

LECTURE NOTES IN PHYSICS

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Jihn E. Kim

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K.-S. Choi and J.E. Kim, *Quarks and Leptons From Orbifolded Superstring*,
Lect. Notes Phys. 696 (Springer, Berlin Heidelberg 2006), DOI 10.1007/b11681670

Library of Congress Control Number: 2006923219

ISSN 0075-8450

ISBN-10 3-540-32763-0 Springer Berlin Heidelberg New York

ISBN-13 978-3-540-32763-9 Springer Berlin Heidelberg New York

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Printed in The Netherlands

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Typesetting: by the authors and techbooks using a Springer L^AT_EX macro package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11681670 54/techbooks 5 4 3 2 1 0

Preface

Using the successful standard model of particle physics but without clear guidance beyond it, it is a difficult task to write a physics book beyond the standard model from a phenomenological point of view. At present, there is no major convincing inner space related experimental evidence against the standard model. The neutrino oscillation phenomena can be considered part of it by including a singlet field in the spectrum. Only the outer space observations on matter asymmetry, dark matter, and dark energy hint at the phenomenological need for an extension; however, there has been theoretical need for almost three decades, chiefly because of the gauge hierarchy problem in the standard model.

Thus, it seems that going beyond the standard model hinges on the desirability of resolving the hierarchy problem. At the field theory level, it is fair to say that the hierarchy problem is not as desperate as the nonrenormalizability problem present in the old V–A theory of weak interactions on the road to the standard model. An extension beyond the standard model can easily be ruled out as witnessed in the case of technicolor. However, a consistent framework with supersymmetry for a resolution of the hierarchy problem has been around for a long time. Even its culprit “superstring” has been around for twenty years, and the most remarkable thing about this supersymmetric extension is that it is still alive. So the time is ripe for phenomenologists to become acquainted with superstring and its contribution toward the minimal supersymmetric standard model in four space-time dimensions.

This book is a journey toward the minimal supersymmetric standard model (MSSM) down the orbifold road. After some field theoretic orbifold attempts in recent years, there has been renewed interest in the physics of string orbifolds and it is time to revisit them. In this book, we take the viewpoint that the chirality of matter fermions is essential toward revealing the secrets of Nature. Certainly, orbifolds are an easy way to get the chirality from higher dimensions.

Strings and their orbifold compactification are presented for the interests of phenomenologists, sacrificing mathematical rigor. They are presented in

such a way that an orbifold model can be constructed by applying the rules included here. At the end of Chap. 10, we construct a $\mathbf{Z}_{12}$ orbifold which contains all imaginable complications. Also, we attempt to correct any incompleteness in the rules presented before in the existing literature. In the final chapter we tabulate the simplest and most widely used orbifold $\mathbf{Z}_3$ with $\mathcal{N}=1$ supersymmetry, completely in the phenomenological sense of obtaining three families. These tables encompass all noteworthy models available with two Wilson lines. Since three Wilson line $\mathbf{Z}_3$ orbifolds do not automatically give three families, in a practical manner these tables close a chapter on $\mathbf{Z}_3$ orbifolds.

This book is not as introductory as a textbook, nor is it as special as a review article on a superstring topic. Instead, we aimed at an interim region so that a phenomenologist can read and directly commence building an orbifold model.

We thank Kyuwan Hwang for his help in constructing the $\mathbf{Z}_3$ orbifold tables. We are also grateful to Kiwoon Choi, Ki-Young Choi, Luis Ibañez, Gordy Kane, Hyung Do Kim, Jewan Kim, Seok Kim, Tatsuo Kobayashi, Bumseok Kyae, Oleg Lebedev, Andre Lukas, Stefan Groot Nibbelink, Hans-Peter Nilles, Fernando Quevedo, Stuart Raby, Michael Ratz, and Hyun Seok Yang for providing valuable suggestions in the course of writing this book. We thank the Korea Research Foundation and the Korean Science and Engineering Foundation for the supports.

Seoul
November, 2005

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Introduction and Summary

During and since the second half of the twentieth century, enormous progress has been made in understanding our universe in terms of fundamental particles and their interactions, namely in the language of quantum field theory. The advent of the standard model (SM) of particle physics has been the culmination of quantum field theory in all its full glory. The dawn of this successful particle physics era was opened with the unexpected discovery of *parity violation* in weak interaction phenomena [1]. It had long been known that weak interactions change the electromagnetic charge, i.e. electron (e) to electron type neutrino (ν_e), neutron (n) to proton (p). But, until the mid-1950s it had never occurred to the leading minds [2] that “parity might be violated”, chiefly because the atomic and nuclear transitions did not reveal any such possibility before that time. For nuclear transitions, both weak and electromagnetic phenomena contribute but at that time there were not sufficient data to fully conclude on the nature of parity operation in weak interactions [1]. For atomic transitions, the fundamental interaction is of electromagnetic origin and the experimental confirmation of parity conservation in atomic phenomena convinced most physicists that parity is conserved in the universe. In hindsight, parity conservation should have been imposed only on electromagnetic interactions, as the discovery of parity violation in weak interactions started a new era for weak interactions. There is still no experimental evidence that strong and electromagnetic interactions violate parity. Therefore, we know that parity violation in weak interactions is at the heart of making our universe as it is now, because the SM assumes from the outset the existence of massless chiral fields.¹

Soon afterward, the parity-violating weak interactions were neatly summarized as a four fermion (charged current) $\times$ (charged current) weak interaction where the charged current J_μ^{CC} is of the “V–A” type [3],

$$\mathcal{H}_{\text{weak}} = \frac{G_F}{\sqrt{2}} J^{CC\mu\dagger}(x) J_\mu^{CC}(x) . \quad (1.1)$$

¹ Massless compared to the Planck mass M_P .

The “V–A” charged current of weak interactions indicates two important things: (1) only the left-handed fermions participate in the charge changing weak interactions, and (2) being current, the fundamental interaction at a deeper level may need a vector boson. Here, we note that the chiral nature of weak interactions is still the mystery among all mysteries of particle physics in the search for a fundamental theory at a very high energy scale using the low energy SM. Several years after this effective low energy (current) $\times$ (current) four-fermion interaction was proposed, a modest attempt via a more fundamental interaction through a heavy spin-1 charged intermediate vector boson (IVB) $W_\mu^\pm$ was put forth [4]. Its coupling to the charged current was given by

$$\frac{g}{2\sqrt{2}} J_\mu^{CC} W^\mu + \text{h.c.} \quad (1.2)$$

The mass of W_μ was supposed to be heavy so that the four-fermion interaction mediated by the IVB is weak compared to the strong interactions. However, this IVB idea had several problems which have since been resolved by the standard model of particle physics.

A spin-1 field coupling to the fermion current had already been known in electromagnetic interactions, i.e. the photon A_μ coupling to the electromagnetic current through $e\bar{\psi}\gamma^\mu\psi A_\mu$. This electromagnetic interaction can be formulated in terms of U(1) gauge theory [5], where one uses the covariant derivative $\mathcal{D}_\mu$ instead of the ordinary partial derivative ∂_μ ,

$$\partial_\mu \longrightarrow \mathcal{D}_\mu \equiv \partial_\mu - ieA_\mu$$

which introduces the minimal gauge coupling of A_μ to charged fields. In quantum mechanics, the additive conservation of electromagnetic charge implies a global U(1) symmetry and generalizing it to a local U(1) leads to the above covariant derivative. This is our first example of how a bigger symmetry might be discovered from a representation of matter, i.e. starting from the electron in the above example of quantum electrodynamics.

Consider the generalization of this gauge principle to the IVB. Since the IVB changes the electromagnetic charge, we must start from a defining state in Hilbert space which contains at least two components differing by one-unit of the electromagnetic charge. This doublet is a kind of matter which, in the doublet representation, necessarily introduces a nonabelian gauge group. This is our second example in which matter can indicate a bigger symmetry. In general, one can introduce the covariant derivative using the nonabelian gauge fields A_μ^i ($i = 1, 2, \dots, N_A$), with the size N_A (e.g. 3 for SU(2)) dependent upon the matter representation. Yang and Mills were the first to show that a consistent construction along this line needs nonlinear couplings between gauge fields.

In the late 1960s the standard model of particle physics was constructed, employing the nonabelian gauge group. The group structure is SU(2) $\times$ U(1) [6], and the covariant derivative is

$$\mathcal{D}_\mu = \partial_\mu - igT^i A_\mu^i - ig'Y B_\mu \quad (1.3)$$

where T^i ($i = 1, 2, 3$) are the SU(2) generators and Y is the electroweak hypercharge generator. The Gell-Mann-Nishijima type definition of the electromagnetic charge is $Q_{\text{em}} = T_3 + Y$. All leptons and quarks are put into left-handed doublets and right-handed singlets, and the charged current IVB mediation violates parity symmetry by construction from the outset. For example, the left-handed electron and its neutrino are put into a doublet $l_L = (\nu_e, e)_L^T$, where $L(R)$ represents the left(right)-handed projection $L = \frac{1+\gamma_5}{2}$, or $\psi_L = \frac{1+\gamma_5}{2}\psi$. Since the quarks carry the additional degree called *color* coming in three varieties, the first family $(l_L, e_R, q_L, u_R, d_R)$ contains 15 two-component chiral fields. In addition, these fifteen fields repeat three times, making a total of 45 chiral fields, all of which have been observed in high energy accelerators.

The SM representation is written in such a way that the intermediate vector boson W^+ transforms the lower elements of l and q to their upper elements. For example e_L to ν_{eL} and d_L to u_L , and hence there exists the coupling

$$\frac{g}{\sqrt{2}} \bar{\nu}_e \gamma^\mu \frac{1+\gamma_5}{2} e W_\mu^+.$$

For each representation, we can assign the Y quantum number to match the electromagnetic charges of the fields in the representation through the following formula,²

$$Q_{em} = T_3 + Y. \quad (1.4)$$

Thus, the standard model is certainly left(L)–right(R) asymmetric in that the interchange $L \leftrightarrow R$ does not give the original representation. This is called a chiral theory.³ In a chiral theory, one cannot write down a mass term for the fermions. Under the SM group SU(2) $\times$ U(1), for example, one cannot write down a gauge invariant mass term for e ($l = \mathbf{2}_{-\frac{1}{2}}, e_R = \mathbf{1}_{-1}$ where the weak hypercharges are written as subscripts in the usual way). The SM is designed such that chiral fermions can obtain mass after the gauge group SU(2) $\times$ U(1) is spontaneously broken down to U(1)_{em}, and then one has to consider only the gauge invariance of the unbroken gauge group U(1)_{em}. This makes it possible to write

$$-m_e \bar{e} e = -m_e (\bar{e}_R e_L + \bar{e}_L e_R).$$

This way of rendering mass to SM chiral fields is assumed throughout this book, and the fundamental question is *how such chiral fields arise in the beginning*. For spontaneous symmetry breaking leading to $G \rightarrow H$,⁴ one needs a singlet member under the Lorentz group and a singlet under the unbroken

² l_L has $Y = -\frac{1}{2}$, e_R has $Y = -1$, q_L has $Y = \frac{1}{6}$, u_R has $Y = \frac{2}{3}$, and d_R has $Y = -\frac{1}{3}$.

³ The converse is not necessarily true; the SU(2)_L $\times$ SU(2)_R $\times$ U(1) model is $L - R$ symmetric but chiral [7].

⁴ In the above example, $G = \text{SU}(2) \times \text{U}(1)_Y$ and $H = \text{U}(1)_{\text{em}}$.

gauge group H , but there should also be a non-singlet under G . In the Hilbert space, such a member as a fundamental field is a neutral scalar transforming nontrivially under both $SU(2)$ and $U(1)$. The simplest such representation is a spin-0 Higgs doublet with $Y = \frac{1}{2}$ [8],

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \quad (1.5)$$

A more complicated mechanism for spontaneous symmetry breaking is the use of a composite field which is a neutral scalar transforming nontrivially under both $SU(2)$ and $U(1)$. The simplest such composite field is one that assumes a new confining force, the so-called *techni-color* confining around the TeV scale, and composites of techni-quarks realize this idea [9]. This neutral scalar component can develop a vacuum expectation value (VEV) which certainly breaks G but leaves H invariant. Breaking the gauge symmetry through the VEV of scalar fields is the *Higgs mechanism* [10]. The SM is a chiral theory based on $SU(2) \times U(1)$ with the above Higgs mechanism employed.

Below the spontaneous symmetry breaking scale, the unbroken gauge symmetry is H , and the gauge bosons of G corresponding to G/H obtain mass of order (gauge coupling) $\times$ (VEV). This process applied to the SM renders three $IVB(W^\pm, Z)$ masses at the electroweak scale: $M_W \simeq 80$ GeV, $M_Z \simeq 91$ GeV. Of course, the photon A_μ remains massless. The origin of the fermion masses in the SM is not from the $SU(2) \times U(1)$ invariant mass term, which cannot be written down anyway, but originates from the gauge invariant Yukawa couplings of the fermions with the spin-0 Higgs doublet. Then, fermion masses are given by (Yukawa couplings) $\times$ (VEV). A variety of fermion masses is attributed to the variety of the Yukawa couplings.

One can glimpse that the essence of the above description of nature in terms of the SM is that *the theory is chiral* until the SM gauge group is spontaneously broken at the electroweak scale of $v \simeq 247$ GeV. Since the fundamental theory may be given at the Planck scale

$$M_P = \frac{1.22 \times 10^{19} \text{ GeV}}{\sqrt{8\pi}} = 2.44 \times 10^{18} \text{ GeV} \quad (1.6)$$

our chief aim in the construction of the SM is to obtain the correct chiral spectrum from a fundamental theory such as string theory given near the Planck scale. As we move toward a chiral theory, the parity violating weak interaction phenomena guide us to the SM.

In the search for a fundamental theory, two approaches can be taken. One can be the accumulation of low energy observed evidence and the building of a theoretically satisfactory gigantic model describing all these phenomena. This is a bottom-up approach in which the model cannot be excluded experimentally and is hence physically sound. The other approach is to find a theoretically satisfactory model given near the Planck scale and compare its low energy manifestation with experimental data. This is known as the top-down approach. Sometimes, the bottom-up approach is mingled together with

the top-down approach because a fundamental theory can never be achieved using the bottom-up approach alone. In any case, one needs guidance for such a theory. From the theoretical point of view, the best guidance is the *symmetry principle*. In recent years, the top-down approach has gained momentum.

Looking back at the construction of the SM, it started from matter representation $|\Psi\rangle$ in the Hilbert space where $|\Psi\rangle$ symbolically stands for the L-handed electron doublet l and the R-handed electron singlet e_R . If we include quarks also in the matter, $|\Psi\rangle$ will include them as well. In this Hilbert space, operations by the weak charge and the electromagnetic charge are treated in a similar fashion, thus the SM is dubbed with the phrase, “unified theory of weak and electromagnetic interactions”. The key point to observe here is the role of matter representation $|\Psi\rangle$. It is the representation on which symmetry charges act. In this book we will generalize this symmetry concept, and adopt the *unification theme: unify all the matter representations if it is possible*.

The first top-down approach toward a more fundamental theory beyond the SM was the grand unified theory (GUT). In one attempt, among the representations in $|\Psi\rangle$ the lepton doublet l and the charge conjugated field d_L^c of the R-handed d_R quark are unified into a single representation [11]. Other SM representations are grouped together. This attempt succeeded in unifying the SM group into a simple group SU(5). Another early attempt was to combine the quark doublet q and the lepton doublet l together into a single representation [7]. Then, the remaining SM representations are matched together with the attempted extended gauge group. This attempt succeeded with a semi-simple group $SU(4) \times SU(2)_L \times SU(2)_R$. It follows that, in these GUTs the strong and electroweak couplings are necessarily the same when the unification is valid.

Apparently, the strong, weak and electromagnetic couplings observed at low energy are not the same at the electroweak scale, and at first glance this idea of unification with the identical gauge couplings seems to contradict the observed phenomena. However, the size of the coupling constant looks different at different energy scales of the probing particle. This is due to the fact that a renormalizable theory intrinsically introduces a mass scale μ , and the energy dependence of the coupling is described by the renormalization group equation. Therefore, one can construct a GUT such that the gauge couplings are unified at a scale, say at M_{GUT} , which is supposed to be super-heavy so that the electroweak coupling and the strong coupling constants are sufficiently separated at the energy scales (~ 100 GeV) probed by the current accelerators [12]. For a significant separation through logarithmic dependence, one needs an exponentially large M_{GUT} [12] which should be smaller than the Planck mass so that gravitational corrections might be insignificant. Here, we should not forget that the construction of the simplest SU(5) was possible after realizing that one can collect all the pieces of the fifteen chiral fields with

one kind of chirality, i.e. in terms of the L-handed fields.⁵ These GUTs render the SM fermions below the GUT symmetry breaking scale so that massless SM fermions can survive down to the electroweak scale. Even though GUTs are basically top-down approaches, they have one notably testable prediction: a proton decays at the experimentally verifiable level with the proton lifetime $\tau_p > 10^{29}$ sec.

Another attempt can be Kaluza–Klein (KK) [14] type higher dimensional theories. But a naive torus compactification of the internal spaces leads to vectorlike fermions (the heavy KK mode and the vectorlike fermions), implying no massless fermions at low energy.

There are several good reasons to go beyond the SM. Probably, the most important reason for the extension is to understand *the family problem* why the fermion families repeat three times. Of course, an extension to understand the family problem must retain the good chiral property. The family problem has been considered with both global and gauge horizontal (or family) symmetries. However, the horizontal symmetry must be broken since different families obviously have different mass scales. If it were a global symmetry, we would expect Goldstone bosons (famillons) after the spontaneous global symmetry breaking. If it were a gauge symmetry, we would expect flavor changing phenomena at some level and one has to be clever enough to forbid gauge anomalies. In general, these horizontal symmetries are very complicated if not impossible. If a horizontal gauge symmetry is considered, it is better to unify it in a GUT-like gauge group so that even the horizontal gauge couplings are also unified with the other gauge couplings. In this respect, the grand unification of families (GUF) has attracted some attention [13]. However, in contrast to GUTs, the grand unification of families in 4 dimensions (4D) has not produced any conspicuous predictions which can be tested. They are hidden at the unknown super heavy mass scales.

Even though the 4D gauge theories with spontaneous symmetry breaking are very successful, as witnessed by the success of the SM, there exists a fundamental problem in 4D gauge models. The problem lies with the introduction of a specific set of representations for the fermions and Higgs fields in the SM, or in GUTs, or in GUFs. Let us call this *the representation problem*, and it becomes more acute when one tries to understand it together with *the family problem*. The specific choice of chiral representations in the GUTs or in the SM cannot be answered in a 4D gauge theory framework. For example, if we try to embed three families in a spinor representation of a big orthogonal group such as in $SO(18)$, there would remain a question, “Why is there only one spinor in $SO(18)$?” On the other hand, the group representation for the spin-1 gauge fields is uniquely fixed by the adjoint representation. There is no other choice. If the matter representation is determined as uniquely as the gauge boson is, then the representation problem would be understood.

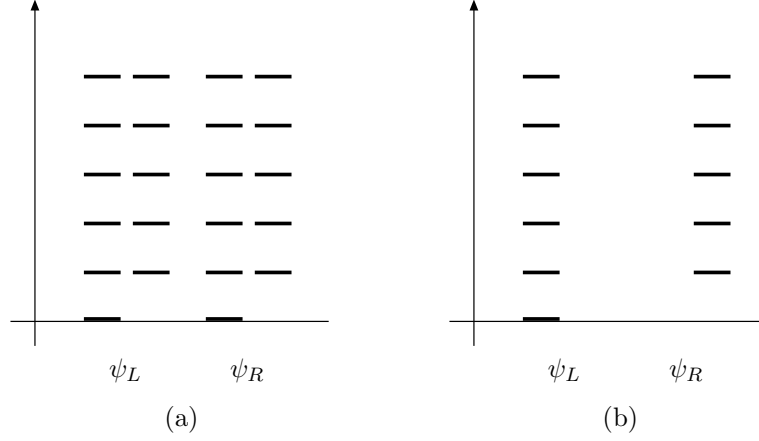
⁵ About the same time as this theory was put forth, the concept of supersymmetry was born from similar knowledge of Weyl spinors.

Even if we go beyond 4D, the representation problem in field theory is still not understood. The only constraint for matter representations is the anomaly cancellation, which does not fix the representation uniquely. Therefore, the theory must be more restrictive than field theory to understand the representation problem.

In this respect, string theory, either bosonic or fermionic, is one alternative to pursue. But to have matter (the spin- $\frac{1}{2}$ fermions), we can only consider the fermionic string or superstring, which is possible in 10D. This superstring theory is far more restrictive than higher dimensional field theories. To apply it to the 4D phenomenology, we must hide the six extra dimensions.

It is known that there are two ways to hide the extra dimensions, one is by the so-called compactification and the other is by introducing a warp factor as in the Randall-Sundrum type-II model [15]. This book is restricted to compactification. Consider the simplest extra dimensional generalization, i.e. to 5D. If one compactifies the extra dimension, the original 5D fields split into 4D fields. The effective 4D fields might be massive, but massless fields might also exist. Consider, for example, a 5D theory with spin-2 graviton ($g_{MN}(M, N = 0, 1, 2, 3, 5)$) only in 5D. By compactifying the fifth dimension on a circle, one obtains the well-known KK spectrum. At low energy the massless modes are a spin-2 graviton ($g_{\mu\nu}(\mu, \nu = 0, 1, 2, 3)$), a spin-1 gauge boson ($g_{\mu 5}$), and a spin-0 dilaton (g_{55}). The spin-1 gauge boson and spin-0 dilaton were originally components of 5D graviton. The compactification radius R determines the masses of the KK modes (integer) $\times R^{-1}$. If one introduces a massless 5D fermion (8 real components in ψ_L or in ψ_R) in addition, then by compactifying on a circle S_1 one obtains two 4D Weyl fermions at each KK level as shown in Fig. 1.1(b). If the representation is vectorlike (in 5D, both 4D ψ_L and 4D ψ_R should be introduced), then it is a general strategy to remove them at a high energy scale due to interactions unless some symmetry forbids such a large mass. So, if two Weyl fermions with opposite quantum numbers are matched at each level, they are removed at high energy. In this sense, two lowest level fermions of Fig. 1.1(a) with zero KK mass can obtain a superheavy mass. Thus, the naive compactification on a circle would not allow a chiral spectrum at low energy. Our hope is to obtain a chiral fermion at low energy through a mechanism in which there remains an unmatched Weyl fermion as shown in Fig. 1.1(b). However, it has been proven that if the gauge bosons originate from the gravity multiplet $g_{MN}(M, N = 0, 1, 2, 3, 5, \dots, D)$ in D dimensions then a chiral theory such as that shown in Fig. 1(b) does not arise at low energy [16]. Therefore, the KK idea, even though beautiful, is not relevant for a low energy chiral theory like the standard model.

If 4D standard model gauge bosons arise from *gauge bosons in higher dimensions*, then a realization of Fig. 1.1(b) does not arise from a simple torus compactification on a circle S_1 . To arrive at Fig. 1.1(b), one must impose careful boundary conditions on the 5D fermion wave functions such that some parts of the spectrum are removed. So, we consider the orbifold, which achieves this idea beautifully, by moding out S_1 by some discrete group such as $\mathbf{Z}_2$.

**Fig. 1.1.** KK mode spectrum

We observed that the standard model gauge bosons must arise from spin-1 gauge bosons in higher dimensions, not from the higher dimensional gravity multiplet. This leads us to the consideration of higher dimensional gauge theories. These higher dimensional gauge fields are required to contain the SM gauge bosons. In higher dimensions, we also introduce fermions which contain the three SM families. Since we introduce these fermions, one has to worry about the gauge anomaly if $D = \text{even}$. Then the gauge anomalies [17] and gravitational anomalies [18] in even dimensions must be taken into account. As mentioned above, however, the higher dimensional field theories do not explain the representation problem except for the requirement of having no anomaly.

In the mid 1980s a breakthrough was made in the search for higher dimensional theories. To introduce fermions, $\mathcal{N}=1$ supersymmetry must be considered. Green and Schwarz observed that in 10D no anomaly appears if the gauge group is $SO(32)$ or $E_8 \times E_8'$ [19]. Basically, these large gauge groups are required to cancel any anomaly arising from the 10D gravitino. The 10D gravitino carries 496 units of anomaly. If one wants to cancel the gravitino anomaly by gauge fermions, the gauge group must match the dimension of the adjoint representation as 496. $SO(32)$ and $E_8 \times E_8'$ fulfil this requirement. In addition, the second rank antisymmetric tensor field B_{MN} is required. Soon, the heterotic string theory was constructed and these large gauge groups are realized as the oscillation modes of the closed strings and winding modes [20].

This string motivated 10D $\mathcal{N}=1$ gauge field theory introduces fermions as superpartners of the gauge bosons which form the adjoint representation of the gauge group. An adjoint representation is a real representation. If one naively compactifies on a torus, say a six dimensional torus T_6 , the real representation keeps the reality and one cannot obtain chiral fermions in 4D. To reach the phenomenologically successful standard model, it is of utmost importance in

extra dimensional field theory to obtain chiral fermions after compactification. Under torus compactification, the 10D $\mathcal{N} = 1$ supersymmetry becomes $\mathcal{N} = 4$ supersymmetry in 4D. Supersymmetries with $\mathcal{N} \geq 2$ introduce vector-like fermions in 4D which are not needed at the electroweak scale. If one must introduce a supersymmetry to understand the gauge hierarchy problem, the only allowable one in 4D is the $\mathcal{N} = 1$ supersymmetry. Therefore, the first set of criteria is to keep only $\mathcal{N} = 1$ supersymmetry after compactification and to allow chiral fermions. In orbifold compactification, which is the main subject of this book, the requirement for the $\mathcal{N} = 1$ supersymmetry in 4D usually allows chiral fermions.

Phenomenologically, the $E_8 \times E'_8$ heterotic string has been discussed much more than the $SO(32)$ heterotic string. The fifteen chiral fields of the SM plus one more chiral field are embeddable in a spinor representation of $SO(10)$. The adjoint representation of the $SO(32)$ group cannot contain the spinor representation of $SO(10)$. Thus, making a reasonable spectrum after compactification from the $SO(32)$ heterotic string is not the obvious first step. On the other hand, the adjoint representation of E_8 contains the spinor representation of $SO(10)$, and it would seem easy to classify the SM fermions through spinor representation of $SO(10)$ (or for that matter, in terms of $\mathbf{10} \oplus \bar{\mathbf{5}}$ of $SU(5)$). For this reason, we will restrict our discussion to the $E_8 \times E'_8$ heterotic string only.

Suppose that 10D compactifies as $M_{10} \rightarrow M_4 \times B_6$, i.e. to a flat 4D Minkowski space M_4 times a compact 6D internal space B_6 . Candelas et al. have shown that the condition for the $\mathcal{N} = 1$ supersymmetry in 4D is to require the $SU(3)$ holonomy of the internal space [21]. This $SU(3)$ symmetry can be embedded in the gauge group also. Spaces with $SU(3)$ holonomy are called the Calabi-Yau spaces. In the example considered in [21], one obtains a 4D $\mathcal{N} = 1$ supersymmetric model with the gauge group $E_6 \times E'_8$. The net number of chiral fermions of the E_6 sector are 36 copies of $\mathbf{27}$. Each $\mathbf{27}$ of E_6 contains one family of the SM. Thus, this Calabi-Yau space gives too many families. Nevertheless, it provides an example of how one obtains chiral fermions and $\mathcal{N} = 1$ supersymmetry in 4D. Actually, Calabi-Yau manifolds are rather complicated spaces.

For reducing the $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 1$, a simpler method known as orbifold was introduced by Dixon, Harvey, Vafa, and Witten [22]. The orbifold method uses discrete groups on top of torus compactification. For a given 6D internal torus, all possible discrete group actions have been classified. Generically they do not act freely. The fixed points lead to singularities of the quotient space which is called an orbifold. The orbifold singularities can be eliminated by cutting out the fixed points and gluing them in smooth “disk-like” surfaces such that the resulting smooth space has $SU(3)$ holonomy. Therefore, an orbifold can be considered a singular limit of a good manifold such as the Calabi-Yau. To a low energy observer, therefore, the orbifold is as good as the Calabi-Yau space. Another merit of the orbifold is that compactification through orbifolding can be systematically found. The simplest orbifold

rendering $\mathcal{N}=1$ supersymmetry in 4D is the $\mathbf{Z}_3$ orbifold with the standard embedding. Here, the $\mathbf{Z}_3$ can be considered as the center of the holonomy group $SU(3)$ of the Calabi-Yau space. One of the flavor questions is *why there are three families of fermions*. Here, the moding by a discrete group can be of help. In a $\mathbf{Z}_3$ orbifold with up to two Wilson lines,⁶ the spectrum is always a multiple of 3. In any other orbifold model, one must calculate the whole spectrum to see whether fermion families result in a multiple of 3 or not.

The chief merit of the orbifold is that it allows chiral fermions at low energy, even though one starts to compactify with such a simple idea as a 6D torus. The existence of chiral fermions is at the heart of the standard model construction and, hence, the making of our universe as it is now before chemistry and biology were able to play their roles.

The models obtained by the orbifold compactification can be considered 4D string models, but their roots are in 10D superstring. There is another method of obtaining a 4D superstring model, using free fermionic formulation [23]. However, the free fermionic formulation lacks the geometrical interpretation present in the orbifold compactification. Yet there exists another method using self-dual lattices [24].

This book is aimed at exposing the most fundamental problems facing the standard model now, and the orbifold is extensively discussed with an emphasis on model building. In the next chapter, we start out our journey by reviewing how the successful chiral field theory known as the standard model was achieved. In Chap. 3, we present the general concept on orbifold. Then, we introduce field theoretic orbifolds in Chap. 4. The basics on string theory is presented in Chap. 5. In Chap. 6, we start to present orbifolding of the 10D string theory. Then, we continue to discuss orbifolds using the Jacobi theta functions, Yukawa couplings, and group theoretic properties of the twisted sector spectra in Chaps. 7, 8 and 9, respectively. In Chap. 10, we review some key aspects of the orbifold phenomenology and present a detailed construction example with the largest discrete moding by $\mathbf{Z}_{12}$ which contains both $\mathbf{Z}_3$ and $\mathbf{Z}_2$. In the last chapter, Chap. 11, we present all the possible $\mathbf{Z}_3$ orbifold models with two Wilson lines. We present five appendices. In Appendix A, we briefly review the issues in the Calabi-Yau space which has the intimate connection with orbifold. In Appendix B, we present two other methods attempting to obtain the SM from superstring, the fermionic construction and intersecting brane models. In Appendix C, we present some relations of Jacobi theta functions, used in Chaps. 7 and 8. In Appendix D, we list useful tables for model building which include the orbifolds with $\mathcal{N}=1$ supersymmetry, the degeneracy factor and vacuum energy needed for orbifold construction, and the conditions on Wilson lines. Finally, some useful numbers in group theory are presented in Appendix E.

⁶ A Wilson line is a closed line integral along the direction tangent to a gauge field, $\oint dx^\mu A_\mu$. A famous example is the Aharonov-Bohm phase. In this book, it is meant as a closed line integral in the compactified space, $\oint dx^i A_i$.

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Standard Model and Beyond

2.1 The Standard Model

The standard model (SM) consists of the confining color gauge theory $SU(3)$ for strong interactions and the spontaneously broken electroweak gauge theory $SU(2)_L \times U(1)_Y$. In this subsection, we introduce the SM, concentrating on the issues relevant for our string orbifold construction.

The symmetry principle is the heart of particle physics. It has its origins in Heisenberg's $SU(2)$ isospin, and flourished in the 1960s under the name of $SU(3)_{\text{flavor}}$. The triple tensor product of the fundamental representation $\mathbf{3}$ of $SU(3)_{\text{flavor}}$ gives

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} .$$

The low lying octet baryons, p, n, Λ , and Σ , are assigned to one $\mathbf{8}$ of the above representation. This classification works also for spin-0 and spin-1 mesons, $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$. This old $SU(3)_{\text{flavor}}$ classification was very successful in classifying low lying hadrons, and the basic ingredients $\mathbf{3}$ for all these classifications are called quarks; up, down and strange quarks, $\mathbf{3} = (u, d, s)^T$ [1]. The quarks were assumed to carry spin- $\frac{1}{2}$ because three quarks make up spin- $\frac{1}{2}$ baryons and quark-antiquark pairs make up integer spin mesons. This development in probing the subnuclear structure was probably the cornerstone of the discovery of the standard model, but a few more new ideas needed to be added. First, a new degree of freedom, color, was introduced. Second, local groups were considered. Third, the flavor group became bigger with discoveries of more quarks, c, b, t . And fourth, the $SU(3)_{\text{flavor}}$ was understood to be nothing but the strong aspect of strong interactions around 1 GeV compared to the current mass of three flavors of quarks, $\mathbf{3} = (u, d, s)^T$. As for the electroweak interaction, only a part of $SU(3)_{\text{flavor}}$ is localized (gauged).

The introduction of color degrees of freedom was motivated from the old $SU(6)$ classification of hadrons. This old $SU(6)$ is a generalization of the above Gell-Mann's flavor $SU(3)_{\text{flavor}}$ together with the rotation group $SU(2)$. For the S-wave composites, the representations of the rotation group are just spins.

Thus, the fundamental representation **6** of SU(6) is composed of the direct product representation of the flavor triplet, i.e. u, d, s quarks, and the spin doublets, $|\uparrow\rangle$ and $|\downarrow\rangle$, $\mathbf{6} = (u^\uparrow, u^\downarrow, d^\uparrow, d^\downarrow, s^\uparrow, s^\downarrow)^T$. Following Gell-Mann, one may try to classify low lying hadrons in terms of tensor products of **6** and $\bar{\mathbf{6}}$, but in the case of SU(6) we note that different spin representations can come in one SU(6) representation. For mesons, we consider $\mathbf{6} \otimes \bar{\mathbf{6}} = \mathbf{1} \oplus \mathbf{35} = \mathbf{1}_{s=0} \oplus (8_{s=0} + 3_{s=1} + 24_{s=1})$ where we indicated the spin s as subscripts. Thus, the representation **35** of SU(6) consists of the pseudoscalar meson octet $(\pi, K^\pm, K^0, \bar{K}^0, \eta)$, the vector meson singlet $(\sim\phi)$,¹ and the vector meson octet $(\rho, K^{*\pm}, K^{*0}, \bar{K}^{*0}, \sim\omega)$. This appears to work rather well.

However, as for baryons, we expect

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{20} \oplus \mathbf{56} \oplus \mathbf{70} \oplus \mathbf{70}' . \quad (2.1)$$

We may try to embed the low lying baryons, the baryon octet (p, n, Λ, Σ) and the baryon decuplet $(\Delta_{(I=\frac{3}{2})}, Y_{(I=1)}^*, \Xi_{(I=\frac{1}{2})}^*, \Omega_{(I=0)})$ in one representation of (2.1). Indeed, the representation **56** can house all these low lying baryons since the baryon octet is spin- $\frac{1}{2}$ and the baryon decuplet is spin- $\frac{3}{2}$. This appears to work as well. But the **56** of SU(6), namely the Young tableaux given in Fig. 2.1, is completely symmetric under any exchange of its composite **6**. Since we interpreted **6** as a spin- $\frac{1}{2}$ quark, the notion that a pair of spin- $\frac{1}{2}$ objects must anticommute under their exchange is grossly violated by **56**. One cannot resolve this dilemma without the introduction of a new degree of freedom [2] which is named *color* in modern theory.

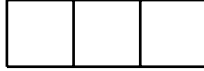


Fig. 2.1. The Young tableaux for **56** of SU(6)

Thus, one assumes that baryons made of three quarks must be completely antisymmetric under the exchange of the color index. The complete symmetric property of **56** of SU(6) then becomes a fortune instead of a disaster when combined with this completely antisymmetric property of color exchange, and the spin-statistics theorem is satisfied. In other words, the baryon wave function must behave as $\Psi_{\text{baryon}} \sim \epsilon_{\alpha\beta\gamma} q^\alpha q^\beta q^\gamma$ where $\alpha\beta\gamma$ are the color indices, 1, 2, 3, or red, green, yellow, respectively. Since a baryon is supposed to be made of three quarks, we need just three indices for color, and $\epsilon_{\alpha\beta\gamma} q^\alpha q^\beta q^\gamma$ must be a singlet under transformation in this new color space. Then we conclude that the new color space is SU(3).² The gauge theory of color SU(3) is called

¹ The symbol $\sim$ implies that the singlet-octet mixing has to be considered.

² Note that it cannot be SO(3) since SO(3) does not allow complex representations the needed property for the existence of both quarks and antiquarks.

quantum chromodynamics, or simply QCD. The gauge bosons of this SU(3)-color are called gluons G_μ^a ($a = 1, 2, \dots, 8$). The reason why only the color singlet states appear as low lying hadrons is the key issue among QCD problems, and is known as the confinement problem. Over the last three decades, strong interactions have successfully been described by QCD. After the initial introduction of SU(3) as the color interaction with integer-charged quarks [2], however, the present day QCD is based on the fractionally charged quarks with unbroken SU(3)_{color} [3, 4, 5] as proven by the $\pi^0 \rightarrow 2\gamma$ decay rate.³

The flavor group we considered for SU(3)_{flavor} is now generalized as SU(6)_{flavor} since there exist at least 6 quarks, u, d, c, s, t , and b . The weak interaction gauges only a part of this SU(6)_{flavor} since it is known that different families exhibit exactly the same kind of tree level charged current (CC) weak interactions. As for the CC weak interactions, the leptons are not different from quarks. Therefore, for gauge interactions it will be enough to consider just the first family doublets, $(\nu_e, e)^T$ and $(u^\alpha, d^\alpha)^T$. Now the electroweak gauge group is taken as SU(2) × U(1) to unify the electromagnetic interaction with the weak interaction [6], resulting in the unified theory aptly named the *electroweak theory*. In the electroweak theory, the interactions are known to be chiral; only the left-handed fields participate in the CC interactions, which was the famous old “V–A” theory introduced in the previous chapter. Thus, the electroweak representation of leptons and quarks are asymmetrical under the chirality exchange, $L \leftrightarrow R$,

$$l_L \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad e_R, \quad q_L \equiv \begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_L, \quad u_R^\alpha, \quad d_R^\alpha \quad (2.2)$$

where α runs over the three color indices and the weak hypercharges Y for the representations are

$$Y(l_L) = -\frac{1}{2}, \quad Y(e_R) = -1, \quad Y(q_L) = \frac{1}{6}, \quad Y(u_R^\alpha) = \frac{2}{3}, \quad Y(d_R^\alpha) = -\frac{1}{3}. \quad (2.3)$$

The electromagnetic charge is given by $Q_{\text{em}} = T_3 + Y$. The chirality of the left-handed doublet and the hypercharge Y are customarily written explicitly as subscripts in the electroweak group, SU(2)_L × U(1)_Y. The gauge sector of this theory has two gauge couplings, the SU(2)_L coupling g and the U(1)_Y coupling g' as shown in (1.3). The ratio of these couplings defines the weak mixing angle $\tan \theta_W = g'/g$. The group SU(2)_L × U(1)_Y is spontaneously broken down to U(1)_{em}, and three gauge bosons obtain mass. These massive spin-1 gauge bosons $W^\pm$ and Z^0 are considered as the mediators of the weak force. If the spontaneous symmetry breaking occurs only through the vacuum expectation value (VEV) of Higgs doublet(s), the tree level ratio of these masses is related to the weak mixing angle by,

³ With a judicious symmetry breaking of SU(3)_{color}, one can obtain integer-charged quarks. For pure strong interaction phenomena, e.g. for QCD β functions, the electromagnetic charges of quarks do not matter.

$$\sin^2 \theta_W = 1 - \frac{M_W^2}{M_Z^2} . \quad (2.4)$$

The same weak mixing angle appears in neutral current (NC) neutrino scattering experiments,

$$\frac{G_F}{\sqrt{2}} \bar{\nu}_\mu \gamma^\alpha (1 + \gamma_5) \nu_\mu \sum_i (\bar{q}_i Q_Z (1 + \gamma_5) q_i + \bar{q}_i Q_Z (1 - \gamma_5) q_i) , \quad (2.5)$$

where $Q_Z = T_3 - Q_{\text{em}} \sin^2 \theta_W$. Phenomenologically, $\sin^2 \theta_W$ can be determined by several independent processes. It turned out that experimentally determined $\sin^2 \theta_W$ is close to the relation (2.4). Thus, any deviation from the tree level relation obtained for the doublet breaking is important and for this purpose we introduce another phenomenologically important parameter ρ defined by

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} . \quad (2.6)$$

ρ becomes 1 at tree level if the spontaneous symmetry breaking occurs only through the vacuum expectation value (VEV) of Higgs doublet(s). The experimentally determined weak mixing angles from the gauge boson mass ratio and the weak NC experiments coincide. The coincidence of these mixing angles given in (2.4) and (2.5) and the ρ parameter being close to 1, implies that the electroweak symmetry breaking occurs through VEVs of a Higgs doublet(s). Large electron-positron (LEP) collider experiments at CERN decisively confirmed this Higgs doublet condition. The dominant loop contribution to ρ parameter is from the heavy top quark [7]. The global fit, including the radiative corrections, gives $\rho = 1.0012_{-0.0014}^{+0.0023}$ and $\sin^2 \theta_W(M_Z) = 0.23113 \pm 0.00015$ [8].

The electroweak representation given in (2.2) repeats three times, thus we say that there are three families of fermions plus a doublet of Higgs bosons:

$$\begin{aligned} & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad e_R, \quad \begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_L, \quad u_R^\alpha, \quad d_R^\alpha \\ & \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \mu_R, \quad \begin{pmatrix} c^\alpha \\ s^\alpha \end{pmatrix}_L, \quad c_R^\alpha, \quad s_R^\alpha \\ & \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \quad \tau_R, \quad \begin{pmatrix} t^\alpha \\ b^\alpha \end{pmatrix}_L, \quad t_R^\alpha, \quad b_R^\alpha \\ & \mathbf{H}_1 = \begin{pmatrix} H^0 \\ H^- \end{pmatrix} \end{aligned} \quad (2.7)$$

which make up the needed matter spectrum of the SM at low energy. The standard model is a gauge theory $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ with the representation (2.7). For nonzero neutrino masses observed by underground detectors, one

usually introduces $SU(3) \times SU(2) \times U(1)$ singlet fields at high energy scale(s). The SM is understood to include this possibility of singlet neutrino introduction.

Naively, one may think that three families seem to be too much for the construction of our universe, chiefly made of e , p , and n . But there exist Sakharov's three conditions to generate non-zero baryon number in the universe starting from a baryon symmetric universe: (1) the existence of baryon number violation, (2) the baryon number violation must accompany $\mathcal{C}$ and $\mathcal{CP}$ violation, and (3) these symmetry violating interactions occurred in the non-equilibrium phase. So for our existence in the universe, $\mathcal{CP}$ violation seems to be necessary. The $SU(2)_L \times U(1)_Y$ gauge theory needs three families to introduce a physically meaningful complex phase, i.e. $\mathcal{CP}$ violation, basically via Yukawa couplings [9], after the spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y$. Even though the $\mathcal{CP}$ phase from the complex Yukawa couplings of the SM turned out to not be the $\mathcal{CP}$ phase for the baryogenesis, still many physicists considered the baryogenesis as a powerful argument for the $\mathcal{CP}$ violation and hence for three families. One may introduce more Higgs doublets to have $\mathcal{CP}$ violation from the Higgs potential, but certainly the $\mathcal{CP}$ violation from the Yukawa couplings is one possibility. Experimentally, the weak $\mathcal{CP}$ violation by Yukawa couplings seems to be the dominant source [8]. One of the most important problems in particle physics is understanding why there appear three families of fermions as shown in (2.7), which is known as the *family problem*. Note that the seemingly innocuous *introduction of color $\mathbf{3}$'s (also $\mathbf{3}$'s) and weak $\mathbf{2}$'s* in (2.7) has simplicity, and probably hides a *profound implication* in the construction of the SM from superstring theory.

The SM gauge group is $SU(3) \times SU(2) \times U(1)$ where the color group $SU(3)$ is unbroken and the electroweak group $SU(2)_L \times U(1)_Y$ is spontaneously broken. Nonabelian gauge theories are known to be asymptotically free if the matter content is not too large [4]. QCD is asymptotically free if the number of quarks is less than 16.5 in the one loop estimation. So, at high energy, the QCD coupling α_c becomes small and QCD is perturbatively calculable. On the other hand, at low energy of $E \leq 1 \text{ GeV}$, QCD becomes strong and all complications due to its strong nature occur, such as the chiral symmetry breaking and confinement. A complete understanding of nonperturbative nature is needed to know the effects of QCD at low energy. One such nonperturbative phenomenon is given by the instanton solution of nonabelian gauge theories [10]. The instanton solution recommends the use of the so-called θ -vacuum [11]. In the θ -vacuum, one must consider the $\mathcal{CP}$ violating interaction, $(g_c^2 \bar{\theta}/16\pi^2) \text{Tr} G \tilde{G}$ where G is the gluon field strength and $\tilde{G}$ is its dual. However, we know that $\mathcal{CP}$ violations have been observed only in weak interaction phenomena, not in strong interactions. This restricts $\bar{\theta}$ phenomenologically, from the observed upper bound of the neutron electric dipole moment, $\bar{\theta} < 10^{-9}$. So, "Why is $\bar{\theta}$ so small?" becomes a good theoretical problem in the SM: *the strong $\mathcal{CP}$ problem* [12]. At the level of the SM with (2.7), this strong $\mathcal{CP}$ problem is not understood.

In this preceding concise review of the SM, we have already exposed three fundamental problems of the SM,

- Coupling unification problem: Why are there three different gauge couplings, or three factors of $SU(3) \times SU(2) \times U(1)$ in the SM?
- Family problem: Why are there three families?
- Strong $\mathcal{CP}$ problem: Why is $\bar{\theta}$ so small?

This book is devoted to some theoretical developments regarding the coupling unification problem and the family problem from the currently popular fundamental theory, superstring. The strong $\mathcal{CP}$ problem can probably be understood from the introduction of a very light axion, also from superstring, however this is outside the focus of this book.

The number of fields in one family appearing in (2.2) is fifteen, and since we have represented them as L- and R-handed fields, these are fifteen chiral fields. In Lagrangian formulation, one can use a field ψ or its charge conjugated field ψ^c . For L- and R-handed quantum fields, $\psi_L = \frac{1+\gamma_5}{2}\psi$ and $\psi_R = \frac{1-\gamma_5}{2}\psi$, the charge conjugated fields have the opposite chiralities, for example $(\psi_R)^c = (\psi^c)_L$, which can easily be checked from the definition of the charge conjugation operation on the Weyl fields $\psi_{L,R}$. So rather than using (2.2), let us use only one chirality of the L-handed fields,

$$l_L \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad e_L^c, \quad q_L \equiv \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad u_L^c, \quad d_L^c \quad (2.8)$$

where color multiplicity must be understood for the quarks, and c refers to the charge conjugated field, i.e. $Y(e^c) = 1, Y(u^c) = -\frac{2}{3}, Y(d^c) = \frac{1}{3}$. This is a simple rewriting of the fifteen chiral fields of (2.2). Note that it is simpler to view all fermions on the same footing, e.g. as left-handed(L), as we have done here. For spin-0 scalars, we do not have such a distinction in terms of chirality.

Chirality seems to be the essential property constructing our universe where the SM fields live long in a vast space. Understanding the chirality of the SM is the overriding theme of this book.

2.2 Grand Unified Theories

The history of particle physics, quantum field theory, and quantum mechanics has been the extension of the realm of symmetry operation in Hilbert space. In the previous section, we have explored this extension of symmetry. Considering only the first quark family members u and d , Heisenberg's isospin symmetry is the unification of individual u and d (or proton and neutron) in one doublet representation of the rotation operation in an extended symmetry space, i.e. the isospin space. In the previous section, we further extended the isospin space to (isospin) $\times$ (color). This extension was applied to matter fields which

can be the bases of the Hilbert space. Thus, we put forward the following in this discussion,

The theme of unification: “*Put all the matter fields on equal footing in an extended space.*”

Under this unification theme, the standard model $SU(3) \times SU(2) \times U(1)$ with 15 chiral fields in (2.8) is not fully unified yet. From the unification theme, we may put the fifteen chiral fields in **15** of $SU(15)$. However with rank 14, this is too large and also has a theoretical problem of having a gauge anomaly. If the 15 chiral fields are extended to 16 chiral fields, by adding one $SU(3) \times SU(2) \times U(1)$ singlet field, one can find an $SO(10)$ gauge theory with a spinor representation.⁴ This perfectly agrees with our unification theme, and also does not have the gauge anomaly. In this $SO(10)$ unified theory, there is only one gauge coupling since all the chiral fields are put in a single representation. In other words, the gauge groups $SU(3) \times SU(2) \times U(1)$ are unified in $SO(10)$.

The unified theories of electromagnetic, weak, and strong forces are called grand unified theories (GUT). Georgi and Glashow looked for the minimal rank gauge group which unifies $SU(3) \times SU(2) \times U(1)$ in one simple group, unifying $SU(3)$ and $SU(2)$, but allows the possibility that there may be more than one representation. Indeed, they succeeded in finding the minimal unification group $SU(5)$ where the fifteen chiral fields of (2.8) are grouped together but still split into two, [14],

$$\bar{\mathbf{5}}_F = \begin{pmatrix} d^c \\ d^c \\ d^c \\ \nu_e \\ -e \end{pmatrix}_L, \quad \mathbf{10}_F = \begin{pmatrix} 0 & u^c & -u^c & u & d \\ -u^c & 0 & u^c & u & d \\ u^c & -u^c & 0 & u & d \\ -u & -u & -u & 0 & e^c \\ -d & -d & -d & -e^c & 0 \end{pmatrix}_L. \quad (2.9)$$

The Higgs doublet $\mathbf{H}_1$ in (2.7), which is actually $\mathbf{H}_2 \sim i\sigma_2 \mathbf{H}_1^*$, is unified into $\mathbf{5}_H$ with a new color triplet of scalars, $\mathbf{h}$,

$$\mathbf{5}_H = \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \\ H^+ \\ H^0 \end{pmatrix}. \quad (2.10)$$

The rank of $SU(5)$ is 4 which is the same as that of $SU(3) \times SU(2) \times U(1)$. Thus, to break $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$, one needs a Higgs field which can have a VEV in the center of $SU(5)$, in order not to reduce the rank. The simplest such choice is an adjoint Higgs field $\mathbf{24}_H$. The next simple choice is $\mathbf{75}_H$. Let us introduce $\mathbf{24}_H$ only for the GUT symmetry breaking,

⁴ H. Georgi found this $SO(10)$ model several hours before the $SU(5)$ [13].

and let its VEV be V_U . Then, below the scale V_U , the effective fields are only those in the SM and the colored scalar $\mathbf{h}$. The heavy fields we introduce in this minimal GUT at the scale V_U are the so-called X and Y gauge bosons each with mass $M_X = M_Y = \frac{1}{2}g_5 V_U$, where g_5 is the unification coupling constant, and twelve real Higgs fields originally introduced in $\mathbf{24}_H$. From (2.9) one can see that X and Y gauge bosons couple to a lepton and a quark, and hence are called “lepto-quark” gauge bosons, and also to two quarks uu or ud , rendering them “di-quark” gauge bosons. Therefore, the lepto-quark gauge bosons mediate proton decay, and their masses should be extremely heavy, $M_{X,Y} \geq 10^{15}$ GeV.

The prototype SU(5) GUT model classifies the 15 chiral fields neatly in $\mathbf{10}_F \oplus \mathbf{5}_F$. But a more important implication is that the electromagnetic charge is quantized, $3Q_{\text{em}}(u) + 3Q_{\text{em}}(d) + Q_{\text{em}}(e) = 0$, i.e. $Q_{\text{em}}(p) = -Q_{\text{em}}(e)$. This is possible because quarks and leptons are put in the same representation. Another equally important implication is that the gauge coupling is unified at the GUT scale $M_{X,Y}$. The running of gauge coupling constants below the GUT scale is [4],

$$\frac{8\pi^2}{g_i^2(M_Z)} = \frac{8\pi^2}{g_U^2(M_U)} + b_i \ln \frac{M_U}{M_Z} \quad (2.11)$$

where we can assume $M_U = M_X$ and g_U is the SU(5) coupling at the unification point M_U , and b_i are the coefficients of β functions of the gauge groups, SU(3), SU(2)_L and U(1)_Y,

$$\begin{aligned} b_1 &= +\frac{3}{5} \text{Tr} Y^2 \delta_{FB} , \\ b_2 &= -\frac{11}{3} \cdot 2 + \frac{2}{3} \sum_j l(R_j(SU(2)_L)) \delta_{FB} , \\ b_3 &= -\frac{11}{3} \cdot 3 + \frac{2}{3} \sum_i l(R_i(SU(3))) \delta_{FB} , \end{aligned} \quad (2.12)$$

where

$$l(R) = \text{Tr} T_R^a T_R^a , \quad (\text{no summation}) \quad (2.13)$$

is the index of representation R . For example, $l(\text{fund. representation}) = \frac{1}{2}$, $l(\text{adj. representation}) = N$ for SU(N) groups. In these formulae, l is normalized for a chiral fermion; thus $\delta_{FB} = 1$ for L and R fermions and summed over L and R. For complex bosons $\delta_{FB} = \frac{1}{2}$, and for real bosons $\delta_{FB} = \frac{1}{4}$. Also, every generator, including the U(1) generator, is normalized for (2.13) to be $\frac{1}{2}$ for the fundamental representation. If the electroweak hypercharge is embedded in a GUT group, the hypercharge is properly normalized. The proportionality of the electroweak hypercharge generator Y to the normalized U(1) generator Y_1 in SU(5) is parametrized by the normalization constant C ,

$$Y = CY_1, \quad \text{with } C^2 = \frac{5}{3} , \quad (2.14)$$

where

$$Y_1 = \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \sqrt{\frac{3}{5}} Y. \quad (2.15)$$

Since the gauge interaction gives $g'Y = g_1 Y_1$, we note that $g_1 = Cg'$. We can calculate the weak mixing angle (2.4)

$$\sin^2 \theta_W^0 = \frac{g'^2}{g_2^2 + g'^2} = \frac{1}{1 + C^2} \quad (2.16)$$

with $g_1 = g_2 = g_3 = g_5 = g_U$, by running the bare values down to the electroweak scale, as we will see in the following section. Since $Q_{\text{em}} = T_3 + Y$, and using the fact that the index (2.13) is independent of a , we arrive at the handy result [15],

$$\sin^2 \theta_W^0 = \frac{\text{Tr } T_3^2}{\text{Tr } Q_{\text{em}}^2}. \quad (2.17)$$

Note that we have assumed that all the SM fields belong to *complete* multiplets over which the trace is taken. If some fields fail to form a multiplet, the relation is no longer valid. As an exercise, we note the following for N_g families of fermions of (2.8),

$$\begin{aligned} b_3 &= -11 + N_g \frac{2}{3} \cdot \frac{1}{2} (1_{u_L} + 1_{d_L} + 1_{u_L^c} + 1_{d_L^c}) = -11 + \frac{4}{3} N_g \\ b_2 &= -\frac{22}{3} + N_g \frac{2}{3} \cdot \frac{1}{2} (3_{(u,d)_L} + 1_{(\nu_e, e)_L}) = -\frac{22}{3} + \frac{4}{3} N_g \\ b_1 &= N_g \frac{3}{5} \cdot \frac{2}{3} \left\{ 2 \left(-\frac{1}{2} \right)_{(\nu_e, e)_L}^2 + 1_{e_L^c} + 2 \cdot 3 \left(\frac{1}{6} \right)_{(u,d)_L}^2 + 3 \left(-\frac{2}{3} \right)_{u_L^c}^2 \right. \\ &\quad \left. + 3 \left(\frac{1}{3} \right)_{d_L^c}^2 \right\} = \frac{4}{3} N_g \end{aligned}$$

where in the subscript of each number we show the representation contributing to that number. Except for the gauge bosons, a complete multiplet (2.8) gives the same contribution to b_i , i.e. $\frac{4}{3} N_g$. Thus, the necessary differentiation of the weak SU(2) coupling and the QCD coupling is implementable in the GUT from the gauge boson contributions [15]. Which GUT actually succeeds in fitting the electroweak data is the key issue in building the GUT model.

In Fig. 2.2, we present a schematic behavior of the running of gauge couplings from the GUT scale M_U down to the electroweak scale M_Z . For a sufficient separation of α_3 and α_2 at the electroweak scale, we need a large logarithm, or

$$\frac{M_U}{M_Z} \geq 10^{12}. \quad (2.18)$$

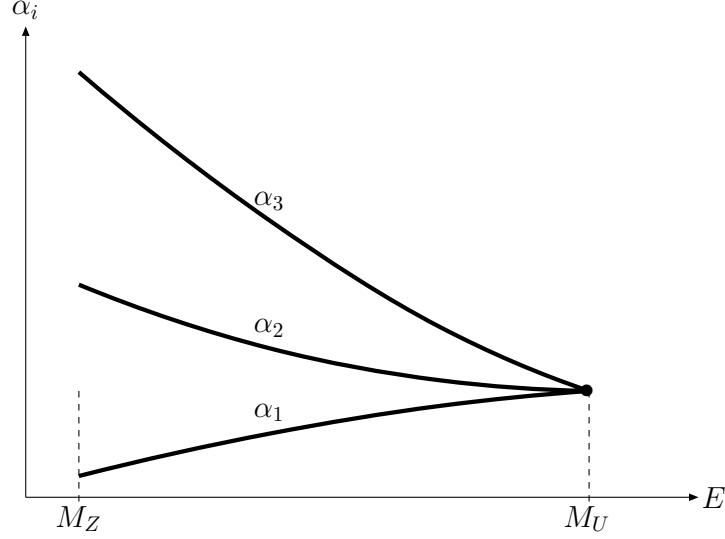


Fig. 2.2. Running of gauge couplings

A good theory must explain this huge number, which is the *gauge hierarchy problem* [16].

The *gauge hierarchy problem* consists of two parts: (1) “How are two different scales implied by (2.18) introduced?”, and (2) “Given those two different scales, are quantum corrections safe to guarantee the hierarchy of (2.18)?” This book is an exploration of bigger theories originally motivated to answer the gauge hierarchy problem, and hence we will come back to this gauge hierarchy question again and again.

But there is one merit to having the large hierarchy (2.18). Since there exist the super heavy X and Y gauge bosons, the proton decay rate is proportional to M_X^{-4} , giving the proton lifetime for the dominant mode as [17, 18]

$$\tau(p \rightarrow e^+ + \pi^0) = O(10^3 - 10^4) \left(\frac{M_X}{m_p} \right)^4 \frac{1}{m_p} \quad (2.19)$$

where m_p is the proton mass.⁵ Thus in fact, a large hierarchy is needed from the requirements of proton longevity and differentiation of strong and electroweak couplings. The current experimental bound on the above partial lifetime is 1.6×10^{33} years [19].

We note that there also exists the colored scalar $\mathbf{h}$ in $\mathbf{5}_H$. This colored scalar $\mathbf{h}$ triggers proton decay through Yukawa couplings $\mathbf{10}_F \mathbf{\bar{5}}_F \mathbf{\bar{5}}_H$ and $\mathbf{10}_F \mathbf{10}_F \mathbf{5}_H$. Even though the Yukawa couplings of the first family is $O(10^{-6})$, mixing with heavier families introduces larger Yukawa couplings for proton

⁵ Note that $\text{GeV}^{-1} \simeq 0.658 \times 10^{-24}$ s in natural units.

decay through $\mathbf{h}$ exchanges. Thus, $\mathbf{h}$ in $\mathbf{5}_H$ and $\bar{\mathbf{h}}$ in $\bar{\mathbf{5}}_H$ must be removed at the GUT scale. Below the GUT scale, there must survive three families of fermions and a Higgs doublet ($\mathbf{H}_1$ and/or $\mathbf{H}_2$).⁶

The unification theme of our discussion does not stop at the minimal GUT, $SU(5)$. As noted before, $SO(10)$ is much nicer than $SU(5)$ from the standpoint of the unification theme. However, $SU(5)$ is favorable for two reasons. First of all, $SO(10)$ requires several intermediate steps in the process of breaking it down to $SU(2)_L \times U(1)_Y$, and hence it is more complicated. These intermediate steps are

$$SO(10) \longrightarrow \begin{cases} SU(5)_{\text{GG}} \\ SU(5)' \times U(1) \text{ [flipped } SU(5)] \\ SU(4) \times SU(2)_L \times SU(2)_R \text{ [Pati-Salam]} \\ SU(4) \times SU(2) \times U(1) \\ SU(3) \times SU(2) \times SU(2) \times U(1) \end{cases} \quad (2.20)$$

Secondly, even though $SO(10)$ is superior to $SU(5)$ in terms of the unification theme, it still cannot include three families in one spinor representation $\mathbf{16}$. $SU(5)$ is favorable because it is minimal. We need to unify families in the GUT scheme. This family unified GUT is an obvious generalization of GUT from the unification theme, “Find a bigger space where all matter fields can be put into a single representation of a GUT group.” Georgi formulated the family unification with the hypothesis [20],

- *Survival hypothesis:* If a GUT group G breaks down to a subgroup G_{sub} at a scale M_G , all real representations of G_{sub} are removed at the scale M_G . The surviving fermions are complex representations of G_{sub} .

Therefore, the search of chiral theories is reduced to the problem of finding groups which allow complex representations. Of course, the full complex fermionic representation(s) should not lead to gauge anomalies. In string theory, every physical parameter is calculable in principle and we may not need the survival hypothesis.

So, because of the anomaly problem, an obvious possibility in 4D is to look for complex representations in groups other than $SU(N)$. There are only two classes which achieve this objective,

$$\begin{aligned} SO(4n+2) &: \text{ with spinor representation } \mathbf{4}^n \\ E_6 &: \text{ with the fundamental representation } \mathbf{27}. \end{aligned} \quad (2.21)$$

Again, the E_6 model [21] houses only one family. As noted before the $\mathbf{16}$ of $SO(10)$ houses one family. Therefore, for the family unification we are left with $SO(14)$, $SO(18)$, etc., in the above category of (2.21). If $SO(14)$ breaks

⁶ The bold faced notation for the color triplet and weak doublet implies that they are not representing one particle. Separating these is the doublet-triplet splitting problem.

down trivially to $SO(10)$ at a scale M_{14} , then below M_{14} the $SO(14)$ spinor **64** reduces to $2(\mathbf{16} \oplus \overline{\mathbf{16}})$ of $SO(10)$ which is real under $SO(10)$ and hence, removed at the scale M_{14} by the survival hypothesis. There remains no family. However, if $SO(14)$ directly breaks down to $SU(3) \times SU(2) \times U(1)$ in a skewed way, there can survive light fermions [22], but they are not of the form given in (2.7). This is the reason why the study of $SU(N)$ groups is necessary. But for the anomaly cancellation, here we cannot build a model with one fermion representation. One interesting GUT model with three $SU(5)$ families in which $SU(N)$ representation appears only once is the $SU(11)$ model [20]

$$SU(11) : \psi^{\alpha\beta\gamma\delta} \oplus \psi_{\alpha\beta\gamma} \oplus \psi_{\alpha\beta} \oplus \psi_{\alpha}$$

where the $SU(11)$ indices α, β , etc. are anti-symmetrized. Generalizing the single appearance of irreducible representations to relative primes for the multiplicities of the representations, there can be numerous GUT models with three $SU(5)$ families after the application of survival hypothesis [23].

So far, we have only discussed simple gauge groups for GUT. But one may consider semi-simple groups for grand unified models. In fact, the Pati-Salam model $SU(4) \times SU(2)_L \times SU(2)_R$ [24] was considered before $SU(5)$. But there is no rationale that gauge couplings of $SU(4)$ and $SU(2)$ are unified above some scale. An intermediate group can be between a real GUT at M_U and the SM at M_Z . Even though this model does not achieve gauge coupling unification, it puts quarks and leptons in the same representation and therefore can be called a GUT. Namely in **4** of $SU(4)$, a quark triplet and a lepton are put together and hence the lepton is called the 4th color. Therefore, the possibility of proton decay exists in principle, but in one version of the Pati-Salam model where $SU(4) \times SU(2)_L \times SU(2)_R$ is unified to $SU(4) \times SU(4)$, the rate is much more suppressed since the nontrivial operator for proton decay appears only at dimension 9 operators, instead of dimension 6 operators shown above in $SU(5)$ [25]. So the breaking scale of $SU(4)$ can be much lower than that of the $SU(5)$ GUT. Another interesting factor group for GUT is $SU(5) \times U(1)$ the so-called flipped $SU(5)$. This factor group is not semi-simple. Here also, the $SU(5)$ coupling and $U(1)$ coupling are not unified, but can be called a GUT since quarks and leptons are put in the same representation. It can be a subgroup of $SO(10)$ [26].

If a semi-simple group is considered as a GUT, then a discrete symmetry must exist for the exchange of factor groups, thus guaranteeing the equality of gauge couplings. The most interesting GUT in this category is the so-called *trinification* [27], $SU(3) \times SU(3) \times SU(3)$, with the representation

$$\mathbf{27}_{\text{tri}} = (\overline{\mathbf{3}}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{3}) \oplus (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{3}) . \quad (2.22)$$

This model cannot separate out any $SU(3)$ and hence there exists an exchange symmetry of gauge groups, and the three $SU(3)$ gauge couplings are unified above the unification scale M_U . In the trinification, quarks and leptons are not put in the same representation, but still it is called a GUT because gauge

couplings are unified. In many aspects, the trinification model is very similar to the E_6 model.

In this book, we will discuss mainly the $SU(5)$ model, the $SO(10)$ model, the Pati-Salam model, the flipped $SU(5)$ model, and the trinification model. We attempt to resolve family unification with a new method appearing in the compactification process.

2.3 Supersymmetry

2.3.1 Global Supersymmetry

Supersymmetry is probably the most spectacular symmetry ever since the discovery of isospin. It can be introduced in many different ways. In our search of an enlarged symmetry of matter spectrum, we note that the SM spectrum (2.7) includes Higgs scalars also. Surprisingly, the chiral (L-handed) lepton doublet l and $\mathbf{H}_1$ have the same quantum numbers except for the spin. Even though it is not necessary, we may try to put a chiral fermion and a complex scalar in a bigger Hilbert space so that they belong to the same representation, in our pursuit of unification.⁷ Due to the fact that the spectrum (2.7) contains spin-0 particles, we open up the possibility of unifying fermions and bosons. The new space is called *superspace* [28] and the supercharge Q is the generator for the transformation of fermions to bosons and vice versa,

$$Q|F\rangle = |B\rangle, \quad Q|B\rangle = |F\rangle. \quad (2.23)$$

From the definition of supercharge in (2.23), we note that Q must be spinorial because both $Q|B\rangle$ and $|F\rangle$ must transform like a spinor under rotation. The spinorial charge Q has two components since our motivation for unification started so that L-handed leptons in (2.7) could be obtained from complex scalars by operating Q . So, Q transforms like a L-handed Weyl fermion. Here follows the chirality which we need desperately for the scalars, and if we succeed in this F and B unification, that alone can be a merit. The Lorentz algebra $SO(3,1)$ is the same as $SU(2) \times SU(2)$; thus the Lorentz group $SO(3,1)$ can be locally viewed as $SU(2) \times SU(2)$. The Weyl fermion transforms as $(\mathbf{2}, \mathbf{1})$ or as $(\mathbf{1}, \mathbf{2})$ under $SU(2) \times SU(2)$. Let $(\mathbf{2}, \mathbf{1})$ be the L-handed field and $(\mathbf{1}, \mathbf{2})$ be the R-handed field. The indices for $SU(2)_L$ are denoted as undotted $\alpha = \{1, 2\}$ and the indices for $SU(2)_R$ are named as dotted $\dot{\alpha} = \{1, 2\}$ [29]. Since Q transforms like a L-handed Weyl spinor, its commutation relation with the angular momentum generator is

$$[J^{\mu\nu}, Q_\alpha] = -i(\sigma^{\mu\nu})_\alpha^\beta Q_\beta. \quad (2.24)$$

⁷ Phenomenologically however, the Higgs fields are known to not be in the same representation as the chiral lepton doublet in the supersymmetrized version.

For the R-handed Weyl fermions and their accompanying complex scalars, the supercharge must transform nontrivially under $SU(2)_R$. Let us call this supercharge $\bar{Q}$ which changes the dotted indices,

$$[J^{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = -i(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} . \quad (2.25)$$

In fact, $\bar{Q}$ is the charge conjugated of Q , since as we have seen in the previous section of this chapter, L changes to R and vice versa under charge conjugation. The total number of complex components in Q and $\bar{Q}$ is four. Thus, we can form the following representation for the supercharge,

$$\mathcal{Q} = \begin{pmatrix} Q \\ \bar{Q} \end{pmatrix} . \quad (2.26)$$

There exists the famous no-go theorem by Coleman and Mandula [30] which says that the fermion-boson symmetry and the space-time symmetry cannot be unified in a naïve way, just by making the internal space bigger, i.e. making the Lie group bigger. Supersymmetry escapes this Coleman-Mandula theorem by not making just the Lie group bigger, but by introducing an algebra outside the Lie algebra. It is called the graded Lie algebra where one adds anti-commutators for some generators. For supercharges Q and $\bar{Q}$, one introduces the following anticommutators

$$\{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^{\mu})_{\alpha\dot{\beta}} P_{\mu} \quad (2.27)$$

$$\{Q_{\alpha}, Q_{\beta}\} = 0, \quad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 . \quad (2.28)$$

With the space translation generators P_{μ} , one introduces commutators as in (2.25),

$$[P_{\mu}, Q_{\alpha}] = 0, \quad [P_{\mu}, \bar{Q}_{\dot{\alpha}}] = 0, \quad [P_{\mu}, P_{\nu}] = 0 . \quad (2.29)$$

Indeed, this expansion of symmetry by introducing anti-commuting supercharges is known to be possible and the fermion-boson symmetry introduced in this way is *supersymmetry*. The parameter for the supersymmetry transformation, ϵ , must transform like a spinor under rotation so that $\bar{Q}\epsilon$ transforms like a scalar. Supersymmetry (SUSY) was introduced in 1971 [31], but the linear realization (2.23) of supersymmetry, what we use today, is due to the work of Wess and Zumino [32].

The time component of the algebra (2.27) gives the Hamiltonian in terms of supercharges

$$H = P^0 = \frac{1}{4}(Q_1 \bar{Q}_1 + \bar{Q}_1 Q_1 + Q_2 \bar{Q}_2 + \bar{Q}_2 Q_2) \quad (2.30)$$

which implies that the energy eigenvalues are nonnegative. If supersymmetry is unbroken, the supercharges annihilate the vacuum $|0\rangle$, and the vacuum energy is zero,

$$\text{Global SUSY : } E_{\text{vac}} \geq 0, \quad \text{equality for unbroken SUSY} . \quad (2.31)$$

The supersymmetry we have discussed above is $\mathcal{N}=1$ supersymmetry. If we introduce more supercharges, they define extended supersymmetry, $\mathcal{N}=1, 2, \dots$. For example, for $\mathcal{N}=2$ the total S_z interval between $(S_z)_{\max}$ and $(S_z)_{\min}$ is $2 \cdot \frac{1}{2} = 1$ because one application of a supercharge changes spin by a half unit. Not to include spin greater than 2, thus we obtain $\mathcal{N}=8$ as the maximum extended supersymmetry. In the remainder of this section, *only $\mathcal{N}=1$ supersymmetry is considered, where the introduction of chirality is possible.*⁸

Although we do not use superfield formalism [28] here, we list its powerful constraints on the form of Lagrangian. Introducing an anticommuting coordinate θ^α ($\alpha = 1, 2$) and $\bar{\theta}_{\dot{\alpha}}$ ($\dot{\alpha} = 1, 2$), one can also introduce a quantum field as a function of x, θ and $\bar{\theta}$ [29]. A polynomial of θ includes only three terms, 1, θ , and θ^2 . A similar polynomial results from $\bar{\theta}$. In view of the $SU(2)_L \times SU(2)_R$ property of the Lorentz group (viz. (2.24, 2.25)), $\theta(\bar{\theta})$ is a doublet under $SU(2)_L(SU(2)_R)$. For fermions, we introduce an L-handed chiral field ψ , which is a singlet under $SU(2)_R$. This ψ can make a singlet of $SU(2)_L$ (viz. $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3}$), by taking the antisymmetric combination with θ , $\theta\psi \equiv \epsilon_{\alpha\beta}\theta^\alpha\psi^\beta$ where $\epsilon_{\alpha\beta}$ is the Levi-Civita tensor of $SU(2)_L$. Thus, $SU(2)_L$ singlets can be a scalar function of $\varphi(x)$ and $\psi(x)\psi(x)$, which is denoted as ϕ , a function of the forms $\theta\psi(x)$ and $F(x)\theta\theta$. A (L-handed) chiral superfield $\Phi(x, \theta)$ is defined as a function of θ only; thus a *chiral superfield* has expansion⁹

$$\Phi(x, \theta) = \phi(x) + \theta^\alpha\psi_\alpha + \theta^\alpha\theta^\beta\epsilon_{\alpha\beta}F^2. \quad (2.32)$$

The chiral superfield Φ contains a spin-0 boson and a spin- $\frac{1}{2}$ fermion. A superfield containing both θ and $\bar{\theta}$ has more degrees. Here, it is possible to introduce a spin-1 boson in the superfield V , with the reality condition $V = V^\dagger$. $V(x, \theta, \bar{\theta})$ is called a vector superfield. This superfield formalism is quite useful in analyzing the form of allowed actions. As a result, first the globally supersymmetric Lagrangian is known to depend only on three functions K, W , and f [29]:

1. Kähler potential $K(\Phi, \Phi^*)$, a hermitian function, determines mainly the kinetic terms.
2. Superpotential $W(\Phi)$, a holomorphic function, determines the potential $V(\Phi)$.
3. Gauge kinetic function $f_{ab}(\Phi)$, a holomorphic function, is the coefficient of gauge kinetic terms.

In particular, the scalar potential is always positive definite

⁸ For example, for $\mathcal{N}=2$ matter fermions are located at $S_z = \frac{1}{2}$ and $S_z = -\frac{1}{2}$ with the same gauge quantum number. These $S_z = \pm\frac{1}{2}$ representations form a vector-like representation under the gauge group.

⁹ Similarly, from the consideration of $SU(2)_R$ an R-handed chiral field $\Phi^\dagger(\bar{z})$ can be given.

$$V(\phi, \phi^*) = \sum_i \left| \frac{\partial W(\phi)}{\partial \phi_i} \right|^2 + \frac{1}{2} f_{ab} D^a D^b, \quad (2.33)$$

where we replaced the superfield Φ with its scalar field ϕ and $D^a = G^i T_i^{aj} \phi_j$ is the D-term. This is another expression of (2.31). Second, there is the following *non-renormalization theorem*. Because of the holomorphicity, the superpotential $W(\Phi)$ does not receive loop corrections other than those of wave function renormalization, at all orders of perturbation theory. Also, the gauge kinetic function $f_{ab}(\Phi)$ does not receive higher order correction beyond one loop order. These are quite restrictive compared to non-supersymmetric models.

2.3.2 Local Supersymmetry, or Supergravity

In this subsection, we introduce some formulae which are needed in later chapters. Thus far, we have considered the *global supersymmetry* where the parameter ϵ for supersymmetry transformation is independent of x_μ . If supersymmetry is the symmetry of the action, it must be severely broken since the superpartner of the electron has not been discovered up to 100 GeV. If the global supersymmetry is broken at the scale M_S , the vacuum energy must be of the order M_S^4 , in view of (2.31), and the cosmological constant problem phenomenologically excludes global supersymmetry from square one. There is no escape from this problem in the global SUSY case.

This leads us to the necessary introduction of gravity through the localization of the supersymmetry parameter, $\epsilon(x)$. The resulting theory is *supergravity*. If the vacuum energy problem is not resolved in supergravity as well, then this would be just an academic exercise. Fortunately, supergravity allows the possibility of introducing a zero cosmological constant after supersymmetry breaking [33].

In the supergravity Lagrangian, K and W appear with a single function G

$$G(\phi, \phi^*) = -3 \log(-K/3) + \log |W|^2 \quad (2.34)$$

where we set the Planck mass $M_P = 1$. The odd looking coefficients are for convenient calculations. The function (2.34) has a symmetry

$$\begin{aligned} -3 \log(-K/3) &\rightarrow -3 \log(-K/3) + h(\phi) + h^*(\phi^*), \\ W &\rightarrow e^{-h} W. \end{aligned} \quad (2.35)$$

Covariant and contravariant indices are used for holomorphic and antiholomorphic scalars, respectively. The Kähler metric of the (sigma model) target space is defined as

$$G^i = \frac{\partial G}{\partial \phi_i}, \quad G_i = \frac{\partial G}{\partial \phi^{*i}}, \quad G_j^i = \frac{\partial^2 G}{\partial \phi_i \partial \phi^{*j}}. \quad (2.36)$$

Then, the bosonic Lagrangian is given by

$$e^{-1}\mathcal{L} = \frac{\mathcal{L}}{\sqrt{g}} = -\frac{1}{2}R + G^i{}_j D_\mu \phi_i D^\mu \phi^{*j} + \frac{1}{4}\text{Re}(f_{ab})F_{\mu\nu}^a F^{b\mu\nu} \\ + \frac{1}{8}\text{Im}(f_{ab})\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^b + V(\phi, \phi^*) \quad (2.37)$$

with the following scalar potential [34, 35]

$$V(\phi, \phi^*) = e^K \left[(G^{-1})^i{}_j D_i W^* D^j W - 3|W|^2 \right] + \frac{1}{2}f_{ab}D^a D^b, \quad (2.38)$$

where

$$D^i W = \frac{\partial W}{\partial \phi_i} + \frac{\partial K}{\partial \phi_i} W, \quad D_i W^* = \frac{\partial W^*}{\partial \phi^{*i}} + \frac{\partial K}{\partial \phi^{*i}} W^*, \quad (2.39)$$

and on-shell D-term

$$D^a = G^i T_i^{aj} \phi_j. \quad (2.40)$$

In contrast with the global supersymmetry case (2.33), here the superpotential is not positive definite.

The supersymmetry breaking condition can be read from the transformation laws of fermionic fields. For supersymmetry breaking, only their scalar component(s) can assume VEV that does not violate the Lorentz symmetry,

$$\delta_\epsilon \Psi \sim -e^{G/2} \frac{1}{W^*} D_i W^* \epsilon - \frac{1}{8} \frac{\partial f_{ab}}{\partial \phi^{*j}} \lambda_a \lambda_b \epsilon \quad (2.41)$$

$$\delta_\epsilon \lambda \sim \frac{i}{2} g \text{Re} f_{ab}^{-1} G^i (T_b)_i^j \phi_j \epsilon. \quad (2.42)$$

The right hand side of (2.41) is called the F-term of Ψ . Supersymmetry can be broken if the F-term is nonvanishing, either by (i) the first term assuming VEV or (ii) the second term through the gaugino (λ_a) condensation by some strong force. The supersymmetry breaking scale for Case (i) is

$$M_S^2 = e^{G/2} \frac{1}{W^*} D_i W^*, \quad (2.43)$$

and a similar expression holds for Case (ii).

Supergravity formulated with an AdS curvature $\bar{\Lambda} < 0$ has a negative vacuum energy $\bar{\Lambda}$. With the supersymmetry broken a positive constant, (viz. (2.31, 2.38)), is added to $\bar{\Lambda}$, making it possible for the vacuum energy in the broken phase to be made zero, $V_0 = 0$. For example, if the SUSY is broken by the nonzero F term only, then the flat space condition requires, $G_{0k} G_0^k = 3$. In this flat limit, the gravitino mass is

$$m_{3/2} = M_P e^{G_0/2}, \quad (2.44)$$

where M_P is the Planck mass, 2.44×10^{18} GeV. Thus, supergravity saves us from the disaster of a huge cosmological constant. But, it must be remembered that *we achieved the flat space by a fine-tuning*, since the initial curvature $\bar{\Lambda}$, with which we formulated the theory, is an arbitrary number.

Now let us introduce the $\mathcal{N} = 1$ supergravity for the matter content (2.7). The R-handed fields are understood to be the charge conjugated L-handed fields as in (2.8). The Yukawa couplings are contained in the cubic terms of the superpotential W . Since all fields in (2.7) have superpartners, we do not bother to distinguish between the complex scalars and L-handed fermions. In the superpotential W , the fields denote the first component, i.e. scalar component. The scalar partners of fermions in (2.7) are called sfermions, e.g. squark, selectron, and sneutrino. On the other hand, the fermionic partners of (2.7) scalars carry the suffix “ino”, e.g. Higgsinos. Supersymmetrization of Higgs bosons require the Higgsinos to be L-handed so that they can couple to other L-handed fields. We said before that scalars do not have a left or right handed distinction. But, here we change this statement. Scalars in supersymmetric theory have chiralities which are determined by the chirality of their fermionic partner. If we consider gauge bosons, their fermionic partners ($s = \frac{1}{2}$) are called gauginos, e.g. gluino, wino, zino, photino, and bino. Here, bino means the partner of the $U(1)_Y$ gauge boson B_μ . To give mass to an electron, we can consider a superpotential, $l_L e_L^c \mathbf{H}_1$. Down-type quarks can obtain mass by the term, $q_L d_L^c \mathbf{H}_1$, however up-type quarks cannot obtain mass by $\mathbf{H}_1$. They need the $Y = +\frac{1}{2}$ Higgsino doublet $\mathbf{H}_2$. Unlike the case without supersymmetry, $i\sigma_2 \mathbf{H}_1^*$ cannot serve for the up-type quark mass since the charge conjugated field is R-handed and W does not allow couplings of both chiralities. Therefore, we need another L-handed Higgs doublet $\mathbf{H}_2$ for the up-type quark masses, $q_L u_L^c \mathbf{H}_2$. It is also needed to cancel the gauge anomaly since we considered the complex fermion through supersymmetrization of $\mathbf{H}_1$. The addition of $\mathbf{H}_2$ makes the representation $\mathbf{H}_1 \oplus \mathbf{H}_2$ real under $SU(3) \times SU(2) \times U(1)$. The $\mathcal{N} = 1$ supersymmetric gauge model with the spectrum given in (2.7) and with the addition of $\mathbf{H}_2$ is the minimal supersymmetric standard model (MSSM). Namely, the Higgs sector becomes a bit bigger

$$\text{MSSM : } \mathbf{H}_1 \oplus \mathbf{H}_2 . \quad (2.45)$$

So far, we have taken the strategy of extending the symmetry and ultimately introducing supersymmetry. Supersymmetry helps in solving some important problems as well, such as the second gauge hierarchy problem. One way to phrase the gauge hierarchy problem is, “Why is the mass of the electroweak Higgs boson $\mathbf{H}_1$ so much smaller than the mass of the GUT Higgs boson $\mathbf{24}_H$?” If we start from a small ratio for these masses for $\mathbf{H}_1$ and $\mathbf{24}_H$, the radiative corrections should not destroy this smallness. With supersymmetry, indeed this stability problem is understood because the relevant Yukawa couplings and quartic couplings are related. Both kinds of couplings are related by the superpotential W . As depicted in Fig. 2.3, boson loops positively contribute and fermion loops negatively contribute to the Higgs boson mass so that the quadratic divergence is cancelled. Thus, the second gauge hierarchy problem is understood with a supersymmetric extension. This leads to the MSSM being a serious contender for interactions around the TeV scale. If the MSSM were to solve the gauge hierarchy problem, the divergence problem of



Fig. 2.3. Fermion and boson loops cancel the quadratic divergence

the Higgs boson mass should only be forced a few TeV above the upper region of the electroweak scale. If the SUSY breaking scale M_S is raised any more than a few TeV, there will be a wide range of energy scales, from M_Z to M_S , where SUSY is not helpful for the stabilization of the Higgs boson mass.

In the MSSM, the superpartners of the three families must be raised to the SUSY scale M_S since they have not yet been discovered. This is realized by making the supersymmetry spontaneously broken. If it is broken softly, then only soft SUSY breaking masses appear below the scale M_S . Soft masses are the non-supersymmetric scalar masses and the gaugino masses. These soft SUSY breaking parameters can be given if the SUSY breaking mechanism is known. One popular scenario is the gravity-mediated SUSY breaking where soft masses at the electroweak scale are given as functions of the gravitino mass, $m_{3/2}$. For a review, see [35].

2.3.3 SUSY GUT

This leads us to the obvious unification, the supersymmetrized GUT or SUSY GUT. Again, the simplest SUSY GUT is SUSY SU(5) [36]. The matter content $\mathbf{10}_F$ and $\bar{\mathbf{5}}_F$ are supersymmetrized. Also, two Higgs representations are needed, one housing $\mathbf{H}_1$ and the other housing $\mathbf{H}_2$,

$$\mathbf{5}_H = \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \\ \mathbf{h}^4 \\ \mathbf{h}^5 \end{pmatrix}, \quad \bar{\mathbf{5}}_H = \begin{pmatrix} \bar{\mathbf{h}}_1 \\ \bar{\mathbf{h}}_2 \\ \bar{\mathbf{h}}_3 \\ \bar{\mathbf{h}}_4 \\ \bar{\mathbf{h}}_5 \end{pmatrix} \quad (2.46)$$

which are all L-handed fields. Here, $\mathbf{h}^4$ and $\mathbf{h}^5$ form the Higgs doublet $\mathbf{H}_1$, and $\bar{\mathbf{h}}_4$ and $\bar{\mathbf{h}}_5$ form the Higgs doublet $\mathbf{H}_2$. To break the GUT group, we also need a L-handed adjoint Higgs field $\mathbf{24}_H$. Then, we can consider a renormalizable superpotential

$$W = m\bar{\mathbf{5}}_H^T \mathbf{5}_H + \bar{\mathbf{5}}_H^T \mathbf{24}_H \mathbf{5}_H + \text{Tr} \mathbf{24}_H^3 + M \text{Tr} \mathbf{24}_H^2. \quad (2.47)$$

where all fields are L-handed, m and M are GUT scale masses and the couplings have been suppressed. After assigning a huge VEV to $\mathbf{24}_H$, we need to make $\mathbf{H}_1$ and $\mathbf{H}_2$ survive to the electroweak scale and $\mathbf{h}$ and $\bar{\mathbf{h}}$ escape at the

GUT scale. This can be done by a fine-tuning of the couplings in the above superpotential. For example, the above superpotential ends up requiring the following form after assigning a VEV m_1 to $\mathbf{24}_H$,

$$\bar{\mathbf{5}}_H^T \begin{pmatrix} m + 2m_1 & 0 & 0 & 0 & 0 \\ 0 & m + 2m_1 & 0 & 0 & 0 \\ 0 & 0 & m + 2m_1 & 0 & 0 \\ 0 & 0 & 0 & m - 3m_1 & 0 \\ 0 & 0 & 0 & 0 & m - 3m_1 \end{pmatrix} \mathbf{5}_H .$$

One must fine-tune m and the VEV of $\mathbf{24}_H$ m_1 so that $m = 3m_1$, in order to obtain the electroweak Higgs doublets $\mathbf{H}_1$ and $\mathbf{H}_2$. This is the problem of separating $\mathbf{H}_1$ and $\mathbf{H}_2$ from the triplets $\mathbf{h}$ and $\bar{\mathbf{h}}$, which is known as the *doublet-triplet splitting problem*. It is one of the most difficult hierarchy problems in SUSY models.

Assuming the following extremely simplified symmetry breaking pattern,

$$\begin{aligned} SU(5) &\longrightarrow [SU(3) \times SU(2) \times U(1)]_{\text{SUSY}} \quad \text{at } M_U , \\ &\longrightarrow [SU(3) \times SU(2) \times U(1)]_{\text{non-SUSY}} \quad \text{at } M_{\text{SUSY}} , \\ &\longrightarrow SU(3) \times U(1)_{\text{em}} \quad \text{at } M_Z , \end{aligned} \quad (2.48)$$

we can estimate how much the gauge couplings are differentiated at the electroweak scale M_Z . Conversely, given the experimentally measured couplings at the electroweak scale, we can check whether or not they meet at the unification point M_U . The current situation is shown in Fig. 2.4 with the one-loop evolution of gauge couplings. Originally, with crude data on α_c [37] they do not seem to meet [38]. However, with the LEP data on α_c [8] they meet within the experimental error bounds [39]. Thus, SUSY GUT models with the MSSM spectrum below M_U seem to have some truth to them.

At present, supersymmetry is the most popular scenario for the solution of the gauge hierarchy problem, and is why the phenomenological aspects of MSSM are so vigorously studied. But the MSSM has some serious theoretical problems related to the following issues;

- 15(+1) *chiral fermions* in one family,
- number of fermion families ≥ 3 , probably exactly 3,
- $\mathcal{N} = 1$ supersymmetry,
- gauge hierarchy,
- doublet-triplet splitting,
- one pair of Higgsinos, $\mathbf{H}_1$ and $\mathbf{H}_2$,
- the hypercharge quantization, $\sin^2 \theta_W \simeq \frac{3}{8}$ at GUT scale,
- absence of strong CP violation,

as well as more detailed questions such as that pertaining to the hierarchies of Yukawa couplings. The Yukawa hierarchy problem is probably the most important issue at the TeV scale, which will likely be verified by future high energy experiments. In this book, however, we focus on searches of theoretical structures with which all or some of the above problems can be understood.

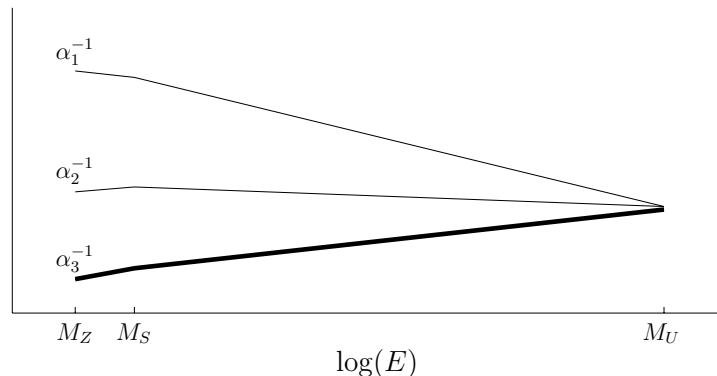


Fig. 2.4. Running of gauge couplings of Fig. 2.2 with one-loop beta function and experimental inputs. We used $M_Z = 91.187 \text{ GeV}$, $M_S = 10^3 \text{ GeV}$, $M_U = 1.4 \times 10^{16} \text{ GeV}$. $\alpha_1^{-1}(M_Z) = 1/0.01681$, $\alpha_2^{-1}(M_Z) = 1/0.03358$, $\alpha_3^{-1}(M_Z) = 1/(0.1200 \pm 0.0028)$, and $\sin^2 \theta_W(M_Z) = 0.231$ [8]

2.4 Extra Dimensions

Field Theory

In the 4D quantum field theory framework, we exhausted most possibilities of unification which are not obviously excluded by experimental data. This leads us to extending space-time itself to higher dimensions, by adding extra dimensions to the familiar 4D model. In fact, the addition of extra dimension(s) beyond 4D is an older idea than any of the extensions we have discussed so far, and was first addressed by Kaluza and Klein in the 1920s [40]. Trying to interpret a photon as a gauge field in the metric, Kaluza said that, “... such an interpretation of $F_{\mu\nu}$ is hardly supported unless otherwise one makes an extremely odd decision of a new fifth dimension of the world.” If we are to introduce *extra dimensions*, they must be cleverly hidden from us. The Kaluza-Klein (KK) idea is to compactify extra dimensions to such a small scale that the resolution power of 4D observers cannot see the structures of the extra dimensions. This is the famous *compactification* idea. Kaluza’s excuse of introducing the extra dimensions was to view gravity and electromagnetism on the same footing. But Kaluza’s view of $g_{\mu\nu}$ and A_μ^a on the same footing has failed in fact because of the chirality problem. Our excuse for introducing six extra dimensions, is to the unification of families.

Starting in the late 1990s, extra dimensions were studied in the quantum field theory framework. The chief motivation for the field theoretic study was to understand the gauge hierarchy problem with extra dimensions.

Arkani-Hamed, Dimopoulos, and Dvali (ADD) [41] introduced a TeV scale fundamental mass M_5 for gravity with extra dimensions, where the full extended space is called *bulk*. ADD assumed a flat bulk. The electroweak scale quantum fields are confined to a 4D boundary called the *brane*. Our perception

of a huge 4D Planck mass M_P is blamed for the large size of the extra dimensions. This idea works even for extra dimensions as large as 100 micrometers because of poor gravity experiments at small scales. This idea is designed to answer the gauge hierarchy problem by introducing only TeV scale masses M_5 and M_Z in the Lagrangian, i.e. the hierarchy problem is not there from the outset. On the other hand, Randall and Sundrum (RS) considered the AdS bulk, i.e. with a negative bulk cosmological constant. In the RS-I model with two brane boundaries [42], the SM fields are put on the SM brane, B2, where the brane tension is negative while there exists another brane, B1, where the brane tension is positive. If the distance between these two branes is d in the 5D example, by two fine-tunings the mass parameters at the SM brane are $\sim M_5 e^{-kd}$ where k is the mass determined by M_5 and the 5D cosmological constant Λ_5 . Thus, the relevant mass parameters at the SM brane can be exponentially small compared to the fundamental mass $M_5 \sim M_P$, and d of $O(100M_5^{-1})$ can give the electroweak scale. This is an attractive proposal for an exponential hierarchy, but serious cosmological problems arise from putting the SM fields in the negative tension brane [43]. Furthermore, from our theme of unification, this proposal can be at best just one side view of the full structure.

The ADD and RS-I proposals use the old compactification idea. Randall and Sundrum proposed another idea of cleverly hiding extra dimensions even though they are not compactified. It is the RS-II model [44] where only one positive tension brane is placed at $y = 0$ which is the brane for the SM fields. Again, the effect of gravity falls off exponentially $e^{-k|y|}$ in the bulk and the deep bulk ($|y| > k^{-1}$) is effectively hidden from us. Since all the fields (the SM fields and GUT fields) are put at the same brane, the RS-II model does not give a rationale for the gauge hierarchy solution. Most likely, this RS-II may be a way to understand the more serious hierarchy problem, the cosmological constant problem, through self-tuning solutions [45].

String Theory

Quantum field theory in four spacetime dimensions (4D) has a limited ability to unify all forces in Nature even though its favorite baby, the SM, is known to be very successful phenomenologically below the electroweak scale ~ 100 GeV. The SM has nineteen free parameters which are tuned such that all the observed electroweak data are explained; at present this tuning is known to be possible without a major discrepancy with the data. But the SM does not explain why the nineteen parameters take those phenomenologically required values. This is the uniqueness problem of why the SM takes the specific set having those required values out of numerous other possible sets. One can envision the existence of a truly unifying fundamental theory which is sometimes called the theory of everything (TOE). Nature may not allow such a theory. However, if such a theory exists, one should be able to calculate those parameters of the SM. We have observed that such a unification is possible in

GUTs for the case of gauge couplings. In addition, 4D quantum field theory at present cannot incorporate gravity in the scheme because of its inability to treat the divergences appearing in quantum gravity. At the very least, these two problems, the uniqueness problem and the quantum gravity problem, hint to another theory beyond the 4D quantum field theory for TOE if it indeed exists.

String theory can be a candidate for TOE, at least for answering the above problems. As for the uniqueness problem, it has only one parameter, the string tension α'^{-1} . If everything works out fine, then all the SM parameters are calculable in principle. As for the gravity problem, all string theories include closed strings which contain an excitation responsible for gravity. Removal of infinities in string theory can be explained after exploiting the one loop amplitude. Intuitively, one can imagine that string is better behaved than a point particle theory because string is an extended object. String theory with $\alpha'^{-1} \sim 1 \text{ GeV}^2$ was initially considered for the theory for strong interactions [46] before the development of QCD. But we know that strong interaction is due to the confining $SU(3)_c$ gauge theory. If string is useful for physics at all, it must be for gravity with $\alpha'^{-1} \sim M_P^2$ [47].

In quantum mechanics (the first quantization), identical particles are treated by the permutation symmetry which is put in by hand. This is similar to declaring that all 0.511 MeV mass fermionic particles in the universe are identical. They are the identical particle, electron. Quantum field theory (the second quantization) gives a cute interpretation for the existence of identical particles in Nature, i.e. why an electron on Earth is identical to an electron in Andromeda Galaxy. Quantum field theory assumes that in the universe there is only one electron quantum field ψ_e . Both the electron on Earth and the electron in Andromeda are created by the same quantum field ψ_e and hence they cannot be different kinds of particles. But why are there so many quantum fields in the SM? String theory answers this question by saying that there is only one string field X in the universe (in bosonic string). Different modes of excitations correspond to different quantum fields and string theory can explain the existence of different kinds of particles in Nature. In this sense, string theory contains the basic logic for TOE.

One predictive power of string theory is that it fixes the number of spacetime dimensions where it lives. Bosonic string theory, which is the simplest string theory of all, is possible only in 26 spacetime dimensions (26D). But, quarks and leptons, which are fermions, are not present in the 26D bosonic string theory. To incorporate fermions, one considers superstring theory which is possible only in ten spacetime dimensions (10D). In the above determination of the number of spacetime dimensions, the quantum idea for string at the level of the first quantization is used for a consistent string theory. Still the second quantization of string is not yet fully developed if it is even needed at all. String theory used in particle physics is the first quantized version. At present, it is not known what kind of bonus on the theme of unification will reveal if one succeeds in further generalizing the first quantized string theory.

Superstring theories accompany supersymmetry. Because quarks and leptons are present, it is better for the fundamental dimension of spacetime to be 10D rather than 26D if string is the fundamental object in the universe, and with this logic supersymmetry is essential at the fundamental level in 10D. But the spacetime around us seems to be 4D and we must find a way out from 10D to 4D. A useful way to obtain 4D is by compactifying six extra dimensions so that its smallness is not perceived by us. The main aim of this book is to discuss this process of compactification, using *orbifolds*. Here, the question why we live in 4D is not answered theoretically. Simply, we use the fact that our universe is 4D. But there exists a cosmological argument that 4D extended to become large while the extra six dimensions did not [48].

There are several superstring theories in 10D. Strings come in two varieties, closed strings and open strings as shown in Fig. 2.5. One cannot construct an open string only theory since A and B of the open string in Fig. 2.5 can be joined together to give a closed string. Thus, any superstring theory must contain closed strings and string theory can be a theory for gravity.

As for the uniqueness problem, superstring does not satisfy this criterion as was hoped in the beginning. Namely, the number of possible superstring theories is not one. It is known that there exist five 10D superstring theories which are distinguished by what kind of string(s) is used: Type-I, Type-IIA, Type-IIB [49], $SO(32)$ heterotic, and most importantly $E_8 \times E_8$ heterotic string [50]. However, in the mid-1990s string duality was found, in which different superstrings are related by duality. In fact, six theories (five 10D superstrings and one 11D supergravity) are related by duality, which is pictorially shown in Fig. 2.6. The six theories are considered to be located at specific corners of their mother theory called the M-theory. At present, we do not know in detail how the M-theory looks.

Among these, the $E_8 \times E_8$ heterotic string has attracted the most attention [51, 52] since the group is sufficiently large and the symmetry breaking chain $E_8 \rightarrow E_6 \rightarrow SO(10) \rightarrow SU(5)$ is the desirable one, at least at the level of classification of the 15 chiral fields. As noted in Sect. 2.2, the spinor **16** of $SO(10)$ correctly houses the 15 chiral fields. Regarding the doublet-triplet

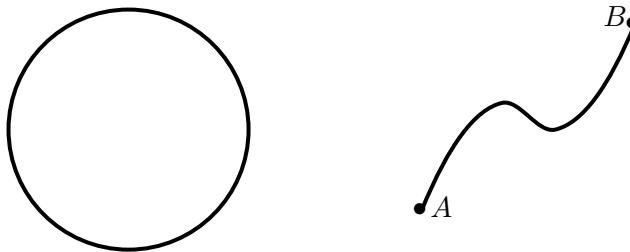


Fig. 2.5. Closed and open strings

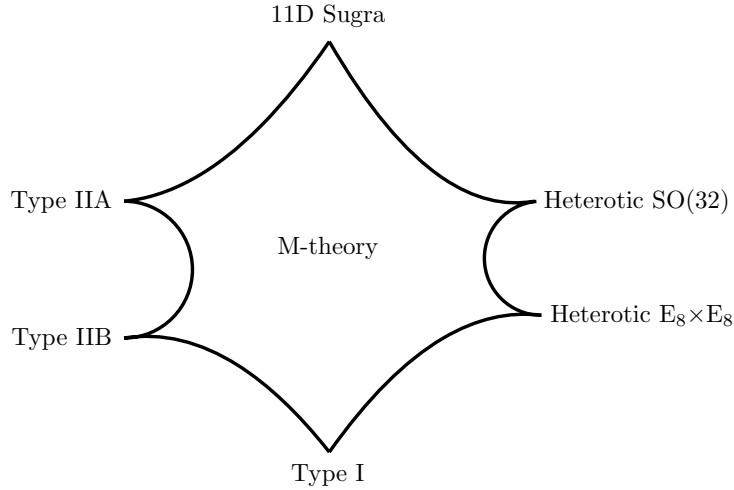


Fig. 2.6. M-theory vacua

splitting problem of Sect. 2.3, it was found that a $\mathbf{Z}_3$ orbifold compactification does not introduce the extra color triplets $\mathbf{h}$ and $\bar{\mathbf{h}}$, thus solving the doublet-triplet splitting problem [53]. In addition, string theory has very rich structures, possibly solving all the problems listed in Sect. 2.3.

Even though string theory escapes the two fundamental problems we mentioned earlier, it still must explain all the nineteen parameters of the SM if it is really the TOE. Therefore, in string theory the search for the MSSM is of utmost importance. But, theoretically this dream has not been realized yet to everyone's satisfaction.

String theory has to be tested by experiments for it to be a good physical theory. But the size of string for gravitation is considered to be extremely small ($\sim 10^{-31}$ cm) if another large parameter is not introduced.¹⁰ In this case, it is impossible to see “string” by exciting it, and hints of string can come only indirectly. Such hints are the SM forbidden processes, e.g. proton decay, some level of flavor changing neutral current processes, etc. But these SM forbidden processes at low energy are discussed basically by effective field theory and hence they cannot pinpoint their origin, even if discovered, to string or GUTs for example. In this sense, we are in dilemma of proving the existence of “string”. Maybe, the best we can hope is that string theory gives a *consistent framework* for the appearance of the SM. On this road of consistency check supplied with some prejudice on the gauge hierarchy problem, we anticipate the first experimental window to string as verification of a supersymmetric standard model.

¹⁰ Reference [41] considers the possibility of a large extra dimension.

Compactification

We conclude this lengthy chapter by showing qualitatively that if we start with extra dimensions, then the chiral fields necessary at the electroweak scale need some kind of orbifold-like compactification.¹¹

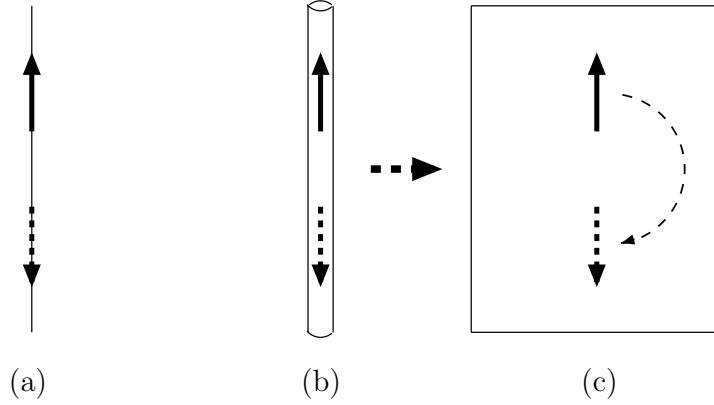


Fig. 2.7. The chirality of a massless particle in even dimension $2n$ embedded in a higher odd dimension $2n+1$: (a) $D=4$, (b) $D=5$, and (c) opening the cylinder of (b)

The chirality is given by $(1 \pm \Gamma_{2n+1})/2$ in even space-time dimensions, $2n$. Consider the chirality in 4D. In Fig. 2.7(a), we consider an arrow in 2D (i.e. one space dimension), which can be considered a $S_z = \frac{1}{2}$.¹² In one space dimension, the arrow can be put in either of two ways, as shown with the solid arrow and the dashed arrow in Fig. 2.7(a). They are different. But if the arrow came from a torus compactification from 3D (i.e. two space dimensions), it is in fact Fig. 2.7(b). It appears that the 2D directions are different in the compactified case 2.7(b), but in fact by opening up the compactified dimension as in Fig. 2.7(c), we note that the solid arrow can be transformed to the dashed arrow by a 3D rotation. Stated in terms of Weyl spinors in 4D, ψ_L (solid) and ψ_R (dashed), the Weyl spinor ψ_L cannot be transformed to ψ_R . But if these are embedded in 5D, a 5D rotation transforms one to the other, which means that they can belong to the same representation in 5D. Thus, a 5D spinor contains two 4D Weyl spinors, ψ_L and ψ_R . The rigorous mathematical statement on this situation can be found in any textbook on superstring [54], and will be dealt with in Sect. 3.2. This shows that even if we start from a chiral theory

¹¹ Some manifolds such as the Calabi-Yau have singular limits which become orbifolds.

¹² The angular momentum can be given from 4D, and the directions in 2D become helicities in 4D.

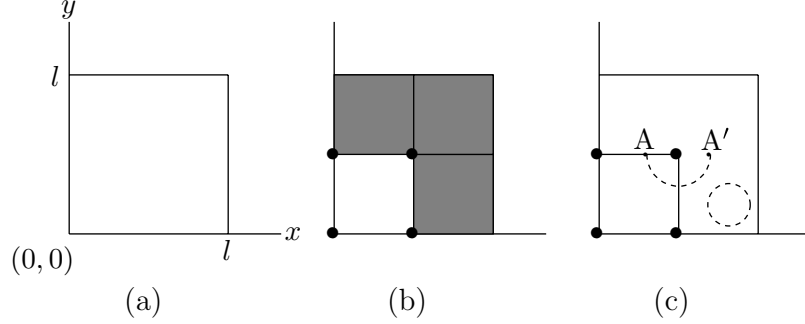


Fig. 2.8. A 2D torus moded out by $\mathbf{Z}_2 \times \mathbf{Z}'_2$: (a) 2D torus, (b) the moded torus with fixed points and fixed lines, and (c) untwisted (*dashed closed loop*) and twisted (*dashed arc A-A'*) strings

in higher dimensions, a *naïve torus compactification washes out the chirality in 4D*, since we must pass through 5D in the process.

To obtain chirality, one must twist tori. Because of the simplicity in drawing figures in 2D, consider two internal dimensions compactified on a 2D torus as shown in Fig. 2.8(a). For the torus compactification Fig. 2.8(a), $x \equiv x + l$ and $y \equiv y + l$, there is no chiral fermions left in 4D as discussed above. So, an attempt should be made to mode out the torus by a discrete group. In Fig. 2.8(b), we show the $\mathbf{Z}_2 \times \mathbf{Z}'_2$ moding: $x \equiv -x$ and $y \equiv -y$. Then, there are 4 fixed points under the shifts and reflections, shown as bullets in 2.8(b). Using the shift and reflection symmetries, it is sufficient to consider the *fundamental domain* which is shown as the white square in Fig. 2.8(b). The fixed points are at the corners of the fundamental domain. This geometry most probably with fixed points is called an *orbifold*.¹³ Closed strings moving in this geometry can be untwisted or twisted as shown in Fig. 2.8(c). The untwisted string is obviously closed. The twisted string in Fig. 2.8(c) does not appear closed at first glance, but it is in fact closed around the fixed point in the orbifold geometry since A and A' are identical, which can be shown by the allowed shifts and reflections. Thus a string appears just sitting around that fixed point. The space-time dimension of fixed points are 4D. We have seen that 4D *does not necessarily require* that a 4D Weyl fermion accompany the opposite chirality partner. Since a 4D chiral field (the effective field for a string) sitting at a fixed point can be present without its chiral partner at the same fixed point, it is possible to have a chiral theory with orbifold geometry after compactification. But the final effective 4D theory must be free of gauge anomalies, which means that the sum of untwisted and twisted sector fields together give an anomaly-free theory. Therefore, as for the untwisted fields also, they can carry an anomaly, i.e. they can be chiral. This happens because the moding group $\mathbf{Z}_2 \times \mathbf{Z}'_2$ keeps only a half of two massless Weyl fields in

¹³ Some orbifolds do not have fixed points.

the compactified 4D, or in other words, a half of the massless spectrum is projected out from the Hilbert space. *In string orbifolds, it is unambiguously determined which chiral fields should be at the fixed points and which should be in the bulk* [52]. Unlike string orbifolds, there is no guiding principle in field theoretic orbifolds, “how to put the chiral fields at the brane and in the bulk”, except for the condition of total anomaly cancellation. For field theoretic orbifolds, the argument is the same as the above until we try to put localized fields at the fixed points and some in the bulk. One can consider the dashed curves of Fig. 2.8(c) as the same altitude points of the wave function sitting at the fixed point. In field theoretic orbifolds, one extra dimension (5D) has been considered extensively. On the other hand, in closed string theory with one compactification scale, it is difficult to achieve an effective 5D theory.

In this book, we aim to explore *the most probable orbifold compactification toward the MSSM* from the $E_8 \times E'_8$ heterotic string. To obtain three families, the moding discrete group is taken as $\mathbf{Z}_3$ with two Wilson lines. In the last chapter, we tabulated all the interesting $\mathbf{Z}_3$ orbifold models with two Wilson lines. Some of these may contain spectrums leading to the MSSM. In Chap. 3, we give the definition of orbifolds. Then, before presenting the full-fledged string theory orbifolds, we will introduce the orbifold in field theory in Chap. 4 in order to first present a general understanding of orbifolds. Recently, field theoretic orbifolds have been studied extensively, after showing that the doublet-triplet splitting is also possible in the field theoretic orbifold compactification.

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Orbifolds and Spinors

In this chapter, we introduce the basic notion of orbifold and spinor properties in higher dimensions. Both of these play crucial roles in obtaining the chiral spectrum in four dimensions. In the discussion, we start with low dimensional examples and then extend them to higher dimensional ones.

3.1 Orbifolds

Roughly speaking, an orbifold is made of a manifold by cutting it with a suitable symmetry and gluing the resulting edges. We are familiar with this cutting and gluing when we make a cone starting with a disc.

In physics with orbifold geometry, the wavefunction or the string coordinate is restricted by the boundary conditions consistent with the orbifold, and some unfitted fields are projected out. This process breaks some of original symmetries. For studying supersymmetry of fields and strings living on the orbifold, further conditions are needed, which drastically reduce the number of orbifolds to be considered.

After defining orbifolds in an abstract form, we will consider some examples and try to understand the geometrical meaning of them. The references [1, 2] contain extensive treatments on this topic.

3.1.1 Orbifold in Abstract

An *orbifold* is obtained from a manifold by identifying points under a discrete symmetry group S [1]. It is called the *space group* and its element $g = (\theta, v)$ transforms a point x in $\mathbf{R}^n$ as,

$$g : x^i \rightarrow \theta^i_j x^j + v^i \quad (3.1)$$

with summation convention understood. A set of images due to such operations is called orbit, hence the name orbifold is given: on an orbifold each

point corresponds to an orbit (of the symmetry group) on the covering space (here $\mathbf{R}^n$). What we mean by modding or identification is that we assign an equivalence relation between two elements related by such an action

$$g \in S \quad (3.2)$$

and count inequivalent points only. Symbolically the orbifold is denoted as $\mathbf{R}^n/S$. Since S is a discrete group, it preserves the flatness of $\mathbf{R}^n$, thus the orbifold inherits the flatness of $\mathbf{R}^n$.

Torus

The most familiar example of orbifold is torus. The set of translations defines the lattice,

$$\Lambda_1 = \{(1, me) \mid m \in \mathbf{Z}\} , \quad (3.3)$$

with a real number e . This means that the lattice Λ_1 is isomorphic to the set of integers $\mathbf{Z}$. The one dimensional torus is obtained by modding out a real axis by such pure translations

$$T^1 = \mathbf{R}/\Lambda_1 . \quad (3.4)$$

It is nothing but a circle $T^1 = S^1$, with the circumference $e = 2\pi R$, which is flat since the curvature scalar is zero. In other words, the torus is made by identifying the endpoints of the unit interval $[0, e]$.

Also, by definition we form a two-torus T^2 by identifying the opposite sides of a square or parallelogram. We stress that *the torus is flat* in the sense that we have constant metric and hence curvature scalar vanishes.¹ This still holds good if we regard the torus as a direct product of circles, $T^2 = S^1 \times S^1$. But, if we consider embedding the two-torus in three dimensions, this does not look flat. It is inevitable to bend it. It merely arises from the extrinsic property of the compulsory embedding 2D torus in three dimensions. If we embed it into four dimensions, one S^1 for each $\mathbf{R}^2$, it becomes flat again [3].

By induction, we can generalize this to an n dimensional torus by either identifying the opposite edges of an n -cube or n direct product of S^1 ,

$$T^n = \mathbf{R}^n/\Lambda_n , \quad (3.5)$$

where the lattice Λ_n is now n dimensional,

$$\Lambda_n = \left\{ \left(1, \sum m^i e^i \right) \mid m^i \in \mathbf{Z} \right\} . \quad (3.6)$$

¹ When we take into account global topology we should consider one more complication. Transporting once around in a direction, the orientation can be changed (by identifying the opposite edges of the parallelogram with reversed orientations as in the Möbius strip). From the Möbius strip, identifying the sides but not changing the orientation, one obtains a Klein bottle while changing the orientation also one obtains a projective plane [3].

The n basis vectors e^i are not necessarily orthogonal nor of the same length. The space is parameterized by the metric tensor

$$G^{ij} = e^i \cdot e^j . \quad (3.7)$$

Therefore, the volume of torus is

$$V = \sqrt{\det G} . \quad (3.8)$$

The important property that a torus is not simply connected will be used later. Think of a path on the torus winding around one direction, for instance specified by e^1 . This circle is not contractible to a point, since it is wound. We refer this as *cycle* or one (dimensional)-cycle. For an n dimensional torus T^n , we have n independent 1-cycles. This is a topological notion since if we deform the circle or the torus in some way, the winding nature is unchanged. We can understand many characteristic features of the topological objects merely by knowing how many such cycles exist.

3.1.2 Orbifold as Moded-Out Torus

It is easy to see that the automorphism (3.1) satisfies the following group properties,

$$\begin{aligned} (\theta, v)(\omega, u) &= (\theta\omega, v + \theta u) , \\ (\theta, v)^{-1} &= (\theta^{-1}, -\theta^{-1}v) . \\ [(\theta, v)(\omega, u)](\rho, w) &= (\theta, v)[(\omega, u)(\rho, w)] \end{aligned} \quad (3.9)$$

with identity element $(1, 0)$. Formally, it looks the same as that of the continuous Poincaré group.

The *point group* $\mathbf{P}$ is the subgroup of $\mathbf{S}$ with automorphism $\theta = (\theta, 0)$. (It is convenient to imagine a pure rotation.) Throughout this book, we restrict our discussion on the orbifold compatible with the torus. In other words, we will make an orbifold by moding out torus. Then, for every θ there is a unique vector v such that (θ, v) is an element in $\mathbf{S}$, up to translation Λ defining the torus. To see it, compare two elements with the same θ ,

$$(\theta, v)(\theta, u)^{-1} = (\theta, v)(\theta^{-1}, -\theta^{-1}u) = (1, v - u) . \quad (3.10)$$

Then $v - u$ should belong to the lattice Λ , otherwise the product is not compatible with torus. Thus we will label the action by θ only. This generalized point group $\bar{\mathbf{P}}$ is identified with $\mathbf{S}/\Lambda$ and commute with Λ . Thus we have another definition for the orbifold

$$\mathbf{R}^n/\mathbf{S} = \mathbf{R}^n/(\Lambda \times \bar{\mathbf{P}}) = T^n/\bar{\mathbf{P}} . \quad (3.11)$$

In what follows, we will restrict the point group $\mathbf{P}$ to a discrete subgroup of $\text{SO}(d)$ generated by a single rotation θ . This is an Abelian group and called cyclic group $\mathbf{Z}_N$, where an order N is the minimum natural number satisfying $\theta^N = 1$.

Conjugacy Class and Twisted Sector

In categorizing the action, a useful concept is the notion of conjugation. Two elements a and b are said to be *conjugate* if there is a relation $b = gag^{-1}$ where g is a group element, $g \in \mathbf{G}$. It is an equivalence relation, $a \sim b$. Accordingly we define the *conjugacy class* of a as

$$[a] = \{gag^{-1} \mid g \in \mathbf{G}\} . \quad (3.12)$$

Consider an action of translation by $(1, v_0)$. Using (3.9), one can easily check that

$$(\omega, u)(1, v_0)(\omega, u)^{-1} = (1, \omega v_0) . \quad (3.13)$$

Therefore, the conjugacy class of $(1, v_0)$ is

$$[(1, v_0)] = \{(1, \omega v_0) \mid \omega \in \mathbf{P}\} , \quad (3.14)$$

which is the set of translations by the amount $|v_0|$ in all possible directions.

Next consider a more general element (θ^k, v_0) and its general conjugation by (ω, u)

$$\begin{aligned} (\omega, u)(\theta^k, v_0)(\omega, u)^{-1} &= (\omega\theta^k\omega^{-1}, -\omega\theta^k\omega^{-1}u + \omega v_0 + u) \\ &= (\theta^k, \omega v_0 + (1 - \theta^k)u), \end{aligned} \quad (3.15)$$

where in the last line we assumed ω and θ commute. Its conjugacy class is

$$[(\theta^k, v_0)] = \{(\theta^k, \omega v_0 + (1 - \theta^k)u) \mid \omega \in \mathbf{P}, u \in \Lambda\} . \quad (3.16)$$

Sometimes one uses a shorthand notation

$$(1 - \theta^k)\Lambda = \{(1 - \theta^k)u \mid u \in \Lambda\} . \quad (3.17)$$

Note that ω in the conjugation (3.15) does not affect the rotational part θ^k . The conjugacy class $[(\theta^k, v_0)]$ parameterize k th *twisted sector*. The *untwisted sector* is parameterized by the conjugacy class (3.14) and can be regard as corresponding to the twisted sector formula with $\theta^0 = 1$, i.e. the 0th twisted sector.

3.1.3 Fixed Points

In general, the action is not freely acting,² i.e. an orbifold has *fixed points* f . After a point group action θ^k , a fixed point comes back to the original location by a certain lattice translation v

$$f = \theta^k f + v . \quad (3.18)$$

² An action is freely acting if its inverse is single-valued.

It will be very convenient to represent a fixed point by the *relation* (θ, v) , not by the coordinate of f . Note that not all the lattice translations of $v \in \Lambda$ determine the equivalent fixed point. We will show that, if we find a particular solution v_0 , all the elements in the conjugacy class $[(\theta^k, v_0)]$ in (3.16) parameterize the same fixed point f . Consider two fixed points f_1 and f_2 in the class conjugated by θ^{j_1} and θ^{j_2} , respectively,

$$\begin{aligned} f_1 &= \theta^k f_1 + \theta^{j_1} v_0 + (1 - \theta^k) u_1 , \\ f_2 &= \theta^k f_2 + \theta^{j_2} v_0 + (1 - \theta^k) u_2 . \end{aligned} \quad (3.19)$$

Subtracting, we have

$$(1 - \theta^k)(f_1 - f_2) = (\theta^{j_1} - \theta^{j_2})v_0 + (1 - \theta^k)(u_1 - u_2) . \quad (3.20)$$

In the $k = 1$ case, for any value of j_1 and j_2 the right hand side is always factorizable by $(1 - \theta)$, so by dividing with $(1 - \theta)$ the right hand side becomes a lattice element. It means that, there is one to one correspondence between the fixed point f and the conjugacy class,

$$[(\theta, v_0)] = \{(\theta, \omega v_0 + (1 - \theta)u) \mid \omega \in \bar{\mathbb{P}}, u \in \Lambda\} . \quad (3.21)$$

From (3.19), we can express ωv_0 in terms of fixed point f ,

$$\omega v_0 = (1 - \theta)f . \quad (3.22)$$

Consequently, if we know the fixed points, we can denote the conjugacy class as

$$(\theta, (1 - \theta)(f + \Lambda)) , \quad (3.23)$$

for each fixed point f . Namely, the twisted sector θ is divided into subsectors depending on the fixed point f . This expression will be extensively used in calculating quantities related with fixed points.

For the conjugacy class of a higher twisted sector θ^k , we cannot factorize the right hand side of (3.19) when j_1 or j_2 is smaller than k , to make a lattice element. In this case, the conjugacy class does not have one to one correspondence with fixed points under θ^k . This is because some of them are not fixed points under θ (rotation by smaller angles). In this sense, the conjugacy class is a more fundamental concept than the fixed point. We will come back to this situation in Subsect. 7.2.1. If the eigenvalue of θ^k is 1 then there are fixed tori rather than fixed points.

These fixed points are clearly singular in the sense that the curvature diverges at the fixed points. Therefore, the orbifold is not a manifold in general. However, in later chapters we will see that fields and strings have well-defined behaviors at the fixed points.

Fundamental Region

We have defined the orbifold as a moded out torus $T^n/\bar{\mathbf{P}}$. This (co)set of points is called *fundamental region* or fundamental domain. For the orbifold, we can think of and draw in its covering space, torus, for practical reasons.

All the points of covering space, which lie outside the *fundamental region* or fundamental domain can be transformed into points in the fundamental region, under some point actions g . Therefore, the fundamental region can be considered to be surrounded by the fixed points. By definition, we can think of the geometry of orbifold by gluing the boundaries of the fundamental region. The volume of an orbifold is defined by that of the fundamental region.

Number of Fixed Points

Consider a point on the lattice site, denoted by a vector $a^i e^i$. Here, e^i is a lattice defining vector so that every entry of a is chosen as an integer. By the action of $\theta \in \bar{\mathbf{P}}$, the image $(\theta a)^i e^i$ is another basis vector and hence has also integral entries. This implies that θ can be represented by a matrix with integer elements (by pullback). Thus, it is guaranteed that the trace and the determinant are also integers. In particular, the following determinant is an integer

$$\chi = \det(1 - \theta) = \prod_i 4 \sin^2(\pi \phi_i) , \quad (3.24)$$

with the index i running over the compact dimensions. Then χ is the number of fixed points due to the Lefschetz fixed point theorem [1, 4]. This relies on the twist vector ϕ only, regardless of the form of torus, i.e. the lattice. If we include noncompact dimensions, or fixed tori than points, then $\chi = 0$ since the corresponding component of ϕ is zero. In Tables 3.2 and 3.3, we list examples of orbifolds and their χ .

We proceed the discussion with some examples.

3.1.4 One Dimensional Orbifold

Recently, one dimensional orbifold has been extensively studied in the context of field theoretical orbifold, which is the main theme of Chap. 4.

$S^1/\mathbf{Z}_2$ Orbifold

The simplest example of nontrivial orbifold is $S^1/\mathbf{Z}_2$. Let us choose $\mathbf{Z}_2$ action as the reflection with respect to the origin,

$$y \rightarrow -y . \quad (3.25)$$

It follows that $\pi R + y$ and $\pi R - y$ are also identified because

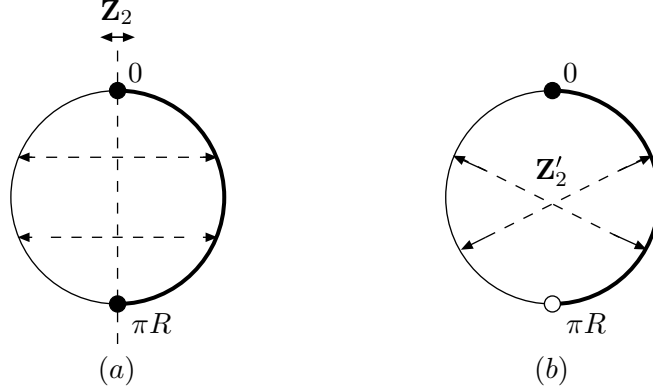


Fig. 3.1. Two possible $S^1/\mathbf{Z}_2$ orbifolds. **(a)** $\mathbf{Z}_2$ action is defined by reflection $y \rightarrow -y$. Fixed points at $y = 0, \pi R$ are denoted by bullets. **(b)** $\mathbf{Z}'_2$ action is defined by reflection $y \rightarrow y + \pi R$. Two points $0, \pi R$ are again identified and there is no fixed point

$$\pi R + y \rightarrow -\pi R - y \rightarrow \pi R - y, \quad (3.26)$$

with the last relation coming from the definition of S^1 . In other words, we make the orbifold $S^1/\mathbf{Z}_2$ by identifying the opposite points of S^1 with respect to vertical axis of S^1 drawn in 2D as shown in Fig. 3.1. Here, we have two fixed points $y = 0$ and $y = \pi R$. Equivalently saying, we have a finite interval $[0, \pi R]$. This interval is the fundamental region; fields can move within this interval $[0, \pi R]$.

Another Moding

There is another $\mathbf{Z}'_2$ symmetry action on S^1 , i.e. the rotation by π ,

$$y \rightarrow y + \pi R. \quad (3.27)$$

It is also viewed as a translation by a half of the original interval. It is of order two because twice the action is an identity.

This orbifold has no fixed points, and equivalently the $\mathbf{Z}'_2$ action is freely acting on the circle S^1 . The topology of this orbifold is not closed interval, but again the circle with circumference reduced to the half πR .

Sometimes, this orbifold is called $\mathbf{RP}^1$, meaning the real, projected interval in one dimensional plane.³ Although it is one dimensional, we can introduce two real numbers (x, y) and identify coordinates up to a scale

$$(x, y) \sim (\lambda x, \lambda y). \quad (3.28)$$

³ It is also known as crosscap.

Since coordinates are the same up to a scale factor λ , we may fix the scale as R . Also, because $\lambda = R$ and $\lambda = -R$ lead to the same point, we can equivalently mod it out by $\mathbf{Z}_2$ action (3.27).

$S^1/(\mathbf{Z}_2 \times \mathbf{Z}'_2)$ Orbifold

We can combine the above two $\mathbf{Z}_2$ actions to define $\mathbf{Z}_2 \times \mathbf{Z}'_2$ action. The first $\mathbf{Z}_2$ action P is a reflection around the vertical hypothetical axis and the second $\mathbf{Z}'_2$ action T generates translation by πR . Combining them, the action TP give rise to the reflection around the horizontal axis

$$y \xrightarrow{P} -y \xrightarrow{T} \pi R - y \quad (3.29)$$

Therefore, we have a new fixed points at $y = \pi R/2$. It is convenient to introduce a new coordinate $y' = y + \pi R/2$. Then the action TP becomes

$$y' \rightarrow -y' . \quad (3.30)$$

We do not count the identical points $y = \pi R, y = -\pi R/2$ because the fundamental region is now $[0, \pi R/2]$. The orbifold has another axis of reflection as in Fig. 3.2. We can associate two projection condition with two actions. This is the most complicated example in one dimension.

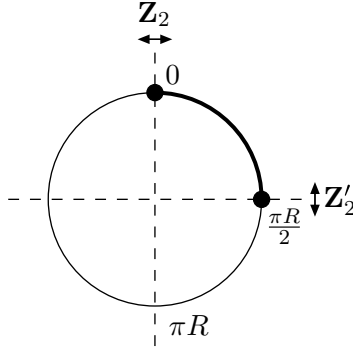


Fig. 3.2. Fixed points at $y = 0, \frac{\pi R}{2}$

No More Complicated Orbifold

Whatever (or more complex) discrete group $\mathbf{Z}_N$ we use in 1D, the effect is essentially the same. Everything reduces to two cases we have considered above. At best, the length of interval reduces to $2\pi R/N$. Nevertheless, when we consider a field or string living on this orbifold, we can associate a more

complicated $\mathbf{Z}_N$ action for the field or string to satisfy. This is a profitable aspect of orbifold when we try to break many symmetries.

Also combining more than one $\mathbf{Z}_2$ actions P and $\mathbf{Z}'_2$ actions T , one obtains the same orbifold as in the previous example because they commute

$$TP = PT . \quad (3.31)$$

Thus, the $\mathbf{Z}_2$ and $\mathbf{Z}_2 \times \mathbf{Z}'_2$ orbifolds discussed above are all the orbifolds in one dimension.

3.1.5 Two Dimensional Orbifolds

$T^2/\mathbf{Z}_2$ Orbifold

Two basis vectors define two dimensional torus T^2 ,

$$x \sim x + e^i, \quad i = 1, 2, \quad (3.32)$$

where unit vectors e^i are not necessarily orthogonal nor of unit length.

We may consider, for example, the orthogonal basis vectors in the Cartesian coordinates as

$$e^1 = (2\pi R_1, 0), \quad e^2 = (0, 2\pi R_2), \quad (3.33)$$

but the discussion below will be bases-independent. Eventually we make a $T^2/\mathbf{Z}_2$ orbifold by the identification

$$x \sim -x \quad (3.34)$$

Be warned that this definition is basis dependent, because it can be reexpressed as

$$x_1 e^1 + x_2 e^2 \rightarrow -x_1 e^1 - x_2 e^2, \quad (3.35)$$

and we have another identification under basis vectors. This orbifold is depicted in Fig. 3.3, where four fixed points are

$$0, \pi R_1 e^1, \pi R_2 e^2, \pi R_1 e^1 + \pi R_2 e^2 \quad (3.36)$$

are marked with bullets. In view of (3.21) we can label fixed points by the following space group elements.

$$(\theta, 0), \quad (\theta, e^1), \quad (\theta, e^2), \quad (\theta, e^1 + e^2), \quad (3.37)$$

with which we come back to the original fixed points.

We can take the fundamental region as $OO'B'B$. This choice is not unique and we have many equivalent ones. For example, we can take $OAA'O''$ or even take the triangle $OO'O''$ as the fundamental region. In any case, the volume of orbifold, $2\pi^2 R_1 R_2$, is the half of the original torus.

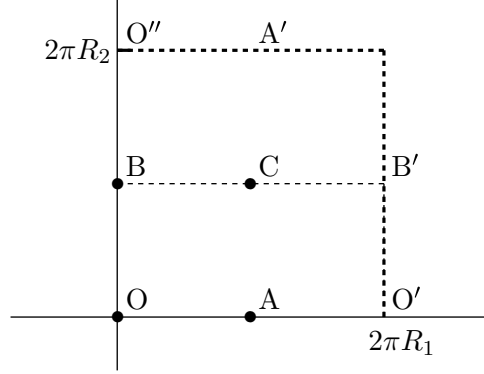


Fig. 3.3. The $T^2/\mathbf{Z}_2$ orbifold in the $z = x_1 + ix_2$ plane. The orbifold fixed points are denoted by bullets. The fundamental region is the lower half-rectangle. The physical space may be taken as the two-sided rectangle formed by folding the boxed region along BB' and then gluing together the touching edges. It is a kind of ravioli

Note that some different choices of basis vectors may define the same torus. But if we consider symmetries on fields living on the torus, they can be different, because the symmetries are described in terms of different unit vectors. As a simple exercise, consider two different choices of lattices, $e^1 = (0, 2), e^2 = (2, 0)$ and $e^1 = (0, 2), e^2 = (1, 2)$, seem to describe the same torus. However, when we mod them out by the $\mathbf{Z}_2$ action defined by $x \rightarrow -x$, we have different orbifolds: they have different fixed points and the shapes of orbifolds are different. Formally speaking, if we consider a complex structure, the equivalent classes of orbifold become small. We will encounter this point again when we discuss the modular invariance in string theory. There the world-sheet is described by torus and its equivalence is a delicate problem.

$T^2/\mathbf{Z}_3$ Orbifold

Now consider the $T^2/\mathbf{Z}_3$ orbifold. We form a torus T^2 by taking basis vectors e^1 and e^2 being of equal lengths $|e^1| = |e^2|$ but not parallel

$$\cos \phi_{12} = \frac{e^1 \cdot e^2}{|e^1||e^2|} = -\frac{1}{2}.$$

The $\mathbf{Z}_3$ action is generated by rotation θ by an angle $2\pi/3$. Then, we obtain the following relations,

$$\theta e^1 = e^2, \quad \theta e^2 = -e^1 - e^2. \quad (3.38)$$

The unit lattice is depicted in Fig. 3.4. The basis vectors are the root vectors of $\text{SU}(3)$, hence it is sometimes called “ $\text{SU}(3)$ lattice”. This amounts to fixing two of three parameter of $G^{ij} = e^i \cdot e^j$ except the overall size of torus $R_1 = R_2$.

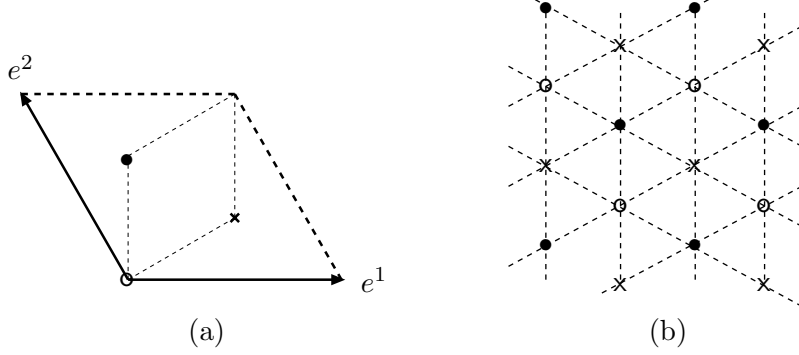


Fig. 3.4. (a) $\mathbf{Z}_3$ orbifold is formed by identification of two edges of $\text{SU}(3)$ torus. Three fixed point are marked by o, x, •. The region inside the lightly dashed parallelogram is the fundamental region. (b) The covering space of $\mathbf{Z}_3$ orbifold. Any two adjacent equilateral triangles form a fundamental region. The unit lattice is a hexagon and respects the $\mathbf{Z}_3$ symmetry

It has three fixed points marked by o, x, • as shown in Fig. 3.4. We may label such points by their coordinates: for example, $f_x = \frac{1}{3}(2e^1 + e^2)$. It is defined up to lattice translation Λ . Therefore, $\theta f_x = \frac{1}{3}(-e^1 + e^2) \sim f_x$ indicating the same fixed point up to a lattice translation by e^1 . Similarly, $f_o = 0$ and $f_\bullet = \frac{1}{3}(e^1 + 2e^2)$ can be shown to be fixed points with shifts by 0 and $e^1 + e^2$, respectively. Instead of using the above coordinate notation, we use the following useful labeling,

$$\text{o} : (\theta, 0), \quad \text{x} : (\theta, e^1), \quad \bullet : (\theta, e^1 + e^2) \quad (3.39)$$

according to (3.21). For instance, the point x comes back to the original point after the rotation θ and translation e^1 . From (3.16) the equivalent fixed point outside the fundamental region is related to the original one by lattice translation $(1 - \theta)\Lambda$.

We can check that the half of the fundamental region is surrounded by the fixed points. The matrix form of (3.38) is

$$\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad (3.40)$$

from which we have θ represented by integral elements

$$\theta = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

The determinant $\chi = \det(1 - \theta) = 3$ is the number of fixed points.

We have another twisted sector. The second twisted sector is generated by $\theta^2 = e^{4\pi i/3} = e^{-2\pi i/3}$, which is the same amount of rotation but in the opposite direction. Thus, we expect the same properties as discussed above for this θ^2 twisted sector also.

3.1.6 Geometry of Orbifold

The orbifold is formed by identifying points under the point group $\bar{P}$. This is achieved by folding the fundamental region of Fig. 3.4 and gluing the edges. This leads to a “ravioli” (or “pillow”) type manifold which has the topology of S^2 .

However the folding, rather than identification, is somewhat misleading. As discussed in the torus case, there is no change of curvature and it is everywhere flat with constant G^{ij} . This situation is depicted in Fig. 3.5.

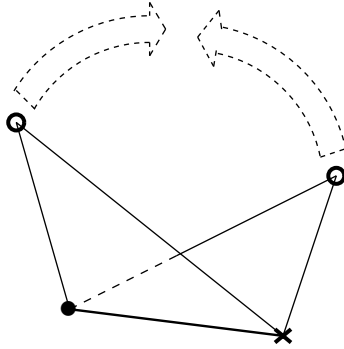


Fig. 3.5. The orbifold is formed by identifying points under the point group $\bar{P}$. This is achieved by folding the fundamental region of Fig. 3.4 and gluing the edges to form a ravioli

We have seen that there is a non-contractible loop in the torus along the direction of vector. In the orbifold case we have another kind of non contractible cycle around the fixed points. We have another kind of closed path, up to a point group element used in the identification. When we contract this loop, it is wound around the fixed point but cannot be shrunk into a point.

$T^2/\mathbf{Z}_6$ Orbifold

So far we have dealt with the examples in which the basis vectors of covering torus obey the point group, for example $\theta e^1 = e^2$. However, this is not mandatory, since the generalized point group $\bar{P}$ action in (3.11) is defined up to lattice translation. In other words, it is only necessary that *the lattice* generated by basis vectors should be compatible with the space group action. A good illustration is provided by $T^2/\mathbf{Z}_6$ orbifold on the G_2 lattice of Fig. 3.6. In the Cartesian coordinates, the basis vectors are taken as

$$e^1 = (1, 0), \quad e^2 = (-3/2, \sqrt{3}/2). \quad (3.41)$$

Note that the basis vectors are not of equal lengths and angle between them is $\phi_{12} = 5\pi/6$,

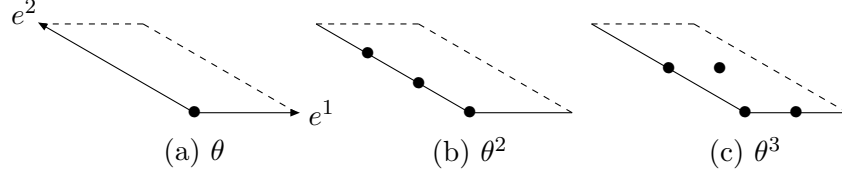


Fig. 3.6. $T^2/\mathbf{Z}_6$ orbifold is defined on a G_2 lattice by $\theta = e^{2\pi i/6}$. Each twisted sector has a different number of fixed points

$$\cos \phi_{12} = \frac{e^1 \cdot e^2}{|e^1||e^2|} = -\frac{\sqrt{3}}{2} \quad (3.42)$$

This defines a torus T^2 . From this torus, we make an orbifold under the identification given in (3.48),

$$z \sim e^{2\pi i/6} z. \quad (3.43)$$

We can readily check that

$$\theta e^1 = 2e^1 + e^2 = e^2 + (e^1 + e^2), \quad (3.44)$$

so that the space group action is compatible.

Fixed points are shown in Fig. 3.6 for three twisted sectors. For the first twisted sector, θ , there is only one fixed point, the origin. In the second twisted sector, θ^2 , the action is equivalent to $\mathbf{Z}_3$ generated by $\theta^2 = e^{2\pi i/3}$, and we have three fixed points which can be read from Fig. 3.4(b). In the third twisted sector, the action is equivalent to $\mathbf{Z}_2$ generated by $\theta^3 = e^{\pi i}$ and there are four fixed points as in the case of Fig. 3.3.

But let us see again (3.44), and reason what does it mean. If we define

$$e^{2'} = e^1 + e^2, \quad (3.45)$$

we see that, by $2\pi/3$ rotation, $\theta^2 e^1 = e^{2'}$. This is the defining condition for the $SU(3)$ lattice as we saw in the previous subsection. *The G_2 lattice generated by e^1 and e^2 and $SU(3)$ lattice generated by e^1 and $e^{2'}$ are the same.* The lattice is the same, therefore the *orbifold is the same*, as is clear from Fig. 3.7.

Crystallography

Now the question is to find out how many lattices are allowed in two dimensions. We have already considered the $SU(3)$ and G_2 lattices. This question can be restated in another form: using only one kind of tile, how many polygon types are allowed for tiling the two dimensional space. The lattice point is defined by vertices of tiles, where the adjacent tiles meet. The tile is well-filled only if at the vertex the sum of the angles of a polygon equals exactly to 2π . In other words every vertex should have n -fold rotational symmetry D_n . A

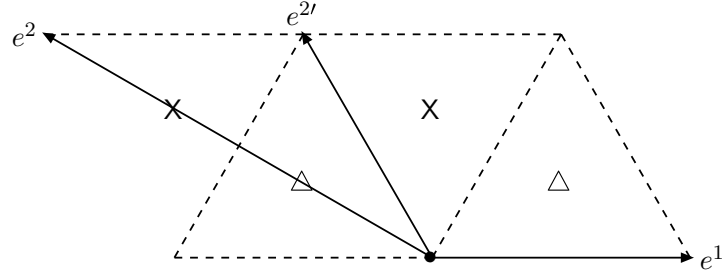


Fig. 3.7. The G_2 lattice is the same as the $SU(3)$ lattice. Fixed points on the e^2 axis of the G_2 lattice are the same, up to shifts, as the fixed points of the $SU(3)$ lattice

n -gon have a side angle $\pi - 2\pi/n$. Decomposing such polygon as n isosceles triangles and using the fact that the angles of a triangle sum up to π , one can easily show the filling condition becomes

$$N \left(\pi - \frac{2\pi}{n} \right) = 2\pi. \quad (3.46)$$

Equation (3.46) admits only four solutions $N = 2, 3, 4, 6$ for which $n = \infty, 6, 4, 3$, respectively. This situation is depicted in Fig. 3.8. Therefore, by discrete rotations we can have only four kinds of orbifold lattices in the two dimensional torus.

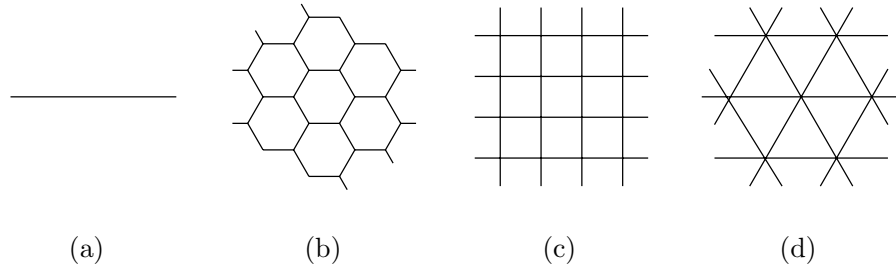


Fig. 3.8. There are only four possible crystals (tilings) in two dimensions, thus there exist four two dimensional orbifolds. Each figure is invariant under two, three, four, and six fold rotations

$T^2/\mathbf{Z}_N$ Orbifold

Now we present a general 2D orbifold $T^2/\mathbf{Z}_N$.

Let us introduce the cyclic group $\mathbf{Z}_N$ as the group generated by a $2\pi/N$ rotation. The positive integer N is called order since N successive $\mathbf{Z}_N$ actions

become identity. In (3.46) we observed that the allowed N s are 2,3,4, and 6. It is convenient to pair the coordinates (x, y) as

$$z \equiv \frac{1}{\sqrt{2}}(x + iy), \quad \bar{z} \equiv \frac{1}{\sqrt{2}}(x - iy). \quad (3.47)$$

By action θ , z transforms to

$$z \rightarrow e^{2\pi i/N} z. \quad (3.48)$$

To make an orbifold, we usually take a torus compatible with this action. That is the torus should possess the symmetry (3.48)

$$z \sim z + e, \quad z \sim z + e^{2\pi i/N} e, \quad (3.49)$$

with the basis vector e . Then, the $T^2/\mathbf{Z}_N$ orbifold is obtained when we identify all the points under the action of (3.48).

In string theory, the coordinate z is interpreted as a world-sheet boson and the $\mathbf{Z}_N$ action on the world-sheet boson is defined in the same way via (3.48). In the supersymmetric extension of the world-sheet boson, we have its superpartner ψ . Then, ψ transforms under the $\mathbf{Z}_N$ action as

$$\psi \rightarrow e^{2\pi i J/N} \psi, \quad (3.50)$$

where J is the rotation generator for the spinorial representation (given in terms of the Dirac gamma matrices). We know that spinors acquire a minus sign when we rotate them by 2π . Therefore, the consistency condition $\theta^N = 1$ is extended so that

$$N\phi = \text{even integer}. \quad (3.51)$$

We relaxed the condition to even integers because the superstring has fermionic coordinates on which we perform the same action.

3.1.7 Higher Dimensional Orbifolds

Our main interest will be to compactify ten dimensional string down to four and hide six internal dimensions. Therefore, we want six compact dimensions.

It is convenient to denote the space group action θ in terms of a complex representation as done above. In six dimensions, it is diagonalized as

$$\theta = \text{diag}(e^{2\pi i\phi_1}, e^{2\pi i\phi_2}, e^{2\pi i\phi_3}). \quad (3.52)$$

Requiring it to be order N , we obtain the similar condition as (3.55)

$$N \sum_i \phi_i = \text{even integer}, \quad (3.53)$$

as well as each $N\phi_i$ being an integer.

As seen in Subsect. 3.1.6 in two dimension, there are only four possible twists ϕ as automorphisms compatible with lattices forming crystals. In general there are only limited number of lattices and twists in a given dimension. The systematic analysis is done in [5, 6, 7]. A twist is said to be *reducible* if it is decomposed into products. Therefore, irreducible twists provide building blocks of higher dimensional twists, which are tabulated in Table 3.1. One can verify that possible combinations of these irreducible twists lead to twists with order $N = 2, 3, \dots, 10, 12, 14, 15, 18, 20, 24, 30$ up to six dimensions.

Table 3.1. Irreducible crystallographic actions. $\frac{1}{2}(1)$ is the irreducible action in one dimensional lattice

$d = 2$	$d = 4$	$d = 6$
$\frac{1}{3}(1)$	$\frac{1}{5}(2, 1)$	$\frac{1}{7}(3, 2, 1)$
$\frac{1}{4}(1)$	$\frac{1}{8}(3, 1)$	$\frac{1}{9}(4, 2, 1)$
$\frac{1}{6}(1)$	$\frac{1}{10}(3, 1)$	$\frac{1}{14}(5, 3, 1)$
	$\frac{1}{12}(5, 1)$	$\frac{1}{18}(7, 5, 1)$

Given a twist, more than one lattices can be related, because the twist is only responsible for the rotational compatibility. The lattice is effectively classified by Lie algebra, which provides the root lattice. See Chap. 9. The simple roots generate the root lattice which sometimes called Coxeter lattices. The twist is naturally realized by Coxeter element (9.18) and its order is determined by Coxeter number (9.2). For example, the $SU(3)$ lattice is compatible with twist $\frac{1}{3}(1)$ as a rotation by $2\pi/3$. Note that some lattices are identical,

$$G_2 = SU(3), \quad SO(8) = SO(9) = F_4.$$

Thus, we are led to define $\mathbf{Z}_N$ action on the d torus with complexified coordinates,

$$z_i \equiv \frac{1}{\sqrt{2}}(x_{2i-1} + ix_{2i}), \quad \bar{z}_i = \bar{z}_i \equiv \frac{1}{\sqrt{2}}(x_{2i-1} - ix_{2i}). \quad (3.54)$$

Hence the orbifold action is naturally extended

$$z_i \rightarrow e^{2\pi i \phi_i} z_i, \quad (3.55)$$

for each i . We form orbifold under identifications

$$z \sim z_i + e^i, \quad z \sim z + e^{2\pi i \phi} e^i, \quad (3.56)$$

with basis vectors e^i . Thinking of z as bosonic coordinate, the total eigenvalue of θ is the sum of eigenvalues of each 2D rotation. On the spacetime spinor ψ

$$\psi \rightarrow e^{2\pi i \sum J_i \phi_i} \psi, \quad (3.57)$$

where J_i are the rotation generators on the spinor.

Supersymmetry Constraint

For a complete discussion, we borrow one physical concept in this abstract chapter on orbifold. For the fields living in the orbifold, a restriction for supersymmetry is imposed, otherwise we have the tachyon problem which is not treatable at the moment.

When we compactify six dimensions, ten dimensional supersymmetry generators can be decomposed into $Q_{(10)} = Q_{(4)} \otimes Q_{(6)}$. Choosing one chirality, the six dimensional part $Q_{(6)}$ transforms as **4** of $SO(6) = SU(4)$. Because the remaining parts $Q_{(4)}$ becomes the four dimensional generators, the dimension **4** counts the number of supersymmetries. Its spinorial representation is represented by $|s\rangle = |s_1 \ s_2 \ s_3\rangle = |\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\rangle$ with even number of minus signs. (See the next section.) Under point group, it transforms as

$$Q_{(6)} \rightarrow \exp(2\pi i s \cdot \phi) Q_{(6)}.$$

The invariant component corresponds to the unbroken supersymmetry generator. For $\mathcal{N} \geq 1$ supersymmetry, we need at least one solution, say if we choose $s = (+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, the argument of exponent vanishes,

$$\phi_1 - \phi_2 - \phi_3 = 0. \quad (3.58)$$

The number of solutions, $\mathcal{N}$, counts the number of unbroken supersymmetry generators from orbifold compactification.

Each entry of ϕ describes a $2\pi\phi_i$ rotation, and hence is a rational number. One can check straightforwardly that among the lattices allowed crystallographically shown in Table 3.1 and also among their combinations, the number of those satisfying the supersymmetry condition (3.58) is 13, which are tabulated in Table 3.2 for $\mathcal{N} = 1$ and Table 3.3 for $\mathcal{N} = 2$. In six compact dimensions, the upper limit of order is $N = 12$. In general, we have many possible different lattices for the same shifts. They are represented in algebraic notation in Table 3.2.

For compactification of four dimensions on $T^4/\mathbf{Z}_N$ shown in Table 3.2 corresponding to the ϕ entries with the vanishing third entry, the allowed values of N are only $N = 2, 3, 4$, and 6. This is exactly the case of two dimensional orbifold because the supersymmetry condition (3.58) implies $\phi_1 = \phi_2$, hence each orbifold is forced to be decomposed into a direct product of two identical $T^2/\mathbf{Z}_N$ orbifolds.

$T^6/\mathbf{Z}_3$ Orbifold

For example, the $T^6/\mathbf{Z}_3$ orbifold is specified by a vector ϕ

Table 3.2. Possible six dimensional orbifolds allowing $\mathcal{N} = 1$ supersymmetry. We used the fact that the $SU(3)$ lattice is identical to the G_2 lattice, and the $SO(8)$ lattice is identical to the F_4 and $SO(9)$ lattices. The lattice within the [] bracket involves the outer automorphisms and thus cannot be decomposed to the product of lower dimensional ones. The number of fixed points is χ

Order	Crystallographic Lattice	ϕ	χ
3	$SU(3)^3$	$\frac{1}{3}(2 \ 1 \ 1)$	27
4	$SU(4)^2$ $SU(4) \times SO(5) \times SU(2)$ $SO(5)^2 \times SU(2)^2$	$\frac{1}{4}(2 \ 1 \ 1)$	16
6-I	$SU(3) \times SU(3)^2$ $SU(3) \times [SU(3)^2]$	$\frac{1}{6}(2 \ 1 \ 1)$	3
6-II	$SU(2) \times SU(6)$ $SU(3) \times SO(8)$ $SU(2)^2 \times SU(3)^2$ $SU(2)^2 \times [SU(3)^2]$	$\frac{1}{6}(3 \ 2 \ 1)$	12
7	$SU(7)$	$\frac{1}{7}(3 \ 2 \ 1)$	7
8-I	$SO(5) \times SO(8)$ $[SU(4)^2]$	$\frac{1}{8}(3 \ 2 \ 1)$	4
8-II	$SU(2) \times SO(10)$ $SU(2)^2 \times SO(8)$	$\frac{1}{8}(4 \ 3 \ 1)$	8
12-I	E_6 $SU(3) \times SO(8)$	$\frac{1}{12}(5 \ 4 \ 1)$	3
12-II	$SU(2)^2 \times SO(8)$	$\frac{1}{12}(6 \ 5 \ 1)$	4

$$\phi = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}. \quad (3.59)$$

This is a direct product of two dimensional lattices $T^2/\mathbf{Z}_3$; therefore it is reducible. We can check that this action has a definite order 3 in view of (3.53). This fixes some parameters of G^{ij} as in the two dimensional case; however we have not done yet for inter-torus parameters, for example G^{13} or G^{14} , etc.

The Lefschetz fixed point theorem still gives that the total number of fixed points is $3^3 = 27$. The rotations in the 1–2, 3–4, and 5–6 tori are commutative because rotation generators are the Cartan generators of $SO(6)$. Namely, it is an *Abelian* orbifold.

Table 3.3. Possible four dimensional orbifolds with $\mathcal{N} = 2$ supersymmetry. It is a direct product of two dimensional orbifolds, whose tiling exists only for four cases, as shown in Fig. 3.8

Order	Lattice	ϕ	χ
2	half plane	$\frac{1}{2}(1\ 1)$	16
3	hexagon	$\frac{1}{3}(1\ 1)$	9
4	square	$\frac{1}{4}(1\ 1)$	1
6	triangle	$\frac{1}{6}(1\ 1)$	1

Holonomy

The orbifold naturally leads to the notion of holonomy group. From a given point of orbifold, we can transport a tangent vector along a closed curve back to the original point. Then, because of geometry, the vector rotates by some amount. A successive transformation rotates such vector further and these rotations form a group. The *holonomy* group is defined as a group containing all the possible rotations. Once geometry is determined, so is the holonomy group.⁴

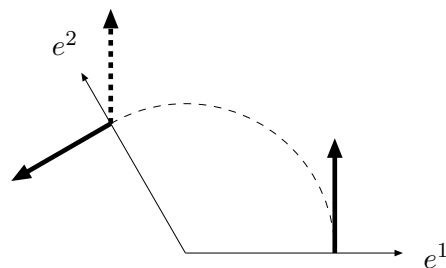


Fig. 3.9. The $\mathbf{Z}_N$ orbifold has a holonomy group $\mathbf{Z}_N$. Transporting a vector, we have an N -fold rotation as the effect of identification (*bold arrows*)

We see that the group $\mathbf{P}$ is a holonomy group. Consider the $\mathbf{Z}_N$ example. When we parallel transport vector once around a fixed point, it rotates by an angle $2\pi/N$ because of the identification. See Fig. 3.9. All the possible rotations form a discrete group $\mathbf{Z}_N$ generated by $2\pi/N$ rotation. Therefore,

⁴ The Levi-Civita transportation does not change the length, the “norm” of tangent vector. The name “holo-” means that the whole “holo-” group preserves the norm.

we can say that the $\mathbf{Z}_N$ orbifold has the holonomy group $\mathbf{Z}_N$. The converse is also true: orbifold can also be defined in terms of a discrete holonomy group $\mathbf{P}$.

The condition (3.58) can be translated to that the manifold should have a $\mathbf{Z}_N$ holonomy. In fact, we will see that as long as the holonomy group $\mathbf{P}$ belongs to $\mathrm{SU}(3)$, we preserve $\mathcal{N} = 1$ supersymmetry. All $\mathbf{Z}_N$ groups of Table 3.2 are subgroups of $\mathrm{SU}(3)$. They will be discussed along with the Calabi-Yau manifold.

3.2 Spinors

In this book, we need the spinor properties in higher dimensions extensively toward obtaining 4D chiral fermions. Therefore, let us define spinors in arbitrary dimensions and study their key properties related to our phenomenological needs.

3.2.1 Rotation and Vector

The group $\mathrm{SO}(4)$ is defined as four dimensional isotropic rotation. Its generator J_{ab} has the following components

$$[J_{ab}]_{jk} = -i(\delta_{aj}\delta_{bk} - \delta_{bj}\delta_{ak}) . \quad (3.60)$$

It has (ab) element $-i$ and (ba) element $+i$ and generates rotation around a - b plane. Exponentiating it we have the familiar form for the rotation matrix.

It satisfies the following commutation relation

$$[J_{ab}, J_{cd}] = i(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc}) . \quad (3.61)$$

It defines a *vector* v^j , which transforms as

$$v^j \rightarrow v'^j = [e^{J_{ab}\theta_{ab}}]^j_k v^k . \quad (3.62)$$

It is straightforward to extend the above 4D rotational symmetry to an arbitrary d dimensional rotation. Simply, the a, b running is extended to $1, \dots, d$ and the resulting group is called $\mathrm{SO}(d)$.

There is another important group similar to $\mathrm{SO}(d)$: the Lorentz group denoted as $\mathrm{SO}(1, d-1)$. In the Lorentz group, the definition of inner product is given by the metric

$$\eta_{ab} = \mathrm{diag}(+1, -1, -1, \dots, -1) . \quad (3.63)$$

Now, the angular momentum commutation relation becomes

$$[J_{ab}, J_{cd}] = i(\eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc}) . \quad (3.64)$$

In what follows, mostly we will deal with the Lorentz group since we are interested in the spacetime spinors, and we can switch to the rotation group $SO(d)$ if needed simply by replacing η s of (3.64) to δ s of (3.61).

Note that the index a of (3.64) runs the number of spacetime dimensions. The representation v in (3.62) is called the “vector representation” which has the dimension d .

3.2.2 Spinors in Arbitrary Dimensions

Another Set of Generators

There exists another kind of “rotational” generators satisfying the same commutation relation, (3.64). Consider the familiar case in four dimensions. Although the entire argument will be basis independent, we work in the *chiral basis* of gamma matrices of the standard textbooks [8]

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (3.65)$$

expressed in terms of the conventional Pauli matrices σ^i and the 2×2 identity matrix $\mathbf{1}$. If we define

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu], \quad (3.66)$$

$S_{\mu\nu}$ satisfies the commutation relation (3.64), replacing $J_{\mu\nu}$ by $S_{\mu\nu}$. Sometimes this group is referred to as $\text{Spin}(1,3)$ and it is a double covering group of $SO(1,3)$. Later we will extend it to an arbitrary dimensional Lorentz group $SO(1, d-1)$.

A *Dirac spinor* is defined to be the object transforming under the $SO(1, d-1)$ group as

$$\psi_\alpha \rightarrow \psi'_\alpha = [e^{S_{ab}\theta_{ab}}]_\alpha{}^\beta \psi_\beta. \quad (3.67)$$

The Dirac spinor is reducible under the transformation (3.67). Let us define the γ^5 matrix in 4D as⁵

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad (3.68)$$

where only the last equality is dependent on the choice of basis. The projections $\frac{1 \pm \gamma^5}{2}$ chooses only the upper or lower two-entries,

$$\psi_L = \frac{1}{2}(1 + \gamma^5)\psi, \quad \psi_R = \frac{1}{2}(1 - \gamma^5)\psi \quad (3.69)$$

which are named as “left-handed” and “right-handed”, respectively. One can check that γ^5 commutes with all the Lorentz group generators, $[\gamma^5, S^{\mu\nu}] = 0$.

⁵ This has the opposite sign from that of Bjorken and Drell [8], and the convention is called “particle physics convention” since the L-handed is $\frac{1+\gamma^5}{2}$.

Therefore, one can decompose the Dirac spinors into two *Weyl spinors* according to the eigenvalues of γ^5 . In other words, each ψ_L and ψ_R transform independently. These are the 4D Weyl spinors.

How can one generalize the 4D spinor to five dimensions? First, note that we need one more gamma matrix because one dimension is increased. The essential property of gamma matrices in d dimensions is the following anti-commutation relation known as the Dirac–Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}, \quad M \text{ and } N = 0, 1, 2, \dots, d-1. \quad (3.70)$$

Fortunately if we define five dimensional gamma matrices as

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma^4 = i\gamma^5 \quad (3.71)$$

they satisfy the relation (3.70). We adopt the convention that the upper case roman character $M = 0, \dots, 4$ is used for a higher dimensional index.⁶ In effect, we introduced five independent gamma matrices, each with a “vector” index, which satisfy the Dirac–Clifford algebra. These five gamma matrices define five dimensional spinors with which a 5D Dirac equation can be written.

The same extension cannot be applied to six dimensions because we have no independent sixth 4×4 gamma matrix. We should double the size of matrices and construct 6D gamma matrices from 4D gamma matrices,

$$\Gamma^\mu = \gamma^\mu \otimes \sigma^3, \quad \mu = 0, \dots, 3, \quad \Gamma^{d-2} = iI \otimes \sigma^1, \quad \Gamma^{d-1} = iI \otimes \sigma^2, \quad (3.72)$$

where γ^μ are the gamma matrices in 4 dimensions. For an even d , by direct products of gamma and Pauli matrices, we obtain d dimensional gamma matrices out of $d-2$ dimensional ones. Forming generators acting on spinors as done in (3.66), we extend the group to $\text{Spin}(1, d-1)$.

By a similarity transformation, we can choose the chiral basis in even- d dimensions. In the chiral basis, the chirality operator is defined as

$$\Gamma = -i^{(d-2)/2} \Gamma^0 \Gamma^1 \dots \Gamma^{d-1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \quad (3.73)$$

The Weyl spinors are the eigenstates of (3.73) with eigenvalues ± 1 , and the left- and right-handed Weyl spinors are given as in (3.69).

Spinorial Basis

It is most useful to work in the *spinorial basis* [9, 10]. We can regroup gamma matrices into ladder operators,

$$\begin{aligned} \Gamma^{0\pm} &= \frac{1}{2}(\Gamma^0 \pm \Gamma^1), \\ \Gamma^{a\pm} &= \frac{1}{2}(i\Gamma^{2a} \pm \Gamma^{2a+1}), \quad a = 1, \dots, (d-2)/2. \end{aligned} \quad (3.74)$$

⁶ Only in the next section, we name index $M = (\mu, 5) = 0, 1, 2, 3, 5$ to highlight the fifth dimension there.

Then the anti-commutation relation becomes

$$\{\Gamma^{\tilde{a}+}, \Gamma^{\tilde{b}-}\} = \delta^{\tilde{a}\tilde{b}}, \quad \tilde{a} = \{0, a\} \quad (3.75)$$

while the other anti-commutators vanish. So, we can represent it as a direct product of number- $\frac{d}{2}$ 2×2 matrices. Now, we have the direct product of number- $\frac{d}{2}$ spin- $\frac{1}{2}$ systems whose eigenstates are represented by a set of respective S_z eigenvalues

$$|s_0, s_1, \dots, s_{d/2-1}\rangle. \quad (3.76)$$

This is the standard method for the construction of spinor states. We can define the “lowest state” by the state annihilated by all the annihilation operators

$$\Gamma^{\tilde{a}-} | - \dots - \rangle = 0, \quad (3.77)$$

where $+$ and $-$ denote $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Now we can construct a tower of states by acting the creation operators on the “lowest state”

$$|s_0, s_1, \dots, s_{d/2-1}\rangle = \prod_{\tilde{a}=0}^{d/2-1} (\Gamma^{\tilde{a}+})^{s_{\tilde{a}}+\frac{1}{2}} | - \dots - \rangle. \quad (3.78)$$

Note that the eigenvalues of the chirality operator Γ in (3.73) are

$$2^{d/2} s_0 s_1 \dots s_{d/2-1}. \quad (3.79)$$

Therefore, the positive (negative) chirality representation has even (odd) number of $-\frac{1}{2}$ eigenvalues. The Lorentz transformation can be written simply as

$$|s_i\rangle \rightarrow e^{2\pi i \sum s_j \theta_j} |s_i\rangle, \quad (3.80)$$

where $\theta_j = \theta_{2j, 2j+1}$, with $j = 0, \dots, \frac{d}{2}-1$. For example, the chiral **16** of $\text{SO}(10)$ has irreducible representation⁷ with even (or odd depending on conventions) number of $-\frac{1}{2}$ s,

$$\begin{aligned} &|++++\rangle \\ &|--+++, \quad (\text{and 9 more permutations}) \\ &|---+ \rangle, \quad (\text{and 4 more permutations}). \end{aligned} \quad (3.81)$$

Majorana Spinors

In four dimensions, the charge conjugation matrix C satisfies

$$C^{-1} \gamma^\mu C = -\gamma^{\mu\top}, \quad (3.82)$$

and the charge conjugated spinor is $\psi^c \equiv C \bar{\psi}^\top = C \psi^*$. The charge conjugated spinor ψ^c satisfies the Dirac equation with the opposite charge from that of ψ .

⁷ The dimension of the representation is the same as that of $\text{SO}(1,9)$

Note that the 4D rotation matrix transforms as $C^{-1}S^{\mu\nu}C = -S^{\mu\nu\top}$. Now let us extend the charge conjugation matrix to arbitrary dimensions. First we note that the eigenvalue s_i is real. Therefore, the eigenvalues of the following operators are either real or pure imaginary,

$$\begin{aligned} C_1\Gamma^0 &= \Gamma^3\Gamma^5 \dots \Gamma^{d-1}, \\ C_2\Gamma^0 &= \Gamma C_1 \end{aligned} \quad (3.83)$$

where C_1 or C_2 is taken as the charge conjugation matrix. We check that

$$C^{-1}S^{MN}C = -S^{MN\top}, \quad (3.84)$$

with C being either C_1 or C_2 . From this, we see that spinors ψ and $C\bar{\psi}^\top$ transforms the same under the Lorentz group. On the other hand, the chirality matrix transforms as

$$C^{-1}\Gamma C = (-1)^{(d-2)/2}\Gamma^*, \quad (3.85)$$

For $d = 2 \pmod{4}$, the chirality of each Weyl representation is not changed under the charge conjugation, i.e. each Weyl field is its own conjugated field. For example, in these dimensions a left-handed Weyl spinor transforms again into a left-handed one under the charge conjugation. In detail, for a left-handed Weyl spinor we have $\Gamma\psi = \psi$, i.e. $\Gamma = 1$. Multiplying C on the complex conjugated relation, we obtain

$$C\Gamma^*\psi^* = C\psi^* \rightarrow C\Gamma^*C^{-1}C\psi^* = \Gamma C\psi^* = \Gamma\psi^c = \psi^c \quad (3.86)$$

where we used the relation (3.85) and the last equality of (3.86) shows that $\Gamma = 1$ for ψ^c also in $d = 2 \pmod{4}$. In these $d = 2 \pmod{4}$ dimensions, the chirality remains so strongly under charge conjugation and hence it appears that the anomaly cancelation needs more care. Especially, there is a potential danger of *gravitational anomalies*. On the other hand, for $d = 0 \pmod{4}$ each Weyl spinor is the charge-conjugated one of the other Weyl spinor.

The Majorana condition defines a Majorana spinor

$$\psi = \psi^c \rightarrow \psi = C\psi^* = C(C\psi^*)^* = CC^*\psi \quad (3.87)$$

whence we obtain the condition $CC^* = 1$ to have a Majorana spinor. Using the properties of gamma matrices, we obtain the following conditions in even dimensions.

$$C_1C_1^* = (-1)^{(d-2)d/8}, \quad C_2C_2^* = (-1)^{(d-2)(d-4)/8}. \quad (3.88)$$

In odd dimensions, we have $C^{-1}\Gamma^\mu C = (-1)^{(d-1)/2}\Gamma^{\mu\top}$ and they satisfy (3.84). Therefore, the Majorana condition can be consistently imposed in $d = 2, 3, 4, 8, 9 \pmod{8}$.

The above results are summarized in Table 3.4.

Table 3.4. Spinors in various dimensions. Same patterns are repeated by the period 8. A spinor in d dimensions has real components $2^{\lfloor d/2 \rfloor}$. Further Majorana and/or Weyl condition reduces the number of components by a half. This table is valid for $\text{SO}(1, d-1)$

Dim. d	Majorana	Weyl	Majorana-Weyl	Min. Rep.
2	yes	self	yes	1
3	yes			2
4	yes	complex		4
5				8
6		self		8
7				16
8	yes	complex		16
9	yes			16
10	yes	self	yes	16
11	yes			32
12	yes	complex		64

Sometimes we call the spinor pseudo-Majorana (example $d = 9$) because the charge conjugation matrix satisfy $C^{-1}\Gamma^M C = +\Gamma^{M\top}$. When the Majorana condition is not available (example $d = 5$), sometimes we introduce the symplectic Majorana condition between a pair of Dirac spinors (although the name “Dirac spinor” is misleading), $\psi_1^c = -\psi_2, \psi_2^c = \psi_1$, which is reasonable when the hypermultiplet is considered.

Invariants

By sandwiching gamma matrices between spinors, we form $\text{SO}(d)$ or $\text{SO}(1, d-1)$ tensor representations

$$\zeta^\top C^{-1} \Gamma^{\mu_1} \Gamma^{\mu_2} \dots \Gamma^{\mu_m} \chi, \quad (3.89)$$

where we note that $\zeta^\top C^{-1}$ transforms like $\bar{\zeta}$. It is reducible and decomposed into antisymmetric tensors. We will encounter these tensors when we consider the $\text{SO}(10)$ GUT Lagrangian and supersymmetry algebra related with D-branes.

It is easily understood that a gamma matrix converts two spinorial indices to one vectorial one

$$\Gamma_{\alpha\dot{\alpha}}^\mu \quad (3.90)$$

and the square of gamma matrix is proportional to the unit matrix

$$(\Gamma^\mu)^2 \propto 1 \quad (3.91)$$

in the spinor basis. Therefore, sandwiching gamma matrices between a pair of spinors with the same (opposite) chirality, only those with even (odd) numbers

of vector indices survive. As an exercise, note that $\overline{\mathbf{16}} \cdot \mathbf{16}$ coupling in $\text{SO}(10)$ contains $\zeta^\top C^{-1} \Gamma_{\mu_1 \mu_2 \mu_3} \chi$ transforming like a tensor of 120 dimensions and $\zeta^\top C^{-1} \Gamma_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \chi$ transforming like a tensor of 126 dimensions. These are useful to form a seesaw neutrino mass matrix in $\text{SO}(10)$.

We observe that multiples of gamma matrices

$$\Gamma^{\mu_1 \mu_2 \dots \mu_d} \equiv \Gamma^{[\mu_1} \Gamma^{\mu_2} \dots \Gamma^{\mu_d]} , \quad (3.92)$$

where the indices inside the square bracket is antisymmetrized, provide a complete set of bases for $2^{d/2} \times 2^{d/2}$ real matrices. We can verify it by counting the number of independent matrices of the form (3.92), $1 + \binom{d}{1} + \binom{d}{2} + \binom{d}{3} + \dots + \binom{d}{d} = 2^{d/2} \times 2^{d/2}$. A general $2^{d/2} \times 2^{d/2}$ matrix has the same number of independent elements and hence matrices (3.92) form a complete set. Matrices with the number of indices greater than $d/2$ in the set made of (3.92) are related to those with less indices by the relation

$$\Gamma^{\mu_1 \dots \mu_s} \Gamma = -\frac{i^{-k+s(s+1)}}{(d-s)!} \epsilon^{\mu_1 \dots \mu_d} \Gamma_{\mu_{s+1} \dots \mu_d} , \quad (3.93)$$

whose four dimensional counterpart is familiar. Taking Dirac spinors on both sides, we have decomposition

$$\mathbf{2}^{d/2} \times \mathbf{2}^{d/2} = [0] + [1] + \dots + [d] . \quad (3.94)$$

where $[m]$ denotes antisymmetric tensor with m indices. In even dimensions, for Weyl spinors with the same chirality, we follow a similar step to arrive at

$$\begin{aligned} \mathbf{2}^{d/2-1} \times \mathbf{2}^{d/2-1} &= [1] + [3] + \dots + [d/2], & d/2 \text{ odd}, \\ \mathbf{2}^{d/2-1} \times \mathbf{2}^{d/2-1} &= [0] + [2] + \dots + [d/2], & d/2 \text{ even}. \end{aligned} \quad (3.95)$$

For those with different chiralities,

$$\begin{aligned} \mathbf{2}^{d/2-1'} \times \mathbf{2}^{d/2-1} &= [1] + [3] + \dots + [d/2], & d/2 \text{ even}, \\ \mathbf{2}^{d/2-1'} \times \mathbf{2}^{d/2-1} &= [0] + [2] + \dots + [d/2], & d/2 \text{ odd}. \end{aligned} \quad (3.96)$$

Adding the numbers in (3.95) and (3.96), we obtain (3.94).

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Field Theoretic Orbifolds

If one starts from a $(4 + n)$ -dimensional spacetime, it is necessary to obtain an effective 4 dimensional (4D) theory by hiding the extra n space-like dimensions. The most widely used method is to make these extra dimensions so small that 4D experiments so far performed could not have proved their existence.

The study of orbifold is started from the dilemma that the simple torus compactification does not allow chiral fermions in 4D. So we need a manifold with a nontrivial geometry. In this regard, in Chap. 3 we have introduced various properties of compact orbifolds.

With the orbifolded n internal space, it is possible to have chiral fermions in an effective 4D low energy theory. Thus, recently the investigation of the extra-dimensional field theory with orbifold singularity got a great deal of interest to obtain 4D chiral fermions toward a realistic construction of the standard model (SM). Even though the orbifold started its appearance in physics in string theory long time ago, this topology can be studied at field theory level also and in fact understanding orbifold is easier in field theory. So, before presenting the orbifold study in string theory [1], we introduce the concept of orbifold in field theory first at least for an educational purpose.

From the viewpoint of low energy effective field theory, the presence of chiral fermions has enough merit to study orbifolds in field theory. Since Kawamura first introduced the embedding of the non-trivial boundary conditions of the orbifolds into the gauge space in point particle theory, there has been a lot of interesting works in this direction, the so-called (field theoretic) *orbifold-GUTs* [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The anomaly structure of these theories on singular manifold has been also analysed [14, 15, 16, 17, 18] and all the models considered above are anomaly-free.

We employ the five dimensional metric $\eta_{MN} = \text{diag}(+1, -1, -1, -1, -1)$, where we index the entire five dimensions (5D) by the upper case Latin characters and our four spacetime dimensions by Greek characters: $X^M = (x^\mu, x^5)$, $M = 0, 1, 2, 3, 5$. Conventionally, we parameterize the extra dimension

by $y \equiv x^5$ instead of x^4 to distinguish the compactified internal space from our spacetime x^μ .

4.1 Fields on Orbifolds

4.1.1 Scalar Fields

Simple Toroidal Compactification

We begin with a study on complex scalar fields living in a 5D space $X^M = (x^\mu, x^5)$. The five dimensional action is

$$S = \int d^5x \frac{1}{2} \partial_M \Phi^* \partial^M \Phi = \int d^4x dy \frac{1}{2} (\partial_\mu \Phi^* \partial^\mu \Phi + \partial_5 \Phi^* \partial^5 \Phi) . \quad (4.1)$$

Let us form a torus $T^1 = S^1$ by the identification of

$$y \rightarrow y + 2\pi R . \quad (4.2)$$

This breaks the Lorentz group $SO(1,4)$ down to $SO(1,3)$; thus the y dependence decouples. We make the mode expansion

$$\Phi(x, y) = \sum_{n=-\infty}^{\infty} \Phi^{(n)}(x^\mu) e^{iny/R} , \quad (4.3)$$

in terms of a complete set of functions on S^1 satisfying

$$\frac{1}{2\pi R} \int_0^{2\pi R} dy e^{iny/R} e^{-imy/R} = \delta_{nm} . \quad (4.4)$$

Note that if the scalar $\Phi(x, y)$ were real, the reality condition relates $\Phi^{(n)*}(x^\mu) = \Phi^{(-n)}(x^\mu)$. For a complex function, they are treated independent. Plugging the mode expansion back to the original action (4.1), and using the completeness relation (4.4), we obtain

$$S = \frac{2\pi R}{2} \int d^4x \sum_{n=-\infty}^{\infty} \left(\partial_\mu \Phi^{(n)*} \partial^\mu \Phi^{(n)} - \frac{n^2}{R^2} \Phi^{(n)*} \Phi^{(n)} \right) . \quad (4.5)$$

Let us redefine four dimensional fields as $\phi^{(n)}(x^\mu) = \sqrt{2\pi R} \Phi^{(n)}(x^\mu)$ to have the canonical normalization. As a result, we have a tower of scalar fields $\phi^{(n)}$ with the mass squared $(n/R)^2$. Note that we have the unique massless complex scalar $\phi^{(0)}$, called *zero mode*. In the small radius limit $R \rightarrow 0$, the nonzero modes become superheavy and they decouple. In the low energy limit, some lowest modes n below the interested energy scale $\Lambda (\gg n/R)$ are relevant.

Up to Symmetry of the Action

The invariance of the action under the translation (4.2) does not imply that the field itself is invariant under (4.2). For the field, it is sufficient for it to be invariant up to a phase under the symmetry of the action

$$\Phi(x, y + 2\pi R) \rightarrow e^{2\pi i R \varphi} \Phi(x, y) . \quad (4.6)$$

This is the symmetry transformation of complex scalar fields. Then, the mode expansion is affected,

$$\Phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \Phi^{(n)}(x^\mu) e^{i(n/R + \varphi)y} . \quad (4.7)$$

Thus the mass is shifted to

$$m^2 = \left(\frac{n}{R} + \varphi \right)^2 \quad (4.8)$$

If we can assign different phase φ to different fields, for example to bosons and fermions, the noninvariant field is projected out so that we can break supersymmetry [19]. We will see that since we have more degrees of freedom such as global or gauged symmetry, we have much freedom to choose the phase.

Introducing Orbifold

Now we introduce $S^1/\mathbf{Z}_2$ orbifold as discussed in Chap. 3, by identifying points on a circle under the action

$$\theta : y \rightarrow -y . \quad (4.9)$$

As shown in Fig. 4.1, it is convenient to think of identification of the opposing points with respect to the central hypothetical line connecting two points of

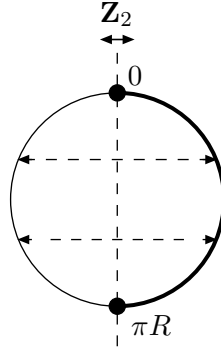


Fig. 4.1. $S^1/\mathbf{Z}_2$ orbifold. The fundamental region $[0, \pi R]$ is surrounded by fixed points (*bullets*)

$y = 0$ and $y = \pi R$. This identification forces the fields to be an eigenstate of θ

$$\Phi(x, -y) = \eta \Phi(x, y) , \quad (4.10)$$

where the prefactor η is an eigenvalue. It follows that $\eta = \pm 1$ from the consistency of $\mathbf{Z}_2$, $\eta^2 = 1$. According to their eigenvalues, we call the eigenstates $\Phi_+(x, y)$ and $\Phi_-(x, y)$. For the $\mathbf{Z}_2$ even and odd states, the mode expansions become

$$\begin{aligned} \Phi_+(x, y) &= \sum_{n=-\infty}^{\infty} \Phi_+^{(n)}(x^\mu) \cos \frac{ny}{R} \\ \Phi_-(x, y) &= \sum_{n=-\infty}^{\infty} \Phi_-^{(n)}(x^\mu) \sin \frac{ny}{R} \end{aligned} \quad (4.11)$$

Applying the 5D differential operator ∂^2 , we observe that the four dimensional field $\Phi_\pm^{(n)}(x^\mu)$ has the squared mass $(n/R)^2$. Note that the zero mode exists only for the even eigenfunction $\Phi_+^{(0)}(x)$. Because we see only lowest modes in the low energy limit, we can explain the absence of some fields, by assigning an appropriate parity η in (4.10).

Bulk and Branes

We call the five dimensional space as *bulk*. We sometimes assume a hypersurface at the fixed points of y which are called *branes*. We use the word p -brane for the p spatial-dimensional objects. In the example of Fig. 4.1, they are 3-branes because the spatial dimensions of $y = 0$ and $y = \pi R$ are three.

The word “brane” is quoted from membrane and originates from Dirichlet brane (D-brane) [20], which we will shortly review in Appendix B. It has the following important features

- *Charge*. An extended supersymmetry naturally possesses spatially extended objects with central charge.
- *Tension*. It has tension, positive or negative, thus acts as a source of gravity.
- *Field localization*. In string theory, an open string can end at the brane thus its low energy state looks as localized field.

A priori, orbifold fixed points do not have these properties but are made to mimic them by hand. There are some known mechanisms for localizing fields.¹

¹ Localization of gravity in the warped background is discussed in [21] and of gauge fields in [22].

4.1.2 Gauge Fields

Similar features obtained above are shared by gauge fields. Consider an Abelian gauge field A_M , whose action is given by

$$\begin{aligned} S &= -\frac{1}{4g_5^2} \int d^5x F_{MN} F^{MN} \\ &= -\frac{1}{4g_5^2} \int d^4x dy (F_{\mu\nu} F^{\mu\nu} + 2(\partial_\mu A_5 - \partial_5 A_\mu)^2) . \end{aligned} \quad (4.12)$$

The five dimensional gauge coupling g_5 has a dimension of $(\text{length})^{1/2}$. We compactify y as before $y \sim y + 2\pi R$. As a consequence of the broken 5D Lorentz symmetry, here A_5 is decoupled to become a scalar in the 4D effective theory,

$$\begin{aligned} A_\mu(x, y) &= \sum_{n=-\infty}^{\infty} A_\mu^{(n)}(x^\mu) e^{iny/R} \\ A_5(x, y) &= \sum_{n=-\infty}^{\infty} A_5^{(n)}(x^\mu) e^{iny/R} , \end{aligned} \quad (4.13)$$

with the Hermiticity condition $A_M^{(n)*} = A_M^{(-n)}$. Integrating out y , the first term simply becomes a tower of kinetic terms

$$-\frac{2\pi R}{4g_5^2} \int d^4x \sum_{n=-\infty}^{\infty} F_{\mu\nu}^{(n)} F^{(n)\mu\nu} \quad (4.14)$$

Note that the canonical normalization determines four dimensional coupling

$$g_4^2 = \frac{g_5^2}{2\pi R} .$$

The second term becomes

$$2\frac{2\pi R}{4g_5^2} \int d^4x \sum_{n=-\infty}^{\infty} \left[\partial_\mu A_5^{(n)} - i\frac{n}{R} A_\mu^{(n)} \right]^2 \quad (4.15)$$

from which we observe that

- The zero mode scalar ($\sim A_5^{(0)}$) survives with the correct sign for the kinetic energy term.
- The nonzero mode $n \neq 0$ gauge field absorbs a scalar via the Higgs mechanism $A_\mu^{(n)} \rightarrow A_\mu^{(n)} + i(R/n)\partial_\mu A_5^{(n)}$ to acquire the squared mass $(n/R)^2$.

This reasoning on the Abelian gauge fields can be readily generalized to non-abelian gauge fields. In the nonabelian case, note that we always have the charged scalar $A_5^{a(0)}$ with the gauge index a in order to transform as the adjoint representation. In what follows, we will use the familiar matrix notation for nonabelian gauge fields, $A_M \equiv A_M^a T^a$, $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$, etc.

Explicit Symmetry Breaking

As in the case of scalar field, a discrete symmetry introduced in orbifold forces the fields to become eigenstates of the associated action, and only the zero mode in the even eigenstates survives.

To be concrete, consider the $S^1/\mathbf{Z}_2$ orbifold again by the identification $\theta : y \sim -y$. Because the gauge field transforms as the adjoint representation, we associate the action θ with the symmetry action P (now it is matrix) as

$$A_\mu(x, -y) = P A_\mu(x, y) P^{-1}, \quad (4.16)$$

but note also that

$$A_5(x, -y) = -P A_5(x, y) P^{-1} \quad (4.17)$$

$$\phi(x, -y) = P \phi(x, y). \quad (4.18)$$

These symmetries are correlated, because under which the Lagrangian should be invariant. Taking care of the covariant derivative, the transformation property of A_5 should be the same as that of $\partial/\partial y$, while the transformation property of A_μ is the same as that of $\partial/\partial x^\mu$. Hence, the transformation properties of A_5 and A_μ are opposite under P .

As an example consider gauge symmetry breaking of $SU(3)$ down to $SU(2) \times U(1)$. The generators of $SU(3)$ are the Gell-Mann matrices λ^a . If we take $P = \text{diag}(1, 1, -1)$, under θ

$$\begin{aligned} \lambda^a &\rightarrow P \lambda^a P^{-1} = +\lambda^a, & a = 1, 2, 3 \text{ and } 8, \\ \lambda^{\hat{a}} &\rightarrow P \lambda^{\hat{a}} P^{-1} = -\lambda^{\hat{a}}, & \hat{a} = 4, 5, 6, 7 \end{aligned} \quad (4.19)$$

The generators λ_1, λ_2 , and λ_3 belong to the $SU(2)$ subgroup and λ_8 belongs to $U(1)$. The corresponding gauge fields $A_\mu^a \lambda^a$ (no summation) are expanded by even functions and have zero modes, providing generators of the unbroken symmetry $SU(2) \times U(1)$ in the low energy theory. Restating the relation (4.19), the unbroken generators satisfy

$$[A_\mu^a \lambda^a, P] = 0, \quad (\text{no summation}) \quad (4.20)$$

as in the breaking by an adjoint Higgs field. Note some features of this kind of symmetry breaking:

- This is an explicit symmetry breaking, by associating the action θ with the global symmetry P that we chose. Also by the Higgs mechanism, the massive fields $A_\mu^{(n)}$ absorbs scalar fields $\partial_\mu A_5^{(n)}$.
- By the relation (4.17), we always have scalar fields A_5 , the off-diagonal elements of A_5 (of which some linear combinations), transforming like $2(\mathbf{2}, \mathbf{1})$.

- From the relation (4.18), the parity for the scalar field ϕ under θ is determined. For example, for the representation **3** which is decomposed into **2** + **1** under $SU(2) \times U(1)$, the doublet **2** is even and hence survives. This relation will be used in the next section.
- We can have the same unbroken gauge group if we take P the opposite to the one discussed above, i.e. $P = \text{diag}(-1, -1, 1)$. In this case, the singlet **1** survives.

This $SU(3)$ was a candidate for unification of the electroweak symmetry $SU(2) \times U(1)$ but ruled out from wrong prediction of unified coupling. However if we consider extra dimensions, this is possibly revived [17, 23].

Symmetry Restoration at a Fixed Point

Another point to note is the symmetry behavior at the fixed point $y = 0$ and πR . The operator P breaks gauge symmetry. The matrix valued gauge field $A_M(x, y)$ has the following expansion in terms of small y

$$A_M(x, y) = A_M(x) + f_M(x)y + O(y^2) \quad (4.21)$$

where f_M is simply a function of x . At a fixed point with a finite y , $y = \pi R$, $A_M(x, y)$ is not invariant under the change $y \rightarrow -y$, i.e. under P . But at the fixed point $y = 0$, $A_M(x, y) = A_M(x)$ and the transformation under P leaves the $SU(3)$ symmetry intact. In a field theoretic orbifold, this is a general feature: at the origin $y = 0$ the gauge symmetry is unbroken. On the other hand, at the fixed point with a finite y the gauge symmetry is a broken one if P breaks the symmetry. It is schematically shown in Fig. 4.2. The effective gauge symmetry below the KK compactification is the common union of the symmetries at the fixed points ($SU(3)$ and G_{SM}). In the above example, the common intersection is the $SU(2) \times U(1)$. For the bulk, one may consider the bulk symmetry as $SU(3)$ since the symmetry is broken only by the boundary conditions. But, the massless gauge bosons in the bulk do not form a complete $SU(3)$ multiplet due to the boundary conditions, and hence one can consider the bulk symmetry as $SU(2) \times U(1)$. This example shows that one may find the effective 4D gauge symmetry by studying the massless gauge bosons in the bulk or by picking up the common union of the symmetries respected at the fixed points.

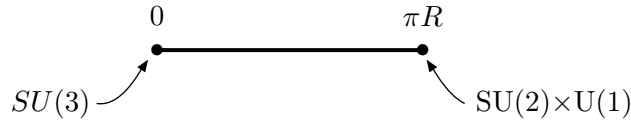


Fig. 4.2. The fundamental region of Fig. 4.1. Symmetries at the fixed points are shown

4.1.3 Chiral Fields

Five Dimensional Spinors

We briefly review the properties of five dimensional spinors along the line discussed in Sect. 3.2.2. The 5D gamma matrices are defined in terms of four dimensional ones $\Gamma^\mu = \gamma^\mu$

$$\Gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad (4.22)$$

with Pauli matrices σ^i and unit matrix $\mathbf{1}$. We use the 4D chirality operator γ^5 as the fifth gamma matrix in 5D

$$\Gamma^5 = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 = i\gamma^5 = \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix}. \quad (4.23)$$

They satisfy the 5D Dirac–Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}. \quad (4.24)$$

The minimal spinor in five dimension has eight real components, which is presented in Table 3.4. The action is

$$S = \int d^5x \bar{\Psi}(x, y) [i\Gamma^M (\partial_M + igA_M) - m(y)] \Psi(x, y), \quad (4.25)$$

where we retained the y dependence of mass term for a later use. The Lorentz invariant bilinears are the same as those in the four dimensional case. They include the Dirac mass term $\bar{\Psi}\Psi$. The symplectic Majorana mass term looks $\Psi_1^\top C_5 \Psi_2$ with the five dimensional charge conjugation matrix $C_5 = \Gamma^0 \Gamma^2 \Gamma^5$.

$\mathbf{Z}_2$ Action on Fermions

For the dimensional reduction, the important issue is to have a complete set of eigenstates in the y coordinate which is conveniently split into even and odd ones. For this purpose, we use the inversion ($\mathbf{Z}_2$) operator $\theta : y \rightarrow -y$, while leaving x^μ invariant. Consider the kinetic energy term first. For a Dirac spinor $\Psi(x, y)$, the inverted one $\Psi(x^\mu, -y)$ does not satisfy the Dirac equation. We should rearrange the components of the Dirac spinor so that the newly obtained spinor satisfies the Dirac equation. It is done by a matrix M satisfying

$$M\Gamma^5 M^{-1} = -\Gamma^5, \quad M\Gamma^\mu M^{-1} = \Gamma^\mu, \quad \text{or} \quad (4.26)$$

$$M\Gamma^5 M^{-1} = \Gamma^5, \quad M\Gamma^\mu M^{-1} = -\Gamma^\mu. \quad (4.27)$$

We see that for M proportional to Γ^5 these conditions are satisfied. Therefore, under the $\mathbf{Z}_2$ action, the associated projection condition leads to

$$\psi(x, -y) = \eta \Gamma^5 \psi(x, y) . \quad (4.28)$$

Because we will deal with the four dimensional effective theory, it is convenient to write them in terms of the Γ^5 eigenstates

$$\begin{aligned} \psi(x, -y)_L &= \eta \psi(x, y)_L , \\ \psi(x, -y)_R &= -\eta \psi(x, y)_R \end{aligned} \quad (4.29)$$

where $L(R)$ corresponds to $\Gamma^5 = 1(-1)$. When we extend this to nonabelian gauge symmetries, the Lagrangian is invariant only when

$$A_\mu(x, -y) = P A_\mu(x, y) P^{-1} , \quad (4.30)$$

$$\Psi(x, -y) = P \Gamma^5 \Psi(x, y) . \quad (4.31)$$

It follows that the *constant* mass term is not allowed by $\mathbf{Z}_2$, although the Lorentz invariance allows it. It is because under the inversion of y , the Dirac mass term acquires a minus sign,

$$\bar{\Psi}(x, -y) \Psi(x, -y) \rightarrow -\bar{\Psi}(x, y) \Gamma^{5\dagger} \Gamma^5 \Psi(x, y) = -\bar{\Psi}(x, y) \Psi(x, y) , \quad (4.32)$$

where we used the fact $\Gamma^{5\dagger} \Gamma^0 = -\Gamma^0 \Gamma^{5\dagger}$. Nevertheless, we can use a kink type mass

$$m(-y) = -m(y) \quad (4.33)$$

which might arise by a soliton background of scalar field or in models with warped geometry. If we neglect the x dependence, the solutions at the orbifold fixed points $y = 0, \pi R$ for *normalizable* fermion wave functions should have a *definite parity*. It is *localized* and used to mimic brane physics.

Mode Expansion

We may decompose fermion as

$$\Psi(x, y) = \sum_{n=-\infty}^{\infty} \Psi^{(n)}(x^\mu) e^{iny/R} . \quad (4.34)$$

But from the relation (4.28), we require

$$\begin{aligned} \Psi(x, -y) &= \sum_{m=-\infty}^{\infty} \Psi^{(m)}(x^\mu) e^{-imy/R} \\ &= \sum_{n=-\infty}^{\infty} \Psi^{(-n)}(x^\mu) e^{iny/R} = \eta \Gamma^5 \Psi(x, y) . \end{aligned} \quad (4.35)$$

Comparing term by term, we obtain

$$\Psi^{(-n)}(x^\mu) = \eta \Gamma^5 \Psi^{(n)}(x^\mu) , \quad (4.36)$$

thus each $\Psi^{(n)}$ is not a Dirac fermion. At this point, it is convenient to introduce chirality L and R ,

$$\Psi(x^\mu, y) = \sum_{n=0}^{\infty} \left(\Psi_L^{(n)}(x^\mu) \phi_L^{(n)}(y) + \Psi_R^{(n)}(x^\mu) \phi_R^{(n)}(y) \right) \quad (4.37)$$

with $\Psi_L = \frac{1}{2}(1 + \gamma^5)\Psi$ and $\Psi_R = \frac{1}{2}(1 - \gamma^5)\Psi$, where

$$\gamma^5 = i\Gamma^5. \quad (4.38)$$

The explicit form of the new harmonic functions are

$$\phi_L^{(n)}(y) = e^{iny/R} + \eta e^{-iny/R}, \quad \phi_R^{(n)}(y) = e^{iny/R} - \eta e^{-iny/R}, \quad (4.39)$$

which are not used but only the following relations are used

$$\frac{d}{dy} \phi_L^{(n)}(y) = +i \frac{n}{R} \phi_R^{(n)}(y), \quad (4.40)$$

$$\frac{d}{dy} \phi_R^{(n)}(y) = -i \frac{n}{R} \phi_L^{(n)}(y). \quad (4.41)$$

Note the sign difference. Evaluating ∂_5 , the sign difference is compensated by Γ^5 thus

$$i\Gamma^5 \partial_5 \Psi(x^\mu, y) = -i \frac{n}{R} \sum_{n=0}^{\infty} \left(\Psi_L^{(n)}(x^\mu) \phi_R^{(n)}(y) + \Psi_R^{(n)}(x^\mu) \phi_L^{(n)}(y) \right). \quad (4.42)$$

Thus, integrating out with respect to y , we have the following four dimensional action

$$S = 2\pi R \int d^4x \sum_{n=0}^{\infty} \bar{\psi}^{(n)}(x^\mu) \left(i\Gamma^\mu \partial_\mu - \frac{n}{R} \right) \psi^{(n)}(x^\mu), \quad (4.43)$$

with the usual redefinition $\psi^{(n)}(x^\mu) = \sqrt{2\pi R} \Psi^{(n)}(x^\mu)$. We have a tower of four dimensional Dirac spinors with masses n/R . In (4.43), we have the zero mode Dirac fermion, which contains a pair of left- and right-handed Weyl fermions; thus the theory is *not chiral* in the simple toroidal compactification.

Orbifold Leads to Chiral Theory

As before, consider the $S^1/\mathbf{Z}_2$ orbifold under identification by (4.9). The field is forced to be an eigenstate of the action (4.9), and the eigenvalues are opposite for left- and right-handed Weyl fields. Choosing $\eta = +1$, (4.29) restricts the left(right)-handed spinor to be the even(odd) eigenstate. Therefore, we have

$$\Psi(x, y) = \sum_{n=0}^{\infty} \Psi_L^{(n)}(x) \cos \frac{ny}{R} + \sum_{n=1}^{\infty} \Psi_R^{(n)}(x) \sin \frac{ny}{R}. \quad (4.44)$$

Comparing with the simple toroidal compactification result (4.37), we have zero mode only for the left handed spinor. Therefore, we obtain a *chiral theory* from the orbifold condition. This is due to the choice of the quantum number of the discrete operator, $\eta = +1$.

4.2 Supersymmetry in Five and Six Dimensions

4.2.1 Five Dimension

The minimal spinor in five dimension has eight real components. Thus, the supercharge of the minimal $\mathcal{N}_{5D} = 1$ supersymmetry in five dimension has the same Lorentz property. Furthermore, it gives $\mathcal{N} = 2$ supersymmetry in four dimensions.

In 5D global supersymmetric theory, two supermultiplets are important:

- *vector multiplet* composed of one vector multiplet and one chiral multiplet of $\mathcal{N} = 1$,

$$V = \{ v_v = (A_\mu, \lambda_2), v_c = (A_5 + i\Sigma, \lambda_1) \}, \quad (4.45)$$

where we use the symplectic Majorana notation $\lambda^i \equiv (\lambda_L^i, \epsilon^{ij} \bar{\lambda}_{Lj})^\top$. We observed earlier that A_5 is decoupled from A_μ in 4D. It forms a complex field with Σ .

- *hypermultiplet* composed of two $\mathcal{N} = 1$ chiral multiplets, for each being charge conjugate to the other,

$$H = \{ h_r = (\phi_1, \psi_L)_r, \hat{h}_{\bar{r}} = (\phi_2, \bar{\psi}_R)_{\bar{r}} \}. \quad (4.46)$$

These are shown in Fig. 4.3. There are two directions for supersymmetry transformation which are denoted as real lines and dashed lines, respectively.

Since the supersymmetry generators commute with all the other symmetries of Lagrangian, we expect that the transformation properties are the same for the components of a supermultiplet. But we note from (4.29) that the transformation is different for the fields with the opposite chirality. Since an $\mathcal{N} = 1$ multiplet has a definite chirality, we conclude that *in terms of $\mathcal{N} = 1$, all component fields in the multiplet have the same transformation properties, including the project action P for the orbifold.*

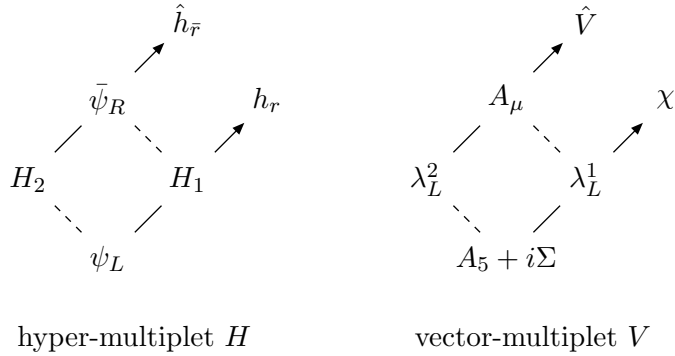


Fig. 4.3. Decomposition of $\mathcal{N} = 2$ multiplets into $\mathcal{N} = 1$ multiplets

There are more relations. The $\mathcal{N} = 2$ supersymmetry without central charge has an R symmetry $SU(2)_R$. Its subgroup is the exchanges of $\psi_L \rightarrow -\psi_R$, $\psi_R \rightarrow \psi_L$ and the same for λ_L and λ_R . So the projection acts as

$$\begin{aligned} h &= Ph \\ \hat{h} &= -P\hat{h} \\ v_v &= Pv_v P^{-1} \\ v_c &= -Pv_c P^{-1}, \end{aligned} \tag{4.47}$$

in notation of (4.45, 4.46). Therefore, all component fields in $\mathcal{N} = 2$ multiplets have definite transformation properties which include the parity. Componentwise they have the form

$$\begin{aligned} \phi_1(x, -y) &= P\phi_1(x, y) \\ \psi_L(x, -y) &= P\psi_L(x, y) \\ \phi_2(x, -y) &= -P\phi_2(x, y) \\ \psi_R(x, -y) &= -P\psi_R(x, y) \\ A_\mu(x, -y) &= PA_\mu(x, y)P^{-1} \\ \lambda_2(x, -y) &= P\lambda_2(x, y)P^{-1} \\ A_5(x, -y) &= -PA_5(x, y)P^{-1} \\ \Sigma(x, -y) &= -P\Sigma(x, y)P^{-1} \\ \lambda_1(x, -y) &= -P\lambda_1(x, y)P^{-1} \end{aligned} \tag{4.48}$$

The Lagrangian will be constructed soon. For more discussion, see [24].

Note that either supercharge of $\mathcal{N} = 2$ generator has the opposite chirality to the other. Seen in both equations above, any projection P acts differently on each supercharge because of the relation (4.29): it inevitably *breaks a half of supersymmetry*. In four dimension, only vector and chiral multiplets with +1 eigenvalue survive, thus we have $\mathcal{N} = 1$ supersymmetry. It is an explicit symmetry breaking: there is no asymptotic limit of symmetry restoration.

4.2.2 Six Dimension

We briefly comment on some properties of six dimensional supersymmetries for later use. Mostly the unique properties rely on the nature of spinors in six dimensions, i.e. the result of Subsect. 3.2.2.

- The minimal spinor, a six dimensional Weyl spinor, is the same as four dimensional Dirac. Thus we have the same minimal supersymmetry as in five dimensional case.
- We can similarly define chirality operator $\Gamma = i \prod_{M=0}^5 \Gamma^M$. Under charge conjugation, the chirality does not change. The supersymmetric charge is also chiral. When it comes to the number of supersymmetries, we should

refer to the chirality. The minimal supersymmetry is called $(1,0)$ supersymmetry.

- Gaugino and gravitino are chiral and its charge conjugation cannot transform into the ones with the opposite chirality, thus the theory has a potential of having chiral and gravitational anomalies. Especially, gauginos contribute to chiral anomaly.

4.3 Realistic GUT Models

GUTs in extra dimensions may shed light on some difficult issues of GUTs when they are compactified to a 4D SM. In this section, we discuss some attempts on extra dimensional GUTs. The well-known GUT groups $SU(5)$ and $SO(10)$ will be discussed in Subsect. 4.3.1 for 5D and Subsect. 4.3.2 for 6D, respectively, not because these groups depend on the number of dimensions but because we want to introduce two GUTs in an economical way. The dimension restricts the GUT group representations only for canceling the gauge anomalies.

4.3.1 $SU(5)$ GUT in Five Dimension

A More $\mathbf{Z}_2$ Action

We introduce $S^1/(\mathbf{Z}_2 \times \mathbf{Z}'_2)$ orbifold, studied in Subsect. 3.1.4. It is obtained by moding out S^1 by two $\mathbf{Z}_2$ actions

$$g \in \mathbf{Z}_2 : y \rightarrow -y , \quad (4.49)$$

$$h \in \mathbf{Z}'_2 : y' \rightarrow -y' . \quad (4.50)$$

where we defined $y' = \frac{\pi R}{2} - y$, and h is also reads as $y \rightarrow \pi R - y$. This situation is drawn in Fig. 4.4. It can be checked that the two actions commute; thus we have the consistency condition for the associated actions P and P' to commute.

When a more complicated orbifold is introduced, there are more associated projections so that more symmetries can be broken. In general, any bulk field $\phi(x, y)$ can have components with the intrinsic $\mathbf{Z}_2 \times \mathbf{Z}'_2$ parities of $(++)$, $(+-)$, $(-+)$ and $(--)$. Denoting those fields as ϕ_{++} , ϕ_{+-} , ϕ_{-+} and ϕ_{--} , they have mode expansions in terms of cosines and sines with the following $\mathbf{Z}_2 \times \mathbf{Z}'_2$ parities [2]

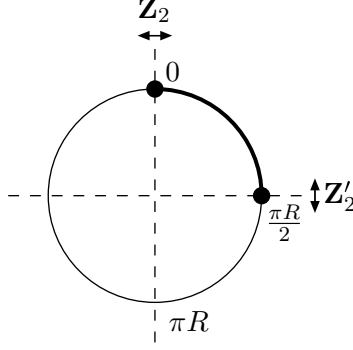


Fig. 4.4. $S^1/(\mathbf{Z}_2 \times \mathbf{Z}'_2)$ orbifold. Fixed points at $y = 0, \frac{\pi R}{2}$ are denoted by bullets. The thick arc is the fundamental region

$$\phi_{++}(x, y) = \sum_{n=0}^{\infty} \phi_{++}^{(2n)}(x^\mu) \cos \frac{2ny}{R}, \quad (4.51)$$

$$\phi_{+-}(x, y) = \sum_{n=0}^{\infty} \phi_{+-}^{(2n+1)}(x^\mu) \cos \frac{(2n+1)y}{R}, \quad (4.52)$$

$$\phi_{-+}(x, y) = \sum_{n=0}^{\infty} \phi_{-+}^{(2n+1)}(x^\mu) \sin \frac{(2n+1)y}{R}, \quad (4.53)$$

$$\phi_{--}(x, y) = \sum_{n=0}^{\infty} \phi_{--}^{(2n+2)}(x^\mu) \sin \frac{(2n+2)y}{R}. \quad (4.54)$$

Note that only ϕ_{++} can have a zero mode, allowing a massless 4D field in the effective low energy theory. It survived from both of projections. In $\mathbf{Z}_2 \times \mathbf{Z}'_2$ orbifold, the resulting zero modes are less than in the $\mathbf{Z}_2$ case.

For $\mathbf{Z}'_2$ being a reflection in the fifth coordinate, the fermionic bulk fields should satisfy the second boundary conditions similar to (4.10), with different overall factor η' :

$$\mathbf{Z}'_2 : \psi(x, y') \longrightarrow \eta' \psi(x, -y'), \quad \psi^c(x, y') \longrightarrow -\eta' \psi^c(x, -y'). \quad (4.55)$$

In order to obtain a chiral theory, one $\mathbf{Z}_2$ was enough as discussed above. Now, we can use the additional $\mathbf{Z}'_2$ in breaking another continuous symmetry of the system, especially the nonabelian gauge symmetry [7].

Field Contents

The 5D $\mathcal{N} = 1$ supersymmetry was reviewed in the previous section. Here, we introduce the minimal content of the supersymmetric SU(5) GUT [25] in 5D.

- One vector multiplet V in the bulk, transforming as **24** of SU(5)

$$V = \{(A_\mu, \lambda^2), (A_5 + i\Sigma, \lambda^1)\} , \quad (4.56)$$

for gauge bosons,

- Two hypermultiplets $H^s (s = 1, 2)$ in the bulk, transforming as $\mathbf{5}$ and $\bar{\mathbf{5}}$

$$\begin{aligned} H^{(1)} &= \{H_5 \equiv (H_1^{(1)}, \psi_L^{(1)}), \hat{H}_{\bar{5}} = (H_2^{(1)}, \bar{\psi}_R^{(1)})\} \\ H^{(2)} &= \{\hat{H}_5 \equiv (H_1^{(2)}, \psi_L^{(2)}), H_{\bar{5}} = (H_2^{(2)}, \bar{\psi}_R^{(2)})\} , \end{aligned} \quad (4.57)$$

for Higgs bosons,

- Three generations of matter multiplets $\Phi_{\bar{\mathbf{5}}} + \Phi_{\mathbf{10}}$ at the fixed point $y = 0$, transforming as $\bar{\mathbf{5}} + \mathbf{10}$.

Note that the gauge symmetry at the fixed point $y = 0$ is $SU(5)$ as commented before. We chose such special setting to solve the doublet-triplet splitting problem addressed in Subsect. 2.3.3. The projection associated with orbifold imposes boundary conditions and hence some bulk fields are projected out at low energy. We will design such projections to remove unwanted fields. The matter fields located at the fixed point(s) remain intact. The gauge invariant action is given by

$$S = \int d^5x \mathcal{L}^{(5)} + \frac{1}{2} \int d^5x \delta(y) \mathcal{L}^{(4)} \quad (4.58)$$

where

$$\begin{aligned} \mathcal{L}^{(5)} &= \mathcal{L}_{YM}^{(5)} + \mathcal{L}_H^{(5)} , \\ \mathcal{L}_{YM}^{(5)} &= -\frac{1}{2} \text{Tr} F_{MN}^2 + \text{Tr} |D_M \Sigma|^2 + \text{Tr} (i \bar{\lambda}_i \gamma^M D_M \lambda^i) - \text{Tr} (\bar{\lambda}_i [\Sigma, \lambda^i]) , \\ \mathcal{L}_H^{(5)} &= |D_M H_i^{(s)}|^2 + i \bar{\psi}_{(s)} \gamma^M D_M \psi^{(s)} - (i \sqrt{2} g_{(5)} \bar{\psi}_{(s)} \lambda^i H_i^{(s)} + \text{h.c.}) \\ &\quad - \bar{\psi}_{(s)} \Sigma \psi^{(s)} - H_{(s)}^{\dagger i} \Sigma^2 H_i^{(s)} - \frac{1}{2} g_{(5)}^2 \sum_{m,A} (H_{(s)}^{\dagger i} (\sigma^m)_i^j T^A H_j^{(s)})^2 , \end{aligned} \quad (4.59)$$

$$\begin{aligned} \mathcal{L}^{(4)} &\equiv \sum_{3 \text{ families}} \int d^2 \bar{\theta} d^2 \theta \left(\Psi_{\bar{5}}^\dagger e^{2g_{(5)} V^A T^A} \Psi_{\bar{5}} + \Psi_{\mathbf{10}}^\dagger e^{2g_{(5)} V^A T^A} \Psi_{\mathbf{10}} \right) \\ &\quad + \sum_{3 \text{ families}} \int d^2 \theta \left(f_{U(5)} H_5 \Phi_{(10)} \Phi_{(10)} + \hat{f}_{U(5)} \hat{H}_5 \Phi_{(10)} \Phi_{(10)} \right. \\ &\quad \left. + f_{D(5)} H_{\bar{5}} \Phi_{(\bar{5})} \Phi_{(10)} + \hat{f}_{D(5)} \hat{H}_{\bar{5}} \Phi_{(\bar{5})} \Phi_{(10)} \right) + \text{h.c.} , \end{aligned}$$

where $\lambda^i = (\lambda_L^i, \epsilon^{ij} \bar{\lambda}_{Lj})^T$, $D_M \equiv \partial - ig_{(5)} A_M(x^\mu, y)$, $g_{(5)}$ is a 5D gauge coupling constant, σ^m are Pauli matrices, the T^A are $SU(5)$ generators, $V^A T^A$ is an $SU(5)$ vector supermultiplet.

Projections

As studied before, if we determine the transformation property P for one field, the property for the rest fields are completely fixed by the invariance property of the action, as shown in (4.48).

We take $S^1/(\mathbf{Z}_2 \times \mathbf{Z}'_2)$ orbifold in the y direction and associate two actions with gauge space projections P and P' ,

$$P = \text{diag}(1, 1, 1, 1, 1) , \quad (4.60)$$

$$P' = \text{diag}(-1, -1, -1, 1, 1) . \quad (4.61)$$

The $SU(5)$ gauge symmetry is broken down to that of the standard model (SM) gauge group, $G_{\text{SM}} = SU(3) \times SU(2) \times U(1)$. As seen in (4.19) the boundary condition acts differently on the $SU(5)$ generators $T^A (A = 1, 2, \dots, 24)$,

$$P' T^a P'^{-1} = T^a, \quad T^a \in G_{\text{SM}} \quad (4.62)$$

$$P' T^{\hat{a}} P'^{-1} = -T^{\hat{a}}, \quad T^{\hat{a}} \in SU(5)/G_{\text{SM}} . \quad (4.63)$$

In Table 4.1, we list the parity assignments and the mass spectrum of the KK modes of the bulk fields. Each Higgs multiplet in $H_5(\hat{H}_5, \hat{H}_5, H_5)$ is divided into the $SU(3)$ -color triplet $H_C(\hat{H}_C, \hat{H}_C, H_C)$ and the $SU(2)$ -weak doublet $H_u(\hat{H}_d, \hat{H}_u, H_d)$. Note that only H_u and H_d have zero modes. All

Table 4.1. The $\mathbf{Z}_2 \times \mathbf{Z}'_2$ parities and the KK masses, in units of $\frac{1}{R}$, of the orbifolded $SU(5)$ bulk fields

4D Fields	Quantum Numbers	$\mathbf{Z}_2 \times \mathbf{Z}'_2$	Mass
$A_\mu^{a(2n)}, \lambda^{2a(2n)}$	$(\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1})$	$(+, +)$	$2n$
$A_\mu^{\hat{a}(2n+1)}, \lambda^{2\hat{a}(2n+1)}$	$(\mathbf{3}, \mathbf{2}) + (\bar{\mathbf{3}}, \mathbf{2})$	$(+, -)$	$2n + 1$
$A_5^{a(2n+2)}, \Sigma^{a(2n+2)}, \lambda^{1a(2n+2)}$	$(\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1})$	$(-, -)$	$2n + 2$
$A_5^{\hat{a}(2n+1)}, \Sigma^{\hat{a}(2n+1)}, \lambda^{1\hat{a}(2n+1)}$	$(\mathbf{3}, \mathbf{2}) + (\bar{\mathbf{3}}, \mathbf{2})$	$(-, +)$	$2n + 1$
$H_C^{(2n+1)}$	$(\mathbf{3}, \mathbf{1})$	$(+, -)$	$2n + 1$
$H_u^{(2n)}$	$(\mathbf{1}, \mathbf{2})$	$(+, +)$	$2n$
$\hat{H}_C^{(2n+1)}$	$(\bar{\mathbf{3}}, \mathbf{1})$	$(-, +)$	$2n + 1$
$\hat{H}_d^{(2n+2)}$	$(\mathbf{1}, \mathbf{2})$	$(-, -)$	$2n + 2$
$\hat{H}_C^{(2n+1)}$	$(\mathbf{3}, \mathbf{1})$	$(-, +)$	$2n + 1$
$\hat{H}_u^{(2n)}$	$(\mathbf{1}, \mathbf{2})$	$(-, -)$	$2n + 2$
$H_C^{(2n+1)}$	$(\bar{\mathbf{3}}, \mathbf{1})$	$(+, -)$	$2n + 1$
$H_d^{(2n)}$	$(\mathbf{1}, \mathbf{2})$	$(+, +)$	$2n$

the color triplet fields have masses of order the KK scale, $\sim 1/R$. Thus the doublet-triplet splitting problem of $SU(5)$ is nicely resolved by assigning the boundary conditions given in (4.48).

Let us scrutinize the roles of two projection P and P' associated with $\mathbf{Z}_2$ and $\mathbf{Z}'_2$ symmetries. Why do we need two discrete symmetries? As seen in Subsect. 4.1.2, P' alone might break gauge symmetry down to SM gauge group G_{SM} , as seen in the $\mathbf{Z}'_2$ parity in Table 4.1. It also breaks a half of the supersymmetries because it chooses only fermions of one chirality in each supermultiplet, for example choosing λ^{2a} but not $\lambda^{2\bar{a}}$. Clearly this is not sufficient, because the unwanted fields still persist, for example charged scalars $A_\mu^{\hat{a}}(\mathbf{3}, \mathbf{2})$ and triplet Higgs $\hat{H}_C(\bar{\mathbf{3}}, \mathbf{1})$ and so on, which are not observed and possibly mediate rapid proton decay. Therefore, we need another projection P to make them heavy.

As discussed at the end of Subsect. 4.2.1, each projection P or P' breaks a half of $\mathcal{N} = 2$ supersymmetries down to $\mathcal{N} = 1$. Although each P and P' breaks a half of $\mathcal{N} = 2$ supersymmetries down to $\mathcal{N} = 1$, their common intersection is also $\mathcal{N} = 1$. It is also observed that the full $SU(5)$ group remains intact at the origin $y = 0$, as seen before, because the projection is not acted at this point.

4.3.2 $SO(10)$ GUT in Six Dimension

In this subsection we discuss the next simplest GUT $SO(10)$ in 6D [8, 13, 18]. One interesting feature of 6D $SO(10)$ is that each fixed point respects different gauge groups. In this case the low energy effective theory is the common intersection of the groups respected at each fixed point. This is the generalization of the symmetry breaking we discussed in the previous subsection and has more complex structure. This kind of the common intersection as the gauge group appears in string orbifold also.

Group theoretically, the $SO(10)$ has some merits over the $SU(5)$ except not being the minimal one. Firstly, the fifteen chiral fields are put in a single representation $\mathbf{16}$ together with an $SU(5)$ singlet neutrino and realizes our *theme of unification*. Second, since it contains an $SU(5)$ singlet neutrino in the spinor $\mathbf{16}$ of $SO(10)$, it is possible to introduce small Majorana neutrino masses through the see-saw mechanism [26]. Of course, one can introduce $SU(5)$ singlets in the $SU(5)$ GUT and introduce a similar see-saw mechanism, but in the $SO(10)$ GUT the see-saw neutrino mass is related to other couplings dictated from the $SO(10)$ symmetry. Third, because the top and bottom quarks are put in the same representation $\mathbf{16}$, it is possible to relate their masses, i.e. the so-called top-bottom unification is possible. Thus, it seems that the $SO(10)$ GUT has its own merit to study [27, 28].

We go beyond five dimension for the following reasons. With a single projection P we can break $SO(10)$ down to one of its maximal groups only, thus cannot directly go to G_{SM} . We need at least two projections. As commented before, the discrete group on a circle can be at most $\mathbf{Z}_2 \times \mathbf{Z}'_2$, as shown in

Subsect. 3.1.4. If we need the doublet-triplet splitting from orbifold, one needs more projections: it is necessary to go beyond five dimension. Moreover, if we use the second $\mathbf{Z}_2$ action to break gauge group further, there remain unwanted massless fields from A_5 components of the vector multiplet [18]. These do not form a complete representation of GUT groups. Thus the gauge coupling unification may not be accomplished. To break $\text{SO}(10)$ down to $G_{\text{SM}}(\times \text{U}(1))$ just by orbifolding, we need to go at least to 6D.

In addition, two extra dimensional compactification shows the essential features of compactification in still higher dimensions. In the following chapters on the string compactification, we encounter more internal dimensions to be compactified. These features include the localization of gauge groups and matter spectra at fixed points due to the presence of Wilson lines.

Subgroups of $\text{SO}(10)$

Before presenting a full orbifold model, let us recapitulate the group theoretical aspects of $\text{SO}(10)$. As pointed out in Sect. 2.2, the interesting rank 5 subgroups of $\text{SO}(10)$ are

- (i) $\text{SU}(5) \times \text{U}(1)$, where $\text{SU}(5)$ is the Georgi–Glashow (GG) group [25]
- (ii) $\text{SU}(4)_c \times \text{SU}(2)_L \times \text{SU}(2)_R$, the Pati–Salam (PS) group [29]
- (iii) $\text{SU}(5)' \times \text{U}(1)_X$, the flipped $\text{SU}(5)$ (F- $\text{SU}(5)$) group [30].

Of course, the differences arise in the way of embedding matter contents. In Table 4.2 the sixteen chiral fields are classified under these three cases. All these groups can be obtained when one nontrivial $\mathbf{Z}_2$ boundary conditions are imposed.

Table 4.2. The chiral fields are L-handed. Note that the discrete $\mathbf{Z}_2$ element of $\text{SU}(2)_R$ exchanges up and down type quarks and leptons, thus this relates GG- $\text{SU}(5)$ and F- $\text{SU}(5)$

G_{SM}	fields	GG- $\text{SU}(5)$	PS 422	F- $\text{SU}(5)$
q	$(\mathbf{3}, \mathbf{2})$	$\mathbf{10}_F$	$(\mathbf{4}, \mathbf{2}, \mathbf{1})$	$\mathbf{10}_F$
u^c	$(\bar{\mathbf{3}}, \mathbf{1})$	$\mathbf{10}_F$	$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$	$\bar{\mathbf{5}}_F$
d^c	$(\mathbf{3}, \mathbf{1})$	$\bar{\mathbf{5}}_F$	$(\mathbf{4}, \mathbf{1}, \mathbf{2})$	$\mathbf{10}_F$
l	$(\mathbf{1}, \mathbf{2})$	$\bar{\mathbf{5}}_F$	$(\mathbf{4}, \mathbf{2}, \mathbf{1})$	$\bar{\mathbf{5}}_F$
e^c	$(\mathbf{1}, \mathbf{1})$	$\mathbf{10}_F$	$(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$	$\mathbf{1}_F$
N	$(\mathbf{1}, \mathbf{1})$	$\mathbf{1}_F$	$(\mathbf{4}, \mathbf{1}, \mathbf{2})$	$\mathbf{10}_F$

Since G_{SM} is a subgroup of each Case (i), (ii), and (iii), G_{SM} is a common union of them. Consider for example, the quark doublet q and the lepton doublet l . Both of these complete the PS representation $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ under the PS group. However, they belong to two different representations under the

GG SU(5). So, we must split $\mathbf{10}$ and $\bar{\mathbf{5}}_F$ so that q and l themselves become a complete representation. It is most easily achieved from chopping off $\bar{\mathbf{5}}_F$ so that l is split. Then, $\mathbf{10}$ is also split to produce q . When we chop off $\bar{\mathbf{5}}_F$ into $3 + 2$, the unbroken group is G_{SM} . For the part of the PS group, the fourth color is separated from the remaining three colors to produce q and l . This means that the common intersection of the $SU(5)_{\text{GG}} \times U(1)$ and the PS group is $G_{\text{SM}} \times U(1)$. In this way, one can confirm that the common intersection of any two columns of Table 4.2 contains G_{SM} . If we considered the PS group and the F-SU(5) group, then the common intersection is again $G_{\text{SM}} \times U(1)$. Similarly, the common intersection of $SU(5)_{\text{GG}} \times U(1)$ and the F-SU(5) is $G_{\text{SM}} \times U(1)$.

The subgroup structure of the SO(10) can be understood more clearly when we classify the 45 generators T^a [8, 18]. They are represented by *imaginary and antisymmetric (thus Hermitian) 10×10 matrices*. To deal with SU(5) subgroup, the standard convention is to embed U(n) group into SO($2n$). Then, it is convenient to write these imaginary and antisymmetric generators as direct products of 2×2 and 5×5 matrices, giving

$$SO(10) : \sigma_0 \otimes A_5, \sigma_1 \otimes A_5, \sigma_2 \otimes S_5, \sigma_3 \otimes A_5 . \quad (4.64)$$

Here σ_0 and $\sigma_{1,2,3}$ are the 2×2 unit matrix and the Pauli matrices; S_n and A_n are real and symmetric $n \times n$ matrices, and imaginary and antisymmetric $n \times n$ matrices, respectively. It is easily checked that the surviving generators are fifteen S_5 s and ten A_5 s. The U(5) subgroup of SO(10) is then generated by

$$U(5) : \sigma_0 \otimes A_5, \sigma_2 \otimes S_5 \quad (4.65)$$

whose total number is 25 the number of U(5) generators. Excluding U(1) generator $\sigma_2 \otimes I_5$, the rest form the generators of SU(5), which are traceless under this convention.

With one of the following projections P_i we can break SO(10) to one of them²

$$P_{\text{GG}} \equiv \sigma_2 \otimes I_5 , \quad (4.66)$$

$$P_{\text{F}} \equiv \sigma_2 \otimes \text{diag}(1, 1, 1, -1, -1) , \quad (4.67)$$

$$P_{\text{PS}} \equiv \sigma_0 \otimes \text{diag}(1, 1, 1, -1, -1) . \quad (4.68)$$

The unbroken generators satisfy the condition (4.20)

$$[T^a, P] = 0 . \quad (4.69)$$

We assigned the name of the projectors according to the resulting subgroup. In what follows we treat each subgroup separately.

² In [18], P_{GG} , P_{F} and P_{PS} are represented as P_2 , P_3 and P_4 , respectively.

Georgi–Glashow SU(5)

The above choice of SU(5) embedding is the Georgi–Glashow type, because in the conventional basis, the hypercharge generator $Y \propto T^{24}$ is diagonal and its eigenvalues are proportional to that of $\bar{\mathbf{5}}$ of GG-SU(5) embedding.

By embedding the SM gauge group into this U(5), we can divide the 5×5 matrix further by choosing the first three indices 1,2,3 for the $SU(3)_c$ and the last two indices 4,5 for the $SU(2)_L$. Then, A_3, S_3, A_2 , and S_2 contain the SM group generators. The total number of these are 13 out of which the identity generator is not belonging to the SM gauge group. The remaining 12 generators are those of the SM. Now, let us denote the left-over pieces of A_5 and S_5 as A_X and S_X . Then, the generators of the Georgi–Glashow $SU(5)_{GG} \times U(1)$ subgroup can be grouped as

$$SU(5)_{GG} \times U(1) : \frac{\sigma_0 \otimes A_3, \sigma_0 \otimes A_2, \sigma_0 \otimes A_X}{\sigma_2 \otimes S_3, \sigma_2 \otimes S_2, \sigma_2 \otimes S_X}. \quad (4.70)$$

The total number of generators in (4.70) is 25.

Pati–Salam SU(4) × SU(2) × SU(2)

It is easy to understand the structures of projectors P_{PS} . Note that the PS group is obtained as

$$SO(10) \rightarrow SO(6) \times SO(4) \simeq SU(4) \times SU(2) \times SU(2).$$

It means that the SO(10) generators are partitioned into blocks of dimensions 6 and 4 and the diagonal blocks survived, i.e. the $SO(6) \times SO(4)$ generators remain, which is equivalent to the PS group. The P_{PS} of (4.68) can be denoted as $I_6 \otimes (-I_4)$ and yield the ones we want, by a similar relation as (4.19). The resulting generators of the Pati–Salam group $SU(4)_c \times SU(2)_L \times SU(2)_R$ are given by

$$PS_{422} : \frac{(\sigma_0, \sigma_1, \sigma_3) \otimes A_3, \sigma_2 \otimes S_3}{(\sigma_0, \sigma_1, \sigma_3) \otimes A_2, \sigma_2 \otimes S_2}. \quad (4.71)$$

where the upper line lists the SU(4) generators and the lower line lists the $SU(2) \times SU(2)$ generators. Here the total number of generators is 21.

Flipped SU(5)' × U(1)_X

Note that P_{PS} is a T_{3R} rotation by angle π since P_{PS} has -1 entries in the T_{3R} direction. Thus, the relation $P_F = P_{PS} P_{GG}$ shows that $SU(5)_{GG}$ is flipped by a T_{3R} . This implies that the representations of $SU(5)_{GG}$ are flipped to those of the flipped SU(5):

$$\begin{aligned} e_L^c &\leftrightarrow N \\ d_L^c &\leftrightarrow u_L^c \end{aligned}$$

from which the name “flipped” has been coined.

If the $U(1)_X$ part appearing in the flipped $SU(5)$ is not the identity as given in the form of (4.66), then Q_{em} being traceless in $SO(10)$ must contain the $U(1)_X$ piece of the flipped $SU(5)$. Here, the commutators of $\sigma_2 \otimes \text{diag}(1, 1, 1, -1, -1)$ with the generators $\sigma_1 \otimes A_X$ and $\sigma_3 \otimes A_X$ are non-vanishing and put again in the set. Note that there are two non-vanishing factors $[\sigma_2, \sigma_{1,3}]$ and $[\text{diag}(1, 1, 1, -1, -1), A_X]$. If we assign one minus for a non-vanishing commutator, we end up with $+$ for the two commutator factors; thus the F- $SU(5)$ gauge bosons carry the $+$ parity. The surviving generators are the following [30, 31].

$$SU(5)_F \times U(1)'_X : \frac{\sigma_0 \otimes A_3, \sigma_0 \otimes A_2, \sigma_1 \otimes A_X}{\sigma_2 \otimes S_3, \sigma_2 \otimes S_2, \sigma_3 \otimes A_X}. \quad (4.72)$$

Here also, the total number of generators is 25.

SM as the Common Intersection

It can be clearly seen from the above decompositions that the intersection of any combination of two of the GG- $SU(5)$, F- $SU(5)$ and PS 422, is $G_{\text{SM}} \times U(1)$, whose common generators are

$$\begin{aligned} &\sigma_0 \otimes A_3, \sigma_0 \otimes A_2, \\ &\sigma_2 \otimes S_3, \sigma_2 \otimes S_2, \end{aligned} \quad (4.73)$$

which are underlined in (4.70, 4.72, 4.71). The common intersection of these $SO(10)$ subgroups is shown schematically in Fig. 4.5. The $U(1)$ generator in the common intersection is $\sigma_2 \otimes I_5$.

Because the Cartan subalgebra always commutes with all the projectors, as in (4.69), we need another method to reduce the rank 5 of $SO(10)$ down

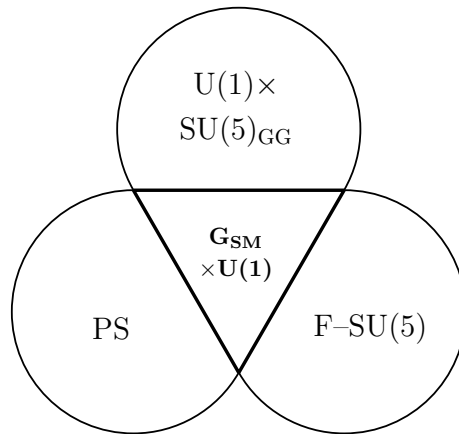


Fig. 4.5. The common intersection of $SO(10)$ subgroups

to the rank 4 of the SM. One way is to employ the Higgs doublet of $SU(2)_R$. For example, 5D $SO(10)$ models can be considered even if one $\mathbf{Z}_2$ is used for the rank preserving breaking $SO(10) \rightarrow SU(4) \times SU(2) \times SU(2)$ which is eventually broken down to G_{SM} by the VEV of Higgs $\langle (\mathbf{\bar{4}}, \mathbf{1}, \mathbf{2}) \rangle$.

Another method is to employ continuous Wilson lines, where the orbifold actions do not commute and hence the projections do not commute. Therefore, some Cartan subalgebra do not remain invariant so that the rank is reduced. This is explained in Subsect. 9.2.3.

Sometimes, it is useful to use the convention given in [28]. The unitary transformation from the notation we used to that of [18, 28] is the following. Writing our 2×2 σ space in the $\{i, 5 + i\}$ coordinates, 10×10 $SO(10)$ generators can be written as

$$M = \begin{pmatrix} A + C & B + S \\ B - S & A - C \end{pmatrix}_{10 \times 10} \quad (4.74)$$

where A, B , and C are antisymmetric and S is symmetric. For example, we have

$$\begin{aligned} \sigma_0 \otimes A_5 &= \begin{pmatrix} A_3 & A_X & 0 & 0 \\ A_X & A_2 & 0 & 0 \\ 0 & 0 & A_3 & A_X \\ 0 & 0 & A_X & A_2 \end{pmatrix}, \\ \sigma_1 \otimes B_5 &= \begin{pmatrix} 0 & 0 & B_3 & B_X \\ 0 & 0 & B_X & B_2 \\ B_3 & B_X & 0 & 0 \\ B_X & B_2 & 0 & 0 \end{pmatrix}, \\ P_{GG} &= \begin{pmatrix} 0 & 0 & -iI_3 & 0 \\ 0 & 0 & 0 & -iI_2 \\ iI_3 & 0 & 0 & 0 \\ 0 & iI_2 & 0 & 0 \end{pmatrix}, \text{ etc.} \end{aligned}$$

The unitary transformation is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{5 \times 5} & iI_{5 \times 5} \\ iI_{5 \times 5} & I_{5 \times 5} \end{pmatrix}. \quad (4.75)$$

Under the unitary transformation, M transforms to

$$M' = \begin{pmatrix} A - iS & B - iC \\ B + iC & A + iS \end{pmatrix}_{10 \times 10}. \quad (4.76)$$

In this case, the 24 $SU(5)$ generators are A and traceless S . The $U(1)$ generator of $U(5)$ is

$$I_{SU(5)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}_{10 \times 10} \quad (4.77)$$

which belongs to M' up to a phase.

$T^2/\mathbf{Z}_2$ Orbifold and Gauge Symmetries at Fixed Points

We compactify two dimensions y^1, y^2 on $T^2/\mathbf{Z}_2$ orbifold considered in Subsect. 3.1.5. The torus T^2 are made under identification

$$y^i \sim y^i + e^i . \quad (4.78)$$

To obtain chiral fermions, we must mod out the torus by some discrete group. We choose $\mathbf{Z}_2$ action

$$\theta : (y^1, y^2) \rightarrow (-y^1, -y^2) . \quad (4.79)$$

The 6D $\mathcal{N}=1$ SO(10) gauge multiplet can be decomposed into 4D $\mathcal{N}=1$ SUSY multiplets as in (4.45): a vector multiplet V and chiral adjoint multiplet Φ . The bulk action is given by

$$S = \int d^6x \left\{ \frac{1}{4g^2} \text{Tr} \left[\int d^2\theta \mathcal{W}^\alpha \mathcal{W}_\alpha + \text{h.c.} \right] + \int d^4\theta \frac{1}{g^2} \text{Tr} \left[(\sqrt{2}\partial^\dagger + \Phi^\dagger) e^{-V} (-\sqrt{2}\partial + \Phi) e^V + \partial^\dagger e^{-V} \partial e^V \right] \right\} , \quad (4.80)$$

where $V = V^a T^a$, $\Phi = \Phi^a T^a$, $\text{Tr}[T^a, T^b] = \delta^{ab}$ and $\partial = \partial_5 - i\partial_6$.

Under the torus translations we identify

$$V(\mathbf{y} + e_i) = T_i V(\mathbf{y}) T_i^{-1} , \quad (4.81)$$

$$\Phi(\mathbf{y} + e_i) = T_i \Phi(\mathbf{y}) T_i^{-1} . \quad (4.82)$$

Now, we require a consistency condition. Successive transportation by $e_1, e_2, -e_1$ and $-e_2$ move a point to the same point,

$$V(y) = T_2^{-1} T_1^{-1} T_2 T_1 V(y) T_1^{-1} T_2^{-1} T_1 T_2 ,$$

thus we have

$$T_2 T_1 = T_1 T_2 \quad (4.83)$$

The projection matrices are chosen as

$$T_1 = P_{\text{GG}}, \quad T_2 = P_{\text{F}} . \quad (4.84)$$

Under the orbifolding $\mathbf{Z}_2$ (π rotation), we identify

$$V(-\mathbf{y}) = Z V(\mathbf{y}) Z^{-1} , \quad (4.85)$$

$$\Phi(-\mathbf{y}) = -Z \Phi(\mathbf{y}) Z^{-1} , \quad (4.86)$$

with $Z = \sigma_0 \otimes I_5$. Note that Z is not belonging to the set of SO(10) gauge generators, viz. (4.64), since it is real and symmetric. It belongs to a discrete group $\mathbf{Z}_2$, and is used to break supersymmetry.

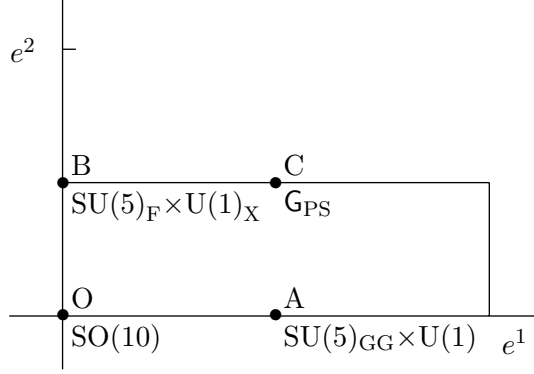


Fig. 4.6. The $T^2/\mathbf{Z}_2$ orbifold in the (y_1, y_2) plane. The orbifold fixed points are denoted by bullets and localized groups at each fixed point are indicated. The fundamental region is the rectangle

At the fixed points on the orbifold, certain gauge transformation parameters are forced to vanish. Remember that the 5D $\mathbf{Z}_2$ example in (4.21) at the fixed point $y = 0$ leaves the $SU(5)$ symmetry intact. But at the fixed point $y = \pi R$, the $\mathbf{Z}_2$ transformation leaves only the SM group G_{SM} invariant. Note that the analysis would be simplified in terms of $y' = \pi R - y$ at the fixed point $y = \pi R$.

This observation is also applicable in the present 6D case. The matter contents and interactions located at the fixed points O, A, B, and C of Fig. 4.6 respect different gauge symmetries. At O, the full $SO(10)$ gauge symmetry is respected.

Recall that in (3.21) and (3.37), we have labeled fixed points by point actions

$$O : (\theta, 0), \quad A : (\theta, e^1), \quad B : (\theta, e^2), \quad C : (\theta, e^1 + e^2), \quad (4.87)$$

with which we come back to the original fixed points. Considering the fixed point A, under action θ we are forced to move along e^1 to come back to the original point A in the covering space. The vector field is transformed as

$$\begin{aligned} \theta : V(\pi R, 0) &= V(-\pi R, 0) \\ e^1 : V(\pi R, 0) &= P_{\text{GG}} V(-\pi R, 0) P_{\text{GG}}^{-1} \end{aligned}$$

where $T_1 = P_{\text{GG}}$. Because of (4.20), generators commuting with P_{GG} survive; thus the gauge symmetry at A is $SU(5)_{\text{GG}} \times U(1)$. Namely, the wave function for every gauge field outside $SU(5)_{\text{GG}} \times U(1)$ vanishes at the fixed point A. Thus, interactions at this fixed point preserve only the $SU(5)_{\text{GG}} \times U(1)$. Similarly, interactions at point B where the projector is P_{F} need only preserve $SU(5)_{\text{F}} \times U(1)_X$.

At the fixed point C, one must apply both e^1 and e^2 translations after $\mathbf{Z}_2$ action. One finds $V(y) = T_2 T_1 V(y) T_1^{-1} T_2^{-1}$. From (4.66) and (4.67), one notes

that $T_2 T_1 = P_F P_{GG} = P_{PS}$. Thus, at the fixed point C the gauge symmetry is the PS group $SU(4)_c \times SU(2)_L \times SU(2)_R$.

From (4.85, 4.86) one notes that the 4D $\mathcal{N} = 1$ supersymmetry is preserved at each fixed point.

We could have taken $T_1 = P_{PS}$ and $T_2 = P_{GG}$. Then, the fixed point A preserves $SU(4)_c \times SU(2)_L \times SU(2)_R$, and the fixed point B preserves $SU(5)_{GG} \times U(1)$. The unbroken gauge group is the $G_{SM} \times U(1)$ as before. This is so because geometrically the four fixed points are equivalent and the different choice of projectors is equivalent to renaming the fixed points.

4.4 Local Anomalies at Fixed Points

We noted that an effective 4D theory compactified from the original 5D theory is anomaly free. Therefore, possible obstruction of gauge symmetry can be present only at fixed points. In this respect, the $U(1)$ gauge anomaly in 5D theories compactified on an $S^1/\mathbf{Z}_2$ orbifold was first discussed in [14]. They considered a single bulk fermion with unit charge and imposed chiral boundary conditions. The anomaly – defined as the five dimensional divergence of the current – lives entirely on the orbifold fixed points,

$$\partial_M J^M = \frac{1}{2} [\delta(y) + \delta(y - \pi R)] \mathcal{Q}(x, y) , \quad (4.88)$$

where J^M is the 5D fermionic current and

$$\mathcal{Q}(x, y) = \frac{g_5^2}{32\pi^2} F_{\mu\nu}(x, y) \tilde{F}^{\mu\nu}(x, y) , \quad (4.89)$$

is the 4D chiral anomaly in the external gauge potential $A_M(x, y)$. We noted that the effective 4D anomaly is absent

$$\int dy \partial_M J^M = 0 \longrightarrow \mathcal{Q}(x, \pi R) = -\mathcal{Q}(x, 0) .$$

But if any anomaly is present in 4D, it must be localized at the fixed points. So, if $\mathcal{Q}(x, 0)$ is nonzero then the only possibility is that it is proportional to $\delta(x)$, and (4.88) gives an anomaly, which is not allowed. Therefore, for the $\mathbf{Z}_2$ case the absence of 4D anomaly is sufficient to ensure $\mathcal{Q}(x, 0) = 0$ and hence the consistency of the higher dimensional orbifold theory.

For the orbifold $S^1/(\mathbf{Z}_2 \times \mathbf{Z}'_2)$, this phenomenon does not persist. Despite the fact that the orbifold projections remove both fermionic zero modes, gauge anomalies localized at the fixed points were found [15],

$$\partial_M J^M = \frac{1}{4} [\delta(y) - \delta(y - \pi R/2) + \delta(y - \pi R) - \delta(y - 3\pi R/2)] \mathcal{Q}(x, y) , \quad (4.90)$$

for gauge fields having the odd parity as the boundary conditions. Even if the 4D effective theory is anomaly-free because anomalies cancel after integration over the fifth dimension, it is possible that the gauge invariance is broken at the fixed points, spoiling the consistency of the 5D theory. Thus, the full 5D anomaly structure must be checked for the models with $S^1/(\mathbf{Z}_2 \times \mathbf{Z}'_2)$. The anomaly structure of the Abelian gauge theory with arbitrary boundary condition on 5D orbifold is analyzed in [16]. For the nonabelian gauge theory, [17] discussed the anomaly structure and the cure by Chern-Simons terms in detail.

Consider a 5D (Dirac) fermion living in $R^4 \times I$ coupled with a U(1) external gauge field. The compact component I is either $S^1/\mathbf{Z}_2$ or $S^1/(\mathbf{Z}_2 \times \mathbf{Z}'_2)$ and it has in general fixed points. Thus the resulting space is an orbifold. The action of 5D U(1) gauge theory is given by (4.25). The two orbifold projections act on the space-time points as

$$\mathbf{Z}_2 : (x, y) \rightarrow (x, -y) \quad (4.91)$$

$$\mathbf{Z}'_2 : (x, y) \rightarrow (x, \pi - y) \quad (4.92)$$

and the points related by these projections are identified, viz. Fig. 4.4. For the 5D spinor, the following boundary condition is imposed,

$$\Psi(x, \mathbf{Z}_2^{(\prime)}(y)) = \eta^{(\prime)} \gamma^5 \Psi(x, y), \quad \eta, \eta' = \pm 1. \quad (4.93)$$

We call η (and η') the parity of the field under $\mathbf{Z}_2$ (and $\mathbf{Z}'_2$). The γ^5 matrix in (4.93) distinguishes the parities of two 4D Weyl spinors in Ψ and allows only one Weyl spinor for having zero mode (the + parity). Notice that a 5D bulk mass term $m \Psi^\dagger \Gamma^0 \Psi$ (with Γ^0 of Eq. (4.22)) is forbidden unless m has a nontrivial profile in the bulk with parities $(-, -)$.

Even though the 5D current

$$J^M = \bar{\Psi} \Gamma^M \Psi \quad (4.94)$$

is classically conserved, it may have a local anomalous divergence at quantum level. One can rewrite the action (4.25) as a collection of 4D massive Dirac fermions by expanding Ψ in terms of the complete set formed by the solutions of free Dirac equation in 5D as in (4.37),

$$\Psi(x, y) = \sum_n [\psi_{nR}(x) \phi_n^R(y) + \psi_{nL}(x) \phi_n^L(y)] \quad (4.95)$$

$$\psi_{nL/R} = P_{L/R} \psi_n \quad (4.96)$$

where ψ_n is a 4D Dirac fermion of mass M_n , and $P_{L/R}$ are the left-hand and right-hand projection operators, $(1 \pm \gamma_5)/2$. The KK mode $\phi_n^{L/R}$ is one of either (4.51–4.54) and satisfy

$$\phi_n^R M_n - \frac{d}{dy} \phi_n^L = 0 \quad (4.97)$$

$$\phi_n^L M_n + \frac{d}{dy} \phi_n^R = 0. \quad (4.98)$$

From (4.93) we get the following transformation rules under the orbifold projections

$$\phi^{R/L}(-y) = \pm \epsilon \phi^{R/L}(y), \quad \phi^{R/L}(\pi R - y) = \pm \epsilon' \phi^{R/L}(y). \quad (4.99)$$

Using the orthogonality of the KK modes, the action can be written as

$$S = \int d^4x \left[\sum_n \left(\bar{\psi}^{(n)} i \gamma^\mu \partial_\mu \psi^{(n)} - M^{(n)} \bar{\psi}^{(n)} \psi^{(n)} \right) - g_5 \sum_{n,m} \left(j_{Lmn}^\mu \mathcal{A}_{\mu mn}^L + j_{Rmn}^\mu \mathcal{A}_{\mu mn}^R - i j_{5mn} \mathcal{A}_{5mn} \right) \right] \quad (4.100)$$

where

$$\mathcal{A}_{\mu mn}^{L/R} = \int_0^{2\pi R} dy \phi_{L/R}^{(m)}(y) \phi_{L/R}^{(n)}(y) A_\mu(x, y) \quad (4.101)$$

$$\mathcal{A}_{5mn} = \int_0^{2\pi R} dy \phi_L^{(m)}(y) \phi_R^{(n)}(y) A_5(x, y) \quad (4.102)$$

and

$$j_{L/R(mn)}^\mu = \bar{\psi}^{(m)} \gamma^\mu P_{L/R} \psi^{(n)} \quad (4.103)$$

$$j_{5mn} = \bar{\psi}_L^{(m)} \psi_R^{(n)} - \bar{\psi}_R^{(m)} \psi_L^{(n)}. \quad (4.104)$$

After choosing the gauge $A_5 = 0$, the well known result for the anomalous divergence of chiral current in 4D gives the relations

$$\begin{aligned} \partial_\mu j_{L(mn)}^\mu &= i \left[\bar{\psi}^{(m)} M^{(m)} P_L \psi^{(n)} - \bar{\psi}^{(m)} P_R M^{(n)} \psi^{(n)} \right] \\ &\quad - \frac{g_5^2}{32\pi^2} \left(\mathcal{F}_{\mu\nu}^L \tilde{\mathcal{F}}^{L\mu\nu} \right)_{mn} \end{aligned} \quad (4.105)$$

$$\begin{aligned} \partial_\mu j_{R(mn)}^\mu &= i \left[\bar{\psi}^{(m)} M^{(m)} P_R \psi^{(n)} - \bar{\psi}^{(m)} P_L M^{(n)} \psi^{(n)} \right] \\ &\quad + \frac{g_5^2}{32\pi^2} \left(\mathcal{F}_{\mu\nu}^R \tilde{\mathcal{F}}^{R\mu\nu} \right)_{mn} \end{aligned} \quad (4.106)$$

On the other hand, the 5D current can be written in terms of the 4D currents

$$\begin{aligned} J^\mu(x, y) &= \sum_{(mn)} \left[\phi_R^{(m)}(y) \phi_R^{(n)}(y) j_{Rmn}^\mu(x) \right. \\ &\quad \left. + \phi_L^{(m)}(y) \phi_L^{(n)}(y) j_{Lmn}^\mu(x) \right] \quad (4.107) \end{aligned}$$

$$J^5(x, y) = -i \sum_{mn} \phi_L^{(m)}(y) \phi_R^{(n)}(y) j_{5mn}(x). \quad (4.108)$$

Noticing that at the classical level

$$\partial_5 J^5 = \sum_{mn} i \left[M^{(n)} \left(\phi_R^{(m)} \phi_R^{(n)} - \phi_L^{(m)} \phi_L^{(n)} \right) \bar{\psi}^{(m)} (P_L - P_R) \psi^{(n)} \right] \quad (4.109)$$

the divergence of the 5D current can be expressed by

$$\partial_M J^M = \frac{g_5^2}{32\pi^2} f(y) F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad f(y) = \sum_m \left(\phi_R^{(m)} \phi_R^{(m)} - \phi_L^{(m)} \phi_L^{(m)} \right). \quad (4.110)$$

It should be stressed that in deriving (4.110) one tacitly assumed that (4.109) is still valid at the quantum level and all the quantum effects are encoded in (4.105–4.106). To discuss the complete anomaly structure, one should also consider the parity anomaly.

The sum over the KK modes in (4.110) can be computed using the completeness property of the mode functions [14]. For the case of $(+, +)$ parity, consider

$$\Delta(y, y') \equiv \sum_{m \geq 0} \phi(y)_R^{(m)} \phi(y')_R^{(m)} - \sum_{m > 0} \phi(y)_L^{(m)} \phi(y')_L^{(m)}, \quad (4.111)$$

where $\phi(y)_R^{(m)} = (1/\sqrt{\pi R}) \cos(2my/R)$ and $\phi(y)_L^{(m)} = (1/\sqrt{\pi R}) \sin(2my/R)$, which reduces to (4.110) if we set $y = y'$. Because the $\phi(y)_L^{(m)}$ functions are antisymmetric at $y = 0$ while the $\phi(y)_R^{(m)}$ are symmetric, we can write

$$\Delta(y, -y') \equiv \sum_{m \geq 0} \phi(y)_R^{(m)} \phi(y')_R^{(m)} + \sum_{m > 0} \phi(y)_L^{(m)} \phi(y')_L^{(m)}. \quad (4.112)$$

Since $\phi(y)_m^R$ and $\phi(y)_m^L$ form a complete set of functions with periodic boundary conditions in the interval $[0, \pi R]$, (although they are still normalized in the interval $[0, 2\pi R]$) we can write³

$$\Delta(y, -y') = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \delta(y - y' - n\pi R), \quad (4.113)$$

which leads to the expression, by setting $y' = -y$,

$$\Delta(y, y) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \delta(2y - n\pi R) = \sum_{n=-\infty}^{+\infty} \frac{1}{4R} \delta(y/R - n\pi/2). \quad (4.114)$$

Using this method for the case of $\mathbf{Z}_2 \times \mathbf{Z}'_2$ orbifold, various choices of the fermion parity lead to

$$f^{(++)}(y) = -f^{(--)}(y) = \frac{1}{4R} \sum_{n=-\infty}^{+\infty} \delta(y/R - n\pi/2); \quad (4.115)$$

$$f^{(+-)}(y) = -f^{(-+)}(y) = \frac{1}{4R} \sum_{n=-\infty}^{+\infty} (-1)^n \delta(y/R - n\pi/2). \quad (4.116)$$

³ Note that the sum of the eigenfunctions in the complete set is a delta function.

In particular, if the fermions have opposite parities $(+, -)$ and $(-, +)$, one recovers (4.90).

For the case of $\mathbf{Z}_2$, one finds

$$f^{(+)}(y) = -f^{(-)}(y) = \frac{1}{2R} \sum_{n=-\infty}^{+\infty} \delta(y/R - \pi n) , \quad (4.117)$$

which reproduces (4.88).

For a consistent quantum theory, there should not appear any fixed point anomaly. In this regard, note that the above fixed point anomalies can be always canceled by adding appropriate Chern-Simons terms in the bulk and some fermions at the fixed point(s) [17, 32].

Bulk and Local Anomalies in Six Dimension

The automatic cancelation of bulk anomaly in 5D is not maintained in 6D. In even dimensions, chiral anomalies can be present. The gauge anomalies in even dimensions were studied by many groups, notably by Frampton and Kephart [33]. In 6D, the chiral anomaly arises from box diagrams. So, the 4D intuition that $\overline{N}$ of $SU(N)$ carries -1 unit of the anomaly of N does not work in 6D. In 6D, unlike in 4D, orthogonal groups can have gauge anomalies. Usually, in even dimensions the $\mathcal{N}=1$ theory can be made anomaly free by adjusting bulk matter contents by hand. Although it is possible to remove the $SO(10)$ gauge anomalies in 6D in this way [8], it is desirable to have some guidance for canceling the gauge anomaly. For an effective 4D theory, the orbifolding has been used in this chapter. One method to remove the gauge anomalies is to consider parity symmetric $\mathcal{N}=2$ SUSY theory [8]. Even if the 4D gauge anomaly is canceled in this way, the local anomaly cancelation has to be checked carefully as we will discuss later. Another method is to form a complete representation of higher rank gauge group for which the anomaly cancelation is guaranteed. In 6D, exceptional groups can be used for an automatic absence of gauge anomalies as $SO(4N+2)$ groups with complex fermion representations do in 4D [34].

For the localized anomaly we have the same result as in five dimensions [35]: the anomaly freedom implies the cancellation of localized anomalies at the fixed points.

4.5 What We Will Do with String Theory

We have learned that the projection associated with orbifold symmetry imposes a boundary conditions on the fields and thus breaks some symmetries. The projection is in accord with the symmetry of Lagrangian, being either global or local.

We can do the same thing by introducing string theory compactified on orbifold. The string theory is consistently defined in higher dimensions, $D = 26$ for bosonic and $D = 10$ for supersymmetric strings, thus they naturally require compact extra dimensions beyond our 4D. The string theory on orbifold also associate the symmetry of orbifold with the boundary conditions; thus breaks some symmetries

$$A_M^w \rightarrow e^{2\pi i w \cdot V} A_M^w, \quad (4.118)$$

where the charges of the field w is labeled differently but the effect is essentially the same $P = e^{2\pi i w \cdot V}$. Indeed, the field theory with extra dimensions provides a low energy limit of string theory. Historically the doublet–triplet splitting from orbifold, discussed in Subsect. 2.3.3, was observed long time ago in string orbifolds in this way.

In this sense, string theory is more restrictive and predictive. In the field theoretic orbifold, there are too many arbitrariness: the number of dimensions, the gauge group, and so on. We cannot determine which fields should live in the bulk or at the fixed points. The only consistency condition is the anomaly cancelation.

We have experienced that anomaly cancelation is a hint of larger symmetry: for example chiral anomaly cancelation of the SM is automatically implemented when we consider GUTs such as SU(5), SO(10), etc. Although the resulting low energy theory is chiral, the whole combination resulting from the fields of the GUT groups cancel the anomaly.

The anomaly cancelation can be explained in string theory with a more fundamental consistency condition: for example, in the closed string theory by the modular invariance. It determines possible gauge symmetries, orbifold actions and spectra in the bulk and also at the fixed points, etc. Most of the physics in the low energy limit is determined by a single vector v of (4.118). Now we move on discussing string theory orbifolds in the subsequent chapters.

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Quantization of Strings

Starting from this chapter, we discuss string theory for particle physics. Namely, our emphasis will be, starting from string theory, understanding low energy physics described by the standard model (SM). As summarized in Chap. 2, the SM is a chiral theory for fermions, and hence obtaining a chiral spectrum from string theory is of utmost importance. This goes with string theory with fermions, i.e. superstring rather than bosonic string. Superstring is written in ten spacetime dimensions (10D) but the effective quantum field theory of the SM is in four spacetime dimensions (4D), and the extra six dimensions (6D) must be cleverly hidden from the 4D observers. For hiding these extra dimensions, we follow the compactification scheme via orbifolds introduced in Chap. 3 and applied to quantum field theory in Chap. 4. It will be exploited fully in string theory in the subsequent chapters. In this chapter, we introduce basics in string theory.

There are excellent books on string theory [1, 2, 3, 4, 5]. In this spirit, we attempt to excerpt the key formulae of string theory in this chapter which will be used in the subsequent chapters for constructing 4D string models. In Sect. 5.1, we introduce bosonic string up to the point that 26 dimensions are necessary for a consistent quantization of the bosonic string, because the number of dimensions is the least thing to be understood in our endeavor of understanding the chirality problem with the theme of unification. In Sect. 5.2, we introduce fermionic string with the same spirit. Here, we will be even more brief since the basic logic is the same as the bosonic string. The Ramond (R) and Neveu-Schwarz (NS) boundary conditions are introduced in Sect. 5.2. In Sect. 5.3, we introduce the $E_8 \times E_8'$ heterotic string which is the theory to be compactified in the subsequent chapters.

Conventions

We mainly follow the conventions of [1] except the metrics. Since our aim is to use the string results for particle physicists, we use the “mostly negative”

convention used by most particle physicists which differs from that of [1]. Our spacetime metric is

$$\eta_{\mu\nu} = \text{diag}(+1, -1, -1, \dots, -1), \quad \mu, \nu = 0, 1, \dots, D-1. \quad (5.1)$$

Also for the world-sheet,

$$\eta_{\alpha\beta} = \text{diag}(+1, -1), \quad \alpha, \beta = 0(\tau), 1(\sigma). \quad (5.2)$$

We set the inverse string tension $\alpha' = \frac{1}{2}$. Frequently the world-sheet light-cone coordinate $\sigma^\pm = \tau \pm \sigma$ will be used. It follows that $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. It is also useful to introduce complex coordinates $z = e^{\tau+i\sigma}$, $\bar{z} = e^{\tau-i\sigma}$, which is called holomorphic (left handed) and antiholomorphic (right handed).

5.1 Bosonic String

5.1.1 Action and Its Invariance Properties

We begin with an example of relativistic point particle. One successful form for the action with the Poincaré invariance is

$$S = m \int d\tau \sqrt{\dot{X}^\mu \dot{X}_\mu} \quad (5.3)$$

where the dot denote $d/d\tau$. We parameterized τ , along the world-line. Physically there should be no dependence on τ thus it has reparametrization invariance $X'_\mu(\tau'(\tau)) = X_\mu(\tau)$ and we can easily check indeed it is. Variation of (5.3) with respect to δX^μ gives

$$\delta S = -m \int d\tau \dot{u}_\mu \delta X^\mu$$

where $u^\mu = \dot{X}^\mu / \sqrt{\dot{X}^\nu \dot{X}_\nu}$ is the D -velocity. The equation of motion is $\dot{u}^\mu = 0$ which is a free particle motion. In the nonrelativistic limit we can identify m as the particle mass.

Let us consider another successful form for the action, by introducing world line metric $e(\tau)$,

$$S' = \frac{1}{2} \int d\tau \left(\frac{1}{e} \dot{X}^\mu \dot{X}_\mu + e m^2 \right) \quad (5.4)$$

which has the Poincaré invariance. Of course, we know that the invariant volume element in the world line is $e(\tau)d\tau$. Variation of (5.4) with respect to $e(\tau)$ relates

$$e^2 = \frac{\dot{X}^\mu \dot{X}_\mu}{m^2}$$

and plugging back, the action S' of (5.4) reduces to the original action S of (5.3). We feel much better for the latter form because it is quadratic in X^μ and successfully describes the massless limit.

Let us extend the above discussion to the string. Strings propagate in space-time manifolds, sweeping out an area. The world volume swept out by a string is a two dimensional surface for whose coordinates we introduce τ and σ . The spatial extent of the string is defined to, in a suitable definition of the parameter σ ,

$$0 \leq \sigma \leq \pi . \quad (5.5)$$

We will treat bosonic degrees X^μ , ($\mu = 0, 1, 2, \dots, D-1$) as D displacements defined on a world-sheet parametrized by τ and σ . On this world-sheet, we introduce a metric $h_{\alpha\beta}(\tau, \sigma)$. Thus, we introduce the following action for the relativistic string of type (5.4),

$$S = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \eta^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu \quad (5.6)$$

where $h = \det h_{\alpha\beta}$ and T is string tension having dimension of mass squared. Here, we can write a world sheet action in analogy with the relativistic point particle action in the world line, and one can write (5.6) blindly and start from there. The action (5.6) has the following symmetries:

1. *General covariance* (world-sheet) which implies that the action is invariant under the reparametrization of coordinates

$$\tau \rightarrow \tau'(\tau, \sigma), \sigma \rightarrow \sigma'(\tau, \sigma) .$$

Therefore,

$$\begin{aligned} X'^\mu(\tau', \sigma') &= X^\mu(\tau, \sigma) , \\ h'_{\gamma\delta}(\tau', \sigma') &= \frac{\partial \sigma^\alpha}{\partial \sigma'^\gamma} \frac{\partial \sigma^\beta}{\partial \sigma'^\delta} h_{\alpha\beta}(\tau, \sigma) . \end{aligned} \quad (5.7)$$

Due to the reparametrization invariance on the world-sheet, we can deal with two dimensional general relativity.

2. *Weyl invariance* (world-sheet) under the local scaling,

$$\begin{aligned} X'^\mu(\tau, \sigma) &= X^\mu(\tau, \sigma) \\ h'_{\alpha\beta}(\tau, \sigma) &= e^{\Lambda(\tau, \sigma)} h_{\alpha\beta}(\tau, \sigma) , \end{aligned} \quad (5.8)$$

which holds only in two dimensions.

3. *Poincaré invariance* (target space),

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu . \quad (5.9)$$

Therefore, we have a unitary representation under Poincaré algebra and the state is labeled by momentum, spin and internal quantum numbers.

The standard way of quantizing the system having gauge symmetry is to fix the gauge and use the gauge condition as a constraint. In this regard, one can remind the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ in Maxwell's theory.

There are three degrees of freedom for the symmetric tensor $h_{\alpha\beta}$. Using reparametrization invariance, we can fix $h_{\alpha\beta}$ as

$$h_{\alpha\beta} = e^\varphi \eta_{\alpha\beta} , \quad (5.10)$$

called the *conformal gauge*. Plugging (5.10) into (5.6), the φ dependence disappears due to the two dimensional nature of the worldsheet. This makes our later analysis easy although the following considerations make it nontrivial.

- We should note that either the scale invariance or the reparameterization invariance is broken in quantum theory. Fortunately, there is no anomaly in a critical dimension (for example $D = 26$ in the bosonic string case).
- Riemann and Roch theorem tells us about fixing of $h_{\alpha\beta}$. It is always possible to fix gauge (5.10) in the tree Feynman diagram (topologically it is reduced to a sphere). However, in the loop diagrams there remain non-fixable degrees in general, which we will see in Subsect. 5.1.4.
- Even if we succeed in fixing gauge $h_{\alpha\beta}$, there is a residual symmetry, which will be discussed in the following subsection.

The string equations are obtained by varying with respect to X^μ . Besides the surface term we have,

$$\partial_\alpha \partial^\alpha X^\mu = \left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^\mu = 0 . \quad (5.11)$$

As is familiar, the solution of this partial differential equation is separated into two independent ones

$$X^\mu(\tau, \sigma) = X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma) . \quad (5.12)$$

Therefore, it is useful to introduce the light-cone variables $\sigma^\pm = \tau \pm \sigma$. Then, (5.11) becomes $\partial_+ X_R^\mu = \partial_- X_L^\mu = 0$; thus we have chiral scalars in two dimensions. This contributes to gravitational and/or Weyl anomalies in two dimensions.

This is complemented by the constraint, that the variation with respect to $h^{\alpha\beta}$ vanishes,

$$T_{\alpha\beta} \equiv \frac{2}{T\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} \bigg|_{h_{\alpha\beta}=\eta_{\alpha\beta}} = -\partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{2} \eta_{\alpha\beta} \partial^\gamma X^\mu \partial_\gamma X_\mu = 0 , \quad (5.13)$$

which is the *world-sheet energy-momentum tensor*. It corresponds to the Gauss law constraint $\nabla \cdot \mathbf{E} = 0$ in Maxwell's theory, originating from $A_0 = 0$ which in turn corresponds to (5.10).

There seems to exist three independent constraints for the symmetric tensor $T_{\alpha\beta}$. However, because of the scaling invariance (5.10) which is equivalent to

$$\text{Tr } T_{\alpha\beta} = \eta^{\alpha\beta} T_{\alpha\beta} = 0 , \quad (5.14)$$

only the following two among them are independent,

$$\begin{aligned} T_{++} &= -\partial_+ X_L^\mu \cdot \partial_+ X_{\mu L} = 0 \\ T_{--} &= -\partial_- X_R^\mu \cdot \partial_- X_{\mu R} = 0 \end{aligned} \quad (5.15)$$

so that T_{++} and T_{--} respectively depend on σ^+ and σ^- only. Explicitly, they have the following expressions,

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) = -\frac{1}{4}(\dot{X} + X')^2 = -\left(\frac{\partial X_L^\mu}{\partial \sigma^+}\right)^2 \quad (5.16)$$

$$T_{--} = \frac{1}{2}(T_{00} - T_{01}) = -\frac{1}{4}(\dot{X} - X')^2 = -\left(\frac{\partial X_R^\mu}{\partial \sigma^-}\right)^2 \quad (5.17)$$

The boundary condition is to abolish the surface term contributions,

$$T \int d\tau \frac{\partial X_\mu}{\partial \sigma} \delta X^\mu \Big|_{\sigma=0}^\pi = 0. \quad (5.18)$$

As in the quantization in gauge theories, we will quantize the theory such that these constraint equations are satisfied between physical states.

Quantization of Closed Strings

The following boundary condition satisfying (5.18) specifies the *closed string*,

$$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma). \quad (5.19)$$

(In fact, we also need $h_{\alpha\beta}(\tau, \sigma + \pi) = h_{\alpha\beta}(\tau, \sigma)$). Let us try to solve the equation of motion, consistent with this condition. We have the following mode expansion

$$X_R^\mu(\tau - \sigma) = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)} \quad (5.20)$$

$$X_L^\mu(\tau + \sigma) = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)} \quad (5.21)$$

where $\alpha' \equiv 1/\pi T$ is called *Regge slope* and has a dimension of length squared; thus α' represents a squared string length scale, $\sim \ell_s^2$.

Now let us quantize X^μ . First we treat them as operators. The momentum conjugate to X^μ is found to be

$$P^\mu(\tau, \sigma) = T \dot{X}^\mu. \quad (5.22)$$

The first quantization of string leads to the following equal τ commutators

$$[P^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = T[\dot{X}^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = i\delta(\sigma - \sigma')\eta^{\mu\nu} \quad (5.23)$$

with others vanishing. They lead to the following commutators for the center of mass variables and oscillator mode operators

$$[x^\mu, p^\nu] = -i\eta^{\mu\nu}, \quad (5.24)$$

$$[\alpha_m^\mu, \alpha_n^{\nu\dagger}] = -m\delta_{mn}\eta^{\mu\nu}, \quad (5.25)$$

$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^{\nu\dagger}] = -m\delta_{mn}\eta^{\mu\nu}, \quad (5.26)$$

with the other commutators vanishing. Using the Hermiticity property of X^μ , operators $\alpha, \tilde{\alpha}$ with negative indices are related to positive indices by

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu, \quad (\tilde{\alpha}_n^\mu)^\dagger = \tilde{\alpha}_{-n}^\mu. \quad (5.27)$$

Thus, the world-sheet Hamiltonian becomes

$$\begin{aligned} H &= - \int_0^\pi d\sigma (\dot{X} \cdot P_\tau - L) \\ &= - \sum_{n \neq 0} (\alpha_{-n}^\mu \alpha_{n\mu} + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu}) - \alpha' p^\mu p_\mu. \end{aligned} \quad (5.28)$$

where sum over μ is understood.

In fact we have an unwanted state of negative norm

$$\|\alpha_{-m}^0|0;p\rangle\|^2 < 0.$$

due to the opposite sign in the commutation relation along the time direction.¹ This negative norm state is pathological and we should decouple these states. It turns out that such unphysical degrees are decoupled when the constraint is appropriately imposed, as we encounter in the Gupta-Bleuler formalism in QED [6]. The remaining part of this section is devoted to removing such an ambiguity.

Open Strings

The open string has the Neumann boundary condition

$$\frac{\partial X^\mu}{\partial \sigma} = 0, \quad \text{at } \sigma = 0, \pi. \quad (5.29)$$

Technically, we can make an open string by $X_R(\sigma + \pi) = X_L(\sigma)$, $X_L(\sigma + \pi) = X_R(\sigma)$. That amounts to $\alpha^\mu = \tilde{\alpha}^\mu$, and we have

¹ Because of the mostly negative metric convention, we define the norm of a state $|||\phi\rangle||^2$ as $-\langle\phi|\phi\rangle$.

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) \quad (5.30)$$

It has the same quantization

$$[x^\mu, p^\nu] = -i\eta^{\mu\nu}, \quad [\alpha_m^\mu, \alpha_n^\nu] = -m\delta_{m+n,0} \eta^{\mu\nu}. \quad (5.31)$$

Therefore, the Hamiltonian becomes

$$H = -\frac{1}{2} \sum_{n \neq 0} \alpha_{-n}^\mu \alpha_{n\mu} - \alpha' p^\mu p_\mu. \quad (5.32)$$

Traditionally, it has been the end of the story. Because of δX^μ in the boundary term (5.18), the Dirichlet boundary condition

$$X^\mu(\sigma = 0) = X_0^\mu, \quad X^\mu(\sigma = \pi) = X_1^\mu \quad (5.33)$$

is also allowed. Although it breaks the 10D Lorentz symmetry, we can make the 4D Lorentz symmetry unbroken. We will see that the endpoint behavior translates into a dynamical object, called the Dirichlet brane [7], which will be dealt with in Appendix B.2. With this discovery of Dirichlet brane, the most interesting physics comes from the open string sector [8]. Any open string theory should contain closed strings also, since a closed string can be created by interactions of two open strings. On the contrary, we can view that theories with closed strings contain open strings also since in the closed string case we can think of D-branes which come from collective modes of open strings.

5.1.2 Conformal Symmetry and Virasoro Algebra

In this section we will consider the so-called old covariant quantization. In the world-sheet point of view, string theory is a two dimensional theory, which has an infinite-dimensional local conformal symmetry, generated by the Virasoro algebra [9]. Thanks to this symmetry, we may have a quantum theory of string since many ambiguities disappear.

Residual Symmetry

We have discussed in the covariant gauge where the world-sheet metric was chosen as $h_{\alpha\beta} = \eta_{\alpha\beta}$. But still there remains a gauge freedom. We can further fix the gauge, exhausting all the gauge freedom, as in the unitary gauge in field theory. This fixing guarantees that we do not expect any unphysical states.

This residual gauge freedom can be seen as follows. After choosing a covariant gauge, any combined reparametrization of (5.7) chosen in the following way

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = -\Lambda \eta^{\alpha\beta} \quad (5.34)$$

is consistent with the Weyl scaling (5.8). Under this, the angle between two vectors is preserved and thus it is called *conformal symmetry*. It seems to be peculiar, but it is consistent with the infinite number of conserved quantities in the covariant gauge, following from $\partial_- T_{++} = 0$. This is always satisfied when we transform coordinates such that

$$\sigma^+ \rightarrow \tilde{\sigma}^+(\sigma^+), \quad \sigma^- \rightarrow \tilde{\sigma}^-(\sigma^-). \quad (5.35)$$

It turns out that the generator for this local transformation is the energy-momentum tensor of (5.15). Since it is local, it is useful to expand it in terms of the Fourier components, by substituting the mode expansions (5.20, 5.21),

$$L_m = \frac{T}{2} \int_0^\pi d\sigma_- e^{2im\sigma_-} T_{--} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n}^\mu \alpha_{\mu n}, \quad (m \neq 0), \quad (5.36)$$

$$\tilde{L}_m = \frac{T}{2} \int_0^\pi d\sigma_+ e^{2im\sigma_+} T_{++} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}^\mu \tilde{\alpha}_{\mu n}, \quad (m \neq 0) \quad (5.37)$$

which are called Virasoro operators [10]. In particular, we have

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu. \quad (5.38)$$

In the above, we have defined the Virasoro operators for the case of $m \neq 0$, otherwise we have the ordering problem since α_n and α_{-n} do not commute. Therefore, for $m = 0$, we *define* $L_0, \tilde{L}_0$ as normal ordered products:

$$\begin{aligned} L_0 &= -\frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{-n}^\mu \alpha_{\mu n} : \equiv -\frac{1}{2} \alpha_0^\mu \alpha_{\mu 0} - \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{\mu n} \\ \tilde{L}_0 &= -\frac{1}{2} \sum_{n=-\infty}^{\infty} : \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{\mu n} : \equiv -\frac{1}{2} \tilde{\alpha}_0^\mu \tilde{\alpha}_{\mu 0} - \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{\mu n} \end{aligned} \quad (5.39)$$

This definition of L_0 and $\tilde{L}_0$ differs by a normal ordering constant from the one having the ambiguity by using (5.36, 5.37) with a brute force substitution $m = 0$. Let us call that constant a . (We have the same a for the left and the right movers since the systems are the same.) Then, the Hamiltonian (5.28) becomes

$$H = 2(L_0 + \tilde{L}_0 - 2a). \quad (5.40)$$

Virasoro Algebra

Using the commutation relation (5.24–5.26), we obtain the Virasoro algebra [10]

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{12} m(m^2 - 1) \delta_{m+n, 0} \quad (5.41)$$

where $D = \eta_{\mu\nu}\eta^{\mu\nu}$ is the number of world-sheet boson. A sketch of the derivation is the following. First, consider the case $m + n \neq 0$,

$$[L_m, L_n] = \frac{1}{4} \left[\sum_{p=-\infty}^{\infty} \alpha_{m-p}^{\mu} \alpha_{\mu p}, \sum_{q=-\infty}^{\infty} \alpha_{n-q}^{\nu} \alpha_{\nu q} \right] = (m-n)L_{m+n}, \quad m+n \neq 0.$$

But for $n = -m \neq 0$, i.e. $m + n = 0$ with $m \neq 0$, we can compute the commutator as $[L_m, L_{-m}] = 2mL_0 + f(m)\delta_{m+n,0}$ where the constant is the vacuum energy contribution. To calculate $f(m)$, we compute the commutator between the vacuum state, so that L_0 part does not contribute. With the definition of L_m given in (5.36), we obtain for $(-n = m > 0)$,

$$\langle 0|[L_m, L_n]|0\rangle = \frac{D}{12}m(m^2 - 1).$$

Combining these, we obtain (5.41).

It is the property of conformal field theory that the Virasoro algebra is totally determined by the central charge, (5.41). We have the same algebra for the left-moving operators $\tilde{L}_m$, since they are basically satisfying the same algebra and independent from the right-moving operators $[L_m, \tilde{L}_n] = 0$. Using (5.44), for a physical state $|\varphi\rangle$ we can obtain

$$\langle \varphi|L_m L_{-m}|\varphi\rangle = 2ma + \frac{D}{12}m(m^2 - 1), \quad \text{for } m > 0. \quad (5.42)$$

For $D = 26$ and $a = 1$, (5.42) prevents us from extending the condition $L_{-n}|\varphi\rangle = 0$ to two or more values of $n \geq 1$.

Physical States

Now, we try to satisfy the constraint equations between physical states $|\varphi\rangle$ in the quantum mechanical Hilbert space, instead of the operator relations $T_{\alpha\beta} = 0$ in (5.15). In the quantum system, the constraints are applied to the physical state $|\varphi\rangle$ as

$$T_{++}|\varphi\rangle = 0, \quad T_{--}|\varphi\rangle = 0 \quad (\text{too strong}) \quad (5.43)$$

which are the only nontrivial constraints because of the Weyl symmetry.

However, this is a too strong condition with which no nontrivial physical states survive. Instead, we employ a much milder constraint, as in the Gupta-Bleuler quantization [6]. Namely, impose vanishing conditions only for positive components

$$\begin{aligned} L_m|\varphi\rangle &= \tilde{L}_m|\varphi\rangle = 0, \quad \text{for } m > 0 \\ (L_0 - a)|\varphi\rangle &= (\tilde{L}_0 - a)|\varphi\rangle = 0. \end{aligned} \quad (5.44)$$

The mass shell condition follows from the last condition, since L_0 contains $\alpha_0^\mu \sim p^\mu$ (5.38). We will obtain the spectrum resulting from this condition in the next section.

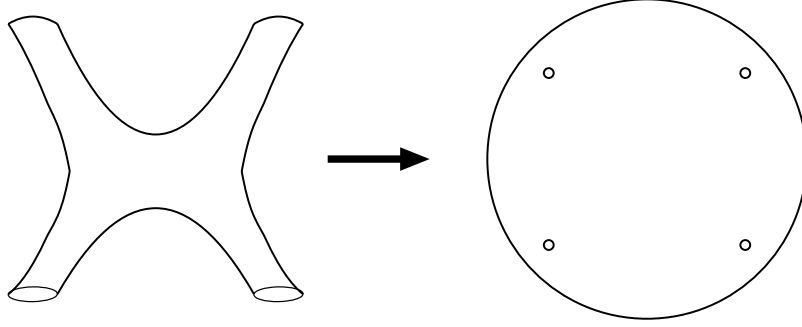


Fig. 5.1. With a suitable transformation, the $|\text{in}\rangle$ or $|\text{out}\rangle$ state can shrink into a single point, keeping the quantum numbers. Especially, a state becomes an operator and the mass is converted into the conformal weight

Vertex Operators

Consider an interaction diagram for closed strings with external “in” and “out” states, as shown in Fig. 5.1. Its complicated form can be simplified by using the rescaling (5.8), $\delta h_{\alpha\beta} \rightarrow (\delta\Lambda)h_{\alpha\beta}$ with a suitable $\delta\Lambda(\tau, \sigma)$. We can make the diagram into a sphere by shrinking external legs into points. If not, practically we cannot calculate the amplitude.

Locally, this is viewed as transforming a closed string propagation diagram (a cylinder) into a two dimensional plane. It is done by a holomorphic transformation

$$w \rightarrow z = e^w, \quad w \equiv \tau + i\sigma. \quad (5.45)$$

We can easily check that the *state* from the infinite past ($\tau \rightarrow -\infty$) corresponds to that on the *point* at the origin. During the transformation, all the information on the state $|\varphi\rangle$ should be kept. Therefore, we are led to have a corresponding local operator $V_\varphi(z, \bar{z})$ at the point. The simplest case is the one carrying no particular quantum number: the tachyon. Since it is an eigenstate of the spacetime translation, it carries a momentum as

$$: e^{ik \cdot X} : \quad (5.46)$$

with the understanding of normal ordering.

Now, let us define the conformal weight J as,

$$V'(z') = \left(\frac{\partial z'}{\partial z} \right)^J V(z). \quad (5.47)$$

Naively speaking, it shows the scale transformation behavior. From now on, we will concentrate on the left mover only with the argument z , since the same can be copied to the right mover also. For example, the fields $\partial_\alpha^n X$ and $: e^{ik \cdot X} :$ have conformal weights n and $-k^2/4$, and so on.

It turns out that the information on the mass of a given state is converted into the conformal weight. The vertex operator will be inserted into the path integral as $\int d^2z \sqrt{-h} V_\phi$. Since it should be conformally invariant also, $d^2z \sqrt{-h}$ has conformal weight -2 under the transformation (5.47). Therefore, V_ϕ should have the conformal weight two, $J = 2$. If V is for the tachyon (5.46), we have

$$-\frac{\alpha' k^2}{4} = -\frac{\alpha' M^2}{4} = 2. \quad (5.48)$$

The operator $g^{\mu\nu} X_\mu X_\nu$ has the conformal weight 0 and reduces to the tachyon case.

We may guess a form for the vertex operator of graviton. Carrying spin two, we need a second rank tensor indices. Likewise, we can make the vertex operator singlet under the world sheet transformation. Thus, we obtain

$$: \partial_\alpha X_\mu \partial^\alpha X_\nu e^{ik \cdot X} : \quad (5.49)$$

Since each ∂X carries conformal weight 1, the total conformal weight is

$$1 + 1 - \frac{k^2}{4} = 2,$$

and it reduces to $k^2 = 0$. The *graviton should be strictly massless*.

5.1.3 Light-Cone Gauge

Here we will consider quantization in the light-cone frame. Although we discard the manifest Lorentz invariance (although not lost) of the theory, we can easily obtain the spectrum explicitly. Due to the residual gauge freedom, we can solve the constraint equation (5.15) explicitly in this light-cone frame, to leave only physical degrees.

In the previous subsection, we observed that there is a residual gauge freedom (5.35). Under it, $\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$ transforms as

$$\tilde{\tau} = \frac{1}{2}[\tilde{\sigma}^+(\tau + \sigma) + \tilde{\sigma}^-(\tau - \sigma)]. \quad (5.50)$$

For $\tilde{\tau}$, this is nothing but a general solution (5.12) of the wave equation (5.11), $(\partial_\sigma^2 - \partial_\tau^2)\tilde{\tau} = 0$. So any linear combination of string coordinates X^μ can be a solution up to a multiplicative and an additive constant.

What choice will be the most useful one? We will show that, by a suitable choice we can solve the constraint equation (5.15) fully to leave only the physical (transverse) degrees of freedom. It is done by introducing the following light-cone coordinates,

$$X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}). \quad (5.51)$$

In this coordinate,² $\eta_{ij} = \text{diag}(-1, -1, \dots, -1)$ with $(i, j = 1, 2, \dots, D-2)$, $\eta_{+-} = 1$, and $\eta_{-+} = 1$, thus the inner products look as

$$V \cdot W = +V^+W^- + V^-W^+ - V^iW^i . \quad (5.52)$$

Now, using the gauge freedom let us set $\tilde{\tau}$ to a spacetime direction as $\tilde{\tau} = X^+/p^+ + (\text{constant})$. In other words, we choose

$$X^+(\tau, \sigma) = x^+ + p^+\tau , \quad (5.53)$$

and p^+ is interpreted as the conjugate momentum to x^+ . Stated differently, the oscillator coefficients α_n^+ are chosen to be zero for $n \neq 0$. It amounts to making the world-sheet time direction and X^+ coincident in the infinite p^+ limit; thus this gauge choice is called the *light-cone gauge*. From (5.15) the constraint equations $(\dot{X} \pm X')^2 = 0$ lead to

$$2p^+(\dot{X}^- \pm X'^-) = \sum_i (\dot{X}^i \pm X'^i)^2 . \quad (5.54)$$

So far, we have solved the constraint equation to eliminate X^- . Namely, $\dot{X}^-$ and X'^- can be expressed in terms of X^i ,

$$2p^+\dot{X}^- = \sum_i [(\dot{X}^i)^2 + (X'^i)^2] , \quad (5.55)$$

$$2p^+X'^- = \sum_i 2\dot{X}^i X'^i . \quad (5.56)$$

Thus, in the light-cone gauge X^i are the only dynamical degrees.

We are left to specify the closed or open string boundary conditions. Here we will consider the closed string case, since the open string case is more straightforward. We have the following mode expansion for X^- ,

$$X^- = x^- + p^-\tau + \frac{i}{2} \sum_{n \neq 0} \left(\frac{1}{n} \alpha_n^- e^{-in(\tau-\sigma)} + \frac{1}{n} \tilde{\alpha}_n^- e^{-in(\tau+\sigma)} \right) \quad (5.57)$$

Inserting this into (5.55, 5.56) and comparing term by term, we can solve α_n^- , $\tilde{\alpha}_n^-$ in terms of α_n^i , $\tilde{\alpha}_n^i$.

$$\alpha_n^- = \frac{1}{2p^+} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} (: \alpha_{n-m}^i \alpha_m^i : + : \tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i : - a\delta_{n0}) . \quad (5.58)$$

In particular, the identification $p^- = \alpha_0^-$ gives the mass shell condition,

² We will denote the transverse coordinates by roman indices i, j . The remaining two coordinates can be called $X^\pm$, or $\tilde{\tau}, \tilde{\sigma}$.

$$\begin{aligned}
\frac{1}{8}M^2 &= \frac{1}{8}p^2 = \frac{1}{8}[2p^+p^- - (p^i)^2] \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i) - a
\end{aligned} \tag{5.59}$$

where we used (5.58) in the second equation. Here, a is the vacuum constant³ arising from commuting α_n^i and α_{-n}^i , and $\tilde{\alpha}_n^i$ and $\tilde{\alpha}_{-n}^i$. Since we have $(D-2)$ physical degrees,

$$a = -(D-2) \sum_{n=1}^{\infty} \frac{n}{2} = -\frac{(D-2)}{2} \zeta(-1) = \frac{(D-2)}{24} . \tag{5.60}$$

The formally divergent $\zeta(-1)$ is calculated by an analytic continuation to give

$$\zeta(-1) = -\frac{1}{12} . \tag{5.61}$$

In fact, this regularization has a manifest physical meaning if we multiply the regulator by a physical quantity $e^{-\epsilon n(\pi/2p^+ \alpha' l)^{1/2}}$ and taking the limit $\epsilon \rightarrow 0$: the scale invariant (i.e. l -independent) quantity is (5.61). Also we will justify this prescription by requiring the modular invariance in Appendix C.

Also by comparing the $n=0$ parts of both sides of (5.56), we have

$$\sum_{n>0} \alpha_{-n}^i \alpha_n^i = \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \tag{5.62}$$

Thus, we can summarize the closed string mass as

$$M^2 = M_L^2 + M_R^2, \quad M_L^2 = M_R^2 \tag{5.63}$$

where

$$\frac{1}{4}M_R^2 = \frac{1}{2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i - \frac{1}{2}a , \tag{5.64}$$

$$\frac{1}{4}M_L^2 = \frac{1}{2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i - \frac{1}{2}a . \tag{5.65}$$

For the open string, instead we obtain

$$\frac{1}{2}M^2 = \sum_{n>0} \alpha_{-n}^i \alpha_n^i - \frac{1}{2}a . \tag{5.66}$$

From the mode expansion, we construct a tower of physical states. Because of the Poincaré invariance in the target space, we have the unitary representation in terms of mass, spins, etc. With oscillators, we define the ground state that is annihilated by all the oscillator annihilation operators,

³ Note (5.36, 5.37).

$$\begin{aligned} p^i|0; k\rangle &= k^i|0; k\rangle, \\ \alpha_m^i|0; k\rangle &= 0, \quad m > 0 \end{aligned} \quad (5.67)$$

where k^μ is the eigenvalue of the momentum operator p^μ .

Note that we have $a = (D-2)/24$ from (5.60). The condition for vanishing Lorentz anomaly fixes $D = 26$ by “no-ghost” theorem [11], thus $a = 1$. Here we just present a heuristic argument. In the light-cone gauge of open strings, all string excitations are created by operating transverse oscillators α_{-n}^i on the vacuum. The ground state $|0\rangle$ is a tachyon since $M^2|0\rangle = -a|0\rangle = -|0\rangle$. As known in the Higgs potential, this signals that this vacuum is not the true vacuum, which requires a study of string field theory. Here, we ignore this tachyon problem. Let us consider the first excited state

$$\alpha_{-1}^i|0\rangle. \quad (5.68)$$

It has $(D-2)$ components ($i = 1, 2, \dots, D-2$) and is a vector representation of the transverse rotation group $\text{SO}(D-2)$. Its mass is $\frac{1}{4}\alpha' M^2 = 1 - a = 1 - (D-2)/24$. If we require the Lorentz symmetry in D dimensions, it should be massless. Therefore, we have $D = 26$ and $a = 1$.

For the closed string, according to (5.64, 5.65), the first excited states are massless states

$$\tilde{\alpha}_{-1}^i|0\rangle_L \otimes \alpha_{-1}^j|0\rangle_R. \quad (5.69)$$

We can decompose this as symmetric traceless, symmetric trace, and antisymmetric parts. They provide graviton G_{ij} , dilaton ϕ , and antisymmetric tensor B_{ij} . Following the above argument, it is easy to see that they are the massless representation of the Lorentz group $\text{SO}(1, 25)$.

Note that it is proven equivalent to the light-cone quantization. Therefore, without confusion we will conveniently use the covariant Lorentz indices μ, ν instead of the physical degree indices i, j .

5.1.4 Partition Function and Modular Invariance

The most important consistency condition that we will inspect every time later is the so-called *modular invariance*, emerging from one loop amplitude. Let us calculate the partition function, or Euclidianized one loop vacuum-to-vacuum amplitude,

$$\mathcal{Z} = \int [Dh][DX] e^{-S_E(h, X)}. \quad (5.70)$$

Topologically its Feynman diagram is torus. This torus is described by one complex number $\tau = \tau_1 + i\tau_2$, called the modular parameter⁴, or in the case of torus, *complex structure*. It is made by identifying sides of a parallelogram parameterized by 1 and τ , as shown in Fig. 5.2 (a). Without loss of generality we can fix $\tau_2 > 0$. For example, $\tau = i$ corresponds to a rectangle.

⁴ The modular parameter should not be confused with the world-sheet coordinate $\tau = \sigma^0$.

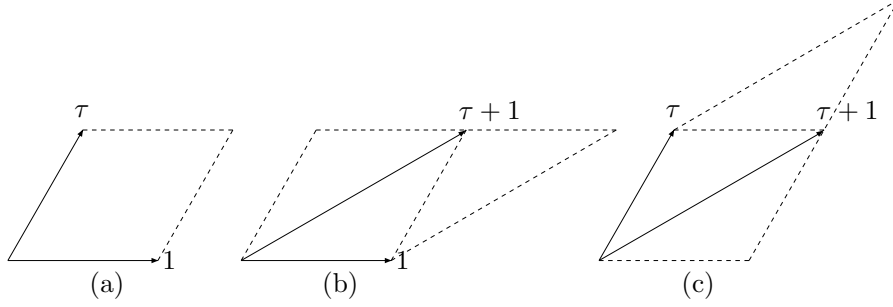


Fig. 5.2. (a) τ defines a torus bounded by the parallelogram. The subsequent figures show those modified by $\mathcal{T}$ (b), and by $\mathcal{TST}$ (c) which is conformally equivalent to the one by $\mathcal{S}$

The problem arises from the measure $[Dh]$ in (5.70). We cannot maintain the covariant gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$ globally *maintaining the original periodicity* [12]: the residual degree of freedom is embedded in τ so that at best we may have the form

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2. \quad (5.71)$$

Such non-fixable parameter(s) is called *modulus* and its number is determined by topology.

The string loop amplitude should not be affected by a reparametrization of τ , belonging to the modular transformation $\text{PSL}(2, \mathbf{Z})$,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1. \quad (5.72)$$

It is generated by two elements, $\mathcal{T}$ and $\mathcal{S}$,

$$\mathcal{T} : \tau \rightarrow \tau + 1, \quad \mathcal{S} : \tau \rightarrow -1/\tau. \quad (5.73)$$

They satisfy the relations $\mathcal{S}^2 = (\mathcal{ST})^3 = 1$. Under $\mathcal{T}$ the torus is deformed as Fig. 5.2(b). The role of $\mathcal{S}$ is not transparent, but the $\mathcal{TST}$ generates the transformation $\tau \rightarrow \tau/(\tau + 1)$ which corresponds to Fig. 5.2(c), which is equivalent to $\mathcal{S}$ up to conformal rescaling. Note that it exchanges world-sheet time σ^0 and space σ^1 directions.

As in the Fadeev–Popov gauge fixing case in gauge theories, we should divide the measure by the overcounting redundancy. We should restrict the integration region to the *fundamental region* $\mathbf{C}^1/\text{PSL}(2, \mathbf{Z})$,

$$|\tau| > 1, \quad |\tau_1| < \frac{1}{2}, \quad \tau_2 > 0 \quad (5.74)$$

as shown in Fig. 5.3. This is an orbifold having fixed points at $\tau = i$ and $\tau = e^{2\pi i/3}$. With the field theory analogy, the loop amplitude has the ultraviolet diverges when the region goes to $\text{Im}\tau \rightarrow 0$ [12], which is absent due to the

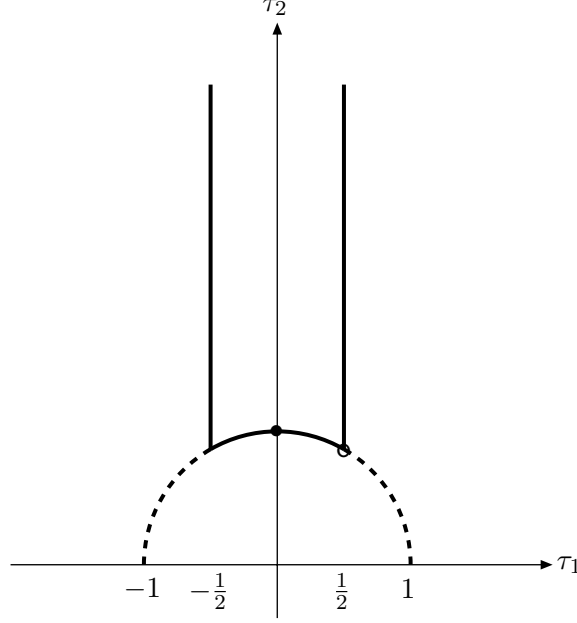


Fig. 5.3. The fundamental region is bounded by the bold lines and arc. Two fixed points are shown as $\bullet$ and $\circ$

nature of the string being an extended object. In the point particle limit, where the torus shrinks to a circle, this can be modified to a hexagonal diagram in ten dimension, which is responsible for the chiral and gravitational anomalies. It turns out that its low energy limit is anomaly free since anomaly comes from the failure of its regularization.

We form such torus [12] by field theory on a circle, evolving the Euclidian time by $2\pi\tau_2$ and translating in the σ direction by $2\pi\tau_1$, and then gluing the ends together. They are generated by the world-sheet Hamiltonian $H = \tilde{L}_0 + L_0 - \tilde{a} - a$ (5.40) and the momentum $P = \tilde{L}_0 - L_0$ respectively. Thus, we have

$$\mathcal{Z}(\tau, \bar{\tau}) = \text{Tr}[\exp(2\pi i\tau_1 P - 2\pi\tau_2 H)] \quad (5.75)$$

$$= \text{Tr}[q^{\tilde{L}_0 - \tilde{a}} \bar{q}^{L_0 - a}] \quad (5.76)$$

where $q = \exp(2\pi i\tau)$. The resulting one loop amplitude has the form

$$\mathcal{Z} = V \int \frac{d\tau d\bar{\tau}}{(\text{Im}\tau)^2} \mathcal{Z}(\tau, \bar{\tau}). \quad (5.77)$$

We will not consider constant volume factor V .

Let us calculate the bosonic string case. From (5.39), $\tilde{L}_0$ and L_0 are separated into continuous (center of momentum) and discrete (oscillator) parts. In

the light cone gauge we have 24 physical degrees of freedom; For each degree it is

$$\int \frac{dk}{2\pi} e^{-\pi\tau_2 k^2} \prod_{n=1}^{\infty} \sum_{N_n, \tilde{N}_n=0}^{\infty} q^{n\tilde{N}_n + \frac{n}{2}} \bar{q}^{nN_n + \frac{n}{2}} . \quad (5.78)$$

where $n\tilde{N}_n = \tilde{\alpha}_{-n}\tilde{\alpha}_n$ and $nN_n = \alpha_{-n}\alpha_n$. The continuous part is simply the Gaussian integral

$$\int \frac{dk}{2\pi} e^{-\pi\tau_2 k^2} = (4\pi^2\tau_2)^{-1/2} . \quad (5.79)$$

The oscillator part is a geometric sum

$$\prod_{n=1}^{\infty} \sum_{\tilde{N}_n=0}^{\infty} q^{\frac{n}{2} + n\tilde{N}_n} = \prod_{n=1}^{\infty} q^{\frac{n}{2}} \sum_{\tilde{N}_n=0}^{\infty} q^{n\tilde{N}_n} \quad (5.80)$$

$$= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \dots) \quad (5.81)$$

$$= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)^{-1} \quad (5.82)$$

$$\equiv \eta(\tau)^{-1}, \quad (5.83)$$

which is the definition of Dedekind eta function. Note that the unclear zeta function regularization $c = -\frac{1}{24}$ of (5.61) was used, but it is the only consistent way when we want modular invariance below. By definition, the partition function is the generating function of the state number d_n with mass squared being $8n$

$$\mathcal{Z}(\tau) = \sum_{n=0}^{\infty} d_n q^n, \quad \frac{M^2}{8} = n . \quad (5.84)$$

The right mover is equivalent to the left one, except the anti-holomorphic argument $\tau \rightarrow \bar{\tau}$. Including also continuous part, with 24 identical such physical degrees of freedom, we have finally

$$\mathcal{Z}(\tau, \bar{\tau}) = \left[(4\pi^2\tau_2)^{-1/2} |\eta(\tau)|^{-2} \right]^{24} . \quad (5.85)$$

One can verify that this is invariant under the modular transformations,

$$\mathcal{T}: \quad \eta(\tau + 1) = e^{i\pi/12} \eta(\tau) \quad (5.86)$$

$$\mathcal{S}: \quad \eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau) . \quad (5.87)$$

5.2 Superstring

Now let us introduce supersymmetry in string theory, which leads to *superstring theory*. Many important features of superstring rely on supersymmetry.

Historically, before the linear realization of space-time supersymmetry in four spacetime dimensions, supersymmetry was introduced in string theory first in two dimensional world-sheet [13, 14].

For our purpose, it will suffice to restrict our discussion to the Neveu–Schwarz–Ramond (NSR) formalism. It has only the world-sheet supersymmetry manifestly, but indeed has the spacetime supersymmetry also when we introduce a projection introduced by Gliozzi, Scherk and Olive (the GSO projection) [15]. As in the previous section, here we will consider the light-cone quantization also.

5.2.1 World-Sheet Action

In addition to the bosonic fields $X^\mu(\tau, \sigma)$, $\mu = 0, 1, \dots, D-1$, let us also introduce the same number of fermionic fields $\Psi^\mu(\tau, \sigma)$ in the world-sheet. The minimal choice is the two-component Majorana–Weyl fermion. The action

$$S = -\frac{1}{2\pi} \int d^2\sigma \left\{ \partial_\alpha X^\mu \partial^\alpha X_\mu - i \bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu \right\} \quad (5.88)$$

is invariant under the two dimensional global supersymmetry in the world-sheet,

$$\delta X^\mu = \xi \bar{\Psi}^\mu, \quad \delta \Psi^\mu = -i \rho^\alpha \partial_\alpha X^\mu \xi. \quad (5.89)$$

The two dimensional gamma matrices are taken as

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (5.90)$$

for which the algebra is

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta} I. \quad (5.91)$$

Since we chose purely imaginary gamma matrices, the Dirac operator is real and so are the components of Majorana spinors $\Psi^* = \Psi$. In (5.88), we formed a Lorentz scalar with $\bar{\Psi} \equiv \Psi^\dagger \rho^0$.

We see that the fermion Ψ^μ has the same vector index as the boson X^μ and has the same $\text{SO}(D-1, 1)$ Poincaré invariance. It seems strange that the fermion transforms as a vector, defying the spin-statistics theorem. However on the world-sheet point of view, the vector transformation property is the symmetry of target space and hence it is merely an internal symmetry. Without violating the spin-statistics theorem, it is a spin half fermion on the world-sheet but interestingly we will see that it can be either boson or fermion in the spacetime.

For our purpose, there is no need to construct an explicit action with reparametrization invariance or local supersymmetry. It turns out [1] that the action (5.88) is understood as gauge fixed form of local supersymmetric one, as in the bosonic case. Thus we only need to take care of additional constraint. The Euler-Lagrange equations are

$$\partial_\alpha \partial^\alpha X^\mu = 0, \quad i\rho^\alpha \partial_\alpha \Psi^\mu = 0. \quad (5.92)$$

They are to be supplemented by the constraints: the vanishing world-sheet energy-momentum tensor

$$\begin{aligned} T_{\alpha\beta} = & -\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{i}{4} \bar{\Psi}^\mu (\rho_\alpha \partial_\beta + \rho_\beta \partial_\alpha) \Psi_\mu \\ & + \frac{1}{2} \eta_{\alpha\beta} \left(\partial_\gamma X^\mu \partial^\gamma X_\mu + \frac{i}{2} \bar{\Psi}^\mu \rho^\gamma \partial_\gamma \Psi_\mu \right), \end{aligned} \quad (5.93)$$

and the vanishing world-sheet supercurrent

$$J_\alpha = \frac{1}{2} \rho^\beta \rho_\alpha \Psi^\mu \partial_\beta X_\mu, \quad (5.94)$$

whose spatial integral gives supersymmetry generator Q_α . They are superpartners of each other, which is natural because two successive supersymmetry transformations generate a translation which is generated by the energy-momentum tensor.

It is convenient to rewrite the fermionic part of the action (5.88) as

$$S_F = \frac{i}{\pi} \int d^2\sigma (\Psi_L^\mu \partial_- \Psi_{\mu L} + \Psi_R^\mu \partial_+ \Psi_{\mu R}), \quad (5.95)$$

with

$$\Psi^\mu = \begin{pmatrix} \Psi_R^\mu(\tau - \sigma) \\ \Psi_L^\mu(\tau + \sigma) \end{pmatrix}. \quad (5.96)$$

We denoted so because the left and right handed fermions Ψ_L, Ψ_R satisfy

$$\partial_- \Psi_L^\mu = 0, \quad \partial_+ \Psi_R^\mu = 0. \quad (5.97)$$

The boundary conditions come from the vanishing boundary term

$$\left[\psi_L \delta \psi_L - \psi_R \delta \psi_R \right]_{\sigma=0}^\pi = 0. \quad (5.98)$$

Since it has two Ψ s with one variation, the surface term goes away for closed strings if the solution is periodic or antiperiodic

$$\text{Closed} \begin{cases} \Psi_L^\mu(\tau, \sigma + \pi) = \pm \Psi_L^\mu(\tau, \sigma), \\ \Psi_R^\mu(\tau, \sigma + \pi) = \pm \Psi_R^\mu(\tau, \sigma). \end{cases} \quad (5.99)$$

Therefore, we have four combinations of boundary conditions. We will see that the spacetime supersymmetry requires that the theory should contain all the four sectors. For open strings,

$$\text{Open} \begin{cases} \Psi_L^\mu(\tau, \sigma) = \Psi_R^\mu(\tau, \sigma), \\ \Psi_L^\mu(\tau, \sigma + \pi) = \pm \Psi_R^\mu(\tau, \sigma + \pi). \end{cases} \quad (5.100)$$

The periodic boundary condition is called the *Ramond boundary condition* (R), and the anti-periodic boundary condition is called the *Neveu-Schwarz boundary condition* (NS). More general boundary conditions are possible for a less symmetric case that we will consider in the following chapter.

Super-Virasoro Algebra

For closed strings, the boundary conditions R or NS are imposed independently for the left and right movers Ψ_L and Ψ_R . Since the left and right moving states are identical, we will deal with the right movers in what follows. The expressions for left movers are easily obtained when we replace $\tau - \sigma$ with $\tau + \sigma$. On the right movers we have two choices for boundary conditions

$$\text{R : } \Psi_R^\mu = \sum_{n \in \mathbf{Z}} d_n^\mu e^{-2in(\tau - \sigma)} \quad (5.101)$$

$$\text{NS : } \Psi_R^\mu = \sum_{r \in \mathbf{Z} + 1/2} b_r^\mu e^{-2ir(\tau - \sigma)} . \quad (5.102)$$

From the action (5.95), the canonical momenta corresponding to $\Psi_{L,R}$ are $\frac{i}{2\pi} \Psi_{L,R}$, whence the canonical quantization of the fermionic degrees,

$$\begin{aligned} \{\Psi_L^\mu(\tau, \sigma), \Psi_L^\nu(\tau, \sigma')\} &= \{\Psi_R^\mu(\tau, \sigma), \Psi_R^\nu(\tau, \sigma')\} = -2\pi\delta(\sigma - \sigma')\eta^{\mu\nu} , \\ \{\Psi_L^\mu(\tau, \sigma), \Psi_R^\nu(\tau, \sigma')\} &= 0 , \end{aligned}$$

leads to the following anti-commutators for the oscillators

$$\text{R : } \{d_m^\mu, d_n^\nu\} = \{\tilde{d}_m^\mu, \tilde{d}_n^\nu\} = -\delta_{m+n,0} \eta^{\mu\nu} \quad (5.103)$$

$$\text{NS : } \{b_r^\mu, b_s^\nu\} = \{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = -\delta_{r+s,0} \eta^{\mu\nu}$$

with the anti-commutator of left and right movers vanishing. For $m = n = 0$, the d_0^μ has the commutation relation

$$\{d_0^\mu, d_0^\nu\} = -\eta^{\mu\nu} . \quad (5.104)$$

This is the 10D gamma matrices if we define Γ^μ as

$$\Gamma^\mu = i\sqrt{2}d_0^\mu . \quad (5.105)$$

Thus, the R sector corresponds to the fermionic sector in 10D space-time. On the other hand, the NS sector corresponds to the bosonic sector in 10D space-time.

Here also, we will consider the closed strings only. We have seen that the bosonic Virasoro algebra was introduced from the constraint equations. In the fermionic case also, the super-Virasoro algebra arises from constraint equations, $T_{\alpha\beta} = 0$ and $J^\alpha = 0$. These constraints become

$$\begin{aligned} T_{++} &= -\partial_+ X_L^\mu \partial_+ X_{\mu L} - \frac{i}{2} \Psi_L^\mu \partial_+ \Psi_{\mu L} = 0 \\ T_{--} &= -\partial_- X_R^\mu \partial_- X_{\mu R} - \frac{i}{2} \Psi_R^\mu \partial_- \Psi_{\mu R} = 0 \\ J_+ &= \Psi_L^\mu \partial_+ X_{\mu L} = 0 \\ J_- &= \Psi_R^\mu \partial_- X_{\mu R} = 0 . \end{aligned} \quad (5.106)$$

As in the bosonic case, the Virasoro operators

$$L_m = \frac{1}{2\pi} \int_0^\pi d\sigma e^{2im\sigma} T_{--} \quad (m \neq 0)$$

are defined. Similarly, the $\tilde{L}_m$ operators are defined from T_{++} . For the supercurrent, the Fourier components of the right-movers are

$$\begin{aligned} \text{R} : F_m &= \frac{1}{2\pi} \int_0^\pi d\sigma e^{2im\sigma} J_- , \quad \text{for } m \in \mathbf{Z}, m \neq 0 \\ \text{NS} : G_r &= \frac{1}{2\pi} \int_0^\pi d\sigma e^{2ir\sigma} J_- , \quad \text{for } r \in \mathbf{Z} + \frac{1}{2} . \end{aligned} \quad (5.107)$$

Similarly, $\tilde{F}_m$ and $\tilde{G}_r$ for the left-movers can be given. Accordingly, for the NS and R sectors we have

$$\begin{aligned} \text{NS} : L_m &= -\frac{1}{2} \sum_{n \in \mathbf{Z}} \alpha_{m-n}^\mu \alpha_{\mu n} + \frac{1}{2} \sum_{r \in \mathbf{Z} + \frac{1}{2}} \left(\frac{m}{2} - r \right) b_{m-r}^\mu b_{\mu r}, \quad m \neq 0, \\ G_r &= - \sum_{n \in \mathbf{Z}} b_{r-n}^\mu \alpha_{\mu n} \\ \text{R} : L_m &= -\frac{1}{2} \sum_{n \in \mathbf{Z}} \alpha_{m-n}^\mu \alpha_{\mu n} + \frac{1}{2} \sum_{n \in \mathbf{Z}} \left(\frac{m}{2} - n \right) d_{m-n}^\mu d_{\mu n}, \quad m \neq 0, \\ F_m &= - \sum_{n \in \mathbf{Z}} d_{m-n}^\mu \alpha_{\mu n} . \end{aligned} \quad (5.108)$$

Also, the same kinds of expressions hold for left movers, $\tilde{L}_m, \tilde{G}_r, \tilde{F}_m$. As in the bosonic case, we need the normal ordering for the $m = 0$ case. In terms of the normal ordered oscillators, the right movers are

$$\text{NS} : L_0 \equiv -\frac{1}{2} \sum_{n \in \mathbf{Z}} : \alpha_{-n}^\mu \alpha_{\mu n} : -\frac{1}{2} \sum_{r \in \mathbf{Z} + \frac{1}{2}} r : b_{-r}^\mu b_{\mu r} : \quad (5.109)$$

$$\text{R} : L_0 \equiv -\frac{1}{2} \sum_{n \in \mathbf{Z}} : \alpha_{-n}^\mu \alpha_{\mu n} : -\frac{1}{2} \sum_{n \in \mathbf{Z}} n : d_{-n}^\mu d_{\mu n} : \quad (5.110)$$

And the left movers are the same. These generate the supersymmetric extension of the conformal field theory. Now the super-Virasoro algebra can be written with G s, F s, and L s,

$$\begin{aligned}
\text{NS : } \quad [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m+n,0} \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} , \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \\
\text{R : } \quad [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n,0} \\
[L_m, F_n] &= \left(\frac{m}{2} - n\right) F_{m+n} , \\
\{F_m, F_n\} &= 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n,0} .
\end{aligned}$$

All the same relations hold for the left-mover operators. We see that the commutators between L_n looks the same as the original Virasoro algebra except for the coefficient of the central charge.

For the physical states, we require the constraint equations to be satisfied. Thus,

$$\begin{aligned}
\text{NS : } \quad (L_n - \delta_{n0}a_{\text{NS}})|\phi\rangle &= 0, \quad n \geq 0; \quad G_r|\phi\rangle = 0, \quad r > 0 \\
\text{R : } \quad (L_n - \delta_{n0}a_{\text{R}})|\phi\rangle &= 0, \quad n \geq 0; \quad F_m|\phi\rangle = 0, \quad m \geq 0
\end{aligned} \quad (5.111)$$

where a_{NS} and a_{R} are constants to be determined in the next subsection. Similar relations hold for the left movers.

5.2.2 Light-Cone Gauge

To draw physical degrees only, we take the light-cone gauge as done in the bosonic case. Taking the light-cone coordinate, we choose the same form for X^+ as given in (5.53),

$$X^+(\sigma, \tau) = x^+ + p^+\tau .$$

All the steps are the same as presented throughout (5.20–5.22). By supersymmetry transformation (5.89) we fix

$$\psi^+ = 0 , \quad (5.112)$$

which is verified as $\delta X^+ = \bar{\xi}\psi^+ = 0$. Then we solve the constraint equations,

$$p^+\partial_+X_L^- = \partial_+X_L^i\partial_+X_L^i + \frac{i}{2}\Psi_L^i\partial_+\Psi_L^i , \quad (5.113)$$

$$p^+\partial_-X_R^- = \partial_-X_R^i\partial_-X_R^i + \frac{i}{2}\Psi_R^i\partial_-\Psi_R^i , \quad (5.114)$$

$$\frac{1}{2}p^+\Psi_L^- = X_L^i\partial_+\Psi_L^i , \quad (5.115)$$

$$\frac{1}{2}p^+\Psi_R^- = X_R^i\partial_+\Psi_R^i , \quad (5.116)$$

for X^- and ψ^- in terms of X^i and ψ^i s. Using the same method as done in the bosonic case, we obtain the oscillators α^- in terms of α_n^i , b^i and d^i as

$$\alpha_n^-(X, \Psi) = \alpha_n^-(X) + \frac{1}{2p^+} \sum_{r=-\infty}^{\infty} \left(r - \frac{1}{2}n \right) : b_{n-r}^i b_r^i : - \frac{a}{2p^+} \delta_{n,0} \quad (5.117)$$

where the first term on the RHS is from the bosonic part of the action (5.58). Also for the fermionic NS degrees, we have

$$b_r^- = \frac{1}{p^+} \sum_{i=1}^{D-2} \sum_{s \in \mathbf{Z} + \frac{1}{2}} \alpha_{r-s}^i b_s^i. \quad (5.118)$$

The same relation is obtained in the R sector, if we set $n = r + 1/2$ and replace b_r with d_n ,

$$d_n^- = \frac{1}{p^+} \sum_{i=1}^{D-2} \sum_{s=-\infty}^{\infty} \alpha_{n-s}^i d_s^i. \quad (5.119)$$

As in the bosonic case, we want to check the hidden Lorentz invariance for the generators,

$$M^{\mu\nu} = \int_0^\pi d\sigma [(X^\mu P^\nu - X^\nu P^\mu) + (\Psi^\mu P_\Psi^\nu - \Psi^\nu P_\Psi^\mu)] \quad (5.120)$$

which should satisfy the Lorentz algebra,

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma}). \quad (5.121)$$

The additional pieces are provided by the fermionic superpartners. Thus, the generator for the Lorentz group has another piece K ,

$$M^{\mu\nu} = -M^{\nu\mu} = M_0^{\mu\nu} + K^{\mu\nu} \quad (5.122)$$

where M_0^{-i} is the part contributed by bosonic coordinate only. The mixed commutator $[M, K]$ should vanish. For the transverse degrees M^{ij} , it is straightforward to verify the Lorentz algebra. The check for the Lorentz algebra in the light-cone gauge is needed for those involving the lightlike coordinates. Thus, we find

$$[M^{-i}, M^{-j}] = \frac{-1}{(p^+)^2} \sum_{i=1}^{\infty} (\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i) (\Delta_n - n) \quad (5.123)$$

where

$$\Delta_n = n \left(\frac{D-2}{8} \right) + \frac{2}{n} \left(a_{\text{NS}} - \frac{D-2}{16} \right) \quad (5.124)$$

where a_{NS} is the normal ordering constant in the NS sector. Since the Lorentz algebra (5.121) gives $[M^{-i}, M^{-j}] = 0$, we need $\Delta_n = n$ for Lorentz symmetry, i.e. for a consistent NSR string we must require

$$D = 10, \quad a_{\text{NS}} = \frac{1}{2}. \quad (5.125)$$

This provides another proof of relations among Lorentz symmetry, conformal anomaly a and consistent spacetime dimension D .

Finally, we get the mass shell conditions, following the method of bosonic string,

$$M^2 = p^2 = M_R^2 + M_L^2 \quad (5.126)$$

In the NS sector, we obtain

$$\frac{1}{4}M_R^2 = N^\alpha + N^b - \frac{1}{2} = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i - \frac{1}{2} \quad (5.127)$$

$$\frac{1}{4}M_L^2 = \tilde{N}^\alpha + \tilde{N}^b - \frac{1}{2} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \sum_{r=1/2}^{\infty} r \tilde{b}_{-r}^i \tilde{b}_r^i - \frac{1}{2} \quad (5.128)$$

We should supplement the constraint,

$$M_R^2 = M_L^2. \quad (5.129)$$

In the Ramond sector the normal ordering constant is not needed since there appears an exact cancellation between bosonic and fermionic degrees,

$$a_{\text{R}} = 0. \quad (5.130)$$

The mass shell condition is also the same, if we let $n = r + \frac{1}{2}$ and replace b_r with d_n .

For a later use, we just summarize the zero point energy a for each physical degree. For each physical boson we have $+\frac{1}{24}$ by (5.60). By supersymmetry, we have the (negative) same contribution $-\frac{1}{24}$ for each fermions in the R sector. In the NS sector, we deduce it is $+\frac{1}{48}$ to have the result (5.125). In Appendix C we will sketch “the proof” for these values by considering more general situations.

5.2.3 Spectrum and GSO Projection

The states are constructed by piling up oscillators in the Fock space. We will consider a single right mover, which is exactly the same as the left mover. The physical conditions (5.111) define the unique vacuum in each sector.

NS Sector

Consider the NS sector. The lowest state is the vacuum

$$|0\rangle. \quad (5.131)$$

From the mass shell condition (5.126), it is tachyonic since the mass is $\frac{1}{8}M^2 = -1$, which is undesirable. The first excited states are

$$b_{-1/2}^i |0\rangle, \quad (5.132)$$

which are massless. They carry the transversal $SO(8)$ index i , and therefore form a massless representation of $SO(1, 9)$. This means, the NS states become *bosonic* states in the spacetime sense. Applying a few more oscillators, we immediately meet the following problem. The Fock space has a tower of states,

$$|\varphi\rangle = b_{-r_1}^{i_1} b_{-r_2}^{i_2} \cdots b_{-r_M}^{i_M} |0\rangle. \quad (5.133)$$

For $b_{-r_k}^{i_k}$ satisfying the world-sheet fermionic relation (5.103), exchanging two indices yields extra factor (-1) , which is not desirable for statistics of spacetime bosons.

A consistent way of projecting out some states from the Fock space has been formulated by Gliozzi, Scherk and Olive (GSO) [15], which is known as *the GSO projection*,

$$\frac{1 - (-1)^F}{2}, \quad (5.134)$$

with fermion oscillator number F in the NS sector

$$F = \sum_{r=1/2}^{\infty} b_{-r}^i b_r^i. \quad (5.135)$$

This projection has an additional advantage. A priori it is not clear whether the state from NSR formulation has 10D space-time supersymmetry, but with the GSO projection, both in the NS and R sectors, only a supersymmetric spectrum is kept.

R Sector

In the R sector, the zero point energy $a_R = 0$; thus the lowest state is massless and this sector is tachyon-free. The commutation relation (5.103) $\{d_0^\mu, d_0^\nu\} = -\eta^{\mu\nu}$ implies that d_0^μ behave as gamma matrices (5.105). In the light-cone gauge, defining

$$d^{i\pm} = \frac{1}{\sqrt{2}}(d^{2i} \pm i d^{2i+1}), \quad i = 1, 2, 3, 4, \quad (5.136)$$

they satisfy another similar algebra

$$\{d^{i+}, d^{j-}\} = \delta_{ij}, \quad (5.137)$$

with other (anti)commutators vanishing. We have spinorial representation that we saw in the previous chapter. Therefore, each represents the creation

(+) and annihilation (−) operator of the spin- $\frac{1}{2}$ system. Thus, we can build up the spinorial state denoted as

$$|s_1 s_2 s_3 s_4\rangle, \quad (5.138)$$

where each s_i can assume $\pm\frac{1}{2}$. The vacuum $|\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}\rangle$ is defined as that annihilated by all the d_0^{i-} . We can make the following $2^4 = 16$ massless states

$$\begin{aligned} &|0\rangle \\ &d_0^{i+}|0\rangle \\ &d_0^{i_1+}d_0^{i_2+}|0\rangle \\ &d_0^{i_1+}d_0^{i_2+}d_0^{i_3+}|0\rangle \\ &d_0^{i_1+}d_0^{i_2+}d_0^{i_3+}d_0^{i_4+}|0\rangle. \end{aligned} \quad (5.139)$$

They make up spacetime fermions. Especially, we can define the chirality operator

$$\Gamma = d_1 d_2 \cdots d_8 = 2^4 s_1 s_2 s_3 s_4, \quad (5.140)$$

which gives $\Gamma = +1$ or $\Gamma = -1$, respectively, for even or odd numbers of $-\frac{1}{2}$ in $s_1 s_2 s_3 s_4$. Accordingly such states are called spinorial $\mathbf{8}_s$ of $\text{SO}(8)$ for $\Gamma = +1$ and conjugate spinorial $\mathbf{8}_c$ of $\text{SO}(8)$ for $\Gamma = -1$.

Then, we encounter a problem if we require the spacetime supersymmetry, since we have too many R states compared to those in the NS sector. To match, we introduce another GSO projector in this R sector

$$\frac{1 + \Gamma(-1)^F}{2}, \quad (5.141)$$

with fermion number in the R sector given as, similarly to (5.135),

$$F = \sum_{m=1}^{\infty} d_{-m}^i d_m^i, \quad (5.142)$$

or in the s basis,

$$\Gamma = \sum_{i=1}^4 s_i = 0, \pmod{2}. \quad (5.143)$$

We can choose Γ , which can be called the chirality, as we wish. We can check that we have the same number of fermions in the R sector as bosons in the NS sector. Furthermore, every state matches at the full massive level.

So far we discussed the 10D spinors with four s_i 's. This argument can be repeated in other dimensions, and spinors in various dimensions obtained in this way are listed in Table 3.4.

Spin Structure

Here we will calculate partition function for superstring. The procedure is the same as before discussed in Subsect. 5.1.4. The alteration is that the generators $L_0, \tilde{L}_0$ of (5.109) and (5.110) now contain fermionic parts in addition, and we have eight physical degrees of freedom.

The fermionic oscillator can be either occupied or unoccupied

$$\text{Tr} q^{N^b} = \prod_{r=1/2}^{\infty} \text{Tr} q^{r^{b-r} \cdot b_r} = \prod_{n=1}^{\infty} (1 + q^{n-1/2})^8, \quad (5.144)$$

$$\text{Tr} (-1)^F q^{N^b} = \prod_{r=1/2}^{\infty} \text{Tr} (-1)^F q^{r^{b-r} \cdot b_r} = \prod_{n=1}^{\infty} (1 - q^{n-1/2})^8, \quad (5.145)$$

where we used dot product for brevity. Therefore, we have the result,

$$\mathcal{Z}_{\text{NS}} = \frac{1}{2} q^{-1/2} \prod_{n=1}^{\infty} (1 - q^n)^{-8} \left[\prod_{n=1}^{\infty} (1 + q^{n-1/2})^8 - \prod_{n=1}^{\infty} (1 - q^{n-1/2})^8 \right] \quad (5.146)$$

where we also included the vacuum energy of the NS state (5.131). Similarly, we have the R sector partition function,

$$\mathcal{Z}_{\text{R}} = 8 \text{Tr} q^N \quad (5.147)$$

where

$$N = \sum_{n=1}^{\infty} (\alpha_n^\dagger \alpha_n + n d_n^\dagger d_n), \quad (5.148)$$

and 8 is the zero mode degeneracy. Just using the results given above, we obtain

$$\mathcal{Z}_{\text{R}} = 8 \prod_{n=1}^{\infty} (1 - q^n)^{-8} (1 + q^n)^8 \quad (5.149)$$

These functions can be rewritten in a more fancy form.

$$\mathcal{Z}_{\text{R}}(\tau) = \left(\frac{\vartheta_4}{\eta} \right)^4 \equiv \left(\frac{\vartheta_{[1/2]}^{[0]}}{\eta} \right)^4 \quad (5.150)$$

$$\mathcal{Z}_{\text{NS}}(\tau) = \left(\frac{\vartheta_3}{\eta} \right)^4 - \left(\frac{\vartheta_2}{\eta} \right)^4 \equiv \left(\frac{\vartheta_{[0]}^{[0]}}{\eta} \right)^4 - \left(\frac{\vartheta_{[0]}^{[1/2]}}{\eta} \right)^4 \quad (5.151)$$

where ϑ_4, ϑ_3 , and ϑ_2 are defined also. Here, we have used the Jacobi theta function. They are defined as

$$\begin{aligned}
\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau) &= \sum_{n \in \mathbf{Z}} q^{\frac{1}{2}(n+\alpha)^2} e^{2\pi i(n+\alpha)\beta} \\
&= \eta(\tau) e^{2\pi i\alpha\beta} q^{\frac{\alpha^2}{2} - \frac{1}{24}} \\
&\quad \times \prod_{n=1}^{\infty} (1 + q^{n+\alpha-\frac{1}{2}} e^{2\pi i\beta}) (1 + q^{n-\alpha-\frac{1}{2}} e^{-2\pi i\beta}) . \quad (5.152)
\end{aligned}$$

The first definition in terms of sum is highly appropriate for our use, since the first factor counts the energy with a shift α and the second piece is needed for some projection by adjusting the phase β .

The theta function satisfies the modular transformation properties

$$\mathcal{T} : \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau + 1) = e^{-i\pi(\alpha^2 - \alpha)} \vartheta \begin{bmatrix} \alpha \\ \alpha + \beta - \frac{1}{2} \end{bmatrix}(\tau) , \quad (5.153)$$

$$\mathcal{S} : \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(-1/\tau) = (-i\tau)^{1/2} e^{2\pi i\alpha\beta} \vartheta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}(\tau) . \quad (5.154)$$

Miraculously, it has been known that $\mathcal{Z}_{\text{NS}} = \mathcal{Z}_{\text{R}}$ as an “abstruse identity” by Jacobi [2]. This relation is a necessary condition for the existence of space-time supersymmetry, although it does not prove supersymmetry. We will see in the next section that, when we construct explicit string models with the GSO projection, it gives the supersymmetric spectrum.

Now we perform modular transformation.

$$\begin{aligned}
\mathcal{T} : \quad \vartheta_2 &\rightarrow e^{\pi i/4} \vartheta_2, \quad \vartheta_3 \leftrightarrow e^{\pi i/4} \vartheta_4 \\
\mathcal{S} : \quad \vartheta_3/\eta &\rightarrow \vartheta_3/\eta, \quad \vartheta_4/\eta \leftrightarrow \vartheta_2/\eta . \quad (5.155)
\end{aligned}$$

With antiholomorphic right movers, the whole partition function is made invariant. Therefore, *we cannot make the modular invariant theory with only one of four sectors*. The consistency forces modular invariance.

We can reinterpret this result as follows. We imposed the R (periodic) and NS (antiperiodic) boundary conditions for the world-sheet fermion $\Psi(\tau, \sigma)$ in the σ direction as given in (5.98, 5.99). However, (5.95) indicates that we may assign equivalent boundary conditions in the τ direction. To see this, it is useful to rename each term in the original partition function (5.146, 5.147) as follows, revealing a regularity,

$$\mathcal{Z}_{\text{NS}}^{\text{NS}} = \text{Tr}(q^{L_0(\text{NS})}) , \quad (5.156)$$

$$\mathcal{Z}_{\text{R}}^{\text{NS}} = \text{Tr}((-1)^F q^{L_0(\text{NS})}) , \quad (5.157)$$

$$\mathcal{Z}_{\text{NS}}^{\text{R}} = \text{Tr}(q^{L_0(\text{R})}) , \quad (5.158)$$

$$\mathcal{Z}_{\text{R}}^{\text{R}} = \text{Tr}((-1)^F q^{L_0(\text{R})}) , \quad (5.159)$$

where $L_0(\text{NS}, \text{R})$ is the Virasoro operator (world-sheet Hamiltonian) constructed with the NS and R oscillators. F in (5.157) is the NS sector F number (5.135) and F in (5.159) is the R sector F number (5.142).

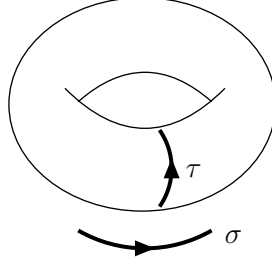


Fig. 5.4. When a fermion is transported around a cycle in the non-simply connected Feynman diagram, it acquires an ambiguous phase, which is remedied by the GSO projection

The GSO projection has an effect of taking into account both boundary conditions, which is referred to as *spin structures* [16]. This is required when we consider loop amplitudes. The corresponding Feynman diagrams are not simply connected (having genera), and there is a subtlety in defining boundary conditions for fermions living on them. The situation is depicted in Fig. 5.4. When rotated once around a closed cycle, a boson remains invariant, i.e. with no extra phase. However, a fermion acquires an ambiguous phase, where there are two possibilities. The GSO projection projects out consistently one of them that we do not want [1].

5.2.4 Superstring Theories

Closed String Theory: Type IIA, IIB

Now we have superstring theories. Let us investigate massless states, formed by the following left and right movers. Recall that when we perform the GSO projection in the R sector (5.141) for each mover, we have two choices of chirality Γ . Therefore, the following two choices are possible

$$\text{Type IIA: } (\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s)$$

$$\text{Type IIB: } (\mathbf{8}_v \oplus \mathbf{8}_c) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c)$$

Their bosonic superpartner $\mathbf{8}_v$ in the NS sector is denoted by X^i .

We can decompose the product in the $SO(8)$ representation. What they have in common in both Type IIA and Type IIB theories are the products

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{35}_v + \mathbf{28} + \mathbf{1}, \quad \text{NS-NS}, \quad (5.160)$$

$$\mathbf{8}_c \otimes \mathbf{8}_c = \mathbf{56}_s + \mathbf{8}_s, \quad \text{NS-R}, \quad (5.161)$$

which correspond to graviton $G_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$ and dilaton ϕ , and their superpartners gravitino ψ_μ and dilatino λ . The selection of chirality in the Ramond sector of either mover leads to different spectrum.

First consider choosing the different chirality $\Gamma_L = 1, \Gamma_R = -1$. The resulting spectrum is

$$\mathbf{8}_c \otimes \mathbf{8}_s = \mathbf{56}_v + \mathbf{8}_v, \quad \text{R-R} \quad (5.162)$$

$$\mathbf{8}_c \otimes \mathbf{8}_v = \mathbf{56}_c + \mathbf{8}_c, \quad \text{R-NS}, \quad (5.163)$$

which leads to *Type IIA theory*. The name comes from the supergravity theory having the same spectrum, namely type IIA ($N = 2, 32$ supercharges) supergravity. The chiral fields are antisymmetric tensor with rank 1(vector), 3 corresponds to $\mathbf{8}_{s,c}, \mathbf{56}_{s,c}$ in the R-R sector. We see that there are exactly the same amount of fields with opposite chiralities in the NS-R and R-NS sectors, therefore the theory is non-chiral and automatically anomaly free.

On the other hand if we select the same chirality $\Gamma_L = \Gamma_R = 1$ we have *Type IIB theory*, whose spectrum is

$$\mathbf{8}_s \otimes \mathbf{8}_s = \mathbf{35}_c + \mathbf{28} + \mathbf{1}, \quad \text{R-R}, \quad (5.164)$$

$$\mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{56}_s + \mathbf{8}_s, \quad \text{R-NS}. \quad (5.165)$$

The R-R antisymmetric tensors account for rank 0(scalar), rank 2, and rank 4 self-dual tensors. The theory is now chiral. Still one can check that the theory is free of anomalies.

Open String Theory: Type I

The open string theory consists of both open and closed strings. Especially, the closed string sector is unoriented, because it is always formed by (unoriented) open strings. The unorientedness is specified by the invariance under parity operation exchanging left and right movers. It is useful to facilitate Type IIB theory because it does not have the chirality and thus we keep the parity even states only to obtain the closed string sector of Type I string. Exchanging the left and right movers corresponds to exchanging two indices of the rank 2 fields; thus the antisymmetric tensor $B_{\mu\nu}$ is projected out. So, we have graviton $G_{\mu\nu}$ and dilaton ϕ . However, we know that in 10D supergravity, the bosonic multiplets are completed by the antisymmetric tensor. Note in this regard that Type IIB string theory has another set of fields from the Ramond-Ramond sector. Anticommuting world-sheet fields multiply an additional minus sign when exchanged, thus the projection only yield another rank two antisymmetric tensor $C^{\mu\nu}$ to complete the supergravity multiplet.

In the open string sector, massless degrees of freedom are gauge bosons, described by the Chan-Paton factors attached at both ends. Consistency of one-loop diagrams imposes a certain condition called the Ramond-Ramond tadpole cancellation [17]. It fixes the gauge group completely as $SO(32)$.

We have another theory the heterotic string which will be discussed in the following section.

5.3 Heterotic String

The later part of this book will be devoted to compactification of heterotic string theory [18] which we will review here in more detail. It describes the gauge group directly by assigning charge along the string. The consistent theory leads to the gauge group as either $SO(32)$ or $E_8 \times E_8$ with the dimension 496 for the adjoint representation, which is expected after the revolutionary work of Green and Schwarz [19]. They showed that in the ten dimensional supergravity theories with these gauge groups the gravitational and gauge anomalies are miraculously canceled. Soon afterward, the heterotic string theory was found [18]. Its low energy limit $\alpha' \rightarrow 0$ describes correctly the desired ten dimensional supergravity with the gauge group suggested in [19].

Heterotic String

As the word heterotic (“hybrid vigor”) implies, the *heterotic string* has different world-sheet theories for the left and right movers. This possibility arises because left and right movers of the closed string behave independently, as in (5.12). Conventionally we define the heterotic string as

a closed string with $(N_L, N_R) = (0, 1)$ world-sheet supersymmetries on the left and right movers, respectively.

Which one is called left is a matter of convention.

At tree level, canceling the conformal anomaly determines the world-sheet theory with $N = 0$ and $N = 1$ supersymmetries should be the same as the 26 dimensional bosonic and the 10 dimensional superstring theories, respectively. A closed string with $(N_L, N_R) = (0, 1)$ world sheet supersymmetry has asymmetric representation for the left and right movers. It is composed of the left half $(\tau + \sigma)$ of bosonic string and the right half $(\tau - \sigma)$ of the fermionic string. The field contents are displayed in Table 5.1. We are mainly interested in this “bosonic description” of the coordinates X^I , since it directly gives the group degrees of freedom and yields a nice geometric interpretation. Yet there is another equivalent description of the theory in 2D. Recall that by the spacetime dimension we mean just how many bosonic (and fermionic for the supersymmetric theory) fields X^I we employ. We know that two fermions play the role of one holomorphic boson. Therefore, there is an alternative and equivalent “fermionic version” of the theory with 16 bosonic left movers replaced by 32 fermions. Also at the loop level we require the modular invariance condition as a consistency condition, which will be discussed in Appendix B.1.

5.3.1 Nonabelian Gauge Group

To have a ten dimensional theory, we compactify the sixteen left moving bosonic degrees on the torus. This leads to nonabelian gauge groups in the spirit of the Kaluza–Klein theory. Let us see how the nonabelian groups arise from the compactified internal space.

Table 5.1. The degrees of freedom of heterotic string

Left	Right	L-Oscil.	R-Oscil.	Indices
$X^\mu(\tau + \sigma)$	$X^\mu(\tau - \sigma)$	$\tilde{\alpha}_n^\mu, p_L^\mu$	$\alpha_n^\mu, p_R^\mu,$	$\mu = 0, 1, \dots, 9$
$X^I(\tau + \sigma)$	$\Psi^\mu(\tau - \sigma)$	$\tilde{\alpha}_n^I$	b_r^μ, d_m^μ	$I = 1, \dots, 16$

Toroidal Compactification

Firstly, let us see what happens by compactifying just one coordinate, say the D^{th} , on a circle of radius R by identifying

$$x^D \equiv x^D + 2\pi R. \quad (5.166)$$

The D dimensional metric G_{MN} can be decomposed generically as

$$ds^2 = \tilde{G}_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + G_{DD} (dx^D + A_\mu dx^\mu)^2. \quad (5.167)$$

Note that $\tilde{G}_{\mu\nu}$ is different from $G_{\mu\nu}$. Now we observe that there exists the position-dependent translational symmetry in the metric,

$$x'^D = x^D + \lambda(x^\mu), \quad (5.168)$$

$$A'_\mu = A_\mu - \partial_\mu \lambda(x^\mu). \quad (5.169)$$

Compactification of many dimensions yield as many Abelian gauge bosons and a generalized isometry to (5.168) might even lead to a nonabelian gauge group, whose possibility will not be pursued here.

What will happen to a D -dimensional scalar field $\phi(x^M)$? For an effective $(D-1)$ dimensional theory, the Klein decomposition (4.3) is used,

$$\phi(x^M) = \sum_{m=-\infty}^{\infty} \phi_m(x^\mu) e^{imx^D/R}. \quad (5.170)$$

We assume $\tilde{G}_{DD} = 1$ and $G_{\mu\nu} = \eta_{\mu\nu}$ then $\det \tilde{G}_{MN} = \det \eta_{\mu\nu} = (-1)^{D-2}$. Plugging the metric (5.167) into the equation of motion $\partial_M \partial^M \phi = 0$, we observe that a “covariant derivative” is appearing,

$$\left(\partial_\mu + i \frac{m}{R} A_\mu \right)^2 \phi_m(x^\mu) = \left(\frac{m}{R} \right)^2 \phi_m(x^\mu). \quad (5.171)$$

The lesson we learn from this simple example is that the momentum m/R in the compact dimension plays the both role of *charge and mass*. Compactification of more dimensions leads to as many Abelian gauge fields $A_\mu^n = G_{\mu n}$. We will see shortly that, because of the stringy effect, in the string theory we have more gauge fields carrying momentum as internal quantum number like n ; therefore they act like charged boson $W^\pm$ in the nonabelian gauge group.

Evidently, the remaining $(D-1)$ dimensional part of the Einstein–Hilbert action still has the general covariance and hence gives rise to the Einstein gravity. This is what Kaluza and Klein (KK) originally obtained [20] as a unification of the U(1) gauge theory and gravity. The main difficulty for the KK theory to be a realistic model for particle physics is that it is hard to fit m/R as charge and mass at the same time.

String Case

Now we proceed to discuss string compactification. Again, we first compactify one dimension of bosonic string theory, with right movers present, as in (5.166). As in the point particle case (5.170), the wave function e^{ipx} (in the center of mass frame) should be single-valued, obeying the same periodicity of (5.166).

$$p = \frac{m}{R}, \quad m = \text{integer} . \quad (5.172)$$

A characteristic feature of string is the existence of the *winding mode*,

$$X(\tau, \sigma + \pi) = X(\tau, \sigma) + 2\pi L . \quad (5.173)$$

Since the string is closed, the winding modes are also quantized, viz. (5.166),

$$2L = Rn, \quad n = \text{integer} , \quad (5.174)$$

and a negative n corresponds to the reverse direction of winding. (The factor 2 is an artifact from the different periods of the worldsheet and spacetime coordinates. Setting the dimensional parameter $\alpha' = 1/2$, it has the same dimension as momentum p .)

The mode expansion obeying this condition reads

$$X(\sigma, \tau) = x + p\tau + 2L\sigma + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left[\alpha_n e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n e^{-2in(\tau+\sigma)} \right] \quad (5.175)$$

which is decomposed to the right and left movers

$$X_R = x_R + \left(\frac{1}{2}p - L \right) (\tau - \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-2in(\tau-\sigma)} , \quad (5.176)$$

$$X_L = x_L + \left(\frac{1}{2}p + L \right) (\tau + \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-2in(\tau+\sigma)} . \quad (5.177)$$

Now the above equations change the mass shell conditions such as (5.64, 5.65) to

$$\begin{aligned} \frac{1}{4}M_L^2 &= \frac{1}{2} \left(\frac{1}{2}p + L \right)^2 + \tilde{N} - 1 , \\ \frac{1}{4}M_R^2 &= \frac{1}{2} \left(\frac{1}{2}p - L \right)^2 + N - 1 . \end{aligned} \quad (5.178)$$

Of course, there is the constraint on the left and right masses, with the same reasoning leading to (5.62),

$$\frac{1}{2}M^2 = M_L^2 = M_R^2.$$

In the light-cone gauge we define the oscillator number as shown in (5.64, 5.65)

$$\begin{aligned}\tilde{N} &= \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \tilde{\alpha}_n + \sum_{i=1}^{23} \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_{in} \\ N &= \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n + \sum_{i=1}^{23} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_{in}.\end{aligned}\tag{5.179}$$

We displayed the index i of the transverse dimension, and separated the oscillators $\alpha, \tilde{\alpha}$ of the compact dimension, i.e. the 26th, for a later use.

Gauge Symmetry Enhancement

It is useful to rewrite the mass condition (5.178), redefining R , as

$$\frac{1}{4}M^2 = \frac{m^2}{4R^2} + n^2 R^2 + N + \tilde{N} - 2\tag{5.180}$$

with the constraint $M_L^2 = M_R^2$, i.e.

$$N - \tilde{N} = mn.$$

At a generic value of R , only the massless states are those with $\tilde{N} = N = 1, n = m = 0$,

$$\tilde{\alpha}_{-1}^M |0\rangle_L \otimes \alpha_{-1}^N |0\rangle_R$$

They correspond to graviton, dilaton and antisymmetric tensor. In particular,

$$\begin{aligned}\tilde{\alpha}_{-1}^i |0\rangle_L \otimes \alpha_{-1} |0\rangle_R \\ \tilde{\alpha}_{-1} |0\rangle_L \otimes \alpha_{-1}^i |0\rangle_R\end{aligned}\tag{5.181}$$

correspond to two U(1) gauge bosons $\tilde{A}_i$ and A_i . It has the right vector index and they correspond to the KK states of G_{MN} and B_{MN} as in (5.167). In the field theory case we expected one gauge field A_μ from G_{MN} , but in the string case there is an additional one from B_{MN} .

We note that the spectrum remains the same if we exchange

$$m \leftrightarrow n, \quad R \leftrightarrow \frac{\alpha'}{R},\tag{5.182}$$

where we explicitly choose $\alpha' = \frac{1}{2}$. We can show that this is not only the symmetry of spectrum but also that of the theory itself, called the *T-duality*.

It means that we cannot tell the difference between the theories compactified on the large or the small radii. In the small radius limit $R \rightarrow 0$, the winding modes remain light, while in the large radius limit $R \rightarrow \infty$ the momenta remain light playing the same role. Again, the existence of winding turns out to be the unique nature of string theory.

For the critical radius $R = 1/\sqrt{2}$, the mass shell condition (5.180) becomes

$$\frac{1}{4}M^2 = \frac{m^2 + n^2}{2} + N + \tilde{N} - 2 \quad (5.183)$$

with $N - \tilde{N} = mn$. Now we have four *more* massless states

$$\begin{aligned} & \tilde{\alpha}_{-1}|0\rangle_L \otimes |1, -1\rangle_R, \\ & \tilde{\alpha}_{-1}|0\rangle_L \otimes |-1, 1\rangle_R, \\ & |1, 1\rangle_L \otimes \alpha_{-1}|0\rangle_R, \\ & |-1, -1\rangle_L \otimes \alpha_{-1}|0\rangle_R. \end{aligned}$$

with integers in the ket understood as $|m, n\rangle$. They provide $W^\pm$ gauge bosons of $SU(2) \times SU(2)$. Note that $W^\pm$ become massless only for the critical radius $R = 1/\sqrt{2}$.

5.3.2 Constructing Heterotic String

Generalizing the above result, let us compactify the extra sixteen dimensional space on a torus T^{16} defined by

$$x^I \equiv x^I + 2\pi L^I, \quad I = 1, 2, \dots, 16. \quad (5.184)$$

The winding (5.173) is naturally extended as

$$X^I(\tau, \sigma + \pi) = X^I(\tau, \sigma) + 2\pi L^I \quad (5.185)$$

The vector L^I on a lattice Γ is defined by

$$2L^I = \sum_{a=1}^{16} n_a R_a e_a^I. \quad (5.186)$$

We have written the basis in terms of non-orthogonal basis vectors e_a^I in the orthonormal space X^a , and the R_a and n_a are the compactification radius and winding number, respectively, in the a^{th} lattice direction.

Also, the momentum p^I is quantized for the same reason of the single valuedness of $e^{ip \cdot x}$. Being conjugate to x , the momentum lattice is spanned in the dual lattice Γ^* with basis vectors e_a^{*I} satisfying

$$\sum_{I=1}^{16} e_a^I e_b^{*I} = \delta_{ab}. \quad (5.187)$$

Then, p^I in the dual lattice Γ^* has the expression

$$p^I = \sum_{a=1}^{16} \frac{m_a}{R_a} e_a^{*I}, \quad m_a = \text{integer} . \quad (5.188)$$

If we had right moving bosons as well, the story might end here. The mass condition for the right movers is a straightforward generalization of (5.178) and the discussion goes parallel to the one presented above. However, we introduced the extra sixteen bosonic degrees only in the *left moving* sector. (A more general discussion will be given in Subsect. 6.4.3.)

Immediately, it gives rise to the following problems: it is hard to imagine momentum nor winding, and moreover, difficult to quantize. In such a case we employ the following. Suppose first that we had both left and right movers, whose mode expansions are given in (5.176, 5.177). Then, the quantization condition is

$$[x^I, p^J] = i\delta^{IJ} \quad (5.189)$$

Naturally, we require that the left moving momentum p_L^I and the center of mass x_L^I commute with right moving ones. This is to assign the following commutation relation,

$$[x_L^I, p_L^J] = [x_R^I, p_R^J] = \frac{i}{2} \delta^{IJ} . \quad (5.190)$$

Then, we employ the right movers as the auxiliary fields so that they are eliminated with constraints,

$$p_R^I = \frac{1}{2} p^I - L^I = 0 . \quad (5.191)$$

Then, the momentum and the winding are forced to be identified; thus we have $p_L^I = p^I$, which we will call P^I hereafter, with the usual commutation relation, $[x^I, p^J] = i\delta^{IJ}$.

Now we apply the constraint condition of (5.191), $p^I = 2L^I$. Then,

$$\sum_{a=1}^{16} \frac{m_a}{R_a} e_a^{*I} = \sum_{a=1}^{16} n_a R_a e_a^I . \quad (5.192)$$

We should find the satisfying solution of this: *self-dual* lattice $\Gamma = \Gamma^*$.

This condition is very restrictive because the lattice e_a^I is related to its dual lattice e_a^{*I} . We immediately observe that the necessary condition is that the product of lattice vectors $p_a \cdot p_b$ (not with dual) gives an integer. In terms of basis vectors,

$$e_a \cdot e_b \equiv \sum_I e_a^I e_b^I = \text{integer} \quad (5.193)$$

which is called the *integral* lattice. In view of (5.192), the integral lattice is possible for the self-dual radii $R_a = 1/\sqrt{2} = \sqrt{\alpha'}$; thus we expect gauge symmetry will be enhanced beyond $U(1)^{16}$. Of course, finite radius is meaningful only when the dimensions is *compact*. We will require further restrictions from the modular invariance.

5.3.3 Allowed Gauge Group

In the heterotic string theory, the modular invariance condition determines the gauge group structure. Here, we require the modular invariance for the gauge degrees of freedom.

It is natural to guess that the partition function is composed of ϑ functions. Also, the modular invariance condition is contained in the metric of the lattice. It is defined by $A_{ab} = e_a \cdot e_b$, which is the Cartan matrix. They are presented in Appendix E.

1. Under $\mathcal{T}$, the amplitude is invariant only if the lattice is *even*, that is, a product $p_L \cdot p_L$ is even. It is shown that if only the diagonal elements A^{IJ} are even then the product is even. This is the nature of Lie algebra diagrams, whose diagonal elements of Cartan matrix are even.
2. $\mathcal{S}$ exchanges coordinate σ^0 and σ^1 , as in Fig. 5.4, whose action on bosons is translated as follows. Under $\mathcal{S}$, the lattice spanned by e^I is changed to its dual spanned by e^{*I} . For the one-loop amplitude to be invariant under this transformation, there should be no change of lattice volume, whose factor is given by $(\det A)^{-1/2}$. Therefore, provided $\det A = 1$, we have the *self-dual* lattice.

Therefore, *for the even and self dual lattice, we have the modular invariance and obtain the solution of (5.192).*

Let us verify the above statements. For simplicity, consider the E_8 momentum states first,

$$\mathcal{Z}^{E_8}(\tau) = \frac{1}{\eta(\tau)^8} \sum_{P \in \Gamma_8} q^{P^2/2} \quad (5.194)$$

where the η function is inserted for the modular invariance. The Γ_8 is the E_8 lattice, (5.110)

$$\Gamma_8 = \left\{ (n_1, n_2, \dots, n_8), \left(n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, \dots, n_8 + \frac{1}{2} \right) \mid n_i \in \mathbf{Z}, \sum n_i \in 2\mathbf{Z} \right\}. \quad (5.195)$$

Observing the lattice structure, we can sum over the lattice by introducing a projection,

$$\begin{aligned} \mathcal{Z}^{E_8}(\tau) &= \frac{1}{2\eta^8} \sum_{\{n_j \in \mathbf{Z}\}} q^{\frac{1}{2} \sum n_j^2} \left[1 + (-1)^{\sum n_j} \right] \\ &\quad + \frac{1}{2\eta^8} \sum_{\{n_j \in \mathbf{Z}\}} q^{\frac{1}{2} \sum (n_j + \frac{1}{2})^2} \left[1 + (-1)^{\sum n_j} \right], \end{aligned} \quad (5.196)$$

where j runs from 1 to 8. In the fermionic construction, this projection corresponds to the GSO projection. Observing the regular pattern, it can be reexpressed in a fancy form,

$$\mathcal{Z}^{E_8}(\tau) = \frac{1}{2\eta(\tau)^8} \left\{ \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^8(\tau) + \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}^8(\tau) + \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}^8(\tau) + i\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}^8(\tau) \right\}, \quad (5.197)$$

where we have included $i\vartheta_{[1/2]}^{[1/2]} = 0$ by taking into account of the last term in (5.196).

The $\mathcal{T}$ transformation gives rise to a nontrivial phase $-i\pi/4$ when $\alpha = 1/2$, so only the theta function with the power $8k$ with integer k leaves the partition function invariant. This is the reason why we have a modular invariant Euclidian lattice in $8k$ dimensions. For the $\mathcal{S}$ transformation, note that the nontrivial τ dependence is the same as that of the Dedekind $\eta(\tau)$. Thus, by dividing $\eta(\tau)$ we expect some linear combination of these gives a modular invariant partition function. Using these properties, now we can check that E_8 partition function (5.197) is invariant under both transformations.

The complete $E_8 \times E_8$ partition function is the square of the single one

$$\mathcal{Z}^{E_8 \times E_8} = (\mathcal{Z}^{E_8})^2.$$

It can be generalized to the $SO(32)$ case also. Observe that the eight dimensional lattice is formed by (5.196) whose indices n_j run over from 1 to 8. Running from 1 to 16, we have a sixteen dimensional lattice. Remarkably, it is known that there is only one modular invariant function of sixteen dimensions [21]; therefore the $E_8 \times E_8$ and $SO(32)$ partition functions are the same. This partitioning corresponds to giving boundary conditions simultaneously to some parts, as we have constructed the original heterotic string theory.

$SO(2n)$ Weights and Lattices, Bosonization

We saw that there can be even and self-dual *Euclidian* lattices only in $8k$ dimensions. Let us define the lattice Γ_{8k} as

$$\Gamma_{8k} = \left\{ (n_1, n_2, \dots, n_{8k}), \left(n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, \dots, n_{8k} + \frac{1}{2} \right) \mid n_i \in \mathbf{Z}, \sum n_i \in 2\mathbf{Z} \right\} \quad (5.198)$$

The former spans the root lattices of $SO(16k)$ and the latter spans the spinorial representations of $SO(16k)$. We know that E_8 roots **248** can be made by giving a suitable commutation relations between the adjoint **120** and the spinorial **128** of $SO(16)$,

$$\mathbf{248} = \mathbf{120} + \mathbf{128}. \quad (5.199)$$

They are vectors of Γ_8 , which is the only 8 dimensional even and self-dual lattice. The only possible lattices of dimension 16 are

$$\Gamma_8 \times \Gamma_8, \quad \Gamma_{16} \quad (5.200)$$

corresponding to those spanned by the $E_8 \times E_8$ and the $SO(32)$ root systems, respectively. In 24 dimensions, there are 24 kinds of lattices, the Niemeier lattices [22], and so on.

Table 5.2. The four conjugacy classes of $SO(2n)$ group

Name	Symbol	Vector	Constraint
scalar	0	$(n_1, \dots, n_n)$	$\sum n_i$ even
vector	v	$(n_1, \dots, n_n)$	$\sum n_i$ odd
spinor	s	$(n_1 + \frac{1}{2}, \dots, n_n + \frac{1}{2})$	$\sum n_i$ even
conj. spinor	c	$(n_1 + \frac{1}{2}, \dots, n_n + \frac{1}{2})$	$\sum n_i$ odd

We are at the moment considering gauge degree of freedom due to the left moving sixteen bosons, which have only Euclidian signatures (5.193). When we compactify more dimensions, we expect the possibility of higher rank gauge groups. Moreover, if we loosen the condition of the Euclidian lattice to *Lorentzian* one due to negative valued metrics from the right movers, we have a more general class of modular invariant lattices, which we will consider in Subsect. 6.4.3.

The weight lattice of $SO(2n)$ can be decomposed into four sublattices, or conjugacy classes, as displayed in Table 5.2. In particular, for $SO(8)$ the lowest elements are three fundamental weights

$$\begin{aligned}
\mathbf{8}_v &= \pm(\underline{1} \ 0 \ 0 \ 0) \\
\mathbf{8}_s &= ([+ \ + \ + \ +]) \\
\mathbf{8}_c &= ([- \ + \ + \ +])
\end{aligned} \tag{5.201}$$

where the underline means permutations and the square bracket means even flips of signs. All these have the same length 1 and can be transformed into each other by the *triality* relation. In Type II string, an $\mathbf{8}_v$ is the spacetime boson and one among $\mathbf{8}_s$ and $\mathbf{8}_c$ is chosen by the GSO projection to be its partner as the spacetime fermion.⁵

In the heterotic string, the right mover has the above representation (5.201). One recognizes that all the vectors s satisfy the mass condition $s^2 = 1$. These spinorial representation is unified to a momentum vector with an aid of *bosonization*; $\mathbf{8}_v$ is NS and $\mathbf{8}_{s,c}$ is R states and we can show that there exists one-to-one correspondence between $\mathbf{8}_v$ and $\mathbf{8}_{s,c}$, due to supersymmetry. Before we had 8 bosons and 8 fermions in the worldsheet, and now we have $8 + 4 = 12$ bosons. The zero point energy is $c = 12 \cdot (-\frac{1}{24}) = -\frac{1}{2}$, therefore the right-mover mass is

$$\frac{M_R^2}{4} = \frac{s^2}{2} - \frac{1}{2} = 0. \tag{5.202}$$

⁵ The Spin (32) is the double covering group of $SO(32)$ that contains all four conjugacy classes. In the strict sense, we should call the group that heterotic string possesses as Spin (32)/ $\mathbf{Z}_2$, containing 0 and s weights, while the original $SO(32)$ is another Spin (32)/ $\mathbf{Z}_2$ containing 0 and v. However in this book, we just stick to the name $SO(32)$ because, whichever we choose to use, the numbers of zero modes are the same.

In later calculations, we will employ these bosonized right movers, because it transparently shows massless spectrum and its chirality.

5.3.4 Spectrum and Current Algebra

As before, let us choose the light-cone gauge. The remaining ten dimensional string has the standard one. The mass squared operator in the light-cone gauge is

$$M^2 = M_L^2 + M_R^2, \quad (5.203)$$

with the constraint $M_L^2 = M_R^2$. For M_L^2 , we use the 26D bosonic string compactified on 16D torus,

$$\frac{1}{4}M_L^2 = \frac{1}{2} \sum_{I=11}^{26} (P^I)^2 + \tilde{N} - 1 \quad (5.204)$$

where

$$\tilde{N} = \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I). \quad (5.205)$$

We are forced to have the critical radius $R = 1/\sqrt{2}$ because we have no corresponding right movers $p_R = 0$.

The right moving string is just like that of ten dimensional Type II superstring. We have M_R^2 from (5.127),

$$\frac{1}{4}M_R^2 = \begin{cases} \sum_{m=0}^{\infty} (\alpha_{-m}^i \alpha_m^i + m d_{-m}^i d_m^i) & \text{R sector} \\ \sum_{n=0}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^i b_r^i - \frac{1}{2} & \text{NS sector} \end{cases} \quad (5.206)$$

with the usual constraint

$$M_L^2 = M_R^2. \quad (5.207)$$

Finally, let us find out the lowest states. From (5.204), the lowest lying state seems to be $P^2 = 0, \tilde{N} = 0$ to have $\frac{1}{4}M_L^2 = -1$. However, this is not so because from (5.206) there is no right moving state with the same mass. Thus, the state $\frac{1}{4}M_L^2 = -1$ is projected out, and this theory is automatically tachyon free.

The right movers allow the following lowest mass states⁶

$$b_{-1/2}^\mu |0\rangle_R \text{ (NS)}, \quad d_0 |0\rangle_R \text{ (R)}. \quad (5.208)$$

Noting the spacetime index, the NS states provide bosonic states and the R states provide fermionic states, forming together superpartners. Combined with massless left movers, they give rise to low energy fields. We will present bosonic states:

⁶ In the remainder of this chapter, μ, ν represent 10D indices.

- $\tilde{N} = 1$ and $P^2 = 0$ of the following kinds.

$$\tilde{\alpha}_{-1}^{\mu}|0\rangle_L \otimes b_{-1/2}^{\nu}|0\rangle_R . \quad (5.209)$$

They make up graviton $G^{\mu\nu}$, dilaton ϕ and the antisymmetric tensor $B^{\mu\nu}$.

- For $\tilde{N} = 1$ and $P^2 = 0$, there are states carrying internal index I

$$\tilde{\alpha}_{-1}^I|0\rangle_L \otimes b_{-1/2}^{\mu}|0\rangle_R , \quad (5.210)$$

which provide sixteen $U(1)$ generators $A^{\mu I}$ of the Cartan subalgebra H^I . They are vectors with the single spacetime index μ in 10D.

- $\tilde{N} = 0$ and $P^2 = 2$ states of the form

$$|P\rangle_L \otimes b_{-1/2}^{\mu}|0\rangle_R , \quad (5.211)$$

where P belonging to the lattice (5.198). The possible forms of P are $(\pm 1 \pm 1 0 \cdots 0)$ with 16 entries, and $(\pm 1 \pm 1 0 \cdots 0)(0 \cdots 0)'$ and $([\frac{1}{2} \frac{1}{2} \cdots \frac{1}{2}]) (0 \cdots 0)'$ with 8 entries in each bracket plus those with the unprimed and primed brackets exchanged.⁷ These provide the root vectors of $SO(32)$ and $E_8 \times E_8$, respectively. The resulting low energy fields are the charged generators E^P , or the ladder operators.

We can make a consistency check. The effective theory made out of these fields is the minimal ($N = 1$) ten dimensional supergravity coupled to Yang-Mills gauge group [23]. This theory has the spin- $\frac{3}{2}$ gravitino and thus has a gravitational anomaly of unit -496 [24]. It can be cancelled by introducing 496 chiral fermions corresponding to gauginos. Another kinds of anomaly are canceled by coupling to the antisymmetric tensor field, etc. [19]. This miraculous cancelation of the anomalies can be tracked back to the modular invariance in string theory.

Current Algebra

The massless spectrum is not sufficient to show the group property, although they are indeed the roots of the algebra. This is because they do not show the relations among them. It turns out that the vertex operators have the exact properties we want in this regard. We claim [25] that the vertex operators of the massless roots and weights considered above are

$$H^I(z) = \frac{1}{\pi} \dot{X}_L^I(z) , \quad (5.212)$$

$$E^P(z) = c_P \frac{1}{\pi} : \exp[2iP \cdot X_L(z)] : , \quad (5.213)$$

⁷ The underline means permutations and the square bracket means even flips of signs.

where P is now interpreted as the root vectors and c_P is ± 1 determined by the commutation relations among them. As a simple check, assume “zero modes” of the above operators

$$H_0^I(z) = P^I, \quad E_0^P = c_P : e^{2iP \cdot X_L(\tau+\sigma)} : \quad (5.214)$$

Then by the commutation relations coming from (5.190), we have

$$\begin{aligned} [H_0^I, H_0^J] &= 0, \\ [H_0^I, E_0^P] &= P^I E_0^P. \end{aligned} \quad (5.215)$$

This is nothing but the relations between generators of a simple Lie algebra, between the Cartan subalgebra and ladder operators. Some properties of the simple Lie algebras are summarized in Appendix E. With this in mind, we have the operator product expansion

$$t^a(z)t^b(0) \sim \frac{\delta^{ab}}{z^2} + \frac{if^{abc}}{z}t^c(0), \quad (5.216)$$

where t^a can be any of (5.212) and (5.213) and f^{abc} is the structure constant of the algebra we deal with. With the mode expansion

$$t^a(z) = \sum_{m=-\infty}^{\infty} \frac{t_m^a}{z^{m+1}}, \quad (5.217)$$

they satisfy the following commutation relation

$$[t_m^a, t_n^b] = if^{abc}t_{m+n}^c + m\delta^{ab}\delta_{m+n,0} \quad (5.218)$$

whose zero mode $m = n = 0$ reduces to that of the conventional Lie algebra. This extended algebra will reveal the structure of spectrum and will be dealt with in Sect. 9.4.

5.3.5 Fermionic Construction

Some readers may feel unsatisfactory in that we have introduced different spacetime dimensions for the left and right movers. Alternatively, we may introduce 32 fermions instead of 16 bosons in *ten dimensions*. In fact, it is an equivalent choice, since in two (world-sheet) dimensions, two holomorphic fermions play the role of one holomorphic boson. Their conformal dimension, correlation functions and spectrum are the same. These 32 real fermions λ^I have an $SO(32)$ internal symmetry. As we have done the GSO projection on the right movers, we introduce the GSO projection (5.134, 5.141) on these fermions also via the operator

$$(-1)^F. \quad (5.219)$$

States with the odd number of fermionic oscillators are projected out. Actually this projection has a counterpart in the bosonic string theory. It is equivalent to our choosing of $(\sum P_I = \text{even only})$ in (5.198).

Since there are two kinds of boundary conditions on each fermions, we have essentially two choices for possible theories. One is that we can assign the same boundary conditions on all of them. This leads to the $SO(32)$ gauge group. Here we briefly present the other less straightforward case yielding the $E_8 \times E_8$ group. Divide the 32 fermions into two partitions, say n and $(32 - n)$ fermions for the R and NS fermions. We know that for each degree of freedom, the zero point energy is $-\frac{1}{24}$ and $\frac{1}{48}$ for the R and NS fermions, respectively. With 24 bosonic degrees of freedom, we have the following total zero point energy $-a$ with

$$a_{\text{NSNS}} = \frac{8}{24} + \frac{n}{48} + \frac{32-n}{48} = 1, \quad (5.220)$$

$$a_{\text{NSR}} = \frac{8}{24} + \frac{n}{48} - \frac{32-n}{24} = \frac{n}{16} - 1, \quad (5.221)$$

$$a_{\text{RNS}} = \frac{8}{24} - \frac{n}{24} + \frac{32-n}{48} = 1 - \frac{n}{16}, \quad (5.222)$$

$$a_{\text{RR}} = \frac{8}{24} - \frac{n}{24} - \frac{32-n}{24} = -1 \quad (5.223)$$

where $\frac{8}{24}$ is from the eight bosonic coordinates of 10D. (Note that considering the right mover, we specify three boundary conditions on a state.) For the NSR or RNS sectors to have massless states, we require $n = 16$, otherwise it reduces to a theory with the same boundary conditions for all the movers.

First, noting that $-a_{\text{NSNS}} = -1$, we can find massless states in the NSNS states with

$$\lambda_{-1/2}^I \lambda_{-1/2}^J |0\rangle, \quad (5.224)$$

which split into the following three cases

$$\begin{aligned} I = 1, \dots, 16, \quad J = 1, \dots, 16 & \quad (\mathbf{120}, \mathbf{1}) \\ I = 1, \dots, 16, \quad J = 17, \dots, 32 & \quad (\mathbf{16}, \mathbf{16}) \\ I = 17, \dots, 32, \quad J = 17, \dots, 32 & \quad (\mathbf{1}, \mathbf{120}). \end{aligned}$$

We should apply the GSO projection. Since we have partitioned the fermions into two sectors, we will see that a reasonable choice is to use an independent projection for each sector

$$(-1)^{F_1}, \quad (-1)^{F_2}. \quad (5.225)$$

In this NSNS case, both partitions are in the NS sector; thus the GSO projection (5.134) is applied. The $(\mathbf{16}, \mathbf{16})$ has an odd fermion number since the number operator $F = \sum_{r=1/2}^{\infty} b_{-r}^i b_r^i$ acts on the first 16 indices and the last 16 indices separately, and hence is projected out.

In addition, we have $a_{\text{RNS}} = 0$ in the RNS sector. These become spinorial massless states, in view of (5.139) with eight s_i s,

$$|s_1 s_2 \cdots s_8\rangle, \quad \text{spinorial made of } \lambda_0^I, \quad I = 1, \dots, 16. \quad (5.226)$$

They lead to the states

$$(\mathbf{256}, \mathbf{1}) = (\mathbf{128}_s, \mathbf{1}) + (\mathbf{128}_c, \mathbf{1}).$$

Again the GSO projection should be applied. The fermions are partitioned into sixteen R and sixteen NS, therefore (5.141) and (5.134) are separately used. They project out one, say $(\mathbf{128}_c, \mathbf{1})$, and there remains $(\mathbf{1}, \mathbf{128}_s)$. By a similar reasoning applied to the NSR sector, we obtain $(\mathbf{1}, \mathbf{128}_s)$. Except for the $\text{SO}(16)$ charges, all the states obtained here have the same quantum numbers; thus they all belong to one $E_8 \times E_8$ multiplet

$$(\mathbf{248}, \mathbf{1}) + (\mathbf{1}, \mathbf{248}). \quad (5.227)$$

Generalizing Boundary Conditions and Lower Dimensional Strings

We have seen that $\text{SO}(32)$ and $E_8 \times E_8$ are the only possible gauge groups in the modular invariant theory. This comes from the requirement that both 16 dimensional group space and 10 dimensional spacetime are *independently* modular invariant. However, if we require that *the whole theory* (the group space and spacetime combined together) is modular invariant, we have more theories. Actually, this arises when we consider more general boundary conditions than NS and R. We can complexify a pair of fermions as $\tilde{\lambda}^{I+} = \tilde{\lambda}^{2I-1} + i\tilde{\lambda}^{2I}$ and consider more general boundary conditions,

$$\tilde{\lambda}^{I+} \rightarrow e^{2\pi i V_I} \tilde{\lambda}^{I+} \quad (5.228)$$

with the 16 component vector V . Of course, we have an equivalent description with those described by the bosonic version of the theory. This naturally gives rise to the theories that will be considered in the next chapter.

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Strings on Orbifolds

As discussed in the previous chapter, the heterotic string possesses very rich symmetries. It naturally describes $SO(32)$ and $E_8 \times E_8$ gauge group, by assigned charges along the string. Also it has sixteen real ($\mathcal{N} = 4$ in four dimension) supersymmetries. However, these symmetries are too large from the phenomenological point of view, by the criteria discussed in Chap. 2. Namely, in weakly coupled string theories, unless the fortuitous appearance of family structure results from the twisted sector as we will see it is better for a big gauge group is given already so that the standard model(SM) gauge group of rank 4 can be embedded there. In addition, as emphasized in Chap. 2, the group must allow “spinor representation” of $SO(10)$. This leads to the $E_8 \times E_8'$ heterotic string as the first choice. In a strongly coupled string, non-perturbative effects may produce defects and we have to consider branes also. Compactification of strongly coupled strings are briefly sketched in Appendix B.2.

As merits and demerits, the string theory is defined on ten dimensional spacetime. By compactifying ten dimensional heterotic string down to four dimension and relating such symmetries with spacetime properties, we can break these symmetries. Here we will deal with the easiest and intuitive compactification scheme known today, the orbifold compactification, such that the resulting four dimensional theory has gauge symmetry of the standard model and only four ($\mathcal{N} = 1$) supersymmetries. Alternatively, on the way to the SM we may meet the grand unified theory (GUT). In this process, the chiral nature of the SM fermions should result, which is one of the most mysterious puzzles of the Planck scale physics.

In this chapter, the basic ingredients on orbifold theory is dealt without rigorous proof which is postponed to the next chapter. The rules needed for obtaining gauge group and particle spectrum is understood here. There are many good treatments on this topic [1, 2, 3] as well as the original paper [4].

Conventions

Still, we use normalization $\alpha' = \frac{1}{2}$ but sometimes it is made explicit if necessary. For the zero point energy, it is renamed $c = -a, \tilde{c} = -\tilde{a}$ to cope with the literature. We compactify some dimensions on complex manifolds. We will denote coordinates as follows

$$x^M = (x^\mu, \underbrace{z^a, z^{\bar{a}}}_{x^i}, x^I) .$$

Greek letters x^μ, x^ν denote our four dimensions. It will be convenient to use Latin and their barred ones $z^a, z^{\bar{a}}$ for holomorphic and antiholomorphic complex indices, respectively. The gauge space index, the upper case Latin in x^I , remains untouched. Mostly, we adopt the standard convention used in [5].

6.1 Strings on Orbifolds

Contrary to the point particle, a string has spatial extent so that it behaves differently in the topologically nontrivial space. In what follows our main interest is the closed string, which is characterized by the boundary condition $X(\sigma + \pi) = X(\sigma)$. It has a nontrivial feature on torus, the winding along the torus, as we have seen already in (5.173)

$$X(\sigma + \pi) = X(\sigma) + 2\pi L .$$

What is changed if one changes the manifold from torus to orbifold? Firstly, the orbifold is moded out torus, and still we expect the above closed string found on torus. We call this the *untwisted string*. We will see that some of them are projected out, because the moding is accompanied by a suitable action on the string Hilbert space. We have the *twisted string* in addition.

6.1.1 Twisted String

We compactify six internal dimensions on orbifold $T^6/\mathbf{Z}_N$ where the order N action belongs to $\bar{\mathbf{P}}$ of Chap. 3. We have an additional modding under the identification

$$z^a \sim \theta z^a \tag{6.1}$$

up to a lattice translation. From now on, we choose the coordinate as the “eigenstate” under θ , satisfying $\theta z^a = e^{2\pi i \phi_a} z^a$,

$$z^a = \frac{1}{\sqrt{2}}(x^a + e^{2\pi i \phi_a} x^{a+1}), \quad z^{\bar{a}} \equiv \overline{z^a} = \frac{1}{\sqrt{2}}(x^a + e^{-2\pi i \phi_a} x^{a+1}) . \tag{6.2}$$

Each component of the vector $\phi \equiv (\phi_1, \phi_2, \phi_3)$ is of a form l/N with l being an integer, since N successive rotations give the identity. The vector ϕ specifies

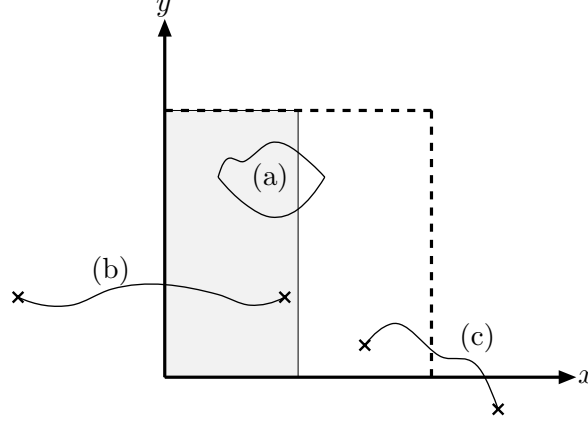


Fig. 6.1. The *shaded* region depicts the fundamental region and *dotted* border bound the covering torus. (a) A normal closed string. (b) A closed string winding around torus. (c) A twisted string in $\mathbf{Z}_2$ orbifold. The $\mathbf{Z}_2$ action θ is identification $(x, y) \sim (-x, -y)$. Also we can label this twisting by a shift $2e_x$ under identification

the point group fully, up to lattice dependence. (We have stressed that the six dimensional crystal itself is not decomposed into products of two dimensional ones, however the orbifold action can be.) The generalization to arbitrary even dimensions is straightforward.

When we act the identity, or equivalently $N\phi$ and NV , on a state, the total phase acquired should be trivial,

$$\sum_i N\phi_i = 0 \pmod{2}, \quad (6.3)$$

where the modulo 2 condition is reserved for spacetime fermions. For spin- $\frac{1}{2}$ fermions, the rotation by 2π is minus the identity thus we should be careful about mentioning the order.

Because of orbifold geometry, we have a closed string modulo $\bar{P}$ of (3.1) since the space is identified under $\bar{P}$,

$$Z^a(\sigma + \pi) = \theta Z^a(\sigma) = e^{2\pi i \phi_a} Z^a(\sigma). \quad (6.4)$$

We call such string as *twisted string*. The rotation element θ labels how much the string is twisted. Fix θ and call the set of strings in (6.4) as first twisted sector. There is also twisted string by θ^2 and so on. We have θ^k twisted strings, for $k = 1, \dots, N-1$, consisting the k^{th} twisted sector. This situation is depicted in Fig. 6.1. We also have projection condition in the twisted sectors. We call $k = 0$ twisted sector as the untwisted sector.

6.1.2 Mode Expansion and Quantization

We now show the mode expansion and quantization. The modification is only slight, arising from the boundary condition (6.4). Comparing with (5.20) and (5.21) the following mode expansions have the desired phase

$$Z^a(\sigma, \tau) = z_f^a + \frac{i}{2} \sum_{m \in \mathbf{Z}} \left(\frac{\tilde{\alpha}_{m+k/N}^a}{m+k/N} e^{-2i(m+k/N)(\tau+\sigma)} + \frac{\alpha_{m-k/N}^a}{m-k/N} e^{-2i(m-k/N)(\tau-\sigma)} \right) \quad (6.5)$$

where we used our convention $\alpha' = \frac{1}{2}$ so that $\sqrt{\alpha'/2} = \frac{1}{2}$. Its complex conjugate is

$$Z^{\bar{a}}(\sigma, \tau) = z_f^{\bar{a}} + \frac{i}{2} \sum_{m \in \mathbf{Z}} \left(\frac{\tilde{\alpha}_{m-k/N}^{\bar{a}}}{m-k/N} e^{-2i(m-k/N)(\tau-\sigma)} + \frac{\alpha_{m+k/N}^{\bar{a}}}{m+k/N} e^{-2i(m+k/N)(\tau+\sigma)} \right). \quad (6.6)$$

These bosonic parts are common to the left and right movers. Here, the center of mass coordinate z_f is the *fixed point*, $z_f(\pi) = z_f(0)$, under the space action (6.4),

$$z_f = (1 - \theta)^{-1} \Lambda.$$

The twisted string does not have a momentum nor a winding mode. We can easily decompose (6.5,6.6) into the left and right movers. The commutation relations of twisted oscillators are

$$[\tilde{\alpha}_{m+k/N}^a, \tilde{\alpha}_{-n-k/N}^{\bar{b}}] = \left(m + \frac{k}{N}\right) \delta^{a\bar{b}} \delta_{mn}, \quad (6.7)$$

$$[\alpha_{m-k/N}^a, \alpha_{-n+k/N}^{\bar{b}}] = \left(m - \frac{k}{N}\right) \delta^{a\bar{b}} \delta_{mn}. \quad (6.8)$$

Therefore, we interpret $\tilde{\alpha}_{-n-k/N}^{\bar{b}} (n \geq 1)$, $\alpha_{-n+k/N}^{\bar{b}} (n \geq 0)$ as creation operators. Also, we define a ground state as the one annihilated by all the annihilation operators

$$\tilde{\alpha}_{m-k/N}^a |\sigma_k\rangle = 0, \quad m \geq 1, \quad \alpha_{m+k/N}^a |\sigma_k\rangle = 0, \quad m \geq 0. \quad (6.9)$$

We will meet a more clarified version of the twisted ground state in Subsect. 8.1.1.

Since string mass is proportional to the length stretched, the massless string state can only be located on the fixed point z_f . We see this from the mass shell condition of the original heterotic string,

$$\frac{1}{4}M_L^2 = \frac{1}{2}P^2 + \tilde{N} + \tilde{c} , \quad (6.10)$$

$$\frac{1}{4}M_R^2 = \frac{1}{2} + N + c \quad (6.11)$$

$$M_L^2 = M_R^2 = M^2/2 . \quad (6.12)$$

The momentum $P^I \equiv p_L^I$ is quantized, which is a sixteen dimensional vector. The fractional mode numbers make the following things nontrivial.

1. The corresponding oscillator number operator

$$\begin{aligned} \tilde{N} = & \sum_{a=1}^4 \sum_{n+\phi^a > 0} \tilde{\alpha}_{-n-\phi^a}^{\bar{a}} \tilde{\alpha}_{n+\phi^a}^a + \sum_{a=1}^4 \sum_{n-\phi^a > 0} \tilde{\alpha}_{-n+\phi^a}^a \tilde{\alpha}_{n-\phi^a}^{\bar{a}} \\ & + \sum_I^{16} \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I \end{aligned} \quad (6.13)$$

can have fractional eigenvalues. For example, an excited state $\tilde{\alpha}_{-1/3}|0\rangle$ has a fractional oscillator number $\tilde{N} = \frac{1}{3}$.

2. When we rewrite (6.5) componentwise, we see that, under the point group action θ , each oscillator transforms as

$$\begin{aligned} \tilde{\alpha}_{m+k/N}^a & \rightarrow e^{2\pi i k/N} \tilde{\alpha}_{m+k/N}^a , \\ \tilde{\alpha}_{m-k/N}^{\bar{a}} & \rightarrow e^{-2\pi i k/N} \tilde{\alpha}_{m-k/N}^{\bar{a}} , \end{aligned} \quad (6.14)$$

and so on. We will see that due to this phase some states are projected out.

3. Consequently the resulting zero point energy $\tilde{c}$, the sum of the normal ordering constant, is changed. For obtaining zero point energy we need a more generalized version of Riemann zeta function,

$$f(\eta) = \frac{1}{2} \sum_{n=0}^{\infty} (n + \eta) = -\frac{1}{24} + \frac{1}{4}\eta(1 - \eta) , \quad (6.15)$$

for each *real, bosonic* degree of freedom with $0 \leq \eta \leq 1$. This is justified by analytic continuation, or also we will present a proof in Appendix C. The fermionic states have the contributions with the same magnitude but with the opposite sign. Therefore, for the bosonic left movers we have the vacuum energy

$$\tilde{c} = 2 \sum_{a=1}^4 f(\phi_a) + \sum_{I=1}^{16} f(0) , \quad (6.16)$$

The factor 2 in the first term comes from the definition (3.55), so that we rotated pairwise. We verify that the untwisted string has $\tilde{c} = -1$.

For the superstring right movers we have the fermionic degree ψ^M . Note that we will use fermions only for formal quantization. We have similar relaxed boundary conditions

$$\psi^a(\sigma + \pi) = e^{2\pi i \phi_a} \psi^a(\sigma) \text{ for the R sector} \quad (6.17)$$

$$\psi^{\bar{a}}(\sigma + \pi) = -e^{2\pi i \phi_a} \psi^{\bar{a}}(\sigma) \text{ for the NS sector ,} \quad (6.18)$$

yielding the following mode expansions,

$$\text{For R : } \begin{cases} \psi_R^a = \sum_n d_{n+\phi_a}^a e^{-2i(n+\phi_a)(\tau-\sigma)} , \\ \psi_R^{\bar{a}} = \sum_n d_{n-\phi_a}^{\bar{a}} e^{-2i(n-\phi_a)(\tau-\sigma)} \end{cases} \quad (6.19)$$

$$\text{For NS : } \begin{cases} \psi_R^a = \sum_r b_{r+\phi_a}^a e^{-2i(r+\phi_a)(\tau-\sigma)} , \\ \psi_R^{\bar{a}} = \sum_r b_{r-\phi_a}^{\bar{a}} e^{-2i(r-\phi_a)(\tau-\sigma)} \end{cases} \quad (6.20)$$

Thus, we have the following modified anticommutation relations

$$\{d_{m+k/N}^a, d_{-n-k/N}^{\bar{b}}\} = \left(m + \frac{k}{N}\right) \delta^{a\bar{b}} \delta_{mn} , \quad (6.21)$$

$$\{b_{m-k/N}^a, b_{-n+k/N}^{\bar{b}}\} = \left(m - \frac{k}{N}\right) \delta^{a\bar{b}} \delta_{mn} . \quad (6.22)$$

The oscillator numbers now have the bosonic part which is similar to (6.13). The oscillator numbers of the fermionic part are for the R and NS sectors, respectively,

$$N_F = \sum_{a=1}^4 \sum_{n+\phi_a > 0} (n + \phi_a) d_{-n-\phi_a}^{\bar{a}} d_{n+\phi_a}^a + \sum_{a=1}^4 \sum_{n-\phi_a > 0} (n + \phi_a) d_{-n+\phi_a}^a d_{n-\phi_a}^{\bar{a}} , \quad (6.23)$$

$$N_F = \sum_{a=1}^4 \sum_{r+\phi_a > 0} (r + \phi_a) b_{-r-\phi_a}^{\bar{a}} b_{r+\phi_a}^a + \sum_{a=1}^4 \sum_{r-\phi_a > 0} (r + \phi_a) b_{-r+\phi_a}^a b_{r-\phi_a}^{\bar{a}} , \quad (6.24)$$

where $n \in \mathbf{Z}$ and $r \in \mathbf{Z} + \frac{1}{2}$. Consequently, we have the following modified zero point energies,

$$c = \begin{cases} 2 \sum_{a=1}^4 f(\phi_a) - 2 \sum_{a=1}^4 f(\phi_a) = 0 , & \text{R} \\ 2 \sum_{a=1}^4 f(\phi_a) - 2 \sum_{a=1}^4 f\left(\phi_a + \frac{1}{2}\right) , & \text{NS} \end{cases} \quad (6.25)$$

For the untwisted superstring, $c = -\frac{1}{2}$ and 0 (where $\phi_i = 0$) for the Neveu–Schwarz and the Ramond states, respectively.

We have seen such extended boundary condition in the context of the fermionic construction, in the Subsect. 5.3.5. In this respect, the left moving fermions spanning the gauge space and the spacetime get the same kind of gauge conditions.

For a practical calculation, we will use the bosonized description for the right movers. Replacing eight fermions by four bosons, we have 12 bosons in total. Twisting the bosonized fermions is identical to that of original bosons. Here, we have the zero point energy

$$c = 3 \sum_{a=1}^4 f(\phi_a),$$

which is tabulated in Appendix D. From Eq. (6.16), we observe

$$\tilde{c} - \frac{1}{2} = c.$$

6.1.3 Shifting

We have another way to make an orbifold, by *translational* symmetry. Consider an action on the torus,

$$\mathcal{V}: \quad x^i \rightarrow x^i + 2\pi v^i, \quad (6.26)$$

where v^i also has the form lR^i/N . (Also it is an abbreviation of $(1, v) \in \bar{\mathbf{P}}$ since there is no rotation element in (6.26).) Certainly, it is an action of order N , and defines an orbifold $T^n/\mathbf{Z}_N$, as seen in Subsect. 3.1.4. The operation (6.26) is freely acting and there is no fixed point. Also we require the following constraint for the gauge modes similarly as (6.3),

$$\sum_I NV_I = 0 \pmod{2} \quad (6.27)$$

where the modulo 2 condition is reserved for spinorial states, e.g. in the $E_8 \times E_8$ theory.

Apart from the usual untwisted string, there is a twisted string up to the shift (6.26)

$$X^i(\sigma + \pi) = X^i(\sigma) + 2\pi L^i + 2\pi v^i. \quad (6.28)$$

Although this is equivalent to (6.26), we have displayed some possible contribution from string winding L^i . This is a generalization of winding: it completely close the string on the *moded torus* along the smaller circumference $2\pi v = 2\pi l/N$. Formally the mode expansion is same as (5.176, 5.177) with L replaced by $L + v$, therefore same are quantization, mass condition and so on.

There is no change of zero point energy $\tilde{c}$, because the action (6.28) does not affect on the oscillators.

We should not forget to include the k^{th} twisted sector shifted by kv with $k = 1, \dots, N-1$.

Every state has the transformation property under $\mathcal{V}$,

$$\mathcal{V}: |p + kv\rangle \rightarrow e^{2\pi i(p+kv)\cdot v} |p + kv\rangle. \quad (6.29)$$

Being familiar in quantum mechanics, the generator of translation (6.26) is the momentum operator $-i\partial_i$ whose eigenvalue is $p + kv$. We may also check that vertex operator $:e^{i(p+kv)x}:$ possesses the desired symmetry.

The mode expansion now takes the form

$$X^i = x^i + (p^i + v^i)\sigma + \frac{i}{2} \sum \left(\frac{1}{n} \tilde{\alpha}_n^i e^{-2in(\tau+\sigma)} + \frac{1}{n} \alpha_n^i e^{-2in(\tau-\sigma)} \right),$$

hence momentum p is changed to $p + v$. Our main purpose is to employ this shift (6.26) for extra sixteen bosonic degrees of freedom, responsible for describing gauge group. They have *only left mover*, so we should act the shift in an *asymmetric* manner

$$X_L^I(\sigma + \pi) = X_L^I(\sigma) + 2\pi L^I + \pi V^I. \quad (6.30)$$

The missing factor 2 in the last term is due to the fact that X^I are only the left movers, which hence has no geometric interpretation. To implement this situation we formally assume the presence of right movers as done before. Then, the mode expansion goes like

$$X_R^I = x_R^I + \left(\frac{1}{2} p^I - L^I \right) (\tau - \sigma) + \frac{i}{2} \sum \frac{1}{n} \alpha_n^I e^{-2in(\tau-\sigma)}, \quad (6.31)$$

$$X_L^I = x_L^I + \left(\frac{1}{2} p^I + L^I + V^I \right) (\tau + \sigma) + \frac{i}{2} \sum \frac{1}{n} \tilde{\alpha}_n^I e^{-2in(\tau+\sigma)}. \quad (6.32)$$

Again we impose the “absence of right mover” constraint as in (5.191) $p_R = \frac{1}{2}p - L$. It is compulsory to choose a compact radius $R = \sqrt{\alpha'}$ and here the interplay between momentum and winding is crucial. Then, the mode expansion becomes

$$X_L^I = x_L^I + (P^I + V^I)(\tau + \sigma) + \frac{i}{2} \sum \frac{1}{n} \tilde{\alpha}_n^I e^{-2in(\tau+\sigma)}, \quad (6.33)$$

where $P = p_L$ as before. Summarizing, the action (6.28) gives rise to a changed momentum

$$P \rightarrow P + V, \quad (6.34)$$

and so does the mass shell condition (6.10),

$$\begin{aligned} \frac{M_L^2}{4} &= \tilde{N} + \frac{(P+V)^2}{2} + \tilde{c}, \\ \frac{M_R^2}{4} &= N + \frac{(s+\phi)^2}{2} + c \end{aligned} \quad (6.35)$$

where $\tilde{c}$ and c depend on the spacetime.

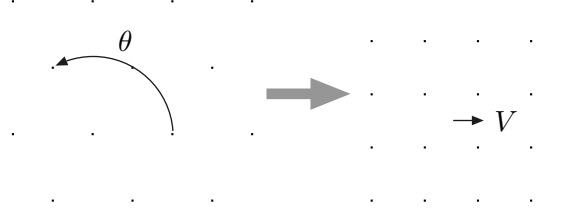


Fig. 6.2. We associate point action θ with lattice shift V in the group space. Then, the momentum state $|P\rangle$ has a different boundary condition and breaks the group

6.2 Breaking Gauge Group

6.2.1 Embedding into Gauge Group

To obtain the four dimensional world, we should make six dimensions out of ten compact and small. The smooth manifold of six torus T^6 is not the one we want since it is well known that we cannot have chiral fermions there [6]. Instead, we compactify on the six dimensional orbifold. Then the modular invariance condition becomes nontrivial and relates this compact space with the internal gauge symmetries, resulting in some projections. This will provide boundary conditions to break the gauge group.

Firstly, we form the orbifold by moding out the torus by the following action

$$O = T_{\text{group},L}^{16}/\mathcal{V} \times T_L^6/\bar{\mathcal{P}} \times T_R^6/\bar{\mathcal{P}}. \quad (6.36)$$

We decomposed the six dimensional torus into two pieces, although the geometry of torus is not decomposable. It only means that the string has independent symmetries on the left- and the right-movers.

Next, we associate the orbifold action θ of $T_L^6 \times T_R^6$ with the group lattice T_G^{16} translated by V , see Fig. 6.2,

$$(\theta, 0) \longrightarrow (1, V). \quad (6.37)$$

This is called *embedding* of the orbifold action in the group space. This just indicates that we are giving boundary conditions differently for each degree of freedom as follows.

	$x^0 \dots x^3$	$x^4 \dots x^9$	$x^{10} \dots x^{25}$
left	noncompact	$(\theta, 0)$	$(1, V)$
right	noncompact	$(\theta, 0)$	not present

We will see this association is required and accomplished in the way that the modular invariance is satisfied. Because the modular invariance is maintained, we have a nontrivial relation between orbifold action ϕ and shift V (embedded in the group space) which is presented by some conditions on ϕ and V so that some states are projected out. It will be shown shortly. The consequence is

that the gauge symmetry is broken and there appear another states. This will give rise to a more general string theory called 4D string models.

This is the simplest setup in the sense that we use only the translational moding $\mathcal{V}$ on the gauge group lattice and use the symmetric moding $\bar{\mathcal{P}}$ on the left and the right movers. We deal with the shift (translation) on the gauge lattice (6.30) because of the following theorem: every inner automorphism of finite order can always be represented by a shift vector [7]. The other automorphism (outer automorphism) is the symmetry of Dynkin diagram (for example, rotation symmetry of $SU(n)$ diagram by 180 degree) up to Weyl transformations. Since the E_8 group has only inner automorphisms, all of its subgroups can be represented by shift vectors.

This is not the only choice. Instead we can make use of the symmetry (isometry) of the lattice, like the point group action of orbifold $\bar{\mathcal{P}}$ of (3.1). For example we can make a lattice rotation, which does not leave the Cartan generators invariant. We will consider this possibility in Sect. 9.2.

6.2.2 Modular Invariance

The consistency condition of the closed string theory comes from the modular invariance. We learned that the modular invariance under reparametrization of τ in (5.72) constrains the group lattice to $E_8 \times E_8$ or $SO(32)$. The spacetime part is *independently* modular invariant. By orbifolding, each part loses its own modular invariance, however we can choose a special combination to make the *whole theory* invariant. Here, we sketch the procedure for obtaining constraints, postponing the proof to the next chapter.

As before, we consider a closed string one-loop diagram, specified by modular parameter τ , since the modular transformation is generated by two generators (5.73)

$$\mathcal{T} : \tau \rightarrow \tau + 1, \quad \mathcal{S} : \tau \rightarrow -1/\tau, \quad (6.38)$$

where τ is the modular invariance. We need to check the modular invariance under these transformations only.

$\mathcal{T}$ Invariance

The real axis is identified with σ , so that the $\mathcal{T}$ generate a shift in the σ direction whose operator is generated by the world-sheet momentum $P = \tilde{L}_0 + \tilde{c} - (L_0 + c)$. In the exponentiated form is

$$U(\sigma) = \exp[2i(\tilde{L}_0 + \tilde{c} - L_0 - c)\sigma]. \quad (6.39)$$

Since the closed string has no preferred origin, these levels should match modulo integer,

$$\tilde{L}_0 + \tilde{c} - L_0 - c = \text{integer}. \quad (6.40)$$

It seems that the level matching condition should be satisfied in the form $\tilde{L}_0 + \tilde{c} - L_0 - c = 0$. However, the phase under orbifold action (6.14, 6.29)

will project out some states, so that we require a somewhat looser condition modulo integer.

It is noted that this simple relation holds in the abelian orbifold case. In the nonabelian orbifold which we will not consider, we have a much more complicated condition [8]. To examine, consider the left movers with the mass shell conditions (6.35) and (6.16)

$$\tilde{L}_0 + \tilde{c} = \tilde{N} + \frac{(P+V)^2}{2} + \tilde{c}, \quad (6.41)$$

$$L_0 + c = N + \frac{(s+\phi)^2}{2} + c. \quad (6.42)$$

It is easy to see that the oscillator number N is a multiple of $\frac{1}{N}$, and the zero point energies, $(P+V)^2$ and $(s+\phi)^2$, are multiples of $\frac{1}{N^2}$. In general, it is not possible to make sum of them integer to satisfy the relation (6.40). Thus, a necessary condition is that the $\frac{1}{N^2}$ dependence should be reduced to an integral multiple of $1/N$,

$$(P+V)^2 - (s+\phi)^2 = \text{integer} \cdot \frac{1}{N}.$$

Since we are in the even and self dual lattice, the first two terms are trivially multiples of $2/N$. Since P and s are in the even lattices, the only nontrivial condition is

$$\sum_i \phi_i^2 = \sum_I V_I^2 \mod \frac{2}{N}. \quad (6.43)$$

Once this condition is satisfied, so is the relation for the k^{th} twisted sector kV for any integer k . Of course this results in the same condition also from the fermionic construction as discussed in [4]. We have shown a necessary condition for the modular invariance. This is the guideline for choosing V for a given ϕ : we can in principle *choose any shift vector* V so long as this condition is satisfied.

The sufficiency of the condition which will be discussed in the next chapter, invariance to all orders of loop diagrams, and its implication for anomaly freedom is discussed in [8].

S Invariance

In fact, we have seen in Subsect. 5.2.3, considering only the boundary condition in the σ direction is not sufficient, for unitarity and supersymmetry, etc. In particular, here in the orbifold case we should specify twisted boundary condition (6.4) and (6.30) in the τ direction also

$$Z(\tau, \sigma + \pi) = hZ(\tau, \sigma), \quad (6.44)$$

$$Z(\tau + \pi\tau_1, \sigma + \pi\tau_2) = gZ(\tau, \sigma). \quad (6.45)$$

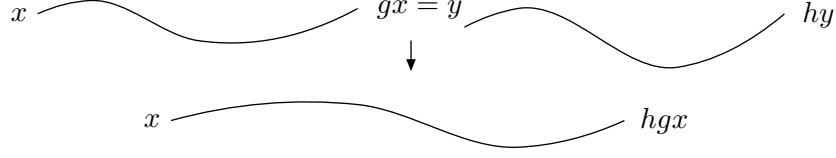


Fig. 6.3. Interactions between twisted strings. The two twisted strings by g and h join together into the string twisted by hg . We see that twisted string maps from a Hilbert space into another

Here, $\tau_1 + i\tau_2$ is the modular parameter and the h, g are elements of the space group $\bar{P}$. We label such sector to be (h, g) . We will briefly learn its consequence and perform an explicit calculation in the next chapter.

$\mathcal{S}$ interchanges the boundary conditions of the two world-sheet coordinates τ and σ ,

$$\mathcal{S} : (h, g) \rightarrow (g^{-1}, h) . \quad (6.46)$$

This means that the *twisted sector is needed* to close the modular transformation. Recall that we defined the twisted sector by the nontrivial boundary condition of the σ coordinate. The part of the untwisted sector $(1, h)$ goes into the twisted sector $(h, 1)$. Also it follows that in the untwisted sector, we should specify the boundary conditions in τ direction.

This situation can be explained pictorially, as in Fig. 6.3. Suppose the interaction between two string twisted by h and g . Merging, the new twisted string is twisted by hg . The twisted string vertex operator maps one twisted string Hilbert space to another Hilbert space.

Under general finite transformation $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$ of (5.72), the boundary condition goes like

$$(h, g) \rightarrow (h^d g^c, h^b g^a) . \quad (6.47)$$

In particular,

$$\mathcal{T} : (h, g) \rightarrow (h, hg) \quad (6.48)$$

$$\mathcal{S} : (h, g) \rightarrow (g^{-1}, h) . \quad (6.49)$$

From this, we also see that $\mathcal{T}$ requires twisted sectors.

6.2.3 The String Hilbert Space

Untwisted Sector

Gauge Sector

First focus on the untwisted sector, where the orbifold and lattice shift is trivial. The mass shell condition (6.35) reduces to the one when no orbifolding is introduced, (5.204). However, whenever a state transforms by a space group element θ , it acquires phase given in (6.29)

$$|P\rangle \rightarrow e^{2\pi i P \cdot V} |P\rangle . \quad (6.50)$$

The unbroken gauge boson should not carry such a phase in (6.29), or we require

$$P \cdot V = 0 \pmod{\text{integer}} . \quad (6.51)$$

It is so because the *gauge boson* transforms one state into another, and during such a gauge transformation the state should not acquire a nontrivial (orbifold) phase. The right movers should not introduce an additional phase, which is possible for the singlet right mover. Thus, the multiplicity calculated from the right movers is 1 for the gauge boson.

Matter Sector

There are other invariant states, which do not satisfy the condition (6.51). This is due to the fact that we have also right movers to form a complete state, in order to make the whole state invariant. We call them the *matter representations*, since they constitute quantum numbers carried by matter. They are again classified by the transformation property under (6.29)

$$P \cdot V = \frac{k}{N} \pmod{\text{integer}} . \quad (6.52)$$

where $k = 1, 2, \dots, N-1$. With right movers with the opposite phase, they can form complete string states.

Right Movers

Recall that the right mover of heterotic string is the same as that of Type II string. We have ten bosons and ten fermions, which is divided into the NS and the R sectors. By the GSO projection, we have the spinorial $\mathbf{8}_s$ only in the R sector and the vector $\mathbf{8}_v$ in the NS sector.

In the NS sector, the ground state $b_{-1/2}^M |0\rangle$ has the vector index $M = (\mu, a, \bar{a})$. The transformation property under the point group is the same as those of the left movers with the same index.

In the R sector, whose states are spinorial, the orbifold action is not so clear. However, when we bosonize them the representation becomes easy to treat. The original state is $\mathbf{8}_s$ of (5.201) and denoted as

$$\mathbf{8}_s : |s\rangle_R = |[++++]_R . \quad (6.53)$$

This is convenient because we can represent the R states like the NS states as following. Under the space group action, specified by ϕ , each state $|s\rangle_R$ transforms as

$$|s\rangle_R \rightarrow e^{-2\pi i s \cdot \phi} |s\rangle_R . \quad (6.54)$$

This is essentially the same as (6.50). The minus sign in the exponent is used, taking account of the right moving state, coming from the sign in $\tilde{L}_0 - L_0$.

Also the NS sector 8_v has the representation $s = (\pm 1 \ 0 \ 0 \ 0)$ which has the same transformation property. Because of supersymmetry, the R and NS sectors have the same branching rules, related to each other by the triality relation.

Twisted Sector

The same method holds for obtaining spectra in the twisted sectors. We have seen that the inclusion of the twisted sectors is compulsory due to the modular invariance requirement. The twisted sectors always provide matter representations. A string in the untwisted sector freely moves in the bulk (the whole space), but a string in the twisted sector is confined (localized) at the fixed point.

We see that we have such twisted sectors associated by $k\phi$ and kV with $k = 1, 2, \dots, N-1$. This affects the zero point energy, the mass shell condition and the projection condition (through the phase).

It is important to note that the zero point energy, with the mass shell conditions (6.16) and (6.25), is nontrivial. Also it acquires a nontrivial phase,

$$\begin{aligned} |P + kV\rangle &\rightarrow e^{2\pi i(P+kV)\cdot V} |P + kV\rangle, \\ \alpha_{-l/N} &\rightarrow e^{2\pi i l/N} \alpha_{-l/N}. \end{aligned} \quad (6.55)$$

Note that a fractional oscillator emerges only in the twisted sector satisfying the projection condition (6.14) and (6.29) so that a state with the left- and right-movers combined together should be invariant.

In this spirit, what so far is called the untwisted sector corresponds to the $k = 0$ *twisted sector*. In the prime order orbifold, $\mathbf{Z}_3$ and $\mathbf{Z}_7$, the projection condition is trivially the consequence of the modular invariance condition (6.43).

We can do the same thing to the right twisted states. We will deal with specific examples in the following sections.

6.3 $\mathbf{Z}_3$ Orbifold: Standard Embedding

The most famous and nontrivial example of embedding is on the $\mathbf{Z}_3$ orbifold, discussed in Subsect. 3.1.7. It is specified by a vector (3.55),

$$\phi = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}.$$

We choose *standard embedding*, that is, the orbifold action is associated with the shift

$$V = \phi \quad (6.56)$$

with other degrees are not touched. Namely, the shift vector is specified by

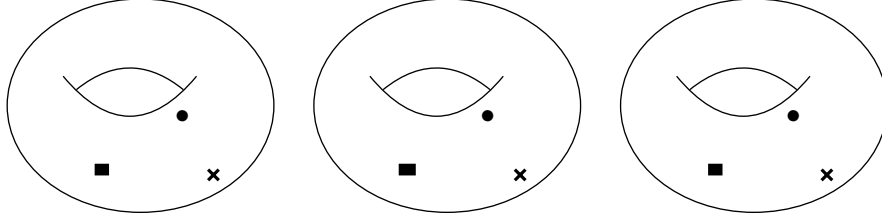


Fig. 6.4. The torus and the fixed points of the $\mathbf{Z}_3$ orbifold. The same symbols in Fig. 3.4 are used

$$V = \left(\frac{2}{3} \frac{1}{3} \frac{1}{3} \ 0 \ 0 \ 0 \ 0 \ 0 \right) (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) .$$

Sometimes repeating zeroes are abbreviated using superscript. In this, the above vector may simply written as

$$V = \left(\frac{2}{3} \frac{1}{3} \frac{1^2}{3} \ 0^5 \right) (0^8) . \quad (6.57)$$

This automatically satisfies modular invariance condition (6.43).

Fixed points of $\mathbf{Z}_3$ orbifold are sketched in Fig. 6.4.

6.3.1 Untwisted Sector

The untwisted states are a part of the original states invariant under the point group action θ and associated shift V . It is not necessary that each left and right mover itself be invariant, but only the combined state should be invariant.

Left Mover

We have massless oscillators of a kind

$$\tilde{\alpha}_{-1}^M |0\rangle_L \quad (6.58)$$

The point group action θ is defined on the orbifolded spacetime. Accordingly we can decompose index $M = (\mu, a, \bar{a}, I)$, transforming as

$$\mathbf{8}_v + \mathbf{16} = \mathbf{1} + \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{16} , \quad (6.59)$$

By (6.1) and (6.4) those states having holomorphic index a ($\mathbf{3}$ of (6.59)) transform as $e^{2\pi i/3} \equiv \alpha$ and antiholomorphic index $\bar{a}$ ($\bar{\mathbf{3}}$ of (6.59)) as α^2 . Those with noncompact four dimension index μ are not affected. The associated shift vector V only translates the momentum lattice, and those with the group space index I are not affected.

Gauge Fields

By the structure we see that (6.57) leaves the second “hidden” E_8 intact, thus our concern is the first E_8 . The massless modes satisfying $P \cdot V = \text{integer}$ of (6.51) include¹

$$\begin{aligned} &\pm(0 \ 1 \ -1 \ 0^5) \\ &\pm(1 \ \underline{1} \ 0 \ 0^5) \end{aligned}$$

and permutations of the underlined elements. With the two Cartan generators, they form the root vectors **8** of $SU(3)$. We can identify the *simple roots* as $(-1 \ -1 \ 0 \ 0^5)$ and $(0 \ 1 \ -1 \ 0^5)$, since we can generate all the roots from them. The inner products among the simple roots defines the Cartan matrix and Dynkin diagram of $SU(3)$. There exists another equivalent choice for positive roots and simple roots.

We know that the remaining subgroup of E_8 is E_6 .

$$\begin{aligned} &(0 \ 0 \ 0 \ \underline{\pm 1} \ \underline{\pm 1} \ 0 \ 0 \ 0) \\ &\pm(- \ + \ + [+ \ + \ + \ + \ -]) \end{aligned}$$

and the square parenthesis denotes even sign flips. Here, the plus and minus represents $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. We can check that, together with the six $U(1)$ generators they form the root vectors **78** of E_6 . So we expect that this untwisted sector determines the unbroken gauge group.

Matter Fields

Now we come to matter representation. States satisfying $\exp(2\pi i P \cdot V) = \exp 2\pi i/3 \equiv \alpha$, or $P \cdot V = \frac{1}{3}$, are

$$P \cdot V = \frac{1}{3} : \left\{ \begin{array}{l} (+ \ \underline{\pm} \ - \ [+ \ + \ + \ + \ -]) \\ (- \ - \ - \ [+ \ + \ + \ + \ -]) \\ (0 \ \underline{1} \ 0 \ \underline{\pm 1} \ 0 \ 0 \ 0 \ 0) \\ (-1 \ 0 \ 0 \ \underline{\pm 1} \ 0 \ 0 \ 0 \ 0) \\ (1 \ \underline{-1} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \\ (0 \ -1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0) \end{array} \right. \quad (6.60)$$

Under a $\mathbf{Z}_3$ rotation, these acquire a phase α . We can check that with an aid from the root vectors these form the weight vectors (**3**, **27**) of $SU(3) \times E_6$.

We have another states transforms like α^2 , represented by the same vector with the opposite sign $|-P\rangle_L$ so that $(-P) \cdot V = -\frac{2\pi i}{3}$. By $(-P)$, they provide conjugate representations. We *should not count them as independent states*, because they turn out to be antiparticles with the opposite helicity.

¹ In the following vectors, if we flip the signs of the first entries we may easily notice regular patterns. Nevertheless we employ this convention because the form manifestly show the group theoretical origin of the standard root and weight vectors, explained in the Appendix E.

Right Mover

For complete string states we need right movers. The right movers are from an independent superstring. The zero point energy is 0 and $-\frac{1}{2}$ for the R and the NS sectors, respectively.

In the NS sector, the ground state $b_{-1/2}^M|0\rangle_R$ has the vector index $M = (\mu, a, \bar{a})$. The transformation property under the point group is the same as those of the left movers with the same index.

In the R sector the spinorial state $|s\rangle_R$ has the representation (6.53). Under the $\mathbf{Z}_3$ action ϕ , the phase factor acquired is

$$|s\rangle_R \rightarrow e^{-2\pi i s \cdot \phi} |s\rangle_R, \quad (6.61)$$

where $\phi = \frac{1}{3}(0 \mid 2 \ 1 \ 1)$. Because of supersymmetry, we have the same branching rule as those in the NS sector,

$$\mathbf{8}_s = \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} + \mathbf{1}. \quad (6.62)$$

where

$$\mathbf{3} : (+ - \underline{+-}), (+ + ++) \sim \alpha^1 \quad (6.63)$$

$$\bar{\mathbf{3}} : (- + \underline{+-}), (- - --) \sim \alpha^2 \quad (6.64)$$

$$\mathbf{1} + \mathbf{1} : (+ + --), (- - ++) \sim \alpha^0. \quad (6.65)$$

We check that superpartners transform same under $\text{SU}(3)$ by triality relation.

6.3.2 Chirality

The complete state is made by combining the left and right movers. In a $\mathbf{Z}_3$ invariant way, the singlet is combined with the left moving gauge multiplet,

$$|P\rangle_L \otimes b_{-1/2}^\mu |0\rangle_R, \quad \alpha_{-1}^I |0\rangle_L \otimes b_{-1/2}^\mu |0\rangle_R. \quad (6.66)$$

where

$$P \in (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}) \text{ of } \text{SU}(3) \times \text{E}_8, \quad s \in \mathbf{1} + \mathbf{1}. \quad (6.67)$$

Their fermionic superpartners are gauginos

$$|P\rangle_L \otimes |s\rangle_R, \quad \alpha_{-1}^I |0\rangle_L \otimes |s\rangle_R. \quad (6.68)$$

In (6.61), ϕ acts on the last three components s_1, s_2, s_3 of s . In particular, the noncompact component s_0 remains untouched and we adopt the convention s_0 as the helicity in $4D$. The low energy fields inherit the chirality from the right movers. The chirality is the first component of $\Gamma = 2^4(s_0 \ s_1 \ s_2 \ s_3)$. Or we can define $s \equiv (\oplus \text{ or } \ominus |s_1 \ s_2 \ s_3)$ where $\oplus$ and $\ominus$ can be called right-handed and left-handed, respectively. Reading components of s above (6.65), we can have either +1 or -1 chirality for gauge bosons. Gauge bosons couple

to the fermion current which carries the minus sign for the product of the eigenvalues of s_0 . Hence, with our definition of the product of chiralities being +1 the gauge boson and their accompanying gauginos must be defined to be left-handed, i.e. $2s_0 = -1$. The $s_0 = \frac{1}{2}$ components correspond to their $\mathcal{CTP}$ conjugates.

For the matter multiplet, we have

$$|P\rangle_L \otimes b_{-1/2}^\mu |0\rangle_R, \quad |P\rangle_L \otimes |s\rangle_R, \quad P \in (\mathbf{3}, \mathbf{27}), s \in \bar{\mathbf{3}} \quad (6.69)$$

and their antiparticles as $\mathcal{CTP}$ conjugates. The multiplicity three in the untwisted sector matter is due to this anti-triplet. Each matter field above has only one four dimensional helicity $s_0 = -\frac{1}{2}$ when combined with the R sector right movers such that it is $\mathbf{Z}_3$ invariant, viz. (6.60) and (6.64). The other helicity states come from the states transforming oppositely under the gauge group. So, we define the chirality for the representation $(\mathbf{3}, \mathbf{27})_-$ as left-handed,

$$\left. \begin{matrix} (\mathbf{3}, \mathbf{27})_- \\ (\bar{\mathbf{3}}, \mathbf{27})_+ \end{matrix} \right\} \Rightarrow (\mathbf{3}, \mathbf{27})_L, \quad (6.70)$$

where $(\bar{\mathbf{3}}, \mathbf{27})_+$ being the antiparticle states of $(\mathbf{3}, \mathbf{27})_-$. The resulting spectrum is *chiral*. We call it a *left-handed* chiral fermion, or a left-handed Weyl fermion $(\mathbf{3}, \mathbf{27})_L$. If we want, we can instead call it a right-handed $(\bar{\mathbf{3}}, \mathbf{27})_R$. Thus, the weights satisfying $P \cdot V = \frac{1}{3}$ are left-handed. [If we interpret the states with $P \cdot V = \frac{2}{3}$ as particles, they are right-handed.]

Counting all the states, we have the $\text{SU}(3) \times \text{E}_8$ gauge bosons and

$$3 (\mathbf{3}, \mathbf{27})_L$$

as matter fermions in the untwisted sector.

6.3.3 Twisted Sector

In the twisted sector, as discussed in (6.13) and below, there are some modifications to oscillator numbers, phases and zero point energies.

Left Mover

The zero point energy is shifted by $f(\eta)$ of (6.15) for each real bosonic degree of freedom. With twist $\phi = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ on the *complex* degrees, we have vacuum energy from (6.16)²

$$\tilde{c} = 18f(0) + 4f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) = -\frac{2}{3}, \quad (6.71)$$

² In Appendix D it is denoted as $1 - \zeta$.

and we associate it with the momentum shift $P \rightarrow P + V$ as well. Thus, the level matching condition becomes

$$\frac{1}{4}M_L^2 = \frac{1}{2}(P + V)^2 + \tilde{N} - \frac{2}{3}. \quad (6.72)$$

Without oscillator, i.e. $\tilde{N} = 0$, the massless states $|P\rangle_L$ satisfying $\frac{1}{2}(P + V)^2 = \frac{2}{3}$ are

$$\begin{pmatrix} 0 & -1 & -1 & 0^5 \\ -1 & 0 & 0 & \pm 1 & 0^4 \\ - & - & -[+ & + & + & -] \end{pmatrix}.$$

Compared with untwisted state, group theoretically they transform as $(\mathbf{1}, \mathbf{27})$. The phase acquired is

$$e^{2\pi i(P+V)\cdot V} = 1. \quad (6.73)$$

From the mode expansion of the twisted states (6.6), there are fractional oscillators $\tilde{\alpha}_{-1/3}^a$ with $\tilde{N} = \frac{1}{3}$. So, there are additional massless states

$$\tilde{\alpha}_{-1/3}^a |P + V\rangle_L, \quad a = \bar{1}, 2, 3, \quad (6.74)$$

with P satisfying $\frac{1}{2}(P + V)^2 = \frac{1}{3}$,

$$\begin{pmatrix} 0^8 \\ -1 & \underline{-1} & 0 & 0^5 \end{pmatrix}$$

which form $(\bar{\mathbf{3}}, \mathbf{1})$. Since a can assume $\bar{1}, 2, 3$, they transform as a triplet under the holonomy group $SU(3)$; thus the multiplicity is 3. The phase acquired is

$$\alpha \quad \text{for } \tilde{\alpha}_{-1/3} \\ \alpha^2 (= e^{2\pi i(P+V)\cdot V}) \quad \text{for } |P + V\rangle_L \quad (6.75)$$

so that the state $\tilde{\alpha}_{-1/3}^a |P + V\rangle$ is invariant.

The second twisted sector is obtained by the shift vector $2V$ which is equivalent to $-V$. As in the case of untwisted sector states, the second twisted sector states are the $\mathcal{CTP}$ conjugates with the opposite charges and helicities of the states we considered so far. The phases in (6.55) are opposite because $(P - V) \cdot V = (-P - V) \cdot V = -(P + V) \cdot V$.

These are all the massless states.

(We present an example how the massive oscillators are formed. The lowest massive oscillator states are in the form

$$\begin{aligned} \alpha_{-2/3}^a |0\rangle, \quad a = 1, \bar{2}, \bar{3}, \quad \bar{\mathbf{3}}, \\ \alpha_{-1/3}^a \alpha_{-1/3}^b |0\rangle, \quad a, b = \bar{1}, 2, 3, \quad \mathbf{6}, \end{aligned} \quad (6.76)$$

and have the oscillator number $\tilde{N} = \frac{2}{3}$. The holonomy quantum number also represent the transformation property under point group element θ .)

Right Mover

We have the same twist on the right movers. As claimed before, above (5.202), we will use the bosonized right movers, among which only the original eight bosons get twisted, while the four bosonized fermions have no twists. Accordingly the zero point energy is

$$c = 6f(0) + 2f\left(\frac{2}{3}\right) + 4f\left(\frac{1}{3}\right) = -\frac{1}{6}.$$

One may check that it is always satisfied $c = \tilde{c} + 1/2$. Resorting to the mass shell condition

$$\frac{M_R^2}{4} = \frac{(s + \phi)^2}{2} + N - \frac{1}{6} \quad (6.77)$$

we find two massless states

$$s = (- - - -), \quad s = (-1 \ 0 \ 0 \ 0)$$

corresponding to R and NS states, respectively, belonging to different conjugacy classes, as in (5.201). In the spacetime, they are superpartners to the others, as it should be. We verify that the phase

$$e^{-2\pi i(s+\phi)\cdot\phi} = 1 \quad (6.78)$$

saves the state which survives the GSO projection. The resulting spectrum has the negative (left-handed) chirality, with untwisted matter defined from $P \cdot V = \frac{1}{3}$. As before, the second twisted sector with shift $2\phi \simeq -\phi$ provides the opposite (positive) helicity states with the opposite weight vectors, to form a complete state including the CTP conjugate. In the above calculation, the phases for the left and right movers vanished independently, but this is not necessary in general. Only the *combined state* should have the vanishing phase. Moreover, there might be a nontrivial “vacuum phase” which cannot be calculated by the operator method, which we will encounter in (7.32) of Chap. 7.

Tensoring both movers, since we have $\chi = 27$ fixed points, we have

$$27(\mathbf{1}, \mathbf{27}) + 81(\bar{\mathbf{3}}, \mathbf{1}).$$

6.3.4 Anomaly Cancellation

In total we have

$$3(\mathbf{3}, \mathbf{27})$$

in the untwisted sector and

$$27(\mathbf{1}, \mathbf{27}) + 81(\bar{\mathbf{3}}, \mathbf{1})$$

in the twisted sector. In addition we have four dimensional $\mathcal{N} = 1$ supergravity multiplet coupled to the $SU(3) \times E_6$ gauge group and there appear some Kaluza–Klein states. Since each $\mathbf{27}$ contains one complete family, we have 36 chiral families after breaking the $SU(3)$ group.

It is a nontrivial check for the anomaly cancellation of the $SU(3)$ part between chiral fermions from the untwisted and twisted sectors. It is known that E_6 is anomaly free. At the 4D string theory level, there are 81 $\mathbf{3}$ s from the untwisted sector and 81 $\mathbf{3}$ s in the twisted sector. Thus, there is no gauge anomaly. In addition, there is no gravitational anomaly. Shortly we will see that in the presence of more than one shift vector there still does not exist any anomaly. This is a generic feature of the orbifold theory, although there are only proofs [9] for the case of toroidal compactification on self-dual lattices. In some models, at most there exists an anomalous $U(1)$ which is canceled by the generalized Green–Schwarz mechanism [10].

If the 4D theory is assumed to result from the 10D theory, there might be local anomalies even if the global anomaly is vanishing. It is a well-known phenomenon as discussed in Chap. 4. The anomaly from the untwisted sector is equally distributed to 27 fixed points, thus each fixed point gets 3 units of anomaly from the untwisted sector. Noting that each twisted sector carries -3 units of anomaly from the twisted sector spectrum, there is no local anomaly at fixed points.³

6.3.5 Fermionic Description

We discussed an alternative description of the heterotic string by 32 fermions, in Subsect. 5.3.5. In obtaining the $E_8 \times E_8$ theory by compactifying 16 dimensions of the left moving bosonic string, we assign different boundary conditions to two sets of sixteen fermions in the fermionic version. Here in the construction of 4D strings, we still expect that we have the identical states at all the mass levels whatever description we use, bosonic or fermionic.

Untwisted Sector

By definition, in the untwisted sector the relevant states are those which remain invariant under the $\mathbf{Z}_3$ action. Certainly, these untwisted states should be the same as those in the bosonic description. In the preceding section, we already saw how the states are formed in the bosonic description. In the fermionic description, having the $E_8 \times E_8$ theory is equivalent to partitioning the 32 fermionic degrees of freedom into two: $(\mathbf{120}, \mathbf{1}) + (\mathbf{1}, \mathbf{120})$ in the NS NS, $(\mathbf{128}, \mathbf{1})$ in the R NS and $(\mathbf{1}, \mathbf{128})$ in the NS R sectors. The R R sector has

³ Including nonperturbative effects, for instance by turning on the instanton background, some states not satisfying this modular invariance can be consistent [11]. The anomaly freedom still can be traced back to the absence of divergences in a given string theory.

zero point energy +1, thus has no massless field. The GSO projection kicks out $(\mathbf{16}, \mathbf{16})$ representation. In terms of $\text{SO}(16) \times \text{SO}(16)$, representations have the same form as P of the bosonic description as in Sect. 6.3.1 so that the transformation property under $\mathbf{Z}_3$ is also the same.

Twisted Sector

We consider more general boundary conditions, or twisting, which is represented by the shift vector,

$$\lambda_I(2\pi) = \pm e^{2\pi i V_I} \lambda_I(0) \quad (6.79)$$

for the R and NS sectors. Then the mode expansion gives the following modified zero point energy (6.15)

$$\tilde{c} = 2f(0) + 2 \sum_{i=1}^3 f(\phi_i) + \sum_{I=1}^{16} f(V_I) .$$

By the definition of the R sector, the original shift vector is used, while in the NS sector the $-$ sign is absorbed into the phase $\pm \frac{1}{2}$. Plugging in the shift vectors, we obtain the zero point energies $\tilde{c} = 0, -\frac{1}{2}, 1, +\frac{1}{2}$ for the R NS, NS NS, RR, NS R sectors, respectively. Therefore,

$$\frac{M_L^2}{4} = \tilde{N} + \tilde{c} = \sum_i^3 \sum_{m \in \mathbf{Z} + \phi_i} \tilde{\psi}_{-m}^i \tilde{\psi}_m^i + \sum_I^{16} \sum_{n \in \mathbf{Z} + V_i} \lambda_{-n}^I \lambda_n^I + \tilde{c} . \quad (6.80)$$

The relevant states in the R NS sector are $\tilde{N} = 0$ and $\lambda_0^I |0\rangle$, $I = 4, \dots, 8$. These are spinors $\mathbf{16}$ and $\overline{\mathbf{16}}$ of $\text{SO}(10)$. The GSO projection project out the former.

In the NS NS sector, relevant massless state is obtained from $\tilde{N} - \frac{1}{2} = 0$,

$$\begin{aligned} & \lambda_{-1/6}^1 \lambda_{-1/6}^2 \lambda_{-1/6}^3 |0\rangle , \\ & \lambda_{-1/2}^I |0\rangle, \quad I = 4, \dots, 8 , \\ & \lambda_{-1/6}^K \tilde{\alpha}_{-1/3}^a |0\rangle, \quad K = 1, 2, 3, \quad a = \bar{1}, 2, 3 \end{aligned}$$

the first two are $\mathbf{1}$ and $\mathbf{10}$ of $\text{SO}(10)$. The superscripts are again holomorphic indices $\lambda^I = \lambda^{2I-1} + i\lambda^{2I}$. Since the adjoint field is that of E_6 , with $\mathbf{16}$, they form a complete multiplet $\mathbf{27}$ of E_6 . The last one is $\mathbf{3}$ of $\text{SU}(3)$ whose multiplicity is 3 corresponding to three oscillators having index a . Forming states with right movers, we have $(\mathbf{1}, \mathbf{27})$ and $3(\mathbf{3}, \mathbf{1})$ at each fixed point. We only focused on the first E_8 representation because the other “hidden” E_8 is not broken.

6.3.6 Need Improvement

In most cases, the number of generations is a multiple of three, which originates either from (1) the right movers forming a 3 under the space group, or (2) from the number of fixed points 27. Thus, the $\mathbf{Z}_3$ orbifold is a natural candidate giving three generation standard models. However, the model presented above needs a further symmetry breaking because:

1. There are too many generations, 36 copies of $\mathbf{27}$ of E_6 .
2. $SU(3) \times E_6$ is still too big. One may be satisfied at this stage as having obtained a GUT group. However, we do not have an adjoint representation to break E_6 down to the standard model gauge group.
3. All the twisted sectors have the same spectrum, because we have no way of distinguishing them. If we can distinguish them in some way, they give different spectra at different fixed points. Then, we may have different Yukawa couplings for different family members and thus can provide the observed SM mass hierarchy.

In the following section, we introduce Wilson lines, which can solve some of the problems. At generic level, apart from the phenomenological requirements, Wilson lines need to be considered. In a sense, this procedure introduces more shift vectors breaking the gauge group further. Also Wilson lines distinguish fixed points for them to have different spectra.

6.4 Turning on Background Fields

6.4.1 Another Shift Vector from Wilson Line

Consider a constant gauge field $A(x) = A$. *Locally*, it is a pure gauge

$$A_i = -i\Lambda^{-1}(x)\frac{\partial\Lambda(x)}{\partial x^i}, \quad \Lambda(x) = \exp(iA_i x^i), \quad (6.81)$$

and can be eliminated by gauge transformation $\Lambda(x)$.

When we have such a gauge field on a circle, removing A by gauge transformation is not true *globally*. The gauge parameter $\Lambda(x)$ does not obey periodicity; thus is not well-defined. Alternatively, we force the gauge field to vanish by gauge transformation

$$A_I \rightarrow A_I + i\Lambda^{-1}\frac{\partial\Lambda(x)}{\partial x} = 0, \quad \psi(x) \rightarrow \Lambda(x)\psi(x) = e^{iA \cdot x}\psi(x). \quad (6.82)$$

Then it is inescapable that the wavefunction $\psi(x)$ acquires a phase. Note that the phase in (6.82) is interpreted as the ‘‘Scherk–Schwarz’’ phase in (4.6).

The gauge invariant measure of this effect is called the *Wilson line*,

$$U = \exp \left[2\pi i \oint A_i^I dx \cdot H^I \right] \equiv \exp (2\pi i a_i^I \cdot H^I). \quad (6.83)$$

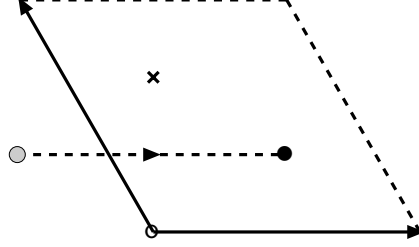


Fig. 6.5. Wilson lines as shifts. Going around to come to the original point, we feel the presence of Wilson line along the path

Here, we made group space index $I(I = 1, \dots, 16)$ explicit and $i(= 1, \dots, 6)$ is the index of compact space. Only the index i is related to the non-contractible cycles. This is the familiar situation we encounter in the Aharonov–Bohm effect. It happens whenever we have a non-contractable cycle on a non-simply connected manifold due to compactness or singularity.

In effect

$$|P\rangle \rightarrow U|P\rangle = \exp(2\pi i a_i \cdot P)|P\rangle. \quad (6.84)$$

This is nothing but a shifted vector with an associated action, viz. Fig. 6.5,

$$(1, e_i) \longrightarrow a_i. \quad (6.85)$$

This restricts the boundary condition of the wave function and thus is used to break the gauge group, as shown in Fig. 6.5. It is the well-known Hosotani mechanism [12].

Let us now associate the Wilson line with such a translation. Recall that the point group action

$$X^i(\sigma + \pi) = (\theta X)^i(\sigma) + \sum_{a=1}^6 m^a e_a^i \quad (6.86)$$

with e_a^i being unit vector along the i th torus. So far we have neglected this translational piece because all the fixed points have been equivalent: there was no way to distinguish a specific fixed point. However, when we turn on the background field, we see an effect of the Wilson line whenever we go along the torus. The fixed points are *not equivalent and can be distinguished*. This is what we have classified in (3.21). Coming back to the original point by (6.86), we associate the effect of Wilson line with the following in the group space

$$X^I(\sigma + \pi) = X^I(\sigma) + V^I + \sum_{i=1}^6 m_i a_i^I$$

so that we have a combined homomorphism of (6.30) and (6.85) to have

$$(\theta, e_i) \longrightarrow V + a_i. \quad (6.87)$$

This is equivalent to the shift in the momentum space

$$P^I \rightarrow P^I + V^I + \sum_{i=1}^6 m_i a_i^I . \quad (6.88)$$

In general, the Wilson line can assume continuous values, however we choose discrete values in order to associate with orbifold actions (6.3). Thus we require that Na_i should belong to the weight lattice and should satisfy the modular invariance condition. For example, we can show it for the $N = 3$ case as

$$\begin{aligned} \theta e_1 &= e_2, \quad \text{implying } a_1 = a_2, \quad \text{and} \\ \theta e_2 &= -e_1 - e_2, \quad \text{implying } a_2 = -a_1 - a_2 = -2a_2 , \end{aligned}$$

which lead to

$$3a_2 = 0, \quad 3a_1 = 0, \quad \text{up to lattice shifts .}$$

In the untwisted sector, the projection (6.29) is modified by additional pieces a_i . The net effect in the untwisted sector is the common intersection,

$$\begin{aligned} P \cdot V &= \text{integer} , \\ P \cdot \left(V + \sum_{i=1}^6 m_i a_i \right) &= \text{integer} . \end{aligned} \quad (6.89)$$

In the twisted sector, the mass shell condition becomes

$$\frac{M_L^2}{4} = \frac{(P^I + V^I + \sum_i m_i a_i^I)^2}{2} + \tilde{N} + \tilde{c} , \quad (6.90)$$

noting that m_i determines the specific fixed point. $\tilde{c}$ is the normal ordering constant from the internal field oscillators; thus it remains the same as that without the Wilson line. With this in mind, we extend the modular invariance condition (6.43) to

$$\phi^2 = \left(V + \sum m_i a_i \right)^2 = 0, \quad \text{mod } 2/N \quad (6.91)$$

where the vector indices are suppressed.

It is noticeable from the above equations that each twisted sector, labeled by $\sum m_i e_i$, is distinguished; thus each sector with $\sum m_i e_i$ is considered as an independent shift, resulting with an independent spectrum. This spectrum is understood as a string localized around the fixed point, drawn in Fig. 6.1(c). Locally, each twisted sector does not see other shift vectors except its own $V + \sum m_i a_i$, thus it has a larger gauge group. This provides a good picture when we consider localized anomalies discussed in Sect. 4.4 [13].

There can exist nontrivial relations between shift vectors. For instance, in the SU(3) lattice of the preceding section, a unit lattice vector transforms like $\theta e_i = e_{i+1}$. Thus, the Wilson line along this lattice should be the same

$$a_{i+1} = a_i . \quad (6.92)$$

Namely, there can exist only three independent Wilson lines when we compactify 6 dimensions. If Wilson lines a_i and a_{i+1} are not related by the lattice transformation, they need not be related. For instance, for a $\mathbf{Z}_4$ orbifold $\phi = \frac{1}{4}(2 \ 1 \ 1)$ the action on the first two-torus is the reflection about the origin, i.e. $\mathbf{Z}_2$. In this case there is no relation between two unit vectors defining the first two-torus, hence the Wilson line shift vectors a_1 and a_2 are independent. It means that we can take the two-torus with the edges having different sizes, $R_1 \neq R_2$. These Wilson lines are suggested for breaking gauge symmetry further [14].

It is convenient to use the following modular invariance form which is equivalent to (6.91)

$$\begin{aligned} \phi^2 - V^2 &= 0 \quad \text{mod } 2/N , \\ a_i^2 &= 0 \quad \text{mod } 2/N , \\ 2V \cdot a_i &= 0 \quad \text{mod } 2/N , \\ 2a_i \cdot a_j &= 0 \quad \text{mod } 2/N, \quad i \neq j \end{aligned} \quad (6.93)$$

for all i, j . Also orbifold projection condition (6.89) is further simplified to obtain gauge group and spectrum.

Gauge group:

$$\begin{aligned} P \cdot V &= \text{integer} , \\ P \cdot a_i &= \text{integer} , \end{aligned} \quad (6.94)$$

for all i . In Sect. 6.3, we have considered in detail the matter fields for the case of no Wilson lines. With Wilson lines, any $V + \sum_i m_i a_i$ is a shift vector. Choose one among them as V . Then in the untwisted sector we just consider V . In the untwisted sector the closed string on the torus is not circling any fixed point. In this case, the background gauge field should not produce any extra phase. Therefore, for the untwisted matter we require

Untwisted matter:

$$\begin{aligned} P \cdot V &= \frac{k}{N} , \\ P \cdot a_i &= \text{integer}, \quad \text{for all } i \end{aligned} \quad (6.95)$$

where $k = 1, 2, \dots, N-1$. This is a simple way of projecting out many matter fields from the untwisted sector.

For the twisted sector, we obtain the massless spectrum exactly in the same way as in Sect. 6.3 but by replacing V with $V + \sum_i m_i a_i$. Now, the degenerate twisted sector is distinguished by the associated Wilson lines. An example is shown below.

6.4.2 $\mathbf{Z}_3$ Example: Looking for SU(5) Model

Consider a $\mathbf{Z}_3$ shift vector associated with the point group action θ

$$V = \left(\frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{2}{3} 0 0 0 \right) \left(\frac{2}{3} 0 0 0 0 0 0 \right)' \quad (6.96)$$

which breaks the gauge group down to $\mathrm{SU}(9) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$.

Every two-torus is topologically characterized by two non-contractible loops. Turn on a Wilson line along the y^1 direction, for example, with a shift

$$a_1 = \left(0 0 0 0 0 0 0 \frac{2}{3} \right) \left(0 \frac{1}{3} \frac{1}{3} 0 0 0 0 0 \right)' \quad (6.97)$$

By the $\mathbf{Z}_3$ symmetry $\theta e^1 = e^2$, we have automatically $a_2 = a_1$ along the y^2 direction.

The resulting group is

$$\mathrm{SU}(5) \times \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{U}(1)^2 \times [\mathrm{SU}(2) \times \mathrm{SO}(10) \times \mathrm{U}(1)^2]',$$

identified the untwisted sector gauge fields satisfying $P \cdot V = \text{integer}$ and $P \cdot a_1 = \text{integer}$, listed on the first row of Table 6.1. They are the common intersection of gauge groups of each fixed point, which however is more easily obtained by considering (6.94) for all the shifts. Matter representations satisfy (6.95).

The center of mass of twisted string is at the fixed point. There are three fixed points shown in Fig. 3.4. Under the point group action, they return to the original point, only accompanied by a translation (6.86). While the origin $\mathbf{o}$ remains invariant, but $\mathbf{x}$ need e_2 translation after the point action

$$X_{\mathbf{x}}^i(\pi) = (\theta X_{\mathbf{x}})^i(0) + e_2. \quad (6.98)$$

Now we have a Wilson line along this direction, returning to the original point; the string is shifted by a_1 as well as V due to θ .

$$P \rightarrow P + V + a_1 \quad (6.99)$$

The effective shift vector in this twisted sector is

$$V + a_1 = \frac{1}{3}(7 \ 3 \ 3 \ 1 \ 1 \ 1 \ 0 \ 0)(0^8) \simeq \frac{1}{3}(1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0)(0^8) \quad (6.100)$$

The mass shell condition (6.90) of the twisted sector spectrum is,

$$\frac{M_L^2}{4} = \frac{(P + V + a_1)^2}{2} + \tilde{N} - \frac{3}{2},$$

where zero point energy is unchanged because it is from the spacetime part.

Table 6.1. Spectrum of flipped SU(5)-like model. The + and - means $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, the underline means permutations and the square bracket [] means permutations with even numbers of minus sign additions. The representation counts the number of weights shown in that line

Sector		State	#	Repr.
U	untwisted	$(\underline{1} \ -1 \ 0^3 \ 0^3)$	20	SU(5)
		$\pm(0^5 \ 1^2 \ 0)$	2	SU(2) ₁
	adjoint	$(0^5 \ \underline{1} \ -1 \ 0)$	2	SU(2) ₂
		$(0 \ \underline{1} \ -1 \ 0^5)'$	2	SU(2)'
		$(0^3 \ \underline{\pm 1} \ \pm 1 \ 0^3)'$	40	SO(10)'
U	untwisted	$(\underline{1} \ 0^3 \ 0 \ \underline{\pm 1} \ 0 \ 0), (0^4 \ -1 \ \underline{\pm 1} \ 0 \ 0)$	20	$2(\mathbf{5}, \mathbf{2}_2)$
		$(-1^2 \ 0^2 \ 0^4), (-1 \ 0^3 \ 1 \ 0^3)$	10	$(\overline{\mathbf{10}}, \mathbf{1})$
	matter	$(-1 \ 0 \ 0 \ \underline{\pm 1} \ 0^4)'$	10	$(\mathbf{1}, \mathbf{10})'$
		$(-+ \ -[- \ +^4])'$	10	$(\mathbf{2}, \mathbf{16})'$
T0	twisted by V	$(0^8), (\underline{-1} \ 0^3 \ -1 \ 0^3)$	5	$(\overline{\mathbf{5}}, \mathbf{1})$
		$(-^5[- \ + \ +])$	4	$(\mathbf{1}, \mathbf{2}_1 + \mathbf{2}_2)$
T1	twisted	$(0^4 \ -1 \ 0^2 \ -1)(-1 \ \underline{-1} \ 0 \ 0^5)'$	2	$(\mathbf{2}, \mathbf{1})'$
		$(0^4 \ -1 \ 0^2 \ -1)(0^8)'$	1	$(\mathbf{1})(\mathbf{1})'$
	by $V + a_1$	$(-^5[+ \ +] -)(-1 \ \underline{-1} \ 0 \ 0^5)'$	4	$(\mathbf{2}_1)(\mathbf{2})'$
		$(-^5[+ \ +] -)(0^8)'$	2	$(\mathbf{2}_1)(\mathbf{1})'$
T2	twisted	$(0^4 \ -1 \ 0 \ 0 \ 1)(-1 \ \underline{1} \ 0 \ 0^5)'$	2	$(\mathbf{2}, \mathbf{1})'$
		$(0^4 \ -1 \ 0 \ 0 \ 1)(0^8)'$	1	$(\mathbf{1})(\mathbf{1})'$
	by $V - a_1$	$(-^5[+ \ -] +)(-1 \ \underline{1} \ 0 \ 0^5)'$	4	$(\mathbf{2}_2)(\mathbf{2})'$
		$(-^5[+ \ -] +)(0^8)'$	2	$(\mathbf{2}_2)(\mathbf{1})'$

Similarly, after θ rotation, the point $\bullet$ goes back to the original point by additional translation by $e_1 + e_2$. The corresponding shift vector is $V + a_1 + a_2$. In this $\mathbf{Z}_3$ case it is identical to $V + 2a_1 \simeq V - a_1$.

It follows that, in the presence of Wilson line, 27 fixed points are not equivalent any more. We have three different kinds of twisted sectors by V , $V + a_1$, $V - a_1$, corresponding to fixed points $\circ$, $\bullet$, $\times$, respectively. This situation is depicted in Fig. 6.6.

Following the rules presented above, we can obtain massless spectrum in the untwisted sector (U) and three twisted sectors (T0, T1, T2). They are tabulated in Table 6.1. There is no oscillator state satisfying $(P + V + m_1 a_1)^2 = \frac{2}{3}$ where $m_1 = 0, 1$ and -1 . The chiral representations written as bold faced numbers are defined as L-handed. The nonabelian anomaly can appear only from the SU(5) factor group. The relevant chiral representations are $3(\mathbf{5}, \mathbf{2}_1 + \mathbf{2}_2)$ and $3(\overline{\mathbf{10}}, \mathbf{1})$ from U and nine $\overline{\mathbf{5}}$ from T0; thus there is no SU(5) anomaly. There is a potential anomalous U(1) that can be cancelled by the Green-Schwarz mechanism. Thus, for a full discussion on the anomaly, we need to know the U(1) charges whose calculation will be postponed until

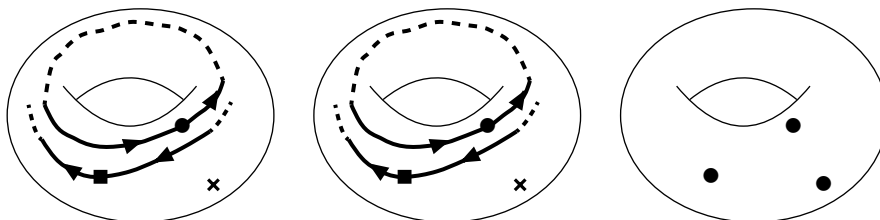


Fig. 6.6. $\mathbf{Z}_3$ fixed points wrapped by two Wilson lines a_1 and a_3 . Wilson line directions are shown by arrows. Thus, three fixed points wrapped by Wilson lines except the third torus are distinguished, which are shown by a bullet, square and cross in two tori. The wrapping by Wilson lines are shown by solid lines in the fundamental domain, while the *dashed lines* are just redundant ones outside the fundamental domain. A loop originating from the square is not completely shown. In fact, the orbifold is like a ravioli, viz. Fig. 3.5

Chap. 10. With the survival hypothesis, this example gives $3(\overline{\mathbf{10}} \oplus \mathbf{5})$, but we lack the adjoint representation $\mathbf{24}_H$ for breaking $SU(5)$ down to the SM.

In low energy effective field theory which results from a large compactification radius for orbifolding, we effectively have an extra dimension. Twisted strings “localized” at a special fixed point can represent localized states. In field theoretic orbifolds, such a localization has been called “brane”, in Sect. 4.3.

In Fig. 6.6, a_1 in the first torus distinguishes the three fixed points in the first torus and a_3 in the second torus distinguishes the three fixed points in the second torus. The product of the first and the second torus gives 9 distinct fixed points. Thus, for the chiral fermions in the $\mathbf{Z}_3$ orbifold, we compute the massless spectrum from these 9 different twisted sectors in addition to the untwisted sector U ,

$$V + m_1 a_1 + m_2 a_2, \quad m_1, m_2 = 0, 1, -1. \quad (6.101)$$

But the fixed points of the third torus is not distinguished, which is explicitly represented as the same kind of bullets in Fig. 6.6, and hence each of the nine twisted sectors has the multiplicity 3. Thus, the Wilson line dictates which field lives at which fixed point. In field theoretic models, there is no corresponding restriction of this behavior of the string twisted sector, and hence there is no principle in field theory, restricting fields at fixed points except for the less strict requirement of the anomaly cancelation.

6.4.3 Other Background Fields

In the bosonic language, we obtained 10 dimensional heterotic strings by compactifying 26 dimensional one down to 10D. The resulting gauge group is of rank $16 = 26 - 10$. In principle, we can compactify more dimensions (or even

all the dimensions except two due to ghost contribution) to obtain higher rank gauge groups.

In $d < 10$ noncompact dimensions, we have other contributions from right movers also carrying momenta, so we denote the entire momentum vector as $p = (p_L, p_R)$, whose number of components is $(26 - d, 10 - d)$. Locality of theory [15] requires that such a momentum has a product of Lorentzian signature $(26 - d, 10 - d)$

$$p \circ p' \equiv p_R \cdot p_R - p_L \cdot p_L . \quad (6.102)$$

Still the modular invariance is an important constraint, which is equivalent for this product to be even and self dual. We have seen in the *Euclidian* case that the invariant lattices can exist in $8n$ dimensions, but there are more possibilities in the Lorentzian lattice.

Narain Compactification

The general string models are classified in [15, 16]. We will try to find a full set of momenta satisfying the condition (6.102) which embeds the modular invariance. Consider a representative lattice Γ_0 . By the Lorentz transformation $\Lambda \in O(26 - d, 10 - d, \mathbf{R})$, we can span all the possible lattice $\Lambda\Gamma_0$ (meaning acting Λ on all the points in the lattice Γ_0). This is an over-counting however, because from the Hamiltonian (the mass shell condition (5.203)) rotations in the left and right movers $\Lambda' \in O(26 - d, \mathbf{R}) \times O(10 - d, \mathbf{R})$ are independent symmetries so that these symmetries lead to equivalent ones. Modding out by such equivalent classes, we have a group space of symmetries of moduli space,

$$\frac{O(26 - d, 10 - d, \mathbf{R})}{O(26 - d, \mathbf{R}) \times O(10 - d, \mathbf{R})} . \quad (6.103)$$

In fact, we have another equivalent class, namely the extended T -duality, that leaves Γ_0 invariant although elements are shuffled. This denoted conventionally as $\Lambda'' \in O(26 - d, 10 - d, \mathbf{Z})$, so that $\Lambda\Gamma_0 \sim \Lambda'\Lambda''\Gamma_0$. The resulting moduli space is smaller, obtained from (6.103) by a further modding

$$O(26 - d, 10 - d, \mathbf{Z}) . \quad (6.104)$$

This symmetry contains the action (5.182) $R \leftrightarrow 1/2R$, and also large Lorentz transformation yielding the original periodicity and axionic symmetry $b_{mn} \rightarrow b_{mn} + 1$.

We can understand what it means, by taking the $d = 4$ example. From the left movers, we expect the group of rank $26 - d = 22$. From the right movers, one obtains $U(1)^6$ which is not enhanced any more due to $p_R^2 = 0$ for the right movers. Counting the number of parameters, we find that it is

$$22 \cdot 6 = \frac{(22 + 6)(22 + 6 - 1)}{2} - \frac{22 \cdot 21}{2} - \frac{6 \cdot 5}{2} .$$

It indicates that the fields in this space can be identified as the Kaluza–Klein modes of graviton, antisymmetric tensor and gauge fields: $\frac{1}{2}6(6+1)$ from G_{ij} , $\frac{1}{2}6(6-1)$ from B_{ij} , and $6 \cdot 16$ from A_i^I corresponding to the Wilson lines. Vacuum expectation values (VEV) of moduli such as $\langle G_{ij} \rangle$ describe the geometry. They are called *moduli field* and determine geometry of the compact space. They span $22 \cdot 6$ dimensional *moduli space*. A certain phase of low energy theory corresponds to some special point in the moduli space. In the construction of heterotic string, a special point of the critical radius was chosen to enhance the gauge symmetry.

The general dependence of action on G_{ij} and B_{ij} can be derived, just plugging them back into the action,

$$S = \frac{1}{2\pi} \int d\sigma d\tau \left(G_{ij} \partial_\alpha X^i \partial^\alpha X^j + \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j \right). \quad (6.105)$$

We have seen that the momentum and the winding affect only the quantization of zero modes of $X^i = 2\sigma n^i + q^i(\tau)$, where $n^i, q^i(\tau)$ are winding and center of mass, respectively. Plugging back into (6.105), we have the following action

$$S = \int d\tau \left(\frac{1}{2} G_{ij} \dot{q}^i \dot{q}^j + 2B_{ij} \dot{q}^i n^j - 2G_{ij} n^i n^j \right)$$

Thus, the canonical momentum is

$$P_i = G_{ij} \dot{q}^j + 2B_{ij} n^j. \quad (6.106)$$

In the compact dimension this momentum should be single-valued $P_i = m_i$. Therefore,

$$\dot{q}^j = G^{jk} m_k - 2B_k^j n^k$$

Putting back we have the momentum lattice,

$$p_R^i = \frac{1}{2} m^i - (B^{ij} + G^{ij}) n_j \quad (6.107)$$

$$p_L^i = \frac{1}{2} m^i - (B^{ij} - G^{ij}) n_j \quad (6.108)$$

Plugging in (6.102), we can check that the inner product of these momenta becomes

$$p \circ p' = (n^i m'_i + n'^i m_i)$$

is integral, and becomes even if we set $p = p'$. Also, we can check that the form of momentum is unimodular, so that we have a modular invariant lattice.

We can do the similar thing for the Wilson line A_i^I to have the above result, by adding the term [15, 16].

$$\epsilon^{\alpha\beta} A_i^I \partial_\alpha X^i \partial_\beta X^I \quad (6.109)$$

In this case, the absence of right movers in X^I make things nontrivial. Then, we can replace $\epsilon^{\alpha\beta}$ with $\eta^{\alpha\beta}$ and impose a constraint $(\partial_\tau - \partial_\sigma)X^I = 0$. Then, the generalized momenta become as follows

$$p_R^i = \frac{1}{2}m^i - (B^{ij} + G^{ij})n_j - \frac{1}{2}m^I A_I^i - \frac{1}{4}A_I^i A_I^j n_j, \quad (6.110)$$

$$p_L^i = \frac{1}{2}m^i - (B^{ij} - G^{ij})n_j - \frac{1}{2}m^I A_I^i - \frac{1}{4}A_I^i A_I^j n_j, \quad (6.111)$$

$$P^I = p_L^I = m^I + A_j^I n^j, \quad (6.112)$$

with quantized momentum m^i , and winding n^i .

$E_8 \times E_8$ -SO(32) Duality

As a consequence we see duality between SO(32) and $E_8 \times E_8$ heterotic string theory. Compactifying the SO(32) theory on a torus and turn on order two Wilson line along it

$$a_9 = \frac{1}{2}(1^8 \ 0^8).$$

This breaks gauge group to $SO(16) \times SO(16)$. This group is also obtainable from $E_8 \times E_8$ theory on the same torus with Wilson line

$$a_9 = \frac{1}{2}(2 \ 0^7)(2 \ 0^7).$$

Now we focus on the neutral states under $SO(16) \times SO(16)$, or $P^I = 0$. The solution is present when $n^9 \equiv 2n$ is even number. Plugging back into (6.110, 6.111) and also Wilson lines above we obtain the momentum is

$$\begin{aligned} p_{L,R}^9 &= \frac{m + 2n}{R} \mp \frac{2nR'}{\alpha'}, \\ p_{L,R}'^9 &= \frac{m' + 2n'}{R} \mp \frac{2n'R'}{\alpha'} \end{aligned} \quad (6.113)$$

where $m^9 \equiv m$ and restored dimensionful parameter R and α' . The primed quantity is of $E_8 \times E_8$. Exchanging $R \leftrightarrow \alpha'/2R'$ and $(m + 2n, n) \leftrightarrow (n', m' + 2n')$, so that $(p_L^9, p_R^9) \leftrightarrow (-p_L'^9, p_R'^9)$. We have the same spectrum, which is called *T-duality*. Note that now the radius does not have the critical value of $R = \sqrt{1/2}$. Their partition functions should be the same, and indeed from the theory of modular form, it is known that there exists only one partition function for the 16 dimensional lattice.

Moduli Fields

The moduli fields discussed above play an important role in low energy physics and cosmology. Before discussing their dynamics in Sect. 8.2, we briefly inspect some properties that can be deduced from the above symmetries.

We will consider the example of toroidal compactification where the compact space is decomposed into the direct products of two-tori T^2 . Consider one of them, say along the y^1 - y^2 plane. Its geometry is determined by metric tensor G_{ij} , but from the four dimensional supersymmetry, it forms a chiral multiplet; thus should be a part of complex scalar. Rewriting the action in the presence of background fields (6.105), we obtain

$$S = \frac{1}{\pi} \int d^2z [G_{12} (\partial Y^1 \bar{\partial} Y^2 + \bar{\partial} Y^1 \partial Y^2) - iB_{12} (\partial Y^1 \bar{\partial} Y^2 - \bar{\partial} Y^1 \partial Y^2)] . \quad (6.114)$$

Collecting terms in the complex coordinate $Z = Y^1 + iY^2$,

$$S = \frac{1}{\pi} \int d^2z (iT \partial Z \bar{\partial} Z + \text{h.c.}) . \quad (6.115)$$

Here

$$\begin{aligned} iT &= R^2 (B_{12} + i\sqrt{\det G}) \\ &\equiv b_{12} + i \frac{V}{4\pi^2} , \end{aligned} \quad (6.116)$$

is called the volume modulus or the Kähler modulus, where R and V are the radius and the volume of torus and the determinant is over the two dimensional metric G . Remarkably, in string dynamics, the antisymmetric tensor field B is also responsible for geometry. Also note that it has a discrete (due to (6.106)) shift symmetry, the axionic symmetry

$$b_{mn} \rightarrow b_{mn} + 1 , \quad (6.117)$$

known to be present to the all orders of perturbation expansion. We have another field

$$\begin{aligned} iU &= \frac{R^2}{G_{11}} (G_{12} + i\sqrt{\det G}) \\ &= R^2 \frac{|e_2|}{|e_1|} e^{i\phi_{12}} \end{aligned} \quad (6.118)$$

called complex structure modulus, whose name will become evident shortly. In the orbifold compactification, this will survive as a field only when the orbifold action is compatible with the rotation by π . Otherwise, it is “frozen” because the side length of the torus and angle between them is fixed by action, for example in the $\mathbf{Z}_3$ case $\theta e_3 = e_4$. We naturally expect these moduli should acquire VEVs dynamically. Anyway, we can rewrite the metric as

$$ds^2 = \frac{\text{Re}T}{R^2 \text{Re}U} |dY^1 + iU dY^2|^2 , \quad (6.119)$$

therefore the U modulus determines complex structure. This is for the target space, whereas the similar one in (5.71) is for the worldsheet.

As discussed above, these moduli fields have the symmetry, which is a subgroup of T-duality group $O(26-d, 10-d, \mathbf{Z})$. It turns out in this case that the full (target space) modular symmetries are

$$\mathrm{PSL}(2, \mathbf{Z}) \times \mathrm{PSL}(2, \mathbf{Z}) \ltimes \mathbf{Z}_2^2. \quad (6.120)$$

The two $\mathrm{PSL}(2, \mathbf{Z})$ s act in the same way to modular transformation on the torus parameter τ . (But in this convention, as in the literature, the form of transformation is slightly tilted as $T \rightarrow (aT - ib)/(icT + d)$ and the same for U .) For example, the symmetries $T \rightarrow T + i$ and $T \rightarrow 1/T$ correspond to axionic symmetry (6.117) and simultaneous T -duality along y^1 and y^2 directions, respectively. The last two $\mathbf{Z}_2$'s are known as the special case of “mirror symmetry” and corresponds to exchanging $T \leftrightarrow U$ and orbifold action. In Sect. 8.2 we will calculate the Kähler potential and the superpotential of these. There we will see that the target space metric of the low energy action takes into account of the above symmetry as a footprint. For a general orbifold, the full modular symmetry is known and corresponding tree-level low energy effective actions are calculated [17].

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Partition Function and Spectrum

We have seen in Subsect. 5.3.4 that the only modular invariant lattices in sixteen dimensions are those of $E_8 \times E_8$ and $SO(32)$, while the ten dimensional spacetime part is *independently* modular invariant. But if we relax this condition such that it is modular invariant in the *entire 26 dimensions*, we have a more general theory. One fruitful result was the discovery of $SO(16) \times SO(16)$ heterotic string without supersymmetry [1], but its applicability is much more profound. This leads to a huge class of orbifold models which we discuss in this book. Furthermore, taking into account the right movers also, we can have a more general modular invariance, leading to the asymmetric orbifold.

The purposes of this chapter is to define orbifold more formally and generalize the theory to nonprime and factor orbifolds. The explicit partition function will provide the proof on the orbifold rules presented in the previous chapter. It gives a deeper understanding on the spectrum and the interactions, which is extended to more generalized types of orbifold. Many references are available [2, 3, 4].

7.1 General Orbifolds

In this section we will calculate the partition function, which contains all the necessary information on the spectrum. The generalized GSO projection condition will be derived, which contains the vacuum phase in nonstandard embedding. Still, the Jacobi elliptic function is a useful tool [5] for studying such a property. We also extend the discussion to more complicated cases, nonprime orbifolds and factor orbifolds: $\mathbf{Z}_N$ with a nonprime N and $\mathbf{Z}_M \times \mathbf{Z}_N$.

7.1.1 General Orbifolds

Now we are ready to define orbifold Hilbert spaces and partition functions. The geometry of orbifold requires a string to be closed up to the space group action, $g = (\theta, v) \in S$,

$$X(\sigma + \pi) = gX(\sigma) = \theta X(\sigma) + v . \quad (7.1)$$

It seems that, for every g there is an independent Hilbert space, but it is not true because $\mathbf{S}$ is a nonabelian group. Consider an additional action $h \in \mathbf{S}$ on (7.1),

$$hX(\sigma + \pi) = hgX(\sigma) = (hgh^{-1})hX(\sigma) . \quad (7.2)$$

Since X and hX are identified, the twist state specified by hgh^{-1} should be equivalent to the one specified by g . The reader may recall that this is the notion of conjugacy class $[g]$, discussed in Subsect. 3.1.2. Therefore, the twisted (including untwisted) Hilbert space is defined by the conjugacy class

$$\mathcal{H}_{[g]} = \{X \mid X(\sigma + \pi) = gX(\sigma), g \sim hgh^{-1}, g, h \in \mathbf{S}\} . \quad (7.3)$$

The point group is an abelian group, therefore we have an independent Hilbert space for each $g \in \bar{\mathbf{P}}$. So, it is sufficient just to deal with the shift vector embedding. We projected out the states with

$$P_g = \frac{1}{N} \sum_{g \in \bar{\mathbf{P}}} g . \quad (7.4)$$

However, if we have Wilson lines, there is no translation symmetry thus we should introduce a nonabelian space group $\mathbf{S}$ and project out the states with only commuting elements in $\mathbf{S}$, otherwise the projection does not make sense. A set of such commuting elements are called centralizer $C(g)$. Therefore,

$$P_g = \frac{1}{|C(g)|} \sum_{g \in \bar{\mathbf{P}}} g , \quad (7.5)$$

where $|C|$ is the order of the centralizer.

In what follows, however we will be content with the theory composed with point group $\bar{\mathbf{P}}$, i.e. without Wilson lines. A modular invariant theory is formed in the following way:

1. Include the h twisted sector via the twisted mode expansion (6.5) to have $\tilde{L}_0(h)$ and $L_0(h)$.
2. Keep only the g invariant states by the projection (7.4).

This is what we introduced as the twisted boundary conditions for both world sheet directions (6.45), so that the partition function has the form

$$\mathcal{Z}(\tau, \bar{\tau}) = \frac{1}{N} \sum_{h \in \mathbf{P}} \sum_{g \in \mathbf{P}} \mathcal{Z}(h, g) , \quad (7.6)$$

$$= \sum_{h \in \bar{\mathbf{P}}} \left[\frac{1}{N} \sum_{g \in \bar{\mathbf{P}}} \text{Tr}(g q^{\tilde{L}_0(h)} \bar{q}^{L_0(h)}) \right] \quad (7.7)$$

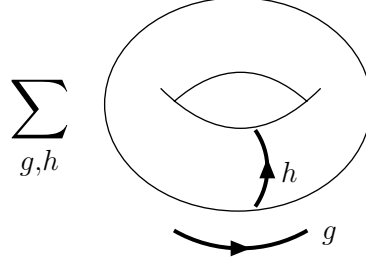


Fig. 7.1. Summing over the spin structure is extended. The phase ambiguity is resolved by the generalized GSO projection g

where

$$q = e^{2\pi i \tau} . \quad (7.8)$$

The $h = 1$ case is our definition of the untwisted sector. The summation is illustrated in Fig. 7.1.

To be more specific, we will consider only orbifold generated by $\theta = e^{2\pi i \phi_a J_a}$, thus we restrict $g = \theta^m$ and $h = \theta^n$ and the above partition function takes the form

$$\mathcal{Z}(\tau, \bar{\tau}) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mathcal{Z}_{(\theta^m, \theta^n)}(\tau, \bar{\tau}) \quad (7.9)$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{m=0}^{N-1} \mathcal{Z}_{(\theta^m, \theta^n)}(\tau, \bar{\tau}) . \quad (7.10)$$

Untwisted Sector

Let us calculate the partition function of the untwisted sector. For the exponent we use the untwisted Lagrangian but we project onto a θ^n invariant state

$$\mathcal{Z}_{(1, \theta^n)}(\tau) = \text{Tr}(\theta^n q^{\tilde{L}_0(1)} \bar{q}^{L_0(1)}) . \quad (7.11)$$

Let us calculate first for left moving oscillators from the mode expansion. The straightforward calculation leads

$$q^{-\frac{1}{12}} (1 + qe^{2\pi i n \phi_a} + qe^{-2\pi i n \phi_a} + \dots) \quad (7.12)$$

where the overall exponent $-\frac{1}{12}$ is the zero point energy for one complex bosonic degree z^a and the next ones come from oscillators α_{-1}^a and $\tilde{\alpha}_{-1}^{\bar{a}}$, respectively, with the phase dependence due to eigenvalues of θ . We can convert this expression into a manifestly modular transforming function,

$$\begin{aligned}
& -2 \sin(n\pi\phi_a) \frac{\eta(\tau)}{\vartheta \left[\frac{1}{2} + n\phi_a \right] (\tau)} \\
& = q^{-\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^k e^{2\pi i n \phi_a})^{-1} (1 - q^k e^{-2\pi i n \phi_a})^{-1}
\end{aligned} \tag{7.13}$$

where

$$\begin{aligned}
\eta(\tau)^{-1} & \equiv \prod_{n=1}^{\infty} \sum_{\tilde{N}_n=0}^{\infty} q^{n/2+n\tilde{N}_n} = \prod_{n=1}^{\infty} q^{n/2} \sum_{\tilde{N}_n=0}^{\infty} q^{n\tilde{N}_n} \\
& = q^{-1/24} \prod_{n=1}^{\infty} (1 - q^n)^{-1} .
\end{aligned} \tag{7.14}$$

We confirm that the leading order expansion yields the desired result (7.12). The prefactor in the first line is as an artifact arising from the definition of theta function, to have coefficient 1 as in (7.12), because there is no degeneracy in the untwisted sector. We can see this explicitly from the definition of theta function

$$\begin{aligned}
\frac{1}{\eta} \vartheta \left[\frac{1}{2} + n\phi_a \right] & = e^{i\pi/2} e^{i\pi n \phi_a} q^{\frac{1}{12}} \prod_{k=1}^{\infty} (1 - q^k e^{2\pi i n \phi_a}) (1 - q^{k-1} e^{-2\pi i n \phi_a}) \\
& = -2 \sin(n\pi\phi_a) q^{\frac{1}{12}} \prod_{k=1}^{\infty} (1 - q^k e^{2\pi i n \phi_a}) (1 - q^k e^{-2\pi i n \phi_a}) .
\end{aligned} \tag{7.15}$$

where we factored out $(1 - e^{-2\pi i n \phi_a})$ from the first factor in the second product of the first line. Note that the twisted string has no center of momentum motion; thus contains no continuous part.

The partition function for the right mover is just the complex conjugation. Gathering all, we obtain

$$\mathcal{Z}_{(1,\theta^n)} = \chi(\theta^n) \left| \prod_{a=1}^4 \frac{\eta}{\vartheta \left[\frac{1}{2} + n\phi_a \right]} \right|^2 (4\pi^2 \tau_2)^{-1} , \tag{7.16}$$

where the prefactor

$$\chi(\theta^n) = \prod_{a=1}^3 2 \sin^2(n\pi\phi_a) \tag{7.17}$$

is from the geometry and the last factor is from the noncompact dimensions (5.85). We comment on the $n \rightarrow 0$ limit. In this limit both the numerator and denominator of (7.13) goes to zero, and we see [5],

$$\lim_{\epsilon \rightarrow 0} \frac{-2 \sin(\pi\epsilon)}{\vartheta \left[\frac{1}{2} + n\phi_a \right] (\tau)} = \eta^{-3}(\tau) , \tag{7.18}$$

so that (7.13) reduces to $1/\eta^2$, reproducing the case with no orbifold introduced.

Twisted Sectors

Let us apply the modular transformation. Under $\mathcal{T}$, the transformation only induces a phase which is canceled by the phase from the complex conjugate. Under $\mathcal{S}$, it gives

$$\begin{aligned} & \chi(\theta^{-n})(q\bar{q})^{\bar{c}+c} \left| \prod_{a=1}^3 \prod_k (1 - q^{k-1-n\phi_a})^{-1} (1 - q^{k+n\phi_a})^{-1} \right| \\ &= \chi(\theta^{-n}) \left| \prod_{a=1}^3 \frac{\eta}{\vartheta \left[\begin{smallmatrix} \frac{1}{2} - n\phi_a \\ \frac{1}{2} \end{smallmatrix} \right]} \right|^2 \frac{4\pi^2 \tau_2}{|\tau|^2} \\ &\equiv \mathcal{Z}_{(\theta^{-n}, 1)}, \end{aligned} \quad (7.19)$$

where the number of fixed points of an action is the same as that of the inverse action $\chi(\theta^{-n}) = \chi(\theta^n)$. The definition in terms of partition function on the last line fits well with the previous one, (7.16). Thus, we obtained the partition function for $(\theta^{-n}, 1)$, which belongs to the θ^{-n} twisted sector. The zero point energy is modified to the correct value, according to the regularization (6.15). The factor $(-i\tau)^{1/2}$ is common to eta and theta functions so that they cancel. Expanding the above equation further, we have

$$(q\bar{q})^{\bar{c}+c} (1 + \dots) \prod_{a=2}^4 4 \sin^2(n\pi\phi_a). \quad (7.20)$$

Therefore, we have the degeneracy factor coinciding with the number of fixed points (3.24). In the untwisted sector, this factor set the overall degeneracy to 1.

We guess the natural form for the general (untwisted or twisted sector) partition function,

$$\mathcal{Z}_{(\theta^m, \theta^n)}^X(\tau) = \chi(\theta^m, \theta^n) \left| \prod_{a=1}^4 \frac{\eta(\tau)}{\vartheta \left[\begin{smallmatrix} 1/2 + m\phi_a \\ 1/2 + n\phi_a \end{smallmatrix} \right](\tau)} \right|^2 (4\pi^2 \tau_2)^{-1}. \quad (7.21)$$

Note that we included noncompact spacetime degrees X (there are two more in the light cone gauge). The prefactor $\chi(\theta^m, \theta^n)$ should be the number of simultaneous fixed points under θ^m and θ^n twists.

Gauge and Fermionic Degrees

Now we include gauge and fermionic degrees of freedom. A lesson from the first definition of theta function in (5.152) is that the lattice shift by $\frac{1}{2}$ in (5.195) is realized as a shift in the argument of the theta function. For the group space twist with $V = (V_1, V_2, V_3, \dots)$, $P \rightarrow P + V$, therefore the theta function looks like

$$\mathcal{Z}_{(\theta^m, \theta^n)}^G(\tau) = \frac{1}{2} \sum_{\alpha, \beta} \eta^{\alpha\beta} \prod_{I=1}^r \frac{1}{\eta(\tau)} \vartheta \left[\begin{matrix} \alpha + mV_I \\ \beta + nV_I \end{matrix} \right] (\tau) . \quad (7.22)$$

Here, α and β can assume 0 and $\frac{1}{2}$, as seen in (5.156)–(5.159). We introduced a phase $\eta^{\alpha\beta} = e^{2\pi i \alpha \beta}$. For $\alpha = \beta = \frac{1}{2}$, this phase is i which appears in the last term of (5.197). The spin structure α and β still assume $\frac{1}{2}$ and 0 for the R and the NS states, respectively.

Also, the partition function for its superpartner can be similarly obtained as done in (7.16),

$$\mathcal{Z}_{(\theta^m, \theta^n)}^\psi(\bar{\tau}) = \sum_{\alpha, \beta} \eta^{\alpha\beta} \prod_{a=1}^4 \frac{1}{\eta(\bar{\tau})} \vartheta \left[\begin{matrix} m\phi_a + \alpha \\ n\phi_a + \beta \end{matrix} \right] (\bar{\tau}) , \quad (7.23)$$

with, again, spin structures.

7.1.2 The Full Partition Function

Combining all the above expressions, we finally arrive at the full partition function

$$\begin{aligned} \mathcal{Z}_{(\theta^m, \theta^n)}(\tau, \bar{\tau}) &= \eta_{mn} \mathcal{Z}_{(\theta^m, \theta^n)}(\tau) \mathcal{Z}_{(\theta^m, \theta^n)}(\bar{\tau}) \\ &= \eta_{mn} \mathcal{Z}_{(\theta^m, \theta^n)}^G(\tau) \mathcal{Z}_{(\theta^m, \theta^n)}^X(\tau) \mathcal{Z}_{(\theta^m, \theta^n)}^X(\bar{\tau}) \mathcal{Z}_{(\theta^m, \theta^n)}^\psi(\bar{\tau}) . \end{aligned} \quad (7.24)$$

where we represented as the product of the holomorphic and antiholomorphic parts.

It is nontrivial to check that the full partition function has the modular invariance, but it is not so hard to investigate that property [6, 7, 8]:

- Under $\mathcal{S}$, for the consistent definition of spin structure the following should be satisfied

$$\begin{aligned} NV^I &= N\phi_a = 0, \quad \text{mod } 1 , \\ N \sum_I V^I &= N \sum_a \phi_a = 0, \quad \text{mod } 2 \end{aligned} \quad (7.25)$$

which reproduce the condition (6.3). If we fermionize the variables, the latter condition for no orbifold case $N = 1$ reproduces the GSO projection for fermions.

- Under $\mathcal{T}$, the untwisted $\mathcal{Z}^G$ of (7.22) and $\mathcal{Z}^X$ of (7.21) should be separately modular invariant. However, the twisted sector $\mathcal{Z}^G$ for example acquires a phase $e^{-\pi i(mnV^2)}$. Thus as in the above case, for the consistent definition of twisted sector, we need the total phase

$$\eta_{mn} = \exp[-mn\pi i(V^2 - \phi^2)] . \quad (7.26)$$

This “vacuum phase” is not seen in the operator formalism, discussed in Chap. 6. For the case of nonstandard embedding, this phase should be taken into account. As a consequence, N successive $\mathcal{T}$ transformations (6.48) change the boundary condition to

$$(h, g) \rightarrow (h, h^N g) = (h, g) . \quad (7.27)$$

This means that we come back to the original twisted sector after N successive $\mathcal{T}$ transformations. For every (h, g) , there should be no extra phase and hence it follows that

$$\eta_{mn} = \exp[-\pi i(V^2 - \phi^2)N] = 1 \quad (7.28)$$

reproducing the modular invariance condition (6.43)

$$\sum (NP)^2 = \sum (N\phi)^2 \mod 2N . \quad (7.29)$$

This condition *relates two quantities from the different sectors, one is V from gauge sector and ϕ from spacetime sector*. Also this condition miraculously relates the anomaly cancellation from different contributions: the information on the multiplicity and the number of fixed points are contained in $\mathcal{Z}^X$ and the information on the chiral representations is contained in $\mathcal{Z}^G$ in (7.24). But this is enough for the full partition function only for prime orbifolds. We will include the extra piece shortly in Sect. 7.2.

Expecting Asymmetric Orbifold

The above statement is equivalently rephrased as follows. Because bosonic spacetime degrees from the left and right states are already modular invariant, the above condition connects *holomorphic and anti-holomorphic* partition functions, which of each is modular noninvariant. We can even construct an orbifold theory by relaxing the common left and right moving bosonic states so that they are asymmetrical.

7.1.3 Generalized GSO Projections

Our objective is to extract invariant states with the orbifold conditions. Looking at the partition function (7.6), given the mass level which is denoted by the power of q , there should be no extra phases. Otherwise the states are washed out by projector $1 + g + g^2 + \dots g^{N-1} = 0$. This means nothing but that the point group invariant states survive.

Thus, we can extract the spectrum in the θ^n twisted sector, by watching out the form (7.9). We rewrite a *projection operator* as

$$P_m = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\chi}(\theta^m, \theta^n) \Delta_{(\theta^m, \theta^n)} . \quad (7.30)$$

This is composed of two parts:

- In the absence of Wilson line, the phase operator

$$\Delta_{(\theta^m, \theta^n)} = (\Delta_{(\theta^m)})^n \quad (7.31)$$

is

$$\begin{aligned} \Delta_{(\theta^m)} = & \exp 2\pi i [\tilde{N} - N + (P + mV) \cdot V - (s + m\phi) \cdot \phi] \\ & \cdot \exp 2\pi i \left[-\frac{1}{2}(mV^2 - m\phi^2) \right] \end{aligned} \quad (7.32)$$

where $\tilde{N}$ and N are the oscillator numbers of the left and right movers, respectively. We included vacuum phase (7.26) as well. We will deal with the case in the presence of Wilson lines, in the next section.

- In (7.30), $\tilde{\chi}(\theta^m, \theta^n)$ is the degeneracy factor [3, 4]. It is basically the same as $\chi(\theta^m, \theta^n)$, the number of simultaneous fixed points in the θ^m and θ^n twisted sectors. It is true for the prime orbifolds $\mathbf{Z}_3$ and $\mathbf{Z}_7$. It differs from the number of fixed points only when $m\phi_a$ is an integer. In this case, there emerges the unwanted factor $\prod 4 \sin^2(m\phi_j \pi)$ in the following expression, and therefore we should divide by it. It is evident that in the untwisted sector $\tilde{\chi}(1, \theta^n) = 1$. Therefore, the multiplicity condition is summarized as

$$\tilde{\chi}(\theta^m, \theta^n) = \begin{cases} \chi(\theta^m, \theta^n) & [\chi(\theta^m) \neq 0] \\ \chi(\theta^m, \theta^n) / \prod_a 4 \sin^2(n\phi_a \pi) & [n \neq 0, \chi(\theta^m) = 0] \\ \prod_a 4 \sin^2(m\phi_a \pi) & [n = 0, \chi(\theta^m) = 0] , \end{cases} \quad (7.33)$$

These numbers are tabulated in Appendix D.

What we extract from the above equation is the following. After obtaining state vectors satisfying the mass shell condition, we may obtain the multiplicity factor by resorting to this projection operator. In the absence of a shift, this is the original GSO projection in the right mover since it determines s with a definite chirality and we saw that the left mover behavior is also interpreted as the GSO projection; hence the projection (7.30) can be called the *generalized GSO projection condition*.

This projector must be used for the case of nonprime orbifolds. The massless states cannot be obtained by the level matching condition alone. For example, in the $\mathbf{Z}_4$ orbifold twisted sectors shuffle the partition functions of $(\theta^2, 1)$, (θ^2, θ^2) and $(1, \theta^2)$ by modular transformations. That affects the effective number of fixed points as we will see later. Also, we will consider a $\mathbf{Z}_{12} - I$ orbifold in detail in Chap. 10 and use this consideration of multiplicity.

7.1.4 $\mathbf{Z}_3$ Example

To see explicitly how the method of the partition function works, consider again the $\mathbf{Z}_3$ orbifold with the standard embedding, discussed in Sect. 6.3. We will restrict to the nontrivial left movers only.

Untwisted Sector

First consider the untwisted sector $(0, m)$ which is an abbreviation of (θ^0, θ^m) . We form the partition function of the $(0, 1)$ sector as $\mathcal{Z}_{(0,1)}(\tau) = \mathcal{Z}_{(0,1)}^{\mathbf{E}_8 \times \mathbf{E}'_8} \mathcal{Z}_{(0,1)}^X = \mathcal{Z}_{(0,1)}^{\mathbf{E}_8} \mathcal{Z}_{(0,1)}^{\mathbf{E}'_8} \mathcal{Z}_{(0,1)}^X$, using the original $\mathbf{E}_8$ partition function $\mathcal{Z}_{(0,1)}^{\mathbf{E}_8}$ given in (5.197).

- In terms of the theta functions

$$\begin{aligned} \mathcal{Z}_{(0,1)}(\tau) &= \frac{\vartheta \begin{bmatrix} 0 \\ 2/3 \end{bmatrix} \vartheta \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}^2 \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^5 + \dots}{2 \eta^8} \mathcal{Z}^{\mathbf{E}_8}(\tau) \cdot \\ &\quad \cdot \left(-2 \sin \frac{\pi}{3}\right)^3 \frac{\eta^4}{\vartheta \begin{bmatrix} 1/2 \\ 7/6 \end{bmatrix} \vartheta \begin{bmatrix} 1/2 \\ 5/6 \end{bmatrix}^2 \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}} \\ &= \left(-2 \sin \frac{\pi}{3}\right)^3 \frac{\vartheta \begin{bmatrix} 0 \\ 2/3 \end{bmatrix} \vartheta \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}^2 \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^5 + \dots}{2 \vartheta \begin{bmatrix} 1/2 \\ 7/6 \end{bmatrix} \vartheta \begin{bmatrix} 1/2 \\ 5/6 \end{bmatrix}^2 \eta^7} \mathcal{Z}^{\mathbf{E}_8}(\tau) \end{aligned} \quad (7.34)$$

where the ellipsis denotes the sum over the spin structures and the first term shown in the numerator is for $\alpha = \beta = 0$. It is further simplified by the relation $\vartheta_{[2/3]}^0 = \vartheta_{[1/3]}^0$ and $\vartheta_{[7/6]}^{1/2} = \vartheta_{[5/6]}^{1/2}$. The number of eta functions (without those in $\mathcal{Z}^{\mathbf{E}_8}$) is $8 - 3 + 2 = 7$ with the limit $\vartheta_{[1/2]}^{1/2}/\eta \rightarrow \eta^2$.

This form itself has a manifest modular invariance, by (5.153, 5.154). Schematically the numerator has a form

$$\vartheta \begin{bmatrix} 0 \\ \beta \end{bmatrix}^8 + \vartheta \begin{bmatrix} 1/2 \\ \beta \end{bmatrix}^8 + \vartheta \begin{bmatrix} 0 \\ \beta + 1/2 \end{bmatrix}^8 + \vartheta \begin{bmatrix} 1/2 \\ \beta + 1/2 \end{bmatrix}^8 \quad (7.35)$$

where $\vartheta_{[\beta]}^0$ implies $\vartheta_{[2/3]}^0 \vartheta_{[1/3]}^0 \vartheta_{[0]}^0$, etc. Under $\mathcal{T}$, each $\vartheta_{[\beta]}^{1/2}$ and $\vartheta_{[\beta+1/2]}^{1/2}$ just acquires a phase $e^{i\pi/4}$, viz. (5.153), which is canceled because of the power 8 of the theta functions. Also under $\mathcal{T}$, $\vartheta_{[\beta]}^0$ and $\vartheta_{[\beta+1/2]}^0$ are interchanged without extra phases. By a similar reasoning we can check that the denominator is modular invariant. It is because the nontrivial

phase $(e^{i\pi/4})^3$ of the theta functions is canceled by the phase of eta functions $(e^{i\pi/12})^{15}$, viz. (5.86).¹

Surveying the modular transformation property under $\mathcal{S}$, (5.154), we learn that ϑ/η combination does not yield nontrivial $(-i\tau)^{1/2}$. Also the two arguments in the theta function is interchanged, which means that $(0,1)$ twisted sector partition function becomes that of $(1,0)$.

- The above form can be expressed as a product representation,

$$\mathcal{Z}_{(0,1)}(\tau) = \frac{\prod_n (1 + q^{n-\frac{1}{2}})^{10} (1 + \alpha q^{n-\frac{1}{2}})^3 (1 + \alpha^2 q^{n-\frac{1}{2}})^3 + \cdots}{2q^{2/3} \prod_n (1 - q^n)^2 (1 - \alpha q^n)^3 (1 - \alpha^2 q^n)^3} \mathcal{Z}^{E_8}(\tau) \quad (7.36)$$

where $\alpha = e^{2\pi i/3}$. It shows the contribution from the sum of the oscillators. The factor $(-2 \sin \frac{\pi}{3})^3$ is dropped off by $\chi_{(\theta^m, \theta^n)}^{1/2}$ in (7.21). Note that the denominator contribution is positive since $(1 - q)^{-1} = 1 + q + q^2 + \cdots$. In particular, this can be directly obtained by the fermionic construction of group, as seen in Sect. 6.3.5.

- The summation representation, (7.34), has the most compact form,

$$\mathcal{Z}_{(0,1)}(\tau) = \frac{\sum_P q^{\frac{1}{2}P^2} e^{2\pi i P \cdot V}}{q \prod_n (1 - q^n)^{10} (1 - \alpha q^n)^3 (1 - \alpha^2 q^n)^3} \quad (7.37)$$

where P s belong to the whole $E_8 \times E'_8$ lattice. The q^{-1} contributes to the zero point energy. For $P^2 = 2$, the leading term in the q expansion has coefficient 1. If an extra phase does not come from the right movers, the invariant states are those with $P \cdot V = \text{integer}$ which has multiplicity 1. Considering the numerator, it is reasonable to define the orbifold shift transformation as $e^{2\pi i P \cdot V}$ as given in (6.29).

Note that in $\mathbf{Z}_3$ the form of the $(1, \theta^2)$ sector is the same as that of the $(1, \theta^{-1})$ sector but with the different signs of the shift vectors. Also by $\mathcal{S}$ transformation, $(1, \theta)$ twisted sector is transformed to $(\theta^{-1}, 1)$ twisted sector which is also equivalent to $(\theta^2, 1)$.

Twisted Sector

Now consider the twisted sector. The only nontrivial one is the $(1,1)$ sector, i.e. the sector with $m \neq 0$ in (m, n) ,

$$\begin{aligned} \mathcal{Z}_{(1,1)}(\tau) &= \left(-\sin \frac{\pi}{3} \right) \frac{\vartheta_{[1/2]}^{2/3} \vartheta_{[1/2]}^{2/3} \vartheta_{[0]}^{0/5} + \cdots}{2\eta^8 \vartheta_{[7/6]}^{7/6} \vartheta_{[5/6]}^{5/6} \vartheta_{[1/2]}^{1/2}} \mathcal{Z}^{E_8}(\tau) \\ &= -3\sqrt{3} \frac{\sum_P q^{(P+V)^2/2} e^{2\pi i (P+V) \cdot V}}{q^{2/3} \prod_n (1 - q^n)^{10} (1 - q^{n-1/3} \alpha^2)^3 (1 - q^{n-2/3} \alpha)^3} \end{aligned} \quad (7.38)$$

¹ In the denominator there are seven η s times extra eight η s from $\mathcal{Z}^{E_8}$.

where we used (7.22) and (7.21). Here, the multiplicity $(-3\sqrt{3})^2$ is not canceled so that with the right mover we have 27 “fixed points”. In the asymmetric orbifold, this multiplicity is not correlated with the right movers. From the above equation we briefly show how to read off the twisted sector mass shell condition $\frac{1}{2}(P+V)^2 + \tilde{N} - \tilde{c}$ given in (6.72), as well as the projection condition related with $(P+V) \cdot V$. The (inverse) denominator has the expansion

$$q^{-2/3}(1 + 3q^{1/3}\alpha + 9q^{2/3}\alpha^2 + \dots).$$

Besides the zero point energy factor $q^{-2/3}$ giving $\tilde{c} = -\frac{2}{3}$, these terms correspond to 3 oscillators of $\tilde{N} = \frac{1}{3}$ and 9 oscillators of $\tilde{N} = \frac{2}{3}$, as shown in (6.76). In the product expression of the ϑ function, the power of q corresponds to the oscillator number $\tilde{N}$ and their coefficient is the multiplicity times the phase. The leading term 1 is projected out since we have no left movers with the same mass.

We have the following equivalent summation expression, viz. (5.152),

$$\mathcal{Z}_{(1,1)}(\tau) = -3\sqrt{3} \frac{\sum_P q^{(P+V)^2/2} e^{2\pi i(P+V) \cdot V}}{\sum_s q^{(s+\phi)^2/2} e^{2\pi i(s+\phi) \cdot \phi} \prod (1 - q^n)^2}, \quad (7.39)$$

where the last factor in the denominator is the oscillator contribution coming from two noncompact dimensions in the light-cone gauge.

7.2 Nonprime Orbifold

7.2.1 $\mathbf{Z}_4$ Orbifold

To see how the projection operator does work, consider the $\mathbf{Z}_4$ orbifold with the standard embedding $\phi = \frac{1}{4}(2 \ 1 \ 1)$, $V = \frac{1}{4}(2 \ 1 \ 1 \ 0^5)(0^8)$ [3]. This also provides a good example for nonprime orbifolds. Though the twisting is denoted by three numbers, the corresponding six dimensional crystal cannot be decomposed into a two-torus T^2 , but a three-torus T^3 of $SU(4)$ lattice as shown in Table 3.2.

In the untwisted sector, by the rule (6.51) $P \cdot V = \text{integer}$, we have gauge group $E_6 \times SU(2) \times U(1) \times E'_8$ and the spectrum $2(\mathbf{27}, \mathbf{2}) + 2(\mathbf{1}, \mathbf{2}) + (\mathbf{27}, \mathbf{1}) + (\mathbf{27}, \mathbf{1})$. It seems that it obeys the branching rule; that is, the original adjoint $\mathbf{248}$ is split into the ones of the subgroup $\mathbf{78} + \mathbf{3} + \mathbf{1}$, plus above untwisted matter. But this is not the case in general. Later we will see that the complicated setup such as higher order orbifolding or the presence of Wilson lines, the resulting spectrum in the untwisted sector do not add to 248.

In the $k = 1$ twisted sector, the first two-torus (corresponding to $\frac{2}{4}$) has four fixed points under π rotation and each remaining torus (corresponding to $\frac{1}{4}$) has two fixed points under $\pi/2$ rotation. Therefore, $T^6/\mathbf{Z}_4$ has $4 \times 2 \times 2 = 16$ fixed points. We have zero point energy $\tilde{c} = 2f(\frac{1}{2}) + 4f(\frac{1}{4}) + 18f(0) = -11/16$ and $(\mathbf{27}, \mathbf{1}) + 2(\mathbf{1}, \mathbf{2})$ satisfy the mass shell condition. It is sufficient to present

for the highest weight vectors, since the whole representation is obtained by successive subtractions by simple roots. They are

$$(P + V) = \left(\frac{1}{2} \ -\frac{1}{4} \ -\frac{1}{4} \ 1 \ 0^4\right) (0^8), \left(-\frac{1}{2} \ \frac{1}{4} \ -\frac{3}{4} \ 0^5\right) (0^8),$$

corresponding to $(\mathbf{27}, \mathbf{1})$, $(\mathbf{1}, \mathbf{2})$, respectively. The right mover has the zero point energy $c = 2f(\frac{1}{2}) + 4f(\frac{1}{4}) + 6f(0) = -3/16$ and from the mass shell condition we obtain $s = (- - - -)$ and $s = (0 \ -1 \ 0 \ 0)$ as R and NS states, respectively. From this we interpret the state has -1 helicity or left-handed as for the untwisted sector states. Therefore, we obtain $\Delta_1 = 1$ and from the projection operator we have the multiplicity (see Appendix D)

$$P_1 = \frac{1}{4}(16 + 16\Delta_1 + 16\Delta_1^2 + 16\Delta_1^3),$$

and hence there result sixteen states of $(\mathbf{27}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$. The multiplicity is coincident to the number of fixed points because there is no fixed torus.

Now consider the $k = 2$ twisted sector, whose effective twist is $2\phi = \frac{1}{2}(2 \ 1 \ 1)$. Because the first entry is integral, we expect that there is a fixed torus with vanishing χ . From the mass shell condition, we have $(\mathbf{27}, \mathbf{1})$ and $(\overline{\mathbf{27}}, \mathbf{1})$. Appearance of this vectorlike representation seems natural because now the $k = 2$ twisted sector is the same as the $(N - k) = 2$ twisted sector. But something counterintuitive happens when we calculate the multiplicity. We obtain $s = (+ \ -\frac{3}{2} \ - \ -)$ and $s = (- - - -)$ as massless right movers in the Ramond sector. We will put all the possible combinations of tensor product into the projector

$$P_2 = \frac{1}{4}(16 + 4\Delta_2 + 16\Delta_2^2 + 4\Delta_2^3)$$

and the result is as follows.

State	Left Mover P	Right Mover s	Δ_2	P_2
1	$\mathbf{27}$	$(- - - -)$	1	10
2	$\mathbf{27}$	$(+ \ -\frac{3}{2} \ - \ -)$	$-\frac{1}{2}$	6
3	$\overline{\mathbf{27}}$	$(- - - -)$	$-\frac{1}{2}$	6
4	$\mathbf{27}$	$(+ \ -\frac{3}{2} \ - \ -)$	1	10

We see that the states 1 and 4 are related by $\mathcal{CPT}$ conjugation and make up ten states of $\mathbf{27}$, and 2 and 3 make up six states of $\overline{\mathbf{27}}$, which are all *left-handed*. In this case, not only the multiplicity does not coincide with the number of fixed points, but also the spectrum is not vectorlike.

We can understand the situation geometrically. In the $k = 2$ sector (the θ^2 sector), the effective shift is $2\phi = \frac{1}{2}(2 \ 1 \ 1) \simeq \frac{1}{2}(0 \ 1 \ 1)$. The first two-torus is invariant, “fixed torus.” Among $4 \times 4 = 16$ fixed points on the remaining four-torus, only 4 of them with (complex) coordinates are invariant under the θ^1 as well. Let us name these states by the corresponding fixed points

$$|0, 0\rangle, \left|\frac{1}{2} + \frac{1}{2}i, 0\right\rangle, \left|0, \frac{1}{2} + \frac{1}{2}i\right\rangle, \left|\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right\rangle \quad (7.40)$$

where the coordinate variable of the first two-torus is ignored. We required that the physical state should have a definite transformation property under θ .

The remaining 12 points can be pairwise combined to make eigenstates under θ ; for instance, since $\theta(\frac{1}{2}, 0) = (\frac{1}{2}i, 0)$ we can pair two fields at such points

$$\frac{1}{\sqrt{2}} \left(\left|\frac{1}{2}, 0\right\rangle \pm \left|\frac{1}{2}i, 0\right\rangle \right). \quad (7.41)$$

Only the six invariant pairs with $+$ sign survive. They correspond to the same number of $(\mathbf{27} + \mathbf{\bar{27}})$ s. The other states having $-$ sign is projected out by orbifold projection. The projection operator P_2 in (7.30) is a mnemonic for counting such degeneracy. We see that not all the $\mathbf{27}$ s are localized at the equivalent fixed points. This can be further separated in the presence of Wilson lines.

Eigenstates of Point Group Element in Higher Twisted Sector

We require the physical states being invariant under the point group action $\theta \in \mathbb{P}$. This statement is meaningful only when each left and right mover has a definite transformation property under θ . Namely, we require that each left and right mover is an eigenstate of θ [9].

As seen in the previous example, in general the twisted string states are not eigenstates of θ , in higher twisted sectors of $\mathbf{Z}_N$ orbifold. However we can always make an eigenstate by a linear combination. For a such state at the fixed point f in the θ^k twisted sector, there is always a finite integer l such that

$$\theta^l f = f + v, \quad v \in \Lambda. \quad (7.42)$$

Then,

$$\frac{1}{\sqrt{l}} (|f\rangle + \gamma^{-1}|\theta f\rangle + \gamma^{-2}|\theta^2 f\rangle + \cdots + \gamma^{l-1}|\theta^{l-1} f\rangle) \quad (7.43)$$

is evidently an eigenstate of θ , with the eigenvalue $\gamma \equiv e^{i2\pi p/l}$ ($p = 1, 2, \dots, l$). The total number of field is not changed, as it should be. Combining with the right movers, the complete state $|L\rangle \otimes |R\rangle$ should be invariant under orbifold action θ , as before. Note that this does not necessarily mean that the surviving state is an invariant state, but the complete state should be. As we will see in Sect. 9.2, this will be the general strategy to make an eigenstate. This provides a geometric realization of the projection operator (7.30).

7.2.2 $\mathbf{Z}_M \times \mathbf{Z}_N$ Orbifolds

Combination of Twisting

Referring to the orbifolds of Table 3.2, there is another way to have $\mathcal{N} = 1$ supersymmetry. Namely, we have two $\mathcal{N} = 2$ twistings in different directions.

For example, we can choose the first two rows of Table 3.2, $\phi = \frac{1}{2}(1 \ 1 \ 0)$ and $\psi = \frac{1}{3}(1 \ 0 \ 1)$ with the associated shift vectors V and W . This results in a new type of orbifold of “order” 6, which is the maximum order of possible twists. For a $\mathbf{Z}_M \times \mathbf{Z}_N$ orbifold, possible twists are

$$k\phi + l\psi, \quad k = 0, \dots, M-1, \quad l = 0, \dots, N-1 \quad (7.44)$$

corresponding to a shift in the group space

$$kV + lW. \quad (7.45)$$

Now we have more than one untwisted sector. If we considered only $\mathbf{Z}_M$, kV for different k s are basically the same. But with $\mathbf{Z}_M \times \mathbf{Z}_N$, we must consider all the cases of (7.44). We easily see that the resulting group is the intersection of the unbroken groups of each action, because now two projection conditions (6.51) should be simultaneously satisfied,

$$P \cdot V = \text{integer}, \quad P \cdot W = \text{integer}. \quad (7.46)$$

The modular invariance condition is basically the same as (6.43) coming from (6.40),

$$\sum [L(k\phi + l\psi)]^2 = \sum [L(kV + lW)]^2 \pmod{2L} \quad (7.47)$$

for all k and l , with the least common multiple L with M and N fixed. As always, the untwisted sector spectrum is invariant combinations of the left and the right movers. In this case, every state has the transformation property (α^k, β^l) under two orbifold actions.

The twisted sector spectrum is obtained by using the shift vectors (7.45). We can make $\mathbf{Z}_M \times \mathbf{Z}_N$ orbifolds with $\mathcal{N} = 1$ supersymmetry by selecting two shifts of Table 3.3.

$\mathbf{Z}_2 \times \mathbf{Z}_2$ Example

Let us illustrate with the simplest $\mathbf{Z}_2 \times \mathbf{Z}_2$ example with the standard embedding. The twists and shift vectors are $\phi = \frac{1}{2}(1 \ 1 \ 0)$, $\psi = \frac{1}{2}(1 \ 0 \ 1)$, $V = \frac{1}{2}(1 \ 1 \ 0^6)(0^8)$, $W = \frac{1}{2}(1 \ 0 \ 1 \ 0^5)(0^8)$. The modular invariance condition (7.47) is automatically satisfied.

In the untwisted sector, the transformation properties are

ϕ	ψ	L-States	R-States
1	1	78 + 1 + 1	1 + 1
1	α	27 + $\overline{\mathbf{27}}$ + 1 + 1	1 + 1
α	1	27 + $\overline{\mathbf{27}}$ + 1 + 1	1 + 1
α	α	27 + $\overline{\mathbf{27}}$ + 1 + 1	1 + 1

where $\alpha^2 = 1$. We have $\mathbf{Z}_2 \times \mathbf{Z}_2$ invariant states. Without a Wilson line, the untwisted sector can be also obtained by the branching rule of **248** for the left movers and **8** for the right movers.

The twisted sector is composed of three subsectors, acted by V, W , and $V + W$. From the mass shell condition $(P + kV + lW)^2/2 + \tilde{N} - 3/4 = 0$, we have $\mathbf{27} + \mathbf{27} + 4(\mathbf{1})$ in each twisted sector. For right movers, there appears a singlet, thus the multiplicity is 1. Counting antiparticles, in each twisted sector we have $\mathbf{27} + 2(\mathbf{1})$ with the multiplicity 16 from the number of fixed points. Therefore, we have 48 families which also agree with the Euler characteristic $\chi_E(T^6/\mathbf{Z}_2 \times \mathbf{Z}_2)/2 = 48$.

Number of Extra Dimensions, Fixed Points, and Consistency

The above $\mathbf{Z}_2 \times \mathbf{Z}_2$ has an interesting interpretation. We see three duplicates coming from the sectors labeled by the twisting $\phi, \psi, \phi + \psi$. For example, ϕ subsector, neglecting ψ twist, is equivalent to a $T^4/\mathbf{Z}_2$ orbifold that has fixed torus, coordinatized by third complex entry; the resulting group is $SU(2) \times E_7$ with matter **56** [10]. We have these fields localized on this hypersurface. By symmetry, the ψ subsector has the equivalent subsector and has the same gauge group. The four dimensional theory is the common intersection, $E_6 \times U(1)^2 \subset E_7 \times SU(2)$. The matter in the untwisted sector can be given by the branching rule, $\mathbf{56} \rightarrow \mathbf{27} + \mathbf{27} + \mathbf{1} + \mathbf{1}$.

Now we can suggest the number of extra dimensions in the low energy limit, or field theory limit, which is discussed in Chap. 3. Contrary to the critical radius in the group space, the compactification scale of spacetime manifold $T^6/\mathbf{Z}_3$ is a free parameter. By adjusting some radii of the i^{th} tori $R_i = \langle G_{ii} \rangle$, we can vary or even de-compactify dimensions sending $R_i \rightarrow \infty$. However, in the $\mathbf{Z}_3$ orbifold we have a symmetry $\theta e_i = e_{i+1}$, and observed that the corresponding radii were forced to be the same $R_i = R_{i+1}$. By enlarging two dimensions, a 6D world instead of the 4D world is considered [10, 11]. We can still adjust such radii large but finite and have a field theoretical orbifold model. If we have $\mathbf{Z}_2$ orbifold, such identification of radii need not be imposed and we can de-compactify only one dimension to have a 5D field theoretic model [12, 13].

The strong constraint from the modular invariance is that the fields at fixed points are determined uniquely. On the other hand, in the field theoretical model we observed in Chap. 4 that the only consistency condition is the anomaly freedom. In string orbifolds, it is now automatically satisfied by modular invariance condition.

7.2.3 Wilson Lines on General Orbifolds

Effectively Reduced Order of Wilson Lines

Recall that in the $\mathbf{Z}_3$ orbifold case we are forced to identify Wilson lines in two directions in a $SU(3)$ lattice, e.g. $a_1 = a_2$, because of lattice symmetry $e^2 = \theta e^1$. From the relation of conjugacy class (3.16), the boundary conditions (θ, v_0) and $(\theta, v_0 + (1 - \theta)\Lambda)$ describe the equivalent fixed points, thus we set Wilson lines along v_0 and θv_0 same,

$$a_{v_0} = a_{\theta v_0} . \quad (7.48)$$

In general orbifold, this constraint becomes stronger [9]. Since the Wilson lines are turned along a basis defining lattice, the relations among them are *lattice dependent*, as seen in Table 7.1.

We will take an example of the first entry in $\mathbf{Z}_4$ orbifold, formed on $SU(4) \times SU(4)$ lattice. Consider first twisted sector. On each lattice, root vectors of $SU(4)$ become basis vectors $e^i \equiv \alpha^i$ and they satisfy the following relations

$$\theta \alpha^1 = \alpha^2 , \quad (7.49)$$

$$\theta \alpha^2 = \alpha^3 , \quad (7.50)$$

$$\theta \alpha^3 = -(\alpha^1 + \alpha^2 + \alpha^3) . \quad (7.51)$$

Table 7.1. Constraints for Wilson lines

	Lattice	Effective Order	Condition
$\mathbf{Z}_3$	$SU(3)^3$	$3a_1 = 0, 3a_3 = 0,$ $3a_5 = 0$	$a_1 = a_2, a_3 = a_4, a_5 = a_6$
$\mathbf{Z}_4$	$SU(4)^2$	$2a_1 = 0, 2a_4 = 0$	$a_1 = a_2 = a_3, a_4 = a_5 = a_6$
	$SU(4) \times SO(5) \times SU(2)$	$2a_1 = 0, 2a_5 = 0,$ $2a_6 = 0$	$a_1 = a_2 = a_3, a_4 = 0$
	$SO(5)^2 \times SU(2)^2$	$2a_2 = 0, 2a_4 = 0,$ $2a_5 = 0, 2a_6 = 0$	$a_1 = a_3 = 0$
$\mathbf{Z}_6$ -I	$SU(3) \times SU(3)^2$	$3a_1 = 0$	$a_1 = a_2, a_3 = a_4 = a_5 = a_6 = 0$
$\mathbf{Z}_6$ -II	$SU(2) \times SU(6)$	$2a_1 = 0$	$a_2 = a_3 = a_4 = a_5 = a_6 = 0$
	$SU(3) \times SO(8)$	$3a_1 = 0, 2a_5 = 0$	$a_1 = a_2, a_3 = a_4 = 0, a_5 = a_6$
	$SU(2)^2 \times SU(3)^2$	$3a_1 = 0, 2a_3 = 0,$ $2a_4 = 0$	$a_1 = a_2, a_5 = a_6 = 0$
		$7a_1 = 0$	$a_1 = a_2 = a_3 = a_4 = a_5 = a_6$
$\mathbf{Z}_8$ -I	$SO(8) \times SO(5)$	$2a_1 = 0, 2a_6 = 0$	$a_1 = a_2 = a_3 = a_4, a_5 = 0$
$\mathbf{Z}_8$ -II	$SO(10) \times SU(2)$	$2a_4 = 0, 2a_6 = 0$	$a_1 = a_2 = a_3 = 0, a_4 = a_5$
	$SU(2)^2 \times SO(8)$	$2a_1 = 0, 2a_5 = 0,$ $2a_6 = 0$	$a_1 = a_2 = a_3 = a_4$
$\mathbf{Z}_{12}$ -I	E_6	no restriction	$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$
	$SU(3) \times SO(8)$	$3a_1 = 0$	$a_1 = a_2, a_3 = a_4 = a_5 = a_6 = 0$
$\mathbf{Z}_{12}$ -II	$SU(2)^2 \times SO(8)$	$2a_1 = 0, 2a_2 = 0$	$a_3 = a_4 = a_5 = a_6 = 0$

Equation (7.51) is the lowest root. We have independent four fixed points, which are the fundamental weights (Λ) of $SU(4)$. From these relations and the conjugacy relation (cf. (3.16)), we require

$$a_1 = a_2 = a_3 . \quad (7.52)$$

Note that this relation has the bases as simple roots and every two neighbors are not orthogonal but are angled by 120 degree; this is what Table 7.1 means.

Now consider the second twisted sector. The bases have the relations

$$\theta^2 \alpha^1 = \alpha^3 \quad (7.53)$$

$$\theta^2 \alpha^2 = -(\alpha^1 + \alpha^2 + \alpha^3) \quad (7.54)$$

$$\theta^2 \alpha^3 = \alpha^1 . \quad (7.55)$$

Note that $\lambda(\alpha^1 + \alpha^3)$ is invariant under θ^2 . This means that points separated by $\lambda(\alpha^1 + \alpha^3)$ are identical: by the action θ^2 there is no way to tell such a separation. We should mode out by the invariant translational element, that is, we identify two points separated by $\lambda(\alpha^1 + \alpha^3)$,

$$\theta^2 \alpha^1 \simeq -\alpha^1, \quad \theta^2 \alpha^2 \simeq -\alpha^2 \quad (7.56)$$

Effectively we have four independent fixed points $0, \frac{1}{2}\alpha^1, \frac{1}{2}\alpha^2, \frac{1}{2}(\alpha^1 + \alpha^2)$. Now we classify these fixed points to tell which ones belong to the same conjugacy class according to (7.56). It turns out that

$$\theta^2 \frac{1}{2}\alpha^1 \simeq -\frac{1}{2}\alpha^1 \quad (7.57)$$

belong the same conjugacy class, we require $a_{\alpha^1} = a_{-\alpha^1} = -a_{\alpha^1}$ up to a lattice translation. In other words,

$$2a_1 \in \Gamma \quad (7.58)$$

and the same holds for $2a_2$ also, where Γ is the group lattice. *We have order two Wilson lines in effect*, although the space group action is order four. This analysis is very similar to what we have calculated in the $\mathbf{Z}_4$ case in Subsect. 7.2.1 and indeed they are related as shown in [3, 9].

Generalized GSO Projection

This affects the degeneracy factor (7.33) and furthermore affects the generalized GSO projection operator (7.30) [9]. In the θ^k twisted sector with Wilson lines described by $m_i a_i^I$, and m_i has a k dependence. Then

$$P_{\theta^k, m} = \frac{1}{N} \sum_{h=0}^{N-1} \sum_{k=0}^{N-1} \tilde{\chi}(\theta^k, m_i; \theta^h, l_i) \Delta_{(\theta^k, m_i; \theta^h, l_i)} , \quad (7.59)$$

where

$$\Delta_{(\theta^k, m_i; \theta^h, l_i)} = \exp 2\pi i \left[k\tilde{N} - kN - \frac{1}{2}(hV + la) \cdot (kV + ma) + \frac{1}{2}hk\phi^2 + (hV + la) \cdot (P + kV + ma) - h\phi \cdot (s + kn\phi) \right], \quad (7.60)$$

where dot product is on the gauge index I and the internal index i . The degeneracy factor, or number of simultaneous fixed points, is obtained in the same way except that now it should be the simultaneous fixed points including Wilson line translations. Unless $kl = mn$, $\tilde{\chi}$ vanishes and it reduces to a simpler form

$$P_{\theta^k, m} = \frac{1}{N} \sum_{h=0}^{N-1} \tilde{\chi}(\theta^k, m_i; \theta^h) \Delta_{(\theta^k, m_i; \theta^h)}, \quad (7.61)$$

where $\Delta_{(\theta^k, m_i; \theta^h)} = \Delta_{(\theta^k, m_i; \theta^h, l_i)}$ and $kl = hm$.

We can do a similar analysis for $\mathbf{Z}_M \times \mathbf{Z}_N$ type orbifolds [14].

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Interactions on Orbifolds

In the conventional, or field theoretical, effective theory approach, we have been content with obtaining the spectrum, and constructed low energy Lagrangian guided by symmetry and naturalness. Since we have the first principle of string theory, we can in principle perform more detailed calculations. It turns out that many “stringy” effects can enhance or suppress the interaction.

In this chapter we try to understand low energy theory taking into account stringy effects. Firstly, we calculate the Yukawa interaction, which is of utmost importance in understanding the low energy physics, using the powerful tool of conformal field theory [1, 2]. Using the dimensional reduction [3, 4] and the symmetry matching [5, 6], we will construct Kähler potentials and gauge kinetic functions. Still many properties are constrained by supersymmetry and the stringy version of nonrenormalization theorem tells us a lot on the form of the action. There are also pedagogical reviews on this topic [7, 8].

We will use the coordinates $z = e^{2(\tau+i\sigma)}$ thus $\partial \equiv \partial_z$, and $Z^i \equiv X^{2i-1} + iX^{2i}$.

8.1 Yukawa Couplings

The heterotic orbifold theory also predicts feasible forms for superpotentials, among which we are interested in the Yukawa couplings. In this section we will take the strategy of calculating string amplitude and compare the supergravity action giving rise to it. There is a very powerful method using conformal field theory [2]. This low-level method is universally applicable to any string theory including intersecting brane models [9] (see for instance Appendix B.2). From three point functions we can extract relative strengths among possible couplings. Then, we will sketch the procedure of obtaining four point functions which will give the absolute normalization.

Knowledge on Kähler potential, whose form we will seek in the next section, is required to have the absolute normalization of physical Yukawa

couplings. However, the qualitative result such as relative strengths will not change.

8.1.1 Vertex Operators for Interactions

For the relevant interactions, especially for Yukawa couplings, we need three and four point correlation functions. The corresponding tree level string diagram is the “sphere” with a number of external legs. The external legs of the sphere representing the “in” and the “out” state particles are shrunk at some points, as shown in Fig. 5.1, and the quantum numbers are contained in the vertex operators.

In this section, we will, instead of light cone gauge, employ a convenient covariant approach with the full Lorentz group $SO(1,9)$ [2]. Thus, we should take into account the conformal ghosts. It is known that, from anomaly cancellation for ghost currents, the sum of ghost charges equals -2 for the sphere diagram [2, 10]. The renormalizable Yukawa type interaction, i.e. boson–fermion–fermion coupling (BFF), is calculated from a correlation function

$$\left\langle V_{-1} V_{-\frac{1}{2}} V_{-\frac{1}{2}} \right\rangle, \quad (8.1)$$

where the bracket represents path integration over all the relevant variables including ghosts and the subscripts denote their charges. The form of the vertex operator V will be evident shortly. This BFF interactions are in the effective Lagrangian, from which we can deduce the superpotential. For a nonrenormalizable interaction of type $B^n FF$, we need a vertex operators with ghost charge 0, and the interaction is given by

$$\left\langle V_{-1} V_{-\frac{1}{2}} V_{-\frac{1}{2}} V_0 \cdots V_0 \right\rangle. \quad (8.2)$$

Now we proceed to construct pertinent vertex operators. Sometimes it is easy to deal with those of left and right movers separately by $V(z, \bar{z}) = V(z)V(\bar{z})$.

Untwisted Fields

Ghost fields (bosonized), $\varphi(\bar{z})$, are required when using the covariant formalism. The general vertex operator has the form

$$V_q(z, \bar{z}) = e^{is \cdot H} e^{ik \cdot X} e^{q\varphi}, \quad (8.3)$$

where q is the ghost charge. The factor $e^{ik \cdot X(z, \bar{z})} \equiv e^{ik_L X_L(z) + ik_R X_R(\bar{z})}$, with only spacetime degrees of freedom, indicates that the vertex operator carries momentum k . The five component antiholomorphic field $H(\bar{z})$ is the bosonization of the right moving worldsheet fermions. Due to this reason, s is sometimes called H -momentum.

From conformal field theory of ghosts, the NS and the R state have -1 and $-\frac{1}{2}$ ghost charges, respectively. Therefore for bosonic states,

$$V_{-1} = e^{is \cdot H} e^{ik \cdot X} e^{-\varphi} , \quad (8.4)$$

Thus, it has $s = (\pm 1 \ 0 \ 0 \ 0 \ 0)$, as the vector representation **10** of $SO(10)$. See (5.201). In this case, we have an explicit expression in terms of the worldsheet fermions $e^{is \cdot H} \simeq \psi^\mu$. For fermions

$$V_{-\frac{1}{2}} = e^{is \cdot H} e^{ik \cdot X} e^{-\frac{1}{2}\varphi} , \quad (8.5)$$

where $s = ([\frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2}])$ with even numbers of minus signs, which constitute the spinorial representation **16**.

To calculate nonrenormalizable interaction (8.2) we should make a zero charge vertex operator. We perform the picture changing operation [10] (here, consider right movers only),

$$V_0(\bar{z}) = \lim_{\omega \rightarrow z} e^{\varphi(\bar{\omega})} J_-(\bar{\omega}) V_{-1}(\bar{z}) , \quad (8.6)$$

where $J_- = \bar{\partial} X^\mu \psi_\mu + \bar{\partial} Z^i \bar{\psi}^i + \bar{\partial} \bar{Z}^i \psi^i$ is the right-moving worldsheet supercurrent (5.106). Therefore, we have the following 0 charge vertex operator

$$V_0(\bar{z}) = e^{ik \cdot X} (ik^\nu \psi_\nu e^{is \cdot H} + \bar{\partial} X) . \quad (8.7)$$

In the zero momentum limit $k \rightarrow 0$, it is simply

$$V_0 = \bar{\partial} X$$

which is the same as the bosonic vertex operator, as it should be, because both describe spacetime bosons, and the k -dependent extra term is the connection in the worldsheet sigma model.

It is also trivial to deduce the form for the gauge degrees of freedom,

$$V_{\text{op}} = e^{ik \cdot X} e^{iP \cdot F} , \quad (8.8)$$

where $F^I(z)$ describes the sixteen, left-handed bosonic degrees of freedom of the gauge fields, previously denoted as $X^I(z)$.

One can understand this with the notation used in Chapter 6 as following. The correlation function (8.1) is the vacuum expectation value or the expectation value with the orbifold lattice $|\Lambda\rangle$. A vertex operator corresponds to a state in this sense

$$V_{\text{op}} \leftrightarrow V_{\text{op}} |\Lambda\rangle .$$

If an orbifold lattice $|\Lambda\rangle$ is given, $V_{\text{op}} |\Lambda\rangle$ describes a mode, an untwisted or twisted. A twisted mode is shown in Fig. 8.1. Any information on this state is contained in $V_{\text{op}} |\Lambda\rangle$ for which we used the notation $|P + V\rangle$ where V is a shift for example. The information on the gauge transformation is contained

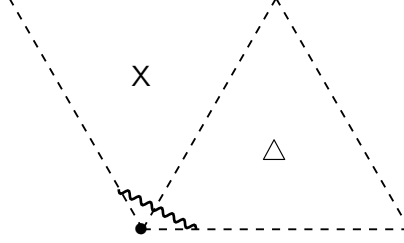


Fig. 8.1. A twisted mode shown as the photon line around the fixed point $\bullet$ in the fundamental region of $\mathbf{Z}_3$ lattice

in V_{op} of (8.8). Let us discuss for untwisted states $|P_{\text{un}}\rangle$. The gauge group is obtained by the spin-1 spectrum in the untwisted sector satisfying $P_{\text{un}} \cdot V = 0$. For nonzero roots, P_{un} s satisfy $P_{\text{un},\pm}^2 = 2$. Our convention in quantum field theory is that we diagonalize the Cartan generators, $Q^I (I = 1, 2, \dots, 16)$ which are mutually commuting. Thus, any nonzero root $|P_{\text{un}}\rangle$ has a set of 16 eigenvalues of Q^I in the sense of quantum field theory. The gauge transformation generated by the Cartan subalgebra on the state $|P_{\text{un}}\rangle$ in gauge theory is given by¹

$$e^{iQ^I \theta^I} |P_{\text{un}}\rangle = e^{i\lambda^I \theta^I} |P_{\text{un}}\rangle, \quad I = \text{not summed} \quad (8.9)$$

where λ^I is the eigenvalue of Q^I . String theory contains all the information in the vertex operator (8.8) where $F^I(z)$ are vectors originally belonged to X^I . If the vertex operator were acted on $|P_{\text{un}}\rangle$, we would obtain a phase $e^{iP_{\text{un}} \cdot F^I(z)}$ where $F^I(z)$ here is $\lambda^I \theta^I$ of (8.9). There are 16 possible numbers of λ_I constructed from the state $|P_{\text{un}}\rangle$ with the quantized set of 16 vectors F^I of the form $(1, 0, 0, \dots)$, $(0, 1, 0, \dots)$, etc. Since Q^I s are diagonalized, there is one to one correspondence between 16 Q^I s and 16 F^I s. Thus, it is proper to interpret the eigenvalue of Q^I being $P_{\text{un}} \cdot F^I$. These provide eigenvalues of 16 Cartan generators. Note that for a U(1) generator say Q_1 , it must give the same quantum number for charged gauge bosons, i.e. those belonging to nonabelian gauge groups with $P_{\text{un},\pm}$, and hence satisfies $P_{\text{un},\pm} \cdot F_1 = 0$. In this way, one can find the U(1) vectors F_1 and consequently U(1) generators Q_1 . Their eigenvalues on the state $|W\rangle$ is simply denoted as $W \cdot F_1$. The normalization of Q_1 and hence that of F_1 will be discussed in Chaps. 9 and 10.

Twisted Fields

We also need a correlation function among *twisted* states. Calculating correlation functions for untwisted states were relatively easy and straightforward.

¹ For a twisted sector V , we consider $|P + V\rangle$ instead of $|P_{\text{un}}\rangle$.

Namely, by using only suitable operators and commutators, we could obtain the correlation functions.

However, it is not the case for the twisted states, since we learned in Sect. 6.1.1 that twisted string maps from one Hilbert space to another Hilbert space. It reminds us of a familiar problem with the fermionic vertex operator. A spin half fermion is characterized by the double valued function, $\psi(e^{2\pi i}z) = -\psi(z)$. To make a vertex operator we introduce “spin fields” S and $\tilde{S}$ obeying

$$\psi(z)S(0) \sim z^{-1/2}\tilde{S}(0) . \quad (8.10)$$

where the $\sim$ means that both sides are the same up to an irrelevant regular part. This should be realized by local fields, because the position of singularity is independent of z . $S(0)$ and $S(\infty)$ are interpreted as making cuts running between 0 and ∞ . This $z^{1/2}$ is double valued function and for single-valuedness we introduce the covering space, called *Riemann surface* [1]. When we transport $\psi(z)$ once around (crossing the branch cut), the coordinate can be thought to be in a different space and once more around we are back to the original space. This is exactly what we do in the complex analysis when we have a multiple-valued function. With the twist fields, the theory is not local any more, however with an appropriate projection such as GSO or orbifold as we will see, the theory remains local.

The two point correlation function now contains spin fields

$$\langle S(\infty)\psi(z)\psi(w)S(0) \rangle = -\frac{1}{2} \frac{1}{z-w} \left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}} \right) . \quad (8.11)$$

Although it has different global structure to the one without spin fields,

$$\langle \psi(z)\psi(w) \rangle = -\frac{1}{z-w}$$

in the $z \rightarrow w$ limit, or locally, the both behave the same. Surveying conformal weights, one notes that it reproduces the mass shell condition [2].

Precisely the same action is done to bosons by a $\mathbf{Z}_2$ twisting $Z(e^{2\pi i}z) = -Z(z)$. We may introduce a double-valued “twist field” σ and “excited twist field” τ performing the same action on the bosonic field

$$\partial Z(z)\sigma(0) \sim z^{-1/2}\tau(0) .$$

This is naturally generalized to $\mathbf{Z}_N$ twist fields σ_k and τ_k (from now on it is understood that the order N is implicit),

$$\partial Z(z)\sigma_k(0) \sim z^{-1+k/N}\tau_k(0) . \quad (8.12)$$

However, now the reason for acquiring a phase is not due to the spin statistics, but to the property of geometry. For patching to a single-valued space we introduce a covering space, which is exactly how we made the $\mathbf{Z}_N$ orbifold.

The multi-valued function $z^{k/N}$ gives the correct orbifold phase. The twisted ground state (6.9) is now interpreted as

$$|\sigma_k\rangle_{L,R} = \sigma_k(0,0)|0\rangle_{L,R} . \quad (8.13)$$

Therefore, the relation (8.12) can be extracted when we consider the (complexified) mode expansion, (6.5). Consider, for example, the ∂Z part, where only the holomorphic left movers are relevant. By the definition of the ground state, $\tilde{\alpha}_{m+k/N}^a$ s with $m \geq 0$ annihilate the ground state (8.13). Therefore, among excited states, the most singular part is

$$\partial Z \sigma_k(0,0)|0\rangle = \partial Z |\sigma_k\rangle \sim -\frac{i}{2} z^{k/N-1} \tilde{\alpha}_{k/N-1}^{\bar{a}} |\sigma_k\rangle \equiv z^{k/N-1} \tau_k(0,0)|0\rangle . \quad (8.14)$$

where we could extract the excited twist field τ_k also, with the suggestive name. Note that derivatives correspond to oscillators. In our notation, the correspondence between (anti-)holomorphic and left or right oscillators are

$$\begin{aligned} \partial^n &\leftrightarrow \tilde{\alpha}_{-n}, \text{ for left mover}(z) \\ \bar{\partial}^n &\leftrightarrow \alpha_{-n}, \text{ for right mover}(\bar{z}) \end{aligned} \quad (8.15)$$

They also affect the conformal weight, since they contribute to the zero point energy. The one to one correspondence between vertex operators and massless $|\text{in}\rangle$ and $|\text{out}\rangle$ states can be made. It can be seen transparently from the fact that the sum of the conformal weights to be 1 is precisely the condition for massless states.

In the similar manner, we obtain

$$\begin{aligned} \partial Z(z) \sigma_k(0,0) &\sim z^{k/N-1} \tau_k(0,0) , \\ \partial \bar{Z}(z) \sigma_k(0,0) &\sim z^{-k/N} \tau'_k(0,0) , \\ \bar{\partial} Z(\bar{z}) \sigma_k(0,0) &\sim \bar{z}^{-k/N} \tilde{\tau}'_k(0,0) , \\ \bar{\partial} \bar{Z}(\bar{z}) \sigma_k(0,0) &\sim \bar{z}^{k/N-1} \tilde{\tau}_k(0,0) , \end{aligned} \quad (8.16)$$

where the twist fields are in general functions of both coordinates ω and $\bar{\omega}$. The excited twisted fields, τ, τ' etc., are related with others with slightly different conformal weights. Also, we have similar expressions for σ_{-k} .

The whole vertex operator for a right mover (in the $k=1$ twisted sector) has the form

$$V_{-1} = e^{i(s+\phi)\cdot H} e^{ik\cdot X} e^{-\varphi} \sigma_1 ,$$

with s being components of vectorial **10** and

$$V_{-\frac{1}{2}} = e^{i(s+\phi)\cdot H} e^{ik\cdot X} e^{-\frac{1}{2}\varphi} \sigma_1 .$$

with s being spinorial **16**. Here ϕ is the twist vector, $\phi = (\frac{2}{3} \frac{1}{3} \frac{1}{3} 0 0)$ for $\mathbf{Z}_3$ example, and σ is the twist field. We have calculated s already, e.g. in (6.63–6.65),

Table 8.1. H -momenta for twisted sector fields in the -1 picture for various $\mathbf{Z}_N$ orbifolds. These are the solutions of (6.77) in our convention on twist given in Table 3.2

k	1	2	3	4	5	6
$\mathbf{Z}_3$	$\frac{1}{3}(-1 \ 1 \ 1)$					
$\mathbf{Z}_4$	$\frac{1}{4}(-2 \ 1 \ 1)$	$\frac{1}{2}(0 \ 1 \ 1)$				
$\mathbf{Z}_6$ -I	$\frac{1}{6}(-4 \ 1 \ 1)$	$\frac{1}{3}(-1 \ 1 \ 1)$	$\frac{1}{2}(0 \ 1 \ 1)$			
$\mathbf{Z}_6$ -II	$\frac{1}{6}(-3 \ 2 \ 1)$	$\frac{1}{3}(0 \ 2 \ 1)$	$\frac{1}{2}(1 \ 0 \ 1)$			
$\mathbf{Z}_7$	$\frac{1}{7}(-4 \ 2 \ 1)$	$\frac{1}{7}(-1 \ 4 \ 2)$	$\frac{1}{7}(2 \ -1 \ -4)$			
$\mathbf{Z}_8$ -I	$\frac{1}{8}(-5 \ 2 \ 1)$	$\frac{1}{4}(-1 \ 2 \ 1)$	$\frac{1}{8}(1 \ -2 \ -5)$	$\frac{1}{2}(1 \ 0 \ 1)$		
$\mathbf{Z}_8$ -II	$\frac{1}{8}(4 \ 3 \ 1)$	$\frac{1}{4}(0 \ 3 \ 1)$	$\frac{1}{4}(4 \ 1 \ 3)$	$\frac{1}{2}(0 \ 1 \ 1)$		
$\mathbf{Z}_{12}$ -I	$\frac{1}{12}(-7 \ 4 \ 1)$	$\frac{1}{6}(-1 \ 4 \ 1)$	$\frac{1}{4}(1 \ 0 \ -3)$	$\frac{1}{3}(-1 \ 1 \ 1)$	$\frac{1}{12}(1 \ -4 \ 7)$	$\frac{1}{2}(1 \ 0 \ 1)$
$\mathbf{Z}_{12}$ -II	$\frac{1}{12}(6 \ 5 \ 1)$	$\frac{1}{6}(0 \ 5 \ 1)$	$\frac{1}{4}(2 \ 1 \ 1)$	$\frac{1}{3}(0 \ 2 \ 1)$	$\frac{1}{12}(-6 \ 1 \ 5)$	$\frac{1}{2}(0 \ 1 \ 1)$

and (6.77) in the $\mathbf{Z}_3$ example (note that the twisted sector H -momentum is not s , but $s + k\phi$). Note also that, the untwisted sector has various massless solutions of (5.202) as H -momentum, depending on what Lorentz components they have. However, in the twisted sector there is essentially a unique solution to (6.77), because a specification on orbifold determines it. We have tabulated them in Table. 8.1.

We can similarly construct higher order twist operators. However, as in the construction of wavefunction in Subsec 7.2.1, there might be twist fields that are not eigenstates of point group action $\theta \in \bar{\mathbf{P}}$ in the higher twisted sectors. If a twisted sector has fixed tori, we can make a physical twist field as an eigenstate of θ by forming a linear combination, as done in (7.43),

$$\frac{1}{\sqrt{l}} \left(\sigma_{k,f} + \gamma^{-1} \sigma_{k,\theta f} + \gamma^{-2} \sigma_{k,\theta^2 f} + \cdots + \gamma^{l-1} \sigma_{k,\theta^{l-1} f} \right) . \quad (8.17)$$

The same argument applies to the twisted left movers. For example, consider the state $(\bar{\mathbf{3}}, \mathbf{1})$ in the standard embedding of $\mathbf{Z}_3$. The massless state (6.74) is formed with the aid of oscillator $\tilde{\alpha}_{-1/3}^i$, therefore we expect the fractional oscillator contribution $\partial Z^i / \partial z^{1/3}$. Therefore, we have

$$V^i = e^{i(P+V) \cdot F} \partial_{z^{1/3}} Z^i e^{ik \cdot X} \sigma_1 . \quad (8.18)$$

It carries the spacetime index i and plays an interesting role in understanding geometry. On the other hand, $(\mathbf{27}, \mathbf{1})$ has no oscillator

$$V = e^{i(P+V) \cdot F} e^{ik \cdot X} \sigma_1 . \quad (8.19)$$

8.1.2 Selection Rules

From the invariance properties, we deduce the following selection rules arising from the properties of the internal space [1, 2, 11]. Without calculating it

explicitly, we can quickly check whether a term with given fields survives or not. We will fully calculate the relevant Yukawa coupling coefficient in the next subsection.

Space Group Selection

In view of (3.21), every twist field (in a vertex operator) is labeled by twisted sector θ^k and fixed point f , which belong to the conjugacy class (3.16)

$$\{(\theta^k, (1 - \theta^k)(f + v)) \mid v \in \Lambda\} . \quad (8.20)$$

As a product of such twist fields, the transformation property of a correlation function is also given by the product of the elements given in (8.20). The amplitude is nonzero only when the product is $(1, 0)$ up to lattice translation. This is so because there should be no net rotation for the whole amplitude. To show the translational part, let the integration contour C encircle all the vertex operators. We can enlarge it to infinity for there is no singularity or twist field outside. Then the integration is as good as that having no vertex operator thus vanishes.

It suffices to consider an example of three point correlation function, which transforms as

$$\begin{aligned} & (\theta^k, (1 - \theta^k)(f_1 + v_1)) (\theta^l, (1 - \theta^l)(f_2 + v_2)) (\theta^m, (1 - \theta^m)(f_3 + v_3)) \\ &= \left(\theta^{k+l+m}, \theta^{k+l}(1 - \theta^m)(f_3 + v_3) \right. \\ & \quad \left. + \theta^k(1 - \theta^l)(f_2 + v_2) + (1 - \theta^k)(f_1 + v_1) \right) , \end{aligned} \quad (8.21)$$

which should be $(1, 0)$ up to lattice translation. Therefore, the rotational part should be identity, leading to $m = -k - l$. By absorbing v_3 into other lattice translation elements (v_1, v_2 are accordingly changed), the condition states that there must exist vectors $\tau_1, \tau_2 \in \Lambda$ such that

$$(1 - \theta^k)(f_1 - \tau_1) + \theta^k(1 - \theta^l)(f_2 - \tau_2) = (1 - \theta^{k+l})f_3 . \quad (8.22)$$

One might worry about the transformation property of the linearly combined twist field (8.17). In fact, if the above selection rule is satisfied by a set of fixed points f_i , each $\theta^{k_i} f_i$ also satisfies it. It means that the linear combination (8.17), $\sum_{k_i} \gamma^{-k_i} \sigma_{k_i, \theta^{k_i}}$, made out of θ^{k_i} has the same transformation property, and hence the linear combination can be really treated as a single field.

Lorentz Invariance

As in the field theory case, the momentum should be conserved. The momentum k is carried by the element $e^{ik \cdot X}$, where $X^i(z, \bar{z})$ is the world sheet

boson carrying the spacetime index i . The integration of its products yields a delta function, which vanishes unless momentum is conserved. Likewise, the gauge degrees of freedom are represented by F^I (or X^I) whose coefficients P s are conserved also. These constitute the invariance under the gauge group, realized as *Lorentz invariance*.

The same is true for the fermionic right movers in the superstring; the worldsheet fermions are represented by $H(\bar{z})$ by bosonization. The coefficients $(s + k\phi)$ play the role of momentum, called the H -momentum, so that they should sum up to zero for a non-vanishing correlation function.

We already noted that this is nothing but representing the weights (10 or 16) of the Lorentz group. Thus, the vanishing sum implies the invariance under the Lorentz group. In fact, H -momentum is not a well-defined notion. Consider again the operation (8.6) changing the picture from (8.4) to (8.7). The former has a well-defined H -momentum, but the latter has an extra term ∂X , which has no H -momentum. However, it does not mean that the H -momentum is lost completely, because still the Lorentz transformation property remains there.

This leads us to introduce another invariant quantity [11, 12],

$$R_a = s_a + N_a - \bar{N}_{\bar{a}} , \quad (8.23)$$

where N_a and $\bar{N}_{\bar{a}}$ are the numbers of holomorphic and antiholomorphic oscillators, respectively, in the a^{th} direction. Namely, R_a can be interpreted as the *generalized* number of holomorphic minus antiholomorphic Lorentz indices in the a^{th} direction. It is because J_- contains a factor $\psi^a \bar{\partial} \bar{Z}^{\bar{a}} + \bar{\psi}^{\bar{a}} \bar{\partial} Z^a = e^{is \cdot H} \bar{\partial} \bar{Z}^{\bar{a}} + e^{-is \cdot H} \bar{\partial} Z^a$, where $s = (1, 0, 0)$ is the internal index.² J_- is a Lorentz scalar, carrying $R = 0$,³ so that the transformation properties of the vertex operator before and after the picture changing are the same.

This quantity (8.23) has another invariance property. We have defined the point group action θ by a simultaneous rotation in the sublattices (sub-two-tori). In fact, the invariance should hold componentwise *in each sublattice*. We have the well defined transformation rules for oscillators and fermions under rotation in the sublattices as given in (6.14) and (6.61). Their vertex operator version is

$$\bar{\partial}^n Z^a \rightarrow e^{2\pi i \phi_a} \bar{\partial}^n Z^a \quad (8.24)$$

$$\bar{\partial}^n \bar{Z}^{\bar{a}} \rightarrow e^{-2\pi i \phi_a} \bar{\partial}^n \bar{Z}^{\bar{a}} \quad (8.25)$$

$$e^{is \cdot H} \rightarrow e^{-2\pi i s \cdot \phi} e^{is \cdot H} , \quad (8.26)$$

where n is an arbitrary integer. Namely, the orbifold transformation property depends only on the Lorentz transformation property. It is readily checked that the charge (8.23) is invariant under these transformations, because we

² J_+ would have $s = (-1, 0, 0)$.

³ E.g. the first term has $s = (1, 0, 0)$ and $\bar{N}_{\bar{1}} = 1$.

have kept invariant states under the *orbifold projections*. Therefore, Lorentz symmetry is a more fundamental symmetry.

Remarkably, this can be interpreted as a discrete *R-symmetry* in the sense of conventional supersymmetric field theories. It is because the supercharge, (2.23), can be written as

$$Q = \int \frac{d\bar{z}}{2\pi i} e^{-\varphi(\bar{z})/2} S(\bar{z}) e^{ir \cdot H(\bar{z})} \quad (8.27)$$

where S is the spin field (8.10), and hence accompanied by $r = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ in (3.58) and $-\frac{1}{2}$ ghost charge. Spin field $S(\bar{z})$ changes a boson (fermion) to a fermion (boson).

The H -momentum transformation properties of bosons and fermions are given by the same formula, (8.26), but with different s 's, the difference being defined as r . In other words, they should be rotated simultaneously and there follows

$$Q \rightarrow e^{-2\pi i r_a \phi^a} Q, \quad a = \text{not summed}.$$

It is consistent with the vertex operator transformation

$$V \rightarrow e^{-2\pi i R_a \phi^a} V, \quad a = \text{not summed} \quad (8.28)$$

and the same for the corresponding low-energy field.

In the orbifold compactification, the continuous *R*-symmetry is broken to a discrete symmetry, because the rotations (8.24–8.26) are discrete rotations. Their orders are determined by twist vector ϕ . For example, invariance for $\mathbf{Z}_6$ -II orbifold specified by $\phi = \frac{1}{6}(3 \ 2 \ 1)$ implies,

$$\begin{aligned} \sum R_1 &= -1 \pmod{2}, \\ \sum R_2 &= 1 \pmod{3}, \\ \sum R_3 &= 1 \pmod{6}, \end{aligned} \quad (8.29)$$

where the summation is over the fields forming the product (8.2), where we have *two* fermion components. So, in the low energy field theory, we can mimic this rule by imposing a discrete symmetry.

Therefore, all the allowed terms can be determined in the effective field theory, by assigning appropriate discrete charges to each field as above. Even, it turns out to be true [12] that the target space modular property (6.120) determines the strength of Yukawa couplings in terms of modular weights, that we will show in the next section, without considering the stringy effect. The important thing here is the spectrum and their discrete quantum numbers.

8.1.3 Three Point Correlation Function

Now let us calculate the three point correlation function of the form (8.1) [1, 2, 13]. Up to overall normalization, it completely determines Yukawa coupling quantitatively.

The integration over X, H or ghosts leads to the well-known result similar to the Veneziano amplitude. The nontrivial part involves a correlation among twist fields

$$\mathcal{Z} \equiv \langle \sigma_{k,f_1}(z_1, \bar{z}_1) \sigma_{l,f_2}(z_2, \bar{z}_2) \sigma_{-k-l,f_3}(z_3, \bar{z}_3) \rangle \quad (8.30)$$

where the subscripts k and f_i denote the twist k/N and the fixed point, respectively. This arises for each two-torus, embedded in the complex plane, but the calculation can be independent for each torus, so we suppressed the Lorentz index. The space group selection rule determines, for nonvanishing twists, the relation between fixed points, which is (8.22) in the case at hand, so hereafter we will suppress the fixed point indices f_i .

The worldsheet field factors out to the classical and quantum parts,

$$Z = Z_{\text{cl}} + Z_{\text{qu}} , \quad (8.31)$$

and so does the correlation function

$$\mathcal{Z} = \mathcal{Z}_{\text{qu}} \cdot \sum_{\{Z_{\text{cl}}\}} \exp(-S_{\text{cl}}) \quad (8.32)$$

where the classical action S_{cl} is given in (5.88). The classical part describes the local minimum of Euclidian action, which we call *instanton*. It is calculated by path integral (here, summation) of Euclidian action with respect to all the possible solutions Z_{cl} satisfying the equation of motion

$$\bar{\partial} \partial Z_{\text{cl}} = \partial \bar{\partial} \bar{Z}_{\text{cl}} = 0 .$$

The classical solutions of the string equations of motion are

$$\begin{aligned} \partial Z_{\text{cl}} &= a(z - z_1)^{-1+k/N} (z - z_2)^{-1+l/N} (z - z_3)^{-(k+l)/N} , \\ \bar{\partial} Z_{\text{cl}} &= b(z - z_1)^{-k/N} (z - z_2)^{-l/N} (z - z_3)^{-1+(k+l)/N} , \end{aligned} \quad (8.33)$$

where we kept the most singular parts. For the symmetric orbifold at hand, automatically there result $\bar{\partial} \bar{Z}_{\text{cl}} = (\partial Z_{\text{cl}})^*$ and $\partial \bar{Z}_{\text{cl}} = (\bar{\partial} Z_{\text{cl}})^*$. We can verify that they are solutions indeed since they are either totally holomorphic or anti-holomorphic, and possess the desired singular behavior of (8.12) as we send z to z_1, z_2 , or z_3 . It turns out that, in the three point function case, only the solutions for a in ∂Z_{cl} is relevant, in the sense that the other solutions give divergent summation in (8.32). So we set $b = 0$.

This solution specifies the local properties only, thus the coefficient a is not determined at this level. To determine it, we should consider the *global property*. As explained, the twist fields $\sigma_k(z_1)$ and $\sigma_l(z_2)$ introduce a branch cut running between the points z_1 and z_2 , which makes a non-simply connected topology. Now consider a contour C that encircles p times around z_1 counter-clockwise and q times around z_2 clockwise, as depicted in Fig. 8.2. Viewing space group action, the phase acquired from the former is $(e^{2\pi i k/N})^p$ and from the latter $(e^{-2\pi i l/N})^q$. Evidently, this will give a nontrivial information on a ,

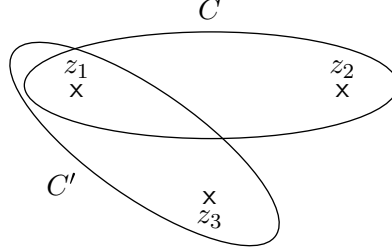


Fig. 8.2. There are two independent choices of contours for three point function for $k \neq l$

but what will be the simple one? One easily guesses $p = l, q = k$ seems to be the best choice, but this is not sufficient. For a moment, just content with the integers p and q satisfying $pk = ql$. Denoting those resulting actions as θ^{pk} and $\theta^{-ql} = \theta^{-pk}$, in view of (3.9), the conjugacy class becomes

$$\begin{aligned} & (\theta^{pk}, (1 - \theta^{pk})(f_1 + \Lambda)) (\theta^{-pk}, (1 - \theta^{-pk})(f_2 + \Lambda)) \\ &= (1, (1 - \theta^{pk})(f_1 - f_2 + \Lambda)) \\ &\ni (1, v), \end{aligned} \quad (8.34)$$

where, recall that, v is a complex number for a two dimensional lattice Λ . This requires,

$$\oint_C dz \partial Z_{\text{cl}} = v, \quad (8.35)$$

with Z_{cl} given in (8.33). It is called the *global monodromy* condition, meaning a phase from the boundary condition. Therefore, the classical solution Z_{cl} is parametrized by lattice translation Λ , implied in v in (8.34); the summation of (8.32) is over entire Λ .

Now plug the classical solutions (8.33) into (8.35). Using the $\text{SL}(2, \mathbf{C})$ transformation, we can always send z_1, z_2, z_3 to $0, 1, \infty$, respectively,

$$\begin{aligned} & a(-z_3)^{-(k+l)/N} \oint_C dz z^{-1+(k/N)} (z-1)^{-1+(l/N)} \\ &= -2ia(-z_3)^{-(k+l)/N} \sin(pk\pi/N) \frac{\Gamma(k/N)\Gamma(l/N)}{\Gamma((k+l)/N)} \\ &= v. \end{aligned} \quad (8.36)$$

Here $z_3 \rightarrow \infty$ is implicit so that $z - z_3 \simeq -z_3$ is decoupled. Thus, we obtain

$$a = \frac{i(-z_3)^{(k+l)/N}}{2} \frac{\Gamma((k+l)/N)}{\sin(pk\pi/N)\Gamma(k/N)\Gamma(l/N)} v. \quad (8.37)$$

Inserting this solution into the action (8.32) and performing integration on d^2z (the method of which is discussed in [17]), we have

$$S_{\text{cl}} = \frac{|v|^2}{4\pi \sin^2(pk\pi/N)} \frac{|\sin(k\pi/N)| |\sin(l\pi/N)|}{|\sin((k+l)\pi/N)|}. \quad (8.38)$$

Plugging it again into (8.32), we can completely determine the size of Yukawa coupling up to the normalization, where the sum is over (8.34).

However, this is not complete. We do not know how to fix integers p, q and in fact there is another way to make a contour, because we have the third singular point at z_3 due to twist field $\sigma_{-k-l}(z_3)$. A contour C' encircling z_1 and z_3 also gives rise to *independent* nontrivial phase. Since that every time we wrap the twist at z_3 we acquires phase $e^{-2\pi i(k+l)}$, we should take an integer r satisfying

$$pk = ql = r(k+l). \quad (8.39)$$

We can always find such p, q, r less than the order N . This is the reason why the naive choice of $p = l, q = k$ does not suffice, since then in general there is no integer solution for r . Fortunately, it turns out that we can keep the solution (8.32) and the consideration on C' just restricts the range of summation of v in (8.32). This fact will be revealed when we calculate the four point correlation function in the next subsection.

Solving the equation similar to (8.35) for C' , we obtain

$$(1 - \theta^k)(f_1 - \lambda_1) + \theta^k(1 - \theta^l)(f_2 - \lambda_2) = (1 - \theta^{k+l})f_3, \quad (8.40)$$

where λ_1, λ_2 are specific lattice vectors satisfying (8.34) for C, C' , respectively. The space group invariance condition (8.22) has the similar form, so that by comparison we may easily guess the solution

$$v \in \tau_1 + \frac{1 - \theta^{k+l}}{1 - \theta^{\text{gcd}(k,l)}} \Lambda, \quad (8.41)$$

which further restricts v . Note that $1 - \theta^{\text{gcd}(k,l)}$ always divide $1 - \theta^{k+l}$ to make the last term belong to the lattice.

What if we consider another choice of contour encircling $\sigma_k(z_1)$ and $\sigma_{-k-l}(z_3)$? To obtain a vanishing phase, we should wind p times clockwise around z_1 and r times counterclockwise around z_2 . Applying the global monodromy and the consistency conditions, it leads to the same condition as before, since the same conditions ((8.39) and (8.22)) should be satisfied. However, the translation element (8.41) has a slightly different form in this picture.

The normalization is determined by the quantum part $\mathcal{Z}_{\text{qu}}$, which is just a constant for three point functions. Because of conformal symmetry, any three points z_1, z_2, z_3 can be mapped into, say $0, 1, \infty$; therefore the correlation function contains no information. It can be fixed, however, by factorizing a four point function which we will see in the next subsection.

8.1.4 Four Point Correlation Function

Now we turn to the calculation of four point correlation functions. Not only this determines the normalization of Yukawa couplings (three point correlation

functions), but also exhibits the modular property clearly [14, 15].

$$\mathcal{Z} \equiv \langle \sigma_{-k, f_1}(z_1, \bar{z}_1) \sigma_{+k, f_2}(z_2, \bar{z}_2) \sigma_{-l, f_3}(z_3, \bar{z}_3) \sigma_{+l, f_4}(z_4, \bar{z}_4) \rangle . \quad (8.42)$$

(Of course, a general four point correlation function has twistings $k, l, m, -k-l-m$.) Basically the procedure is the same as above, however the calculation is more involved. We briefly sketch the steps of general strategy and the results [2, 13, 16].

The Classical Part

Firstly, we compute the classical part. The form of the classical solutions are same as the case of three point function,

$$\partial Z_{\text{cl}} = a \omega_{k,l}(z), \quad \bar{\partial} Z_{\text{cl}} = b \omega_{1-k, 1-l}(\bar{z}) \quad (8.43)$$

where,

$$\omega_{k,l}(z) \equiv (z - z_1)^{-k/N} (z - z_2)^{k/N-1} (z - z_3)^{-l/N} (z - z_4)^{l/N-1} . \quad (8.44)$$

In this case, both in (8.43) contribute finite amounts so that both a and b survive.

Using the same method as done before for three point functions, we have the global monodromy condition

$$\int_{C_i} dz \partial Z_{\text{cl}} + \int_{C_i} d\bar{z} \bar{\partial} Z_{\text{cl}} = v_i \quad (8.45)$$

along the contours C_i of Fig. 8.3. The best choice is to connect $z_1 - z_2$ and $z_3 - z_4$, since each pair of twist fields have the same order so that each branch cut ends. In other words, outside the branch cut, there is no phase acquisition. As discussed before, although there are two independent contours C_1 and

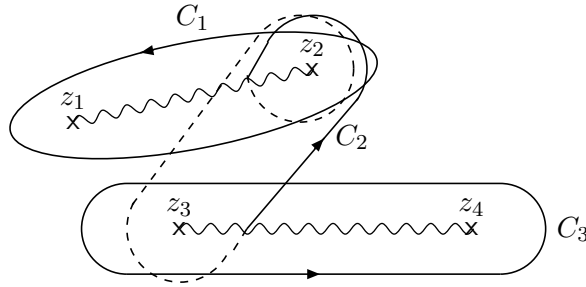


Fig. 8.3. To calculate four point correlation function for $\mathbf{Z}_N$ orbifold, we encircle the vertex several times to have zero net twist. We take C_2 by encircling θ^l vertex q times and θ^k vertex p times, so that $pk = ql$

C_2 , we will require the consistency condition again. They yield the following conditions

$$v_1 \in (1 - \theta^k)(f_2 - f_1 + \Lambda) \quad (8.46)$$

$$v_2 \in (1 - \theta^{pk})(f_3 - f_2 + \Lambda) . \quad (8.47)$$

The contour C_1 is simple, because encircling z_1 and z_2 once counterclockwise will cancel phase θ^{-k} and θ^k . The nontrivial one is C_2 that encircle z_2 clockwise p times and z_3 counterclockwise q times, so that we require $pk = ql$. The solutions for a and b are given by linear combinations

$$a = a_1 v_1 + a_2 v_2, \quad b = b_1 v_1 + b_2 v_2 \quad (8.48)$$

with a_i and b_i being expressed by some complicated hypergeometric functions [13, 16].

Plugging the classical solution requires the integration (again see [17])

$$\begin{aligned} \int_C |\omega_{k,k}|^2 &= \int_C d^z |z|^{-2k/N} |z - x|^{-2(1-k/N)} |z - 1|^{-2k/N} \\ &= \frac{\pi^2}{\sin(\pi k/N)} I(x, \bar{x}) \end{aligned} \quad (8.49)$$

so that

$$S_{\text{cl}} = \frac{\pi}{4\tau_2 \sin(\pi k/N)} \left[|\tau|^2 |v_1|^2 + |v_2|^2 \tau_1 (v_1 v_2^* e^{-i\pi k/N} + v_1^* v_2 e^{i\pi k/N}) \right] . \quad (8.50)$$

Here $\tau = \tau_1 + i\tau_2 \equiv iF(1-x)/F(x)$, $I(x, \bar{x}) = F(x)\bar{F}(1-\bar{x}) + F(1-x)\bar{F}(\bar{x})$ and $F(x) = F_2(\frac{k}{N}, 1 - \frac{k}{N}; 1; x)$ is hypergeometric function. It is commented that some limit of hypergeometric function goes to Jacobi theta function with the argument τ , and this is related to the modular properties of the target space moduli, (6.120). In the partition function, the sum runs over lattice points, v_1 and v_2 .

Here we use the consistency condition again (although we presented the $k = l$ case above, where the consistency condition was satisfied automatically). Take another contour C_3 as in Fig. 8.3, and consider C_2 and C_3 . In fact, C_3 over-determines the condition, so we survey how does it make the solution space narrower. Then, v_2 belongs to (8.47) and v_3 now belongs to $(1 - \theta^l)(f_4 - f_3 + \Lambda)$ by the similar reasoning discussed for C_1 . We observe that, if we regard the complex plane as a stereographic projection of two-sphere $S^2 \simeq \mathbf{C} \cup \{\infty\}$, we can continuously deform C_1 to C_3 with the opposite orientation (see Fig. 8.4),

$$C_1 = -C_3 . \quad (8.51)$$

Therefore, we have

$$v_1 = -v_3 . \quad (8.52)$$

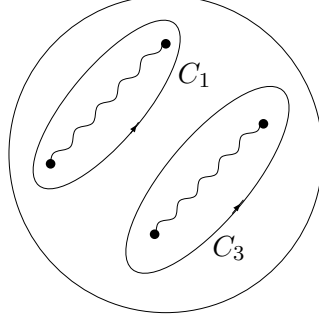


Fig. 8.4. If we compactify the complex plane $\mathbf{C}$ by pulling infinity into a point, it becomes a two-sphere $S^2 \simeq \mathbf{C} \cup \{\infty\}$. On it, we can deform C_1 continuously to C_3 , so that $C_3 = -C_1$

Thus, the calculation with C_2 and C_3 , requiring $pk = ql$ here, agrees with the previous calculation. However, the translation part is now restricted to the *intersection of v_1 and v_3 regions*. Therefore, we restrict the summation region,

$$v \in (1 - \theta^k) \left(f_2 - f_1 - T_1 + \frac{1 - \theta^l}{1 - \theta^{\gcd(k,l)}} \Lambda \right), \quad (8.53)$$

where T_1 is, again, a lattice vector satisfying the space group selection rule,

$$(1 - \theta^k)(f_2 - f_1 - T_1) + (1 - \theta^l)(f_4 - f_3 - T_2) = 0. \quad (8.54)$$

It is easily checked that when $k = l$, the consistency condition is automatically satisfied. An ambiguity occurs when pk/N is an integer, for example $k = 2, l = 3$ in $\mathbf{Z}_6$ orbifold [15], then the condition $pk = ql$ is trivially satisfied. In this case, we modify C_2 such that it encircles z_2 and z_3 in the opposite directions (one clockwise and the other counterclockwise), which gives a non-trivial condition again, rather than the one drawn in Fig. 8.3.

The Quantum Part

Now consider the quantum part. Although we cannot calculate the four point correlation function (8.42) directly, we can calculate the following twisted “Green’s function”,

$$g(z, w; z_i) \equiv \frac{\langle -\frac{1}{2} \partial_z Z \partial_w \bar{Z} \sigma_{+k}(z_1) \sigma_{-k}(z_2) \sigma_{+l}(z_3) \sigma_{-l}(z_4) \rangle}{\mathcal{Z}_{\text{qu}}}. \quad (8.55)$$

Note that the energy-momentum tensor $T(z)$ is the normal-ordered version of the free Lagrangian, having the operator product expansions (OPE),

$$\begin{aligned} -\frac{1}{2} \partial_z Z \partial_w \bar{Z} &\sim \frac{1}{(z - w)^2} + T(z), \\ T(z) \sigma_k(w, \bar{w}) &\sim \frac{h_k \sigma_k(w, \bar{w})}{(z - w)^2} + \frac{\partial_w \sigma_k(w, \bar{w})}{z - w}. \end{aligned} \quad (8.56)$$

To calculate the RHS of (8.55), take the expectation value after multiplying $\sigma_{-k}(z_2)\sigma_{+l}(z_3)\sigma_{-l}(z_4)$ on the second equation of (8.56). Then, comparing it with Green's function (8.55), and tending $w \rightarrow z_2$, we have a differential equation for the quantum part of (8.42),

$$\partial_{z_2} \log \mathcal{Z}_{\text{qu}} = \lim_{z \rightarrow z_2} \left(\frac{z - z_2}{\mathcal{Z}_{\text{qu}}} \langle T(z) \sigma_{-k}(z_1) \sigma_{+k}(z_2) \sigma_{-l}(z_3) \sigma_{+l}(z_4) \rangle - \frac{h_k}{z - z_2} \right). \quad (8.57)$$

Now the problem is translated to calculating $g(z, w)$, thus T , and solve this differential equation. It is possible because the OPE for $\partial_z X \sigma_k$, etc. is known in (8.16). Therefore, one can derive various limits when z goes to w, z_i . For example,

$$\begin{aligned} g(z, w; z_i) &\sim (z - w)^{-2}, & \text{as } z \rightarrow w \\ &\sim (z - z_1)^{-k/N}, & \text{as } z \rightarrow z_1 \\ &\sim (z - z_2)^{-1+k/N}, & \text{as } z \rightarrow z_2 \\ &\vdots \end{aligned} \quad (8.58)$$

From these, we can guess that the function having the desired property is proportional to $\omega_{k,l}(z)$ and $\omega_{N-k,N-l}(w)$ of (8.44). Therefore, we have

$$g(z, w; z_i) = \omega_{k,l}(z) \omega_{N-k,N-l}(w) \left[\frac{P(z, w)}{(z - w)^2} + A(z_i) \right], \quad (8.59)$$

with a bi-polynomial $P(z, w)$ in z and w , and a constant $A(z_i)$. The polynomial is completely determined using the property (8.56) up to constant A , which is again determined by applying the global monodromy condition. Because the classical part in (8.31) took the spatial translation v_i in (8.45), the remaining quantum part has a condition as

$$\int_{C_i} dz \partial Z_{\text{qu}} + \int_{C_i} d\bar{z} \bar{\partial} Z_{\text{qu}} = 0. \quad (8.60)$$

Performing the contour integration as in the calculation of classical part we obtain $g(z, w; z_i)$, and then solving differential equation, we have [13]

$$\mathcal{Z}_{\text{qu}}(x, \bar{x}) = c |x|^{-2k/N(1-k/N)} (1 - x)^{-l/N(1-k/N)} (1 - \bar{x})^{k/N(1-l/N)} \times I(x, \bar{x})^{-1} \quad (8.61)$$

with undetermined constant c , and $I(x, \bar{x})$ is given in (8.49). This is a general result even for $k \neq l$.

Factorization and Normalization

Now we can determine the normalization. The calculation is quite involved [2, 13], but the procedure is evident. When we move $z_2 \rightarrow z_4$, or $x \rightarrow \infty$, the

products of twists at this points $\sigma_{-k/N}(z_2)\sigma_{-k/N}(z_4)$ become doubly twisted function $\sigma_{-2k/N}(z_4)$; therefore the four point correlation function (8.42) factorizes to the sum of three point functions. Again, the three point function is completely determined by standard OPE of conformal field theory as

$$\mathcal{Z}(x, \bar{x}) \sim |x|^{-h} |z_\infty|^{-h'} \sum_f Y_{f_2, f_4, f}^k (Y_{f_1, f_3, f}^k)^* \quad (8.62)$$

where the sum is over the fixed points, and h, h' are some combinations of conformal weights. It is interpreted as the sum of the products of Yukawa couplings, because they are

$$Y_{f_2, f_4, f}^k = \lim_{|x| \rightarrow \infty} |x|^{k/N(1-k/N)} \langle \sigma_{k, f_2}(x, \bar{x}) \sigma_{l, f_4}(1, 1) \sigma_{-k-l, f}(0, 0) \rangle \quad (8.63)$$

and we can compare it with the known result on the three point function.

When we move z_2 to z_1 , or $x \rightarrow 0$, the twists $\sigma_{k/N}(z_2)\sigma_{-k/N}(z_1)$ neutralizes and is proportional to identity; the entire correlation function becomes essentially two point correlation function, or Green's function. Comparing it with the known two point correlation function, we can obtain the absolute normalization of (8.61),

$$c = 2\pi \sqrt{\det G_{ij}} \equiv 2\pi A_\Lambda \quad (8.64)$$

where A_Λ is the area of two-torus.

Multiplying all the two-torus contributions, the final result turns out to be

$$Y_{f_a, f_b, f_c}^{k, l} = \left[\prod_{j=1}^{d/2} 2\pi A_{\Lambda, j} \Gamma_{k_j, l_j} \right]^{1/2} \sum_v \exp(-\pi v^\dagger \cdot M v) \quad (8.65)$$

where we view $v = (v_1, \dots, v_{d/2})$ as a complex vector. The summation range is restricted as in (8.41),

$$v \in \tau_1 - \tau_2 + \frac{1 - \theta^{k+l}}{1 - \theta^{\gcd(k, l)}} \Lambda, \quad (8.66)$$

and

$$\Gamma_{k, l} = \frac{\Gamma(1 - k/N) \Gamma(1 - l/N) \Gamma((k + l)/N)}{\Gamma(k/N) \Gamma(l/N) \Gamma(1 - (k + l)/N)} \quad (8.67)$$

where we make k and l lie in $0 < k/N, l/N < 1$ by subtracting integers.

The above result can be extended, including antisymmetric tensor background [15, 18]. When we include contributions from antisymmetric tensor field as in (6.105) and Wilson lines, it leads to a modification

$$|v|^2 \rightarrow \frac{T + A \cdot \bar{A}}{\text{Re} U} (|v|^2 - 2 \text{Re} v \text{Im} v \text{Im} U). \quad (8.68)$$

Neglecting Wilson lines $A = 0$ and letting U modulus $U = 1$, it amounts to a simple modification; $|v|^2$ becomes $T|v|^2$. It is because T modulus (6.116) unifies the metric and the antisymmetric tensor field which together describes the volume. Therefore, we expect that the Yukawa coupling has the target space modular transformation property (6.120) [14, 15]. As explained, the U modulus gets a fixed VEV if the orbifold action is not compatible with the rotation by π .

8.1.5 Yukawa Couplings from $\mathbf{Z}_3$ Orbifold

From now we will deal with specific examples.

Selection Rules

The selection rules are so strong that many Yukawa couplings are forbidden. Let us discuss further for the case of $\mathbf{Z}_3$ [19].

Consider first the space group selection rule. By definition, only the twisted sector fields are subjected, because they transform as (8.20) under space group action. For the case at hand we have only one kind, the first twisted sector field T transforming as θ with $\theta^3 = 1$. Note that we do not over-count “antiparticle” states transforming as θ^2 . Representing untwisted fields as U , the following terms are not allowed,

$$T, UT, UUT, TT, UTT \quad (\text{not allowed}).$$

The Lorentz invariance rule is stronger, since it forbids all the above interactions and additionally prohibits

$$U, UU \quad (\text{not allowed}).$$

Therefore, in particular, there is no way to form invariant mass term. We see that the mass can only be generated by radiative correction, which is suppressed by string unification scale. Thus we expect a solution of the μ -problem since UU and TT terms for the Higgs masses are forbidden (See Subsect. 10.6.1.). The superpotential should be invariant under the sublattice action. The only allowed dimension 3 superpotentials are

$$UUU, TTT. \quad (8.69)$$

Recall that we have 27 fixed points and hence as many T fields. According to the Lorentz invariance rule, in each sublattice the term should be invariant. In terms of fixed point, the rule translates into the following. Denoting the

different fixed points of Fig. 6.5 as $\bullet$, $\circ$ and $\times$, we have the following kinds of combinations only for the couplings,

$$\bullet \circ \times, \bullet \bullet \bullet, \times \times \times, \circ \circ \circ \quad (8.70)$$

and some permutations in each two-torus. In other words, if we choose two fixed points, the third fixed point is determined. Thus the number of such combinations are $27 \times 27 = 729$, but we will see that the actual possibility reduces.

Also higher dimensional interactions are strongly constrained by the Lorentz invariance rule. Consider for simplicity the fields without oscillators, in which case the Lorentz transformation property is just the H -momentum condition. The untwisted fields U_1, U_2, U_3 have $(-1, 0, 0), (0, 1, 0), (0, 0, 1)$, respectively and the twisted fields have $\frac{1}{3}(-1, 1, 1)$. For the coupling $U_1^k U_2^l U_3^m T^n$, the similar conservation condition (8.29) applies,

$$\left(-k - \frac{n}{3}, l + \frac{n}{3}, m + \frac{n}{3}\right) \equiv (-1, 1, 1) \pmod{(3, 3, 3)}. \quad (8.71)$$

We can easily check that the solutions are only of the following two types

$$U_1^{3p+1} U_2^{3q+1} U_3^{3r+1} T^{9n}, \quad U_1^{3p} U_2^{3q} U_3^{3r} T^{9n+3}, \quad (8.72)$$

where p, q, r, s are nonnegative integers and every combination of three T s satisfies the above point group selection rule (8.70). The importance of non-renormalizable operators depends on the the singlet VEVs.

The Lorentz invariance rule simply implies the local symmetry under both the gauge group and the spacetime Lorentz group. Also, fields with excitation are constrained. In the standard embedding, $(\mathbf{1}, \mathbf{27})$ contains no derivative, but $(\bar{\mathbf{3}}, \mathbf{1})$ does due to the oscillator mode $\alpha_{-1/3}^i$. Therefore the $(\bar{\mathbf{3}}, \mathbf{1})^n$ vanishes unless n is a multiple of 3. We can make the interaction of the type $(\partial Z^1)^a (\partial Z^2)^b (\partial Z^3)^c$ survive when $a + b + c = 0 \pmod{3}$ by multiplying suitable moduli fields $M_{i\bar{j}}$.

Mass Hierarchy, Area Rule

Consider a untwisted sector coupling UUU (or a twisted sector coupling TTT all sitting at the same fixed points). If we use one of them as Higgs field and give a VEV, then the remaining two fields have the degenerate mass. Due to this fact, it is in general hard to explain the quark mass hierarchy with this simple model of U matter only [21].

For the twisted sector fields TTT located at the different fixed points we have a very fruitful interpretation of the quark mass hierarchy in terms of geometry. The Yukawa coupling for TTT type is totally determined by the given twist and the shape of lattice. For the $\mathbf{Z}_3$ orbifold, we have $k = l = 1$, thus take $p = q = 2, r = 1$ in (8.39). Inserting them into (8.38), we obtain

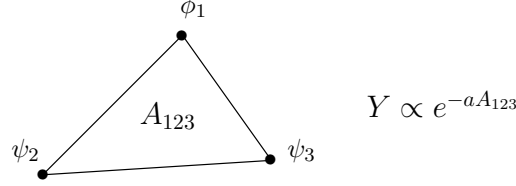


Fig. 8.5. The area rule. The Yukawa coupling is proportional to the exponential of the minus area of triangle formed by three fields

$$S_{\text{cl}}^i = \frac{|v|^2}{2\sqrt{3}}. \quad (8.73)$$

Therefore,

$$\mathcal{Z}_{\text{cl}} \sim \sum_v \exp \left(-\frac{|v_i|^2}{2\sqrt{3}\pi} \right). \quad (8.74)$$

where $v \in (1 - \theta)(f_1 - f_2 + \Lambda)$. The fixed point dependence is given only by $f_1 - f_2$. The result obtained so far is concentrated on the two dimensional hypersurface connecting just two fixed points. This exponentially suppressed behavior e^{-av^2} is understood as instanton effects in the language of sigma model, for there the coupling is proportional to v . Since the exponent in the suppression factor is proportional to the distance squared between two points, the whole interaction among three points is proportional to the *area of the triangle* they form, which is the famous area rule for Yukawa couplings of the type *TTT*.

We can generalize the result to the case of six dimensional orbifold. If we rewrite $v \equiv u + f_1 - f_2$ in the lattice unit basis (then now the components are real), we can introduce a coefficient matrix M in (8.75), so that

$$Y = gN \sum_{u \in \mathbf{Z}^6} \exp \left[-\frac{1}{2\sqrt{3}\pi} (f_1 - f_2 + u)^\top M (f_1 - f_2 + u) \right] \quad (8.75)$$

where the full numerical coefficient is given in (8.65),

$$N = V_\Lambda^{1/2} \frac{3^{3/4} \Gamma(2/3)}{\Gamma^3(1/3)}.$$

In this case, the space group selection rule (8.22) constrains the fixed points to be all at the same point or all at the different points

$$f_1 = f_2 = f_3, \quad \text{or} \quad f_1 \neq f_2, f_2 \neq f_3, f_3 \neq f_1,$$

up to a lattice translation. Among 729 possible couplings, only the following 8 combinations $f_1 - f_2$ give different size,

$$\begin{aligned}
& (0, 0, 0, 0, 0, 0) , \\
& \left(\frac{1}{3}, \frac{2}{3}, 0, 0, 0, 0 \right), \left(0, 0, \frac{1}{3}, \frac{2}{3}, 0, 0 \right), \left(0, 0, 0, 0, \frac{1}{3}, \frac{2}{3} \right), \\
& \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 0, 0 \right), \left(\frac{1}{3}, \frac{2}{3}, 0, 0, \frac{1}{3}, \frac{2}{3} \right), \left(0, 0, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right), \\
& \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right),
\end{aligned} \tag{8.76}$$

depending on how the three fields are distributed.

The geometry is determined by the gravitational moduli G_{ij} having 21 components, viz. Sect. 6.4.3. Among them some are redundant; for the above $\mathbf{Z}_3$ example, we have the $\theta e_4 = e_5$; thus the compactifying radii along these directions are the same $R_4 = R_5$ and the angle between them is $\phi_{45} = 2\pi/3$. It turns out that only 9 parameters are independent which are parametrized by three radii and six relative angles between unit vectors

$$R_4, R_6, R_8, \phi_{46}, \phi_{48}, \phi_{68}, \phi_{47}, \phi_{49}, \phi_{69} . \tag{8.77}$$

The simplest case is to set the six angles to be zero, in which case the matrix M in (8.75) can take the simple form

$$M = \begin{pmatrix} T_4 & -\frac{1}{2}T_4 & & & & \\ -\frac{1}{2}T_4 & T_4 & & & & \\ & & T_6 & -\frac{1}{2}T_6 & & \\ & & -\frac{1}{2}T_6 & T_6 & & \\ & & & & T_8 & -\frac{1}{2}T_8 \\ & & & & -\frac{1}{2}T_8 & T_8 \end{pmatrix} \tag{8.78}$$

with other entries being zero. T_i s given in (8.78) measure the length of unit lattice $\text{Re}T_i = \sqrt{3}R_i^2$ associated with Λ . We assumed that each two torus is orthogonal to others; thus the matrix is block diagonal.

We can expand the exponential form (8.75) in powers of $\lambda_i = 3e^{-2\pi T_i/3}$, which explains the exponential suppression along the distance between two fields. In the simplest case, setting all T_i s equal to T we have a mass hierarchy $1, \lambda, \lambda^2$, and λ^3 [22]. As a bottom-up approach, we can try to put some fields on some specific fixed points to explain the observed mass hierarchy. However, from the top-down approach, namely beginning with $E_8 \times E_8$ theory down to the SM, it is very hard just using renormalizable couplings to compromise with phenomenological needs. The Higgs field is located at the fixed point f_1 and M is a 6×6 matrix, taking into account the VEV and couplings. The appearance of $f_1 - f_2$ accounts the fact that the Yukawa couplings are dependent on two fixed points by (8.70).

Table 8.2. Renormalizable vertices in the $\mathbf{Z}_N$ orbifold. θ^n and U_i denotes the number of twists and the number of (spacetime Lorentz symmetry) $\text{SO}(10)$ vector components, respectively

Orbifold	Pure twisted	Mixed
$\mathbf{Z}_3$	$\theta\theta\theta$	
$\mathbf{Z}_4$	$\theta\theta\theta^2$	$\theta^2\theta^2U_3$
$\mathbf{Z}_6\text{-I}$	$\theta\theta^2\theta^3, \theta^2\theta^2\theta^2$	$\theta^3\theta^3U_3$
$\mathbf{Z}_6\text{-II}$	$\theta\theta^2\theta^3, \theta\theta\theta^4$	$\theta^2\theta^4U_3, \theta^3\theta^3U_2$
$\mathbf{Z}_7$	$\theta\theta^2\theta^4$	
$\mathbf{Z}_8\text{-I}$	$\theta\theta^2\theta^5, \theta^2\theta^2\theta^4$	$\theta^4\theta^4U_2$
$\mathbf{Z}_8\text{-II}$	$\theta\theta^3\theta^4, \theta\theta\theta^6$	$\theta^2\theta^6U_3, \theta^4\theta^4U_3$
$\mathbf{Z}_{12}\text{-I}$	$\theta\theta^2\theta^9, \theta\theta^4\theta^7$	$\theta^3\theta^9U_2, \theta^6\theta^6U_2$
	$\theta^2\theta^3\theta^7, \theta^2\theta^4\theta^6$	
	$\theta^4\theta^4\theta^4$	
$\mathbf{Z}_{12}\text{-II}$	$\theta\theta^3\theta^8, \theta\theta\theta^{10}$	$\theta^2\theta^{10}U_3, \theta^4\theta^8U_3$
	$\theta^3\theta^3\theta^6, \theta^2\theta^5\theta^5$	$\theta^6\theta^6U_3$

One may expect the target space modular invariance (6.120), and indeed (8.75) can be expressed by the Jacobi theta function,

$$Y = g_s N \vartheta \begin{bmatrix} f_1 - f_2 \\ 0 \end{bmatrix} (0, M) \quad (8.79)$$

which has the manifest target space modular invariance.

Renormalizable Couplings in $\mathbf{Z}_N$ Orbifolds

In Table 8.2, we have listed possible renormalizable couplings arising from general $\mathbf{Z}_N$ orbifolds, satisfying the selection rules given above [23]. There, θ^n means a twisted field belonging to the n th twisted sector fields and U_i means the untwisted fields with a nonzero spacetime Lorentz $\text{SO}(10)$ vector component in the i th direction. Not only the purely twisted ones, but also the mixed ones are restricted by the Lorentz group selection rules, from the H -momentum conservation. For the example of $\mathbf{Z}_6\text{-I}$ orbifold, $\theta\theta\theta^4$ coupling is forbidden by the Lorentz group selection rule, although it is allowed by the space group rule.

The strength of coupling is determined by (8.32) and (8.38). The universal normalization is given in (8.65). The case study of a complete $\mathbf{Z}_N$ Yukawa coupling is done and tabulated in [20]. However, we should be careful on the summation range of the lattice vector, restricted by (8.41) [15], as we have seen before.

As a result, in more general orbifolds, properties of Yukawa couplings can be quite different from those of the simple $\mathbf{Z}_3$ orbifold case discussed above. For example, in the $\mathbf{Z}_4$ orbifold the space group selection rule does not forbid “mixed” TTU type couplings, as long as both twisted fields belong to the

second twisted sector. However, the main difference is that, in the general $\mathbf{Z}_N$ orbifold, the renormalizable couplings would have *off-diagonal* components [20]. In the $\mathbf{Z}_3$ case, the space group selection rule completely determines the third fixed point given two fixed points. This $\mathbf{Z}_3$ case would imply the mass degeneracy of all the fields arising from the Higgs mechanism, and no $\mathcal{CP}$ phase. Still in some cases, for example $\theta^2\theta^2\theta^2$ coupling in $\mathbf{Z}_6$ -I orbifold, also the third fixed point is determined by the space group selection rule. However, $\theta\theta^2\theta^3$ can have off-diagonal components.

8.1.6 Toward Realistic Yukawa Couplings

A Texture

The most famous model for the Yukawa couplings is provided by the Weinberg–Fritzsch ansatz [24]

$$M = \begin{pmatrix} 0 & A & 0 \\ A & 0 & B \\ 0 & B & C \end{pmatrix}, \quad |A| \ll |B| \ll |C|, \quad (8.80)$$

which naturally explains the Cabbibo angle from the quark mass hierarchy

$$\sin \theta_C \sim \sqrt{m_d/m_s}. \quad (8.81)$$

In the following, we try to show that this form is impossible in string models [25].

In the prime order orbifolds, we can prove it easily. We note that the KM mixing can arise from the off-diagonal term(s) of mass matrix in three complex dimensions. It is noted that in the $\mathbf{Z}_3$ and $\mathbf{Z}_7$ orbifolds, i.e. in the prime orbifolds, there do not appear off-diagonal terms among renormalizable couplings. The following explanation is called as “box closing rule” [25]. If we have a subblock filled except one entry, it should be always completely filled,

$$\begin{bmatrix} \times & \\ \times & \times \end{bmatrix} \rightarrow \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}. \quad (8.82)$$

This is because if the three entries on LHS should satisfy the space group selection rule in the original matrix, then it can be shown that the blank position (1, 1) also satisfies the rule, thus we should fill the blank position. This rule shows that at the renormalizable level, we cannot obtain the matrix of the Weinberg–Fritzsch type (8.80). However, we may use the fact that off-diagonal terms can appear from nonrenormalizable couplings.

Fortunately, we have an alternative version [19, 25] naturally emerging from string theory and it also explains angles in terms of mass ratios

$$M = \begin{pmatrix} \epsilon & a & b \\ \tilde{a} & A & c \\ \tilde{b} & \tilde{c} & B \end{pmatrix}, \quad (8.83)$$

while those with lower case letters are very small compared to $|A| \ll |B|$, to explain large masses for the third family members. To take this form, only A, B should emerge from renormalizable operators while others from nonrenormalizable ones for them to get suppressed. This also explains the Cabbibo angle relation (8.81) and qualitatively gives natural ratios of fermion mass scales.

Non-prime orbifolds allow off-diagonal terms even in the renormalizable couplings. In Table 8.2, we see that there can be more than one coupling.

To conclude, the nonrenormalizable couplings are very crucial ingredients for having realistic Yukawa interactions as shown in [11, 26, 27].

Constructing Flat Direction

Although renormalizable operators in odd order orbifolds are diagonal, we can form off-diagonal entries using nonrenormalizable operators. If some mechanism breaks the gauge symmetry and the SM nonsinglet fields develop VEVs, then we can fit nonrenormalizable off-diagonal entries and fit the mass matrix toward realistic couplings [11, 26, 27]. Of course, this idea can also be applied to nonprime orbifolds. This method is very useful in the top-down approach. Accompanying this method, there possibly arises the Fayet–Iliopoulos term generated by the anomalous $U(1)$, breaking several unwanted $U(1)$ s.

8.2 Low Energy Action

In the previous section, we calculated correlation functions directly with string theory and matched the corresponding low energy action. In this section, we will take a different route. The ten dimensional *supersymmetry* is restrictive enough to completely determine all the possible terms [28, 29]. Also if coupled, from the condition of anomaly cancelation the gauge group is determined as $SO(32)$ or $E_8 \times E'_8$ [30]. Therefore, to acquire the low energy limit of superstring ($\alpha' \rightarrow 0$) it suffices to match the supermultiplets. The worldsheet scale invariance and the vanishing beta function convert to a set of equations. This turns out to be 10D supergravity in string frame. For our purpose, we just note the redefinition of fields.

8.2.1 Dimensional Reduction

The 10D supergravity Lagrangian coupled to Yang–Mills fields is completely fixed by local supersymmetry. The bosonic part is

$$e_{(10)}^{-1} \mathcal{L} = -\frac{1}{2\kappa} R_{(10)} + \frac{9}{16} \phi^{-2} \partial_\mu \phi \partial^\mu \phi + \frac{3}{4} \phi^{3/2} H_{MNP}^2 - \frac{1}{4} \phi^{3/4} \text{Tr} F_{MN}^2 \quad (8.84)$$

where subscript (10) indicates a 10D object. We have dilaton ϕ , and the field strength of the antisymmetric tensor field B_{MN} with the Chern–Simons form

$$H_{MNP} = \partial_{[M} B_{NP]} - \frac{\kappa}{\sqrt{2}} \text{Tr} \left(A_{[M} F_{NP]} - \frac{2}{3} A_{[M} A_N A_{P]} \right), \quad (8.85)$$

where κ is the 10D gravitational constant and the trace is normalized in units of the vector of $\text{SO}(16)$. As a fully consistent theory, anomaly cancellation requires an additional Chern–Simons type coupling of curvature tensor similar to the last term, but it is not required by supersymmetry and does not affect low energy theory, so we neglect it here. The field strength of the gauge field A_M is,

$$F_{MN} = \partial_{[M} A_{N]} + [A_M, A_N]. \quad (8.86)$$

This provides the low energy limit of Type-I and heterotic string theories, with a suitable field redefinition and normalization.

How about the four dimensional theory? In the early days, for the simple toroidal compactification of extra six dimensions a four dimensional Lagrangian was obtained simply by truncating the unwanted fields [31, 32]. After the advent of string theory, orbifolds and Calabi–Yau manifolds are used to compactify extra dimensions, and here only the space group invariant states are kept in the low energy theory and the resulting couplings are matched with the four dimensional Lagrangian [3, 4]. This is called *inheritance principle*, stating that every interaction from untwisted sector is same as that of toroidal compactification. However we should take into account twisted sector contribution by considering symmetries they obey.

Let us take the following compactification ansatz

$$G_{\mu\nu}^{(10)} = e^{-3\sigma} G_{\mu\nu}, \quad G_{a\mu}^{(10)} = G_{\mu a}^{(10)} = 0, \quad (8.87)$$

and we take three T^2 sublattices being isotropic,

$$G_{a\bar{b}}^{(10)} = \delta_{a\bar{b}} e^\sigma, \quad B_{a\bar{b}} = \epsilon_{a\bar{b}} \eta \quad (8.88)$$

where $\epsilon_{a\bar{b}} \rightarrow \epsilon_{abc}$ is the $\text{SU}(3)$ invariant tensor. We will see that the compact spacetime has a large $\text{SU}(3)$ holonomy. Evidently e^σ determines the volume of manifold. Since η has the translation symmetry of axion (6.117), it is identified *model-dependent axion*. In general compactification the number of η -like fields are model dependent. The resulting four dimensional Lagrangian has the form

$$\begin{aligned} e^{-1} \mathcal{L} = & -\frac{1}{2} R + 3 \partial_\mu \sigma \partial^\mu \sigma + \frac{9}{16} \frac{\partial_\mu \phi \partial^\mu \phi}{\phi^2} \\ & - \frac{1}{4} \phi^{3/4} e^{3\sigma} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \theta \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \\ & + \frac{3}{2} e^{-2\sigma} \phi^{3/2} \left(\partial_\mu \eta - \frac{1}{\sqrt{2}} i Q_I^* \overleftrightarrow{D}_\mu Q^I \right)^2 + \frac{1}{4} \phi^{3/2} e^{-6\sigma} (\partial_\mu \theta)^2 \\ & + 3 e^{-\sigma} \phi^{-3/4} D_\mu Q_I^* D^\mu Q^I \\ & - 8 \kappa^2 g^2 \phi^{-3/2} e^{-6\sigma} |W|^2, \end{aligned} \quad (8.89)$$

with the superpotential

$$W = 8\sqrt{2}gd_{IJK}Q^IQ^JQ^K . \quad (8.90)$$

Here, we do not present the tedious calculation but we can infer the origins for the terms as follows. By the index structure, we see that the term $F_{\mu a}^2$ contains the kinetic term for Q_I , where the group index I is correlated to the holonomy index a . The $H_{\mu ab}^2$ gives rise to the quadratic term and also contains two Q_I s with derivatives. In H_{abc}^2 , the Chern–Simons term contains commutators between three A_i s; thus gives rise to the Yukawa type coupling $d_{IJK}Q_IQ_JQ_K$. F_{ab}^2 contains $d_{IJK}Q^JQ^K$ and $Q_I^*Q^I$ terms. We easily see that the last two terms are derived from the superpotential. We make the dual transformation

$$H_{\mu\nu\rho} \propto \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \theta . \quad (8.91)$$

The pseudoscalar θ is called the model-independent axion since it is always present in string theory.

Equation (8.89) looked rather messy, and we hope that it would match with the standard form of four dimensional supergravity Lagrangian. The following reparametrization will make things look better. We can instantly read off the coefficient of terms involving $F_{\mu\nu}$ to have a complex modulus

$$S = e^{3\sigma} \phi^{-3/4} + 3\sqrt{2}i\theta . \quad (8.92)$$

Therefore, the gauge kinetic function is $f_{AB} = \delta_{AB}S$. Also employ the modulus introduced in (6.116),

$$T = e^\sigma \phi^{3/4} + \sqrt{2}i\eta + Q_I^*Q^I \quad (8.93)$$

where S, T, Q^I are complex chiral fields. Therefore,

$$K = -\ln(S + S^*) - 3\ln(T + T^* - 2|Q^I|^2) . \quad (8.94)$$

gives the desired Lagrangian.

Already this simple dimensional reduction reveals so much, but there need more improvements. First, At best this describes the untwisted sector fields, since we have truncated only the bulk spectrum. As we did in the previous section, however, we can calculate string diagrams to include the nonperturbative results. Second, For the same reason, this contains perturbative effects only (in the field theory point of view). The result of the superpotential on the twisted sector fields, comes from the nonperturbative effects, interpreted via instantons. Also, The matching we used is in the zeroth order of α' of the curvature tensor R . For example, the simplest Green–Schwarz mechanism requires a modification of order α' so that the supergravity includes higher order terms of R^2 , etc, which becomes highly nonlinear.

Twisted Fields and Blowing Up

More fields are left massless, i.e. the twisted sector fields. Due to their purely stringy origin, their contributions are not caught by supergravity. Again, the $\mathbf{Z}_3$ standard embedding is the simplest example [1]. We observed in the previous section that they have the trilinear superpotential $(\bar{\mathbf{3}}, \mathbf{1})^3$ and $(\mathbf{1}, \mathbf{27})^3$ with the coefficient given in (8.75).

Let us concentrate on $(\bar{\mathbf{3}}, \mathbf{1})$ field and name it as $M_b^{\bar{a}}$ where the group index of $SU(3)$ is $\bar{a}$. Since it is charged under $SU(3)$ there appears a D -term potential with

$$D^A \propto M_b^{\bar{a}*} t_{bc}^A M_c^{\bar{a}} = \text{Tr}(M^\dagger t^A M) . \quad (8.95)$$

Using the completeness of generators t^A , it can always be diagonalized and assume VEVs as

$$\langle M_b^{\bar{a}} \rangle = \rho_f \delta_b^{\bar{a}} , \quad (8.96)$$

at each fixed point labeled by f . At this vacuum $SU(3)$ is completely broken⁴ but supersymmetry is preserved (provided the D -term potential vanish; we can easily check that F -terms do vanish). We interpret ρ_f as the radius of “blown-up sphere”, by which the singularity of the fixed point is resolved to be smooth. It is the only massless modulus field and other two fields become massive, since $(D^A)^2$ contains only ρ_f . In Appendix A, we will see that this blowing up operation transits an orbifold to a Calabi–Yau manifold. There the role of $(\bar{\mathbf{3}}, \mathbf{1})$ is naturally expected, because the $SU(3)$ gauge group is identified with the holonomy group.

8.2.2 General Dependence on Moduli Fields

Kähler Potential

The strategy for calculating general Kähler form is to make Taylor expansions in terms of matter fields Q^I ,

$$K = -\log(S + S^*) + K^M(T, T^*, U, U^*) + K^Q(T, T^*, U, U^*)QQ^* \\ + Z(T, T^*, U, U^*)(QQ + Q^*Q^*) + \dots \quad (8.97)$$

The explicit calculation is possible for the standard-embedding orbifolds [5]. The tree level S dependence is obtained as we have done before, (8.94), and in fact it is model independent, since it is completely determined by the classical symmetry of string theory.

Of course, the coefficients are determined case by case. For the simplest case of factorizable torus $T^6 = (T^2)^3$, the Kähler potential is calculated to be

$$K = -\log(S + S^*) - \sum_a \log(T_a + T_a^*) + \sum_I |Q_i|^2 \prod_a (T_a + T_a^*)^{n_a^I} \quad (8.98)$$

⁴ If there were only one fixed point, $SU(3)$ breaks down to $SU(2)$. But in general there are more than one fixed point and $SU(3)$ is completely broken.

where the fractional number n_a^I is known as modular weight of the matter field Q^I , which is the power appearing in the modular transformation

$$Q^I \rightarrow (ic_a T_a + d_a)^{n_a^I} Q^I . \quad (8.99)$$

(A field possessing such transformation behavior are called modular form.) There are other possible contributions from the U moduli in the $\mathbf{Z}_2$ orbifold, which can be inferred by the mirror symmetry.

Superpotential

The superpotential have the form

$$W = \chi_{IJK}(T) Q^I Q^J Q^K + \dots \quad (8.100)$$

where the ellipsis denote higher order couplings of Q^I , which are exponentially suppressed. We have already calculated this in the first section. By the symmetry of classical equation, especially the axionic symmetry of T field in (8.94), we can show that it receives no perturbative corrections for T to all orders of sigma model expansion. Because it does not depend on S modulus which appears in the string coupling as $\text{Re } S = 1/g_s$, there is no correction from string perturbation. This is the stringy version of the renowned non-renormalization theorem [33]. There is no self coupling of moduli S, T, U ; thus they remain in flat directions, which explains the name “*moduli fields*”. The only correction, which becomes the holomorphic coefficient $\chi_{IJK}(T)$, comes from nonperturbative corrections, and it is exponentially suppressed $\sim e^{-T}$, but still obeys modular symmetry (6.120). The superpotential W should transform as a modular form of a given weight. Since $K \rightarrow K + \log |c_i T_i + d_i|^2$ should be Kähler by supergravity,

$$W \rightarrow W(c_i T_i + d_i)^{-1} \quad (8.101)$$

for each i th complex plane. The overall modular weight for superpotential should be -3 for the factorizable tori case. There is no perturbative correction. We can take into account the mirror symmetry $T \leftrightarrow U$ if there is a U field present.

Gauge Kinetic Function

We also obtained the tree level form of gauge kinetic function.

$$f_{ab}(S, T, U, Q^I) = S \delta_{ab} , \quad (8.102)$$

with S given in (8.92). Also, the same nonperturbative nonrenormalization theorem is applied and there is, at most, only one-loop correction present, because of the holomorphicity of f_{ab} . The total dependence has the form [5],

$$f_a = k_a S - \sum_i \frac{\alpha_a^i}{4\pi^2} \log \eta(iT^i) + \text{const} , \quad (8.103)$$

where k_a is the level of algebra. Dedekind eta function (5.83) always enters in the modular invariant function. In addition to the beta function coefficient, we have more, as follows

$$\alpha_a^i = \sum_I T_a(Q^I)(1 - 2n_I^i) - C(G_a) , \quad (8.104)$$

The full Kähler potential has the form [34]

$$K_{1\text{-loop}} = -\log \left(S + S^* + \frac{1}{4\pi^2} \sum_{i=1}^3 \delta_i^{\text{GS}} \log(T_i + T_i^*) \right) - \sum_{i=1}^3 \log(T + T^*) \quad (8.105)$$

for toroidal compactification. Here δ^{GS} is possible contribution from anomalous U(1).

8.2.3 Threshold Correction

For estimating gauge coupling constants [35], we assume that the masses of the fields taking part in the running are the same as the unification scale M_U . By the decoupling theorem [36], we can neglect massive fields heavier than the running scale. However, around the unification scale the effects of mass differences become significant because the heavy masses themselves are of order of the unification scale M_U . We should take into account this effect which is known as the *threshold correction*. In this section, we briefly consider the threshold correction due to string [37, 38]. The running gauge coupling is

$$\frac{16\pi^2}{g_a^2(\mu)} = S k_a \frac{16\pi^2}{g_4^2} + b_a \log \frac{M_s^2}{\mu^2} + \Delta_a \quad (8.106)$$

where g_4 is the 4D unification coupling at the string unification scale M_s^2 above which extra dimensions beyond 4D appear.⁵

We calculate the stringy threshold correction Δ_a by string one loop amplitude we considered in Chap. 5, (5.77),

$$\Delta_a = \int \frac{d^2\tau}{\tau_2} [\mathcal{B}_a(\tau, \bar{\tau}) - b_a] , \quad (8.107)$$

integrated over the $\text{SL}(2, \mathbf{Z})$ fundamental region. The modular invariant amplitude $\mathcal{B}_a$ is just a partition function (7.6) weighted by the squared charge generator Q_a^2 ,

⁵ In fact, there is an extra contribution of a form $16\pi^2 k_a Y$ which can be absorbed into the first term.

$$\mathcal{B}_a(\tau, \bar{\tau}) = \sum_{h \in \bar{\mathbb{P}}} \left[\frac{1}{N} \sum_{g \in \bar{\mathbb{P}}} \text{Tr}(Q_a^2 g q^{\bar{L}_0(h)} \bar{q}^{L_0(h)}) \right]. \quad (8.108)$$

It is analogous to the field theory case where b_a is given by the trace of the squared charge generators over the charged fields leading to $-\frac{11}{3}\text{Tr}Q_a^2$. In the field theory limit, note that $\tau_2 \rightarrow \infty$, and $\mathcal{B}_a(\tau, \bar{\tau})$ reduces to b_a , to match the low energy running (8.106).

In the orbifold theory, we can calculate Δ_a explicitly. The important result obtained in [38] is that the only nontrivial threshold correction emerges *in the twisted sector having fixed tori*. This is so because the threshold correction is concentrated at the orbifold fixed points. For instance, consider the second twisted sector of the $\mathbf{Z}_4$ orbifold of Subsect. 7.2.1. Here, we have a twist $2\phi = \frac{1}{2}(2 \ 1 \ 1) \simeq \frac{1}{2}(0 \ 1 \ 1)$. The first two-torus remain untouched. The moduli can freely move in this direction, thus has the corresponding coordinate dependence. As a corollary, the $\mathbf{Z}_3$ orbifold having no fixed torus receives no threshold correction except for a trivial constant correction of order 5%.

The result is simply,

$$\Delta_a = - \sum_{\text{moduli}} \sum_i b_a^i \frac{N'}{N} \log [(T_i + T_i^*) |\eta(T_i)|^4] + c_a, \quad (8.109)$$

where c_a is a constant term independent of the moduli, N' is the order of the subsector action (2 for the second twisted sector of the above $\mathbf{Z}_4$ example), and η is the Dedekind eta function (5.83), a regular customer in the modular form. All the moduli fields have the same dependence [38].

The most important issue is whether this threshold correction is large enough to fill the gap between grand unification scale and string scale (or the Planck scale). In a large T limit, from the asymptotic behavior of eta function, it behaves as

$$\Delta_a \sim \sum_i b_a^i \frac{N'}{N} \frac{\pi(T + T^*)}{6}. \quad (8.110)$$

However, it seems that the threshold correction alone is not enough to make the three couplings meet at the string scale. But, we note that in field theoretic calculation of the running we must take into account the above string threshold corrections which are present near the string scale.

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String Orbifold Spectra

One of the most important issues in string theory is to understand the structure of group and the pattern of symmetry breaking. In this chapter, we discuss how the orbifolding of the heterotic string renders a low energy group. In fact, we can see the underlying symmetries of the spectrum just from the theory of Lie algebra and its generalization called the *affine Lie algebra*.

First, we discuss the automorphism and generalize it to break symmetry. It can reduce the rank and provide a basis on asymmetric orbifolds. Next, we discuss the structure of shift vectors. Analysis based on Lie algebra will reveal symmetry breaking patterns so that the classification become possible. We will generalize this idea to the case with Wilson lines.

We will also discuss the twisted sector spectrum in orbifold theory. With the aid of affine Lie algebra, we can analyze the massive and twisted sector spectra. It turns out that an isomorphic algebra describes the twisted sector spectrum. Here, we will restrict the discussion to self-dual lattices only since they are all the needed for the study of E_8 and $SO(32)$ appearing in heterotic string. A generalization will be straightforward. For a general discussion on group properties, there exists good references [1].

Basic Tools

In Lie algebra, the first notion to introduce is *simple roots* α^i . The root vectors are formed by linear combinations of simple roots. The highest root θ is uniquely determined by

$$\theta = \sum_{i=1}^r a_i \alpha^i \quad (9.1)$$

where r is the rank of the algebra. The positive integer a_i , called the *Coxeter label* is assigned to every simple root α^i . It is useful to define $a_0 = 1$. For the E_8 case, the Coxeter labels are shown inside circles in Fig. 9.1. The sum

$$g \equiv 1 + \sum_{i=1}^r a_i \quad (9.2)$$

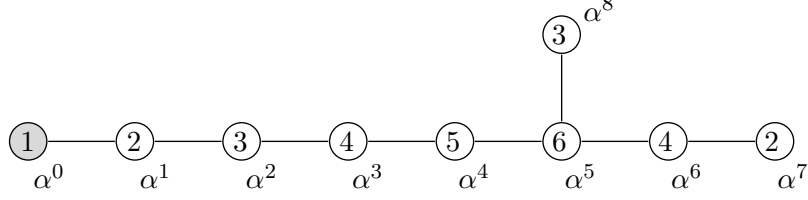


Fig. 9.1. The extended Dynkin diagram of E_8 . The Coxeter labels are shown inside circles, and the extended root is grey colored

is called the *Coxeter number* and coincides with the quadratic Casimir

$$2g\delta^{ab} = f^{acd}f^{bcd} . \quad (9.3)$$

For E_8 , we have $g = 30$.

Let us define the fundamental weights Λ_j such that

$$\alpha^i \cdot \Lambda_j = \delta_j^i . \quad (9.4)$$

In this way, the dual vector space of the root space is defined.¹ They provide another complete basis system $\{\Lambda_i\}$ which is called the *Dynkin basis*. A vector in the Dynkin basis will be denoted by a square bracket $[\]$. There is the well known theorem that every highest weight representation has integral nonnegative entries in the Dynkin basis.

From (9.4), we obtain the relation between the roots and the weights

$$\alpha^i = A^{ij} \Lambda_j . \quad (9.5)$$

It means that the Cartan matrix provides the components of the simple roots for fixed i in the Dynkin basis which is a convenient basis in obtaining irreducible representations from the highest weight vector. Moreover, it provides the “change of basis” which appears in the modular transformation $\mathcal{S}$ of Sect. 7.1 and Subsect. 5.3.2. The covariant factor is the volume change factor or Jacobian, and the invariance condition translates into $\det A = 1$. Typical ones satisfying this are E_8 and $SO(32)$.

If we consider an extended root $\alpha^0 \equiv -\theta$, then its inner products with the original simple roots give the extended Dynkin diagram shown in Fig. 9.1. Then, the simple roots and the extended root are linearly dependent and projecting out one of them gives a maximal regular subalgebra of the original algebra E_8 . However, it is known that there are five exceptions to this rule on “maximality”: when one projects out the 3rd root of F_4 , the 3rd root of E_7 , the 3rd or the 5th or the 6th root of E_8 [2].

¹ But we do not bother about this fact since we will be working in the self-dual space.

9.1 Automorphism

The actions we performed in the previous chapters are characterized by a set of shift vectors $V, a_1, a_2, \dots$. These actions are understood as automorphisms we will discuss here. Gauge symmetry is broken because only the invariant states under the projection survive. We will also see that the Weyl group provides an automorphism and plays an important role in understanding it. So, the Weyl group can be used for symmetry breaking, and its application even reduces the rank of the gauge group. The symmetry breaking by Weyl group is generalized to the case of asymmetric orbifolds, where the action on left and right movers differ.

9.1.1 Inner Automorphisms of Finite Order

An automorphism ω of an algebra $\mathfrak{g}$ is an isomorphism from $\mathfrak{g}$ to itself (hence the prefix *auto*) [3]. In other words, it preserves the action

$$\omega([x, y]) = [\omega(x), \omega(y)] \quad (9.6)$$

and the mapping is a one-to-one correspondence. Roughly speaking, it is a permutation among roots. An *order* of automorphism, N , is defined to be the minimum integer such that ω^N becomes identity. We are interested in a finite automorphism, where N is a finite number.

By linear combinations with complex coefficients, we can always decompose the set of generators into eigenstates having definite orders

$$\mathfrak{g} = \bigoplus_{k=0}^{N-1} \mathfrak{g}_k \quad (9.7)$$

with the following definition of $\mathfrak{g}_k$

$$\omega(x) = e^{2\pi i k/N} x, \quad x \in \mathfrak{g}_k. \quad (9.8)$$

This is so because every generator x , comes back to itself after a finite number of operation ω .

Imagine an action on a ladder operator, then the resulting eigenvalue is $e^{2\pi i P \cdot V}$, where V is a shift (6.29). Then gauge fields come from $\mathfrak{g}_0$, which is called the *fixed point algebra* (What we call “fixed point” here is for algebraic elements, so it has nothing to do with the fixed point in orbifold). Matter representations can come also from eigenstates $\mathfrak{g}_k, k \neq 0$.

In general, we can make a state into a definite eigenstate $\mathfrak{g}_k$ by forming a linear combination

$$x_k = \frac{1}{N} \sum_{j=0}^{N-1} e^{-2\pi i k j/N} \omega^j(x), \quad (9.9)$$

for which we can show $\omega(x_k) = e^{2\pi i k/N} x_k$.

There is an important theorem that *a finite inner automorphism is always represented by the shift vector (6.29)*. The key idea is that, for the fixed point algebra under ω , i.e. $\mathfrak{g}_0$ of $\mathfrak{g}$, we can *always find the elements of the Cartan subalgebra of $\mathfrak{g}$, invariant under ω* . A sketch of the proof is as follows. They are some special linear combinations of elements of $\mathfrak{g}$, such that they commute each other and form a maximal Abelian subalgebra of $\mathfrak{g}_0$. It implies that $\omega(E^\alpha)$ is still eigenstate with respect to H^i , since

$$[H^i, \omega(E^\alpha)] = [\omega(H^i), \omega(E^\alpha)] = \omega([H^i, E^\alpha]) = \alpha^i \omega(E^\alpha) . \quad (9.10)$$

Because there is only one E^α corresponding to the root α , $\omega(E^\alpha)$ should be proportional to E^α up to a phase. Evidently this ω is mapped by $\text{ad}_H \omega$ where $\text{ad}_A B = [A, B]$ and the phase is determined by a linearly combined Cartan subalgebra; thus

$$\omega = \exp \left(\frac{2\pi i}{N} \sum_{i=1}^r v^i \text{ad}_{H^i} \right) \quad (9.11)$$

which is nothing but the definition of the shift vector (6.29).

9.1.2 Weyl Group

How can we parameterize and classify such group actions? Consider a reflection σ_α on the plane perpendicular to a root α ,

$$V \rightarrow \sigma_\alpha V = V - 2 \frac{\alpha \cdot V}{\alpha^2} \alpha = V - (\alpha \cdot V) \alpha \quad (9.12)$$

where V is a root, but it is clear that it acts on weights and shift vectors also in the same manner. The last equality comes from the self-duality. Being a reflection, we have $\sigma_\alpha = \sigma_{-\alpha} = \sigma_\alpha^{-1}$. We see that these reflections form a group, called *Weyl group*. Like the root system, every Weyl reflection is decomposed by a product of fundamental reflections σ_{α^i} for simple roots α^i , but not every element of Weyl group is a reflection. Also, each Weyl reflection acts as an automorphism since it just permutes the roots.

A Weyl reflection in orthogonal bases will provide a good example. With Weyl reflections we can understand many operations on the weight lattices, and on the shift vectors also. For instance, the Weyl reflections of V with respect to the roots of the type $(1, -1, 0^8)$,

$$\begin{aligned} V &= (V_1, V_2, \dots, V_8) \\ &\rightarrow (V_1, V_2, \dots, V_8) - [(1, -1, 0^6) \cdot (V_1, V_2, \dots, V_8)](1, -1, 0^6) \\ &= (V_2, V_1, \dots, V_8) \end{aligned} \quad (9.13)$$

correspond to exchanging two components. This statement can be generalized. With respect to the roots of the type $(1, 1, 0^8)$, the Weyl reflections give

$$(V_1, V_2, \dots, V_8) \rightarrow (-V_2, -V_1, \dots, V_8) . \quad (9.14)$$

Then, if we apply both actions together, we have the result that two elements just obtain minus signs.

What will be the case when we apply Weyl reflections on the half-integral elements? In general, it results in a quite complicated action. Fortunately, it is known that three such actions are redundant to two actions up to integral reflections. This fact is crucially used when we check an equivalences between two shift vectors. Later we will see that the Weyl reflection of affine Lie algebra accompanies a translation. Thus, we confirm that all the symmetries are generated by Weyl reflections.

If an automorphism is generated by the algebraic elements of $\mathfrak{g}$ itself, then the automorphism is called the *inner automorphism*, otherwise it is called the *outer automorphism*. Weyl reflection is inner automorphism, because it is generated by generator themselves $\sigma_\alpha = \exp[i\pi(E^\alpha + E^{-\alpha})/2]$, so that

$$\sigma_\alpha H^I \sigma_\alpha^{-1} = H^I - \alpha^I \alpha \cdot H . \quad (9.15)$$

Therefore, an automorphism is a group of permutating roots. Since the Weyl group is a group of reflections, we can always rearrange the roots to have bases of simple roots from an initially given set of simple roots. The only remaining ambiguity is rearranging simple roots within themselves. This cannot be arbitrary, but should be in the way that preserves the relations between the simple roots. That is, it is a symmetry that leaves the Cartan matrix or the Dynkin diagram invariant. Therefore, we have

$$\text{Automorphism of } \mathfrak{g} = (\text{Symmetries of Dynkin diagram}) \ltimes (\text{Weyl group}) \quad (9.16)$$

Since the E_8 Dynkin diagram has no symmetry, the only automorphism is the Weyl reflections.

9.2 General Action on Group Lattice

9.2.1 Another Embedding

In Chaps. 5 and 6, we have broken a group by associating the space action θ with a translation V in the group space, as in (6.28),

$$(\theta, v)x \longrightarrow (1, V)X = X + V .$$

The most general possibility is that with an automorphism on the group space Θ [4],

$$(\theta, v)x \longrightarrow (\Theta, V)X = \Theta X + V . \quad (9.17)$$

As with the shift vector (6.51), the unbroken gauge group is obtained by the invariance condition under this action. We have matter fields also which are given definite transformation properties under this unbroken group.

Therefore, we can form a point action Θ on the group lattice, by a suitable product of Weyl reflections. In fact, then the form and, in particular, *the order of Θ is dependent on the choice of lattice*. Happily, there is a special element, called the *Coxeter element*, defined by the product of all the fundamental reflections

$$w = \prod \sigma_{\alpha^i} . \quad (9.18)$$

Although its form is also dependent on the lattice choice, it has a definite order, which turns out to be the Coxeter number g given in (9.2), i.e. $w^g = 1$ [1].

For the adjoint representation, every element x has a corresponding state labeled by a root vector $|x\rangle$. For the heterotic string, all the states begin from roots, thus the above action is applicable. The action is defined by a bracket relation $y|x\rangle = |[y, x]\rangle$. The string states can be treated in this way. In this case, the group sum $+$ is a linear superposition of the states, which is clearly understood in terms of vertex operators (5.213) and (5.212).

For an action w , the invariant states with the invariant right movers become gauge bosons. The noninvariant states with suitable right movers form matter representations.

9.2.2 Z_3 Example: $E_8 \times E_8$ Through $SU(3)^8$

We want to associate the space action with an automorphism of order 3, which should be identical to the Coxeter number [4]. A typical example with the Coxeter number n is $SU(n)$ [1]. The $SU(3)$ element $w = \sigma_{\alpha^1} \sigma_{\alpha^2}$ is a sequential reflections, first on the α^2 axis and then on the α^1 axis; it turns out to be a rotation by $2\pi/3$ and an order 3 action.

The elements of E_8 can be represented by those of $SU(3)^4$, which is one of the maximal subgroups. Here, the **248** are given by four adjoints,

$$\{(\mathbf{8}, \mathbf{1}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{8}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{8})\}$$

which constitute 32 states, and eight kinds of multi fundamentals in the form

$$\{(\mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{1}), (\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}, \mathbf{3}), (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}, \mathbf{3}), (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{3}, \mathbf{3}), (\text{complex conjugates})\} \quad (9.19)$$

which are 216 states. All the $SU(3)$ s are equivalent, manifestly seen by a suitable complex conjugation $\mathbf{3} \leftrightarrow \bar{\mathbf{3}}$.

It is another way of representing *the same* E_8 , resulting to the same lattice Γ_8 with a different basis. Each element can be represented as a vector in the weight space: for example the weight vectors of $(\mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{1})$ are the eight-dimensional vectors of the form $(d_a, d_b, d_c, 0)$ with each d being the $SU(3)$ weight $\mathbf{3}$

$$\left\{ \pm \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}}i, -\frac{2}{\sqrt{6}}i \right\}. \quad (9.20)$$

Those corresponding to $\bar{\mathbf{3}}$ have the opposite sign. Their lengths are chosen such that $d_a^2 + d_b^2 + d_c^2 = 3d^2 = 2$. Also, we can verify that the roots ρ , or the difference of weights, satisfy $\rho^2 = 2$.

Analogously to the spacetime twisting, it is convenient to use complexified coordinates. The modular invariance condition again restricts the possible automorphisms. Essentially, it comes from (6.40), by the way of guaranteeing massless states. Now the only crucial difference is that we have a twisting Θ on the group space; thus states got twisted by Θ cannot have a momentum contribution to mass (but have a fractional oscillator number). To guarantee a massless state, the number of action w on the eight $SU(3)$ lattices should be a multiple of three. For example, states such as $(d, 0, 0, 0)(0, 0, 0, 0)$ and $(d, d, 0, 0)(0, 0, 0, 0)$ do not have length squared as 2. The only possibilities are listed in Table 9.4.

Untwisted Sector

We act the Coxeter element on the *first three* $SU(3)$ s of eight $SU(3)$ s of $E_8 \times E_8$, focusing on the first E_8 , as its schematics drawn in Fig. 9.2.

Consider the oscillator states $\alpha_{-1}^I |0\rangle$ under Θ . Six oscillators $(1_+, 2_+, 3_+, 1_-, 2_-, 3_-)$ transform like $\alpha \equiv e^{2\pi i/3}$ and the others $(I = 7, 8)$ remain invariant. So we are tempted to say that those six $U(1)$ generators are projected out so that the rank is reduced. However, there are new states invariant under Θ among the winding states

$$\begin{aligned} & |\rho_1\rangle + |\rho_2\rangle + |\rho_3\rangle \\ & |-\rho_1\rangle + |-\rho_2\rangle + |-\rho_3\rangle \end{aligned} \quad (9.21)$$

for each $SU(3)$. The roots $\pm\rho_i$ s are shown in Fig. 9.3. By these, we mean that the states are represented by vertex operators like those given in (5.213), for example for the first state

$$V \simeq: e^{i\rho_1 Z^1} + e^{i\rho_2 Z^1} + e^{i\rho_3 Z^1} : \quad (9.22)$$

up to coefficients, where $Z^1 = X^1 + iX^2$ for the first $SU(3)$. Checking commutation relations, it can be shown that they again form another six $U(1)$

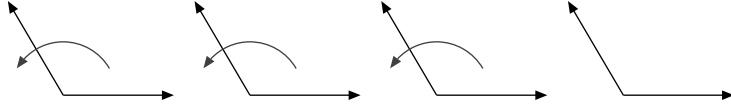


Fig. 9.2. E_8 is decomposed into four $SU(3)$ s. We associate $\mathbf{Z}_3$ action θ with the rotation on the first three $SU(3)$ lattices by $2\pi/3$. It is realized by products $\Theta = w_1 w_2 w_3 w_5 w_6 w_7$ of $SU(3)$ Coxeter elements $w_i = \sigma_{\alpha^i}^i \sigma_{\alpha^2}^i$

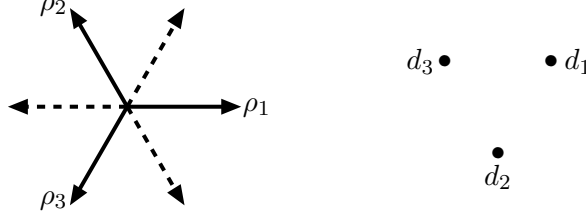


Fig. 9.3. $SU(3)$ roots and weights of $\mathbf{3}$

generators, so that the rank is not reduced. Of course, the charge is newly assigned according to these new Cartan generators.

For the charged generators, we can form invariant states with the following linear combination

$$|P\rangle + |\Theta P\rangle + |\Theta^2 P\rangle. \quad (9.23)$$

This corresponds to $|x_0\rangle$ in (9.9). First consider the states $(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}, \mathbf{3})$. Under the rotation Θ , only the first three entries d_a, d_b, d_c are affected. Moreover the third entry d_c is zero. We want an invariant state by forming a linear combination of the form

$$|(-d_1, d_2, 0, d_d)\rangle + |(-d_2, d_3, 0, d_d)\rangle + |(-d_3, d_1, 0, d_d)\rangle, \quad (9.24)$$

with an arbitrary d_d . Cyclic permutations are allowed. To avoid over-counting, we fix the first entry $d_a = d_1$ and allow permutations like

$$|(-d_1, d_b, 0, d_d)\rangle + (\text{permutations on first two } ds). \quad (9.25)$$

We can freely choose any weights for d_b and d_d . So we have $3 \times 3 = 9$ states. With complex conjugation counted, we have 2×3 such states; thus 54 states in total. Similarly, for the states of $(\mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{1})$ we have $3^3/3$ states in each. With complex conjugation counted, we have $2 \times 3^3/3 = 18$ states.

Collecting them all along with the Cartan generators, we have $8 + 78$ states which form the adjoint representation of

$$SU(3) \times E_6.$$

Now consider the untwisted sector matter transforming like α . They are the states of the form

$$|S\rangle = |P\rangle + \alpha^2 |\Theta P\rangle + \alpha |\Theta^2 P\rangle \quad (9.26)$$

transforming like α , since the action $\Theta|S\rangle = |\Theta S\rangle = \alpha|S\rangle$ permutes three terms in (9.26) except for the additional factor α . This corresponds to $|x_1\rangle$ in (9.9). Using the same elements as in (9.24)

$$\begin{aligned} & |(-d_1, d_2, 0, d_d)\rangle + \alpha^2 |\Theta(-d_2, d_3, 0, d_d)\rangle + \alpha |\Theta^2(-d_3, d_1, 0, d_d)\rangle \\ &= |(-d_1, d_2, 0, d_d)\rangle + \alpha^2 |(-d_3, d_1, 0, d_d)\rangle + \alpha |(-d_2, d_3, 0, d_d)\rangle \end{aligned} \quad (9.27)$$

we obtain $8 \times 3^2 = 72$ states, whose form of vertex operators are easily deduced from (9.22)

$$V \simeq: e^{i(-d_1, d_2, 0, d_d) \cdot Z} + \alpha^2 e^{i(-d_3, d_1, 0, d_d) \cdot Z} + \alpha e^{i(-d_2, d_3, 0, d_d) \cdot Z} :$$

Similarly, for each $SU(3)$ we have two states of the following form,

$$\begin{aligned} & |\rho_1\rangle + \alpha^2 |\rho_2\rangle + \alpha |\rho_3\rangle , \\ & |-\rho_1\rangle + \alpha^2 |-\rho_2\rangle + \alpha |-\rho_3\rangle . \end{aligned} \quad (9.28)$$

Including oscillators in total, we have 81 states which form **(3, 27)**.

Twisted Sector

We have considered a twist in the gauge degrees of freedom in (9.17),

$$X^I(\pi) = (\Theta X(0))^I + V^I , \quad (9.29)$$

where we admitted the lattice translation by lattice vectors of E_8 . This is a space group action, for which we have fixed points. Naively the number of fixed points is 9, but there is a subtlety. The number of fixed points is a geometric notion; thus with the left movers only we do not know how to define an orbifold action. However, from the partition function we learned that we gave up the geometric meaning and adopt the multiplicity as given in (7.30). There is another way out. We can introduce an auxiliary lattice to make it symmetric and project out the right movers again. Then we have a well-defined meaning for the orbifolding action. For the moment, we just take its square root

$$n_f = \sqrt{9} = 3 . \quad (9.30)$$

We will justify it later.

Since we twist gauge degrees of freedom as well, the zero point energy is $\tilde{c} = 12f(0) + 4f(\frac{2}{3}) + 8f(\frac{1}{3}) = -\frac{1}{3}$. The mass shell condition becomes

$$\frac{1}{4}M_L^2 = \frac{1}{2}P^2 + \tilde{N} - \frac{1}{3} . \quad (9.31)$$

Note that the twisted string, even in the group space, cannot have a nontrivial P . The non-vanishing components of P^I are for $I = 7, 8$ and we have fractional oscillators in the $I = 1, \dots, 6$ directions.

The oscillator contributions for $P^2 = 0$ and $\tilde{N} = \frac{1}{3}$ make up the state of a form $\alpha_{-1/3}^{M_+}|0\rangle_L$, where M_+ is a complexified index. If M_+ assumes a spacetime holomorphic index $M_+ = a$, it transforms as **3** under the holonomy group. It is important to note that there are three fixed points in the group space. Indeed, it has the same structure of three copies for **(3, 1)**. Remarkably, the fixed points act as representations and transform nontrivially under nonabelian

Table 9.1. Twisted sector states for the “standard” $\mathbf{Z}_3$ orbifold by automorphism. Only the left movers are shown. d_d can be any six $\text{SU}(3)$ weights

State	Group Spc. Mult.	Spacetime Mult.	Representation
$ 0, 0, 0, d_d\rangle$	3	27	part of $27(\mathbf{1}, \mathbf{27})$
$\alpha_{-1/3}^{I_+} 0\rangle$	3	27	part of $27(\mathbf{1}, \mathbf{27})$
$\alpha_{-1/3}^a 0\rangle$	3	27	$27(\bar{\mathbf{3}}, \mathbf{1})$

gauge group. This is clearly seen by commutation relations of zero mode parts of vertex operators.

The remaining oscillators in the group space $\alpha_{-1/3}^{I_+}|0\rangle$ with $I_+ = 1, 2, 3$ gives nine states because of the fixed points. Considering the state with $P^2 = 2/3$ and $\tilde{N} = 0$, we have the possibility of $P = (0, 0, 0, \pm d_d)$. Six states with multiplicity 3 count $3 \times 6 = 18$ states. With nine states discussed above, we obtain $(\mathbf{1}, \mathbf{27})$. The above states are tabulated in Table 9.1.

So far we have not considered the spacetime multiplicity, namely 27 fixed points. We can confirm that the model is identical to that with the shift vector with the standard embedding $V = \frac{1}{3}(2 \ 1^2 \ 0^5)(0^8)$.

9.2.3 Reducing the Rank by Orbifolding

With general automorphism Θ , when we have more than one such actions on the group space, generally they do not commute. Then, if the Cartan subalgebra is not invariant under all such actions, the rank is reduced [4]. This is the case when we include the Wilson line of the type $(1, v) \rightarrow (1, a)$, so that

$$(\Theta, 0)(1, a) \neq (1, a)(\Theta, 0). \quad (9.32)$$

In other words, this happens when the rotated Wilson line Θa is not equal to a : $\Theta a \neq a$. In this case, this Wilson line is not subject to any order N condition, that is, the strength of Wilson line $\langle a \rangle$ is free. For this reason, we call it *continuous Wilson line*. Note that in a specific decomposition like (9.19), the possible Wilson line shift a_i is provided by weight vectors of the unbroken group. In this case, it is the translation part in (9.29). When a vector a is invariant under a rotation Θ , they can be treated as independent shift vectors, and even the action Θ can be converted into an equivalent shift vector V .

As always, the modular invariance condition for the additional Wilson lines (6.91) should be checked. We note that only the part of a invariant under Θ fulfills this condition, whereas the noninvariant part does not.

The rank reduction is important in the process of obtaining the SM. In addition to the above mechanism of rank reduction by the non-commuting point rotation and the Wilson line shift, there are simpler field theoretic methods.

For example, we can use the Higgs mechanism via VEVs of some massless fields in the spectrum and also the Fayet-Illiopulous term generated by the anomalous U(1).

9.3 Asymmetric Orbifold

So far the twisting (6.4) acted equally on the left and right movers. We may relax this condition and have a different twisting on each mover.

	$x^0 \dots x^3$	$x^4 \dots x^9$	$x^{10} \dots x^{25}$
left		(θ_L, v_L)	(Θ, V)
right		(θ_R, v_R)	

We can even think of an action $\theta_L \oplus \Theta$ acting on the whole left movers. This is an *asymmetric orbifold* [5].

9.3.1 Extending Group Lattice

We take six dimensional radius to the critical one $R = 1$ to enhance the rank of group to $16 + 6 = 22$. We can choose any group which has rank 22 satisfying the Narain lattice condition (6.110–6.112). Let us fix such extra lattice coming from the rank 6 part to that of $SU(3)^3$. It corresponds to fixing G_{ij} as,

$$\langle G_{ij} \rangle = \frac{1}{4} \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & 2 & -1 \\ & & -1 & 2 \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \quad (9.33)$$

In view of (6.110, 6.111), we want to simplify the mass shell condition. The choice for B_{ij} is

$$\langle B_{ij} \rangle = \frac{1}{4} \begin{pmatrix} 0 & -1 & & \\ +1 & 0 & & \\ & & 0 & -1 \\ & & +1 & 0 \\ & & & & 0 & -1 \\ & & & & +1 & 0 \end{pmatrix} \quad (9.34)$$

We can always choose $p_{Ri} = 0$ by assigning relations between m_i and n_i . This is the constraint that in the original construction of heterotic string we have no right mover. The mass shell condition is

$$\frac{M_L^2}{4} = \frac{P^2}{2} + \frac{p_L^2}{2} + \tilde{N} - 1 = 0, \quad \frac{M_R^2}{4} = \frac{p_R^2}{2} + N - 1 = 0. \quad (9.35)$$

As always P is the momentum in the group space and p_L is the momentum in the compact dimension. Besides the solution $P^2 = 2$, $p_L^2 = 0$, we have another class of solutions with $P^2 = 0$, $p_L^2 = 2$, which has the form $p_L = (\rho, 0, 0)$, with the $SU(3)$ roots ρ and weight d_i . We have three **8**s to have an enhanced gauge group

$$E_8 \times E_8 \times SU(3)^3 \times U(1)^6$$

with rank $22 + 6$, where rank 6 comes from right movers whose generators take the form of (9.21),

$$b_{-\frac{1}{2}}^\mu (|\pm \rho_1\rangle + |\pm \rho_2\rangle + |\pm \rho_3\rangle)$$

where signs are correlated. We have the same situation as in the previous subsection and obtain the same kinds of twisted states.

Of course, we can adjust VEVs of B_{ij} to have another theory with a different gauge group. Also, following the argument of $E_8 \times E_8$ - $SO(32)$ duality, we expect that there exists some duality among them.

9.3.2 Asymmetric $\mathbf{Z}_3$ Example

Consider asymmetric orbifolds such that left movers are untwisted and right movers are twisted by $\mathbf{Z}_3$. Then, under the $\mathbf{Z}_3$ action θ_R , let the lattice get twisted as

$$|P, p_L, p_R\rangle \rightarrow \exp(2\pi i P \cdot V) |P, p_L, \theta_R p_R\rangle. \quad (9.36)$$

In (9.36) we have explicitly shown the spacetime momenta p_L, p_R , because here they are treated equally with P . Therefore in the untwisted sector, the invariant L-moving states with $\alpha_{-1}^\mu |0\rangle$ and $|P\rangle$ with $P \cdot V = \text{integer}$ makes up the adjoint representation. It breaks E_8 down to $E_6 \times SU(3)$. Including the original $SU(3)^3$ adjoint $\alpha_{-1}^i |0\rangle$ and $|p_L, p_R\rangle$ with $p_L^2 = 2$, $p_R^2 = 0$, $N = 1$, we have

$$E_6 \times E_8 \times SU(3)^4 \times U(1)^6$$

with the same matter representation as in the standard $\mathbf{Z}_3$ case.

In the twisted sector, the mass shell condition becomes

$$\frac{M_L^2}{4} = \frac{(P+V)^2}{2} + \frac{p_L^2}{2} + \tilde{N} - 1 = 0, \quad \frac{M_R^2}{4} = \frac{p_R^2}{2} + N - \frac{2}{3} = 0. \quad (9.37)$$

Note that we have no twist on the left mover, thus no shift on the zero point energy. While P can become the original vectors in the $E_8 \times E_8$ lattice, the p_L can assume the $SU(3)$ roots ρ and weights d with $d^2 = 2/3$ in particular. The relevant spectrum is listed in Table 9.2. The right mover has the same spectrum as in the standard $\mathbf{Z}_3$ case, since the twisting and hence the zero point energy are the same.

Table 9.2. Spectrum of the asymmetric $\mathbf{Z}_3$ orbifold acted on the right mover only. The underline denotes permutations. The twisted sector has multiplicity 1

State	Representation
Untwisted sector:	
$\tilde{N} = 0, n^2 = 0, P^2 = 2$	$3(\mathbf{27}, \mathbf{3}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
Twisted sector:	
$\tilde{N} = 0, p_L^2 = 0, (P + V)^2 = 2$	$(\overline{\mathbf{27}}, \mathbf{3}; \mathbf{1}, \mathbf{1}, \mathbf{1})$
$\tilde{N} = 0, p_L^2 = \frac{2}{3}, (P + V)^2 = \frac{4}{3}$	$(\mathbf{27}, \mathbf{1}; \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{27}, \mathbf{1}; \overline{\mathbf{3}}, \mathbf{1}, \mathbf{1})$
$\tilde{N} = 0, p_L^2 = \frac{4}{3}, (P + V)^2 = \frac{2}{3}$	$(\mathbf{1}, \overline{\mathbf{3}}; \mathbf{3}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \overline{\mathbf{3}}; \overline{\mathbf{3}}, \overline{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \overline{\mathbf{3}}; \overline{\mathbf{3}}, \mathbf{3}, \mathbf{1})$

9.3.3 Symmetrizing Lattice

The asymmetric orbifold suffers from some ambiguities such as in defining the notion of fixed point. Here, we give up the geometric meaning since, for example, we cannot imagine fixed points present only on the left mover. The cure comes from introducing a mirror lattice to make the theory *symmetric and Euclidian*; then after a consistent calculation we project out the right movers. This was the idea from which the heterotic string was first formulated.

Let us begin with the original, even and self-dual lattice $\Gamma^{p,q}$ (think of $(p, q) = (22, 6)$). To treat all the coordinates equally, consider a Euclidianized lattice denoted by $\tilde{\Gamma}^{p,q}$, by treating the R-handed part in the same way as the L-handed part. Then, it is not self-dual any more. Nevertheless, for vector $(p_1, p_2) \in \tilde{\Gamma}^{p,q}$, vector $(p_1, -p_2)$ belongs to the dual lattice since it comes from the Euclidianization. We will consider windings on $\frac{1}{2}\tilde{\Gamma}^{p,q}$ and momenta on its dual lattice. They have the form for momenta $p = 2(k_1, -k_2)$ and windings $w = \frac{1}{2}(k_3, k_4)$ with (k_1, k_2) and (k_3, k_4) belonging to $\tilde{\Gamma}^{p,q}$.

Now let us construct a symmetrized lattice $\Gamma^{p+q;p+q}$, whose element consists of momenta (6.110, 6.111),

$$(p_L; p_R) = \left(\frac{1}{2}p - Bw + Gw; \frac{1}{2}p - Bw - Gw \right) \quad (9.38)$$

where we renamed the quantized momenta m and windings n as p and w , respectively, and for brevity we suppressed the vector and matrix indices. It is convenient to consider the $\frac{1}{2}p$ part only in p_L in scalar products. So we choose B_{ij} such that,

$$e \cdot Be = e \circ Ge, \quad \text{mod } 2, \quad (9.39)$$

as we did just above. G_{ij} is already fixed by orbifolding. The unit lattice of $\tilde{\Gamma}^{p,q}$ is e_i . G_{ij} has the Lorenzian signature; having $+$ sign for p entries (thus $Bw = Gw$), and $-$ for q entries (thus $Bw = -Gw$). Then vectors $(p_L; p_R)$ span a Euclidian lattice $\Gamma^{p+q;p+q}$ which are generated by vectors having the form

$$(k_1, 0; 0, -k_2), \quad (0, k_2; -k_1, 0) . \quad (9.40)$$

In other words, we have a well-defined projection to select just one of them to make the asymmetric lattice

$$\Gamma_{p,q} \ni (k_1, k_2) \equiv (k_1, 0; 0, -k_2) . \quad (9.41)$$

For the oscillators, we use only the first p left-moving and the last q right-moving ones.

Number of Fixed Points

Since $\tilde{\Gamma}$ is a Euclidian lattice, we can define the number of fixed points without ambiguity. Let I denote an invariant sublattice of $\tilde{\Gamma}$ by the operation θ , and let N be the sublattice orthogonal to I . Every fixed point x_f satisfy $(1 - \theta)x_f = v$ by virtue of (3.18). Also for every $w \in I$ we have $(1 - \theta)w = 0$. It follows that $v \cdot w = 0$, which means that the fixed points lie in N . We find the number of fixed points as

$$n_f^{\text{symm}} = \left| \frac{N}{(1 - \theta)\tilde{\Gamma}} \right| , \quad (9.42)$$

in view of (3.21).

Since we have the mirror twin of the original asymmetric lattice, we may reasonably define the number of fixed points as [6]

$$n_f^{\text{asymm}} = \sqrt{n_f^{\text{symm}}} . \quad (9.43)$$

In the group lattice, we have the same result just by replacing θ with Θ . This reasoning can be easily generalized to the twisted sector.

Although this has a direct geometric interpretation, it is hard to apply it in the practical sense. Rather we use a simple formula

$$n_f^{\text{asymm}} = \sqrt{\frac{\det(1 - \theta)}{|I^*/I|}} . \quad (9.44)$$

Here $\det(1 - \theta)$ is the number of fixed points of invariant normal lattice N (assuming it is symmetric lattice), given by the Lefschetz fixed point theorem, (3.24). I^* is the dual lattice of I , generated by the fundamental weights. $|I^*/I|$ is called the index of I in I^* , and it is the inverse of the volume of unit lattice I .

They are the same because the relations $|N^*/(1 - \theta)N^*| = \det(1 - \theta)$ and $(1 - \theta)\Gamma^{p,q} = (1 - \theta)N^*$ hold [5]. Its interpretation is simple, as shown below. By a modular transformation $\mathcal{S}$, the untwisted sector goes to the twisted sector, which has a number of fixed points we are counting. There appears an extra factor $|I^*/I|$ as the volume difference factor of the partition functions. From (9.5), we observe that it is given by

Table 9.3. Possible $\mathbf{Z}_3$ shift vectors for each E_8 sector, in the Dynkin basis $[s_i|s_0]$ and in the Cartan–Weyl basis $3V$. The number of $SU(3)$ s under the Coxeter action w given in (9.18) is also listed. The convention on fundamental weights are presented in Appendix E

Case	$[s_i s_0]$	$3V$	Group	Number of w
0	[00000000 0]	(0^8)	E_8	0
1	[01000000 0]	$(2 \ 1 \ 1 \ 0^5)$	$SU(3) \times E_6$	3
2	[00000001 0]	$(\frac{3}{2} \ \frac{1}{2} \ 7)$	$SU(9)$	4
3	[10000000 1]	$(1 \ 1 \ 0^6)$	$E_7 \times U(1)$	2
4	[00000010 1]	$(2 \ 0^7)$	$SO(14) \times U(1)$	1

$$|I^*/I| = \det A \quad (9.45)$$

where A is the Cartan matrix for the root system generating I . When I is semisimple and/or contains an abelian group, $\det A$ is simply the product of simple subgroup determinants.

In the example discussed in Subsect. 9.2.2, we have rotated three $SU(3)$ subgroups of visible E_8 , counting $SU(3)$ fixed points for three lattices, and we have $\det(1 - \theta) = 27$. Now let us check the invariant lattice I . There remains one $SU(3)$ lattice in this E_8 thus $\det A_{SU(3)} = 3$. The hidden sector E_8 is untouched, so there is no fixed point, and $\det A_{E_8} = 1$, E_8 being the famous self-dual lattice. Therefore, we have

$$n_f^{\text{asymm}} = \sqrt{\frac{27}{3} \cdot \frac{1}{1}} = 3.$$

We can verify the number of fixed points in Table 9.3. Note that $\det A_{E_6} = 3$.

9.3.4 Equivalence to Symmetric Orbifold

In the literature, the asymmetric orbifold is used not only for those with the asymmetric orbifolding action, but also for those with the radius of the compact internal space taking the critical value $R = \sqrt{\alpha'}$, as introduced in the beginning of this subsection. With the critical radius, we have the extra gauge group $SU(3)^3$. Of course, this critical radius is not necessary for asymmetric orbifolds.

However then, there is *no way of distinguishing* whether this is the compactified internal space or the original gauge group space. It is just a matter of assigning coordinates. Note that we have used the Coxeter action for breaking the gauge group, e.g. breaking $SU(3)$ in Subsect. 9.2.2. We can use the same trick for breaking $SU(3)_{\text{st}}$ arising from spacetime in asymmetric orbifold. Indeed, if we do exchange one of $SU(3)$ s from group space and $SU(3)_{\text{st}}$, we have a situation that there appears a symmetric orbifold with three Coxeter elements with $SU(3)^3$ in the group space. Also, the number of fixed points and

the mass shell condition is the same, as it should be. However, this equivalence is not always valid [5].

More Four Dimensional String Models

A generalization of the above asymmetric orbifold scheme with an arbitrary number of (auxiliary) noncompact dimensions is straightforward. In this case, of course we do not have a symmetry like T -duality in the field theory limit because the T -duality is a characteristic feature of the string winding. The 4D string from the fermionic construction, which will be briefly discussed in Appendix B.1, is based on the same philosophy. In fact, it is obtained just by fermionizing the bosons of the asymmetric orbifolds [7].

9.4 Group Structure

An apt reader might have noticed that the *massless spectrum from untwisted sector*, resulting from breaking by shifts, obeys the *branching rule* of Lie algebra. We will observe here its group theoretical origin. In particular, if there is an Abelian algebra in the unbroken algebra, there are some subtleties, which is cured only by understanding the group structure.

Moreover, we are interested in the massive states also. Interestingly, they still form the representations of a certain algebra, which we will discuss here. In usual case we neglected them, because they have mass of order $1/\alpha'$ so that decoupled in the low energy limit. Nevertheless, these massive states are relevant for the following reasons.

In the twisted sector we have observed that the spectrum does not belong to the untwisted sector spectrum, since it is in a different Hilbert space. For example, there appears a state like $P = (1^4 \ 0^4)(0^8)$ not belonging to $E_8 \times E_8$ roots since $P^2 \neq 2$. It belongs to the root *lattice*. These states appear to be mixed with massive states because the twisting (6.4) and the shift $P \rightarrow P + V$ give massless states, like (6.72). We will see that this originates from just isomorphic algebra in $P + V$ basis [8].

In this section we will also show that there cannot be an adjoint matter representation from the *level one* algebra. Thus, the higher level algebra must be used if the adjoint Higgs of $SU(5)$, $SO(10)$ or E_6 GUTs are used for gauge symmetry breaking. It can be obtained by projecting out some components of massive states with $P^2 > 2$ to $P^2 = 2$ states [9, 10, 11, 12].

There are good references for the topics discussed in this section [1, 13].

9.4.1 Classification of the Gauge Group

The information on group breaking is entirely contained in the shift vector V . By representing V in basis of the fundamental weights, we can track the group theoretical origin. Armed with it we can have a deeper understanding of its structure and handily classify all the possible breaking.

Shift Vector

The shift vector of the $\mathbf{Z}_3$ example given in Sect. 6.3 is,

$$V = \frac{1}{3}(\Lambda_2)(0) \quad (9.46)$$

where

$$\Lambda_2 = (2 \ 1 \ 1 \ 0^5) \quad (9.47)$$

which can be read from Appendix E. Then, by definition of the fundamental weights, the condition (6.51) for unbroken roots is satisfied for every (simple) root except α^2 . The remaining root vectors make up the root system of the unbroken algebra $\text{SU}(3) \times \text{E}_6$. This is nothing but Cartan's general procedure of obtaining the maximal subalgebra and explains why the untwisted sector spectrum obeys the branching rule. We can generalize this method as follows.

Denoting the shift vector in terms of the fundamental weights in the Dynkin basis,

$$V = \frac{1}{N} \sum_{i=1}^r s_i \Lambda_i = \frac{1}{N} [s_i], \quad (9.48)$$

we can always satisfy the following condition [14, 15]

$$N = s_0 + \sum_{i=1}^r a_i s_i, \quad s_0, s_i \in \mathbf{Z}_{\geq 0} \quad (9.49)$$

with *nonnegative integers* s_0, s_i . Surveying the condition (6.51), one observes that it is sufficient to consider only simple root vectors since any root vector is made of successive addition of simple roots. Now consider a product

$$V \cdot \alpha^i = \frac{1}{N} \sum_{j=1}^r s_j (\alpha^i \cdot \Lambda_j) = \frac{1}{N} s_i. \quad (9.50)$$

If $s_i = 0$, then the condition (6.51) is satisfied. If not, the corresponding i th root is not a root of the unbroken group. The unbroken group is obtained from the extended Dynkin diagram: remove this i th circle from the extended Dynkin diagram. However the Cartan generators are invariant so that the rank is preserved as in the original group.

Actually, only a few set of integers $[s_i]$ can satisfy the relation (9.49). We present the result of this study for the $\mathbf{Z}_3$ orbifold in Table 9.3. The same is applied to the other E'_8 .

What happens for the shift vector not satisfying (9.49), for example $V = \frac{1}{3}(\Lambda_5)(0)$? $\Lambda_5 = (5 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0)$ is given in Appendix E. Then it is redundant, and it can be explicitly checked that it yields again $\text{SU}(3) \times \text{E}_6$. The α seems to be projected out, however there is another “simple root” $\alpha = \alpha^3 + 2\alpha^4 + 3\alpha^5 + 2\alpha^6 + \alpha^7 + \alpha^8$ connected to α^8 and α^2 . With a number of Weyl reflections,

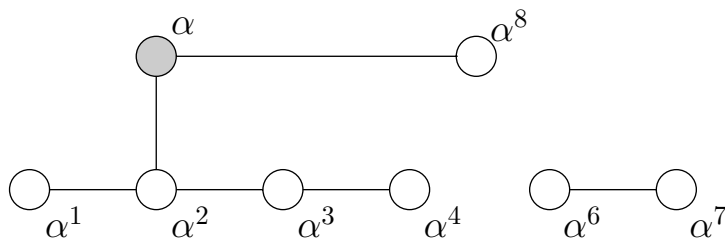


Fig. 9.4. Some shift vectors (here $V = \frac{1}{3}\Lambda_5$) are redundant to the “standard form” ($V = \frac{1}{3}\Lambda_2$)

Table 9.4. The only modular invariant combinations without Wilson lines, breaking $E_8 \times E'_8$. The numbers in the first column are from Table 9.3. We used prime to discriminate two E_8 's, which can be exchanged. The number of fixed points in the group space emerges when we mod by the point group action Θ , not by the shift vector

Combination	Group	# Fixed Points
0×0	$E_8 \times E'_8$	1
1×0	$SU(3) \times E_6 \times E'_8$	3
1×1	$SU(3) \times E_6 \times SU(3)' \times E'_6$	9
2×3	$SU(9) \times E'_7 \times U(1)'$	9
3×4	$E_7 \times U(1) \times SO(14)' \times U(1)'$	1

this shift becomes $V = \frac{1}{3}(\Lambda_2)(0)$, satisfying (9.49). This situation is depicted in Fig. 9.4.

So far we have just analyzed the pattern of symmetry breaking in terms of group theory. Combining with another E'_8 , the modular invariance condition (6.43) restricts possible combinations. The only possible cases are those presented in Table 9.4.

With Wilson Lines

With Wilson lines, we can generalize the above Dynkin diagram technique. The main difference is that we have another shift vectors providing additional projection conditions (7.60). One may expect that the unbroken group is the common intersection of the groups obtained by the shift vector V and Wilson lines a_i , which is not true however. The unbroken group is most probably the common intersection of the fixed point gauge groups which are given by $V + \sum m_i a_i$. But there may survive extra roots giving rise to a larger group than the intersection group.

At every stage we begin with the unbroken Dynkin diagram and apply essentially the same rules as explained in [15]. Here, we just summarize the following result.

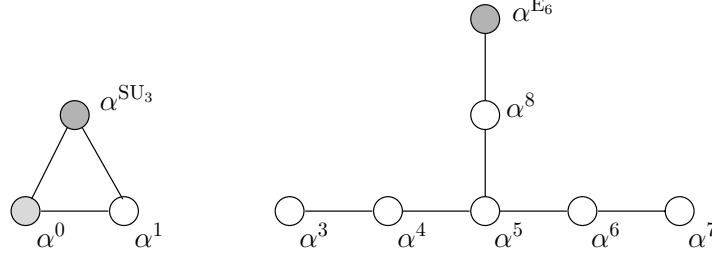


Fig. 9.5. With a further breaking with a Wilson line, we just apply the same rule to the subgroups

1. Break the group according the rule above (9.49).
2. For a further breaking, apply the same rule to each subgroup, now with the dual Coxeter labels a'_i of the subalgebra,

$$N = s'_0 + \sum_{i=1}^r a'_i s'_i. \quad (9.51)$$

This will give a possible shift vector corresponding to the Wilson line. See Fig. 9.5 as an example.

3. However, we cannot satisfy the previous condition with these new s'_i at this stage. We relax the rule of step 1, by

$$N = s'_0 + \sum_{i=1}^r a_i s'_i, \quad \text{mod } N. \quad (9.52)$$

4. Repeat the procedure iteratively.

9.4.2 Matter: The Highest Weight Representation

The matter spectra can be understood also in terms of the highest weight representation. For example, for the untwisted matter $(\mathbf{3}, \mathbf{27})$, the highest weight vector is $[10][100000]$ in the $SU(3) \times E_6$ Dynkin basis. It is annihilated by all the simple roots, among which the nontrivial conditions are

$$E^{\alpha^1} |[1 \ 0][1 \ 0^5]\rangle = 0 \quad (9.53)$$

$$E^{\alpha^3} |[1 \ 0][1 \ 0^5]\rangle = 0 \quad (9.54)$$

where we used the original name of E_8 simple roots. Then, by successive subtractions of simple roots, according to Cartan matrices, we list all the states in the representation. In fact, this can be obtained by the branching rule, since they come from the adjoint **248** of E_8 . So in principle we have the untwisted sector representations by reading off the branching tables [16].

Can we do the same thing for the twisted sector representations? That should be, because they are charged under $SU(3) \times E_6$ and we can decompose the representations under $SU(3) \times E_6$. Indeed, for example, the vectors forming $(\mathbf{3}, \mathbf{1})$ in the twisted sector are obtained from

$$(-1 \ 0 \ -1 \ 0^5) \xrightarrow{-\alpha^1} (-1 \ -1 \ 0 \ 0^5) \xrightarrow{-\alpha^0} (0^8) \quad (9.55)$$

where α^0 is now one of the simple roots of the $SU(3)$ subgroup. The weight (0^8) *without* oscillator α_{-1}^I belongs to the E_8 *lattice*, but is not a root. This implies that this representation does not come from representation **248**. In obtaining this representation, we just resorted to the twisted mass shell condition (6.72), and it seems that we have another representation something like **248**, whose broken representation gives such state as $(\mathbf{3}, \mathbf{1})$. We will see in the next section that this is easily understood when we use the algebra in a twisted form.

Affine Lie Algebra

We have seen that the group generators, (5.213, 5.212), are represented by vertex operators [17]. The operator product expansion between two currents (whose expectation value is two point correlation function) is given as $z \rightarrow 0$,

$$T^a(z)T^b(0) \sim \frac{k\delta^{ab}}{z^2} + \frac{if^{abc}}{z}T^c(0) . \quad (9.56)$$

By “triangular decomposition”, T^a is decomposed to H^I, E^α and $E^{-\alpha}$. The current can be expanded as $T^a(z) = \sum T_n^a z^{-n-1}$ to give

$$[T_m^a, T_n^b] = if^{abc}T_{m+n}^c + m\delta_{m+n,0}\delta^{ab}K \quad (9.57)$$

where $a = 1, 2, \dots, d$ with the dimension of the adjoint representation d . This extended algebra is called the *affine Lie algebra* or the *Kac-Moody algebra*. The zero mode of this algebra $m = n = 0$ reduces to the ordinary simple Lie algebra. We have introduced two additional generators. One is the *central element* K , commuting with all the generators

$$[K, T_m^a] = 0 , \quad (9.58)$$

whose eigenvalue k is called the *level* of the algebra. The other is the *grade* D , whose eigenvalue is the Kaluza-Klein mode number, and satisfies the following commutation relations

$$[D, T_m^a] = mT_m^a, \quad [D, K] = 0 . \quad (9.59)$$

Without the central extension, we just have replicas of the simple Lie algebra. In our formulation, the relative normalization is fixed as $k = 1$ if we identify z^{-1} coefficient as the structure constant f^{abc} of Lie algebra with normalization (9.2).

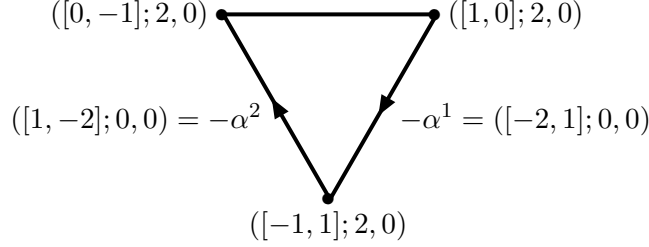


Fig. 9.6. Representation **3** of $SU(3)$ in the $k = 2$ level. The highest weight is $([1\ 0]; 2, 0)$. We obtain all the weights by applying $-\alpha^1 = ([-2\ 1]; 0, 0)$ to obtain $([-1\ 1]; 2, 0)$, and then applying $-\alpha^2 = ([1\ -2]; 0, 0)$ to obtain $([0\ -1]; 2, 0)$

With the central extension, now we have another ladder operator T_n with $n \neq 0$, raising and lowering the grade number n . In addition, the elements of the Cartan subalgebra are not mutually commuting

$$[H_n^I, H_{-n}^J] = \delta^{IJ} nK. \quad (9.60)$$

The Cartan subalgebra elements also raise and lower the eigenvalues of D by nk .

Since the generators $(H_0; K, D)$ are mutually commuting, we can consider their simultaneous eigenvector $(\lambda; k, n)$. We define the inner product as

$$(\lambda; k, n) \cdot (\lambda'; k', n') = \lambda \cdot \lambda' + kn' + k'n. \quad (9.61)$$

With this definition, still the gauge generators belong to the $k = 0$ level. For the simple $SU(3)$ case, it is illustrated in Fig. 9.6.

Integrability and No-Adjoint Theorem

It is sufficient to consider the highest weight vector only, of a form

$$\Lambda = \sum_{i=1}^r t_i \Lambda_i, \quad (9.62)$$

with nonnegative integers t_i . In the “ket” state notation it is $|\hat{\Lambda}\rangle \equiv |\Lambda; k, n\rangle$ of (9.60). Looking at the eigenvalue in the $(\alpha^0; 0, 1)$ direction, we have

$$\hat{H}^{(\alpha^0; 0, 1)} |\hat{\Lambda}\rangle = (\alpha^0 \cdot \Lambda + k) |\Lambda\rangle \quad (9.63)$$

where α^0 is the extended root we discussed before.

In analogy with the simple harmonic oscillator algebra, one can see that the eigenvalues of $\hat{H}$ are nonnegative integers. In the simple Lie algebra, note that the eigenvalues of $\Lambda \cdot \theta$ are already integers. Therefore, the eigenvalue for level

k is a nonnegative integer, which translates into the so-called “integrability” condition

$$k \geq \sum_{i=1}^r a_i t_i \geq 0 ,$$

or, using the extended root,

$$k = t_0 + \sum_{i=1}^r a_i t_i \quad (9.64)$$

where t_0 is a nonnegative integer. It is noted that for the level one ($k = 1$) algebra only a few can satisfy this condition because $a_i \geq 1$. As shown in Appendix E, for the $SU(n)$ algebra the Coxeter label corresponding to every fundamental weight² is always 1; thus every antisymmetric representation Λ_i with the dimension $\binom{n}{i}$, is possible. However, for the other groups $a_i = 1$ is possible only for the outer most nodes of the Dynkin diagram. For E_6 for example, Λ_1 and Λ_6 have $a_i = 1$. In other words, **27** and **$\overline{27}$** can satisfy the $k = 1$ condition.

A corollary of this observation is the “no-adjoint theorem” that the adjoint representation, needed for breaking the $SU(5)$, $SO(10)$ and E_6 GUTs, cannot satisfy this condition. Look at Dynkin diagrams of Appendix E. The highest weight vector for the adjoint representation of $SU(n)$ is $\Lambda_1 + \Lambda_{n-1}$, hence it has the sum of the Coxeter level greater than 1: $a_1 + a_{n-1} = 2 > 1$. For $SO(n)$, the adjoint representation $\mathbf{n}(\mathbf{n}-1)/2$ is Λ_2 , and the level is $a_2 = 2$. For the other groups, the adjoint representations have the Coxeter label greater than 1.

When $k > 1$, this constraint is relaxed; however there exist some upper limit for dimension of a highest weight. We will come back to this point later.

Mass is a Conformal Weight

Now let us consider the matter representations. As in (9.53, 9.54), it satisfies

$$E_0^\alpha |\Lambda\rangle = 0, \text{ for all } \alpha > 0 , \quad (9.65)$$

$$T_n^\alpha |\Lambda\rangle = 0, \text{ for all } n > 0 . \quad (9.66)$$

The last equation is also the requirement for the highest weight along the direction of D .

The Virasoro operators for group degree of freedom are constructed by the method of Sugawara [18],³

$$\tilde{L}_n = \frac{1}{2(k+g)} \sum_{m \in \mathbf{Z}} \sum_{a=1}^d : T_{m+n}^a T_{-m}^a : . \quad (9.67)$$

² Completely antisymmetric representation in this case

³ T_{m+n}^a are coming from the energy-momentum tensor.

The above normalization gives the correct Virasoro algebra (5.41). So we have the minimal nonnegative eigenvalue of Hamiltonian operator L_0 as

$$[\tilde{L}_m, T_{-n}^a] = nT_{m-n}^a . \quad (9.68)$$

It means that the eigenvalue of $\tilde{L}_0$ is nothing but n , or the eigenvalue D .

The mass shell condition (6.10) is translated into

$$\frac{1}{4}M_L^2 = \frac{P^2}{2} + \tilde{N} + \tilde{c} = h_\Lambda + \tilde{c} = 0 . \quad (9.69)$$

In other words, from this condition we can find the highest weight state having the above h_Λ . This is the only masslessness condition and all the possible combinations of h_Λ satisfying this condition is the highest weight spectrum. The conformal weight h of $|\varphi\rangle$ appears as

$$\tilde{L}_0|\varphi\rangle = (h + \tilde{c})|\varphi\rangle = \left[\frac{1}{2}(P + V)^2 + \tilde{c} \right] |\varphi\rangle . \quad (9.70)$$

Being normal ordered, $\tilde{L}_0$ is proportional to $\sum_a^d T_0^a T_0^a$ which is nothing but the Casimir operator $C_2(G)$,

$$C_2(G) = (\Lambda + 2\rho) \quad (9.71)$$

where ρ is the sum of fundamental weights [1]. Since Λ is given by (9.62), (9.67) becomes

$$\tilde{L}_0|\varphi\rangle = \frac{\Lambda + \rho}{2(k+g)}|\varphi\rangle = \frac{1}{2(k+g)} \left(\sum_i t_i \Lambda_i + 2 \sum_i \Lambda_i \right) \sum_j |\Lambda_j\rangle .$$

Thus, the eigenvalue is given by

$$h_\Lambda = \frac{1}{2(k+g)} \sum_{i,j=1}^r (t_i + 2)t_j A_{ij}^{-1} , \quad (9.72)$$

where $A_{ij}^{-1} = \Lambda_i \cdot \Lambda_j$. Consider the case that Λ is composed of one fundamental weight, i.e. $\Lambda = \Lambda_i$. In this case $k = 1$ and using (9.2) we obtain

$$h_{\Lambda_i} = \frac{1}{2}\Lambda_i^2 = \frac{1}{2}A_{ii}^{-1} \quad (\text{no summation of } i) . \quad (9.73)$$

Although it looks like a nontrivial task, only a few Λ can satisfy the above condition. Indeed, we have a very few candidates because of (9.64).

Given a group, the (inverse) Cartan matrix A_{ij}^{-1} is completely determined, regardless of the basis. Consider a simple group $\mathbf{h}$ as a part of unbroken group. If an element belonging to $\mathbf{h}$ satisfies the mass shell condition, i.e. those in (9.73) satisfy (9.69), then P is given by

$$P = (A^{\mathbf{h}})^{-1}_{ij} \alpha^j \quad (9.74)$$

where $A^{\mathbf{h}}$ is the Cartan matrix for the subgroup $\mathbf{h}$, but now α^j is a simple root of E_8 .

In fact this relation, especially (6.35), is oversimplified because a generic state is charged under more than one simple or Abelian algebra. Typically, it is the unbroken algebra resulting from breaking of $E_8 \times E'_8$ or $SO(32)$. Denote the whole unbroken algebra as $\bigoplus \mathbf{h}$ and the conformal weight of each simple algebra as $h_{\Lambda^{\mathbf{h}}}$. Since the conformal weight is additive, we may replace the h_{Λ} in (9.73) as the sum over the whole algebra,

$$h_{\Lambda} = \sum h_{\Lambda^{\mathbf{h}}} . \quad (9.75)$$

Also because such algebras are disconnected, we can easily write the weight vector P in (9.74) as the sum of the weight vectors of each algebra

$$P = \sum \Lambda_i^{\mathbf{h}} . \quad (9.76)$$

We will see shortly that this still holds for the Abelian groups.

9.4.3 Twisted Algebra

The twisting (6.4) make the algebra also twisted

$$T^a(\sigma + \pi) = \omega T^a(\sigma) = e^{2\pi i \eta^a} T^a(\sigma) \quad (9.77)$$

with $N\eta^a = \mathbf{Z}$. This is an automorphism so that it preserves the commutation relation, $[\omega T^a, \omega T^b] = \omega([T^a, T^b])$. Hence, we have the twisted affine Lie algebra

$$[T^a_{m+\eta^a}, T^b_{n+\eta^b}] = if^{abc} T^c_{m+n+\eta^a+\eta^b} + (m + \eta^a) \delta_{m+n+\eta^a+\eta^b, 0} \delta^{ab} K . \quad (9.78)$$

We have seen that we can always make the Cartan generator invariant. In this basis the condition (9.77) becomes the shift (6.29),

$$\omega : |P\rangle \rightarrow \exp(2\pi i P \cdot V) |P\rangle . \quad (9.79)$$

This automorphism is realized by the shift vectors given in (6.29), and the weight vector is shifted as $P + V$.

Since the Cartan subalgebra is not affected by twisting in view of (9.77) with $P \cdot V \in \mathbf{Z}$, we have still integer modded algebra. However, we can find a set of newly defined operators,

$$\tilde{E}_n^\alpha = E_{n+V \cdot \alpha}^\alpha , \quad (9.80)$$

$$\tilde{H}_n^I = H_n^I + V^I \delta_{n,0} K , \quad (9.81)$$

$$\tilde{K} = K , \quad (9.82)$$

$$\tilde{D} = D - V \cdot H_0 . \quad (9.83)$$

These tilded operators satisfy the same commutation relations (9.57). That means, the twisted algebra is isomorphic to the untwisted algebra. The adjoint representations of gauge bosons should belong to the common intersection. The grade of the ladder operator is shifted to $n + V \cdot \alpha$.

Looking at the (9.81) we see that the Cartan subalgebra of grade zero ($n = 0$) is shifted by the shift vector V . It follows that their eigenvalues, the weight vectors P are shifted into $P + V$. The other elements with $n \neq 0$ in the Cartan subalgebra are merely the grade ladder operators, in view of (9.60). Since we have the same algebra as the untwisted algebra, we also have the highest weight representation $|\tilde{\Lambda}\rangle$ for the vector $P + V$, whose eigenvalue is (9.72). The explicit highest vector representation should be

$$P + V = (A^{\mathbf{h}})_{ij}^{-1} \alpha^j \quad (9.84)$$

just as (9.74).

Now we can understand the $(\mathbf{3}, \mathbf{1})$ representation of (9.55) in the twisted sector, in terms of the highest weight representation. From (9.69), we should have $h_{\Lambda} = -\tilde{c} = \frac{2}{3}$. Since the oscillator $\alpha_{-1/3}^i$ has conformal weight $\frac{1}{3}$ (from $\tilde{N} = \frac{1}{3}$) we expect that $\frac{1}{3}$ comes from the $\mathbf{3}$ of $SU(3)$ with as many multiplicity. Indeed, by reading the inverse Cartan matrix $(A^{SU(3)})_{11}^{-1} = \frac{2}{3}$ presented in Appendix E, we obtain the corresponding highest weight representation (here $P + V$)

$$P + V = (A^{SU(3)})_{1j}^{-1} \alpha^j = \left(-\frac{1}{3} \frac{1}{3} \quad -\frac{2}{3} \quad 0^5; 0^8 \right) \quad (9.85)$$

where α^j are the E_8 simple roots. Indeed, $P = (P + V) - V = (-1 \ 0 \ -1 \ 0^5)$ is the right vector given in Subsect. 6.3.3.

9.4.4 Abelian Charge

Recall that under the shift (6.29) the Cartan generators $\{H^I\}$ remain invariant, even when their roots are prevented by the condition (6.29). This means that in the unbroken group, their linear combination $\sum_I q_i^I H^I$ plays the role of the $U(1)_i$ generator and the rank is preserved. The corresponding charge generator that projects the state vector to give the $U(1)$ charge is

$$Q_i = q_i \cdot P, \quad (9.86)$$

or $P + kV$ instead of P in the k^{th} twisted sector. The Abelian generators are proportional to the fundamental weights used (corresponding to $s_i \neq 0$) in the shift vector (9.48),

$$q_i \propto \Lambda_i \quad (9.87)$$

if the extended root of the original algebra is projected out $s_0 \neq 0$. This is true because q_i should be orthogonal to the rest of the (simple) roots, otherwise this vector would be the root vector of the corresponding nonabelian group.

We use the same index i since we have one to one correspondence between Λ_i and $U(1)$ subgroups. If the extended root survives $s_0 = 0$, we can always find the following Abelian generators q (as many as the number of Abelian groups in the fixed point algebra). By making linear combinations between the fundamental weights used in the shift vector (9.48), allowing the negative coefficient we have

$$q \propto \sum s'_i \Lambda_i, \quad s'_i \in \mathbf{Z} \quad (9.88)$$

satisfying

$$q \cdot \theta = 0 ,$$

for it should be orthogonal to the extended root $-\theta$ of the original algebra.

The normalization of q_i is related to the level k and determined by normalization of the current $T^a(z)$ [19]. The corresponding vertex operator in this direction is $q_i \cdot \partial_z X$, and has a different coefficient from (9.56). From (9.56), by fixing normalization of f^{abc} , as in (9.3), the relative normalization of the z^{-2} term should be $k = q_i^2$ in this direction. For Abelian groups, the structure constants vanish, and the normalization has to be fixed in another way. However, at the compactification scale of an orbifold, this $U(1)$ generator is embedded in $E_8 \times E'_8$ and thus has a definite normalization

$$q_i^2 = k \quad (9.89)$$

to 1, as discussed before. The conformal weight for a state is

$$h_{Q_i} = \frac{1}{2} Q_i^2 = \frac{1}{2} (q_i \cdot P)^2. \quad (9.90)$$

Comparing to the similar relation (9.73), we can determine the $U(1)$ charged piece of vector P . Interestingly, it is also proportional to q_i : The other parts of P are fundamental weights of the unbroken nonabelian group, which should not be charged under this $U(1)$,

$$q_i \cdot P = q_i \cdot r, \quad r \propto \Lambda_i \propto q_i. \quad (9.91)$$

This means that we can decompose the shift vector into completely disconnected parts. The resulting state vector is

$$P = \sum A_{ij}^{-1} \alpha^{jV} + r. \quad (9.92)$$

The normalization of q is fixed by (9.90). In general, states may be charged under more than one $U(1)$ s: then the vector is simply the addition of each $U(1)$ part.

There are potential anomalous $U(1)$ s. Since they are embedded in the original group $SO(32)$ or $E_8 \times E_8$, they can be rearranged to one $U(1)$. This is cancelled by the Green-Schwarz mechanism, if the $U(1)$ charges of the whole spectrum satisfy a specific “universality” condition. It also fixes normalization [19, 20, 21], and our normalization gives the correct answer. Using this [22],

if we have at least one anomalous $U(1)$, the GS mechanism fixes the normalization in four dimensional theory, regardless of the origin of group breaking, which in this case is orbifolding.

This is the only way to find Abelian generator systematically. The information is extracted, from which Cartan subalgebras became Abelian generators after symmetry breaking.

Example

Consider the $T^6/\mathbf{Z}_3$ example with the shift vector

$$V = \frac{1}{3}(2 \ 0^7)(1 \ 1 \ 0^6) = \frac{1}{3}(\Lambda_7)(\Lambda_1) .$$

We can check that the modular invariance condition is satisfied and the resulting gauge group is $SO(14) \times U(1) \times E_7 \times U(1)$. The two $U(1)$ generators are $q_7 = \frac{1}{2}(\Lambda_7; 0)$ and $q'_1 = \frac{1}{\sqrt{2}}(0; \Lambda_1)$ by the normalization (9.89). Note that this gives the correct normalization [21] for the Green-Schwarz mechanism. In view of the branching rule, in the untwisted sector we obtain

$$3(\mathbf{14})(\mathbf{1}) + 3(\mathbf{64})(\mathbf{1}) + 3(\mathbf{56})(\mathbf{1}) + 3(\mathbf{1})(\mathbf{1}) . \quad (9.93)$$

In the twisted sector, the zero point energy is still $\tilde{c} = -\frac{2}{3}$. The $SO(14)$ vector with $h_{\mathbf{14}} = \frac{1}{2}$ alone cannot be massless, but should have other components to fulfill the mass shell condition. The missing mass is provided by other vectors r_7 and r'_1 charged under $U(1)$ s.

The corresponding highest weight vector has the form

$$P + V = \sum_j (A^{SO(14)})_{1j}^{-1} \alpha^j + r_7 + r'_1 . \quad (9.94)$$

The first term is Λ_1 of $SO(14)$. The r_7 and r'_1 are also proportional to $(\Lambda_7)(0)$ and $(0)(\Lambda_1)$, respectively. They are completely fixed by the condition

$$h_Q = \frac{1}{2}(q_7 \cdot r_7)^2 + \frac{1}{2}(q'_1 \cdot r'_1)^2 = \frac{1}{6} , \quad (9.95)$$

and the generalized GSO projection condition. The resulting vector is

$$P + V = (0 \ 1 \ 0^6)(0^8) + \left(-\frac{1}{3}\right)(1 \ 0^7)(0^8) + \frac{1}{3}(0^8)(1 \ 1 \ 0^6) , \quad (9.96)$$

and charged as $(\mathbf{14})(\mathbf{1})$. The Lorentz $\mathbf{3}$ of $SU(3)$ (by $\alpha_{-1/3}^i$) can contribute $h = \frac{1}{3}$ and it provides another charged state, $(\mathbf{1})(\mathbf{1})$. In addition, there is a state which is a singlet under the whole nonabelian group $(\mathbf{1})(\mathbf{1})$. They all have multiplicity $\chi = 27$.

9.4.5 Complete Spectrum of SO(32) String

The group structure of SO(32) is particularly simple. It is because, there are roots with integral elements only. Hence the Weyl reflection, which is the only meaningful automorphism, rearranges their positions and signs only, by (9.13) and (9.14). As a consequence, the shift vectors are classified only by the number of elements. Any vectorial shift of $\mathbf{Z}_{2N}$ orbifold can be brought to the following form

$$V = \frac{1}{2N} (0^{n_0}, \dots, N^{n_N}), \quad \text{with} \quad \sum_{k=0}^N n_k = 16, \quad (9.97)$$

(up to subtraction of $(1, 0^{15})$ and change of one minus sign) and lead to the symmetry breaking pattern

$$SO(32) \rightarrow SO(2n_0) \times U(n_1) \times \dots \times U(n_{N-1}) \times SO(2n_N). \quad (9.98)$$

We employ a subscript to make the distinction between the various factors, for example $U(n_k) = U(1)_k \times SU(n_k)$ and $U(1)_0 = SO(2n_0)$ when $n_0 = 1$. Of course these shift vectors should satisfy conditions on the modular invariance and the evenness of the sum.

Table 9.5. The twisted matter of SO(32) orbifold models: the k -form representations $[\mathbf{n}]_k^\pm$ of $U(n)$ and the vector $\mathbf{2n}$ and spinor $\mathbf{2}_\pm^{n-1}$ representations of SO(2n). For the three cases, $\frac{1}{2}(P + \tilde{V})^2$ are $\frac{k}{2}, k(\frac{1}{2} + \alpha\tilde{V}) + \frac{1}{2}n\tilde{V}^2$, and $\frac{n}{8}$, respectively

$\tilde{V}$	Group	Repr.	Weights	Prop.
$\tilde{V} = 0$	SO (2n)	$[\mathbf{2n}]^k$	$(\pm 1^k, 0^{n-k})$	$k = 0, 1$
$0 < \tilde{V} < \frac{1}{2}$	U (n)	$[\mathbf{n}]_k^\alpha$	$\alpha(1^k, 0^{n-k})$	$\alpha = \pm$ $k \geq 0$
$\tilde{V} = \frac{1}{2}$	SO (2n)	$\mathbf{2}_\alpha^{n-1}$	$(-\frac{1}{2}^k, \frac{1}{2}^{n-k}) - (\frac{1}{2}^n)$	$\alpha = (-)^k$

Looking at the mass shell condition, we conclude that only a few kinds of representations can appear. The result is tabulated in Table 9.5. All $U(n)$ representations are totally anti-symmetric k -form tensor representations of the vector $\mathbf{n}$ or its complex conjugate $\bar{\mathbf{n}}$, denoted by $[\mathbf{n}]_k^+$ and $[\mathbf{n}]_k^-$, respectively. (In particular, $[\mathbf{n}]_0^\pm = \mathbf{1}$, $[\mathbf{n}]_1^+ = \mathbf{n}$, $[\mathbf{n}]_1^- = \bar{\mathbf{n}}$, and $[\mathbf{n}]_k = [\bar{\mathbf{n}}]_{n-k}$.) The representations of SO(2n) that arise are the fundamental representation $[\mathbf{2n}]^k$ or the spinor representation $\mathbf{2}_\alpha^{n-1}$ of $\alpha = \pm$ chirality. The index $k = 0, 1$ is used to simultaneously treat the fundamental and the singlet representation. In the SO(32) heterotic string case, we can explicitly prove the no-adjoint theorem by showing that only the weights, whose entries in the Dynkin basis does not exceed level k , can appear.

Using these representations we can identify the irreducible *twisted states*. The vectorial weights give rise to representations of the form

$$\mathbf{R} = ([2\mathbf{n}_0]^{k_0}, [\mathbf{n}_1]_{k_1}^{\alpha_1}, \dots, [\mathbf{n}_{m-1}]_{k_{m-1}}^{\alpha_{m-1}}, 2\mathbf{n}_m^{\alpha_m}), \quad (9.99)$$

where $\alpha_a = \pm$, $k_0 = 0, 1$ and $k_a \geq 0$. The mass contribution of this state reads

$$\frac{M_L^2}{4} = \frac{k_0}{2} + \sum_{a=1}^{m-1} k_a \left(\frac{1}{2} + \alpha_a \tilde{v}_{pa} \right) + \frac{1}{2} (\tilde{v}_p)^2. \quad (9.100)$$

The GSO projection on the vectorial weights require that

$$\frac{1 - \alpha_m}{4} + \frac{1}{2} \sum_{a=0}^{m-1} k_a \in 2\mathbf{Z}. \quad (9.101)$$

So all we need to do is to solve these linear equations. As a result we obtain general spectrum including twisted sector, which is presented in [23].

9.4.6 Higher Level Algebra

In the ordinary setup of string embedding, only the level one algebra is allowed, because the whole normalization is fixed by (9.3). The relative normalization of z^{-2} term in (9.56) should be $k = 1$ [24]. To have an adjoint representation based on $SU(5)$, $SO(10)$ or E_6 , evading the no-adjoint theorem (9.64), we should have higher level algebra $k \geq 2$.

Although the normalization is fixed from the beginning, we can modify the theory by embedding the group into a larger group and changing the GSO projection to have a different normalization [9]. We take one heuristic example of level 2 $SU(2)_2$ algebra from two $SU(2)_1$'s, where the subscript denotes the level of algebra [25]. Let us call them $SU(2)^A \times SU(2)^B$ and form root space orthogonal to each other. Simple root vectors α^A and α^B of both groups have length squared two as before. Projecting the root vectors of each group by keeping symmetrical combinations,

$$j^v(z) = j^A(z) + j^B(z), \quad (9.102)$$

we have another $SU(2)_2$. Since they come from the same group, so that j^v have twice the length, thus they lead to a new normalization $k = 2$. It is easily viewed in the root space depicted in Fig. 9.7. The $\bullet$'s indicate the original roots of each $SU(2)$ s. After applying some suitable GSO projection (dashed line), we have new $SU(2)$ roots $\circ$ having length squared 1.

This shows a typical *regular diagonal embedding*, $\mathfrak{g}_k \subset \mathfrak{g}_1^k$. In this notation, the example in the preceding paragraph is $SU(2)_2 \subset SU(2)_1^2$. However, there is another kind of embedding such as $SU(2)_4 \subset SU(3)_1$ or $SU(2)_{28} \subset (G_2)_1$ [1].

Beginning with $E_8 \times E_8'$ and $SO(32)$, a limited number of semi-simple groups can be obtained. In other words, an embedding into an algebra with

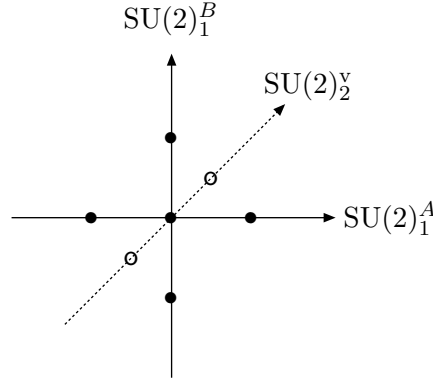


Fig. 9.7. The level 2 algebra $SU(2)_2$ is obtained by projecting to symmetrical currents from two level 1 groups $[SU(2)_1]^2$

rank $r > 16$ is impossible, when we require level 3 algebra $(E_6)_3$ diagonally embedded in $(E_6)^3$. Most embeddings in these semi-simple groups are introduced in [1]. A systematic analysis is done in [9] and there are some models in such higher level algebra realizing $SO(10)$ [11] and E_6 [12]. Also there is a very powerful model-independent criterion on $SO(10)$ GUT from higher level algebra [10]. A more detailed study revealed more strong constraint: although **126** of $SO(10)$ is a tensor representation and thus is not ruled out by the no go theorem, there is no way to obtain it [10]. This representation **126** is needed to allow see-saw neutrino masses purely from $SO(10)$ representations and R -parity as well.

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Orbifold Phenomenology

In this chapter, we review several attempts toward obtaining a supersymmetric standard model from the $E_8 \times E'_8$ heterotic string and some phenomenological implications. In Chaps. 6–9, we discussed in length the theoretical framework for orbifolded superstrings. We restricted our discussion to the $E_8 \times E'_8$ heterotic string, chiefly because breaking the chain of E_8 down to the $SU(5)$ or $SU(3)^3$ goes through the intermediate **27** of E_6 . This is an important observation since the fifteen chiral fields (2.8) of the standard model (SM) are contained in the fundamental **27** of E_6 or in the spinor **16** of $SO(10)$. By restricting ourselves to only this chain, we can automatically achieve the correct charge assignments, coming from **16** of $SO(10)$. On the other hand, if the intermediate step cannot contain **27** of E_6 , it is most probable that exotically charged particles (exotics) would appear.¹ But this argument utilizes matter only from the untwisted sector. For twisted sector matter, $SO(32)$ breaking can be useful as commented in Sect. 10.4.

In a sense, a true phenomenological study of orbifold models is premature since there is presently no *standard orbifold model* (SOM) to compare different predictions from different models. Therefore, in this chapter we will be satisfied with just showing that the theoretical problems encountered in Chap. 2 can have solutions in orbifold models.

Among the various aspects of heterotic strings on orbifolds, we list three grand unification (GUT) items which are needed at the outset.

- *Adjoint problem:* The affine Lie algebra or Kac–Moody algebra with level $k = 1$ does not allow an adjoint representation as discussed in Chap. 9. Thus, to break a string derived GUT group from level 1 algebra down to the SM, one needs a GUT group with rank $r \geq 5$. If a string derived SM is achieved rather than through an intermediate GUT step, this adjoint problem is not present.

¹ Exotics can come in two varieties: chiral exotics and vectorlike exotics. The more dangerous chiral exotics has the problem to be observed below the electroweak scale.

- *Hypercharge quantization:* The weak mixing angle $\sin^2 \theta_W$ at the GUT scale seems to be close to $\frac{3}{8}$. If the fifteen chiral fields are put in a GUT without exotic particles, the weak mixing angle is $\frac{3}{8}$. This kind of GUT has the appropriate hypercharge quantization. Some GUTs allow automatic hypercharge quantization.
- *Three families:* The GUTs obtained from superstring theory must allow three families.
- *Flavor problem:* A model should not be excluded from the obvious violation of mass and/or mixing angle relations.

Because of the gauge hierarchy problem, it is of utmost importance to derive either the minimal supersymmetric standard model (MSSM) or just the SM itself from a compactification of 10D superstring theory. At present, it is fair to say that there is no universally accepted MSSM from superstrings. If the MSSM is obtained by compactification, it goes without saying that the model unifies gravity with elementary particle forces.

In this chapter, we review attempts toward obtaining the MSSM from superstring theory and some phenomenological aspects relevant to orbifold compactification. In addition, we will explore applications of the toolkit given in Chap. 6. Hence, in the last section we construct a $\mathbf{Z}_{12}$ orbifold in detail to show; a) how the non-prime orbifolds can be constructed with nontrivial χ ; b) how three families arise without Wilson lines; and c) to see whether a flipped $SU(5)$ is possible or not in this specific example.

10.1 String Unification

10.1.1 Relation Between Gauge Couplings

In heterotic string, we have simple relation between coupling constants at the ten dimensional Planck scale [1]

$$g_{10}^2 = \frac{4\kappa_{10}^2}{k\alpha'} . \quad (10.1)$$

Here $\alpha' \sim M_s^{-2}$ is the string scale squared and k is the level of affine Lie algebra. It is fixed by supersymmetry and extra term responsible for Green–Schwarz mechanism, and also by calculating three point correlation functions of gravitons and gauge bosons. This relation holds the same when we reduce the space dimensions by compactifying the internal space and also absorbing the dilaton VEV,

$$g_{YM}^2 = \frac{4\kappa^2}{k\alpha'} . \quad (10.2)$$

Phenomenologically, the unification scale M_U is around $(2-3) \times 10^{16}$ GeV. The standard folklore is that M_U is better to be realized around the string scale M_s . Although they are very close, they still differ by a factor of $O(10)$,

assuming order one g_{10} . Many ideas are suggested to remedy this discrepancy, including threshold correction (8.109). Also it is noted that the Type II construction can pull down the string scale, as we will see in Appendix B.2.

10.1.2 Gauge Coupling Unification

From (10.2), we can deduce a relation between gauge couplings,

$$k_3 g_3^2 = k_2 g_2^2 \quad (10.3)$$

where k_3, k_2 are the levels of $SU(3)$ and $SU(2)$ algebras, respectively. For non-abelian gauge groups, therefore, it is a simple matter to relate gauge couplings. Also it is shown that if we have at least one anomalous $U(1)$, we can even relate the coupling of this $U(1)$ the same as (10.3).

In heterotic string, we have always $k = 1$ for the unified gauge group $E_8 \times E_8$ or $SO(32)$, and they stay unchanged under the usual GSO and orbifold projections by shift vectors. Therefore, we can choose $k_1 = 1$ and thus the relation

$$g_1 = g_2 = g_3 = g_{10} \quad (10.4)$$

reduces to the conventional one, as discussed in Sect. 2.2. It implies a possibility that we can achieve the gauge unification with the value g_{10} at the compactification scale even without an intermediate GUT. This also follows from the *inheritance principle* that gauge couplings are determined by gauge bosons appearing in the untwisted sector. However, determination of gauge couplings at the compactification scale is not so trivial because there is a small difference between the compactification scale and the Planck scale. The general automorphism discussed in Sect. 9.2 does not give $k = 1$ in general since this method mixes two or more groups which affects the normalization of currents as discussed in Subsect. 9.4.6.

We discussed the possibility of higher level algebras in Subsect. 9.4.6, which required nontrivial projections other than the GSO projection.

10.2 Standard-Like Models

One step toward the unification of elementary particle forces near the Planck scale is the observation of drastically different magnitudes of the QCD and weak couplings at the electroweak scale: $\alpha_3 \simeq 0.120$ and $\alpha_2 \simeq 0.0336$. This difference can be explained most naturally by the Georgi-Quinn-Weinberg (GQW) mechanism [2]. With the GQW mechanism, GUTs unifying coupling constants can be considered near the Planck scale. In 4D string models also, the nonabelian gauge couplings can be unified at the string scale even if a GUT group is not assumed as explained in the previous section.

Even though one obtains factor groups such as those in the SM, there is a possibility that the nonabelian couplings are unified at the string scale. Thus,

it is not a bad idea to obtain an SM or MSSM directly via some orbifold compactification of string theory.

With the simplest and most widely used level $k = 1$, it is known that the adjoint representation is not allowed in the spectrum as discussed in (9.64). This is the *problem of adjoint representation in string models*. If one goes beyond $k = 1$, then the resulting spectrum must be much more abundant and would not be a minimal model. Models with differing k_i do not unify coupling constants. Therefore, it has been hoped that a reasonable MSSM can be obtained at the level $k = 1$. In this spirit, the initial motivation was to obtain the factor group $SU(3) \times SU(2)$ directly by compactification.

Soon after orbifold compactification was introduced, the first model building attempt was to try $SU(3) \times SU(2) \times \cdots$ [3]. This class of models is known as *standard-like models*. The standard-like models required the following features toward a supersymmetric standard model.

- It must already contain the SM gauge group $SU(3) \times SU(2) \times U(1)_Y$, i.e. there is no need for an adjoint representation to obtain the SM.
- Fifteen chiral fermions with the correct electroweak hypercharges are contained in the spectrum.
- Three families are in the spectrum.
- There must exist at least a (pair of) Higgs doublet(s) $\mathbf{H}_1$ (and $\mathbf{H}_2$ for the case of supersymmetry) for the electroweak symmetry breaking.

Supersymmetry may be added to this list. Because of the three families requirement, $\mathbf{Z}_3$ orbifold models with two Wilson lines are helpful from the outset. In some orbifold models, it is possible to exclude extra colored scalars [3], which achieves the doublet-triplet splitting. But a standard-like model has some flaws since it is not quite a supersymmetric standard model yet for the following reasons.

- It contains too many extra $U(1)$ s which have to be broken.
- There are too many extra chiral fields with exotic electroweak hypercharges.
- There are too many Higgs doublets (a minimum of six) in $\mathbf{Z}_3$ orbifold models with two Wilson lines. It is hoped that with three Wilson lines, there is a possibility to obtain just two Higgs doublets in the twisted sector; however such a model has not been found yet.

The standard-like models obtain $SU(3)$ and $SU(2)$ directly by compactification. All orbifold models considered toward this objective used the following type of shift vector and Wilson line(s) [3, 4, 5, 6, 7],

$$\begin{aligned} V^I &= \left(\frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{2}{3} \cdots \right) (\cdots) \\ a_1^I &= \left(\frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{2}{3} \frac{1}{3} \cdots \right) (\cdots) \end{aligned} \tag{10.5}$$

where $\dots$ are chosen to satisfy the modular invariance conditions discussed in Chap. 6 but not to enhance $SU(3)$ and $SU(2)$ to some larger groups. A kind of skewing results in the $SU(5)$ group space, as implied by the first five entries of (10.5), along with many extra particles beyond those of the SM. Even though we ambitiously began without the need for the adjoint representation of some GUT group, we may have reached the limitation of the standard-like models.

A similar situation is encountered in the family unification GUT with skewing of the $SO(4n+2)$ group space. Since it is intuitive to understand why these kinds of skewing lead inevitably to queerly charged particles, let us consider the spinor representation **64** of $SO(14)$ [8]. When $SO(14)$ breaks down to $SU(5)$, the branching rule of **64** is $2 \cdot (\mathbf{10} + \mathbf{\bar{5}} + \mathbf{1} + \mathbf{10} + \mathbf{5} + \mathbf{\bar{1}})$, i.e. there are two families of $SU(5)$ but with a vector-like form, so no chiral family results at low energy. In terms of the SM quantum numbers, these vector-like representations are

$$\begin{aligned} & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \\ & \begin{pmatrix} \nu'_e \\ e' \end{pmatrix}_R, \quad \begin{pmatrix} u' \\ d' \end{pmatrix}_R, \quad \begin{pmatrix} \nu'_\mu \\ \mu' \end{pmatrix}_R, \quad \begin{pmatrix} c' \\ s' \end{pmatrix}_R \end{aligned} \quad (10.6)$$

where we have not shown 32 $SU(2)$ -singlet charge-conjugated fields with the same chiralities (L for the charge-conjugated unprimed singlets and R for the primed ones). The aforementioned skewing raises the electromagnetic charges by one unit for the right-handed e' family and lowers them by one unit for the right-handed μ' family, which is allowed in the $SO(14)$ model. These shifts give [8],

$$\begin{aligned} & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \\ & \begin{pmatrix} \tau^+ \\ \bar{\nu}_\tau \end{pmatrix}_R, \quad \begin{pmatrix} q_{5/3} \\ t \end{pmatrix}_R, \quad \begin{pmatrix} E^- \\ E^{--} \end{pmatrix}_R, \quad \begin{pmatrix} b \\ q_{-4/3} \end{pmatrix}_R. \end{aligned} \quad (10.7)$$

The model (10.7) gives three standard lepton families and two standard quark families, with the rest being queer states. In particular, t and b , even though their electromagnetic charges coincide with those in the SM, do not belong to the same doublet, which was proven wrong by the decay modes of b in models with t and b in different doublets [9]. Also, there appear queerly charged particles, E^{--} , $q_{5/3}$, and $q_{-4/3}$.

As in the above skewed model, in most standard-like models the appearance of queer particles is ubiquitous, which is one of the reasons these models are *not called standard but called standard-like*.

10.3 Three Families

10.3.1 $\mathbf{Z}_3$ Orbifolds

In $\mathbf{Z}_3$ orbifold models, the chiral fermions arise in multiples of 3 if we use two Wilson lines. This is why two Wilson line models are much more important than any other $\mathbf{Z}_3$ orbifold, and we will tabulate all the relevant two Wilson line models in the next chapter.

However, for three families we observe that if the needed quark doublets $(\mathbf{3}, \mathbf{2})$ from a $\mathbf{Z}_3$ orbifold appear in the untwisted sector, then the resulting spectrum which includes those from the twisted sectors will most probably lead to three generations. The key point is that the multiplicity in the untwisted sector is 3 and the spectrum from the other sectors wherever they come from may be adjusted to a three family model if the hypercharge Y in $(\mathbf{3}, \mathbf{2})$ is correctly given. Indeed, this was shown in cases with three Wilson line models of [6, 10]. There is a potential problem however, on the mass spectrum with three families from the untwisted sector. If the quark singlets also appear from the untwisted sector, then we have the wrong relation $m_c = m_t$ [11], not overcoming the flavor problem listed in the beginning of this chapter. It is better for the quark doublets to appear from twisted sectors. Then, three families arising from twisted sectors are automatically obtained from models with two Wilson lines. But two Wilson line models have a high level of degeneracy. Recently, this degeneracy was used to obtain a massive third family [12, 13].

If all three families appear from the twisted sectors in three Wilson line models, it will be of great interest because any degeneracy among families can be avoided. For example, it was pointed out in (8.75) that the Yukawa couplings can be exponentially suppressed, $e^{-l_{ij}}$ where l_{ij} is proportional to the distance between the fixed points T_i and T_j housing two respective chiral fermions. This can be a geometrical understanding of the quark mass hierarchy [14].²

10.3.2 Other Three-Family Schemes

There are other models in which three families are possible. One early example was the so-called stringy flipped SU(5) with the Antoniadis-Ellis-Hagelin-Nanopoulos (NAHE) set and its follow-ups with the standard-like models where the fermionic construction of 4D string was employed [17]. The NAHE set in the fermionic construction will be briefly reviewed in Appendix B.1. It has been argued that the three family models with fermionic construction were possible because of the underlying $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry in their construction [18]. As noted in Appendix B.1 regarding b_1, b_2 , and b_3 boundary conditions,

² In intersecting brane models also, the geometrical interpretation can be considered [15].

the NAHE set in the fermionic construction is at the root of three-family model construction. The set b_1, b_2 , and b_3 can represent a $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry. But from the fermionic construction, it is difficult to visualize the symmetry geometrically. Recently, a $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold was constructed which resulted in three families [19].

Other three-family models were obtained recently with the Pati-Salam group in $\mathbf{Z}_6$ orbifold models [20].³ The motivation of [20] was to insert a $\mathbf{Z}_2$ symmetry so that the recent 5D field theoretic orbifolds using $\mathbf{Z}_2$ discussed in Chap. 4 are derivable from string construction. Here, different scales of tori are necessarily introduced since one is trying to achieve intermediate 5D physics. It seems that these kinds of three families use non-prime or $\mathbf{Z}_M \times \mathbf{Z}_N$ orbifolds. In fact, it was argued long ago that three families might arise from $\mathbf{Z}_6 = \mathbf{Z}_3 \times \mathbf{Z}_2$ orbifolds [4]. These non-prime or $\mathbf{Z}_M \times \mathbf{Z}_N$ orbifolds try to distinguish three 2-tori, as shown in Fig. 10.1 for the case of $\mathbf{Z}_2 \times \mathbf{Z}_2$. It must be so. Referring to Table 3.2, the $\mathcal{N}=1$ SUSY results from $\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_6, \mathbf{Z}_7, \mathbf{Z}_8$ and $\mathbf{Z}_{12}$. Except $\mathbf{Z}_3$ and $\mathbf{Z}_7$, the others are non-prime orbifolds. These non-prime orbifolds reduce to $\mathcal{N}=1$ even though their factors started with $\mathcal{N}=2$ supersymmetry of Table 3.2, thus the situation of Fig. 10.1 arises; each $\mathbf{Z}_N$ reduces $\mathcal{N}=4$ to $\mathcal{N}=2$ and the common intersection is $\mathcal{N}=1$.

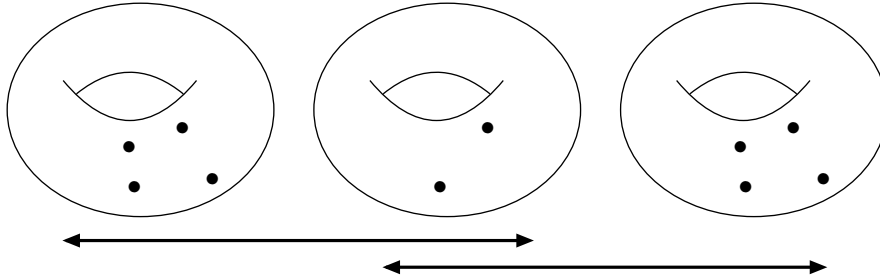


Fig. 10.1. A $\mathbf{Z}_2$ applied to the internal space (45) and (67) tori and another $\mathbf{Z}_2$ applied to (67) and (89) tori

In these non-prime or $\mathbf{Z}_M \times \mathbf{Z}_N$ orbifolds, the reason for three families is not so obvious. One can certainly check the result later, after all the dust settles. For the case of $\mathbf{Z}_2 \times \mathbf{Z}_2$, the representation can be $(\mathbf{2}, \mathbf{2}), (\mathbf{2}, \mathbf{1}), (\mathbf{1}, \mathbf{2})$, and $(\mathbf{1}, \mathbf{1})$. The untwisted sector and twisted sectors can have these multiplicities and three families can actually result from their combinations [20, 19]. As this example shows, the reasons for three families can be numerous. One speculation, given long ago, is that it may be due to $3 = (\text{number of compactified dimensions})/2$ [21]. In a sense, this can be glimpsed with the NAHE set with b_1, b_2 , and b_3 which are represented by ω and ϕ in Appendix B.1. Looking at b_1, b_2 , and b_3 , one notices that they result from 6 real dimensions, i.e. $3 = 6/2$. But, the three families with the NAHE set need several more

³ In intersecting brane models, the Pati-Salam group is obtained easily [16].

boundary conditions which do not necessarily satisfy the $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry. If this speculation on $3 = 6/2$ is to be of any use, it must be supplied with other powerful conditions since we know that not all string compactification models lead to three families.

In a fermionic construction, Kakushadze and Tye [22] obtained an $\text{SO}(10)$ model with an adjoint Higgs. As is well known, the adjoint Higgs does not come from the level one algebra; rather their construction is realized in the level three current algebra. In this case, if one considers standard-like models then there are too many extra fields, and colored particles are not guaranteed to be just $\mathbf{3}$ and $\bar{\mathbf{3}}$ only.⁴ But GUTs may not introduce representations allowing color sextets since the dimension of such representations will be huge since string models are limited to have a few hundred chiral fields.

Before concluding, we briefly remark on some possibilities from other construction methods. In the Calabi–Yau compactification, which we will briefly present in Appendix A, the number of families is given by topological numbers. For example, in the simplest case of the standard embedding with gauge group E_6 , the number of families is equal to half the number of genus. This property might be changed in the nonstandard embeddings with the presence of fluxes. Also, in the intersecting brane models, where the quarks and leptons are localized at intersection points, the number of families is the number of intersection points, which again is given by the geometry one sets up.

10.4 U(1) Charges

10.4.1 Hypercharge Identification

To distinguish the SM chiral fields among the plethora of fields, one has to identify the $\text{U}(1)$ quantum numbers. Many $\text{U}(1)$ s arise from the left-over Cartan subalgebras, and all the $\text{U}(1)$ charges are determined at the compactification scale. Among them, identifying the hypercharge $\text{U}(1)_Y$ is of utmost importance. The method of obtaining $\text{U}(1)$ generators is described in Subsect. 9.4.4. Conveniently, they are chosen to be orthogonal to each other.

A weight in the lattice can have simultaneous eigenvalues for mutually commuting diagonalized generators. We have 16 mutually commuting generators. Here the Dynkin technique, where $\text{SU}(2)$ roots are the building blocks, is quite useful. The rank is the number of independent $\text{SU}(2)$ directions and also is the number of mutually commuting generators. Consider an $\text{SU}(2)$ subgroup whose raising and lowering operators are $E_1^{(1)}$ and $E_{-1}^{(1)}$ with the commutator $[E_1^{(1)}, E_{-1}^{(1)}] = 2T_3^{(1)}$. The vector corresponding to $T_3^{(1)}$ sits at the origin in the root diagram. But we know that the root vector $E_{\pm 1}^{(1)}$ has the $T_3^{(1)}$ eigenvalue ± 1 . The trick to obtain this eigenvalue is to introduce the Dynkin basis, the

⁴ Exclusion of color $\mathbf{6}$, etc. is an important rule toward bigger symmetries. For an example, see [23].

weights $\{\Lambda_i\}$ dual to the simple roots $\{\alpha_i\}$ as done in (9.4). If $E_1^{(1)}$ is a simple root, its $T_3^{(1)}$ eigenvalue is 1, obtained by $E_1^{(1)} \cdot \Lambda^{(1)}$, which is nonvanishing since $\Lambda^{(1)}$ is belonging to the nonabelian group in consideration. The dual space dimension being the same as the number of simple roots, we can completely identify 16 mutually commuting operators in the dual space. In this way, we can obtain 16 eigenvalues of $T_3^{(i)}$ ($i = 1, 2, \dots, 16$) for nonvanishing roots. This method of obtaining the eigenvalues of the diagonalized $T_3^{(i)}$ can be generalized for any weight W as $W \cdot \Lambda_i$. In terms of string states, it was discussed in Subsect. 8.1.1.

Orbifolding removes some of simple roots. If the rank is not reduced, then there appears some U(1) operators from the original 16 mutually commuting operators. Since a U(1) generator in consideration is chosen such that it commutes with all the other nonabelian generators, any root vector of the nonabelian group has the vanishing U(1) eigenvalue. These eigenvalues are identified as the scalar product, $\alpha_{\text{nonabelian}} \cdot \Lambda_{\text{U(1)}} = 0$, where Λ is a fundamental weight not belonging to a set of duals of $\alpha_{\text{nonabelian}}$, viz. (9.4). Thus, the essential feature is that U(1) generators are constructed as linear combinations of fundamental weights, satisfying

$$\alpha_{\text{nonabelian}} \cdot \Lambda_{\text{U(1)}} = 0 \quad (10.8)$$

for all nonvanishing roots, $\alpha_{\text{nonabelian}}$. The above vanishing scalar product condition does not fix the normalization of U(1) generators. But, the OPE fixes the normalization of hypercharges to be, (9.89),

$$q_i^2 = k = 1. \quad (10.9)$$

In this language, there is one to one correspondence between charge raising and lowering $E_{\pm\alpha}$ generators (and U(1) generators) and lattice vectors satisfying $P^2 = 2$ (and fundamental weights). The *U(1) eigenvalue of any weight*, say W , is similarly defined as a scalar product

$$W \cdot \Lambda_{\text{U(1)}}. \quad (10.10)$$

Now consider heterotic string. For gauge charges, consider the vertex operators of the left-movers,

$$V = e^{ik \cdot X} e^{i(P+V) \cdot F} \quad (10.11)$$

where $P + V$ is the weight we consider and F corresponds to the U(1) gauge group in the above sense. For the untwisted sector, we set $V = 0$ in (10.11). Thus, the eigenvalue of Q^I is⁵

$$Q^I |\psi_{\text{un}}(P)\rangle = P \cdot F^I |\psi_{\text{un}}(P)\rangle.$$

⁵ Note that Q^I on the LHS is an operator and $P \cdot F^I$ on the RHS is a number. For $|\psi_{\text{tw}}(P + V)\rangle$, the eigenvalue is $(P + V) \cdot F^I$.

The eigenvalue $P \cdot F^I$ is the scalar product of two vectors, P and F^I . In the $E_8 \times E'_8$ heterotic string, the simplest 16 vectors corresponding to 16 Cartan generators can be $(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)(\cdots)'$, $(0\ 1\ 0\ 0\ 0\ 0\ 0\ 0)(\cdots)'$, etc. The number of $U(1)$ generators is ≤ 13 because we need at least the nonabelian group, $SU(3) \times SU(2)$. These $U(1)$ generators are denoted as Q_i ($i = 1, 2, \dots, n$) with $i_{\max} = n \leq 13$. Thus, for the untwisted sector $|P\rangle \equiv |\psi_{\text{un}}(P)\rangle$,⁶

$$Q_i|P\rangle = P \cdot q_i|P\rangle. \quad (10.12)$$

Similarly, the eigenvalues of Q_i on the twisted states $|P_{\text{twisted}} + V\rangle$ are given by $(P_{\text{twisted}} + V) \cdot q_i$. Summarizing, the eigenvalues of the $U(1)$ charges are

$$U \text{ sector : } Q_i(P_{\text{untwisted}}) = q_i \cdot P_{\text{untwisted}} \quad (10.13)$$

$$T(\{n_j\}) \text{ sector : } Q_i(P_{\text{twisted}}) = q_i \cdot \left(P + V + \sum_{n,j} \epsilon_j a_n \right) \quad (10.14)$$

where $\epsilon_j = 0, +1$, and -1 , and a_n is the Wilson line.

As an example, consider the Wilson lines of the standard-like model (10.5), which contains the gauge group $SU(3) \times SU(2) \times U(1)$ and a number of other $U(1)$ s. In this model, the untwisted sector has the following weights for quark doublets,

$$\begin{cases} P_u = (\underline{1\ 0\ 0\ 1\ 0\ 0\ 0\ 0})(0^8)' \\ P_d = (\underline{1\ 0\ 0\ 0\ -1\ 0\ 0\ 0})(0^8)' \end{cases} \quad (10.15)$$

We find that, the quark doublets are charged under the following basis vectors

$$q_1 = \frac{1}{\sqrt{3}}(1\ 1\ 1\ 0\ 0\ 0\ 0\ 0)(0^8)' \quad (10.16)$$

$$q_2 = \frac{1}{\sqrt{2}}(0\ 0\ 0\ 1\ -1\ 0\ 0\ 0)(0^8)' \quad (10.17)$$

which are properly normalized as in (9.89). We identify hypercharge generator Q_Y as

$$Q_Y = \frac{1}{\sqrt{3}}q_1 - \frac{1}{\sqrt{2}}q_2. \quad (10.18)$$

One verifies that the corresponding generator is nothing but Y of (2.15). The up and down quark representations appear in the right places as in (2.9). Indeed, it gives rise to the desired hypercharges

$$\begin{aligned} Q_{Y,u} &= q_Y \cdot P_u = \frac{1}{6} \\ Q_{Y,d} &= q_Y \cdot P_d = \frac{1}{6}, \end{aligned} \quad (10.19)$$

and hence we expect the correct normalizations for C and $\sin^2 \theta_W^0$.

⁶ Q_i on the LHS is an operator and q_i on the RHS is a vector.

It seems that we should look for the hypercharge with a clever insight. However, the regular pattern inherits a structure of the unified group SU(5). One is easily convinced that the above weight vectors of quark doublet (10.15) is a part of representation **10** of SU(5)

$$(\underline{1} - 1 \ 0 \ 0 \ 0 \ 0^3)(0^8) . \quad (10.20)$$

Indeed, without Wilson line vector a in (10.5), the unbroken group contains SU(5). Without such a unified group, we cannot expect the desired charge pattern for quarks and leptons. However, since the problem arises if some fields are outside the complete multiplet such as **5** or **10** of SU(5), the hypercharge quantization is not the same as that of the SU(5) GUT. Not only this kind of model cannot give the desired hypercharge, but also gives rise to “exotic” particles having strange charges.

10.4.2 Weak Mixing Angle

The U(1) charge normalization is an important issue. At the GUT scale, the weak mixing angle is

$$\sin^2 \theta_W^0 = \frac{1}{1 + C^2} \quad (10.21)$$

which was given in (2.16).

The discussion of the weak mixing angle is somehow intricate, since the spectrum of heterotic string is described by the affine Lie algebra with diverse twisted sectors. In standard-like models, we have no simple picture of conventional grand unification. For example, the convenient form (2.17) cannot be used here.

The untwisted sector is described by zero modes which is simple Lie algebra, and the spectrum pattern is rather simple. So, if there is no matter fields from the twisted sector, then the particle spectrum is essentially given by the original **248** + **248'** of $E_8 \times E'_8$. In this case, the U(1) charges are determined at the string scale just by the branching rule since the nonabelian groups and the U(1) groups come from the same $E_8 \times E'_8$. Nonzero roots of $E_8 \times E'_8$ are defined to have $(\text{length})^2 = 2$. The U(1)s come from the center and basically they are defined by the sixteen independent Q s of the previous subsection, or by any sixteen independent combinations.

The desirable U(1) charges for unbroken subgroup and spectrum will be obtained if it belongs to the chain of the E-type unification group $SU(5) \subset E_8$. It amounts to the unification assumption (here the GUT group as E_8) and the bare weak mixing angle is related to the untwisted sector spectrum by (2.17).

The pertinent question is how to normalize the relative Q_i s. For the Cartan subalgebra generator H_i ($i = 1, \dots, 16$), we can find the charge raising and lowering operators, $E_{\pm\alpha_i}$, satisfying

$$[E_{\alpha_i}, E_{-\alpha_i}] = 2H_i \quad (10.22)$$

where α_i is not necessarily a simple root. For an $SU(2)$ representation $|\psi\rangle$, one can check that (10.22) is satisfied. For Hermitian $SU(2)$ generators T_a , we have $\text{Tr } T_a T_b = l(R) \delta_{ab}$ for the representation R with the index $l(R)$.⁷ Noting that charge raising and lowering operators of $SU(2)$ are $T_{\pm} = T_1 \pm iT_2$, the trace of $E_{\alpha}^{\dagger} E_{\alpha}$ is two times the trace of H_i^2 . Since $E_{+\alpha_i}|\beta\rangle = |\beta + \alpha_i\rangle$ and $\langle\beta|E_{-\alpha_i} = \langle\beta + \alpha_i|$, we obtain,

$$\text{Tr } E_{\alpha_i}^{\dagger} E_{\alpha_i} = \sum_{m_j} \langle j, m_j | E_{\alpha_i}^{\dagger} E_{\alpha_i} | j, m_j \rangle = \sum_{m_j} ||E_{\alpha_i} | j, m_j \rangle||^2$$

where j is the isospin of representation R . This can be summed to give

$$\begin{aligned} \sum_{m_j=-j}^{j-1} ||E_{\alpha_i} | j, m_j \rangle||^2 &= 2 \sum_{m_j=-j}^{j-1} (j - m_j)(j + m_j + 1) \\ &= 2 \sum_{m=1}^{2j} m(2j + 1 - m) = 4 \cdot \frac{1}{3} j(j + 1)(2j + 1) \end{aligned} \quad (10.23)$$

where the factor 2 comes from normalization of nonzero root $P^2 = \langle\beta|\beta\rangle = 2$ in the untwisted sector. Thus, the index of an $SU(2)$ representation becomes

$$l(j) = \frac{1}{3} j(j + 1)(2j + 1) . \quad (10.24)$$

For example, we obtain $l(\mathbf{2}) = \frac{1}{2}, l(\mathbf{3}) = 2$, etc. For the Cartan generator H_i , we have $\text{Tr } H_i^2 = \sum_{\beta} (P_{\beta} \cdot H)^2 \langle\beta|\beta\rangle = 2 \sum_{\beta} (P_{\beta} \cdot H_i)^2$ since H_i eigenvalue of $|\beta\rangle$ is $(P_{\beta} \cdot H_i)$. Thus, $\sum_{\beta} (P_{\beta} \cdot H_i)^2$ becomes the index,

$$\sum_{\beta} (P_{\beta} \cdot H_i)^2 = \frac{1}{3} j(j + 1)(2j + 1) . \quad (10.25)$$

For $Q_2 = \sqrt{2}q_2$ (This identification is possible for diagonalized Q_2) with q_2 given in (10.17), $j = \frac{1}{2}$ members ($\uparrow$ and $\downarrow$) of Q_2 can, for example, be

$$|P_1\rangle \rightarrow (0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1)(\cdots) , \quad |P_2\rangle \rightarrow (0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1)(\cdots)$$

which gives $\sum_{i=1}^2 (P_i \cdot Q_2)^2 = 2$, yet we know (10.25) is $\frac{1}{2}$ for this doublet. Therefore, a correct normalization of Q_2 is achieved by multiplying by a factor $\frac{1}{2}$ to Q_2 , i.e. $q_2/\sqrt{2}$ from (10.17). A similar argument leads to multiplying $\frac{1}{\sqrt{6}}$ to $Q_1 = (1 \ 1 \ 1 \ 0 \ \cdots)$ to obtain $q_1/\sqrt{2}$ from (10.16). Namely, we multiply $\frac{1}{\sqrt{2}}$ to unit length operators. Rather than using the fractional numbers in Q s, we will be interested only in the relative normalization of these $U(1)$ generators and $SU(2)_W$ generators. The $SU(2)$ generator belonging to the Cartan subalgebra

⁷ We use H_i and Q_i interchangeably, but try to represent the Cartan center with H_i and $U(1)$ charge with Q_i .

is T_3 . So, T_3 and $\tilde{Q}_2$ are defined to be the same magnitude since both of them come from E_8 . If we use $\tilde{Q}_2$ as $T_3 = (0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0)(\cdots)$, we must choose the other corresponding U(1) charges as $\tilde{Q}_1 = \sqrt{\frac{2}{3}}Q_1$, etc. instead of the correctly normalized $q_1/\sqrt{2}$, etc.

After the U(1) normalization from the untwisted sector is known, one looks for the locations of the SM fields. The gauge group and charge assignments are not changed even if there are fields in the twisted sector. *The problem becomes that whether a set of given fields belongs to a GUT multiplet belonging to a unified group.*

To see it how it works, let us consider the following example with three Wilson lines [6],

$$\begin{aligned} V &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}' \\ a_1 &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \end{pmatrix}' \\ a_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}' \\ a_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}' \end{aligned} \quad (10.26)$$

which gives the gauge group $SU(3) \times SU(2) \times U(1)^{13}$. In this above explicit example, three quark doublets are in the untwisted sector, yet the lepton doublets are in the twisted sectors. If all the SM fields were in the untwisted sector, then one could be sure that $\sin^2 \theta_W^0 = \frac{3}{8}$. However, the model (10.26) has the possibility of $\sin^2 \theta_W^0 \neq \frac{3}{8}$. To show it explicitly, one has to identify these SM fields and express their electroweak hypercharges in terms of the above normalized (length = $\sqrt{2}$) U(1) charges.

After determination of hypercharge by surveying the spectrum,

$$q_Y = Cq_u = \frac{1}{\sqrt{2}} \sum_i c_i q_i \quad (10.27)$$

where q_u is the charge of the unified group and the factor $1/\sqrt{2}$ comes from the normalization convention $l(\text{fund}) = \frac{1}{2}$. Since all the q_i s are orthonormal $C^2 = \sum_i c_i^2$, and from (10.21) we have

$$\sin^2 \theta_W^0 = \frac{1}{1 + \sum_i c_i^2} \quad (10.28)$$

As the prototype example, consider the case where the $SU(5)$ space is spanned by the first five entries of (10.26). Then, the electroweak hypercharge is given only by two U(1)s (10.18),

$$C^2 = 2 \left(\frac{1}{2} + \frac{1}{3} \right) = \frac{5}{3} \quad (10.29)$$

which leads to $\sin^2 \theta_W^0 = \frac{3}{8}$. Of course, this result is due to the fact q_Y in (10.18) and (10.27) has the desirable structure. We apply the above formula to the standard-like model (10.26) of [6] to obtain

$$q_Y = -\frac{32}{75} \sqrt{\frac{3}{2}} q_1 + \frac{13}{50} q_2 - \frac{19}{25} \frac{1}{\sqrt{2}} q_4 + \frac{23}{25} \frac{1}{\sqrt{2}} q_7 + \frac{52}{25} \frac{1}{\sqrt{2}} q_8 + \frac{21}{25} \frac{1}{\sqrt{2}} q_{12} \quad (10.30)$$

leading to $C^2 = \frac{5353}{1500}$. Thus, we obtain $\sin^2 \theta_W^0 = \frac{1500}{6853} \sim 0.219$. Another example is the model discussed in [7] where $q_Y = \frac{1}{\sqrt{3}} q_1 - \frac{1}{\sqrt{2}} q_2 + q_4$. By the same method we employed above, this model gives $C^2 = \frac{11}{3}$, and hence, $\sin^2 \theta_W^0 = \frac{3}{14}$.

In general, most standard-like models have a small $\sin^2 \theta_W^0$. From the preceding discussion, it is obvious that $\sin^2 \theta_W^0 = \frac{3}{8}$ requires $C^2 = \frac{5}{3}$ which is possible if q_Y is given with the form seen in (10.18), with q_1 and q_2 only. If extra pieces appear beyond (10.18), then $C^2 > \frac{5}{3}$ and $\sin^2 \theta_W^0$ is smaller than $\frac{3}{8}$. But this statement is not valid if the coefficients of q_1 and q_2 are more complicated than those given in (10.18). A simple flipped SU(5) model may be given with a hypercharge $Y = -\frac{1}{\sqrt{3}} q_1 + \frac{1}{\sqrt{2}} q_2 + \sum_{i \geq 3} x_i q_i$ with at least one nonvanishing x_i to have e^+ in an SU(5) singlet representation. In this case, $\sin^2 \theta_W^0 < \frac{3}{8}$. On the other hand, it will be extremely lucky if nonvanishing x_i s conspire to give $\sin^2 \theta_W^0 = \frac{3}{8}$ with more complicated coefficients in front of q_1 and q_2 .

Note that we cannot use the simple formula (2.17) in general. To recapitulate how it is obtained, the basic reason stems from the unification assumption where the ratio g^2/g'^2 is traded for $\text{Tr } Y^2 / \text{Tr } \tilde{Y}^2$. This formula is very powerful only when we know the spectrum of the model under the unification assumption. Of course, in GUTs such as SU(5), only the knowledge of the fundamental representation is needed and hence, this formula is already very powerful. But in standard-like models the formula is not necessarily helpful.

We restricted our discussion to the $E_8 \times E'_8$ heterotic string, to use the branching chain of E_8 down to SO(10) or SU(3)³ going through the intermediate **27** of E_6 . It utilizes the fundamental **27** of E_6 or the spinor **16** of SO(10) in the process of reducing the rank. If there is no more matter particle beyond three families, we can automatically achieve the correct charge assignments. If the intermediate step cannot contain **27** of E_6 , it is most probable that exotics would appear. Exotics, vector-like or chiral, have problems in the unification of coupling constants and in the low energy phenomenology.

Strictly speaking, however, the above branching chain argument considers the possibility of **16** of SO(10) *only from the untwisted sector*. Indeed, **16** of SO(10) from the twisted sector has been obtained from the heterotic SO(32) [24], which is a new trail to be explored. But the number of SO(10) families from SO(32) turns out to be even. On the other hand, the number of families in the Pati-Salam group from SO(32) needs not be even.

10.4.3 Anomalous U(1)_X

If the compactification produces U(1)s, there is a possibility that an anomalous U(1) is present [25]. This anomalous U(1), called U(1)_X, can be used to break gauge symmetries further. The D -term for the anomalous U(1)_X is

$$D^{(X)} = c^{(X)} + \sum_i Q_i^{(X)} \phi_i^* \phi_i \quad (10.31)$$

where ϕ_i are the scalar fields with the the anomalous U(1)_X charges $Q_i^{(X)}$, and

$$c^{(X)} = \frac{2g}{192\pi^2} \sum_i Q_i^{(X)} = \frac{2g}{192\pi^2} \text{Tr } Q^{(X)} \quad (10.32)$$

where g is understood to include the dilaton VEV. If there exist some scalar(s) ϕ_i whose sign for the U(1)_X charge is opposite to that of $\text{Tr } Q^{(X)}$, the D-term forces ϕ_i to develop a VEV, breaking the U(1) symmetry [5]. It is possible to have several independent scalars acquiring VEVs from the effects of the D-term, in which case several U(1) symmetries are broken. Therefore it is also used to reduce the rank.

Ten dimensional string theory has the Green-Schwarz term for the anomaly cancelation. The Green-Schwarz term is [26]

$$\begin{aligned} & \sim \epsilon^{MNL_1L_2L_3L_4L_5L_6L_7L_8} B_{MN} F_{L_1L_2} \langle F_{L_3L_4} F_{L_5L_6} F_{L_7L_8} \rangle \\ & \rightarrow \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} (\partial_\rho A_\sigma - \partial_\sigma A_\rho) \end{aligned} \quad (10.33)$$

where in the second line we denote the effective 4D coupling after inserting the vacuum expectation values $\langle F_{L_3L_4} F_{L_5L_6} F_{L_7L_8} \rangle$. In terms of the dual field $\partial^\mu a \propto \epsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma}$, its coupling can be written in the form,

$$mA_\mu \partial^\mu a \quad (10.34)$$

which is reminiscent of the gauge boson mass. This term shows that the gauge boson A^μ absorbs the longitudinal degree $\partial^\mu a$, as noted in the spontaneous breaking of a U(1) gauge symmetry where the gauge boson is made massive by a similar term. In the present case, this mechanism appears along with the anomalous U(1) symmetry in the orbifold compactification. Thus, the gauge boson mass is of the order of the compactification scale, and hence below the compactification scale one obtains a global symmetry U(1)_X. Namely, the gauged anomalous U(1)_X charge translates to the charge of a global symmetry U(1)_Γ below the compactification scale [10]. This global symmetry U(1)_Γ can be used for the Peccei-Quinn symmetry, which is needed for a very light axion at the intermediate scale $\sim 10^{10-12}$ GeV [6, 10].

The anomalous U(1)_X was used to fix the bare value of $\sin^2 \theta_W^0$ in field theoretic models from the requirement of canceling the U(1)³ anomaly [27]. The U(1)_X charge is related to the U(1)³ and U(1)-(gravity)² anomalies, which is determined as a part of U(1) normalization in string models as discussed in some detail in the previous subsection.

10.5 Symmetries

10.5.1 Global and Discrete Symmetries

There is a belief that there does not exist global symmetries in string theory. It is not conserved by Planck scale suppressed operators from gravitational interactions, including the blackhole formation and evaporation and wormhole effects [28]. Because a black hole has “no-hair” it cannot carry a global charge. Therefore, the global charge of a field thrown into a black hole is not conserved. Most of all, the remarkable prediction from string theory is that there is no global symmetry except that corresponding to the model-independent axion [29]. Any global symmetry introduced in the worldsheet is lifted to a gauge symmetry. It is because in the target space, this symmetry is coordinate (X) dependent and becomes the local symmetry as discussed in Appendix B.2.

In contrast to global symmetries, discrete “gauge” symmetries are possible in string theory as a subgroup of continuous symmetries of string theory. Such discrete charge survives well below the Planck scale. In supersymmetry, the R parity is $(-1)^{3B-L}$, and is sometimes traced back to a part of gauge symmetry such as Pati–Salam group or trinification or $SO(10)$. If so, then it would be a good quantum number, too.

There are “fundamental” discrete symmetries in quantum field theory. The parity $\mathcal{P}$ is the inversion of spatial direction. It exchanges helicities of fermions. From discussions below (1.4), $\mathcal{P}$ is not a good symmetry, and the resulting theory is called a *chiral theory*. As can be deduced from the definition, (3.81) and below, the charge conjugation operation $\mathcal{C}$ conjugates the gauge charges. Still $\mathcal{C}$ is not a good symmetry in the SM. Also, time reversal $\mathcal{T}$ is not a good symmetry, but there is the renowned $\mathcal{CPT}$ theorem that any local, Lorentz invariant quantum field theory preserves the product $\mathcal{CPT}$.

The discrete symmetries $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ can be a part of *gauge symmetry*, or Lorentz symmetry in higher dimensional theory [30]. It is possible if the spacetime dimensions are $8k+1$, $8k+2$ and $8k+3$ so that Majorana fermions and invariant gauge groups such as E_8 or $SO(4n)$ are allowed. These naturally arise in the string theory framework. In terms of weights P , charge conjugation operation $\mathcal{C}$ amounts to changing $P^I \rightarrow -P^I$. Consequently, in the bosonic description, it has the form

$$X^I \rightarrow -X^I, \quad I = 1, 2, \dots, 16. \quad (10.35)$$

This is the property of spacelike (extended version of four dimensional) $\mathcal{P}$ operation

$$\begin{aligned} X^M &\rightarrow X^M, & M = 0; 4, 6, 8, \\ X^M &\rightarrow -X^M, & M = 1, 2, 3; 5, 7, 9. \end{aligned} \quad (10.36)$$

Note that the 4D part ($M = 0, 1, 2, 3$) is the familiar parity operation. The above $\mathcal{C}$ and $\mathcal{P}$ operations are the elements of proper Lorentz transformation $SO(1,9) \times SO(16)$, thus they are gauge symmetries. This means that the

symmetries are *exact* and survives even if nonperturbative and gravitational interactions are considered. Therefore, its breaking (for example the $\mathcal{CP}$ violation) must be achieved spontaneously if introduced at the 10D heterotic string level. We cannot do a similar reasoning for $\mathcal{T}$, but at least perturbatively it is a good symmetry in string theory.

Besides $\mathcal{P}$ being four dimensional inversion in $M = 1, 2, 3$ directions, we can freely choose the inversions in extra dimensions, as long as the transformation is proper. However, taking fermions into account, it turns out that the choice of (10.36) is unique, otherwise it does not commutes with the GSO projection. In terms of complexified coordinates, it corresponds to complex conjugation.

10.5.2 $\mathcal{CP}$ Violation

Whatever four dimensional $\mathcal{CP}$ is given from a string theory, its observed violation can be realized as the choice of vacuum in the process of compactification. The symmetry (10.36) is not preserved in the presence of background field as in (8.68), such as antisymmetric tensor field B or T modulus. Generically, it can be block-diagonalized and one can easily verify that it is not preserved under (10.36). Showing its dependence on Wilson line A is nontrivial. It turned out [31] that, by continuous Wilson lines, $\mathcal{CP}$ is not violated, because the space group selection rule associates the gauge group action to make the interaction term invariant. Also it is argued that discrete Wilson lines can violate that symmetry.

Strong $\mathcal{CP}$

The effective strong $\mathcal{CP}$ parameter

$$\bar{\theta} = \theta_{\text{QCD}} + \arg \det M_{\text{quark}} \quad (10.37)$$

is the coefficient of gluon topological term

$$\frac{\bar{\theta}}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} . \quad (10.38)$$

Here, θ_{QCD} is the parameter introduced at the scale, presumably a GUT scale, below which QCD becomes an exact confining gauge symmetry. M_{quark} is the quark mass matrix (including heavy quarks) introduced at the $\mathcal{CP}$ violating scale. The observed absence of neutron electric dipole moment gives a bound on $\bar{\theta}$

$$|\bar{\theta}| < 10^{-9} \quad (10.39)$$

which constitutes the strong $\mathcal{CP}$ problem, “Why is the observed value of $\bar{\theta}$ so small?” [32]

At field theory level, there are following three types of solutions to the strong $\mathcal{CP}$ problem, which need some kind of symmetry. Indeed, all of them

are implementable in superstring. But the true indication of the strong $\mathcal{CP}$ solution can be accessed only after a realistic MSSM from superstring is found.

- **The axion solution:** The axion solution needs a very light axion derivable from the Peccei-Quinn global symmetry [33] at an intermediate scale [34]. The second rank antisymmetric tensor in the transverse direction $B_{\mu\nu}$ is an axion, which is always present in string theory. Thus, it is called the *model-independent axion* [35]. The three form field strength H satisfies [26]

$$dH = \text{tr} R^2 - \frac{1}{30} \text{Tr} F^2 . \quad (10.40)$$

Defining the dual of $H_{\mu\nu\rho}$ as

$$F_1 \partial^\mu a_{MI} = \frac{1}{96\pi^2} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} , \quad (10.41)$$

we obtain

$$\partial^2 a_{MI} = \frac{1}{32\pi^2 F_1} \left(\text{tr} R_{\mu\nu} \tilde{R}^{\mu\nu} - \frac{1}{30} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \quad (10.42)$$

which shows that a_{MI} is an axion with the decay constant F_1 . F_1 has been estimated as [36],

$$F_1 \simeq \frac{\tilde{g}^2}{192\pi^{5/2}} M_P \sim 1.5 \times 10^{15} \text{ GeV} \quad (10.43)$$

which is too large compared to the allowed axion window [32]. In (8.94), we have seen that the imaginary part of T modulus have the axionic shift symmetry and hence more additional pseudoscalars can appear depending on the compactification scheme. They are B_{ij} . In generic Calabi-Yau compactification, the second Hodge number counts these massless modes as in (A.21)

$$h_{0,2} = h_{2,0} = 0 , \quad h_{1,1} \geq 1 \quad (10.44)$$

where the inequality is valid for all Kähler manifold due to the Kähler two form. Then, the coupling (10.38) is possible due to the Green-Schwarz term. These pseudoscalars are called the *model-dependent axion* [37]. For $h_{0,2} = h_{2,0} = 0$ and $h_{1,1} = 1$, there exists a model-dependent axion a_{MD} , and the axion couplings in a Calabi-Yau compactification are calculated as [38]

$$\frac{1}{32\pi^2 F_1} a_{MI} (F\tilde{F} + F'\tilde{F}') + \frac{1}{32\pi^2 F_2} a_{MD} (F\tilde{F} - F'\tilde{F}') \quad (10.45)$$

where F_2 is the decay constant of the model-dependent axion. It is again at the compactification scale. Axions in M-theory has been discussed in [39].

Nevertheless, it has been argued that even if the model-dependent axions are introduced, the world sheet instantons would make them massive by generating a superpotential violating the shift symmetry of the model-dependent axions [40].

The axion solution seems to have a problem because there is no global symmetry except that corresponding to the model-independent axion $H_{\mu\nu\rho} \sim \partial_\mu B_{\nu\rho}$ [40]. The problem with the model-independent axion is that its decay constant is at a GUT scale [36]. But, as discussed above, one global symmetry $U(1)_F$ can survive down to low energy in models with the anomalous $U(1)_X$ [6, 10]. This $U(1)_X$ can be used also for the massless up-quark solution. In some models, there are no anomalous $U(1)_X$. In such a case, it was pointed out that $H_{\mu\nu\rho}$ is not the QCD axion, but can be considered as a quintessential axion [41].

- **The massless up-quark solution:** The massless up-quark solution can be addressed in orbifold compactification, just by observing $\det M_{\text{quark}} = 0$ from the spectrum obtained by orbifold compactification [42]. The problem is whether this leads to a phenomenologically viable mass matrix or not [43]. At the 4D field theory level, this solution also has an axial global $U(1)$ symmetry which is unbroken, however.
- **The set- θ -zero solution:** The set- θ -zero solution sets $\theta_{QCD} = 0$ at the tree level, presumably by a symmetry, and requires that loop corrections to θ_{QCD} are sufficiently small. As we have done in the orbifold compactification, discrete symmetries can be introduced in string theory. So in principle, the Nelson-Barr type solution with the discrete $\mathcal{CP}$ symmetry is realizable in string theory.

The recently favored model among the set- θ -zero solutions is the Nelson-Barr type solution [44]. The Nelson-Barr type solution needs heavy vector-like representations. At this heavy scale, $\mathcal{CP}$ is spontaneously broken and the source for the weak $\mathcal{CP}$ violation is introduced. Below the spontaneous $\mathcal{CP}$ violation scale but above the electroweak scale, the Yukawa couplings become complex due to the introduction of $\mathcal{CP}$ violation. The massless up-quark solution makes the vacuum angle unphysical. The axion solution chooses $\theta = 0$, via the axion potential. Since the weak $\mathcal{CP}$ violation seems to prefer the Kobayashi-Maskawa type weak $\mathcal{CP}$ violation, then the Nelson-Barr type, massless up-quark and the axion solutions are the allowable ones. But, the spontaneous $\mathcal{CP}$ violation at the electroweak scale is not favored.

The set- θ -zero solution needs a discrete $\mathcal{CP}$ symmetry so that the θ_{QCD} of QCD is vanishing at the bare Lagrangian level. Assuming a $\mathcal{CP}$ invariant Lagrangian, the discrete $\mathcal{CP}$ symmetry chooses the weak $\mathcal{CP}$ violation as a spontaneously broken one. Then, θ_{QCD} is calculable below the scale of the spontaneous $\mathcal{CP}$ violation, and is required in order to have the limit (10.39). Both the massless up quark solution and the axion solution need a global symmetry $U(1)_{PQ}$ where the divergence of the corresponding current, which is proportional to (10.38) is nonvanishing.

Another method to gain a strong $\mathcal{CP}$ solution in string models is an accidental PQ symmetry [45]. There can be discrete symmetries in string models which may allow a PQ symmetry up to some level, for example up to dimension 9 operators in the superpotential. Then the θ parameter is within the limit 10^{-9} . This model is similar to the Nelson-Barr type in that discrete symmetries are used. String constructions of these types of solutions may contain both features.

Electroweak $\mathcal{CP}$

Electroweak $\mathcal{CP}$ violation is implemented by complex phases in the Yukawa couplings, which are not invariant under $\mathcal{CP}$. The CKM type complex Yukawa couplings can introduce the needed electroweak $\mathcal{CP}$. In general texture, there is a basis independent $\mathcal{CP}$ measure, called the Jarlskog invariant [46]

$$J \propto \text{Im det}[Y^u Y^{u\dagger}, Y^d Y^{d\dagger}] . \quad (10.46)$$

String models are constructed in 10D ($8k + 2 = 10$) and hence they provide the possibility that the electroweak $\mathcal{CP}$ can be a discrete gauge symmetry [47] where the covering continuous symmetry is the 10D Lorentz symmetry times the 10D gauge symmetry. Or it can appear as a subgroup of continuous symmetries of string theory. Whatever four dimensional $\mathcal{CP}$ is given from a 10D theory, the observed electroweak $\mathcal{CP}$ violation can be realized as the choice of vacuum in the process of compactification of 10D string models. So the phenomenological consideration of the electroweak $\mathcal{CP}$ violation reduces to the study of Yukawa couplings and the flavor problem from string models. The above KM type complex Yukawa couplings can introduce the needed electroweak $\mathcal{CP}$.

The Nelson-Barr type introduction of electroweak $\mathcal{CP}$ violation is possible with this discrete gauge symmetry spontaneously broken below 10^9 GeV so that it is consistent with the $\mathcal{CP}$ violation in the kaon system [30]. But one drawback is that one introduces another small mass parameter here.

The main ingredients for Yukawa couplings are string selection rules and their moduli dependence. Therefore, it is crucial to know how such moduli are stabilized to have nontrivial complex phases.

Although highly model dependent, many properties are known for renormalizable interactions having only the T modulus dependence, thanks to the target space modular properties (6.120). In Subsects. 8.1.5 and 8.1.6 we discussed the structure of Yukawa couplings. In the simplest $\mathbf{Z}_3$ orbifold, at the renormalizable level the mass matrix is always diagonal by the selection rules, therefore we cannot expect nontrivial phases. By the standard procedure, a number of phases can be absorbed by redefining fermion fields

$$Y^u \rightarrow U^\dagger Y^u V^u, \quad Y^d \rightarrow U^\dagger Y^d V^d \quad (10.47)$$

where matrices $U, V^{u,d}$ are unitary matrices. From Table 8.2, we observe that Yukawa couplings have a similar structure for prime orbifolds and for some

nonprime orbifolds. In non-prime orbifolds, nontrivial $\mathcal{CP}$ phase can emerge at the renormalizable level [48]. In the simplest case, where the dependence is only from T moduli, there is always vanishing $\mathcal{CP}$ phase when $\text{Im}T = \pm\frac{1}{2}$ as a consequence axionic symmetry in (8.94). In general, we expect that the boundary of fundamental region has such properties, if the Jarlskog parameter is invariant under the moduli transformation, but the proof is tricky [48]. This interval includes fixed point at $T = e^{i\pi/6}$ of $\text{PSL}(2, \mathbf{Z})$ action (6.120) which can be likely the extremum of the potential of the T modulus. An analysis was done [49] by assuming empirical form for a superpotential having the desired modular properties.

Of course, the possibility of nonrenormalizable interactions are always open, whose strength can still be sizable. In general, a scalar can acquire a VEV with a nontrivial phase. It is also related to supersymmetry breaking and the above discussion could not take into account supersymmetry breaking effects. There is a scenario proposing that some charged scalars acquire VEVs and F terms have complex phases from the FI term (10.31) generated by the GS mechanism [50].

10.6 Supersymmetry Breaking

At the level of orbifold construction, the source for SUSY breaking is not touched upon since the construction aimed to obtain $\mathcal{N}=1$ supersymmetric models. Supersymmetry breaking must go beyond the orbifold construction, i.e. in the field theory limit below the compactification scale. One notable exception to this rather general statement is in cases with an anomalous $\text{U}(1)_X$ discussed in the previous section. The D-term for the anomalous $\text{U}(1)_X$ is given in (10.31). For a supersymmetric vacuum, $D^{(X)}$ must vanish. However, if the signs of $c^{(X)}$ and all $Q_i^{(X)}$ are the same, then it is not possible to satisfy this supersymmetry condition and hence supersymmetry is broken spontaneously [25]. Nevertheless, this mechanism will be not so useful phenomenologically unless the compactification scale is in the intermediate region $M_I \sim 10^{12}$ GeV, since the needed electroweak scale masses are of order $M_I^2 M_P^{-1}$. String theory might be constructed in the intermediate scale [51] toward this kind of $\text{U}(1)_X$ supersymmetry breaking.

The gaugino condensation mechanism for supersymmetry breaking is always applicable in string theory because of the antisymmetric tensor field H_{MNP} [52]. For this mechanism to work, there must appear a hidden-sector confining group G_h , presumably bigger than $\text{SU}(3)_c$, or just $\text{SU}(3)_h$, but with very few matter representations so that it confines at the intermediate scale M_I . For the gauge mediated supersymmetry breaking, one has to check explicitly that another sector appears for supersymmetry breaking and a messenger field(s) for transferring the information of supersymmetry breaking to the observable sector. In orbifold models, the gauge mediation has to be studied on a model-by-model basis.

No-Scale Supergravity

One feature of the effective potential from superstring [53] is that the potential takes the no scale form [54]. In our example of $T^6/\mathbf{Z}_3$ orbifold, all the two-tori have the same geometry, therefore all the volume moduli are the same $T_{i\bar{j}} = T\delta_{i\bar{j}}$. Therefore, we have the Kähler potential (8.94) of a form

$$K = -3 \log [T + T^* - h(A^a, A^{a*})] - \tilde{K}(S^n, S^{n*}) . \quad (10.48)$$

and the superpotential W , *not depending on T* ,

$$W = W_1(A^a) + W_2(S^n) . \quad (10.49)$$

Here, we consider a general form for matter field function $h(A^a, A^{a*})$ and S moduli $\tilde{K}(S^n, S^{n*})$. It is important to note that the *coefficient* -3 of the first term has a consequence of canceling some terms from the $-3|W|^2$ term. Plugging K into (2.38), it has the form [54]

$$V = \frac{1}{3(T + T^* - h)^2} (N^{-1})^i{}_j \left(\frac{\partial W}{\partial A_i} \right) \left(\frac{\partial W^*}{\partial A^{*j}} \right) + (G^{-1})^n{}_m (D_n W) (D_m W)^* \quad (10.50)$$

where

$$N^i{}_j = \frac{\partial^2 h}{\partial A_i \partial A^{*j}} .$$

Here, $G^n{}_m$ and $N^a{}_b$ are positive definite. Even if we include the D -term scalar potential, (the last term in (2.38)), the scalar potential (10.50) is positive definite and has the minimum at zero. This is similar to the global supersymmetry case, but the minimum *does not* correspond to the supersymmetry preserving one, since in view of (2.41)

$$\delta_\epsilon \Psi^i \sim D^i W = - \frac{3}{|T + T^* - h|} \frac{\partial h}{\partial A_i} W \quad (10.51)$$

need not vanish. It is better to have the zero minimum of the scalar potential with broken supersymmetry. This helps fitting with the almost zero cosmological constant [55].

Without S , the supersymmetry breaking scale M_{SUSY} (2.43) and the gravitino mass $m_{3/2}$ (2.44) are not determined at tree level, which is then generated by radiative corrections. Because of this absence of scale parameter, it is called *no scale supergravity*. This scale invariance property is the characteristic feature in the string interactions [53]. However, the supersymmetry transformation of the S field relates gravitino mass to the supersymmetry breaking scale.

Cosmological Moduli Problem

As discussed in detail in [7, 56], there are several restrictions on possible Yukawa couplings. Because of these restrictions, it is likely that numerous massless modes or the so-called flat directions would appear. Flat directions in string compactifications are extensively discussed in [57]. These flat directions are lifted only by gravity and thus, the masses of the modes corresponding to flat directions are $O(m_{3/2}) \sim \text{TeV}$ with the minimal form for the Kähler potential. We call the scalar fields obtaining the electroweak scale mass, but with couplings suppressed by the Planck mass, are collectively called moduli. Here, a severe cosmological problem known as the moduli problem [58] exists, which is similar to the cosmological gravitino problem [59]. A cosmological solution to the gravitino and moduli problems might be an introduction of heavy particle(s) whose decay products generate huge entropy, such that the number densities of gravitino and moduli are sufficiently diluted after the heavy particle decay. But this kind of diluting cosmological scenario must accompany an acceptable baryogenesis mechanism so that the baryon asymmetry is not overly washed out.

10.6.1 The μ and Doublet-Triplet Splitting Problems

As introduced in Chap. 2, the μ problem exists in supersymmetric models [60]. It is basically a problem on the quadratic couplings of chiral fields. A more severe problem is the MSSM problem introduced in Chap. 2. One of the MSSM problems we discussed in Chap. 2 is, “Why is there only one pair of Higgs doublets without any extra color triplets?”

Since the low energy supergravity is defined in terms of Kähler potential $K(\phi, \phi^*)$, gauge kinetic function $f(\phi)$ and superpotential $W(\phi)$, one must make sure that the μ term,

$$W_\mu = \mu \mathbf{H}_1 \mathbf{H}_2$$

does not arise at the string scale. For a solution at the supergravity level, it was argued that some kind of symmetry is needed for this purpose [61].

If W_μ is not allowed at the string scale, the next question is what the leading contribution to μ will be. It can arise as a nonrenormalizable term in the superpotential [60] or in the Kähler potential [62].

In string models, there is a suggestion from the Kähler potential to generate an electroweak scale μ . In [63], this idea was shown explicitly using the $\mathbf{Z}_3$ orbifolds. With the $\mathbf{Z}_3$ orbifolds, the bare mass term $\mathbf{H}_1 \mathbf{H}_2$ is not allowed because UU and TT , as discussed in Chap. 7, are not allowed from $\mathbf{Z}_3$ where U and T are chiral fields from the untwisted sector and twisted sector, respectively. But, terms of the form UUU and TTT are allowed, which however is not complete in that it is not explained why one U or T in UUU or TTT in the superpotential does not develop a string scale VEV. If they develop VEVs,

then a large μ term would be possible by $\langle U \rangle UU$ and/or $\langle T \rangle TT$. Again, one needs a symmetry for this explanation. Assuming that the Planck scale Higgsino mass term is forbidden from the superpotential, then the μ term is generated if the Kähler potential mixes Higgs doublets with neutral scalars which acquire vacuum expectation values of order the Planck scale [63]. At the effective supergravity level, it was shown that this calculation corresponds to the Giudice-Masiero mechanism [62].

Another attempt for the electroweak scale μ term is by strong dynamics at the intermediate scale. In $\mathbf{Z}_3$ orbifold construction, this possibility was suggested in [10].

Relation with Doublet-Triplet Splitting Problem

The μ problem reappears as the *doublet-triplet splitting problem* as a disguise. As noted in [3], it is known that some standard-like models do not allow extra colored scalars. Basically, this is one possible solution of the doublet-triplet splitting problem. In field theoretic orbifolds, the doublet-triplet splitting problem is realized in the bulk Higgs multiplets such that $\mathbf{Z}_2$ quantum numbers of color triplets and electroweak doublets differ, as discussed in Chap. 4. This situation corresponds to allowing Higgs multiplets in the untwisted sector in the orbifolded superstring. For $\mathbf{Z}_3$ orbifolds, then the multiplicity of the Higgs pairs is 3. So far, there has not appeared any phenomenologically successful model allowing just one pair of Higgs doublets from string models. Thus, the standard-like models from $\mathbf{Z}_3$ orbifolds have another problem; there are too many pairs of Higgs doublets. Since Higgs doublets must appear in pairs $\mathbf{H}_1 \oplus \mathbf{H}_2$, the minimum number of Higgs doublets in a $\mathbf{Z}_3$ orbifold is six for one and two Wilson line models, i.e. three pairs of $\mathbf{H}_1$ and $\mathbf{H}_2$. With three Wilson lines, one pair of Higgs doublets is a logical possibility, but such a model has not yet been constructed. It is also possible that the minimum number is six even with three Wilson lines, which also has not been proven yet. Six doublets are too many. One trouble with having so many Higgs doublets is that it does not unify the gauge couplings at $\sim 3 \times 10^{16}$ GeV. The multiple number of Higgs doublets seems to be a difficult problem for string models to overcome.

Thus, we have the “MSSM” problem [12] which is more difficult than the μ problem or the doublet-triplet splitting problem. It has been suggested that one pair of Higgs doublets may remain light while two pairs are massive if a strong electroweak gauge group is obtained near the Planck scale. Since there seem to exist many vector-like matter representations above the GUT scale as depicted in Fig. 10.2, a strong electroweak force at high energy can be easily realized from compactifications of superstring. In this case, small size instantons with the Pontryagin number 4 of strong electroweak gauge group G containing a semisimple group $SU(2)_R \times SU(2)_L$ at high energy are shown to dictate the Higgsino mass matrix to satisfy $\det M_{\tilde{H}} = 0$ which allows one or two light pairs of Higgs doublets [64]. This choice of light Higgs doublets

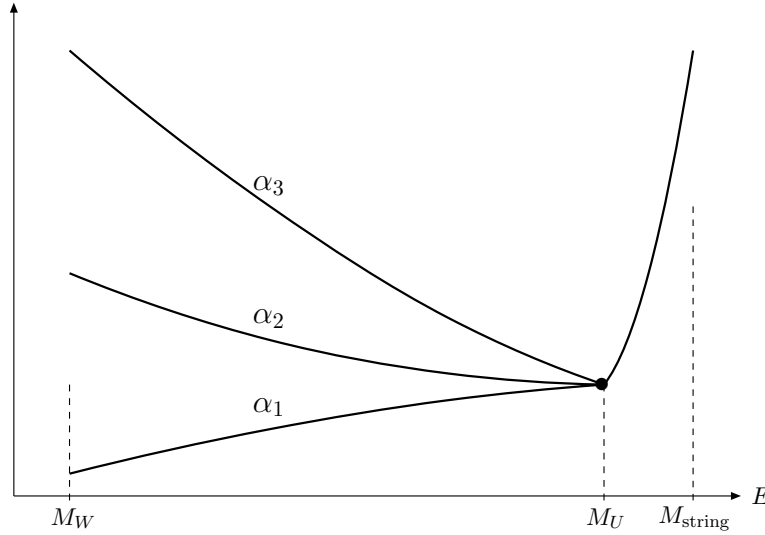


Fig. 10.2. Gauge couplings above the GUT scale

does not rely on a symmetry, but arises from the dynamics as the axion dynamics chooses $\theta = 0$. The μ is a dynamical scalar coming from flat directions of string compactification. There it has been also argued that one pair is a cosmologically favorable choice. The needed interaction from small scale instantons should not allow emission of the SM quarks and leptons. If they are allowed, the effect is negligible since the quark and lepton masses appearing in the determinantal interaction is sub-TeV scale. GUTs with semisimple groups such as $SU(2)_R \times SU(2)_L \in SU(3)^3$ and $SU(2)_R \times SU(2)_L \times SU(4)$ have Higgs doublets with the representation $(\mathbf{2}, \mathbf{2})$. As will be discussed below, GUTs with semisimple groups are attractive because they may not need adjoint representation for breaking the GUT down to the SM gauge group. Here, quarks and leptons behave differently from Higgs doublets for some instanton configurations and we can achieve the condition $\det M_{\tilde{H}} = 0$.

10.7 Grand Unifications with Semi-Simple Groups

10.7.1 Hypercharge Quantization

The $\sin^2 \theta_W^0$ problem seems to be an intriguing one. It may be quantized at $\sin^2 \theta_W^0 = \frac{3}{8}$ as some unification models such as $SU(5)$, $SO(10)$ and E_6 imply. Quantization to this specific value of $\sin^2 \theta_W^0$ is called the correct *hypercharge quantization*. The hypercharge quantization can be a useful guide in selecting a few GUT models from a multitude of possibilities [65].

As observed in a previous section, the hypercharge quantization is not understood in standard-like models. This strongly suggests that we should

look for new models where the hypercharge is quantized. Some of these are simply GUTs such as $SU(5)$, $SO(10)$, and E_6 . But simple GUT constructions with level $k = 1$, do not give the adjoint representation needed for further symmetry breaking of GUTs down to the SM. In this regard, the flipped $SU(5)$ model, $SU(5) \times U(1)$, attracted a great deal of attention because it can be broken down to the SM without an adjoint representation [66]. Indeed, as noted before, flipped $SU(5)$ models were constructed [17] for this purpose in the fermionic construction scheme [67]. However, this flipped $SU(5)$ has the hypercharge quantization problem also, since the electric charge leaks to $U(1)$ [65]. If GUTs with simple groups and the flipped $SU(5)$ are not allowed for the hypercharge quantization, then the remaining possibilities are semi-simple groups such as the Pati-Salam type or trinification. The tables listed in the next chapter will be helpful in picking up possible semi-simple groups from Z_3 orbifolds.

10.7.2 Trinification $SU(3)^3$ from Z_3

The GUT called trinification is the $SU(3)^3$ gauge theory with the following matter [68],

$$\Psi_{\text{tri}} \equiv (\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{3}) + (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}). \quad (10.52)$$

For the matter spectrum, this is very similar to **27** of E_6 . Even though it is semi-simple, a permutation symmetry of three $SU(3)$ s makes the respective gauge coupling constants identical and the trinification is a proper GUT even at field theory level. This is in contrast to the Pati-Salam group unification in $SU(4) \times SU(2) \times SU(2)$ where the strong and electroweak couplings are different in principle. String theory, however, gives a rationale for the unification of couplings since these couplings are basically related to the gravitational coupling, and the Pati-Salam model can be considered as a GUT.

Out of $SU(3)_1 \times SU(3)_2 \times SU(3)_3$ in the trinification, let us identify $SU(3)_3$ as QCD and $SU(3)_2$ as the weak gauge group. Then the trinification spectrum (10.52) has definite properties under color: quarks are in $(\mathbf{1}, \bar{\mathbf{3}}, \mathbf{3})$, anti-quarks are in $(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$, and leptons and Higgs doublets are in $(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1})$. Therefore, these representations are named as quark humor, anti-quark humor, and lepton humor representations, respectively.

SUSY trinification models have two difficult problems. One is the difficulty of assigning the R-parity, and the other is the difficulty of rendering a small neutrino mass [69]. For neutrino masses, it is required to have an additional vectorlike lepton humor, $(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}) + (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$. In superstring models, these two problems must be checked on model by model basis.

Since the ranks of $SU(3)^3$ and the SM gauge group are 6 and 4, respectively, we need two complex scalars for the Higgs mechanism for breaking $SU(3)^3$ down to the SM. The VEVs of two complex scalars are in the directions $\bar{N}_5 = (1, \bar{3}, 0) \in (\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1})^*$ and $N_{10} = (\bar{3}, 3, 0) \in (\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1})$, where numbers in $(1, \bar{3}, 0)$ and $(\bar{3}, 3, 0)$ denote the weights of the triplet and anti-triplet. In

general, for the symmetry breaking one needs VEVs of at least one from each set of $\{N_5, \bar{N}_5\}$ and $\{N_{10}, \bar{N}_{10}\}$.

By inspecting the entries of the tables given in Chap. 11, one observes that there are many models with only the trinification spectrum but without the required vector-like lepton humors. For example, No. 19 in the table of Chap. 11 has the trinification spectrum in the E_8 . The requirement of the vector-like lepton humor reduces the number of possible models drastically. In fact, there is no trinification spectrum with additional vectorlike lepton humors in two Wilson line models. This indicates that there might be no acceptable trinification model with three Wilson lines either. However, note that there exist models with many $SU(3)$ s [70]. In such models with multiple $SU(3)$ s, it is possible to link two $SU(3)$ s by VEVs of the so-called linkage fields carrying quantum numbers of both $SU(3)$ s [12]. Then, one can find trinification models from some entries listed in the tables of Chap. 11. Because there is a limited number of such multiple $SU(3)$ s, the number of possible trinification models is also very limited.

As shown above, the orbifold construction of a trinification model is very restrictive. The two trinification problems, the R-parity and the neutrino mass problems, must be checked with a specific model. For example, [13] can allow the vacuum where appropriate discrete parity or R parity can be assigned, if one chooses VEVs from $\bar{N}_5, N_{10}$, and $\bar{N}_{10}$.⁸ Assigning $\langle N_5 \rangle = 0$ is the key for the discrete R-parity from trinification models.

Another requirement of trinification is to have at least one extra vector-like lepton humor for acceptable neutrino masses. Trinification with just vector-like lepton humors is acceptable in this regard, but it does not achieve the hypercharge quantization because the vector-like lepton humor $\{(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}) + (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})\}$ is not a complete multiplet of the trinification GUT. It gives $\sin^2 \theta_W^0 = \frac{1}{4}$ [12]. For the hypercharge quantization, therefore, one needs a vectorlike complete trinification spectrum $\Psi_{\text{tri}} + \bar{\Psi}_{\text{tri}}$. The required number of fields for the vectorlike complete trinification spectrum is very large in the $\mathbf{Z}_3$ orbifold compactification, $3(27 + 27) = 162$. With the trinification spectrum already present, $3(27)$, together they count 216 fields. Fortunately, there exist one entry in the tables of Chap. 11 [13] which achieves the hypercharge quantization, $\sin^2 \theta_W^0 = \frac{3}{8}$, and acceptable neutrino masses by identifying $SU(3)$ s by the aforementioned linkage fields.

10.7.3 Non-Prime and $\mathbf{Z}_M \times \mathbf{Z}_N$ Orbifolds

Another semi-simple group realizing the gauge symmetry breaking through VEVs of the fields by orbifold compactification is the Pati-Salam group $SU(4) \times SU(2) \times SU(2)$ with the matter representation $(\mathbf{4}, \mathbf{2}, \mathbf{1}) + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ and the Higgs representation $(\mathbf{1}, \mathbf{2}, \mathbf{2})$ [71]. Here color is embedded in $SU(4)$ in the direction 1, 2, and 3, and the fourth direction is colorless. So, lepton is called

⁸ N_5 and N_{10} are the SM singlet fields of the trinification spectrum (10.52).

the fourth color. This Pati-Salam model can be embedded in $SO(10)$, as we discussed at some length in Chap. 4. Then the matter representation belongs to **16** of $SO(10)$ and the Higgs representation belongs to **10** of $SO(10)$. One merit here is that the matter and the Higgs are distinguished and the introduction of the R-parity is easy. However, it is not as much symmetrical as the trinification discussed in the previous subsection. Since the rank of the Pati-Salam group is 5, one complex field not belonging to the center is sufficient for the spontaneous symmetry breaking of the Pati-Salam group down to the SM group. For this purpose, an $SU(3) \times SU(2) \times U(1)$ singlet member $\langle \bar{\mathbf{4}}, \mathbf{1}, \mathbf{2} \rangle$ is sufficient for breaking both $SU(4)$ and $SU(2)$, since the rank is reduced just by one unit.

One interesting class leading to the Pati-Salam group is non-prime orbifolds or product orbifolds [4]. From the different perspective of obtaining a 5D intermediate field theory model as discussed in Chap. 4, it seemed necessary to introduce $\mathbf{Z}_2$ which was so important for 5D field theoretic orbifolds. Basically, one can then consider $\mathbf{Z}_4$, $\mathbf{Z}_6$, $\mathbf{Z}_8$, and $\mathbf{Z}_{12}$ for string non-prime orbifolds and $\mathbf{Z}_2 \times \mathbf{Z}_n$ for product orbifolds. It was argued that the fermionic models with the NAHE set have the symmetry $\mathbf{Z}_2 \times \mathbf{Z}_2$ [18]. A $\mathbf{Z}_6$ orbifold model was constructed in [20]. With the $\mathbf{Z}_2$, certainly it is possible to consider an intermediate step of 5D ($\mathcal{N}=2$ in 4D sense) theory. For this 5D theory however, one has to consider a hierarchy of compactification scales for two sets of orbifolds. See, for example, Fig. 10.1. If there is no hierarchy of compactification radii, then it amounts to considering a 4D effective theory resulting with $\mathcal{N}=1$ supersymmetry.

This two radii orbifold was considered in [20] for a $\mathbf{Z}_6$ -II model, but going beyond Fig. 10.1, i.e. to three different lattices. This interesting three lattices embedding is $G_2 \oplus SU(3) \oplus SO(4)$. [Note that the G_2 lattice is identical to the $SU(3)$ lattice.] The lattice $G_2 \oplus SU(3) \oplus SO(4)$ can be understood as follows. From Table 3.2, the G_2^2 lattice embedding for the first two tori gives $\mathcal{N}=2$ supersymmetry. An $SU(3)$ sublattice of G_2 is taken from the second torus. Then identify this $SU(3)$ lattice as the one in $SU(3) \times SO(8)$ in 6-II of Table 3.1. At the third torus, take only an $SO(4)$ sublattice of $SO(8)$ since we are now left with rank 2 at the third torus. Thus, this consideration leads to $\mathcal{N}=1$ with the lattice $G_2 \oplus SU(3) \oplus SO(4)$ [20].

Reference [20] found a three family Pati-Salam model after removing vector-like representations. Here, the reason for three families is not so obvious. Except in $\mathbf{Z}_3$, one must check the whole spectrum and conclude whether the model contains three families or not. Since the fermionic construction initiated with the flipped $SU(5)$ starts from the $SO(10)$ spectrum [17], the fermionic construction is somewhat akin to the Pati-Salam. In Appendix B.1, we briefly discuss the fermionic construction. Since the three family model with the NAHE set has the symmetry $\mathbf{Z}_2 \times \mathbf{Z}_2$ [18], an important question has been whether one can construct such three family models from $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold compactifications. Recently, a three family model was constructed with a $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold with Wilson lines [19]. Here, the reason for 3

families is not obvious, and one can only make a conclusion after obtaining the massless spectrum. It may be that $3 = (\text{number of compactified dimension})/2$ [21], which may be easier to understand in the fermionic construction however.

10.7.4 Relation to GUTs and Coupling Constants

In the previous subsection, we considered three gauge groups $SU(5) \times U(1)$, $SU(3)^3$, and $SU(4) \times SU(2) \times SU(2)$. The common feature of these groups is that an adjoint representation is not necessary to obtain the SM gauge group by spontaneous symmetry breaking. The reason for this follows from the fact that the ranks of these groups are greater than the rank of the SM group 4, so that the scalar field(s) not belonging to the center is needed for breaking those GUT groups down to the SM group.

The $SO(10)$ with **16** and E_6 with **27** are closely related to these product form groups. Therefore, the above product groups are almost equivalent to considering $SO(10)$ or E_6 . But, the $SU(5)$ GUT has rank 4 and thus cannot have a corresponding product group. In sum, the following correspondence exists between a simple GUT group and its maximal-subgroup GUT group,

$$\begin{aligned} \text{none} &\leftrightarrow SU(5) \\ \text{flipped } SU(5) &\leftrightarrow SO(10) \\ \text{PS } SU(4) \times SU(2) \times SU(2) &\leftrightarrow SO(10) \\ \text{trinification} &\leftrightarrow E_6 . \end{aligned}$$

Namely, for the level one Kac-Moody algebra the $SU(5)$ GUT does not have the same rank sibling and has the difficulty of obtaining the SM group. For the gauge coupling unification of these orbifold GUTs from the weakly coupled heterotic string, the situation is schematically shown in Fig. 10.2. Here, M_{string} is the string scale of order 6×10^{17} GeV [72]. Since M_{string} does not coincide with the unification point M_U obtained from running the observed couplings from the electroweak scale to high energy scale, M-theory with an additional parameter (the distance between two branes) was introduced to bring M_{string} down to M_U [73]. From Fig. 10.2, we note that the weakly coupled heterotic string is also acceptable if the gauge coupling running between M_U and M_{string} is as that given in Fig. 10.2. But one expects the gauge coupling running between M_U and M_{string} as given in Fig. 10.3 where the dashed lines represent the strong, weak and $U(1)$ couplings. The solid line behavior is required if the gauge couplings unify at M_U . This solid line behavior is possible only with the trinification model since three gauge couplings of $SU(3)^3$ run in the identical manner between M_{string} and M_U . For the Pati-Salam and the flipped $SU(5)$, even if the hypercharge quantization is somehow realized, the evolution of coupling constants near M_{string} generically behaves as that shown with dashed lines. But one may construct models with the Pati-Salam and flipped $SU(5)$ groups with appropriate matter fields between M_{string} and M_U .

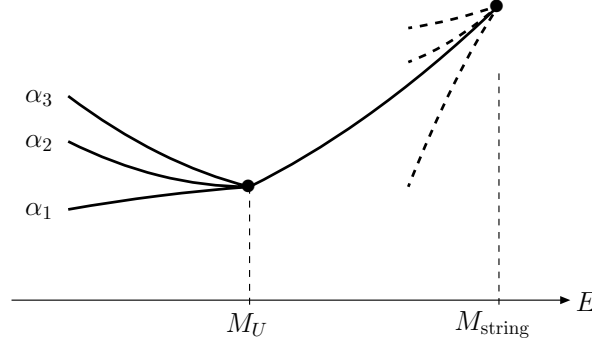


Fig. 10.3. Gauge couplings between M_U and M_{string}

As discussed at the end of Chap. 8, there exist string threshold effects Δ_a on the coupling constants, as given in (8.109). In the large T limit, it looks like

$$\Delta_a \sim \sum_i b_a^i \frac{N'}{N} \frac{\pi(T + T^*)}{6} \quad (10.53)$$

where N is the order of $\mathbf{Z}_N$ and N' is the order of the subsector action. The differences of coupling constants at M_U are given by

$$4\pi \left(\frac{1}{\alpha_a} - \frac{1}{\alpha_b} \right) = (b_a - b_b) \left(\ln \frac{M_s^2}{M_U^2} + \frac{\Delta_a - \Delta_b}{b_a - b_b} \right). \quad (10.54)$$

When we are given a specific model with shift vectors, we can explicitly calculate the above threshold correction, since we know all the particle contents and their charges below the string unification scale M_s . But the final integral with τ is extremely difficult. So, such analysis has been done numerically in [74] for the standard-like model of [3]. Their estimate for threshold correction is given for the difference, $\Delta_a - \Delta_b$. Remarkably, it was observed that all relative threshold corrections are almost identical and small after division by the difference of the β -function coefficients,

$$\frac{\Delta_a - \Delta_b}{b_a - b_b} \simeq 0.079, \quad (10.55)$$

where

$$b_3 = 9, \quad b_2 = 18, \quad b_{10} = -18, \quad b_1 = 32. \quad (10.56)$$

This result shows that the string threshold correction does not seriously affect the coupling evolution obtained purely from the field theory limit. In [74], it was speculated that this minor threshold correction for the difference may be due to the modular invariance of string theory. For the standard $\mathbf{Z}_3$ orbifold with gauge group $E_8 \times \text{SU}(3) \times E_6$, the relative string threshold correction

between $SU(3)$ and E_6 (between $SU(3)$ and E_8) is zero ($\sim(10.55)$), which is consistent with the conclusion [74], (10.55).

So the evolution of couplings seems to be given approximately by field theoretic calculation considering the particle contents between M_s and M_U .

10.8 $\mathbf{Z}_{12}$ Orbifold

In this section, we construct a $\mathbf{Z}_{12}$ orbifold model in detail. The chief motivation considering this complicated example is to exercise a model building with many possible ingredients we discussed in this book. One missing aspect here is a various lattice embedding in three two-tori. Table 3.2 contains lattice embedding up to two kinds of lattices. An interesting embedding of three different kinds of lattices in three two-tori is $G_2 \oplus SU(3) \oplus SO(4)$ with $\mathbf{Z}_6$ -II⁹ which has been used in [20] as discussed in Sect. 10.7.3. As expected, different lattices at different tori makes the calculation even more complicated. The $\mathbf{Z}_6$ -I model uses either the E_6 , or $SU(3) \times F_4$, or $SU(3) \times SO(8)$ lattice as shown in Table 3.2. Without Wilson lines, there is no dependence of specific lattice.

We take the shift vector as $\mathbf{Z}_{12}$ -I or $\mathbf{Z}_3 \times \mathbf{Z}_4$ of Table 3.2,¹⁰

$$\phi = \left(\frac{5}{12}, \frac{4}{12}, \frac{1}{12} \right) \quad (10.57)$$

with $\phi^2 = \frac{1}{12} \cdot \frac{7}{2}$.

Let us look for flipped $SU(5)$ models. There are two interesting E_8 shift vectors v in the tables of [75], allowing $SU(5)$ groups, which are shown below. For a flipped $SU(5)$, we need just $SU(5)$ since there appear many $U(1)$ s. In the end we will show that the model is not successful phenomenologically, because the massless spectrum does not contain $\mathbf{10} \oplus \bar{\mathbf{10}}$. But this explicit construction illustrates the method in detail.

	E_8 shift	E'_8 shift	4D gauge group
(i)	$\left(\frac{1}{4} \frac{1}{4} \frac{1}{4} \quad -\frac{1}{4} \frac{1}{2} \frac{1}{3} \frac{1}{3} \frac{1}{3} \right)$	$(\cdots)$	$SU(5) \times SU(3) \times U(1)^2 \times \cdots$
(ii)	$\left(\frac{1}{2} \frac{5}{12} \frac{5}{12} \frac{5}{12} \frac{1}{6} \frac{1}{12} 0 0 \right)$	$(\cdots)$	$SU(5) \times SU(3) \times U(1)^2 \times \cdots$

We study Case (i) of the above table. The reference to the tables of [75] is not necessary, since the following discussion is enough to look for $SU(5) \times SU(3)$. Let us try to include $SU(5)$ toward a possibility of flipped $SU(5)$. For a possibility of three families, let us also include $SU(3)$. From

⁹ $\mathbf{Z}_6$ -II corresponds to a pure $\mathbf{Z}_6$ orbifold while $\mathbf{Z}_6$ -I is a product $\mathbf{Z}_3 \times \mathbf{Z}_2$ orbifold.

Similarly, $\mathbf{Z}_{12}$ -I is $\mathbf{Z}_3 \times \mathbf{Z}_4$ and $\mathbf{Z}_{12}$ -II is a pure $\mathbf{Z}_{12}$.

¹⁰ There is another shift called $\mathbf{Z}_{12}$ -II which allows four fixed points.

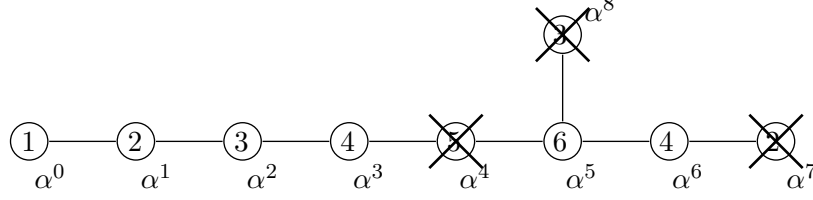


Fig. 10.4. The $SU(5) \times SU(3)$ subgroup of E_8 . The Coxeter labels are shown inside circles

the Dynkin diagram technique, we find that one possibility is choosing $a_4 + 2a_7 + a_8 = 12$ where a_i s are the Coxeter levels of simple roots α_i . In Fig. 10.4, we strike out α_4, α_7 and α_8 , and obtain the remaining $SU(5)$ and $SU(3)$ Dynkin diagrams. Now, let us obtain the shift vector. Choosing the shift vector as $V = \frac{1}{12}(\Lambda_4 + 2\Lambda_7 + \Lambda_8)$ (see Appendix E), we have a $\mathbf{Z}_{12}$ orbifold with

$$v = \frac{1}{12} \begin{pmatrix} 21 & 3 & 3 & 3 & 3 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} (\cdots).$$

By Weyl transformations, we can transform v to

$$\begin{aligned} v' &= \frac{1}{12} \begin{pmatrix} 21 & 3 & 3 & 3 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} (\cdots) \\ &\equiv \frac{1}{12} \begin{pmatrix} 21 & 3 & 3 & 3 & 1 & 1 & 1 & 51 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} (\cdots) \end{aligned}$$

where a lattice shift is applied in the last line. Now let us Weyl-reflect v' with respect to the plane orthogonal to α_7 with $\alpha_7 \cdot v' = \frac{15}{12}$,

$$\begin{aligned} v'' &= v' - 2 \frac{\alpha_7 \cdot v'}{\alpha_7^2} \alpha_7 = v' - \frac{15}{12} \alpha_7 \\ &= \frac{1}{12} (3 \ 9 \ 9 \ 9 \ 8 \ 8 \ 8 \ 18) (\cdots) \\ &= \frac{1}{12} (3 \ -3 \ -3 \ -3 \ -4 \ -4 \ -4 \ -6) (\cdots). \end{aligned}$$

Weyl reflections of the above shift lead to Case (i) except the overall negative sign. The negative sign can be changed via the interchange of the chirality $L \leftrightarrow R$. Thus, with the E'_8 shift included, let us study

$$V = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{-1}{4} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V^2 = \frac{23}{24} \quad (10.58)$$

which satisfies modular invariance condition of (6.43), i.e. $V^2 - \phi^2 = 0$ modulo $2 \cdot (\text{integer})/12$, because

$$V^2 - \phi^2 = \frac{2}{3}. \quad (10.59)$$

Table 10.1. Root vectors P_I in untwisted sector satisfying $P \cdot V = 0$. The underlined entries allow permutations. The $+$ and $-$ in the spinor part denote $\frac{1}{2}$ and $-\frac{1}{2}$, respectively

Vector	Number of States	Representation
(0 0 0 0 0 <u>1</u> -1 0)	6	8 of SU(3)
(<u>1</u> -1 0 0 0 0 0 0)	6	
(<u>1</u> 0 0 1 0 0 0 0)	3	
(-1 0 0 -1 0 0 0 0)	3	
(+ + + + + + + +)	1	24 of SU(5)
(- - - - - - - -)	1	
(- + + - + + + +)	3	
(- - + + - - - -)	3	

10.8.1 Gauge Group from Untwisted Sector

From the above Dynkin diagram construction of the shift vector, it is obvious that $SU(5) \times SU(3)$ results from the first E_8 . Here, let us observe this explicitly. Among the original $E_8 \times E_8$ roots P with $P^2 = 2$, we seek ones satisfying (6.51)

$$P \cdot V = 0 \pmod{1}. \quad (10.60)$$

These form the E_8 gauge multiplets shown in Table 10.1.

In Table 10.1, we have the same convention as in the previous chapter. There are 6 winding states (nonzero roots) in the first row and adding two oscillators we have the adjoint 8 of SU(3). The rest of Table 10.1 has 20 nonzero roots which form the adjoint representation of SU(5) together with four oscillators. The remaining 2 oscillators out of 8 generate $U(1)^2$. These, coming from E_8 , form the adjoint representations of $SU(5) \times SU(3) \times U(1)^2$.

Let us take two simple roots of SU(3) as

$$\alpha_a = (0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1 \ 0), \quad \alpha_b = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1) \quad (10.61)$$

and four simple roots of SU(5) as

$$\begin{aligned} \alpha_1 &= (1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \\ \alpha_2 &= (0 \ 1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0) \\ \alpha_3 &= (0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0) \\ \alpha_4 &= (- \ - \ - \ - \ - \ - \ - \ -) \end{aligned} \quad (10.62)$$

Then, the highest weights of some representations are

$$\begin{aligned} \mathbf{3} &: (0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0) \\ \bar{\mathbf{3}} &: (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -1) \\ \mathbf{24} &: (+ \ - \ - \ + \ - \ - \ - \ -) \\ \mathbf{5} &: (1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0) \\ \bar{\mathbf{5}} &: (- \ - \ - \ - \ - \ - \ - \ -) \\ \mathbf{10} &: (+ \ + \ + \ + \ - \ - \ - \ -) \\ \bar{\mathbf{10}} &: (0 \ 0 \ -1 \ 1 \ 0 \ 0 \ 0 \ 0) \end{aligned} \quad (10.63)$$

In the same way, we obtain the gauge group $\text{SO}(10) \times \text{SU}(2) \times \text{U}(1)^2$ from the other E_8 sector.

10.8.2 Chirality

The chirality is determined by the 8 component $\text{SO}(8)$ spinor of the Ramond sector of right movers, viz. Table 3.4. It is labeled by $\{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\}$ with an even number of minus signs. By compactifying 6 internal spaces, three entries are killed, and there remains one $\pm\frac{1}{2}$ which is represented as $\oplus$ and $\ominus$. Let us choose this remaining one as the first entry as done in Subsect. 6.3.2. We start with $\omega \equiv \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ which is called right-handed since the first entry is $+\frac{1}{2}$. Depending on the compactification, the three internal spaces may have $+\frac{1}{2}$ or $-\frac{1}{2}$ which we denote as 0 and -1 , respectively. So a three component r is $\{0 \text{ or } -1, 0 \text{ or } -1, 0 \text{ or } -1\}$. Construct a three component vector $\tilde{s} \equiv r + \tilde{\omega}$ where $\tilde{\omega}$ is the last three components of ω . Then $\tilde{s}$ has eight possibilities $\{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\}$. If $\tilde{s}$ has an even number of minus signs, then the chirality is defined to be right-handed (R) $\oplus$ as s_0 of Subsect. 6.3.2 with $s = \{s_0, \tilde{s}\}$, and if $\tilde{s}$ has an odd number of minus sign then the chirality is defined as left-handed (L) $\ominus$. With this definition, in view of the gauge boson coupling to the fermion current the gauge boson and hence gauginos must be defined better to be L-handed to choose a total plus sign.

10.8.3 Matter from Untwisted Sector

From the untwisted sector, the $P^2 = 2$ vectors can be candidates for the matter. Those with $P \cdot V \equiv 0$ are the gauge multiplets. The massless matter fields are those with $P \cdot V \not\equiv 0$ and satisfy the masslessness condition. We have to consider $P \cdot V \equiv \frac{k}{12}$ where $k = 1, 2, \dots, 11$. We will consider only $k = 1, 2, \dots, 6$, since the $CT\bar{P}$ conjugates of $k = 1, 2, \dots, 5$ sectors will appear in $k = 7, 8, \dots, 11$ sectors. Thus, we obtain the following spectrum.

	Vector	Degeneracy	$\text{SU}(5) \times \text{SU}(3)$
$p \cdot V = \frac{1}{12}(\text{L})$	$(-1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0)$	3×3	$(\bar{\mathbf{5}}, \mathbf{3})$
	$(0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0)$	1×3	
	$(- \ - \ - \ + \ - \ + \ - \ -)$	1×3	
$p \cdot V = \frac{1}{12}(\text{L})$	$(+ \ + \ + \ - \ - \ + \ - \ -)$	1×3	$(\mathbf{1}, \mathbf{3})$
	Vector	Degeneracy	$\text{SU}(5) \times \text{SU}(3)$
$p \cdot V = \frac{4}{12}(\text{L})$	$(0 \ 0 \ 0 \ 0 \ 0 \ -1 \ -1 \ 0)$	1×3	$(\bar{\mathbf{5}}, \mathbf{3})$
	$(+ \ + \ - \ - \ + \ + \ - \ -)$	3×3	
	$(+ \ + \ + \ + \ + \ + \ - \ -)$	1×3	

	Vector	Degeneracy	$\mathrm{SU}(5) \times \mathrm{SU}(3)$
$p \cdot V = \frac{5}{12}(\mathrm{R})$	$(0 \ 0 \ 0 \ 1 \ 0 \ \underline{-1 \ 0 \ 0})$	1×3	$(\overline{\mathbf{10}}, \mathbf{\bar{3}})$
	$(\underline{-1 \ 0 \ 0 \ 0 \ 0} \ \underline{-1 \ 0 \ 0})$	3×3	
	$(\underline{+ \ + \ -} \ + \ + \ \underline{+ \ + \ -})$	3×3	
	$(\underline{+ \ - \ -} \ - \ + \ \underline{+ \ + \ -})$	3×3	

Let α denote the phase $e^{2\pi i/12}$. For $k = 1$ or $P \cdot V = \frac{1}{12}$, the left-movers obtain a phase α . We need an extra phase α^{-1} from the right movers, which is accomplished by $e^{-2\pi i \tilde{s} \cdot \phi}$ where $\phi = \frac{1}{12}(5 \ 4 \ 1)$ and $s = (\ominus + - +)$. It is left-handed (see Table 10.3). The four dimensional chirality is read of from the s_0 components of right mover, denoted as (s_0, s_1, s_2, s_3) . We make it a convention for $s = -\frac{1}{2}$ to be left-handed. For $k = 4$, $s = (\ominus + + -)$ provides the needed α^{-4} ; thus it is left-handed. For $k = 5$, $s = (\oplus + + +)$ provides α^{-5} ; thus it right-handed.

α from the right movers is provided by $s = (\oplus - + -)$ which thus will couple to $k = 11$. It is right-handed. α^4 from the right movers is provided by $s = (\oplus - - +)$ which will couple to $k = 8$. It is right-handed. α^5 from the right movers is provided by $s = (\oplus - - -)$ which will couple to $k = 7$. It is left-handed. These give the antiparticle spectra of those obtained in the previous paragraph.

So far we considered six chirality operators from $\mathbf{8}$ of the Ramond sector. There are two more chirality operators: $s = (\oplus + + +)$ and $s = (\ominus - - -)$. These from the right movers do not provide any phase; so they must couple to the $k = 0$ (or $k = 12$) sector. In fact we considered them already in obtaining the gauge sector which is chosen as left-handed through $s = (\ominus - - -)$.

Table 10.2 summarizes matter from the untwisted sector. Note that for $p \cdot V = \frac{5}{12}$ the original R-handed field is changed to the charge conjugated L-handed field in Table 10.2.

10.8.4 Twisted Sector

The Z_{12} -I with the twist vector $\phi = \frac{1}{12}(5 \ 4 \ 1)$ has three fixed points in the prime, i.e. θ^1 and θ^5 twisted sectors. This is because it has three fixed points in the second torus since it is the same as $\mathbf{Z}_3$, and for the first and the third torus the origin is the only fixed point. For the other nonprime order twists

Table 10.2. Summary of massless matter from the untwisted-sector

$p \cdot v$	$[\mathrm{SU}(5) \times \mathrm{SU}(3)]_{\text{chirality}}$
$\frac{1}{12}$	$(\mathbf{\bar{5}}, \mathbf{3})_L \oplus (\mathbf{1}, \mathbf{3})_L$
$\frac{4}{12}$	
$\frac{5}{12}$	$(\mathbf{\bar{5}}, \mathbf{3})_L$
$\frac{5}{12}$	$(\mathbf{10}, \mathbf{3})_L$

such as $k = 4$ and 6 , counting the number of massless states involves a more complicated nonvanishing projector P_k , (7.30).

The masslessness condition for left movers is

$$\frac{(P + kV)^2}{2} + \tilde{N}_L - \tilde{c} = 0 \quad (10.64)$$

with $\tilde{c} = 1 - \frac{1}{4}\eta_k(1 - \eta_k)$ of Chaps. 5 and 6. These are tabulated in Appendix D. For the θ^k twist ($k = 1, 2, \dots, 6$), we have

$$2\tilde{c} = \begin{cases} \frac{210}{144}, & k = 1; & \frac{192}{144}, & k = 4 \\ \frac{216}{144}, & k = 2; & \frac{210}{144}, & k = 5 \\ \frac{234}{144}, & k = 3; & \frac{216}{144}, & k = 6 \end{cases} \quad (10.65)$$

On the other hand, the right mover masslessness condition leads to

$$(\tilde{s} + k\phi_s)^2 = 2c = 2\tilde{c} - 1 - 2\tilde{N}_R \quad (10.66)$$

which is again tabulated in Appendix D. For $\mathbf{Z}_{12}$ -I, they are

$$2c = \frac{11}{24}, \frac{1}{2}, \frac{5}{8}, \frac{1}{3}, \frac{11}{24}, \frac{1}{2}, \quad (10.67)$$

for $k = 1, 2, \dots, 6$, respectively.

The following table summarizes matter from the twisted sector. In the θ^6 sector of the table, we omitted, except in the first row, hidden sector representations $(-1^2 - 2 \ 0^5)$ and $(0^2 - 2 \ 0^5)$ which provide $\mathcal{CPT}$ conjugate to each other.

The number of zero modes in the twisted sector are given by projector

$$P_m = \frac{1}{N} \sum \tilde{\chi}(\theta^m, \theta^n) \Delta_{(\theta^m)}^n \quad (10.68)$$

where Δ is the phase of the specified twisted sector and $\tilde{\chi}(\theta^i, \theta^j)$ is the Euler number (7.33) of the internal space both given in Chap. 7. The formula for multiplicity, (10.68), is the GSO allowed number of states. For the prime orbifold $\mathbf{Z}_3$, the multiplicity is just $\frac{1}{3}(1 + \Delta + \Delta^2)$ which can be either 1 (for $\Delta = 1$) or 0 (for $\Delta = e^{\pm 2\pi i/3}$). So in Subsect. 6.3.3 it was sufficient to count those with the vanishing phase. But for nonprime orbifolds such as $\mathbf{Z}_{12}$, the multiplicity (10.68) is nonvanishing even if Δ were not 1. Only for θ^1 and θ^5 twists, the multiplicity is given by the vanishing Δ phase.

Twist	$\tilde{N}_R(\chi)$	Vector Satisfying (10.64)	$[SU(5) \times SU(3)]$ rep.
θ	0 (L)	$(-1\ 0\ 0\ 0\ -1\ 0\ 0\ 0)(0^8)$ $(0\ 0\ 0\ 1\ -1\ 0\ 0\ 0)(0^8)$ $(-\ -\ -\ +\ -\ \frac{3}{2}\ -\ -\ -)(0^8)$ $(-\ -\ -\ +\ +\ -\ -\ -)(0^8)$	$3\ (\bar{\mathbf{5}}, \mathbf{1})_L$ $3\ (\mathbf{1}, \mathbf{1})_L$
θ	$\frac{1}{4}$ (L)	$(0^8)(0^8)$ $(-\ -\ +\ +\ -^4)(0^8)$ $(-^8)(0^8)$	$3(\mathbf{5}, \mathbf{1})_L$
θ^2	0 (L)	$(-^3\ +\ -\ \frac{3}{2}\ -^3)(0^2\ -1\ \pm 1\ 0^4)$ $(-^3\ +\ -\ -\ \frac{3}{2}\ -^2)(0^8)$	$3(\mathbf{10}, \mathbf{1})'_L$ projected out
θ^2	$\frac{1}{6}$	$(-^3\ +\ -\ \frac{3}{2}\ -^3)(-1\ 0\ -1\ 0^5)$	projected out
θ^3	0 (L)	$(-1^3\ 0\ -2\ -1^3)(-1\ 0\ -1\ 0^5)$ $(-1^2\ 0\ 1\ -2\ -1^3)(-1\ 0\ -1\ 0^5)$ $(-^3\ +\ -\ \frac{3}{2}\ -^3)(-1\ 0\ -1\ 0^5)$ $(-1^3\ 1\ -1\ -1^3)(0^2\ -2\ 0^5)$ $(-1^3\ 1\ -1\ -1^3)(0^2\ -1^2\ -1\ 0^4)$ $(-^3\ +\ -\ \frac{3}{2}\ -\ \frac{3}{2}\ -)(-1\ 0\ -1\ 0^5)$ $(-1^3\ 1\ -1^4)(0^8)$	$(\mathbf{5}, \mathbf{1})(\mathbf{1}, \mathbf{2})'_L$ $(\mathbf{1}, \mathbf{1})'_L$ $(\mathbf{10}, \mathbf{1})'_L$ $(\mathbf{1}, \bar{\mathbf{3}})(\mathbf{1}, \mathbf{2})'_L$ $(\mathbf{1}, \mathbf{1})_L$
θ^3	$\frac{1}{4}$ (L)	$(-1^3\ 1\ -1^4)(-1\ 0\ -1\ 0^5)$	$(\mathbf{1}, \mathbf{2})'_L$
θ^4	0 (L)	$(-1^3\ 1\ -2\ -2\ -1^2)(-1\ 0\ -1\ 0^5)$ $(-1^3\ 1\ -2\ -2\ -1^2)(0^2\ -2\ 0^5)$	$6(\mathbf{1}, \bar{\mathbf{3}})(\mathbf{1}, \mathbf{2})'_L$ $6(\mathbf{1}, \bar{\mathbf{3}})(\mathbf{1}, \mathbf{1})_L$
θ^5		no zero mass lattice	
θ^6	0 (L)	$(-2\ -1\ -1\ 1\ -3\ -2\ -2\ -2)$ $\cdot (0\ 0\ -2\ \dots)$ $(-1\ -1\ -1\ 2\ -3\ -2\ -2\ -2)$ $(\frac{-3}{2}\ \frac{-3}{2}\ \frac{-3}{2}\ \frac{3}{2}\ \frac{-7}{2}\ \frac{-5}{2}\ \frac{-5}{2}\ \frac{-5}{2})$ $(-1\ -2\ -2\ 2\ -3\ -2\ -2\ -2)$ $(-2\ -2\ -2\ 1\ -3\ -2\ -2\ -2)$ $(\frac{-3}{2}\ \frac{-3}{2}\ \frac{-3}{2}\ \frac{3}{2}\ \frac{-5}{2}\ \frac{-3}{2}\ \frac{-3}{2}\ \frac{-3}{2})$ $(\frac{-3}{2}\ \frac{-3}{2}\ \frac{-3}{2}\ \frac{3}{2}\ \frac{-5}{2}\ \frac{-3}{2}\ \frac{-5}{2}\ \frac{-5}{2})$ $(\frac{-3}{2}\ \frac{-3}{2}\ \frac{-3}{2}\ \frac{3}{2}\ \frac{-7}{2}\ \frac{-5}{2}\ \frac{-3}{2}\ \frac{-3}{2})$	included hidden mul. and $\mathcal{CTP}$ s in this box $(\bar{\mathbf{5}}, \mathbf{1})_{(5R+6L)} \rightarrow \mathcal{CTP}$ of A $(\mathbf{5}, \mathbf{1})_{(5R+6L)} \rightarrow \text{rep. A}$ $(\mathbf{1}, \mathbf{3})_{(5R+6L)} \rightarrow \mathcal{CTP}$ of B $(\mathbf{1}, \bar{\mathbf{3}})_{(5R+6L)} \rightarrow \text{rep. B}$

Since the calculation is straightforward even though tedious, here we show only the θ^4 twist in detail. The case of θ^4 includes all possible complications one can anticipate. The θ^4 twist vectors are

$$\begin{aligned}
4\phi &= \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \\
4V &\equiv \left(1\ 1\ 1\ -1\ 2\ \frac{4}{3}\ \frac{4}{3}\ \frac{4}{3}\right) \left(\frac{1}{3}\ \frac{1}{3}\ \frac{4}{3}\ 0\ 0\ 0\ 0\ 0\right). \quad (10.69)
\end{aligned}$$

In the θ^4 twisted sector, consider $P = (0^8)(0^8)$. In the above convention, P should be $(-1^3, 1, -2, -2, -1^2)(-1, 0, -1, 0^5)$. Then, from (10.64) we have $(P + 4V)^2 = 2\tilde{c} = \frac{4}{3}$ which is (10.65) with $\tilde{N}_L = 0$. Therefore, the state $P = (0^8)(0^8)$ is massless. The GSO projection is given when combined with the right movers. The masslessness condition for right movers is $(s + 4\phi)^2 = \frac{1}{3}$. But it is easier to calculate the multiplicity first and look for the masslessness condition for the multiplicity nonzero weights. Anyway, we should look for all the massless states in the end.

Note that for $P = (0^8)(0^8)$,

$$(P + 4V) \cdot V = \frac{5}{6}, \quad \tilde{\phi} \cdot \phi_s = \frac{7}{6}. \quad (10.70)$$

To obtain the chirality χ , we look at $s = (\frac{1}{2}\chi, \tilde{s})$ allowing nonvanishing multiplicities. See Subsect. 6.3.2 and (7.32). Then the phase of Δ_{θ^4} is found as $(P + \tilde{V}) \cdot V - (\tilde{s} + \tilde{\phi}) \cdot \phi_s - \frac{1}{2}m(V^2 - \phi_s^2)$. With $m = 4$, we have $\frac{1}{2}m(V^2 - \phi_s^2) = \frac{4}{3}$, and the phase of Δ is 2π times $(\frac{1}{3} - \tilde{s} \cdot \phi_s)$, and hence we obtain the multiplicity listed in Table 10.3. Then, we read the chirality χ , or $\oplus$ or $\ominus$, from the first entry of $s = (\frac{1}{2}\chi | s_1 \ s_2 \ s_3)$, which is shown in the first column of Table 10.3. The components of the vector $\tilde{s} \equiv (r + \tilde{\omega})$ are the last three $\pm\frac{1}{2}$'s of s . The number of massless states of the table is given by P_{θ^4} for each case. For this, we use the Euler numbers given in Appendix D [76],

$$\begin{aligned} \tilde{\chi}(\theta^4, 1) &= 27, \quad \tilde{\chi}(\theta^4, \theta^1) = 3, \quad \tilde{\chi}(\theta^4, \theta^2) = 3, \quad \tilde{\chi}(\theta^4, \theta^3) = 3, \\ \tilde{\chi}(\theta^4, \theta^4) &= 27, \quad \tilde{\chi}(\theta^4, \theta^5) = 3, \quad \tilde{\chi}(\theta^4, \theta^6) = 3, \quad \tilde{\chi}(\theta^4, \theta^7) = 3, \\ \tilde{\chi}(\theta^4, \theta^8) &= 27, \quad \tilde{\chi}(\theta^4, \theta^9) = 3, \quad \tilde{\chi}(\theta^4, \theta^{10}) = 3, \quad \tilde{\chi}(\theta^4, \theta^{11}) = 3. \end{aligned} \quad (10.71)$$

Table 10.3. $+$ and $-$ denote $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and $\oplus(\ominus)$ is R(L)-handed. Note that $(P + \tilde{V}) \cdot V - \frac{1}{2}m(V^2 - \phi_s^2) = -\frac{1}{2} \rightarrow \frac{1}{2}$

$\frac{1}{2}\chi$	$\tilde{s} = (r + \tilde{\omega})$	$\tilde{s} \cdot \phi_s$	$(\tilde{s} + \tilde{\phi}) \cdot \phi_s$	Δ Phase	Multiplicity
$\oplus$	$+$ $+$ $+$	$\frac{5}{12}$	$\frac{19}{12}$	$-\frac{1}{12} \cdot 2\pi$	0
$\oplus$	$+$ $-$ $-$	0	$\frac{14}{12}$	$\frac{1}{3} \cdot 2\pi$	0
$\oplus$	$-$ $+$ $-$	$-\frac{1}{12}$	$\frac{13}{12}$	$\frac{5}{12} \cdot 2\pi$	0
$\oplus$	$-$ $-$ $+$	$-\frac{4}{12}$	$\frac{10}{12}$	$\frac{2}{3} \cdot 2\pi$	0
$\ominus$	$-$ $-$ $-$	$-\frac{5}{12}$	$\frac{9}{12}$	$\frac{3}{4} \cdot 2\pi$	6
$\ominus$	$-$ $+$ $+$	0	$\frac{14}{12}$	$\frac{1}{3} \cdot 2\pi$	0
$\ominus$	$+$ $-$ $+$	$\frac{1}{12}$	$\frac{15}{12}$	$\frac{1}{4} \cdot 2\pi$	6
$\ominus$	$+$ $+$ $-$	$\frac{4}{12}$	$\frac{18}{12}$	$0 \cdot 2\pi$	9

Therefore, we obtain

$$P_4 = P_{\theta^4} = \frac{1}{12} \{3(1 + \Delta^4 + \Delta^8) \cdot [8 + 1 + \Delta(1 + \Delta + \Delta^2)]\} \quad (10.72)$$

which becomes 9, 6, 6 for $\Delta = 1, -1, \pm i$, respectively, and 0 for the other cases. Now consider the masslessness condition (10.67) for $s = \{\ominus, ---\}, \{\ominus, ++\}$ and $\{\ominus, +- -\}$. For $s^2 = 1$, the masslessness condition $(s + \tilde{\phi})^2 = \frac{1}{3}$ leads to $\tilde{s} \cdot \phi = -\frac{8}{12}$ which does not appear in Table 10.3. So we look for $s^2 = 3, 5, \dots$ with entries of $\pm \frac{3}{2}$. For $s^2 = 3$, the masslessness condition is $\tilde{s} \cdot \phi = -\frac{11}{12}$, which is not satisfied for any $\tilde{s}$ with one $\pm \frac{3}{2}$. For $s^2 = 5$, the masslessness condition is $\tilde{s} \cdot \phi = -\frac{14}{12}$ which is realized for $s = (\ominus \frac{-3}{2} \frac{-3}{2} -)$. It shows that the chirality is left-handed. The phase of Δ becomes $\frac{3}{4} \cdot 2\pi$, and the multiplicity becomes 6.

The masslessness condition for left movers becomes

$$(P + \tilde{V})^2 = \frac{4}{3}. \quad (10.73)$$

Thus, the vectors satisfying (10.73) constitute the representation $(\mathbf{1}, \bar{\mathbf{3}})_L$. This constitutes the E_8 part of the θ^4 twisted sector.

Note that in the θ^6 twisted sector, the $\mathcal{CTP}$ conjugate states are again in the θ^6 twisted sector, which is always the case in the θ^N twisted sector in a $\mathbf{Z}_{2N}$ orbifold model. In calculating the multiplicity, we have to consider the $\mathbf{Z}_2$ symmetry in the θ^N twisted sector, i.e. dividing by 2. In our case of θ^6 twisted sector, there appears another factor 2 coming from the hidden sector multiplicity 2. It is shown only in the first line of the θ^N twisted sector of the twisted sector table. In the θ^6 twisted sector, the $\mathcal{CTP}$ conjugate of A and B are explicitly shown.

In Table 10.4 we summarize the number of observable sector fields except singlets. Adding the representations from the untwisted sector, Table 10.2, we can easily check that there do not exist any nonabelian anomalies of $SU(5)$ and $SU(3)$. Even though we have not shown explicitly, there does not exist any gauge anomalies except for one $U(1)$ anomaly which is cancelled by the Green-Schwarz mechanism.

Table 10.4. Summary of massless observable matter from the twisted-sectors

Twist	$[SU(5) \times SU(3)]_{\text{chirality}}$
θ^1	$3(\bar{\mathbf{5}}, \mathbf{1})_L \oplus 3(\mathbf{5}, \mathbf{1})_L$
θ^3	$2(\mathbf{5}, \mathbf{1})_L \oplus 2(\mathbf{1}, \bar{\mathbf{3}})_L$
θ^4	$18(\mathbf{1}, \bar{\mathbf{3}})_L$
θ^6	$6(\mathbf{5}, \mathbf{1})_L \oplus 5(\bar{\mathbf{5}}, \mathbf{1})_L \oplus 6(\mathbf{1}, \bar{\mathbf{3}})_L \oplus 5(\mathbf{1}, \mathbf{3})_L$

10.8.5 No Flipped SU(5)

When we break the SU(3) group by the Higgs mechanism, the massless spectrum we obtain are

$$3(\mathbf{10} \oplus \bar{\mathbf{5}} \oplus \mathbf{1})_L \quad (10.74)$$

plus $(\mathbf{5} + \bar{\mathbf{5}})$'s and singlets. It is quite interesting to note that (10.74) is the SO(10) spinor or the flipped SU(5) matter spectrum. For it to describe a flipped SU(5) model, the U(1) charges of the flipped SU(5) group must be in the ratio $1 : -3 : 5$. But even before checking the U(1) charges, the spectrum lacks $\mathbf{10} \oplus \bar{\mathbf{10}}$ for the Higgs fields. Thus, we do not succeed in obtaining a flipped SU(5) model from the above $\mathbf{Z}_{12} - I$ model. The kind of difficulty of obtaining an adjoint representation at the level 1 algebra reappears here as the difficulty of obtaining the representation $\mathbf{10} \oplus \bar{\mathbf{10}}$.

Exercise 1: Show that the states with nonvanishing multiplicities in Table 10.3 do not satisfy the mass shell condition $M_R^2 = 0$ but $s = (\ominus, \frac{-3}{2}, \frac{-3}{2}, -)$ satisfies the mass shell condition.

Exercise 2: Prove that the chirality of the θ^1 twisted sector is -1 , i.e. left-handed.

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Code Manual and $\mathbf{Z}_3$ Tables

11.1 Introduction

In this chapter, the $\mathbf{Z}_3$ orbifold models with two Wilson lines with all the interesting gauge groups, which can allow chiral fermions at low energy, are tabulated by computer output. Thus, $\mathbf{Z}_3$ orbifolds with two Wilson lines is a closed chapter now.¹ We restricted to two Wilson line models since here the multiplicity 3 is automatic. We have not included models with three Wilson lines, since here it is not guaranteed from the outset that the resulting spectrum has the multiplicity 3 in addition to the difficulty of too much computing time required. Our strategy is the following. First, we program such that a minimum computing time is required at our present knowledge. How this is achieved will be explained below. The process discussed in this chapter produces more than 60 thousand two Wilson line models. Then, we go through all the models to choose physically interesting ones. For example, we exclude all models which give vectorlike representations. Also, we exclude by inspection the models with gauge groups which cannot give standard-like models (after gauge symmetry breaking). These procedure reduces the number to ~ 1000 . Still some of these can be excluded, but that work is left for the reader who uses our table.

In order to classify all the possible heterotic string models originating from $E_8 \times E'_8$ gauge group orbifolded by shift vectors and Wilson line backgrounds, one must first develop a useful tool. The number of such models are expected to be of order of 10^7 for three Wilson lines, hence it is impossible to do it by hand. So, given the input in terms of the shift vector V and three Wilson lines a_1, a_2 , and a_3 , our computer tool should be able to

- identify the gauge group,
- identify the untwisted and twisted matter representations,
- print the model in terms of a human readable form,

¹ For standardlike models, there already exists a study on two Wilson lines [1].

and also be able to

- traverse over all possible different combinations of the shift vectors and 3 Wilson lines,
- avoid duplication of the same model by correctly identifying the equivalent classes imposed by various symmetries and redundancies in the formulation of the theory.

Thus, the tool is divided largely by above two parts. The first part will be called “Group” coded mainly in `group.c`, while the second part will be called “Classify” coded mainly in `wyle.c` and `classify.c`. For the most part of the code, the first priority is the efficiency and the least computing time. Below, we explain the features of this tool developed from the scratch by Hwang [2].

11.2 Group

The *group part* provides functions to identify the gauge group, the untwisted sector matter spectrum and the twisted sector matter spectrum when a shift vector and (if any) Wilson line vectors²

$$V, a_1, a_3, a_5 \quad (11.1)$$

are given. The gauge group and the *untwisted* (U) sector matter spectrum are determined (from the U sector) by inspecting the dot product between the E_8 root vector $r_i, i = 1 \dots 240$ and V and a_j :

$$\text{gauge boson : } r_i \cdot V = \text{integer} \quad (11.2)$$

$$k - \text{untwisted matter : } r_i \cdot V = k/N \pmod{\text{integer}} \quad (11.3)$$

$$r_i \cdot a_j = \text{integer for all } j.$$

The states r_i are categorized according to transformation property under the orbifold, or the dot product value $r_i \cdot V$ and should be inspected to find out which representation of the group they would form.

Unlike the untwisted sector, for the *twisted sector* (T) there is no finite pre-defined set of vectors which will be inspected. Every vectors P^I on the $E_8 \times E'_8$ root lattice should be tested to see if they satisfy the certain mass shell conditions for a given branch of the twisted sector described by the modified shift vector $V' = V \pm a_1, \dots$, twist number k , and the moded oscillator $\tilde{N}$. After the states satisfying the mass shell conditions are collected, they are inspected to find out which representation of the group they would form.

Technically, finding out which 16-dimensional state P^I survives is quite easy: just scan over some well defined region of integer numbers. Although the number of dimensions of the vectors, 16, is not small, a well developed

² We use ϕ for the twist of the internal space and V for the shift in the group lattice. Note $a_1 = a_2, a_3 = a_4$, and $a_5 = a_6$.

algorithm makes it not a big problem. However, it is at a totally different level of difficulty, that one can identify which representation of the given gauge group those such survived states belong to and then makes us enable to print the output in a human-recognizable form. For the untwisted sector, the states are just a subset³ of the set of all E_8 roots r_i , so the process is rather simple. For the twisted sectors, there is no such luck, and it makes quite a big job. A relatively easy old attempt was to find out the $U(1)$ charges of the survived states and categorize states with the same charges and naming them by the number of such states [3]. Actually, this method can shorten the process drastically. But our process is more transparent since the representations are obtained from the weight system directly.

A straightforward approach is to use the following relation between the weight vectors of the irreducible representation,

$$\alpha = \alpha' \pm r_i . \quad (11.4)$$

By adding (or subtracting) each identified simple root r_i to each survived state α in the list of the survived states $\mathcal{Q}$, one tests whether the resulting vector α' is in $\mathcal{Q}$. If there is such a root vector r_i between α' , α and α' are in the same representation. While repeating this to every states in $\mathcal{Q}$, one eventually finds a state θ which cannot be connected further by adding:

$$\begin{aligned} \theta &= \alpha'' + r_k \in \mathcal{Q} \\ \theta + r_j &\notin \mathcal{Q} \quad \text{for all } j . \end{aligned} \quad (11.5)$$

Then θ is the *highest weight state* and the index k of the last simple root r_k which connects α'' to θ is the *Coxeter label* of that representation. For the case for a general semi-simple group, perform this process for each simple factor group and combine them in order to identify the total representation. This process is quite time-consuming, mainly in testing whether the added (or subtracted) state is in the list of the survived states $\mathcal{Q}$ or not. For the untwisted sector, a survived state is an E_8 root which is readily identified. However, for the twisted sector, there is no such simple identification, which means that one should compare all the survived states to make sure that the test succeeds or fails. Since we are talking about a 16 dimensional vector, the required computing time is huge.

In the beginning, the code was developed in terms of **MAPLE**, which provides the graph theory package which makes it easy to identify the Lie group structure. But the high-level language has the general problem of slowness in the simple operation of *comparing*. The heavy requirement of comparing mentioned above makes the **MAPLE** code very slow: it can calculate one model in

³ We have seen that without Wilson lines it is just a branching rule.

30 seconds in 2.4 GHz Pentium4 machine, which means that we need months to calculate all the 2 Wilson line models. Hence, we start to port it into C, where we need to develop some packages which high-level programming languages provide by default. In this way, the final C code is developed which is able to calculate 1000 models in about 30 seconds. This C code enables us to calculate the whole two Wilson line models in an order of hours.

11.2.1 Gauge Group Identification

Before we identify the representation of matter states, we need to identify the gauge group. The states satisfying (11.2) are the root vectors of the surviving gauge group. Thus, all we need to do to identify the gauge group is to identify these simple roots. The choice for simple roots depends on the choice of “positive roots” [4]. One consistent way to define positiveness of the E_8 root vectors is to choose a hypersurface of the 8 dimensional root space, norm of which is given by P_+ , the “positivity vector”. Then a root r_i is positive if $P_+ \cdot r_i > 0$, and $-r_i$ is automatically negative since $P_+ \cdot (-r_i) < 0$, consistent with the property of the Lie algebra roots. Under this definition of positivity, the set of the simple roots of E_8 , $\{\alpha_i\}$, consists of roots *having the smallest positive dot product values between them and P_+* . The conventional selection of simple roots of E_8 are the following:

$$\begin{aligned}
\alpha_1 = r_{31} &= (0, 1, -1, 0, 0, 0, 0, 0) \\
\alpha_2 = r_{55} &= (0, 0, 1, -1, 0, 0, 0, 0) \\
\alpha_3 = r_{75} &= (0, 0, 0, 1, -1, 0, 0, 0) \\
\alpha_4 = r_{91} &= (0, 0, 0, 0, 1, -1, 0, 0) \\
\alpha_5 = r_{103} &= (0, 0, 0, 0, 0, 1, -1, 0) \\
\alpha_6 = r_{109} &= (0, 0, 0, 0, 0, 0, 1, -1) \\
\alpha_7 = r_{233} &= (+, -, -, -, -, -, -, +) \\
\alpha_8 = r_{111} &= (0, 0, 0, 0, 0, 0, 1, 1)
\end{aligned} \tag{11.6}$$

Here $+$ means $\frac{1}{2}$ while $-$ means $-\frac{1}{2}$ and the index of in the subscript of r_i is the index of E_8 roots which our C code actually uses. These roots can be the simple roots of E_8 in our mechanism if we choose the positivity vector as

$$P_+ = (22, 6, 5, 4, 3, 2, 1, 0) . \tag{11.7}$$

The dot product value $\alpha_i \cdot P_+$ is all 2 except for α_7 which is 1, up to the overall scale factor 2 which is there to represent all states by integers. All the other roots have dot product value larger than 2. We call these 8 root states as the “absolute simple roots” in the sense that the dot product value is the smallest under any circumstances. The *absolute highest root*, which is the highest root of E_8 is

$$-\alpha_0 = r_1 = (1, 1, 0, 0, 0, 0, 0, 0), \quad (11.8)$$

dot product value of which with P_+ is 28.

Note that it is quite difficult to write a code to find the simple roots directly using the definition in the text books on group theory. Using the concept of “positivity vector” above simplifies it charmingly.

It is easy to generalize the above method for the case of surviving gauge groups from E_8 . First, collect the surviving *absolute simple roots*. Then, remove every surviving root which can be expressed as a positive sum of them. Find the smallest root (meaning “having the smallest dot product value with the positivity vector P_+ ”). It is a simple root. Remove again any root which can be expressed as a positive sum of the identified simple roots. Do this repeatedly until all the surviving roots are removed.

After identifying all the simple roots, we analyze the dot product structure between them (effectively speaking, drawing the Dynkin diagram), sorting their orders in such a way that a bigger simple group comes first while for the labelling of the simple roots we use Georgi’s conventions [4]. At this stage, we can use this index of the simple root for the identification of the Dynkin index of the zero mode representation.

11.3 Classification

The overall flow of *classification part* is the following:

- restrict to the fundamental region (the first step of removing translational equivalence)
- pick up a “standard list”,
- collecting Weyl classes
- remove using additional Weyl reflections
- remove by additional translational equivalences (the second step of removing by translational equivalences)
- for more than 2 Wilson line cases, remove the redundancy of interchanging Wilson line vectors one another.

What the *classification part* does is to scan over all the possible input vectors and remove all the redundancies and symmetries. Since there are so many symmetries and redundancies, removing all of them is extremely non-trivial. We solve this problem by many different steps.

11.3.1 Translational Equivalence

A state is equivalent to another if they are connected by a vector in the $E_8 \times E'_8$ root lattice Γ .

$$P^{I'} = P^I + \Lambda_1, \quad \Lambda_1 \in \Gamma. \quad (11.9)$$

Basically, this type of redundancies can be most easily removed by fixing the fundamental area and considering the states only inside this area. However, since Γ is not a simple rectangular lattice, it is difficult to express this condition for the fundamental area. Hence, we used a little bit relaxed condition⁴

$$(P^I)^2 \leq 1. \quad (11.10)$$

This condition still allows two vectors which are equivalent up to lattice translation of Γ . However, they can be different by only one E_8 root vector, for example,

$$\left(\frac{2}{3}, \frac{2}{3}, 0, 0, 0, 0, 0, 0\right) \quad \text{and} \quad \left(-\frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0, 0\right). \quad (11.11)$$

For this case, one can build a very effective criterion for reducibility of the shift vector.

Removing the redundancy of the symmetry is the same as choosing a representative vector among the equivalent vectors. (This can be viewed as one way of choosing the vacuum.) Thus, let us build a well defined criterion of choosing the representative vectors when there are translationally equivalent vectors. For the symmetry of translational equivalence, the most intuitive way is to choose a “smaller” vector as a representative one. However, for a 16 dimensional vector, it is not enough since there are too many equivalent vectors. Fortunately, there are other stuff to be used as a criterion. For the general case of presence of an arbitrary number of Wilson line vectors, there are a few predefined vectors from the earlier stage. For example, if we want to discuss the second Wilson line vector a_3 , there are already predefined shift vector V and the first Wilson line vector a_1 . Thus, we can use these vectors to choose a representative vector. We choose the vector P^I as a representative vector among P^I and $P^{I'}$ if P^I has the *bigger* dot product value with the predefined vectors. Thus, for simplicity, let us enlarge the notion of “smallness” such that we call P^I is the smaller than $P^{I'}$ if P^I turns out to be more adequate for a representative vector than $P^{I'}$.

Now, let us introduce some terminology which are useful for discussing removing the redundancy of translational equivalence.

Definition of Reducibility: We will call a vector P^I *super-reducible* if there is another translationally equivalent vector $P^{I'}$ which has a smaller norm. We will call a vector P^I *reducible* if there exists another translationally equivalent vector $P^{I'}$ which has the same norm as P^I but has a *bigger* dot product value with the predefined vectors $V, a_1, \dots$. We will call a vector P^I *translationally degenerate* if there are other translationally equivalent vectors $P^{I'}$ which have the same norm and the same dot product value with the

⁴ We can transform the $P^2 = 2$ normalization of previous chapters to $|P| \leq 1$ region.

predefined vectors as P^I . We will call a vector P^I *irreducible* if there is no translationally equivalent vector which has the same norm as P^I .

Clearly, if a vector P^I is irreducible, it is the representative vector of its translationally equivalent class. If P^I is super-reducible or reducible, it cannot be a representative vector, so one can just ignore it when scanning. The non-trivial part is P^I is translationally degenerate. In that case, one should compare all the translationally degenerate vectors and choose one vector as a representative vector. We judiciously define the criterion such that P^I is smaller than $P^{I'}$ if P^I has a bigger element than $P^{I'}$ if one scans from the first element to last element. With this refinement, one can choose an indisputable representative vector for every translational equivalent class.

From the definition of the reducibility property of any vector P^I from the previous paragraph, it looks like one must compare all the vectors in the equivalent class of P^I . However, the following observation enables one that he can see whether P^I is reducible or not without directly comparing to the other vectors.

Theorem of Reducibility: Let s be the biggest value of the dot product between the vector P^I and every E_8 root vector α . If $s > 1$ then P^I is super-reducible. If $s < 1$ then P^I is irreducible. If $s = 1$, follow the next step.

Let t be the smallest value of the dot product between the predefined shift vector V and every E_8 root vector α which gives the smallest s . If $t > 0$ then P^I is reducible. If $t < 0$ then P^I is irreducible. If $t = 0$ repeat this step with next predefined Wilson line vector a_1, a_3 or a_5 until $t \neq 0$. If $t = 0$ for every existing predefined vector, then P^I is translationally degenerate to some other equivalent vector $P^{I'}$.

(Proof) Assume that there is another translationally equivalent vector $P^{I'} = P^I - \alpha$ for some E_8 root α . Then

$$(P^{I'})^2 = (P^I)^2 - 2P^I \cdot \alpha + \alpha^2 \quad (11.12)$$

Since $\alpha^2 = 2$ for any E_8 root α ,

$$P^I \cdot \alpha - 1 = \frac{1}{2} \left((P^I)^2 - (P^{I'})^2 \right) \quad (11.13)$$

If $(P^I)^2$ is smaller than any other $(P^{I'})^2$, the RHS is always smaller than zero and $P^I \cdot \alpha$ is always less than 1 for any E_8 root α . If $(P^I)^2$ is bigger than some other $(P^{I'})^2$, the RHS can be bigger than zero and $P^I \cdot \alpha$ can be bigger than 1 for some E_8 root α . Thus the first step of the theorem is proved. On the other hand, if $(P^I)^2$ is the same as some other $(P^{I'})^2$, the RHS can be as big as 0 and $P^I \cdot \alpha$ can be as big as 1 for some E_8 root α . For this case, one must compare $P^I \cdot V$ and $P^{I'} \cdot V$ for the predefined shift vector V , if any. Let us denote the set of α which gives $P^I \cdot \alpha = 1$ as A . By the same spirit, if

$$V \cdot \alpha = V \cdot (P^I - P^{I'}) \quad (11.14)$$

is always bigger than zero for all $\alpha \in A$, P^I is the *smallest* one in the sense defined in this section. If the LHS can be smaller than zero for some $\alpha \in A$, P^I is *bigger* than some $P^{I'}$. If the smallest value of the LHS is zero, then one must repeat this using the next unused predefined Wilson line vector $a_1, \dots$. If there is no unused predefined Wilson line vector, P^I is translationally degenerate to $P^{I'} = P^I - \alpha$ for that α .

Besides the fact that one does not need to compare every vector in the equivalent class, one major advantage of this theorem is that one can easily get the biggest value of $P^I \cdot \alpha$ since the E_8 root α has the special simple form. Every integral E_8 root has two non-zero entry whose absolute value is one. Thus the biggest $P^I \cdot \alpha$ for integral α is the sum of the largest absolute values of entries of P^I . For the spinorial E_8 root, every entry has the same absolute value $\frac{1}{2}$. Thus, the biggest $P^I \cdot \alpha$ for spinorial α is just the sum of the absolute values of all entries of P^I divided by 2. The overall biggest $P^I \cdot \alpha$ is the bigger one between those two. The only thing one should do to implement this in the code, is checking which entry of P^I is the biggest, the second biggest, and the smallest one. Then maximum value of $P^I \cdot \alpha$ can be readily read off. If this value is 1, just compare dot product values with predefined shift vectors and $\alpha \in A$, the number of which are just a few. By this way, one can determine that a given P^I is a translational representative one or not without consuming much time. All the steps of removing translational redundancy can be done at this process except the cumbersome case that P^I is translationally degenerate to some other $P^{I'}$. For that case, one treats them in another part of the code (denoted by additional removing of translational equivalence in the flowchart of the previous page).

11.3.2 Weyl Reflection Equivalence

The E_8 root lattice has many automorphisms known as Weyl reflections. If one enlists every Weyl reflected vector whenever one needs, it requires a huge amount of memory and time. One *must* invent some way to deal with this problem by a simpler method.

Weyl reflection by an integral E_8 root can be viewed as a sorting. We treat these by the notion of *standard list*, which means just a sorted list.

Using the standard list, all the other equivalent vectors by the Weyl reflection by spinorial E_8 roots can be described by a few other standard lists. In this way, one can catalogue every vector in a Weyl equivalent class and Weyl chamber, which is the set of Weyl equivalent classes.

Once one makes this catalogue, one can pick up a representative vector for each equivalent class by some well defined rule. Our choice is to choose one with the largest number of zero entries. If two vectors have the same number of zero entries, one compare each non-zero entries and choose a bigger one. We call the preferred one as the *simpler* one, another terminology useful in dealing with the Weyl equivalence.

In practice, in order to speed up the time to access each Weyl class,⁵ one can use a few ID values which are invariant under the Weyl reflection. The ID values actually used are the norm $(P^I)^2/2$ and the dot product value with predefined shift vectors $P^I \cdot V$, etc.

If there exist predefined shift vectors, one must consider only the Weyl reflections which leave the predefined shift vectors invariant. Combined with the translational equivalence, however, another complication occurs at this stage. If the Weyl reflection turns a predefined shift vector V into another V' which is different from V only by a lattice translation, then still it is the valid Weyl reflection. However, it does change the ID value of the dot product with the predefined shift vector V . Thus, one should care about this in the later stage. At the later stage, one compares each representative vector of each Weyl equivalent class of the earlier stage and chooses the *simplest* one, ignoring the others. By this way, one can remove all the redundancy due to the Weyl reflections.

11.3.3 Interchanging between Wilson Line Vectors a_1, a_3 and a_5

If there are more than one Wilson line vector, the model is irrelevant to their ordering. For the sake of simplicity required in building computer code, however, one distinguishes the order of a_i 's maximally. Thus, one must treat the equivalence of the interchanged set by hand after choosing representative vectors for all translational and Weyl equivalent classes. Since every process uses the information of the predefined shift vectors, changing the order of two Wilson line vectors are very nontrivial. Especially, choosing a representative vector for the equivalent classes are not compatible with changing the orders since it is an act of symmetry breaking. Thus, implementing the removing of interchanging Wilson line vectors is a powerful independent check for the integrity of the code which deals translational redundancy and Weyl reflection redundancy. We can manage to build this code and succeeded in checking, which makes us very confident about the integrity of our code.

11.4 Tables for Two Wilson Line $\mathbf{Z}_3$ Models

Below we tabulate all interesting two Wilson line $\mathbf{Z}_3$ models. This encompasses most physically interesting three family models from $\mathbf{Z}_3$ orbifold. With two Wilson lines, every spectrum is a multiple of three and hence the multiplicity 3 is automatic. The search for two Wilson line $\mathbf{Z}_3$ models produced the output with more than 60,000. Then, we inspected all the output to represent one out of $O(100)$ models giving the same kind of matter spectrum with the same gauge group. The process discussed in the previous sections could not identify

⁵ The only way to access them is to compare all of them to the probing vector, which needs a huge amount of time again.

these. In the following tables we already used the exchange symmetry of $E_8 \leftrightarrow E'_8$, and hence the assignment of the standard model can be in E_8 or E'_8 . In the tables, we use the following convention,

- The model number at the top left corner of the box is just one out of $O(100)$ models. But this model may not actually represent all of them since they may differ by $U(1)$ charges even though it is not likely.
- The shift vectors relevant for the E_8 and E'_8 locations are separately shown. These define the model.
- We used the following abbreviations for the representations

$*$: complex conjugate representation of $SU(n)$
 v : vector representation of $SO(8)$
 s : spinor representation of $SO(n)$
 c : conjugate spinor representation of $SO(n)$
 b : conjugate fundamental representation of E_6

- In the untwisted sector, the coefficient in front of the representation is the multiplicity. For example, $3(\mathbf{1}, \mathbf{3})$ means that we have three copies of $(\mathbf{1}, \mathbf{3})$. We keep the multiplicity in the tables.
- In the twisted sector, the multiplicity is written in front of the representation. But, with the limited space available we neglected the $U(1)$ charges. For example, $4T3 + 4T4 + 4T8$ ($2, 16, 1$) means that we have four copies of $(\mathbf{2}, \mathbf{16}, \mathbf{1})$ each from T3, T4, and T8 sectors, but a possible differentiation by the $U(1)$ charges is not shown.
- We name the twisted sector according to the following local shift vectors,

U : untwisted sector
 $T0$: twisted sector with twist V
 $T1$: twisted sector with twist $V + a_1$
 $T2$: twisted sector with twist $V - a_1$
 $T3$: twisted sector with twist $V + a_3$
 $T4$: twisted sector with twist $V - a_3$
 $T5$: twisted sector with twist $V + a_1 + a_3$
 $T6$: twisted sector with twist $V + a_1 - a_3$
 $T7$: twisted sector with twist $V - a_1 + a_3$
 $T8$: twisted sector with twist $V - a_1 - a_3$.

References

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Tables

The entries in the tables are ordered as,

Table number

gauge group from E_8	UT matter from E_8
gauge group from E'_8	UT matter from E'_8
shift vectors $6V, 6a_1, 6a_3$ in the E_8 direction	
shift vectors $6V, 6a_1, 6a_3$ in the E'_8 direction	
matter from twisted sector	
$3T_2+3T_4+3T_5$ (1, 3, 1, 3^*, 1) means the representation (1, 3, 1, 3^*, 1) appears 3 times in T_2, 3 times in T_4, and 3 times in T_5.	

1	
$SU(3) \times SU(3) \times SU(3) \times SU(3)$	E_8 $3(3, 1, 3^*, 3^*)$ $v = (00000000) a_1 = (00000000) a_3 = (00000000)$ $v = (42200000) a_1 = (00042200) a_3 = (00042200)$
$3T_2+3T_4+3T_5$ (1, 3, 1, 3^* , 1) $3T_1+3T_3+3T_8$ (1, 3, 1, 1, 3^*) $3T_2+3T_4+3T_5$ (1, 3, 3, 1, 1) $3T_1+3T_3+3T_8$ (1, 3, 3, 1, 1) $3T_0+3T_6+3T_7$ (1, 1, 1, 3, 3, 1) $3T_1+3T_3+3T_8$ (1, 1, 3, 1, 3) $3T_2+3T_4+3T_5$ (1, 1, 3, 3, 1) $3T_0+3T_6+3T_7$ (1, 1, 3, 3, 1) $3T_0+3T_6+3T_7$ (1, 1, 1, 3, 3, 3) $9T_0+9T_6+9T_7$ (1, 3, 3, 1, 1, 1) $9T_1+9T_3+9T_8$ (1, 1, 1, 3, 1, 1) $9T_2+9T_4+9T_5$ (1, 1, 1, 1, 3, 1)	
2	
$SU(6)$	E_7 $3(15^*) + 3(6) + 3(6)$ $v = (00000000) a_1 = (22000000) a_3 = (22000000)$ $v = (42200000) a_1 = (22022000) a_3 = (22022000)$
$6T_0+3T_1+3T_2+3T_3+3T_4+3T_5+6T_6+6T_7+3T_8$ (1, 6) $3T_0+3T_6+3T_7$ (1, 15) $3T_1+3T_2+3T_3+3T_4+3T_5+3T_8$ (1, 6) $27T_0+18T_1+18T_2+18T_3+18T_4+18T_5+27T_6+27T_7+18T_8$ (1, 1)	
3	
$E_6 \times SU(3)$ $E_6 \times SU(3)$	$(none)$ $3(27^*, 3)$ $v = (00000000) a_1 = (42200000) a_3 = (42200000)$ $v = (42200000) a_1 = (00000000) a_3 = (00000000)$
$3T_2+3T_4+3T_5$ (1, 3, 1, 3^*) $3T_1+3T_3+3T_8$ (1, 3^* , 1, 3^*) $3T_0+3T_6+3T_7$ (1, 1, 27, 1) $9T_0+9T_6+9T_7$ (1, 1, 1, 3^*)	
4	
$E_6 \times SU(3)$ $E_6 \times SU(3)$	$(none)$ $(none)$ $v = (00000000) a_1 = (42200000) a_3 = (42200000)$ $v = (42200000) a_1 = (42200000) a_3 = (42200000)$
$3T_1+3T_3+3T_8$ (1, 3^* , 1, 3) $3T_2+3T_4+3T_5$ (27, 1, 1, 1) $9T_2+9T_4+9T_5$ (1, 3, 1, 1) $3T_0+3T_6+3T_7$ (1, 1, 27, 1) $9T_0+9T_6+9T_7$ (1, 1, 1, 3^*)	
5	
$E_6 \times SU(3)$ $SO(8)$	$3(27^*, 3)$ $(none)$ $v = (42200000) a_1 = (00000000) a_3 = (00000000)$ $v = (42200000) a_1 = (42200000) a_3 = (40 - 220000)$
$3T_2$ (27, 1, 1) $9T_0+9T_1+9T_2+9T_3+9T_4+9T_5+9T_6+9T_7+9T_8$ (1, 3^* , 1)	

6	
$E_6 \times SU(3)$ $SO(8)$	(none) (none)
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0\ 0)$
	$3T2(1\ 27^*, 1\ ;1) \quad 9T3+9T5+9T7(1\ 3\ ;1) \quad 9T0+9T1+9T2(1\ 3^*, 1) \quad 3T4+3T6+3T8(1\ 1\ ;8c) \quad 3T4+3T6+3T8(1\ 1\ ;8s)$ $3T4+3T6+3T8(1\ 1\ ;8v) \quad 36T4+36T6+36T8(1\ 1\ ;1)$

7	
$E_6 \times SU(3)$ $SU(3) \times SU(3) \times SU(3) \times SU(3)$	(none) (none)
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$
	$3T8(1\ 3\ ;1, 3, 1, 1) \quad 3T6(1\ 3\ ;1, 3^*, 1, 1) \quad 3T3(1\ 3^*, 3, 1, 1, 1) \quad 3T7(1\ 3^*, 1, 1, 3^*, 1) \quad 3T5(1\ 3^*, 1, 1, 1, 3^*) \quad 3T2(1\ 1\ ;3, 1, 3^*, 1) \quad 3T1(1\ 1\ ;3, 1, 1, 3^*) \quad 3T2(1\ 1\ ;3^*, 3, 1, 1) \quad 3T1(1\ 1\ ;3^*, 3^*, 1, 1) \quad 3T0(1\ 1\ ;1, 3, 3, 1) \quad 3T1(1\ 1\ ;1, 3, 1, 3) \quad 3T2(1\ 1\ ;1, 3^*, 3, 1) \quad 3T0(1\ 1\ ;1, 3^*, 1, 3) \quad 3T0(1\ 1\ ;1, 1, 3^*, 3^*) \quad 3T4(27\ 1\ ;1, 1, 1, 1) \quad 9T4(1\ 3\ ;1, 1, 1, 1) \quad 9T0(1\ 1\ ;3^*, 1, 1, 1) \quad 9T1(1\ 1\ ;1, 1, 1, 3, 1) \quad 9T2(1\ 1\ ;1, 1, 1, 3)$

8	
$E_6 \times SU(3)$ $SU(3) \times SU(3) \times SU(3) \times SU(3)$	(none) $3(3, 1, 3^*, 3^*)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$
	$3T6(1\ 3\ ;3^*, 1, 1, 1) \quad 3T8(1\ 3\ ;1, 1, 3, 1) \quad 3T4(1\ 3\ ;1, 1, 1, 3) \quad 3T7(1\ 3^*, 3^*, 1, 1, 1) \quad 3T3(1\ 3^*, 1, 1, 3, 1) \quad 3T5(1\ 3^*, 1, 1, 1, 3) \quad 3T2(1\ 1\ ;3, 1, 3^*, 1) \quad 3T1(1\ 1\ ;3, 1, 1, 3^*) \quad 3T2(1\ 1\ ;3^*, 3, 1, 1) \quad 3T1(1\ 1\ ;3^*, 3^*, 1, 1) \quad 3T0(1\ 1\ ;1, 3, 3, 1) \quad 3T1(1\ 1\ ;1, 3, 1, 3) \quad 3T2(1\ 1\ ;1, 3^*, 3, 1) \quad 3T2(1\ 1\ ;1, 3^*, 3, 1) \quad 3T0(1\ 1\ ;1, 3^*, 1, 3) \quad 3T0(1\ 1\ ;1, 1, 3^*, 3^*) \quad 3T4(27\ 1\ ;1, 1, 1, 1) \quad 9T4(1\ 3\ ;1, 1, 1, 1) \quad 9T0(1\ 1\ ;3^*, 1, 1, 1) \quad 9T1(1\ 1\ ;1, 1, 1, 3, 1) \quad 9T2(1\ 1\ ;1, 1, 1, 3)$

9	
$E_6 \times SU(3)$ $SU(3) \times SU(3) \times SU(3) \times SU(3)$	$3(27^*, 3)$ (none)
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$
	$3T1(1\ 3^*, 3, 1, 1, 1) \quad 3T0(1\ 3^*, 3^*, 1, 1, 1) \quad 3T8(1\ 3^*, 1, 3, 1, 1) \quad 3T7(1\ 3^*, 1, 3^*, 1, 1) \quad 3T6(1\ 3^*, 1, 1, 3^*, 1) \quad 3T3(1\ 3^*, 1, 1, 3, 1) \quad 3T4(1\ 3^*, 1, 1, 1, 3) \quad 3T5(1\ 3^*, 1, 1, 1, 3^*) \quad 3T2(27^*, 1\ ;1, 1, 1, 1) \quad 9T2(1\ 3^*, 1, 1, 1, 1)$

10	
$E_6 \times SU(3)$ $SU(3) \times SU(3) \times SU(3) \times SU(3)$	$3(27^*, 3)$ $3(3, 1, 3^*, 3^*)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$
	$3T0+3T6+3T7(1\ 3^*, 3^*, 1, 1, 1) \quad 3T1+3T3+3T8(1\ 3^*, 1, 1, 3, 1) \quad 3T2+3T4+3T5(1\ 3^*, 1, 1, 1, 3)$

11	
$SO(14)$ E_6	(none) $3(27^*)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)$
	$3T1+3T2+3T3+3T4+3T5+3T8(14v\ ;1) \quad 3T0+3T6+3T7(1\ ;27^*) \quad 27T0+12T1+12T2+12T3+12T4+12T5+27T6+27T7+12T8(1\ ;1)$

12	
$SO(14)$ $SU(6) \times SU(3)$	(none) $3(15^*, 3)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (-4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ -2\ -2\ 0\ 0\ 0)$
	$6T0+3T1+3T2+3T3+3T4+6T5+3T6+3T7+6T8(1\ ;6, 1) \quad 3T0+3T5+3T8(1\ ;15^*, 1) \quad 9T0+3T1+3T2+3T3+3T4+9T5+3T6+3T7+9T8(1\ ;1, 3^*)$

13	
$SO(14)$ $SU(6) \times SU(3)$	(none) $3(6, 3)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$
	$3T2+3T4+3T5(14v\ ;1, 1) \quad 6T0+6T6+6T7(1\ ;6, 1) \quad 3T0+3T6+3T7(1\ ;15^*, 1) \quad 3T1+3T3+3T8(1\ ;6^*, 1) \quad 9T0+9T6+9T7(1\ ;1, 3^*) \quad 3T1+3T3+3T8(1\ ;1, 3) \quad 12T2+12T4+12T5(1\ ;1, 1)$

14	
$SU(5) \times SU(2)$ $SU(3) \times SU(2)$	$3(1, 1)$ $3(1, 1) + 3(3, 1) + 3(1, 1) + 3(3^*, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ -2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 3\ 1\ 1\ 1\ 1\ -1) a_3 = (4\ -2\ 2\ 2\ 0\ 0\ 0\ 2)$
	$3T3(5\ 1\ ;1, 1) \quad 3T2(5^*, 1\ ;1, 1) \quad 3T0+3T1+3T6+3T7(1\ 1\ ;3, 1) \quad 3T0+3T5+3T6+3T7(1\ 1\ ;3^*, 1) \quad 6T0+3T1+3T5+6T6+6T7(1\ 1\ ;1, 2) \quad 3T2+3T3+9T4+9T8(1\ 2\ ;1, 1) \quad 24T0+12T1+6T2+6T3+9T4+12T5+24T6+24T7+9T8(1\ 1\ ;1, 1)$

15	
$SU(3) \times SU(3) \times SU(3) \times SU(3)$ $E_6 \times SU(3)$	(none) $3(27^*, 3)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$
	$3T2(3\ 1\ ;1, 1, 1, 1, 3^*) \quad 3T1(3^*, 1, 1, 1, 1, 3^*) \quad 3T3(1\ 3^*, 1, 1, 1, 3^*) \quad 3T4(1\ 3, 1, 1, 1, 3^*) \quad 3T5(1\ 1, 3, 1, 1, 3^*) \quad 3T8(1\ 1, 3^*, 1, 1, 3^*) \quad 3T6(1\ 1, 1, 3, 1, 3^*) \quad 3T7(1\ 1, 1, 3^*, 1, 3^*) \quad 3T0(1\ 1, 1, 1, 27^*, 1) \quad 9T0(1\ 1, 1, 1, 1, 1, 3^*)$

16	
$SU(3) \times SU(3) \times SU(3) \times SU(3)$ $SO(8)$	$3(3, 1, 3^*, 3^*)$ (none)
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0)$	
$3T2(1, 3^*, 1, 3; 1)\ 3T2(1, 3, 3, 1; 1)\ 3T2(1, 1, 3^*, 3^*; 1)\ 9T0+9T1+9T2(3^*, 1, 1, 1; 1)\ 9T3+9T5+9T7(1, 1, 3, 1; 1)\ 9T4+9T6+9T8(1, 1, 1, 3; 1)$	
17	
$SU(3) \times SU(3) \times SU(3) \times SU(3)$ $SU(3) \times SU(3) \times SU(3) \times SU(3)$	(none) $3(3, 1, 3^*, 3^*)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$	
$3T2(3, 1, 1, 1; 3^*, 1, 1, 1)\ 3T1(3^*, 1, 1, 1; 3^*, 1, 1, 1)\ 3T3(1, 3^*, 1, 1; 1, 1, 3, 1)\ 3T4(1, 3, 1, 1; 1, 1, 1, 3)\ 3T5(1, 1, 3, 1; 1, 1, 3, 1)\ 3T8(1, 1, 3^*, 1; 1, 1, 1, 3)\ 3T6(1, 1, 1, 3; 1, 1, 1, 3)\ 3T7(1, 1, 1, 3^*; 1, 1, 3, 1)\ 3T0(1, 1, 1, 1; 1, 1, 1, 1; 3^*, 1, 3)\ 3T0(1, 1, 1, 1; 1, 3^*, 3^*)\ 9T0(1, 1, 1, 1; 3^*, 1, 1, 1)$	
18	
$SU(3) \times SU(3) \times SU(3) \times SU(3)$ $SU(3) \times SU(3) \times SU(3) \times SU(3)$	(none) (none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$	
$3T1(3^*, 1, 1, 1; 3, 1, 1, 1)\ 3T2(1, 3^*, 3^*, 1; 1, 1, 1, 1)\ 3T3(1, 3^*, 1, 1; 1, 1, 3, 1)\ 3T2(1, 3, 1, 3^*; 1, 1, 1, 1)\ 3T4(1, 3, 1, 1; 1, 1, 1, 3)\ 3T2(1, 1, 3, 3; 1, 1, 1, 1)\ 3T5(1, 1, 3, 1; 1, 1, 1, 3^*)\ 3T8(1, 1, 3^*, 1; 1, 3, 1, 1)\ 3T6(1, 1, 1, 3; 1, 1, 3^*, 1)\ 3T7(1, 1, 1, 3^*; 1, 3^*, 1, 1)\ 3T0(1, 1, 1, 1; 1, 1, 1, 1; 3, 3, 1)\ 3T0(1, 1, 1, 1; 1, 3^*, 1, 3)\ 3T0(1, 1, 1, 1; 1, 3^*, 3^*)\ 9T2(3, 1, 1, 1; 1, 1, 1, 1)\ 9T0(1, 1, 1, 1; 3^*, 1, 1, 1)$	
19	
$SU(3) \times SU(3) \times SU(3)$ $SO(8) \times SU(3)$	(none) (none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0)$	
$3T2(3, 3, 1; 1, 1)\ 3T2(3^*, 1, 3; 1, 1)\ 3T2(1, 3^*, 3^*; 1, 1)\ 3T4+3T6+3T8(3, 1, 1; 1, 1)\ 3T3+3T5+3T7(3^*, 1, 1; 1, 1)\ 3T4+3T6+3T8(1, 3^*, 1; 1, 1)\ 3T3+3T5+3T7(1, 3, 1; 1, 1)\ 3T4+3T6+3T8(1, 1, 3; 1, 1)\ 3T3+3T5+3T7(1, 1, 3^*; 1, 1)\ 3T0(1, 1, 1; 8c, 1)\ 3T0(1, 1, 1; 8s, 1)\ 3T0(1, 1, 1; 8v, 1)\ 9T1(1, 1, 1; 1, 3)\ 9T0(1, 1, 1; 1, 3^*)\ 9T0+27T2(1, 1, 1; 1, 1)$	
20	
$SU(3) \times SU(3) \times SU(3)$ $SO(8) \times SU(3)$	$3(1, 3^*, 3^*)$ $3(1, 3^*) + 3(8s, 3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T5(1, 3, 1; 1, 3)\ 3T6(1, 1, 3; 1, 3)\ 3T4+3T8(3, 1, 1; 1, 1)\ 3T3+3T7(3^*, 1, 1; 1, 1)\ 3T4+3T8(1, 3^*, 1; 1, 1)\ 3T3+3T7(1, 3, 1; 1, 1)\ 3T4+3T8(1, 1, 3; 1, 1)\ 3T2(1, 1, 1; 8c, 1)\ 3T0(1, 1, 1; 8v, 1)\ 3T0+9T1+3T2(1, 1, 1; 1, 3^*)\ 12T0+12T2(1, 1, 1; 1, 1)$	
21	
$SU(3) \times SU(3) \times SU(3)$ $SO(8) \times SU(3)$	$3(3, 1, 3^*)$ (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T4(3^*, 3, 1; 1, 1)\ 3T7(3^*, 1, 1; 1, 3^*)\ 9T4+3T6+3T8(3, 1, 1; 1, 1)\ 3T3+3T5(3^*, 1, 1; 1, 1)\ 9T4+3T6+3T8(1, 3^*, 1; 1, 1)\ 3T3+3T5(1, 3, 1; 1, 1)\ 9T4+3T6+3T8(1, 1, 3; 1, 1)\ 3T3+3T5(1, 1, 3^*; 1, 1)\ 3T2(1, 1, 1; 8c, 1)\ 3T0(1, 1, 1; 8v, 1)\ 3T0+9T1+3T2(1, 1, 1; 1, 3)\ 3T0+3T2(1, 1, 1; 1, 3^*)\ 12T0+12T2(1, 1, 1; 1, 1)$	
22	
$SU(3) \times SU(3) \times SU(3)$ $SO(8) \times SU(3)$	$3(3^*, 3^*, 1)$ (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ -2\ 0\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (-4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T1(3, 1, 3^*; 1, 1)\ 3T8(3, 1, 1; 1, 3^*)\ 3T1(3^*, 3^*, 1; 1, 1)\ 3T1(1, 3, 3; 1, 1)\ 3T3(1, 3, 1; 1, 3)\ 3T4+3T6(3, 1, 1; 1, 1)\ 3T5+3T7(3^*, 1, 1; 1, 1)\ 3T4+3T6(1, 3^*, 1; 1, 1)\ 3T5+3T7(1, 3, 1; 1, 1)\ 3T4+3T6(1, 1, 3; 1, 1)\ 3T5+3T7(1, 1, 3^*; 1, 1)\ 3T0+3T2(1, 1, 1; 8v, 1)\ 3T0+3T2(1, 1, 1; 1, 3)\ 3T0+3T2(1, 1, 1; 1, 3^*)\ 12T0+27T1+12T2(1, 1, 1; 1, 1)$	
23	
$SU(3) \times SU(3) \times SU(3)$ $SO(8) \times SU(3)$	(none) (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T6(3, 1, 1; 1, 3^*)\ 3T4(3^*, 3, 1; 1, 1)\ 3T8(1, 3^*, 1; 1, 3)\ 9T4(3, 1, 1; 1, 1)\ 3T3+3T5+3T7(3^*, 1, 1; 1, 1)\ 9T4(1, 3^*, 1; 1, 1)\ 3T3+3T5+3T7(1, 3, 1; 1, 1)\ 9T4(1, 1, 3; 1, 1)\ 3T3+3T5+3T7(1, 1, 3^*; 1, 1)\ 3T2(1, 1, 1; 8c, 1)\ 3T1(1, 1, 1; 8s, 1)\ 3T0(1, 1, 1; 8v, 1)\ 3T0+3T1+3T2(1, 1, 1; 1, 3)\ 3T0+3T1+3T2(1, 1, 1; 1, 3^*)\ 12T0+12T1+12T2(1, 1, 1; 1, 1)$	
24	
$SU(3) \times SU(3) \times SU(3)$ $SO(8) \times SU(3)$	$3(3, 1, 3^*) + 3(1, 3, 3) + 3(3^*, 3^*, 1)$ (none)
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0)$	
$3T2(3, 1, 3^*; 1, 1)\ 3T2(3^*, 3^*, 1; 1, 1)\ 3T2(1, 3, 3; 1, 1)\ 3T4+3T6+3T8(3, 1, 1; 1, 1)\ 3T3+3T5+3T7(3^*, 1, 1; 1, 1)\ 3T4+3T6+3T8(1, 3^*, 1; 1, 1)\ 3T3+3T5+3T7(1, 3, 1; 1, 1)\ 3T4+3T6+3T8(1, 1, 3; 1, 1)\ 3T3+3T5+3T7(1, 1, 3^*; 1, 1)\ 9T1(1, 1, 1; 1, 3)\ 9T0(1, 1, 1; 1, 3^*)\ 27T2(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(3) \times SU(3)$ $SO(8) \times SU(3)$	$3(3^*, 3^*, 1)$ (none)
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0)\ a_3 = (2\ -2\ -2\ -2\ 0\ 0\ 0\ 0)$	
3T7 (3, 1, 3;1, 1) 3T6 (3, 1, 1;1, 3) 3T4+3T8 (3, 1, 1;1, 1) 3T3+3T5+9T7 (3*, 1, 1;1, 1) 3T4+3T8 (1, 3*, 1;1, 1) 3T3+3T5+9T7 (1, 3, 1;1, 1) 3T4+3T8 (1, 1, 3;1, 1) 3T3+3T5+9T7 (1, 1, 3*;1, 1) 3T2 (1, 1, 1;8c, 1) 3T1 (1, 1, 1;8s, 1) 3T1+3T2 (1, 1, 1;1, 3) 9T0+3T1+3T2 (1, 1, 1;1, 3*) 12T1+12T2 (1, 1, 1;1, 1)	

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$SU(3) \times SU(3) \times SU(3)$ $SU(6)$	(none) $3(15^*) + 3(6) + 3(6)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
3T4+3T6+3T8 (3, 1, 1;1) 3T3+3T5+3T7 (3*, 1, 1;1) 3T4+3T6+3T8 (1, 3*, 1;1) 3T3+3T5+3T7 (1, 3, 1;1) 3T4+3T6+3T8 (1, 1, 3;1) 3T3+3T5+3T7 (1, 1, 3*;1) 6T0 (1, 1, 1;6) 3T0 (1, 1, 1;15*) 27T0+27T1+27T2 (1, 1, 1;1)	

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$SU(3) \times SU(3) \times SU(3)$ $SU(6)$	(none) (none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
3T2 (3, 3, 1;1) 3T2 (3*, 1, 3;1) 3T2 (1, 3*, 3*;1) 3T4+3T6+3T8 (3, 1, 1;1) 3T3+3T5+3T7 (3*, 1, 1;1) 3T4+3T6+3T8 (1, 3*, 1;1) 3T3+3T5+3T7 (1, 3, 1;1) 3T4+3T6+3T8 (1, 1, 3;1) 3T3+3T5+3T7 (1, 1, 3*;1) 6T0 (1, 1, 1;6) 3T0 (1, 1, 1;15*) 27T0+27T1+27T2 (1, 1, 1;1)	

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$SU(3) \times SU(3) \times SU(3)$ $SU(6)$	$3(1, 1, 1) + 3(1, 1, 1) + 3(1, 1, 1)$ $3(1) + 3(1) + 3(1) + 3(20) + 3(1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
3T4+3T6+3T8 (3, 1, 1;1) 3T3+3T5+3T7 (3*, 1, 1;1) 3T4+3T6+3T8 (1, 3*, 1;1) 3T3+3T5+3T7 (1, 3, 1;1) 3T4+3T6+3T8 (1, 1, 3;1) 3T3+3T5+3T7 (1, 1, 3*;1) 3T0+3T1+3T2 (1, 1, 1;6) 3T0+3T1+3T2 (1, 1, 1;6*) 18T0+18T1+18T2 (1, 1, 1;1)	

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$SU(3) \times SU(3) \times SU(3)$ $SU(6)$	(none) (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
3T4 (3*, 3, 1;1) 9T4+9T6 (3, 1, 1;1) 3T3+3T5+3T7 (3*, 1, 1;1) 9T4+9T8 (1, 3*, 1;1) 3T3+3T5+3T7 (1, 3, 1;1) 9T4 (1, 1, 3;1) 3T3+3T5+3T7 (1, 1, 3*;1) 3T0+3T1+3T2 (1, 1, 1;6) 3T0+3T1+3T2 (1, 1, 1;6*) 18T0+18T1+18T2 (1, 1, 1;1)	

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$SU(3) \times SU(3) \times SU(3)$ $SU(6)$	$3(3, 1, 3^*) + 3(1, 3, 3) + 3(3^*, 3^*, 1)$ (none)
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
3T2 (3, 1, 3*;1) 3T2 (3*, 3*, 1;1) 3T2 (1, 3, 3;1) 3T4+3T6+3T8 (3, 1, 1;1) 3T3+3T5+3T7 (3*, 1, 1;1) 3T4+3T6+3T8 (1, 3*, 1;1) 3T3+3T5+3T7 (1, 3, 1;1) 3T4+3T6+3T8 (1, 1, 3;1) 3T3+3T5+3T7 (1, 1, 3*;1) 27T0+27T1+27T2 (1, 1, 1;1)	

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$SU(3) \times SU(3) \times SU(3)$ $SU(6)$	$3(3, 3^*, 3^*)$ $3(15^*) + 3(6) + 3(6)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 2\ 0\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)\ a_3 = (-2\ -2\ 0\ -2\ -2\ 0\ 0\ 0)$	
9T0+3T1+3T2+3T3+3T4+3T6+3T7 (3*, 1, 1;1) 3T1+3T2+3T3+3T4+9T5+3T6+3T7 (1, 3, 1;1) 3T1+3T2+3T3+3T4+3T6+3T7+9T8 (1, 1, 3;1)	

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$SU(3) \times SU(3) \times SU(3) \times SU(3)$ E_8	(none) (none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$	
3T6 (1;3, 3, 1, 1) 3T5 (1;3, 3*, 1, 1) 3T4 (1;3, 1, 3*, 1) 3T7 (1;3, 1, 3, 1) 3T8 (1;3, 1, 1, 3) 3T3 (1;3, 1, 1, 3*) 3T4 (1;3*, 3, 1, 1) 3T3 (1;3*, 3*, 1, 1) 3T8 (1;3*, 1, 3*, 1) 3T5 (1;3*, 1, 3, 1) 3T6 (1;3*, 1, 1, 3) 3T7 (1;3*, 1, 1, 3*) 3T5 (1;1, 3, 3*, 1) 3T0 (1;1, 3, 3, 1) 3T3 (1;1, 3, 1, 3) 3T1 (1;1, 3, 1, 3*) 3T1 (1;1, 3*, 3*, 1) 3T4 (1;1, 3*, 3, 1) 3T0 (1;1, 3*, 1, 3) 3T6 (1;1, 3*, 1, 3*) 3T7 (1;1, 1, 3*, 3) 3T0 (1;1, 1, 3*, 3*) 3T1 (1;1, 1, 3, 3) 3T8 (1;1, 1, 3, 3*) 9T1 (1;3, 1, 1, 1) 9T0 (1;3*, 1, 1, 1) 9T8 (1;1, 3, 1, 1) 9T7 (1;1, 3*, 1, 1) 9T6 (1;1, 1, 3*, 1) 9T3 (1;1, 1, 3, 1) 9T4 (1;1, 1, 1, 3) 9T5 (1;1, 1, 1, 3*)	

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$SU(3) \times SU(3) \times SU(3) \times SU(3)$ E_8	(none) $3(3, 1, 3^*, 3^*)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$	
3T2+3T4+3T5 (1;3, 1, 3*, 1) 3T1+3T3+3T8 (1;3, 1, 1, 3*) 3T2+3T4+3T5 (1;3*, 3, 1, 1) 3T1+3T3+3T8 (1;3*, 3*, 1, 1) 3T0+3T6+3T7 (1;1, 3, 3, 1) 3T1+3T3+3T8 (1;1, 3, 1, 3) 3T2+3T4+3T5 (1;1, 3*, 3, 1) 3T0+3T6+3T7 (1;1, 3*, 1, 3) 3T0+3T6+3T7 (1;1, 1, 3*, 3*) 9T0+9T6+9T7 (1;3*, 1, 1, 1) 9T1+9T3+9T8 (1;1, 1, 3, 1) 9T2+9T4+9T5 (1;1, 1, 1, 3)	

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$SO(8) \times SU(3)$ E_7	$3(1) + 3(56)$ $3(8v, 1) + 3(8c, 1) + 3(8s, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)\ a_3 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)$	
3T2+3T4+3T5 (1;8c, 1) 3T1+3T3+3T8 (1;8s, 1) 3T0+3T6+3T7 (1;8v, 1) 3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8 (1;1, 3) 3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8 (1;1, 3*) 12T0+12T1+12T2+12T3+12T4+12T5+12T6+12T7+12T8 (1;1, 1)	

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E_7	(none)
$SU(6)$	(none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$6T_0+3T_3+3T_4+3T_5+3T_6+3T_7+3T_8\ (1\ ;6)\quad 3T_1\ (1\ ;15)\quad 3T_0\ (1\ ;15^*)\quad 6T_1+3T_3+3T_4+3T_5+3T_6+3T_7+3T_8\ (1\ ;6^*)$ $27T_0+27T_1+18T_3+18T_4+18T_5+18T_6+18T_7+18T_8\ (1\ ;1)$	

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E_7	(none)
$SU(6)$	$3(15^*) + 3(6) + 3(6)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)\ a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$6T_0+3T_1+3T_2+3T_3+3T_4+3T_5+6T_6+6T_7+3T_8\ (1\ ;6)\quad 3T_0+3T_6+3T_7\ (1\ ;15^*)\quad 3T_1+3T_2+3T_3+3T_4+3T_5+3T_8\ (1\ ;6^*)$ $27T_0+18T_1+18T_2+18T_3+18T_4+18T_5+27T_6+27T_7+18T_8\ (1\ ;1)$	

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E_7	$3(1) + 3(56)$
$SU(6)$	$3(1) + 3(1) + 3(1) + 3(20) + 3(1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)\ a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T_0+3T_1+3T_2+3T_3+3T_4+3T_5+3T_6+3T_7+3T_8\ (1\ ;6)\quad 3T_0+3T_1+3T_2+3T_3+3T_4+3T_5+3T_6+3T_7+3T_8\ (1\ ;6^*)$ $18T_0+18T_1+18T_2+18T_3+18T_4+18T_5+18T_6+18T_7+18T_8\ (1\ ;1)$	

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$E_6 \times SU(3)$	(none)
$E_6 \times SU(3)$	(none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$	
$3T_8\ (1, 3; 1, 3)\quad 3T_6\ (1, 3; 1, 3^*)\quad 3T_3\ (1, 3^*; 1, 3)\quad 3T_7\ (1, 3^*; 1, 3^*)\quad 3T_5\ (27^*, 1; 1, 1)\quad 3T_4\ (27, 1; 1, 1)\quad 9T_4\ (1, 3; 1, 1)$ $9T_5\ (1, 3^*; 1, 1)\quad 3T_0\ (1, 1; 27^*, 1)\quad 3T_1\ (1, 1; 27, 1)\quad 9T_1\ (1, 1; 1, 3)\quad 9T_0\ (1, 1; 1, 3^*)$	

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$E_6 \times SU(3)$	(none)
$E_6 \times SU(3)$	$3(27^*, 3)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$	
$3T_2+3T_4+3T_5\ (1, 3; 1, 3^*)\quad 3T_1+3T_3+3T_8\ (1, 3^*; 1, 3^*)\quad 3T_0+3T_6+3T_7\ (1, 1; 27^*, 1)\quad 9T_0+9T_6+9T_7\ (1, 1; 1, 3^*)$	

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$E_6 \times SU(3)$	(none)
$E_6 \times SU(3)$	(none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$	
$3T_1+3T_3+3T_8\ (1, 3^*; 1, 3)\quad 3T_2+3T_4+3T_5\ (27, 1; 1, 1)\quad 9T_2+9T_4+9T_5\ (1, 3; 1, 1)\quad 3T_0+3T_6+3T_7\ (1, 1; 27^*, 1)\quad 9T_0+9T_6+9T_7\ (1, 1; 1, 3^*)$	

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$E_6 \times SU(3)$	(none)
$SO(8)$	(none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0\ 0)$	
$3T_4\ (27, 1; 1)\quad 9T_4+9T_6+9T_8\ (1, 3; 1)\quad 9T_3+9T_5+9T_7\ (1, 3^*; 1)\quad 3T_0+3T_1+3T_2\ (1, 1; 8c)\quad 3T_0+3T_1+3T_2\ (1, 1; 8s)\quad 3T_0+3T_1+3T_2\ (1, 1; 8v)\quad 36T_0+36T_1+36T_2\ (1, 1; 1)$	

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$E_6 \times SU(3)$	$3(27^*, 3)$
$SO(8)$	(none)
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0\ 0)$	
$3T_2\ (27^*, 1; 1)\quad 9T_0+9T_1+9T_2+9T_3+9T_4+9T_5+9T_6+9T_7+9T_8\ (1, 3^*; 1)$	

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$E_6 \times SU(3)$	$3(27^*, 3)$
$SO(8)$	$3(1) + 3(1) + 3(8v) + 3(1) + 3(8s) + 3(8c)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0\ 0)$	
$9T_0+9T_1+9T_2+9T_3+9T_4+9T_5+9T_6+9T_7+9T_8\ (1, 3^*; 1)$	

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$E_6 \times SU(3)$	(none)
$SU(3) \times SU(3) \times SU(3) \times SU(3)$	(none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\ a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)\ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$	
$3T_8\ (1, 3; 1, 3, 1, 1)\quad 3T_6\ (1, 3; 1, 3^*, 1, 1)\quad 3T_3\ (1, 3^*; 3, 1, 1, 1)\quad 3T_7\ (1, 3^*; 1, 1, 3^*, 1)\quad 3T_5\ (1, 3^*; 1, 1, 1, 3^*)\quad 3T_2\ (1, 1; 3, 1, 3^*, 1)$ $3T_1\ (1, 1; 3, 1, 1, 3^*)\quad 3T_2\ (1, 1; 3^*, 3, 1, 1)\quad 3T_1\ (1, 1; 3^*, 3^*, 1, 1)\quad 3T_0\ (1, 1; 1, 3, 3, 1)\quad 3T_1\ (1, 1; 1, 3, 1, 3)\quad 3T_2\ (1, 1; 1, 3^*, 3, 1)\quad 3T_0\ (1, 1; 1, 3^*, 1, 3)\quad 3T_0\ (1, 1; 1, 1, 1, 3^*, 3^*)\quad 3T_4\ (27, 1; 1, 1, 1, 1)\quad 9T_4\ (1, 3; 1, 1, 1, 1)\quad 9T_0\ (1, 1; 3^*, 1, 1, 1)$ $9T_1\ (1, 1; 1, 1, 3, 1)\quad 9T_2\ (1, 1; 1, 1, 1, 3)$	

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$E_6 \times SU(3)$	(none)
$SU(3) \times SU(3) \times SU(3) \times SU(3)$	$3(3, 1, 3^*, 3^*)$
$v = (00000000) a_1 = (00000000) a_3 = (42200000)$	
$v = (42200000) a_1 = (00042200) a_3 = (00042200)$	
$3T6(1, 3; 3^*, 1, 1, 1) \quad 3T8(1, 3; 1, 1, 3, 1) \quad 3T4(1, 3; 1, 1, 1, 3) \quad 3T7(1, 3^*; 3^*, 1, 1, 1) \quad 3T3(1, 3^*; 1, 1, 3, 1) \quad 3T5(1, 3^*; 1, 1, 1, 3) \quad 3T2(1, 1; 3, 1, 3^*, 1) \quad 3T1(1, 1; 3, 1, 1, 3^*) \quad 3T2(1, 1; 3^*, 3, 1, 1) \quad 3T1(1, 1; 3^*, 3^*, 1, 1) \quad 3T0(1, 1; 1, 3, 3, 1) \quad 3T1(1, 1; 1, 3, 1, 3) \quad 3T2(1, 1; 1, 3^*, 3, 1) \quad 3T0(1, 1; 1, 3^*, 1, 3) \quad 3T0(1, 1; 1, 1, 3^*, 3^*) \quad 9T0(1, 1; 3^*, 1, 1, 1) \quad 9T1(1, 1; 1, 1, 3, 1) \quad 9T2(1, 1; 1, 1, 1, 3)$	

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$E_6 \times SU(3)$	$3(27^*, 3)$
$SU(3) \times SU(3) \times SU(3) \times SU(3)$	(none)
$v = (42200000) a_1 = (00000000) a_3 = (00000000)$	
$v = (42200000) a_1 = (42200000) a_3 = (00042200)$	
$3T1(1, 3^*; 3, 1, 1, 1) \quad 3T0(1, 3^*; 3^*, 1, 1, 1) \quad 3T8(1, 3^*; 1, 3, 1, 1) \quad 3T7(1, 3^*; 1, 3^*, 1, 1) \quad 3T6(1, 3^*; 1, 1, 3^*, 1) \quad 3T3(1, 3^*; 1, 1, 3, 1) \quad 3T4(1, 3^*; 1, 1, 1, 3) \quad 3T5(1, 3^*; 1, 1, 1, 3^*) \quad 3T2(27^*, 1; 1, 1, 1, 1) \quad 9T2(1, 3^*; 1, 1, 1, 1)$	

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$E_6 \times SU(3)$	$3(27^*, 3)$
$SU(3) \times SU(3) \times SU(3) \times SU(3)$	(none)
$v = (42200000) a_1 = (00000000) a_3 = (00000000)$	
$v = (42200000) a_1 = (24 - 200000) a_3 = (00042200)$	
$3T2(1, 3^*; 3, 1, 1, 1) \quad 3T0(1, 3^*; 3^*, 1, 1, 1) \quad 3T6(1, 3^*; 1, 3, 1, 1) \quad 3T5(1, 3^*; 1, 3^*, 1, 1) \quad 3T8(1, 3^*; 1, 1, 3^*, 1) \quad 3T3(1, 3^*; 1, 1, 3, 1) \quad 3T4(1, 3^*; 1, 1, 1, 3) \quad 3T7(1, 3^*; 1, 1, 1, 3^*) \quad 3T1(27^*, 1; 1, 1, 1, 1) \quad 9T1(1, 3^*; 1, 1, 1, 1)$	

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$SO(14)$	(none)
E_6	(none)
$v = (00000000) a_1 = (00000000) a_3 = (40000000)$	
$v = (42200000) a_1 = (42200000) a_3 = (22000000)$	
$3T3+3T4+3T5+3T6+3T7+3T8(14v; 1) \quad 3T0(1; 27^*) \quad 3T1(1; 27) \quad 27T0+27T1+12T3+12T4+12T5+12T6+12T7+12T8(1; 1)$	

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$SO(14)$	(none)
E_6	$3(27^*)$
$v = (00000000) a_1 = (40000000) a_3 = (40000000)$	
$v = (42200000) a_1 = (22000000) a_3 = (22000000)$	
$3T1+3T2+3T3+3T4+3T5+3T8(14v; 1) \quad 3T0+3T6+3T7(1; 27^*) \quad 27T0+12T1+12T2+12T3+12T4+12T5+27T6+27T7+12T8(1; 1)$	

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$SO(14)$	(none)
$SU(6) \times SU(3)$	(none)
$v = (00000000) a_1 = (00000000) a_3 = (40000000)$	
$v = (42200000) a_1 = (42200000) a_3 = (00022000)$	
$3T7+3T8(14v; 1, 1) \quad 6T0+3T3+3T4(1; 6, 1) \quad 3T1(1; 15, 1) \quad 3T0(1; 15^*, 1) \quad 6T1+3T5+3T6(1; 6^*, 1) \quad 9T0+3T3+3T4(1; 1, 3^*) \quad 9T1+3T5+3T6(1; 1, 3) \quad 12T7+12T8(1; 1, 1)$	

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$SO(14)$	(none)
$SU(6) \times SU(3)$	$3(15^*, 3)$
$v = (00000000) a_1 = (40000000) a_3 = (40000000)$	
$v = (42200000) a_1 = (00022000) a_3 = (00022000)$	
$6T0+3T1+3T2+3T3+3T4+3T5+6T6+6T7+3T8(1; 6, 1) \quad 3T0+3T6+3T7(1; 15^*, 1) \quad 9T0+3T1+3T2+3T3+3T4+3T5+9T6+9T7+3T8(1; 1, 3^*)$	

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$SO(14)$	(none)
$SU(6) \times SU(3)$	$3(6, 3)$
$v = (00000000) a_1 = (40000000) a_3 = (40000000)$	
$v = (42200000) a_1 = (42220000) a_3 = (42220000)$	
$3T2+3T4+3T5(14v; 1, 1) \quad 6T0+6T6+6T7(1; 6, 1) \quad 3T0+3T6+3T7(1; 15^*, 1) \quad 3T1+3T3+3T8(1; 6^*, 1) \quad 9T0+9T6+9T7(1; 1, 3^*) \quad 3T1+3T3+3T8(1; 1, 3) \quad 12T2+12T4+12T5(1; 1, 1)$	

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$SO(14)$	$3(14v) + 3(64s)$
$SU(6) \times SU(3)$	$3(1, 1) + 3(20, 1)$
$v = (40000000) a_1 = (00000000) a_3 = (00000000)$	
$v = (42222000) a_1 = (42200000) a_3 = (42200000)$	
$3T2+3T4+3T5(14v; 1, 1) \quad 3T0+3T6+3T7(1; 6, 1) \quad 3T1+3T3+3T8(1; 6^*, 1) \quad 3T0+3T6+3T7(1; 1, 3^*) \quad 3T1+3T3+3T8(1; 1, 3) \quad 12T2+12T4+12T5(1; 1, 1)$	

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$SO(8) \times SU(3)$	(none)
$SU(2) \times SU(2) \times SU(2)$	$3(2, 1, 1) + 3(1, 1, 2) + 3(1, 2, 1)$
$v = (40000000) a_1 = (40000000) a_3 = (04220000)$	
$v = (42222000) a_1 = (2400 - 2000) a_3 = (4022 - 2200)$	
$3T2(1, 1; 1, 2, 2) \quad 3T2(1, 1; 2, 1, 2) \quad 3T2(1, 1; 2, 2, 1) \quad 3T3(8s, 1; 1, 1, 1) \quad 3T3+9T8(1, 3; 1, 1, 1) \quad 3T3+9T7(1, 3^*; 1, 1, 1) \quad 3T0+3T1+6T2+3T4+3T5+3T6(1, 1; 1, 1, 2) \quad 3T0+3T1+6T2+3T4+3T5+3T6(1, 1; 1, 2, 1) \quad 3T0+3T1+6T2+3T4+3T5+3T6(1, 1; 1, 1, 2) \quad 9T0+9T1+36T2+12T3+9T4+9T5+9T6(1, 1; 1, 1, 1)$	

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$SO(14)$ $SU(3) \times SU(3) \times SU(3)$	(none) (none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_3 = (4\ 0\ -2\ 2\ 2\ 0\ 0)$	
$3T1(1;3,3,1) \quad 3T0(1;3,1,3^*) \quad 3T0(1;3^*,3^*,1) \quad 3T1(1;3^*,1,3) \quad 3T0(1;1,3,3) \quad 3T1(1;1,3^*,3^*) \quad 3T4+3T6+3T8$ $(1;3,1,1) \quad 3T3+3T5+3T7(1;3^*,1,1) \quad 3T3+3T5+3T7(1;1,3,1) \quad 3T4+3T6+3T8(1;1,3^*,1) \quad 3T4+3T6+3T8(1;1,1,3)$ $3T3+3T5+3T7(1;1,1,3^*) \quad 27T0+27T1(1;1,1,1)$	

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$SO(14)$ $SU(3) \times SU(3) \times SU(3)$	(none) $3(1,3^*,3^*)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$	
$3T0(1;3,3,1) \quad 3T1(1;3,1,3) \quad 3T2(1;3^*,3,1) \quad 3T0(1;3^*,1,3) \quad 3T0(1;1,3^*,3^*) \quad 3T3+3T4(14v;1,1,1) \quad 9T2+3T7+3T8$ $(1;3,1,1) \quad 9T1+3T5+3T6(1;3^*,1,1) \quad 9T1+3T5+3T6(1;1,3,1) \quad 9T2+3T7+3T8(1;1,3^*,1) \quad 9T2+3T7+3T8(1;1,1,3)$ $9T1+3T5+3T6(1;1,1,3^*) \quad 27T0+12T3+12T4(1;1,1,1)$	

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$SO(14)$ $SU(3) \times SU(3) \times SU(3)$	(none) $3(3,3^*,3^*)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0) \ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$	
$3T2(1;3,3^*,1) \quad 3T1(1;3,1,3^*) \quad 3T0(1;1,3^*,3^*) \quad 9T0+9T1+9T2+3T3+3T4+3T5+3T6+3T7+3T8(1;3^*,1,1)$ $9T0+9T1+9T2+3T3+3T4+3T5+3T6+3T7+3T8(1;1,3,1) \quad 9T0+9T1+9T2+3T3+3T4+3T5+3T6+3T7+3T8(1;1,1,3)$	

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$SO(14)$ $SU(3) \times SU(3) \times SU(3)$	$3(14v) + 3(64s)$ $3(1,1,1) + 3(1,1,1) + 3(1,1,1)$
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) \ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$	
$3T2+3T7+3T8(14v;1,1,1) \quad 3T1+3T5+3T6(1;3,1,1) \quad 3T0+3T3+3T4(1;3^*,1,1) \quad 3T0+3T3+3T4(1;1,3,1) \quad 3T1+3T5+3T6$ $(1;1,3^*,1) \quad 3T0+3T3+3T4(1;1,1,3) \quad 3T1+3T5+3T6(1;1,1,3^*) \quad 12T2+12T7+12T8(1;1,1,1)$	

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$SO(14)$ $SU(3) \times SU(3) \times SU(3)$	(none) $3(3^*,3^*,1) + 3(3,1,3) + 3(1,3,3^*)$
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) \ a_1 = (4\ 2\ 2\ 0\ -2\ 2\ 0\ 0) \ a_3 = (4\ 2\ 2\ 0\ -2\ 2\ 0\ 0)$	
$3T2+3T4+3T5(1;3,1,3) \quad 3T2+3T4+3T5(1;3^*,3^*,1) \quad 3T2+3T4+3T5(1;1,3,3^*) \quad 3T1+3T3+3T8(1;3,1,1) \quad 3T0+3T6+3T7$ $(1;3^*,1,1) \quad 3T0+3T6+3T7(1;1,3,1) \quad 3T1+3T3+3T8(1;1,3^*,1) \quad 3T0+3T6+3T7(1;1,1,3) \quad 3T1+3T3+3T8(1;1,1,3^*)$ $27T2+27T4+27T5(1;1,1,1)$	

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$SU(9)$ $SU(6)$	(none) (none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$3T4+3T6+3T8(9;1) \quad 3T3+3T5+3T7(9^*;1) \quad 6T0(1;6) \quad 3T1(1;15) \quad 3T0(1;15^*) \quad 6T1(1;6^*) \quad 27T0+27T1(1;1)$	

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$SU(9)$ $SU(6)$	(none) $3(15^*) + 3(6) + 3(6)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) \ a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_1 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0) \ a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$3T2+3T4+3T5(9;1) \quad 3T1+3T3+3T8(9^*;1) \quad 6T0+6T6+6T7(1;6) \quad 3T0+3T6+3T7(1;15^*) \quad 27T0+27T6+27T7(1;1)$	

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E_7 $SU(4)$	$3(1) + 3(56)$ $3(6)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0) \ a_3 = (0\ 4\ 0\ -2\ 2\ 0\ 0\ 0)$	
$3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8(1;4) \quad 3T0(1;6) \quad 3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8(1;4^*)$ $36T0+30T1+30T2+30T3+30T4+30T5+30T6+30T7+30T8(1;1)$	

63

E_7 $SU(3) \times SU(3)$	(none) $3(3,1) + 3(3,1) + 3(3,1)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) \ a_3 = (0\ 0\ 0\ 0\ 4\ 2\ 2\ 0)$	
$3T2(1;3^*,3) \quad 3T1(1;3^*,3^*) \quad 9T1+9T2+3T3+3T4+3T5+3T6+3T7+3T8(1;3,1) \quad 9T0+3T3+3T4+3T5+3T6+3T7+3T8$ $(1;3^*,1) \quad 9T0+9T1+3T3+3T4+3T5+3T6+3T7+3T8(1;1,3) \quad 9T0+9T2+3T3+3T4+3T5+3T6+3T7+3T8(1;1,3^*)$ $27T0+27T1+27T2+18T3+18T4+18T5+18T6+18T7+18T8(1;1,1)$	

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E_7 $SU(3) \times SU(3)$	(none) $3(3,3^*)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \ a_3 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) \ a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) \ a_3 = (1\ -1\ -1\ 3\ 3\ 1\ 1\ 1)$	
$3T1(1;3,3^*) \quad 9T2+3T3+3T4+3T5+3T6+3T7+3T8(1;3,1) \quad 9T0+9T1+9T2+3T3+3T4+3T5+3T6+3T7+3T8(1;3^*,1)$ $9T0+9T1+9T2+3T3+3T4+3T5+3T6+3T7+3T8(1;1,3) \quad 9T0+3T3+3T4+3T5+3T6+3T7+3T8(1;1,3^*)$ $27T0+27T1+27T2+18T3+18T4+18T5+18T6+18T7+18T8(1;1,1)$	

65

E_7 $SU(3) \times SU(3)$	$3(1) + 3(56)$ $3(1, 1) + 3(1, 1) + 3(1, 1) + 3(1, 1) + 3(1, 1) + 3(1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 4\ 2\ 2\ 0)$
	$3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8 \ (1\ ;3, 1) \quad 3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8 \ (1\ ;3^*, 1)$ $3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8 \ (1\ ;1, 3) \quad 3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8 \ (1\ ;1, 3^*)$ $18T0+18T1+18T2+18T3+18T4+18T5+18T6+18T7+18T8 \ (1\ ;1, 1)$

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$SU(9)$ $SU(3) \times SU(3)$	(none) $3(3, 1) + 3(3, 1) + 3(3, 1)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 4\ 2\ 2\ 0)$
	$3T2 \ (1\ ;3^*, 3) \quad 3T1 \ (1\ ;3^*, 3^*) \quad 3T4+3T6+3T8 \ (9\ ;1, 1) \quad 3T3+3T5+3T7 \ (9^*\ ;1, 1) \quad 9T1+9T2 \ (1\ ;3, 1) \quad 9T0 \ (1\ ;3^*, 1) \quad 9T0+9T1 \ (1\ ;1, 3) \quad 9T0+9T2 \ (1\ ;1, 3^*) \quad 27T0+27T1+27T2 \ (1\ ;1, 1)$

67

$SU(9)$ $SU(3) \times SU(3)$	(none) $3(3, 3^*)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (1\ -1\ -1\ 3\ 3\ 1\ 1\ 1)$
	$3T1 \ (1\ ;3, 3^*) \quad 3T4+3T6+3T8 \ (9\ ;1, 1) \quad 3T3+3T5+3T7 \ (9^*\ ;1, 1) \quad 9T2 \ (1\ ;3, 1) \quad 9T0+9T1+9T2 \ (1\ ;3^*, 1) \quad 9T0+9T1+9T2 \ (1\ ;1, 3) \quad 9T0 \ (1\ ;1, 3^*) \quad 27T0+27T1+27T2 \ (1\ ;1, 1)$

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$E_6 \times SU(3)$ $SU(3)$	$3(27^*, 3)$ $3(1) + 3(1) + 3(1) + 3(1) + 3(1) + 3(1) + 3(1) + 3(1) + 3(1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 4\ 2\ 2\ 0)$
	$9T0+9T1+9T2+9T3+9T4+9T5+9T6+9T7+9T8 \ (1, 3^*\ ;1)$

69

$E_6 \times SU(3)$ $SU(3)$	$3(27^*, 3)$ $3(3) + 3(3) + 3(3)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 2\ 2\ 0)$
	$3T6 \ (1, 3^*\ ;3^*) \quad 9T0+9T1+9T2+9T3+9T4+9T5+9T7+9T8 \ (1, 3^*\ ;1)$

70

$SO(14)$ $SU(2) \times SU(2) \times SU(2)$	$3(14v) + 3(64s)$ $3(1, 1, 1) + 3(2, 2, 2)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 0\ 0\ 2\ 0\ 0)$
	$3T2 \ (14v\ ;1, 1, 1) \quad 3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8 \ (1\ ;1, 1, 2) \quad 3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8 \ (1\ ;1, 2, 1)$ $3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8 \ (1\ ;2, 1, 1) \quad 9T0+9T1+12T2+9T3+9T4+9T5+9T6+9T7+9T8 \ (1\ ;1, 1, 1)$

71

$SO(14)$	$3(14v) + 3(64s)$ (none)
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ -2\ 2\ 0\ 0) a_3 = (0\ 4\ 2\ 0\ 0\ 0\ 2\ 0)$
	$27T0+27T1+27T2+27T3+27T4+27T5+27T6+27T7+27T8 \ (1)$

72

E_6 $SO(8) \times SU(3)$	(none) (none)
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0)$
	$3T4 \ (27\ ;1, 1) \quad 3T0+3T2+3T5 \ (1\ ;8c, 1) \quad 3T0+3T1+3T7 \ (1\ ;8s, 1) \quad 3T0+3T6+3T8 \ (1\ ;8v, 1) \quad 3T1+3T2+9T3+3T5+3T6+3T7+3T8 \ (1\ ;1, 3) \quad 9T0+3T1+3T2+3T5+3T6+3T7+3T8 \ (1\ ;1, 3^*) \quad 9T0+12T1+12T2+27T4+12T5+12T6+12T7+12T8 \ (1\ ;1, 1)$

73

E_6 $SO(8) \times SU(3)$	$3(27^*)$ (none)
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (-4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)$
	$3T3 \ (27^*\ ;1, 1) \quad 3T2+3T6 \ (1\ ;8c, 1) \quad 3T1+3T8 \ (1\ ;8s, 1) \quad 3T0+3T4 \ (1\ ;8v, 1) \quad 3T0+3T1+3T2+3T4+3T6+9T7+3T8 \ (1\ ;1, 3) \quad 3T0+3T1+3T2+3T4+9T5+3T6+3T8 \ (1\ ;1, 3^*) \quad 12T0+12T1+12T2+27T3+12T4+12T6+12T8 \ (1\ ;1, 1)$

74

E_6 $SO(8) \times SU(3)$	$3(27^*)$ $3(1, 3^*) + 3(8s, 3^*)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0)$
	$3T2+3T4+3T5 \ (1\ ;8c, 1) \quad 3T0+3T6+3T7 \ (1\ ;8v, 1) \quad 3T0+9T1+3T2+9T3+3T4+3T5+3T6+3T7+9T8 \ (1\ ;1, 3) \quad 3T0+3T2+3T4+3T5+3T6+3T7 \ (1\ ;1, 3^*) \quad 12T0+12T2+12T4+12T5+12T6+12T7 \ (1\ ;1, 1)$

75

E_6 $SO(8) \times SU(3)$	(none) (none)
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0)$
	$3T4 \ (27\ ;1, 1) \quad 3T0+3T2+3T5 \ (1\ ;8c, 1) \quad 3T0+3T1+3T7 \ (1\ ;8s, 1) \quad 3T0+3T6+3T8 \ (1\ ;8v, 1) \quad 3T1+3T2+9T3+3T5+3T6+3T7+3T8 \ (1\ ;1, 3) \quad 9T0+3T1+3T2+3T5+3T6+3T7+3T8 \ (1\ ;1, 3^*) \quad 9T0+12T1+12T2+27T4+12T5+12T6+12T7+12T8 \ (1\ ;1, 1)$

76

E_6 $SO(8) \times SU(3)$	$3(27^*)$ (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (-4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T3(27^*, 1, 1) \quad 3T2+3T6(1, 8c, 1) \quad 3T1+3T8(1, 8s, 1) \quad 3T0+3T4(1, 8v, 1) \quad 3T0+3T1+3T2+3T4+3T6+9T7+3T8(1, 1, 3)$ $3T0+3T1+3T2+3T4+9T5+3T6+3T8(1, 1, 3^*) \quad 12T0+12T1+12T2+27T3+12T4+12T6+12T8(1, 1, 1)$	

77

E_6 $SO(8) \times SU(3)$	$3(27^*)$ $3(1, 3^*) + 3(8s, 3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T2+3T4+3T5(1, 8c, 1) \quad 3T0+3T6+3T7(1, 8v, 1) \quad 3T0+9T1+3T2+9T3+3T4+3T5+3T6+3T7+9T8(1, 1, 3)$ $3T0+3T2+3T4+3T5+3T6+3T7(1, 1, 3^*) \quad 12T0+12T2+12T4+12T5+12T6+12T7(1, 1, 1)$	

78

E_6 $SU(6)$	(none) $3(15^*) + 3(6) + 3(6)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$6T0+3T1+3T2+3T5+3T6+3T7+3T8(1, 1, 6) \quad 3T0(1, 15^*) \quad 3T1+3T2+3T5+3T6+3T7+3T8(1, 1, 6^*)$ $27T0+18T1+18T2+27T3+27T4+18T5+18T6+18T7+18T8(1, 1)$	

79

E_6 $SU(6)$	(none) (none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$3T4(27, 1) \quad 6T0+3T1+3T2+3T5+3T6+3T7+3T8(1, 1, 6) \quad 3T0(1, 15^*) \quad 3T1+3T2+3T5+3T6+3T7+3T8(1, 1, 6^*)$ $27T0+18T1+18T2+27T3+27T4+18T5+18T6+18T7+18T8(1, 1)$	

80

E_6 $SU(6)$	$3(27)$ (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ -2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T4(27, 1) \quad 3T0+3T1+3T2+3T3+3T5+3T7(1, 1, 6) \quad 3T0+3T1+3T2+3T3+3T5+3T7(1, 1, 6^*)$ $18T0+18T1+18T2+18T3+27T4+18T5+27T6+18T7+27T8(1, 1)$	

81

E_6 $SU(6)$	$3(1) + 3(1) + 3(1)$ $3(1) + 3(1) + 3(1) + 3(20) + 3(1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8(1, 1, 6) \quad 3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8(1, 1, 6^*)$ $18T0+18T1+18T2+18T3+18T4+18T5+18T6+18T7+18T8(1, 1)$	

82

E_6 $SU(6)$	(none) $3(6^*) + 3(15) + 3(6^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (1\ 3\ 1\ 1\ 1\ 1\ 1\ -1)$	
$3T0+3T1+3T3+3T4+3T5+3T6(1, 1, 6) \quad 3T2(1, 15) \quad 3T0+3T1+6T2+3T3+3T4+3T5+3T6(1, 1, 6^*)$ $18T0+18T1+27T2+18T3+18T4+18T5+18T6+27T7+27T8(1, 1)$	

83

$SU(6) \times SU(3)$ $SO(8) \times SU(3)$	(none) (none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0)$	
$3T3(1, 3^*, 1, 3) \quad 6T4+3T6+3T8(6, 1, 1, 1) \quad 3T4(15^*, 1, 1, 1) \quad 3T5+3T7(6^*, 1, 1, 1) \quad 9T4+3T6+3T8(1, 3, 1, 1) \quad 3T5+3T7(1, 3^*, 1, 1) \quad 3T0+3T2(1, 1, 8c, 1) \quad 3T0+3T1(1, 1, 8s, 1) \quad 3T0(1, 1, 8v, 1) \quad 3T1+3T2(1, 1, 1, 3) \quad 9T0+3T1+3T2(1, 1, 1, 3^*)$ $9T0+12T1+12T2(1, 1, 1, 1)$	

84

$SU(6) \times SU(3)$ $SO(8) \times SU(3)$	$3(6^*, 3^*)$ (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T8(1, 3, 1, 3) \quad 3T6(1, 3, 1, 3^*) \quad 3T4(15, 1, 1, 1) \quad 3T3+3T5+3T7(6, 1, 1, 1) \quad 6T4(6^*, 1, 1, 1) \quad 9T4(1, 3, 1, 1) \quad 3T3+3T5+3T7(1, 3^*, 1, 1) \quad 3T2(1, 1, 8c, 1) \quad 3T1(1, 1, 8s, 1) \quad 3T0(1, 1, 8v, 1) \quad 3T0+3T1+3T2(1, 1, 1, 3) \quad 3T0+3T1+3T2(1, 1, 1, 3^*)$ $12T0+12T1+12T2(1, 1, 1, 1)$	

85

$SU(6) \times SU(3)$ $SO(8) \times SU(3)$	$3(1, 1) + 3(20, 1)$ $3(8v, 1) + 3(8c, 1) + 3(8s, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T4+3T6+3T8(6, 1, 1, 1) \quad 3T3+3T5+3T7(6^*, 1, 1, 1) \quad 3T4+3T6+3T8(1, 3, 1, 1) \quad 3T3+3T5+3T7(1, 3^*, 1, 1) \quad 3T2(1, 1, 8c, 1) \quad 3T1(1, 1, 8s, 1) \quad 3T0(1, 1, 8v, 1) \quad 3T0+3T1+3T2(1, 1, 1, 3) \quad 3T0+3T1+3T2(1, 1, 1, 3^*) \quad 12T0+12T1+12T2(1, 1, 1, 1)$	

86	
$SU(6) \times SU(3)$	$3(6^*, 3^*)$
$SO(8) \times SU(3)$	$3(1, 3) + 3(8c, 3)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 2\ 2\ 2\ 2\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (-2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (-2\ 2\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T2+3T4+3T5(1, 3; 1, 3^*) \quad 3T1+3T3+3T8(6, 1; 1, 1) \quad 3T1+3T3+3T8(1, 3^*; 1, 1) \quad 3T0+3T6+3T7(1, 1; 8v, 1) \quad 3T0+3T6+3T7(1, 1; 1, 3) \quad 3T0+3T6+3T7(1, 1; 1, 3^*) \quad 12T0+12T6+12T7(1, 1; 1, 1)$	
87	
$SU(6) \times SU(3)$	$3(15^*, 3)$
$SO(8) \times SU(3)$	(none)
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0)$	
$3T1(1, 3^*; 1, 3) \quad 3T0(1, 3^*; 1, 3^*) \quad 6T2+3T3+3T4+3T5+3T6+3T7+3T8(6, 1; 1, 1) \quad 3T2(15^*, 1; 1, 1) \quad 9T2+3T3+3T4+3T5+3T6+3T7+3T8(1, 3^*; 1, 1)$	
88	
$SU(6) \times SU(3)$	(none)
$SU(6)$	$3(15^*) + 3(6) + 3(6)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$3T6+3T8(6, 1; 1) \quad 3T5+3T7(6^*, 1; 1) \quad 9T4+3T6+3T8(1, 3; 1) \quad 9T3+3T5+3T7(1, 3^*; 1) \quad 6T0+3T1+3T2(1, 1; 6) \quad 3T0(1, 1; 15^*) \quad 3T1+3T2(1, 1; 6^*) \quad 27T0+18T1+18T2(1, 1; 1)$	
89	
$SU(6) \times SU(3)$	(none)
$SU(6)$	(none)
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$6T4+3T6+3T8(6, 1; 1) \quad 3T4(15^*, 1; 1) \quad 3T5+3T7(6^*, 1; 1) \quad 9T4+3T6+3T8(1, 3; 1) \quad 9T3+3T5+3T7(1, 3^*; 1) \quad 6T0+3T1+3T2(1, 1; 6) \quad 3T0(1, 1; 15^*) \quad 3T1+3T2(1, 1; 6^*) \quad 27T0+18T1+18T2(1, 1; 1)$	
90	
$SO(12)$	$3(12v)$
$SU(6)$	$3(1) + 3(20)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (3\ 3\ -1\ -1\ -1\ -1\ -1\ -1)$	
$3T5(12v; 1) \quad 3T0+3T1+3T2+3T3+3T4+3T6+3T8(1; 6) \quad 3T0+3T1+3T2+3T4+3T6+3T7+3T8(1; 6^*) \quad 18T0+18T1+18T2+9T3+18T4+18T5+18T6+9T7+18T8(1; 1)$	
91	
$SU(6) \times SU(3)$	(none)
$SU(6)$	$3(6^*) + 3(15) + 3(6^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (1\ 3\ 1\ 1\ 1\ 1\ -1)$	
$3T4+3T6(6, 1; 1) \quad 3T3+3T5(6^*, 1; 1) \quad 3T4+3T6+9T8(1, 3; 1) \quad 3T3+3T5+9T7(1, 3^*; 1) \quad 3T0+3T1(1, 1; 6) \quad 3T2(1, 1; 15) \quad 3T0+3T1+6T2(1, 1; 6^*) \quad 18T0+18T1+27T2(1, 1; 1)$	
92	
$SU(6) \times SU(3)$	$3(6^*, 3^*)$
$SU(6)$	$3(6^*) + 3(15) + 3(6^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 2\ 2\ 2\ 2\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 3\ 1\ 1\ 1\ 1\ -1) a_3 = (1\ 3\ 1\ 1\ 1\ 1\ -1)$	
$3T1+3T3+3T8(6, 1; 1) \quad 9T2+9T4+9T5(1, 3; 1) \quad 3T1+3T3+3T8(1, 3^*; 1) \quad 3T0+3T6+3T7(1, 1; 6) \quad 3T0+3T6+3T7(1, 1; 6^*) \quad 18T0+18T6+18T7(1, 1; 1)$	
93	
$SU(6) \times SU(3)$	(none)
$SU(6)$	$3(6^*) + 3(15) + 3(6^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 2\ 2\ 2\ 2\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ -4\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 3\ 1\ 1\ 1\ 1\ -1) a_3 = (-1\ -3\ -1\ -1\ -1\ -1\ -1\ 1)$	
$3T1+3T8(6, 1; 1) \quad 3T4+3T5(6^*, 1; 1) \quad 9T2+3T4+3T5(1, 3; 1) \quad 3T1+9T3+3T8(1, 3^*; 1) \quad 3T0+3T7(1, 1; 6) \quad 3T6(1, 1; 15) \quad 3T0+6T6+3T7(1, 1; 6^*) \quad 18T0+27T6+18T7(1, 1; 1)$	
94	
$SU(6) \times SU(3)$	$3(15^*, 3)$
$SU(6)$	$3(15^*) + 3(6) + 3(6)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$3T1+3T2+3T3+3T4+3T5+3T8(6, 1; 1) \quad 9T0+3T1+3T2+3T3+3T4+3T5+9T6+9T7+3T8(1, 3^*; 1)$	
95	
$SU(6) \times SU(3)$	$3(15^*, 3)$
$SU(6)$	(none)
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$6T2+3T3+3T4+3T5+3T6+3T7+3T8(6, 1; 1) \quad 3T2(15^*, 1; 1) \quad 9T0+9T1+9T2+3T3+3T4+3T5+3T6+3T7+3T8(1, 3^*; 1)$	
96	
$SU(6) \times SU(3)$	$3(6, 3)$
$SU(6)$	(none)
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$6T2(6, 1; 1) \quad 3T2(15^*, 1; 1) \quad 3T3+3T5+3T7(6^*, 1; 1) \quad 3T3+3T5+3T7(1, 3; 1) \quad 9T0+9T1+9T2(1, 3^*; 1) \quad 3T4+3T6+3T8(1, 1; 6) \quad 3T4+3T6+3T8(1, 1; 6^*) \quad 18T4+18T6+18T8(1, 1; 1)$	

97

$SO(12)$ $SU(6)$	(none) $3(6)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$	
$3T_3+3T_4+3T_8\ (12v, 1)$ $6T_0+3T_1+3T_2+3T_6+3T_7\ (1, 6)$ $3T_0\ (1, 15^*)$ $3T_1+3T_2+3T_5+3T_6+3T_7\ (1, 6^*)$ $27T_0+18T_1+18T_2+18T_3+18T_4+9T_5+18T_6+18T_7+18T_8\ (1, 1)$	

98

$SO(12)$ $SU(6)$	(none) $3(15^*)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (-2\ -2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$3T_3+3T_4\ (12v, 1)$ $6T_0+3T_1+3T_2+3T_5+3T_6+3T_7+3T_8\ (1, 6)$ $3T_0\ (1, 15^*)$ $3T_1+3T_2+3T_6+3T_7\ (1, 6^*)$ $27T_0+18T_1+18T_2+18T_3+18T_4+9T_5+18T_6+18T_7+9T_8\ (1, 1)$	

99

$SO(12)$ $SU(6)$	$3(1) + 3(32s)$ (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 4\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T_3+3T_4+3T_5+3T_7\ (12v, 1)$ $3T_0+3T_1+3T_2+3T_6\ (1, 6)$ $3T_0+3T_1+3T_2+3T_8\ (1, 6^*)$ $18T_0+18T_1+18T_2+18T_3+18T_4+18T_5+9T_6+18T_7+9T_8\ (1, 1)$	

100

$SO(12)$ $SU(6)$	$3(12v)$ $3(1) + 3(20)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ -4\ 0\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T_4\ (12v, 1)$ $3T_0+3T_1+3T_2+3T_3+3T_5+3T_6+3T_7\ (1, 6)$ $3T_0+3T_1+3T_2+3T_3+3T_5+3T_7+3T_8\ (1, 6^*)$ $18T_0+18T_1+18T_2+18T_3+18T_4+18T_5+9T_6+18T_7+9T_8\ (1, 1)$	

101

$SO(10) \times SU(2)$ $SO(8)$	$3(1, 2) + 3(16s^*, 1)$ (none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T_4+3T_6+3T_8\ (10v, 1, 1)$ $3T_2\ (1, 1, 8c)$ $3T_1\ (1, 1, 8s)$ $3T_0\ (1, 1, 8v)$ $9T_3+6T_4+9T_5+6T_6+9T_7+6T_8\ (1, 2, 1)$ $30T_0+30T_1+30T_2+9T_3+12T_4+9T_5+12T_6+9T_7+12T_8\ (1, 1, 1)$	

102

$SO(10) \times SU(2)$ $SO(8)$	$3(10v, 1) + 3(16s^*, 2)$ $3(1) + 3(1) + 3(1)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T_4+3T_6+3T_8\ (10v, 1, 1)$ $9T_0+9T_1+9T_2+9T_3+6T_4+9T_5+6T_6+9T_7+6T_8\ (1, 2, 1)$ $9T_0+9T_1+9T_2+9T_3+12T_4+9T_5+12T_6+9T_7+12T_8\ (1, 1, 1)$	

103

$SO(10) \times SU(2)$ $SO(8)$	$3(10v, 1) + 3(16s^*, 2)$ $3(1) + 3(1) + 3(1)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ -4\ 0\ 0\ 0\ 0\ 0\ 0)$	
$3T_4+3T_6+3T_8\ (10v, 1, 1)$ $9T_0+9T_1+9T_2+9T_3+6T_4+9T_5+6T_6+9T_7+6T_8\ (1, 2, 1)$ $9T_0+9T_1+9T_2+9T_3+12T_4+9T_5+12T_6+9T_7+12T_8\ (1, 1, 1)$	

104

$SU(7)$ $SU(6)$	(none) $3(15^*)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$	
$3T_3+3T_4+3T_5\ (7, 1)$ $3T_3+3T_4+3T_8\ (7^*, 1)$ $6T_0+3T_1+3T_2+3T_6+3T_7\ (1, 6)$ $3T_0\ (1, 15^*)$ $3T_1+3T_2\ (1, 6^*)$ $27T_0+18T_1+18T_2+12T_3+12T_4+6T_5+9T_6+9T_7+6T_8\ (1, 1)$	

105

$SU(7)$ $SU(6)$	(none) $3(6)$
$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ -2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$3T_3+3T_4+3T_5+3T_7\ (7, 1)$ $3T_3+3T_4+3T_7+3T_8\ (7^*, 1)$ $6T_0+3T_1+3T_2\ (1, 6)$ $3T_0\ (1, 15^*)$ $3T_1+3T_2+3T_6\ (1, 6^*)$ $27T_0+18T_1+18T_2+12T_3+12T_4+6T_5+9T_6+12T_7+6T_8\ (1, 1)$	

106

$SU(7)$ $SU(6)$	$3(21^*)$ $3(1) + 3(20)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T_3+3T_4+3T_5+3T_7\ (7, 1)$ $3T_4\ (7^*, 1)$ $3T_0+3T_1+3T_2+3T_6\ (1, 6)$ $3T_0+3T_1+3T_2+3T_8\ (1, 6^*)$ $18T_0+18T_1+18T_2+6T_3+12T_4+6T_5+9T_6+6T_7+9T_8\ (1, 1)$	

107

$SU(7)$	$3(21)$
$SU(6)$	$3(1) + 3(20)$
$v = (22000000) a_1 = (00000000) a_3 = (0-2222000)$	
$v = (40000000) a_1 = (11111111) a_3 = (33111111)$	
$3T7 (7;1)$	$3T4+3T6+3T7+3T8 (7^*;1)$
$18T0+18T1+18T2+9T3+6T4+9T5+6T6+12T7+6T8 (1;1)$	$3T0+3T1+3T2+3T3 (1;6)$
	$3T0+3T1+3T2+3T5 (1;6^*)$

108

$SU(7)$	$3(21^*)$
$SU(6)$	$3(1) + 3(20)$
$v = (22000000) a_1 = (00000000) a_3 = (20222000)$	
$v = (40000000) a_1 = (-11111111-1) a_3 = (3-1-1-1-1-1-13)$	
$3T3+3T5+3T7+3T8 (7;1)$	$3T8 (7^*;1)$
$18T0+18T1+18T2+6T3+9T4+6T5+9T6+6T7+12T8 (1;1)$	$3T0+3T1+3T2+3T6 (1;6)$
	$3T0+3T1+3T2+3T4 (1;6^*)$

109

$SU(7)$	$3(7^*) + 3(35^*)$
$SU(6)$	(none)
$v = (40000000) a_1 = (00000000) a_3 = (11111111)$	
$v = (42222000) a_1 = (22220000) a_3 = (42002000)$	
$3T2+3T3+3T4+3T5+3T6+3T7+3T8 (7;1)$	$3T2+3T4+3T6+3T8 (7^*;1)$
$9T0+9T1+12T2+6T3+12T4+6T5+12T6+6T7+12T8 (1;1)$	$3T0 (1;6)$
	$3T1 (1;6^*)$

110

$SO(10) \times SU(2)$	$3(1,2) + 3(16s^*,1)$
$SU(4)$	$3(6)$
$v = (22000000) a_1 = (00000000) a_3 = (40220000)$	
$v = (40000000) a_1 = (42200000) a_3 = (31-131111-1)$	
$3T4 (10v,1;1)$	$3T0+3T1+3T2+3T8 (1,1;4)$
$30T0+30T1+30T2+9T3+12T4+9T5+15T6+9T7+15T8 (1,1;1)$	$3T0+3T1+3T2+3T6 (1,1;4^*)$
	$9T3+6T4+9T5+9T7 (1,2;1)$

111

$SO(10) \times SU(2)$	$3(10v,1) + 3(16s^*,2)$
$SU(4)$	$3(1) + 3(4) + 3(4^*)$
$v = (42200000) a_1 = (00000000) a_3 = (20020000)$	
$v = (42200000) a_1 = (40000000) a_3 = (31-131111-1)$	
$3T4 (10v,1;1)$	$3T8 (1,1;4)$
$9T0+9T1+9T2+9T3+12T4+9T5+15T6+9T7+15T8 (1,1;1)$	$3T6 (1,1;4^*)$
	$9T0+9T1+9T2+9T3+6T4+9T5+9T7 (1,2;1)$

112

$SO(10) \times SU(2)$	$3(1,2) + 3(16s^*,1)$
$SU(2) \times SU(2) \times SU(2)$	$3(1,1,1) + 3(2,2,2)$
$v = (22000000) a_1 = (00000000) a_3 = (40220000)$	
$v = (40000000) a_1 = (33111111) a_3 = (113311-1-1)$	
$3T5 (1,2;1,1,2)$	$3T7 (1,2;1,2,1)$
$6T0+6T1+6T2+3T4+3T6+3T7+3T8 (1,1;1,2,1)$	$3T3 (1,2;1,1)$
$6T0+6T1+6T2+3T4+3T6+3T7+3T8 (1,1;1,2,1)$	$6T0+6T1+6T2+3T3+3T4+3T6+3T8 (1,1;2,1,1)$
$18T0+18T1+18T2+3T3+9T4+3T5+9T6+3T7+9T8 (1,1;1,1,1)$	$3T3+3T5+3T7 (1,2;1,1)$

113

$SO(10) \times SU(2)$	$3(1,2) + 3(16s^*,1)$
$SU(2) \times SU(2) \times SU(2)$	$3(1,1,1) + 3(1,1,1) + 3(2,1,1) + 3(1,1,2) + 3(1,2,1) + 3(1,1,1)$
$v = (22000000) a_1 = (00000000) a_3 = (40220000)$	
$v = (40000000) a_1 = (33111111) a_3 = (1-1311-1-1-3)$	
$6T0+6T1+6T2+3T4+3T6+3T8 (1,1;1,1,2)$	$6T0+6T1+6T2+3T4+3T6+3T8 (1,1;1,2,1)$
$6T0+6T1+6T2+3T4+3T6+3T8 (1,1;1,2,1)$	$6T0+6T1+6T2+3T4+3T6+3T8 (1,1;2,1,1)$
$9T3+9T5+9T7 (1,2;1,1,1)$	$18T0+18T1+18T2+9T3+9T4+9T5+9T6+9T7+9T8 (1,1;1,1,1)$

114

$SO(12)$	(none)
$SU(3) \times SU(3)$	$3(3^*,3^*) + 3(3,1) + 3(1,3)$
$v = (00000000) a_1 = (22000000) a_3 = (40000000)$	
$v = (42200000) a_1 = (00022000) a_3 = (22020200)$	
$3T0 (1;3^*,3^*)$	$9T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7 (1;3,1)$
$9T0+3T1+3T2+3T3+3T4+3T6+3T7+3T8 (1;1,3)$	$3T1+3T2+3T6+3T7+3T8 (1;3^*,1)$
$27T0+18T1+18T2+9T3+9T4+9T5+18T6+18T7+9T8 (1;1,1)$	$3T1+3T2+3T5+3T6+3T7 (1;1,3^*)$

115

$SO(12)$	(none)
$SU(3) \times SU(3)$	$3(3,1) + 3(1,3)$
$v = (00000000) a_1 = (22000000) a_3 = (40000000)$	
$v = (42200000) a_1 = (42222000) a_3 = (22020200)$	
$3T0 (1;3^*,3^*)$	$3T4 (12v;1,1)$
$9T0+3T1+3T2+3T3+3T4+3T6+3T7+3T8 (1;1,3)$	$9T0+3T1+3T2+3T5+3T6+3T7 (1;3,1)$
$27T0+18T1+18T2+9T3+18T4+9T5+18T6+18T7+9T8 (1;1,1)$	$3T1+3T2+3T3+3T6+3T7+3T8 (1;1,3^*)$

116

$SO(12)$	$3(1) + 3(32s)$
$SU(3) \times SU(3)$	$3(3^*,3^*)$
$v = (22000000) a_1 = (00000000) a_3 = (2-20000000)$	
$v = (40000000) a_1 = (31111111-1) a_3 = (331111-1-11)$	
$3T4 (12v;1,1)$	$3T0+3T1+3T2+3T3+3T5+3T6+3T7 (1;3,1)$
$3T0+3T1+3T2+3T6 (1;1,3^*)$	$18T0+18T1+18T2+9T3+18T4+9T5+9T6+9T7+9T8 (1;1,1)$

117

$SU(3) \times SU(3)$	$SO(12)$	$3(12\nu)$
		$3(1, 1) + 3(1, 1) + 3(3^*, 3^*) + 3(1, 1)$
		$\nu = (22000000) a_1 = (00000000) a_3 = (40000000)$
		$\nu = (40000000) a_1 = (11111111) a_3 = (3111 - 1 - 1 - 1 - 3)$
		$3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8 \begin{pmatrix} 1 & ;3, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T4+3T6+3T8 \begin{pmatrix} 1 & ;3^*, & 1 \end{pmatrix}$
		$3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8 \begin{pmatrix} 1 & ;1, & 3 \end{pmatrix} \quad 3T0+3T1+3T2+3T4+3T6+3T8 \begin{pmatrix} 1 & ;1, & 3^* \end{pmatrix}$
		$18T0+18T1+18T2+9T3+18T4+9T5+18T6+9T7+18T8 \begin{pmatrix} 1 & ;1, & 1 \end{pmatrix}$

118

$SU(3) \times SU(3)$	$SU(7)$	(none)
		$3(3^*, 3^*) + 3(3, 1) + 3(1, 3)$
		$\nu = (00000000) a_1 = (22000000) a_3 = (20222000)$
		$\nu = (42200000) a_1 = (00022000) a_3 = (2200 - 2200)$
		$3T0 \begin{pmatrix} 1 & ;3^*, & 3^* \end{pmatrix} \quad 3T5 \begin{pmatrix} 7 & ;1, & 1 \end{pmatrix} \quad 3T8 \begin{pmatrix} 7^* & ;1, & 1 \end{pmatrix} \quad 9T0+3T1+3T2+3T3+3T4+3T6 \begin{pmatrix} 1 & ;3, & 1 \end{pmatrix} \quad 3T1+3T2+3T7 \begin{pmatrix} 1 & ;3^*, & 1 \end{pmatrix}$
		$9T0+3T1+3T2+3T3+3T4+3T7 \begin{pmatrix} 1 & ;1, & 3 \end{pmatrix} \quad 3T1+3T2+3T6 \begin{pmatrix} 1 & ;1, & 3^* \end{pmatrix} \quad 27T0+18T1+18T2+9T3+9T4+6T5+9T6+9T7+6T8 \begin{pmatrix} 1 & ;1, & 1 \end{pmatrix}$

119

$SU(3) \times SU(3)$	$SU(7)$	(none)
		$3(1, 3) + 3(3, 1)$
		$\nu = (00000000) a_1 = (22000000) a_3 = (20222000)$
		$\nu = (42200000) a_1 = (42222000) a_3 = (2200 - 2200)$
		$3T0 \begin{pmatrix} 1 & ;3^*, & 3^* \end{pmatrix} \quad 3T4+3T5 \begin{pmatrix} 7 & ;1, & 1 \end{pmatrix} \quad 3T4+3T8 \begin{pmatrix} 7^* & ;1, & 1 \end{pmatrix} \quad 9T0+3T1+3T2+3T6 \begin{pmatrix} 1 & ;3, & 1 \end{pmatrix} \quad 3T1+3T2+3T3+3T7 \begin{pmatrix} 1 & ;3^*, & 1 \end{pmatrix}$
		$9T0+3T1+3T2+3T7 \begin{pmatrix} 1 & ;1, & 3 \end{pmatrix} \quad 3T1+3T2+3T3+3T6 \begin{pmatrix} 1 & ;1, & 3^* \end{pmatrix} \quad 27T0+18T1+18T2+9T3+12T4+6T5+9T6+9T7+6T8 \begin{pmatrix} 1 & ;1, & 1 \end{pmatrix}$

120

$SU(3) \times SU(3)$	$SU(7)$	$3(1) + 3(7^*) + 3(7)$
		$3(3^*, 3^*)$
		$\nu = (22000000) a_1 = (00000000) a_3 = (42222000)$
		$\nu = (40000000) a_1 = (31111111 - 1) a_3 = (331111 - 1 - 1 - 1)$
		$3T4 \begin{pmatrix} 7 & ;1, & 1 \end{pmatrix} \quad 3T4 \begin{pmatrix} 7^* & ;1, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T3+3T5+3T6+3T7 \begin{pmatrix} 1 & ;3, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T8 \begin{pmatrix} 1 & ;3^*, & 1 \end{pmatrix}$
		$3T0+3T1+3T2+3T3+3T5+3T7+3T8 \begin{pmatrix} 1 & ;1, & 3 \end{pmatrix} \quad 3T0+3T1+3T2+3T6 \begin{pmatrix} 1 & ;1, & 3^* \end{pmatrix} \quad 18T0+18T1+18T2+9T3+12T4+9T5+9T6+9T7+9T8 \begin{pmatrix} 1 & ;1, & 1 \end{pmatrix}$

121

$SU(3) \times SU(3)$	$SU(7)$	$3(21)$
		$3(1, 1) + 3(1, 1) + 3(1, 1)$
		$\nu = (22000000) a_1 = (00000000) a_3 = (0 - 2222000)$
		$\nu = (40000000) a_1 = (15111111) a_3 = (313111 - 1 - 1)$
		$3T3 \begin{pmatrix} 7 & ;1, & 1 \end{pmatrix} \quad 3T3+3T4+3T6+3T8 \begin{pmatrix} 7^* & ;1, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T5 \begin{pmatrix} 1 & ;3, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T7 \begin{pmatrix} 1 & ;3^*, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T5 \begin{pmatrix} 1 & ;1, & 3 \end{pmatrix}$
		$3T0+3T1+3T2+3T7 \begin{pmatrix} 1 & ;1, & 3^* \end{pmatrix} \quad 18T0+18T1+18T2+12T3+6T4+9T5+6T6+9T7+6T8 \begin{pmatrix} 1 & ;1, & 1 \end{pmatrix}$

122

$SU(3) \times SU(3)$	$SU(7)$	$3(21^*)$
		$3(1, 1) + 3(3, 3) + 3(1, 1) + 3(1, 1)$
		$\nu = (22000000) a_1 = (00000000) a_3 = (20222000)$
		$\nu = (40000000) a_1 = (-11111111 - 1) a_3 = (331111 - 1 - 1 - 1)$
		$3T3+3T5+3T7 \begin{pmatrix} 7 & ;1, & 1 \end{pmatrix} \quad 3T0+3T1+3T2 \begin{pmatrix} 1 & ;3, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T4+3T6+3T8 \begin{pmatrix} 1 & ;3^*, & 1 \end{pmatrix} \quad 3T0+3T1+3T2 \begin{pmatrix} 1 & ;1, & 3 \end{pmatrix}$
		$3T0+3T1+3T2+3T4+3T6+3T8 \begin{pmatrix} 1 & ;1, & 3^* \end{pmatrix} \quad 18T0+18T1+18T2+6T3+9T4+6T5+9T6+6T7+9T8 \begin{pmatrix} 1 & ;1, & 1 \end{pmatrix}$

123

$SU(3) \times SU(3)$	$SU(7)$	$3(7^*) + 3(35^*)$
		$3(3^*, 3^*)$
		$\nu = (40000000) a_1 = (00000000) a_3 = (11111111)$
		$\nu = (42222000) a_1 = (22220000) a_3 = (04200200)$
		$3T3+3T4+3T5+3T7 \begin{pmatrix} 7 & ;1, & 1 \end{pmatrix} \quad 3T4 \begin{pmatrix} 7^* & ;1, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T6 \begin{pmatrix} 1 & ;3, & 1 \end{pmatrix} \quad 3T8 \begin{pmatrix} 1 & ;3^*, & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T8 \begin{pmatrix} 1 & ;1, & 3 \end{pmatrix} \quad 3T6 \begin{pmatrix} 1 & ;1, & 3^* \end{pmatrix}$
		$9T0+9T1+9T2+6T3+12T4+6T5+9T6+6T7+9T8 \begin{pmatrix} 1 & ;1, & 1 \end{pmatrix}$

124

$SU(5) \times SU(2) \times SU(2)$	$SO(8)$	$3(5, 1, 2) + 3(5^*, 2, 1) + 3(1, 2, 2)$
		$3(1) + 3(1) + 3(1)$
		$\nu = (42200000) a_1 = (00000000) a_3 = (42022200)$
		$\nu = (42200000) a_1 = (40000000) a_3 = (04220000)$
		$3T4+3T6+3T8 \begin{pmatrix} 1 & , & 2 & , & 1 \end{pmatrix} \quad 3T4+3T6+3T8 \begin{pmatrix} 5 & , & 1 & , & 1 \end{pmatrix} \quad 3T4+3T6+3T8 \begin{pmatrix} 5^* & , & 1 & , & 1 \end{pmatrix} \quad 9T0+9T1+9T2 \begin{pmatrix} 1 & , & 1 & , & 2 & ; & 1 \end{pmatrix} \quad 9T3+9T5+9T7 \begin{pmatrix} 1 & , & 2 & ; & 1 & ; & 1 \end{pmatrix}$
		$9T0+9T1+9T2+9T3+12T4+9T5+12T6+9T7+12T8 \begin{pmatrix} 1 & , & 1 & , & 1 & ; & 1 \end{pmatrix}$

125

$SU(5) \times SU(2) \times SU(2)$	$SO(8)$	$3(5^*, 2, 2) + 3(10, 1, 1)$
		$3(8\nu) + 3(8s) + 3(8c)$
		$\nu = (42200000) a_1 = (00000000) a_3 = (20 - 222000)$
		$\nu = (42200000) a_1 = (20020000) a_3 = (40 - 220000)$
		$3T4+3T6+3T8 \begin{pmatrix} 5 & , & 1 & , & 1 & ; & 1 \end{pmatrix} \quad 9T0+9T1+9T2+3T4+3T6+3T8 \begin{pmatrix} 1 & , & 1 & , & 2 & ; & 1 \end{pmatrix} \quad 9T3+3T4+9T5+3T6+9T7+3T8 \begin{pmatrix} 1 & , & 2 & , & 1 & ; & 1 \end{pmatrix}$
		$9T0+9T1+9T2+9T3+9T5+9T7 \begin{pmatrix} 1 & , & 1 & , & 1 & ; & 1 \end{pmatrix}$

126

$SU(5) \times SU(2) \times SU(2)$	$SO(8)$	$3(1, 1, 2) + 3(5, 1, 1) + 3(10, 2, 1)$
		(none)
		$\nu = (42200000) a_1 = (00000000) a_3 = (40022000)$
		$\nu = (42200000) a_1 = (42020000) a_3 = (40 - 2 - 20000)$
		$3T4+3T6+3T8 \begin{pmatrix} 1 & , & 2 & , & 1 \end{pmatrix} \quad 3T4+3T6+3T8 \begin{pmatrix} 5 & , & 1 & , & 1 \end{pmatrix} \quad 3T3+3T4+3T5+3T6+3T7+3T8 \begin{pmatrix} 5^* & , & 1 & , & 1 \end{pmatrix} \quad 3T3+3T5+3T7 \begin{pmatrix} 1 & , & 1 & , & 2 & ; & 1 \end{pmatrix}$
		$9T0+9T1+9T2+3T3+3T5+3T7 \begin{pmatrix} 1 & , & 2 & , & 1 \end{pmatrix} \quad 9T0+9T1+9T2+12T4+12T6+12T8 \begin{pmatrix} 1 & , & 1 & , & 1 & ; & 1 \end{pmatrix}$

127

$SU(5) \times SU(2) \times SU(2)$ $SU(3) \times SU(2) \times SU(2) \times SU(2)$	(none) $3(3, 1, 1, 1) + 3(3, 1, 2, 2)$
$v = (00000000) a_1 = (42200000) a_3 = (42022200)$ $v = (42200000) a_1 = (00040000) a_3 = (00042200)$	
$3T0(1, 1, 1; 1, 1, 2, 2) \quad 3T0(1, 1, 1; 1, 2, 1, 2) \quad 3T0(1, 1, 1; 1, 2, 2, 1) \quad 3T2(1, 1, 2; 1, 1, 1, 2) \quad 3T1(1, 1, 2; 1, 1, 2, 1) \quad 3T5(1, 2, 1; 1, 1, 1, 2) \quad 3T8(1, 2, 1; 1, 1, 2, 1) \quad 3T3(5, 1, 1; 1, 1, 1, 1) \quad 3T4(5^*, 1, 1; 1, 1, 1, 1) \quad 9T0+3T6+3T7(1, 1, 1; 3^*, 1, 1, 1) \quad 6T0+3T2+3T5+3T6+3T7(1, 1, 1; 1, 1, 1, 2) \quad 6T0+3T1+3T6+3T7+3T8(1, 1, 1; 1, 1, 2, 1) \quad 6T0+3T6+3T7(1, 1, 1; 1, 1, 2, 1, 1) \quad 3T1+3T2+3T3+3T4(1, 1, 2; 1, 1, 1, 1) \quad 3T3+3T4+3T5+3T8(1, 2, 1; 1, 1, 1, 1) \quad 9T0+3T1+3T2+3T5+3T8(1, 1, 1; 1, 1, 1, 1)$	

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$SU(5) \times SU(2) \times SU(2)$ $SU(3) \times SU(2) \times SU(2) \times SU(2)$	(none) $3(3, 1, 2, 1)$
$v = (00000000) a_1 = (42200000) a_3 = (42022200)$ $v = (42200000) a_1 = (00040000) a_3 = (1-1-113311)$	
$3T0(1, 1, 1; 1, 1, 2, 2) \quad 3T0(1, 1, 1; 1, 2, 1, 2) \quad 3T0(1, 1, 1; 1, 2, 2, 1) \quad 3T2(1, 1, 2; 1, 1, 1, 2) \quad 3T1(1, 1, 2; 1, 1, 2, 1) \quad 3T5(1, 2, 1; 1, 1, 1, 2) \quad 3T8(1, 2, 1; 1, 1, 2, 1) \quad 3T7(1, 2, 2; 1, 1, 1, 1) \quad 3T3+3T7(5, 1, 1; 1, 1, 1, 1) \quad 3T4+3T7(5^*, 1, 1; 1, 1, 1, 1) \quad 3T6(1, 1, 1; 3, 1, 1, 1) \quad 9T0(1, 1, 1; 3^*, 1, 1, 1) \quad 6T0+3T5+3T6(1, 1, 1; 1, 1, 1, 2) \quad 6T0+3T2+3T6+3T8(1, 1, 1; 1, 1, 2, 1) \quad 6T0+3T1+3T6(1, 1, 1; 1, 1, 2, 1) \quad 3T1+3T2+3T3+3T4(1, 1, 2; 1, 1, 1, 1) \quad 3T3+3T4+3T5+3T8(1, 2, 1; 1, 1, 1, 1) \quad 9T0+3T1+3T2+3T5+12T7+3T8(1, 1, 1; 1, 1, 1, 1)$	

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$SU(5) \times SU(2) \times SU(2)$ $SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(5^*, 2, 1) + 3(5, 1, 2) + 3(1, 2, 2)$ $3(1, 1, 1, 1) + 3(1, 2, 2, 2)$
$v = (42200000) a_1 = (00000000) a_3 = (4-2-222000)$ $v = (42200000) a_1 = (40000000) a_3 = (00042200)$	
$3T5(1, 1, 2; 1, 1, 1, 2) \quad 3T7(1, 1, 2; 1, 1, 2, 1) \quad 3T3(1, 1, 2; 1, 2, 1, 1) \quad 3T2(1, 2, 1; 1, 1, 1, 2) \quad 3T1(1, 2, 1; 1, 1, 2, 1) \quad 3T0(1, 2, 1; 1, 2, 1, 1) \quad 3T4(1, 2, 2; 1, 1, 1, 1) \quad 3T4(5, 1, 1; 1, 1, 1, 1) \quad 3T4(5^*, 1, 1; 1, 1, 1, 1) \quad 3T8(1, 1, 1; 3, 1, 1, 1) \quad 3T6(1, 1, 1; 3^*, 1, 1, 1) \quad 3T2+3T5+3T6+3T8(1, 1, 1; 1, 1, 1, 2) \quad 3T1+3T6+3T7+3T8(1, 1, 1; 1, 1, 2, 1) \quad 3T0+3T3+3T6+3T8(1, 1, 1; 1, 1, 2, 1, 1) \quad 3T3+3T5+3T7(1, 1, 2; 1, 1, 1, 1) \quad 3T0+3T1+3T2(1, 2, 1; 1, 1, 1, 1) \quad 3T0+3T1+3T2+3T3+12T4+3T5+3T7(1, 1, 1; 1, 1, 1, 1)$	

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$SU(5) \times SU(2) \times SU(2)$ $SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(5^*, 2, 2) + 3(10, 1, 1)$ $3(1, 1, 2, 2) + 3(1, 2, 1, 2) + 3(1, 2, 2, 1)$
$v = (42200000) a_1 = (00000000) a_3 = (20-222000)$ $v = (42200000) a_1 = (20020000) a_3 = (00004220)$	
$3T1(1, 1, 2; 1, 1, 1, 2) \quad 3T2(1, 1, 2; 1, 1, 2, 1) \quad 3T0(1, 1, 2; 1, 2, 1, 1) \quad 3T5(1, 2, 1; 1, 1, 1, 2) \quad 3T7(1, 2, 1; 1, 1, 2, 1) \quad 3T3(1, 2, 1; 1, 2, 1, 1) \quad 3T4+3T6+3T8(5, 1, 1; 1, 1, 1, 1) \quad 3T1+3T5(1, 1, 1; 1, 1, 1, 2) \quad 3T2+3T7(1, 1, 1; 1, 1, 2, 1) \quad 3T0+3T3(1, 1, 1; 1, 2, 1, 1) \quad 3T0+3T1+3T2+3T4+3T6+3T8(1, 1, 2; 1, 1, 1, 1) \quad 3T3+3T4+3T5+3T6+3T7+3T8(1, 2, 1; 1, 1, 1, 1) \quad 3T0+3T1+3T2+3T3+3T5+3T7(1, 1, 1; 1, 1, 1, 1)$	

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$SU(5) \times SU(2) \times SU(2)$ $SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(5^*, 1, 1) + 3(1, 2, 1) + 3(10^*, 1, 2)$ $3(1, 1, 1, 2) + 3(1, 2, 1, 1) + 3(1, 1, 2, 1)$
$v = (40000000) a_1 = (22222200) a_3 = (11111133)$ $v = (42222000) a_1 = (42000200) a_3 = (00422000)$	
$3T2(1, 1, 2; 1, 1, 1, 2) \quad 3T3(1, 1, 2; 1, 1, 2, 1) \quad 3T6(1, 1, 2; 1, 2, 1, 1) \quad 3T8(1, 2, 2; 1, 1, 1, 1) \quad 3T1+3T4+3T7+3T8(5, 1, 1; 1, 1, 1, 1) \quad 3T8(5^*, 1, 1; 1, 1, 1, 1) \quad 3T0(1, 1, 1; 3, 1, 1, 1) \quad 3T5(1, 1, 1; 3^*, 1, 1, 1) \quad 3T0+3T2+3T5(1, 1, 1; 1, 1, 1, 2) \quad 3T0+3T3+3T5(1, 1, 1; 1, 1, 2, 1) \quad 3T0+3T5+3T6(1, 1, 1; 1, 1, 2, 1) \quad 3T1+3T2+3T3+3T4+3T6+3T7(1, 1, 2; 1, 1, 1, 1) \quad 3T1+3T4+3T7(1, 2, 1; 1, 1, 1, 1) \quad 3T2+3T3+3T6+12T8(1, 1, 1; 1, 1, 1, 1)$	

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$SU(3) \times SU(3) \times SU(3)$ $SU(3) \times SU(3)$	(none) $3(3, 3^*)$
$v = (00000000) a_1 = (42200000) a_3 = (40-222200)$ $v = (42200000) a_1 = (00040000) a_3 = (1-1-113311)$	
$3T4+3T6+3T8(3, 1, 1; 1, 1) \quad 3T3+3T5+3T7(3^*, 1, 1; 1, 1) \quad 3T4+3T6+3T8(1, 3^*, 1; 1, 1) \quad 3T3+3T5+3T7(1, 3, 1; 1, 1) \quad 3T4+3T6+3T8(1, 1, 3, 1, 1) \quad 3T3+3T5+3T7(1, 1, 3^*, 1, 1) \quad 9T0(1, 1, 1; 3^*, 1) \quad 9T0+9T2(1, 1, 1; 1, 3) \quad 9T0(1, 1, 1; 1, 3^*) \quad 27T0+27T1(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(3) \times SU(3)$ $SU(3) \times SU(3)$	(none) $3(3^*, 1) + 3(3^*, 1) + 3(3^*, 1)$
$v = (00000000) a_1 = (42200000) a_3 = (40-222200)$ $v = (42200000) a_1 = (22022000) a_3 = (00000422)$	
$3T0(1, 1, 1; 3, 3^*) \quad 3T4+3T6+3T8(3, 1, 1; 1, 1) \quad 3T3+3T5+3T7(3^*, 1, 1; 1, 1) \quad 3T4+3T6+3T8(1, 3^*, 1; 1, 1) \quad 3T3+3T5+3T7(1, 3, 1; 1, 1) \quad 3T4+3T6+3T8(1, 1, 3, 1, 1) \quad 3T3+3T5+3T7(1, 1, 3^*, 1, 1) \quad 9T2(1, 1, 1; 3, 1) \quad 9T0(1, 1, 1; 3^*, 1) \quad 9T0+9T2(1, 1, 1; 3^*, 1) \quad 27T0+27T1(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(3) \times SU(3)$ $SU(3) \times SU(3)$	$3(1, 3^*, 3^*)$ $3(1, 3^*) + 3(1, 3^*) + 3(1, 3^*)$
$v = (22000000) a_1 = (20200000) a_3 = (00042200)$ $v = (40000000) a_1 = (04220000) a_3 = (20002220)$	
$3T4+3T8(3, 1, 1; 1, 1) \quad 3T3+3T7(3^*, 1, 1; 1, 1) \quad 3T4+3T8(1, 3^*, 1; 1, 1) \quad 3T3+9T5+3T7(1, 3, 1; 1, 1) \quad 3T4+9T6+3T8(1, 1, 3; 1, 1) \quad 3T3+3T7(1, 1, 3^*, 1, 1) \quad 3T0+3T2(1, 1, 1; 3, 1) \quad 3T0+3T2(1, 1, 1; 3^*, 1) \quad 3T0+9T1+3T2(1, 1, 1; 1, 3) \quad 3T0+3T2(1, 1, 1; 1, 3^*) \quad 18T0+18T2(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(3) \times SU(3)$ $SU(3) \times SU(3)$	$3(1, 3^*, 3^*)$ $3(3^*, 3)$
$v = (22000000) a_1 = (20200000) a_3 = (00042200)$ $v = (40000000) a_1 = (22220000) a_3 = (31113111)$	
$3T5(1, 3, 1; 1, 3^*) \quad 3T4+3T8(3, 1, 1; 1, 1) \quad 3T3+3T7(3^*, 1, 1; 1, 1) \quad 3T4+3T8(1, 3^*, 1; 1, 1) \quad 3T3+3T7(1, 3, 1; 1, 1) \quad 3T4+9T6+3T8(1, 1, 3; 1, 1) \quad 3T3+3T7(1, 1, 3^*, 1, 1) \quad 3T0+9T1+3T2(1, 1, 1; 3, 1) \quad 3T0+3T2(1, 1, 1; 3^*, 1) \quad 3T0+3T2(1, 1, 1; 1, 3) \quad 3T0+3T2(1, 1, 1; 1, 3^*) \quad 18T0+18T2(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(3) \times SU(3)$ $SU(3) \times SU(3)$	$3(3, 1, 3^*) + 3(1, 3, 3) + 3(3^*, 3^*, 1)$ $3(3, 1) + 3(3, 1) + 3(3, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 4\ 2\ 2\ 0)$
$3T4+3T6+3T8\ (3, 1, 1; 1, 1)$	$3T3+3T5+3T7\ (3^*, 1, 1; 1, 1)$
$3T4+3T6+3T8\ (1, 1, 3; 1, 1)$	$3T3+3T5+3T7\ (1, 1, 3^*; 1, 1)$
	$9T0\ (1, 1, 1; 3^*, 1)$
	$27T1+27T2\ (1, 1, 1; 1, 1)$

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$SU(3) \times SU(3) \times SU(3)$ $SU(3) \times SU(3)$	$3(3, 1, 3^*) + 3(1, 3, 3) + 3(3^*, 3^*, 1)$ $3(3, 3^*)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (1\ -1\ -1\ 3\ 3\ 1\ 1\ 1)$
$3T4+3T6+3T8\ (3, 1, 1; 1, 1)$	$3T3+3T5+3T7\ (3^*, 1, 1; 1, 1)$
$3T4+3T6+3T8\ (1, 1, 3; 1, 1)$	$3T3+3T5+3T7\ (1, 1, 3^*; 1, 1)$
	$9T0\ (1, 1, 1; 3^*, 1)$
	$9T2\ (1, 1, 1; 1, 3)$
	$27T1\ (1, 1, 1; 1, 1)$

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$SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(15^*)$ $3(1, 1, 1) + 3(1, 2, 1) + 3(2, 1, 2) + 3(1, 2, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 0\ -2\ 2\ 2\ 0\ 0)$
$3T0\ (1; 1, 2, 2)$	$3T2\ (1; 2, 2, 1)$
$3T5+3T6\ (6; 1, 1, 1)$	$6T2+3T3+3T4+3T7+3T8\ (1; 1, 1, 2)$
$3T0+3T3+3T4+3T7+3T8\ (1; 2, 1, 1)$	$30T0+27T1+30T2+9T3+9T4+9T5+9T6+9T7+9T8\ (1; 1, 1, 1)$

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$SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(1) + 3(20)$ $3(1, 2, 2) + 3(2, 2, 1) + 3(2, 1, 2)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ -3\ 1\ 1\ 1\ 1\ 1\ -1)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 0\ 2\ 2\ 0\ 0)$
$3T0\ (1; 1, 2, 2)$	$3T3\ (6; 1, 1, 1)$
$3T4\ (6^*; 1, 1, 1)$	$3T1+3T2+3T5+3T6+3T7+3T8\ (1; 1, 1, 2)$
$3T1+3T2+3T5+3T6+3T7+3T8\ (1; 1, 2, 1)$	$6T0+3T1+3T2+3T5+3T6+3T7+3T8\ (1; 2, 1, 1)$
	$30T0+9T1+9T2+9T3+9T4+9T5+9T6+9T7+9T8\ (1; 1, 1, 1)$

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$SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(15^*) + 3(6) + 3(6)$ $3(1, 1, 1) + 3(1, 1, 1) + 3(1, 2, 2) + 3(1, 1, 1) + 3(2, 2, 1) + 3(2, 1, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 2\ 2\ 0\ 0)$
$3T3+3T4+3T5+3T6+3T7+3T8\ (1; 1, 1, 2)$	$3T3+3T4+3T5+3T6+3T7+3T8\ (1; 2, 1, 1)$
$27T0+27T1+27T2+9T3+9T4+9T5+9T6+9T7+9T8\ (1; 1, 1, 1)$	

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$SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(15^*) + 3(6) + 3(6)$ $3(1, 1, 1) + 3(2, 1, 2) + 3(2, 2, 1) + 3(1, 1, 1) + 3(1, 2, 2) + 3(1, 1, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ -2\ 0\ 2\ 0\ 0\ 2\ -2)$
$3T1+3T2+3T3+3T4+3T5+3T8\ (1; 1, 1, 2)$	$3T1+3T2+3T3+3T4+3T5+3T8\ (1; 1, 2, 1)$
$3T1+3T2+3T3+3T4+3T5+3T8\ (1; 2, 1, 1)$	$27T0+9T1+9T2+9T3+9T4+9T5+27T6+27T7+9T8\ (1; 1, 1, 1)$

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$SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(1) + 3(1) + 3(1) + 3(20) + 3(1)$ $3(1, 1, 1) + 3(2, 2, 2)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 0\ 0\ 2\ 0\ 0)$
$3T2\ (6; 1, 1, 1)$	$3T2\ (6^*; 1, 1, 1)$
$3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8\ (1; 1, 1, 2)$	$3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8\ (1; 1, 2, 1)$
$3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8\ (1; 1, 2, 1)$	$9T0+9T1+18T2+9T3+9T4+9T5+9T6+9T7+9T8\ (1; 1, 1, 1)$

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$SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(1) + 3(20)$ $3(1, 1, 1) + 3(1, 1, 1) + 3(1, 1, 1) + 3(2, 2, 2) + 3(1, 1, 1)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (-2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ -3\ 1\ 1\ 1\ 1\ 1\ -1)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (0\ 4\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 0\ -2\ 2\ 0\ 0)$
$3T5\ (6; 1, 1, 1)$	$3T6\ (6^*; 1, 1, 1)$
$3T0+6T1+3T2+3T3+3T4+3T7+3T8\ (1; 1, 1, 2)$	$3T0+6T1+3T2+3T3+3T4+3T7+3T8\ (1; 1, 2, 1)$
$3T0+6T1+3T2+3T3+3T4+3T7+3T8\ (1; 2, 1, 1)$	$9T0+18T1+9T2+9T3+9T4+9T5+9T6+9T7+9T8\ (1; 1, 1, 1)$

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$SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(1) + 3(1) + 3(1) + 3(20) + 3(1)$ $3(1, 1, 1) + 3(2, 2, 2)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 0\ 0\ 2\ 0\ 0)$
$3T2\ (6; 1, 1, 1)$	$3T2\ (6^*; 1, 1, 1)$
$3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8\ (1; 1, 1, 2)$	$3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8\ (1; 1, 2, 1)$
$3T0+3T1+3T3+3T4+3T5+3T6+3T7+3T8\ (1; 2, 1, 1)$	$9T0+9T1+18T2+9T3+9T4+9T5+9T6+9T7+9T8\ (1; 1, 1, 1)$

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$SU(5) \times SU(2) \times SU(2)$ $SU(4)$	$3(5, 1, 2) + 3(5^*, 2, 1) + 3(1, 2, 2)$ $3(1) + 3(4) + 3(4^*)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 0\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 1\ -1\ 3\ 1\ 1\ 1\ -1)$
$3T4\ (1, 2, 2; 1)$	$3T4\ (5, 1, 1; 1)$
$3T4\ (5^*, 1, 1; 1)$	$3T8\ (1, 1, 1; 4)$
$3T6\ (1, 1, 1; 4^*)$	$9T0+9T1+9T2\ (1, 1, 2; 1)$
$9T3+9T5+9T7\ (1, 2, 1; 1)$	$9T0+9T1+9T2+9T3+12T4+9T5+15T6+9T7+15T8\ (1, 1, 1; 1)$

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$SU(5) \times SU(2) \times SU(2)$ $SU(4)$	$3(5^*, 2, 2) + 3(10, 1, 1)$ $3(1) + 3(1) + 3(1) + 3(4^*) + 3(4)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 0\ 2\ 0\ 0\ 0)$
$3T4+3T6+3T8\ (5, 1, 1; 1)$ $9T0+9T1+9T2+9T3+9T5+9T7\ (1, 1, 1; 1)$	$9T0+9T1+9T2+3T4+3T6+3T8\ (1, 1, 2; 1)$ $9T3+3T4+9T5+3T6+9T7+3T8\ (1, 2, 1; 1)$

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$SU(5) \times SU(2) \times SU(2)$ $SU(4)$	$3(1, 1, 2) + 3(5, 1, 1) + 3(10, 2, 1)$ $3(6)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ -2\ 0\ 0\ 2\ 0\ 0\ 0)$
$3T4\ (1, 2, 2; 1)$ $3T4\ (5, 1, 1; 1)$ $3T3+3T4+3T5+3T7\ (5^*, 1, 1; 1)$ $3T6\ (1, 1, 1; 4)$ $3T8\ (1, 1, 1; 4^*)$ $3T3+3T5+3T7\ (1, 1, 2; 1)$ $9T0+9T1+9T2+3T3+3T5+3T7\ (1, 2, 1; 1)$ $9T0+9T1+9T2+12T4+15T6+15T8\ (1, 1, 1; 1)$	

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$SU(5) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(1, 1, 2) + 3(5, 1, 1) + 3(10, 2, 1)$ $3(1, 1, 1) + 3(1, 1, 1) + 3(2, 1, 1) + 3(1, 1, 1) + 3(1, 2, 1) + 3(1, 1, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 0\ 0\ 2\ 2\ 0\ 0)$
$3T3+3T5+3T7\ (5^*, 1, 1; 1, 1)$ $3T4+3T6+3T8\ (1, 1, 1; 1, 1, 2)$ $3T4+3T6+3T8\ (1, 1, 1; 2, 1)$ $3T4+3T6+3T8\ (1, 1, 1; 2, 1, 1)$ $3T3+3T5+3T7\ (1, 1, 2; 1, 1, 1)$ $9T0+9T1+9T2+3T3+3T5+3T7\ (1, 2, 1; 1, 1)$ $9T0+9T1+9T2+9T4+9T6+9T8\ (1, 1, 1; 1, 1)$	

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$SU(5) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(10, 1, 1) + 3(5^*, 2, 2)$ $3(2, 1, 1) + 3(1, 2, 1) + 3(1, 1, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ -2\ 2\ 0\ 2\ 2\ 0\ 0)$
$3T6\ (1, 1, 2; 1, 1, 2)$ $3T8\ (1, 1, 2; 1, 2, 1)$ $3T4\ (1, 1, 2; 2, 1, 1)$ $3T3+3T5+3T7\ (5, 1, 1; 1, 1, 1)$ $3T6\ (1, 1, 1; 1, 1, 2)$ $3T8\ (1, 1, 1; 1, 2, 1)$ $3T4\ (1, 1, 1; 2, 1, 1)$ $3T3+3T4+3T5+3T6+3T7+3T8\ (1, 1, 2; 1, 1, 1)$ $9T0+9T1+9T2+3T3+3T5+3T7\ (1, 2, 1; 1, 1, 1)$ $9T0+9T1+9T2+3T4+3T6+3T8\ (1, 1, 1; 1, 1, 1)$	

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$SU(5) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(1, 1, 2) + 3(5^*, 1, 1) + 3(10^*, 2, 1)$ $3(1, 1, 1) + 3(2, 2, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 4\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$
$3T2\ (1, 2, 1; 1, 1, 2)$ $3T1\ (1, 2, 1; 1, 2, 1)$ $3T0\ (1, 2, 1; 2, 1, 1)$ $3T4+3T6+3T8\ (5, 1, 1; 1, 1, 1)$ $3T2+3T3+3T5+3T7\ (1, 1, 1; 1, 1, 2)$ $3T1+3T3+3T5+3T7\ (1, 1, 1; 1, 2, 1)$ $3T0+3T3+3T5+3T7\ (1, 1, 1; 2, 1, 1)$ $3T4+3T6+3T8\ (1, 1, 2; 1, 1, 1)$ $3T0+3T1+3T2+3T4+3T6+3T8\ (1, 2, 1; 1, 1, 1)$ $3T0+3T1+3T2+9T3+9T5+9T7\ (1, 1, 1; 1, 1, 1)$	

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$SU(3) \times SU(3) \times SU(3) \times SU(3)$ $SU(3)$	$3(3, 1, 3^*, 3^*)$ $3(1) + 3(1) + 3(1) + 3(1) + 3(1) + 3(1) + 3(1) + 3(1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 4\ 2\ 2\ 0)$
$9T0+9T1+9T2\ (3^*, 1, 1, 1; 1)$ $9T3+9T5+9T7\ (1, 1, 3, 1; 1)$ $9T4+9T6+9T8\ (1, 1, 1, 3; 1)$	

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$SU(3) \times SU(3) \times SU(3) \times SU(3)$ $SU(3)$	$3(3, 1, 3^*, 3^*)$ $3(3) + 3(3) + 3(3)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 2\ 0\ 0)$
$3T6\ (1, 1, 1, 3; 3^*)$ $9T0+9T1+9T2\ (3^*, 1, 1, 1; 1)$ $9T3+9T5+9T7\ (1, 1, 3, 1; 1)$ $9T4+9T8\ (1, 1, 1, 3; 1)$	

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$SU(6)$	$3(15^*) + 3(6) + 3(6)$ (none)
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ 2\ 0\ 2\ 0\ 0)$
$27T0+27T1+27T2+27T3+27T4+27T5+27T6+27T7+27T8\ (1)$	

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$SU(4) \times SU(4)$ $SU(4) \times SU(3)$	(none) $3(4, 3)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (5\ 1\ 1\ -1\ 1\ 1\ 1\ -1)$
$3T2+3T3+3T5\ (4, 1; 1, 1)$ $3T2+3T4+3T8\ (4^*, 1; 1, 1)$ $3T4+3T5\ (1, 4^*; 1, 1)$ $3T3+3T8\ (1, 4; 1, 1)$ $3T2\ (1, 6; 1, 1)$ $6T0\ (1, 1; 4, 1)$ $3T0\ (1, 1; 6, 1)$ $6T0+3T1+3T6+3T7\ (1, 1; 4^*, 1)$ $3T1\ (1, 1; 1, 3)$ $9T0+3T6+3T7\ (1, 1; 1, 3^*)$ $15T0+6T1+12T2+3T3+3T4+3T5+6T6+6T7+3T8\ (1, 1; 1, 1)$	

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$SU(4) \times SU(4)$ $SU(4) \times SU(3)$	$3(4, 1) + 3(1, 4^*) + 3(4^*, 4)$ $3(6, 1)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ -2\ -2\ 0\ 0\ 0\ 0)$
$3T1+3T3+3T5+3T7\ (4, 1; 1, 1)$ $3T4\ (6, 1; 1, 1)$ $3T1\ (4^*, 1; 1, 1)$ $3T3+3T4+3T5+3T7\ (1, 4^*; 1, 1)$ $3T4\ (1, 4; 1, 1)$ $3T1\ (1, 6; 1, 1)$ $3T0+3T6\ (1, 1; 4, 1)$ $3T2+3T8\ (1, 1; 4^*, 1)$ $3T0+3T8\ (1, 1; 1, 3)$ $3T2+3T6\ (1, 1; 1, 3^*)$ $6T0+12T1+6T2+3T3+12T4+3T5+6T6+6T7+6T8\ (1, 1; 1, 1)$	

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$SU(4) \times SU(4)$	$3(4, 1) + 3(4^*, 4^*) + 3(1, 4)$
$SU(2) \times SU(2) \times SU(2)$	$3(1, 1, 1) + 3(1, 1, 1) + 3(1, 1, 1) + 3(1, 1, 1) + 3(2, 2, 2)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 4\ 2\ 2\ 2\ 0\ 0\ 0)$
	$v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 0\ 0\ 2\ 0\ 0)$
$3T_4+3T_6+3T_8(4, 1; 1, 1, 1) \ 3T_4+3T_6+3T_8(1, 4; 1, 1, 1) \ 3T_0+3T_1+3T_2+3T_3+3T_5+3T_7(1, 1; 1, 1, 2) \ 3T_0+3T_1+3T_2+3T_3+3T_5+3T_7(1, 1; 1, 2, 1) \ 3T_0+3T_1+3T_2+3T_3+3T_5+3T_7(1, 1; 2, 1, 1) \ 9T_0+9T_1+9T_2+9T_3+3T_4+9T_5+3T_6+9T_7+3T_8(1, 1; 1, 1, 1)$	

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$SU(4) \times SU(4)$	$3(4, 1) + 3(4^*, 4^*) + 3(1, 4)$
$SU(2) \times SU(2) \times SU(2)$	$3(2, 2, 1) + 3(2, 1, 2) + 3(1, 2, 2)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 4\ 2\ 2\ 2\ 0\ 0\ 0)$
	$v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 1\ 1\ -1\ -1\ 3\ 1\ 1)$
$3T_4+3T_6+3T_8(4, 1; 1, 1, 1) \ 3T_4+3T_6+3T_8(1, 4; 1, 1, 1) \ 3T_0+3T_1+3T_2+3T_3+3T_5+3T_7(1, 1; 1, 1, 2) \ 3T_0+3T_1+3T_2+3T_3+3T_5+3T_7(1, 1; 1, 2, 1) \ 3T_0+3T_1+3T_2+3T_3+3T_5+3T_7(1, 1; 2, 1, 1) \ 9T_0+9T_1+9T_2+9T_3+3T_4+9T_5+3T_6+9T_7+3T_8(1, 1; 1, 1, 1)$	

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$SU(4) \times SU(4)$	(none)
$SU(3) \times SU(3)$	$3(3^*, 3^*) + 3(1, 3) + 3(3, 1)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ -2\ 2\ 0\ 0)$
$3T_0(1, 1; 3^*, 3^*) \ 3T_5+3T_7(4, 1; 1, 1) \ 3T_6+3T_8(4^*, 1; 1, 1) \ 3T_7+3T_8(1, 4^*; 1, 1) \ 3T_5+3T_6(1, 4; 1, 1) \ 9T_0+3T_1+3T_2+3T_3(1, 1; 3, 1) \ 3T_4(1, 1; 3^*, 1) \ 9T_0+3T_1+3T_2+3T_4(1, 1; 1, 3) \ 3T_3(1, 1; 1, 3^*) \ 27T_0+9T_1+9T_2+9T_3+9T_4+3T_5+3T_6+3T_7+3T_8(1, 1; 1, 1)$	

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$SU(4) \times SU(4)$	(none)
$SU(3) \times SU(3)$	$3(1, 3) + 3(3, 1)$
	$v = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ -2\ 2\ 0\ 0)$
$3T_0(1, 1; 3^*, 3^*) \ 3T_2+3T_5+3T_7(4, 1; 1, 1) \ 3T_2+3T_6+3T_8(4^*, 1; 1, 1) \ 3T_7+3T_8(1, 4^*; 1, 1) \ 3T_5+3T_6(1, 4; 1, 1) \ 3T_2(1, 6; 1, 1) \ 9T_0+3T_3(1, 1; 3, 1) \ 3T_1+3T_4(1, 1; 3^*, 1) \ 9T_0+3T_4(1, 1; 1, 3) \ 3T_1+3T_3(1, 1; 1, 3^*) \ 27T_0+9T_1+12T_2+9T_3+9T_4+3T_5+3T_6+3T_7+3T_8(1, 1; 1, 1)$	

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$SU(4) \times SU(4)$	$3(1, 6) + 3(6, 4^*)$
$SU(3) \times SU(3)$	$3(3^*, 3^*) + 3(1, 1) + 3(1, 1) + 3(1, 1)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$
	$v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 0\ 2\ 0\ 0)$
$3T_3+3T_5+3T_7(4, 1; 1, 1) \ 3T_4+3T_6+3T_8(4^*, 1; 1, 1) \ 3T_3+3T_4+3T_5+3T_6+3T_7+3T_8(1, 4; 1, 1) \ 3T_0+3T_1+3T_2(1, 1; 3, 1) \ 3T_0+3T_1+3T_2(1, 1; 1, 3) \ 9T_0+9T_1+9T_2+3T_3+3T_4+3T_5+3T_6+3T_7+3T_8(1, 1; 1, 1)$	

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$SU(4) \times SU(4)$	$3(4, 1) + 3(1, 4^*) + 3(4^*, 4)$
$SU(3) \times SU(3)$	$3(3^*, 3^*)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$
	$v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 2\ 0\ 0\ 2\ 0\ 0)$
$3T_3+3T_5+3T_7(4, 1; 1, 1) \ 3T_4(6, 1; 1, 1) \ 3T_3+3T_4+3T_5+3T_7(1, 4^*; 1, 1) \ 3T_4(1, 4; 1, 1) \ 3T_0+3T_1+3T_2+3T_6(1, 1; 3, 1) \ 3T_8(1, 1; 3^*, 1) \ 3T_0+3T_1+3T_2+3T_8(1, 1; 1, 3) \ 3T_6(1, 1; 1, 3^*) \ 9T_0+9T_1+9T_2+3T_3+12T_4+3T_5+9T_6+3T_7+9T_8(1, 1; 1, 1)$	

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$SO(12)$	$3(12_v)$
$SO(12)$	$3(1) + 3(32_{sp})$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (-4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (-2\ -2\ 0\ 0\ 0\ 0\ 0\ 0)$
$3T_1+3T_4+3T_7(12_v; 1) \ 3T_0+3T_2+3T_3+3T_5+3T_6+3T_8(1; 12_v) \ 18T_0+18T_1+18T_2+18T_3+18T_4+18T_5+18T_6+18T_7+18T_8(1; 1)$	

163

$SO(12)$	$3(12_v)$
$SU(4) \times SU(4)$	$3(4, 1) + 3(1, 4^*) + 3(4^*, 4)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$
$3T_0+3T_1+3T_3+3T_6+3T_7+3T_8(1; 4, 1) \ 3T_2+3T_4+3T_5(1; 6, 1) \ 3T_0+3T_6+3T_7(1; 4^*, 1) \ 3T_2+3T_4+3T_5(1; 1, 4) \ 3T_0+3T_6+3T_7(1; 1, 6) \ 3T_1+3T_2+3T_3+3T_4+3T_5+3T_8(1; 1, 4^*) \ 12T_0+3T_1+12T_2+3T_3+12T_4+12T_5+12T_6+12T_7+3T_8(1; 1, 1)$	

164

$SO(12)$	$3(1) + 3(32_s)$
$SU(4) \times SU(4)$	$3(1, 6) + 3(6, 4^*)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$
$3T_0+3T_1+3T_3+3T_6+3T_7+3T_8(1; 4, 1) \ 3T_0+3T_2+3T_4+3T_5+3T_6+3T_7(1; 4^*, 1) \ 3T_1+3T_2+3T_3+3T_4+3T_5+3T_8(1; 1, 4) \ 3T_0+3T_6+3T_7(1; 1, 6) \ 12T_0+3T_1+3T_2+3T_3+3T_4+3T_5+12T_6+12T_7+3T_8(1; 1, 1)$	

165

$SO(10) \times SU(2)$	$3(1, 1) + 3(10_v, 2)$
$SU(5) \times SU(2) \times SU(2)$	$3(1, 2, 2) + 3(5, 1, 2) + 3(5^*, 2, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (3\ 3\ 3\ 1\ 1\ 1\ 1\ -1) a_3 = (3\ 3\ 3\ 1\ 1\ 1\ 1\ -1)$
$3T_0+3T_6+3T_7(1, 1; 1, 2, 2) \ 3T_1+3T_3+3T_8(1, 2; 1, 2, 1) \ 3T_2+3T_4+3T_5(1, 2; 1, 1, 2) \ 3T_0+3T_6+3T_7(1, 1; 5, 1, 1) \ 3T_0+3T_6+3T_7(1, 1; 5^*, 1, 1) \ 3T_1+3T_3+3T_8(1, 1; 1, 2, 1) \ 3T_2+3T_4+3T_5(1, 1; 1, 1, 2) \ 3T_1+3T_2+3T_3+3T_4+3T_5+3T_8(1, 2; 1, 1, 1) \ 12T_0+3T_1+3T_2+3T_3+3T_4+3T_5+12T_6+12T_7+3T_8(1, 1; 1, 1, 1)$	

166

$SO(10) \times SU(2)$	$3(1, 2) + 3(16s^*, 1)$
$SU(5) \times SU(2) \times SU(2)$	$3(5^*, 1, 1) + 3(1, 2, 1) + 3(10^*, 1, 2)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 4\ -2\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (1\ 1\ 1\ 1\ 1\ 1\ 3\ 3)$	
$3T0+3T5+3T8\ (1, 1; 1, 2, 2)\ 3T1+3T4+3T7\ (1, 2; 1, 1, 2)\ 3T0+3T2+3T3+3T5+3T6+3T8\ (1, 1; 5, 1, 1)\ 3T0+3T5+3T8\ (1, 1; 5^*, 1, 1)\ 3T2+3T3+3T6\ (1, 1; 1, 2, 1)\ 3T1+3T2+3T3+3T4+3T6+3T7\ (1, 1; 1, 1, 2)\ 3T1+3T4+3T7\ (1, 2; 1, 1, 1)\ 12T0+3T1+3T4+12T5+3T7+12T8\ (1, 1; 1, 1, 1)$	

167

$SU(7)$	$3(21^*)$
$SO(12)$	$3(1) + 3(32_{sp})$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0)$	
$3T1+3T2+3T3+3T4+3T5+3T8\ (7\ ;1)\ 3T2+3T4+3T5\ (7^*\ ;1)\ 3T0+3T6+3T7\ (1\ ;12v)\ 18T0+6T1+12T2+6T3+12T4+12T5+18T6+18T7+6T8\ (1;1)$	

168

$SU(7)$	$3(21^*)$
$SU(7)$	$3(7) + 3(1) + 3(7^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (5\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (5\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T1+3T2+3T3+3T4+3T5+3T8\ (7\ ;1)\ 3T2+3T4+3T5\ (7^*\ ;1)\ 3T0+3T6+3T7\ (1\ ;7)\ 3T0+3T6+3T7\ (1\ ;7^*)\ 12T0+6T1+12T2+6T3+12T4+12T5+12T6+12T7+6T8\ (1;1)$	

169

$SU(7)$	$3(21^*)$
$SU(7)$	$3(7^*) + 3(35^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (-2\ 0\ -2\ -2\ -2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (-1\ -1\ -1\ -1\ -1\ -1\ -1\ -1)$	
$3T1+3T4+3T7\ (7;1)\ 3T0+3T2+3T3+3T5+3T6+3T8\ (1;7)\ 3T0+3T5+3T8\ (1;7^*)\ 12T0+6T1+6T2+6T3+6T4+12T5+6T6+6T7+12T8\ (1;1)$	

170

$SU(7)$	$3(21^*)$
$SU(4) \times SU(4)$	$3(4, 1) + 3(1, 4^*) + 3(4^*, 4)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (-2\ 0\ -2\ -2\ -2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (2\ 4\ -2\ -2\ -2\ 0\ 0\ 0)$	
$3T1+3T4+3T7\ (7;1, 1)\ 3T0+3T2+3T3+3T5+3T6+3T8\ (1;4, 1)\ 3T0+3T5+3T8\ (1;4^*, 1)\ 3T0+3T5+3T8\ (1;1, 6)\ 3T2+3T3+3T6\ (1;1, 4^*)\ 12T0+6T1+3T2+3T3+6T4+12T5+3T6+6T7+12T8\ (1;1, 1)$	

171

$SU(7)$	$3(7) + 3(1) + 3(7^*)$
$SU(4) \times SU(4)$	$3(6, 1) + 3(4^*, 6)$
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (5\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (5\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$ $v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$	
$3T0+3T1+3T3+3T6+3T7+3T8\ (1\ ;4, 1)\ 3T2+3T4+3T5\ (1\ ;6, 1)\ 3T0+3T2+3T4+3T5+3T6+3T7\ (1\ ;1, 4)\ 3T1+3T2+3T3+3T4+3T5+3T8\ (1;1, 4^*)\ 3T0+3T1+12T2+3T3+12T4+12T5+3T6+3T7+3T8\ (1;1, 1)$	

172

$SU(6)$	$3(15^*)$
$SU(4) \times SU(4)$	(none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (-4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$6T1+3T5+3T6\ (6;1, 1)\ 3T1\ (15^*;1, 1)\ 3T0+3T2+3T3+3T7\ (1;4, 1)\ 3T0+3T2+3T4+3T8\ (1;4^*, 1)\ 3T3+3T4\ (1;1, 4)\ 3T0+3T2\ (1;1, 6)\ 3T7+3T8\ (1;1, 4^*)\ 12T0+27T1+12T2+3T3+3T4+9T5+9T6+3T7+3T8\ (1;1, 1)$	

173

$SU(6)$	$3(6)$
$SU(4) \times SU(4)$	(none)
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (-4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$6T1\ (6;1, 1)\ 3T1\ (15^*;1, 1)\ 3T7\ (6^*;1, 1)\ 3T0+3T2+3T3+3T5\ (1;4, 1)\ 3T4\ (1;6, 1)\ 3T0+3T2+3T6+3T8\ (1;4^*, 1)\ 3T4+3T5+3T8\ (1;1, 4)\ 3T0+3T2\ (1;1, 6)\ 3T3+3T4+3T6\ (1;1, 4^*)\ 12T0+27T1+12T2+3T3+12T4+3T5+3T6+9T7+3T8\ (1;1, 1)$	

174

$SU(6)$	(none)
$SU(4) \times SU(4)$	$3(4, 1) + 3(1, 4^*) + 3(4^*, 4)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$3T6\ (6;1, 1)\ 3T8\ (6^*;1, 1)\ 3T0+3T1+3T2+3T3+3T5+3T7\ (1;4, 1)\ 3T4\ (1;6, 1)\ 3T0+3T1+3T2\ (1;4^*, 1)\ 3T4\ (1;1, 4)\ 3T0+3T1+3T2\ (1;1, 6)\ 3T3+3T4+3T5+3T7\ (1;1, 4^*)\ 12T0+12T1+12T2+3T3+12T4+3T5+9T6+3T7+9T8\ (1;1, 1)$	

175

$SU(6)$	$3(1) + 3(20)$
$SU(4) \times SU(4)$	$3(1, 6) + 3(6, 4^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ -3\ 1\ 1\ 1\ 1\ 1\ -1)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$3T3\ (6;1, 1)\ 3T4\ (6^*;1, 1)\ 3T0+3T1+3T5+3T6\ (1;4, 1)\ 3T0+3T2+3T7+3T8\ (1;4^*, 1)\ 3T1+3T2+3T5+3T6+3T7+3T8\ (1;1, 4)\ 3T0\ (1;1, 6)\ 12T0+3T1+3T2+9T3+9T4+3T5+3T6+3T7+3T8\ (1;1, 1)$	

176

$SU(6)$	$3(1) + 3(1) + 3(1)$
$SU(4) \times SU(4)$	$3(6, 1) + 3(4^*, 6)$
$v = (40000000) a_1 = (22000000) a_3 = (3 - 311111 - 1)$ $v = (42222000) a_1 = (00000000) a_3 = (40000000)$	
$3T0+3T1+3T3+3T4+3T5+3T6 \begin{pmatrix} 1 & 4 & 1 \end{pmatrix} \quad 3T2+3T7+3T8 \begin{pmatrix} 1 & 6 & 1 \end{pmatrix} \quad 3T0+3T2+3T3+3T4+3T7+3T8 \begin{pmatrix} 1 & 1 & 4 \end{pmatrix}$ $3T1+3T2+3T5+3T6+3T7+3T8 \begin{pmatrix} 1 & 1 & 4^* \end{pmatrix} \quad 3T0+3T1+12T2+3T3+3T4+3T5+3T6+12T7+12T8 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$	

177

$SU(6)$	(none)
$SU(4) \times SU(4)$	$3(4^*, 1) + 3(1, 4) + 3(4, 4^*)$
$v = (40000000) a_1 = (3111111 - 1) a_3 = (31111113)$ $v = (42222000) a_1 = (00000000) a_3 = (4222 - 2000)$	
$3T1(6; 1, 1) \quad 3T5(6^*; 1, 1) \quad 3T2+3T7+3T8(1; 4, 1) \quad 3T6(1; 6, 1) \quad 3T0+3T2+3T3+3T4+3T7+3T8(1; 4^*, 1) \quad 3T0+3T3+3T4+3T6(1; 1, 4) \quad 3T2+3T7+3T8(1; 1, 6) \quad 3T6(1; 1, 4^*) \quad 3T0+9T1+12T2+3T3+3T4+9T5+12T6+12T7+12T8(1; 1, 1)$	

178

$SO(10)$	$3(16s^*)$
$SO(8) \times SU(2)$	$3(1, 2)$
$v = (22000000) a_1 = (20200000) a_3 = (2 - 2000000)$ $v = (40000000) a_1 = (04000000) a_3 = (22220000)$	
$3T3+3T4+3T5 \begin{pmatrix} 10v & 1 & 1 \end{pmatrix} \quad 3T2 \begin{pmatrix} 1 & 8c & 1 \end{pmatrix} \quad 3T7 \begin{pmatrix} 1 & 8s & 1 \end{pmatrix} \quad 3T0 \begin{pmatrix} 1 & 8v & 1 \end{pmatrix} \quad 6T0+9T1+6T2+9T6+6T7+9T8 \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}$ $18T0+9T1+18T2+24T3+24T4+24T5+9T6+18T7+9T8 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$	

179

$SO(10)$	$3(16s^*)$
$SU(5)$	$3(5)$
$v = (22000000) a_1 = (20200000) a_3 = (2 - 2000000)$ $v = (40000000) a_1 = (04000000) a_3 = (-1 - 131111 - 1)$	
$3T3+3T4 \begin{pmatrix} 10v & 1 \end{pmatrix} \quad 3T0+3T2+3T7 \begin{pmatrix} 1 & 5 \end{pmatrix} \quad 3T0+3T2+3T5+3T7 \begin{pmatrix} 1 & 5^* \end{pmatrix} \quad 24T0+27T1+24T2+24T3+24T4+12T5+27T6+24T7+27T8 \begin{pmatrix} 1 & 1 \end{pmatrix}$	

180

$SO(10)$	$3(1)$
$SU(5)$	$3(1) + 3(10^*)$
$v = (22000000) a_1 = (20200000) a_3 = (0 - 4000000)$ $v = (40000000) a_1 = (22000000) a_3 = (-11311111)$	
$3T4 \begin{pmatrix} 10v & 1 \end{pmatrix} \quad 3T0+3T2+3T3+3T6+3T7+3T8 \begin{pmatrix} 1 & 5 \end{pmatrix} \quad 3T0+3T2+3T3+3T6+3T7 \begin{pmatrix} 1 & 5^* \end{pmatrix}$ $24T0+27T1+24T2+24T3+24T4+27T5+24T6+24T7+12T8 \begin{pmatrix} 1 & 1 \end{pmatrix}$	

181

$SO(10)$	$3(10v)$
$SU(5)$	$3(5) + 3(5) + 3(10^*)$
$v = (22000000) a_1 = (20200000) a_3 = (00400000)$ $v = (40000000) a_1 = (02200000) a_3 = (-1 - 1 - 311111)$	
$3T0+3T2+3T5+3T7 \begin{pmatrix} 1 & 5 \end{pmatrix} \quad 3T0+3T2+3T5+3T7+3T8 \begin{pmatrix} 1 & 5^* \end{pmatrix} \quad 24T0+27T1+24T2+27T3+27T4+24T5+27T6+24T7+12T8 \begin{pmatrix} 1 & 1 \end{pmatrix}$	

182

$SU(6)$	$3(15^*)$
$SU(6)$	$3(6)$
$v = (22000000) a_1 = (20200000) a_3 = (00022000)$ $v = (40000000) a_1 = (04000000) a_3 = (-13111111)$	
$3T3+3T4+3T5+3T6+3T8 \begin{pmatrix} 6 & 1 \end{pmatrix} \quad 3T3+3T4+3T8 \begin{pmatrix} 6^* & 1 \end{pmatrix} \quad 3T0+3T2+3T7 \begin{pmatrix} 1 & 6^* \end{pmatrix} \quad 3T0+3T2 \begin{pmatrix} 1 & 6 \end{pmatrix}$ $18T0+27T1+18T2+18T3+18T4+9T5+9T6+9T7+18T8 \begin{pmatrix} 1 & 1 \end{pmatrix}$	

183

$SU(6)$	$3(15^*)$
$SU(6)$	$3(15^*)$
$v = (22000000) a_1 = (20200000) a_3 = (20222000)$ $v = (40000000) a_1 = (22000000) a_3 = (-1 - 311111 - 1)$	
$3T3+3T4+3T7+3T8 \begin{pmatrix} 6 & 1 \end{pmatrix} \quad 3T4+3T7 \begin{pmatrix} 6^* & 1 \end{pmatrix} \quad 3T0+3T2 \begin{pmatrix} 1 & 6^* \end{pmatrix} \quad 3T0+3T2+3T5+3T6 \begin{pmatrix} 1 & 6 \end{pmatrix}$ $18T0+27T1+18T2+9T3+18T4+9T5+9T6+18T7+9T8 \begin{pmatrix} 1 & 1 \end{pmatrix}$	

184

$SU(6)$	(none)
$SU(6)$	$3(1) + 3(20)$
$v = (22000000) a_1 = (42200000) a_3 = (20222000)$ $v = (40000000) a_1 = (22000000) a_3 = (33111111)$	
$3T3+3T4 \begin{pmatrix} 6 & 1 \end{pmatrix} \quad 3T4+3T5 \begin{pmatrix} 6^* & 1 \end{pmatrix} \quad 3T0+3T1+3T2+3T7+3T8 \begin{pmatrix} 1 & 6^* \end{pmatrix} \quad 3T0+3T1+3T2+3T6+3T7 \begin{pmatrix} 1 & 6 \end{pmatrix}$ $18T0+18T1+18T2+9T3+18T4+9T5+9T6+18T7+9T8 \begin{pmatrix} 1 & 1 \end{pmatrix}$	

185

$SU(6)$	$3(15^*)$
$SU(4) \times SU(3)$	$3(1, 3^*)$
$v = (22000000) a_1 = (20200000) a_3 = (00022000)$ $v = (40000000) a_1 = (04000000) a_3 = (20222000)$	
$3T3+3T4+3T5+3T6 \begin{pmatrix} 6 & 1 & 1 \end{pmatrix} \quad 3T3+3T4 \begin{pmatrix} 6^* & 1 & 1 \end{pmatrix} \quad 3T2+3T8 \begin{pmatrix} 1 & 4 & 1 \end{pmatrix} \quad 3T0 \begin{pmatrix} 1 & 6 & 1 \end{pmatrix} \quad 3T2+3T7 \begin{pmatrix} 1 & 4^* & 1 \end{pmatrix} \quad 3T0+9T1+3T2 \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}$ $3T0+3T2+3T7+3T8 \begin{pmatrix} 1 & 1 & 3^* \end{pmatrix} \quad 18T0+12T2+18T3+18T4+9T5+9T6+6T7+6T8 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$	

186

$SU(6)$	$3(15^*)$
$SU(4) \times SU(3)$	$3(4, 3^*)$
$v = (22000000) a_1 = (20200000) a_3 = (20222000)$ $v = (40000000) a_1 = (22000000) a_3 = (20222000)$	
$3T3+3T4+3T8 \begin{pmatrix} 6 & 1 & 1 \end{pmatrix} \quad 3T4 \begin{pmatrix} 6^* & 1 & 1 \end{pmatrix} \quad 3T2 \begin{pmatrix} 1 & 4 & 1 \end{pmatrix} \quad 3T0 \begin{pmatrix} 1 & 6 & 1 \end{pmatrix} \quad 3T2+3T5+3T6+3T7 \begin{pmatrix} 1 & 4^* & 1 \end{pmatrix} \quad 3T0+9T1+3T2+3T5+3T6 \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}$ $3T0+3T2+3T7 \begin{pmatrix} 1 & 1 & 3^* \end{pmatrix} \quad 18T0+12T2+9T3+18T4+6T5+6T6+6T7+9T8 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$	

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$SU(6)$ $SU(4) \times SU(3)$	$3(6)$ $3(4, 3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ -2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$	
$3T4(6; 1, 1) \ 3T3+3T4(6^*; 1, 1) \ 3T2+3T8(1; 4, 1) \ 3T0(1; 6, 1) \ 3T2+3T5+3T6+3T7+3T8(1; 4^*, 1) \ 3T0+9T1+3T2+3T5+3T6+3T8(1; 1, 3) \ 3T0+3T2+3T7+3T8(1; 1, 3^*) \ 18T0+12T2+9T3+18T4+6T5+6T6+6T7+12T8(1; 1, 1)$	

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$SU(6)$ $SU(4) \times SU(3)$	$3(15^*)$ $3(1, 3^*) + 3(6, 3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 2\ 2\ 2\ 2)$	
$3T5+3T6(6; 1, 1) \ 3T0+3T2+3T3+3T8(1; 4, 1) \ 3T0+3T2+3T4+3T7(1; 4^*, 1) \ 3T0+9T1+3T2+3T3+3T4+3T7+3T8(1; 1, 3) \ 3T0+3T2(1; 1, 3^*) \ 12T0+12T2+6T3+6T4+9T5+9T6+6T7+6T8(1; 1, 1)$	

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$SU(6)$ $SU(4) \times SU(3)$	$3(1) + 3(20)$ $3(6, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ -3\ 1\ 1\ 1\ 1\ 1\ -1)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 4\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0\ 0)$	
$3T1+3T2+3T3(6; 1, 1) \ 3T1+3T2+3T4(6^*; 1, 1) \ 3T5+3T8(1; 4, 1) \ 3T0(1; 6, 1) \ 3T6+3T7(1; 4^*, 1) \ 3T0+3T6+3T8(1; 1, 3) \ 3T0+3T5+3T7(1; 1, 3^*) \ 18T0+18T1+18T2+9T3+9T4+6T5+6T6+6T7+6T8(1; 1, 1)$	

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$SU(5) \times SU(2)$ $SO(8) \times SU(2)$	$3(5^*, 2)$ $3(1, 1) + 3(8s, 2)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 2\ 0\ 0\ 0\ 0\ 0)$	
$3T6+3T7+3T8(1, 2; 1, 2) \ 3T3+3T4+3T5(5, 1; 1, 1) \ 3T4(5^*, 1; 1, 1) \ 3T2(1, 1; 8c, 1) \ 3T0(1, 1; 8v, 1) \ 6T0+9T1+6T2+3T6+3T7+3T8(1, 1; 1, 2) \ 3T3+6T4+3T5+3T6+3T7+3T8(1, 2; 1, 1) \ 18T0+9T1+18T2+6T3+12T4+6T5+3T6+3T7+3T8(1, 1; 1, 1)$	

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$SU(5) \times SU(2)$ $SO(8) \times SU(2)$	$3(10, 1)$ $3(8c, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 2\ 2\ 0\ 0\ 0\ 0)$	
$3T5(1, 2; 1, 2) \ 3T4+3T6+3T8(5, 1; 1, 1) \ 3T3+3T4+3T7+3T8(5^*, 1; 1, 1) \ 3T2(1, 1; 8s, 1) \ 3T0(1, 1; 8v, 1) \ 6T0+9T1+6T2+3T5(1, 1; 1, 2) \ 3T3+6T4+3T5+3T6+3T7+6T8(1, 2; 1, 1) \ 18T0+9T1+18T2+6T3+12T4+3T5+6T6+6T7+12T8(1, 1; 1, 1)$	

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$SU(5) \times SU(2)$ $SO(8) \times SU(2)$	$3(5, 1) + 3(10, 2)$ $3(1, 2)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ -4\ 0\ 0\ 0\ 0\ 0)$	
$3T0+3T1+3T7(1, 2; 1, 2) \ 3T2+3T4+3T6(5, 1; 1, 1) \ 3T2+3T3+3T4+3T5+3T6+3T8(5^*, 1; 1, 1) \ 3T0+3T1+3T7(1, 1; 1, 2) \ 3T0+3T1+6T2+3T3+6T4+3T5+6T6+3T7+3T8(1, 2; 1, 1) \ 3T0+3T1+12T2+6T3+12T4+6T5+12T6+3T7+6T8(1, 1; 1, 1)$	

193

$SU(5) \times SU(2)$ $SU(5)$	$3(5^*, 2)$ $3(1) + 3(5) + 3(5^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (-1\ -1\ -1\ 3\ 1\ 1\ 1\ -1)$	
$3T3+3T4+3T5(5, 1; 1) \ 3T4(5^*, 1; 1) \ 3T0+3T2(1, 1; 5) \ 3T0+3T2(1, 1; 5^*) \ 3T3+6T4+3T5+9T6+9T7+9T8(1, 2; 1) \ 24T0+27T1+24T2+6T3+12T4+6T5+9T6+9T7+9T8(1, 1; 1)$	

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$SU(5) \times SU(2)$ $SU(5)$	$3(10, 1)$ $3(1) + 3(10^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (-1\ 1\ 3\ 1\ 1\ 1\ 1\ 1)$	
$3T4+3T6(5, 1; 1) \ 3T3+3T4+3T7(5^*, 1; 1) \ 3T0+3T2+3T8(1, 1; 5) \ 3T0+3T2(1, 1; 5^*) \ 3T3+6T4+9T5+3T6+3T7(1, 2; 1) \ 24T0+27T1+24T2+6T3+12T4+9T5+6T6+6T7+12T8(1, 1; 1)$	

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$SU(5) \times SU(2)$ $SU(5)$	$3(10, 1)$ $3(1) + 3(1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (-1\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T4+3T6+3T8(5, 1; 1) \ 3T4+3T5+3T6+3T7(5^*, 1; 1) \ 3T0+3T2(1, 1; 5) \ 3T0+3T2(1, 1; 5^*) \ 9T3+6T4+3T5+6T6+3T7+3T8(1, 2; 1) \ 24T0+27T1+24T2+9T3+12T4+6T5+12T6+6T7+6T8(1, 1; 1)$	

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$SU(5) \times SU(2)$ $SU(5)$	$3(5, 2)$ $3(1) + 3(5) + 3(5^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ -2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 3\ 1\ 1\ 1\ 1\ -1) a_3 = (3\ 3\ -1\ -1\ -1\ -1\ -1\ -13)$	
$3T5(5, 1; 1) \ 3T5+3T6+3T8(5^*, 1; 1) \ 3T0+3T1(1, 1; 5) \ 3T0+3T1(1, 1; 5^*) \ 9T3+9T4+6T5+3T6+9T7+3T8(1, 2; 1) \ 24T0+24T1+27T2+9T3+9T4+12T5+6T6+9T7+6T8(1, 1; 1)$	

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$SU(5) \times SU(2)$ $SU(5)$	$3(1, 1)$ $3(1) + 3(10^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ -1\ 3\ 1\ 1\ 1\ -1)$	
$3T3+3T4(5, 1; 1) \quad 3T4+3T8(5^*, 1; 1) \quad 3T0+3T1+3T2+3T7(1, 1; 5) \quad 3T0+3T1+3T2(1, 1; 5^*) \quad 3T3+6T4+9T5+9T6+3T8(1, 2; 1) \quad 24T0+24T1+24T2+6T3+12T4+9T5+9T6+12T7+6T8(1, 1; 1)$	

198

$SU(5) \times SU(2)$ $SU(5)$	$3(5, 1) + 3(10, 2)$ $3(5^*)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 0\ 2\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 2\ 2\ 0\ 0\ 0)$	
$3T4+3T6(5, 1; 1) \quad 3T3+3T4+3T5+3T6+3T8(5^*, 1; 1) \quad 3T2(1, 1; 5) \quad 9T0+9T1+3T3+6T4+3T5+6T6+9T7+3T8(1, 2; 1) \quad 9T0+9T1+12T2+6T3+12T4+6T5+12T6+9T7+6T8(1, 1; 1)$	

199

$SO(10)$ $SU(4) \times SU(2)$	$3(10_v)$ $3(1, 2) + 3(4^*, 2) + 3(4, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 0\ 0\ 0\ 0)$	
$3T2+3T5+3T7+3T8(1; 4, 1) \quad 3T0(1; 6, 1) \quad 3T2+3T5+3T7(1; 4^*, 1) \quad 6T0+9T1+6T2+6T5+9T6+6T7+3T8(1; 1, 2) \quad 24T0+9T1+18T2+27T3+27T4+18T5+9T6+18T7+9T8(1; 1, 1)$	

200

$SO(10)$ $SU(4) \times SU(2)$	$3(16s^*)$ $3(4, 2)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$3T4(10_v; 1, 1) \quad 3T2+3T6(1; 4, 1) \quad 3T0(1; 6, 1) \quad 3T2+3T5+3T6+3T8(1; 4^*, 1) \quad 6T0+9T1+6T2+3T5+6T6+9T7+3T8(1; 1, 2) \quad 24T0+9T1+18T2+27T3+24T4+9T5+18T6+9T7+9T8(1; 1, 1)$	

201

$SO(10)$ $SU(4) \times SU(2)$	$3(16s^*)$ $3(1, 2) + 3(6, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (-1\ 1\ 1\ 3\ 1\ 1\ 1\ 1)$	
$3T4(10_v; 1, 1) \quad 3T0+3T2+3T6+3T8(1; 4, 1) \quad 3T0+3T2+3T5+3T6(1; 4^*, 1) \quad 6T0+6T2+9T3+3T5+6T6+3T8(1; 1, 2) \quad 18T0+27T1+18T2+9T3+24T4+9T5+18T6+27T7+9T8(1; 1, 1)$	

202

$SO(8)$ $SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(1) + 3(8_v)$ $3(3^*, 1, 1, 1) + 3(3^*, 2, 1, 2)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 2\ 2\ 0\ 0)$	
$3T0(1; 1, 2, 2) \quad 3T2(1; 1, 2, 2, 1) \quad 3T0+9T1+3T2+3T5+3T6(1; 3, 1, 1, 1) \quad 3T0+3T2(1; 3^*, 1, 1, 1) \quad 6T2+3T5+3T6+9T7+9T8(1; 1, 1, 1, 2) \quad 3T5+3T6(1; 1, 1, 2, 1) \quad 6T0+9T3+9T4+3T5+3T6(1; 1, 2, 1, 1) \quad 12T0+12T2+9T3+9T4+9T7+9T8(1; 1, 1, 1, 1)$	

203

$SO(8)$ $SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(1) + 3(8_v)$ $3(3^*, 1, 2, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 2\ 2\ 0\ 0)$	
$3T0(1; 1, 1, 2, 2) \quad 3T2(1; 1, 2, 2, 1) \quad 3T4(8_s; 1, 1, 1, 1) \quad 3T0+9T1+3T2(1; 3, 1, 1, 1) \quad 3T0+3T2+3T7(1; 3^*, 1, 1, 1) \quad 6T2+9T5+3T7(1; 1, 1, 1, 2) \quad 9T6+3T7+9T8(1; 1, 1, 2, 1) \quad 6T0+9T3+3T7(1; 1, 2, 1, 1) \quad 12T0+12T2+9T3+30T4+9T5+9T6+9T8(1; 1, 1, 1, 1)$	

204

$SU(5) \times SU(2)$ $SU(4) \times SU(2)$	$3(5^*, 2)$ $3(1, 2) + 3(4^*, 2) + 3(4, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 2\ 2\ 0\ 0\ 0)$	
$3T5(1, 2; 1, 2) \quad 3T6+3T8(5, 1; 1) \quad 3T2+3T7(1, 1; 4, 1) \quad 3T0(1, 1; 6, 1) \quad 3T2(1, 1; 4^*, 1) \quad 6T0+9T1+6T2+3T5+3T7(1, 1; 1, 2) \quad 9T3+9T4+3T5+3T6+3T8(1, 2; 1, 1) \quad 24T0+9T1+18T2+9T3+9T4+3T5+6T6+9T7+6T8(1, 1; 1, 1)$	

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$SU(5) \times SU(2)$ $SU(4) \times SU(2)$	$3(10, 1)$ $3(1, 1) + 3(4^*, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 2\ 2\ 0\ 0\ 0)$	
$3T4+3T8(5, 1; 1, 1) \quad 3T4+3T5+3T7(5^*, 1; 1, 1) \quad 3T2+3T6(1, 1; 4, 1) \quad 3T0(1, 1; 6, 1) \quad 3T2(1, 1; 4^*, 1) \quad 6T0+9T1+6T2+3T6(1, 1; 1, 2) \quad 9T3+6T4+3T5+3T7+3T8(1, 2; 1, 1) \quad 24T0+9T1+18T2+9T3+12T4+6T5+9T6+6T7+6T8(1, 1; 1, 1)$	

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$SU(5) \times SU(2)$ $SU(4) \times SU(2)$	$3(5^*, 1) + 3(1, 2)$ $3(4, 2)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 0\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$3T7(1, 2; 1, 2) \quad 3T4+3T6(5, 1; 1, 1) \quad 3T4(5^*, 1; 1, 1) \quad 3T2(1, 1; 4, 1) \quad 3T0(1, 1; 6, 1) \quad 3T2+3T5+3T8(1, 1; 4^*, 1) \quad 6T0+9T1+6T2+3T5+3T7+3T8(1, 1; 1, 2) \quad 9T3+6T4+3T6+3T7(1, 2; 1, 1) \quad 24T0+9T1+18T2+9T3+12T4+9T5+6T6+3T7+9T8(1, 1; 1, 1)$	

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$SU(5) \times SU(2)$	$3(10^*, 1)$
$SU(4) \times SU(2)$	$3(4, 1) + 3(4, 1) + 3(6, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 4\ 2\ 2\ 0\ 0\ 0) a_3 = (-1\ 3\ 1\ 1\ 1\ 1\ 1)$
$3T4+3T6(5, 1; 1, 1) \quad 3T3(5^*, 1; 1, 1) \quad 3T0+3T2(1, 1; 4, 1) \quad 3T0+3T2+3T5+3T7(1, 1; 4^*, 1) \quad 6T0+6T2+3T5+3T7(1, 1; 1, 2)$ $3T3+3T4+3T6+9T8(1, 2; 1, 1) \quad 18T0+27T1+18T2+6T3+6T4+9T5+6T6+9T7+9T8(1, 1; 1, 1)$	

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$SU(5) \times SU(2)$	$3(5^*, 2)$
$SU(4) \times SU(2)$	$3(1, 2) + 3(1, 2) + 3(1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 0\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (3\ 3\ 1\ -3\ 1\ 1\ 1)$
$3T4(1, 2; 1, 2) \quad 3T6+3T7+3T8(5, 1; 1, 1) \quad 3T7(5^*, 1; 1, 1) \quad 3T0+3T2(1, 1; 4, 1) \quad 3T0+3T2(1, 1; 4^*, 1) \quad 6T0+9T1+6T2+3T4$ $(1, 1; 1, 2) \quad 9T3+3T4+9T5+3T6+6T7+3T8(1, 2; 1, 1) \quad 18T0+9T1+18T2+9T3+3T4+9T5+6T6+12T7+6T8(1, 1; 1, 1)$	

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$SU(5) \times SU(2)$	$3(10, 1)$
$SU(4) \times SU(2)$	$3(1, 1) + 3(6, 2) + 3(1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ -2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 0\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (-2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (1\ 3\ 3\ -1\ 1\ 1\ 1\ -1)$
$3T3(1, 2; 1, 2) \quad 3T6(5, 1; 1, 1) \quad 3T5+3T7(5^*, 1; 1, 1) \quad 3T0+3T1+3T4(1, 1; 4, 1) \quad 3T0+3T1+3T8(1, 1; 4^*, 1)$ $6T0+6T1+9T2+3T3+3T4+3T8(1, 1; 1, 2) \quad 3T3+3T5+3T6+3T7(1, 2; 1, 1) \quad 18T0+18T1+9T2+3T3+9T4+6T5+6T6+6T7+9T8(1, 1; 1, 1)$	

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$SU(5) \times SU(2)$	$3(5, 1) + 3(10, 2)$
$SU(4) \times SU(2)$	$3(1, 2) + 3(6, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 0\ 2\ 0\ 0\ 0) a_3 = (2\ -2\ 0\ -2\ 2\ 0\ 0)$
$3T0(1, 2; 1, 2) \quad 3T4(5, 1; 1, 1) \quad 3T3+3T4+3T5+3T8(5^*, 1; 1, 1) \quad 3T6(1, 1; 4, 1) \quad 3T2(1, 1; 4^*, 1) \quad 3T0+3T2+3T6(1, 1; 1, 2)$ $3T0+9T1+3T3+6T4+3T5+9T7+3T8(1, 2; 1, 1) \quad 3T0+9T1+9T2+6T3+12T4+6T5+9T6+9T7+6T8(1, 1; 1, 1)$	

211

$SU(5) \times SU(2)$	$3(5^*, 1) + 3(10^*, 2)$
$SU(4) \times SU(2)$	$3(4^*, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 4\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ -2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 2\ 2\ 0\ 0)$
$3T1+3T8(1, 2; 1, 2) \quad 3T3+3T4+3T6+3T7(5, 1; 1, 1) \quad 3T3(5^*, 1; 1, 1) \quad 3T2+3T5(1, 1; 4, 1) \quad 3T1+3T2+3T5+3T8(1, 1; 1, 2)$ $9T0+3T1+6T3+3T4+3T6+3T7+3T8(1, 2; 1, 1) \quad 9T0+3T1+9T2+12T3+6T4+9T5+6T6+6T7+3T8(1, 1; 1, 1)$	

212

$SO(10)$	$3(1)$
$SU(3) \times SU(2)$	$3(1, 1) + 3(3^*, 2) + 3(1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 4\ 0\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 0\ 0\ 2\ 2\ 0)$
$3T0+3T2+3T3+3T4+3T5+3T7+3T8(1; 3, 1) \quad 3T0+3T2+3T4+3T5+3T8(1; 3^*, 1) \quad 6T0+9T1+6T2+3T3+6T4+6T5+3T7+6T8(1; 1, 2)$ $24T0+9T1+24T2+12T3+24T4+24T5+27T6+12T7+24T8(1; 1, 1)$	

213

$SO(10)$	$3(16s^*)$
$SU(3) \times SU(2)$	$3(1, 2) + 3(3^*, 2) + 3(3, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (-1\ 1\ -1\ -1\ 3\ 1\ 1\ 1)$
$3T0+3T2+3T4+3T6+3T8(1; 3, 1) \quad 3T0+3T2+3T4+3T7(1; 3^*, 1) \quad 6T0+9T1+6T2+6T4+3T6+3T7+3T8(1; 1, 2)$ $24T0+9T1+24T2+27T3+24T4+27T5+12T6+12T7+12T8(1; 1, 1)$	

214

$SO(10)$	$3(16s^*)$
$SU(3) \times SU(2)$	$3(3^*, 1) + 3(1, 2) + 3(3^*, 1) + 3(3^*, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ -2\ -2\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 3\ 1\ 1\ 1\ 1\ -1) a_3 = (-1\ -1\ 1\ 1\ 1\ -1\ -1\ -13)$
$3T0+3T2+3T3+3T5+3T7+3T8(1; 3, 1) \quad 3T0+3T2+3T3(1; 3^*, 1) \quad 6T0+6T2+6T3+3T5+3T7+3T8(1; 1, 2)$ $24T0+27T1+24T2+24T3+27T4+12T5+27T6+12T7+12T8(1; 1, 1)$	

215

$SO(10)$	$3(16s^*)$
$SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(1, 1, 1, 1) + 3(2, 2, 1, 2) + 3(1, 1, 2, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 0\ 2\ 2\ 0\ 0)$
$3T0(1; 1, 1, 2, 2) \quad 3T2(1; 1, 2, 2, 1) \quad 3T4(1; 2, 1, 2, 1) \quad 6T2+6T4+9T5+3T6+3T7+3T8(1; 1, 1, 1, 2)$ $3T6+3T7+3T8(1; 1, 1, 1, 2, 1) \quad 6T0+9T3+6T4+3T6+3T7+3T8(1; 1, 2, 1, 1) \quad 6T0+9T1+6T2+3T6+3T7+3T8(1; 2, 1, 1, 1)$ $18T0+9T1+18T2+9T3+18T4+9T5+3T6+3T7+3T8(1; 1, 1, 1, 1)$	

216

$SU(5) \times SU(2)$	$3(10^*, 1)$
$SU(3) \times SU(2)$	$3(3, 1) + 3(3^*, 1) + 3(1, 1) + 3(1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 4\ 2\ 2\ 0\ 0\ 0) a_3 = (-1\ 1\ 1\ -1\ 3\ 1\ 1\ -1)$
$3T4+3T6(5, 1; 1, 1) \quad 3T3(5^*, 1; 1, 1) \quad 3T0+3T2+3T7(1, 1; 3, 1) \quad 3T0+3T2+3T5(1, 1; 3^*, 1) \quad 6T0+6T2+3T5+3T7(1, 1; 1, 2)$ $3T3+3T4+3T6+9T8(1, 2; 1, 1) \quad 24T0+27T1+24T2+6T3+6T4+12T5+6T6+12T7+9T8(1, 1; 1, 1)$	

217

$SU(5) \times SU(2)$	$3(10, 1)$
$SU(3) \times SU(2)$	$3(1, 1) + 3(3^*, 2) + 3(1, 1)$
$v = (22000000) a_1 = (20200000) a_3 = (40022000)$	
$v = (40000000) a_1 = (22220000) a_3 = (13 -1 -13111)$	
$3T8(5, 1; 1, 1) \quad 3T5+3T7(5^*, 1; 1, 1) \quad 3T0+3T2+3T4+3T6(1, 1; 3, 1) \quad 3T0+3T2(1, 1; 3^*, 1) \quad 6T0+9T1+6T2+3T4+3T6(1, 1; 1, 2) \quad 9T3+3T5+3T7+3T8(1, 2; 1, 1) \quad 24T0+9T1+24T2+9T3+12T4+6T5+12T6+6T7+6T8(1, 1; 1, 1)$	

218

$SU(5) \times SU(2)$	$3(5^*, 1) + 3(1, 2)$
$SU(3) \times SU(2)$	$3(1, 2) + 3(3^*, 2) + 3(3, 1)$
$v = (22000000) a_1 = (20200000) a_3 = (04022000)$	
$v = (40000000) a_1 = (22220000) a_3 = (13113111)$	
$3T6(5, 1; 1, 1) \quad 3T0+3T2+3T4+3T8(1, 1; 3, 1) \quad 3T0+3T2+3T5(1, 1; 3^*, 1) \quad 6T0+9T1+6T2+3T4+3T5+3T8(1, 1; 1, 2) \quad 9T3+3T6+9T7(1, 2; 1, 1) \quad 24T0+9T1+24T2+9T3+12T4+12T5+6T6+9T7+12T8(1, 1; 1, 1)$	

219

$SU(5) \times SU(2)$	$3(5^*, 2)$
$SU(3) \times SU(2)$	$3(1, 1) + 3(3^*, 1) + 3(1, 2) + 3(1, 2)$
$v = (22000000) a_1 = (20200000) a_3 = (20022200)$	
$v = (40000000) a_1 = (22220000) a_3 = (22004220)$	
$3T3+3T5(5, 1; 1, 1) \quad 3T0+3T2+3T4(1, 1; 3, 1) \quad 3T0+3T2(1, 1; 3^*, 1) \quad 6T0+9T1+6T2+3T4(1, 1; 1, 2) \quad 3T3+3T5+9T6+9T7+9T8(1, 2; 1, 1) \quad 24T0+9T1+24T2+6T3+12T4+6T5+9T6+9T7+9T8(1, 1; 1, 1)$	

220

$SU(5) \times SU(2)$	$3(1, 1)$
$SU(3) \times SU(2)$	$3(1, 1) + 3(1, 1) + 3(3^*, 2)$
$v = (22000000) a_1 = (42200000) a_3 = (4 - 2222000)$	
$v = (40000000) a_1 = (22220000) a_3 = (311 - 1311 - 1)$	
$3T3(1, 2; 1, 2) \quad 3T4(5, 1; 1, 1) \quad 3T5(5^*, 1; 1, 1) \quad 3T0+3T1+3T2+3T7+3T8(1, 1; 3, 1) \quad 3T0+3T1+3T2(1, 1; 3^*, 1) \quad 6T0+6T1+6T2+3T3+3T7+3T8(1, 1; 1, 2) \quad 3T3+3T4+3T5+9T6(1, 2; 1, 1) \quad 24T0+24T1+24T2+3T3+6T4+6T5+9T6+12T7+12T8(1, 1; 1, 1)$	

221

$SU(5) \times SU(2)$	$3(5^*, 1) + 3(10^*, 2)$
$SU(3) \times SU(2)$	$3(1, 2) + 3(3^*, 2) + 3(3, 1)$
$v = (42200000) a_1 = (20020000) a_3 = (00022000)$	
$v = (40000000) a_1 = (00040000) a_3 = (11 - 1 - 1311 - 1)$	
$3T0(1, 2; 1, 2) \quad 3T3+3T4+3T8(5, 1; 1, 1) \quad 3T2+3T7(1, 1; 3, 1) \quad 3T5(1, 1; 3^*, 1) \quad 3T0+3T2+3T5+3T7(1, 1; 1, 2) \quad 3T1+3T3+3T4+9T6+3T8(1, 2; 1, 1) \quad 3T0+9T1+12T2+6T3+6T4+12T5+9T6+12T7+6T8(1, 1; 1, 1)$	

222

$SU(5) \times SU(2)$	$3(5^*, 1) + 3(10^*, 2)$
$SU(3) \times SU(2)$	$3(3, 1) + 3(3, 1) + 3(3, 1) + 3(1, 2)$
$v = (42200000) a_1 = (20020000) a_3 = (00022000)$	
$v = (42200000) a_1 = (22022000) a_3 = (20 - 220200)$	
$3T3+3T4+3T8(5, 1; 1, 1) \quad 3T2+3T5+3T7(1, 1; 3^*, 1) \quad 3T2+3T5+3T7(1, 1; 1, 2) \quad 9T0+9T1+3T3+3T4+9T6+3T8(1, 2; 1, 1) \quad 9T0+9T1+12T2+6T3+6T4+12T5+9T6+12T7+6T8(1, 1; 1, 1)$	

223

$SU(5) \times SU(2)$	$3(10^*, 1)$
$SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(2, 1, 1, 1) + 3(1, 1, 2, 1) + 3(2, 1, 2, 1)$
$v = (22000000) a_1 = (20200000) a_3 = (00422000)$	
$v = (40000000) a_1 = (04220000) a_3 = (20202200)$	
$3T2(1, 1; 1, 1, 2, 2) \quad 3T0(1, 1; 1, 2, 2, 1) \quad 3T8(1, 2; 1, 1, 2, 1) \quad 3T4+3T6(5, 1; 1, 1, 1) \quad 3T3(5^*, 1; 1, 1, 1) \quad 6T0+3T5+3T7(1, 1; 1, 1, 2) \quad 3T5+3T7+3T8(1, 1; 1, 1, 2, 1) \quad 6T2+3T5+3T7(1, 1; 1, 2, 1, 1) \quad 6T0+9T1+6T2+3T5+3T7(1, 1; 2, 1, 1, 1) \quad 3T3+3T4+3T6+3T8(1, 2; 1, 1, 1, 1) \quad 18T0+9T1+18T2+6T3+6T4+3T5+6T6+3T7+3T8(1, 1; 1, 1, 1, 1)$	

224

$SU(5) \times SU(2)$	$3(5^*, 2)$
$SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(1, 2, 1, 2) + 3(2, 1, 2, 1)$
$v = (22000000) a_1 = (20200000) a_3 = (20022200)$	
$v = (40000000) a_1 = (22220000) a_3 = (240 - 22200)$	
$3T0(1, 1; 1, 2, 2, 1) \quad 3T2(1, 1; 2, 2, 1, 1) \quad 3T6(1, 2; 1, 1, 2, 1) \quad 3T7(1, 2; 1, 2, 1, 1) \quad 3T8(1, 2; 2, 1, 1, 1) \quad 3T3+3T5(5, 1; 1, 1, 1) \quad 6T0+9T1+6T2+3T4(1, 1; 1, 1, 1, 2) \quad 6T2+3T4+3T6(1, 1; 1, 1, 2, 1) \quad 3T4+3T7(1, 1; 1, 2, 1, 1) \quad 6T0+3T4+3T8(1, 1; 2, 1, 1, 1) \quad 3T3+3T5+3T6+3T7+3T8(1, 2; 1, 1, 1, 1) \quad 18T0+9T1+18T2+6T3+3T4+6T5+3T6+3T7+3T8(1, 1; 1, 1, 1, 1)$	

225

$SU(5) \times SU(2)$	$3(5^*, 1) + 3(10^*, 2)$
$SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(1, 1, 1, 1) + 3(1, 2, 2, 2) + 3(2, 1, 1, 1)$
$v = (42200000) a_1 = (20020000) a_3 = (00022000)$	
$v = (42200000) a_1 = (00040000) a_3 = (02022200)$	
$3T0(1, 2; 1, 1, 1, 2) \quad 3T6(1, 2; 1, 1, 2, 1) \quad 3T1(1, 2; 1, 2, 1, 1) \quad 3T3+3T4+3T8(5, 1; 1, 1, 1, 1) \quad 3T0+3T2+3T5+3T7(1, 1; 1, 1, 1, 2) \quad 3T2+3T5+3T6+3T7(1, 1; 1, 1, 2, 1) \quad 3T1+3T2+3T5+3T7(1, 1; 1, 2, 1, 1) \quad 3T2+3T5+3T7(1, 1; 2, 1, 1, 1) \quad 3T0+3T1+3T3+3T4+3T6+3T8(1, 2; 1, 1, 1, 1) \quad 3T0+3T1+3T2+6T3+6T4+3T5+3T6+3T7+6T8(1, 1; 1, 1, 1, 1)$	

226

$SU(5) \times SU(2)$	$3(5, 1) + 3(10, 2)$
$SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(1, 2, 1, 1) + 3(1, 1, 1, 1) + 3(2, 1, 2, 2)$
$v = (42200000) a_1 = (20 - 200000) a_3 = (-1111111 - 1)$	
$v = (42200000) a_1 = (00040000) a_3 = (200 - 22200)$	
$3T8(1, 2; 1, 1, 1, 2) \quad 3T2(1, 2; 1, 1, 2, 1) \quad 3T0(1, 2; 2, 1, 1, 1) \quad 3T3+3T4+3T6(5^*, 1; 1, 1, 1, 1) \quad 3T1+3T5+3T7+3T8(1, 1; 1, 1, 1, 2) \quad 3T1+3T2+3T5+3T7(1, 1; 1, 1, 2, 1) \quad 3T0+3T1+3T5+3T7(1, 1; 2, 1, 1, 1) \quad 3T0+3T2+3T3+3T4+3T6+3T8(1, 2; 1, 1, 1, 1) \quad 3T0+3T1+3T2+6T3+6T4+3T5+6T6+3T7+3T8(1, 1; 1, 1, 1, 1)$	

227

$SU(6)$	$3(15^*)$
$SU(3) \times SU(3)$	$3(3,1) + 3(1,3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (5\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (-1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T3+3T4+3T8\ (6;1,1)\ 3T4\ (6^*;1,1)\ 3T0+3T2+3T5\ (1;3,1)\ 3T0+3T2+3T6+3T7\ (1;3^*,1)\ 3T0+3T2+3T5+3T7\ (1;1,3)$	
$3T0+3T2+3T6\ (1;1,3^*)\ 18T0+27T1+18T2+9T3+18T4+9T5+9T6+9T7+9T8\ (1;1,1)$	

228

$SU(6)$	$3(6)$
$SU(3) \times SU(3)$	$3(1,3^*) + 3(3^*,3) + 3(3,1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (-1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T7\ (6^*;1,1)\ 3T0+3T2+3T3+3T4+3T5+3T6\ (1;3,1)\ 3T0+3T2+3T4+3T8\ (1;3^*,1)\ 3T0+3T2+3T3+3T4\ (1;1,3)$	
$3T0+3T2+3T4+3T5+3T6+3T8\ (1;1,3^*)\ 18T0+27T1+18T2+9T3+18T4+9T5+9T6+9T7+9T8\ (1;1,1)$	

229

$SU(6)$	$3(15^*)$
$SU(3) \times SU(3)$	$3(1,3^*) + 3(3^*,3) + 3(3,1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (-2\ 0\ -2\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (-1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T4+3T7\ (6;1,1)\ 3T0+3T2+3T3+3T5+3T6\ (1;3,1)\ 3T0+3T2+3T8\ (1;3^*,1)\ 3T0+3T2+3T3\ (1;1,3)\ 3T0+3T2+3T5+3T6+3T8\ (1;1,3^*)\ 18T0+27T1+18T2+9T3+9T4+9T5+9T6+9T7+9T8\ (1;1,1)$	

230

$SU(6)$	$3(1) + 3(1) + 3(1)$
$SU(3) \times SU(3)$	$3(3^*,3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (3\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T4\ (6;1,1)\ 3T4\ (6^*;1,1)\ 3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7\ (1;3,1)\ 3T0+3T1+3T2+3T8\ (1;3^*,1)$	
$3T0+3T1+3T2+3T3+3T5+3T7+3T8\ (1;1,3)\ 3T0+3T1+3T2+3T6\ (1;1,3^*)\ 18T0+18T1+18T2+9T3+18T4+9T5+9T6+9T7+9T8\ (1;1,1)$	

231

$SU(6)$	(none)
$SU(3) \times SU(3)$	$3(1,1) + 3(1,1) + 3(3^*,3^*) + 3(1,1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (3\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T6\ (6;1,1)\ 3T8\ (6^*;1,1)\ 3T0+3T1+3T2+3T3+3T4+3T5+3T7\ (1;3,1)\ 3T0+3T1+3T2+3T4\ (1;3^*,1)$	
$3T0+3T1+3T2+3T3+3T4+3T5+3T7\ (1;1,3)\ 3T0+3T1+3T2+3T4\ (1;1,3^*)\ 18T0+18T1+18T2+9T3+18T4+9T5+9T6+9T7+9T8\ (1;1,1)$	

232

$SU(6)$	$3(1) + 3(20)$
$SU(3) \times SU(3)$	$3(3^*,3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ -3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (3\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1)$	
$3T2+3T3\ (6;1,1)\ 3T2+3T4\ (6^*;1,1)\ 3T0+3T1+3T5+3T6+3T7\ (1;3,1)\ 3T0+3T8\ (1;3^*,1)\ 3T0+3T1+3T5+3T6+3T8\ (1;1,3)\ 3T0+3T7\ (1;1,3^*)\ 18T0+9T1+18T2+9T3+9T4+9T5+9T6+9T7+9T8\ (1;1,1)$	

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$SO(8) \times SU(2)$	$3(1,2)$
$SU(4) \times SU(2) \times SU(2)$	$3(1,1,2) + 3(4^*,2,1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (-2\ -2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$	
$3T0+3T3\ (1,1;1,2,2)\ 3T5\ (1,2;1,1,2)\ 3T7+3T8\ (1,2;1,2,1)\ 3T4\ (8s,1;1,1,1)\ 3T0+3T1+3T3+3T6\ (1,1;4,1,1)\ 3T2\ (1,1;6,1,1)\ 3T0+3T3\ (1,1;4^*,1,1)\ 3T1+6T2+3T5+3T6\ (1,1;1,1,2)\ 3T1+6T2+3T6+3T7+3T8\ (1,1;1,2,1)\ 6T4+3T5+3T7+3T8\ (1,2;1,1,1)\ 18T0+3T1+12T2+18T3+18T4+3T5+3T6+3T7+3T8\ (1,1;1,1,1)$	

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$SO(8) \times SU(2)$	$3(1,1) + 3(8c,2)$
$SU(4) \times SU(2) \times SU(2)$	$3(6,1,1) + 3(4^*,2,2)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ -2\ -2\ 2\ 2\ 0\ 0\ 0)$	
$3T5+3T8\ (1,2;1,1,2)\ 3T3+3T4\ (1,2;1,2,1)\ 3T0+3T1+3T2+3T6+3T7\ (1,1;4,1,1)\ 3T0\ (1,1;6,1,1)\ 6T0+3T1+3T2+3T5+3T6+3T7+3T8\ (1,1;1,1,2)\ 6T0+3T1+3T2+3T3+3T4+3T6+3T7\ (1,1;1,2,1)\ 3T3+3T4+3T5+3T8\ (1,2;1,1,1)\ 12T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8\ (1,1;1,1,1)$	

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$SU(5)$	$3(5^*)$
$SU(3) \times SU(2) \times SU(2)$	$3(3,1,1) + 3(1,1,2) + 3(3^*,1,2) + 3(1,2,1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (1\ -1\ 3\ 1\ 1\ 1\ 1\ 1)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0\ 0)$	
$3T0\ (1;1,2,2)\ 3T8\ (5;1,1,1)\ 3T0+3T1+3T2+3T5\ (1;3,1,1)\ 3T0+3T2+3T7\ (1;3^*,1,1)\ 3T1+6T2+3T5+9T6+3T7\ (1;1,1,2)\ 3T1+6T2+3T5+3T7\ (1;1,2,1)\ 24T0+6T1+12T2+27T3+27T4+6T5+9T6+6T7+12T8\ (1;1,1,1)$	

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$SU(5)$	$3(1) + 3(1)$
$SU(3) \times SU(2) \times SU(2)$	$3(1,1,1) + 3(3^*,2,1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (1\ 3\ 3\ 1\ 1\ 1\ 1\ 1)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 0\ 2\ 2\ 2\ 0\ 0)$	
$3T0\ (1;1,2,2)\ 3T4\ (5;1,1,1)\ 3T4\ (5^*;1,1,1)\ 3T0+3T1+3T2+3T5+3T6\ (1;3,1,1)\ 3T0+3T2+3T7\ (1;3^*,1,1)\ 3T1+6T2+3T5+3T6+3T7\ (1;1,1,2)\ 3T1+6T2+3T5+3T6+3T7+9T8\ (1;1,2,1)\ 24T0+6T1+12T2+27T3+24T4+6T5+6T6+6T7+9T8\ (1;1,1,1)$	

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$SU(3) \times SU(2) \times SU(2)$	$SU(5)$
	$3(1) + 3(5) + 3(5^*)$
	$3(1, 2, 2) + 3(3, 2, 1) + 3(3^*, 1, 2)$
	$v = (22000000) a_1 = (2 - 20000000) a_3 = (1 - 13111111)$
	$v = (40000000) a_1 = (02200000) a_3 = (00 - 222200)$
$3T0(1;1,2,2) \quad 3T0+3T1+3T5(1;3,1,1) \quad 3T0+3T2+3T8(1;3^*,1,1) \quad 3T1+3T2+3T5+9T7+3T8(1;1,1,2) \quad 3T1+3T2+3T5+9T6+3T8(1;1,2,1) \quad 24T0+6T1+6T2+27T3+27T4+6T5+9T6+9T7+6T8(1;1,1,1)$	

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$SU(3) \times SU(2) \times SU(2)$	$SU(5)$
	$3(1) + 3(10^*)$
	$3(1, 1, 1) + 3(3^*, 2, 1)$
	$v = (22000000) a_1 = (2 - 20000000) a_3 = (31311111)$
	$v = (40000000) a_1 = (42200000) a_3 = (00 - 222200)$
$3T0(1;1,2,2) \quad 3T4+3T8(5;1,1,1) \quad 3T4(5^*;1,1,1) \quad 3T0+3T1+3T5+3T6(1;3,1,1) \quad 3T0+3T2(1;3^*,1,1) \quad 3T1+3T2+3T5+3T6(1;1,1,2) \quad 3T1+3T2+3T5+3T6+9T7(1;1,2,1) \quad 24T0+6T1+6T2+27T3+24T4+6T5+6T6+9T7+12T8(1;1,1,1)$	

239

$SU(3) \times SU(2) \times SU(2)$	$SU(5)$
	$3(1) + 3(10^*)$
	$3(3^*, 1, 1) + 3(3, 1, 1) + 3(1, 1, 1)$
	$v = (22000000) a_1 = (2 - 20000000) a_3 = (-1 - 131111 - 1)$
	$v = (40000000) a_1 = (15111111) a_3 = (0222000 - 2)$
$3T3+3T4(5;1,1,1) \quad 3T3(5^*;1,1,1) \quad 3T0+3T6+3T8(1;3,1,1) \quad 3T0+3T1+3T2(1;3^*,1,1) \quad 6T0+3T1+3T2+3T6+3T8(1;1,1,2) \quad 6T0+3T1+3T2+3T6+3T8(1;1,2,1) \quad 12T0+6T1+6T2+24T3+12T4+27T5+6T6+27T7+6T8(1;1,1,1)$	

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$SU(3) \times SU(2) \times SU(2)$	$SU(5)$
	$3(1) + 3(10^*)$
	$3(3^*, 1, 1) + 3(1, 1, 1) + 3(3^*, 2, 2)$
	$v = (22000000) a_1 = (2 - 20000000) a_3 = (20222220)$
	$v = (40000000) a_1 = (22222200) a_3 = (02200022)$
$3T5(5;1,1,1) \quad 3T0+3T1+3T2+3T3+3T7+3T8(1;3,1,1) \quad 3T0(1;3^*,1,1) \quad 6T0+3T1+3T2+3T3+9T4+3T7+3T8(1;1,1,2) \quad 6T0+3T1+3T2+3T3+9T6+3T7+3T8(1;1,2,1) \quad 12T0+6T1+6T2+6T3+9T4+12T5+9T6+6T7+6T8(1;1,1,1)$	

241

$SU(3) \times SU(2) \times SU(2)$	$SU(5)$
	$3(1) + 3(1)$
	$3(1, 1, 1) + 3(3^*, 1, 1) + 3(3^*, 2, 2)$
	$v = (42200000) a_1 = (42020000) a_3 = (202 - 22000)$
	$v = (42200000) a_1 = (00022000) a_3 = (4000 - 2200)$
$3T1+3T2+3T3+3T4+3T5+3T7+3T8(1;3,1,1) \quad 3T2+3T8(1;3^*,1,1) \quad 3T1+6T2+3T3+3T4+3T5+9T6+3T7+6T8(1;1,1,2) \quad 9T0+3T1+6T2+3T3+3T4+3T5+3T7+6T8(1;1,2,1) \quad 9T0+6T1+12T2+6T3+6T4+6T5+9T6+6T7+12T8(1;1,1,1)$	

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$SU(4) \times SU(2) \times SU(2)$	$SU(4) \times SU(2)$
	$3(1, 2, 2)$
	$3(1, 2) + 3(4^*, 2) + 3(4, 1)$
	$v = (22000000) a_1 = (40000000) a_3 = (00222200)$
	$v = (40000000) a_1 = (02200000) a_3 = (42 - 222000)$
$3T7+3T8(1,1,2;1,2) \quad 3T5(4,1,1;1,1) \quad 3T6(4^*,1,1;1,1) \quad 3T1+3T2(1,1,1;4,1) \quad 3T0(1,1,1;6,1) \quad 3T2(1,1,1;4^*,1) \quad 6T0+3T1+6T2+3T7+3T8(1,1,1;1,2) \quad 3T5+3T6+3T7+3T8(1,1,2;1,1) \quad 9T3+9T4+3T5+3T6(1,2,1;1,1) \quad 24T0+9T1+18T2+9T3+9T4+3T5+3T6+3T7+3T8(1,1,1;1,1)$	

243

$SU(4) \times SU(2) \times SU(2)$	$SU(4) \times SU(2)$
	$3(1, 2, 1) + 3(4^*, 1, 2)$
	$3(4, 2)$
	$v = (22000000) a_1 = (2 - 20000000) a_3 = (22222200)$
	$v = (40000000) a_1 = (42200000) a_3 = (02 - 222000)$
$3T6+3T8(1,1,2;1,2) \quad 3T5+3T7(4,1,1;1,1) \quad 3T4(6,1,1;1,1) \quad 3T0(1,1,1;6,1) \quad 3T1+3T2(1,1,1;4^*,1) \quad 6T0+3T1+3T2+3T6+3T8(1,1,1;1,2) \quad 6T4+3T5+3T6+3T7+3T8(1,1,2;1,1) \quad 9T3+6T4+3T5+3T7(1,2,1;1,1) \quad 9T0+9T1+9T2+9T3+12T4+3T5+3T6+3T7+3T8(1,1,1;1,1)$	

244

$SU(4) \times SU(2) \times SU(2)$	$SU(4) \times SU(2)$
	$3(1, 2, 1) + 3(4^*, 1, 2)$
	$3(1, 2) + 3(6, 1)$
	$v = (22000000) a_1 = (2 - 20000000) a_3 = (22222200)$
	$v = (40000000) a_1 = (42200000) a_3 = (00022220)$
$3T3(1,2,1;1,2) \quad 3T5+3T7(4,1,1;1,1) \quad 3T4(6,1,1;1,1) \quad 3T0+3T1(1,1,1;4,1) \quad 3T0+3T2(1,1,1;4^*,1) \quad 6T0+3T1+3T2+3T3(1,1,1;1,2) \quad 6T4+3T5+9T6+3T7+9T8(1,1,2;1,1) \quad 3T3+6T4+3T5+3T7(1,2,1;1,1) \quad 18T0+9T1+9T2+3T3+12T4+3T5+9T6+3T7+9T8(1,1,1;1,1)$	

245

$SU(4) \times SU(2) \times SU(2)$	$SU(4) \times SU(2)$
	$3(6, 1, 1) + 3(4^*, 2, 2)$
	$3(1, 2) + 3(4^*, 2) + 3(4, 1)$
	$v = (42200000) a_1 = (20020000) a_3 = (20022200)$
	$v = (42200000) a_1 = (20 - 222000) a_3 = (42 - 2 - 2 - 2000)$
$3T4(1,1,2;1,2) \quad 3T1(1,2,1;1,2) \quad 3T3+3T6+3T7+3T8(4,1,1;1,1) \quad 3T2(1,1,1;4,1) \quad 3T1+3T2+3T4(1,1,1;1,2) \quad 3T3+3T4+9T5+3T6+3T7+3T8(1,1,2;1,1) \quad 9T0+3T1+3T3+3T6+3T7+3T8(1,2,1;1,1) \quad 9T0+3T1+9T2+3T3+3T4+9T5+3T6+3T7+3T8(1,1,1;1,1)$	

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$SU(4) \times SU(2) \times SU(2)$	$SU(4) \times SU(2)$
	$3(6, 1, 1) + 3(4^*, 2, 2)$
	$3(1, 1) + 3(1, 2) + 3(1, 2)$
	$v = (42200000) a_1 = (20020000) a_3 = (-200 - 22200)$
	$v = (42200000) a_1 = (42022200) a_3 = (-22 - 200020)$
$3T0+3T1(1,2,1;1,2) \quad 3T4+3T5+3T7+3T8(4,1,1;1,1) \quad 3T2(6,1,1;1,1) \quad 3T0+3T1(1,1,1;1,2) \quad 6T2+9T3+3T4+3T5+9T6+3T7+3T8(1,1,2;1,1) \quad 3T0+3T1+6T2+3T4+3T5+3T7+3T8(1,2,1;1,1) \quad 3T0+3T1+12T2+9T3+3T4+3T5+9T6+3T7+3T8(1,1,1;1,1)$	

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$SU(4) \times SU(3)$ $SU(2) \times SU(2) \times SU(2)$	$3(4^*, 3^*)$ $3(1, 1, 1) + 3(1, 2, 1) + 3(2, 1, 2) + 3(1, 2, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ -2\ 2\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 0\ -2\ 2\ 2\ 0\ 0)$
	$3T0(1, 1, 1; 1, 2, 2) \quad 3T4+3T5+3T7(4, 1; 1, 1, 1) \quad 9T3+3T5+3T7(1, 3; 1, 1, 1) \quad 3T4(1, 3^*; 1, 1, 1) \quad 3T1+3T2+3T6+3T8(1, 1; 1, 1, 2) \quad 3T1+3T2+3T6+3T8(1, 1; 1, 2, 1) \quad 6T0+3T1+3T2+3T6+3T8(1, 1; 2, 1, 1) \quad 30T0+9T1+9T2+6T4+6T5+9T6+6T7+9T8(1, 1; 1, 1, 1)$

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$SU(4) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(4^*, 1, 1)$ $3(1, 1) + 3(3^*, 2) + 3(1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ -1\ 3\ 3\ 1\ 1\ 1\ 1)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ 2\ 0\ 0\ 2\ 2\ 2\ 0)$
	$3T3(1, 2, 1; 1, 2) \quad 3T4+3T8(4, 1, 1; 1, 1) \quad 3T5(4^*, 1, 1; 1, 1) \quad 3T0+3T1+3T2+3T6(1, 1, 1; 3, 1) \quad 3T0+3T2(1, 1, 1; 3^*, 1) \quad 6T0+3T1+6T2+3T3+3T6(1, 1, 1; 1, 2) \quad 3T4+3T5+9T7+3T8(1, 1, 2; 1, 1) \quad 3T3+3T4+3T5+3T8(1, 2, 1; 1, 1) \quad 24T0+12T1+24T2+3T3+3T4+3T5+12T6+9T7+3T8(1, 1, 1; 1, 1)$

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$SU(4) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(1, 1, 1) + 3(6, 2, 1)$ $3(1, 1) + 3(1, 1) + 3(3^*, 2)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 2\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (3\ 3\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (0\ 2\ 2\ 0\ 0\ -2\ -2)$
	$3T7(1, 2, 1; 1, 2) \quad 3T3+3T5(4, 1, 1; 1, 1) \quad 3T4+3T8(4^*, 1, 1; 1, 1) \quad 3T0+3T1+3T2(1, 1, 1; 3, 1) \quad 3T0(1, 1, 1; 3^*, 1) \quad 6T0+3T1+3T2+3T7(1, 1, 1; 1, 2) \quad 3T3+3T4+3T5+3T8(1, 1, 2; 1, 1) \quad 3T3+3T4+3T5+9T6+3T7+3T8(1, 2, 1; 1, 1) \quad 24T0+12T1+12T2+3T3+3T4+3T5+9T6+3T7+3T8(1, 1, 1; 1, 1)$

250

$SU(4) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(4^*, 2, 1) + 3(1, 1, 2)$ $3(1, 2) + 3(3, 1) + 3(3^*, 2)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 4\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ -2\ 2\ 2\ 2\ 0)$
	$3T3(1, 2, 1; 1, 2) \quad 3T4+3T6(4, 1, 1; 1, 1) \quad 3T0+3T1+3T5(1, 1, 1; 3, 1) \quad 3T0+3T2(1, 1, 1; 3^*, 1) \quad 6T0+3T1+3T2+3T3+3T5(1, 1, 1; 1, 2) \quad 3T4+3T6+9T8(1, 1, 2; 1, 1) \quad 3T3+3T4+3T6+9T7(1, 2, 1; 1, 1) \quad 24T0+12T1+12T2+3T3+3T4+12T5+3T6+9T7+9T8(1, 1, 1; 1, 1)$

251

$SU(4) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(4, 1, 2) + 3(1, 2, 1)$ $3(3^*, 1) + 3(1, 2) + 3(3^*, 1) + 3(3^*, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 2\ 2\ 2\ 2\ 0\ 0) a_3 = (4\ 0\ 2\ 2\ 2\ 0\ 2\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 3\ 1\ 1\ 1\ 1\ -1) a_3 = (1\ 1\ 3\ 1\ 1\ -1\ -1)$
	$3T1+3T3(4^*, 1, 1; 1, 1) \quad 3T0+3T5+3T6+3T7(1, 1, 1; 3, 1) \quad 3T0(1, 1, 1; 3^*, 1) \quad 6T0+3T5+3T6+3T7(1, 1, 1; 1, 2) \quad 3T1+9T2+3T3+9T4(1, 1, 2; 1, 1) \quad 3T1+3T3+9T8(1, 2, 1; 1, 1) \quad 24T0+3T1+9T2+3T3+9T4+12T5+12T6+12T7+9T8(1, 1, 1; 1, 1)$

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$SU(4) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(6, 1, 1) + 3(4^*, 2, 2)$ $3(1, 1) + 3(3, 1) + 3(1, 2) + 3(1, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 0\ -2\ 2\ 0\ 0)$
	$3T7(1, 1, 2; 1, 2) \quad 3T3+3T4+3T5+3T6(4, 1, 1; 1, 1) \quad 3T2(1, 1, 1; 3^*, 1) \quad 3T2+3T7(1, 1, 1; 1, 2) \quad 3T3+3T4+3T5+3T6+3T7+9T8(1, 1, 2; 1, 1) \quad 9T0+9T1+3T3+3T4+3T5+3T6(1, 2, 1; 1, 1) \quad 9T0+9T1+12T2+3T3+3T4+3T5+3T6+3T7+9T8(1, 1, 1; 1, 1)$

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$SU(4) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(4^*, 2, 1) + 3(1, 1, 2)$ $3(1, 1, 1, 1) + 3(1, 1, 2, 1) + 3(2, 2, 1, 2)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 4\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 0\ 2\ 2\ 0\ 0)$
	$3T0(1, 1, 1; 1, 2, 2) \quad 3T8(1, 1, 2; 1, 1, 1, 2) \quad 3T7(1, 2, 1; 1, 2, 1, 1) \quad 3T3(1, 2, 1; 2, 1, 1, 1) \quad 3T4+3T6(4, 1, 1; 1, 1, 1) \quad 3T1+3T2+3T5+3T8(1, 1, 1; 1, 1, 1, 2) \quad 3T1+3T2+3T5(1, 1, 1; 1, 1, 2, 1) \quad 6T0+3T1+3T2+3T5+3T7(1, 1, 1; 1, 2, 1, 1) \quad 6T0+3T1+3T2+3T3+3T5(1, 1, 1; 2, 1, 1, 1) \quad 3T4+3T6+3T8(1, 1, 2; 1, 1, 1, 1) \quad 3T3+3T4+3T6+3T7(1, 2, 1; 1, 1, 1, 1) \quad 18T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8(1, 1, 1; 1, 1, 1, 1)$

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$SU(4) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(6, 1, 1) + 3(4^*, 2, 2)$ $3(2, 1, 2, 1) + 3(1, 2, 1, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ -2\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ -2\ 2\ 0\ 0\ 2\ 2\ 0)$
	$3T4(1, 1, 2; 1, 2, 1, 2) \quad 3T5(1, 1, 2; 2, 1, 1, 1) \quad 3T0(1, 2, 1; 1, 1, 1, 2) \quad 3T1(1, 2, 1; 1, 2, 1, 1) \quad 3T3+3T6+3T7+3T8(4, 1, 1; 1, 1, 1) \quad 3T0+3T2(1, 1, 1; 1, 1, 2) \quad 3T2+3T4(1, 1, 1; 1, 2, 1) \quad 3T1+3T2(1, 1, 1; 1, 2, 1, 1) \quad 3T2+3T5(1, 1, 1; 2, 1, 1, 1) \quad 3T3+3T4+3T5+3T6+3T7+3T8(1, 1, 2; 1, 1, 1, 1) \quad 3T0+3T1+3T3+3T6+3T7+3T8(1, 2, 1; 1, 1, 1, 1) \quad 3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8(1, 1, 1; 1, 1, 1, 1)$

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$SU(4) \times SU(3)$ $SU(3) \times SU(3)$	$3(6, 1)$ $3(1, 1) + 3(1, 1) + 3(3^*, 3^*) + 3(1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (3\ 1\ 1\ 1\ -1\ -1\ -1\ -3)$
	$3T3+3T8(4, 1; 1, 1) \quad 3T4+3T7(4^*, 1; 1, 1) \quad 3T4+3T8(1, 3; 1, 1) \quad 3T3+3T7(1, 3^*; 1, 1) \quad 3T0+3T1+3T2+3T5+3T6(1, 1; 3, 1) \quad 3T0+3T2(1, 1; 3^*, 1) \quad 3T0+3T1+3T2+3T5+3T6(1, 1; 1, 3) \quad 3T0+3T2(1, 1; 1, 3^*) \quad 18T0+9T1+18T2+6T3+6T4+9T5+9T6+6T7+6T8(1, 1; 1, 1)$

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$SU(4) \times SU(3)$	$3(1, 3)$
$SU(3) \times SU(3)$	$3(3^*, 1) + 3(3, 3^*) + 3(1, 3)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (2\ 4\ 2\ 2\ 2\ 0\ 0\ 0)$	
$3T3(4, 1; 1, 1) \quad 3T7(4^*, 1; 1, 1) \quad 3T3+3T7(1, 3; 1, 1) \quad 9T6(1, 3^*; 1, 1) \quad 3T0+3T1+3T2(1, 1; 3, 1) \quad 3T0+3T2+3T4+3T5+3T8(1, 1; 3^*, 1) \quad 3T0+3T1+3T2+3T4+3T8(1, 1; 1, 3) \quad 3T0+3T2+3T5(1, 1; 1, 3^*) \quad 18T0+9T1+18T2+6T3+9T4+9T5+6T7+9T8(1, 1; 1, 1)$	

257

$SU(4) \times SU(3)$	$3(4^*, 3^*)$
$SU(3) \times SU(3)$	$3(3, 1) + 3(1, 3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (5\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (0\ 2\ 0\ 0\ 0\ -2\ -2\ -2)$	
$3T3+3T4+3T7+3T8(4, 1; 1, 1) \quad 3T4(4^*, 1; 1, 1) \quad 3T3+3T4+9T5+3T7(1, 3; 1, 1) \quad 3T4+3T8(1, 3^*; 1, 1) \quad 3T0+3T6(1, 1; 3, 1) \quad 3T0+3T1+3T2(1, 1; 3^*, 1) \quad 3T0+3T2+3T6(1, 1; 1, 3) \quad 3T0+3T1(1, 1; 1, 3^*) \quad 18T0+9T1+9T2+6T3+12T4+9T6+6T7+6T8(1, 1; 1, 1)$	

258

$SU(4) \times SU(3)$	$3(1, 1) + 3(4, 1) + 3(4^*, 1)$
$SU(3) \times SU(3)$	$3(3^*, 3^*)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 4\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (3\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (3\ 3\ 1\ 1\ 1\ -1\ -1\ -1)$	
$3T3(4, 1; 1, 1) \quad 3T2(6, 1; 1, 1) \quad 3T4(4^*, 1; 1, 1) \quad 3T2+3T4(1, 3; 1, 1) \quad 3T2+3T3(1, 3^*; 1, 1) \quad 3T0+3T1+3T5+3T6+3T7(1, 1; 3, 1) \quad 3T0+3T8(1, 1; 3^*, 1) \quad 3T0+3T1+3T5+3T6+3T8(1, 1; 1, 3) \quad 3T0+3T7(1, 1; 1, 3^*) \quad 18T0+9T1+18T2+6T3+6T4+9T5+9T6+9T7+9T8(1, 1; 1, 1)$	

259

$SU(4) \times SU(3)$	$3(4, 3)$
$SU(3) \times SU(3)$	$3(3^*, 1) + 3(3, 3^*) + 3(1, 3)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ -2\ 2\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (0\ 2\ 0\ 0\ 0\ -2\ -2\ -2)$	
$3T4+3T5+3T6(4^*, 1; 1, 1) \quad 3T5(1, 3; 1, 1) \quad 3T4+3T6+9T8(1, 3^*; 1, 1) \quad 3T0+3T3(1, 1; 3, 1) \quad 3T0+3T1+3T2+3T7(1, 1; 3^*, 1) \quad 3T0+3T2+3T3+3T7(1, 1; 1, 3) \quad 3T0+3T1(1, 1; 1, 3^*) \quad 18T0+9T1+9T2+9T3+6T4+6T5+6T6+9T7(1, 1; 1, 1)$	

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$SU(4) \times SU(3)$	$3(1, 3) + 3(6, 3)$
$SU(3) \times SU(3)$	$3(3^*, 1) + 3(3, 3) + 3(1, 3^*)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (0\ 0\ 0\ 0\ 4\ 0\ 0\ 0)$	
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 0\ 0\ 2\ 2\ -2)$	
$3T5+3T8(4, 1; 1, 1) \quad 3T6+3T7(4^*, 1; 1, 1) \quad 9T0+3T5+3T6+3T7+3T8(1, 3^*; 1, 1) \quad 3T4(1, 1; 3, 1) \quad 3T1+3T2+3T3(1, 1; 3^*, 1) \quad 3T3(1, 1; 1, 3) \quad 3T1+3T2+3T4(1, 1; 1, 3^*) \quad 9T1+9T2+9T3+9T4+6T5+6T6+6T7+6T8(1, 1; 1, 1)$	

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$SU(5)$	$3(1) + 3(10^*)$
$SU(2) \times SU(2)$	$3(2, 1) + 3(1, 1) + 3(2, 2) + 3(1, 1) + 3(1, 2)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ -2\ 0\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 2\ 2\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 2\ 2\ 2\ 2\ 2\ 0\ 0) a_3 = (0\ 2\ 0\ 0\ -2\ -2\ 2\ 0)$	
$3T5(5; 1, 1) \quad 6T0+3T1+3T2+3T3+3T7+3T8(1; 1, 2) \quad 6T0+3T1+3T2+3T3+3T7+3T8(1; 2, 1) \quad 30T0+15T1+15T2+15T3+27T4+12T5+27T6+15T7+15T8(1; 1, 1)$	

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$SU(5)$	$3(5) + 3(5) + 3(10^*)$
$SU(2) \times SU(2)$	$3(1, 2) + 3(1, 1) + 3(1, 2) + 3(2, 1) + 3(2, 1) + 3(1, 1)$
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 0\ 0\ 2\ 0\ 0\ 0)$	
$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 2\ 2\ 0\ 0) a_3 = (2\ 0\ -2\ 0\ 0\ -2\ 2\ 0)$	
$3T7(5^*; 1, 1) \quad 3T2+3T3+3T5+3T8(1; 1, 2) \quad 3T2+3T3+3T5+3T8(1; 2, 1) \quad 27T0+27T1+15T2+15T3+27T4+15T5+27T6+12T7+15T8(1; 1, 1)$	

263

$SU(4) \times SU(2)$	$3(1, 2) + 3(6, 1)$
$SU(3) \times SU(2) \times SU(2)$	$3(3^*, 1, 1) + 3(3, 2, 1) + 3(1, 1, 2) + 3(1, 2, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0) a_3 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (1\ 1\ -1\ 3\ 3\ 1\ 1\ -1)$	
$3T0(1, 1; 1, 2, 2) \quad 3T3(1, 2; 1, 2, 1) \quad 3T8(4, 1; 1, 1) \quad 3T6(4^*, 1; 1, 1, 1) \quad 3T0+3T4(1, 1; 3, 1, 1) \quad 3T0+3T5+3T7(1, 1; 3^*, 1, 1) \quad 3T4+3T5+3T7(1, 1; 1, 1, 2) \quad 3T3+3T4+3T5+3T7(1, 1; 1, 2, 1) \quad 3T3+3T6+3T8(1, 2; 1, 1, 1) \quad 24T0+27T1+27T2+3T3+6T4+6T5+9T6+6T7+9T8(1, 1; 1, 1, 1)$	

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$SU(4) \times SU(2)$	$3(1, 1) + 3(1, 2) + 3(1, 2)$
$SU(3) \times SU(2) \times SU(2)$	$3(1, 2, 2) + 3(3, 1, 2) + 3(3^*, 2, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 0\ 2\ 2\ 0\ 0\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 1\ -1\ 3\ 3\ 1\ 1\ 1)$	
$3T0(1, 1; 1, 2, 2) \quad 3T4(1, 2; 1, 1, 2) \quad 3T3(1, 2; 1, 2, 1) \quad 3T0+3T6+3T8(1, 1; 3, 1, 1) \quad 3T0+3T5+3T7(1, 1; 3^*, 1, 1) \quad 3T4+3T5+3T6+3T7+3T8(1, 1; 1, 1, 2) \quad 3T3+3T5+3T6+3T7+3T8(1, 1; 1, 2, 1) \quad 3T3+3T4(1, 2; 1, 1, 1) \quad 24T0+27T1+27T2+3T3+3T4+6T5+6T6+6T7+6T8(1, 1; 1, 1, 1)$	

265

$SU(4) \times SU(2)$	$3(4, 2)$
$SU(3) \times SU(2) \times SU(2)$	$3(3^*, 1, 1) + 3(3, 2, 1) + 3(1, 1, 2) + 3(1, 2, 1)$
$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 2\ 2\ 0)$	
$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (1\ 1\ -1\ 3\ 3\ 1\ 1\ -1)$	
$3T0(1, 1; 1, 2, 2) \quad 3T6+3T8(4^*, 1; 1, 1, 1) \quad 3T0+3T4(1, 1; 3, 1, 1) \quad 3T0+3T5+3T7(1, 1; 3^*, 1, 1) \quad 3T4+3T5+3T7(1, 1; 1, 1, 2) \quad 9T3+3T4+3T5+3T7(1, 1; 1, 2, 1) \quad 9T1+9T2+3T6+3T8(1, 2; 1, 1, 1) \quad 24T0+9T1+9T2+9T3+6T4+6T5+9T6+6T7+9T8(1, 1; 1, 1, 1)$	

266	
$SU(4) \times SU(2)$	$3(4, 2)$
$SU(3) \times SU(2) \times SU(2)$	$3(1, 1, 2) + 3(1, 2, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ 0\ 2\ 2\ 2\ 2\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 1\ -1\ 3\ 3\ 1\ 1\ 1)$
$3T0(1, 1; 1, 2, 2) \quad 3T2(1, 2; 1, 1, 2) \quad 3T1(1, 2; 1, 2, 1) \quad 3T4(6, 1; 1, 1, 1) \quad 3T6+3T8(4^*, 1; 1, 1, 1) \quad 3T0+3T7(1, 1; 3, 1, 1)$ $3T0+3T5(1, 1; 3^*, 1, 1) \quad 3T2+3T5+3T7(1, 1; 1, 1, 2) \quad 3T1+3T5+3T7(1, 1; 1, 2, 1) \quad 3T1+3T2+6T4+3T6+3T8(1, 2; 1, 1, 1)$ $24T0+3T1+3T2+27T3+24T4+6T5+9T6+6T7+9T8(1, 1; 1, 1, 1)$	

267	
$SU(4) \times SU(2)$	$3(1, 1) + 3(4, 1)$
$SU(3) \times SU(2) \times SU(2)$	$3(1, 1, 1) + 3(3^*, 2, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (4\ -2\ 0\ 0\ 2\ 0\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ -2\ 2\ 2\ 0\ 0\ 0)$
$3T0(1, 1; 1, 2, 2) \quad 3T2(4, 1; 1, 1, 1) \quad 3T2+3T6(4^*, 1; 1, 1, 1) \quad 3T0+3T3+3T5+3T7(1, 1; 3, 1, 1) \quad 3T0+3T8(1, 1; 3^*, 1, 1)$ $3T3+3T5+3T7+3T8(1, 1; 1, 1, 2) \quad 3T3+9T4+3T5+3T7+3T8(1, 1; 1, 2, 1) \quad 9T1+6T2+3T6(1, 2; 1, 1, 1)$ $24T0+9T1+18T2+6T3+9T4+6T5+9T6+6T7+6T8(1, 1; 1, 1, 1)$	

268	
$SU(4) \times SU(2)$	$3(1, 1) + 3(4, 1)$
$SU(3) \times SU(2) \times SU(2)$	$3(3^*, 1, 1) + 3(1, 1, 1) + 3(3^*, 2, 2)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 2\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ -2\ 2\ 0\ 2\ 2\ 0\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (1\ 1\ 1\ 1\ 1\ 1\ 1) a_3 = (1\ 3\ 3\ 1\ 1\ 1\ -1\ -1)$
$3T8(1, 2; 1, 1, 2) \quad 3T4(4^*, 1; 1, 1, 1) \quad 3T0+3T2+3T3+3T5+3T6+3T7(1, 1; 3, 1, 1) \quad 3T0(1, 1; 3^*, 1, 1)$ $6T0+3T2+3T3+3T5+3T6+3T7+3T8(1, 1; 1, 1, 2) \quad 6T0+9T1+3T2+3T3+3T5+3T6+3T7(1, 1; 1, 2, 1) \quad 3T4+3T8(1, 2; 1, 1, 1)$ $12T0+9T1+6T2+6T3+9T4+6T5+6T6+6T7+3T8(1, 1; 1, 1, 1)$	

269	
$SU(4) \times SU(2)$	$3(4, 1) + 3(4^*, 2) + 3(1, 2)$
$SU(3) \times SU(2) \times SU(2)$	$3(3, 1, 2) + 3(3^*, 2, 1) + 3(1, 2, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 0\ -2\ 2\ 2\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 0\ 2\ 2\ 0\ 0)$
$3T7(1, 2; 1, 1, 2) \quad 3T4(4, 1; 1, 1, 1) \quad 3T2+3T8(1, 1; 3, 1, 1) \quad 3T5+3T6(1, 1; 3^*, 1, 1) \quad 3T2+3T5+3T6+3T7+3T8(1, 1; 1, 1, 2)$ $9T1+3T2+3T5+3T6+3T8(1, 1; 1, 2, 1) \quad 9T3+3T4+3T7(1, 2; 1, 1, 1) \quad 27T0+9T1+6T2+9T3+9T4+6T5+6T6+3T7+6T8(1, 1; 1, 1, 1)$	

270	
$SU(4) \times SU(2)$	$3(4^*, 1) + 3(6, 1) + 3(4^*, 1)$
$SU(3) \times SU(2) \times SU(2)$	$3(1, 1, 1) + 3(3^*, 2, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 0\ 2\ 2\ 2\ -2)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ 0\ 2\ 2\ 0\ 0)$
$3T3+3T4+3T6(4, 1; 1, 1, 1) \quad 3T4(4^*, 1; 1, 1, 1) \quad 3T2+3T7+3T8(1, 1; 3, 1, 1) \quad 3T5(1, 1; 3^*, 1, 1) \quad 3T2+3T5+3T7+3T8(1, 1; 1, 1, 2)$ $9T1+3T2+3T5+3T7+3T8(1, 1; 1, 2, 1) \quad 3T3+6T4+3T6(1, 2; 1, 1, 1) \quad 27T0+9T1+6T2+9T3+18T4+6T5+9T6+6T7+6T8(1, 1; 1, 1, 1)$	

271	
$SU(4) \times SU(2)$	$3(1, 1) + 3(1, 1) + 3(6, 2)$
$SU(3) \times SU(2) \times SU(2)$	$3(1, 1, 1) + 3(3, 2, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 0\ 2\ 4\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ -2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 0\ 2\ 2\ 2\ 0\ 0)$
$3T1(1, 2; 1, 2, 1) \quad 3T3+3T4(4, 1; 1, 1, 1) \quad 3T3+3T5(4^*, 1; 1, 1, 1) \quad 3T6(1, 1; 3, 1, 1) \quad 3T2+3T7+3T8(1, 1; 3^*, 1, 1)$ $3T2+3T6+3T7+3T8(1, 1; 1, 1, 2) \quad 3T1+3T2+3T6+3T7+3T8(1, 1; 1, 2, 1) \quad 9T0+3T1+6T3+3T4+3T5(1, 2; 1, 1, 1)$ $9T0+3T1+6T2+18T3+9T4+9T5+6T6+6T7+6T8(1, 1; 1, 1, 1)$	

272	
$SU(4)$	$3(4) + 3(1) + 3(1)$
$SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(3^*, 1, 1, 1) + 3(3^*, 2, 1, 2)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ 2\ 2\ 2\ 2\ 2\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ -2\ 2\ 2\ 2\ 0\ 0)$
$3T0(1; 1, 1, 2, 2) \quad 3T8(4^*; 1, 1, 1, 1) \quad 3T0+3T3+9T5+3T7(1; 3, 1, 1, 1) \quad 3T0(1; 3^*, 1, 1, 1) \quad 3T3+9T4+9T6+3T7(1; 1, 1, 1, 2)$ $3T3+3T7(1; 1, 1, 1, 2) \quad 6T0+9T1+9T2+3T3+3T7(1; 1, 2, 1, 1) \quad 12T0+9T1+9T2+9T4+9T6+15T8(1; 1, 1, 1, 1)$	

273	
$SU(4)$	$3(1) + 3(1) + 3(4)$
$SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(3, 1, 1, 2)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 1\ 1\ 3\ 1\ 1\ 1\ 1)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ -1\ -1\ 3\ 3\ 1\ 1\ -1)$
$3T0(1; 1, 1, 2, 2) \quad 3T4(4; 1, 1, 1, 1) \quad 3T4+3T7(4^*; 1, 1, 1, 1) \quad 3T0+3T8(1; 3, 1, 1, 1) \quad 3T0+9T6(1; 3^*, 1, 1, 1) \quad 9T2+9T5+3T8(1; 1, 1, 1, 2)$ $9T1+3T8(1; 1, 1, 2, 1) \quad 6T0+9T3+3T8(1; 1, 2, 1, 1) \quad 12T0+9T1+9T2+9T3+30T4+9T5+15T7(1; 1, 1, 1, 1)$	

274	
$SU(4)$	$3(6) + 3(1) + 3(4) + 3(4)$
$SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(3^*, 1, 1, 1) + 3(3^*, 2, 2, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ 0\ -2\ -2\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ -2\ -2\ 0\ 2\ 2\ 0\ 0)$
$3T4+3T6(4^*; 1, 1, 1, 1) \quad 3T2+3T7+9T8(1; 3, 1, 1, 1) \quad 3T2+3T7(1; 1, 1, 1, 2) \quad 9T1+3T2+9T5+3T7(1; 1, 1, 2, 1) \quad 9T0+3T2+9T3+3T7(1; 1, 2, 1, 1)$ $9T0+9T1+9T3+15T4+9T5+15T6(1; 1, 1, 1, 1)$	

275	
$SU(4)$	$3(1) + 3(6) + 3(4^*) + 3(4^*)$
$SU(3) \times SU(2) \times SU(2) \times SU(2)$	$3(3, 1, 2, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 0\ 4\ 2\ 0\ 0\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 2\ 0\ 2\ 0\ 0\ 0\ 0) a_3 = (2\ -2\ 0\ 0\ 2\ 2\ 0\ 0)$
$3T3+3T4+3T7(4; 1, 1, 1, 1) \quad 3T4(4^*; 1, 1, 1, 1) \quad 3T2(1; 3, 1, 1, 1) \quad 9T5(1; 3^*, 1, 1, 1) \quad 9T0+3T2(1; 1, 1, 1, 2) \quad 3T2+9T6+9T8(1; 1, 1, 1, 2)$ $9T1+3T2(1; 1, 2, 1, 1) \quad 9T0+9T1+15T3+30T4+9T6+15T7+9T8(1; 1, 1, 1, 1)$	

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$SU(3) \times SU(2) \times SU(2) \times SU(2)$ $SU(3)$	$3(3, 1, 1, 2)$ $3(1) + 3(1) + 3(3^*) + 3(3) + 3(1)$
$v = (22000000) a_1 = (00400000) a_3 = (1133311-1)$ $v = (40000000) a_1 = (22220000) a_3 = (331-13111)$	
$3T4(3, 1, 1, 1; 1) \quad 9T3(3^*, 1, 1, 1; 1) \quad 3T0+3T7(1, 1, 1, 1; 3) \quad 3T0+3T5(1, 1, 1, 1; 3^*) \quad 9T1+9T2+3T4(1, 1, 1, 2; 1) \quad 3T4+9T6(1, 1, 2, 1; 1) \quad 3T4+9T8(1, 2, 1, 1; 1) \quad 36T0+9T1+9T2+18T5+9T6+18T7+9T8(1, 1, 1, 1; 1)$	

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$SU(3) \times SU(2) \times SU(2) \times SU(2)$ $SU(3)$	$3(3^*, 1, 1, 1) + 3(3^*, 2, 1, 2)$ $3(1) + 3(1) + 3(1) + 3(3) + 3(3^*)$
$v = (42200000) a_1 = (20020000) a_3 = (00042200)$ $v = (42200000) a_1 = (40-220000) a_3 = (-20002220)$	
$9T4+3T6+3T8(3, 1, 1, 1; 1) \quad 3T5(1, 1, 1, 1; 3) \quad 3T2(1, 1, 1, 1; 3^*) \quad 9T3+3T6+9T7+3T8(1, 1, 1, 2; 1) \quad 3T6+3T8(1, 1, 2, 1; 1) \quad 9T0+9T1+3T6+3T8(1, 2, 1, 1; 1) \quad 9T0+9T1+18T2+9T3+18T5+9T7(1, 1, 1, 1; 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(1, 1, 1) + 3(3, 1, 1) + 3(3^*, 1, 1)$ $3(3, 2) + 3(1, 1) + 3(1, 1)$
$v = (22000000) a_1 = (00400000) a_3 = (2-2222200)$ $v = (40000000) a_1 = (33111111) a_3 = (31331-1-1-1)$	
$3T3+3T7(3, 1, 1; 1) \quad 3T4+3T6(3^*, 1, 1; 1) \quad 3T0(1, 1, 1; 3, 1) \quad 3T0+3T5+3T8(1, 1, 1; 3^*, 1) \quad 6T0+9T1+3T5+3T8(1, 1, 1; 1, 2) \quad 3T3+3T4+3T6+3T7(1, 1, 2; 1, 1) \quad 3T3+3T4+3T6+3T7(1, 2, 1; 1, 1) \quad 24T0+9T1+27T2+6T3+6T4+12T5+6T6+6T7+12T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(1, 1, 1) + 3(3^*, 1, 2)$ $3(3^*, 1) + 3(1, 1) + 3(1, 1) + 3(3, 1)$
$v = (22000000) a_1 = (40220000) a_3 = (00402200)$ $v = (40000000) a_1 = (04220000) a_3 = (13113111)$	
$3T4+3T6+3T8(3, 1, 1; 1) \quad 3T3(3^*, 1, 1; 1) \quad 3T0+3T2(1, 1, 1; 3, 1) \quad 3T0+3T5(1, 1, 1; 3^*, 1) \quad 6T0+3T2+3T5(1, 1, 1; 1, 2) \quad 3T3+3T4+3T6+9T7+3T8(1, 1, 2; 1, 1) \quad 3T3+3T4+3T6+3T8(1, 2, 1; 1, 1) \quad 24T0+27T1+12T2+6T3+6T4+12T5+6T6+9T7+6T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(1, 1, 1) + 3(3, 1, 2)$ $3(1, 1) + 3(1, 1) + 3(3^*, 2)$
$v = (22000000) a_1 = (40220000) a_3 = (402-22200)$ $v = (40000000) a_1 = (-22220000) a_3 = (13113111)$	
$3T4(1, 1, 2; 1, 2) \quad 3T6(3, 1, 1; 1) \quad 3T3+3T5+3T7(3^*, 1, 1; 1) \quad 3T0+3T2+3T8(1, 1, 1; 3, 1) \quad 3T0(1, 1, 1; 3^*, 1) \quad 6T0+3T2+3T4+3T8(1, 1, 1; 1, 2) \quad 3T3+3T4+3T5+3T6+3T7(1, 1, 2; 1, 1) \quad 3T3+3T5+3T6+3T7(1, 2, 1; 1, 1) \quad 24T0+27T1+12T2+6T3+3T4+6T5+6T6+6T7+12T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(1, 1, 2) + 3(1, 2, 1)$ $3(1, 2) + 3(3, 2) + 3(3^*, 1)$
$v = (22000000) a_1 = (40220000) a_3 = (400-22220)$ $v = (40000000) a_1 = (1311111-1) a_3 = (13311-1-11)$	
$3T5(1, 2, 1; 2) \quad 3T3(3, 1, 1; 1) \quad 3T8(3^*, 1, 1; 1) \quad 3T0+3T7(1, 1, 1; 3, 1) \quad 3T0+3T2+3T6(1, 1, 1; 3^*, 1) \quad 6T0+3T2+3T5+3T6+3T7(1, 1, 1; 1, 2) \quad 3T3+9T4+3T8(1, 1, 2; 1, 1) \quad 3T3+3T5+3T8(1, 2, 1; 1, 1) \quad 24T0+27T1+12T2+6T3+9T4+3T5+12T6+12T7+6T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(1, 1, 2) + 3(1, 2, 1) + 3(3^*, 2, 1) + 3(3, 1, 1)$ $3(1, 2) + 3(3^*, 2) + 3(3, 1)$
$v = (42200000) a_1 = (20020000) a_3 = (40-202220)$ $v = (42200000) a_1 = (00040000) a_3 = (11-113111)$	
$3T3+3T7(3, 1, 1; 1) \quad 3T4(3^*, 1, 1; 1) \quad 3T2+3T8(1, 1, 1; 3, 1) \quad 3T6(1, 1, 1; 3^*, 1) \quad 9T0+3T2+3T6+3T8(1, 1, 1; 1, 2) \quad 3T3+3T4+3T7(1, 1, 2; 1, 1) \quad 3T3+3T4+9T5+3T7(1, 2, 1; 1, 1) \quad 9T0+27T1+12T2+6T3+6T4+9T5+12T6+6T7+12T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(3, 1, 2) + 3(3^*, 2, 1) + 3(1, 2, 2)$ $3(1, 1) + 3(3, 1) + 3(1, 2) + 3(1, 2)$
$v = (42200000) a_1 = (20020000) a_3 = (40-222200)$ $v = (42200000) a_1 = (22022000) a_3 = (20-20-2200)$	
$3T7(1, 2, 1; 2) \quad 3T4+3T6(3, 1, 1; 1) \quad 3T3+3T5(3^*, 1, 1; 1) \quad 3T2(1, 1, 1; 3^*, 1) \quad 3T2+3T7(1, 1, 1; 1, 2) \quad 3T3+3T4+3T5+3T6+9T8(1, 1, 2; 1, 1) \quad 3T3+3T4+3T5+3T6+3T7(1, 2, 1; 1, 1) \quad 27T0+27T1+12T2+6T3+6T4+6T5+6T6+3T7+9T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(3, 1, 1) + 3(1, 2, 1) + 3(3^*, 1, 2) + 3(1, 1, 2)$ $3(1, 2) + 3(3, 1) + 3(3, 1) + 3(3, 1)$
$v = (42200000) a_1 = (20020000) a_3 = (04002200)$ $v = (42200000) a_1 = (40022000) a_3 = (02000222)$	
$3T4+3T6(3, 1, 1; 1) \quad 3T7(3^*, 1, 1; 1) \quad 3T2+3T3+3T5(1, 1, 1; 3^*, 1) \quad 3T2+3T3+3T5(1, 1, 1; 1, 2) \quad 3T4+3T6+3T7+9T8(1, 1, 2; 1, 1) \quad 3T4+3T6+3T7(1, 2, 1; 1, 1) \quad 27T0+27T1+12T2+12T3+6T4+12T5+6T6+6T7+9T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(1, 1, 1) + 3(3^*, 1, 1) + 3(3^*, 2, 2)$ $3(1, 1) + 3(1, 1) + 3(3^*, 2)$
$v = (42200000) a_1 = (00022000) a_3 = (40020200)$ $v = (42200000) a_1 = (22022000) a_3 = (20020400)$	
$3T5(1, 1, 2; 1, 2) \quad 3T1+3T2+3T3+3T7+3T8(3, 1, 1; 1, 1) \quad 3T4+3T6(1, 1, 1; 3, 1) \quad 3T4+3T5+3T6(1, 1, 1; 1, 2) \quad 3T1+3T2+3T3+3T5+3T7+3T8(1, 1, 2; 1, 1) \quad 9T0+3T1+3T2+3T3+3T7+3T8(1, 2, 1; 1, 1) \quad 9T0+6T1+6T2+6T3+12T4+3T5+12T6+6T7+6T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(3) \times SU(2)$	$3(3^*, 1, 2) + 3(1, 2, 2) + 3(3, 2, 1)$ $3(1, 1) + 3(3, 1) + 3(1, 2) + 3(1, 2)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (0\ 0\ 0\ 2\ 2\ 0\ 0) \mathbf{a}_3 = (2\ 0\ 0\ 2\ 0\ 2\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (2\ 2\ 0\ 2\ 2\ 0\ 0) \mathbf{a}_3 = (4\ 2\ -2\ 0\ -2\ 2\ 0\ 0)$
$3T1+3T2(3, 1, 1; 1, 1) \quad 3T3+3T7(3^*, 1, 1; 1, 1) \quad 3T8(1, 1, 1; 3^*, 1) \quad 9T6+3T8(1, 1, 1; 1, 2) \quad 3T1+3T2+3T3+9T5+3T7(1, 1, 2; 1, 1) \quad 9T0+3T1+3T2+3T3+3T7(1, 2, 1; 1, 1) \quad 9T0+6T1+6T2+6T3+27T4+9T5+9T6+6T7+12T8(1, 1, 1; 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(1, 1, 1) + 3(3^*, 1, 2)$ $3(1, 2, 1, 1) + 3(2, 2, 1, 1) + 3(2, 1, 1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (4\ 0\ 2\ 2\ 0\ 0\ 0) \mathbf{a}_3 = (0\ 0\ 4\ 0\ 2\ 2\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (0\ 4\ 2\ 2\ 0\ 0\ 0) \mathbf{a}_3 = (1\ 1\ 1\ -1\ 3\ 3\ 1\ -1)$
$3T0(1, 1, 1; 2, 1, 2, 1) \quad 3T7(1, 1, 2; 1, 2, 1, 1) \quad 3T4+3T6+3T8(3, 1, 1; 1, 1, 1, 1) \quad 3T3(3^*, 1, 1; 1, 1, 1, 1) \quad 6T0+3T2+3T5(1, 1, 1; 1, 1, 1, 2) \quad 3T2+3T5(1, 1, 1; 1, 1, 2, 1) \quad 6T0+3T2+3T5+3T7(1, 1, 1; 2, 1, 1) \quad 9T1+3T2+3T5(1, 1, 1; 2, 1, 1, 1) \quad 3T3+3T4+3T6+3T7+3T8(1, 1, 2; 1, 1, 1, 1) \quad 3T3+3T4+3T6+3T8(1, 2, 1; 1, 1, 1, 1) \quad 18T0+9T1+3T2+6T3+6T4+3T5+6T6+3T7+6T8(1, 1, 1; 1, 1, 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(3, 1, 2) + 3(3^*, 2, 1) + 3(1, 2, 2)$ $3(2, 1, 1, 2) + 3(1, 2, 2, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0) \mathbf{a}_3 = (4\ 0\ -2\ 2\ 2\ 2\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0) \mathbf{a}_3 = (2\ 0\ -2\ 0\ 2\ 2\ 0)$
$3T8(1, 1, 2; 1, 1, 2, 1) \quad 3T7(1, 2, 1; 1, 2, 1, 1) \quad 3T4+3T6(3, 1, 1; 1, 1, 1, 1) \quad 3T3+3T5(3^*, 1, 1; 1, 1, 1, 1) \quad 9T0+3T2(1, 1, 1; 1, 1, 2) \quad 3T2+3T8(1, 1, 1; 1, 2, 1) \quad 3T2+3T7(1, 1, 1; 1, 2, 1, 1) \quad 9T1+3T2(1, 1, 1; 2, 1, 1, 1) \quad 3T3+3T4+3T5+3T6+3T8(1, 1, 2; 1, 1, 1, 1) \quad 3T3+3T4+3T5+3T6+3T7(1, 2, 1; 1, 1, 1, 1) \quad 9T0+9T1+3T2+6T3+6T4+6T5+6T6+3T7+3T8(1, 1, 1; 1, 1, 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(1, 1, 2) + 3(1, 2, 1) + 3(3^*, 2, 1) + 3(3, 1, 1)$ $3(1, 1, 1, 1) + 3(1, 2, 2, 2) + 3(2, 1, 1, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0) \mathbf{a}_3 = (4\ 0\ -2\ 0\ 2\ 2\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0) \mathbf{a}_3 = (0\ 2\ 0\ -2\ 2\ 2\ 0)$
$3T5(1, 2, 1; 1, 2, 1, 1) \quad 3T3+3T7(3, 1, 1; 1, 1, 1, 1) \quad 3T4(3^*, 1, 1; 1, 1, 1, 1) \quad 9T0+3T2+3T6+3T8(1, 1, 1; 1, 1, 2) \quad 9T1+3T2+3T6+3T8(1, 1, 1; 1, 2, 1) \quad 3T2+3T5+3T6+3T8(1, 1, 1; 1, 2, 1, 1) \quad 3T2+3T6+3T8(1, 1, 1; 2, 1, 1, 1) \quad 3T3+3T4+3T7(1, 1, 2; 1, 1, 1, 1) \quad 3T3+3T4+3T5+3T7(1, 2, 1; 1, 1, 1, 1) \quad 9T0+9T1+3T2+6T3+6T4+3T5+3T6+6T7+3T8(1, 1, 1; 1, 1, 1, 1)$	

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$SU(3) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2) \times SU(2)$	$3(1, 1, 1) + 3(3, 1, 1) + 3(3, 2, 2)$ $3(1, 1, 2, 1) + 3(2, 1, 1, 1) + 3(2, 1, 2, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (4\ 0\ 0\ 2\ 2\ 0\ 0) \mathbf{a}_3 = (4\ 0\ 0\ 2\ 0\ 2\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (2\ 0\ 0\ 4\ 2\ 0\ 0) \mathbf{a}_3 = (2\ 0\ 0\ 0\ 0\ 4\ 2\ 0)$
$3T8(1, 1, 2; 1, 1, 2, 1) \quad 3T0(1, 2, 1; 2, 1, 1, 1) \quad 3T1+3T3+3T5+3T6+3T7(3^*, 1, 1; 1, 1, 1, 1) \quad 3T2+3T4(1, 1, 1; 1, 1, 2) \quad 3T2+3T4+3T8(1, 1, 1; 1, 1, 2, 1) \quad 3T2+3T4(1, 1, 1; 1, 2, 1, 1) \quad 3T0+3T2+3T4(1, 1, 1; 2, 1, 1, 1) \quad 3T1+3T3+3T5+3T6+3T7+3T8(1, 1, 1; 2, 1, 1, 1) \quad 3T0+3T1+3T3+3T5+3T6+3T7(1, 2, 1; 1, 1, 1, 1) \quad 3T0+6T1+3T2+6T3+3T4+6T5+6T6+6T7+3T8(1, 1, 1; 1, 1, 1, 1)$	

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$SU(3) \times SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(3^*, 1, 1, 2)$ $3(2, 1, 1) + 3(1, 1, 1) + 3(2, 2, 1) + 3(1, 2, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0) \mathbf{a}_3 = (2\ 2\ 0\ 4\ 2\ 2\ 0)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (1\ 3\ 1\ 1\ 1\ 1\ -1) \mathbf{a}_3 = (3\ 1\ 3\ 3\ 1\ 1\ -1)$
$3T1(1, 1, 1, 2; 2, 1, 1) \quad 3T7(1, 1, 2, 1; 1, 2, 1) \quad 9T4(3, 1, 1, 1; 1, 1, 1) \quad 3T3(3^*, 1, 1, 1; 1, 1, 1) \quad 6T0+3T6+3T8(1, 1, 1, 1; 1, 2) \quad 6T0+3T6+3T7+3T8(1, 1, 1, 1; 1, 2, 1) \quad 6T0+3T1+3T6+3T8(1, 1, 1, 1; 2, 1, 1) \quad 3T1+9T2+3T3(1, 1, 1, 2; 1, 1, 1) \quad 3T3+3T7(1, 1, 2, 1; 1, 1, 1) \quad 3T3+9T5(1, 2, 1, 1; 1, 1, 1) \quad 18T0+3T1+9T2+9T5+9T6+3T7+9T8(1, 1, 1, 1; 1, 1, 1)$	

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$SU(3) \times SU(2) \times SU(2) \times SU(2)$ $SU(2) \times SU(2) \times SU(2)$	$3(3^*, 1, 1, 1) + 3(3^*, 2, 1, 2)$ $3(1, 1, 1) + 3(2, 1, 1) + 3(1, 2, 1) + 3(2, 2, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0) \mathbf{a}_3 = (0\ 0\ 0\ 4\ 2\ 2\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (4\ 0\ -2\ 2\ 0\ 0\ 0) \mathbf{a}_3 = (0\ 2\ 0\ 2\ 2\ 2\ 0)$
$3T7(1, 1, 1, 2; 2, 1, 1) \quad 3T1(1, 2, 1, 1; 1, 2, 1) \quad 9T4+3T6+3T8(3, 1, 1, 1; 1, 1, 1) \quad 3T2+3T5(1, 1, 1, 1; 1, 2) \quad 3T1+3T2+3T5(1, 1, 1, 1; 1, 2, 1) \quad 3T2+3T5+3T7(1, 1, 1, 1; 2, 1, 1) \quad 9T3+3T6+3T7+3T8(1, 1, 1, 2; 1, 1, 1) \quad 3T6+3T8(1, 1, 2, 1; 1, 1, 1) \quad 9T0+3T1+3T6+3T8(1, 2, 1, 1; 1, 1, 1) \quad 9T0+3T1+9T2+9T3+9T5+3T7(1, 1, 1, 1; 1, 1, 1)$	

293

$SU(4) \times SU(2)$ $SU(2) \times SU(2)$	$3(4, 2)$ $3(1, 1) + 3(1, 1) + 3(1, 1) + 3(1, 2) + 3(2, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (0\ 0\ 4\ 0\ 0\ 0\ 0) \mathbf{a}_3 = (1\ 1\ 3\ 3\ 1\ 1\ 1)$ $v = (4\ 0\ 0\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (2\ 2\ 2\ 2\ 2\ 0\ 0) \mathbf{a}_3 = (3\ 3\ 1\ 1\ -1\ -1\ 3\ -1)$
$3T5+3T7(4^*, 1; 1, 1) \quad 6T0+3T3+3T6+3T8(1, 1; 1, 2) \quad 6T0+3T3+3T6+3T8(1, 1; 2, 1) \quad 9T1+9T2+3T5+3T7(1, 2; 1, 1) \quad 30T0+9T1+9T2+15T3+27T4+9T5+15T6+9T7+15T8(1, 1; 1, 1)$	

294

$SU(4) \times SU(2)$ $SU(2) \times SU(2)$	$3(4^*, 1) + 3(4, 2) + 3(1, 2)$ $3(1, 2) + 3(1, 1) + 3(1, 2) + 3(2, 1) + 3(2, 1) + 3(1, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (2\ 0\ 0\ 2\ 0\ 0\ 0) \mathbf{a}_3 = (4\ 0\ 0\ 2\ 2\ 2\ 0)$ $v = (4\ 2\ 2\ 0\ 0\ 0\ 0) \mathbf{a}_1 = (2\ 0\ 0\ 2\ 2\ 2\ 0) \mathbf{a}_3 = (2\ 0\ -2\ 0\ 0\ -2\ 2\ 0)$
$3T7(4^*, 1; 1, 1) \quad 3T2+3T3+3T5+3T8(1, 1; 1, 2) \quad 3T2+3T3+3T5+3T8(1, 1; 2, 1) \quad 9T0+9T1+3T7(1, 2; 1, 1) \quad 9T0+9T1+15T2+15T3+27T4+15T5+27T6+9T7+15T8(1, 1; 1, 1)$	

295

$SU(3) \times SU(2)$	$3(1, 2) + 3(3^*, 2) + 3(3, 1)$
$SU(2) \times SU(2)$	$3(1, 1) + 3(1, 2) + 3(2, 1) + 3(1, 1) + 3(1, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (2\ 2\ -2\ 0\ 2\ 2\ 2\ 0)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 0\ -2\ 2\ 2\ 0)$
$3T5+3T7(3, 1; 1, 1) \quad 3T4(3^*, 1; 1, 1) \quad 3T1+3T2+3T3(1, 1; 1, 2) \quad 3T1+3T2+3T3(1, 1; 2, 1) \quad 9T0+3T4+3T5+3T7(1, 2; 1, 1)$	
$9T0+15T1+15T2+15T3+12T4+12T5+27T6+12T7+27T8(1, 1; 1, 1)$	

296

$SU(3) \times SU(2)$	$3(1, 1) + 3(1, 1) + 3(3, 2)$
$SU(2) \times SU(2)$	$3(1, 1) + 3(2, 2) + 3(1, 1) + 3(1, 2) + 3(2, 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 2\ 0)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ 0\ 0\ 2\ 2\ 0\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ -2\ 2\ 0\ 0)$
$3T4+3T6(3^*, 1; 1, 1) \quad 3T1+3T2+3T3+3T7+3T8(1, 1; 1, 2) \quad 3T1+3T2+3T3+3T7+3T8(1, 1; 2, 1) \quad 9T0+3T4+3T6(1, 2; 1, 1)$	
$9T0+15T1+15T2+15T3+12T4+27T5+12T6+15T7+15T8(1, 1; 1, 1)$	

297

$SU(3) \times SU(3)$	$3(3, 3^*)$ (none)
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 0\ 0\ 0\ 0) a_3 = (1\ -1\ -1\ 3\ 3\ 1\ 1\ 1)$
	$v = (4\ 2\ 2\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 0\ 0\ 4\ 2\ 2\ 0\ 0) a_3 = (4\ 0\ -2\ 2\ 2\ 0\ 2\ 0)$
$9T0(3^*, 1) \quad 9T4(1, 3) \quad 27T1+27T2+27T3+27T5+27T6+27T7+27T8(1, 1)$	

298

$SU(4) \times SU(2)$	$3(1, 1) + 3(4^*, 1)$
$SU(4) \times SU(2) \times SU(2)$	$3(4, 1, 1)$
	$v = (2\ 2\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (4\ -2\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (3\ 3\ 1\ 3\ 1\ 1\ 1\ -1)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (2\ 4\ 2\ 0\ 0\ 0\ 0\ 0) a_3 = (0\ 2\ -2\ 2\ 2\ 0\ 0\ 0)$
$3T2(1, 2; 1, 1, 2) \quad 3T1+3T6(4, 1; 1, 1, 1) \quad 3T5(6, 1; 1, 1, 1) \quad 3T1(4^*, 1; 1, 1, 1) \quad 3T7(1, 1; 4, 1, 1) \quad 3T0(1, 1; 6, 1, 1) \quad 3T3+3T4(1, 1; 4^*, 1, 1) \quad 6T0+3T2+3T3+3T4+3T7(1, 1; 1, 1, 2) \quad 6T0+3T3+3T4+3T7+9T8(1, 1; 1, 2, 1) \quad 6T1+3T2+6T5+3T6(1, 2; 1, 1, 1) \quad 12T0+18T1+3T2+3T3+3T4+24T5+9T6+3T7+9T8(1, 1; 1, 1, 1)$	

299

$SU(4) \times SU(4)$	$3(1, 6) + 3(6, 4^*)$
$SU(4) \times SU(4)$	$3(4^*, 1) + 3(1, 4) + 3(4, 4^*)$
	$v = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0) a_1 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_3 = (0\ 2\ 2\ 2\ 2\ 0\ 0\ 0)$
	$v = (4\ 2\ 2\ 2\ 2\ 0\ 0\ 0) a_1 = (4\ 2\ 2\ 2\ -2\ 0\ 0\ 0) a_3 = (4\ 2\ 2\ 2\ -2\ 0\ 0\ 0)$
$3T1+3T3+3T8(4, 1; 1, 1) \quad 3T2+3T4+3T5(4^*, 1; 1, 1) \quad 3T1+3T2+3T3+3T4+3T5+3T8(1, 4; 1, 1) \quad 3T0+3T6+3T7(1, 1; 4^*, 1)$	
$3T0+3T6+3T7(1, 1; 1, 4) \quad 3T0+3T1+3T2+3T3+3T4+3T5+3T6+3T7+3T8(1, 1; 1, 1)$	

A

Calabi–Yau Manifold

An important class of manifolds permitting $\mathcal{N} = 1$ supersymmetry in four dimension are the Calabi–Yau manifolds [1]. For a gauge hierarchy solution at least one supersymmetry is needed, however for the chiral nature of the SM fermions it is restricted to exactly one supersymmetry.

Orbifold may be regarded as the singular limit of Calabi–Yau manifold. Also the opposite can be made hold good, since we now have a mathematical tool for resolving singularity to recover a smooth manifold geometry.

In this appendix, we sketch the line of thought one needed for such a setup. Many features of Calabi–Yau manifold are shared by orbifold, which is the subject of this book. A more detailed treatment is given in [1, 2, 3], especially on K3 manifold in [4], differential geometry and index theorem in [5] and on special holonomy in [6, 7]. Here the basic knowledge of differential form is required [3, 8].

Geometry Breaks Supersymmetry

We begin with the minimal ten dimensional supergravity. It is the low energy limit $\alpha' \rightarrow 0$ of heterotic string and has 16 real supercharges. The bosonic Lagrangian looks as

$$e^{-1}\mathcal{L} = \frac{1}{2\kappa^2}R + \frac{3}{4}\phi^{-3/2}H_{MNP}^2 - \frac{1}{2g^2}F_{MN}^2 \quad (\text{A.1})$$

The fermionic Lagrangian is completely fixed from this by supersymmetry. To examine the condition of supersymmetry breaking, we need to look at the transformation properties of fermions, the gravitino ψ_M , gaugino χ^a and dilatino λ . It is because their superpartners, the scalars, can develop VEV to break supersymmetry. We present them schematically,

$$\delta\psi_M = \frac{1}{\kappa}\nabla_M\epsilon + \phi^{-3/4}(\Gamma_M^{NPQ} - 9\delta_M^N\Gamma^{PQ})\epsilon H_{NPQ}, \quad (\text{A.2})$$

$$\delta\chi^a = \phi^{-1/2}\Gamma^{MN}F_{MN}^a\epsilon \quad (\text{A.3})$$

$$\delta\lambda = \Gamma^M\frac{\partial_M\phi}{\phi}\epsilon + \phi^{-3/4}\Gamma^{MNP}\epsilon H_{MNP}, \quad (\text{A.4})$$

up to four-fermion terms. Here, ∇_M is the covariant derivative on the metric connection and the gamma matrices are antisymmetrized, e.g. $\Gamma^{PQ} = \Gamma^{[P}\Gamma^{Q]}$, etc.

In the simple setup where $H_{MNP} = 0$ and the dilaton VEV is constant, the conditions for supersymmetry reduce to

$$\nabla_M\epsilon(x) = 0, \quad (\text{A.5})$$

$$\Gamma^{MN}F_{MN}^a(x) = 0. \quad (\text{A.6})$$

For a moment, let us concentrate on (A.5). The second condition will be discussed in Subsect. A.1.4. The covariant derivative has the connection term from metric; so it is purely a *geometric* condition. The amount of preserved supersymmetry is converted into the number of invariants, when we transport the fermion ϵ along an arbitrary closed path.

Next let us compactify six dimensions. The analysis becomes convenient in the language of group theory. We can decompose the parameter ϵ under the spacetime little group $\text{SO}(6) \times \text{SO}(2) \subset \text{SO}(8)$, or,

$$\epsilon = \epsilon^{(6)}(y^i) \times \epsilon^{(2)}(x^\mu). \quad (\text{A.7})$$

This amounts to the branching rule of the $\text{SO}(8)$ spinor $\mathbf{8} \rightarrow \mathbf{4} \times \mathbf{2}$ and the chiralities are correlated. The number of invariants constructed out of $\mathbf{4}$ leaves the same number of 4D supersymmetries. As we encounter in the Higgs mechanism, the generic nonzero components of $\mathbf{4}$ can be rotated by an appropriate $\text{SU}(4)$ transformation to a singlet component of $\text{SU}(3) \subset \text{SU}(4)$, say to the fourth entry of $\langle \mathbf{4} \rangle$. If the six dimensional manifold has a shape such that spinor $\epsilon^{(6)}$ transforms as $\mathbf{3}$ under arbitrary transportation, going around a closed loop in this six dimension leaves $\langle \mathbf{4} \rangle$ invariant and one nonvanishing parameter is left in $\langle \mathbf{4} \rangle$. The branching of $\mathbf{4}$ is

$$\mathbf{4} \rightarrow \mathbf{3} + \mathbf{1}, \quad (\text{A.8})$$

one fourth of the parameters is guaranteed to be preserved. This leaves exactly the same number of 4D supersymmetries, i.e., $\mathcal{N} = 4/4 = 1$. Such geometric transformations form a group called *holonomy*. For the present purpose, we need exactly the $\text{SU}(3)$ holonomy. A manifold admitting an $\text{SU}(n)$ holonomy is called Calabi–Yau manifold.

These considerations guide us to analyze the spacetime group $\text{SO}(2n)$ in terms of $\text{SU}(n)$. We have familiarized with this decomposition in grand unified theory in Chap. 4. This notion is naturally analyzed in the context of complex manifold that we will discuss here.

Also note that, this unbroken supersymmetry condition, or the special holonomy condition is just continuous version of (3.58). The one fourth of original supersymmetry survives because discrete $\mathbf{Z}_3$ is the center of $\mathrm{SU}(3)$.

A.1 Calabi–Yau Manifold

Throughout this book, we have used implicitly the notion of complex manifolds. It is more than a convenience, since the holonomy group $\mathrm{SU}(n)$ and its subgroup, the point group of the orbifold, act on complex representations.

A.1.1 Complex Manifold

We need a transition from a real manifold to a complex manifold. Complex coordinates are naturally introduced as

$$z^a \equiv \frac{1}{\sqrt{2}}(y^{2a-1} + iy^{2a}), \quad z^{\bar{a}} \equiv \overline{z^a} = \frac{1}{\sqrt{2}}(y^{2a-1} - iy^{2a}). \quad (\text{A.9})$$

Which object does then play the role of imaginary number? We see that the block-diagonal matrix defined on a point

$$J = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \\ & & & & \ddots \end{pmatrix}. \quad (\text{A.10})$$

satisfies $J^2 = -1$. We can verify this and it is diagonalizable to $J^a_b = i\delta^a_b$ in the complex basis $\{z^a, z^{\bar{a}}\}$. (In the toroidal compactification, periodicity gives rise to some ambiguity, as seen in Subsect. 5.1.4.) We will use letter $a, b, \dots$ to denote the complex basis and $i, j, \dots$ the real basis in this Appendix A. This complexification is nothing but to decompose an $\mathrm{SO}(2n)$ vector representation into an $\mathrm{SU}(n)$ fundamental representation. Every vector, spinorial and antisymmetric representation of $\mathrm{SO}(2n)$ can be decomposed under suitable tensor products of fundamental $\mathbf{n}$ and anti-fundamental $\bar{\mathbf{n}}$ of $\mathrm{SU}(n)$. The example of $\mathrm{SO}(10) \rightarrow \mathrm{SU}(5)$ is presented in (4.64) in Chap. 4 and below.

We may think of a tensor $J(x^i)$ that has the same components $J(x^i) = J$ at every point x^i . At a point, every antisymmetric tensor can be always adjusted to have the form (A.10). The problem is whether we can do this in the local neighborhood or not. In general relativity, we have a criterion that a manifold is flat if its Riemann tensor vanishes. We have a similar counterpart tensor, called *Nijenhuis* tensor, for the case of complex structure. The complex structure is “flat” when it vanishes by the Newlander–Nirenberg theorem [8]. Not every almost complex manifold admits a complex structure, e.g. S^4 does not admit a complex structure [6].

It is convenient to make J in terms of holonomy singlet spinor ϵ of (A.5),

$$J^i_j(y^l) = G^{ik} \epsilon^\dagger(y^l) \Gamma_{kj} \epsilon(y^l) . \quad (\text{A.11})$$

We can check that this is the desired complex structure from the singlet property of ϵ .

Kähler Manifold

The metric of a form $G_{ab} = G_{\bar{a}\bar{b}} = 0$ always gives a positive definite norm. Then, the manifold is called *Hermitian*.

The index of a metric is raised by the almost complex structure $k_{ij} = G_{ik} J^k_j$. This makes a metric tensor into a differential form, a $(1,1)$ form called the Kähler form. In the complex basis, it is

$$k_{a\bar{b}} = -i G_{a\bar{b}} . \quad (\text{A.12})$$

When the metric is Hermitian and the Kähler form is closed

$$dk = 0 , \quad (\text{A.13})$$

the manifold is called *Kähler manifold*. Since $d = \partial + \bar{\partial}$, it implies,

$$\partial k = \bar{\partial} k = 0 . \quad (\text{A.14})$$

Since k is a total derivative in the holomorphic and antiholomorphic derivative, G of (A.12) is obtained from a single function ϕ called *Kähler potential*,

$$G_{a\bar{b}} = -\partial_a \bar{\partial}_{\bar{b}} \phi . \quad (\text{A.15})$$

This is an important feature. As a consequence, the nonvanishing connection is either purely holomorphic or antiholomorphic,

$$\Gamma_{bc}^a = G^{a\bar{d}} \partial_b G_{c\bar{d}}, \quad \Gamma_{\bar{b}\bar{c}}^{\bar{a}} = G^{\bar{a}d} \partial_{\bar{b}} G_{c\bar{d}} . \quad (\text{A.16})$$

This means that a holomorphic(an antiholomorphic) index transforms again into a holomorphic(an antiholomorphic) one. Thus, an n - (complex) dimensional Kähler manifold admits a $U(n)$ holonomy.

Homology and Cohomology

How can one specify the manifolds that we have discussed? Here only topology is important.¹ Recall that two real-dimensional (compact and orientable)

¹ For physical applications, it is known that the Kähler potential for the kinetic terms in supersymmetric theory and the Yukawa couplings are independent of a specific metric.

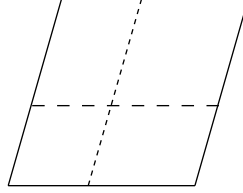


Fig. A.1. One-cycles on a two-torus shown as *dashed lines*. The two homology cycles shown above are different, as shown by *short dashes* and *long dashes*

manifolds are completely classified by their genus. Likewise, the knowledge on the homology class matters here.

We have effective quantities to extract the topological information, called the *cohomology* class. For the real differential operator d , F is closed form if $dF = 0$ and is exact if it is expressed as $F = dA$. Every exact form is closed but not vice versa. This property is called the nilpotence $d^2 = 0$. We can define the cohomology class of k -form H^k as

$$H^k = \frac{\text{closed } k\text{-form}}{\text{exact } k\text{-form}}. \quad (\text{A.17})$$

Namely, it is the *closed class of differential k -forms, not overcounting those connected by gauge transformation*, $F_{[k]} \sim F_{[k]} + d\Lambda_{[k-1]}$.

Interestingly, there is a similar notion in the geometrical side. We can introduce a boundary operator δ . The boundary of a manifold has no boundary, therefore δ is also a nilpotent operator. The cycle is defined as a sub-manifold annihilated by δ . If the manifold is already a boundary δM , it is also cycle but not vice versa. *Homology* class of k -cycle H_k is defined in the same way. The independent one cycles on two-torus are depicted in Fig. A.1.

These similar objects, the closed and the exact forms have one to one correspondence to the boundary and the cycle, respectively. They are dual by the theorem of de Rham. It follows that $\dim H^k = \dim H_k$.

Similarly, we have complex differential operators ∂ and $\bar{\partial}$ generate holomorphic and antiholomorphic indices. Naturally we can think of the (p, q) form having p and q holomorphic and antiholomorphic indices, respectively, and similarly the closed and exact forms. We define p, q cohomology class as

$$H_{\partial}^{p,q} = \frac{\partial\text{-closed } (p, q)\text{-form}}{\partial\text{-exact } (p, q)\text{-form}}. \quad (\text{A.18})$$

In this case also, we have the de Rham counterpart, $H_{p,q}$. Therefore, $h^{p,q} \equiv \dim H^{p,q}$ is the same as $h_{p,q} \equiv \dim H_{p,q}$.

The complex manifold is specified by the topological parameters $h^{p,q}$. Since we have the identity $H^{p,q} = H^{q,p} = H^{n-p, n-q}$ in n -complex dimensions, these numbers are tabulated in the form of a diamond called the Hodge diamond. The Euler number is

$$\chi = \sum_{p,q} (-1)^{p+q} h^{p,q} . \quad (\text{A.19})$$

We have connected the geometric quantity (in the left hand side) with some quantity characteristic of gauge fields defined on the manifold (in the right hand side).

Calabi–Yau Manifold

Since we have a $U(n)$ holonomy in the Kähler manifold, we must check its $U(1)$ part. The additional conditions are equivalently stated as follows:

- Ricci-flatness: $R_{ij} = 0$
- Vanishing first Chern class: $c_1 = 0$
- There is a everywhere covariantly constant volume form of type $(n,0)$.

A Kähler manifold with the vanishing $U(1)$ part is called *Calabi–Yau manifold*. In other words, the Calabi–Yau (CY) manifold admits an $SU(n)$ holonomy.

Note that the $U(1)$ generator of $SO(2n)$ is nothing but the almost complex structure J . Therefore, given an $SO(2n)$ generator M , we can extract the $U(1)$ part as $\text{Tr} JM$. Viewing the Riemann tensor as a matrix element $R_{ijkl} = R_{(ij)kl}$, we can see that the Ricci flat condition implies the vanishing $U(1)$ part

$$\text{Tr} J R_{(ij)} = 0 , \quad (\text{A.20})$$

which is translated into Ricci flat condition by some algebra.

In this standard, the torus T^n is the simplest CY manifold since it is everywhere flat $R_{ijkl} = 0$. However, its holonomy group consists of only the identity. The conditions we have discussed above are the only necessary ones to be satisfied. Below, we require the holonomy group neither bigger nor smaller than $SU(n)$.

A.1.2 Mode Expansion

Calabi–Yau Threefold

Since we are interested in the compactification of six real dimensions, here we focus on the CY threefold. Some numbers are fixed in the CY threefold. Since all the points are equivalently “homologous” and there is only one volume form, we have $h^{0,0} = h^{3,3} = 1$. Since the CY threefold has the nowhere-vanishing $(3,0)$ form, we have $h^{3,0} = h^{0,3} = 1$. Other numbers vanish except $h^{2,1} = h^{1,2}$, $h^{1,1} = h^{2,2}$ by the property of the vanishing first Chern class, etc. They are shown in the following Hodge diamond

$$\begin{array}{ccccccc}
& & h^{0,0} & & & & 1 \\
& & h^{1,0} & h^{0,1} & & & 0 & 0 \\
& h^{2,0} & h^{1,1} & h^{0,2} & & & 0 & h^{1,1} & 0 \\
h^{3,0} & h^{2,1} & h^{1,2} & h^{0,3} & = & 1 & h^{2,1} & h^{2,1} & 1 & . \\
& h^{3,1} & h^{2,2} & h^{1,3} & & & 0 & h^{1,1} & 0 \\
& & h^{3,2} & h^{2,3} & & & 0 & 0 & \\
& & & h^{3,3} & & & & 1 &
\end{array} \quad (A.21)$$

Since Kähler form $k_{a\bar{b}}$ is $(1,1)$ form, we have at least $h^{1,1} \geq 1$. It corresponds to the number of parameters describing deformations of complex structure $G_{ij}\delta J^j_k$, as in (A.11). In typical cases we have $h^{1,1} = 1$. The $(2,1)$ forms describe the deformations of the shape of manifold. Usually the Calabi–Yau manifold is described by *algebraic variety*, that is, a (set of) polynomials in the projective spaces and, with them, the shape deformations are readily countable [3]. Finally, it is getting certain that there always exists equivalent physics even if we choose a dual theory with another compact manifold to exchange $h^{1,2}$ and $h^{1,1}$. This phenomenon is known as *mirror symmetry*.

Associating Action

As in the orbifold case, the breaking of group is achieved by relating a holonomy action in the compact space (a generalized point action of the orbifold case) with a group element. The space action of Calabi–Yau threefold belongs to the $SU(3)$ holonomy.

Let us choose the *standard embedding*,² that is to associate the space action into the same one in the group space,

$$R_{ab} = F_{ab} , \quad (A.22)$$

where the indices of R and F are the $SU(3)$ parts of the internal space $SO(6)$ and the group space $E_8 \times E_8$, respectively. This naturally breaks $E_8 \times E_8$ into

$$E_6 \times E_8.$$

There is no extra $SU(3)$ as a gauge group in the sense that it became the holonomy group. We easily understand that the adjoint $(\mathbf{248}, \mathbf{1}) + (\mathbf{1}, \mathbf{248})$ branches into

$$\begin{aligned}
A_{M,a} : (\mathbf{3}, \mathbf{27}, \mathbf{1}), \quad A_{M,\bar{a}} : (\bar{\mathbf{3}}, \bar{\mathbf{27}}, \mathbf{1}), \quad A_{M,a} : (\mathbf{8}, \mathbf{1}, \mathbf{1}), \\
A_{M,I} : (\mathbf{1}, \mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248}), \quad I \neq a, b.
\end{aligned} \quad (A.23)$$

Again, the first entries pretend as representations of the gauge group but are of the $SU(3)$ holonomy group. Their indices are attached, in addition to the Lorentz index M . Since $\mathbf{3}$ and $\mathbf{27}$ are correlated, we suppressed indices of the latter. Thus, the entries in the first line of (A.23) give $(\mathbf{27}, \mathbf{1}), (\bar{\mathbf{27}}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{1})$ of $E_6 \times E_8$, respectively.

² For nonstandard embedding, a discussion is given in [3].

The consistency condition comes from the Bianchi identity for antisymmetric tensor field H ,

$$dH = \text{Tr}_{\text{fund}} R^2 - \frac{1}{30} \text{Tr}_{\text{adj}} F^2 = 0 . \quad (\text{A.24})$$

We can verify that this condition is satisfied. We only need to know the number of fundamental $\mathbf{3}$ of $\text{SU}(3)$ in both R and F . From the above decomposition of $\mathbf{248}$ of E_8 into $\text{SU}(3) \times E_6$, we have 3 copies of $\mathbf{3}$ s and $\bar{\mathbf{3}}$ s in the adjoint $\mathbf{8}$ and 27 copies of $\mathbf{3}$ s and $\bar{\mathbf{3}}$ s in the $(\mathbf{3}, \mathbf{27}, 1)$, and $(\bar{\mathbf{3}}, \mathbf{27}, 1)$ to make up 30 in total.

The above conditions (A.22) naturally remind us of the relation between the space twist and the shift vector (6.43). It follows that (A.24) corresponding to the modular invariance condition (6.43) guarantees the anomaly freedom in the low energy theory. Certainly this Calabi–Yau case is the more general and continuous version. The correspondence is studied in [10].

Mode Expansion

Now, we make the Klein decomposition on the Calabi–Yau manifold. The basic strategy is the same as in the toroidal case. We can decompose the Laplacian and the Dirac operator into our 4D spacetime part (x^μ) and the 6D internal space part (y^i),

$$\nabla_M \nabla^M = \partial_\mu \partial^\mu + \nabla_i \nabla^i , \quad (\text{A.25})$$

$$\Gamma_M \nabla^M = \Gamma_\mu \partial^\mu + \Gamma_i \nabla^i . \quad (\text{A.26})$$

The mode expansion of a (bosonic or fermionic) field ϕ has a form³

$$\phi = \sum \phi_4(x^\mu) \omega_6(y^i) . \quad (\text{A.27})$$

Here, the harmonic form ω is the solution of the Laplace equation, $(\partial + \bar{\partial})^2 \omega = 0$, or equivalently

$$\partial \omega = \bar{\partial} \omega = 0 . \quad (\text{A.28})$$

The harmonic forms are the bases of homology cycle $H^{p,q}$, which gives the topological number $h^{p,q} = \dim H^{p,q}$.

Now let us consider the standard embedding. Consider first the field transforming as $(\bar{\mathbf{3}}, \mathbf{27})$. As in (A.23), if the index M assumes the holomorphic value the one represented as $A_{a,\bar{b}}$ is a (1,1) form; therefore we have a similar mode expansion,

$$A_{a,\bar{b}} = \sum_{p=1}^{h^{1,1}} A^{[p]}(x^\mu) \omega_{a\bar{b}}^{[p]}(y^i) . \quad (\text{A.29})$$

³ In general $\Gamma_i \partial^i$ and $\Gamma_\mu \partial^\mu$ do not commute. However, we can multiply a suitable gamma matrix to make them commute. Even, for the consideration of the zero modes of $\Gamma_i \partial^i$, the eigenvalue vanishes so there is no such ambiguity.

It gives the $h_{1,1}$ zero modes. The “Kähler” moduli $T_{a\bar{b}}$ of (8.78) is also a (1,1) form; therefore it has the same mode expansion and the same number of zero modes.

Next consider $(\mathbf{3}, \mathbf{27})$, denoted by $A_{a,b}$. This is not a differential form, which can be understood by the fact that the indices a and b are symmetric because they are associated to the metric. We can make it a differential form as

$$A_{a,\bar{d}\bar{e}} = A_{a,b} G^{b\bar{c}} \Omega_{\bar{c}\bar{d}\bar{e}} \quad (\text{A.30})$$

This is merely a change of basis, since G and Ω are invariant forms. Then, we have the following mode expansion

$$A_{a,\bar{d}\bar{e}} = \sum_{p=1}^{h^{1,2}} A^{[p]}(x^\mu) \omega_{a\bar{d}\bar{e}}^{[p]}(y^i) . \quad (\text{A.31})$$

Therefore, the number of chiral $\mathbf{27}$ s is $h^{1,2}$. The same argument applies to the complex structure moduli T_{ab} , yielding the same number of zero modes.

Number of Families and the Index Theorem

We know that $\mathbf{27}$ of E_6 houses one complete family including a pair of Higgs doublets, as discussed in (2.21). Because of the Survival Hypothesis, every pair of $\mathbf{27}$ and $\bar{\mathbf{27}}$ can form a mass term around the unification scale; thus they decouple at low energy if there is no symmetry forbidding it to happen. Thus, net number of families is the difference $|h^{2,1} - h^{1,1}|$. It is remarkable to see, by the definition of Euler number χ in (A.19)

$$\text{Number of families : } |h^{2,1} - h^{1,1}| = \frac{|\chi|}{2} . \quad (\text{A.32})$$

The number of families reduces to the *topological property*. This relies on the assumptions that (1) the Survival Hypothesis (SH) holds,⁴ (2) one representation, here $\mathbf{27}$, houses a complete family, and (3) as seen in (A.23) the field transforming as $\mathbf{27}$ carries the internal space index a of the special holonomy $SU(3)$. In particular, these conditions are in general violated in the presence of Wilson lines. A typical construction of Calabi–Yau manifold gives a very large Euler number of order 100 so that we need a further sophistication. The one constructed by Tian and Yau gives $|\chi| = 6$ to allow three families [11].

There is a remarkable approach in interpreting the above result. One notes the branching rule of the adjoint E_8 given in (A.23). The complete generation $\mathbf{27}$ of E_6 , which is our interest, is correlated with $\mathbf{3}$ of $SU(3)$. The observation is that, when we calculate the vanishing eigenstate of Dirac operator ∇ , $(\mathbf{3}, \mathbf{27})$,

⁴ One may envision that this SH is evaded by a stringy calculation, for example in cases where a mass term cannot be formed. But massless $\mathbf{27}$ must be made heavy around or above the electroweak scale; so the number of chiral families is still $\frac{|\chi|}{2}$.

with the $SU(3)$ gauge connection in the internal direction, it is understood as massless multiplet **27** in four dimension.

Let us define a chirality operator in the internal space $\Gamma^{(6)} = \Gamma^4 \Gamma^5 \dots \Gamma^9$. Actually $\epsilon^{(6)}$ in (A.7) is the eigenstate of $\Gamma^{(6)}$. Take ten dimensional spinor (**3**, **27**) with chirality +1. Then it is a special property of $2k + 2$ (which is 6 in the case at hand) dimensions that the chirality of internal dimension is identified with the chirality of our four dimension.

We know that the Klein–Gordon operator $H = \nabla_i \nabla^i$, in terms of the internal space Dirac operator $\nabla \equiv \Gamma_i \nabla^i$, is a Hamiltonian (energy) operator. Since it commutes with $\Gamma^{(6)}$, we can set every eigenstate of H as an eigenstate of $\Gamma^{(6)}$. For every nonzero solution ψ of the Dirac equation, $\nabla \psi$ is also a solution with the opposite chirality. Thus, a nonzero eigenvalue appears as a pair. Note that this does not apply to vanishing solutions $\nabla \psi = 0$. However, the following quantity, called *index*, being the difference of the zero eigenstate numbers of left and right movers,

$$\text{index}_Q \nabla = n_+ - n_- , \quad (\text{A.33})$$

is invariant. Here, the subscript means that the covariant derivative is with respect to the representation Q of $SU(3)$. We also observe that in four dimension, charge conjugation changes Q to its conjugate Q^* and exchanges the chirality as well; therefore $n_+ \leftrightarrow n_-$ and $\text{index}_{Q^*} \nabla = -\text{index}_Q \nabla$. It follows that the real representation is not counted, but only complex representation **3**, thus **27** (correlated!) is counted.

If we continuously change the value of gauge field contained in ∇ , it may be that the number of zero modes can be changed by acquiring a mass. However, still nonzero mode eigenstates always make up pairs (A.33) invariant. A quantity that is invariant under continuous change of parameter is referred to as *topological invariant*. We can rewrite the index as

$$\text{index}_Q \nabla = \text{Tr}(\Gamma^{(6)} e^{-\beta H}) \quad (\text{A.34})$$

where we will send β to infinity. Since a massive eigenvalue always emerges as a pair with the opposite charges, this indeed counts the difference of the number of zero modes. Interestingly, this is reminiscent of the heat kernel regularization of anomaly and the right hand side is the anomaly polynomial [8]. In six dimensions, the surviving terms of the RHS are

$$n_+ - n_- = \frac{1}{2^3 3! (2\pi)^3} \int \left(\text{Tr}_Q F^3 - \frac{1}{8} \text{Tr}_Q F \text{Tr} R^2 \right) = \frac{\chi}{2} . \quad (\text{A.35})$$

This is an integer like the instanton number, and turns out to be the same as half the Euler number of (A.19).

Further Breaking, Wilson Lines

With the embedding (A.22), we do not have a realistic model, for the same reason as in the orbifold case. We need a further symmetry breaking. Still

there is another object, Wilson line, related with non-contractible cycles dealt in Sect. 6.4.

We generalize the Wilson line U as the object associating one-cycle (instead of translational symmetry that the torus possesses) with the action in the group space. As in the orbifold case, the Wilson line should satisfy the consistency condition (A.24). This condition is exhausted by the identification of the $SU(3)$ in the group space and the holonomy group, leaving the field strength F of the other group E_6 being zero. This is exactly the situation where the Wilson line emerges.

To be compatible with the standard embedding, we do not touch on the $SU(3)$ subgroup associated with the holonomy group, but consider the unbroken E_6 . As an example, let us work in the $SU(3)^3$ basis which is one of the maximal subgroups of E_6 . Consider a matrix acting on the generator of this group. Specifically, we choose U of (6.83) as

$$U = \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha \end{pmatrix} \otimes \begin{pmatrix} \beta & & \\ & \beta & \\ & & \beta^{-2} \end{pmatrix} \otimes \begin{pmatrix} \gamma & & \\ & \delta & \\ & & (\gamma\delta)^{-1} \end{pmatrix}, \quad (\text{A.36})$$

where $\alpha, \beta, \dots$ are the unit roots of 1. Since only the commuting generators of (A.23) survive, this breaks E_6 down to

$$SU(3) \times SU(2) \times U(1)^3.$$

It contains the desired standard model group, but at least two unwanted Abelian groups.

We may go on to turn on more Wilson lines and choose another CY manifold to find a realistic model. There has been endeavors along this line [13].

A.1.3 Relation with Orbifold

K3 Manifold

The simplest nontrivial Calabi–Yau manifold is a two complex dimensional one. It is known that, topologically speaking, the Calabi–Yau twofold is unique, and is named as the K3 manifold.⁵ K3 has the following Hodge numbers

$$\begin{array}{ccccc} & h^{0,0} & & & 1 \\ & h^{1,0} & h^{0,1} & & 0 & 0 \\ h^{2,0} & h^{1,1} & h^{0,2} & = & 1 & 20 & 1 \\ & h^{2,1} & h^{1,2} & & 0 & 0 \\ & h^{2,2} & & & & 1 \end{array} \quad (\text{A.37})$$

Therefore, the Euler number is 24.

⁵ After the name of three mathematicians Kummer, Kähler and Kodaira, inspired by the similar names of mountain summits K1, K2, K3, ... in India. [4]

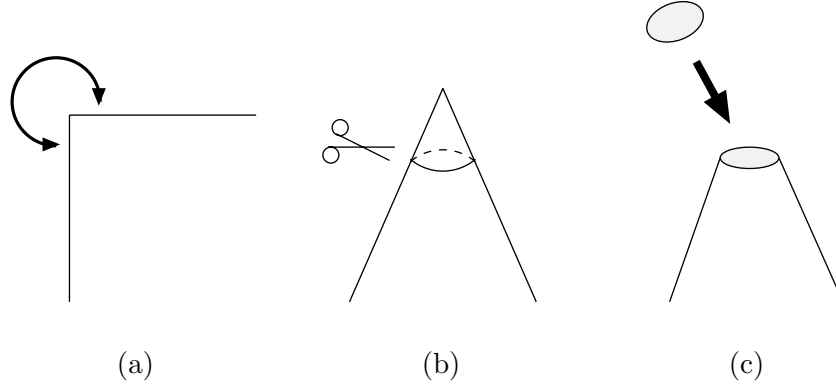


Fig. A.2. The Calabi–Yau twofold is obtained by blowing up $T^4/\mathbf{Z}_2$ orbifold. This is achieved by cutting around the singular fixed points and replacing them with a smooth manifold, which is also moded out by the same $\mathbf{Z}_2$

This manifold has an interesting connection with four dimensional manifolds $T^4/\mathbf{Z}_N$ with $N = 2, 3, 4, 6$ [14]. By “blowing-up” the singularities of the orbifolds, all of them become the K3 manifold. This should be due to the uniqueness of K3. Naively, this correspondence is understood by excising around the apex of the cone and replacing it with a “well-patched” smooth manifold, that we will see shortly as in Fig. A.2.

Here, we focus on the case of $T^4/\mathbf{Z}_2$ orbifold. The easiest criterion for the equivalence comes from topology. The Euler number becomes

$$\chi = \frac{\chi(M) - \chi(F)}{N} + \chi(N) , \quad (\text{A.38})$$

with the notation to be understood as the following. The original two-torus has genus one and has the same topology as a donut, thus has the Euler number $\chi(M) = 0$. A cutoff region around the fixed point can be stretched to a disk which has Euler number 1, and we have 16 such fixed points; so $\chi(F) = 16$. Because of moding with $\mathbf{Z}_N$, the region for integration to calculate the topological number on the orbifold is reduced to $1/N$ of the torus integration region, thus it is divided by N . $N = 2$ is the order of orbifold action. Shortly, we will see that the patched region corresponds to a smooth manifold of sphere with Euler number 2; so $\chi(N) = 16 \cdot 2$. We therefore have

$$\chi = -\frac{16}{2} + 16 \cdot 2 = 24 ,$$

in accord with (A.37).

Blowing Up

In this four dimensional space, we can solve the Ricci flat constraint. From (A.5), the integrability condition becomes

$$[\nabla_m, \nabla_n]\epsilon = \frac{1}{4}R_{mn}{}^{pq}\Gamma_{pq}\epsilon = 0 \quad (\text{A.39})$$

It is certain that, if not Ricci-flat there is an additional piece proportional to $\text{Tr}JR$ in (A.20). Since a chiral spinor can be written as $(\Gamma - 1)\epsilon = 0$, the constraint equation becomes

$$R_{mnpq} = \frac{1}{2}\epsilon^{abcd}R_{mn}{}^{pq} \quad (\text{A.40})$$

We may check that the following metric given in [9] is a solution,

$$ds^2 = \left(1 - \left(\frac{a}{r}\right)^4\right)^{-1} dr^2 + r^2 \left(1 - \left(\frac{a}{r}\right)^4\right) \sigma_3^2 + r^2(\sigma_1^2 + \sigma_2^2) \quad (\text{A.41})$$

where we parameterized the line element by radial variable r and these Euler angles (θ, ϕ, ψ) , with $0 \leq \theta < \pi, 0 \leq \phi < 2\pi, 0 \leq \psi < 4\pi$,

$$\begin{aligned} \sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi \\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \\ \sigma_3 &= d\psi + \cos\theta d\phi. \end{aligned} \quad (\text{A.42})$$

This looks like a strange convention, but we can easily check that this is a differential form version of the Pauli matrix (Maurer–Cartan equation) $\sigma_a \wedge \sigma_b = 2\epsilon_{abc}d\sigma_c$. We verify that $\sigma_1^2 + \sigma_2^2$ is the metric of two-sphere $d\theta^2 + \sin^2\theta d\phi^2$ and $\sigma_1^2 + \sigma_2^2 + \sigma_3^2$ is that of three-sphere.

The metric (A.41) has one parameter a corresponding to the radius of two-cycle (In the similar case of $\mathbf{Z}_3$, explicitly we identify a with ρ of (8.96)). Let us look at the local property near $r = a$. Parameterizing $r \equiv a + \epsilon^2/a$ we have

$$ds^2 = d\epsilon^2 + \epsilon^2(d\psi + \cos\theta d\phi)^2 + a^2 d\Omega_2^2 \quad (\text{A.43})$$

in the limit $\epsilon \rightarrow 0$. Locally (neglecting $d\phi$) we see that the space locally looks like $R^2(\epsilon, \psi) \times S^2(\theta, \phi)$ whose Euler number is that of the two-sphere. For this space to have no singularity at $r = a$, there we need to mod out by a $\mathbf{Z}_2$ action $\psi \rightarrow 4\pi - \psi$ on S^2 . Fortunately this is the same action we have. Also the curvature drops as $1/r$ so is asymptotically flat. This property is called asymptotically locally Euclidian (ALE).

Therefore, if we cut out the cone around the fixed point, put this metric, and glue the edges, then it is well glued. At each fixed point, in addition to a parameterizing the blown-up radius, we have two more parameters (θ, ϕ) determining the orientation of gluing. Therefore, we have $16 \times 3 = 48$ real parameters. In addition, the 4D metric has $4 \cdot 5/2 = 10$ parameters, determining geometry. Thus, we have 58 real parameters. Among them, only 24 are relevant from the topological point of view, which are Hodge diamond (A.37). Besides the trivial zero cycle(point) and four cycle(volume), the non-trivial ones are 22 two-cycles. The original torus had the holomorphic cycle

$dy^m \wedge dy^n$ giving $4 \cdot 3/2 = 6$ independent two-cycles. When we blow up all sixteen fixed points, there appear 16 parameters since each fixed point has one parameter a . Therefore, there are 24 relevant parameters in total, which coincides with the Euler number we obtained above.

There is a well-established mathematical procedure of treating singularities (such as blowing up and down, collectively called surgery) of manifold as shown in Fig. A.2. A detailed explanation can be found in [4].

A.1.4 More General Construction

Another Equation

There is another supersymmetry preserving condition (A.6),

$$\Gamma^{ij} F_{ij} \epsilon = (\Gamma^{ab} F_{ab} + \Gamma^{\bar{a}\bar{b}} F_{\bar{a}\bar{b}} + 2\Gamma^{a\bar{b}} F_{a\bar{b}}) \epsilon = 0, \quad (\text{A.44})$$

where we decomposed the two-form F_{ij} into $(2, 0), (0, 2), (1, 1)$ parts. Although nontrivial, it is easy to see that (A.44) reduces to

$$F_{ab} = F_{\bar{a}\bar{b}} = 0 \quad (\text{A.45})$$

$$G_{a\bar{b}} F^{a\bar{b}} = 0. \quad (\text{A.46})$$

The first equation (A.45) states that locally the field strength F_{ab} is a pure gauge and globally the transition function of $A_{\bar{a}}$ is holomorphic. In the standard embedding of Subsect. A.1.2 this is automatically satisfied since we have identified the gauge connection with the spin connection which is holomorphic.

Now consider the second equation (A.46). For a $U(1)$ part, it can be converted into $H = \text{Tr} F k^{n-1} = 0$ condition where n is the number of complex dimensions. Interestingly, the integral $\int H$ over the compact manifold is a topological invariant, and easily one might see that F is a topological invariant, the first Chern class. For the nonabelian part, vanishing $G_{a\bar{b}} F^{a\bar{b}}$ requires F to be a “stable fiber bundle” in some condition, by the theorem by Donaldson and by the one generalized by Uhlenbeck and Yau [6, 16].

Other Manifolds

According to Berger [15], a simply-connected, irreducible and nonsymmetric manifold of dimension D with Riemannian metric, is uniquely classified according to Table A.1. In our heterotic string case, the Calabi–Yau threefold is our main interest in compactifying heterotic string, since it leaves $1/4$ of the original supersymmetry. In view of string duality, unified theories require more dimensions, implying more compact dimensions and different special holonomies. For example M- and F- theories are living in eleven and twelve dimensions, thus we introduce manifold of G_2 and $SU(4)$ holonomy, respectively. The other side of the coin is that we can see string duality when we compactify merely one or two dimensions in these generalized theories.

Table A.1. According to Berger, a simply-connected, irreducible and nonsymmetric manifold of dimension D with Riemannian metric, is totally classified according to this table [6, 15]

Real Dimension	Holonomy Group	Decomposition
D	$SO(D)$	Trivial
$D = 2m \geq 4$	$U(m) \subset SO(2m)$	ex. 3+1
$D = 2m \geq 4$	$SU(m) \subset SO(2m)$	
$D = 4m \geq 8$	$Sp(m) \times Sp(1) \subset SO(4m)$	
$D = 4m \geq 8$	$Sp(m) \subset SO(4m)$	7+1
$D = 7$	$G_2 \subset SO(7)$	
$D = 8$	$Spin(7) \subset SO(8)$	

Note that in orbifold theory we have applied this condition, which yields the condition in Subsect. 3.1.7. In other words, orbifold has a discrete holonomy. We will see that this fact reflects that the orbifold is the discrete limit of Calabi–Yau manifold.

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B

Other Constructions

In this book, we emphasized the chiral nature of the SM. The orbifold construction has been discussed toward obtaining three family models from the $E_8 \times E'_8$ heterotic string. But there are other methods also for obtaining 4D string models. Among these, we review very briefly on the fermionic construction and the interesting brane setup. Since obtaining three families is one of the most important objectives going beyond 4D, we discuss these other constructions up to the point of introducing the possibility for three families. We do not attempt to review the whole ideas and realistic model buildings.

B.1 Fermionic Construction

We focused on the bosonic description so far. On the other hand, there exists an equivalent fermionic description: in 2D two Majorana-Weyl fermions, λ_1 and λ_2 , are equivalent to one periodic, holomorphic boson $\partial X(z)$. We compactify all dimensions but four, with the critical radius $\sqrt{\alpha'}$. The resulting fermions are $44(= 2 \cdot 26 - 2 \cdot 4)$ left movers and $18(= 2 \cdot 6 + 6)$ right movers. The rank is enhanced to $22 + 6$. Note that fermions in the noncompact dimensions should be considered as well.

We may regard the orbifolding as simply assigning nontrivial boundary conditions. In this context, the $E_8 \times E'_8$ heterotic string in 10D is the first example of fermionic construction: we partitioned 32 fermions into two sets of 16 fermions each and assigned different boundary conditions. That was the only way of modular invariant sets with 32 fermions. With more and more fermions, the number of possibility grows in a geometrical progression.

B.1.1 Rules

Conventionally we define the twist vector $W = (w_i)$ as $w_i = 1 - 2V_i$, where V_i is the shift vector as before

$$\lambda_i(2\pi) = e^{\pi i(1-w_i)} \lambda_i(0) . \quad (\text{B.1})$$

The fermions are Majorana-Weyl thus assumes $w_i = 1$ or 0 only, corresponding to Ramond (R) or Neveu-Schwarz (NS) sector respectively. We will later complexify them and generalize the boundary condition. Now we have vectors with $20 + 44$ entries and name the twisted sector with them. Sometimes we add two more right movers for noncompact space to check the supersymmetry condition. We divide the vector using a bar and the standard convention is to place the right mover boundary conditions on the left side of the bar,

$$(w_1 \ w_2 \ \cdots \ w_{20} | \tilde{w}_1 \ \tilde{w}_2 \ \cdots \ \tilde{w}_{44}) . \quad (\text{B.2})$$

The nontrivial restrictions are the GSO projection and the modular invariance. These conditions are extracted in a most straightforward way from the partition function [1, 2]. We just present the rules in their simplest forms [3] and point out their origins:

1. **Modular invariance:** If there are sectors denoted by W_1 and W_2 , then there must exist a sector

$$W_1 + W_2 . \quad (\text{B.3})$$

We define the sum modulo 2 by the periodicity of w_i , thus $1 + 1 = 0$. This seems contradictory, since two successive periodic (R) boundary condition gives antiperiodic (NS) one. However this take into account of the spin structure (6.45): the extra phase $e^{\pi i}$ is needed for the periodic (R) boundary condition in the τ direction.

In addition, like the untwisted sector discussed in Chap. 5, we require that there exist the most trivial states, or untwisted states, where all are in the NS sector or in the R sector,

$$(0^{20} | 0^{44}), \quad (1^{20} | 1^{44}) . \quad (\text{B.4})$$

From the above condition, two sectors are described by just one vector

$$W_0 = (1^{20} | 1^{44}) . \quad (\text{B.5})$$

2. **GSO projection:** The Gliozzi-Scherk-Olive (GSO) projection projects out half of the spinors of definite chirality. Before orbifolding it is defined by (5.134, 5.141). Now in the presence of many shift vectors, we have an additional rule: if there are the coinciding Ramond boundary conditions “1” between two sectors, in each sector we just project out half of the corresponding spinors. Which half we should project out is our freedom of choice. In the literature, it is expressed as the choice of correlation coefficient of the GSO projection. We will show this in the following examples.
3. **Supersymmetry condition (Triplet constraint):** For the right movers, the supercurrent in (5.106) is

$$J_- = 2i \sum_{\mu=1}^2 \psi^\mu \partial_{\bar{z}} X^i + i \sum_{i=1}^6 \lambda_1^i \lambda_2^i \lambda_3^i , \quad (\text{B.6})$$

The first term comes from the noncompact two dimensions and the second term comes from the compact six dimensions. Thus we partition 20 right moving fermions into

$$\{(\psi^1\psi^2) (\lambda_1^1\lambda_2^1\lambda_3^1) \cdots (\lambda_1^6\lambda_2^6\lambda_3^6)\} . \quad (\text{B.7})$$

It is consistent only when the entire energy-momentum tensor have the definite boundary condition; thus each term has the same

$$w^{\psi^\mu} = w_1^i + w_2^i + w_3^i, \quad \text{mod } 2 \quad (\text{B.8})$$

for each i . Here, w^{ψ^μ} is the boundary condition for both ψ^μ , and w_j^i 's are those of λ_j^i .

In the symmetric orbifold, we assign the same boundary condition on the corresponding left movers $\tilde{\lambda}_j^i$.

The spectrum is obtained from the mass shell condition

$$M_L^2 = \tilde{N} + \tilde{c} = 0 , \quad (\text{B.9})$$

$$M_R^2 = N + c = 0 , \quad (\text{B.10})$$

with the zero point energies $\tilde{c}$ and c are calculated from (6.15). Note that the sign is the opposite of that of the bosonic case given in (6.15). We have contribution $-\frac{1}{48}$ and $\frac{1}{24}$ from the entries of 0 and 1, respectively.¹

B.1.2 Models

Original $E_8 \times E'_8$ Heterotic String

As an example, we first treat the original ten dimensional $E_8 \times E'_8$ heterotic string. W_0 is always present. We specify two vectors, W_1 and W_2 ,

$$\begin{aligned} W_0 &= (1^8|1^{32}) \\ W_1 &= (1^8|0^{32}) \\ W_2 &= (0^8|1^{16}0^{16}) . \end{aligned} \quad (\text{B.11})$$

We have indicated 32 boundary conditions out of 44 to highlight the gauge degrees of freedom, as discussed in Subsect. 5.3.5. The other 12 are understood to be filled with 1. We have $2^3 = 8$ sectors in total. The W_1 and $2W_i \equiv W_i + W_i = (0^8|0^{32})$ sectors have the same left movers, which is our primary interest. Its zero point energy is $-1 = -\frac{1}{24} \cdot 8 + (-\frac{1}{48}) \cdot 32$ from the left sector only. When combined with the right movers, we have the adjoint $(\mathbf{120}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{120})$ of $SO(16) \times SO(16)$,

$$\tilde{\lambda}_{-1/2}^i \tilde{\lambda}_{-1/2}^j |0\rangle, \quad i, j = 1, \dots, 16, \quad \text{or} \quad i, j = 17, \dots, 32 ,$$

¹ It is $\frac{1}{24} - \frac{1}{4}\eta(1-\eta)$. Here, 0 corresponds to $\eta = \frac{1}{2}$, from (B.1).

and in addition the conventional graviton, dilation and antisymmetric tensor fields. We have no crossed sectors such as the set one from $i = 1, \dots, 16$, and the other from $j = 17, \dots, 32$, because the original GSO projection (5.134) before orbifolding. Adding W_1 to each sector, we have super-partners because of the change in the right mover boundary conditions.

In the W_2 sector, we have the zero point energy $0 = -\frac{1}{24} \cdot 8 + (-\frac{1}{48}) \cdot 16 + \frac{1}{24} \cdot 16$. The states are constructed by successive applications of creation operators of the form

$$|s_i\rangle, \quad i = 1, \dots, 8,$$

where the spinorial s is generated by $\tilde{\lambda}_0^i$. They seem to be spinorial of dimension 2^8 , **(256, 1)**. However, this W_2 has a Ramond overlap with W_0 (some entries have the common 1s), thus half the states are projected out. We have **(128, 1)**. For the $W_0 + W_1 + W_2$ sector, we obtain **(1, 128)** in a similar manner. As we know, these representations **(120, 1)**, **(128, 1)**, **(1, 120)**, and **(1, 128)** are not separable and we have an enhanced gauge symmetry with the adjoint **248s**. Namely, the gauge group is $E_8 \times E'_8$.

Four Dimensional Superstring

As an example, let us consider the flipped SU(5) model of [4], the so-called “revamped” version. They are the nine vectors $1, S, \zeta, b_1, b_2, b_3, b_4, b_5$, and α with one constraint.² Let us start the discussion with the following three vectors

$$\begin{aligned} 1 &= ((1^2)(1^6)(1^{12})|(1^{16})(1^{12})(1^{16})) \\ S &= ((1^2)(1^6)(0^{12})|(0^{16})(0^{12})(0^{16})) \\ \zeta &= ((0^2)(0^6)(0^{12})|(0^{16})(0^{12})(1^{16})) \end{aligned}$$

This is almost the same as (B.11). Focusing on the *left movers*, the only difference from the 10D example is that in 4D we have twelve more entries. We interpret that the first and the last 16 entries are going to describe $E_8 \times E'_8$. The 12 entries in the middle are interpreted as compactification of six dimensions, equivalent to 12 fermionic degrees of freedom. Thus we may interpret this piece as *orbifolding*. The S sector³ just provide the superpartners, as W_1 did in the 10D example. The ζ sector has zero point energy 0, as W_2 did in 10D, providing **(1, 128)** of $SO(28) \times SO(16)$. The untwisted sector has the zero

² For comparison, we follow the names of fields used in [4]

$$((\chi^\mu)(\chi^1 y^1 \omega^1 \dots \chi^6)(y^1 \dots y^6 \omega^1 \dots \omega^6)|(\bar{\psi}^1 \dots \bar{\psi}^5 \bar{\eta}^1 \dots \bar{\eta}^3)(\bar{y}^1 \dots \bar{y}^6 \bar{\omega}^1 \dots \bar{\omega}^6)(\bar{\phi}^1 \dots \bar{\phi}^8))$$

The fields $\chi, \bar{\chi}, y, \bar{y}, \omega$, and $\bar{\omega}$ are real fields and the $\bar{\eta}, \bar{\psi}$, and $\bar{\phi}$ are complex fields counted up to 20 for the right movers and up to 44 for the left movers.

³ It should be the S boundary condition, but “sector” is used here to follow the terminology in the fermionic construction.

point energy -1 providing $(\mathbf{378}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{120})$ of $\mathrm{SO}(28) \times \mathrm{SO}(16)$. These together make up the adjoint representations of $\mathrm{SO}(28) \times \mathrm{E}_8$. Note that we have an enhanced gauge group $\mathrm{SO}(28) \times \mathrm{E}_8$ from $\mathrm{E}_8 \times \mathrm{E}'_8$ by taking the critical radius $R = \sqrt{\alpha'}$. Note that there are always six $\mathrm{U}(1)$ s or enhanced rank six group from the right movers which we will neglect for brevity.

Now introduce more vectors to break the group

$$\begin{aligned} b_1 &= ((1^2)(1^2 0^2 0^2)(1^4 0^4 0^4)|(1^{10})(1^2 0^2 0^2)(\mathbf{1}^4 \mathbf{0}^4 \mathbf{0}^4)(0^{16})) \\ b_2 &= ((1^2)(0^2 1^2 0^2)(0^4 1^4 0^4)|(1^{10})(0^2 1^2 0^2)(\mathbf{0}^4 \mathbf{1}^4 \mathbf{0}^4)(0^{16})) \\ b_3 &= ((1^2)(0^2 0^2 1^2)(0^4 0^4 1^4)|(1^{10})(0^2 0^2 1^2)(\mathbf{0}^4 \mathbf{0}^4 \mathbf{1}^4)(0^{16})) \end{aligned}$$

where we used bold-faced numbers for the middle 12 entries corresponding to the left mover boundary conditions. Conditions b_1, b_2 and b_3 have a cyclic symmetry and note that every right mover has the same 12 components with the corresponding left mover entries. This corresponds to a symmetric orbifold. Note that there exists a relation $\zeta = b_1 + b_2 + b_3 + 1$, thus there will be eight sectors instead of nine. We may not need ζ any more, but its explicit form simplifies the discussion. Only $b_i + b_j$ and $1 + b_i + b_j$ sectors, with $(i \neq j)$, thus six in total), have non-positive zero point energy $\tilde{c} = -1$ and 0 , respectively, and hence a possibility for massless spectra. In fact, we can check by considering $\lambda_{-1/2}^i$ and $\lambda_{-1/2}^j$ that the group is broken to $\mathrm{SO}(10) \times \mathrm{SO}(6)^3 \times \mathrm{E}_8$.

Considering the matter spectrum, we have three equivalent sectors b_1, b_2 and b_3 . In each sector, there are $(2^5, 2^3) = (32, 8)$ states, $(\mathbf{16}, \mathbf{4}) + (\mathbf{16}, \bar{\mathbf{4}}) + (\bar{\mathbf{16}}, \mathbf{4}) + (\bar{\mathbf{16}}, \bar{\mathbf{4}})$. Combined with right movers, half of them are the antiparticles of the other half, say $(\mathbf{16}, \mathbf{4}) + (\mathbf{16}, \bar{\mathbf{4}})$.

Similarly the right movers have 2^4 multiplets, yet we count only half of them because a complete Weyl fermion is made of two helicity states. We see that each b_i , due to the non-vanishing 1^2 in the middle bracket, reduces the number the half and at the same time, breaks half of the supersymmetries. However, only two of them among three b_i are independent directions, leading to $\mathcal{N} = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$ supersymmetry and 2 multiplet. We have two $(\mathbf{16}, \mathbf{4}) + (\mathbf{16}, \bar{\mathbf{4}})$ s in each b_1, b_2 and b_3 sector, thus we have total 48 generations $(\mathbf{16})$ s of $\mathrm{SO}(10)$.

We have geometric interpretation: we can check that in the 12 component orbifold section (= the bold-faced ones) in the left movers, each b_i corresponds to a $\mathbf{Z}_2$ orbifold action. The b_3 and b_1 action correspond to

$$\theta = \frac{1}{2}(1 \ 1 \ 0), \quad \omega = \frac{1}{2}(0 \ 1 \ 1), \quad (\text{B.12})$$

respectively.⁴ Here, b_2 is generated by the other two, $\theta\omega = \frac{1}{2}(1 \ 0 \ 1)$: we have a $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry. The important thing to note is that we have the family

⁴ The $(1, 1, 0)$ corresponds to a $\mathbf{Z}_2$ orbifold in the 4–5 and 6–7 tori and $(0, 1, 1)$ corresponds to another $\mathbf{Z}_2$ orbifold in the 6–7 and 8–9 tori, as discussed in Chap. 5.

number as a multiple of three because of the triple symmetries in exchanging b_i 's. Basically, this can be interpreted as three fixed tori of $\omega, \theta, \omega\theta$. In view of the forms b_1, b_2 and b_3 , 6 real dimensions are split in such a way that they are grouped into three by complexifying them and furthermore they should give only three fixed tori. Then, the number of families is a multiple of three. Here, the number of complex compact dimensions appears as $\frac{1}{2} \cdot 6 = 3$. Still we have not obtained three families yet. This set b_1, b_2 , and b_3 is called the NAHE set [4]. So far the NAHE set respects the $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry.

We proceed to introduce more vectors to have a realistic model,

$$\begin{aligned} b_4 &= ((1^2)(1^2 0^2 0^2)(110010001000)|(1^{10})(1^2 0^2 0^2)(110010001000)(0^{16})) \\ b_5 &= ((1^2)(0^2 1^2 0^2)(100011000100)|(1^{10})(0^2 1^2 0^2)(100011000100)(0^{16})) \\ \alpha &= \left((1^2)(0^2 0^2 0^2)(100000100000)|(\frac{1}{2}^{10})(\frac{1}{2}^2 \frac{1}{2}^2 \frac{1}{2}^2)(100000100000)(\frac{1}{2}^8 1^4 0^4) \right) \end{aligned}$$

Here, we extended boundary conditions by complexifying the Majorana-Weyl fermions $\lambda(z) = \lambda_1(z) + i\lambda_2(z)$, etc. to have a half-integer shift $\frac{1}{2}$. Conditions b_4 and b_5 give rise to Higgs bosons and the last shift α breaks $\text{SO}(10)$ down to the flipped $\text{SU}(5)$. But, we can see that the $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry is not present anymore. Note that the right movers have the same form as before, and hence do not touch upon the $\mathcal{N} = 1$ supersymmetry. But the other right moving entries are different, reducing the number of generations down from 48 to 6. However there is an independent GSO projection provided by 2α which also reduce the number of generations by two, thus we have three generations.

The final gauge group contains flipped $\text{SU}(5)$:

$$[\text{SU}(5) \times \text{U}(1)] \times \text{U}(1)^4 \times \text{SO}(10) \times \text{SO}(6) .$$

The rank is reduced by 2, by the shift α . Now the spectrum contains complete families $\mathbf{1} + \mathbf{5} + \mathbf{10}$, and $\bar{\mathbf{1}} + \bar{\mathbf{5}} + \bar{\mathbf{10}}$. By a GSO projection, we keep $\mathbf{1} + \bar{\mathbf{5}} + \mathbf{10}$ so that they form a flipped $\text{SU}(5)$ spectrum. It has been shown that the spectrum $\mathbf{1} + \bar{\mathbf{5}} + \mathbf{10}$ has the $\text{U}(1)$ charge ratio $5 : -3 : 1$ of the flipped $\text{SU}(5)$ [4]. A flipped $\text{SU}(5)$ model can be broken down to the SM by VEVs of $\mathbf{10}_H$ and $\bar{\mathbf{10}}_H$. The Higgs fields $\mathbf{10}_H + \bar{\mathbf{10}}_H$ needed for the GUT breaking and $\mathbf{5}_H + \bar{\mathbf{5}}_H$ needed for the electroweak symmetry breaking are also present. This model, however, has a $\text{U}(1)$ and hence there appear exotic multiplets (singlets and $\mathbf{10}$ of different $\text{SO}(10)$ and $\mathbf{4}, \mathbf{6}$ etc.) with unfamiliar $\text{U}(1)$ charges. Of course, the extra $\text{U}(1)$'s beyond the $\text{U}(1)$ in the flipped $\text{SU}(5)$ may be required to be broken. One such example, the Fayet-Illiopoulos mechanism with the the anomalous $\text{U}(1)$ was discussed in Chap. 9.

In this kind of compactification, 44 fermions are equal, and we are not obliged to use the same twisting to both left and right movers. Then, we have the fermionic version of the asymmetric orbifold.

B.2 Intersecting Branes

B.2.1 Charged Open String

In this section, we briefly discuss another possibility of describing group degrees of freedom by open strings of Type I (and Type II) theory. Recall that the heterotic string has uniform charge distribution along the closed string, whose local relation is given by (9.56). Alternatively, we can assign charges on both ends of an open string. This was the original old string idea of 1960s attempted for describing strong interactions.

Just tag a new degree of freedom at each end labeled by indices i and j as shown in Fig. B.1(a), called *the Chan-Paton factors*, running from 1 to n . As a representation, we introduce $n \times n$ matrix λ_{ij}^a . A string state with momentum k can carry an additional degree a ,

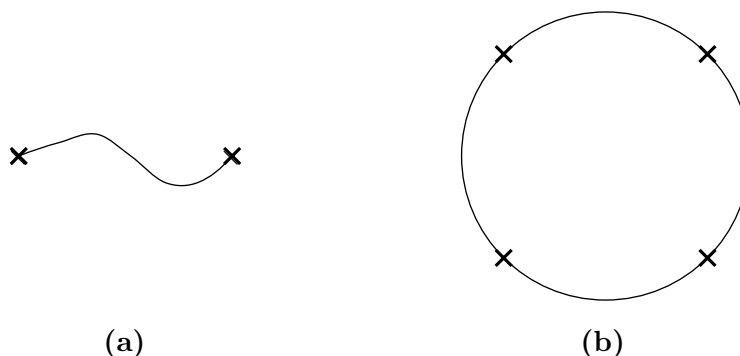


Fig. B.1. (a) Assigning the charges on both ends. (b) The four point tree level diagram

$$|k, a\rangle = \sum_{i,j=1}^n |k, ij\rangle \lambda_{ij}^a \quad (\text{B.13})$$

In the open string Feynman diagram, in and out states have definite Chan-Paton factors. For example, the four-point tree level diagram have ends with the same indices, as shown in Fig. B.1(b).

Summing up all the possible states, the amplitude is proportional to the trace of the products λ^a ,

$$\lambda_{ij}^1 \lambda_{jk}^2 \lambda_{kl}^3 \lambda_{li}^4 = \text{Tr} \lambda^1 \lambda^2 \lambda^3 \lambda^4$$

Generally, the amplitude is invariant under a global $U(n)$ transformation

$$\lambda^a \rightarrow U \lambda^a U^{-1}$$

under which the endpoint ($\times$) transforms as a fundamental–anti-fundamental representation, $\mathbf{n}$ and $\bar{\mathbf{n}}$. This $U(n)$ is the world sheet global symmetry. Now in the target space, this is coordinate (X^i) dependent and hence is escalated to a *local symmetry*. This is in fact the general argument in string theory: in string theory every symmetry is a local symmetry, i.e. there is no global symmetry. Similarly, when we consider un-oriented strings we can also describe $SO(n)$ and $Sp(2n)$ groups.

B.2.2 Dirichlet Brane

The closed string compactified on a torus has the symmetry (5.182), called T -duality,

$$n \leftrightarrow w, \quad R \leftrightarrow \frac{\alpha'}{R}. \quad (\text{B.14})$$

It means that physics is the same in the dual space, by changing the roles of momentum and winding. It turns out that this is the symmetry of string theory itself, and the open string theory possesses such a symmetry as well. In the dual space, the Neumann boundary condition (5.29) becomes the Dirichlet boundary condition: the open string endpoints should be placed in some specific hyper-surfaces. This endpoint behavior is now understood as *Dirichlet brane*, or D-brane in short. On it the open string dynamics can be understood alternatively as the dynamics of D-branes.

In the beginning, Type II theory was defined as a theory of closed strings. However, inserting D-branes is also consistent: far from the brane the massless spectrum contains closed and oriented strings only, i.e. Type II string theory. However, the D-brane is an object for an open string to be attached: near the brane open strings emerge. Moreover, the Ramond–Ramond antisymmetric field strength tensors couple to these D-branes. In view of M theory, this unified picture is appealing.⁵

In the presence of D-branes, only “unoriented” half of the $\mathcal{N} = 8$ supersymmetries $\tilde{Q} + Q$ of Type II is conserved. In the T -dual picture without the brane, where the D-brane is present,

$$\tilde{Q}_\alpha + PQ_\alpha$$

where $P = \prod P_m$ is a parity action and the product is over the real dualized dimension.

The D-brane is a Bogomolnyi–Prasad–Sommerfield (BPS) state. Supersymmetry guarantees some stable configurations, even nonperturbatively. With one brane, a string with both ends at the brane, the resulting massless state is charged under $U(1)$. BPS state guarantees the stability when more than one branes are coincident. Alternatively we can check that there is

⁵ Type I theory can be seen as a Type II string theory in presence of space filling orientifold plane O9 [7].

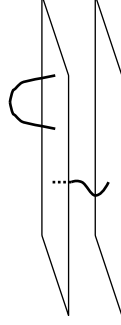


Fig. B.2. String stretched between two intersecting branes. These give $SU(2)$

no force between parallel D-branes; by the worldsheet supersymmetry NS–NS force (containing gravitational attraction) is cancelled by the R–R force. Now there is a symmetry enhancement: a string can end at *different branes*, whose 4D positions are the same, i.e. the same x^μ as shown in Fig. B.2. It is like a charged boson $W^\pm$. With n branes, the Chan–Paton factor runs over n , so that the resulting symmetry is extended to $U(n)$. Its dynamics is described by Dirac–Born–Infeld action,

$$S = -T_p \int d^{p+1}\xi \text{Tr} e^{-\Phi} \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} . \quad (\text{B.15})$$

Expanding to the quadratic order in $\alpha' F_{MN}$, it reduces to $(p+1)$ -dimensional Yang–Mills (YM) action,

$$S_p = -\frac{T_p(2\pi\alpha')^2}{4g_s} \int d^{p+1}x \text{Tr} F_{MN} F^{MN} \quad (\text{B.16})$$

with a potential of transverse scalar degrees. Here T_p and g_s are the tension and Type II string coupling fixed by the vacuum expectation value (VEV) of dilaton. Therefore, the YM coupling is

$$g_{p+1}^2 = g_s T_p^{-1} (2\pi\alpha')^{-2} \quad (\text{B.17})$$

Every BPS object has a conserved charge, seen by the supersymmetry commutation relations. Inspecting Type II theory spectrum, the massless state contains the antisymmetric tensor in the R–R sector. Type IIA and Type IIB theories contain $(p+1)$ -form fields $C_{p+1} \equiv C_{\mu_1 \dots \mu_{p+1}}$ with $p+1$ being odd and even, respectively. They couple to the D-brane as

$$\int C_{p+1}$$

where the integration is over the $p+1$ dimensional world volume spanned by the Dp brane. So Type IIA and Type IIB theories have Dp -brane with p even and odd, respectively.

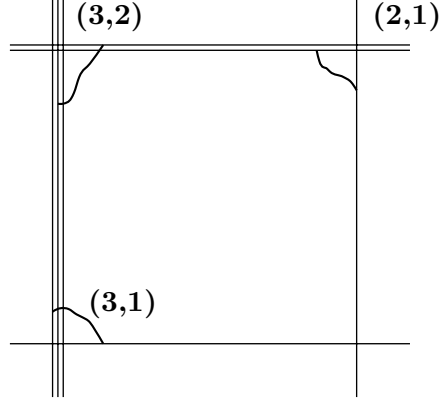


Fig. B.3. Example for a standard-like model configuration

B.2.3 Intersecting Branes

The resulting fields belong to an adjoint representation, transforming like $\mathbf{n} \times \bar{\mathbf{n}}$ of $U(n)$. For quarks and leptons, we need *chiral* representations. Thus, consider two sets of branes intersecting with an angle, viz. Fig. B.3. Each set has n, m bunches of coincident D-branes. The string stretched between the two are localized at the intersecting point to minimize the energy.

Then, we have $(\mathbf{m}, \mathbf{n})$ bi-fundamental representation. The $U(1)$ charge is provided by some linear combination of extra $U(1)$ factor in $U(n) = U(1) \times SU(n)$. If the SM fields were charged under more than two nonabelian groups, we cannot make an intersecting brane model, contrary to the case in heterotic string models. It follows from a simple observation that a nonabelian horizontal (family) symmetry cannot be introduced from intersecting brane models since one cannot introduce an additional set of branes intersecting at the same intersection.

Now, to have our 4D we need to compactify six dimensions. In a typical realistic setup, we compactify six dimension as before and consider 3-cycle $[\pi_a]$ which is provided by a D6 brane. Our world is four dimensional intersection. With some nontrivial cycle, we can make more than one intersection points. The effective 4D interaction is obtained by integrating out with the internal dimension coordinates as we have done in Chaps. 3 and 5, $\int dy^{(6)} \dots$. Decomposing six torus as $T^2 \times T^2 \times T^2$ and denoting the wrapping number (l_a^i, m_a^i) along each unit direction a, b of the i th 2-torus, we have the intersection number

$$I_{ab} = \prod (l_a^i m_b^i - m_a^i l_b^i) \quad (\text{B.18})$$

where the product runs over the compact dimensions.

For instance, three intersection points can explain *three families*, as in the heterotic string case. This also has the same merit: the distance explains the hierarchy between Yukawa interactions. The more distant the intersection

points are separated, the more (exponentially) suppressed the interaction between them.

At the moment there is no stabilization mechanism on how many branes should be present. The model building is a bottom-up approach.

Ramond–Ramond Tadpole Cancellation

Recall that in the heterotic string the consistency condition came from the modular invariance, which gives no divergence. In the open string, a potential divergence lies in the R–R tadpole diagram. For vanishing divergence, the sum of the R–R charge should be cancelled in the transverse compact dimension to the cycle. The condition is [6]

$$\sum_a n_a [\pi_a] = 0 . \quad (\text{B.19})$$

It is obtained from the consistency condition of equations of motion. Physically, it is understood in terms of the (R–R) charge conservation in the compact space. The flux lines from charge should be connected between charges.

With D-branes only, it is hard to cancel the R–R charges in each direction. We can use anti-D-brane which has the opposite R–R charge; however the brane then attract each other, destined to be annihilated. The cure can come from the introduction of *orientifold*. Consider the worldsheet parity operation

$$\Omega : \quad \sigma \rightarrow -\sigma \quad (\text{B.20})$$

Indeed this operation interchanges the left and right mover. In the T -dual coordinate $X' = X'_L(\tau + \sigma) - X'_R(\tau - \sigma)$, this operation accompanies a space-time $\mathbf{Z}_2$ reflection

$$X' \rightarrow -X' . \quad (\text{B.21})$$

We understand this discrete symmetry behavior (with fixed point) as orientifold. In fact, the orientifold dynamics is effect of the D-brane and its orbifold image. Like D-branes, the orientifold is charged under NS–NS and R–R. We can calculate the brane tension by the amplitude of closed string exchange between the orientifold and the D-brane, or between D-brane and mirror brane. The resulting tension is

$$T_p^O = \pm 2^{p-5} T_p^D . \quad (\text{B.22})$$

When we compactify one dimension, the world sheet parity $\mathbf{Z}_2$ induces two fixed points as discussed in Chap. 3, thus we have a double $O(p-1)$ s. So, for every p , we always need sixteen D-branes to cancel the R–R charges of orientifolds. The non-vanishing sum of NS–NS charges simply implies the time dependent dynamics.

The R–R charge cancellation leads to the anomaly cancelation in the low energy theory. For example, $SU(n_a)^3$ anomaly is proportional to the number of $\mathbf{n}_a$ minus number of $\bar{\mathbf{n}}_a$, hence proportional to

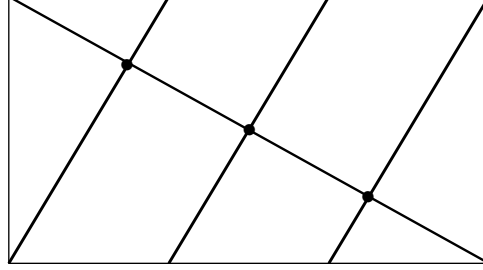


Fig. B.4. The intersection number accounts for the number of families

$$\sum_b I_{ab} n_b = [\pi_a] \cdot \sum_b n_b [\pi_b] = 0$$

coming from the above condition (B.19). Generically, the rest of anomalies such as $U(1)^3$ or $SU(n_a)^2 \times U(1)$ are not cancelled in the same way because like in the heterotic string case there are potential anomalous $U(1)$ s. They are canceled by a generalized Green–Schwarz mechanism, here the antisymmetric tensors responsible for axions are provided by the R–R tensor fields in the compact space. There are more than one such anomalous $U(1)$ s.

Supersymmetry

We comment that the supersymmetry condition [5] is the same with that of the heterotic string on orbifold (3.58). Here the condition is parametrized by three angles ϕ_i , tilted angle from the unit direction in each i th two-torus. This tilts the brane and the supersymmetry generator as

$$\tilde{Q} + RPR^{-1}Q \quad (\text{B.23})$$

with the eigenvalue of R being $e^{2\pi i \phi_i}$. The unbroken symmetry is the common intersection $P = RPR^{-1}$. The condition reduces to the invariant part of Q under R^2 , in the spinorial representation s

$$s \cdot 2\phi = 0 \quad (\text{B.24})$$

for at least one combination of $s = ([\frac{1}{2}^4])$ with even numbers of minus sign. For example, taking a basis $s = (\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2})$,

$$\phi_1 + \phi_2 + \phi_3 = 0$$

We can rewrite this in terms of radii R_i of six-torus and three cycle numbers

$$\arctan \frac{R_2 m_1}{R_1 l_1} + \arctan \frac{R_4 m_2}{R_3 l_2} + \arctan \frac{R_6 m_3}{R_5 l_3} = 0. \quad (\text{B.25})$$

The merit of non-supersymmetric model is that we can interpret the Higgs field as tachyon. It signals instability and triggers geometric transition to the

setup of lower energy which typically has smaller gauge symmetry. This is interpreted as the tachyon condensation or the brane recombination. There is no hierarchy problem since the Yang-Mills field coupling can be arbitrarily lowered by adjusting the compactification size

$$M_{\text{Pl}}^2 g_{\text{YM}}^2 = \frac{M_s^{11-p} V_X}{g_s}$$

with V_X being the volume of the space spanned by 3-cycle. However, it suffers another hierarchy problem because of the size V_X .

Of course, instead of orbifold, we can use the Calabi–Yau manifold, but then the system is not explicitly solvable.

B.2.4 Models

Without SUSY

First, we discuss the simple non-super-symmetric model given in [6]. The setup, with Type IIA theory with D6 branes, is given in Table B.1 and its schematics are drawn in Fig. B.3. With Relation (B.18), we also have the intersection number I giving the number of families. The resulting spectrum is shown in Table B.2. With a number of intersecting branes, we have the gauge group,

$$U(3)_a \times U(2)_b \times U(1)_c \times U(1)_d .$$

To cancel the total R–R charge, O6 orientifold planes are introduced. Instead of O6, alternatively we can consider a mirror brane for which we use a star(*). They sit at the mirror image points of the original branes, and are complex conjugate representations. Compare (ab) intersection with (ab*).

With the $U(1)$ charge assignments $Q_{a,b,c,d}$ of Table B.2, we can interpret them as the baryon number, the lepton number, the right-handed isospin, and a Pecci–Quinn type charge, respectively. This is a generic feature because, for example, the $U(1)$ from the strong interaction group $U(3) = U(1) \times SU(3)$ must be a significant contributor to the baryon number symmetry $U(1)_B$, because only strongly interacting quarks are transforming nontrivially under the strong gauge group. The lepton number in the above model is conserved, so Majorana type neutrino masses are not possible, and hence a seesaw mechanism is not allowed. Such $U(1)$ symmetries are local, and a potential anomaly

Table B.1. An example of D6-brane, [6], with wrapping numbers giving a SM spectrum

# Coincidence	(l_i^1, m_i^1)	(l_i^2, m_i^2)	(l_i^3, m_i^3)
$n_a = 3$	(1, 0)	(2, 1)	(1, 1/2)
$n_b = 2$	(0, -1)	(1, 0)	(1, 3/2)
$n_c = 1$	(1, 3)	(1, 0)	(0, 1)
$n_d = 1$	(1, 0)	(0, -1)	(1, 3/2)

Table B.2. The standard model spectrum of Table B.1 and their U(1) charges. The intersection bd has vanishing intersection number

Intersection	I	Matter		Q_a	Q_b	Q_c	Q_d	Y
(ab)	1	Q_L	(3, 2)	1	-1	0	0	1/6
(ab*)	2	q_L	(3, 2)	1	1	0	0	1/6
(ac)	-3	U_R	($\bar{3}$, 1)	-1	0	1	0	-2/3
(ac*)	-3	D_R	($\bar{3}$, 1)	-1	0	-1	0	1/3
(bd*)	-3	L	(1, 2)	0	-1	0	-1	-1/2
(cd)	-3	E_R	(1, 1)	0	0	-1	1	1
(cd*)	3	N_R	(1, 1)	0	0	1	1	0

is cancelled by the GS mechanism. The corresponding gauge bosons become massive but the symmetries still persists as global symmetries, which suppress rapid proton decay. The non-anomalous hypercharge generator is

$$Y = \frac{1}{6}Q_a - \frac{1}{2}Q_c + \frac{1}{2}Q_d.$$

With SUSY

The model presented in [8, 9, 10] achieved a supersymmetric realization by using an orbifold on top of the intersecting branes. Recall that supersymmetry condition is the same with that of the heterotic string. The orbifold action is $\mathbf{Z}_2 \times \mathbf{Z}_2$ with shift vectors, $\theta = \frac{1}{2}(1 \ 1 \ 0)$ and $\omega = \frac{1}{2}(0 \ 1 \ 1)$. The common supersymmetry is $\mathcal{N} = 1$. Under orbifolding, the orientifold plane is also nontrivial. Notice here that the D-brane has an orientation, because open string endpoints are charged differently, thus $(1 \ 1 \ 0)$ are different from $(1 \ -1 \ 0)$.

Here, we just take an example of a four family model where all the two-tori are rectangular. In order to have three family model they tilted one of tori. The resulting group is $U(3) \times U(2) \times USp(2) \times USp(8) \times USp(8)$.

We check that this can satisfy the supersymmetry condition (B.24, B.25) leaving $\mathcal{N} = 1$ supersymmetry. This type of $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold makes it easy because each cycle is just 2-cycle not three. In general, it is hard to satisfy the condition (B.25) even though we can freely adjust R_i 's. The resulting spectrum is straightforward. As in the heterotic string case, in general there are many exotic particles charged under the SM gauge group.

In conclusion, the intersecting D-brane models share many common features with the heterotic string theory on orbifold. In fact, the intersecting brane model has the duality with an M theory compactified on the G_2 manifold [11]. With such a string duality, we will be able to see more connections between different theories considered today.

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C

Elliptic Functions

In this appendix, we present a brief summary on some properties of elliptic functions. For a detailed discussion, see [1].

The Dedekind eta function is

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) . \quad (\text{C.1})$$

It has the following modular transformation property

$$\mathcal{T} : \eta(\tau + 1) = e^{i\pi/12} \eta(\tau) \quad (\text{C.2})$$

$$\mathcal{S} : \eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau) . \quad (\text{C.3})$$

To deal with a more general boundary condition, we introduce the Jacobi theta function

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) = \sum_{n \in \mathbf{Z}} q^{\frac{1}{2}(n+\alpha)^2} e^{2\pi i(n+\alpha)\beta} \quad (\text{C.4})$$

$$= \eta(\tau) e^{2\pi i \alpha \beta} q^{\alpha^2/2 - 1/24} \prod_{n=1}^{\infty} (1 + q^{n+\alpha-1/2} e^{2\pi i \beta}) (1 + q^{n-\alpha-1/2} e^{-2\pi i \beta}) . \quad (\text{C.5})$$

In Chap. 5, we have seen that the superstring partition function is described by the following functions,

$$\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\tau) = \vartheta_1(\tau) = 0 , \quad (\text{C.6})$$

$$\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (\tau) = \vartheta_2(\tau) = 2q^{1/4} (1 + q^2 + q^6 + q^{12} + q^{20} + \dots) , \quad (\text{C.7})$$

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) = \vartheta_3(\tau) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots , \quad (\text{C.8})$$

$$\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (\tau) = \vartheta_4(\tau) = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} + \dots . \quad (\text{C.9})$$

Jacobi's "abstruse identity" used in Chap. 5 is

$$(\vartheta_3(\tau))^4 - (\vartheta_4(\tau))^4 - (\vartheta_2(\tau))^4 = 0 \quad (\text{C.10})$$

which can be checked for each q^{2n} term in the above expansion. The modular transformations are

$$\mathcal{T}: \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau + 1) = e^{-i\pi(\alpha^2 - \alpha)} \vartheta \begin{bmatrix} \alpha \\ \alpha + \beta - 1/2 \end{bmatrix}(\tau), \quad (\text{C.11})$$

$$\mathcal{S}: \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\left(\frac{-1}{\tau}\right) = (-i\tau)^{1/2} e^{2\pi i \alpha \beta} \vartheta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}(\tau). \quad (\text{C.12})$$

In calculating a loop diagram of an *unoriented* string, we need more general boundary conditions. Sometimes an oriented string also requires such more general boundary conditions. Thus we can extend this to define

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu|\tau) \equiv \vartheta \begin{bmatrix} \alpha \\ \nu + \beta \end{bmatrix}(\tau) \quad (\text{C.13})$$

by replacing β with $\nu + \beta$. It has similar transformation property

$$\mathcal{T}: \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu|\tau + 1) = e^{-i\pi(\alpha^2 - \alpha)} \vartheta \begin{bmatrix} \alpha \\ \alpha + \beta - 1/2 \end{bmatrix}(\nu|\tau), \quad (\text{C.14})$$

$$\mathcal{S}: \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\left(\frac{\nu}{\tau} \middle| \frac{-1}{\tau}\right) = (-i\tau)^{1/2} e^{2\pi i(\nu^2/2\tau + \alpha\beta)} \vartheta \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}(\nu|\tau). \quad (\text{C.15})$$

Under transformation of argument ν

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu + 1|\tau) = e^{2\pi i \alpha} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu|\tau) \quad (\text{C.16})$$

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu + \tau|\tau) = e^{-2\pi i(\beta + \nu + \tau/2)} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu|\tau). \quad (\text{C.17})$$

In dealing with shift vectors, it is convenient to use

$$\vartheta \begin{bmatrix} \alpha + 1 \\ \beta \end{bmatrix}(\nu|\tau) = \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu|\tau) \quad (\text{C.18})$$

$$\vartheta \begin{bmatrix} \alpha \\ \beta + 1 \end{bmatrix}(\nu|\tau) = e^{2\pi i \alpha} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu|\tau). \quad (\text{C.19})$$

They are easily read off from the sum and the product definitions, respectively.

Zeta Function Regularization

By reading off the partition function, we can rationalize zero point energy $f(\eta)$ of (6.15). Expressing partition function form by sum representation like (7.37), the numerator does not have fractional power of q . From the denominator,

besides the oscillator part, the fractional power emerges from the product definition of $q^{\alpha^2/2}$, which is from the shift $\alpha = kV + 1/2$ of the k th twisted sector. From the simple property of (C.19), note that α modulo integer is the same as α . Thus, we can always make $-1/2 \leq \alpha - d < 1/2$, or

$$0 \leq kV - d < 1$$

by subtracting a vector d with integral elements. This is nothing but the restriction on η in (6.15). Now we count the total fractional power, coming only from $\eta(\tau)$, beside the oscillator. Each *complex* degree of freedom has contribution

$$2f(\eta) = \frac{1}{24} - \frac{1}{2} \left(\eta - \frac{1}{2} \right)^2, \quad (\text{C.20})$$

which is precisely (6.15). In other words, the zeta function regularization is consistent if we require the modular invariance.

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D

Useful Tables for Model Building

In this Appendix, we collect useful tables for model building. Firstly, we list Tables 3.2 and 3.3 discussed in Chap. 3. Then, we list the degeneracy factor $\tilde{\chi}(\theta^i, \theta^j)$ which counts the GSO allowed multiplicity. Compared to the geometrical study for each case, using it is rather straightforward. Finally, we repeat Table 7.1, constraining Wilson line conditions.

The shift vectors ϕ and the number of fixed points χ for possible six dimensional orbifolds allowing $\mathcal{N} = 1$ supersymmetry are given in the following table,

Order	Crystallographic Lattice	ϕ	χ
3	$SU(3)^3$	$\frac{1}{3}(2 \ 1 \ 1)$	27
4	$SU(4)^2$ $SU(4) \times SO(5) \times SU(2)$ $SO(5)^2 \times SU(2)^2$	$\frac{1}{4}(2 \ 1 \ 1)$	16
6-I	$SU(3) \times SU(3)^2$ $SU(3) \times [SU(3)^2]$	$\frac{1}{6}(2 \ 1 \ 1)$	3
6-II	$SU(2) \times SU(6)$ $SU(3) \times SO(8)$ $SU(2)^2 \times SU(3)^2$ $SU(2)^2 \times [SU(3)^2]$	$\frac{1}{6}(3 \ 2 \ 1)$	12
7	$SU(7)$	$\frac{1}{7}(3 \ 2 \ 1)$	7
8-I	$SO(5) \times SO(8)$ $[SU(4)^2]$	$\frac{1}{8}(3 \ 2 \ 1)$	4
8-II	$SU(2) \times SO(10)$ $SU(2)^2 \times SO(8)$	$\frac{1}{8}(4 \ 3 \ 1)$	8
12-I	E_6 $SU(3) \times SO(8)$	$\frac{1}{12}(5 \ 4 \ 1)$	3
12-II	$SU(2)^2 \times SO(8)$	$\frac{1}{12}(6 \ 5 \ 1)$	4

The $\mathcal{N} = 1$ supersymmetry can be obtained also by a product orbifold $\mathbf{Z}_M \times \mathbf{Z}_N$, which is shown pictorially in Fig. 10.1, where $\mathbf{Z}_M$ and $\mathbf{Z}_N$ are chosen from the following table,

Orbifold	Grid Unit	ϕ	χ
$\mathbf{Z}_2$	half plane	$\frac{1}{2}(1\ 1)$	16
$\mathbf{Z}_3$	hexagon	$\frac{1}{3}(1\ 1)$	9
$\mathbf{Z}_4$	square	$\frac{1}{4}(1\ 1)$	1
$\mathbf{Z}_6$	triangle	$\frac{1}{6}(1\ 1)$	1

For prime orbifolds, the degeneracy factor for the allowed massless fields is simply given by the number of fixed points. Massless fields are located either in the bulk or at the fixed points. So, it is clear in $\mathbf{Z}_3$ and $\mathbf{Z}_7$ orbifolds to locate geometrically the massless fields. For nonprime orbifolds, however, the allowed massless states are some combinations of functions located at fixed points. Thus, the calculation of the degeneracy factor is more involved, as given in the $\mathbf{Z}_{12}$ example of Chap. 10.

The i th twisted sector has the multiplicity

$$P_{\theta^i} = \frac{1}{N} \sum_{j=0}^{N-1} \tilde{\chi}(\theta^i, \theta^j) \Delta(i)^j \quad (\text{D.1})$$

where Δ is the phase of the state, $|\text{L} - \text{movers}\rangle \otimes |\text{R} - \text{movers}\rangle$. In the standard case of taking $i = 0$ as the untwisted sector, the i^{th} twisted sector has

$$\Delta(i) = e^{2\pi i[(P+iV) \cdot V - (\tilde{s}+i\phi) \cdot \phi]}$$

where $\tilde{s}$ is the last three entries of $s = \{s_0; \tilde{s}\}$. The helicity $\oplus$ or $\ominus$ is determined from $|\text{R} - \text{movers}\rangle$, by the first component of $s = \{s_0; \tilde{s}\}$. s has half integers such as $[++ +]$ with $[\]$ indicating even number of sign flips, i.e. in this example even number of minus signs and all possible permutations.

As noted above, for the prime orbifolds $\mathbf{Z}_N$ ($N = 3, 7$) we have the (i, j) independent $\tilde{\chi}(\theta^i, \theta^j) = \chi$, and hence the multiplicity is $\chi(\Delta^0 + \Delta^1 + \dots + \Delta^{N-1})/N$ which is 0 for $\Delta = e^{2\pi i k/N}$ ($k = 1, \dots, N-1$) and equivalent to the Euler number χ for the case of $\Delta = 1$.

For the nonprime orbifolds, one has to perform a nontrivial calculation based on (D.1). For this purpose, the degeneracy factors $\tilde{\chi}(\theta^i, \theta^j)$ calculated by (7.33) are given below:

$$\tilde{\chi}(\theta^i, \theta^j)$$

	$j =$
i	0 1 2
$\mathbf{Z}_3$	1 27 27 27

	$j =$
i	0 1 2 3
$\mathbf{Z}_4$	1 16 16 16 16
	2 16 4 16 4

	$j =$
i	0 1 2 3 4 5
$\mathbf{Z}_6\text{--I}$	1 3 3 3 3 3 3
	2 27 3 27 3 27 3
	3 16 1 1 16 1 1
$\mathbf{Z}_6\text{--II}$	1 12 12 12 12 12 12
	2 9 3 9 3 9 3
	3 16 4 4 16 4 4

	$j =$
i	0 1 2 3 4 5 6
$\mathbf{Z}_7$	1 7 7 7 7 7 7
	2 7 7 7 7 7 7
	3 7 7 7 7 7 7

	$j =$
i	0 1 2 3 4 5 6 7
$\mathbf{Z}_8\text{--I}$	1 4 4 4 4 4 4 4
	2 16 4 16 4 16 4 16
	3 4 4 4 4 4 4 4
	4 16 2 4 2 16 2 4
$\mathbf{Z}_8\text{--II}$	1 8 8 8 8 8 8 8
	2 4 2 4 2 4 2 4
	3 8 8 8 8 8 8 8
	4 16 2 4 2 16 2 4

	$j =$
i	0 1 2 3 4 5 6 7 8 9 10 11
$\mathbf{Z}_{12}\text{--I}$	1 3 3 3 3 3 3 3 3 3 3 3
	2 3 3 3 3 3 3 3 3 3 3 3
	3 4 1 1 4 1 1 4 1 1 4 1
	4 27 3 3 3 27 3 3 3 27 3 3
	5 3 3 3 3 3 3 3 3 3 3 3
	6 16 1 1 4 1 1 16 1 1 4 1
$\mathbf{Z}_{12}\text{--II}$	1 4 4 4 4 4 4 4 4 4 4 4
	2 1 1 1 1 1 1 1 1 1 1 1
	3 16 4 4 16 4 4 16 4 4 16 4
	4 9 1 1 1 9 1 1 1 9 1 1
	5 4 4 4 4 4 4 4 4 4 4 4
	6 16 1 1 4 1 1 16 1 1 4 1

For calculating masses in the k^{th} twisted sector, we need the vacuum energy contribution. The weights for massless states satisfy

$$(P + \tilde{V})^2 = 2(1 - \zeta_{N,k}) - 2\tilde{N}_L. \quad (\text{D.2})$$

where $\zeta_{N,k} = \frac{1}{4} \sum_i \phi_i(1 - \phi_i)$ with ϕ_i lattice-shifted such that $0 \leq \phi_i \leq 1$. The present $(1 - \zeta)$ corresponds to $\tilde{c}$ of Chaps. 5 and 6. Below we list -2 times the vacuum energy $-2(1 - \zeta_{N,k})$ for the left movers,

$2\tilde{c} = 2(1 - \zeta_{N,k})$						
$k =$	1	2	3	4	5	6
$\mathbf{Z}_{12}\text{-I}$	$\frac{210}{144}$	$\frac{216}{144}$	$\frac{234}{144}$	$\frac{192}{144}$	$\frac{210}{144}$	$\frac{216}{144}$
$\mathbf{Z}_{12}\text{-II}$	$\frac{206}{144}$	$\frac{248}{144}$	$\frac{198}{144}$	$\frac{224}{144}$	$\frac{206}{144}$	$\frac{216}{144}$
$\mathbf{Z}_8\text{-I}$	$\frac{94}{64}$	$\frac{88}{64}$	$\frac{94}{64}$	$\frac{96}{64}$		
$\mathbf{Z}_8\text{-II}$	$\frac{90}{64}$	$\frac{104}{64}$	$\frac{90}{64}$	$\frac{96}{64}$		
$\mathbf{Z}_7$	$\frac{10}{7}$	$\frac{10}{7}$	$\frac{10}{7}$			
$\mathbf{Z}_6\text{-I}$	$\frac{54}{36}$	$\frac{48}{36}$	$\frac{54}{36}$			
$\mathbf{Z}_6\text{-II}$	$\frac{50}{36}$	$\frac{56}{36}$	$\frac{54}{36}$			
$\mathbf{Z}_4$	$\frac{22}{16}$	$\frac{24}{16}$				
$\mathbf{Z}_3$	$\frac{4}{3}$					
$\mathbf{Z}_2(6\text{D})$	1					

For -2 times the *bosonized* vacuum energy of right movers, we find

$-2c = -(2\tilde{c} - 1)$						
$k =$	1	2	3	4	5	6
$\mathbf{Z}_{12}\text{-I}$	$\frac{11}{24}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{1}{3}$	$\frac{11}{24}$	$\frac{1}{2}$
$\mathbf{Z}_{12}\text{-II}$	$\frac{31}{72}$	$\frac{13}{18}$	$\frac{3}{8}$	$\frac{5}{9}$	$\frac{31}{72}$	$\frac{1}{2}$
$\mathbf{Z}_8\text{-I}$	$\frac{15}{32}$	$\frac{3}{8}$	$\frac{15}{32}$	$\frac{1}{2}$		
$\mathbf{Z}_8\text{-II}$	$\frac{13}{32}$	$\frac{5}{8}$	$\frac{13}{32}$	$\frac{1}{2}$		
$\mathbf{Z}_7$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{7}$			
$\mathbf{Z}_6\text{-I}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$			
$\mathbf{Z}_6\text{-II}$	$\frac{50}{36}$	$\frac{56}{36}$	$\frac{54}{36}$			
$\mathbf{Z}_4$	$\frac{3}{8}$	$\frac{1}{2}$				
$\mathbf{Z}_3$	$\frac{1}{3}$					
$\mathbf{Z}_2(6\text{D})$	$\frac{1}{2}$					

where the vacuum energy contributions from the R and NS sectors add up. For example, the $k = 1$ sector of $\mathbf{Z}_{12} - I$ leads to

$$c = -\frac{1}{2} + 4f(0) + 2f(0) + 2f\left(\frac{5}{12}\right) + 2f\left(\frac{4}{12}\right) + 2f\left(\frac{1}{12}\right) = -\frac{11}{48}.$$

Finally, the Wilson line conditions are given in the following table,

	Lattice	Effective Order	Condition
$\mathbf{Z}_3$	$\text{SU}(3)^3$	$3a_1 = 0, 3a_3 = 0,$ $3a_5 = 0$	$a_1 = a_2, a_3 = a_4, a_5 = a_6$
$\mathbf{Z}_4$	$\text{SU}(4)^2$	$2a_1 = 0, 2a_4 = 0$	$a_1 = a_2 = a_3, a_4 = a_5 = a_6$
	$\text{SU}(4) \times \text{SO}(5) \times \text{SU}(2)$	$2a_1 = 0, 2a_5 = 0,$ $2a_6 = 0$	$a_1 = a_2 = a_3, a_4 = 0$
	$\text{SO}(5)^2 \times \text{SU}(2)^2$	$2a_2 = 0, 2a_4 = 0,$ $2a_5 = 0, 2a_6 = 0$	$a_1 = a_3 = 0$
$\mathbf{Z}_6\text{-I}$	$\text{SU}(3) \times \text{SU}(3)^2$	$3a_1 = 0$	$a_1 = a_2, a_3 = a_4 = a_5 = a_6 = 0$
$\mathbf{Z}_6\text{-II}$	$\text{SU}(2) \times \text{SU}(6)$	$2a_1 = 0$	$a_2 = a_3 = a_4 = a_5 = a_6 = 0$
	$\text{SU}(3) \times \text{SO}(8)$	$3a_1 = 0, 2a_5 = 0$	$a_1 = a_2, a_3 = a_4 = 0, a_5 = a_6$
	$\text{SU}(2)^2 \times \text{SU}(3)^2$	$3a_1 = 0, 2a_3 = 0,$ $2a_4 = 0$	$a_1 = a_2, a_5 = a_6 = 0$
	$\text{SU}(7)$	$7a_1 = 0$	$a_1 = a_2 = a_3 = a_4 = a_5 = a_6$
$\mathbf{Z}_8\text{-I}$	$\text{SO}(8) \times \text{SO}(5)$	$2a_1 = 0, 2a_6 = 0$	$a_1 = a_2 = a_3 = a_4, a_5 = 0$
$\mathbf{Z}_8\text{-II}$	$\text{SO}(10) \times \text{SU}(2)$	$2a_4 = 0, 2a_6 = 0$	$a_1 = a_2 = a_3 = 0, a_4 = a_5$
	$\text{SU}(2)^2 \times \text{SO}(8)$	$2a_1 = 0, 2a_5 = 0,$ $2a_6 = 0$	$a_1 = a_2 = a_3 = a_4$
	E_6	no restriction	$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$
$\mathbf{Z}_{12}\text{-I}$	$\text{SU}(3) \times \text{SO}(8)$	$3a_1 = 0$	$a_1 = a_2, a_3 = a_4 = a_5 = a_6 = 0$
$\mathbf{Z}_{12}\text{-II}$	$\text{SU}(2)^2 \times \text{SO}(8)$	$2a_1 = 0, 2a_2 = 0$	$a_3 = a_4 = a_5 = a_6 = 0$

E

Some Algebraic Elements of Lie Groups

In this book, we mainly dealt with subgroups embedded in E_8 or $\text{Spin}(32)/\mathbf{Z}_2$, which arise in heterotic string. Thus, roots α are normalized as $\alpha^2 = 2$ and are called self-dual¹ $\alpha^* = \frac{2\alpha}{\alpha^2} = \alpha$. The Cartan matrices A are symmetric and they allow only the following “simply-laced” groups.

Simple Root and Fundamental Weight

Firstly, we present orthogonal representation of various roots. The simple roots of $\text{SU}(n+1)$ and $\text{SO}(2n)$ are

$$A_n \text{ or } \text{SU}(n+1) : \begin{cases} \alpha^1 = (1, -1, 0, 0, \dots, 0, 0) \\ \alpha^2 = (0, 1, -1, 0, \dots, 0, 0) \\ \alpha^3 = (0, 0, 1, -1, \dots, 0, 0) \\ \vdots \\ \alpha^n = (0, 0, 0, 0, \dots, 1, -1) \end{cases}$$

with $n+1$ entries, and

$$D_n \text{ or } \text{SO}(2n) : \begin{cases} \alpha^1 = (1, -1, 0, 0, \dots, 0, 0) \\ \alpha^2 = (0, 1, -1, 0, \dots, 0, 0) \\ \alpha^3 = (0, 0, 1, -1, \dots, 0, 0) \\ \vdots \\ \alpha^{n-1} = (0, 0, 0, 0, \dots, 1, -1) \\ \alpha^n = (0, 0, 0, 0, \dots, 1, 1) \end{cases}$$

with n entries. For E_8 , we have

¹ Do not confuse with the other usage of “self-dual” in lattices in the sense of Subsect. 5.3.3. There the self-duality meant $\det A = 1$.

$$E_8 : \begin{cases} \alpha^0 = (-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0) \\ \alpha^1 = (0 & 1 & -1 & 0 & 0 & 0 & 0 & 0) \\ \alpha^2 = (0 & 0 & 1 & -1 & 0 & 0 & 0 & 0) \\ \alpha^3 = (0 & 0 & 0 & 1 & -1 & 0 & 0 & 0) \\ \alpha^4 = (0 & 0 & 0 & 0 & 1 & -1 & 0 & 0) \\ \alpha^5 = (0 & 0 & 0 & 0 & 0 & 1 & -1 & 0) \\ \alpha^6 = (0 & 0 & 0 & 0 & 0 & 0 & 1 & -1) \\ \alpha^7 = (+ & - & - & - & - & - & - & +) \\ \alpha^8 = (0 & 0 & 0 & 0 & 0 & 0 & 1 & 1) \end{cases}$$

where $+$ and $-$ denote $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Comparing these, one can easily see which subalgebra survives by eliminating some roots.

The fundamental weights Λ_j are defined by

$$\alpha^i \cdot \Lambda_j = \delta_j^i.$$

For E_8 , they are

$$\begin{aligned} \Lambda_1 &= (1 & 1 & 0 & 0 & 0 & 0 & 0 & 0) \\ \Lambda_2 &= (2 & 1 & 1 & 0 & 0 & 0 & 0 & 0) \\ \Lambda_3 &= (3 & 1 & 1 & 1 & 0 & 0 & 0 & 0) \\ \Lambda_4 &= (4 & 1 & 1 & 1 & 1 & 0 & 0 & 0) \\ \Lambda_5 &= (5 & 1 & 1 & 1 & 1 & 1 & 0 & 0) \\ \Lambda_6 &= (\frac{7}{2} & + & + & + & + & + & + & -) \\ \Lambda_7 &= (2 & 0 & 0 & 0 & 0 & 0 & 0 & 0) \\ \Lambda_8 &= (\frac{5}{2} & + & + & + & + & + & + & +) \end{aligned}$$

from which the extended root is found to be the first fundamental weight $\alpha^0 \equiv -\Lambda_1$ and satisfies the relations $(\alpha^0)^2 = 2$ and $\alpha^0 \cdot \alpha^1 = -1$. Λ_0 has zero elements in the above notation, and it is $\Lambda_0 = (0, 0, 1)$ in the notation of (9.61). Note that $-\alpha^0 \cdot \Lambda_i = (\text{Coxeter level})$. Such orthonormal property is useful in dealing with shift vectors, as discussed in Sect. 9.4. For instance, the standard $\mathbf{Z}_3$ shift vector was $V = \frac{1}{3}(\Lambda_2)(0^8)$.

Cartan Matrix

We also define the Cartan matrix as

$$A^{ij} = \alpha^i \cdot \alpha^j = 2 \cos \theta_{ij}$$

where θ_{ij} is the angle between α^i and α^j . For simply laced groups, the Cartan matrices are shown below.

$$A_n : \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad D_n : \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}$$

$$E_6 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \quad E_8 : \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

and the Cartan matrix of E_7 can be obtained from that of E_8 by removing the first row and the first column. From the relation

$$\alpha^i = A^{ij} \Lambda_j$$

we can use contravariant and covariant index. Indeed, in the root space the inverse Cartan matrix $A_{ij}^{-1} = \Lambda_i \cdot \Lambda_j$ plays the role of a metric tensor, whose terminology is sometimes used in the literature.

$$(A^{SU(n+1)})^{-1} = \frac{1}{n+1} \begin{pmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & \cdots & 6 & 3 \\ & & & \ddots & & \\ 2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{pmatrix}$$

$$(A^{SO(2n)})^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 & 3 \\ & & & \ddots & & & \\ 2 & 4 & 6 & \cdots & 2(n-2) & n-2 & n-2 \\ 1 & 2 & 3 & \cdots & n-2 & n/2 & (n-2)/2 \\ 1 & 2 & 3 & \cdots & n-2 & (n-2)/2 & n/2 \end{pmatrix}$$

$$(A^{E_6})^{-1} = \frac{1}{3} \begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}$$

$$(A^{E_8})^{-1} = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix}$$

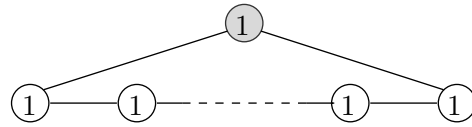
In terms of the fundamental weights, we can reduce the *highest weight vector* to a simple form. The highest weight representation in the main text P (or $P + V$ in the twisted sector) in the simple root basis has always has the form, viz. (9.66),

$$P = A^{-1} \alpha^i .$$

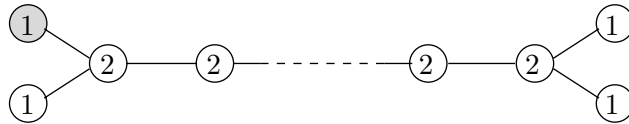
Dynkin Diagram

The above Cartan matrix elements can be simply represented by Dynkin diagrams which are composed of circles and lines. The usual convention is that an open circle denotes the $(\text{length})^2 = 2$ of a simple root and a single line represents 120 degrees between the simple roots connected to the line. If all the simple roots are open circles and all the connecting lines are single lines, then it is called a simply laced algebra. The angles 135 and 150 degrees between simple roots are represented by the double and triple lines, respectively. The ratio of simple roots connected to the double(triple) line is $\sqrt{2}(\sqrt{3})$ among which the smaller simple root is represented by a bullet. If any of these occurs, then it is not a simply laced group.

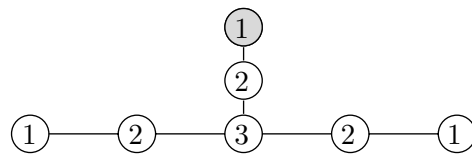
Below we show the Dynkin diagrams with the Coxeter labels of simply laced groups inside the circles. For simply laced groups, the extended roots are shown with grey color. In principle, Dynkin diagram contains all the information of a given algebra. Among them, for example we can extract easily the outer automorphism, or the automorphism not generated by the group elements. It is simply the geometric symmetry of a Dynkin diagram. For example, the simple $SU(n+1)$ Dynkin diagram possesses a $\mathbf{Z}_2$ symmetry along the vertical axis, which corresponds to a complex conjugation. Some extended diagram has an enhanced symmetry. For example, the extended E_6 has the $\mathbf{S}_3$ symmetry the order 3 permutation symmetry, which is called triality. In classifying group breaking, such a symmetry is not readily seen by the symmetry of a shift vector.



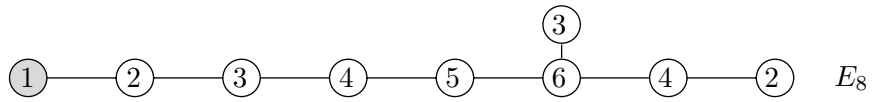
A_n



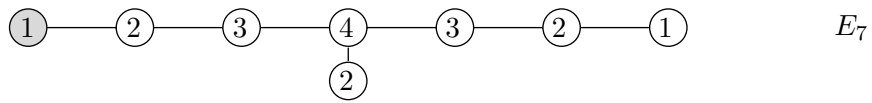
D_n



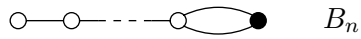
E_6



E_8



E_7



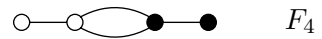
B_n



G_2



C_n



F_4

Symbol	Name, Equation	Value
$g_{\mu\nu}$	spacetime metric tensor	$\eta_{\mu\nu} = \text{diag.}(1, -1, -1, \dots, -1)$
$h_{\alpha\beta}$	worldsheet metric tensor	$\eta_{\alpha\beta} = \text{diag.}(1, -1)$
	GeV·s	1.519255×10^{24}
	GeV·cm	0.5067689×10^{14}
c	speed of light = 1	299 792 453 m/s
$\hbar$	Planck constant = 1	$1.064\,571\,596(82) \times 10^{-34}$ J s
M_P	Planck mass = $(8\pi G_N)^{-1/2}$	2.436×10^{18} GeV
M_{GUT}	grand unification scale $\simeq M_U$	$\sim 3 \times 10^{16}$ GeV
v	electroweak symmetry breaking scale	248 GeV
G_F	Fermi coupling constant	$1.166\,39 \times 10^{-5}$ GeV $^{-2}$
$\sin^2 \theta_W$	weak mixing angle	0.23113(15)
M_Z	Z boson mass = $\frac{1}{2}gv \sec \theta_W$	91.1876(21) GeV
ρ	rho parameter ($= \frac{M_W^2}{M_Z^2 \cos^2 \theta_W}$)	1.0012(23)
M_S	scale for SUSY breaking source	$\sim 0.5 \times 10^{11}$ GeV in SUGRA
M_{SUSY}	observable sector SUSY splitting scale	$\sim$ TeV
H_0	Hubble constant ($\sqrt{\rho_{\text{energy}}/3M_P^2}$)	$2.1332h \times 10^{-42}$ GeV ($h = 0.71_{-0.03}^{+0.04}$)
ρ_c	critical density ($0.81h^2 \times 10^{-46}\text{GeV}^4$)	$1.88h^2 \times 10^{-29}$ g cm $^{-3}$
t	cosmic time in RW universe	$\sim (10^{-3} \text{ GeV}/T)^2$ s
$\mathbf{Z}_N$	discrete symmetry of order N	
$\tilde{N}, N$	oscillator	
N	order	
y	extra dimension coordinate	
$\mathcal{N}$	number of 4D supersymmetries	
θ	point group (orbifold) action	
ϕ	orbifold action vector in spacetime	
α'	inverse tension, Regge slope = ℓ^2	
L_n	Virasoro operator	
P	momentum lattice vector	
V	shift vector from point group action	
a	shift vector from Wilson line	
$f(\eta)$	zero point energy per a real degree	$f(0) = -\frac{1}{24}, f(\frac{1}{2}) = \frac{1}{48}$
$\tilde{c}, \tilde{\alpha}, z$	for left movers	(6.16)
$c, \alpha, \bar{z}$	for right movers	(6.25)
τ	modulus defining torus	
$\mathcal{T}, \mathcal{S}$	modular transformations	
Γ	gauge group root lattice	
ϑ	Jacobi elliptic (theta) function	
η	Dedekind (eta) function	
P_m	projection operator	
$\tilde{\chi}_{mn}$	multiplicity in (θ^m, θ^n) twisted sector	
α	root, made of simple roots α^i	
Λ_i	fundamental weight	
Λ	orbifold lattice	
k	level of affine Lie algebra	

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