}{\partial t} (\rho s) + \frac{\partial}{\partial x_i} (\rho s v_i) = 0 \quad (5.6.3)$$

where s is entropy per unit mass. The equilibrium density, temperature, entropy and pressure are denoted by ρ_0 , T_0 , s_0 and P_0 . The equilibrium velocity is zero. The fluctuations in ρ , T , s and P are $\delta\rho$, δT , δs and δP . We linearize Eqns.(5.6.1)-(5.6.3) in these variables and $\vec{v}$ which is already a fluctuation.

$$\frac{\partial}{\partial t} \delta \rho + \frac{\partial}{\partial x_i} (\rho_0 v_i) = 0 \quad (5.6.4)$$

$$\frac{\partial}{\partial t} (\rho_0 v_i) = -\frac{\partial}{\partial x_i} \delta P \quad (5.6.5)$$

$$s_0 \frac{\partial}{\partial t} (\delta \rho) + \frac{\partial}{\partial t} (\rho_0 \delta s) + \frac{\partial}{\partial x_i} (\rho_0 s_0 v_i) = 0 \quad (5.6.6)$$

Using Eq.(5.6.4), we can reduce Eq.(5.6.6) to

$$\frac{\partial}{\partial t} \delta s = 0 \quad (5.6.7)$$

In the absence of dissipation, the entropy is constant and thus the process is isentropic. We choose $\delta \rho$ and δT as our independent variables and write

$$\delta s = \left(\frac{\partial s}{\partial \rho} \right)_T \delta \rho + \left(\frac{\partial s}{\partial T} \right)_\rho \delta T \quad (5.6.8)$$

$$\delta P = \left(\frac{\partial P}{\partial \rho} \right)_T \delta \rho + \left(\frac{\partial P}{\partial T} \right)_\rho \delta T \quad (5.6.9)$$

We can take the constant value of δs to be at its initial value which is zero and hence Eq.(5.6.8) leads to

$$\delta T = -\left(\frac{\partial T}{\partial \rho} \right)_s \delta \rho \quad (5.6.10)$$

We can now write Eq.(5.6.5) as

$$\begin{aligned} 0 &= \rho_0 \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_i} \left[\left(\frac{\partial P}{\partial \rho} \right)_T \delta \rho + \left(\frac{\partial P}{\partial T} \right)_\rho \delta T \right] \\ &= \rho_0 \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_i} \left[\left(\frac{\partial P}{\partial \rho} \right)_T + \left(\frac{\partial P}{\partial T} \right)_\rho \left(\frac{\partial T}{\partial \rho} \right)_s \right] \delta \rho \\ &= \rho_0 \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_i} \left[\left(\frac{\partial P}{\partial \rho} \right)_s \delta \rho \right] \\ &= \rho_0 \frac{\partial v_i}{\partial t} + \left(\frac{\partial P}{\partial \rho} \right)_s \frac{\partial}{\partial x_i} \delta \rho \end{aligned} \quad (5.6.11)$$

A time derivative takes us to

$$\begin{aligned} \rho_0 \frac{\partial^2 v_i}{\partial t^2} &= -\left(\frac{\partial P}{\partial \rho} \right)_s \frac{\partial}{\partial x_i} \frac{\partial}{\partial t} \delta \rho \\ &= \left(\frac{\partial P}{\partial \rho} \right)_s \rho_0 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} v_j \\ &= \left(\frac{\partial P}{\partial \rho} \right)_s \rho_0 \frac{\partial^2}{\partial x_j \partial x_j} v_i \end{aligned} \quad (5.6.12)$$

where in the last line we have used the fact that in the linearized equations, the velocity field is curl free. Thus

$$\frac{\partial^2}{\partial t^2} v_i = \left(\frac{\partial P}{\partial \rho} \right)_s \nabla^2 v_i \quad (5.6.13)$$

and we have a wave propagation with speed v_0 given by

$$v_0^2 = \left(\frac{\partial P}{\partial \rho} \right)_s \quad (5.6.14)$$

We now show that near the critical point, the response $\left(\frac{\partial P}{\partial \rho} \right)_s$ vanishes weakly. The vanishing shows the impossibility of sound propagation in a critical fluid. One can write

$$\begin{aligned} \left(\frac{\partial P}{\partial \rho} \right)_s &= \left(\frac{\partial P}{\partial \rho} \right)_T + \left(\frac{\partial P}{\partial T} \right)_\rho \left(\frac{\partial T}{\partial \rho} \right)_s \\ &\simeq \left(\frac{\partial P}{\partial T} \right)_\rho \left(\frac{\partial T}{\partial \rho} \right)_s \end{aligned} \quad (5.6.15)$$

near the critical point, since the isothermal response $\left(\frac{\partial \rho}{\partial P} \right)_T$ diverges strongly near the critical point. Further,

$$\begin{aligned} \left(\frac{\partial P}{\partial \rho} \right)_s &\simeq \left(\frac{\partial P}{\partial T} \right)_\rho \left(\frac{\partial T}{\partial \rho} \right)_s = - \left(\frac{\partial P}{\partial T} \right)_\rho \left(\frac{\partial T}{\partial s} \right)_\rho \left(\frac{\partial s}{\partial \rho} \right)_T \\ &= \frac{V^2}{m} \left(\frac{\partial P}{\partial T} \right)_V \left(\frac{\partial s}{\partial V} \right)_T \left/ \left(\frac{\partial s}{\partial T} \right)_V \right. \\ &= \frac{TV}{\rho} \left(\frac{\partial P}{\partial T} \right)_V^2 / C_V \end{aligned} \quad (5.6.16)$$

The derivative $\left(\frac{\partial P}{\partial T} \right)_V$ is non critical. The constant volume specific heat has a weak divergence $C_V \sim \xi^{\alpha/\nu}$, and thus the velocity has a weak zero in the zero frequency limit.

The sound propagation occurs at finite frequencies and thus the response function C_V does not actually diverge. So long as the relaxation time of the critical fluctuations, which is proportional to ξ^z is shorter than ω^{-1} , where ω is the frequency of the sound wave, the specific heat C_V behaves as the zero frequency response should i.e. $C_V \sim \xi^{\alpha/\nu}$. Once $\xi^z \sim \frac{1}{\omega}$, $C_V \sim (\frac{1}{\omega})^{\alpha/z\nu}$ and if the temperature is brought closer to the critical point i.e. ξ increases, the system cannot respond any more and the specific heat remains at $\omega^{-\alpha/z\nu}$. This means that for a sound wave of frequency ω , the propagation velocity decreases as the liquid vapour critical point is approached and reaches a minimum at the critical point. The minimum is lower, the lower the frequency of the sound wave.

The specific heat is a response function and hence the frequency dependence is determined by the combination $-i\omega$. This means that the velocity given by Eq.(5.6.16) is going to be complex, which is exactly as it should be since energy from the sound wave will be transferred to the critical fluctuations. The plane wave (in one dimension e.g.) is given by e^{ikx} , where the wave number $k = \omega/u$, and where u is the sound speed. Dissipation corresponds to k having an imaginary part since a damped wave is written as e^{ikx-bx} . Thus the wave number k for a sound wave with dissipation is complex and follows from a complex sound speed. The relation between complex sound speed and wave number is

$$(k_1 + ik_2)^2 = \frac{\omega^2}{(u_1 + iu_2)^2} \simeq \frac{\omega^2}{u_1^2} - \frac{2iu_2\omega^2}{u_1^3} \quad (5.6.17)$$

giving

$$\begin{aligned} \frac{\omega^2}{u_1^2} &= k_1^2 - k_2^2 \simeq k_1^2 \\ \text{and} \quad \frac{k_2}{k_1} &= -\frac{u_2}{u_1} \end{aligned} \quad (5.6.18)$$

The complex sound speed near the liquid vapour critical point can be written as

$$u^2 = \frac{C_0}{C_v} \quad (5.6.19)$$

Where C_0 is a noncritical constant and C_v is the response function which is sensitive to the temperature difference from the critical point and the frequency. For $\xi^z \ll \omega^{-1}$ the zero frequency specific heat has the usual form

$$C_v \propto \xi^{\alpha/v} = A(\xi^z)^{\alpha/v} + B \quad (5.6.20)$$

where B is a noncritical constant. For $\xi^z \gg \frac{1}{\omega}$, the response fixed at its value for $\xi^z \sim \frac{1}{\omega}$ and hence the frequency dependent specific heat at $T = T_c$, i.e. $\xi = \infty$, is

$$C_v = C_1 \left(\frac{1}{-i\omega} \right)^{\alpha/zv} + B \quad (5.6.21)$$

the i coming from the causal nature of the response function (Kramers-Kronig relation). The velocity can be written as

$$u^{-2} = \frac{1}{C_0} \left[C_1 (-i\omega)^{-\alpha/zv} + B \right] = \frac{1}{C_0} \left[\frac{C_1}{\omega^{\alpha/zv}} \left(\cos \frac{\alpha\pi}{2zv} + i \sin \frac{\alpha\pi}{2zv} \right) + B \right]$$

From Eq.(5.6.18)

$$\frac{k_2}{k_1} = \frac{1}{2} \frac{\sin \frac{\alpha\pi}{2zv}}{\cos \frac{\alpha\pi}{2zv} + \frac{B}{C_1} \omega^{\alpha/zv}} \quad (5.6.22)$$

The attenuation per wavelength α_λ at $T = T_c$ is

$$\begin{aligned} \alpha_\lambda = k_2 \lambda &= 2\pi \frac{k_2}{k_1} = \frac{\pi \sin \frac{\alpha\pi}{2zv}}{\cos \frac{\alpha\pi}{2zv} + \frac{B}{C_1} \omega^{\alpha/zv}} \\ &\simeq \frac{\pi^2 \alpha / zv}{1 + \frac{B}{C_1} \omega^{\alpha/zv}} \end{aligned} \quad (5.6.23)$$

the last line following because the specific heat exponent α is small i.e. $\alpha \ll 1$. As we have seen in the previous section for the liquid-vapour systems $z \simeq 3$ in $D = 3$, $v \simeq 2/3$ for $\alpha \ll 1$ and thus

$$\alpha_\lambda = \frac{\alpha \pi^2 / 4}{1 + \frac{B}{C_1} \omega^{\alpha/zv}} \quad (5.6.24)$$

For low frequencies, the critical attenuation per wavelength attain the universal value $\alpha \pi^2 / 4 \simeq 0.29$. For $T \neq T_c$, the specific heat is a function of ξ and ω , and is given by the scaling function

$$C_v(\xi, \omega) = A \xi^{\alpha/v} f(\omega \xi^z) + B \quad (5.6.25)$$

where $f(0) = 1$ and for $x \gg 1$, $f(x) \propto (-ix)^{-\alpha/zv}$. In the intermediate range the function $f(x)$ needs to be calculated from a complete theory of frequency dependent specific heat.

Returning to the binary liquid, the order parameter is not directly connected to the sound propagation. Consequently, the velocity does not show the critical behaviour of Eq.(5.6.16) near the consolute point. Instead, near the consolute point the isentropic derivative $\left. \frac{\partial P}{\partial \rho} \right)_s$ can be written as

$$\left. \frac{\partial P}{\partial \rho} \right)_s = C_0 + C_1 / C_p \quad (5.6.26)$$

- the proof of which is being left to one of the problems. The constant C_0 is of the order of a few hundreds of metres and the additional part which has a strong temperature dependence due to critical fluctuations is about one percent change on this large background. The constant pressure specific heat for the consolute point has the structure shown in Eq.(5.6.25). For a binary liquid the background contribution B dominates the critical part, and linearizing in this part, we can write Eq.(5.6.26) in the form,

$$u^2 = C_0 + \frac{C_1}{B} - \frac{C_1 A}{B^2} \xi^{\alpha/v} f(\omega \xi^z) \quad (5.6.27)$$

and thus from Eq.(5.6.18), the attenuation per wavelength is given by

$$\alpha_\lambda = 2\pi \frac{k_2}{k_1} \simeq \frac{\pi C_1 A_1}{C_0 B^2} \xi^{\alpha/v} \text{Im } f(\omega \xi^z) \quad (5.6.28)$$

At the critical point, the attenuation

$$\begin{aligned} \alpha_\lambda^c &= \text{Im} \frac{\pi C_1 A_1}{C_0 B^2} A_0 (-i\omega)^{-\alpha/zv} \\ &= \frac{\pi C_1 A_1 A_0}{C_0 B^2} \omega^{-\alpha/zv} \end{aligned} \quad (5.6.29)$$

where A_0 is a constant. The ratio

$$\alpha_\lambda / \alpha_\lambda^c = G(\Omega) \quad (5.6.30)$$

is a function of the scaled variable $\Omega = \omega \xi^z$, which is an important prediction of these considerations. We have experimental evidence of this in the last section. The actual calculation of $G(\Omega)$ requires the theory of frequency dependent specific heat.

5.7 The Lambda Transition

This is where dynamic scaling began. At $2.172^0 K$, liquid He^4 undergoes a second order phase transition to a state characterized by infinite thermal conductivity. It also has a very low shear viscosity (it is only the rotation viscometer which picks up the viscosity, the liquid flows without apparent viscosity) which decreases strongly as the temperature decreases. This new phase of the liquid is called the superfluid and the transition is from a normal to a superfluid state. The specific heat diverges almost at the transition temperature and its temperature dependence resembles the Greek letter λ , which gives the transition its name. The order parameter of the transition is a mesoscopic quantum wavefunction for the ground state. This has a magnitude and a phase and this is a complex number. In the *Ginzburg-Landau* parlance it is a two component order parameter and the static properties of the transition are well understood in terms of the *Ginzburg-Landau* free energy.

In order to set up the equation of motion for the dynamics of the superfluid we need to worry about the conservation laws with the different densities. For conservation laws we need currents and for the superfluid phase there is a special current which is quantum mechanical in nature. A quantum mechanical wavefunction ψ is associated with a current which we call the supercurrent $\vec{j}_s$ and we write in the usual fashion

$$\vec{j}_s = \frac{\hbar}{m} \text{Im} (\psi^* \vec{\nabla} \psi) \quad (5.7.1)$$

With $\psi = |\psi|e^{i\phi}$, we can see that the superfluid velocity $\vec{v}_s$ is given by the gradient of the phase

$$\vec{v}_s = \vec{\nabla}\phi \quad (5.7.2)$$

The normal fluid carries the entropy and the normal current $\vec{j}_n = \rho_n \vec{v}_n$, where $\vec{v}_n$ is the normal fluid velocity and ρ_n is the normal fluid density. The total current is $\vec{j} = \vec{j}_n + \vec{j}_s$. The total density $\rho = \rho_n + \rho_s$ is not the critical mode and to the first approximation we can ignore the fluctuations in the total density. This implies no total current in the system so that $\vec{j} = 0$. Entropy fluctuations are associated with the normal fluid and hence they will be sensitive to the critical fluctuations. If we write S as the entropy density, the entropy current is $\sigma \vec{j}_n$ where σ is the entropy per unit mass then the conservation law for entropy takes the form

$$\frac{\partial S}{\partial t} + \sigma \vec{\nabla} \cdot \vec{j}_n = 0 \quad (5.7.3)$$

With no total current in the system

$$\begin{aligned} \frac{\partial S}{\partial t} &= -\sigma \vec{\nabla} \cdot \vec{j}_n = \sigma \vec{\nabla} \cdot \vec{j}_s \\ &= \frac{\sigma \hbar}{m} (\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1) \end{aligned} \quad (5.7.4)$$

where $\psi = \psi_1 + i\psi_2$.

Now, the gradient of the chemical potential μ drives the superfluid current and the acceleration equals the gradient of μ . The chemical potential fluctuation is induced by the entropy fluctuations and this leads to the equation of motion

$$\dot{\phi} = \frac{\sigma \hbar}{m} S. \quad (5.7.5)$$

Noting that $\psi = |\psi|e^{i\phi}$ and thus $\dot{\psi} = i\psi\dot{\phi}$, we have

$$\dot{\psi} = \frac{\sigma \hbar}{m} i\psi S \quad (5.7.6)$$

Writing in terms of the components

$$\begin{aligned} \dot{\psi}_1 &= -\frac{\sigma \hbar}{m} \psi_2 S \\ \dot{\psi}_2 &= \frac{\sigma \hbar}{m} \psi_1 S \end{aligned} \quad (5.7.7)$$

The *Ginzburg-Landau* free energy which describes the equilibrium fluctuations is given by

$$F = \int d^D x \left[\frac{\kappa^2}{2} \psi^2 + \frac{1}{2} (\vec{\nabla} \psi)^2 + \frac{1}{2} S^2 \right] \quad (5.7.8)$$

where we have ignored the quartic term in the free energy and included its effect in the statics through the phenomenological procedure of using a renormalized mass κ which vanishes at the critical point with the correct correlation length exponent ν . We have made another drastic assumption. The equilibrium correlation of the entropy field is a constant according to the free energy functional of Eq.(5.7.8). However, the entropy correlation is supposed to be the constant pressure specific heat. In reality the specific heat has a weak (logarithmic) divergence at the lambda point. The effect of this will be lost in the dynamics when we use Eq.(5.7.). However, this is an extremely small effect. The nonlinear terms that we have used in Eqns(5.7.) and (5.7.) conserve the free energy of Eq.(5.7.8) as the reversible terms should. To complete the equations for ψ and S we now need to introduce the dissipative part as the functional derivative of F and the noise term that will ensure the existence of the equilibrium distribution function. It should be noted that there is no conservation law associated with the order parameter, but the entropy is a conserved variable. We thus have the dynamics prescribed by

$$\dot{\psi}_1 = -g_0\psi_2S - \Gamma(\kappa^2 - \nabla^2)\psi_1 + N_1 \quad (5.7.9)$$

$$\dot{\psi}_2 = g_0\psi_1S - \Gamma(\kappa^2 - \nabla^2)\psi_2 + N_2 \quad (5.7.10)$$

$$\dot{S} = g_0(\psi_1\nabla^2\psi_2 - \psi_2\nabla^2\psi_1) + \lambda\nabla^2S + N_s \quad (5.7.11)$$

where $g_0 = \frac{\sigma\hbar}{m}$ and

$$\begin{aligned} \langle N_i(\vec{x}_1, t_1)N_j(\vec{x}_2, t_2) \rangle &= 2\Gamma\delta^D(\vec{x}_1 - \vec{x}_2)\delta(t_1 - t_2)\delta_{ij} \\ \langle N_s(\vec{x}_1, t_1)N_s(\vec{x}_2, t_2) \rangle &= 2\Gamma\delta^D(\vec{x}_1 - \vec{x}_2)\delta(t_1 - t_2)\delta_{ij} \end{aligned} \quad (5.7.12)$$

In momentum space, the equations are

$$\dot{\psi}_1 = -g_0 \sum_{\vec{p}} S(p)\psi_2(k-p) - \Gamma(k^2 + \kappa^2)\psi_1(k) + N_1(k) \quad (5.7.13)$$

$$\dot{\psi}_2 = g_0 \sum_{\vec{p}} S(p)\psi_1(k-p) - \Gamma(k^2 + \kappa^2)\psi_2(k) + N_2(k) \quad (5.7.14)$$

$$\dot{S}(k) = g_0 \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} (p_1^2 - p_2^2)\psi_1(p_1)\psi_2(p_2) - \lambda k^2 S(k) + N_s(k) \quad (5.7.15)$$

with the appropriate noise correlations. Let us use these equations first for $T < T_c$ without the dissipative and stochastic terms. The situation for $T < T_c$, is characterized by the existence of a non zero expectation value of the order parameter field. Suppose this symmetry breaking occurs in ψ_1 and $\langle \psi_1 \rangle = m_0 \neq 0$. We write $\psi_1 = m_0 + \tilde{\psi}_1$, where m_0 is independent of time and space. In Fourier space, m_0 has only a $k=0$ component and we can write Eqns.(5.7.10) and (5.7.11), linearized in ψ_2 and S as

$$\dot{\psi}_2(k) = g_0 m_0 S(k) \quad (5.7.16)$$

$$\text{and } \dot{S}(k) = -g_0 m_0 k^2 \psi_2(k) \quad (5.7.17)$$

This leads to the oscillations

$$\ddot{\psi}_2 + g_0^2 m_0^2 k^2 \psi_2 = 0 \quad (5.7.18)$$

$$\ddot{S} + g_0^2 m_0^2 k^2 S = 0 \quad (5.7.19)$$

The temperature fluctuations and the transverse fluctuations of the order parameter field (the *Goldstone mode*) propagate with frequency

$$\omega = g_0 m_0 k \quad (5.7.20)$$

This is the second sound in superfluid Helium - it corresponds to the entropy and hence temperature fluctuations. The order parameter expectation value has the critical behaviour $m \sim \kappa^{\beta/\nu}$. Ignoring the small anomalous dimension exponent η ,

$$\alpha + 2\beta + 2\nu = 2,$$

which combined with the Josephson relation

$$D\nu = 2 - \alpha,$$

leads to

$$\frac{\beta}{\nu} = \frac{D}{2} - 1.$$

From Eq.(5.7.20)

$$\omega \propto k \kappa^{\beta/\nu} = k \kappa^{D/2-1} \quad (5.7.21)$$

We now imagine approaching the lambda point with the wavenumber kept finite. The correlation length increases and once $\kappa \simeq k$, the frequency does not respond any more and hence at $T = T_c$ (i.e. $\kappa = 0$)

$$\omega \propto k^{D/2} \quad (5.7.22)$$

Thus, $z = D/2$ for the superfluid transition and it is the same for the order parameter and entropy fields. The argument, that we have just given is the dynamic scaling argument. The basic assumption that there is only one characteristic time in the system and it is a homogeneous function of k and κ , i.e.

$$\omega = k^z f\left(\frac{k}{\kappa}\right) \quad (5.7.23)$$

The function f will be different for ψ and S but z is the same.

If we now include the dissipation then frequency is complex and can be written as

$$\omega = \omega_0 + i D k^2 \quad (5.7.24)$$

where D is the damping coefficient. For $\kappa \ll k$, $\omega_0 \propto k^{D/2}$ according to Eq.(5.7.22) and if there is only one frequency scale then $D k^2$ must behave as $k^{D/2}$ and hence $D(k) \propto k^{D/2-2}$ for $\kappa \ll k$. Dynamic scaling hypothesis now asserts that

$$D(k, \kappa) = k^{\frac{D}{2}-2} \mathcal{D}(k/\kappa) \quad (5.7.25)$$

and for $\kappa \gg k$

$$D(\kappa) \propto \kappa^{D/2-2} = \xi^{2-D/2} = \xi^{\epsilon/2} \quad (5.7.26)$$

Thus the damping diverges as the correlation length becomes infinite.

If we now go through the transition to the normal phase, then the second sound mode disappears and becomes a diffusive mode - the diffusion corresponding to the thermal conductivity. The continuity of the entropy mode ensures that the thermal conductivity diverges as $\xi^{\epsilon/2}$ due to superfluid fluctuations in the normal state near the transition point. The definite predictions for $D = 3$, are

$$\begin{array}{ll} \text{Thermal conductivity} & \lambda \propto t^{-1/3} \quad (T > T_\lambda) \\ \text{Second sound damping} & D \propto t^{-1/3} \quad (T < T_\lambda) \end{array}$$

The thermal conductivity measurement is extremely precise. It showed a clear cut divergence, but the exponent appeared about 20% higher. The resolution of this problem was an interesting tour de force. We will provide some feeling for the issues involved in the next section.

5.8 Generalized n-Vector Model

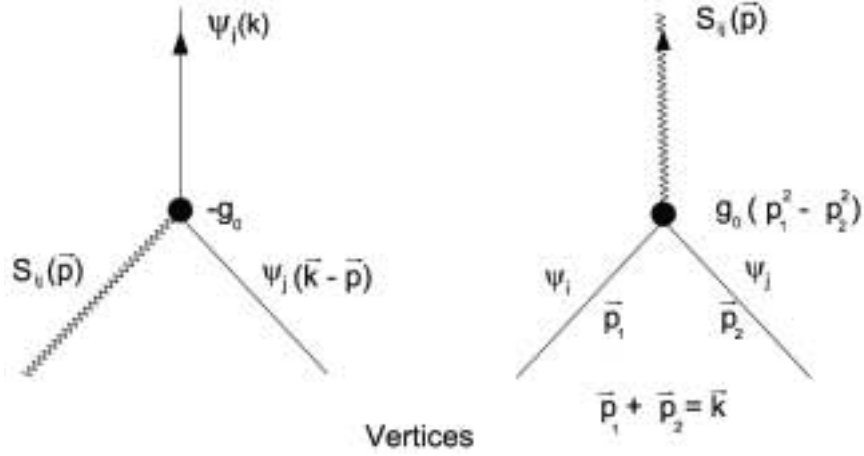
In this section, we introduce a slightly generalized model which will help us make a qualitative point very easily. We introduce a n -component vector order parameter field $\vec{\psi}(\psi_1, \psi_2, \dots, \psi_n)$, and a $n(n-1)/2$ component antisymmetric tensor entropy field S_{ij} with the equation of motion

$$\dot{\psi}_i(k) = -g_0 \sum_{\vec{p}} S_{ij}(\vec{p}) \psi_j(\vec{k} - \vec{p}) - \Gamma(k^2 + \kappa^2) \psi_i(k) + N_i(k) \quad (5.8.1)$$

$$\dot{S}_{ij} = g_0 \sum_{\vec{p}_1 + \vec{p}_2 = \vec{k}} (p_1^2 - p_2^2) \psi_i(p_1) \psi_j(p_2) - \lambda k^2 S_{ij}(k) + N_{S_{ij}}(k) \quad (5.8.2)$$

There is no cross correlation either between the noise for the order parameter and entropy fields or between the different components of noise for the same field. For $n = 2$, we get back the equations of motion for the superfluid transition as written down in Eqns.(5.7.13), (5.7.14) and (5.7.15).

Our task now is to construct the relaxation rates for the order parameter and entropy fields. At the zeroth order (i.e. dropping the nonlinear terms)



$$[G_{\psi}^{(0)}]^{-1} = -i\omega + \Gamma(k^2 + \kappa^2) \quad (5.8.3)$$

$$[G_S^{(0)}]^{-1} = -i\omega + \lambda k^2 \quad (5.8.4)$$

$$C_{\psi}^{(0)} = \frac{2\Gamma}{\omega^2 + \Gamma^2(k^2 + \kappa^2)^2} = \frac{1}{k^2 + \kappa^2} [G_{\psi}^{(0)} + G_{\psi}^{(0)*}] \quad (5.8.5)$$

$$C_S^{(0)} = \frac{2\lambda k^2}{\omega^2 + \lambda^2 k^4} = G_S^{(0)} + G_S^{(0)*} \quad (5.8.6)$$

To find the contribution from the nonlinear terms we need to calculate the self energies of ψ and S fields. The self energies can be constructed from the elements shown in Fig. 5.7. The entropy line is wavy and the order parameter line is straight. The order parameter noise is a circle with a dot and the entropy noise is a circle with a cross. Translating the diagrammatics to algebraic expressions,

$$\begin{aligned} G_{\psi}^{(2)}(k) &= \{G_{\psi}^{(0)}(k, \omega)\}^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} (-g_0^2) [G_{\psi}^{(0)}(k-p, \omega-\omega') \\ &\quad \times C_S^{(0)}(p, \omega') + \{k^2 - (\vec{k} - \vec{p})^2\} G_S^{(0)}(p, \omega') C_{\psi}^{(0)}(k-p, \omega-\omega')] \\ &= -g_0^2 \{G_{\psi}^{(0)}(k, \omega)\}^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} G_{\psi}^{(0)}(k-p, \omega-\omega') \\ &\quad \times G_S^{(0)}(p, \omega') \frac{k^2 + \kappa^2}{\kappa^2 + (\vec{k} - \vec{p})^2} \end{aligned} \quad (5.8.7)$$

and

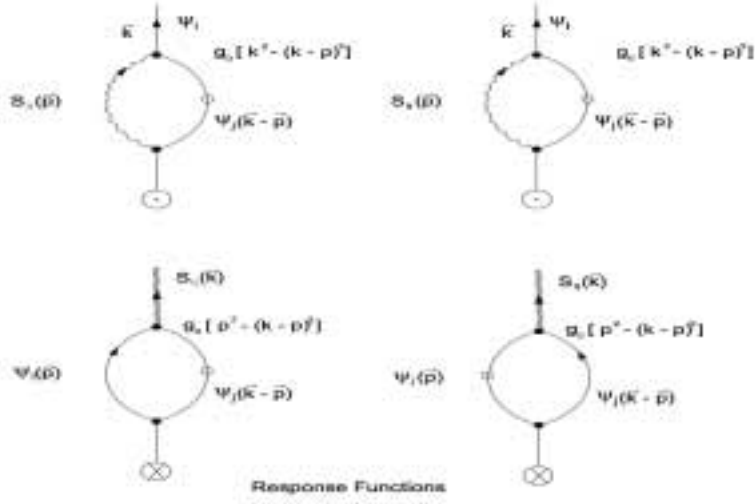


Figure 5.7. Order parameter and entropy response functions for the superfluid transition

$$\begin{aligned}
 G_S^{(2)}(k) &= \{G_S^{(0)}(k, \omega)\}^2 g_0^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} [p^2 - (\vec{k} - \vec{p})^2] G_\psi^{(0)}(p, \omega') \\
 &\quad \times G_\psi^{(0)}(k - p, \omega - \omega') \left\{ \frac{1}{p^2 + \kappa^2} - \frac{1}{\kappa^2 + (\vec{k} - \vec{p})^2} \right\} \\
 &= -\{G_S^{(0)}(k, \omega)\}^2 g_0^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{(p^2 + \kappa^2)[\kappa^2 + (\vec{p} - \vec{k})^2]} \\
 &\quad \times G_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k - p, \omega - \omega') \quad (5.8.8)
 \end{aligned}$$

For the ψ_i there will be $(n - 1)$ diagrams of the kind shown in Fig. 5.7b and hence there will be a factor of $(n - 1)$ multiplying $G_\psi^{(2)}$. The self energies are

$$\begin{aligned}
 \Sigma_\psi(k, \kappa, \omega) &= g_0^2 (k^2 + \kappa^2) (n - 1) \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} \frac{1}{\kappa^2 + (\vec{k} - \vec{p})^2} \\
 &\quad \times G_\psi^{(0)}(k - p, \omega - \omega') G_S^{(0)}(p, \omega') \quad (5.8.9)
 \end{aligned}$$

$$\begin{aligned}
 \Sigma_S(k, \kappa, \omega) &= -g_0^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{(p^2 + \kappa^2)[\kappa^2 + (\vec{p} - \vec{k})^2]} \\
 &\quad \times G_\psi^{(0)}(p, \omega') G_\psi^{(0)}(k - p, \omega - \omega') \quad (5.8.10)
 \end{aligned}$$

A self consistent approximation replaces $G_\psi^{(0)}$ and $G_S^{(0)}$ with G_ψ and G_S and assuming that Σ_ψ and Σ_S dominate the bare relaxation rate $\gamma(k^2 + \kappa^2)$ and λk^2 , we have the following

$$G_\psi^{-1} = -i\omega + \Sigma_\psi(k, \kappa, \omega) \quad (5.8.11)$$

$$G_S^{-1} = -i\omega + \Sigma_S(k, \kappa, \omega) \quad (5.8.12)$$

where

$$\begin{aligned} \Sigma_\psi(k, \kappa, \omega) = & g_0^2(k^2 + \kappa^2)(n-1) \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} \frac{1}{\kappa^2 + (\vec{k} - \vec{p})^2} \\ & \times G_\psi(k-p, \omega-\omega') G_S(p, \omega') \end{aligned} \quad (5.8.13)$$

$$\begin{aligned} \Sigma_S(k, \kappa, \omega) = & -g_0^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega'}{2\pi} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{(p^2 + \kappa^2)[\kappa^2 + (\vec{p} - \vec{k})^2]} \\ & \times G_\psi(p, \omega') G_\psi(k-p, \omega-\omega') \end{aligned} \quad (5.8.14)$$

Using Lorentzian approximations for G_ψ and G_S to do the frequency integration in Eqns.(5.8.13) and (5.8.14)

$$\begin{aligned} \Sigma_\psi(k, \kappa) = & g_0^2(k^2 + \kappa^2)(n-1) \int \frac{d^D p}{(2\pi)^D} \frac{1}{\kappa^2 + (\vec{k} - \vec{p})^2} \\ & \times \frac{1}{\Sigma_S(p) + \Sigma_\psi(\vec{k} - \vec{p})} \end{aligned} \quad (5.8.15)$$

$$\begin{aligned} \Sigma_S(k, \kappa) = & g_0^2 \int \frac{d^D p}{(2\pi)^D} \frac{[p^2 - (\vec{k} - \vec{p})^2]^2}{[\kappa^2 + (\vec{k} - \vec{p})^2](p^2 + \kappa^2)} \\ & \times \frac{1}{\Sigma_\psi(p) + \Sigma_S(\vec{k} - \vec{p})} \end{aligned} \quad (5.8.16)$$

We now make the dynamic scaling Ansatz for the Onsager coefficients Γ_ψ and λ_S , which are defined as

$$\Gamma_\psi = \frac{\Sigma_\psi(k, \kappa)}{k^2 + \kappa^2} = k^x f(k/\kappa) \quad (5.8.17)$$

$$\lambda_S = \Sigma_S(k, \kappa)/k^2 = k^x g(k/\kappa) \quad (5.8.18)$$

Power counting of Eqns.(5.8.15) and (5.8.16) confirm that

$$x = \frac{D}{2} - 2 = \frac{\epsilon}{2}.$$

For $\epsilon > 0$ i.e. $D < 4$, the Onsager coefficients in the long wavelength limit diverges and the nonlinear contribution to the Onsager coefficients dominate the background contribution. Our calculations have thus confirmed the dynamic scaling expectations of §(5.7).

We now want to look at the scaling behaviour of Eqns.(5.8.17) and (5.8.18) a little more closely. Focussing on the critical point (i.e. $\kappa = 0$) relaxation, we have

$$\Gamma_\psi = a_\psi k^{-\frac{\epsilon}{2}} \quad (5.8.19)$$

$$\lambda_S = a_S k^{-\frac{\epsilon}{2}} \quad (5.8.20)$$

where a_S and a_ψ are finite constants. From Eqns(5.8.15) and (5.8.16)

$$a_S a_\psi = \frac{C_D}{(2\pi)^D} g_0^2 I(w, \epsilon) = \frac{C_D}{(2\pi)^D} g_0^2 J(w, \epsilon) \quad (5.8.21)$$

where $w = \frac{a_\psi}{a_S}$ and

$$I(w, \epsilon) = (n-1) \int \frac{d^D p}{C_D} \frac{1}{(\vec{1} - \vec{p})^2} \frac{1}{p^{D/2} + |\vec{1} - \vec{p}|^{D/2} w} \quad (5.8.22)$$

$$J(w, \epsilon) = \int \frac{d^D p}{C_D} \frac{[(\vec{1} - \vec{p})^2 - p^2]^2}{(\vec{1} - \vec{p})^2 p^2} \frac{1}{p^{D/2} + |\vec{1} - \vec{p}|^{D/2} w} \quad (5.8.23)$$

We can find the value of w from

$$I(w, \epsilon) = J(w, \epsilon) \quad (5.8.24)$$

We note that I and J have logarithmic divergences at $D = 4$ and hence as a function of $\epsilon = 4 - D$, there will be poles at $\epsilon = 0$, which we can evaluate from the region $p \geq 1$. Clearly,

$$I(w, \epsilon) = (n-1) \left[\frac{2}{\epsilon} \frac{1}{1+w} + 0(1) \right] \quad (5.8.25)$$

$$J(w, \epsilon) = \frac{4}{D} \frac{1}{\epsilon} + 0(1) \quad (5.8.26)$$

To the lowest order in ϵ , we find that

$$w = 2n - 3. \quad (5.8.27)$$

The important thing with the above result is that for $n = 3/2$, $w = 0$ and hence the dynamic scaling Ansatz does not work. This is true very close to $D = 4$. At $D = 3$, a different value of n may be obtained and indeed a two loop calculation shows that at $D = 3$, $n \simeq 2$, the relevant value of n for the superfluid transition. The ratio w is consequently very small for the lambda transition. This is not a violation of dynamic scaling but can have significant observational consequences as we will see.

The dynamic scaling relations shown in Eqns.(5.7.25) and (5.7.26) are valid in the long wavelength limit i.e. as $k \rightarrow 0$. If one is concerned with larger values of k , then there will be corrections to the scaling forms. This is expressed as

$$\Gamma_\psi = a_\psi k^{-\epsilon/2} [1 + b_\psi k^{\delta_1} + c_\psi k^{\delta_2} + \dots] \quad (5.8.28)$$

$$\lambda_S = a_S k^{-\epsilon/2} [1 + b_S k^{\delta_1} + c_S k^{\delta_2} + \dots] \quad (5.8.29)$$

where the exponents δ_1, δ_2 are positive, which makes these terms vanish as $k \rightarrow 0$. Consequently, terms of the kind k^{δ_1} and k^{δ_2} are called *correction-to-scaling* terms. We now demonstrate how the correction-to-scaling terms can be calculated. To do so, one introduces the above forms for Γ_ψ and λ_S in Eqns.(5.8.15) and (5.8.16) with $\kappa = 0$ and we have

$$\begin{aligned} a_\psi k^{-\epsilon/2} &= \frac{(n-1)}{(1+b_\psi k^\delta + \dots)} \int \frac{d^D p}{C_D} \frac{1}{(\vec{k} - \vec{p})^2} \\ &\quad \times \frac{1}{a_s p^{D/2} [1 + b_s p^\delta + \dots] + a_\psi |\vec{k} - \vec{p}|^{D/2} [1 + b_\psi |\vec{k} - \vec{p}|^\delta + \dots]} \\ &= \frac{(n-1)}{(1+b_\psi k^\delta + \dots)} \int \frac{d^D p}{C_D} \frac{1}{(\vec{k} - \vec{p})^2} \left[\frac{1}{a_s p^{D/2} + a_\psi |\vec{k} - \vec{p}|^{D/2}} \right. \\ &\quad \left. - \frac{a_s b_s p^{\delta+D/2} + a_\psi b_\psi |\vec{k} - \vec{p}|^{\delta+D/2}}{[a_s p^{D/2} + a_\psi |\vec{k} - \vec{p}|^{D/2}]^2} + \dots \right] \end{aligned} \quad (5.8.30)$$

and

$$\begin{aligned} a_s k^{-\epsilon/2} (1 + b_s k^\delta + \dots) &= \int \frac{d^D p}{C_D} \frac{[p^2 - |\vec{k} - \vec{p}|^2]^2}{p^2 |\vec{k} - \vec{p}|^2} \left[\frac{1}{a_\psi p^{D/2} + a_\psi |\vec{k} - \vec{p}|^{D/2}} \right. \\ &\quad \left. - \frac{a_\psi b_\psi (p^{\delta+D/2} + |\vec{k} - \vec{p}|^{\delta+D/2})}{a_\psi^2 [(p^{D/2} + |\vec{k} - \vec{p}|^{D/2})]^2} + \dots \right] \end{aligned} \quad (5.8.31)$$

leading to

$$\begin{aligned} a_s a_\psi &= (n-1) \int \frac{d^D p}{C_D} \frac{1}{|\vec{1} - \vec{p}|^2} \frac{1}{p^{D/2} + w |\vec{1} - \vec{p}|^{D/2}} \\ &= \int \frac{d^D p}{C_D} \frac{[p^2 - |\vec{1} - \vec{p}|^2]^2}{p^2 |\vec{1} - \vec{p}|^2 [p^{D/2} + w |\vec{1} - \vec{p}|^{D/2}]^2} \end{aligned} \quad (5.8.32)$$

$$\begin{aligned} a_s a_\psi b_\psi &= -(n-1) b_s \int \frac{d^D p}{C_D} \frac{1}{|\vec{1} - \vec{p}|^2} \frac{p^{\delta+D/2}}{[p^{D/2} + w |\vec{1} - \vec{p}|^{D/2}]^2} \\ &\quad - (n-1) b_\psi w \int \frac{d^D p}{C_D} \frac{1}{|\vec{1} - \vec{p}|^2} \frac{|\vec{1} - \vec{p}|^{\delta+D/2}}{[p^{D/2} + w |\vec{1} - \vec{p}|^{D/2}]^2} + \dots \end{aligned} \quad (5.8.33)$$

and

$$a_s a_\psi b_s = -b_\psi \int \frac{d^D p}{C_D} \frac{[p^2 - |\vec{1} - \vec{p}|^2]^2}{p^2 |\vec{1} - \vec{p}|^2} \frac{p^{\delta+D/2} + |\vec{1} - \vec{p}|^{\delta+D/2}}{[p^{D/2} + w |\vec{1} - \vec{p}|^{D/2}]^2} \quad (5.8.34)$$

Making the reasonable assumption that δ vanishes at $D = 4$, all our integrals have a logarithmic divergence at $D = 4$ (i.e. $\epsilon = 0$) and we evaluate them in the pole approximation i.e. pick up the ϵ^{-1} part. From Eq.(5.8.32), this yields

$$a_s a_\psi = \frac{1}{\epsilon} = \frac{2(n-1)}{\epsilon(1+w)} \quad (5.8.35)$$

while

$$\begin{aligned} & \int \frac{d^D p}{C_D} \frac{1}{|\vec{1} - \vec{p}|^2} \frac{p^{\delta+D/2}}{[p^{D/2} + w|\vec{1} - \vec{p}|^{D/2}]^2} \\ &= \int \frac{d^D p}{C_D} \frac{1}{|\vec{1} - \vec{p}|^2} \frac{|\vec{1} - \vec{p}|^{\delta+D/2}}{[p^{D/2} + w|\vec{1} - \vec{p}|^{D/2}]^2} = \frac{1}{(1+w)^2} \frac{2}{\epsilon - 2\delta} + O(1) \end{aligned} \quad (5.8.36)$$

and

$$\begin{aligned} & \int \frac{d^D p}{C_D} \frac{1}{|\vec{1} - \vec{p}|^2} \frac{[p^2 - |\vec{1} - \vec{p}|^2]^2}{p^{D/2} + |\vec{1} - \vec{p}|^{D/2}]^2} \times [p^{\delta+D/2} + |\vec{1} - \vec{p}|^{\delta+D/2}] \\ &= \frac{4}{D} \frac{1}{\epsilon - 2\delta} + O(1) = \frac{1}{\epsilon - 2\delta} + O(1) \end{aligned} \quad (5.8.37)$$

Using the above integrals, Eqns. (5.8.33) and (5.8.34) can be rewritten as

$$\begin{aligned} b_\psi &= -\frac{\epsilon}{\epsilon - 2\delta} \frac{1}{(1+w)} (b_s + w b_\psi) \\ b_s &= -\frac{\epsilon}{\epsilon - 2\delta} b_\psi \end{aligned} \quad (5.8.38)$$

leading to

$$1 + w = -\frac{\epsilon}{\epsilon - 2\delta} w + \left(\frac{\epsilon}{\epsilon - 2\delta} \right)^2 \quad (5.8.39)$$

yielding

$$\delta = \frac{\epsilon w/2}{1+w}, \quad (5.8.40)$$

The vital point of the above calculation is that one of the correction-to-scaling exponents is proportional to w . If w is small, then this exponent is small and the *correction-to-scaling term* is going to persist even for very small values of k . Near $D = 4$, w is small for $n = 3/2$, but detailed two loop calculations show that $D = 3$, w is small for $n = 2$, i.e. it is small for the lambda transition. We also note that for small w the amplitudes of b_ψ and b_s differ in sign. For negative values of the amplitude b_s , we have for the k -dependent thermal diffusion (we are working with $w \ll 1$, so that $1 + w \simeq 1$)

$$\begin{aligned} \lambda_s &\simeq a_s k^{-\epsilon/2} [1 - |b_s| k^{\epsilon w/2} + \dots] \\ &= a_s k^{-\epsilon/2} \left[1 - \left(\frac{k}{k_c} \right)^{\epsilon w/2} \right] \end{aligned} \quad (5.8.41)$$

Since the correction term does not disappear until $k \ll k_c$, we have for not so small values of k , an effective exponent

$$\begin{aligned} X_\lambda &= -\frac{d \ln \lambda_s}{d \ln k} = \frac{\epsilon}{2} + \frac{(\frac{k}{k_c})^{\epsilon w/2}}{1 - (\frac{k}{k_c})^{\epsilon w/2}} \frac{\epsilon w}{2} \\ &= \frac{\epsilon}{2} \left[1 + w \frac{(\frac{k}{k_c})^{\epsilon w/2}}{1 - (\frac{k}{k_c})^{\epsilon w/2}} \right] \end{aligned} \quad (5.8.42)$$

For $\frac{k}{k_c} \simeq 10^{-3}$ and $w \simeq 0.1$, the effective exponent is about 24% greater than the dynamic scaling exponent of $\epsilon/2$. Note that for $\frac{k}{k_c} \rightarrow 0$, we would see the exponent $\epsilon/2$, but real experiments can never get down to this asymptotic range, and hence in the k -range accessible to the experiments, the asymptotic dynamic scaling exponent is still not accessible due to the existence of a small *correction-to-scaling* exponent. Note that our discussion in k -space simply carries over to κ -space when we discuss the thermal conductivity measurements as a function of temperature near the lambda point. While the above gives a correct qualitative feel for the way the things work, to get quantitative results one needs to investigate the renormalization group flow equations.

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Systems Far from Equilibrium

6.1 Introduction

In this chapter our focus will be on situations which are not close to thermal equilibrium. To understand the problems associated with it, we first discuss the situation close to equilibrium in a somewhat generalized setting. What we have repeatedly seen in the previous chapters is that small excursions from the equilibrium distribution relax to the equilibrium with time scales set by relaxation rates which we are familiar with in hydrodynamics - the mass or concentration diffusivity, the thermal diffusivity, momentum diffusivity. The balance between the fluctuating forces (obtained by averaging over small scale processes in the critical phenomena) and the dissipative processes ensured that the equilibrium distribution would be attained at $t \rightarrow \infty$. Describing the system in terms of a fluctuating field $\psi(\vec{x})$, we have repeatedly used the free energy functional $F(\psi)$ to describe the probability distribution $e^{-F(\psi)}$ for the weighting of the fluctuations and to describe the dynamics we have used a Langevin equation of the form

$$\dot{\psi} = -\Gamma \frac{\delta F}{\delta \psi} + N \quad (6.1.1)$$

Where Γ is the relaxation rate and N is the noise with the correlation

$$\langle N(t_1)N(t_2) \rangle = 2\Gamma \delta(t_1 - t_2) \quad (6.1.2)$$

This structure ensures that the equilibrium distribution is always attained as $t \rightarrow \infty$. In adding the reversible terms to the above dissipative equation, we have always taken care to ensure that the terms are such that they preserve the equilibrium distribution. Thus we always ensured that the system will attain equilibrium. If we probe the equilibrium fluctuations by applying a very weak external force, then it

should be possible to get at the transport coefficients by studying the response of the system. This is what we do now.

We assume a time dependent infinitesimal force $F(t)$ applied to the system and the field ψ describing the fluctuations responds via the response function $\chi(t - t')$, so that the average value in the presence of the external force is

$$\langle \psi(t) \rangle_F = \int_{-\infty}^{\infty} \chi(t - t') f(t') dt' \quad (6.1.3)$$

The response $\chi(\tau)$ is real, and satisfies the causality condition

$$\chi(\tau) = 0 \quad \text{if } \tau < 0 \quad (6.1.4)$$

In terms of the Fourier transforms, Eq.(6.1.4) reads

$$\psi(\omega) = \chi(\omega) F(\omega) \quad (6.1.5)$$

The response of the system will be assumed to be finite, which can be ensured by

$$\int_0^{\infty} \chi(\tau) d\tau < \infty .$$

The frequency dependent susceptibility $\chi(\omega)$ of Eq.(6.1.5) is given by (keeping in mind Eq.(6.1.4))

$$\chi(\omega) = \int_0^{\infty} d\tau \chi(\tau) e^{i\omega\tau} \quad (6.1.6)$$

In the above equation, it is possible to consider ω to be complex, although only in the upper half plane. Denoting the complex ω by z , we can have $z = \omega + i\eta$ with $\eta > 0$, and Eq.(6.1.6) would be well defined yielding finite values of $\chi(z)$ everywhere in the upper half plane. Thus $\chi(z)$ has no poles in the upper half plane. For $\eta \rightarrow \infty$ $\chi(z) \rightarrow 0$. For real ω , $\chi(\omega)$ is a complex valued function of ω , and the above considerations can be used to relate the real and the imaginary parts of $\chi(\omega)$. This is the usual Kramers-Kronig relation and we offer a quick derivation. For the complex function

$$f(z) = \frac{\chi(z)}{z - \omega_0} \quad (6.1.7)$$

where ω_0 is a real number, there are no poles of the function $\chi(z)$ in the upper half plane and for the contour shown in Fig. 6.1

$$\oint_C f(z) dz = 0 \quad (6.1.8)$$

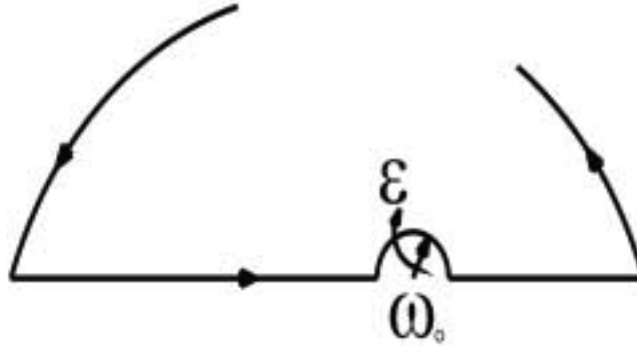


Figure 6.1. The contour in the complex plane

On the semicircle of infinite radius the contribution to the integral is zero since $f(z)$ goes to zero at large values of the argument faster than $\frac{1}{|z|}$. Thus

$$\lim_{\epsilon \rightarrow 0} \left[\int_{-\infty}^{\omega_0 - \epsilon} f(\omega) d\omega + \int_{\text{small semicircle}} f(z) dz + \int_{\omega_0 + \epsilon}^{\infty} f(\omega) d\omega \right] = 0$$

$$\int_{-\infty}^{\omega_0 - \epsilon} f(\omega) d\omega + \int_{\omega_0 + \epsilon}^{\infty} f(\omega) d\omega = \mathbf{P} \int_{-\infty}^{\infty} f(\omega) d\omega \quad (6.1.9)$$

where $\mathbf{P}$ denotes the principal value. For the remaining part,

$$\begin{aligned} \int_{\substack{\text{semicircle} \\ \text{of radius } \epsilon}} f(\omega_0 + \epsilon e^{i\theta}) d(\omega_0 + \epsilon e^{i\theta}) &= -i\epsilon \int f(\omega_0 + \epsilon e^{i\theta}) e^{i\theta} d\theta \\ &= -i\epsilon \int \frac{\chi(\omega_0 + \epsilon e^{i\theta})}{\epsilon e^{i\theta}} e^{i\theta} d\theta \\ &= -i\pi \chi(\omega_0) \end{aligned} \quad (6.1.10)$$

leading to

$$\chi(\omega_0) = \frac{1}{i\pi} \mathbf{P} \int_{-\infty}^{\infty} \frac{\chi(\omega)}{\omega - \omega_0} d\omega \quad (6.1.11)$$

Writing

$$\chi(\omega_0) = \chi_1(\omega_0) + i \chi_2(\omega_0)$$

we obtain

$$\chi_1(\omega_0) = \frac{1}{\pi} \mathbf{P} \int_{-\infty}^{\infty} \frac{\chi_2(\omega)}{\omega - \omega_0} d\omega \quad (6.1.12)$$

and

$$\chi_2(\omega_0) = -\frac{1}{\pi} \mathbf{P} \int_{-\infty}^{\infty} \frac{\chi_1(\omega)}{\omega - \omega_0} d\omega \quad (6.1.13)$$

The above connections between the real and imaginary parts of $\chi(\omega)$ are called the Krammers-Kronig relations.

The time dependent response is given, according to Eq.(6.1.5) by

$$\left\langle \psi(t) \right\rangle_F = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \chi(\omega) F(\omega) d\omega \quad (6.1.14)$$

We now specialize to a particular kind of force,

$$\begin{aligned} F(t) &= F & \text{for } t < 0 \\ &= 0 & \text{for } t > 0 \end{aligned}$$

For the Fourier transform of $F(t)$, we have

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} F(t) = F \int_{-\infty}^0 dt e^{i\omega t} \\ &= F \lim_{\epsilon \rightarrow 0} \int_{-\infty}^0 dt e^{i\omega t} e^{\epsilon t} \\ &= \lim_{\epsilon \rightarrow 0} \frac{F}{\epsilon + i\omega} \\ &= F \lim_{\epsilon \rightarrow 0} \left[\frac{\epsilon}{\epsilon^2 + \omega^2} - \frac{i\omega}{\epsilon^2 + \omega^2} \right] \end{aligned} \quad (6.1.15)$$

Recalling that

$$\mathbf{P}\left(\frac{1}{\omega}\right) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon^2 + \omega^2}$$

and

$$\delta(\omega) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon^2 + \omega^2}$$

Eq.(6.1.15) can be rewritten as

$$F(\omega) = F \left[\mathbf{P}\left(\frac{1}{i\omega}\right) + \pi \delta(\omega) \right] \quad (6.1.16)$$

The response of Eq.(6.1.14) can then be written as

$$\left\langle \psi(t) \right\rangle_F = \frac{F}{2\pi i} \mathbf{P} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega} \chi(\omega) d\omega + \frac{F}{2} \chi(0) \quad (6.1.17)$$

Note that for $t > 0$, the complex function

$$f_1(z) = \frac{e^{izt} \chi(z)}{z - \omega_0}$$

is analytic in the upper half plane and the arguments between Eqns.(6.1.7) and (6.1.11) can be repeated to yield

$$e^{i\omega_0 t} \chi(\omega_0) = \frac{1}{i\pi} \mathbf{P} \int_{-\infty}^{\infty} \frac{e^{i\omega t} \chi(\omega)}{\omega - \omega_0} d\omega \quad (6.1.18)$$

For $t < 0$,

$$e^{-i\omega_0 t} \chi(\omega_0) = \frac{1}{i\pi} \mathbf{P} \int_{-\infty}^{\infty} \frac{e^{-i\omega t} \chi(\omega)}{\omega - \omega_0} d\omega \quad (6.1.19)$$

Thus for $t > 0$

$$\chi(0) = \frac{1}{i\pi} \mathbf{P} \int_{-\infty}^{\infty} \frac{e^{i\omega t} \chi(\omega)}{\omega} d\omega$$

and for $t < 0$

$$\chi(0) = \frac{1}{i\pi} \mathbf{P} \int_{-\infty}^{\infty} \frac{e^{-i\omega t} \chi(\omega)}{\omega} d\omega$$

leading to (from Eq.(6.1.17))

$$\left\langle \psi(t) \right\rangle_F = F \chi(0) \quad \text{for } t < 0 \quad (6.1.20)$$

and

$$\left\langle \psi(t) \right\rangle_F = \frac{F}{i\pi} \mathbf{P} \int_{-\infty}^{\infty} d\omega \frac{\chi(\omega)}{\omega} \cos \omega t \quad \text{for } t > 0 \quad (6.1.21)$$

So long as the force acts on the system, the response is a constant - it is only when the force is removed that the system relaxes.

If $\langle \psi(0) \rangle_F$ is the expectation value at time $t = 0$, when the force is switched off, then this relaxation can be expressed as,

$$\left\langle \psi(t) \right\rangle_F = \left\langle \psi(0) \right\rangle_F e^{-\Gamma t} = F \chi(0) e^{-\Gamma t} \quad (6.1.22)$$

where we have used Eq.(6.1.20). In an identical fashion the correlation $C(t) = \langle \psi(t)\psi(0) \rangle$ can be written as

$$C(t) = \langle \psi(t)\psi(0) \rangle = e^{-\Gamma t} C(0) \quad (6.1.23)$$

where $C(0)$ is the equal time correlation function. Comparing Eqns.(6.1.23), (6.1.22) and (6.1.21)

$$C(t) = C(0)\chi(0)^{-1} \frac{1}{i\pi} \mathbf{P} \int_{-\infty}^{\infty} d\omega \frac{\chi(\omega)}{\omega} \cos \omega t \quad (6.1.24)$$

The zero frequency susceptibility $\chi(0)$ is simply the static susceptibility which is related to the equal time correlation function $C(0)$ by the standard relation ($h\psi$ is the additional term in the free energy when an external field h is present making the weighting function

$$e^{\beta(F(h=0)+h\psi)}$$

and the susceptibility χ is $\langle \frac{\partial \psi}{\partial h} \rangle$)

$$\chi(0) = \frac{1}{k_B T} C(0) \quad (6.1.25)$$

Thus,

$$C(t) = \frac{k_B T}{i\pi} \mathbf{P} \int_{-\infty}^{\infty} d\omega \frac{\chi(\omega)}{\omega} \cos \omega t \quad (6.1.26)$$

which is the well known fluctuation dissipation theorem (FDT).

6.2 Ginzburg-Landau Model

We return to one of the models we looked at briefly in **Chapter 2** - the time dependent Ginzburg-Landau model with a conserved current. The Ginzburg-Landau free energy (equilibrium) is

$$F = \int d^D x \left[\frac{m^2}{2} \phi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{\lambda}{4} \phi^4 \right] \quad (6.2.1)$$

with the mass term m^2 proportional to $T - T_0$. The chemical potential

$$\vec{j} = -\Gamma \vec{\nabla} \frac{\delta F}{\delta \phi} \quad (6.2.2)$$

The mesoscopic models as written down in Eq.(6.2.1) involve coarse graining over microscopic scales and this averaging over the small scale degrees of freedom gives a random contribution $\vec{\xi}$ to the current

$$\vec{j} = -\Gamma \vec{\nabla} \frac{\delta F}{\delta \phi} + \vec{\xi} \quad (6.2.3)$$

with the correlation of ξ prescribed as

$$\begin{aligned} \langle \xi_i \rangle &= 0 \quad \text{and} \\ \langle \xi_i(\vec{x}, t) \xi_j(\vec{x}', t') \rangle &= 2\Gamma k_B T \delta_{ij} \delta(\vec{x} - \vec{x}') \delta(t - t') \end{aligned} \quad (6.2.4)$$

The Langevin equation is given by

$$\frac{\partial \phi}{\partial t} = -\vec{\nabla} \cdot \vec{j} = \Gamma [(m^2 - \nabla^2) \nabla^2 \phi + \lambda \nabla^2 \phi^3] + \vec{\nabla} \cdot \vec{\xi} \quad (6.2.5)$$

and with the correlation of ξ specified as in Eq.(6.2.4), FDT will be satisfied.

we now want to add a driving field. The current thus generated will be proportional to $\vec{E}$. If ϕ is uniform in a region of space (the uniform value being normalized to unity), the current will be

$$\vec{j}_E = (1 - \phi^2) \vec{E} \quad (6.2.6)$$

In writing down the Langevin equation with j_E included in the current the symmetry breaking among the different spatial directions has to be taken into account. The direction along $\vec{E}$ will be termed the parallel direction and the other directions will be perpendicular directions. The gradient in the parallel direction is denoted by ∂ and in the perpendicular direction by $\nabla_\perp$. The generalization of Eq.(6.2.5) taking into account this breaking of symmetry is

$$\begin{aligned} \frac{\partial \phi}{\partial t} = \Gamma & \left[(m_\perp^2 - \alpha_\perp \nabla_\perp^2) \nabla_\perp^2 \phi + (m_\parallel^2 - \alpha_\parallel \partial^2) \partial^2 \phi \right. \\ & \left. - 2\alpha \nabla_\perp^2 \partial^2 \phi + \lambda (\nabla_\perp^2 \phi^3 + \beta \partial^2 \phi^3) + E \partial \phi^2 \right] \\ & - (\vec{\nabla}_\perp \cdot \vec{\xi}_\perp + \partial \xi_\parallel) \end{aligned} \quad (6.2.7)$$

with the correlation given by

$$\langle \vec{\nabla}_\perp \cdot \vec{\xi}_\perp(\vec{x}, t) \vec{\nabla}'_\perp \cdot \vec{\xi}_\perp(\vec{x}', t') \rangle = N_\perp (-\nabla_\perp^2) \delta(\vec{x} - \vec{x}') \delta(t - t') \quad (6.2.8)$$

$$\langle \partial \xi_\parallel(\vec{x}, t) \partial' \xi_\parallel(\vec{x}', t') \rangle = N_\parallel (-\partial^2) \delta(\vec{x} - \vec{x}') \delta(t - t') \quad (6.2.9)$$

It is possible for the above system to show the FDT if certain conditions are satisfied. At the least these conditions require $m_{\perp}^2/m_{\parallel}^2 = N_{\perp}/N_{\parallel}$. In general this would not be valid for our system and so we cannot expect a FDT. For the system in the absence of driving, $m^2 \rightarrow 0$ as the phase transition approaches. In the presence of the drive, we have two mass terms $m_{\perp}^2$ and $m_{\parallel}^2$ and the question is whether both are involved or only one (in this case one needs to specify which one). If there is a FDT, then both $m_{\perp}^2$ and $m_{\parallel}^2$ will tend to zero at the transition. In the presence of the field, we do not expect this to happen. In the direction of the field, there should be an additional mass term, i.e. we do not expect the mass term to vanish at the previous temperature but at a lower temperature. Thus $m_{\parallel}^2 > 0$ at the transition. On the other hand the transverse direction will be unaffected and we would have $m_{\perp}^2$ still vanishing at the transition. These arguments are at best heuristic but prompt us to consider the situation

$$m_{\perp}^2 \rightarrow 0, \quad m_{\parallel}^2 > 0 \quad \text{at} \quad T = T_0 \quad (6.2.10)$$

Away from the transition, the correlations in our system are short ranged in the absence of the external field. The signature of the driving field is to make the correlation functions long ranged and power law like. We will try to show this explicitly by working in the easiest of regimes $T \gg T_0$. In this range the fluctuations ϕ are small and we can drop the nonlinear terms in the Langevin equation of Eq.(6.2.3) which then becomes

$$\begin{aligned} \frac{\partial \phi}{\partial t} = \Gamma \left[(m_{\perp}^2 - \alpha_{\perp} \nabla_{\perp}^2) \nabla_{\perp}^2 \phi + (m_{\parallel}^2 - \alpha_{\parallel} \delta^2) \delta^2 \phi \right. \\ \left. - 2\alpha \nabla_{\perp}^2 \partial^2 \phi \right] - (\vec{\nabla}_{\perp} \cdot \vec{\xi}_{\perp} - \partial \xi_{\parallel}) \end{aligned} \quad (6.2.11)$$

In Fourier space

$$\begin{aligned} -i\omega \phi(k, \omega) = -\Gamma \left[(m_{\perp}^2 + \alpha_{\perp} k_{\perp}^2) k_{\perp}^2 + (m_{\parallel}^2 + \alpha_{\parallel} k_{\parallel}^2) k_{\parallel}^2 \right. \\ \left. + 2\alpha k_{\perp}^2 k_{\parallel}^2 \right] \phi(k, \omega) - i(\vec{k}_{\perp} \cdot \vec{\xi}_{\perp} - k_{\parallel} \xi_{\parallel}) \end{aligned} \quad (6.2.12)$$

with the correlation of $\vec{\xi}$ appropriately transformed into Fourier space. From Eq.(6.2.12) we have

$$\phi(k, \omega) = \frac{-i(\vec{k}_{\perp} \cdot \vec{\xi}_{\perp} - k_{\parallel} \xi_{\parallel})}{-i\omega + \Gamma [(m_{\perp}^2 + \alpha_{\perp} k_{\perp}^2) k_{\perp}^2 + (m_{\parallel}^2 + \alpha_{\parallel} k_{\parallel}^2) k_{\parallel}^2 + 2\alpha k_{\perp}^2 k_{\parallel}^2]} \quad (6.2.13)$$

The correlation function

$$\begin{aligned} C(k, \omega) &= \langle \phi(k, \omega) \phi(-k, -\omega) \rangle \\ &= \frac{N_{\perp} k_{\perp}^2 + N_{\parallel} k_{\parallel}^2}{\omega^2 + \Gamma^2 [(m_{\perp}^2 + \alpha_{\perp} k_{\perp}^2) k_{\perp}^2 + (m_{\parallel}^2 + \alpha_{\parallel} k_{\parallel}^2) k_{\parallel}^2 + 2\alpha k_{\perp}^2 k_{\parallel}^2]} \end{aligned} \quad (6.2.14)$$

The equal time correlation function is

$$\begin{aligned} C(k) &= \int \frac{d\omega}{2\pi} C(k, \omega) \\ &= \frac{N_{\perp} k_{\perp}^2 + N_{\parallel} k_{\parallel}^2}{2\Gamma[(m_{\perp}^2 + \alpha_{\perp} k_{\perp}^2)k_{\perp}^2 + (m_{\parallel}^2 + \alpha_{\parallel} k_{\parallel}^2)k_{\parallel}^2 + 2\alpha k_{\perp}^2 k_{\parallel}^2]} \end{aligned} \quad (6.2.15)$$

The interesting thing is the behaviour for small k . To find this, we can neglect the k^4 terms in the denominator in Eq.(6.2.15)

$$C(k) = \frac{N_{\perp} k_{\perp}^2 + N_{\parallel} k_{\parallel}^2}{2\Gamma(m_{\perp}^2 k_{\perp}^2 + m_{\parallel}^2 k_{\parallel}^2)} \quad (6.2.16)$$

The limit of $k \rightarrow 0$ can now be attained in two different ways

$$\lim_{k_{\perp} \rightarrow 0} \lim_{k_{\parallel} \rightarrow 0} C(k) = \frac{N_{\perp}}{2\Gamma m_{\perp}^2} \quad (6.2.17)$$

and

$$\lim_{k_{\parallel} \rightarrow 0} \lim_{k_{\perp} \rightarrow 0} C(k) = \frac{N_{\parallel}}{2\Gamma m_{\parallel}^2}. \quad (6.2.18)$$

If FDT holds, then the two limits are equal. In general though the two limits are not equal and we will define

$$R = \frac{N_{\perp} m_{\parallel}^2}{N_{\parallel} m_{\perp}^2} \quad (6.2.19)$$

as the quantity which is an index of the violation of the FDT. For $R = 1$, FDT is valid. The biggest violation occurs as $T \rightarrow T_c$. At T_c , $\tau_{\perp} \rightarrow 0$, while $\tau_{\parallel}$ is finite. Thus $R \rightarrow \infty$ as $T \rightarrow T_c$.

The quantity $C(k)$ is the structure factor which can be probed experimentally by light scattering. Consequently, it is interesting to look at the effect of R on $C(k)$. $C(k)$ is a function of two independent variables $k_{\perp}$ and $k_{\parallel}$ and it is best exhibited as a contour plot in the $k_{\perp} - k_{\parallel}$ plane. In Fig. 6.2a, we show the contours for $R = 1$ ($N_{\perp} = 2$, $N_{\parallel} = 1$, $m_{\perp}^2 = 2$, $m_{\parallel}^2 = 1$, $\alpha = 1.5$) and Fig. 6.2b for $R = 2$ ($N_{\perp} = 2$, $N_{\parallel} = 1$, $m_{\perp}^2 = 1$, $m_{\parallel}^2 = 1$, $\alpha_{\perp} = 2$, $\alpha_{\parallel} = 1$, $\alpha = 1.5$). For $R \neq 1$, the contour has a lobed structure.

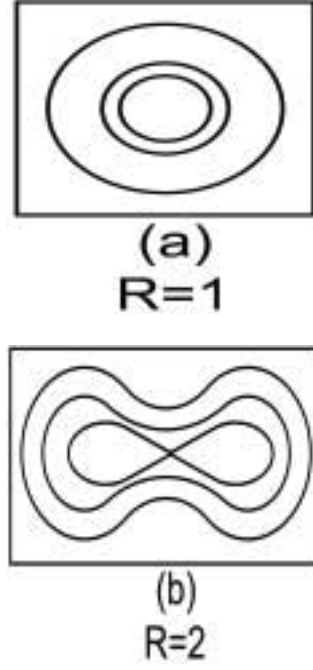


Figure 6.2. Structure factor as a function of R a) R=1 b) R=2

It is instructive to look at the correlation function in the coordinate space

$$\begin{aligned}
 C(\vec{x}) &= \int \frac{d^D k}{(2\pi)^D} e^{-i\vec{k} \cdot \vec{x}} C(k) \\
 &= \frac{1}{2\Gamma} \int \frac{d^D k}{(2\pi)^D} \frac{N_{\perp} k_{\perp}^2 + N_{\parallel} k_{\parallel}^2}{(m_{\perp}^2 k_{\perp}^2 + m_{\parallel}^2 k_{\parallel}^2)} e^{-i\vec{k} \cdot \vec{x}} \\
 &= \frac{1}{2\Gamma} \int \frac{d^D k}{(2\pi)^D} \int_0^{\infty} ds (N_{\perp} k_{\perp}^2 + N_{\parallel} k_{\parallel}^2) e^{-i\vec{k} \cdot \vec{x}} e^{-sm_{\perp}^2 k_{\perp}^2} e^{-sm_{\parallel}^2 k_{\parallel}^2} \\
 &= \frac{1}{(2\pi)^D} \int_0^{\infty} \frac{dS}{2\Gamma} \int_{-\infty}^{\infty} dk_{\parallel} \int_{-\infty}^{\infty} \Pi_{i=1}^{D-1} (dk_{\perp})_i (N_{\perp} k_{\perp}^2 + N_{\parallel} k_{\parallel}^2) \\
 &\quad \times e^{-i\vec{k}_{\perp} \cdot \vec{r}_{\perp}} e^{-ik_{\parallel} \cdot r_{\parallel}} e^{-sm_{\perp}^2 k_{\perp}^2} e^{-sm_{\parallel}^2 k_{\parallel}^2} \\
 &= \frac{1}{(2\pi)^D} \int_0^{\infty} \frac{dS}{2\Gamma} e^{-\frac{r_{\perp}^2}{4Sm_{\perp}^2}} e^{-\frac{r_{\parallel}^2}{4Sm_{\parallel}^2}} \left(\frac{1}{m_{\perp} S^{1/2}} \right)^{D-1} \frac{1}{m_{\parallel} S^{1/2}} \\
 &\quad \times \left\{ N_{\perp} \left[\frac{D-1}{2m_{\perp}^2 S} - \frac{r_{\perp}^2}{4m_{\perp}^4 S^2} \right] + N_{\parallel} \left[\frac{1}{2m_{\parallel}^2 S} - \frac{r_{\parallel}^2}{4m_{\parallel}^4 S^2} \right] \right\}
 \end{aligned}$$

Expanding,

$$\begin{aligned}
C(\vec{x}) &= \frac{1}{(2\pi)^D} \int_0^\infty \frac{dS}{2\Gamma} \frac{1}{m_\perp^{D-1}} \frac{1}{m_\parallel} \frac{1}{S^{D/2}} e^{-\frac{1}{4S}(\frac{r_\perp^2}{m_\perp^2} + \frac{r_\parallel^2}{m_\parallel^2})} \\
&\times \left[\left(N_\perp \frac{D-1}{2m_\perp^2} + \frac{N_\parallel}{2m_\parallel^2} \right) \frac{1}{S} - \frac{1}{4} \left(\frac{N_\perp r_\perp^2}{m_\perp^4} + \frac{N_\parallel r_\parallel^2}{m_\parallel^4} \right) \frac{1}{S^2} \right] \\
&\frac{1}{(2\pi)^D} \frac{1}{2\Gamma} \frac{1}{m_\perp^{D-1}} \frac{1}{m_\parallel} \left(\frac{4m_\perp^2 m_\parallel^2}{m_\parallel^2 r_\perp^2 + m_\perp^2 r_\parallel^2} \right)^{\frac{D}{2}+1} \Gamma\left(\frac{D}{2}\right) \\
&\times \left[\left(N_\perp \frac{D-1}{2m_\perp^2} + \frac{N_\parallel}{2m_\parallel^2} \right) \frac{m_\parallel^2 r_\perp^2 + m_\perp^2 r_\parallel^2}{4m_\perp^2 m_\parallel^2} - \frac{D}{2} \times \frac{1}{4} \left(\frac{N_\perp r_\perp^2}{m_\perp^4} + \frac{N_\parallel r_\parallel^2}{m_\parallel^4} \right) \right]
\end{aligned}$$

Finally, after simplification,

$$\begin{aligned}
C(\vec{x}) &= \frac{1}{(2\pi)^D} \frac{1}{16\Gamma} \frac{1}{m_\perp^4 m_\parallel^4} \left(\frac{4m_\perp^2 m_\parallel^2}{m_\parallel^2 r_\perp^2 + m_\perp^2 r_\parallel^2} \right)^{\frac{D}{2}+1} \Gamma\left(\frac{D}{2}\right) \left[\left(N_\perp (D-1) m_\parallel^2 \right. \right. \\
&\quad \left. \left. + N_\parallel m_\perp^2 \right) \times \left(m_\parallel^2 r_\perp^2 + m_\perp^2 r_\parallel^2 \right) - D \left(N_\perp r_\perp^2 m_\parallel^4 + N_\parallel r_\parallel^2 m_\perp^4 \right) \right] \\
&= \frac{1}{(2\pi)^D} \frac{1}{16\Gamma} \frac{1}{m_\perp^4 m_\parallel^4} \left(\frac{4m_\perp^2 m_\parallel^2}{m_\parallel^2 r_\perp^2 + m_\perp^2 r_\parallel^2} \right)^{\frac{D}{2}+1} \Gamma\left(\frac{D}{2}\right) N_\parallel m_\perp^2 (R-1) \\
&\quad \times \left[(D-1) m_\perp^2 r_\parallel^2 - m_\parallel^2 r_\perp^2 \right] \tag{6.2.20}
\end{aligned}$$

The vital thing about the result is that it is proportional to $R-1$, i.e. it exists only if FDT is violated. For $R=1$, Eq.(6.2.16) leads to $C(k) = \text{constant}$ and hence the Fourier transform would be a delta function at $\vec{x}=0$ - including the k^4 term in $C(k)$, which would yield the usual exponential decay.

For $R \neq 1$, the typical behaviour of $C(r)$ is $\frac{1}{r^D}$ - the power law behaviour that we said in the beginning can be generated in the presence of a driving force. However, the correlation function as obtained in Eq.(6.2.20) is anisotropic. If we call θ the angle between the direction of the applied field $\vec{E}$ and the radius vector $\vec{r}$, then $r_\parallel = r \cos \theta$ and $r_\perp = r \sin \theta$. We can now write Eq.(6.2.20) as

$$\begin{aligned}
C(r, \theta) &= \frac{1}{(2\pi)^D} \frac{1}{16\Gamma} \frac{1}{m_\perp^4 m_\parallel^4} \left(\frac{4m_\perp^2 m_\parallel^2}{m_\parallel^2 \sin^2 \theta + m_\perp^2 \cos^2 \theta} \right)^{\frac{D}{2}+1} \frac{1}{r^D} \Gamma\left(\frac{D}{2}\right) \\
&\quad \times N_\parallel m_\perp^2 (R-1) [(D-1) m_\perp^2 \cos^2 \theta - m_\parallel^2 \sin^2 \theta] \tag{6.2.21}
\end{aligned}$$

If we average over θ , then we would get zero and $C(\vec{r})$ would decay exponentially.

One of the interesting things about the model of Eq.(6.2.7) is the existence of the quadratic term $E \partial \phi^2$ in the equation of motion. This term will cause the three point correlation function, which is usually zero, to be nonzero and we will now

point out how this comes about. The three point correlation function in the Fourier space is

$$C(1, 2, 3) = \langle \phi(p, \omega_1) \phi(q, \omega_2) \phi(r, \omega_3) \rangle \quad (6.2.22)$$

Writing Eq.(6.2.7) in $k - \omega$ space and keeping only the nonlinear term involving E , we have

$$G_0^{-1} \phi(k, \omega) = -i(\vec{k}_\perp \cdot \vec{\xi}_\perp + k_\parallel \xi_\parallel) + i\Gamma E k_\parallel \sum_{p, \omega'} \phi(p, \omega') \phi(k - p, \omega - \omega') \quad (6.2.23)$$

where

$$G_0^{-1} = -i\omega + \Gamma\{(m_\perp^2 + \alpha_\perp k_\perp^2)k_\perp^2 + (m_\parallel^2 + \alpha_\parallel k_\parallel^2)k_\parallel^2 + 2\alpha k_\perp^2 k_\parallel^2\} \quad (6.2.24)$$

For $E = 0$, the zeroth order solution $\phi^{(0)}(k, \omega)$ is given by Eq.(6.2.13) i.e.

$$\phi^{(0)}(k, \omega) = -iG_0(\vec{k}_\perp \cdot \vec{\xi}_\perp + k_\parallel \xi_\parallel).$$

To first order in E , the field $\phi(k, \omega)$ is

$$\phi(k, \omega) = \phi^{(0)}(k, \omega) + i\Gamma E k_\parallel G_0(k, \omega) \sum_{p, \omega'} \phi^{(0)}(p, \omega') \phi^{(0)}(k - p, \omega - \omega') \quad (6.2.25)$$

and to the first order in E , the three point correlation is

$$C(1, 2, 3) = i\Gamma E p_\parallel G_0(p, \omega_1) \left\langle \sum_{p_1, \omega'} \phi^{(0)}(p_1, \omega') \phi^{(0)}(p - p_1, \omega_1 - \omega') \right. \\ \left. \times \phi^{(0)}(q, \omega_2) \phi^{(0)}(r, \omega_3) \right\rangle + \text{similar terms for } q \text{ and } r \quad (6.2.26)$$

(Note that $\langle \phi^{(0)} \phi^{(0)} \phi^{(0)} \rangle = 0$.) Forming the correlations,

$$C(1, 2, 3) = i\Gamma E \left\{ p_\parallel G_0(p, \omega_1) C_0(q, \omega_2) C_0(r, \omega_3) \right. \\ + q_\parallel G_0(q, \omega_2) C_0(p, \omega_1) C_0(r, \omega_3) \\ + r_\parallel G_0(r, \omega_3) C_0(p, \omega_1) C_0(q, \omega_2) \left. \right\} \\ \times \delta(\vec{p} + \vec{q} + \vec{r}) \delta(\omega_1 + \omega_2 + \omega_3) \quad (6.2.27)$$

Integrating over the frequencies, we have

$$\begin{aligned}
C(p, q, r) &= \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \int \frac{d\omega_3}{2\pi} C(1, 2, 3) \\
&= \frac{iE\delta(\vec{p} + \vec{q} + \vec{r})}{\gamma(p) + \gamma(q)\gamma(r)} \left\{ p_{\parallel} C_0(q)c_0(r) + q_{\parallel} C_0(r)c_0(p) \right. \\
&\quad \left. + r_{\parallel} C_0(q)c_0(p) \right\}
\end{aligned} \tag{6.2.28}$$

where

$$\gamma(k) = (m_{\perp}^2 + \alpha_{\perp} k_{\perp}^2) k_{\perp}^2 + (m_{\parallel}^2 + \alpha_{\parallel} k_{\parallel}^2) k_{\parallel}^2 + 2\alpha k_{\perp}^2 k_{\parallel}^2 + 2\alpha k_{\perp}^2 k_{\parallel}^2 \tag{6.2.29}$$

If the FDT holds, for small momenta, $C_0(k)$ goes to a constant and we have $C(p, q, r) \propto \delta(\vec{p} + \vec{q} + \vec{r})(p_{\parallel} + q_{\parallel} + r_{\parallel})$ which is zero. If the FDT does not hold, $C(p, q, r)$ can be zero if $p_{\parallel} = q_{\parallel} = r_{\parallel} = 0$. On the other hand it can typically be infinity as momenta tend to zero. This is because $C(k)$ goes to a constant while $\gamma(k)$ vanishes as k^2 . In coordinate space $C(x_1, x_2, x_3)$ depends on $x_1 - x_2, x_2 - x_3$ and for $|x_1 - x_2| = |x_2 - x_3| = r$, dimensional counting shows $C(r) \sim r^{1-2D}$, which is a power law behaviour.

6.3 Phase Ordering Kinetics

In this section we return to the time dependent Ginzburg-landau (TDGL) model as we discuss the formation of phase ordering. The model, as we have discussed several times before, is governed by the free energy functional

$$F = \int d^D x \left[\frac{m^2}{2} \phi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{\lambda}{4} (\phi^2)^2 \right] \tag{6.3.1}$$

with $m^2 = a_0(T - T_c)$. The high temperature phase ($T > T_c$) is symmetric and the expectation value $\langle \phi \rangle$ of ϕ is zero. In this section we imagine the temperature suddenly being lowered at $t = 0$ from $T > T_c$ to a temperature $T < T_c$. For $T < T_c$, the ground state of the system governed by the above free energy has a non zero expectation value. Thus at $t = 0$, the system is in a state with zero expectation value but is being governed by a free energy whose ground state has a non zero expectation value. So the system is out of equilibrium at $t=0$, and as time goes on it has to gradually move towards equilibrium as $t \rightarrow \infty$. This means the order parameter field ϕ has to acquire expectation value $\langle \phi \rangle$ - this is the growth of order. Domains with $\langle \phi \rangle \neq 0$ will gradually be formed in the system and this growth is referred to as *phase ordering dynamics*. One of the best ways to characterize this growth is through the typical domain size $L(t)$. A probe of the domain size is the structure factor $C(r, t)$ defined as

$$C(r, t) = \langle \phi(\vec{x}, t) \phi(\vec{x} + \vec{r}, t) \rangle \tag{6.3.2}$$

or its Fourier transform

$$C(k, t) = \int C(r, t) e^{i\vec{k} \cdot \vec{r}} d^D r \quad (6.3.3)$$

The evaluation of ϕ is governed by the TDGL equation, which reads

$$\dot{\phi} = -\Gamma \frac{\delta F}{\delta \phi} + N \quad (6.3.4)$$

for non conserved order parameter, and

$$\dot{\phi} = \Gamma \nabla^2 \frac{\delta F}{\delta \phi} + N \quad (6.3.5)$$

for conserved order parameter. The existence of a characteristic scale can be seen easily in the linearized version of Eq.(6.3.5), where for negative values of $m^2 (= -m_0^2$, say), the relaxation rate is

$$\omega = k^2(k^2 - m_0^2)$$

For small k , i.e. $k^2 < m_0^2$, the relaxation rate is negative, signalling instability. The growth rate is a maximum for $k = k_c$, where $k_c^2 = m_0^2/2$. Thus a characteristic scale k_c emerges. The above discussion serves only to emphasize that a characteristic length scale is not an unusual happening. In reality, the solution of the linearized equation is never seen. A typical time development after a quench to an ordered phase is seen in simulations where the dark regions correspond to one phase (positive $\langle \phi \rangle$) and the white regions to the other (negative $\langle \phi \rangle$). The interfaces between the two phases are known as *domain walls*.

It should be noted that the role of the noise in Eqns.(6.3.2) and (6.3.3) is severely limited. The quench to the low temperature phase does not in any significant sense depend on the precise temperature to which the system has been quenched. For all practical purposes, one would quench to $T = 0$ and the thermal noise would be absent. By the same token, the precise value of the temperature in the initial state is irrelevant and we could as well consider the system to be initially at $T = \infty$, i.e. a totally disordered phase. In this case, one could choose the initial condition to be specified by the correlation function

$$\langle \phi(\vec{x}, 0) \phi(\vec{x}', 0) \rangle = \Delta \delta(\vec{x} - \vec{x}') \quad (6.3.6)$$

and have the evolution equation as completely deterministic, i.e.

$$\frac{\partial \phi}{\partial t} = -\frac{\delta F}{\delta \phi} \quad (6.3.7)$$

for the non conserved order parameter, and

$$\frac{\partial \phi}{\partial t} = \nabla^2 \frac{\delta F}{\delta \phi} \quad (6.3.8)$$

for the conserved case (we have scaled the time variable by Γ).

We now bring in the scaling hypothesis. This is the expectation arrived at on the basis of the pictures of the phase separation process, namely, that for large enough time, there is a single characteristic scale $L(t)$ such that the domain structure (in a statistical sense) is independent of time. For the correlation function shown in Eq.(6.3.2), this implies that

$$C(r, t) = f\left(\frac{r}{L(t)}\right) \quad (6.3.9)$$

and

$$C(k, t) = L^D g(kL) \quad (6.3.10)$$

where D is the spatial dimensionality (for quenches down to $T=0$, $f(0)=1$; for quenches to arbitrary $T < T_c$, $f(0)=M(T)^2$) and we can write $C(r, t) = M^2 f(r/L)$. For two different times,

$$C(r, t, t') = f\left(\frac{r}{L}, \frac{r}{L'}\right) \quad (6.3.11)$$

where L and L' stand for $L(t)$ and $L(t')$. For $L \gg L'$

$$C(r, t, t') \rightarrow \left(\frac{L'}{L}\right)^{\bar{\lambda}} h\left(\frac{r}{L}\right) \quad (6.3.12)$$

where the exponent $\bar{\lambda}$ is a nontrivial exponent associated with the phase ordering kinetics. Writing the free energy in Eq.(6.3.1) in general as

$$F = \int d^D x \left[\frac{1}{2} (\nabla \phi)^2 + V(\phi) \right] \quad (6.3.13)$$

the equation of motion for non conserved order parameter is

$$\frac{\partial \phi}{\partial t} = (\nabla^2 \phi - \frac{\partial V}{\partial \phi}) \quad (6.3.14)$$

In the absence of dynamics, the profile of the order parameter (i.e. the wall profile) is

$$\nabla^2 \phi = \frac{\partial V}{\partial \phi} \quad (6.3.15)$$

If we consider a domain wall which is flat and z is the direction perpendicular to the domain wall, then

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial V}{\partial \phi} \quad (6.3.16)$$

with $\phi(\pm\infty) = \pm 1$. The wall where $\phi = 0$ is taken to have its centre at $z = 0$, so that $\phi(0) = 0$. Integrating Eq.(6.3.16) once

$$\left(\frac{\partial \phi}{\partial z}\right)^2 = 2V(\phi) \quad (6.3.17)$$

and hence the energy per unit area of the wall which is the surface tension is given by,

$$\sigma = \int_{-\infty}^{\infty} dz \left(\frac{\partial \phi}{\partial z}\right)^2 = \int_{-1}^{+1} d\phi [2V]^{1/2} \quad (6.3.18)$$

Linearizing Eq.(6.3.17) about $\phi = \pm 1$ gives

$$1 \mp \phi \sim e^{-[V''(\pm 1)]^{1/2}|z|} \quad z \rightarrow \pm\infty \quad (6.3.19)$$

The order parameter exponentially moves away from the walls. The excess energy is localized in the domain walls (where the order changes rapidly) and the driving force for domain growth is the wall curvature since the system energy can decrease only through a reduction in the total wall area. The existence of a surface tension means a force per unit area proportional to the mean curvature at each point of the wall. If the force per unit area is F then the work done in reducing the area by dR is $F 4\pi R^2 dR$ and this must equal the reduction in surface energy which is $8\pi\sigma R dR$ and thus $F = 2\sigma/R$. For non conserved order parameter this force will cause the wall to move with a velocity proportional to the local curvature. The force is expressible as $\eta \frac{dR}{dt}$ where η is a friction coefficient and hence

$$\eta \frac{dR}{dt} = -2\sigma/R \quad (6.3.20)$$

This same result follows from the equation of motion given in Eq.(6.3.14). Considering a spherical domain of $\phi = -1$ immersed in a sea of $\phi = +1$, we note that in a D -dimensional space

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial r^2} + \frac{D-1}{r} \frac{\partial \phi}{\partial r} - v'(\phi) \quad (6.3.21)$$

If the droplet radius R is much greater than the interface width ξ , then we expect a solution of the form

$$\phi(r, t) = f[r - R(t)] \quad (6.3.22)$$

Inserting in Eq.(6.3.21)

$$f'' + \left(\frac{D-1}{r} + \frac{dR}{dt}\right)f' - V'(f) = 0 \quad (6.3.23)$$

Now f changes from $+1$ to -1 in a small region of width ξ near the interface. Its derivative is consequently zero away from the surface $r = R(t)$ and highly peaked near $r = R(t)$. Multiplying Eq.(6.3.23) by f' and integrating from one side of the surface to the other

$$\frac{D-1}{R} + \frac{dR}{dt} = 0 \quad (6.3.24)$$

since away from $r = R(t)$, $f' = 0$ on either side and $V(\phi)$ being symmetric in ϕ also has the same value on either side of $r = R(t)$. Integration of Eq.(6.3.23) yields

$$R^2(t) = R^2(0) - 2(D-1)t \quad (6.3.25)$$

which shows that the time of collapse, t_0 , of the wrong kind of droplet scales with the initial size $R(0)$ of the droplet as $t_0 \sim R^2(0)$. If $\eta = \sigma$, then Eqns.(6.3.24) and (6.3.20) give identical results. But why should this equality hold? Let us consider the rate of energy dissipation for a plane wall moving under the influence of some external driving force with speed v . The rate of energy dissipation per unit area is

$$\begin{aligned} \frac{dE}{dt} &= \int_{-\infty}^{\infty} dz \frac{\delta F}{\delta \phi} \frac{\partial \phi}{\partial t} \\ &= - \int_{-\infty}^{\infty} dz \left(\frac{\partial \phi}{\partial t}\right)^2 \end{aligned} \quad (6.3.26)$$

using Eq.(6.3.7). If the wall profile has the form

$$\phi(z, t) = \phi(z - vt),$$

then

$$\frac{\partial \phi}{\partial t} = -v \frac{\partial \phi}{\partial z}$$

and Eq.(6.3.26) reduces to

$$\frac{dE}{dt} = -v^2 \int_{-\infty}^{\infty} dz \left(\frac{\partial \phi}{\partial z}\right)^2 = -\sigma v^2 \quad (6.3.27)$$

from Eq.(6.3.18). If η is a friction coefficient, then by definition

$$\frac{dE}{dt} = -\eta v^2$$

and we have $\eta = \sigma$. The quantity $1/R$ in Eq.(6.3.24) is the curvature of the spherical surface and hence Eq.(6.3.24) reads

$$v = -K \quad (6.3.28)$$

where v is the wall velocity and K is the curvature multiplied by $(D - 1)$. Although we have derived it for spherical surfaces, this is a general result for any curved surface - as was first established by Allen and Cahn.

For the conserved order parameter it is difficult to consider interfaces independently because there is actual diffusion of order parameter from interfaces of high curvature to regions of low curvature through the intervening bulk phase. To begin with let us consider the field ϕ inside a bulk phase and write $\phi = 1 + \tilde{\phi}$ where $\tilde{\phi}$ is a small deviation from the bulk value. Linearizing in Eq.(6.3.8) in $\tilde{\phi}$

$$\frac{\partial \tilde{\phi}}{\partial t} = -\nabla^4 \tilde{\phi} + V''(1) \nabla^2 \tilde{\phi} \quad (6.3.29)$$

For large length scales $\nabla^4 \tilde{\phi}$ is negligible and $\tilde{\phi}$ satisfies a diffusion equation with $V''(1)$ as the diffusion coefficient. One can make a further simplification. Due to the conservation law, the interface moves only slightly during the period over which the diffusive field $\tilde{\phi}$ relaxes. If the domain size is L , then the diffusive field relaxes over a time scale L^2 . A typical surface velocity is L^{-2} and hence in that time scale the interface moves a distance of order unity, which is much less than L . Thus the relaxation of the diffusion field is fast and it can be taken to be in equilibrium which means $\tilde{\phi}$ satisfies $\nabla^2 \tilde{\phi} = 0$ in the bulk.

To find the boundary condition at the interface, we introduce the chemical potential $\mu = \frac{\delta F}{\delta \phi}$. Eq.(6.3.8) is the continuity equation which can be rewritten as

$$\dot{\phi} = -\vec{\nabla} \cdot \vec{j} \quad (6.3.30)$$

$$\vec{j} = -\vec{\nabla} \mu \quad (6.3.31)$$

$$\mu = V'(\phi) - \nabla^2 \phi \quad (6.3.32)$$

In the bulk

$$\mu = V''(0) \tilde{\phi} - \nabla^2 \tilde{\phi} \simeq V''(1) \tilde{\phi}$$

for the large length scales and hence μ satisfies

$$\nabla^2 \mu = 0 \quad (6.3.33)$$

If we once again imagine a spherical droplet

$$\mu = V'(\phi) - \frac{\partial^2 \phi}{\partial r^2} - \frac{D-1}{r} \frac{\partial \phi}{\partial r}$$

Multiplying by $\frac{\partial \phi}{\partial r}$ and integrating across the interface (symmetric $V(\phi)$)

$$\mu = -\frac{\sigma(D-1)}{2R} = -\frac{\sigma K}{2} \quad (6.3.34)$$

where the second step is the general result for all curved surfaces.

Thus, at the interfaces μ is given in terms of the curvatures. Between the interfaces, μ satisfies Laplace's equation. The motion of an interface is determined by the current imbalance between what is flowing in and what is flowing out and integrating Eq.(6.3.30)

$$\begin{aligned} v \Delta \phi &= j_{out} - j_{in} \\ &= -\frac{\partial \mu}{\partial n} \end{aligned} \quad (6.3.35)$$

the second step following from Eq.(6.3.31) where n is in the direction of normal to the surface.

We now apply this to the spherical domain of size R of negative ϕ ($\phi = -1$) embedded in a sea of positive ϕ . The definition of μ in Eq.(6.3.32) gives $\mu = 0$ at $r = \infty$. For $r > R(t)$ μ is to be found from the solution of $\nabla^2 \mu = 0$. At $r = R$, in $D=3$, the value of μ is $-\sigma/R$ and hence the solution for $r > R$ is $\mu = -\sigma/r$. For $r < R$, the solution r^{-1} is not admissible being singular. Hence, $\mu = \text{constant}$ inside $r = R$ and has to match the value $-\sigma/R$ at $r = R$. Thus

$$\begin{aligned} \mu &= -\frac{\sigma}{r} \quad \text{for } r > R \\ &= -\frac{\sigma}{R} \quad \text{for } r \leq R \end{aligned} \quad (6.3.36)$$

The velocity follows from Eq.(6.3.35) as

$$2 \frac{dR}{dt} = -\frac{\partial \mu}{\partial r} \Big|_{R-\epsilon}^{R+\epsilon} = -\frac{\sigma}{R^2}$$

Integrating

$$R^3(t) = R^3(0) - \frac{3}{2} \sigma t \quad (6.3.37)$$

and the collapse time scales as $R^3(0)$ as opposed to $R^2(0)$ of the non conserved case.

We now turn to the 'growth' law for the domain size L , which is a generalization of the discussion of the isolated spherical domains. For the non conserved order parameters, if there is a single characteristic length L , which is the content of the scaling hypothesis, then the wall velocity has to be of the order of $\frac{dL}{dt}$. On the other hand, the curvature is $O(L^{-1})$ and hence Eq.(6.3.28) gives

$$\begin{aligned} \frac{dL}{dt} &\sim \frac{1}{L} \\ L &\sim t^{1/2} \end{aligned} \quad (6.3.38)$$

Another way of looking at it is via Eq.(6.3.25). A domain of size R collapses in a time of the order of R^2 and hence after time t , there will be no domains of size smaller than $t^{1/2}$, so that the characteristic domain size is $L(t) \sim t^{1/2}$.

For the conserved order parameter, at the interface the chemical potential is $\mu \sim \frac{\sigma}{L}$. It varies over a distance whose characteristic size is L and hence $|\vec{\nabla}\mu| \sim \frac{\sigma}{L^2}$. The current and hence the velocity is proportional to $\vec{\nabla}\mu$ and consequently,

$$\frac{dL}{dt} \sim \frac{\sigma}{L^2}$$

or $L \sim (\sigma t)^{1/3}$ (6.3.39)

We would arrive at a similar result using Eq.(6.3.7) as in the case of the non conserved order parameter.

6.4 Topological Defects

So far, we have discussed systems with a scalar order parameter and where an ‘up’ phase changed to a ‘down’ phase, we had a domain wall, which can be viewed as a defect in the topology. Consequently, a domain wall can be called a ‘topological’ defect. In general by a topological defect one means a surface on which the order parameter vanishes and which separates domains of two equilibrium phases.

The generalized free energy functional, which has $O(n)$ symmetry in the order parameter space is

$$F = \int d^D x \left[\frac{1}{2} (\partial_\mu \vec{\phi} \partial^\mu \vec{\phi}) + V(\vec{\phi}) \right] \quad (6.4.1)$$

The function $V(\vec{\phi})$ depends on the magnitude of $\vec{\phi}$ alone and has the general shape as shown in Fig. 6.3 for $n = 2$ For non conserved field

$$\frac{\partial \phi_i}{\partial t} = \nabla^2 \phi_i - \frac{\partial V}{\partial \phi_i} \quad (6.4.2)$$

The topological defects can be generated by studying the stationary situation of Eq.(6.4.2) with the proper boundary conditions. For the $O(n)$ theory in a D -dimensional space, the requirement that all n components of $\vec{\phi}$ vanish at the defect core defines a surface of dimension $D - n$. For a defect to exist $n \leq D$. A domain wall is a surface of dimension $D - 1$. For $n = 2$, the defects are points in $D = 2$ (vortices) and lines (*strings or vortex lines*) in $D = 3$. For $n = 3$, in $D = 3$ they are points (*hedgehogs or monopoles*). The fields in the various cases are as shown in Fig. 6.4

For $n < D$ the field $\vec{\phi}$ varies only in the n dimensions orthogonal to the vortex core, while remaining stationary in the $D - n$ dimensions parallel to the core. The coarsening occurs by reduction in radius of curvature which reduces the sharp

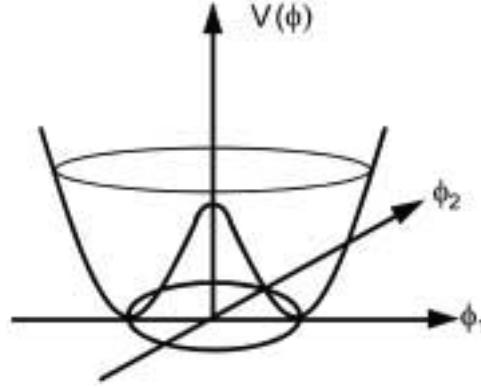


Figure 6.3. A Mexican hat potential

features of the defect. These processes diminish the area of a domain wall or the length of a vortex line. For point defects, coarsening occurs by the mutual annihilation of defect-antidefect pairs. An ‘anti-defect’ has a different topological charge in the direction of the field as one goes around the defect.

For the radially symmetric field $\vec{\phi} = \hat{r} f(r)$ and $f(r)$ is the profile function which satisfies

$$\frac{d^2 f}{dr^2} + \frac{n-1}{r} \frac{df}{dr} - \frac{n-1}{r^2} f - V'(f) = 0 \quad (6.4.3)$$

with boundary conditions $f(0)=0$ and $f(\infty)=1$. The behaviour for large r , can be obtained by setting $f = 1 - \epsilon(r)$ and linearizing in ϵ to find

$$\epsilon(r) \approx \frac{n-1}{V''(1)} \frac{1}{r^2}$$

The approach to saturation is power law like for $n > 1$ as opposed to the exponential behaviour in Eq.(6.3.17) for $n = 1$.

For this radially symmetric defect the energy per unit core volume is

$$E = S_n \int dr r^{n-1} \left[\frac{n-1}{2r^2} f^2 + \frac{1}{2} (\vec{\nabla} f)^2 + V(f) \right] \quad (6.4.4)$$

where $S_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of a n -dimensional sphere. For $n \geq 2$, the first term in the above expression dominates at large r , since its asymptotic behaviour is r^{-2} . Those of the other two terms are r^{-6} and r^{-4} . The energy is clearly, $E \sim \ln(\frac{L_{sys}}{\xi})$ for $n = 2$ and $E \sim L_{sys}^{n-2}$ for $n > 2$.

For a phase ordering system, we have several defects and the field of a single defect will be screened out beyond a characteristic scale $L(t)$ due to the other defects. So the integration in Eq.(6.4.4) will have to be cut off at $L(t)$. The dynamics

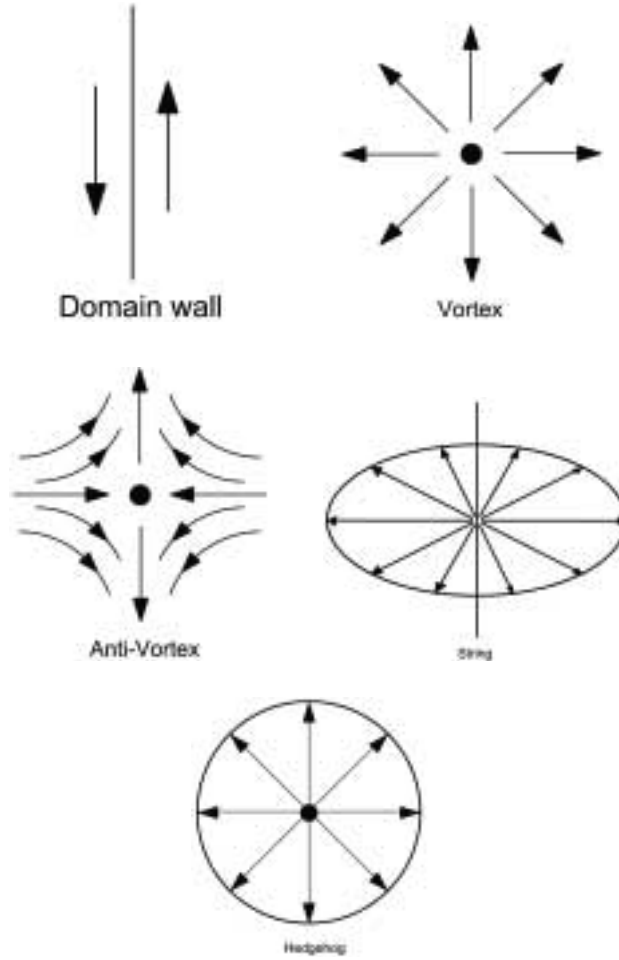


Figure 6.4. Topological singularities in the order parameter field

of defect structures much smaller than $L(t)$ are of particular interest since these are the analogues of the small domains of the scalar system. For $D = n = 2$, these are the vortex-anti vortex pairs, for $D = n = 3$ these are the monopole-antipole pairs, for $D = 3, n = 2$, these are the vortex rings and so on. Such structures are characterized by r , the pair separation or the size of the ring. The volume of the defect core being r^{D-n} , the energy of such a structure is

$$\begin{aligned}
 E &\sim r^{D-2} \ln \frac{r}{\xi} & D \geq n = 2 \\
 E &\sim r^{D-2} & D \geq n > 2
 \end{aligned} \tag{6.4.5}$$

The derivative $\frac{dE}{dr}$ gives the force which leads to the collapse of the structure. The force per unit volume is

$$F(r) \sim \begin{cases} -\frac{1}{r} & D = n = 2 \\ -r^{n-3} \ln \frac{r}{\xi} & D > n = 2 \\ -r^{n-3} & D \geq n > 2 \end{cases} \quad (6.4.6)$$

The total energy density ϵ can be calculated by using Eq.(6.4.5) with $r = L(t)$ and dividing by the value L^D , which gives

$$\begin{aligned} \epsilon &= \frac{1}{L^2} \ln \frac{L}{\xi} & D \geq n = 2 \\ \epsilon &= \frac{1}{L^2} & D \geq n > 2 \end{aligned} \quad (6.4.7)$$

Let us now look at the dynamics of the defects. For extended defects, the defects occupy the last $D - n$ dimensions of a cartesian coordinate system defined by $x_1, x_2, \dots, x_D$. Let the velocity of the defect be v in the x_1 direction. The field ϕ depends on the coordinates $x_1, \dots, x_n$ and the rate of change of energy is given by

$$\begin{aligned} \frac{dE}{dt} &= \int (\Pi_{i=1}^n dx_i) \frac{\delta F}{\delta \phi} \cdot \frac{\partial \phi}{\partial t} \\ &= - \int (\Pi_{i=1}^n dx_i) \left(\frac{\partial \phi}{\partial t} \right)^2 \end{aligned} \quad (6.4.8)$$

If the profile is

$$\vec{\phi}(x_1, \dots, x_n) = \vec{f}(x_1 - vt, x_2, \dots, x_n) \quad (6.4.9)$$

then

$$\frac{\partial \vec{\phi}}{\partial t} = -v \frac{\partial \vec{\phi}}{\partial x_1} \quad (6.4.10)$$

and Eq.(6.4.8) becomes

$$\frac{dE}{dt} = -v^2 \int dx_1 \dots dx_n \left(\frac{\partial \vec{\phi}}{\partial x_1} \right)^2 \quad (6.4.11)$$

If we make a small velocity approximation then it is permissible to set $v = 0$ in the integrand in Eq.(6.4.8) and using the equivalence of all n -directions

$$\frac{dE}{dt} = -\frac{v^2}{n} \int d^n x (\vec{\nabla} \vec{\phi})^2 = -\eta v^2 \quad (6.4.12)$$

where

$$\eta = \frac{1}{n} \int d^n x (\vec{\nabla} \vec{\phi})^2 \quad (6.4.13)$$

which is the energy per unit volume so far as the leading term is concerned (it is a part of $\int d^n x (\vec{\nabla} \vec{\phi})^2$ which is divergent and dominates the energy - see the discussion after Eq.(6.)) and diverges with the system size for $n \geq 2$. For small defect size r , the integral will be cut off at r and we would have a scale dependent friction coefficient

$$\begin{aligned} \eta(r) &\sim r^{n-2} \ln \frac{r}{\xi} & D \geq n = 2 \\ &\sim r^{n-2} & D \geq n > 2 \end{aligned} \quad (6.4.14)$$

The scaling hypothesis now gives the growth law. From Eq.(6.4.6), the force is

$$F(L) \sim 1/L \quad \text{for} \quad D = n = 2$$

or in terms of η

$$F(L) \sim \frac{dL}{dt} \eta(L) \sim \frac{dL}{dt} \ln \frac{L}{\xi} \quad \text{for} \quad D = n = 2.$$

Consequently, for $D = n = 2$,

$$L \sim \left(\frac{t}{\ln t} \right)^{1/2} \quad (6.4.15)$$

For all other situations the same argument yields

$$L \sim t^{1/2} \quad (6.4.16)$$

For the conserved order parameter, this argument would not work - a more general argument for deriving $L(t)$ would be given later.

The existence of defects has a profound effect on the short term behaviour of the two point correlation function $C(r) = \langle \phi(\vec{x}) \phi(\vec{x} + \vec{r}) \rangle$. This can be appreciated from the fact that from the scaling hypothesis $\nabla \phi(r)$ would be estimated as $O(1/L)$. Near the defect core, $\xi \ll r \ll L$, the gradient is much larger since $(\vec{\nabla} \phi)^2 \sim \frac{1}{r^2}$. For the defect field to saturate, we need $r \gg \xi$, while for it to be unaffected by the other defect fields, we need $r \ll L$. Thus short distance is specified by $\xi \ll r \ll L$. Alternatively, large momentum is specified by $\frac{1}{L} \ll k \ll \frac{1}{\xi}$.

For a scalar field, we consider two points $\vec{x}$ and $\vec{x} + \vec{r}$ with $\xi \ll r \ll L$. The product $\phi(\vec{x}) \phi(\vec{x} + \vec{r})$ will be -1 if a wall passes between them and $+1$ if there is no wall. Since $r \ll L$, we neglect the probability of finding more than one wall. To find $C(r, t)$, we need to know the probability with which a randomly placed rod of length r cuts a domain wall. This probability is of the order $\frac{r}{L}$ and hence

$$C(r, t) \approx (-1) \frac{r}{L} + 1(1 - \frac{r}{L}) = 1 - \frac{2r}{L} \quad (6.4.17)$$

The correlation function is non-analytic in $\vec{r}$ at $\vec{r} = 0$, being linear in $|\vec{r}|$ (note that although the above argument breaks down in the core region, we are interested in the region $r \gg \xi$, $L \gg \xi$ with $\frac{r}{L}$ arbitrary, which is to say that the non-analyticity is in the scaling variable $\frac{r}{L}$). Fourier transforming Eq.(6.), we get the structure factor $S(k, t)$, which has, from dimensional analysis, the form

$$S(k, t) \sim \frac{1}{Lk^{D+1}} \quad kL \gg 1 \quad (6.4.18)$$

This is known as Porod's law. The L -dependence of the result is interesting. The factor L^{-1} is the area of the domain wall per unit volume. For $kL \gg 1$, the scattering function (i.e. the structure factor) probes structure on scales much shorter than typical interwall spacing or radius of curvature. In this regime, the structure factor scales as the area of the wall since each element of the wall with linear dimension larger than k^{-1} contributes independently to the structure factor.

The above observation allows one to generalize Eq.(6.) to vector fields. The idea is that for $kL \gg 1$, the structure factor should scale as the total volume of the defect core. Since the dimension of the defect is $D - n$, the amount of defect per unit volume scales as L^{-n} . This immediately yields

$$S(k, t) \sim \frac{1}{L^n k^{D+n}} \quad kL \gg 1 \quad (6.4.19)$$

which is a generalized Porod's law for $O(n)$ field.

6.5 The Structure Factor

An extremely important and at the same time difficult issue is the calculation of the full structure factor $S(k, t)$. In general this can be done exactly only in a few specialized cases, all of which are somewhat far removed from reality. However, some of the qualitative features that these somewhat fanciful situations show persist in the real situations. Hence, it is worthwhile to explore the exact solutions. The one that we will explore here is the spherical limit. For this purpose, we take

$$V(\phi) = -\frac{\phi^2}{2} + \frac{(\phi^2)^2}{4n}$$

where

$$\phi^2 = \sum_i \phi_i^2.$$

The equation of motion as given by Eq.(6.4.2) becomes

$$\begin{aligned}
\dot{\phi}_i &= \nabla^2 \phi_i + \phi_i - \frac{\phi_i^2}{n} \\
&= \nabla^2 \phi_i + \phi_i - \frac{\langle \phi^2 \rangle}{n} \phi_i - \left(\frac{\phi_i^2 - \langle \phi^2 \rangle}{n} \right) \phi_i
\end{aligned} \tag{6.5.1}$$

Since $\langle \phi^2 \rangle = n \langle \phi_i^2 \rangle$, in the limit of $n \rightarrow \infty$, the third term in the R.H.S of Eq.(6.5.1) is $O(1)$, while the last term is $O(n^{-1})$. Consequently, the last term does not survive the $n \rightarrow \infty$ limit and we have a linearized equation of motion

$$\dot{\phi}_i = \nabla^2 \phi_i + \phi_i - \langle \phi_i^2 \rangle \phi_i \tag{6.5.2}$$

where $\langle \phi_i^2 \rangle$ has to be determined self consistently. Writing $a(t) = 1 - \langle \phi_i^2 \rangle$, the above equation of motion in momentum space is

$$\frac{\partial \phi_i(\vec{k})}{\partial t} + k^2 \phi_i(\vec{k}) = a(t) \phi_i(\vec{k}) \tag{6.5.3}$$

which has the solution

$$\phi_i(k, t) = \phi_i(k, 0) e^{-k^2 t} e^{b(t)} \tag{6.5.4}$$

where

$$b(t) = \int_0^t a(t') dt' \tag{6.5.5}$$

Differentiating Eq.(6.5.5),

$$\begin{aligned}
\frac{db}{dt} &= a(t) = 1 - \sum_k \left\langle \phi_i(k, t) \phi_i(-k, t) \right\rangle \\
&= 1 - \Delta \sum_k e^{-2k^2 t} e^{2b(t)}
\end{aligned} \tag{6.5.6}$$

where Eq.(6.3.5) has been used to bring in Δ for the strength of the initial distribution of ϕ_i . As $t \rightarrow \infty$, we expect that $a = 1 - \langle \phi_i \rangle$ will tend to zero and thus for large t , the L.H.S of Eq.(6.5.6) is negligible, and we have

$$\begin{aligned}
1 &= \Delta e^{2b(t)} \sum_k e^{-2k^2 t} = \Delta e^{2b(t)} \left(\frac{1}{8\pi t} \right)^{D/2} \\
\text{or } b(t) &= \frac{D}{4} \ln \left(\frac{8\pi t}{\Delta^{2/D}} \right) = \frac{D}{4} \ln \frac{t}{t_0}
\end{aligned} \tag{6.5.7}$$

The above $b(t)$ leads to

$$a(t) \propto \frac{1}{t}$$

which indeed vanishes as $t \rightarrow \infty$. The solution of Eq.(6.5.4) now becomes

$$\phi_i(k, t) = \phi_i(k, 0) \left(\frac{t}{t_0} \right)^{D/4} e^{-k^2 t} \quad (6.5.8)$$

The structure factor $S(k, t)$ follows as

$$\begin{aligned} S(k, t) &= \left\langle \phi_i(k, t) \phi_i(-k, t) \right\rangle = \Delta \left(\frac{t}{t_0} \right)^{D/2} e^{-k^2 t} \\ &= (8\pi t)^{D/2} e^{-k^2 t} \end{aligned} \quad (6.5.9)$$

and in the coordinate space

$$C(\vec{r}, t) = e^{-r^2/8t} \quad (6.5.10)$$

The results show the expected scaling form of Eqns.(6.3.9) and (6.3.10) with $L(t) \propto t^{1/2}$. The structure factor has a Gaussian tail, rather than the power law tail that we saw occurring for $n \leq D$. The power law tail was due to the existence of defects. In systems with $n \geq D + 1$, where there would be no topological defects, we expect the Gaussian tail to be the correct result.

For the two time correlation function,

$$\begin{aligned} S(k, t, t') &= \left\langle \phi_i(k, t) \phi_i(-k, t') \right\rangle = \left[8\pi \sqrt{t t'} \right]^{D/2} e^{-k^2(t+t')} \\ \text{and } C(r, t, t') &= \left[\frac{4t t'}{(t+t')^2} \right]^{D/4} e^{-\frac{r^2}{4(t+t')}} \end{aligned} \quad (6.5.11)$$

In the limit $t \gg t'$

$$\begin{aligned} C(r, t, t') &= \left(\frac{4t'}{t} \right)^{D/4} e^{-\frac{r^2}{4t}} \\ &= \left(\frac{t'}{L} \right)^{\bar{\lambda}} h(r/L) \end{aligned} \quad (6.5.12)$$

the last line following from Eq.(6.3.12), when we define the exponent $\bar{\lambda}$ through the dependence on the later time. Clearly, $\bar{\lambda} = D/2$ in this spherical limit. If we were to set one of the times equal to zero, then $C(\vec{r}, t, 0)$ would be the correlation with the initial condition and we would get from Eqns.(6.5.9) and (6.5.11)

$$S(\vec{k}, t, 0) = [8\pi \sqrt{t t_0}]^{D/2} e^{-k^2 t} \quad (6.5.13)$$

which is just the same as Eq.(6.5.11) with the shorter time t' set equal to some short time cut off t_0 . One can define a response to the initial condition as

$$G(k, t) = \left\langle \frac{\partial \phi_i(k, t)}{\partial \phi_i(k, 0)} \right\rangle \quad (6.5.14)$$

In the spherical limit

$$G(k, t) = \left(\frac{t}{t_0} \right)^{D/4} e^{-k^2 t} \quad (6.5.15)$$

Clearly,

$$S(k, t, 0) = \Delta G(k, t) \quad (6.5.16)$$

This result holds in general. The general scaling form for $G(k, t)$, namely

$$G(k, t) = L^\lambda g(kL) \quad (6.5.17)$$

defines a new exponent λ which is $D/2$ in the spherical limit. Since correlation with the initial correlation has the scaling form

$$C(\vec{r}, t, 0) = L^{-\bar{\lambda}} f\left(\frac{r}{L}\right),$$

we have from Eq.(6.5.12)

$$\bar{\lambda} = D - \lambda \quad (6.5.18)$$

For the conserved order parameter, there is an extra $-\nabla^2$ on the R.H.S of the equation of motion. Instead of Eq.(6.5.2), one now finds

$$\dot{\phi}_i = -\nabla^4 \phi_i - \nabla^2 (1 - \langle \phi_i^2 \rangle) \phi_i \quad (6.5.19)$$

in the spherical limit. As before, we work in the momentum space and get instead of Eq.(6.5.4)

$$\begin{aligned} \phi_i(k, t) &= \phi_i(k, 0) e^{-k^4 t + k^2 b(t)} \\ \frac{db}{dt} &= a(t) = 1 - \Delta \sum_k e^{-2k^4 t + 2k^2 b(t)} \end{aligned} \quad (6.5.20)$$

At late times, we anticipate $a(t) \rightarrow 0$ and then

$$\begin{aligned} 1 &= \Delta \sum_k e^{-2k^4 t + 2k^2 b(t)} \\ &= \Delta \int \frac{d^D k}{(2\pi)^D} e^{-2k^4 t + 2k^2 b(t)} \\ &= \Delta \frac{C_D}{(2\pi)^D} \int k^{D-1} dk e^{-2 \frac{b^2(t)}{t} [(\frac{k^2 t}{b(t)})^2 - \frac{k^2 t}{b(t)}]} \\ &= \left[\frac{b(t)}{t} \right]^{D/2} \Delta \frac{C_D}{(2\pi)^D} \int e^{-2\beta(t)[x^4 - x^2]} x^{D-1} dx \end{aligned} \quad (6.5.21)$$

where $C_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is the area of a sphere in D-dimensions and $\beta(t) = b^2(t)/t$. The integral in Eq.(6.5.21) can be evaluated by the method of steepest descent if $\beta(t)$ is very large. Anticipating this, we expand

$$x^4 - x^2 = -\frac{1}{4} + 2\left(x - \frac{1}{\sqrt{2}}\right)^2 + \dots$$

and performing the Gaussian integration in Eq.(6.5.21)

$$\begin{aligned} 1 &= \Delta A_D \left[\frac{b(t)}{t} \right]^{D/2} e^{\frac{1}{2}\beta(t)} \left[\frac{1}{\beta(t)} \right]^{1/2} \\ &= A_D \left(\frac{\beta}{t} \right)^{D/4} \frac{1}{\sqrt{\beta}} e^{\beta/2} \end{aligned} \quad (6.5.22)$$

The asymptotic solution can be found by writing

$$0 = \ln A_D + \left(\frac{D}{2} - \frac{1}{2} \right) \ln \beta - \frac{D}{4} \ln t + \frac{\beta}{2} \quad (6.5.23)$$

assuming $\beta \gg \ln \beta$,

$$\beta \simeq \frac{D}{2} \ln t \quad (6.5.24)$$

justifying the use of steepest descent for $t \rightarrow \infty$. The structure factor is given by

$$\begin{aligned} S(k, t) &= \Delta e^{-2k^4 t} e^{2k^2 b(t)} \\ &= \Delta e^{\frac{b^2(t)}{t} [2(\frac{k}{k_m})^2 - (\frac{k}{k_m})^4]} \\ &= \Delta e^{\frac{1}{2}\beta(t) [2x^2 - x^4]} \\ &= \Delta e^{\frac{D}{4} [2x^2 - x^4] \ln t} \\ &= \Delta t^{\frac{D}{4} \phi(x)} \end{aligned} \quad (6.5.25)$$

where

$$\begin{aligned} \phi(x) &= 1 - (1 - x)^2 \\ x &= \frac{k}{k_m} \\ \text{and } k_m &= \frac{D}{8} \left(\frac{\ln t}{t} \right)^{D/4} \end{aligned}$$

is the wavenumber at which the structure factor has a maximum. Had we retained the term $\frac{D-2}{4} \ln \beta$ in Eq.(6.5.23), we would have found a prefactor of $O[(\ln t)^{\frac{2-D}{4}}]$ in Eq.(6.5.25). The result of Eq(6.5.25) is interesting because it clearly exhibits two length scales instead of one required for simple scaling. There is a length scale

$L \sim t^{1/4}$, while there is another $k_m^{-1} \sim (\frac{t}{\ln t})^{1/4}$. For simple scaling, the two scales would have been the same, but here they are logarithmically different. This is an example of multiscaling. It should be noted, though, that this is a special feature of the spherical limit.

Exact solutions for the scaling function can also be found for the one dimensional Ising model and the one dimensional XY model. Instead of discussing these we will now proceed to discuss the approximate theories for finding the scaling functions.

6.6 Approximate Techniques

Approximate techniques for obtaining the scaling functions revolve around the replacement of the physical field $\phi(\vec{x}, t)$ by an auxiliary field $m(\vec{x}, t)$. The field $\phi(\vec{x}, t)$ is essentially ± 1 except at the domain walls where it varies rapidly from +1 to -1 or vice versa. This discontinuous variation is to be replaced by a smoothly varying function m . This is generally achieved by a nonlinear transformation $\phi(m)$ which is a function of the 'sigmoid' shape. Turning to Eq.(6.3.28), we can write the velocity of point on the interface as

$$v = -K = -\vec{\nabla} \cdot \hat{n} \quad (6.6.1)$$

where $\hat{n}$ is the unit vector normal to the wall. The domain wall or the interface is defined by the zeroes of $m(\vec{x})$ i.e. the equation of the interface is $m(\vec{x}) = 0$. Consequently, $\hat{n} = \frac{\vec{\nabla} m}{|\vec{\nabla} m|}$ and from Eq.(6.6.1)

$$v = \frac{-\nabla^2 m + n_\alpha n_\beta \vec{\nabla}_\alpha \vec{\nabla}_\beta m}{|\vec{\nabla} m|} \quad (6.6.2)$$

To find the equation of motion for m , we note that in a frame of reference moving with the interface, we must have

$$\frac{dm}{dt} = \frac{\partial m}{\partial t} + (\vec{v} \cdot \vec{\nabla})m = 0 \quad (6.6.3)$$

Now since $\vec{v}$ is parallel to $\vec{\nabla} m$ and defined in the same direction

$$(\vec{v} \cdot \vec{\nabla})m = v|\vec{\nabla} m| \quad (6.6.4)$$

leading to

$$v = -\frac{1}{|\vec{\nabla} m|} \frac{\partial m}{\partial t} \quad (6.6.5)$$

Combining Eqns.(6.6.2) and (6.6.5), we have

$$\frac{\partial m}{\partial t} = \nabla^2 m - n_\alpha n_\beta \vec{\nabla}_\alpha \vec{\nabla}_\beta m \quad (6.6.6)$$

Since $n_\alpha = \frac{\partial_\alpha m}{|\nabla m|}$, the above equation is nonlinear. To make progress, Ohta, Jasnow and Kawasaki (OJK) made the simplification of replacing $n_\alpha n_\beta$ by its spherical average $\frac{\delta_{\alpha\beta}}{D}$ where D is the dimensionality of space. The equation of motion acquires the simple form

$$\frac{\partial m}{\partial t} = \tilde{D} \nabla^2 m \quad (6.6.7)$$

where $\tilde{D} = \frac{D-1}{D}$ is a diffusion constant.

The solution of Eq.(6.6.7) requires the specification of initial conditions. In the absence of long-range correlations, the form of the random initial conditions will not play a major part in the late-stage scaling. A convenient choice is the Gaussian distribution for m at $t = 0$, which implies

$$\begin{aligned} \langle m(\vec{x}, 0) \rangle &= 0 \\ \text{and} \quad \langle m(\vec{x}, 0) m(\vec{x}', 0) \rangle &= \Delta \delta(\vec{x} - \vec{x}') \end{aligned} \quad (6.6.8)$$

Solving Eq.(6.6.7) and averaging over initial conditions, leads to the correlator

$$\langle m(1)m(2) \rangle = \frac{\Delta}{(8\pi \tilde{D}t)^{D/2}} e^{-\frac{r^2}{8\tilde{D}t}} \quad (6.6.9)$$

where 1 and 2 represent space points separated by r . It is important to define the normalized correlator.

$$\gamma(12) = \frac{\langle m(1)m(2) \rangle}{\sqrt{\langle m^2(1) \rangle} \sqrt{\langle m^2(2) \rangle}} = e^{-\frac{r^2}{8\tilde{D}t}} \quad (6.6.10)$$

To calculate the pair correlation function for the original field, we need to know the joint probability distribution for $m(1)$ and $m(2)$. Given that the distribution is Gaussian this is straight forward and can be written as

$$P(m(1), m(2)) = N e^{-\frac{1}{2(1-\gamma^2)} \left[\frac{m^2(1)}{S_0(1)} + \frac{m^2(2)}{S_0(2)} - 2\gamma \frac{m(1)m(2)}{[S_0(1)S_0(2)]^{1/2}} \right]} \quad (6.6.11)$$

where $\gamma = \gamma(12)$, $S_0(1) = \langle m^2(1) \rangle$, $S_0(2) = \langle m^2(2) \rangle$ and the normalization constant N is given by

$$N = \frac{1}{2\pi} [(1 - \gamma^2) S_0(1) S_0(2)]^{-1/2} \quad (6.6.12)$$

We note that 1 and 2 are arbitrary space time points and here $\gamma = \gamma(12)$ has to be thought of as at two different times t_1 and t_2 and it is straightforward to see that (see Eq.(6.5.11))

$$\gamma = \left[\frac{4t_1 t_2}{(t_1 + t_2)^2} \right]^{D/4} e^{-\frac{r^2}{4\bar{D}(t_1+t_2)}} \quad (6.6.13)$$

The pair correlation function is to be found as

$$C(12) = \left\langle \phi(m_1) \phi(m_2) \right\rangle \quad (6.6.14)$$

In the scaling regime, one can replace $\phi(m)$ by $\text{sgn}(m)$ since the walls occupy a negligible volume fraction. Thus,

$$C(12) = \left\langle \text{sgn}[m(1)] \text{sgn}[m(2)] \right\rangle = \frac{2}{\pi} \sin^{-1} \gamma \quad (6.6.15)$$

and together with Eq.(6.6.13) or (6.6.10) as the case may be (unequal time or equal time), this expression gives the two point correlation function. This expression fits experimental and simulation data very well. For $t_1 \gg t_2$ in the two time correlation function, γ is small and $C(\vec{r}, t_1, t_2)$ becomes approximately,

$$\frac{2}{\pi} \gamma \sim \frac{2}{\pi} \left(\frac{4t_2}{t_1} \right)^{D/4} e^{-r^2/4\bar{D}t_1},$$

which gives $\bar{\lambda} = \frac{D}{2}$ in this approximation.

In a different approach due to Mazenko, The function $\phi(m)$ is chosen in a clever fashion. It is taken to be the equilibrium interface profile function defined by

$$\phi''(m) = v'(\phi) \quad (6.6.16)$$

with boundary conditions $\phi(\pm\infty) = \pm 1$, $\phi(0) = 0$. Near the wall, the field 'm' has the physical significance of a coordinate normal to the wall. Consequently, the interface length scale does not appear in the problem anymore. Only the domain scale $L(t)$ is relevant. In this approach the TDGL equation becomes,

$$\dot{\phi} = \nabla^2 - \phi''(m) \quad (6.6.17)$$

Multiplying by ϕ at a different space time point and averaging over initial conditions, we get

$$\frac{1}{2} \partial_t C(12) = \nabla^2 C(12) - \left\langle \phi''(m(1)) \phi(m(2)) \right\rangle \quad (6.6.18)$$

The above relation is exact. To simplify, it was assumed by Mazenko that m can be treated as Gaussian. This assumption relates the last term to $C(12)$ and one arrives at a closed system.

To end this section we look at a variant of the above technique due to Mazenko. Using

$$\dot{\phi}(m) = \dot{m}\phi'(m)$$

and

$$\begin{aligned}\nabla^2\phi(m) &= \vec{\nabla} \cdot (\vec{\nabla}\phi(m)) = \vec{\nabla} \cdot (\phi'(m)\vec{\nabla}m) \\ &= \phi''(m)(\nabla m)^2 + \phi'(m)\nabla^2m\end{aligned}$$

we can write the equation of motion as

$$\begin{aligned}\dot{\phi} &= \nabla^2\phi - V'(\phi) \\ \text{or } \phi'(m)\dot{m} &= \nabla^2m\phi'(m) + \phi''(m)(\nabla m)^2 - \phi''(m) \\ \text{or } \dot{m} &= \nabla^2m - \frac{\phi''(m)}{\phi'(m)}(1 - (\nabla m)^2)\end{aligned}\quad (6.6.19)$$

For a general potential, this is a complicated nonlinear equation. However, the scaling function is going to be independent of the precise form of the potential and of the particular choice of initial conditions. Physically, the motion of the interface is determined by the curvature. The potential $V(\phi)$ determines the interface profile which is irrelevant to the large scale structure. The key step in the approach to be discussed is the anticipation of the irrelevance of $V(\phi)$ and choosing a $V(\phi)$ to simplify Eq.(6.6.19). The choice is

$$\phi''(m) = -m\phi'(m) \quad (6.6.20)$$

with the boundary condition $\phi(\pm\infty) = \pm 1$ and $\phi(0) = 0$. The solution is

$$\phi(m) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^m dx e^{-x^2/2} = \text{Erf}\left(\frac{m}{\sqrt{2}}\right) \quad (6.6.21)$$

With this choice,

$$\dot{m} = \nabla^2m + m(1 - (\nabla m)^2) \quad (6.6.22)$$

Which is still a nonlinear equation, but simpler than the original TDGL equation.

We now show what is the approximation on Eq.(6.6.22) that leads to the OJK scaling function of Eq.(6.6.15). We introduce an internal *colour* index α for m and write the equation of motion as

$$\dot{m}_\alpha = \nabla^2m_\alpha + m_\alpha \left[1 - \frac{1}{N} \sum_{\beta=1}^N (\vec{\nabla}m_\beta)^2\right] \quad (6.6.23)$$

N is the number of *colour* indices and our situation corresponds to $N = 1$. The OJK result is obtained as $N \rightarrow \infty$, when $\frac{1}{N} \sum_{\beta=1}^N (\vec{\nabla} m_{\beta})^2$ may be replaced by its average value. In this limit,

$$\dot{m} = \nabla^2 m + a(t)m \quad (6.6.24)$$

where

$$a(t) = 1 - \langle (\nabla m)^2 \rangle \quad (6.6.25)$$

The above equations provide a self consistent linear set for $m(t)$.

Taking the initial condition for m to be Gaussian with zero mean and the correlator prescribed in the Fourier space as

$$\left\langle m_{\vec{k}}(0) m_{-\vec{k}'}(0) \right\rangle = \Delta \delta_{\vec{k}\vec{k}'} \quad (6.6.26)$$

the solution of Eq.(6.6.24) is

$$m_{\vec{k}}(t) = m_{\vec{k}}(0) e^{-k^2 t + b(t)} \quad (6.6.27)$$

where

$$b(t) = \int_0^t dt' a(t')$$

leading to

$$a(t) = \frac{db}{dt} = 1 - \Delta \sum k^2 e^{-2k^2 t + 2b(t)} \quad (6.6.28)$$

After evaluating the sum, one obtains for large t (when $\frac{db}{dt}$ can be neglected)

$$e^{2b} \approx \frac{4t}{\Delta D} (8\pi t)^{D/2} \quad (6.6.29)$$

and hence

$$a(t) = \frac{D+2}{4t} \quad (6.6.30)$$

Combining Eqns.(6.6.29) and (6.6.27), we get

$$m_{\vec{k}}(t) = m_{\vec{k}}(0) \left(\frac{4t}{D\Delta} \right)^{1/2} (8\pi t)^{D/4} e^{-k^2 t} \quad (6.6.31)$$

and hence

$$\left\langle m_{\vec{k}}(t) m_{-\vec{k}}(t) \right\rangle = \frac{4t}{D} (8\pi t)^{D/2} e^{-2k^2 t}$$

leading to

$$\langle m(1)m(2) \rangle = \frac{4t}{D} e^{-r^2/8t} \quad (6.6.32)$$

Turning to the evaluation of the correlation function of the original fields ϕ , we note that from Eq.(6.6.32), m is typically $t^{1/2}$ for large t and hence at late times $\phi(m)$ takes the values 1 and -1 . Consequently, we can take

$$\phi = \text{sgn}(m)$$

and

$$C(12) = \langle \text{sgn}[m(1)]\text{sgn}[m(2)] \rangle$$

as we had before. This leads to the OJK result that we had obtained.

6.7 Renormalization Group for Late Stage Behaviour

This section describes the work of Bray which clarified a host of issues regarding the late stage scaling laws. As we have seen before, the RG is an useful tool whenever scaling laws hold. Consequently, the late stage scaling behaviour should be amenable to a RG treatment. The idea is to associate the scaling behaviour with a fixed point of the equation of motion under a RG procedure consisting of a coarse graining step combined with a simultaneous rescaling of length and time. Underlying such an idea is the schematic RG flow for the temperature, depicted in Fig. 6.5 The critical point T_c corresponds to a fixed point of the RG transfor-

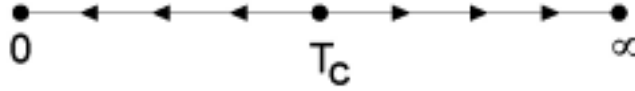


Figure 6.5. Schematic renormalization group flows

mation. At temperatures above T_c , coarse graining the system leads to a more disordered system while coarse graining below T_c leads to a more ordered system. This schematic flow is illustrated by arrows in Fig. 6.5. It follows from this that a quench from any $T > T_c$ to any $T < T_c$ should give the same asymptotic scaling behaviour. Any short range correlation present at the initial temperature will become irrelevant when $L(t) \gg \xi_0$ where ξ_0 is the correlation length for the initial condition. It can be understood from Fig. 6.5 that asymptotic scaling will be governed by the zero temperature fixed point. This implies that thermal noise can be dropped from the equation of motion.

We will illustrate the use of RG for the conserved order parameter where it yields an exact answer. The equation of motion (including the thermal noise for the moment) is given by Eq.(6.3.5)

$$\dot{\phi}_i = \Gamma \nabla^2 \frac{\delta F}{\delta \phi_i} + N_i$$

Fourier transforming, dividing throughout by λk^2 and including a part representing non conserved order parameter evolution, we have

$$\left(\frac{1}{\Gamma k^2} + \frac{1}{\lambda} \right) \frac{\partial \phi_i(k)}{\partial t} = - \frac{\delta F}{\delta \phi_i(-k)} + \frac{N_i}{\Gamma k^2} \quad (6.7.1)$$

If $\Gamma \rightarrow \infty$, we are left with a non conserved order parameter, but for any Γ which is finite, the λ -term in Eq.(6.7.1) is negligible in the long wavelength limit. Requiring that the canonical distribution be recovered in equilibrium i.e. $P(\phi) \propto e^{-F(\phi)/k_B T}$, the usual FDT fixes the correlator of the noise as

$$\left\langle \xi_i(\vec{k}, t_1) \xi_j(-\vec{k}, t_2) \right\rangle = 2T \delta_{ij} \delta(t_1 - t_2) \left(\frac{1}{\Gamma k^2} + \frac{1}{\lambda} \right) \quad (6.7.2)$$

where

$$\xi_i(k) = N_i(k) / \Gamma k^2.$$

The RG transformation consists of the following steps:

- a) The fourier components $\phi_i(k, t)$ for the hard modes with $\frac{\Lambda}{b} < k < \Lambda$ are eliminated by solving Eq.(6.7.2) for such modes and substituting the result into the equation of motion for the soft modes $k < \frac{\Lambda}{b}$. Here Λ is a high momentum cut off and b is a scale factor for the RG.
- b) A scale change is made $k = k'/b$ in order to reinstate the ultraviolet cut off for the soft modes at Λ . Additionally time is rescaled as $t = b^z t'$. The requirement that the domain morphology be invariant under this procedure (if the scaling hypothesis holds) fixes z as the reciprocal of the growth exponent i.e. $L(t) \sim t^{1/z}$. Finally the field $\phi_i(k, t)$ for $k < \frac{\Lambda}{b}$ is rewritten as

$$\vec{\phi}(\vec{k}, t) = \vec{\phi}\left(\frac{k'}{b}, b^z t'\right) = b^\zeta \vec{\phi}(\vec{k}', t') \quad (6.7.3)$$

The scaling form for the structure factor is

$$S(\vec{k}, t) = t^{D/z} g(kt^{1/z})$$

from Eq.(6.3.10). From the definition of S and Eq.(6.7.3),

$$\begin{aligned} S(\vec{k}, t) &= b^{2\zeta} \left\langle \vec{\phi}(\vec{k}', t') \vec{\phi}(-\vec{k}', t') \right\rangle \\ &= b^{2\zeta} t'^{D/z} g(k' t'^{1/z}) \\ &= b^{2\zeta - D t^{D/z}} g(kt^{1/z}) \end{aligned} \quad (6.7.4)$$

where $\zeta = D/2$.

- c) The new equation of motion for the soft modes is interpreted in terms of a new transport coefficient Γ' and a free energy F' . In addition, terms not originally present in Eq.(6.7.2) will be included in subsequent RG steps. Similarly, one must allow for a more general structure for the thermal noise and the distribution P_0 of initial conditions will also be affected.
- d) Scaling behaviour is associated with a fixed point for which both equation of motion and P_0 are invariant. In particular, the fixed point free energy is appropriate to the zero temperature fixed point.

The above procedures are difficult to implement in practice. However, if we assume the existence of a fixed point which is the same as the assumption of scaling the recursion relation for the transport coefficient Γ and the temperature T can be written down exactly. This is sufficient to fix the exponent z . We begin by noting that the term $(\Gamma k^2)^{-1}$ in Eq.(6.7.1) is singular at $k = 0$. Now elimination of large momentum modes will not generate any term in the equation of motion which is singular at $k = 0$. This elimination can and does affect λ the regular part of the transport coefficient in Eq.(6.7.1). It is only the rescaling part of the transformation (step 'b') which will affect λ . (This is analogous to our observation in **Chapter 2**, that for model **B**, the relation between z and η is exact, namely $z = 4 - \eta$. This was because the mode elimination did not produce any logarithmic divergence and the exponent was fixed by writing down the transformation under scale transformation alone and looking for the fixed point).

The scale transformation is now carried out on Eq.(6.7.1). In the process, the free energy scales as $F[\{\vec{\phi}(k'/b)\}] = b^Y F[\{\vec{\phi}(k)\}]$. This follows from the fact that the strong coupling fixed point $T = 0$ is attractive and hence $T' = b^{-Y}T$ and since Z , the partition function, is invariant, the free energy F scales as mentioned above. We can now write Eq.(6.7.1) as

$$[b^{\zeta+2-z} \frac{1}{\Gamma k'^2} + b^{\zeta-z} \frac{1}{\lambda} (1 + \dots)] \frac{\partial \vec{\phi}'(k')}{\partial t'} + \dots = -b^{y-\zeta} \frac{\delta F}{\delta \vec{\phi}'(-k')} + \vec{\xi}(\vec{k}'/b, b^z t') \quad (6.7.5)$$

Dividing throughout by $b^{y-\zeta}$,

$$[b^{2\zeta+2-z-y} \frac{1}{\Gamma k'^2} + b^{2\zeta-z-y} \frac{1}{\lambda} (1 + \dots)] \frac{\partial \vec{\phi}'(k')}{\partial t'} + \dots = - \frac{\delta F}{\delta \vec{\phi}'(-k')} + \vec{\xi}'(k'/b, b^z t') \quad (6.7.6)$$

where the new noise term is

$$\vec{\xi}'(k', t') = b^{\zeta-y} \vec{\xi}(\vec{k}'/b, b^z t') \quad (6.7.7)$$

with the correlator

$$\left\langle \xi'^i(k', t'_1) \xi'^j(-k', t'_2) \right\rangle = 2b^{2\zeta-2y-z} T \delta_{ij} \delta(t'_1 - t'_2) \left(\frac{b^2}{\Gamma k'^2} + \frac{1}{\lambda} (1 + \dots) \right) \quad (6.7.8)$$

Writing Eq.(6.7.6) as

$$\left(\frac{1}{\Gamma' k'^2} + \frac{1}{\lambda'} \right) \frac{\partial \vec{\phi}'(k')}{\partial t'} = \frac{\delta F}{\delta \vec{\phi}'(-k')} + \vec{\xi}'(\vec{k}'/b, b^z t') \quad (6.7.9)$$

we have

$$\Gamma' = \Gamma b^{y+z-2-2\zeta}$$

At the fixed point

$$z = 2 + 2\zeta - y \quad (6.7.10)$$

This evolution is absolutely general. If we work at the ordinary critical point, then T does not change under the scale transformations, i.e. $y = 0$. This makes $z = 2 + 2\zeta$. In this case, we have the structure factor given by

$$S(k, L) = k^{-2+\eta} \tilde{g}(kL) = L^{2-\eta} g(kL).$$

This implies

$$S(k, t) = t^{\frac{2-\eta}{z}} g(kt^{1/z})$$

and with this choice Eq.(6.7.4) leads to $\zeta = 1 - \frac{\eta}{2}$. The dynamic exponent becomes $z = 2 + 2\zeta = 4 - \eta$ which is the model **B** answer. Since Γ does not renormalize at any order and that is the basis for our derivation, $z = 4 - \eta$ is an exact answer independent of the order of perturbation theory.

If we work at the zero temperature fixed point which is the relevant one for the problem of quench, then $\zeta = D/2$ as seen from Eq.(6.7.4). This yields

$$z = 2 + D - y \quad (6.7.11)$$

The exponent y which describes the scaling of the Hamiltonian (hence the energy) is determined by the excitations in the system. For the scalar order parameter (i.e. $n = 1$) the excitations are the domain walls of dimension $D - 1$, while for the vector order parameter (i.e. $n \geq 2$), the excitations are spin waves with dimension $D - 2$. Hence $y = D - 1$ for $n = 1$ and $y = D - 2$ for $n \geq 2$. Consequently,

$$\begin{aligned} z &= 3 & \text{for} & & n &= 1 \\ z &= 4 & \text{for} & & n &\geq 2 \end{aligned} \quad (6.7.12)$$

The derivation requires no assumptions about the details of the domain growth kinetics and the central result (Eq.(6.7.11)) is of general validity. It is not restricted to the $O(n)$ model, but is valid for any symmetry group. One needs only to insert the appropriate value of η .

References

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Surface Growth

7.1 Introduction

We begin with a simple model of growth which is called ballistic deposition. In this model (a one-dimensional version) one begins with a line on which growth will take place. Particles are now dropped vertically down from a randomly chosen point, high above, towards the line in question. Gravity plays no role in this process. The particle stops as soon as it finds a site which has a particle in the nearest neighbour position. At any instant the interface is the line joining the topmost particle of a given site.

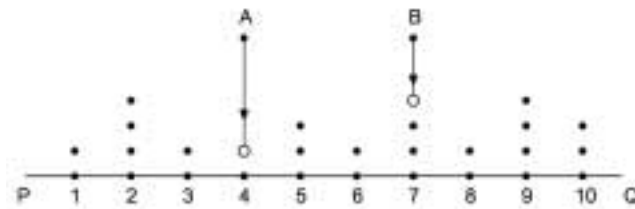


Figure 7.1. Ballistic Deposition Model

In Fig. 7.1 we show a ballistic deposition model with ten sites along the line PQ . particles A and B are dropped from above and will stick in the positions shown according to the rules in the text. The mean height of the interface at any instant is

$$\bar{h}(t) = \frac{1}{N} \sum_{i=1}^N h_i(t) \quad (7.1.1)$$

where $h_i(t)$ is the height of the interface at the site i and N is the total number of sites. In Fig. 7.1 $N = 10$.

The interface will not be flat. There will be fluctuations around $\bar{h}(t)$ and this fluctuation is defined as

$$W(t) = \left[\frac{1}{N} \sum_{i=1}^N (h_i(t) - \bar{h}(t))^2 \right]^{1/2} \quad (7.1.2)$$

It is interesting to observe what the width does as a function of time. In the initial stage, the width increases with time. This increase is expressed as

$$W(t) \propto t^\beta \quad (t \ll t_c) \quad (7.1.3)$$

For $t \gg t_c$, it is seen that the width saturates, i.e. it no longer increases with time and reaches a constant value. This constant value depends upon the number of sites N in the problem or in other words on L , the size of the system. As can be seen from Fig. 7.2, the time t_c at which the width begins to roll over to a constant value also depends on the system size L .

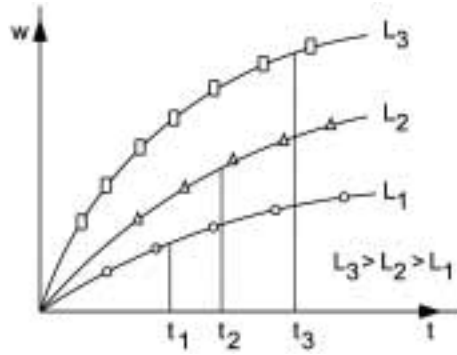


Figure 7.2. Interface width as a function of time for different system sizes

Here we plot the width as a function of time for different system sizes. The times at which the different sized systems roll over towards saturation values are shown as t_1 , t_2 and t_3 . These dependences on the system size can be expressed as

$$W(L) \propto L^\alpha \quad (t \gg t_c) \quad (7.1.4)$$

and

$$t_c \propto L^z \quad (7.1.5)$$

It is now observed that the data on different system sizes (squares, triangles and circles) collapse on a single curve if we choose to plot W/L^α against t/L^z . This is shown in Fig. 7.3 and is mathematically expressed as

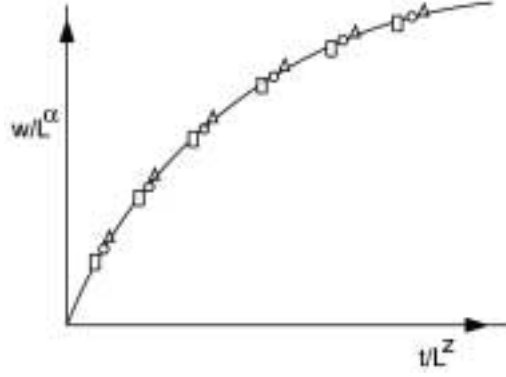


Figure 7.3. Scaling plot for the width of the interface

$$\begin{aligned} W &= L^\alpha f(t/t_c) \\ &= L^\alpha f(t/L^z) \end{aligned} \quad (7.1.6)$$

where $f(x)$ is called a scaling function. It is clear that as $x \rightarrow \infty$, $f(x)$ will tend to a constant. For $x \rightarrow 0$, i.e. $t \ll t_c$, we must have $W \sim t^\beta$, and this can be achieved only if $f(x) \sim x^\beta$, with $\beta z = \alpha$. This means that the three exponents α , β and z are not independent. They are related by the relation,

$$\alpha = \beta z \quad (7.1.7)$$

The interesting question is why the system shows the saturation of the width at large times. This has to do with the existence of correlations between the different sites. The correlation comes from the fact that a particle can stop at any site which has a nearest neighbour. This builds up correlation in the lateral direction. The correlation function can be calculated as

$$C(j, t) = \frac{1}{N} \sum_{i=1}^N (h_i - \bar{h})(h_{i+j} - \bar{h}) \quad (7.1.8)$$

One finds

$$C(j, t) \propto j^{2\alpha} \quad (t \gg t_c) \quad (7.1.9)$$

and

$$C(j, t) \propto j^{2\beta} \quad (t \ll t_c) \quad (7.1.10)$$

One can define a correlation length $\xi \propto t^{1/z}$, such that

$$C(j, t) \propto j^{2\alpha} g(j/\xi) \quad (7.1.11)$$

For $j \gg \xi$ (i.e. $t \ll t_c$), $g(x) \sim x^{-2\alpha}$, so that Eq.(7.1.10) may be regained, with $\alpha/z = \beta$ as in Eq.(7.1.7). For $j \ll \xi$ (i.e. $t \gg t_c$), we need to have $g(\infty) \rightarrow \text{constant}$ to reproduce Eq.(7.1.9). The correlation length ξ expresses the correlation developed in the lateral direction. For $\alpha > 0$, the fluctuation at site i is correlated with fluctuation at site $i + j$ and the correlation grows with growing j . This means fluctuations exist all along the surface and the surface is said to be rough.

7.2 Edwards Wilkinson(EW) Model

We now make the model even simpler. A particle falls from above from a randomly chosen site but sticks only when it comes to the top of the column already standing at that site. This means there is no lateral correlation. The growth of the height $h(x, t)$ at any point x is determined by the flux $\phi(x, t)$ at that point and we have

$$\frac{\partial}{\partial t} h(x, t) = \phi(x, t) \quad (7.2.1)$$

Now $\phi(x, t)$ will have a mean and a fluctuating part and we can decompose it as $\phi(x, t) = F + \eta(x, t)$. clearly, $\langle \phi(x, t) \rangle = F$ and $\langle \eta(x, t) \rangle = 0$. The mean rate at which the particles fall is F and η is the random fluctuation about that mean. This random fluctuation $\eta(x, t)$ is prescribed through its correlation which can be taken as Gaussian

$$\langle \eta(x, t) \eta(x', t') \rangle = 2N \delta(x - x') \delta(t - t') \quad (7.2.2)$$

We now have

$$\frac{\partial h}{\partial t} = F + \eta(x, t) \quad (7.2.3)$$

with the solution

$$h(x, t) = Ft + \int_0^t \eta(x, t') dt' \quad (7.2.4)$$

Clearly,

$$\bar{h} = \langle h(x, t) \rangle = Ft$$

and the fluctuation

$$\begin{aligned} W(t) &= \langle (h(x, t) - \bar{h})^2 \rangle = \langle \int_0^t \eta(x, t') dt' \int_0^t \eta(x, t'') dt'' \rangle \\ &= \int_0^t \int_0^t 2N \delta(t' - t'') dt' dt'' \\ &= 2Nt \end{aligned}$$

The correlation

$$\begin{aligned} C(r) &= \langle (h(\vec{x} + \vec{r}, t) - \bar{h})(h(\vec{x}, t) - \bar{h}) \rangle \\ &= \int_0^t \int_0^t 2dt' dt'' \langle \eta(\vec{x} + \vec{r}, t') \eta(\vec{x}, t'') \rangle \\ &= 0 \quad \text{if } r \neq 0 \end{aligned} \quad (7.2.5)$$

which shows the lack of correlation that we talked about. Correlations will develop if the particles are allowed to slide after sticking at the top of the column and come to rest when it lands in a local minimum. This is equivalent to saying that there can be smoothening in the lateral direction and measuring $h(t)$ from its mean position $\bar{h}(t)$ we can now write the equation of motion in this model as

$$\frac{\partial h}{\partial t} = v \frac{\partial^2 h}{\partial x^2} + \eta \quad (7.2.6)$$

This is the EW model that we have already talked about in **Chapter 2**.

In writing down equations of motion of the above variety it is generally necessary to pay heed to certain symmetries. For the structure

$$\frac{\partial h}{\partial t} = G(x, h, t) + \eta(x, t) \quad (7.2.7)$$

these are

- i) time translation invariance: The growth equation should not depend on time. This rules out explicit time dependence in G
- ii) translation invariance in the growth direction: The growth rule should be independent of the origin of h , i.e. G cannot contain h .
- iii) translational invariance perpendicular to the direction of growth: this rules out explicit x -dependence in G .
- iv) rotation and inversion symmetry about the growth direction: this ensures that vectors such as $\vec{\nabla} h$ and $\vec{\nabla} \nabla^2 h$ do not enter the function G .

- v)up/down symmetry in h : the interface fluctuations are similar with respect to the mean interface height. For non-equilibrium properties of surfaces this symmetry may be broken as it is for the Kardar-Parisi-Zhang (KPZ) system discussed briefly in Chapter 1.

The above arguments lead us to expect only derivatives of h in the function G . Which derivatives are we going to keep? In general we expect $h(x, t)$ to be a slowly varying function of x and hence it is the lowest order derivatives which should get priority. In other words, one does a long wavelength approximation. After this digression, we return to the EW model and carry out the scale transformations

$$\begin{aligned}x &\rightarrow x' = bx \\t &\rightarrow t' = tb^z \\h &\rightarrow h' = b^\alpha h \text{ and} \\ \eta &\rightarrow \eta' = b^\mu \eta\end{aligned}$$

The equation of motion should remain unchanged under these transformations. Carrying out this set of transformations on Eq.(7.1.17)

$$\begin{aligned}b^{z-\alpha} \frac{\partial h'}{\partial t'} &= \nu b^{2-\alpha} \frac{\partial^2 h'}{\partial x'^2} + \eta' b^{-\mu} \\ \frac{\partial h'}{\partial t'} &= \nu b^{2-z} \frac{\partial^2 h'}{\partial x'^2} + \eta' b^{\alpha-\mu-z}\end{aligned} \quad (7.2.8)$$

with Eq.(7.1.13) becoming

$$\begin{aligned}b^{-2\mu} \langle \eta'(x'_1, t'_1) \eta'(x'_2, t'_2) \rangle &= 2N \delta(x'_1 - x'_2) \delta(t'_1 - t'_2) b^{1+z} \\ \langle \eta'(x'_1, t'_1) \eta'(x'_2, t'_2) \rangle &= 2N b^{1+2\mu+z} \delta(x'_1 - x'_2) \delta(t'_1 - t'_2)\end{aligned} \quad (7.2.9)$$

The equation of motion in the primed variables will be

$$\begin{aligned}\frac{\partial h'}{\partial t'} &= \nu \frac{\partial^2 h'}{\partial x'^2} + \eta' \\ \langle \eta'(x'_1, t'_1) \eta'(x'_2, t'_2) \rangle &= 2N \delta(x'_1 - x'_2) \delta(t'_1 - t'_2)\end{aligned} \quad (7.2.10)$$

if

$$\begin{aligned}z &= 2 \\ 1 + 2\mu &= -z = -2 \text{ or } \mu = -3/2 \\ \text{and } \alpha - \mu &= z = 2 \text{ or } \alpha = 1/2\end{aligned} \quad (7.2.11)$$

The behaviour of the width $W(t)$ can be found as

$$\begin{aligned}W(t) &= \langle h(t)^2 \rangle^{1/2} = b^{-\alpha} [\langle h'(tb^z)^2 \rangle]^{1/2} \\ &= b^{-\alpha} W'(tb^z)\end{aligned} \quad (7.2.12)$$

setting $tb^z = 1$, we have

$$W(t) = t^{\alpha/z} = t^{1/4} \quad (7.2.13)$$

From the definition of Eq.(7.1.3), we find $\beta = 1/4$ and from Eq.(7.1.7), we have $\alpha = 1/2$.

If the substrate happens to be D -dimensional, then $\vec{x}$ is a D -dimensional vector and the effect of D shows up in Eq.(7.2.9), which now reads

$$\langle \eta'(x'_1, t'_1) \eta'(x'_2, t'_2) \rangle = 2N b^{D+2\mu+z} \delta(\vec{x}'_1 - \vec{x}'_2) \delta(t'_1 - t'_2) \quad (7.2.14)$$

The invariance under the scale transformation now means

$$\begin{aligned} z &= 2 \\ D + 2\mu &= -z = -2 \quad \text{or} \quad \mu = -\frac{1}{2}(2 + D) \\ \text{and} \quad \alpha - \mu &= z = 2 \quad \text{or} \quad \alpha = 2 + \mu = 1 - \frac{D}{2} \end{aligned} \quad (7.2.15)$$

We then have, by comparing with Eq.(7.2.13)

$$\begin{aligned} \beta &= \frac{1}{4}(2 - D) \\ \alpha &= \frac{1}{2}(2 - D) \\ z &= 2 \end{aligned} \quad (7.2.16)$$

The above conclusion could also have been arrived at by a direct integration of the EW equation. This exact solution is possible because it is a linear equation. In the Fourier space in D -dimension we can write Eq.(7.2.6) as

$$\frac{\partial}{\partial t} h(\vec{k}, t) = -vk^2 h(\vec{k}, t) + \eta(\vec{k}, t) \quad (7.2.17)$$

with

$$\langle \eta(\vec{k}_1, t_1) \eta(\vec{k}_2, t_2) \rangle = 2N \delta(\vec{k}_1 + \vec{k}_2) \delta(t_1 - t_2) \quad (7.2.18)$$

The solution of Eq.(7.2.17)

$$h(\vec{k}, t) = e^{-vk^2 t} \int_0^t \eta(\vec{k}, t') e^{vk^2 t'} dt' \quad (7.2.19)$$

with the initial condition that $h(\vec{k}, t) = 0$ at $t=0$. The correlation function in Fourier space is

$$\begin{aligned}
C(k, t_1, t_2) &= \langle h(\vec{k}_1, t_1) h(-\vec{k}_2, t_2) \rangle \\
&= e^{-\nu k^2(t_1+t_2)} \int_0^{t_1} \int_0^{t_2} \langle \eta(\vec{k}, t') \eta(-\vec{k}, t'') \rangle dt' dt'' \\
&= 2N e^{-\nu k^2(t_1+t_2)} \int_0^{t_1} dt' \int_0^{t_2} dt'' \delta(t' - t'') e^{\nu k^2(t' + t'')} \quad (7.2.20)
\end{aligned}$$

If $t_2 > t_1$, then we need to do the integration over t'' first and this leads to

$$\begin{aligned}
C(k, t_1, t_2) &= 2N e^{-\nu k^2(t_1+t_2)} \int_0^{t_1} dt' e^{2\nu k^2 t'} \\
&= \frac{N}{\nu k^2} [e^{-\nu k^2(t_2-t_1)} - e^{-\nu k^2(t_1+t_2)}] \quad (7.2.21)
\end{aligned}$$

For large values of t_1 and t_2 we can drop the second term and we get

$$C(k, t_1, t_2) = \frac{N}{\nu k^2} e^{-\nu k^2 |t_{12}|} \quad (7.2.22)$$

where $t_{12} = t_2 - t_1$ and the mod sign appears because in Eq.(7.2.22), the first term would involve $t_1 - t_2$ if t_1 were to be greater than t_2 . In coordinate space, the correlation function will be

$$C(r_{12}, t_{12}) = \frac{1}{(2\pi)^{D/2}} \int \frac{d^D k}{k^2} e^{-\nu k^2 |t_{12}|} e^{-i\vec{k} \cdot \vec{r}_{12}} \quad (7.2.23)$$

For $t_{12} = 0$ (equal time correlation function), we have

$$\begin{aligned}
C(r_{12}) &= \frac{1}{(2\pi)^{D/2}} \int \frac{d^D k}{k^2} e^{-i\vec{k} \cdot \vec{r}_{12}} \\
&= A(D) r_{12}^{2-D} \quad (7.2.24)
\end{aligned}$$

where $A(D)$ is a constant whose value depends on the dimensionality of space. Comparing with Eq.(7.1.9) we find $\alpha = 1 - \frac{D}{2}$ in agreement with Eq.(7.2.16). For arbitrary t_{12} ,

$$\begin{aligned}
C(r_{12}, t_{12}) &= \frac{1}{(2\pi)^{D/2}} \int \frac{d^D k}{k^2} e^{-\nu |t_{12}| [k^2 + \frac{i\vec{k} \cdot \vec{r}_{12}}{\nu |t_{12}|}]} \\
&= \frac{1}{(2\pi)^{D/2}} \int \frac{d^D k}{k^2} e^{-\frac{r_{12}^2}{4\nu |t_{12}|}} e^{-\nu |t_{12}| [\vec{k} + \frac{i\vec{r}_{12}}{\nu |t_{12}|}]^2} \\
&= |t_{12}|^{1-D/2} f\left(\frac{r_{12}^2}{\nu |t_{12}|}\right) \\
&= r_{12}^{2-D} g\left(\frac{r_{12}^2}{\nu |t_{12}|}\right) \quad (7.2.25)
\end{aligned}$$

The scaling function $g(x)$ has the property that $g(x) \rightarrow \text{constant}$ as $x \rightarrow \infty$, which ensures that the conclusions of Eq.(7.2.25) and Eq.(7.1.24) agree. The fact that $\alpha = 1 - D/2$ implies that α changes sign as we vary the dimensionality of the substrate. The exponent α is a measure of the roughness of the interface. If we have a fluctuation δh at a point $\vec{r}_1$ of the interface, then the correlation function $C(r_{12})$ tells us what is the likelihood that there will be fluctuation at a point $\vec{r}_2$, a distance $\vec{r}_{12}$ away from $\vec{r}_1$. If $C(r_{12})$ grows with r_{12} , then fluctuations exist all over the interface and the interface is rough. If $C(r_{12})$ decays with r_{12} then the fluctuation at $\vec{r}_1$ is localized and its effect is not felt at a point somewhat removed from $\vec{r}_1$. This makes the interface smooth. If $\alpha > 0$, then $C(r_{12})$ grows with r_{12} and we have a rough interface. If $\alpha < 0$, the correlation function decays and we have a smooth interface. Thus for $D \leq 2$, we have a rough interface and for $D > 2$, the interface is smooth. Larger the dimension of the substrate, more effective is the smoothening effect of $\nu \nabla^2 h$ which smoothes out the growing surface. The smoothening term is like a surface tension.

7.3 Kardar-Parisi-Zhang (KPZ) Model

We now want to include nonlinear terms in the EW model and see if the predictions about the exponents can be changed by the nonlinearity. The logic behind adding the nonlinear term is that as the interface grows, the growth at any particular point should occur preferentially along the local normal. As shown in Fig. 7.4, the height changes by an amount δh at point Q in time δt due to local growth at the point P. The interface velocity at the point P is v in the normal direction and hence the right angled triangle gives

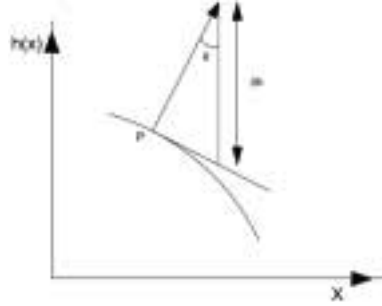


Figure 7.4. Direction of velocity of the growing interface

$$\begin{aligned}
\delta h &= v \delta t \sec \theta \\
&= v \delta t \sqrt{1 + \tan^2 \theta} \\
&= v \delta t \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2} \\
&\simeq v \delta t + \frac{1}{2} v \delta t \left(\frac{\partial h}{\partial x}\right)^2
\end{aligned} \tag{7.3.1}$$

This leads to

$$\frac{\partial h}{\partial t} = v + \frac{1}{2} v \left(\frac{\partial h}{\partial x}\right)^2 + \dots \tag{7.3.2}$$

which implies that the growth equation for h should contain $(\vec{\nabla} h)^2$ as the most important nonlinear contribution. The effect is clearly to induce lateral correlation since what has happened is that the growth in the normal direction at a given point has induced growth at a neighboring point. We, consequently, generalize Eq.(7.2.6) to

$$\frac{\partial h}{\partial t} = v \nabla^2 h + \frac{\lambda}{2} (\vec{\nabla} h)^2 + \eta \tag{7.3.3}$$

which is the D -dimensional KPZ equation.

We first carry out a scale transformation so that the coordinates are rescaled by a factor b as in section 7.2 and the corresponding h , t and η by factors of b^α , b^z and b^μ respectively. Under these rescalings Eq.(7.3.3) becomes

$$b^{z-\alpha} \frac{\partial(hb^\alpha)}{\partial(tb^z)} = v b^{2-\alpha} \nabla'^2(hb^\alpha) + \frac{\lambda}{2} b^{2-2\alpha} [\vec{\nabla}(bh)]^2 + \eta b^\mu b^{-\frac{1}{2}(D+2)}$$

In the above μ remains at $\mu = -\frac{1}{2}(D+2)$ as in Eq.(7.2.15) to keep the noise correlator unchanged. Writing $h' = hb^\alpha$ and $t' = tb^z$, we now have

$$\frac{\partial h'}{\partial t'} = v b^{2-z} \nabla'^2 h' + \frac{\lambda}{2} b^{2-z-\alpha} (\vec{\nabla}' h')^2 + \eta' b^{\frac{D-z}{2}+\alpha} \tag{7.3.4}$$

If the form of this equation is to remain invariant, then we must have $z = 2$, $z = 2 - \alpha$ and $\alpha = \frac{z-D}{2}$. This clearly cannot be as we have three equations for two unknowns. The reason should be apparent from our discussions in **Chapters 3** and **4**. In the presence of nonlinear terms, coupling constant and diffusion coefficients change under the scale transformation. Consequently, we should not have started with the assumption that v , λ and N do not change.

To test the relevance of the nonlinear term, we let z and α remain at the EW values (v and N unchanged) and ask how λ changes. From Eq.(7.3.4), it is immediately clear that

$$\lambda' = b^{2-z-\alpha} \lambda = b^{-\alpha} \lambda = b^{\frac{D-2}{2}} \lambda \tag{7.3.5}$$

Our interest lies in the long distance properties, i.e. when the primed equation probes a distance scale which is higher and for that b has to be smaller than unity. Accordingly $\lambda' > \lambda$ if $D < 2$. Under a scale transformation, the nonlinear term grows for substrate dimension less than 2. On the other hand for $D > 2$, the nonlinear term is irrelevant for long distance behaviour and the exponents of the KPZ equation should asymptotically be the same as for the EW equation.

We can look at the KPZ equation in yet another fashion. If we define $\vec{v} = -\vec{\nabla}h$ then Eq.(7.3.3) becomes

$$-\frac{\partial \vec{v}}{\partial t} = -v \nabla^2 \vec{v} + \frac{\lambda}{2} \vec{\nabla} v^2 + \vec{\nabla} \eta \quad (7.3.6)$$

which is

$$\left(\frac{\partial}{\partial t} + \lambda \vec{v} \cdot \vec{\nabla}\right) \vec{v} = v \nabla^2 \vec{v} - \vec{\nabla} \eta \quad (7.3.7)$$

If $\lambda = 1$, then this is the same as a pressure-less Navier Stokes equation which is known as Burger's equation. Burger's equation shows Galilean invariance i.e. it remains unchanged under $x \rightarrow x' = x - v_0 t$ and $t' = t$ and $v \rightarrow v' = v - v_0$. This of course requires that λ be kept at unity and it suggests that λ does not renormalize in Eq.(7.3.7) which from Eq.(7.3.5) would yield

$$\alpha + z = 2 \quad (7.3.8)$$

This is a consequence of Galilean invariance of Eq.(7.3.3) which can be written as an invariance under the transformation $h \rightarrow h' + \epsilon x$, $x' = x - \epsilon \lambda t$, $t' = t$

Finally note that the Langevin equation of the EW variety i.e.

$$\frac{\partial h}{\partial t} = v \frac{\partial^2 h}{\partial x^2} + \eta$$

with

$$\langle \eta(x, t) \eta(x', t') \rangle = 2N \delta(x - x') \delta(t - t')$$

can be cast as

$$\frac{\partial h}{\partial t} = \frac{\delta F}{\delta h} + \eta$$

where $F = v \int dx \left(\frac{\partial h}{\partial x}\right)^2$ and $\frac{\delta F}{\delta h}$ is the functional derivative. This leads to the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\delta}{\delta h} \left(P \frac{\delta F}{\delta h} \right) + N \frac{\partial^2 P}{\partial h^2} \quad (7.3.9)$$

according to our considerations in Chapter 1. We see that this Fokker-Planck equation leads to an equilibrium distribution

$$P_{eq} \propto e^{-\frac{v}{N} \int (\frac{\partial h}{\partial x})^2 dx} \quad (7.3.10)$$

Thus the EW model has an equilibrium distribution associated with it. The KPZ equation cannot be cast in the form

$$\dot{h} = \frac{\delta F}{\delta h} + \eta$$

and hence one suspects that there are no equilibrium distributions associated with it. We note that for a Langevin equation

$$\dot{h} = G(h) + \eta$$

One can always write a Fokker Planck equation

$$\frac{\partial P}{\partial t} = \frac{\delta}{\delta h} (PG) + N \frac{\delta^2 P}{\delta h^2} \quad (7.3.11)$$

If G does not have the structure $\frac{\delta F}{\delta h}$, then it is not clear if an equilibrium distribution can be found. For the KPZ equation with

$$G = v \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left(\frac{\partial h}{\partial x} \right)^2$$

we can see the equilibrium distribution of Eq.(7.3.10) is also the equilibrium distribution in this case. This is a special feature of a one dimensional substrate. It is not true in higher dimensions.

7.4 KPZ Equation and the Renormalization Group

In this section, we will implement the RG procedure on the KPZ equation to find the scaling behaviour of the diffusion coefficient, coupling constants etc. This will follow very closely the procedure shown in **Chapter 3**. Consequently, we will start out directly by writing the Fourier transform of Eq.(7.3.3) as

$$\begin{aligned} -i\omega h(\vec{k}, \omega) &= -vk^2 h(\vec{k}, \omega) \\ &\quad - \frac{\lambda}{2} \sum_{\vec{p}, \omega_1} \vec{p} \cdot (\vec{k} - \vec{p}) h(\vec{p}, \omega_1) h(\vec{k} - \vec{p}, \omega - \omega_1) + \eta(\vec{k}, \omega) \end{aligned} \quad (7.4.1)$$

where

$$\begin{aligned} h(\vec{r}, t) &= \frac{1}{(2\pi)^{D+1}} \int d^D k d\omega h(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ \text{and } \eta(\vec{r}, t) &= \frac{1}{(2\pi)^{D+1}} \int d^D k d\omega \eta(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned} \quad (7.4.2)$$

with

$$\langle \eta(\vec{k}, \omega) \eta(\vec{k}', \omega') \rangle = 2N \delta(\vec{k} + \vec{k}') \delta\omega + \omega'$$

The range of momentum integration is from 0 to λ and the frequency ranges from 0 to Ω . We split, as usual, the momentum range from 0 to $\frac{\Lambda}{b}$ and $\frac{\Lambda}{b}$ to Λ , while the frequency range is split from 0 to $\frac{\Omega}{b^z}$ and $\frac{\Omega}{b^z}$ to Ω , where b is a number larger than unity. In the momentum range 0 to $\frac{\Lambda}{b}$ and frequency range 0 to $\frac{\Omega}{b^z}$ the height variable is $h_<$. As before the idea is to first find the equation of motion for $h_<$. Writing

$$\begin{aligned} h(\vec{r}, t) = & \frac{1}{(2\pi)^{D+1}} \int_0^{\frac{\Lambda}{b}} d^D k \int_0^{\frac{\Omega}{b^z}} d\omega h_<(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ & + \frac{1}{(2\pi)^{D+1}} \int_{\frac{\Lambda}{b}}^{\Lambda} d^D k \int_{\frac{\Omega}{b^z}}^{\Omega} d\omega h_>(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{aligned} \quad (7.4.3)$$

we have in the range $0 < k < \frac{\Lambda}{b}$, $0 < \omega < \frac{\Omega}{b^z}$

$$\begin{aligned} -i\omega h_<(\vec{k}, \omega) = & -vk^2 h_<(\vec{k}, \omega) - \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_<(\vec{p}, \omega_1) h_<(\vec{k} - \vec{p}, \omega - \omega_1) \\ & - \lambda \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_<(\vec{p}, \omega_1) h_>(\vec{k} - \vec{p}, \omega - \omega_1) \\ & - \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_>(\vec{p}, \omega_1) h_>(\vec{k} - \vec{p}, \omega - \omega_1) + \eta_<(\vec{k}, \omega) \end{aligned} \quad (7.4.4)$$

In the other range,

$$\begin{aligned} -i\omega h_>(\vec{k}, \omega) = & -vk^2 h_>(\vec{k}, \omega) - \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_>(\vec{p}, \omega_1) h_>(\vec{k} - \vec{p}, \omega - \omega_1) \\ & - \lambda \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_<(\vec{p}, \omega_1) h_>(\vec{k} - \vec{p}, \omega - \omega_1) \\ & - \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_<(\vec{p}, \omega_1) h_<(\vec{k} - \vec{p}, \omega - \omega_1) + \eta_>(\vec{k}, \omega) \end{aligned} \quad (7.4.5)$$

The procedure now is to solve for $h_>$ from Eq.(7.4.5) and insert the result in Eq.(7.4.4). Since the solution for $h_>(\vec{k}, \omega)$ will depend on $\eta_>$, the final step will have to be an averaging over $\eta_>$ before we have an equation of motion for $h_<$. Accordingly, we find

$$h_>^{(0)} = \frac{\eta_>}{-i\omega + vk^2} \quad (7.4.6)$$

if we expand $h_>$ as $h_> = h_>^{(0)} + \lambda h_>^{(1)} + \lambda^2 h_>^{(2)} + \dots$. Clearly,

$$\begin{aligned}
(-i\omega + vk^2)h_{>}^{(1)} = & - \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& - \frac{1}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{>}^{(0)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& - \frac{1}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \quad (7.4.7)
\end{aligned}$$

$$\begin{aligned}
(-i\omega + vk^2)h_{>}^{(2)} = & - \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) h_{>}^{(1)}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& - \frac{1}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{>}^{(0)}(\vec{p}, \omega_1) h_{>}^{(1)}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& - \frac{1}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{>}^{(1)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \quad (7.4.8)
\end{aligned}$$

and so on. Using Eq.(7.4.7) in Eq.(7.4.4), we have to $O(\lambda^2)$

$$\begin{aligned}
-i\omega h_{<}(\vec{k}, \omega) = & -vk^2 h_{<}(\vec{k}, \omega) - \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& + \lambda \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) [h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& + \lambda h_{>}^{(1)}(\vec{k} - \vec{p}, \omega - \omega_1)] \\
& + \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) [h_{>}^{(0)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& + \lambda h_{>}^{(0)}(\vec{p}, \omega_1) h_{>}^{(1)}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& + \lambda h_{>}^{(1)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1)] + \eta_{<}(\vec{k}, \omega) \quad (7.4.9)
\end{aligned}$$

where $h_{>}^{(1)}$ is to be substituted from Eq.(7.4.7). Instead of writing it out fully, we recall the third step that needs to be done. We need to average over η and from Eq.(7.4.6), we see that $h_{>}^{(0)}$ is directly proportional to η . We will write out only the surviving terms after this process of averaging (ignoring a constant term which can be removed by a shift of the mean value)

$$\begin{aligned}
-i\omega h_{<}(\vec{k}, \omega) = & vk^2 h_{<}(\vec{k}, \omega) - \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& + \frac{\lambda^2}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{q}, \omega_2) \frac{\vec{q} \cdot (\vec{p} - \vec{q})}{-i\omega_1 + vp^2} \langle h_{>}^{(0)}(\vec{p} - \vec{q}, \omega_1 - \omega_2) \\
& \times h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \rangle \\
& + \frac{\lambda^2}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{q}, \omega_2) \frac{\vec{q} \cdot (\vec{k} - \vec{p} - \vec{q})}{-i(\omega - \omega_1)^2 + v(\vec{k} - \vec{p})^2} \\
& \times \langle h_{>}^{(0)}(\vec{k} - \vec{p} - \vec{q}, \omega - \omega_1 - \omega_2) h_{>}^{(0)}(\vec{p}, \omega_1) \rangle \\
& + \text{trilinear terms in } h_{<} + \eta_{<} \quad (7.4.10)
\end{aligned}$$

We now note that

$$\begin{aligned}
& \sum_{\omega_1, \omega_2, \vec{p}, \vec{q}} \frac{\vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{q}, \omega_2) \vec{q} \cdot (\vec{p} - \vec{q})}{(-i\omega_1 + vp^2)} \\
& \quad \times \langle h_{>}^{(0)}(\vec{p} - \vec{q}, \omega_1 - \omega_2) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \rangle \\
& = \sum_{\vec{p}, \omega_1} 2N \frac{\vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{k}, \omega)}{(-i\omega_1 + vp^2)} \times \frac{\vec{k} \cdot (\vec{p} - \vec{k})}{v^2(\vec{k} - \vec{p})^4 + (\omega - \omega_1)^2} \\
& = \sum_{\vec{p}} \frac{N}{v} \frac{h_{<}(\vec{k}, \omega)}{(\vec{k} - \vec{p})^2} \times \frac{\vec{p} \cdot (\vec{k} - \vec{p}) \vec{k} \cdot (\vec{p} - \vec{k})}{-i\omega + vp^2 + v(\vec{k} - \vec{p})^2}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{\omega_1, \omega_2, \vec{p}, \vec{q}} \frac{2N \vec{p} \cdot (\vec{k} - \vec{p}) \vec{k} \cdot (-\vec{p}) h_{<}(\vec{k}, \omega)}{[\omega^2 + v^2 p^4][-i(\omega - \omega_1) + v(\vec{k} - \vec{p})^2]} \\
& = - \sum_{\vec{p}} \frac{N}{v} \frac{h_{<}(\vec{k}, \omega)}{(\vec{k} \cdot \vec{p})^2} \times \frac{\vec{p} \cdot (\vec{k} - \vec{p}) (\vec{k} \cdot \vec{p})}{-i\omega + vp^2 + v(\vec{k} - \vec{p})^2}
\end{aligned}$$

Dropping the trilinear term in Eq.(7.4.10), we can write

$$\begin{aligned}
-i\omega h_{<}(\vec{k}, \omega) & = -vk^2 h_{<}(\vec{k}, \omega) + \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& \quad - h_{<}(\vec{k}, \omega) \frac{\lambda^2}{2} \frac{N}{v} \int \frac{d^D p}{(2\pi)^D} \frac{\vec{p} \cdot (\vec{k} - \vec{p})}{-i\omega + vp^2 + v(\vec{k} - \vec{p})^2} \\
& \quad \times \left[\frac{\vec{k} \cdot \vec{p}}{p^2} + \frac{\vec{k} \cdot (\vec{k} - \vec{p})}{(\vec{k} - \vec{p})^2} \right] + \eta_{<}(\vec{k}, \omega)
\end{aligned} \tag{7.4.11}$$

The momentum integral needs to be performed over the range $\frac{\Lambda}{b} < |\vec{p}| < \Lambda$ and hence we need only the small- k part. Accordingly, one works to the lowest order in $|\vec{k}|$ to write the integrand as

$$\begin{aligned}
& \frac{\vec{p} \cdot (\vec{k} - \vec{p})}{-i\omega + vp^2 + v(\vec{k} - \vec{p})^2} \times \left[\frac{\vec{k} \cdot \vec{p}}{p^2} + \frac{\vec{k} \cdot (\vec{k} - \vec{p})}{(\vec{k} - \vec{p})^2} \right] \\
& \approx [(\vec{p} \cdot \vec{k})p^2 - 2(\vec{p} \cdot \vec{k})^2 + p^2 k^2 - p^2(\vec{p} \cdot \vec{k})] \times \frac{(\vec{p} \cdot \vec{k} - p^2)}{2vp^2 p^4} \\
& \approx \frac{2(\vec{p} \cdot \vec{k})^2 - p^2 k^2}{2vp^4}
\end{aligned}$$

We note that this term is $O(k^2)$ and hence the third term on the R.H.S is an addition to the first term. In doing the integral, the averaging over $\cos^2 \theta$ produces a factor D^{-1} and Eq.(7.4.11) becomes

$$\begin{aligned}
-i\omega h_{<}(\vec{k}, \omega) = & - \left[v + \frac{\lambda^2 N (2-D)}{4v^2 D} \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} \right] k^2 h_{<}(\vec{k}, \omega) \\
& + \frac{\lambda}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\
& + \eta_{<}(\vec{k}, \omega)
\end{aligned} \tag{7.4.12}$$

The elimination of small scales leads to an effective viscosity $\bar{\nu}u$ given by

$$\bar{\nu} = v + \frac{\lambda^2 N (2-D)}{4v^2 D} \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} \tag{7.4.13}$$

The correlation function for $h(k, \omega)$ has the expansion (to $O(\lambda^2)$)

$$\begin{aligned}
\langle h(\vec{k}, \omega) h(\vec{k}', \omega') \rangle = & \langle h^{(0)}(\vec{k}, \omega) h^{(0)}(\vec{k}', \omega') \rangle + \lambda \langle h^{(0)}(\vec{k}, \omega) h^{(1)}(\vec{k}', \omega') \rangle \\
& + \lambda \langle h^{(1)}(\vec{k}, \omega) h^{(0)}(\vec{k}', \omega') \rangle + \lambda^2 \langle h^{(0)}(\vec{k}, \omega) h^{(2)}(\vec{k}', \omega') \rangle \\
& + \lambda^2 \langle h^{(2)}(\vec{k}, \omega) h^{(0)}(\vec{k}', \omega') \rangle + \lambda^2 \langle h^{(1)}(\vec{k}, \omega) h^{(1)}(\vec{k}', \omega') \rangle \\
= & \frac{2N \delta(\vec{k} + \vec{k}') \delta(\omega + \omega')}{\omega^2 + v^2 k^4} + \lambda^2 [\langle h^{(2)}(\vec{k}, \omega) h^{(0)}(\vec{k}', \omega') \rangle \\
& + \langle h^{(1)}(\vec{k}, \omega) h^{(1)}(\vec{k}', \omega') \rangle + \langle h^{(0)}(\vec{k}, \omega) h^{(2)}(\vec{k}', \omega') \rangle]
\end{aligned} \tag{7.4.14}$$

where

$$h(\vec{k}, \omega) = h^{(0)}(\vec{k}, \omega) + \lambda h^{(1)}(\vec{k}, \omega) + h^{(2)}(\vec{k}, \omega) + \dots$$

and by inserting this in Eq.(7.4.1) and equating like powers of λ

$$\begin{aligned}
h^{(0)}(\vec{k}, \omega) &= \frac{\eta(k, \omega)}{-i\omega + vk^2} \\
(-i\omega + vk^2)h^{(1)}(\vec{k}, \omega) &= -\frac{1}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) h^{(0)}(\vec{p}, \omega_1) h^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \\
(-i\omega + vk^2)h^{(2)}(\vec{k}, \omega) &= -\sum \vec{p} \cdot (\vec{k} - \vec{p}) h^{(1)}(\vec{p}, \omega_1) h^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1)
\end{aligned}$$

All possible contractions of the string of $h^{(0)}$ are considered and finally we find

$$\begin{aligned}
\langle h(\vec{k}, \omega) h(\vec{k}', \omega') \rangle &= \delta(\vec{k} + \vec{k}') \delta(\omega + \omega') \left[\frac{2N}{\omega^2 + v^2 k^4} + \frac{\lambda^2}{v^2} \frac{N^2}{\omega^2 + v^2 k^4} \right. \\
&\quad \left. + \int \frac{d^D p}{p^2 (\vec{k} - \vec{p})^2} \left\{ \frac{[\vec{p} \cdot (\vec{k} - \vec{p})]^2}{-i\omega + v[p^2 + (\vec{k} - \vec{p})^2]} + c.c \right\} \right]
\end{aligned} \tag{7.4.15}$$

To see what kind of dressing of N is implied, it is best to check the integral over ω , and then we find that N is changed to a N_{eff} , where

$$N_{eff} = N + \frac{\lambda^2 N^2}{2v^3} \frac{d^D p}{p^2 (\vec{k} - \vec{p})^2} \frac{[\vec{p} \cdot (\vec{k} - \vec{p})]^2}{k^2 + p^2 + (\vec{k} - \vec{p})^2} \quad (7.4.16)$$

This relation makes apparent the effect of thinning of degrees of freedom. The coefficient N will change to $\bar{N}$ given by

$$\bar{N} = N + \frac{\lambda^2 N^2}{4v^3} \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^D p}{p^2} \quad (7.4.17)$$

To see the effect of the removal of degrees of freedom in λ , we need to go to the third order in λ in Eq.(7.4.9). The $O(\lambda^3)$ terms in Eq.(7.4.9) would be

$$\begin{aligned} & \sum \vec{p} \cdot (\vec{k} - \vec{p}) h_{<}(\vec{p}, \omega_1) h_{>}^{(2)}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + \frac{1}{2} \sum \vec{p} \cdot (\vec{k} - \vec{p}) [h_{>}^{(0)}(\vec{p}, \omega_1) h_{>}^{(2)}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + h_{>}^{(1)}(\vec{p}, \omega_1) h_{>}^{(1)}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + h_{>}^{(2)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1)] \end{aligned}$$

The fields $h_{>}^{(1)}$ and $h_{>}^{(2)}$ have to be substituted from Eqns.(7.4.7) and (7.4.8) and the $h_{>}^{(0)}$ averaged over. We want those terms which have the structure

$$\int \frac{d^D p}{(2\pi)^D} h_{<}(\vec{p}) h_{<}(\vec{k} - \vec{p}) \Lambda(\vec{k}) \vec{p} \cdot (\vec{k} - \vec{p})$$

after the averaging. It will be seen that $\Lambda(\vec{k}) = 0$ for small k .

At the end of the above process, we have removed the degrees of freedom at the high momentum end. The cut off now is $\frac{\Lambda}{b}$ which has to be restored to Λ by a scale transformation. The effect of a scale transformation can be read off from Eq.(7.3.4). If the scale of measuring is changed by b as has been done in changing the cut off to $\frac{\Lambda}{b}$, then the b occurring in Eq.(7.3.4) is to be interpreted as b^{-1} in the present context and we have the total change as ($K_D = \frac{S_D}{(2\pi)^D}$), S_D being the area of the $D - 1$ dimensional hypersurface)

$$\nu' = b^{z-2} \nu \left[1 + \frac{\lambda^2 N (2-D) K_D}{4v^3 D} \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^D p}{p^{3-D}} \right] \quad (7.4.18)$$

$$N' = b^{-2\alpha-(D-z)} N \left[1 + \frac{\lambda^2 N K_D}{4v^3} \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^D p}{p^{3-D}} \right] \quad (7.4.19)$$

$$\lambda' = b^{z+\alpha-2} \lambda \quad (7.4.20)$$

For $b \simeq 1$, we can expand $b^{z-2} \simeq 1 + (z-2) \ln b + O(\ln b)^2$ and similarly for the other powers of b . The integrals give a term proportional to $\ln b$, when we evaluate them in $D = 2$ and thus

$$\begin{aligned}
v' &= v \left[1 + \left\{ (z-2) + \frac{\bar{\lambda}^2(2-D)}{4D} \right\} \ln b \right] \\
N' &= N \left[1 + \left\{ z - D - 2\alpha + \frac{\bar{\lambda}^2}{4} \right\} \ln b \right] \\
\lambda' &= \lambda [1 + \{z + \alpha - 2\} \ln b]
\end{aligned}$$

Writing $b = 1 + \delta l$, we have

$$\frac{dv}{dl} = v \left[z - 2 + \bar{\lambda}^2 \frac{2-D}{4D} \right] \quad (7.4.21)$$

$$\frac{dN}{dl} = N \left[z - D - 2\alpha + \frac{\bar{\lambda}^2}{4} \right] \quad (7.4.22)$$

$$\frac{d\lambda}{dl} = \lambda [z + \alpha - 2] \quad (7.4.23)$$

where

$$\bar{\lambda}^2 = \frac{\lambda^2 N K_2}{v^3}$$

The flow of $\bar{\lambda}$ is clearly,

$$\frac{d\bar{\lambda}}{dl} = \bar{\lambda} \left[\frac{2-D}{2} - \frac{\bar{\lambda}^2}{4D} (3-2D) \right] \quad (7.4.24)$$

This modified coupling constant can flow to two different fixed points. One is $\bar{\lambda} = 0$. This is clearly unstable if $D < 2$. Consequently, the KPZ answers will be different from EW if the substrate dimension is below 2. For $D > 2$, $\bar{\lambda} = 0$ is a stable fixed point and one expects to see a EW like phase but with $\alpha = 0$ since $z=2$. If the substrate dimension is $D = 1$, then one gets a non trivial fixed point $\bar{\lambda}^{*2} = 2$. The stability can be tested by evaluating the derivative of $\frac{\bar{\lambda}}{2} - \frac{\bar{\lambda}^3}{4}$ at the fixed point. It is $\frac{1}{2} - \frac{3}{4}\bar{\lambda}^{*2} = -1$ and hence the fixed point is stable. Thus at $D = 1$, the interface behaviour will be governed by the non trivial fixed point $\bar{\lambda}^{*2} = 2$. From Eq.(7.4.21), we find

$$z = 2 - \frac{1}{2} \quad (7.4.25)$$

$$\alpha = 2 - z = \frac{1}{2} \quad (7.4.26)$$

at this non trivial fixed point. Consequently, we have a rough interface in $D=1$ determined by the strong coupling fixed point. The values obtained in Eqns.(7.4.25) and (7.4.26) happen to be exact. This is fortuitous. For $D > 2$, while the trivial fixed point with a flat interface is stable, the flow diverges for large $\bar{\lambda}$ and hence we anticipate a rough-smooth transition for some characteristic value of $\bar{\lambda}$. The upper critical dimension for the problem would be one for which the rough phase would cease to exist.

7.5 KPZ Equation and Mode Coupling Theories

In this section, we show how to treat the KPZ equation within the self consistent mode coupling scheme shown in **Chapter 4**. To do this we note that the Green's function $G(k, \omega)$ can be written as

$$\left\langle \frac{\delta h(\vec{k}, \omega)}{\delta \eta(\vec{k}', \omega')} \right\rangle \frac{1}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')}$$

or as the correlation

$$\frac{\langle h(\vec{k}, \omega) \eta(\vec{k}', \omega') \rangle}{2N \delta(\vec{k} + \vec{k}') \delta(\omega + \omega')}$$

From the expansion written down following Eq.(7.4.14), we find in the expansion,

$$\begin{aligned} G(k, \omega) &= G_0(k, \omega) + \lambda G_1(k, \omega) + \lambda^2 G_2(k, \omega) + \dots \\ G_0^{-1}(k, \omega) &= -i\omega + vk^2 \\ G_1(k, \omega) &= 0 \\ G_2(k, \omega) &= \frac{1}{2} \frac{1}{-i\omega + vk^2} \left[\sum \frac{\vec{p} \cdot (\vec{k} - \vec{p})}{-i\omega_1 + vk^2} \vec{q} \cdot (\vec{p} - \vec{q}) \langle h_0(\vec{p} - \vec{q}, \omega_1 - \omega_2) \right. \\ &\quad \left. \times h_0(\vec{q}, \omega_2) h_0(\vec{k} - \vec{p}, \omega - \omega_1) \eta(k', \omega') \rangle \right] \\ &= -\frac{1}{2} \frac{1}{-i\omega + vk^2} \left[\sum \vec{p} \cdot (\vec{k} - \vec{p}) \vec{k} \cdot (\vec{k} - \vec{p}) G_0(\vec{p}, \omega_1) \right. \\ &\quad \left. \times C_0(\vec{k} - \vec{p}, \omega - \omega_1) \right] \frac{1}{-i\omega + vk^2} \\ &= G_0(k, \omega) \left[\sum_{\vec{p}, \omega_1} \vec{p} \cdot (\vec{k} - \vec{p}) \vec{k} \cdot (\vec{k} - \vec{p}) G_0(\vec{p}, \omega_1) \right. \\ &\quad \left. \times C_0(\vec{k} - \vec{p}, \omega - \omega_1) \right] G_0(k, \omega) \\ &= -G_0(k, \omega) \Sigma(k, \omega) G_0(k, \omega) / \lambda^2 \end{aligned} \quad (7.5.1)$$

where

$$\begin{aligned} \Sigma(k, \omega) &= \lambda^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} \vec{p} \cdot (\vec{k} - \vec{p}) \vec{k} \cdot (\vec{k} - \vec{p}) G_0(\vec{p}, \omega_1) \\ &\quad \times G_0(\vec{p}, \omega_1) C_0(\vec{k} - \vec{p}, \omega - \omega_1) \end{aligned} \quad (7.5.2)$$

The full Green's function has the form

$$\begin{aligned}
G &= G_0 - G_0 \Sigma G_0 + \dots \\
&= G_0 (1 - G_0 \Sigma + \dots) \\
&= G_0 (1 + G_0 \Sigma)^{-1} \\
\Rightarrow G^{-1} &= G_0^{-1} + \Sigma
\end{aligned} \tag{7.5.3}$$

The self consistent approximation involves replacing the G_0 and C_0 in the expression for Σ by the dressed quantities G and C and we have (as in Chapter 3)

$$\begin{aligned}
G^{-1}(k, \omega) &= -i\omega + \nu k^2 + \lambda^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} \vec{p} \cdot (\vec{k} - \vec{p}) \vec{k} \cdot (\vec{k} - \vec{p}) \\
&\quad \times G(\vec{p}, \omega_1) C(\vec{k} - \vec{p}, \omega - \omega_1)
\end{aligned} \tag{7.5.4}$$

We now turn to the correlation function $C(k, \omega)$ and write down the corresponding expansion for it.

$$C(k, \omega) = C_0(k, \omega) + \lambda C_1(k, \omega) + \lambda^2 C_2(k, \omega) + \dots \tag{7.5.5}$$

where

$$C_0 = \frac{2N}{\omega^2 + \nu^2 k^4} = 2N G_0(k, \omega) G_0(-k, -\omega) \tag{7.5.6}$$

$$C_1 = \langle h_1(k, \omega) h_0(-k, -\omega) \rangle = 0 \tag{7.5.7}$$

$$C_2 = 2 \langle h_2(k, \omega) h_0(-k, -\omega) \rangle + \langle h_1(k, \omega) h_1(-k, -\omega) \rangle \tag{7.5.8}$$

We note that the first term on the R.H.S of Eq.(7.5.8) dresses the term $G(k, \omega)$ in Eq.(7.5.6). It is the second term on the R.H.S of Eq.(7.5.8), that dresses the noise term and the mode coupling approximation rests on the expansion

$$\begin{aligned}
C(k, \omega) &= \left[2N + \frac{\lambda^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} [\vec{p} \cdot (\vec{k} - \vec{p})]^2 \right. \\
&\quad \left. \times C_0(p, \omega_1) C_0(\vec{k} - \vec{p}, \omega - \omega_1) \right] |G_0(k, \omega)|^2
\end{aligned} \tag{7.5.9}$$

Once again the self-consistent approximation involves replacing G_0 by G and C_0 by C , so that

$$\begin{aligned}
C(k, \omega) &= \left[2N + \frac{\lambda^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} [\vec{p} \cdot (\vec{k} - \vec{p})]^2 \right. \\
&\quad \left. \times C(p, \omega_1) C(\vec{k} - \vec{p}, \omega - \omega_1) \right] |G(k, \omega)|^2
\end{aligned} \tag{7.5.10}$$

Now returning to Eq.(7.5.4), we note that the self energy dresses the diffusion rate ν and if $z < 2$, then the self energy dominates νk^2 for small wave numbers and we can write Eq.(7.5.4) as

$$G^{-1}(k, \omega) = -i\omega + \lambda^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} \vec{p} \cdot (\vec{k} - \vec{p}) \times \vec{k} \cdot (\vec{k} - \vec{p}) C(p, \omega_1) C(\vec{k} - \vec{p}, \omega - \omega_1) \quad (7.5.11)$$

The equal time coordinate space correlation function scales as $r^{2\alpha}$ and hence $\langle h(k, t)h(-k, t) \rangle$ scales as $k^{-D-2\alpha}$ which implies that $\langle h(k, \omega)h(-k, -\omega) \rangle$ scales as $k^{-D-2\alpha-z}$. Looking at the second term on the bracketed part of the R.H.S. of Eq.(7.5.10), it scales as $k^{-D+2\alpha-z}$ and for small k will dominate the first term so long as $D + 2\alpha > z$. In this situation, we will have

$$C(k, \omega) = |G(k, \omega)|^2 \frac{\lambda^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} [\vec{p} \cdot (\vec{k} - \vec{p})]^2 \times C(p, \omega_1) C(\vec{k} - \vec{p}, \omega - \omega_1) \quad (7.5.12)$$

The mode coupling approximation is defined through the coupled integral equations for $C(k, \omega)$ and $G(k, \omega)$ in Eqns.(7.5.11) and (7.5.12). These equations can also be viewed as the spherical limit of a N -component model defined by treating $h(\vec{x}, t)$ as a N -component vector in the manner shown in **Chapter 8**.

To exploit these equations fully, we proceed to make a scaling Ansatz for the Green's function and the correlation function. The Green's function has the dimension of the relaxation rate and we can write

$$G^{-1}(k, \omega) = -i\omega + \Sigma(k, \omega) = -i\omega + \Gamma k^z \bar{f}\left(\frac{\omega}{\Gamma k^z}\right) = \gamma k^z f\left(\frac{\omega}{\Gamma k^z}\right) \quad (7.5.13)$$

The correlation function has the form

$$C(k, \omega) = C k^{-(D+2\alpha+z)} g\left(\frac{\omega}{\Gamma k^z}\right) \quad (7.5.14)$$

Using Eq.(7.5.11) at zero frequency, we get

$$\Sigma(\vec{k}, 0) = \lambda^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} [\vec{p} \cdot (\vec{k} - \vec{p})][\vec{k} \cdot (\vec{k} - \vec{p})] \times G(\vec{p}, \omega_1) C(\vec{k} - \vec{p}, -\omega_1)$$

which leads to

$$\bar{f}(0) = C \frac{\lambda^2}{\Gamma^2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} \frac{[\vec{p} \cdot (\vec{k} - \vec{p})][\vec{k} \cdot (\vec{k} - \vec{p})]}{p^z |\vec{1} - \vec{p}|^{D+2\alpha+z}} \times f\left(\frac{\omega_1}{\Gamma p^z}\right) g\left(\frac{\omega_1}{\Gamma |\vec{1} - \vec{p}|^z}\right) \quad (7.5.15)$$

while from Eq.(7.5.12) under the same conditions

$$C(\vec{k}, 0) = |G(k)|^2 \frac{\lambda^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} [\vec{p} \cdot (\vec{k} - \vec{p})]^2 C(\vec{p}, \omega_1) C(\vec{k} - \vec{p}, -\omega_1) \quad (7.5.16)$$

leading to

$$g(0) = C \frac{\lambda^2}{2\Gamma^2} \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} \frac{[\vec{p} \cdot (\vec{1} - \vec{p})]^2}{p^{D+2\alpha+z} |\vec{1} - \vec{p}|^{D+2\alpha+z}} \times g\left(\frac{\omega_1}{\Gamma p^z}\right) g\left(\frac{-\omega_1}{\Gamma |\vec{1} - \vec{p}|^z}\right) \quad (7.5.17)$$

It should be noted that in Eqns.(7.5.15) and (7.5.16), both ω_1 and p are scaled variables, scaled by Γk^z and k respectively. These equations would yield solutions for $f(x)$ and $g(x)$ if α , z and λ are known. More interesting would be to treat α and z and λ as unknowns and make a reasonable Ansatz about $f(x)$ and $g(x)$. The obvious candidate for the frequency dependent shape is Lorentzian. This corresponds to $\tilde{f}(x) = 1$ and $g(x) = (1 + x^2)^{-1}$. Under these approximations, Eq.(7.5.15) becomes,

$$1 = C \frac{\lambda^2}{2\Gamma^2} \int \frac{d^D p}{(2\pi)^D} \frac{[\vec{p} \cdot (\vec{1} - \vec{p})][\vec{1} \cdot (\vec{1} - \vec{p})]}{|\vec{1} - \vec{p}|^{D+2\alpha} (p^z + |\vec{1} - \vec{p}|^z)} \quad (7.5.18)$$

while Eq.(7.5.16) leads to

$$1 = C \frac{\lambda^2}{4\Gamma^2} \int \frac{d^D p}{(2\pi)^D} \frac{[\vec{p} \cdot (\vec{1} - \vec{p})]^2}{|\vec{1} - \vec{p}|^{D+2\alpha} p^{D+2\alpha}} \left[\frac{1}{p^z + |\vec{1} - \vec{p}|^z} \right] \quad (7.5.19)$$

Eliminating $C \frac{\lambda^2}{\Gamma^2}$ from Eqns.(7.5.17) and (7.5.18), we get

$$\int \frac{d^D p}{(2\pi)^D} \frac{[\vec{p} \cdot (\vec{1} - \vec{p})]^2}{|\vec{1} - \vec{p}|^{D+2\alpha} p^{D+2\alpha}} \times \left[\frac{1}{p^z + |\vec{1} - \vec{p}|^z} \right] = 2 \int \frac{d^D p}{(2\pi)^D} \frac{[\vec{p} \cdot (\vec{1} - \vec{p})][\vec{1} \cdot (\vec{1} - \vec{p})]}{|\vec{1} - \vec{p}|^{D+2\alpha} (p^z + |\vec{1} - \vec{p}|^z)} \quad (7.5.20)$$

The above relationship gives a connection between α , z and D and using $\alpha + z = 2$ which is a consequence of Galilean invariance, we can determine α and z separately. One of the interesting issues is at what D does α become 0. We note that the R.H.S of Eq.(7.5.19). has a logarithmic divergence at $\alpha = 0$. We explore this fact to evaluate the integrals as expansions about $\alpha = 0$ and retain the first term only. The first term involves evaluating the integrals in the high momentum limit i.e. $p \gg 1$ and extracting the appropriate 'pole' terms.

The integrand on the L.H.S of Eq.(7.5.19) is approximately

$$\frac{p^4}{2p^{2D+4\alpha+z}} = \frac{1}{2p^{2D+3\alpha-2}}.$$

The divergence occurs if $D=2$ and $\alpha = 0$ and hence in the leading log approximation, we can write the R.H.S. as $\frac{K_D}{2(D+3\alpha-2)}$. The integral on the R.H.S is more tricky. It is not symmetric in $\vec{p}$ and $\vec{1} - \vec{p}$ and hence to correctly extract the 'leading log', one needs to work with the symmetric form which can be done by adding to the integral, the one with $\vec{p}$ and $\vec{1} - \vec{p}$ interchanged and dividing by two. Alternatively, one can work with the symmetric variables $\frac{1}{2} - \vec{p}$ and $\frac{1}{2} + \vec{p}$ instead of $\vec{p}$ and $\vec{1} - \vec{p}$. The integrand becomes $\frac{\alpha}{2D} \frac{1}{p^{D+\alpha}}$ and the integral in the 'pole' approximation becomes $\frac{K_D}{2D}$, where $K_D = \frac{S_D}{(2\pi)^D}$, S_D being the surface area of the D-dimensional hypersphere. We now have from Eq.(7.5.19)

$$\begin{aligned} [2(D+3\alpha-2)]^{-1} &= D^{-1} \\ \text{or } \alpha &= \frac{4-D}{6} \end{aligned} \quad (7.5.21)$$

This shows that there is no rough phase for $D > 4$ or the upper critical dimension is 4. The upper critical dimension is controversial. A set of authors have obtained 4 as the upper critical dimension. Incidentally, Eq.(7.5.20) is exact in $D = 1$.

7.6 Growth with Surface Diffusion

We now consider a variant of the KPZ problem, which is relevant for the growth of crystals by a process of deposition. In this case there has to be a conservation law for the number of particles. The atoms fall on a substrate and the crystal grows over the region covered by the atomic beam. The random fluctuations in the beam intensity would make the surface rough, but surface diffusion is capable of smoothening out holes and bumps. This is important. If we imagine the growth process occurring on a substrate of dimension R (in atomic units) then for a mean height h , the number of atoms deposited is proportional to $R^2 h$. For random fluctuation in the beam intensity the fluctuation in the mean number is $\sqrt{r^2 h}$ and this will reflect in a fluctuation δh in h in the absence of any surface diffusion. Consequently, $\delta h \sim \frac{\sqrt{h}}{R}$. For $R \simeq h \simeq 100$ atomic units, $\delta h \sim 0.1$ atomic unit, which is significant. Hence, the rate of surface diffusion is worth studying.

The conservation law for particles will have the structure

$$\dot{h} + \vec{\nabla} \cdot \vec{j} = 0 \quad (7.6.1)$$

where the current is the gradient of a chemical potential and can be written as $\vec{j} = \vec{\nabla} \mu$. The chemical potential will be determined by the propensity of the particle

to diffuse which will be controlled by the local curvature. The local curvature will be determined by $\nabla^2 h$ and hence $\vec{j} = \kappa \vec{\nabla} \nabla^2 h$, where κ is a constant. Thus Eq.(7.6.1) becomes

$$\dot{h} = -\kappa \nabla^4 h \quad (7.6.2)$$

We now need to add the effect random fluctuations and we have the stochastic growth law,

$$\dot{h} = -\kappa \nabla^4 h + f \quad (7.6.3)$$

where

$$\langle f(\vec{r}, t) f(\vec{r}', t') \rangle = 2N \delta(\vec{r} - \vec{r}') \delta(t - t') \quad (7.6.4)$$

The above growth equation is known as the Mullin's-Sekerka equation. Scaling distances by b and t by b^z , so that

$$\begin{aligned} x &\rightarrow x' = bx \\ t &\rightarrow t' = b^z t \\ h &\rightarrow h' = b^\alpha h \\ \text{and } f &\rightarrow f' = b^\mu f \end{aligned}$$

the primed variables satisfy

$$\frac{\partial h'}{\partial t'} = -\kappa b^{4-z} \nabla'^4 h' + b^{-z+\alpha-\mu} f' \quad (7.6.5)$$

where

$$b^{-2\mu} \langle f(\vec{r}'_1, t'_1) f(\vec{r}'_2, t'_2) \rangle = 2b^{D+z} N \delta(\vec{r}'_1 - \vec{r}'_2) \delta(t'_1 - t'_2) \quad (7.6.6)$$

The noise is unchanged in character if

$$2\mu + D + z = 0 \quad (7.6.7)$$

and Eq.(7.6.5) remains the same as Eq.(7.6.3) if

$$\begin{aligned} z &= 4 \\ \text{and } \alpha &= z + \mu = 4 - \left(\frac{4+D}{2}\right) = 2 - \frac{D}{2} \end{aligned} \quad (7.6.8)$$

Thus we find that for substrate dimensions less than 4, the interface is rough.

The same results can be obtained by writing Eq.(7.6.3) in momentum space as

$$\dot{h}(k, t) = -\kappa k^4 h(k, t) + f(k, t)$$

The propagator $G(k, \omega)$ is $(-i\omega + \Gamma k^4)^{-1}$ and the correlation function

$$\begin{aligned}
C(k, t_{21}) &= \langle h(k, t_2) h(-k, t_1) \rangle \\
&= e^{-\kappa k^4(t_2+t_1)} \int_0^{t_2} dt' \int_0^{t_1} dt'' e^{\kappa k^4(t_2+t_1)} \langle f(k, t') h(-k, t'') \rangle \\
&= e^{-\kappa k^4(t_2+t_1)} \int_0^{t_2} dt' \int_0^{t_1} dt'' 2N e^{\kappa k^4(t_2+t_1)} \delta(t' - t'') \\
&= \frac{N}{\kappa} \frac{1}{k^4} e^{-k^4|t_1-t_2|}
\end{aligned} \tag{7.6.9}$$

In real space,

$$\begin{aligned}
C(r_{12}, t_{12}) &= \frac{N}{\kappa} \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\vec{k} \cdot \vec{r}_{12}}}{k^4} e^{-k^4|t_1-t_2|} \\
&= r_{12}^{4-D} f\left(\frac{t_{12}}{r_{12}^z}\right)
\end{aligned} \tag{7.6.10}$$

which when compared with the scaling form

$$C(r_{12}, t_{12}) = r_{12}^{2\alpha} f\left(\frac{t_{12}}{r_{12}^z}\right)$$

shows that $\alpha = 2 - \frac{D}{2}$ and $z = 4$, as in Eq.(7.6.8)

We now ask the question about the nonlinear terms in Eq.(7.6.3). Recalling, that the structure of the deterministic term in Eq.(7.6.3) is $\nabla^2 \mu$, we note that in the dependence of μ on derivatives of h , $\vec{\nabla} h$ did not figure because the sign of the slope cannot determine the probability of falling into or escaping from that local region. However, when nonlinear terms are allowed, $(\vec{\nabla} h)^2$ is the obvious candidate for the most relevant nonlinear term. Accordingly, the nonlinear equation for growth with surface diffusion becomes

$$\frac{\partial h}{\partial t} = -\kappa \nabla^4 h + \lambda \nabla^2 (\vec{\nabla} h)^2 + f \tag{7.6.11}$$

Carrying out the scalings

$$\begin{aligned}
x &\rightarrow x' = bx \\
t &\rightarrow t' = b^z t \\
h &\rightarrow h' = b^\alpha h \\
\text{and } f &\rightarrow f' = b^\mu f
\end{aligned}$$

we have

$$\frac{\partial h'}{\partial t'} = -\kappa b^{4-z} \nabla'^4 h' + \lambda b^{4-\alpha-z} \nabla'^2 (\vec{\nabla}' h')^2 + b^{-z+\alpha-\mu} f' \tag{7.6.12}$$

As in the KPZ equation, the exponents α and z can no longer be determined by requiring scale invariance. Instead one has to argue that the relaxation rate κ ,

the coupling constant λ and the noise correlator N will renormalize under scale transformation. In keeping with the KPZ case, if the coupling constant does not renormalize (the extra ∇^2 in the interaction term removes the guarantee that there will be no renormalization of the coupling constant in this case either) we have from Eq.(7.6.12)

$$\alpha + z = 4 \quad (7.6.13)$$

We now provide a simple argument for deriving α . Let us imagine the formation of a bump or a hole of lateral dimension R and height h . The gradients in Eq.(7.6.11) are in the substrate and hence they will be proportional to R^{-1} . If τ is the time to form the bump, then the action of the deterministic part of Eq.(7.6.11) yields τ from

$$\frac{h}{\tau} \sim \frac{\kappa h + \lambda h^2}{R^4} \quad (7.6.14)$$

Turning to the stochastic part, the mean number of particles falling over an area R^D in time τ is $\bar{f}\tau R^D$. The fluctuation in this number gives rise to the bump which is $R^D h$. Thus

$$\begin{aligned} [\bar{f}\tau R^D]^{1/2} &\sim R^D h \\ \text{yielding } \tau &\sim R^D h^2 \end{aligned} \quad (7.6.15)$$

Combining with Eq.(7.6.14), we find

$$\frac{R^4}{\kappa + \lambda h} \sim R^D h^2$$

If $\lambda = 0$, then $h \sim R^{\frac{4-D}{2}}$, giving $\alpha = 2 - \frac{D}{2}$ as in Eq.(7.6.8) If the nonlinear term dominates i.e. $\kappa \approx 0$, then (provided λ has no dependence)

$$\begin{aligned} \alpha &= \frac{4-D}{3} \\ \text{with } z &= 4 - \alpha = \frac{8+D}{3} \end{aligned} \quad (7.6.16)$$

We now explore the effect of carrying out a RG treatment on this model. The equations of motion in momentum space are

$$\dot{h}(\vec{k}, t) = -\kappa k^4 h(\vec{k}, t) + \lambda \sum_{\vec{p}} k^2 \vec{p} \cdot (\vec{k} - \vec{p}) h(\vec{p}, t) h(\vec{k} - \vec{p}, t) + f(\vec{k}, t) \quad (7.6.17)$$

with

$$\langle f(\vec{k}_1, t_1) f(\vec{k}_2, t_2) \rangle = 2N \delta(\vec{k}_1 + \vec{k}_2) \delta(t_1 - t_2) \quad (7.6.18)$$

Splitting $h(\vec{k}, t)$ into $h_{<}(\vec{k}, t)$ and $h_{>}(\vec{k}, t)$ as before and working in frequency space

$$\begin{aligned} -i\omega h_{<}(\vec{k}, \omega) = & -\kappa k^4 h_{<}(\vec{k}, \omega) + \lambda \sum [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] \{h_{<}(\vec{p}, \omega_1) \\ & \times h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) + 2h_{>}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + h_{>}(\vec{p}, \omega_1) h_{>}(\vec{k} - \vec{p}, \omega - \omega_1)\} + f_{<}(\vec{k}, \omega) \end{aligned} \quad (7.6.19)$$

and

$$\begin{aligned} -i\omega h_{>}(\vec{k}, \omega) = & -\kappa k^4 h_{>}(\vec{k}, \omega) + \lambda \sum [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] \{h_{<}(\vec{p}, \omega_1) \\ & \times h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) + 2h_{>}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + h_{>}(\vec{p}, \omega_1) h_{>}(\vec{k} - \vec{p}, \omega - \omega_1)\} + f_{>}(\vec{k}, \omega) \end{aligned} \quad (7.6.20)$$

The task is to solve $h_{>}(\vec{k}, \omega)$ perturbatively from Eq.(7.6.20) and insert in Eq.(7.6.19) to obtain an effective equation of motion for $h_{<}(\vec{k}, \omega)$. We expand

$$h_{>}(\vec{k}, \omega) = h_{>}^{(0)}(\vec{k}, \omega) + h_{>}^{(1)}(\vec{k}, \omega) + h_{>}^{(2)}(\vec{k}, \omega) + \dots \quad (7.6.21)$$

and find

$$h_{>}^{(0)}(\vec{k}, \omega) = \frac{f_{>}(\vec{k}, \omega)}{-i\omega + \kappa k^4} \quad (7.6.22)$$

$$\begin{aligned} (-i\omega + \kappa k^4) h_{>}^{(1)}(\vec{k}, \omega) = & \sum [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] \{h_{<}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + 2h_{<}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + h_{>}^{(0)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1)\} \end{aligned} \quad (7.6.23)$$

and

$$\begin{aligned} (-i\omega + \kappa k^4) h_{>}^{(2)}(\vec{k}, \omega) = & \sum [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] \{h_{<}(\vec{p}, \omega_1) h_{>}^{(1)}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + 2h_{>}^{(1)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1)\} \end{aligned} \quad (7.6.24)$$

Up to $O(\lambda^2)$ Eq.(7.6.19) becomes

$$\begin{aligned} -i\omega h_{<}(\vec{k}, \omega) = & -\kappa k^4 h_{<}(\vec{k}, \omega) + \lambda \sum [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] \\ & \times \{h_{<}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + 2\lambda \sum [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] h_{>}^{(0)}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + \lambda \sum_{\vec{p}, \omega_1} [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] h_{>}^{(0)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + 2\lambda^2 \sum [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] h_{>}^{(1)}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1) \\ & + 2\lambda^2 \sum [k^2 \vec{p} \cdot (\vec{k} - \vec{p})] h_{>}^{(1)}(\vec{p}, \omega_1) h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) + f_{<} \end{aligned} \quad (7.6.25)$$

Substituting from Eqns.(7.6.23) and (7.6.22) for $h_{>}^{(1)}$ and $h_{>}^{(0)}$, we average over $f_{>}$. The nonvanishing term of interest is

$$\begin{aligned}
&= 4\lambda^2 \left\langle \sum_{\vec{p}, \omega_1} \sum_{\vec{q}, \omega_2} \frac{[k^2 \vec{p} \cdot (\vec{k} - \vec{p})][p^2 \vec{q} \cdot (\vec{p} - \vec{q})]}{-i\omega + \kappa p^4} h_{<}(\vec{p}, \omega_2) \right. \\
&\quad \left. \times h_{>}^{(0)}(\vec{k} - \vec{p}, \omega - \omega_1) h_{>}^{(0)}(\vec{p} - \vec{q}, \omega_1 - \omega_2) \right\rangle \\
&= 4\lambda^2 \sum_{\vec{p}, \omega_1} \sum_{\vec{q}, \omega_2} \frac{N[k^2 \vec{p} \cdot (\vec{k} - \vec{p})][p^2 \vec{q} \cdot (\vec{p} - \vec{q})]}{-i\omega + \kappa p^4} \\
&\quad \times \frac{\delta(\vec{k} - \vec{q})\delta(\omega - \omega_2)}{[(\omega - \omega_1)^2 + \kappa^2(\vec{k} - \vec{p})^8]} \\
&= -8\lambda^2 N \sum_{\vec{p}, \omega_1} \frac{[k^2 \vec{p} \cdot (\vec{k} - \vec{p})][p^2 \vec{q} \cdot (\vec{p} - \vec{q})]}{(-i\omega + \kappa p^4)[(\omega - \omega_1)^2 + \kappa^2(\vec{k} - \vec{p})^8]} \\
&= -4\lambda^2 N \sum_{\vec{p}} \frac{[k^2 \vec{p} \cdot (\vec{k} - \vec{p})][p^2 \vec{q} \cdot (\vec{p} - \vec{q})]}{(-i\omega + \kappa p^4)[-i\omega + \kappa p^4 + \kappa(\vec{k} - \vec{p})^4]} \\
&= -\frac{2\lambda N k^2}{\kappa} \sum_{\vec{p}} \frac{\vec{p} \cdot (\vec{k} - \vec{p})}{-i\omega + \kappa p^4 + \kappa(\vec{k} - \vec{p})^4} \\
&\quad \times \left[\frac{\vec{k} \cdot (\vec{k} - \vec{p}) p^2}{(\vec{k} - \vec{p})^4} + (\vec{k} - \vec{p})^2 \frac{(\vec{k} \cdot \vec{p})}{p^4} \right] \tag{7.6.26}
\end{aligned}$$

To this order, Eq.(7.6.25) can be cast as

$$\begin{aligned}
-i\omega h_{<}(k, \omega) &= -k^4(\kappa + \delta\kappa)h_{<}(k, \omega) + f_{<}(k, \omega) \\
&\quad + \lambda \sum_{\vec{p}} [\vec{p} \cdot (\vec{k} - \vec{p})] h_{<}(\vec{p}, \omega_1) h_{<}(\vec{k} - \vec{p}, \omega - \omega_1)
\end{aligned}$$

where

$$\begin{aligned}
\delta\kappa(k, \omega) &= \frac{2}{k^2} \frac{\lambda^2 N}{\kappa^2} \int \frac{d^D p}{(2\pi)^D} \frac{\vec{p} \cdot (\vec{k} - \vec{p})}{-i\omega + \kappa[p^4 + (\vec{k} - \vec{p})^4]} \\
&\quad \times \left[\frac{\vec{k} \cdot (\vec{k} - \vec{p}) p^2}{(\vec{k} - \vec{p})^4} + (\vec{k} - \vec{p})^2 \frac{(\vec{k} \cdot \vec{p})}{p^4} \right] \tag{7.6.27}
\end{aligned}$$

The low frequency, low momentum contribution to $\delta\kappa$ is

$$\begin{aligned}
\delta\kappa(k, \omega) &\simeq \frac{2}{k^2} \frac{\lambda^2 N}{\kappa^2} \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{6(\vec{k} \cdot \vec{p})^2 - p^2 k^2}{p^2 2p^4} \\
&= \frac{\lambda^2 N}{\kappa^2} \left(\frac{6}{D} - 1 \right) \int_{\frac{\Lambda}{b}}^{\Lambda} \frac{d^D p}{p^4} \tag{7.6.28}
\end{aligned}$$

By looking at the correlation function, we find that the change δN in the coefficient of the noise correlation is

$$\delta N = \frac{\lambda^2 n^2}{2\kappa^3} \frac{d^D p}{p^4 |\vec{k} - \vec{p}|^4} \frac{[\vec{p} \cdot (\vec{k} - \vec{p})]^2}{p^4 + |\vec{k} - \vec{p}|^4} \quad (7.6.29)$$

From the structure of Eq.(7.6.28), it is clear that as $b \rightarrow 1$, a perturbative change occurs in κ for $D = 4$ and we have

$$\delta \kappa = \frac{\lambda^2 N}{\kappa^2} \left(\frac{6}{D} - 1 \right) \ln b \quad (7.6.30)$$

The noise correlator on the other hand is not affected and as in the KPZ equation the vertex correction is assumed to be unaffected (this is not exactly true, it is true to a very good approximation because the coefficient of the correction term happens to be very small). We now need to restore the cutoff to Λ by carrying out a scale transformation. The result of that can be read off from Eq.(7.6.12) and we find

$$\begin{aligned} \frac{d\kappa}{dl} &= \kappa \left[z - 4 + \frac{\lambda^2 N}{\kappa^3} S_D \left(\frac{6-D}{D} \right) \right] \\ \frac{dN}{dl} &= N[z - 2\alpha - D] \\ \frac{d\lambda}{dl} &= \lambda[z + \alpha - 4] \end{aligned} \quad (7.6.31)$$

It should be noted that unlike the first of Eq.(7.6.31) above the two following equations are exact. The diagrammatic expansion in powers of λ do not contribute to N at all and very insignificantly to λ and hence flow is determined by naïve dimensional scaling. Hence we have the exact ‘results’

$$\begin{aligned} \alpha + z &= 4 \\ 2\alpha - z &= -D \end{aligned}$$

which lead to $\alpha = \frac{1}{3}(4 - D)$ and $z = \frac{1}{3}(8 + D)$ as had been obtained before (Eq.(7.6.16)) by a completely different argument.

It is instructive to look at the problem from the self-consistent mode coupling point of view as well. One proceeds exactly as in the case of the KPZ equation and obtains in analogy with Eq.(7.5.4)

$$G^{-1}(k, \omega) = -i\omega + \kappa k^4 + \Sigma(k, \omega) \quad (7.6.32)$$

where

$$\begin{aligned} \Sigma(k, \omega) &= 4\lambda^2 k^2 \int \frac{d^D p}{(2\pi)^D} \frac{d\omega_1}{2\pi} p^2 [\vec{p} \cdot (\vec{k} - \vec{p})][\vec{k} \cdot (\vec{k} - \vec{p})] \\ &\quad \times G(\vec{p}, \omega_1) C(\vec{k} - \vec{p}, \omega - \omega_1) \end{aligned} \quad (7.6.33)$$

Once again it is tacitly assumed that λ does not renormalize. We first enquire in what situation will $\Sigma(k, \omega)$ dominate κk^4 . If $\Sigma(k, \omega)$ scales as k^z , then a power count of the R.H.S of Eq.(7.6.31) shows that the integral scales as $k^{8-2\alpha-z} = k^{4-\alpha}$. Consequently, so long as $\alpha > 0$, $\Sigma(k, \omega)$ will dominate k^4 .

Let us now explore the region of the integrand where $\vec{p} \simeq \vec{k}$. In this approximation the integral is

$$\begin{aligned} k^2 G(\vec{k}) &= \int \frac{d^D p'}{(2\pi)^D} \frac{d\omega_1}{2\pi} (\vec{k} \cdot \vec{p})^2 C(p', \omega - \omega') \\ &\propto k^2 G(k) \int \frac{d^D p'}{(2\pi)^D} \frac{(\vec{k} \cdot \vec{p})^2}{(p')^{D+2\alpha}} \\ &= k^4 G(k) \frac{C_D}{(2\pi)^D} \frac{1}{D} \int \frac{dp'}{p'^{2\alpha-1}} \end{aligned} \quad (7.6.34)$$

The integral diverges (ultraviolet i.e. short distance divergence because k is small) if $\alpha \geq 1$, i.e. $D \leq 1$. Consequently, the scaling laws that we have obtained will not be valid if $D \leq 1$.

How does this difficulty show up in the RG treatment of the model? If we include higher order terms in the equation of motion, then we would have

$$\begin{aligned} \dot{h} &= -\kappa \nabla^4 h + \lambda \nabla^2 (\vec{\nabla} h)^2 + \lambda_1 \nabla^2 (\vec{\nabla} h)^4 + \dots \\ &\quad + \lambda_n \nabla^2 (\vec{\nabla} h)^{2n} + \dots + f \end{aligned} \quad (7.6.35)$$

If $\alpha = 1$, the dimensionality of all the operators $(\vec{\nabla} h)^{2n}$ become the same and hence all the terms are potentially relevant in the RG sense for $\alpha > 1$ (i.e. $D < 1$). It is clear that we can no longer have $\alpha = 1$ at $D=1$. There is some indication that $z=4$ for $D=1$ and that $\alpha = 3/2$. The difficulty that we see over here is interestingly enough similar to the difficulty that one encounters in turbulence (see **Chapter 8**).

7.7 Discrete Models

The first of the discrete models for surface growth in the presence of diffusion was introduced by Wolf and Villain and by Das Sarma and Tamborenea. The two models differ in the sticking rules for atoms. We illustrate the action of the model for a one dimensional substrate. If we imagine the interface at any instant as shown in Fig. 7.5 The solid circles indicate the particles which have been deposited. The randomly dropped new particle (open circle) has to decide which site it will settle on. For this it looks at the nearest neighbor sites. If at least one of the nearest neighbour sites is occupied it stays at the position at which it has landed. If none of the nearest neighbours are occupied the particle will move. If only one nearest neighbour site has its neighbouring site occupied, then the particle moves to that nearest neighbour position. If both the nearest neighbours have neighbouring sites

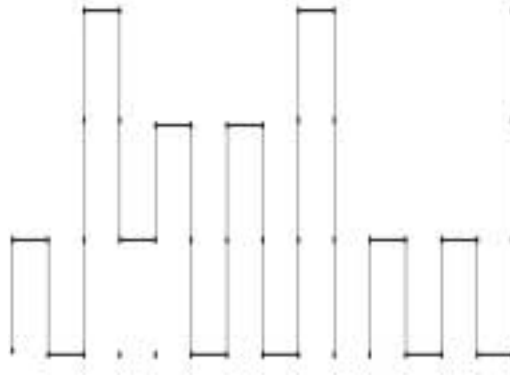


Figure 7.5. The growth rule in the DasSarma-Tamborenea model

occupied or unoccupied, then the particle will move randomly to its left or right. These are the rules for Das Sarma - Tamborenea model. In the Wolf-Villain model, the particle moves to the site which has the largest number of nearest neighbours. An unrealistic aspect of such growth models could be the formation of large local slopes. The particles have the freedom to travel as far down as possible to maximize the number of nearest neighbours. This generally does not happen in the growth of crystals by Molecular Beam Epitaxy (MBE) which the discrete models are supposed to mimic.

The numerical simulations study the width W of the interface which is the mean square deviation from the average height

$$W(L, t) = \left[\frac{1}{N} \sum_{i=1}^N (h_i - \bar{h})^2 \right]^{1/2} \quad (7.7.1)$$

The expected scaling behaviour is $W(L, t) \sim L^\alpha f(\frac{t^{1/z}}{L})$ with $f(x) \sim x^\alpha$ for small x and saturating for $x \gg 1$. Consequently, a typical data set for the width appears shown in Fig. 7.6 The small time measurement yields $\beta = \frac{\alpha}{z} \simeq 0.365$ for the Wolf-Villain Model. The scaling of the saturation width with $\bar{L}$ gives $\alpha \simeq 1.4$. These results are in agreement with those obtained in Eq.(7.6.8) for the linear theory and disagree with the results of the nonlinear theory expressed in Eq.(7.6.16). A two dimensional generalization of the Wolf-Villain model yielded $\beta \simeq 0.20$ and $\alpha \simeq 0.66$ in agreement with the nonlinear theory and disagreeing with the predictions of the linear theory. The surprising result that the nonlinear theory works in $D = 2$ and does not work in $D = 1$ has to do with the fact that the nonlinear model that we are working with cannot be trusted at $D = 1$ where an infinite number of nonlinear terms are equally important.

The unusual features associated with $D = 1$ are further stressed by the work of Krug who went on to study the distribution of step sizes. The step size is defined

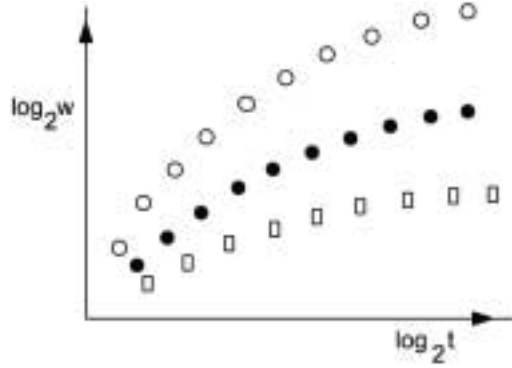


Figure 7.6. Interface width as a function of time in the DasSarma Tamborenea model

as the nearest neighbour height difference. The various moments of the step size are defined as

$$\sigma_q = \langle |h(x+1) - h(x)|^q \rangle^{1/q} \quad (7.7.2)$$

which is clearly a local quantity. There is the width variable which is a global quantity and the generalized width fluctuation moment W_q is defined as

$$W_q = \langle (h - \bar{h})^q \rangle^{1/q} \quad (7.7.3)$$

The scaling of both sets of moments is in terms of a lateral correlation length ξ which increases as $t^{1/z}$, where t is the deposition time. This holds for small times when $\xi \ll L$, the system size. For $\xi \gg L$, the scaling is controlled by L . The local quantity σ_q scales as ξ^{α_q} where α_q is an increasing function of q , while the global W_q scales as $\xi^{3/2}$ independent of q . The local and global regimes are connected by the r -dependent correlation function

$$G_q(r) = \langle |h(x+r) - h(x)|^q \rangle^{1/q} = \xi^{\alpha_q} r^{\zeta_q} f_q(r/\xi) \quad (7.7.4)$$

The property of $f_q(x)$ is that for $x \rightarrow 0$ $f_q(x)$ tends to a constant, while for $x \rightarrow \infty$ $f_q(x) \sim x^{-\zeta_q}$. Now if $r = 1$, $x = 1/\xi \rightarrow 0$ then $f(x) \rightarrow \text{constant}$. Consequently, $G_q(1) \sim \xi^{\alpha_q}$. If $r \gg \xi$, so that one is in the global regime, we have

$$\begin{aligned} G_q(r \gg \xi) &\sim \xi^{\alpha_q} r^{\zeta_q} \left(\frac{\xi}{r}\right)^{\zeta_q} = \xi^{\alpha_q + \zeta_q} \\ &= \xi^{3/2} \end{aligned} \quad (7.7.5)$$

provided $\alpha_q + \zeta_q = 3/2$. Thus ζ_q becomes q dependent because of α_q . The G_q are the analogues of the structure factors in turbulence and the nonlinear q -dependence of ζ_q is the signature of turbulence in this case. The step size distribution was found to be non-Gaussian (a stretched exponential), once again in analogy with intermittency in turbulence. A clear understanding of the basic issues here is still not absolutely clear.

7.8 Growth Models with Correlated Noise

The KPZ model is certainly of great importance as a generic model for non equilibrium phase transitions. However, one has not yet come up with an experimental system that confirms the predictions of the KPZ model satisfactorily. This discrepancy has motivated various modifications on the original model. First of all, one may question the validity of the uncorrelated Gaussian noise. In a real system, the noise could be correlated, non-gaussian or even quenched. Quite generally, in a genuine non equilibrium system, the form of the noise correlations in an effective Langevin type description is a crucial ingredient of the modelling. This is in contrast with equilibrium dynamics, where the functional form of the noise as well as its strength are fixed via an Einstein relation which ensures that asymptotically the probability distribution will be an equilibrium one. Uncorrelated white noise is often a straightforward choice but the sensitivity and stability of the ensuing results need to be carefully tested against modifications of the noise correlations. The motivation is not to introduce a realistic growth model but to see how much change can such perturbations cause over the KPZ results. We can focus on $D = 1$.

In momentum space the model reads (compare Eqns.(7.4.1) and (7.4.2))

$$-i\omega h(\vec{k}, \omega) = -vk^2 h(\vec{k}, \omega) - \frac{\lambda}{2} \sum_{\vec{p}, \omega_1} \vec{p} \cdot (\vec{k} - \vec{p}) h(\vec{p}, \omega_1) h(\vec{k} - \vec{p}, \omega - \omega_1) + \eta(\vec{k}, \omega) \quad (7.8.1)$$

with

$$\langle \eta(\vec{k}, \omega) \eta(\vec{k}', \omega') \rangle = \frac{2N}{k^{2-\beta}} \delta(\vec{k} + \vec{k}') \delta(\omega + \omega') \quad (7.8.2)$$

In coordinate space, the noise correlation reads

$$\langle \eta(\vec{x}_1, t_1) \eta(\vec{x}_2, t_2) \rangle = 2N |\vec{x}_1 - \vec{x}_2|^{\beta-2-1} \delta(t_1 - t_2) \quad (7.8.3)$$

In D -dimensional space the generalization would be

$$\langle \eta(\vec{x}_1, t_1) \eta(\vec{x}_2, t_2) \rangle = 2N |\vec{x}_1 - \vec{x}_2|^{\beta-2-D} \delta(t_1 - t_2) \quad (7.8.4)$$

For $\beta = 2$, we have the KPZ system, while for $\beta < 2$ we have the generalized KPZ system. The basic ingredients of the calculation are the Green's function

$$\left\langle \frac{\delta h(\vec{k}, \omega)}{\delta \eta(\vec{k}', \omega')} \right\rangle \frac{1}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')}$$

and the correlation function

$$\frac{\langle h(\vec{k}, \omega) h(\vec{k}', \omega') \rangle}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')}$$

Dyson's equation gives

$$G^{-1} = -i\omega + vk^2 + \Sigma(k, \omega)$$

and the self-consistent scheme (or the spherical limit) gives

$$\Sigma(k, \omega) = \lambda^2 \int \frac{dp}{2\pi} \frac{d\omega'}{2\pi} pk(k-p)^2 G(p, \omega') C(k-p, \omega-\omega') \quad (7.8.5)$$

The correlation function at the same order of accuracy is

$$\begin{aligned} C(k, \omega) = & |G(k, \omega)|^2 2Nk^{\beta-2} + \lambda^2 |G(k, \omega)|^2 \int \frac{dp}{2\pi} \frac{d\omega'}{2\pi} p^2 (k-p)^2 \\ & \times C(p, \omega') C(k-p, \omega-\omega') \end{aligned} \quad (7.8.6)$$

We note that a similar approximation for the KPZ equation with white noise has been studied by Frey et.al, who not only considered the scaling behaviour but the entire scaling function. We now make the following scaling Ansatz for $\Sigma(k, \omega)$ and $C(k, \omega)$:

$$\Sigma(k, \omega) = k^z g\left(\frac{\omega}{k^z}\right) \quad (7.8.7)$$

$$C(k, \omega) = k^{-(1+2\alpha+z)} f\left(\frac{\omega}{k^z}\right) \quad (7.8.8)$$

where z is the dynamic scaling exponent and α is the roughness exponent. The self-energy $\Sigma(k, \omega)$ will determine the relaxation rate in the long wavelength limit if $z < 2$. Similarly in Eq.(7.8.6) the noise strength $2N$ will not be renormalized (i.e. acquire k, ω dependence) if the second term is no more singular than the first. If the second term dominates in the long wavelength limit, Eq.(7.8.6) reduces to

$$\begin{aligned} C(k, \omega) = & \lambda^2 |G(k, \omega)|^2 \int \frac{dp}{2\pi} \frac{d\omega'}{2\pi} p^2 (k-p)^2 \\ & \times C(p, \omega') C(k-p, \omega-\omega') \end{aligned} \quad (7.8.9)$$

With $G^{-1}(k, \omega) = -i\omega + \Sigma(k, \omega)$, we can solve for α and z between Eqns.(7.8.5) and (7.8.9). This leads to $\alpha = 1/2$ and $z = 3/2$.

On the other hand, if N is not renormalized, then Eq.(7.8.6) shows $2\alpha = 1 - \beta + z$ and Eq.(7.8.5) leads to $z = 1 + \frac{\beta}{3}$. We now need to settle when N will be renormalized. This follows from a simple power count of Eq.(7.8.6). The second term on the R.H.S dominates the first if $z \leq 1 + \frac{\beta}{3}$. The dominance of the second term implies $z = 3/2$ and hence one will see $z = 3/2$ for $\beta \geq 3/2$.

We now check for the finiteness of $\Sigma(k, \omega)$ as given by Eq.(7.8.5). There is no divergence coming from the region where $p > k$. We next check the region $p \simeq k$, and find

$$\Sigma(k, 0) \simeq \lambda^2 k^2 G(k, 0) \int \frac{dq}{2\pi} \frac{d\omega'}{2\pi} q^2 C(q, \omega') \quad (7.8.10)$$

$$\begin{aligned} \Sigma(k, 0) &\simeq \lambda^2 k^2 [\Sigma(k)]^{-1} \int \frac{dq}{2\pi} q^2 \frac{N}{q^{3-2\beta/3}} \\ &= \lambda^2 k^2 [\Sigma(k)]^{-1} N \int \frac{dq}{q^{1-2\beta/3}} \end{aligned} \quad (7.8.11)$$

For $\beta > 0$, the integral converges and we have $z = 1 + \beta/3$, but for $\beta < 0$, the integral needs to be cut off at a lower limit k_0 and hence $\Sigma(k) \sim k k_0^{\beta/3}$, giving $z=1$. Thus,

$$\begin{aligned} z &= 3/2, \quad \alpha = 1/2, \quad \text{for } 2 \geq \beta \geq 3/2 \\ z &= 1 + \beta/3, \quad \alpha = 1 - \beta/3, \quad \text{for } 3/2 \geq \beta \geq 0 \\ z &= 1, \quad \text{for } \beta < 0 \end{aligned} \quad (7.8.12)$$

These are precisely the results obtained by Medina et.al. For $\beta < 0$, the dynamics is governed by the sweeping of small-scale fluctuations by the large-scale ones.

We see that for a range of β around the KPZ value (i.e. if the spatial correlation is weak) the KPZ value for z is obtained. In this case, we may safely say that the long ranged nature of the noise correlation is an irrelevant perturbation. For the range of β between $3/2$ and 0 , one settles on a result dominated by the noise correlation. The surprising thing is that for $\beta < 0$ Eq.(7.8.12) gives $z = 1$ i.e. One has reached the ballistic limit. However, the numerical work of Hayot and Jayaprakash which verified these predictions in the ranged $2 \leq \beta \leq 0$, showed that $z = 1 + \beta/3$ holds for $\beta < 0$. This is once again reminiscent of Kolmogorov scaling in turbulence ($z = 2/3$ there) and like the problem of turbulence this situation is not fully understood.

7.9 Growth models with Nonlocality

The fact that experiments never record a KPZ like exponent has given rise to modifications in the deterministic part of the equation as well. Many of the experimental situations involve complex processes which go beyond the idealization of the deposition of non interacting particles. This could be especially true if medium or fluctuation induced interactions interfere with the process as for example in systems involving proteins, colloids or latex particles. The major interaction that is involved is long ranged hydrodynamic interaction. Can such interactions be relevant to the roughness of the surface? Once again this question is akin to the one asked in the last section where the issue was whether spatial correlation in the noise could be a relevant perturbation.

The first question is how does one incorporate such interactions. The answer proposed by Mukherji and Bhattacharjee is to view the gradient of the height as a measure of the local density and incorporate the long range interaction by coupling

the gradient at two different points. This means that the nonlinear term $(\vec{\nabla}h)^2$ in the KPZ equation gets replaced by

$$\int d^D r' \vec{\nabla}h(\vec{r}) \cdot \vec{\nabla}h(\vec{r}') V(\vec{r} - \vec{r}').$$

The usual model Eq.(7.3.3) obtains if we set

$$V(\vec{r} - \vec{r}') = \frac{\lambda}{2} \delta(\vec{r} - \vec{r}')$$

i.e. make the interaction absolutely short ranged. The proposed model is

$$\dot{h}(\vec{r}, t) = \nu \nabla^2 h + \frac{1}{2} \int d^D r' \vec{\nabla}h(\vec{r}) \cdot \vec{\nabla}h(\vec{r}') V(\vec{r} - \vec{r}') + \eta \quad (7.9.1)$$

$$\langle \eta(\vec{r}, t) \eta(\vec{r}', t') \rangle = 2N \delta(\vec{r} - \vec{r}') \delta(t - t') \quad (7.9.2)$$

$$V(\vec{r} - \vec{r}') = \lambda \delta(\vec{r} - \vec{r}') + \lambda_1 |\vec{r} - \vec{r}'|^{\rho-D} \quad (7.9.3)$$

The rescaling of r by b , t by b^z and h by b^α i.e.

$$\begin{aligned} r' &= \frac{r}{b} \\ t' &= \frac{t}{b^z} \\ h' &= \frac{h}{b^\alpha} \end{aligned}$$

leads to

$$\begin{aligned} \dot{h}'(\vec{r}'_1, t') &= \nu b^{z-2} \nabla'^2 h' + \frac{\lambda}{2} b^{\alpha+z-2} (\vec{\nabla}' h')^2 + \frac{\lambda_1}{2} b^{\alpha+z-2+\rho} \\ &\times \int |\vec{r}'_1 - \vec{r}'_2|^{\rho-D} (\vec{\nabla}' h'(\vec{r}'_1)) \cdot (\vec{\nabla}' h'(\vec{r}'_2)) d^D r'^2 + b^{\mu+z} \eta'(\vec{r}'_1, t') \end{aligned} \quad (7.9.4)$$

The coupling constant λ renormalizes according to $\lambda' = \lambda b^{\alpha+z-2}$ and the constant λ_1 according to $\lambda'_1 = \lambda_1 b^{\alpha+z-2+\rho}$. At the Gaussian fixed point, $\alpha = 1 - \frac{D}{2}$ and $z = 2$, which means $\lambda' = \lambda b^{1-\frac{D}{2}}$ and $\lambda'_1 = \lambda_1 b^{1-\frac{D}{2}+\rho}$. Consequently, for $D < 2$, λ is relevant (as before) and the long range interaction is relevant for $D < 2(1 + \rho)$. For $2 < D < 2(1 + \rho)$, only the long range interactions give a relevant perturbation for $\rho > 0$. Thus it is expected that if $\rho > 0$, the roughening exponent will be determined by the long-range-interaction-fixed-point and consequently the exponents α and z will depend upon the parameter ρ . The complete picture can be obtained only if we

carry out a full RG transformation. We refer the reader to the original publication for the details and end this section by quoting the flow equations in $D = 1$

$$\frac{dv}{dl} = v \left[z - \left(2 + \frac{NS_D}{v^3} \tilde{V}(2) \tilde{V}(1) \frac{(D-2) + 3f(1)}{4D} \right) \right] \quad (7.9.5)$$

$$\frac{dN}{dl} = \left(z - D - 2\alpha \right) N + \frac{N^2 S_D}{4v^3} \left[\tilde{V}(2) \right]^2 \quad (7.9.6)$$

where $\tilde{V}(k) = \lambda + \lambda_1 k^{-\rho}$ is the Fourier transform of the interaction $V(\vec{r}_1 - \vec{r}_2)$. The function $f(x)$ is the logarithmic derivative $\frac{\partial \ln \tilde{V}(k)}{\partial \ln k}$ evaluated at $k = x$. The analysis of the fixed point structure of the above flow equations confirms the qualitative statements made above.

7.10 Roughening Transition

The mechanism for roughening so far has been the random deposition process. Crystal surfaces can, however, be rough under the equilibrium conditions. At low temperatures, thermal fluctuations have no effect on the shape of the crystal with all atoms remaining at their mean positions and the crystal planes looking quite flat. As temperature increases, the probability of an atom breaking its bond with its neighbours increases and atoms can hop onto neighbouring sites, causing roughness on an atomic scale. One can expect a gradual transition to a rough morphology with more atoms hopping from their positions and eventually causing the surface to melt. This is true for the short range correlation but the long range correlation gets broken much earlier. If one focuses on the long distance scales, then there exists a critical temperature T_R (much below the melting temperature) above which the crystal facet is no longer smooth.

Fluctuating about the flat surface cause the surface area and hence the surface energy to increase. The energy of a two dimensional surface is proportional to the area and can be written as

$$H = v \int d^2x \sqrt{(1 + \vec{\nabla}h)^2} \quad (7.10.1)$$

In the absence of height fluctuations (no local slopes) $\vec{\nabla}h = 0$ and the flat surface area is $H_0 = vL^2$ with v being the 'surface tension' or the stiffness of the surface. If the gradient is not too large anywhere, we can expand and get

$$H = H_0 + \frac{1}{2}v \int d^2x (\vec{\nabla}h)^2 + \dots \quad (7.10.2)$$

This Hamiltonian is the same as that of the EW model. The lesson from that model is that the equilibrium (saturated i.e $t \gg L^z$) system is rough with the height

correlation increasing logarithmically. This is independent of the length scale and hence it cannot account for the roughening temperature.

The shortcoming of Eq.(7.10.2) can be seen from the fact that this energy expression has continuous symmetry but the crystal has a discrete symmetry in terms of the lattice spacing a_0 . This can be incorporated by adding a periodic term to Eq.(7.10.2) and writing the excess energy

$$\delta H = \int \left[\frac{1}{2} v (\vec{\nabla} h)^2 - V_0 \cos \left(\frac{2\pi \hbar}{a_0} \right) \right] d^2 x \quad (7.10.3)$$

The equilibrium statistical mechanics of the system is done by writing down the partition function

$$Z = \int \mathcal{D}[h] e^{-\frac{\delta H}{k_B T}} \quad (7.10.4)$$

The minimization of δH yields h constant from the first term in Eq.(7.10.3) which requires $\vec{\nabla} h = 0$ for a minimum. The second term is minimized if h is an integral multiple of a_0 . Thus, the minimization of δH yields a flat surface which is the mean field result. It is the fluctuation around this mean field value that needs to be studied. This is done by Eq.(7.10.4) in the momentum space modes of $h(\vec{r})$, which yields

$$Z = \Pi_{k=0}^{\Lambda} \int dh(k) e^{-\frac{1}{2} [\sum_k v k^2 |h_k|^2 + V(h_k)]} \quad (7.10.5)$$

One now writes down the RG flows for v/T and V_0/T by eliminating the modes lying between Λ/b and Λ and then rescaling the coordinates and fields so that the original cut off Λ restored. These flow equations are quite difficult to derive but can now be found in standard text books and review articles on critical phenomena. It is customary to define the two variables

$$\begin{aligned} x &= \frac{2v a_0^2}{\pi k_B T} \\ \text{and } y &= \frac{4\pi V_0}{k_B T \lambda^2} \end{aligned} \quad (7.10.6)$$

and write the flow equations in terms of them as

$$\frac{dy}{dl} = 2y \frac{x-1}{x} \quad (7.10.7)$$

$$\frac{dx}{dl} = \frac{y^2}{2x} F\left(\frac{2}{x}\right) \quad (7.10.8)$$

where $F(z)$ is a complicated function.

We note that Eq.(7.10.7) has a fixed point at $x = 1$, corresponding to $T = T_R$ given by

$$T_R = \frac{2\nu a_0^2}{\pi k_B} \quad (7.10.9)$$

This fixed point divides two phases - the flat and the rough. The line $y = 0$ is a fixed line of the entire system. Trajectories flow to the line or flow away from it. The part of $y = 0$, corresponding to $x < 1$ i.e. $T > T_R$ is stable and trajectories approach this part of the line under iteration. For $T < T_R$ and $x > 1$ and this part of $y=0$ is unstable. Under iteration the system flows away to $y = \infty$. Flowing to $y = 0$ means that effectively the periodic potential vanishes and the surface tension dominates. This corresponds to the rough phase ($T > T_R$). Flowing to $y = \infty$ means that the periodic part dominates and this corresponds to a smooth surface ($T < T_R$). The flow can be understood also by writing Eqns.(7.10.7) and (7.10.8) as

$$\frac{dy}{dx} = 4 \frac{x-1}{y} \frac{1}{F(\frac{2}{x})} \quad (7.10.10)$$

For $x \simeq 1$, $F(\frac{2}{x}) \simeq F(2)$ and if we use the fact that $F(z)$ is slowly varying, then an integral of Eq.(7.10.10) can be written as

$$y^2 = \frac{4}{F(2)} (x-1)^2 + C \quad (7.10.11)$$

where C is a constant of integration. The curve which passes through $x = 1$, $y = 0$ corresponds to $C = 0$ and represents a pair of straight lines shown as the heavy lines in Fig. 7.7. This pair represents the separatrix of flows that are drawn to or emanate from $y = 0$ and those that do not touch $y = 0$.

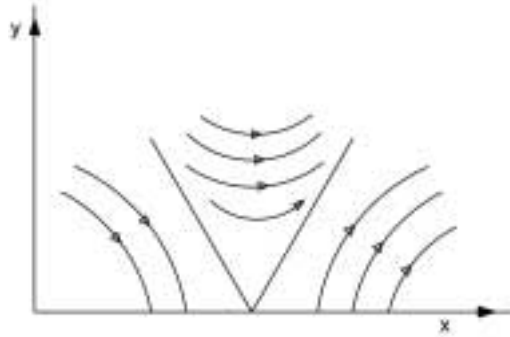


Figure 7.7. Kosterlitz Thouless kind of flow

The Langevin dynamics corresponding to the Hamiltonian of Eq.(7.10.3) can be written as

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h - \frac{2\pi}{a_0} V_0 \sin\left(\frac{2\pi}{a_0} h\right) + \eta \quad (7.10.12)$$

Chin and Weeks had in addition to η , which is a random deposition term, a regular part F which would help to define the mobility of the surface. The mobility could be defined as $\lim_{F \rightarrow 0} h/F$. The term F is analogous to the external magnetic field in the magnetization problem. Just as the field rounds out the magnetic transition, the force F over here smoothes out the roughening transition. In the absence of F , the interface is pinned to the substrate for $T < T_R$ and there is a sharp transition at T_R to a phase where the interface is free to wander. In the presence of F , the interface has a finite velocity even for $T < T_R$.

A RG calculation generates a KPZ type nonlinearity in the equation of motion. Consequently, the model that has been much studied is

$$\frac{\partial h}{\partial t} = v \nabla^2 h - \frac{2\pi}{a_0} V_0 \sin\left(\frac{2\pi}{a_0} h\right) + \frac{\lambda}{2} (\vec{\nabla} h)^2 + F + \eta \quad (7.10.13)$$

7.11 Quenched Noise

For a class of interface phenomena, one does not have the growth of an interface by a deposition process. Instead one has an interface moving in a disordered medium. The resistance of the medium against the flow is different from point to point. This noise is called quenched noise because it does not depend on time. Partial wetting of a paper towel and the spreading of the fluid interface is a good example. Fluid pressure and capillary force drive the interface while disorder in the medium slow down the propagation. In the ensuing competition if disorder dominates then the interface becomes pinned. If the driving force dominates then the interface is free to wander - it is depinned. The transition from a pinned to a depinned state by changing the driving force is called a depinning transition.

A critical example of a depinning transition involves a particle on a rough plane which is acted on by a force F as shown. If the friction coefficient of the plane is μ , then the maximum friction force that the plane can produce is μmg where m is the mass of the particle. For the particle to move the external force F has to exceed the critical value $F_c = \mu mg$. Once the particle starts moving there is a dynamic function which is proportional to the velocity and the equation of motion can be written as

$$m \frac{dv}{dt} = F - F_c - kv \quad \text{for} \quad F > F_c \quad (7.11.1)$$

For $F < F_c$, $v=0$. The solution of Eq.(7.11.1) clearly, shows that after a while (lying out of transients), the motion will be one of uniform velocity given by

$$v = \frac{F - F_c}{k} = \frac{f}{k} \quad (7.11.2)$$

where f is the excess force above the threshold. What happens at $F = F_c$ is an example of depinning transition. Near the threshold, the velocity is determined by the excess force and one can in general write

$$v \propto f^\theta \quad (7.11.3)$$

where θ is the exponent for the depinning transition.

A more general situation is that of an interface in a porous medium as shown in Fig. 7.8. A driving force F acts normal to the flat configuration of the interface.

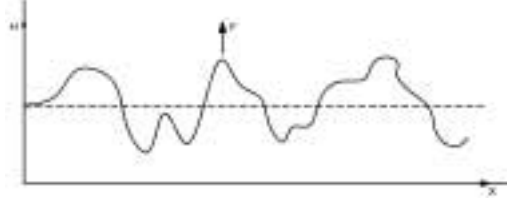


Figure 7.8. Interface in a porous medium

Instead of the single particle of the previous paragraph, we now have a spatially extended string which has surface tension and hence tries to remain straight. The most general equation describing the motion of the driven interface is the KPZ type equation of motion

$$\frac{\partial h}{\partial t} = F + \nu \nabla^2 h + \frac{\lambda}{2} (\vec{\nabla} h)^2 h + \eta(\vec{x}, h) \quad (7.11.4)$$

The big difference with Eq.(7.3.3) is in the noise term. This noise is independent of time. Its correlation is given by

$$\langle \eta(\vec{x}, h) \eta(\vec{x}', h') \rangle = \delta^D(\vec{x} - \vec{x}') R(h - h') \quad (7.11.5)$$

where R is some function of $h - h'$. The temporal evolution of Eq.(7.12.4) is deterministic.

If $F = 0$, then the interface fluctuations find the closest configuration which gives a local minimum for the energy and stays pinned there. A small value of F tends to give the interface a velocity but it is eventually pinned. At a critical value F_c of F , the interface acquires a finite velocity and for $F \geq F_c$, the scaling law of Eq.(7.11.3) is observed.

A dimensional analysis of Eq.(7.11.4) provides information about the length scale over which the interface is pinned. If ' l ' is the length of the pinned domain as shown in Fig. 7.9, then dropping the nonlinear term, the balance of Eq.(7.11.4) gives

$$F + \nu h/L^2 - [R(0)]^{1/2} L^{-D/2} = 0 \quad (7.11.6)$$

where we have assumed $R(h - h')$ is dominant for $|h - h'| < h_0$ and zero otherwise. If the drive is weak, then the surface is pinned if the 'surface tension' term is weaker than the pinning action of the quenched noise, i.e

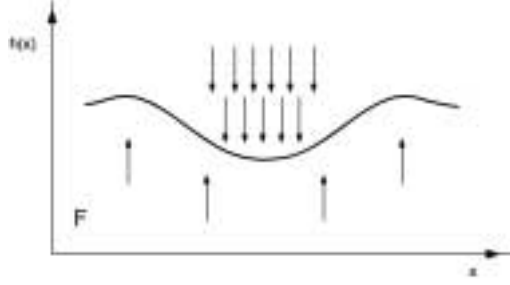


Figure 7.9. Pinning of an interface

$$\frac{vh_0}{L^2} < \frac{[R(0)]^{1/2}}{L^{D/2}}$$

This gives a characteristic scale L_c given by

$$L_c \sim \left[\frac{v^2 h_0^2}{R(0)} \right]^{\frac{1}{4-D}} \quad (7.11.7)$$

For $D \leq 4$ L_c is a critical length below which the surface is smoothed by the surface tension and above which it becomes free to wave. The maximum pinning force F_c can be found by equating the driving force to the pinning force and hence $F_c \sim [R(0)]^{1/2}/L_c^{D/2}$ and using L_c from Eq.(7.11.7)

$$F_c \sim [R(0)]^{\frac{2}{4-D}} (h_0 v)^{-\frac{D}{4-D}} \quad (7.11.8)$$

For $F > F_c$, the interface moves with a steady velocity. One can define a correlation length ξ over which a domain remains pinned. This length diverges as one reaches F_c from above. This is characterized by

$$\xi \sim (F - F_c)^{-\nu} \quad (7.11.9)$$

For $F > F_c$, we have four exponents, the roughness exponent α , the dynamic scaling exponent z , the correlation length exponent ν and the velocity exponent θ . The velocity can be written as

$$v \sim \frac{h}{t} \sim \frac{\xi^\alpha}{\xi^z} \sim \xi^{\alpha-z} \sim (F - F_c)^{-\nu(\alpha-z)} \quad (7.11.10)$$

which is to be compared with Eq.(7.11.3). This gives

$$s\theta = \nu(z - \alpha) \quad (7.11.11)$$

This scaling result reduces the number of independent exponents to three. All these considerations hold in the zero temperature limit, when the thermal noise is negligible. In the presence of the thermal noise, the transition from an unpinned to a pinned surface is somewhat rounded.

7.12 Coupled Growth Models

So far in this chapter, we have been considering a single variable - the height of the interface $h(\vec{x}, t)$ - to characterize the dynamics of the interface. Sometimes it may so happen that this field may couple to another fluctuating field (in our discussion of critical phenomena this occurred when the order parameter for the liquid-gas transition coupled to the auxiliary field - the velocity). For a fluid displacement in a porous medium this secondary field could be the local density or pressure of the fluid. In such situations we have a system of two coupled fields $h_0(\vec{r}, t)$ and $h_1(\vec{r}, t)$. A KPZ like coupled field system is

$$\begin{aligned}\frac{\partial h_0}{\partial t} &= v_0 \nabla^2 h_0 + \lambda_0 (\vec{\nabla} h_0)^2 + \gamma_0 (\vec{\nabla} h_0) \cdot (\vec{\nabla} h_1) + \phi_0 (\vec{\nabla} h_1)^2 + \eta_0 \\ \frac{\partial h_1}{\partial t} &= v_1 \nabla^2 h_1 + \lambda_1 (\vec{\nabla} h_1)^2 + \gamma_1 (\vec{\nabla} h_0) \cdot (\vec{\nabla} h_1) + \phi_1 (\vec{\nabla} h_0)^2 + \eta_1\end{aligned}\quad (7.12.1)$$

There are four scaling exponents characterizing the growth α_0 and α_1 the two roughness exponents and z_0 and z_1 - the two time scale exponents. Strong dynamic scaling implies a single time scale $z_0 = z_1$. Weak scaling means $z_0 \neq z_1$.

A simpler system

$$\begin{aligned}\frac{\partial h_0}{\partial t} &= v_0 \nabla^2 h_0 + \lambda_0 (\vec{\nabla} h_0)^2 + \phi_0 (\vec{\nabla} h_1)^2 + \eta_0 \\ \frac{\partial h_1}{\partial t} &= v_1 \nabla^2 h_1 + \lambda_1 (\vec{\nabla} h_1)^2 + \gamma_1 (\vec{\nabla} h_0) \cdot (\vec{\nabla} h_1) + \eta_1\end{aligned}\quad (7.12.2)$$

where the up down symmetry is preserved in the h_1 equation was proposed by *Ertas and Kardar* to describe the motion of a flux line in a three dimensional superconductor, driven in preferred direction by an external field. The following results were obtained from a RG calculation :

- a) $\gamma_1, \phi_0 > 0, z=3/2, \alpha_0 = \alpha_1 = 1/2$
- b) $\phi_0 = 0, \alpha_1 = 3/4$ if $\gamma_1 > 0$. γ_1 scales to zero if initially negative.
- c) $\gamma_1 = 0$ RG and numerics disagree.
- d) $\gamma_1 < 0, \phi_0 > 0, \Gamma_1 \rightarrow 0$
- e) $\gamma_1 > 0, \phi_0 \geq 0, z_0 = z_1 = 3/4, \alpha_0 = 1/2, \alpha_1$ unknown.

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Turbulence

8.1 Description of the Turbulent State

Turbulence is generally considered to be some kind of a disordered flow. In the first section we will try to clarify as much as possible what sort of an irregular flow is going to be termed turbulent. It will be taken for granted that turbulence will be described by the Navier-Stokes equation. This equation is Newton's law for fluid motion and is written as

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} P + \nu \nabla^2 \vec{v} + \vec{f} \quad (8.1.1)$$

where $\vec{v}(\vec{r}, t)$ is the velocity field, $P(\vec{r}, t)$ is the pressure, ρ is the density, ν the kinematic viscosity and $\vec{f}$ is an external force (e.g. gravity, *Coriolis's* force for a rotating system...). This has to be supplemented by the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad (8.1.2)$$

and the equation of state

$$P = F(\rho, T) \quad (8.1.3)$$

T being the temperature. An important class of fluids are the incompressible ones for which $\rho(\vec{r}, t)$ can be taken to be constant and we find that Eq.(8.1.2) reduces to

$$\vec{\nabla} \cdot \vec{v} = 0 \quad (8.1.4)$$

which is the incompressibility condition. In an incompressible fluid the speed of sound would be infinity and hence incompressibility is a good assumption so long

as one is considering flow speeds which are much smaller than the speed of sound. Since ρ is constant for an incompressible flow we will henceforth set $\rho = 1$ and write the Navier-Stokes equation in the form

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = -\partial_\alpha P + \nu \nabla^2 v_\alpha + f_\alpha \quad (8.1.5)$$

$$\partial_\beta v_\beta = 0 \quad (8.1.6)$$

where the external force will be taken to be solenoidal i.e. $\partial_\beta f_\beta = 0$. The difficulty in solving Eq.(8.1.5) comes from the nonlinear term $v_\beta \partial_\beta v_\alpha$ - a term which is called the inertial force. To be able to solve for the velocity field from Eqns. (8.1.5) and (8.1.6) one needs to know the initial conditions and the boundary conditions. We first note that for the incompressible fluid, the pressure P is not an independent quantity. It is linked to the velocity field because of the relation

$$\nabla^2 P = \partial_\alpha \partial_\alpha P = -\partial_\alpha (v_\beta \partial_\beta v_\alpha) \quad (8.1.7)$$

which follows on taking a divergence of Eq.(8.1.5). Consequently, the Navier-Stokes equation for the incompressible fluid is a nonlocal equation. It is best to introduce the spatial Fourier transform at this point and write the nonlocal Navier-Stokes equation in the momentum space. We define the Fourier transform of the velocity field in D-dimensions as

$$v_\alpha(k, t) = \int d^D r e^{i\vec{k} \cdot \vec{r}} v_\alpha(\vec{r}, t) \quad (8.1.8)$$

and straightforward algebra allows us to express Eqns.(8.1.5) and (8.1.6) as

$$\partial_t v_\alpha(k) + \nu k^2 v_\alpha(k) = \sum_{\vec{p}} M_{\alpha\beta\gamma}(\vec{k}) v_\beta(\vec{p}) v_\gamma(\vec{k} - \vec{p}) + f_\alpha(k) \quad (8.1.9)$$

where

$$M_{\alpha\beta\gamma}(\vec{k}) = i[k_\beta P_{\alpha\beta}(k) + k_\gamma P_{\alpha\beta}(k)] \quad (8.1.10)$$

with the projection operator P given as

$$P_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \quad (8.1.11)$$

We now note that the solution to the Navier-Stokes equation (Eq.(8.1.5)) will be controlled mainly by the inertial term and the viscous force term, $\nu \nabla^2 \vec{v}$. For a characteristic length L and a characteristic velocity v_0 , the ratio of these two effects is given by the dimensionless number called *Reynold's* number R , which is

$$R = \frac{\text{Inertial force}}{\text{Viscous force}} = \frac{(\vec{v} \cdot \vec{\nabla}) \vec{v}}{\nu \nabla^2 \vec{v}} \simeq \frac{v_0^2/L}{\nu v_0/L^2} = \frac{v_0 L}{\nu} \quad (8.1.12)$$

For a flow where a characteristic length L exists (e.g. the flow about a sphere or cylinder or flow between two parallel plates), one can scale all distances by L , velocity by some characteristic speed v_0 and time by L/v_0 . In terms of the dimensionless variables $V_\alpha = v_\alpha/v_0$, $X_\alpha = x_\alpha/L$, and $\tau = t v_0/L$, we have the dimensionless form of the Navier-Stokes equation ($f_i = 0$ for convenience)

$$\partial_\tau V_\alpha + V_\beta \partial_\beta V_\alpha = \partial_\alpha \frac{\partial_\gamma}{\nabla^2} (V_\beta \partial_\beta V_\gamma) + \frac{1}{R} \nabla^2 V_\alpha \quad (8.1.13)$$

with boundary conditions expressed as $V_\alpha = \text{some dimensionless number}$ at $X_\beta = \text{some dimensionless number}$. The characteristic of the flow will be determined completely by R . It should be noted that this is what is exploited when a scale modelling of flow is carried out.

We now imagine a specific flow - a uniform flow field with velocity $U_0 \hat{x}$ in which is inserted an infinitely long cylinder of radius L with its axis along the z -axis. For a low *Reynold's* number, the flow occurs with streamlines following

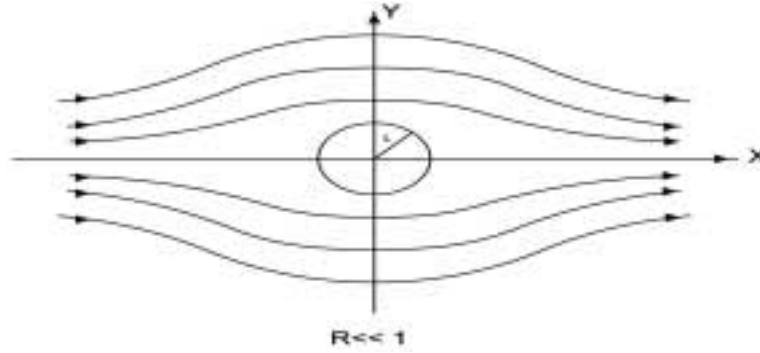


Figure 8.1. Flow past a static cylinder at low speeds

the symmetry of the Navier-Stokes equation and the symmetry dictated by the boundary conditions (e.g. up-down symmetry about the x -axis). It is a stationary flow and time translational invariance is trivially satisfied. As *Reynold's* number is increased, vortices are produced downstream. They are at first stationary, but above a critical *Reynold's* number become time dependent via a *Hopf* bifurcation. Instead of a continuous time translational invariance one has a discrete invariance. At higher values of R , vortices drift downstream, and at some R (around 100) the up-down symmetry about the x -axis is broken. The flow is very complicated and beyond some value of R definitely chaotic. At very high values of R , the flow appears random but there is a statistical restoration of symmetries - time translational invariance, up-down symmetry etc. are restored in a statistical sense. It is this extremely high *Reynold's* number flow, which is irregular but shows the symmetries associated with the Navier-Stokes equation in a statistical sense and

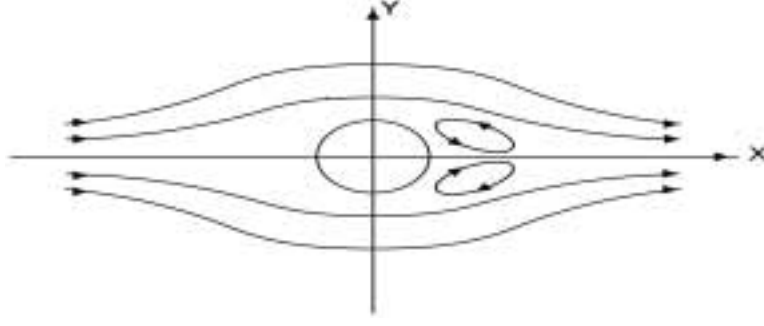


Figure 8.2. Flow past a static cylinder at high speeds

for which there is a finite amount of dissipation (no matter how high the Reynolds number) that we will call turbulent.

In this range of very high *Reynold's* numbers, where dissipation none-the-less is finite, we need to know what do the solutions of the Navier-Stokes equation look like. We first demonstrate that finite dissipation implies potential problems with the derivatives of the velocity field. If we call $\frac{1}{2} \int v^2 d^D r$ the total energy E of the fluid, then from Eq.(8.1.6) with $\vec{f}=0$, we have

$$\begin{aligned} \frac{\partial E}{\partial t} = & - \int v_\alpha v_\beta \partial_\beta v_\alpha d^D r - \int v_\alpha \partial_\alpha P d^D r \\ & + \int \nu v_\alpha \nabla^2 v_\alpha d^D r \end{aligned} \quad (8.1.14)$$

Now, because of the incompressibility,

$$v_\alpha \partial_\alpha P = \partial_\alpha (P v_\alpha) \quad (8.1.15)$$

and

$$v_\alpha v_\beta \partial_\beta v_\alpha = \partial_\beta \left(\frac{v^2}{2} v_\beta \right) \quad (8.1.16)$$

which makes the first two terms on the R.H.S of Eq.(8.1.14) surface terms and hence they vanish when the surfaces are far away (velocity field decaying at infinity) or when the surfaces are stationary ($\vec{v}=0$ on the surface). We thus have,

$$\frac{\partial E}{\partial t} = -\nu \sum_{\alpha, \beta} (\partial_\beta v_\alpha)^2 d^D r \quad (8.1.17)$$

which shows that energy gets dissipated. For $D \leq 3$, we can define a vorticity field $\vec{\omega}$ as

$$\vec{\omega} = \vec{\nabla} \times \vec{v} \quad (8.1.18)$$

and we can write Eq.(8.1.17) for an incompressible fluid as,

$$\frac{\partial E}{\partial t} = -\nu \int d^3r \omega^2 \quad (8.1.19)$$

As $R \rightarrow \infty$ at fixed L , one has $\nu \rightarrow 0$ and hence a finite dissipation in this limit implies that ω^2 has to blow up. Hence the derivatives of the velocity field are going to be ill defined and in the turbulent regime, one expects the velocity field to be non-smooth. The solutions for the velocity field are then thought of as distributions and the notion of weak solutions of Navier-Stokes equation was accordingly introduced. If ϕ_α and ψ are smooth functions with compact support and $\partial_\alpha \phi_\alpha = 0$, then $v_\alpha(\vec{r})$ will be called a weak solution of the Navier-Stokes equation if,

$$\int v_\alpha \partial_\alpha \psi d^3r dt = 0 \quad (8.1.20)$$

$$\int d^3r dt (v_\alpha \partial_t + v_\alpha v_\beta \partial_\beta + v_\alpha \nabla^2 + f_\alpha) \phi_\alpha = 0 \quad (8.1.21)$$

In $D = 3$, weak solutions of the Navier-Stokes equation are known to exist.

We now return to the issue of energy balance in the case of turbulence and including the external force $\vec{f}$, write Eq.(8.1.17) as

$$\frac{\partial E}{\partial t} = -\nu \int \omega^2 d^3r + \int f_\alpha v_\alpha d^3r \quad (8.1.22)$$

If the total energy in turbulence is to remain constant in time, then the viscous dissipation has to be balanced by the energy input $\int f_\alpha v_\alpha d^3r$. If ϵ is the rate at which energy is injected and dissipated, then

$$\epsilon = \nu \int \omega^2 d^3r = \int f_\alpha v_\alpha d^3r \quad (8.1.23)$$

The feeding in of energy obviously occurs at the boundaries, i.e. at large length scales L . The dissipation due to molecular viscosity occurs at short length scales s where the viscous drag dominates the advective (nonlinear) term. The balance leads to $s \sim \nu^{3/4} / \epsilon^{1/4}$. Between the scales L and s , the energy is transported by the nonlinear term in the Navier-Stokes equation. If L_n is the n^{th} scale with $l_n = L \xi^n$, $\xi < 1$, and the velocity at that scale is v_{L_n} , then the total energy is $v_{L_n}^2 V_n$ where V_n is the total volume occupied by the energy containing modes at the n^{th} stage. The typical turn over time for the eddy (mode) at the scale L_n is $\tau = L_n / v_{L_n}$ and hence the rate Π_n at which energy crosses the n^{th} stage is

$$\Pi_n = v_{L_n}^2 V_n / \tau_n = v_{L_n}^3 V_n / L_n \quad (8.1.24)$$

In the *Richardson-Kolmogorov* scenario for this transfer process, Π_n is independent of n and so is V_n . Consequently, $\epsilon = v_l^3 V / l$ at the scale l , or

$$v_l = (\bar{\epsilon} l)^{1/3} \quad (8.1.25)$$

where $\bar{\epsilon}$ is the rate at which the energy density is dissipated. The scaling law in Eq.(8.1.25) once again points towards a problem with the derivative at the small scales.

The above discussion shows that between the outermost scale L and the innermost scale s is a range of scales l where the energy transfer is thought to occur without any loss. This range is termed the inertial range and is the range in space where the scaling law of Eq.(8.1.25) is valid. The ratio of the large to small scale is given by

$$\frac{L}{s} = \frac{L}{v^{3/4}} \epsilon^{1/4} \propto \frac{L}{v^{3/4}} \left(\frac{v^3}{L} \right)^{1/4} = R^{3/4} \quad (8.1.26)$$

For large R , the ratio of the two length scales is big and hence there will be a significant inertial range. The scale L serves as an infrared cutoff in the problem and the scale s is an ultraviolet cutoff.

The picture of the velocity field that is beginning to emerge is that of random variable in time with non-smooth behaviour in space. Associated with the randomness is a distribution - one can think of averaging over time or one can envisage an ensemble of systems differing in the initial conditions. The averages over time and the ensemble averages will be equal due to ergodicity. In the inertial range, the distribution function is universal. This is known through careful experiments, where the velocity at a particular point of space is measured as a function of time to generate a long time series. In Fig. 8.3, we show two parts of the time series over the same interval of time. The time series from $t = 1000$ to 5000 does not seem to bear any correlation to the time series between $t = 11000$ to 15000. However, if one makes a histogram about how many times a particular velocity is repeated then one finds that the two histograms are almost identical, demonstrating while the exact times series differ, the probability distribution remains the same (Fig. 8.4).

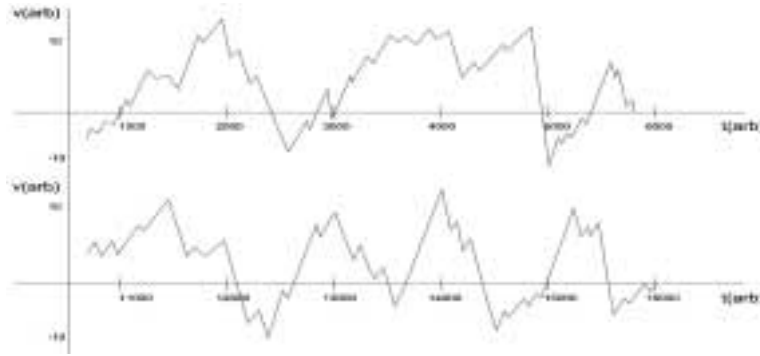


Figure 8.3. Time series for the velocity at a point

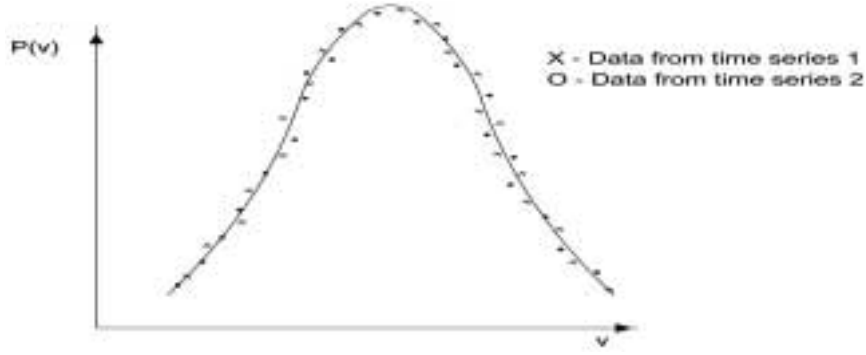


Figure 8.4. Universality of the probability distribution

The probability distribution in the inertial range does not depend upon the way energy is injected into the system or the way it is dissipated.

The central problem in the theory of turbulence is to start from the Navier-Stokes equation and obtain the distribution function $P(v(r, t))$. If not the probability distribution function, the moments of the velocity field need to be calculated. There is plausability argument for the existence of a distribution function. This is a very crude way of looking at the problem but gives some feeling for what is involved. We work at a given spatial point and discretize the Navier-Stokes equation in time. The structure is

$$v_{t+1} - v_t = A v_t - B v_t^2 \quad (8.1.27)$$

or after appropriate scalings

$$u_{t+1} = r u_t (1 - u_t) \quad (8.1.28)$$

The series generated from an initial condition u_0 remains bounded is $r \leq 4$. Since the velocity is bounded we require the u -variable in the discretized problem to be bounded as well. We will analyze Eq.(8.1.28) for the limiting case of $r = 4$. In this case the substitution

$$u_t = \sin^2(\theta_t) \quad (8.1.29)$$

leads to

$$\sin^2(\theta_{t+1}) = 4 \sin^2(2\theta_t) \quad (8.1.30)$$

which means

$$\begin{aligned} \theta_{t+1} &= 2\theta_t && \text{if } \theta_t \leq \pi/4 \\ \theta_{t+1} &= 2\left(\frac{\pi}{2} - \theta_t\right) && \text{if } \pi/4 \leq \theta_t \leq \pi/2 \end{aligned} \quad (8.1.31)$$

Calling $\theta_t = \frac{\pi}{2} X_t$, we have

$$\begin{aligned} X_{t+1} &= 2X_t & \text{if } 0 \leq X_t \leq 1/2 \\ X_{t+1} &= 2(1 - X_t) & \text{if } 1/2 \leq X_t \leq 1 \end{aligned} \quad (8.1.32)$$

which is the well known tent-map. We write this as $X_{t+1} = f(X_t)$ and if the starting point is X_0 , the sequence that is generated is $f(X_0), f^{(2)}(X_0), f^{(3)}(X_0), \dots, f^{(N)}(X_0), \dots$ where $f^{(N)}(X) = f(f(f(\dots N \text{ times}(X))))$. The plot of the successive iterates is shown in Fig. 8.5 It is clear from the above figure that if

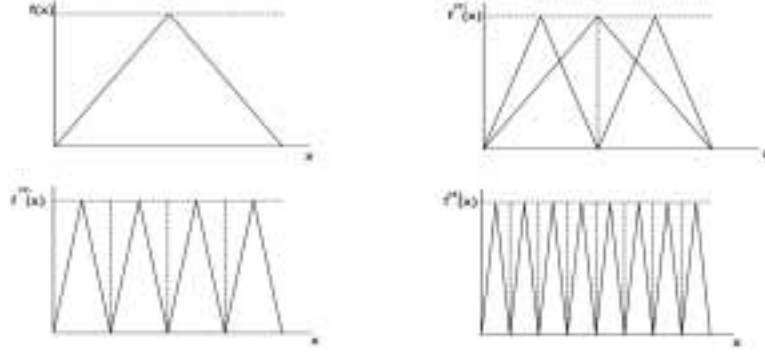


Figure 8.5. Iterations of the tent map

the difference in initial conditions is $\frac{1}{2^N}$, the difference in the iterates after N -steps is 1. This is the sensitive dependence on initial conditions, which is the signature of chaos. The usual picture for chaotic evolution, which quantifies the sensitivity to initial conditions, is the existence of a positive *Lyapunov* exponent λ . The *Lyapunov* exponent is defined as

$$\lambda = \lim_{N \rightarrow \infty, \epsilon \rightarrow 0} \frac{1}{N} \ln \left| \frac{f^{(N)}(X_0 + \epsilon) - f^{(N)}(X_0)}{\epsilon} \right| \quad (8.1.33)$$

From the above discussion it should be clear that

$$\lambda = \ln 2 \quad (8.1.34)$$

for the above system.

A look at the iterates for the tent map shows that the sequence $f^{(N)}(X_0)$ fills the unit interval completely with equal measure. We can define an invariant density for the iterates as

$$\rho(X) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta(X - f^{(n)}(X_0)) \quad (8.1.35)$$

the δ -function implying that we count one every time $f^{(N)}(X_0) = X$. For the tent map $\rho(X) = \text{constant}$. The relation between X and V leads to a function for V . The above argument makes it plausible why temporally the velocity field is going to show sensitive dependence on initial conditions which makes a statistical description possible.

We summarize the basic features of the turbulent state as follows:

- i) It is the state of flow as $R \rightarrow \infty$.
- ii) The velocity field is random in time and hence a probabilistic description with an ensemble consisting of systems with differing initial conditions is appropriate.
- iii) The spatial dependence of the velocity field is non-smooth in the sense that derivatives may be singular.
- iv) There are two basic scales L and s . The large scale L is the scale at which the system receives the energy from outside agencies and the small scale s is the scale at which molecular diffusion becomes dominant. The ratio $L/s = R^{3/4}$ and hence for $R \rightarrow \infty$ the scales are well separated.
- v) The intermediate scales $L \gg l \gg s$ constitute the inertial range and in this range universal features are obtained since things are independent of both the feeding and dissipating mechanisms.

8.2 Kolmogorov Phenomenology

Kolmogorov phenomenology has to do with the short distance behaviour of the correlation functions in the inertial range. We have already seen that assuming a lossless energy transfer through the inertial range and the fact that energy containing eddies fill the entire volume, one has the result that $v_l = \bar{\epsilon}^{1/3} l^{1/3}$, where v_l is the velocity at the length scale l . The extraction of short distance behaviour of correlation functions in the inertial range $s \ll l \ll L$, involves first taking $\nu \rightarrow 0$ (i.e. $s \rightarrow 0$) at fixed positions of the fields and then taking a short distance limit with L fixed. For a n -point correlation function this means

$$\lim_{\lambda \rightarrow 0, \nu \rightarrow 0} \lambda^{-\xi_n} \langle u(\lambda x_1, t) u(\lambda x_2, t) \dots u(\lambda x_n, t) \rangle = \text{finite} \quad (8.2.1)$$

Since in the inviscid limit, there are no other scales except the positions and L , taking the last short distance limit is the same as $L \rightarrow \infty$. *Kolmogorov* assumed that for Galilean invariant correlation functions i.e. correlation functions formed from the differences $v_i(\vec{x}) - v_i(\vec{0})$, the $L \rightarrow \infty$ limit exists. Since they are to be determined from ϵ and x alone, the equal time correlation function

$$S_n(r) = \langle \{[\vec{v}(\vec{x}) - \vec{v}(\vec{0})] \cdot \hat{x}\}^n \rangle \quad (8.2.2)$$

follows from dimensional analysis and

$$S_n(r) = C_n (\bar{\epsilon} r)^{n/3} \quad (8.2.3)$$

for small r . In general since in the inviscid limit, there are three quantities $\bar{\epsilon}$, r and L available, we expect on dimensional grounds,

$$S_n(r) = (\bar{\epsilon} r)^{n/3} F_n(r/L) \quad (8.2.4)$$

Kolmogorov assumption is that as $L \rightarrow \infty$, $F_n(r/L)$ tends to C_n which are universal constants. For the energy spectrum, the above statements lead to the usual 5/3 law. This can be seen by writing the total energy as,

$$\begin{aligned} E &= \frac{1}{2} \int d^3r \langle v(r)^2 \rangle = \frac{1}{2} \int d^3k \langle v(k)v(-k) \rangle \\ &= \int E(k) dk \end{aligned} \quad (8.2.5)$$

since $\langle v(r)^2 \rangle \sim r^{2/3}$, the fourier transform $\langle v(k)v(-k) \rangle$ goes as $k^{-3-2/3}$ and from Eq.(8.2.5) it follows that $E(k) \sim k^{-5/3}$. For $n=3$, one has the only exact result in three dimensional turbulence, namely

$$S_3(r) = -\frac{4}{5} \bar{\epsilon} r \quad (8.2.6)$$

For $n \geq 4$, derivatives from the *Kolmogorov* result begins to show up and the correct phenomenology is

$$S_n(r) = A_n (\bar{\epsilon} r)^{n/3} \left(\frac{r}{L}\right)^{\zeta_n - n/3} \quad (8.2.7)$$

as $L \rightarrow \infty$. Experiments show that $\zeta_n < n/3$. Consequently, the difference of $S_n(r)$ from the Gaussian value $\zeta_n = n/3$ increases as one to shorter and shorter scales. Hence the probability to have large velocity difference increases at short scales. This is the phenomenon of *intermittency*.

The phenomenology is different in $D=2$. The vorticity vector $\vec{\omega}$ in $D=3$ obeys the equation of motion

$$\frac{d\vec{\omega}}{dt} + (\vec{v} \cdot \vec{\nabla}) \vec{\omega} - (\vec{\omega} \cdot \vec{\nabla}) \vec{v} = \nu \nabla^2 \vec{\omega} + \vec{\nabla} \times \vec{f} \quad (8.2.8)$$

In $D=2$, $(\vec{\omega} \cdot \vec{\nabla}) \vec{v} = 0$ since $\vec{\omega}$ happens to be in the z -direction and hence is not a two dimensional vector. Consequently,

$$\partial_t \omega + (\vec{v} \cdot \vec{\nabla}) \omega = \nu \nabla^2 \omega + \nabla \times f \text{ forcing term} \quad (8.2.9)$$

This means that in the absence of forcing $\int \frac{\omega^2}{2} d^2r$ is conserved in the inviscid limit, i.e.

$$\frac{\partial}{\partial t} \int \frac{1}{2} \omega^2 d^2r = -\nu \int (\vec{\nabla} \omega)^2 d^2r \quad (8.2.10)$$

This quantity is known as the enstrophy. The fact that in the inviscid limit ω^2 cannot change means that it remains a constant as $\nu \rightarrow 0$. The total energy is then not dissipated at short scales. It is the enstrophy which gets dissipated and hence there is an enstrophy flux from larger to shorter scales. We have an enstrophy flux $\bar{\epsilon}_s$ given by,

$$\lim_{\nu \rightarrow 0} \nu \langle (\vec{\nabla} \omega)^2 \rangle = \bar{\epsilon}_s \quad (8.2.11)$$

In the inviscid limit, $\lim_{\nu \rightarrow 0} \nu \langle (\vec{\nabla} v)^2 \rangle = 0$ and hence the energy balance equation in $D = 2$ is

$$\frac{d}{dt} \langle \frac{v^2}{2} \rangle = \bar{\epsilon} \quad (8.2.12)$$

So $\langle \frac{v^2}{2} \rangle = \bar{\epsilon} t$ upto a constant and all energy injected into the system is transferred to the fluid. Steady state is not reached as a consequence of this. This energy is transferred to the larger scales which is necessary to keep $\partial_t \int E(k) dk = 0$ while $\partial_t \int k^2 E(k) dk < 0$ due to transfer of enstrophy to smaller scales. The picture in $D = 2$ is that there are two cascades - a forward enstrophy cascade and a backward energy cascade. The enstrophy flux $\bar{\epsilon}_s$ has the dimensions of T^{-3} and hence the energy spectrum $E(k)$ has the form,

$$E(k) = \bar{\epsilon}_s^{2/3} k^{-3} \quad (8.2.13)$$

In D -dimensions, the relation between the velocity correlation function and $E(k)$ is

$$C(k) = \langle v(k)v(-k) \rangle = \frac{E(k)}{k^{D-1}} \quad (8.2.14)$$

and hence in $D = 2$, $C(k) \sim k^{-4}$. This leads to

$$\begin{aligned} \langle \delta v(r)^2 \rangle &= \int \frac{d^2k}{(2\pi)^2} C(k) (e^{i\vec{k} \cdot \vec{r}} - 1) \\ &\sim \int \frac{d^2k}{(2\pi)^2} \frac{(e^{i\vec{k} \cdot \vec{r}} - 1)}{k^4} \\ &\sim r^2 \ln r \end{aligned} \quad (8.2.15)$$

A generalization of the cascade argument given in Eq.(8.1.25), would imply $\delta v(r) \sim r$ which gives a smoother velocity field than implied by Eq.(8.2.15). The occurrence of logarithmic corrections is typical of $D = 2$ and has been the subject of a variety of analysis.

8.3 The Correlation Function

We begin with the two point function

$$C_{ij}(r) = \left\langle [v_i(\vec{x} + \vec{r}) - v_i(\vec{x})][v_j(\vec{x} + \vec{r}) - v_j(\vec{x})] \right\rangle \quad (8.3.1)$$

Isotropy implies that C_{ij} cannot depend on any particular direction in space and so C_{ij} has to be formed from $\hat{n}$, the unit vector in the direction of $\vec{r}$. Thus,

$$C_{ij}(r) = A(r)\delta_{ij} + B(r)n_i n_j \quad (8.3.2)$$

Let us choose the coordinate axes along $\vec{r}$. We call the velocity component along $\vec{r}$ as v_r and the transverse ones v_t . Clearly,

$$\begin{aligned} C_{rr} &= A + B \\ C_{tt} &= A \\ C_{rt} &= 0 \end{aligned} \quad (8.3.3)$$

We can write Eq.(8.3.1) as

$$C_{ij}(r) = \langle 2[v_i(\vec{x})v_j(\vec{x})] - 2\langle v_i(\vec{x} + \vec{r})v_j(\vec{x}) \rangle \rangle \quad (8.3.4)$$

Using the incompressibility conditions and differentiating w.r.t r_i ,

$$\frac{\partial C_{ij}}{\partial r_i} = 0 \quad (8.3.5)$$

Noting that

$$\frac{\partial r}{\partial r_i} = \frac{r_i}{r}$$

and that

$$n_i = \frac{r_i}{r},$$

we have by differentiating Eq.(8.3.2)

$$0 = \frac{\partial C_{ij}(r)}{\partial r_i} = A' \frac{n_i}{r} \delta_{ij} + B' \frac{n_i}{r} n_i n_j + B \left(4 \frac{n_j}{r^2} - \frac{2n_i n_j}{r^3} \frac{n_i}{r} \right) \quad (8.3.6)$$

leading to

$$A' + B' + \frac{2B}{r} = 0 \quad (8.3.7)$$

It follows that

$$\begin{aligned} rC'_{rr} &= 2(C_{tt} - C_{rr}) \\ \text{or} \quad C_{tt} &= \frac{1}{2r} \frac{d}{dr} (r^2 C_{rr}) \end{aligned} \quad (8.3.8)$$

In the inertial range, where $C_{ij} \sim r^{2/3}$

$$C_{tt} = \frac{4}{3} C_{rr} \quad (8.3.9)$$

For the dissipative range, the velocity difference is an analytic function of r and we expect that for small r , $\Delta v_r \propto r$, which implies $C_{ij} \sim r^2$. consequently,

$$C_{tt} = 2C_{rr} \quad (8.3.10)$$

in the dissipative range. From Eq.(8.3.4)

$$\begin{aligned} \langle v_i(\vec{x} + \vec{r}) v_j(\vec{x}) \rangle &= \langle v_i(\vec{x}) v_j(\vec{x}) \rangle - \frac{1}{2} C_{ij} \\ &= \langle v_i(\vec{x}) v_j(\vec{x}) \rangle - \frac{1}{2} A(r) \delta_{ij} - \frac{1}{2} B(r) n_i n_j \\ &= \langle v_i(\vec{x}) v_j(\vec{x}) \rangle + B(r) \delta_{ij} - \frac{1}{2} B(r) n_i n_j \\ &= \frac{v^2}{3} \delta_{ij} + B(r) \delta_{ij} - \frac{1}{2} B(r) n_i n_j \end{aligned} \quad (8.3.11)$$

If we write $B(r) = ar^2$, then differentiating

$$\left\langle \frac{\partial v_i(\vec{x}_1)}{\partial X_{1j}} \frac{\partial v_j(\vec{x}_2)}{\partial X_{2i}} \right\rangle = 15a \left\langle \frac{\partial v_i(\vec{x}_1)}{\partial X_{1l}} \frac{\partial v_l(\vec{x}_2)}{\partial X_{2i}} \right\rangle = 0 \quad (8.3.12)$$

If we set $\vec{r} = \vec{X}_1 - \vec{X}_2 = 0$, then

$$\begin{aligned} \left\langle \left(\frac{\partial v_i}{\partial X_j} \right)^2 \right\rangle &= 15a \\ \text{and} \quad \left\langle \frac{\partial v_i}{\partial X_j} \frac{\partial v_j}{\partial X_i} \right\rangle &= 0 \end{aligned} \quad (8.3.13)$$

For the mean energy dissipation,

$$\bar{\epsilon} = \frac{1}{2} v \left\langle \left(\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \right)^2 \right\rangle = 15av \quad (8.3.14)$$

leading to $a = \frac{\vec{\epsilon}}{15\nu}$. Consequently, in the dissipation range, we have

$$\begin{aligned} C_{tt} &= \frac{2}{15} \frac{\vec{\epsilon}}{\nu} r^2 \\ \text{and} \quad C_{rr} &= \frac{1}{15} \frac{\vec{\epsilon}}{\nu} r^2 \end{aligned} \quad (8.3.15)$$

We now turn to the three point function

$$\begin{aligned} C_{ijk}(r) &= \langle [v_i(\vec{x} + \vec{r}) - v_i(\vec{x})][v_j(\vec{x} + \vec{r}) - v_j(\vec{x})] \\ &\quad \times [v_k(\vec{x} + \vec{r}) - v_k(\vec{x})] \rangle \end{aligned} \quad (8.3.16)$$

First, we focus on $\langle v_i(\vec{x})v_j(\vec{x})v_k(\vec{x} + \vec{r}) \rangle$ and note that in the homogeneous, isotropic situation this rank three tensor has to be constructed from δ_{ij} and n_i , the unit vector in the direction of $\vec{r}$. The form is

$$\begin{aligned} \langle v_i(\vec{x})v_j(\vec{x})v_k(\vec{x} + \vec{r}) \rangle &= C(r)\delta_{ij}n_k + D(r)[\delta_{ik}n_j + \delta_{jk}n_i] \\ &\quad + F(r)n_in_jn_k \end{aligned} \quad (8.3.17)$$

The divergence free condition implies that

$$\frac{\partial}{\partial r_k} \langle v_i(\vec{x})v_j(\vec{x})v_k(\vec{x} + \vec{r}) \rangle = 0 \quad (8.3.18)$$

This derivative leads to two kinds of terms, those which are coefficients of δ_{ij} and those which are coefficients of n_in_j . The coefficient of δ_{ij} leads to

$$C' + 2\frac{C + D}{r} = 0 \quad (8.3.19)$$

The coefficient of n_in_j in conjunction with Eq.(8.3.19) leads to

$$\frac{d}{dr} [r^2(3C + 2D + F)] = 0 \quad (8.3.20)$$

Integrating

$$3C + 2D + F = \frac{\text{constant}}{r^2}$$

If we want a solution which remains finite at $r = 0$, then

$$3C + 2D + F = 0 \quad (8.3.21)$$

leading to

$$F = rC' - C \quad (8.3.22)$$

We now expand C_{ijk} as the sum of 8 terms and note that two of them vanish since $\langle v_i(\vec{x})v_j(\vec{x})v_k(\vec{x}) \rangle$ cannot be expressed in terms of δ_{ij} . Further, $\langle v_i(\vec{x})v_j(\vec{x})v_k(\vec{x} + \vec{r}) \rangle$ is the negative of $\langle v_i(\vec{x} + \vec{r})v_j(\vec{x} + \vec{r})v_k(\vec{x}) \rangle$ and consequently,

$$\begin{aligned} C_{ijk} &= 2[\langle v_i(\vec{x})v_j(\vec{x})v_k(\vec{x} + \vec{r}) \rangle + \langle v_i(\vec{x})v_k(\vec{x})v_j(\vec{x} + \vec{r}) \rangle \\ &\quad + \langle v_k(\vec{x})v_j(\vec{x})v_i(\vec{x} + \vec{r}) \rangle] \\ &= -2(rC' + C)(\delta_{ij}n_k + \delta_{ik}n_j + \delta_{jk}n_i) + 6(rC' - C)n_in_jn_k \end{aligned} \quad (8.3.23)$$

where use has been made of Eqns.(8.3.17), (8.3.21) and (8.3.12) in arriving at the last line. The different components of C_{ijk} can be written down from Eq.(8.3.23) as

$$\begin{aligned} C_{rrr} &= -12C \\ C_{rtt} &= -2(C + rC') \\ C_{irt} &= C_{itt} = 0 \\ C_{rtt} &= \frac{1}{6} \frac{d}{dr}(rC_{rrr}) \end{aligned} \quad (8.3.24)$$

To proceed further, we need to use the Navier-Stokes equation. We start with the two point function and take a time derivative to write

$$\begin{aligned} \partial_t \langle v_i(\vec{x})v_j(\vec{x} + \vec{r}) \rangle &= \langle \dot{v}_i(\vec{x})v_j(\vec{x} + \vec{r}) \rangle + \langle v_i(\vec{x})\dot{v}_j(\vec{x} + \vec{r}) \rangle \\ &= -\langle v_l \partial_l v_i(\vec{x})v_j(\vec{x} + \vec{r}) \rangle - \langle v_i(\vec{x})v_l(\vec{x} + \vec{r})\partial_l v_j(\vec{x} + \vec{r}) \rangle \\ &\quad + v \langle v_j(\vec{x} + \vec{r})\nabla^2 v_i(\vec{x}) \rangle + v \langle v_i(\vec{x})\nabla^2 v_j(\vec{x} + \vec{r}) \rangle \\ &\quad - \langle v_j(\vec{x} + \vec{r})\partial_i P(\vec{x}) \rangle - \langle v_i(\vec{x})\partial_j P(\vec{x} + \vec{r}) \rangle \end{aligned} \quad (8.3.25)$$

The term $\langle v_j(\vec{x} + \vec{r})\partial_i P(\vec{x}) \rangle$ can be written as $\partial_i \langle v_j(\vec{x} + \vec{r})P(\vec{x}) \rangle$. The quantities $\langle v_j(\vec{x} + \vec{r})P(\vec{x}) \rangle$ are the components of a divergence free vector and since there is no centro-symmetric divergence free vector which is finite at the origin, we infer $\langle v_j(\vec{x} + \vec{r})P(\vec{x}) \rangle = 0$. Using translational invariance, whenever needed, we can then write Eq.(8.3.25) as

$$\begin{aligned} \partial_t \langle v_i(\vec{x})v_j(\vec{x} + \vec{r}) \rangle &= -2\partial_l \langle v_i(\vec{x})v_l(\vec{x})v_j(\vec{x} + \vec{r}) \rangle \\ &\quad + 2v\nabla^2 \langle v_i(\vec{x})v_j(\vec{x} + \vec{r}) \rangle \end{aligned} \quad (8.3.26)$$

We now use

$$\langle v_i(\vec{x})v_j(\vec{x} + \vec{r}) \rangle = \frac{1}{3}v^2\delta_{ij} - \frac{1}{2}C_{ij} \quad (8.3.27)$$

and

$$\begin{aligned} \langle v_i(\vec{x})v_l(\vec{x})v_j(\vec{x} + \vec{r}) \rangle &= -\frac{1}{12}C_{rrr}\delta_{il}n_j + \frac{1}{12}\left(\frac{r}{2}C'_{rrr} + C_{rrr}\right)(\delta_{ij}n_l + \delta_{lj}n_i) \\ &\quad - \frac{1}{12}(rC'_{rrr} - C_{rrr})n_in_jn_l \end{aligned} \quad (8.3.28)$$

and work with $i = j = l = r$ (i.e. the longitudinal component) to write

$$-\frac{2}{3}\bar{\epsilon} - \frac{1}{2}C_{rr} = \frac{1}{6r^4} \frac{\partial}{\partial r} (r^4 C_{rrr}) - \frac{\nu}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial C_{rr}}{\partial r} \right) \quad (8.3.29)$$

where $\bar{\epsilon} = \partial_t (\frac{v^2}{2})$. For small values of r , We can safely drop C_{rr} with respect to $\bar{\epsilon}$ and we have the result,

$$C_{rrr} = -\frac{4}{5}\bar{\epsilon}r + 6\nu \frac{d}{dr} C_{rr} \quad (8.3.30)$$

Being in the inertial range is equivalent to setting $\nu = 0$ and then

$$C_{rrr} = -\frac{4}{5}\bar{\epsilon}r \quad (8.3.31)$$

which is the result that would be anticipated on the basis of *Kolmogorov's* picture (Eq.(8.2.3)), with $C_3 = -\frac{4}{5}$. This is the only exact inertial range result in three dimensional turbulence.

What if we were in the dissipation range? We anticipate that C_{rrr} would vanish faster than r for small r in the dissipation range and hence

$$C_{rrr} = \frac{1}{15} \frac{\bar{\epsilon}}{\nu} r^2 \quad (8.3.32)$$

for very small r . This is in accordance with Eq.(8.3.15) derived from a different standpoint.

We end this section by noting the relation between the Fourier transform of $\langle v_i(\vec{k})v_j(-\vec{k}) \rangle$ and $C_{ij}(r)$

$$\begin{aligned} C_{ij}(r) &= 2\langle v_j(x) \rangle^2 - 2\langle v_i(\vec{x})v_j(\vec{x} + \vec{r}) \rangle \\ &= 2 \int \frac{d^D k}{(2\pi)^D} \langle v_j(\vec{k})v_j(-\vec{k}) \rangle - 2 \int \frac{d^D k}{(2\pi)^D} e^{i\vec{k} \cdot \vec{x}} \langle v_j(\vec{k})v_j(-\vec{k}) \rangle \\ &= 2 \int \frac{d^D k}{(2\pi)^D} (1 - e^{i\vec{k} \cdot \vec{x}}) \langle v_j(\vec{k})v_j(-\vec{k}) \rangle \end{aligned} \quad (8.3.33)$$

If $C_{ij} \propto r^\alpha$, then the dimensionally consistent $\langle v_j(\vec{k})v_j(-\vec{k}) \rangle$ is proportional to $k^{-\alpha-D}$. If $C_{ij}(r) \sim r^{2/3}$ as in the inertial range according to *Kolmogorov*, $\langle v_j(\vec{k})v_j(-\vec{k}) \rangle \sim k^{-11/3}$. The resulting integral in Eq.(8.3.33) is convergent in both the upper and lower ends of k . On the other hand if $C_{ij}(r) \sim r^2$ as is true in the dissipative range, then $\langle v_j(\vec{k})v_j(-\vec{k}) \rangle$ would be proportional to k^{-5} from a dimensional argument. However, it is clear from Eq.(8.3.33) that $\langle v_j(\vec{k})v_j(-\vec{k}) \rangle \sim k^{-5}$ does not give a convergent integral at the small k limit. This gives the indication

that the momentum space correlation is scale dependent and we bring in a scale k_0 to write in the dissipation range

$$\langle v_j(\vec{k}) v_j(-\vec{k}) \rangle = \frac{C_0}{k^n} e^{-k/k_0} \quad (8.3.34)$$

with $n \leq 5$. For $n = 1$, the Fourier transform gives

$$\begin{aligned} C_{jj}(r) &= \frac{8\pi}{(2\pi)^3} \frac{C_0 k_0^2 r^2}{1 + k_0^2 r^2} \\ &\simeq \frac{c_0}{\pi^2} k_0^2 r^2 \end{aligned} \quad (8.3.35)$$

when $k_0 r \ll 1$. If we set k_0 equal to the dissipation scale s^{-1} then

$$k_0 = \frac{\bar{\epsilon}^{1/4}}{\nu^{3/4}}$$

and comparing with Eq(8.3.32)

$$C_0 = \frac{\bar{\epsilon}}{15\nu} \pi^2 \frac{\nu^3/2}{\bar{\epsilon}^{1/2}} = \frac{\pi^2}{15} (\nu \bar{\epsilon})^{1/2} \quad (8.3.36)$$

8.4 Randomly Stirred Model

It is customary to split the velocity field into a mean part and a fluctuating part as

$$\vec{v} = \vec{u} + \vec{V} \quad (8.4.1)$$

where $\langle \vec{v} \rangle = V$ and $\vec{u}$ is the fluctuation. Inserting this in the Navier-Stokes equation (no external force) and taking an average we have

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V} = -\vec{\nabla} P_0 + \nu \nabla^2 \vec{V} - \langle (\vec{u} \cdot \vec{\nabla}) \vec{u} \rangle \quad (8.4.2)$$

where P_0 is the average part of the pressure term. Subtracting Eq(8.4.2) from the full Navier-Stokes equation leads to

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} &= -\vec{\nabla} P + \nu \nabla^2 \vec{u} - (\vec{V} \cdot \vec{\nabla}) \vec{u} \\ &\quad - (\vec{u} \cdot \vec{\nabla}) \vec{u} + \langle (\vec{u} \cdot \vec{\nabla}) \vec{u} \rangle \end{aligned} \quad (8.4.3)$$

The mean flow as expressed in Eq.(8.4.2) is affected by the fluctuating flow through the term $\langle (\vec{u} \cdot \vec{\nabla}) \vec{u} \rangle$. One of the standard techniques of obtaining the mean flow is to make an Ansatz about $\langle (\vec{u} \cdot \vec{\nabla}) \vec{u} \rangle$, e.g *Prandtl's* mixing length hypothesis, and

solving for $\vec{V}$ from Eq.(8.4.2). It is then possible to make a consistency check from Eq.(8.4.3). Our interest, here, however is to note that Eq.(8.4.3) can be written as

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\vec{\nabla} P + \nu \nabla^2 \vec{u} + \vec{f} \quad (8.4.4)$$

where $\vec{f}$ is a force which expresses the interaction between the fluctuating and the mean flow. There is a definite expression for this force as can be seen from Eq.(8.4.3), however, one can take the view that since f is virtually impossible to know from first principles we can replace it by a stochastic force (considering that the velocity field u is random) with a prescribed correlation. We thus write the correlator of f as

$$\langle f_i(\vec{x}, t) f_j(\vec{y}, t) \rangle = C_{ij}^{(L)}(\vec{x} - \vec{y}) \delta(t - s) \quad (8.4.5)$$

The delta function expresses the fact that the noise is temporally white. The spatial function $C_{ij}^{(L)}(\vec{x} - \vec{y})$ must be such that it must correctly portray the injection of energy into the length scale L .

We want to explore in this model the consequences of stationarity, i.e. the fact that the correlation function $\langle u_i(\vec{x}, t) u_j(\vec{y}, t) \rangle$ is independent of time. To do this we need to set the derivative of $\langle u_i(\vec{x}, t) u_j(\vec{y}, t) \rangle$ equal to zero. We note that

$$\begin{aligned} u_i(\vec{x}, t + \delta t) &= u_i(\vec{x}, t) + \int_t^{t+\delta t} (-u_l \partial_l u_i(\vec{x}, s) - \partial_i P + \nu \nabla^2 u_i) ds \\ &\quad + \int_t^{t+\delta t} f_i(\vec{x}, s) ds \\ &= u_i(t) + [-u_l \partial_l u_i(\vec{x}, t) - \partial_i P + \nu \nabla^2 u_i] \delta t \\ &\quad + \int_t^{t+\delta t} f_i(\vec{x}, s) ds + O((\delta t)^2) \end{aligned} \quad (8.4.6)$$

We note that f_i is $O((\delta t)^{-1/2})$ because of Eq.(8.4.5). The stationarity condition then takes the form (upto $O(\delta t)$)

$$\begin{aligned} 0 &= -\langle u_l(\vec{x}, t) \partial_l u_i(\vec{x}, t) u_i(\vec{y}, t) \rangle - \langle \partial_i P(\vec{x}, t) u_i(\vec{x}, t) \rangle \\ &\quad + \nu \langle u_i(\vec{y}, t) \nabla^2 u_i(\vec{x}, t) \rangle + (x \leftrightarrow y) \\ &\quad + 2 \langle u_i(t) \int_t^{t+\delta t} f_i(s) ds \rangle + \frac{1}{\delta t} \int_t^{t+\delta t} \langle f_i(\vec{x}, s) f_i(\vec{y}, s') \rangle ds ds' \end{aligned} \quad (8.4.7)$$

Now

$$\begin{aligned} \frac{1}{\delta t} \int_t^{t+\delta t} \langle f_i(\vec{x}, s) f_i(\vec{y}, s') \rangle ds ds' &= \frac{1}{\delta t} \int_t^{t+\delta t} C_{ii}^{(L)}(\vec{x} - \vec{y}) \delta(s - s') \\ &\quad \frac{1}{\delta t} \int_t^{t+\delta t} Tr C^{(L)}(\vec{x} - \vec{y}) ds \\ &= Tr C^{(L)}(\vec{x} - \vec{y}) \end{aligned} \quad (8.4.8)$$

$$\int_t^{t+\delta t} \langle u_i(\vec{x}, t) f_i(\vec{y}, s) \rangle ds = 0 \quad (8.4.9)$$

because s is always greater than t and hence by causality the relation vanishes. Because of translation invariance and divergence free condition

$$\langle \partial_i P(\vec{x}, t) u_i(\vec{x}, t) \rangle = 0 \quad (8.4.10)$$

$$\langle u_i(\vec{x}, t) \nabla^2 u_i(\vec{y}, t) \rangle = -\langle \partial_j u_i(\vec{x}, t) \partial_j u_i(\vec{y}, t) \rangle \quad (8.4.11)$$

$$\begin{aligned} \langle u_j(\vec{x}, t) \partial_j u_i(\vec{x}, t) u_i(\vec{y}, t) \rangle + x \leftrightarrow y &= -\frac{1}{4} \nabla_x \langle (u_i(\vec{x}, t) - u_i(\vec{y}, t)) \\ &\quad \times (u(\vec{x}, t) - u(\vec{y}, t))^2 \rangle \end{aligned} \quad (8.4.12)$$

With the above identities, Eq.(8.4.7) becomes

$$\begin{aligned} \frac{1}{2} Tr C^{(L)}(\vec{x} - \vec{y}) &= -\frac{1}{4} \nabla_x \langle (u_i(\vec{x}, t) - u_i(\vec{y}, t)) [u(\vec{x}, t) - u(\vec{y}, t)]^2 \rangle \\ &\quad + v \langle \partial_i u_i(\vec{x}, t) \partial_j u_j(\vec{x}, t) \rangle \end{aligned} \quad (8.4.13)$$

We now can take the limits $v \rightarrow 0$ and $x \rightarrow y$ in two different orders. If v is held fixed and the limit $x \rightarrow y$ taken, then the first term on the left hand side of Eq.(8.4.13) vanishes and we have,

$$\frac{1}{2} Tr C^{(L)}(0) = \lim_{\substack{\vec{x} \rightarrow \vec{y} \\ v \rightarrow 0}} v \langle (\partial_i u_j(\vec{x})^2) \rangle = \bar{\epsilon} \quad (8.4.14)$$

If we take the limit of $v \rightarrow 0$ first and then consider $\vec{x} \rightarrow \vec{y}$

$$-\frac{1}{4} \nabla_x \langle (u_i(\vec{x}, t) - u_i(\vec{y}, t)) [u(\vec{x}, t) - u(\vec{y}, t)]^2 \rangle = \bar{\epsilon} \quad (8.4.15)$$

We now invoke isotropy to write,

$$\begin{aligned} -\frac{4}{D+2} \bar{\epsilon} [\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ki} x_j] &= \langle [u_i(\vec{x}, t) - u_i(\vec{0}, t)] [u_j(\vec{x}, t) - u_j(\vec{0}, t)] \\ &\quad \times [u_k(\vec{x}, t) - u_k(\vec{0}, t)] \rangle \end{aligned} \quad (8.4.16)$$

from which it follows that

$$\langle [\{\vec{u}(\vec{x}, t) - \vec{u}(\vec{0}, t)\} \cdot \hat{x}]^3 \rangle = -\frac{12}{D(D+2)} \bar{\epsilon} r \quad (8.4.17)$$

The above result is valid for $D \geq 3$ and in $D = 3$, gives the usual result

$$S_3 = -\frac{4}{5} \bar{\epsilon} r.$$

The important thing to note in this derivation is the anomaly that the operator

$$\lim_{\vec{x} \rightarrow \vec{y}, v \rightarrow 0} \vec{\nabla}_x \cdot [\langle \vec{u}(\vec{x}) - \vec{u}(\vec{y}) \rangle [\vec{u}(\vec{x}) - \vec{u}(\vec{y})]^2]$$

is non-zero. Had $\vec{u}(\vec{x})$ been a differentiable field then this term would have been zero.

This kind of anomaly occurs for arbitrary correlation functions $\langle \Pi_n F_n(u(x_n)) \rangle$ where $F_n(u(x_n))$ are arbitrary functions of $u(x_n)$ without any derivatives. We work out the case of $n = 2$ in detail. The stationarity condition for a two point function is

$$\partial_t \langle F_1(u(x_1)) F_2(u(x_2)) \rangle = 0 \quad (8.4.18)$$

Noting that $\partial_t F(u(x)) = (\partial_t u) \cdot \delta F(u)$ by the chain rule we have

$$\langle \partial_t F_1(u(x_1)) F_2(u(x_2)) \rangle + x_1 \leftrightarrow x_2 = 0$$

or

$$\begin{aligned} 0 &= \langle \partial_t u_j(x_1) \partial_j F_1(u(x_1)) F_2(u(x_2)) \rangle + x_1 \leftrightarrow x_2 \\ &= \langle [-\partial_l (u_l(x_1) u_j(x_1)) - \partial_j P(x_1) + v \nabla^2 u_j(x_1)] \partial_j F_1(u(x_1)) F_2(u(x_2)) \rangle \\ &\quad + \langle f_j \partial_j F_1(u(x_1)) F_2(u(x_2)) \rangle + (x_1 \leftrightarrow x_2) \end{aligned} \quad (8.4.19)$$

The first two terms represent correlation functions which are well defined in the inviscid limit. The viscosity involving term which would naively vanish in the $v \rightarrow 0$ limit is actually nonzero due to the dissipative anomaly. Using translational invariance, we can write

$$\begin{aligned} \langle v \nabla^2 \partial_j F_1[u(x_1)] F_2[u(x_2)] \rangle &= v \langle \partial_{kx_1} [\partial_j F_1[u(x_1)] \partial_k u_j(x_1)] F_2(u(x_2)) \rangle \\ &\quad - v \langle \partial_k u_j(x_1) \partial_{kx_1} \partial_j F_1[u(x_1)] F_2[u(x_2)] \rangle \\ &= v \langle \partial_{kx_1} \partial_k F_1[u(x_1)] F_2[u(x_2)] \rangle \\ &\quad - v \langle \partial_k u_j(x_1) \partial_k u_l(x_1) \partial_j \partial_l F_1[u(x_1)] F_2[u(x_2)] \rangle \\ &= -v \langle \partial_k F_1[u(x_1)] \partial_k F_2[u(x_2)] \rangle \\ &\quad - \langle \epsilon_{lj}(x_1) \partial_l \partial_j F_1[u(x_1)] F_2[u(x_2)] \rangle \\ &= -v \nabla_{x_1}^k \nabla_{x_2}^k \langle F_1[u(x_1)] F_2[u(x_2)] \rangle \\ &\quad - \langle \epsilon_{lj}(x_1) \partial_l \partial_j F_1[u(x_1)] F_2[u(x_2)] \rangle \end{aligned} \quad (8.4.20)$$

where

$$\epsilon_{lj} = \nu \partial_k u_j(x) \partial_k u_l(x) \quad (8.4.21)$$

In the inviscid limit, the correlation $\langle F_1[u(x_1)] F_2[u(x_2)] \rangle$ is finite and hence the derivatives are finite at $\nu = 0$. Consequently, the first term on the R.H.S of Eq.(8.4.20) vanishes in the inviscid limit and we have

$$\lim_{\nu \rightarrow 0} \nu \langle \nabla^2 u_j(x_1) \partial_j F_1[u(x_1)] F_2[u(x_2)] \rangle = -\langle \epsilon_{lj}(x_1) \partial_l \partial_j F_1[u(x_1)] F_2[u(x_2)] \rangle \quad (8.4.22)$$

Finally we need to evaluate the correlation function involving the stochastic force. To do this we note that

$$\langle f_i(x) K(u(y)) \rangle = \int d^3z C_{ij}(x-z) \left\langle \frac{\delta K(u(y))}{\delta f_j(z)} \right\rangle \quad (8.4.23)$$

for any functional of G , when we integrate by parts and use the fact that the force is Gaussian. To evaluate $\frac{\delta K(u(y))}{\delta f_j(z)}$, we really need to evaluate $\frac{\delta u_k(y,t)}{\delta f_j(x,s)}$ in the equal time limit $t = s$. This derivative is zero if $t < s$. Starting from the Navier-Stokes equation and considering a differential δu_k caused by a differential δf_j , we have

$$\left[\frac{\partial}{\partial t} + L(u(y)) \right] \frac{\delta(u_k(y,t))}{\delta f_j(x,s)} = \delta_{jk} \delta(y-x) \delta(t-s) \quad (8.4.24)$$

where L is some differential operator depending on u . From Eq.(8.4.24), it is clear that

$$\frac{\delta u_k(y,t)}{\delta f_j(x,s)} = \theta(t-s) G^{jk}(y, x|t, s) \quad (8.4.25)$$

where $G^{jk}(y, x|t, s)$ is the solution of

$$\left[\frac{\partial}{\partial t} + L(u(y)) \right] G^{jk}(y, x|t, s) = 0$$

with the initial condition that at $t = s$,

$$G^{jk}(y, x|t, s) = \delta_{jk} \delta(y-x).$$

Taking the equal time limit Eq.(8.4.25) and setting $\theta(t \rightarrow s) = \frac{1}{2}$, we have

$$\lim_{t \rightarrow s} \frac{\delta u_k(y, t)}{\delta f_j(x, s)} = \frac{1}{2} \delta_{jk} \delta(x - y) \quad (8.4.26)$$

From Eq.(8.4.23), we then have

$$\begin{aligned} \langle f_i(x) K(u(y)) \rangle &= \int d^3 z C_{ij}(x - z) \left\langle \frac{\partial K(y)}{\partial u_k} \frac{\partial u_K(y)}{\partial f_j(z)} \right\rangle \\ &= \int d^3 z \left\langle \frac{\partial K(y)}{\partial u_k} \right\rangle C_{ij}(x - z) \frac{1}{2} \delta_{jk} \delta(y - z) \\ &= \frac{1}{2} C_{ij}(x - y) \left\langle \frac{\partial K(y)}{\partial u_k} \right\rangle \end{aligned} \quad (8.4.27)$$

Putting the information of the Eqns.(8.4.27) and (8.4.22) into the stationarity condition of Eq.(8.4.19), we have after a little rearrangement

$$\begin{aligned} \sum \left\langle \left[\{(\vec{u} \cdot \vec{\nabla}) u_j + \vec{\nabla} p\}(x) \frac{\partial}{\partial u_j} + \epsilon_{ij} \frac{\partial^2}{\partial u_i \partial u_j} \right] F_1[u(\vec{x}_1)] F_2[u(\vec{x}_2)] \right\rangle \\ = \frac{1}{2} \sum C_{ij}(\vec{x}_1 - \vec{x}_2) \left\langle \frac{\partial^2}{\partial u_i \partial u_j} F_1[u(\vec{x}_1)] F_2[u(\vec{x}_2)] \right\rangle \end{aligned} \quad (8.4.28)$$

For the n-point correlation function $\Pi_n F_n(u(x_n))$, we need to replace $F_1[u(\vec{x}_1)] F_2[u(\vec{x}_2)]$ by $\Pi_n F_n(u(x_n))$.

We end this section by exhibiting the long distance and short distance correlations for a driven diffusion equation, i.e. a diffusing scalar field $\phi(\vec{x}, t)$ driven by a stochastic noise f , such that

$$\frac{\partial \phi}{\partial t} = D_1 \nabla^2 \phi + f \quad (8.4.29)$$

In momentum space, we have

$$\frac{\partial}{\partial t} \phi(k, t) + D_1 k^2 \phi(k, t) = f(k) \quad (8.4.30)$$

with the solution

$$\phi(k, t) = \int dt' e^{-D_1 k^2 (t-t')} f(k, t') \quad (8.4.31)$$

The two point correlation function is given by

$$\begin{aligned}
 \langle \phi(k, t_1) \phi(-k, t_2) \rangle &= \int_{-\infty}^{t_1} dt' \int_{-\infty}^{t_2} dt'' e^{-D_1 k^2(t_1-t')} \\
 &\quad \times e^{-D_1 k^2(t_2-t'')} \langle f(k, t') f(-k, t'') \rangle \\
 &= \int dt' dt'' C(k) \delta(t' - t'') e^{-D_1 k^2(t_1+t_2)} e^{D_1 k^2(t'+t'')} \\
 &= \int_{-\infty}^{t_2} C(k) e^{-D_1 k^2(t_1+t_2)} e^{2D_1 k^2 t''} \\
 &= \frac{C(k)}{2D_1 k^2} e^{-D_1 k^2 |t_1 - t_2|} \quad (8.4.32)
 \end{aligned}$$

If we are to form the spatial correlation

$$\langle \phi(\vec{x}_1, t_1) \phi(\vec{x}_2, t_2) \rangle = \int \frac{d^D k}{(2\pi)^D} \frac{C(k)}{2D_1 k^2} e^{-D_1 k^2 |t_1 - t_2|} e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} \quad (8.4.33)$$

Equal time correlation function is

$$\langle \phi(\vec{x}_1, t) \phi(\vec{x}_2, t) \rangle = \int \frac{d^D k}{(2\pi)^D} \frac{C(k)}{2D_1 k^2} e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}$$

If large $|\vec{x}_1 - \vec{x}_2|$ behaviour is required, then we look for the k-part alone and

$$\begin{aligned}
 \langle \phi(\vec{x}_1, t) \phi(\vec{x}_2, t) \rangle &\sim \frac{C(0)}{2D_1} \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{k^2} \\
 &\simeq |\vec{x}_1 - \vec{x}_2|^{2-D} \quad (8.4.34)
 \end{aligned}$$

Now suppose we want to extract the small distance behaviour from Eq.(8.4.33). We then expand the exponential and have

$$\begin{aligned}
 \langle \phi(\vec{x}_1, t) \phi(\vec{x}_2, t) \rangle &= \int \frac{d^D k}{(2\pi)^D} \frac{C(k)}{2D_1 k^2} \left[1 - D_1 k^2 |t_1 - t_2| - \frac{k^2}{2D} (\vec{x}_1 - \vec{x}_2)^2 \right] \\
 &= \int \frac{d^D k}{(2\pi)^D} \frac{C(k)}{2D_1 k^2} - \int \left(\frac{d^D k}{(2\pi)^D} \frac{C(k)}{2D_1} \right) \\
 &\quad \times \left[D_1 |t_1 - t_2| + \frac{(\vec{x}_1 - \vec{x}_2)^2}{2D} \right] \\
 &= \int \frac{d^D k}{(2\pi)^D} \frac{C(k)}{2D_1 k^2} - \frac{C(r=0)}{2D_1} \left[D_1 |t_1 - t_2| + \frac{(\vec{x}_1 - \vec{x}_2)^2}{2D} \right] \quad (8.4.35)
 \end{aligned}$$

The first term is a diverging constant and hence the scaling shown in the second term is exhibited only by the difference variable $\phi(x_1, t) - \phi(x_2, t)$. This is the typical turbulence related problem - the presence of a blowing up constant mode which may be eliminated only by considering field differences.

8.5 Advection of a Passive Scalar

We have seen that the central problem in the theory of homogeneous isotropic turbulence is the determination of the scaling exponents ζ_n associated with the structure functions as shown in Eq.(8.2.7). These exponents differ from the *Kolmogorov* assertion which was based on purely dynamical grounds. To obtain these non-trivial exponents, we wrote down the most general relation between various correlation functions in Eq.(8.4.28). However, apart from $n = 3$, where it is possible to obtain an exact answer as shown in Eq.(8.4.17), it is still not practical to use these equations. In this section we will treat a much simpler problem and explicitly show how the anomalous scaling arises. This is the problem of advection of a scalar field ϕ by a turbulent velocity field $\vec{v}$. The equation of motion is,

$$\frac{\partial \phi}{\partial t} + (\vec{v} \cdot \vec{\nabla})\phi = \kappa \nabla^2 \phi + f \quad (8.5.1)$$

where κ is the diffusion coefficient and random force f has the correlation

$$\langle f(\vec{x}, t) f(\vec{y}, s) \rangle = C \left(\frac{\vec{x} - \vec{y}}{L} \right) \delta(t - s) \quad (8.5.2)$$

The random velocity field has the correlation

$$\langle v_\alpha(\vec{x}, t) v_\beta(\vec{y}, s) \rangle = D_{\alpha\beta}(\vec{x} - \vec{y}) \delta(t - s) \quad (8.5.3)$$

with $\partial_\alpha D_{\alpha\beta} = 0$. An explicit form for $D_{\alpha\beta}$ that we choose is

$$D_{\alpha\beta} = D_0 \delta_{\alpha\beta} - d_{\alpha\beta} \quad (8.5.4)$$

and

$$d_{\alpha\beta} = D_0 \left[(D + \zeta - 1) \delta_{\alpha\beta} - \zeta \frac{r^\alpha r^\beta}{r^2} \right] r^\zeta \quad (8.5.5)$$

where ζ is a parameter which is less than 2. A more rigorous form for $D_{\alpha\beta}(r)$ is

$$D_{\alpha\beta}(r) = d_0 \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\vec{k} \cdot \vec{r}}}{(k^2 + m^2)^{(D+\xi)/2}} \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) \quad (8.5.6)$$

We will work with the form in Eq.(8.5.5) which is more convenient. The parameter ζ fixes the naive dimension under the rescaling $x \rightarrow \mu x$ and $t \rightarrow \mu^\zeta t$. From Eq.(8.5.5), $d_{\alpha\beta}(\mu r) = \mu^\zeta d_{\alpha\beta}(r)$ and hence from Eq.(8.5.3), the scaling of the velocity field by the factor $\mu^{\frac{\zeta}{2} - \frac{\xi}{2}}$. From Eq.(8.5.2), we note that f rescales as $1/\sqrt{T}$. In, Eq.(8.5.1), comparing the terms on the L.H.S, we see that T rescales as $\mu^{2-\zeta}$ and comparing $\frac{\partial \phi}{\partial t}$ with f , we have ϕ rescaling as $\mu^{1-\zeta/2}$. We note that the *Kolmogorov* velocity field satisfies $\langle (\delta v_r)^2 \rangle \sim r^{2/3} \sim r^{4/3}/\tau_r$. This is to be compared with $\langle (\delta v_r)^2 \rangle \sim r^\zeta \delta(t)$ in the above model. Hence in the above $\zeta = 4/3$ is

the correct *Kolmogorov* limit. In this problem, one has the dissipative anomaly because $\lim_{\kappa \rightarrow 0} \langle \kappa (\vec{\nabla} \phi)^2 \rangle$ would be finite and dissipating the scalar energy density $E = \frac{1}{2} \phi^2$. The energy balance reads,

$$\partial_t \int E d^D r = \int [\langle f \phi \rangle - \kappa \langle \vec{\nabla} \phi^2 \rangle] d^D r \quad (8.5.7)$$

The stationarity condition allows one to form equations for the n -point correlation functions analogous to those exhibited in Eq.(8.4.28). In this case they are simpler and have the form

$$\begin{aligned} M_N^\kappa \langle \phi(x_1) \phi(x_2) \dots \phi(x_N) \rangle \\ = \frac{1}{2} \sum_{j,k} C(\vec{x}_j - \vec{x}_k) \langle \phi(x_1) \phi(x_2) \dots \overline{\phi(x_j)} \dots \overline{\phi(x_k)} \dots \phi(x_N) \rangle \end{aligned} \quad (8.5.8)$$

where symbols which have a bar over them are missing in the correlation function. The operator M_N^κ is given by

$$M_N^\kappa = -\kappa \sum_{j=1}^N \nabla_{x_j}^2 + \frac{1}{2} \sum_{j,k} d^{ab}(\vec{x}_j - \vec{x}_k) \nabla_{x_j}^a \nabla_{x_j}^b \quad (8.5.9)$$

In the inertial range $\kappa \rightarrow 0$ and

$$M_N^0 = \frac{1}{2} \sum_{j,k} d^{ab}(\vec{x}_j - \vec{x}_k) \nabla_{x_j}^a \nabla_{x_j}^b \quad (8.5.10)$$

Let us now look at the two point correlation function $\langle \phi(\vec{x}) \phi(\vec{0}) \rangle$. Exploiting translational invariance

$$M_2^0 = -D_0(D-1) \frac{1}{r^{D-1}} \frac{d}{dr} r^{D+\zeta-1} \frac{d}{dr} \quad (8.5.11)$$

In this case, it is now easy to see that $\langle [\phi(\vec{x}) - \phi(\vec{0})]^2 \rangle$ will scale as $r^{2-\zeta}$, which is exactly as the naive dimensional analysis predicts. The non-trivial nature of the problem shows up in the higher order structure factors when it is seen that $\langle [\phi(\vec{x}) - \phi(\vec{0})]^N \rangle \neq r^{(2-\zeta)N/2}$. Instead, they behave as

$$\langle [\phi(\vec{x}) - \phi(\vec{0})]^N \rangle \simeq A_N \left(\frac{L}{r} \right)^{\rho_N} r^{(2-\zeta)N/2} \quad (8.5.12)$$

where

$$\rho_N = \zeta \frac{N(N-2)}{2(D+2)} + O(\zeta^2) \quad (8.5.13)$$

or

$$\rho_N = \zeta \frac{N(N-2)}{2D} + O\left(\frac{1}{D^2}\right) \quad (8.5.14)$$

for large D .

Instead of describing the calculations leading to Eqns.(8.5.13) and (8.5.14) we show how non-trivial exponent can arise even for $N=2$ in a slightly different model. Here an additional friction term is used and we have

$$\partial_t \phi + (\vec{v} \cdot \vec{\nabla}) \phi + \frac{\phi}{\tau} - \kappa \nabla^2 \phi = f \quad (8.5.15)$$

For $N=2$, the correlation function $F_2 = \langle \phi(\vec{x}) \phi(\vec{0}) \rangle$ satisfies in the limit $\kappa=0$ (see Eq.(8.5.8)) for $\zeta=2$,

$$C(r) = -D_0(D-1) \frac{1}{r^{D-1}} \left[\frac{d}{dr} r^{D+1} \frac{d}{dr} \right] F_2 + \frac{2}{\tau} F_2 \quad (8.5.16)$$

This can be solved by factorizing

$$F_2(r) = G(r)H(r) \quad (8.5.17)$$

where $G(r)$ satisfies the homogeneous equation

$$-D_0(D-1) \frac{1}{r^{D-1}} \left[\frac{d}{dr} r^{D+1} \frac{d}{dr} \right] G(r) + \frac{2}{\tau} G(r) = 0 \quad (8.5.18)$$

This has the solution $G(r) \sim r^\alpha$, where

$$\begin{aligned} -D_0(D-1)\alpha(D+\alpha) + \frac{2}{\tau} &= 0 \\ \text{or} \quad \alpha^2 + D\alpha - \frac{2}{\tau D_0(D-1)} &= 0 \\ \text{giving the roots} \quad \alpha_{\pm} &= -\frac{D}{2} \pm \sqrt{\frac{D^2}{4} + \frac{2}{D_0(D-1)\tau}} \end{aligned} \quad (8.5.19)$$

The boundary condition that $F_2(r)$ is finite at $r=0$, eliminates α_- and now $H(r)$ can be found as the solution of

$$\frac{D_0(D-1)}{r^{D-1+\alpha}} \frac{d}{dr} \left(r^{D+1+2\alpha} \frac{dH}{dr} \right) = C(r) \quad (8.5.20)$$

The solution for

$$F_2(r) = r^{\alpha_+} \int_r^\infty \frac{1}{D_0(D-1)} \frac{d\rho}{\rho^{D+1+2\alpha}} \int_0^\rho r^{D-1+\alpha} C(r) dr \quad (8.5.21)$$

For small r , this has the behaviour

$$F_2(r) \sim r^{\alpha_+} \quad (8.5.22)$$

which is to be contrasted with the naïve scaling behaviour which for $\zeta = 2$ is $F_2(r) \sim r^{(0)}$ (at most logarithmic terms). The anomalous dimension has arisen from the zero mode of the stationarity condition. This illustrates the general result that the root cause of the anomalous dimensions lies in the zero modes of the stationarity condition.

8.6 Intermittency Phenomenology

As we have already discussed before, the phenomenon of intermittency has to do with the occurrence of the anomalous scaling (Eq.(8.2.7)) in the velocity structure functions. Soon after the publication of *Kolmogorov's* work in 1941 it was pointed out by Landau that the dissipation which is determined by the fluctuations of the derivative of the random velocity field may not be constant. The local dissipation can be defined as

$$\epsilon(\vec{x}) = \nu \int_{\substack{\text{over a ball} \\ \text{surrounding } \vec{x}}} d^D r \left\langle \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \right\rangle \quad (8.6.1)$$

and experimentally measured with great accuracy. The correlation function $\langle \epsilon(\vec{x}) \epsilon(\vec{x} + \vec{r}) \rangle$ in the *Kolmogorov* analysis has to be determined by $\bar{\epsilon}$ and r alone and hence

$$\langle \epsilon(\vec{x}) \epsilon(\vec{x} + \vec{r}) \rangle = (\bar{\epsilon})^2 \quad (8.6.2)$$

The experimental results clearly supported a power law

$$\langle \epsilon(\vec{x}) \epsilon(\vec{x} + \vec{r}) \rangle = (\bar{\epsilon})^2 \left(\frac{L}{r} \right)^\mu \quad (8.6.3)$$

where μ is a small exponent ($\mu \simeq 0.20$) and is called the intermittency exponent. The experimentally observed signal shows rare but large fluctuations in the dissipation and hence the phenomenon is known as intermittency. The effect of this fluctuation on the velocity correlation function was first considered by *Obukhov and Kolmogorov* in 1962, when it was assumed that the dissipation rate $\bar{\epsilon}$ has a log-normal distribution. The width of the distribution was conjectured from a perturbative calculation to be of the form

$$\sigma^2 = A + 9\delta \ln \left(\frac{L}{r} \right) \quad (8.6.4)$$

where δ is a universal constant. With the mean of the distribution given by m , it follows that

$$\left\langle \epsilon^{p/3} \right\rangle = e^{pm/3} e^{p^2\sigma^2/18} \quad (8.6.5)$$

The velocity difference Δv_r at scale r is related to the local ϵ_r by $\Delta v_r = (r\epsilon_r)^{1/3}$ and hence

$$\begin{aligned} \left\langle |\Delta v_r|^p \right\rangle &= \text{Const} \times r^{p/3} (\epsilon_r)^{p/3} \\ &= \text{Const} \times r^{p/3} e^{pm/3} e^{p^2(A+9\delta \ln \frac{L}{r})} \end{aligned} \quad (8.6.6)$$

For $p = 3$, we have an exact result and that fixes m as follows:

$$\begin{aligned} \left\langle |\Delta v_r|^3 \right\rangle &= \text{Const} \times r e^m e^{\sigma^2/2} \\ &= \text{Const} \times \bar{\epsilon} r \end{aligned} \quad (8.6.7)$$

which identifies

$$\bar{\epsilon} = e^{m + \frac{\sigma^2}{2}}$$

Returning to Eq.(8.6.6) and using Eq.(8.6.7)

$$\begin{aligned} \left\langle |\Delta v_r|^p \right\rangle &= \text{Const} \times r^{p/3} e^{\frac{m+\sigma^2/2}{p/3}} e^{-\frac{p\sigma^2}{6} + \frac{p^2\sigma^2}{18}} \\ &= \text{Const} \times r^{p/3} \bar{\epsilon}^{p/3} e^{p(p-3)(A+9\delta \ln \frac{L}{r})/18} \\ &= \text{Const} \times r^{p/3} \bar{\epsilon}^{p/3} e^{\frac{p(p-3)}{2} \delta \ln \frac{L}{r}} \\ &= C_p \bar{\epsilon}^{p/3} r^{p/3} \left(\frac{L}{r} \right)^{\frac{p(p-3)}{2} \delta} \end{aligned} \quad (8.6.8)$$

For $p = 6$

$$\left\langle |\Delta v|^6 \right\rangle = C_6 \bar{\epsilon}^2 r^2 \left(\frac{L}{r} \right)^{9\delta}$$

Noting that

$$\epsilon_r \sim |\Delta v_r|^3 / r,$$

the dissipation correlation function goes as

$$\langle \epsilon_r^2 \rangle \propto r^{-9\delta}$$

which gives

$$\mu = 9\delta \quad (8.6.9)$$

In the above phenomenology, it is clear that once δ is fixed, all other exponents follow. It is reasonably safe to say that experiments show deviation from *Obukhov-Kolmogorov* phenomenology for high value of p .

The information regarding the dissipation spectrum can be looked at from another point of view. In the inviscid limit, the Navier-Stokes equation is invariant under the rescaling,

$$\begin{aligned} x &\rightarrow x' = \lambda x \\ t &\rightarrow t' = \lambda^{1-\alpha/3} t \\ v &\rightarrow v' = \lambda^{\alpha/3} v \end{aligned} \quad (8.6.10)$$

for any α . Now the local dissipation rate is $\epsilon_r \sim v_r^3/r$ and hence scales as $\lambda^{\alpha-1}$. This would mean that

$$\frac{\epsilon_r}{\epsilon_L} \sim \left(\frac{r}{L}\right)^{\alpha-1} \quad (8.6.11)$$

The constancy of ϵ_r in the *Kolmogorov* picture now suggests $\alpha = 1$ in three dimensional space.

The reality however, is that α is not constant and α takes on different values on different interwoven fractal subsets of the three dimensional space in which the dissipation field is embedded. The set on which the scaling exponent lies α and $\alpha + d\alpha$ is characterized by the fractal (*Hausdroff*) dimension $f(\alpha)$. Experimentally $f(\alpha)$ vs. α can be measured. We would like to indicate how this is done.

The total dissipation in a volume r^D is $E_{r,D}$ and can be written as

$$E_{r,D} = \epsilon_r r^D \sim r^{\alpha-1+D} \quad (8.6.12)$$

We divide the entire space into boxes of size r . What is measured is $E_{r,D}$ in each box and what is constructed from it is the sum of $\sum_{boxes} E_{r,D}^q$. This quantity is supposed to scale with r in the fashion

$$\begin{aligned} \sum_{boxes} E_{r,D}^q &\sim r^{(q-1)D_q} \\ \text{or} \quad \sum_{boxes} r^{(\alpha-1+D)q} &\sim r^{(q-1)D_q} \end{aligned} \quad (8.6.13)$$

The generalized dimensions D_q defined through Eq.(8.6.13) constitute a measure of the spottiness of the dissipation process. Uniform dissipation over all space would correspond to D_q being independent of q . The sum over boxes can be replaced by an integration if we note that the number of boxes in which α lies between α and $d\alpha$ is proportional to $r^{-f_D(\alpha)}$ and thus Eq.(8.6.13) becomes

$$\int \rho(\alpha) r^{q(\alpha-1+D)} r^{-f_D(\alpha)} d\alpha \sim r^{(q-1)D_q} \quad (8.6.14)$$

As $r \rightarrow 0$, we can evaluate the integral by steepest descent and one then has

$$\begin{aligned} q(\alpha - 1 + D) - f_D(\alpha) &= (q - 1)D_q \\ \text{if } \frac{\partial f_D(\alpha)}{\partial \alpha} &= q \\ \text{and } \frac{\partial^2 f_D(\alpha)}{\partial \alpha^2} &< 0 \end{aligned} \quad (8.6.15)$$

From Eq.(8.6.15)

$$\begin{aligned} f_D(\alpha) &= \alpha q - (q - 1)(D_q - D + 1) + D - 1 \\ \text{and } \alpha &= \frac{d}{dq} [(q - 1)(D_q - D + 1)] \end{aligned} \quad (8.6.16)$$

The experiments measure D_q and from that one infers α and $f(\alpha)$. Note that for $D = 1$

$$\begin{aligned} \alpha &= \frac{d}{dq} [(q - 1)D_q] \\ f(\alpha) &= \alpha q - (q - 1)D_q \end{aligned} \quad (8.6.17)$$

The experiments probe the projection along a given direction. A line segment is divided into bins of size n and E_r is determined from each bin. One then forms $Z_q = \sum E_r^q$ for each value of r (it is seen that the slope of $\ln E_r$ vs $\ln r$ is different for different bin locations showing α has a distribution) and $\ln z_q^{\frac{1}{q-1}}$ vs $\ln r$ is plotted. The slope yields D_q from which α and $f(\alpha)$ are obtained using Eq.(8.6.17) because a one dimensional section is being considered. A typical $f(\alpha)$ curve is shown in Fig. 8.6 We note from Eq.(8.6.11), that

$$\alpha = \frac{\ln \frac{\epsilon_r}{\epsilon_L}}{\ln \frac{r}{L}} + 1 \quad (8.6.18)$$

and consequently the probability distribution $P_\epsilon(\epsilon_r)$ for the dissipation can be written as

$$P_\epsilon(\epsilon_r) = P_\alpha(\alpha) \frac{d\alpha}{d\epsilon_r} / P(\epsilon_r \neq 0) \quad (8.6.19)$$

where $P(\epsilon_r \neq 0)$ is the probability that ϵ_r is non-zero for non-zero r i.e. a box of size r always contains a part of the multifractal. In terms of the fractal dimension D_0 , the probability of having a box of size r being occupied by the multifractal is

$$P(\epsilon_r \neq 0) = C_1 \left(\frac{r}{L} \right)^{(D-D_0)} \quad (8.6.20)$$

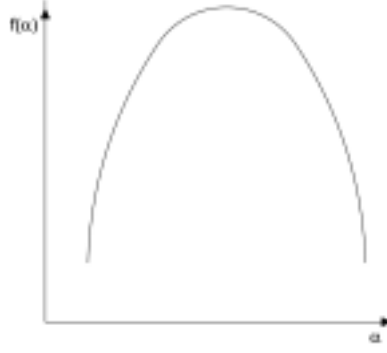


Figure 8.6. A typical distribution of the scaling indices

In a box of length r , the probability density for α lying between α and $\alpha + d\alpha$ is

$$P_\alpha = C_2 \left(\frac{r}{L} \right)^{D - f_D(\alpha)} \quad (8.6.21)$$

Using Eqns.(8.6.17)-(8.6.20)

$$P_\epsilon(\epsilon_r) = C \left(\frac{r}{L} \right)^{D_0 - f_D \left[\frac{\ln \frac{\epsilon_r}{L}}{\ln \frac{r}{L}} + 1 \right]} \times \left(\epsilon_r \ln \frac{r}{L} \right)^{-1}$$

Since $f_D(\alpha)$ is known from the experiment, this is the way the dissipation probability distribution is experimentally found out. How close is this to the log-normal distribution assumed by *Obukhov and Kolmogorov*? To understand this, we assume that the $f(r)$ vs α curve shown in Fig. 8.6 can be approximated by a parabola as

$$f(\alpha) = f(\alpha_0) + \frac{1}{2}(\alpha - \alpha_0)^2 f''(\alpha_0) \quad (8.6.22)$$

where, $\alpha = \alpha_0$ is the point where the $f(\alpha)$ vs α curve has its maximum. The maximum value is the fractal dimension D_0 of the support and hence

$$f(\alpha) = D_0 + \frac{f''(\alpha_0)}{2}(\alpha - \alpha_0)^2 \quad (8.6.23)$$

We note that from Eq.(8.6.15) that for $q \rightarrow 1$, $\frac{\partial f}{\partial \alpha} = 1$ where $\alpha_1 = \lim_{q \rightarrow 1} \frac{d}{dq} [(q - 1)D_q]$. In $D = 1$, Eq.(8.6.17) shows $f(\alpha_1) = \alpha_1$. Thus at the point $\alpha = \alpha_1$, the $f(\alpha)$ vs α has to be tangent to the line $f(\alpha) = \alpha$. For the parabolic approximation,

$$q = \frac{df}{d\alpha} = f''(\alpha_0)(\alpha - \alpha_0)$$

$$\text{leading to } \alpha = \frac{q}{f''(\alpha_0)} + \alpha_0$$

$$\text{and } \alpha_1 = \frac{1}{f''(\alpha_0)} + \alpha_0$$

At $\alpha = \alpha_1$, the parabolic approximation leads to

$$\begin{aligned}\alpha_1 = f(\alpha_1) &= D_0 + \frac{f''(\alpha_0)}{2}(\alpha_1 - \alpha_0)^2 \\ &= D_0 + \frac{1}{2} \frac{1}{f''(\alpha_0)}\end{aligned}$$

On the other hand $\alpha_1 = \alpha_0 + \frac{1}{f''(\alpha_0)}$ and thus we are led to

$$\frac{1}{2f''(\alpha_0)} = -(\alpha_0 - D_0)$$

giving

$$f(\alpha) = D_0 - \frac{1}{4} \frac{(\alpha - \alpha_0)^2}{\alpha_0 - D_0} \quad (8.6.24)$$

If we substitute the above $f(\alpha)$ into Eq.(8.6.21), then the distribution $P_\epsilon(\epsilon_r)$ is log-normal with the mean m and the variance σ^2 given by

$$\begin{aligned}m &= \ln \epsilon_L - (\alpha_0 - 1) \ln \frac{L}{r} \\ \sigma^2 &= 2(\alpha_0 - D_0) \ln \frac{L}{r}\end{aligned} \quad (8.6.25)$$

In this approximation the intermittency exponent μ is given by

$$\mu = 2(\alpha_0 - D_0) \quad (8.6.26)$$

Thus the deviation of the log-normal distribution from the real distribution is measured by the deviation of $f(\alpha)$ vs α curve from a parabolic form.

Realizing that the underlying set on which the dissipation is occurring is a multifractal, *Sreenivasan and Meneveau* came up with a physical picture of energy transmission process which produces a multifractal distribution. This is pictured in Fig. 8.7 We have taken a one dimensional slice of the process and a large (scale L) energy containing eddy is shown dividing its energy into two eddies (scale $L/2$). The division however, does not occur equally - the left hand receives a fraction p_1 while the right hand side receives a fraction p_2 with $p_1 + p_2 = 1$.

To determine the characteristics of this process, we need to compute the quantity $\sum E_r^q$ where the sum is over the different eddies of size r at a scale $r = \frac{L}{2^n}$ and q is an arbitrary integer. We expect,

$$\sum E_r^q = E^q \left(\frac{r}{L} \right)^{(q-1)D_q} \quad (8.6.27)$$

The set is a standard fractal if D_q is independent of q and multifractal if D_q is a non-trivial function of q . From the figure, it is clear that

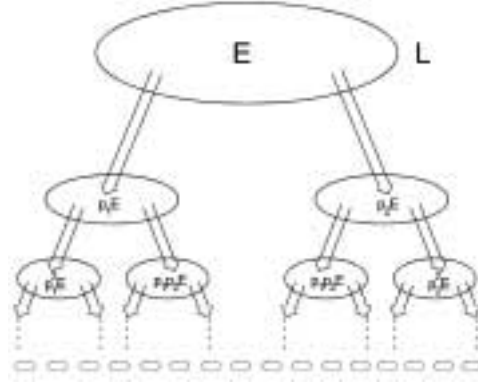


Figure 8.7. The breaking up of an energy containing eddy

$$\begin{aligned}\sum E_r^q &= \sum_m E^q {}^nC_m \left(p_1^q\right)^m \left(p_2^q\right)^{n-m} \\ &= E^q \left(p_1^q + p_2^q\right)^n\end{aligned}\quad (8.6.28)$$

From Eq.(8.6.27)

$$\sum E_r^q = E^q \left(\frac{1}{2^n}\right)^{(q-1)D_q}$$

leading to

$$D_q = -\frac{1}{q-1} \ln_2 \left(p_1^q + p_2^q\right) \quad (8.6.29)$$

With p_1 chosen to fit one of the moments, the indices D_q are known for all the moments and it was found by *Meneveau and Sreenivasan* that $p_1 = 0.7$ accounts for the data on the all the different correlation functions.

A formula for the scaling behaviour of the n -point correlation function was found by *She and Leveque* based on heuristic arguments depending on a picture of the coherent structures in the problem. The velocity correlation can be written

$$\langle |v(\vec{x} + \vec{r}) - \vec{v}(\vec{x})|^n \rangle \propto \epsilon^n r^{n/3} r^{\zeta_n} \quad (8.6.30)$$

here the anomalous dimension exponents ζ_n are given by

$$\zeta_n = -\frac{2}{9}n + 2 \left[1 - \left(\frac{2}{3}\right)^n \right] \quad (8.6.31)$$

and are seen to be universal numbers. The above formula for ζ_n fits the experimental result very well and it is a theoretical challenge to derive ζ_n from the Navier-Stokes equation.

8.7 Perturbation Theory

In this section, we show what happens in different approaches to diagrammatic perturbation theory. For this purpose, we return to Eqns.(8.1.9)-(8.1.11). The two quantities we focus on are the response function $G(k, \omega)$ and the correlation function $C(k, \omega)$ defined as

$$P_{\alpha\beta} = \left\langle \frac{\partial v_\alpha(k, \omega)}{\partial f_\beta(k', \omega')} \right\rangle \left[\delta(\vec{k} + \vec{k}') \delta(\omega + \omega') \right]^{-1} \quad (8.7.1)$$

and

$$P_{\alpha\beta} C(k, \omega) = \langle v_\alpha(k, \omega) v_\beta(k', \omega') \rangle \frac{1}{\delta(\vec{k} + \vec{k}') \delta(\omega + \omega')} \quad (8.7.2)$$

In the absence of the nonlinearity in Eq.(8.1.9), the response function is $G_0(k, \omega)$ given by

$$G_0^{-1}(k, \omega) = -i\omega + \nu k^2 \quad (8.7.3)$$

The nonlinearity changes G_0 to the full G by *Dyson's* equation which reads

$$G^{-1}(k, \omega) = -i\omega + \nu k^2 + \Sigma(k, \omega) \quad (8.7.4)$$

The physical interpretation of Σ is a relaxation rate. Treating the nonlinearity perturbatively, the one loop contribution $\Sigma^{(1)}(k, \omega)$ to Σ is given by

$$\begin{aligned} \Sigma^{(1)}(k, \omega) = & \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega'}{2\pi} M_{\alpha\beta\gamma}(k) M_{\sigma\nu\alpha}(k) P_{\beta\sigma}(p) P_{\nu\gamma}(\vec{k} - \vec{p}) \\ & \times G_0(p, \omega') C_0(\vec{k} - \vec{p}, \omega - \omega') \end{aligned} \quad (8.7.5)$$

where $C_0(k, \omega)$ is the zeroth order correlation function, which is clearly $|G(k, \omega)|^2 \langle ff \rangle$.

We now introduce an approximation known as self-consistent mode coupling and anticipate that in the *Kolmogorov* (scaling) limit, the contribution of the non-linear terms to the self-energy will dominate the molecular viscosity diffusion rate νk^2 . Thus Eq.(8.7.4) becomes

$$G^{-1} = -i\omega + \Sigma(k, \omega) \quad (8.7.6)$$

For the self-consistent mode coupling approximation, one makes the assumption that replacing G_0 and C_0 on the R.H.S of Eq.(8.7.5) by the full response function G and the full correlation function C leads to the full self-energy Σ . Thus the self-consistent mode coupling involves in the diagrammatic language a sum over a class of diagrams of all orders. The self-energy is

$$\Sigma(k, \omega) = \int \frac{d^D p}{(2\pi)^D} \int \frac{d\omega'}{2\pi} M_{\alpha\beta\gamma}(k) P_{\beta\sigma}(p) M_{\sigma\nu\alpha}(p) P_{\nu\gamma}(\vec{k} - \vec{p}) \times G(p, \omega') C(\vec{k} - \vec{p}, \omega - \omega') \quad (8.7.7)$$

The diagrammatic representations of this approximation is shown in Fig. 8.9. There must be a similar approximation for the correlation function for Eq.(8.7.7) to be useful. This complication could be avoided for a system with *fluctuation dissipation theorem* (FDT) where the correlations and response functions are related. However, a FDT can be found only if the equation of motion leads to an equilibrium distribution function associated with Eq.(8.1.9) and hence we need an approximation for $C(k, \omega)$ and in the above scheme this is shown in Fig. 8.8. Let us now

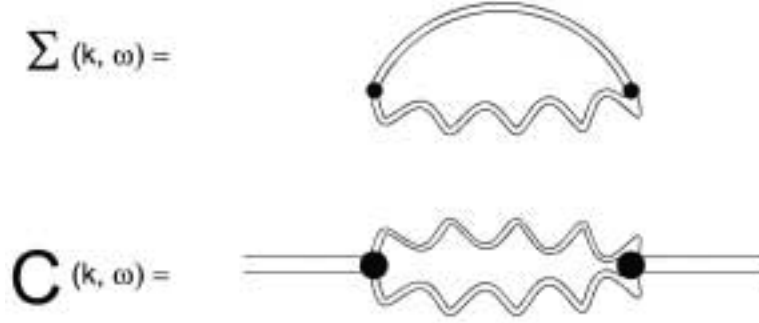


Figure 8.8. The one loop self consistent self energy and correlation function

examine the integrand I of Eq.(8.7.7) when the momentum $\vec{p}$ becomes nearly the same as $\vec{k}$ or equivalently $\vec{q} = \vec{k} - \vec{p} \rightarrow \vec{0}$. In this limit the integrand for the zero frequency self-energy (i.e. $\omega = 0$) becomes

$$I = M_{\alpha\beta\gamma}(k) P_{\beta\sigma}(k) M_{\sigma\nu\alpha}(k) P_{\nu\gamma}(\vec{q}) G(k, \omega') C(\vec{q}, -\omega') \quad (8.7.8)$$

Since the momentum q is small i.e. $q \ll k$, in the above integrand, the integration over ω' , will sample mainly small frequencies i.e. frequencies of the order of q^z , where z is some as yet unknown dynamical exponent. This ensures that $G(k, \omega')$ reduces to $[\Sigma(k)]^{-1}$. Thus Eq.(8.7.7) reads

$$\begin{aligned} \Sigma(k) &= M_{\alpha\beta\gamma}(k) P_{\beta\sigma}(k) M_{\sigma\nu\alpha}(k) \frac{1}{\Sigma(k)} \times \int \frac{d^D q}{(2\pi)^D} \int \frac{d\omega'}{2\pi} P_{\nu\gamma}(\vec{q}) C(q, \omega') \\ &= \frac{M_{\alpha\beta\gamma}(k) P_{\beta\sigma}(k) M_{\sigma\nu\alpha}(k)}{\Sigma(k)} \int \frac{d^D q}{(2\pi)^D} P_{\nu\gamma}(\vec{q}) C(q) \end{aligned} \quad (8.7.9)$$

where $C(q)$ is the equal time correlation function. According to the *Kolmogorov* phenomenology, expressed as

$$C(q) \sim q^{-(\frac{2}{3}+D)}$$

and using this in Eq.(8.7.9), the self-energy $\Sigma(k)$ follows from Eq.(8.7.9) as

$$[\Sigma(k)]^2 \propto k^2 \int \frac{dq}{q^{1+\frac{2}{3}}} \quad (8.7.10)$$

an integral which diverges for $q \rightarrow 0$ and has to be cut off at some low momentum scale k_0 . It follows that $\Sigma(k) \sim k$ instead of $k^{2/3}$ which would be consistent with *Kolmogorov*. Thus, the procedure outlined above for using the Navier-Stokes equation to arrive at the *Kolmogorov* spectrum does not work. It fails because the large effect in the dynamics of the Navier-Stokes fluid is the advection of small eddies by the large ones and this has nothing to do with the *Kolmogorov* spectrum. It is only after this sweeping effect of the larger eddies has been explicitly removed that one can hope to obtain the *Kolmogorov* results. The important thing is to arrive at a strategy for removing the sweeping effect.

We can do so by looking at a high frequency dynamic screening effect. The self-energy of Eq.(8.7.6) has the behaviour $\Sigma(k, 0) \sim k^{2/3}$ at low frequencies. At high frequencies this behaviour is suppressed and power counting in Eq.(8.7.7) shows that

$$\Sigma(k, \omega \gg \Sigma(k)) \sim k^2 / (-i\omega)^2 \quad (8.7.11)$$

A one parameter scaling function for $\Sigma(k, \omega)$ is

$$\Sigma(k, \omega) = \Gamma_0 k^{2/3} \left(1 + \alpha \frac{-i\omega}{\Gamma_0 k^{2/3}} \right)^{-2} \quad (8.7.12)$$

where α is a number of order unity.

The corresponding correlation function which is normalized to the correct low frequency behaviour is

$$C(k, \omega) = \frac{1}{k^{11/3} \Sigma(k)} \left[\frac{1}{1 + \frac{-i\omega}{\Sigma(k, \omega)}} + c.c. \right] \quad (8.7.13)$$

The screening is evident in the frequency integration which would give zero since $\Sigma(k, \omega) \sim (-i\omega)^{-2}$ for high frequencies. Consequently, the difficulty encountered in Eq.(8.7.9) would not arise. The above form is valid for frequencies lower than a cut-off frequency ω_0 , above which the correlation function becomes a Lorentzian once again which restores the equal time form of the time dependent correlation function. This crossover from the form shown above to the Lorentzian is important for the static correlation function but can be ignored in the frequency integration of Eq.(8.7.7). The frequency integration in Eq.(8.7.7) can be written as (for zero external frequency)

$$\begin{aligned}
\int d\omega' G(p, \omega') C(q, -\omega') &= \int d\omega' \frac{[-i\omega' + \Sigma(p)]^{-1}}{q^{11/3} \Sigma(q)} \left\{ \frac{1}{1 + \frac{i\omega'}{\Sigma(q, -\omega')}} + c.c. \right\} \\
&= \frac{1}{q^{11/3} \Sigma(q)} \frac{1}{1 + \frac{\Sigma(p)}{\Sigma(q, -i\Sigma(p))}} \\
&= \frac{1}{q^{11/3}} \frac{1}{q^{2/3} + p^{2/3}} S
\end{aligned} \tag{8.7.14}$$

where

$$S = \frac{q^{2/3} + p^{2/3}}{1 + \frac{p^{2/3}}{q^{2/3}} \left(1 + \alpha \frac{p^{2/3}}{q^{2/3}} \right)^2} \tag{8.7.15}$$

is the screening factor, which is unity when $\alpha = 0$, i.e. the frequency dependence of Σ is ignored. For $q = 0$, the asymptotic behaviour of S is $p^{2/3} \alpha^{-2} q^2 / p^2$ and prevents the integral in Eq.(8.7.7) from diverging. In the above manipulations, we have maintained the pole approximations for the response function G since its role is secondary. We have essentially made the statement that in the time dependent correlation function $\langle v(k, t) v(-k, t + \tau) \rangle$, the small τ behaviour is to be screened out in order to arrive at the *Kolmogorov* spectrum. This has to be done to remove the sweeping effect and unravel the behaviour which we are interested in.

A different point of view due to *Proccacia and L'vov* is to work entirely in coordinate space in a quasi-Lagrangian approach. The short distance singularity that we discussed at the beginning of this section is taken care of by defining a ball of locality in which the correlation function is properly cut off to make the theory finite, i.e. to remove the sweeping contribution. We speculate that what the earlier method does in time, the approach of *Proccacia and L'vov* does in space and it should be possible to explore the connection between the two keeping in mind Taylor's *frozen turbulence hypothesis*.

A different approach was tried by *Yakhot and Orszag* who used the standard renormalization group technique to circumvent the problem. The idea was that one splits the velocity field into two parts - one with high momentum Fourier modes and the other with low momentum Fourier modes, integrates out the high momentum components and studies the effect of that on the low momentum ones. This leads to a new viscosity and after the rescalings of space time and the velocity field a flow equation for the viscosity emerges. The fixed point of the flow corresponds to a scale dependent viscosity. By construction this yields long wavelength, low frequency properties as stressed by *Forster et.al.* In the limit of extremely high *Reynold's* number, one is not particularly worried by this limitation. The process of integrating over the high momentum modes bypasses the difficulty that we encountered with the low- p divergence. We sketch the steps in the following:

- i) split the velocity field as,

$$u_i(x) = u_i^<(x) + u_i^>(x)$$

where

$$u_i^<(x) = \sum_{p < \frac{\Lambda}{b}} u_i(\vec{p}) e^{i\vec{p} \cdot \vec{x}}$$

and

$$u_i^>(x) = \sum_{\frac{\Lambda}{d} < p < \lambda} u_i(\vec{p}) e^{i\vec{p} \cdot \vec{x}}$$

- ii) Find the equations satisfied by $u_i^<(x)$ and $u_i^>(x)$.
- iii) Integrate perturbatively the equation for $u_i^>$
- iv) Insert the solution for $u_i^>$ into the equation for $u_i^<$ and find the equation of motion of $u_i^<$. The linear part gives $\Delta\nu$, the change in viscosity.
- v) Rescale the space and length and the velocity field, so that the new viscosity is given by

$$\begin{aligned} \nu' = & \nu b^{z-2} - 8b^{4-y-2z} F_0 k^{-2} \int_{p=\frac{\Lambda}{b}}^{p=\Lambda} \frac{d^D p}{(2\pi)^D} \int \frac{d\omega}{2\pi} M_{\rho\beta\gamma}(\vec{k}) \\ & \times M_{\sigma\delta\gamma}(\vec{k} - \vec{p}) P_{\beta\delta}(\vec{k} - \vec{p}) P_{\sigma\rho}(\vec{p}) G(\vec{k} - \vec{p}, -\omega) |G(\vec{p}, \omega)|^2 p^{4-D-y} \end{aligned} \quad (8.7.16)$$

Requiring that the two terms on the R.H.S of Eq.(8.7.16) have the same dimensions ensures

$$z = 2 - \frac{y}{3} \quad (8.7.17)$$

and hence

$$\nu(k) = \nu k^{-y/3} \quad (8.7.18)$$

Evaluating the integral in Eq.(8.7.16) in the limit of $k \rightarrow 0$ gives

$$\nu' = \nu \left[1 + \ln b \left(\frac{D_0}{2\nu^3} \frac{D-1}{D+2} \frac{S_D}{(2\pi)^D} - \frac{y}{3} \right) + .. \right] \quad (8.7.19)$$

where S_D is the surface area of a D -dimensional sphere. The fixed point condition yields

$$\frac{\nu^3}{D_0} = \frac{3}{2y} \frac{S_D}{(2\pi)^D} \frac{D-1}{D+2} \quad (8.7.20)$$

One can determine the value of $\frac{D_0^2}{\nu^3}$ and thus $\frac{d_0}{\nu}$ and thus which sets the scale for the *Kolmogorov* spectrum is obtained. For $y = 4$ and $D = 3$, one does find a value very

close to the experimental results and similar success in calculating other universal numbers certainly indicates that this is a successful programme.

The primary difficulties of this approach were pointed out by *De Dominicis and Martin* at the time they introduced the model at the *Kolmogorov* limit i.e. $y = 4$, there is an infinite number of marginal operators and the self-consistent perturbation theory has an infrared divergence at $y = 3$. In the last few years, both these effects have been investigated in a different context - the problem of growth by deposition of atoms on a substrate. The existence of an infinite number of marginal operators and infrared divergences seem to change the roughening and the dynamical exponents of the problem as we have already seen in **Chapter 7**.

The *Kolmogorov* picture in two dimensions is special because there are two cascades to contend with - energy and enstrophy. Enstrophy is defined via Eq. (8.2.10) $(\vec{\nabla} \times v)^2$. The direction of the cascades can be determined from an argument due to *Kraichnan*. With two conserved quantities

$$E = \frac{1}{2} \int v^2(x) d^2x = \frac{1}{2} \sum_k |v(k)|^2 = \int_0^\infty E(k) dk$$

and

$$N = \frac{1}{2} \int (\vec{\nabla} \times v)^2 d^2x = \frac{1}{2} \sum_k k^2 |v(k)|^2 = \int_0^\infty W(k) dk$$

Where $E(k)$ and $W(k)$ are the energy and enstrophy spectrum respectively. The canonical probability distribution for fluctuating E and N is clearly

$$P \sim e^{-\beta E - \mu N} = e^{-\frac{1}{2} \sum (\beta + \alpha k^2) |v_k|^2} \quad (8.7.21)$$

This leads to

$$\langle |v_k|^2 \rangle = (\beta + \alpha k^2)^{-1} \quad (8.7.22)$$

and consequently

$$E(k) = \frac{k}{\beta + \alpha k^2}$$

and

$$W(k) = \frac{k^3}{\beta + \alpha k^2}$$

The enstrophy spectrum $W(k) \sim k$ for high wave number and hence is clearly concentrated towards high wave number side. A spectrum with $W(k) \sim k^x$, $x < 1$, will be out of equilibrium and will proceed towards equilibrium by cascading enstrophy from low to high wave numbers. Energy conservation would then demand an inverse cascade of energy. The enstrophy cascade is dissipated by molecular viscosity at high wave vectors, while the inverse energy cascade causes a condensation phenomenon at low wave numbers. The *Kolmogorov* argument holds

for the inverse energy cascade and one ends up with $E(k) \sim k^{-5/3}$, while for the enstrophy cascade one has to set up the dimensional argument once more, postulating that in the inertial range $E(k)$ is determined by the rate of injection of enstrophy (ϵ_s say) and the local wave number k . This leads to $E(k) \sim k^{-3}$. The infrared difficulties that we talked about previously will appear here as well and a screening approximation will produce a self-energy. For the energy cascade in $D = 2$, $m = 8/3$ and $n = 2/3$ and the inequality $m + n < D + 2$ is satisfied which makes the theory finite. For the enstrophy cascade on the other hand, $m = 4$ and $n = 0$ and we have $m + n = D + 2$. That means a genuine logarithmic divergence and hence for the enstrophy cascade

$$E(k) = C_0(\epsilon_s)^{2/3} k^{-3} \ln \left(\frac{k}{k_0} \right)^{-1/3} \quad (8.7.23)$$

Yet another problem where there exist multiple cascades is the magnetohydrodynamic turbulence. The conserved quantities in the inviscid limit are the kinetic energy and the magnetic energy. Consequently there is a magnetic energy flux and a kinetic energy flux and one can determine the *Kolmogorov* constants etc., for this flow in a manner analogous to that for the pure fluid. The binary liquid is another example of situation with different fluxes - an energy flux and a concentration flux and a renormalization group programme that has been carried out recently.

8.8 Dynamical Systems and Turbulence

Over the last couple of decades an important technique that has been developed in the study of nonlinear system is the reduction to a dynamical system, which is a set of coupled ordinary nonlinear differential equations. By some clever choice of a basis set, the number of equations constituting the dynamical system is kept small. The advantages of the dynamical system are that one can generate a qualitative feel for the solution of the differential equations and also the numerical work becomes considerably simpler. We illustrate the technique by a trivial one dimensional example

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - u^3 \quad (8.8.1)$$

where we imagine that u vanishes outside a strip of length L i.e. $u = 0$ at $x = 0$ and at $x = L$. Clearly, we can expand

$$u(x, t) = \sum_{m=1}^L C_m(t) \sin \frac{m\pi x}{L} \quad (8.8.2)$$

If we insert this expansion in Eq.(8.8.1), then

$$\sum_{m=1}^L \dot{C}_m(t) \sin \frac{m\pi x}{L} = \sum_{m=1}^L (a - m^2) C_m(t) \sin \frac{m\pi x}{L} - \left[\sum_{m=1}^L C_m(t) \sin \frac{m\pi x}{L} \right]^3 \quad (8.8.3)$$

We now need to equate the coefficient of $\sin \frac{m\pi x}{L}$ from either side of Eq.(8.8.3). For $m = 1$, we see immediately that the nonlinear term u^3 is capable of contributing $C_1 C_2^2$, $C_1 C_3^2$, $C_1 C_4^2$ etc and hence the dynamical system will be infinitely large unless there is a truncation at some point. Let us decide to keep only two terms in the above expression and now we have

$$\begin{aligned} \dot{C}_1 &= (a - 1)C_1 - \frac{3}{4} C_1^3 - \frac{3}{2} C_1 C_2^2 \\ \dot{C}_2 &= (a - 4)C_2 - \frac{3}{4} C_2^3 - \frac{3}{2} C_1^2 C_2 \end{aligned} \quad (8.8.4)$$

The above equations constitute 2-dimensional dynamical system which is supposed to mimic the partial differential equation of Eq.(8.8.1). The analysis of a dynamical system proceeds by looking at fixed points i.e. points in (C_1, C_2) space for which $\dot{C}_1 = \dot{C}_2 = 0$. Now, clearly $C_1 = 0, C_2 = 0$ is a fixed point of the above system. There is a totally nontrivial fixed point which is to be obtained as the solution of

$$\begin{aligned} \frac{3}{4} C_1^2 + \frac{3}{2} C_2^2 &= a - 1 \\ \frac{3}{2} C_1^2 + \frac{3}{4} C_2^2 &= a - 4 \end{aligned} \quad (8.8.5)$$

and two other fixed points

$$C_1^2 = \frac{4}{3}(a - 1), C_2 = 0$$

and

$$C_1 = 0, C_2^2 = \frac{4}{3}(a - 4).$$

If the fixed point is stable, then it determines the long time dynamics of the system. To test for stability, one causes a small perturbation around the fixed point and linearizes in the perturbation. For the $(0, 0)$ fixed point e.g. the perturbation $\delta C_1, \delta C_2$ satisfies

$$\begin{aligned} \delta \dot{C}_1 &= (a - 1)\delta C_1 \\ \delta \dot{C}_2 &= (a - 4)\delta C_2 \end{aligned} \quad (8.8.6)$$

Clearly, if $a < 1$, δC_1 and δC_2 both decay to zero. Consequently, the trivial solution $C_1 = C_2 = 0$ is the only stable solution for $a < 1$. For Eq.(8.8.1) this means that

under the boundary conditions $u = 0$ at $x = 0$ and $x = L$, the only stable solution is $u = 0$ for $a < 1$. For $a > 1$, δC_1 grows and δC_2 decays. Thus, the mode $\sin \frac{2\pi x}{L}$ is not excited. The fixed point $(\sqrt{\frac{4}{3}}(a - 1), 0)$ is stable and governs the dynamics. For a significantly greater than unity the dimension of the dynamical system has to be increased for the solution to make sense.

In the interesting issue of whether the study of dynamical systems will offer insights into the problem of fully developed turbulence, shell models have come in extremely handy. In the problem of hydrodynamic instabilities like convection, *Taylor-Couette* instability etc. dynamical systems have been known to be extremely useful and general methods for arriving at the dynamical system have been known for a long time. These techniques almost invariably make use of a Fourier expansion. In the problem of turbulence Fourier decomposition does not help, since the nonlinear term remains equally strong for all Fourier components and truncation is not meaningful. This is where wavelet expansion forms a good tool for obtaining shell models, starting from the Navier-Stokes equation. However, the kind of shell model that we can arrive at from the wavelet expansion are the ones with real coefficients- the models of *Carbone, Gleaques et.al*. We cannot address the question of deriving the **GOY** type models which uses complex coefficients. Here we systemize the earlier work of *Siggia* and *Nakano*. In the process of derivation, it will become clear where exactly the discarding of coefficients in the complete set expansion, the coarse graining procedure and simplification of the cascade process occur to arrive at the structure of the shell model. In shell models, the Fourier space is divided into shells, each shell spanning one octave with wave number k such that for the n^{th} shell $\frac{k_0}{2^n} > |k| > \frac{k_0}{2^{n+1}}$. Wavelets are ideal for deriving shell models since,

- i) wavelets very naturally fit into the idea of dividing k -space into octaves
- ii) the wavelet basis is complete
- iii) the wavelet basis function are localized and thus contain information about local structure.

The wavelet basis is complete and orthonormal, obtained by dilatation and translation of a single function $\psi(x)$. This function $\psi(x)$ is not unique. In one dimension, a function $f(x)$ can be expanded in terms of these basis functions as

$$f(x) = \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \psi^{(m)}(x - 2^m i) \omega^m[i] \quad (8.8.7)$$

where

$$\psi^m(x) = 2^{-m/2} \psi(x/2^m) \quad (8.8.8)$$

Orthonormality implies

$$\int_{-\infty}^{\infty} \psi^{(m)}(x - 2^m i) \psi^{(n)}(x - 2^m j) dx = \delta_{mn} \delta_{ij} \quad (8.8.9)$$

The wavelet coefficients are given by

$$\omega^{(m)}[i] = \int_{-\infty}^{\infty} \psi^{(m)}(x - 2^m i) f(x) dx \quad (8.8.10)$$

and orthonormality implies

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \{\omega^{(m)}[i]\}^2 \quad (8.8.11)$$

The wavelet coefficients represent a decomposition of the function into its contribution to different scales and different positions. The wavelet $\psi^{(m)}(x - 2^m i)$ is localized around $x = 2^m i$ and has a width of the order of 2^m . So $\omega^{(m)}[i]$ represents the behaviour of the function $f(x)$ around the point $2^m i$ on a scale of 2^m . In higher dimensions, the set of basis functions need to be extended but the general structure remains the same.

We begin by considering the one dimensional problem where the velocity field $U(x, t)$ can be expanded as

$$U(x, t) = \sum_{m=1}^N \sum_i U_i^{(m)}(t) \psi_i^{(m)}(x) \quad (8.8.12)$$

Substituting in the one dimensional Burgers equation (no dissipation)

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = 0 \quad (8.8.13)$$

$$\begin{aligned} \frac{\partial U_j^n(t)}{\partial t} &= - \sum_{m_1, m_2, i_1, i_2} \int dx \psi_j^{(n)}(x) \psi_{i_1}^{(m_1)}(x) \frac{d}{dx} \psi_{i_2}^{(m_2)}(x) U_{i_1}^{m_1}(t) U_{i_2}^{m_2}(t) \\ &= \sum_{m_1, m_2, i_1, i_2} C_{j i_1 i_2}^{n m_1 m_2} U_{i_1}^{m_1}(t) U_{i_2}^{m_2}(t) \end{aligned} \quad (8.8.14)$$

For the D -dimensional case, the structure is almost identical (the pressure term for an incompressible fluid has the same structure as $\vec{U} \cdot \vec{\nabla} \vec{U}$) and hence $C_{j i_1 i_2}^{n m_1 m_2}$ is to be found from

$$C_{j i_1 i_2}^{n m_1 m_2} = \int \psi_j^n(x) \psi_{i_1}^{m_1}(x) \frac{d}{dx} \psi_{i_2}^{m_2}(x) d^D x \quad (8.8.15)$$

The prefactors of ψ_j^n yield a factor $2^{-(n+m_1+m_2)D/2}$. In doing the integration, we note that there must be overlap between ψ_j^n and $\psi_{i_1}^{m_1}$ and the delta function coming from the derivative must be non zero in the overlapping part. These considerations lead to the value $2^{(D-1)\text{Min}(n, m_1, m_2)}$, so that

$$C_{j i_1 i_2}^{n m_1 m_2} \sim 2^{-(n+m_1+m_2)D/2} 2^{(D-1)\text{Min}(n, m_1, m_2)} \quad (8.8.16)$$

If n is the smallest of n, m_1, m_2 , then

$$C_{j i_1 i_2}^{n m_1 m_2} \sim 2^{(n-m_1-m_2)D/2} 2^{-n} \quad (8.8.17)$$

The largest coefficients in this case are

$$C_{j i_1 i_2}^{n n n} \sim 2^{-n(1+D/2)} \quad (8.8.18)$$

$$C_{j i_1 i_2}^{n n n+1} \sim 2^{-n(1+D/2)} 2^{-D/2} \quad (8.8.19)$$

$$C_{j i_1 i_2}^{n n+1 n+1} \sim C_{j i_1 i_2}^{n n n+2} \sim 2^{-(n+2)D/2} 2^{-n} \quad (8.8.20)$$

Similarly, if m_1 or m_2 is the smallest of n, m_1, m_2 the largest coefficients are

$$C_{j i_1 i_2}^{n m m} \sim 2^{-nD/2} 2^{-m} \quad (8.8.21)$$

$$C_{j i_1 i_2}^{n m m+1} \sim 2^{-(n+1)D/2} 2^{-m} \quad (8.8.22)$$

$$C_{j i_1 i_2}^{n m+1 m+1} \sim C_{j i_1 i_2}^{n m m+2} \sim 2^{-nD/2} 2^{-m} \sim 2^{-(n+2)D/2} 2^{-m} \quad (8.8.23)$$

We now rewrite Eq.(8.8.14) keeping terms upto $O(2^{-D})$. Our approximation obviously improves as the spatial dimension increases. This yields

$$\begin{aligned} \frac{\partial U_j^n}{\partial t} = \sum_{m=1, i_1, i_2}^n [& C_{j i_1 i_2}^{n m m} U_{i_1}^m U_{i_2}^m + C_{j i_1 i_2}^{n m m+1} U_{i_1}^m U_{i_2}^{m+1} \\ & + C_{j i_1 i_2}^{n m+1 m+1} U_{i_1}^{m+1} U_{i_2}^{m+1} + C_{j i_1 i_2}^{n m m+2} U_{i_1}^m U_{i_2}^{m+2}] \end{aligned} \quad (8.8.24)$$

There is a lot of local structure in the above equation since there is a sum over the lattice points where the wavelets are centred. To simplify, one must coarse-grain. This requires a summation over i_1 and i_2 with the assumption that U_i^m is sufficiently weakly dependent on i , so that a replacement by the average value U^m is sensible. This reduces Eq.(8.8.14) to

$$\begin{aligned} \frac{\partial U_j^n}{\partial t} = \sum_{m=1}^n 2^{-nD/2} [& a^{n m m} U^m U^m + a^{n m m+1} U^m U^{m+1} \\ & + a^{n m+1 m+1} U^{m+1} U^{m+1} + a^{n m m+2} U^m U^{m+2}] \end{aligned} \quad (8.8.25)$$

where a factor of $2^{-nD/2}$ has been pulled out of each $C^{n m m}$.

For the inviscid and unforced flow, the total energy is conserved. This condition can be written as

$$0 = \frac{d}{dt} \frac{1}{2} \sum_{m=1}^N \sum_i U_i^m U_i^m = \sum_{m=1}^N \sum_i U_i^m \dot{U}_i^m = \sum_{m=1}^N 2^{N-m} U^m \dot{U}^m \quad (8.8.26)$$

Imposition of this condition leads to

$$\begin{aligned} \dot{U}^n = 2^{-n(1+D/2)} & \left[\left(U^{n-1} U^{n-1} - \frac{1}{2} 2^{-3D/2} U^n U^{n+1} \right) \right. \\ & + \alpha \left(U^{n-2} U^{n-2} - 2^{-3D/2-2} U^n U^{n+2} \right) \\ & + \beta \left(U^n U^{n-1} - \frac{1}{2} 2^{-3D/2} U^{n+1} U^{n+1} \right) \\ & \left. + \gamma \left(U^{n-1} U^{n-2} - \frac{1}{2} 2^{-3D/2} U^{n-1} U^{n+1} \right) \right] \quad (8.8.27) \end{aligned}$$

Now, suppose that at time $t = 0$, $U^n = 0$ for all $n \leq m$. In that case, Eq.(8.8.27) becomes

$$\dot{U}^m = 2^{-m-m\frac{D}{2}} \left[-\frac{\beta}{2} 2^{-3D/2} U^{m+1} U^{m+1} \right] \quad (8.8.28)$$

We see that U^m will be excited by U^{m+1} and there will be a transfer of energy from large scale modes to smaller scale modes - a forward energy cascade. In a similar fashion if $U^n = 0$ for all $n \geq m$ at $t = 0$, then

$$\dot{U}^m = 2^{-m-mD/2} \left[U^{m-1} U^{m-1} + \alpha U^{m-2} U^{m-2} + \gamma U^{m-1} U^{m-2} \right] \quad (8.8.29)$$

This represents a flow of energy from lower scales to higher scales - a backward cascade of energy. We now introduce a further simplification in Eq.(8.8.27) - only one forward cascade and one backward cascade will be kept. This leads to,

$$\begin{aligned} \dot{U}^n = 2^{-n(1+D/2)} & \left[\left(U^{n-1} U^{n-1} - \frac{1}{2} 2^{-3D/2} U^n U^{n+1} \right) \right. \\ & \left. + \beta \left(U^n U^{n-1} - \frac{1}{2} 2^{-3D/2} U^{n+1} U^{n+1} \right) \right] \quad (8.8.30) \end{aligned}$$

We now choose $\beta = -2^p$ which assures us that if $U^n > 0$ at any time, it will remain non negative at all subsequent times. This can be seen by considering a point of time at which $U^n = 0$. Then,

$$\dot{U}^n = 2^{-n(1+D/2)} \left[U^{n-1} U^{n-1} + \frac{1}{2} 2^{p-3D/2} U^{n+1} U^{n+1} \right] > 0 \quad (8.8.31)$$

Clearly, Eq.(8.8.30) has a fixed point solution, $U^n \propto 2^{\frac{nD}{2} + \frac{n}{3}}$ independent of the value of p . This fixed point corresponds to the *Kolmogorov* spectrum since the total energy contained in a shell of scale n is

$$\int_{k_0 2^{-n}}^{k_0 2^{-(n+1)}} E(k) dk = \frac{1}{2} \sum_n [U^n]^2 \sim 2^{(N-n)D} 2^{nD + \frac{2n}{3}} \quad (8.8.32)$$

If $E(k) \sim k^{-\alpha}$, then $\alpha = 5/3$.

We can absorb the factor $2^{-3D/2}$ by a redefinition of the U as $U 2^{nD/2}$ and the resulting equation is precisely that used as a starting point by *Carbone* (a superficial difference in the position of the factor 2 in the coefficient of $U^{n+1} U^{n+1}$ occurs because our $k_n = k_0/2^n$ as opposed to $k_n = k_0 2^n$ used by others) when the linear energy dissipating term and the forcing term are introduced. Since the *Kolmogorov* spectrum is obtained for all p , we can choose $p = 0$ and write for $n \leq N$

$$\begin{aligned} \dot{U}^n &= k_n (U^{n-1} U^{n-1} - \frac{1}{2} U^n U^{n+1}) - k_n (U^n U^{n-1} - \frac{1}{2} U^{n+1} U^{n+1}) \\ &= -\nu k_n^2 U^n + \delta_{n,N} \end{aligned} \quad (8.8.33)$$

Where $-\nu k_n^2 U^n$ represents the viscous loss and $\delta_{n,N}$ is the term which signifies that in the N^{th} shell (large length scale) there is an extra forcing coming from outside.

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Polymers

9.1 Introduction

A polymer is a large molecule made up of many small chemical units joined together by some chemical bond. Most artificially produced polymers are a repetitive sequence of a particular atomic group and take the form -A-A-A-. The basic unit (A in the example) is called a monomer and the number of monomer units in a polymer is usually called its degree of polymerization. A polymer molecule has a large number of internal degrees of freedom and so can take many different configurations. It is because of these configurations, that statistical mechanics can be applied even to a single polymer chain. An easy way of picturing the polymer is to imagine it lying on a regular lattice.

The circles at some of the lattice points represent the monomer units and the heavy lines along the lattice connecting the circles are the bonds. We assume that the bonds are free to orient themselves in any manner. So all directions have equal probability and the statistics of the polymer configurations is the same as that of a random walk. One of the main issues is the average end to end distance R on the degree of polymerization N . The polymer in the Fig. 9.1 is made up of N bonds. If the radius vector of the n^{th} bond is $\vec{r}_n$, then the “end-to end” vector $\vec{R}$ will be

$$\vec{R} = \sum_n \vec{r}_n \quad (9.1.1)$$

Clearly, $\langle \vec{R} \rangle = 0$ and hence it is $\langle R^2 \rangle$ which one calculates.

$$\begin{aligned} \langle R^2 \rangle &= \sum_m \sum_n \langle \vec{r}_m \cdot \vec{r}_n \rangle \\ &= b^2 \delta_{mn} \\ &= Nb^2 \end{aligned} \quad (9.1.2)$$

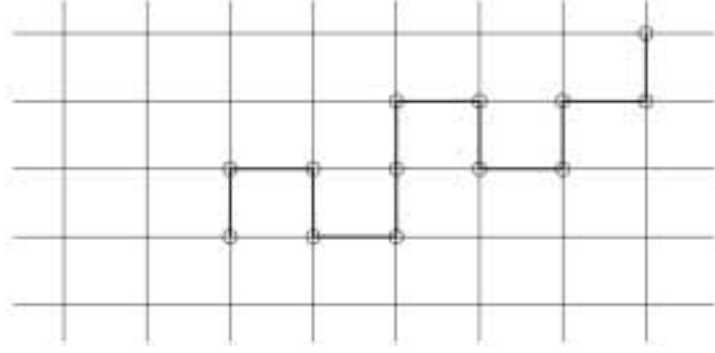


Figure 9.1. A polymer lying along a regular lattice

One can calculate the probability distribution of $\vec{R}$. We consider a N -monomer polymer with one end pegged at $\vec{r} = 0$. The probability of the other end being at $\vec{R}$ is $P(\vec{R}, N)$. If the coordination number of the lattice (i.e. number of nearest neighbours) is Z , and if b_i ($i = 1, 2, \dots, Z$) are the possible bond vectors that a polymer may take then if the polymer has reached $\vec{R}$ after N steps, then after $N - 1$ steps it could have been at any of $\vec{R} - \vec{b}_i$ and the probability of being at any of these positions is $1/Z$. Thus,

$$P(\vec{R}, N) = \frac{1}{Z} \sum_{i=1}^Z P(\vec{R} - \vec{b}_i, N - 1) \quad (9.1.3)$$

If $N \gg 1$ and $|\vec{R}| \gg |\vec{b}_i|$, we can expand the R.H.S and write

$$\begin{aligned} P(\vec{R}, N) &= P(\vec{R}, N) - \frac{1}{Z} \sum_{i=1}^Z \frac{\partial P}{\partial N} - \frac{1}{Z} \sum_{i=1}^Z b_{i\alpha} \frac{\partial P}{\partial R_\alpha} \\ &\quad + \frac{1}{2Z} \sum_{i=1}^Z b_{i\alpha} b_{i\beta} \frac{\partial^2 P}{\partial R_\alpha \partial R_\beta} + \dots \end{aligned} \quad (9.1.4)$$

Now,

$$\begin{aligned} \frac{1}{Z} \sum_{i=1}^Z b_{i\alpha} &= 0 \\ \text{and } \frac{1}{Z} \sum_{i=1}^Z b_{i\alpha} b_{i\beta} &= \frac{b^2}{3} \delta_{\alpha\beta} \end{aligned}$$

From Eq.(9.1.4), we then get the differential equation,

$$\frac{\partial P}{\partial N} = \frac{b^2}{6} \frac{\partial^2 P}{\partial R^2} \quad (9.1.5)$$

with the boundary condition that $\vec{R} = 0$, if $N = 0$, we have the solution

$$P(\vec{R}, N) = \left(\frac{3}{2\pi N b^2} \right)^{3/2} e^{-\frac{3}{2} \frac{R^2}{N b^2}} \quad (9.1.6)$$

The probability distribution of R is Gaussian.

The assumption that the orientation of each bond is random is clearly not realistic. It allows the polymer to fold back on to itself at certain locations which is not allowed. One can remedy this by disallowing doubling back, which means that the bond vector $\vec{r}_{n+1}$ is not allowed to point back, that is it cannot be $-\vec{r}_n$, but can take any of the other $Z - 1$ configurations with equal probability. With this constraint the average used in deriving Eq.(9.1.2) becomes

$$\langle \vec{r}_n \cdot \vec{r}_m \rangle = \frac{b^2}{(z - 1)^{|n-m|}} \quad (9.1.7)$$

With the correlations falling off one still gets

$$\langle R^2 \rangle = N b_{eff}^2 \quad (9.1.8)$$

where b_{eff}^2 can be obtained from b^2 .

The distribution for R is always Gaussian for such chains. Interactions which may occur between two monomer units which are in close proximity along the length of the chain are called short range interactions. In ideal chains (no long range interactions), the relation $\langle R^2 \rangle \propto N$ always holds.

The major change occurs when we put in long range interactions. The main interaction is the excluded volume effect, which means no two monomers can be at the same point in space. Consequently, as the chain winds around two monomer units, however far they may be along the length of the chain, they repel each other equally strongly when they come close to each other. This excluded volume interaction which is genuinely long ranged changes the relation between $\langle R^2 \rangle$ and N .

We now move away from lattice models and introduce the gaussian model based on the experience of the lattice. The Gaussian model assumes that the bond vector $\vec{r}$ itself possesses some flexibility and follows a Gaussian distribution

$$p(r) = \left(\frac{3}{2\pi b^2} \right)^{3/2} \exp \left\{ -\frac{3}{2} \frac{r^2}{b^2} \right\} \quad (9.1.9)$$

If the position of the n^{th} monomer is $\vec{R}_n$ and that of the $(n+1)^{th}$ monomer is $\vec{R}_{n+1}$, the probability distribution of the set of position vectors $\{\vec{R}_n\} = (\vec{R}_0 \vec{R}_1 \dots \vec{R}_N)$ is

$$P(\{\vec{R}_n\}) = \left(\frac{3}{2\pi b^2} \right)^{3N/2} \exp \left\{ -\frac{3}{2b^2} \sum_{n=1}^N (\vec{R}_n - \vec{R}_{n+1})^2 \right\} \quad (9.1.10)$$

One can think of the Gaussian chain as a linkage of monomers consisting of harmonic springs. If K is the spring constant, the energy of the chain can be written as

$$E = \frac{1}{2} K \sum_{n=1}^N (\vec{R}_n - \vec{R}_{n+1})^2 \quad (9.1.11)$$

with the equilibrium state of the chain given by $\exp\{-E/K_B T\}$, if the spring constant K can be chosen to be $K = \frac{3K_B T}{b^2}$. Because of the connection with springlike behaviour, the gaussian chain, pictured in Fig 9.2, is often called a bead spring model. We can write a continuum form of the probability distribution given by



Figure 9.2. A bead and spring model of the polymer

identifying $(\vec{R}_n - \vec{R}_{n+1})^2$ with $(\frac{\partial \vec{R}}{\partial \tau})^2$ where τ is a coordinate along the length of the chain. The probability distribution for a particular $\vec{R}(\tau)$ is then given by the path integral

$$P(\vec{R}(\tau)) \propto \exp \left\{ -\frac{3}{2K_B T} \int_0^N d\tau \left(\frac{\partial \vec{R}}{\partial \tau} \right)^2 \right\} \quad (9.1.12)$$

This is the probability distribution for the ideal chain. Now we need to include the excluded volume effect. The interaction is between $\vec{R}(\tau)$ and $\vec{R}(\tau')$, when the two vectors are very close regardless of where τ and τ' are. We put in an absolutely short range interaction which would be a delta function interaction represented by $v_0 \delta(\vec{R}(\tau) - \vec{R}(\tau'))$. The probability distribution of Eq.(9.1.12) can now be written as

$$P \propto \exp \left\{ -\frac{F}{K_B T} \right\} \quad (9.1.13)$$

where

$$F = \frac{3}{2K_B T} \int_0^N d\tau \left(\frac{\partial \vec{R}}{\partial \tau} \right)^2 + \frac{v_0}{2} \int_0^N d\tau \int_0^N d\tau' \delta(\vec{R}(\tau) - \vec{R}(\tau')) \quad (9.1.14)$$

With the statics set up as shown, we can now write the dynamics around the equilibrium state in the form of a Langevin equation

$$\dot{R}_\alpha(\tau) = -\Gamma \frac{\delta F}{\delta R_\alpha} + N_\alpha$$

where $\langle N_\alpha(\tau, t) N_\alpha(\tau', t) \rangle = 2\Gamma K_B T \delta_{\alpha\beta} \delta(\tau - \tau') \delta(t - t')$ (9.1.15)

This dynamics can become extremely complicated because of the long range interaction.

We will also be considering the question of interaction of the polymer with the solvent and the resulting inter monomer interaction mediated by the solvent. For hydrophobic monomers, this interaction can cause the polymer to fold itself into a ball to avoid the water contact, while for hydrophilic contacts, the polymer chain is expected to swell.

Returning to the bead spring model if we focus on the dynamics of a particular bead, say the bead located at $\vec{R}_n$, then the Langevin dynamics of that bead will be governed by the equation of motion

$$\frac{d\vec{R}_n}{dt} = -\frac{1}{\zeta} \frac{\partial E}{\partial \vec{R}_n} + \vec{g}_n \quad (9.1.16)$$

where ζ is the coefficient of friction of a bead. Using E from Eq.(9.1.11) we can write, for the intermediate beads ($n = 1, 2, \dots, N-1$)

$$\frac{d\vec{R}_n}{dt} = -\frac{K}{\zeta} (\vec{R}_{n+1} + \vec{R}_{n-1} - 2\vec{R}_n) + \vec{g}_n \quad (9.1.17)$$

For the beads at the two end-points $n = 0$ and $n = N$,

$$\frac{d\vec{R}_0}{dt} = -\frac{K}{\zeta} (\vec{R}_1 - \vec{R}_0) + \vec{g}_0 \quad (9.1.18)$$

$$\frac{d\vec{R}_N}{dt} = -\frac{K}{\zeta} (\vec{R}_{N-1} - \vec{R}_N) + \vec{g}_N \quad (9.1.19)$$

In the above

$$K = \frac{3K_B T}{b^2}.$$

In order to cast the above equation in the same form as Eq.(9.1.17), we can write

$$\begin{aligned} \vec{R}_{-1} &= \vec{R}_0 \\ \vec{R}_{N+1} &= \vec{R}_N \end{aligned} \quad (9.1.20)$$

in which case Eq.(9.1.18) holds for $n = 0, 1, \dots, N$ with the end-point (boundary) conditions

$$\frac{\partial \vec{R}_N}{\partial t} = 0, \quad \text{for } n = 0, N \quad (9.1.21)$$

if we think of n as continuous rather than a discrete variable. In this picture of continuously varying n , the equation of motion becomes

$$\frac{\partial \vec{R}}{\partial t} = \frac{K}{\zeta} \frac{\partial^2 \vec{R}}{\partial n^2} + \vec{g}(n, t) \quad (9.1.22)$$

with the above boundary conditions.

The solution for $\vec{R}(n, t)$ can be obtained in terms of the normal modes $\vec{X}_p(t)$. The normal modes are introduced through the relation

$$\vec{R}(n, t) = \sum_p \vec{X}_p(t) \cos \frac{np\pi}{N} \quad (9.1.23)$$

where $p = 0, 1, 2, \dots$. The inversion of this relation yields

$$\vec{X}_p(t) = \frac{1}{N} \int_0^N dn \vec{R}(n, t) \cos \frac{np\pi}{N} \quad (9.1.24)$$

Inserting the normal mode expansion in Eq.(9.1.22), we find

$$\begin{aligned} \dot{\vec{X}}_p(t) &= -\frac{p^2 \pi^2}{N^2} \frac{K}{\zeta} \vec{X}_p + \vec{g}_p \\ &= -\frac{3\pi^2 K_B T}{N^2 b^2} \frac{p^2}{\zeta} \vec{X}_p + \vec{g}_p \\ &= -\frac{\vec{X}_p}{\tau_p} + \vec{g}_p \end{aligned} \quad (9.1.25)$$

where we have used the value of K from Eq.(9.1.19) and the relaxation rate τ_p for the p^{th} mode is written as

$$\tau_p = \frac{\zeta N^2 b^2}{3\pi^2 k_B T} \frac{1}{p^2} \quad (9.1.26)$$

The random force $\vec{g}_p$ has zero mean and its variance is prescribed as

$$\left\langle g_{p\alpha}(t) g_{q\beta}(t') \right\rangle = k_B T \delta_{pq} \delta_{\alpha\beta} \delta(t - t') \quad (9.1.27)$$

The solution of Eq.(9.1.25) can be written as

$$X_{p\alpha}(t) = \int_0^t e^{-\frac{(t-t')}{\tau_p}} g_{p\alpha}(t') dt' + X_{p\alpha}(0) e^{-t/\tau_p} \quad (9.1.28)$$

The correlation functions can be found as

$$\begin{aligned} \left\langle X_{p\alpha}(t_1) X_{q\beta}(t_2) \right\rangle &= \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{-\frac{(t_1-t')}{\tau_p}} e^{-\frac{(t_2-t'')}{\tau_q}} \left\langle g_{p\alpha}(t') g_{q\beta}(t'') \right\rangle \\ &\quad + e^{-\left(\frac{t_1}{\tau_p} + \frac{t_2}{\tau_q}\right)} \left\langle X_{p\alpha}(0) X_{q\beta}(0) \right\rangle \\ &= \int_0^{t_2} dt'' e^{-\frac{(t_1-t'')}{\tau_p}} e^{-\frac{(t_2-t'')}{\tau_q}} \\ &\quad + e^{-\left(\frac{t_1}{\tau_p} + \frac{t_2}{\tau_q}\right)} \left\langle X_{p\alpha}(0) X_{q\beta}(0) \right\rangle \\ &= \frac{\tau_p}{2} \left[e^{-\frac{(t_1-t_2)}{\tau_p}} - e^{-\frac{(t_1+t_2)}{\tau_p}} \right] \\ &\quad + e^{-\left(\frac{t_1}{\tau_p} + \frac{t_2}{\tau_q}\right)} \left\langle X_{p\alpha}(0) X_{q\beta}(0) \right\rangle \end{aligned} \quad (9.1.29)$$

If $t_1 - t_2 = \tau$ and τ is large then,

$$\left\langle X_{p\alpha}(t + \tau) X_{q\beta}(t) \right\rangle = e^{-\tau/\tau_p} \delta_{pq} \delta_{\alpha\beta} \quad (9.1.30)$$

This is true if $p \neq 0$. If $p = 0$, then $\tau_p \rightarrow \infty$, and Eq.(9.1.28) yields

$$X_{0\alpha}(t) - X_{0\alpha}(0) = \int_0^t g_{0\alpha}(t') dt' \quad (9.1.31)$$

Consequently,

$$\begin{aligned} \left\langle [X_{0\alpha}(t) - X_{0\alpha}(0)][X_{0\beta}(t) - X_{0\beta}(0)] \right\rangle &= \int_0^t dt' \int_0^t dt'' \left\langle g_{0\alpha}(t') g_{0\beta}(t'') \right\rangle \\ &= \delta_{\alpha\beta} \int_0^t dt' \\ &= \delta_{\alpha\beta} t \end{aligned} \quad (9.1.32)$$

We can now investigate various physical quantities associated with the polymer dynamics. First let us look at the centre of mass motion. The centre of mass coordinate $\vec{R}_{CM}$ is

$$\begin{aligned}
\vec{R}_{CM}(t) &= \frac{1}{N} \int_0^N dn \vec{R}(n, t) \\
&= \frac{1}{N} \int_0^N dn \sum_p X_p(t) \cos \frac{n\pi p}{N} \\
&= \frac{1}{N} \sum_p X_p(t) \frac{N}{p\pi} \delta_{p0} \\
&= X_0(t)
\end{aligned} \tag{9.1.33}$$

Hence

$$\left\langle [\vec{R}_{CM}(t) - \vec{R}_{CM}(0)]^2 \right\rangle = \left\langle [X_0(t) - X_0(0)]^2 \right\rangle = t \tag{9.1.34}$$

The centre of mass diffuses with a diffusion constant

$$D_{CM} = \frac{k_B T}{N\zeta},$$

which is inversely proportional to N .

If one wants to look at the rotational motion of the polymer as a whole, then one needs to look at the end-to-end distance vector $\vec{R}(t)$ defined as

$$\begin{aligned}
\vec{R}(t) &= \vec{R}(N, t) - \vec{R}(0, t) \\
&= \sum_p X_p(t) (\cos p\pi - 1) \\
&= - \sum_p X_p(t) 2 \sin^2 \frac{p\pi}{2} \\
&\quad \text{positive} \\
&= -4 \sum_{p, \text{ odd}} X_p(t)
\end{aligned} \tag{9.1.35}$$

The correlation function for $\vec{R}(t)$ can be written as

$$\begin{aligned}
\left\langle R_\alpha(t) R_\alpha(0) \right\rangle &= 16 \sum_{p \text{ odd}} \left\langle X_{p\alpha}(t) X_{p\alpha}(0) \right\rangle \\
&= N b^2 \sum_{p \text{ odd}} \frac{8}{p^2 \pi^2} e^{-t/\tau_p}
\end{aligned} \tag{9.1.36}$$

The smallest τ_p will dominate in the above series and hence the rotational relaxation rate is

$$\tau_1 = \frac{\zeta N^2 b^2}{3\pi^2 k_B T} \simeq \frac{N b^2}{D_{CM}}$$

which is the time required for the centre of mass to diffuse through a distance comparable to the size of the polymer.

To study the internal motion of the polymer chain, we consider the mean square displacement of the n^{th} segment, i.e.

$$d^2 = \left\langle [\vec{R}(n, t) - \vec{R}(n, 0)]^2 \right\rangle$$

Using Eq.(9.1.23) we can write this as

$$\begin{aligned} d^2 &= \left\langle \left[X_{0\alpha}(t) - X_{0\alpha}(0) + 2 \sum_{p=1}^{\infty} \cos \frac{np\pi}{N} \{X_{p\alpha}(t) - X_{p\alpha}(0)\} \right] \right. \\ &\quad \times \left. \left[X_{0\alpha}(t) - X_{0\alpha}(0) + 2 \sum_{q=1}^{\infty} \cos \frac{nq\pi}{N} \{X_{q\alpha}(t) - X_{q\alpha}(0)\} \right] \right\rangle \\ &= \left\langle (X_{0\alpha}(t) - X_{0\alpha}(0))(X_{0\alpha}(t) - X_{0\alpha}(0)) \right\rangle \\ &\quad + 4 \sum_{p=1}^{\infty} \cos^2 \frac{np\pi}{N} \left[(X_{p\alpha}(t) - X_{p\alpha}(0))(X_{p\alpha}(t) - X_{p\alpha}(0)) \right] \\ &= 6D_{CM}t + \frac{4N^2b^2}{\pi^2} \sum_{p=1}^{\infty} \cos^2 \frac{\pi pn}{N} (1 - e^{-p^2t/\tau_1}) \frac{1}{p^2} \end{aligned} \quad (9.1.37)$$

where we have written τ_p as τ_1/p^2 . If $t \gg \tau_1$ (the relaxation time for rotational motion), the centre of mass motion dominates. For $t \ll \tau_1$, the internal modes dominate and the second term in Eq.(9.1.37) can be significant. This can be evaluated by replacing the sum by an integral and substituting the average value for $\cos^2 n\pi p/N$. Since

$$\langle \cos^2 \theta \rangle = 1/2,$$

we have

$$\begin{aligned} \langle d^2 \rangle &= 6D_{CM}t + \frac{2N^2b^2}{\pi^2} \sum_{p=1}^{\infty} \cos^2 \frac{\pi pn}{N} \frac{(1 - e^{-p^2t/\tau_1})}{p^2} \\ &= 6D_{CM}t + \frac{2N^2b^2}{\pi^2} \left(\frac{t}{\tau_1} \right) \int_0^{\infty} dy \frac{1 - e^{-y^2}}{y^2} \end{aligned} \quad (9.1.38)$$

This expression clearly shows the crossover from a linear in t behaviour to a square root t behaviour on the time scale τ_1 . The above picture of the dynamics of a polymer chain is called the *Rouse Model*. Its predictions are

- i) $D_{CM} \propto N^{-1} \propto M^{-1}$
- ii) $\tau \propto M^2$

where M is the mass of the polymer. The experimental measurements yield

$$\begin{aligned} D_{CM} &\propto M^{-\nu} \\ \tau &\propto M^{3\nu} \end{aligned} \quad (9.1.39)$$

In an ideal state $\nu = 1/2$ and in a good solvent $\nu \simeq 3/5$. The discrepancy could come from three different sources

- a) *The response of a monomer to an applied force has been taken to be local. However, the monomer will distort the velocity field all over the fluid and this can affect a distant monomer. This is called the backflow effect.*
- b) *Ideal chain elasticity, as expressed through the energy expression fails for a good solvent.*
- c) *Rouse model ignores the fact that real chains do not cross. The repulsion expressed in the second term of Eq.(9.1.38) is absent in the model just discussed.*

We will include the correlation due to the backflow effect by looking at the velocity V_n of the bead at $\vec{n}$ due to a force $\vec{F}_m$ at $\vec{m}$. We will write this response in the form of a mobility matrix μ_{nm} defined as

$$V_n = \sum_m \mu_{nm} F_m \quad (9.1.40)$$

We picture the beads as a set of spheres in a fluid of viscosity η . The fluid velocity $\vec{v}$ satisfies

$$\eta \nabla^2 \vec{v} = \vec{\nabla} P \quad (9.1.41)$$

where we have assumed a steady state situation and the velocity is low enough for the nonlinear term in the Navier-Stokes equation to be dropped. It is an incompressible flow with

$$\vec{\nabla} \cdot \vec{v} = 0$$

Since μ_{nm} depends on the positions of all the spheres, calculating it is very difficult. If we make the simplifying assumption that the average distance between neighbouring spheres is much larger than the radius of the sphere, we can make a few simplifying approximations. Focussing on the n^{th} sphere whose velocity is $\vec{V}_n$, we can write the viscous drag on it as

$$-6\pi\eta a(\vec{V}_n - \vec{v}'(\vec{R}_n))$$

The velocity field $\vec{v}'$ is created by all spheres except the n^{th} one. Consequently, this is a flow field under the action of forces $\vec{F}_m$ at all locations ' m ' other than ' n ' and we have

$$\eta \nabla^2 \vec{v}'(\vec{r}_n) = \vec{\nabla} P - \sum_{m \neq n} \delta(\vec{r} - \vec{R}_m) \vec{F}_m \quad (9.1.42)$$

with

$$\vec{\nabla} \cdot \vec{v}' = 0 \quad (9.1.43)$$

In Fourier space

$$-\eta k^2 v'_\alpha(\vec{k}) = i k_\alpha P - \sum_{m \neq n} e^{i\vec{k} \cdot \vec{R}_m} F_{m\alpha} \quad (9.1.44)$$

The divergence-free condition $k_{al} v_{al}(\vec{k}) = 0$ leads to

$$k^2 P = -i \sum_{m \neq n} e^{i\vec{k} \cdot \vec{R}_m} F_{m\alpha} k_\alpha \quad (9.1.45)$$

Inserting in Eq.(9.1.44)

$$\eta k^2 v'_\alpha(\vec{k}) = \sum_{\beta, m \neq n} e^{i\vec{k} \cdot \vec{R}_m} \left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right] F_{m\beta} \quad (9.1.46)$$

From Eq.(9.1.41)

$$V_{n\alpha} - v'_\alpha(\vec{R}_n) = \frac{F_{n\alpha}}{6\pi\eta a}$$

or

$$\begin{aligned} V_{n\alpha} &= \frac{F_{n\alpha}}{6\pi\eta a} + \frac{1}{\eta} \sum_{\beta, m \neq n} \int d^3k \frac{e^{i\vec{k} \cdot (\vec{R}_m - \vec{R}_n)}}{k^2} \left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right] \\ &= \frac{F_{n\alpha}}{6\pi\eta a} + \sum_{\beta, m \neq n} T_{\alpha\beta}(\vec{R}_n - \vec{R}_m) F_{m\beta} \end{aligned} \quad (9.1.47)$$

where Fourier- transforming yields

$$T_{\alpha\beta}(\vec{r}) = \frac{1}{8\pi\eta r} \left[\frac{r_\alpha r_\beta}{r^2} + \delta_{\alpha\beta} \right] \quad (9.1.48)$$

The mobility matrix μ_{mn} is

$$\begin{aligned} \mu_{mn} &= \frac{1}{6\pi\eta a} \quad \text{for } m = n \\ &= T(\vec{R}_m - \vec{R}_n) \quad \text{for } m \neq n \end{aligned} \quad (9.1.49)$$

The Langevin equation of motion now becomes

$$\frac{d}{dt} R_{n,\alpha} = - \sum_m \mu_{mn} \frac{\partial E}{\partial R_{m\alpha}} + g_{n\alpha} \quad (9.1.50)$$

(There would have been an additional term but it does not show up because

$$\sum_m \frac{\partial \mu_{mn}}{\partial R_m} = 0).$$

The correlation of g_n is specified as

$$\begin{aligned}\langle g_{n\alpha} \rangle &= 0 \\ \langle g_{n\alpha}(t_1) g_{m\beta}(t_2) \rangle &= 2(\mu_{mn})_{\alpha\beta} k_B T \delta(t_1 - t_2)\end{aligned}\quad (9.1.51)$$

Substituting for E , the equation of motion is

$$\dot{R}_{n\alpha} = k \sum_{m,\beta} (\mu_{mn})_{\alpha\beta} (R_{m+1,\beta} + R_{m-1,\beta} - 2R_{m,\beta}) + g_{n\alpha} \quad (9.1.52)$$

Since μ_{mn} depends on $R_{n\alpha}$, this is nonlinear and hence extremely complicated. The simplification that is done is to use $\langle (\mu_{mn})_{\alpha\beta} \rangle$ which is defined as

$$\begin{aligned}\langle (\mu_{mn})_{\alpha\beta} \rangle &= \int d^3r \left(\frac{3}{2\pi|n-m|b^2} \right)^{3/2} \exp \left(-\frac{3}{2} \frac{r^2}{|n-m|b^2} \right) \\ &\quad \times \frac{1}{8\pi\eta} \frac{1}{r} \left(\frac{r_\alpha r_\beta}{r^2} + \delta_{\alpha\beta} \right) \\ &= \bar{\mu}_{mn} \delta_{\alpha\beta}\end{aligned}\quad (9.1.53)$$

where

$$\bar{\mu}_{mn} = \frac{1}{\eta b} \frac{1}{\sqrt{6\pi^3|m-n|}}$$

An analysis by decomposition into normal modes now leads to the more realistic result

$$D_{CM} \propto M^{-1/2}, \quad \tau \propto M^{3/2} \quad (9.1.54)$$

References

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Appendix

Functional Integration

The ultimate mathematical step that one has to perform in the problems that we have discussed so far is an average. This is expressed in terms of the N-point correlation function

$$\begin{aligned} G_N(x_1, x_2, \dots, x_N) &= \langle \phi(x_1) \phi(x_2) \dots \phi(x_N) \rangle \\ &= \frac{1}{Z} \int \mathcal{D}[\phi] \phi(x_1) \phi(x_2) \dots \phi(x_N) e^{-\int F d^D x}. \end{aligned} \quad (\text{A.0.1})$$

Here we have used a scalar field in defining G_N for simplicity but extension to vector fields is quite straightforward. In the above expression, the partition function Z and the free energy F are

$$Z(h(x)) = \int \mathcal{D}[\phi] e^{-\int F(h(x)) d^D x} \quad (\text{A.0.2})$$

$$F(h(x)) = \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} (\phi^2)^2 - h(x) \phi \quad (\text{A.0.3})$$

the $h \rightarrow 0$ limits of the above expressions. It follows that

$$G_N(x_1, x_2, \dots, x_N) = \frac{1}{Z} \frac{\delta^N Z(h(x))}{\delta h(x_1) \delta h(x_2) \dots \delta h(x_N)} \Big|_{h=0} \quad (\text{A.0.4})$$

The integral (A.0.1) can be solved exactly only if $\lambda = 0$. For $\lambda \neq 0$ one can make use of perturbation technique to achieve the desired accuracy for small λ , otherwise one has to fall back on computers for numerical evaluation.

Here we evaluate Z for $\lambda = 0$.

$$\begin{aligned}
 Z &= \int \mathcal{D}[\phi] e^{-\int d^D x [\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2]} \\
 &= \int \mathcal{D}[\phi] e^{-\int d^D x \phi [-\frac{1}{2}\nabla^2 + m^2]\phi} \\
 &= \int \mathcal{D}[\phi] e^{-\int d^D x \phi \mathcal{B} \phi} \tag{A.0.5}
 \end{aligned}$$

where $\mathcal{B}$ represents the operator $-\frac{1}{2}\nabla^2 + m^2$. Every integral, in principle can be thought of as the limit of a sum. Hence to evaluate this integral we replace the continuum by a lattice and at each point i of the lattice we replace $\phi(x)$ by ϕ_i where the variable ϕ_i can range from $-\infty$ to ∞ . Under this transformation

$$\int d^D x \phi \mathcal{B} \phi \rightarrow \sum_{ij} \phi_i B_{ij} \phi_j$$

with the operational meaning of $\mathcal{D}[\phi]$ being

$$\mathcal{D}[\phi] = \prod_{i=1}^N \int_{-\infty}^{\infty} d\phi_i$$

Therefore the discretized version of Eq.(A.0.5) appears as

$$Z = \prod_{i=1}^N \int_{-\infty}^{\infty} d\phi_i e^{-\sum_{jk} \phi_j B_{jk} \phi_k} \tag{A.0.6}$$

where the matrix B_{jk} is symmetric. Here, antisymmetry would imply vanishing of the sum. This allows for diagonalization by an orthogonal transformation. From now on we will use Einstein's summation convention to imply that repeated indices are summed over. We consider an orthogonal transformation S that diagonalizes B such that

$$B_{jk} = S_{jm} D_{mn} S_{nk}^T \tag{A.0.7}$$

where D is the diagonal matrix

$$D_{mn} = D_{mn} \delta_{mn}.$$

Starting from

$$\begin{aligned}
 \phi_j B_{jk} \phi_k &= \phi_j S_{jm} D_{mn} S_{nk}^T \phi_k \\
 &= \psi_m D_{mn} \phi_n \tag{A.0.8}
 \end{aligned}$$

where

$$\begin{aligned} \phi_j S_{jm} &= \psi_m \\ \text{and} \quad \phi_m &= S_{mn} \psi_n \end{aligned} \quad (\text{A.0.9})$$

Using the fact that the Jacobian of the transformation is unity, i.e. $\text{Det}(S) = 1$, we have

$$\Pi_k d\phi_k = \Pi_k d\psi_k. \quad (\text{A.0.10})$$

Taking into account that S is orthogonal, i.e. $SS^T = 1$ we have

$$\begin{aligned} Z &= \Pi_{i=1}^N \int_{-\infty}^{\infty} d\psi_j e^{-\frac{1}{2} D_{jj} \psi_j^2} \\ &= \left[\int_{-\infty}^{\infty} d\psi_j e^{-\frac{1}{2} D_{jj} \psi_j^2} \right]^N \\ &= \left(2\pi \right)^{N/2} \left[\frac{1}{\text{Det}(D)} \right]^{1/2} \\ &= \left(2\pi \right)^{N/2} \left[\frac{1}{\text{Det}(B)} \right]^{1/2} \end{aligned} \quad (\text{A.0.11})$$

It is apparent from the given structure of F that

$$\langle \phi(x_1) \phi(x_2) \dots \phi(x_{2m+1}) \rangle = 0 \quad (\text{A.0.12})$$

in the limit of $\hbar \rightarrow 0$.

For the two point function we have

$$\begin{aligned} \left[\Pi_{i=1}^N \int_{-\infty}^{\infty} d\phi_i \right] \phi_p \phi_q e^{-\frac{1}{2} \phi_j B_{jk} \phi_k} &= \left[\Pi_{i=1}^N \int_{-\infty}^{\infty} d\phi_i \right] \\ &\quad \phi_q B_{pr}^{-1} \frac{\delta}{\delta \phi_r} e^{-\frac{1}{2} \phi_j B_{jk} \phi_k} \\ &= \left[\Pi_{i=1}^N \int_{-\infty}^{\infty} d\phi_i \right] B_{pr}^{-1} \delta_{rq} e^{-\frac{1}{2} \phi_j B_{jk} \phi_k} \\ &= B_{pr}^{-1} \left(2\pi \right)^{N/2} \left[\frac{1}{\text{Det}(B)} \right]^{1/2} \end{aligned} \quad (\text{A.0.13})$$

where in the penultimate step we have integrated by parts. Hence, we can readily write (by induction)

$$\left[\Pi_{i=1}^N \int_{-\infty}^{\infty} d\phi_i \right] \phi_{i_1} \phi_{i_2} \dots \phi_{i_{2n+1}} e^{-\frac{1}{2} \phi_j B_{jk} \phi_k} = 0 \quad (\text{A.0.14})$$

and

$$\begin{aligned} & \left[\prod_{i=1}^N \int_{-\infty}^{\infty} d\phi_i \right] \phi_{i_1} \phi_{i_2} \dots \phi_{i_{2n}} e^{-\frac{1}{2} \phi_j B_{jk} \phi_k} \\ &= \left(2\pi \right)^{N/2} (Det B)^{-1/2} \left[B_{i_1, i_2}^{-1} B_{i_3, i_4}^{-1} \dots B_{i_{2n-1}, i_{2n}}^{-1} + \text{all possible pairings} \right] \end{aligned} \quad (\text{A.0.15})$$

Thus,

$$\langle \phi_{i_1} \phi_{i_2} \dots \phi_{i_{2n}} \rangle = B_{i_1, i_2}^{-1} B_{i_3, i_4}^{-1} \dots B_{i_{2n-1}, i_{2n}}^{-1} + \text{all possible pairings} \quad (\text{A.0.16})$$

This is the Wick's theorem in the parlance of functional integration.

Appendix.1 Gaussian Theory

In this theory $\lambda = 0$. To calculate the correlation function, we go to momentum space which facilitates calculation. We define Fourier transform pair as

$$\begin{aligned} \phi(x) &= \int \frac{d^D p}{(2\pi)^D} \tilde{\phi}(p) e^{-i \vec{p} \cdot \vec{x}} \\ \tilde{\phi}(p) &= \int d^D x \phi(x) e^{i \vec{p} \cdot \vec{x}} \end{aligned} \quad (\text{A.1.1})$$

which allows us to write

$$\begin{aligned} \int d^D x \phi(x) [-\nabla^2 + m^2] \phi(x) &= \int d^D x \int \frac{d^D p}{(2\pi)^D} \tilde{\phi}(p) e^{-i \vec{p} \cdot \vec{x}} [-\nabla^2 + m^2] \\ &\quad \times \int \frac{d^D q}{(2\pi)^D} \tilde{\phi}(q) e^{-i \vec{q} \cdot \vec{x}} \\ &= \int \frac{d^D p}{(2\pi)^D} \tilde{\phi}(p) [p^2 + m^2] \tilde{\phi}(-p) \end{aligned} \quad (\text{A.1.2})$$

The operator $\mathcal{B}$ is thus diagonal in momentum space and hence $\mathcal{B}^{-1} = (p^2 + m^2)^{-1}$. The Green's function $G_0(p)$ of the Gaussian theory is thus

$$G_0(p) = \frac{1}{p^2 + m^2} \quad (\text{A.1.3})$$

Higher order Green's functions can also be found from the Eq.(A.0.16).

We now try to find the susceptibility for this model which follows from the fluctuation-dissipation theorem and is given by

$$\chi = G_0(p=0) = (m^2)^{-1} \propto (T - T_c)^{-1} \quad (\text{A.1.4})$$

Thus $\gamma = 1$. At $T = T_c$

$$G_0(p) \propto \frac{1}{p^2}$$

and the Fourier transform has a conspicuous scaling behaviour

$$G_0(x) \propto \frac{1}{|x|^{D-2}}.$$

Therefore, $\eta = 0$. The Fourier transform at non-zero m^2 is

$$G(x) = \int \frac{d^D p}{(2\pi)^D} \frac{e^{-i\vec{p} \cdot \vec{x}}}{p^2 + m^2}. \quad (\text{A.1.5})$$

This integral is easily done for $D = 3$ but is nontrivial in other dimensions. But the correlations decay exponentially with a scale m^{-1} irrespective of the dimensionality D . Hence one gets as the correlation length

$$\xi = m^{-1} \propto (T - T_c)^{-1/2} \quad (\text{A.1.6})$$

resulting in $\nu = 1/2$. The specific heat exponent *alpha* is found from

$$\begin{aligned} C &\propto \int d^D x d^D y \langle \phi^2(x) \phi^2(y) \rangle \\ &= \int d^D x d^D y \left\langle \int \frac{d^D p}{(2\pi)^D} \tilde{\phi}(p) e^{-i\vec{p} \cdot \vec{x}} \int \frac{d^D q}{(2\pi)^D} \tilde{\phi}(q) e^{-i\vec{q} \cdot \vec{x}} \right. \\ &\quad \times \left. \int \frac{d^D r}{(2\pi)^D} \tilde{\phi}(r) e^{-i\vec{r} \cdot \vec{y}} \int \frac{d^D s}{(2\pi)^D} \tilde{\phi}(s) e^{-i\vec{s} \cdot \vec{y}} \right\rangle_c \\ &= \int \frac{d^D p}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \langle \tilde{\phi}(p) \tilde{\phi}(-p) \tilde{\phi}(q) \tilde{\phi}(-q) \rangle_c \\ &= \int \frac{d^D p}{(2\pi)^D} \left[G_0(p) \right]^2 \\ &= \int \frac{d^D p}{(2\pi)^D} [p^2 + m^2]^{-2} \propto m^{D-4} \propto (T - T_c)^{-\frac{4-D}{2}} \end{aligned} \quad (\text{A.1.7})$$

Hence $\alpha = (4 - D)/2$. The spontaneous magnetization index β cannot be obtained within the Gaussian framework as the theory becomes unstable in the broken symmetry state. This is a first step towards resolving the complicated issue of critical exponents and universality. This theory does not yield correct exponents as the interaction term is missing in the free energy though it satisfies the scaling laws.

Appendix

Field Theoretic RG

We briefly sketch the basic ideas of Field Theoretic RG for a scalar field. The Free energy density can be written as

$$F = \frac{r_0}{2}\phi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{u_0}{4!}\phi^4. \quad (\text{B.0.1})$$

We focus on the two point correlation function $\Gamma^{(2)}(k)$ at the critical point. The crucial point is that we must have a finite $\Gamma^{(2)}(k)$ as the cut-off Λ on the momentum space variable goes to infinity (or equivalently the lattice spacing tending to zero in real space). The behaviour of $\Gamma^{(2)}(k)$ at the critical point is

$$\Gamma^{(2)}(k) \propto k^{2-\eta}$$

for $k \ll \Lambda$. Dimensional analysis suggests that $\Gamma^{(2)}(k)$ should have the form

$$\Gamma^{(2)}(k) = k^2 \left(\frac{k}{\Lambda} \right)^{-\eta} \left[1 + B \left(\frac{k}{\Lambda} \right)^{\omega_1} + \dots \right] \quad (\text{B.0.2})$$

Naïve power counting yields k^2 as the dimension of $\Gamma^{(2)}(k)$ which is independent of the dimensionality of space. Since we are interested in the long-wavelength behaviour of $\Gamma^{(2)}(k)$, we would obviously like to take the limit $k \rightarrow 0$. But that also involves the limit $\Lambda \rightarrow \infty$ and therein lies the problem since it renders the leading term meaningless. To overcome the difficulty we introduce an arbitrary length scale μ in the problem and write

$$\left(\frac{\mu}{\Lambda} \right)^{\eta} \Gamma^{(2)}(k) = k^2 \left(\frac{k}{\mu} \right)^{-\eta} \left[1 + B \left(\frac{k}{\Lambda} \right)^{\omega_1} + \dots \right] \quad (\text{B.0.3})$$

The limit $\Lambda \rightarrow \infty$ will cause no problem on the R.H.S of Eq.(B.0.3). The L.H.S defines the renormalized vertex function

$$\Gamma_R^{(2)}(k) = \left(\frac{\mu}{\Lambda}\right)^\eta \Gamma^{(2)}(k) = Z_\phi^{-1} \Gamma^{(2)}(k) \quad (\text{B.0.4})$$

We define the renormalized field as

$$\phi_R(x) = Z_\phi^{-1/2} \phi(x)$$

to obtain

$$\langle \phi_R(x) \phi_R(x') \rangle = Z_\phi^{-1} \langle \phi(x) \phi(x') \rangle$$

which remains finite as $\Lambda \rightarrow \infty$. The factor Z_ϕ is the wave-function renormalization.

To appreciate the difficulty let us calculate the correlation functions to one loop order directly by using the Free energy given in Eq.(B.0.1). To one loop order $\Gamma^{(2)}(k)$ becomes

$$\Gamma^{(2)}(k) = k^2 + r_0 \frac{u_0}{2} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + r_0}. \quad (\text{B.0.5})$$

In $D = 4$ with the upper cut-off Λ we obtain

$$\Gamma^{(2)}(k) = k^2 + r_0 + \frac{u_0}{2} \left[\frac{\Lambda^2}{16\pi^2} - \frac{r_0}{16\pi^2} \ln \frac{\Lambda^2}{r_0} - \frac{1}{16\pi^2} \frac{r_0}{\Lambda^2} \right] + O(u_0^2) \quad (\text{B.0.6})$$

It can be seen that the two point function does not have a well defined limit as $\Lambda \rightarrow \infty$. Z_ϕ cannot remove the divergence in the above limit. For the massive theory

$$\begin{aligned} \Gamma_R^{(2)}(k) &= k^2 + \text{other terms} \\ \text{or} \quad \frac{\partial \Gamma_R^{(2)}(k)}{\partial k^2} \Big|_{k^2=0} &= 1. \end{aligned} \quad (\text{B.0.7})$$

For a massless theory this differentiation cannot be performed at $k^2 = 0$. Thus Z_ϕ is determined from the equation

$$\begin{aligned} Z_\phi^{-1} \frac{\partial \Gamma^{(2)}(k)}{\partial k^2} \Big|_{k^2=0} &= 1 \\ \text{or} \quad Z_\phi &= 1 + O(u_0^2) \end{aligned} \quad (\text{B.0.8})$$

This equation predicts that there is no contribution to Z_ϕ at $O(u_0)$.

To restore the finiteness of the R.H.S of Eq.(B.0.6) we introduce a renormalized mass defined through the relation

$$\Gamma_R^{(2)}(k^2=0) = r = Z_\phi Z_R^{-1} r_0 \quad (\text{B.0.9})$$

To one loop order we can write,

$$Z_r^{-1} = 1 + \frac{u_0}{32\pi^2} \left[\frac{\Lambda^2}{r_0} - \ln \frac{\Lambda^2}{r_0} - \frac{r_0}{\Lambda^2} + \dots \right] \quad (\text{B.0.10})$$

The next non-trivial correlation function is the one corresponding to the four point vertex function $\Gamma^{(4)}(\{k_i\}, u_0, r_0, \Lambda)$. To one loop order the contribution is given by

$$\begin{aligned} \Gamma^{(4)}(\{k_i\}, u_0, r_0, \Lambda) = & -u_0 + \frac{u_0^2}{2} \left[\int^\Lambda \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + r_0)[(\vec{k}_1 + \vec{k}_2 - \vec{p})^2 + r_0]} \right. \\ & \left. + \text{two permutations} \right] \end{aligned} \quad (\text{B.0.11})$$

Evaluation in $D=4$ yields

$$\begin{aligned} \Gamma^{(4)}(\{k_i\}, u_0, r_0, \Lambda) = & -u_0 + \frac{u_0^2}{2} \left[\frac{3}{16\pi^2} \ln \frac{\Lambda^2}{r_0} + \mathcal{F}\left(\frac{(\vec{k}_1 + \vec{k}_2)^2}{r_0}\right) \right. \\ & \left. + \mathcal{F}\left(\frac{(\vec{k}_1 + \vec{k}_3)^2}{r_0}\right) + \mathcal{F}\left(\frac{(\vec{k}_2 + \vec{k}_3)^2}{r_0}\right) + O\left(\frac{r_0}{\Lambda^2}, \frac{k^2}{\Lambda^2}\right) \right] \end{aligned} \quad (\text{B.0.12})$$

where the function $\mathcal{F}(x)$ is independent of Λ . The wave function renormalization leads to

$$\Gamma_R^{(4)} = Z_\phi^{-2} \Gamma^{(4)} = (1 + O(u_0^2)) \Gamma^{(4)}. \quad (\text{B.0.13})$$

as Z_ϕ does not contain any $O(u_0)$ term. Now even Z_r and Z_ϕ together cannot resolve the problem of taking the limit $\Lambda \rightarrow 0$ in the Eq.(B.0.12). Hence, one introduces yet another renormalization constant through the relation

$$\Gamma_R^{(4)}(\{k_i\}=0) = -g \quad (\text{B.0.14})$$

This implies

$$\begin{aligned} -g &= -u_0 + \frac{u_0^2}{2} \left[\frac{3}{16\pi^2} \ln \frac{\Lambda^2}{r_0} - \frac{3}{16\pi^2} + O\left(\frac{r_0}{\Lambda^2}\right) \right] \\ g &= u_0 - \frac{3u_0^2}{32\pi^2} \left[\ln \frac{\Lambda^2}{r_0} - 1 + O\left(\frac{r_0}{\Lambda^2}\right) \right] + O(u_0^3) \\ \text{or} \quad u_0 &= g + \frac{3g^2}{32\pi^2} \left[\ln \frac{\Lambda^2}{r_0} - 1 + O\left(\frac{r_0}{\Lambda^2}\right) \right] \\ &= Z_\phi^2 Z_u^{-1} g \end{aligned} \quad (\text{B.0.15})$$

Substituting u in terms of u_R in Eq.(B.0.12)

$$\Gamma^{(4)}(\{k_i\}, g, r, \Lambda) = -g - \frac{g^2}{32\pi^2} \left[\mathcal{F}\left(\frac{(\vec{k}_1 + \vec{k}_2)^2}{r_0}\right) + \mathcal{F}\left(\frac{(\vec{k}_1 + \vec{k}_3)^2}{r_0}\right) + \mathcal{F}\left(\frac{(\vec{k}_2 + \vec{k}_3)^2}{r_0}\right) + \frac{3}{16\pi^2} + O\left(\frac{k^2}{r}\right) \right] \quad (\text{B.0.16})$$

Now if we take the limit of $\Lambda \rightarrow \infty$, the result is finite. Z_u is the coupling constant renormalization. This makes the two point and four point correlation functions finite in this limit. It now remains to be shown that this method will automatically make the higher order correlation functions finite.

To do that, we consider the general interaction of the form ϕ^r and a graph with E external legs. The number of internal lines I is given by

$$I = \frac{1}{2}(r n - E) \quad (\text{B.0.17})$$

where n is the number of times the interaction acts. Of the $r n$ lines it produces, E are joined to the external legs. The remaining $r n - E$ lines have to be paired which gives rise to the Eq.(B.0.17). The degree of primitive divergence δ of a graph with l loops and n interaction vertices is

$$\delta = l D - 2I \quad (\text{B.0.18})$$

where there is no divergence due to subintegrations. We now relate l to the number of internal lines I . For each internal line there is an associated momentum, of which not all are independent. At each of the n interaction vertices there is a momentum conserving delta function. But one of the conservation laws corresponds to the overall momentum conservation and thus only $I - (n - 1)$ are independent. So,

$$l = I - (n - 1). \quad (\text{B.0.19})$$

Thus, the degree of divergence becomes,

$$\delta = \left(\frac{rD}{2} - r - D\right)n + D + E - \frac{DE}{2} \quad (\text{B.0.20})$$

The above formula has a n -dependent part. For a given r , there is a choice of D for which the coefficient of n vanishes. This is the critical dimension D_c given by

$$D_c = \frac{2r}{r-2}. \quad (\text{B.0.21})$$

Clearly, for $D = D_c$, δ is independent of the number of loops. For $D > D_c$, δ increases with the number of loops and it would be impossible to absorb all the divergences by introducing a finite number divergences. For $D \leq D_c$, it is possible to absorb all the divergences in a finite number of renormalization constants

rendering a finite theory in the limit of $\Lambda \rightarrow 0$. The theory then becomes renormalizable for $D = D_c$, Super-renormalizable for $D < D_c$ and non-renormalizable for $D > D_c$. For the quartic interaction $r = 4$ and $D_c = 4$. Thus δ in $D = 4$ in ϕ^4 theory is independent of the number of loops. It is clear from Eq.(B.0.20) that only primitive divergence comes from $E \leq 4$ and thus once the $\Gamma^{(2)}$ and $\Gamma^{(4)}$ are rendered finite in the infinite cut-off limit. For the massive ϕ^4 theory, the $\Gamma^{(2)}$ and $\Gamma^{(4)}$ are rendered finite if

$$\begin{aligned} \left. \frac{\delta \Gamma_R^{(2)}}{\delta k^2} \right|_{k=0} &= 1 \\ \Gamma_R^{(2)}(k=0) &= r \\ \Gamma_R^{(4)}(\{k_i\}=0) &= -g \end{aligned} \quad (\text{B.0.22})$$

For the massless theory these renormalization conditions have to change because of the presence of the anomalous dimension index η . Therefore we must evaluate derivative at $k^2 = \mu^2$ and accordingly the first of Eq.(B.0.22) is changed to

$$\left. \frac{\delta \Gamma_R^{(2)}}{\delta k^2} \right|_{k^2=\mu^2} = 1. \quad (\text{B.0.23})$$

Similarly, The condition of second of Eq.(B.0.22) is to be modified to ensure that $\Gamma_R^{(2)}(k=0) = 0$. Finally, we focus on the four point correlation function where it is clear from Eq.(B.0.11) that for $r_0 = 0$ we do not have a well defined integral when the external momentum vanishes. Hence the condition on $\Gamma_R^{(4)}(\{k_i\})$ has to be changed to a condition at finite momentum and we choose the symmetric point μ_S , where

$$k_1^2 = k_2^2 = k_3^2 = k_4^2 = \mu^2.$$

Since

$$\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 = 0$$

because of overall momentum conservation, we have

$$0 = \sum_i k_i^2 = 4\mu^2 + 12(\vec{k}_i \cdot \vec{k}_j)$$

or

$$\vec{k}_i \cdot \vec{k}_j = -\frac{\mu^2}{3} \quad \text{for } i \neq j$$

For $i = j$ $k_i^2 = \mu^2$ and thus for μ_S

$$\vec{k}_i \cdot \vec{k}_j = \frac{\mu^2}{3}(4\delta_{ij} - 1) \quad (\text{B.0.24})$$

Thus the renormalization conditions for the massless theory are

$$\begin{aligned}\frac{\delta \Gamma_R^{(2)}}{\delta k^2} \Big|_{k^2=\mu^2} &= 1 \\ \Gamma_R^{(2)}(k=0) &= 0 \\ \Gamma_R^{(4)}(\mu_S) &= -g.\end{aligned}\tag{B.0.25}$$

To end this discussion, we point out an apparent contradiction. The field theoretic tools we have developed so far are supposed to deal with critical phenomena. However, critical phenomena is uncomplicated for $D > 4$ and is non-trivial for $D < 4$, while the process of renormalization shows that the theory is non-renormalizable for $D > 4$ and super-renormalizable for $D < 4$. If this appears to be a contradiction, we must remember that the question of renormalizability just raised involves an ultraviolet cut-off, while for the critical phenomena the relevant behaviour is in the infrared region. To make things clear, we consider the integral appearing in $\Gamma^{(4)}(k=0)$

$$I(m) = \int_0^\Lambda \frac{d^D p}{(2\pi)^D} \left(\frac{1}{p^2 + m^2} \right)^2 \tag{B.0.26}$$

First, we demonstrate that this integral can be made ultraviolet finite by successive partial integrations involving the 't Hooft and Veltman technique. We note that

$$\frac{1}{D} \sum_{i=1}^D \frac{\partial}{\partial p_i} = 1 \tag{B.0.27}$$

and insert it in Eq.(B.0.26) to obtain

$$\begin{aligned}I(m) &= \sum_{i=1}^D \frac{1}{D} \int_0^\Lambda \frac{\partial}{\partial p_i} \frac{p_i}{(p^2 + m^2)^2} \frac{d^D p}{(2\pi)^D} \\ &\quad + \sum_{i=1}^D \frac{4}{D} \int_0^\Lambda \frac{p_i p_i}{(p^2 + m^2)^3} \frac{d^D p}{(2\pi)^D}\end{aligned}\tag{B.0.28}$$

but

$$\int_0^\Lambda \frac{p_i}{(p^2 + m^2)^2} \frac{d^D p}{(2\pi)^D} = 0,$$

so

$$\begin{aligned}I(m) &= \frac{4}{D} I(m) - \frac{4}{D} m^2 \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + m^2)^3} \\ \text{or } I(m) \left[1 - \frac{4}{D} \right] &= -\frac{4m^2}{D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + m^2)^3}\end{aligned}\tag{B.0.29}$$

The integral on the R.H.S of Eq.(B.0.29) is ultraviolet convergent for $D < 6$. We can now take the limit $\Lambda \rightarrow \infty$ for $D < 6$ and hence $I(m)$ scales as m^{D-4} . The integral diverges for $D < 4$ as $m \rightarrow 0$ which is the problem of critical phenomena - the infrared divergence. It is worthwhile to note that as the integral is made ultraviolet finite by successive partial- p 's, the scaling behaviour w.r.t m is exactly the simple power count m^δ . Thus the infrared behaviour of the loop integrals is m^δ with δ given by Eq.(B.0.20). If we consider the second term of Eq.(B.0.20), then we find that it is the term associated with the free field theory. The free field will dominate i.e. give the correct infrared singularity if the first term is positive. The first term is positive if $D > D_c$ and thus for the infrared question, it is the free field theory which is relevant for $D > D_c$, whereas for $D < D_c$, the infrared behaviour is changed as the first term lowers the values of δ and one observes non-trivial critical singularities.

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