

C. M. Becchi • G. Ridolfi

AN INTRODUCTION TO RELATIVISTIC PROCESSES AND THE STANDARD MODEL OF ELECTROWEAK INTERACTIONS



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Carlo M. Becchi, Giovanni Ridolfi

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Preface

The natural framework of high-energy physics is relativistic quantum field theory. This is a complex subject, and it is difficult to illustrate it in all its aspects within a normal undergraduate course in particle physics, while devoting a sufficient attention to phenomenological aspects. However, in the small-wavelength limit, the semi-classical approximation is, in many cases of practical relevance, accurate enough to provide reliable predictions without entering the technicalities connected with radiative corrections. In particular, in the framework of the semi-classical approximation it is possible to obtain, in a limited number of pages, the expressions for relativistic cross sections and decay rates in a self-contained and rigorous presentation, starting from the basic principles of Quantum Mechanics. Furthermore, even in the case of the standard model of Electroweak Interactions, the construction of the theory in the semi-classical approximation is exhausted by the study of the classical Lagrangian; many difficult problems, such as those related to the unphysical content of gauge theories, can be dealt with by means of simple prescriptions.

These are the reasons that have determined our choice to base these lecture notes on the semi-classical approximation to relativistic quantum field theory. We believe that this approach leads to a description of the most relevant physical processes in high-energy physics, which is adequate to an undergraduate level course on fundamental interactions.

Of course, the lack of control on radiative corrections has some drawbacks: for example, the role of anomalies, and the limitations to the Higgs mechanism, cannot be discussed in this context. These issues, however, are beyond the scope of the present text.

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Introduction

The study of relativistic processes is based on collision phenomena at energies much larger than the rest energies of the particles involved. In this regime, a large number of new particles is typically produced, with large momenta, or, equivalently, small wavelengths. For this reason, the scheme of ordinary Quantum Mechanics, based on the Schrödinger equation for wave functions that depend on a fixed number of variables, is no longer applicable. A suitable framework is rather provided by electromagnetism, that describes radiation phenomena, and therefore the production and absorption of photons. This analogy leads in a natural way to field theory, in which the dynamical variables that describe a given physical system are fields, i.e. variables labelled by the space coordinates, and independent of each other.

These lecture notes aim at presenting a self-contained introduction to the theory of electroweak interactions based on the semi-classical approach to relativistic quantum field theory. This allows a rather detailed discussion of the most relevant aspects of this field.

The main drawbacks of the use of this approximation essentially amount to the lack of any reference to the problem of anomalies, and the corresponding constraints, and of any access to processes that appear at the level of radiative corrections, such as the Higgs decay into two photons, and the whole sector of flavour mixing through neutral currents. These subjects are widely illustrated in the existing literature; see for example [1, 2, 3]. For the same reason, the applications of Quantum Chromodynamics to problems of phenomenological interest are only marginally discussed; a complete review can be found for example in [4]. However, we consider a self-contained analysis of these subjects very difficult to present in the framework of an undergraduate course.

These lectures are organized as follows. Chapter 2 is devoted to a review of basic facts in field theory. We illustrate the simple case of spinless neutral particles, described by a single scalar field. We build an action functional for such a system, and derive the corresponding dynamical equations. These are studied explicitly in the small-field regime, that corresponds to asymptotic states in collision processes. Next, we illustrate the role of symmetries in field

theories, and their relation with conservation laws. Finally, we illustrate the interpretation of the physical states of a quantum scalar field theory as particle states, and we introduce the concept of antiparticle.

In the following Chapter we discuss the calculation of transition amplitudes for collision and decay processes in the semi-classical approximation; we determine the asymptotic conditions to be imposed on the dynamical variables in the remote past and in the far future, and we show how to compute the transition amplitude in the semi-classical limit. We obtain a general formula, which is applied to the case of elastic collisions of spinless particle; the differential cross section for this process is computed in full detail. Chapter 4 is devoted to an introduction to the method of Feynman diagrams.

The results obtained for scalar particles are then extended to spinor particles in Chapter 5, starting from Weyl's basic construction of spinor representations of the Lorentz group. We discuss in some detail the construction of a general class of free Lagrangian densities for spinor fields, including the choices of mass terms that are relevant to neutrino physics.

Spinor fields find their natural application in Quantum Electrodynamics, which is studied in Chapter 6. In this context, we introduce the concept of gauge invariance, which is then extended to the case of non-commutative charges. This allows us to present the Lagrangian density of QCD.

The standard model of electroweak interactions is introduced in Chapter 7. The following Chapter is devoted to the description of the Higgs mechanism, which is first illustrated in the simple case of an abelian symmetry, and then extended to the standard model. We have not attempted an exhaustive review of present literature on the experimental foundations of the standard model and of the present status of its experimental tests. The foundations of the standard model are described for example in [5], while for a review of precision tests we refer the reader to [6]. In Chapter 9 we discuss fermion masses and flavour-mixing phenomena. An exhaustive analysis of these subjects can be found in [7]. Finally, in Chapter 10 we present the full Lagrangian density of the standard model, in a form which allows a direct derivation of the Feynman rules.

The full computation of a few decay rates and cross section is presented in Chapter 11. The results obtained are useful, both for the purpose of illustration of the standard computation techniques in field theory, and for their phenomenological relevance.

Chapter 12 contains an elementary introduction to the important subject of the extensions of the standard model that include neutrino masses. We describe the see-saw mechanism, and we present a simple description of the related phenomenon of lepton flavour oscillations (see [8] for a review on this subject.)

Relativistic field theory

2.1 Scalar fields

The dynamical variables that describe relativistic systems are *fields*, that is, functions defined in each point of ordinary space. Important examples are the electromagnetic fields, and Dirac and Yukawa fields. The field description of a physical system allows a direct implementation of the principle of covariance, that guarantees the invariance of the equations of motion under changes of reference frame, and of the principle of causality, which is connected to the principle of locality, namely, the independence of variables associated to different points in space at the same time.

The equations of motion for a field theory follow from the principle of stationarity of the action functional, which is defined, on the basis of covariance and locality, as an integral over the three-dimensional space and over the relevant time interval, of a Lagrangian density:

$$S = \int_{t_1}^{t_2} dt \int d\mathbf{r} \mathcal{L}(\mathbf{r}, t). \quad (2.1)$$

The Lagrangian density $\mathcal{L}$ is assumed to be a local function of the fields and their first derivatives, so that the equations of motion contain at most second derivatives.

Lorentz invariance of the action functional is achieved if $\mathcal{L}$ transforms as a scalar field: going from a reference frame O to O' through a Lorentz transformation $x' = \Lambda x$, such as for example

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{\beta}{\sqrt{1-\beta^2}} \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ \frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (2.2)$$

the Lagrangian density must transform as

$$\mathcal{L}'(x) = \mathcal{L}(\Lambda^{-1}x). \quad (2.3)$$

We begin by considering the simple case of a single real field ϕ , which is assumed to transform as a scalar under Lorentz transformations. By analogy with classical mechanics, the Lagrangian density has the form

$$\mathcal{L} = \frac{1}{2c^2} (\partial_t \phi)^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi), \quad (2.4)$$

where ∂_t is the partial derivative with respect to t , and $V(\phi)$, the *scalar potential*, is independent of the field derivatives, and bounded from below. We recognize the typical structure of a Lagrangian, difference between a kinetic term and a potential term. A fundamental constraint, called *renormalizability*, requires that $V(\phi)$ be a polynomial in ϕ of degree not larger than four; this point will be discussed in more detail in Section 4.4.

Equation (2.4) can be cast in a manifestly covariant form:

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi), \quad (2.5)$$

where ∂^μ is the four-vector formed with the partial derivatives with respect to space-time coordinates.¹

Let us now consider the variation of the action functional under a generic infinitesimal variation $\delta\phi$ of ϕ , localized in space, with the conditions $\delta\phi(t_1, \mathbf{r}) = \delta\phi(t_2, \mathbf{r}) = 0$. We find

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \int d\mathbf{r} \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \delta \partial^\mu \phi \right] \\ &= \int_{t_1}^{t_2} dt \int d\mathbf{r} \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \right] \delta\phi. \end{aligned} \quad (2.6)$$

Requiring stability of S under such field variations, and exploiting the arbitrariness of $\delta\phi$, we get the field equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} = 0. \quad (2.7)$$

In the case of the Lagrangian density eq. (2.5) we find

$$\partial^2 \phi + \frac{dV(\phi)}{d\phi} = 0. \quad (2.8)$$

If the scalar potential $V(\phi)$ is a second-degree polynomial, eq. (2.8) becomes linear, and can be made homogeneous by a field translation:

$$\partial^2 \phi + a\phi = 0 \quad (2.9)$$

¹ Throughout these lectures, we adopt the convention $a^\mu a_\mu = a_0^2 - |\mathbf{a}|^2$.

where a is a real constant. If $a \geq 0$, eq. (2.9) is a harmonic equation, multidimensional extension of the equation of the harmonic oscillator. As we shall see in detail in Section 2.3, the quantization of this equation is straightforward; it describes a system of non-interacting bosons of mass $\frac{\hbar\sqrt{a}}{c}$. For this reason, the quadratic part of the Lagrangian density will be called the *free* part, while the rest will be called the *interaction*.

From the point of view of classical field theory we show in Appendix A that a generic solution of eq. (2.9) vanishes, for large t , as $t^{-3/2}$. As a consequence of this behaviour, if $V(\phi)$ has its absolute minimum in $\phi = 0$, one can show that the large-time solutions of eq. (2.8) approach those of the corresponding harmonic (free) equation for sufficiently small initial values of the field and its derivatives. In other words, the asymptotically vanishing harmonic solutions become attractors, since the non-linear terms in the field equation vanish faster than the linear terms. Of course, if the minimum of the potential is unique the set of fields with harmonic asymptotic behaviour widens.

In the context of high energy physics, one studies scattering processes of relativistic particles, that involve asymptotically free particle behaviour. This implies a selection of potentials leading to asymptotically free, and hence asymptotically linear, field equations. Therefore in the case of scalar fields we are led to require the potential to have its absolute minimum in $\phi = 0$. The simplest non-trivial example of such choice is

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \quad (2.10)$$

with λ positive. The corresponding free equation is

$$\partial^2\phi + \mu^2\phi = 0. \quad (2.11)$$

2.2 Symmetries in field theory

Let us consider an infinitesimal deformation of the field:

$$\phi \rightarrow \phi + \alpha \delta\phi, \quad (2.12)$$

where α is a constant infinitesimal parameter. We will call this a *global symmetry* transformation if it leaves unaltered the equations of motion, that is, if under (2.12) the action S is unchanged, or equivalently if

$$\mathcal{L} \rightarrow \mathcal{L} + \alpha \partial^\mu K_\mu, \quad (2.13)$$

where K_μ is some function of x . In this case, we have

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi} \alpha\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial^\mu\phi} \alpha\partial^\mu\delta\phi \\ &= \alpha\delta\phi \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial^\mu \frac{\partial\mathcal{L}}{\partial\partial^\mu\phi} \right] + \alpha\partial^\mu \left[\frac{\partial\mathcal{L}}{\partial\partial^\mu\phi} \delta\phi \right] \\ &= \alpha\partial^\mu K_\mu. \end{aligned} \quad (2.14)$$

Using the equations of motion eq. (2.7), we find that the vector current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi - K^\mu \quad (2.15)$$

obeys the continuity equation

$$\partial^\mu J_\mu = 0, \quad (2.16)$$

or equivalently

$$\frac{\partial}{\partial t} Q = 0; \quad Q = \int d\mathbf{r} J^0 \quad (2.17)$$

as a consequence of the symmetry property of the Lagrangian density eq. (2.13).

As an example, let us consider constant space-time translations:

$$x^\mu \rightarrow x^\mu + a^\mu \quad (2.18)$$

$$\delta \phi = a^\mu \partial_\mu \phi. \quad (2.19)$$

If the Lagrangian density does not depend explicitly on x , $\mathcal{L} = \mathcal{L}(\phi(x), \partial\phi(x))$, then

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} a^\mu \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} a^\mu \partial^\nu \partial_\mu \phi = a^\mu \partial^\nu \left[\frac{\partial \mathcal{L}}{\partial \partial^\nu \phi} \partial_\mu \phi \right] \quad (2.20)$$

by the equations of motion. On the other hand,

$$\delta \mathcal{L} = a^\mu \partial_\mu \mathcal{L} = a^\mu \partial^\nu (g_{\mu\nu} \mathcal{L}), \quad (2.21)$$

and therefore

$$\partial^\mu T_{\mu\nu} = 0 \quad (2.22)$$

where

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \partial_\nu \phi - g_{\mu\nu} \mathcal{L} \quad (2.23)$$

is called the *energy-momentum tensor*, because the physical quantities that are conserved as a consequence of space-time translation invariance are the total energy and momentum. For example, the total energy of the system is given by

$$E = \int d\mathbf{r} T_{00} = \int d\mathbf{r} \left(\frac{\partial \mathcal{L}}{\partial \partial^0 \phi} \partial_0 \phi - \mathcal{L} \right). \quad (2.24)$$

In the case of the field theory defined by (2.10) this gives

$$U = \frac{1}{2c^2} (\partial_t \phi)^2 + \frac{1}{2} \left[(\nabla \phi)^2 + \mu^2 \phi^2 \right] + \frac{\lambda}{4!} \phi^4. \quad (2.25)$$

As a second example, let us consider a system described by two real scalar fields ϕ_1, ϕ_2 . The Lagrangian density is a function of the fields and their

derivatives, $\mathcal{L}(\phi_i, \partial\phi_i)$, and the condition of stationarity of S leads to a system of field equations:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0 \quad , \quad i = 1, 2. \quad (2.26)$$

Let us assume that the Lagrangian density is invariant under the global transformation

$$\begin{pmatrix} \phi'_1(x) \\ \phi'_2(x) \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}. \quad (2.27)$$

To first order in α , eq. (2.27) becomes

$$\phi'_i(x) - \phi_i(x) = \alpha \sum_{j=1}^2 \epsilon_{ij} \phi_j(x), \quad (2.28)$$

where ϵ_{ij} is the antisymmetric tensor in two dimensions:

$$\epsilon_{12} = 1, \quad \epsilon_{21} = -1, \quad \epsilon_{11} = \epsilon_{22} = 0. \quad (2.29)$$

This transformation is a rotation in the two-dimensional space whose points are identified by the 'coordinates' ϕ_1, ϕ_2 ; this will be called the *isotopic space*, and has obviously nothing to do with the physical space. The assumed invariance of the Lagrangian density implies

$$\sum_{i,j=1}^2 \left[\partial_\mu (\epsilon_{ij} \alpha \phi_j(x)) \frac{\partial \mathcal{L}(x)}{\partial \partial_\mu \phi_i(x)} + \epsilon_{ij} \alpha \phi_j(x) \frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} \right] = 0 \quad (2.30)$$

(assuming $K^\mu = 0$.) Since α is arbitrary and x -independent, this becomes

$$\begin{aligned} & \sum_{i,j=1}^2 \epsilon_{ij} \left[\partial_\mu \phi_j(x) \frac{\partial \mathcal{L}(x)}{\partial \partial_\mu \phi_i(x)} + \phi_j(x) \frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} \right] \\ &= \sum_{i,j=1}^2 \epsilon_{ij} \left[\partial_\mu \phi_j(x) \frac{\partial \mathcal{L}(x)}{\partial \partial_\mu \phi_i(x)} + \phi_j(x) \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \right] \\ &= \partial_\mu \left[\sum_{i,j=1}^2 \epsilon_{ij} \phi_j(x) \right] = 0, \end{aligned} \quad (2.31)$$

where we have used eq. (2.26). Hence

$$\partial_\mu J^\mu(x) = 0; \quad J^\mu(x) = \sum_{i,j=1}^2 \epsilon_{ij} \phi_j(x) \frac{\partial \mathcal{L}(x)}{\partial \partial_\mu \phi_i(x)}. \quad (2.32)$$

The most general renormalizable Lagrangian density invariant under the transformation eq. (2.27) is

$$\mathcal{L} = \frac{1}{2} \left[(\partial\phi_1)^2 + (\partial\phi_2)^2 - \mu^2 (\phi_1^2 + \phi_2^2) \right] - \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2, \quad (2.33)$$

which gives

$$J_\mu = \phi_2 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_2. \quad (2.34)$$

This example simplifies considerably if we introduce a complex scalar field

$$\Phi(x) = \frac{1}{\sqrt{2}} [\phi_1(x) + i\phi_2(x)]. \quad (2.35)$$

The transformation in eq. (2.27) becomes

$$\Phi'(x) = e^{i\alpha} \Phi(x), \quad (2.36)$$

that is, it reduces to the multiplication of the complex field by a phase factor. Furthermore, we find

$$\mathcal{L} = \partial\Phi^* \partial\Phi - m^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2, \quad (2.37)$$

and the conserved current takes the form

$$J_\mu = i [\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*]. \quad (2.38)$$

The current eJ may be interpreted as the electromagnetic four-current associated to the complex field Φ , which is therefore to be interpreted as a charged scalar field.

The correspondence between conserved currents and invariance under continuous symmetries, that reduce to transformations like eq. (2.28) for infinitesimal parameters, is guaranteed by a general theorem, originally proved by E. Noether.

2.3 Particle interpretation

The scalar field theory outlined in Section 2.1 leads naturally to a particle interpretation. To show this, we consider the energy density in the free limit:

$$U_0 = \frac{1}{2c^2} (\partial_t \phi)^2 + \frac{1}{2} [(\nabla \phi)^2 + \mu^2 \phi^2]. \quad (2.39)$$

We restrict the system to a finite volume V of the three-dimensional space, imposing suitable boundary conditions of the field, and we compute the total energy $E = \int_V d\mathbf{r} U_0$ in terms of the Fourier modes of the field:

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{r}}. \quad (2.40)$$

We obtain

$$E = \frac{1}{2c^2} \sum_{\mathbf{k}} \left[|\partial_t \tilde{\phi}_{\mathbf{k}}|^2 + \omega^2 |\tilde{\phi}_{\mathbf{k}}|^2 \right], \quad (2.41)$$

where

$$\frac{\omega^2}{c^2} = k^2 + \mu^2. \quad (2.42)$$

Equation (2.41) has the same form as the energy of a system of simple harmonic oscillators, one for each value of $\mathbf{k}$. Quantization is therefore straightforward: we define creation and annihilation operators in term of $\tilde{\phi}_{\mathbf{k}}$ and $\partial_t \tilde{\phi}_{\mathbf{k}}$, now interpreted as operators in the Schrödinger picture, as

$$\tilde{\phi}_{\mathbf{k}} = c \sqrt{\frac{\hbar}{2\omega}} \left(A_{\mathbf{k}} + A_{-\mathbf{k}}^\dagger \right), \quad \partial_t \tilde{\phi}_{\mathbf{k}} = -ic \sqrt{\frac{\hbar\omega}{2}} \left(A_{\mathbf{k}} - A_{-\mathbf{k}}^\dagger \right). \quad (2.43)$$

and we impose the commutation rules

$$[A_{\mathbf{k}}, A_{\mathbf{q}}^\dagger] = \delta_{\mathbf{k}, \mathbf{q}}. \quad (2.44)$$

The quantum states of the system are obtained operating with $A_{\mathbf{k}}^\dagger$ on a vacuum state $|0\rangle$. From the expression of the total energy we obtain the Hamiltonian

$$H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} A_{\mathbf{k}}^\dagger A_{\mathbf{k}} + \text{constant}. \quad (2.45)$$

The constant term must be set to zero, in order that the vacuum state be Lorentz-invariant. Using the results of Section 2.2, one can compute the total momentum components

$$P_i = \int d\mathbf{r} T_{0i} \quad (2.46)$$

in terms of creation and annihilation operators. The result is

$$\mathbf{P} = \sum_{\mathbf{k}} \hbar\mathbf{k} A_{\mathbf{k}}^\dagger A_{\mathbf{k}}. \quad (2.47)$$

Clearly, the states $A_{\mathbf{k}}^\dagger |0\rangle$ are eigenvectors of both H and $\mathbf{P}$ with eigenvalues

$$\hbar\omega_{\mathbf{k}} = \sqrt{c^2 \hbar^2 (k^2 + \mu^2)} \quad (2.48)$$

and

$$\hbar\mathbf{k} \quad (2.49)$$

respectively. It is therefore natural to interpret them as states of free particles with momentum $\hbar\mathbf{k}$ and mass $m = \hbar\frac{\mu}{c}$. This is the reason why the quadratic part of the Lagrangian density is called the free part. The non-quadratic interaction terms will be collectively denoted by $\mathcal{L}_I$.

We denote with $|\{N_{\mathbf{q}}\}\rangle$ the state with $N_{\mathbf{q}}$ particles with momentum $\hbar\mathbf{q}$ for each value of $\mathbf{q}$. Then, the commutation rules (2.44) give

$$A_{\mathbf{k}}|\{N_{\mathbf{q}}\}\rangle = \sqrt{N_{\mathbf{q}}} |\{N_{\mathbf{q}} - \delta_{\mathbf{k},\mathbf{q}}\}\rangle \quad (2.50)$$

$$A_{\mathbf{k}}^{\dagger}|\{N_{\mathbf{q}}\}\rangle = \sqrt{N_{\mathbf{q}} + 1} |\{N_{\mathbf{q}} + \delta_{\mathbf{k},\mathbf{q}}\}\rangle. \quad (2.51)$$

Notice that the particles described by the real scalar field obey Bose statistics: each state can be occupied by an arbitrary number of particles, and the wave functions are symmetric under coordinate permutations. Fermi statistics will appear in Section 5.1, where fields describing spin-1/2 particles are introduced.

The expression of the field in the Heisenberg picture evolved at time t is immediately obtained from eq. (2.40):

$$\phi(\mathbf{r}, t) = c\sqrt{\frac{\hbar}{V}} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{k}}}} \left(A_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}t} + A_{-\mathbf{k}}^{\dagger} e^{i\omega_{\mathbf{k}}t} \right). \quad (2.52)$$

In the infinite-volume limit the discrete variable $\mathbf{k}$ takes values in the continuum, and sums over $\mathbf{k}$ must be replaced by three-dimensional integrals:

$$\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int d\mathbf{k}. \quad (2.53)$$

Correspondingly,

$$\delta_{\mathbf{k},\mathbf{q}} \rightarrow \frac{(2\pi)^3}{V} \delta(\mathbf{k} - \mathbf{q}). \quad (2.54)$$

We now define operators $A(\mathbf{k})$, functions of the continuum variable $\mathbf{k}$, as

$$A(\mathbf{k}) = \sqrt{\frac{V}{(2\pi)^3}} A_{\mathbf{k}}, \quad (2.55)$$

so that

$$[A(\mathbf{k}), A^{\dagger}(\mathbf{q})] = \delta(\mathbf{k} - \mathbf{q}). \quad (2.56)$$

Hence,

$$\begin{aligned} \phi(\mathbf{r}, t) &= c\sqrt{\frac{\hbar}{V}} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{k}}}} \left(A_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}t} + A_{-\mathbf{k}}^{\dagger} e^{i\omega_{\mathbf{k}}t} \right) \\ &= c\sqrt{\frac{\hbar}{V}} \frac{V}{(2\pi)^3} \int d\mathbf{k} \sqrt{\frac{(2\pi)^3}{V}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{k}}}} [A(\mathbf{k}) e^{-i\omega_{\mathbf{k}}t} + A^{\dagger}(-\mathbf{k}) e^{i\omega_{\mathbf{k}}t}] \\ &= \frac{c\sqrt{\hbar}}{(2\pi)^{3/2}} \int d\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{2\omega_{\mathbf{k}}}} [A(\mathbf{k}) e^{-i\omega_{\mathbf{k}}t} + A^{\dagger}(-\mathbf{k}) e^{i\omega_{\mathbf{k}}t}]. \end{aligned} \quad (2.57)$$

Using the four-vector notation

$$k \cdot x = \omega_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{r}, \quad (2.58)$$

we finally obtain

$$\phi(\mathbf{r}, t) = \frac{c\sqrt{\hbar}}{(2\pi)^{3/2}} \int d\mathbf{k} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} [A(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + A^\dagger(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}]. \quad (2.59)$$

In the second-quantized formalism, the transition between an initial state with n particles and a final state with m particles, both corresponding to well distinguished and orthogonal wave functions, is induced by a linear combination of operators that annihilate n particles and create m . Correspondingly, the transition operator matrix element is the sum of $n!m!$ terms; in each term, the contribution of one of the annihilation operators is given by one of the initial-particle momentum wave functions, and the contribution of one creation operator is given by a complex conjugate final-particle wave function. Therefore, the transition amplitude is a sum of terms proportional to the symmetric product of the initial wave functions and of the complex conjugate final wave functions. The situation with overlapping wave functions is different because of the square root factors appearing in eqs. (2.50,2.51). In high energy physics this situation only arises in heavy ion interactions.

2.4 Antiparticles

The quantization of the charged scalar field Φ introduced at the end of Section 2.2 proceeds along the same lines. Since the equation of motion for the free field is the same as for the real scalar field, the corresponding free-field solution is also of the same kind, with the only difference that in this case the coefficients of the plane waves with positive and negative frequency are not hermitian conjugates of each others, since the field is complex:

$$\Phi(\mathbf{r}, t) = \frac{c\sqrt{\hbar}}{(2\pi)^{3/2}} \int d\mathbf{k} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} [A(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + B^\dagger(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}]. \quad (2.60)$$

Thus, we have now particles of two different species, created by the operators $A^\dagger(\mathbf{k})$ and $B^\dagger(\mathbf{k})$, with equal masses.

The values of the electric charge of the two species of particles can be obtained by computing the total electric charge

$$Q = e \int d\mathbf{r} J^0 = ie \int d\mathbf{r} [\Phi^* \partial^0 \Phi - \Phi \partial^0 \Phi^*] \quad (2.61)$$

in terms of creation and annihilation operators. A straightforward calculation yields

$$Q = e \int d\mathbf{k} [A^\dagger(\mathbf{k})A(\mathbf{k}) - B^\dagger(\mathbf{k})B(\mathbf{k})]. \quad (2.62)$$

We conclude that, for each charged particle, the theory predicts the existence of another charged particle with equal mass and opposite charge, which is usually called the *antiparticle*. This is part of a very general result, called the CPT theorem, that establishes the correspondence between matter and antimatter.

Scattering theory

3.1 Cross sections and decay rates

Before discussing scattering processes in field theory, we briefly recall a few general results in scattering theory in the traditional Lippman-Schwinger formulation.

The purpose of scattering theory is the description of asymptotic-time transitions between different states of a system of particles that undergo short-range interactions; in other words, the typical situation is the transition from a state in the remote past composed of particles largely separated in space, and therefore non-interacting, to another state with similar features in the far future. It is natural to think of the full Hamiltonian H , that describes the evolution of the system at all times, as the sum of an asymptotic Hamiltonian H_0 , that describes the asymptotic evolution of non-interacting particles, and an interaction term V . Both H and H_0 are assumed to have the same spectrum, which, for simplicity, is assumed to be purely continuum.

In the Lippman-Schwinger formulation, an asymptotic state is in correspondence to a generic state $\phi_{\mathbf{k}}$ of n non-interacting particles with definite momenta $\mathbf{k}_i$, $i = 1, \dots, n$. For simplicity, we shall use the short-hand notations $d\mathbf{k}$, $\delta(\mathbf{k} - \mathbf{k}')$, $E_{\mathbf{k}}$ for $\prod_i d\mathbf{k}_i$, $\prod_i \delta(\mathbf{k}_i - \mathbf{k}'_i)$, $\sum_i E_{\mathbf{k}_i}$, respectively. The states $\phi_{\mathbf{k}}$ are assumed to form a generalized orthonormal basis in the space of free-particle states, that is, of generalized eigenstates of H_0 . Next, one builds two distinct bases of generalized eigenstates of H by

$$\Psi_{\mathbf{k}}^{(\pm)} = \phi_{\mathbf{k}} + \frac{1}{E_{\mathbf{k}} - H_0 \pm i\epsilon} V \Psi_{\mathbf{k}}^{(\pm)}, \quad (3.1)$$

where ϵ is a positive infinitesimal. These are *incoming* (+) and *outgoing* (−) states of our system. Indeed, considering incoming and outgoing wave packets

$$\Psi_f^{(\pm)} = \int d\mathbf{k} f(\mathbf{k}) \Psi_{\mathbf{k}}^{(\pm)} \quad (3.2)$$

and using the identity between distributions

$$\int_{\mp\infty}^0 d\tau e^{-ix\tau} = \frac{i}{x \pm i\epsilon}, \quad (3.3)$$

eq. (3.1) gives

$$\Psi_f^{(\pm)} = \int d\mathbf{k} f(\mathbf{k}) \left(\phi_{\mathbf{k}} - i \int_{\mp\infty}^0 d\tau e^{-i(E_{\mathbf{k}} - H_0)\tau} V \Psi_{\mathbf{k}}^{(\pm)} \right). \quad (3.4)$$

The time evolution of these wave packets is therefore

$$\begin{aligned} \Psi_f^{(\pm)}(t) &= \int d\mathbf{k} f(\mathbf{k}) e^{-iE_{\mathbf{k}}t} \Psi_{\mathbf{k}}^{(\pm)} \\ &= \int d\mathbf{k} f(\mathbf{k}) e^{-iH_0t} \left(\phi_{\mathbf{k}} - i \int_{\mp\infty}^t d\tau e^{-i(E_{\mathbf{k}} - H_0)\tau} V \Psi_{\mathbf{k}}^{(\pm)} \right). \end{aligned} \quad (3.5)$$

Clearly, as $t \rightarrow \mp\infty$ the wave packets $\Psi_f^{(\pm)}(t)$ approach the analogous wave packets built with the non-interacting states $\phi_{\mathbf{k}}$, therefore explaining the correspondence of incoming states with $\Psi_f^{(+)}(t)$ and of outgoing states with $\Psi_f^{(-)}(t)$.

The scalar product between a free-evolution state

$$\phi_{\mathbf{p}}(t) = e^{-iH_0t} \phi_{\mathbf{p}} \quad (3.6)$$

and $\Psi_f^{(+)}(t)$ is given by

$$\begin{aligned} \left(\phi_{\mathbf{p}}(t), \Psi_f^{(+)}(t) \right) &= f(\mathbf{p}) - i \int d\mathbf{k} f(\mathbf{k}) \int_{-\infty}^t d\tau e^{-i(E_{\mathbf{k}} - E_{\mathbf{p}})\tau} \left(\phi_{\mathbf{p}}, V \Psi_{\mathbf{k}}^{(+)} \right) \\ &\xrightarrow{t \rightarrow \infty} f(\mathbf{p}) - 2\pi i \int d\mathbf{k} f(\mathbf{k}) \delta(E_{\mathbf{k}} - E_{\mathbf{p}}) \left(\phi_{\mathbf{p}}, V \Psi_{\mathbf{k}}^{(+)} \right). \end{aligned} \quad (3.7)$$

This is the main result of the Lippman-Schwinger formulation of scattering theory: at large times, the wave packet $\Psi_f^{(+)}(t)$ is decomposed as the sum of the initial packet and a scattered packet, with amplitude

$$\begin{aligned} -2\pi i \delta(E_{\mathbf{k}} - E_{\mathbf{p}}) \left(\phi_{\mathbf{p}}, V \Psi_{\mathbf{k}}^{(+)} \right) &= -2\pi i \delta(E_{\mathbf{k}} - E_{\mathbf{p}}) \left(\Psi_{\mathbf{p}}^{(-)}, V \phi_{\mathbf{k}} \right) \\ &\equiv -2\pi i \delta(E_{\mathbf{k}} - E_{\mathbf{p}}) \mathcal{F}_{\mathbf{p}, \mathbf{k}}, \end{aligned} \quad (3.8)$$

where the first identity is a direct consequence of eq. (3.1). The quantity

$$A = -2\pi i \int d\mathbf{k} f(\mathbf{k}) \delta(E_{\mathbf{k}} - E_{\mathbf{p}}) \mathcal{F}_{\mathbf{p}, \mathbf{k}} \quad (3.9)$$

is the *scattering amplitude*.

The two different bases of the Hilbert space $\Psi_{\mathbf{k}}^{(-)}, \Psi_{\mathbf{k}}^{(+)}$ are related by a unitary transformation; one can define a unitary matrix that relates the

coefficients of the decomposition of any state vector in the two bases. This matrix is called the *scattering matrix*, or simply $\mathcal{S}$ matrix, and its elements are given by

$$\mathcal{S}_{\mathbf{k},\mathbf{p}} = (\Psi_{\mathbf{k}}^{(-)}, \Psi_{\mathbf{p}}^{(+)}). \quad (3.10)$$

For any state vector Ψ one has

$$(\Psi_{\mathbf{k}}^{(-)}, \Psi) = \sum_{\mathbf{p}} \mathcal{S}_{\mathbf{k},\mathbf{p}} (\Psi_{\mathbf{p}}^{(+)}, \Psi). \quad (3.11)$$

Using eqs. (3.1) and (3.8) one can check that

$$\mathcal{S}_{\mathbf{k},\mathbf{p}} = \delta(\mathbf{k} - \mathbf{p}) - 2\pi i \delta(E_k - E_p) \mathcal{F}_{\mathbf{k},\mathbf{p}}. \quad (3.12)$$

The unitarity of $\mathcal{S}$, that follows from the completeness of both bases (asymptotic completeness), is expressed by the equation

$$\int d\mathbf{q} \mathcal{S}_{\mathbf{k},\mathbf{q}} \mathcal{S}_{\mathbf{p},\mathbf{q}}^* = \delta(\mathbf{k} - \mathbf{p}). \quad (3.13)$$

Using eq. (3.12), we obtain the unitarity condition

$$\delta(E_k - E_p) [\mathcal{F}_{\mathbf{k},\mathbf{p}} - \mathcal{F}_{\mathbf{p},\mathbf{k}}^*] = 2\pi i \delta(E_k - E_p) \int d\mathbf{q} \delta(E_k - E_q) \mathcal{F}_{\mathbf{k},\mathbf{q}} \mathcal{F}_{\mathbf{p},\mathbf{q}}^*. \quad (3.14)$$

In the cases of interest, invariance under space translations implies momentum conservation, and we may write

$$\mathcal{F}_{\mathbf{p},\mathbf{k}} = \delta(\mathbf{p} - \mathbf{k}) T(\mathbf{p}, \mathbf{k}). \quad (3.15)$$

Therefore,

$$A = -2\pi i \int d\mathbf{k} f(\mathbf{k}) \delta(E_k - E_p) \delta(\mathbf{p} - \mathbf{k}) T(\mathbf{p}, \mathbf{k}). \quad (3.16)$$

In explicit calculations, the scattering amplitude from an initial state i of two non-interacting particles with average momenta $\mathbf{p}_1, \mathbf{p}_2$ to a final state f of n non-interacting particles with average momenta $\mathbf{k}_1, \dots, \mathbf{k}_n$ is given by

$$A_{i \rightarrow f} = \int d\mathbf{q}_1 \dots d\mathbf{q}_n d\mathbf{Q}_1 d\mathbf{Q}_2 \quad (3.17)$$

$$\psi_{\mathbf{k}_1}^*(\mathbf{q}_1) \dots \psi_{\mathbf{k}_n}^*(\mathbf{q}_n) \mathcal{T}(\mathbf{q}_1, \dots, \mathbf{q}_n; \mathbf{Q}_1, \mathbf{Q}_2) \psi_{\mathbf{p}_1}(\mathbf{Q}_1) \psi_{\mathbf{p}_2}(\mathbf{Q}_2),$$

where, enforcing energy-momentum conservation as in eqs. (3.9,3.15),

$$\mathcal{T}(\mathbf{q}_1, \dots, \mathbf{q}_n; \mathbf{Q}_1, \mathbf{Q}_2) = -2\pi i T(\mathbf{q}_1, \dots, \mathbf{q}_n; \mathbf{Q}_1, \mathbf{Q}_2) \quad (3.18)$$

$$\times \delta\left(\sum_i \mathbf{q}_i - \mathbf{Q}_1 - \mathbf{Q}_2\right) \delta\left(\sum_i E_{q_i} - E_{Q_1} - E_{Q_2}\right).$$

For simplicity, we shall assume a Gaussian shape for the wave packets:

$$\psi_{\mathbf{p}_0}(\mathbf{p}) = \frac{1}{(\sqrt{\pi}\sigma)^{3/2}} e^{-\frac{(\mathbf{p}-\mathbf{p}_0)^2}{2\sigma^2}}; \quad \int d\mathbf{p} |\psi_{\mathbf{p}_0}(\mathbf{p})|^2 = 1, \quad (3.19)$$

where the momentum spread of the packets σ (and hence their space dimension $1/\sigma$) is taken to be the same for all particles. A further simplification is achieved by taking the limit $\sigma \rightarrow 0$ in eq. (3.19); we get

$$\psi_{\mathbf{p}_0}(\mathbf{p}) \xrightarrow{\sigma \rightarrow 0} \frac{(\sqrt{2\pi}\sigma)^3}{(\sqrt{\pi}\sigma)^{3/2}} \delta(\mathbf{p} - \mathbf{p}_0) = \left(\sqrt{4\pi}\right)^{3/2} \delta(\mathbf{p} - \mathbf{p}_0), \quad (3.20)$$

that corresponds to the plane-wave representation.

In the typical scattering experiment, two colliding beams, described by initial wave packets, are prepared by means of suitable acceleration and collimation devices. The final state is then analyzed by a system of detectors. In many (but not all) physically relevant cases one of the two initial beams is at rest in the laboratory reference frame; we will however discuss the scattering process in full generality. The initial-state wave packets are determined by the momentum and energy resolutions of the accelerator, and by the geometric parameters that define its luminosity. On the other hand, the final state selection is limited to a certain domain in the space of final momenta, corresponding to the acceptance of the detectors. Therefore, what one is really interested in is the production probability of a given number of final particles in a given region of their momentum space. This is obtained from the production probability density in the generic point $\mathbf{k}_1, \dots, \mathbf{k}_n$ of the n final particle momentum space originated by a certain two-particle initial state, given by

$$W(\mathbf{k}_1, \dots, \mathbf{k}_n) = \left| \int d\mathbf{Q}_1 d\mathbf{Q}_2 T(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{Q}_1, \mathbf{Q}_2) \psi_{\mathbf{p}_1}(\mathbf{Q}_1) \psi_{\mathbf{p}_2}(\mathbf{Q}_2) \right|^2. \quad (3.21)$$

In order to relate $W(\mathbf{k}_1, \dots, \mathbf{k}_n)$ to $|A_{i \rightarrow f}|^2$, one has to shrink the final-state packets to momentum-space delta functions, while keeping the width of the initial-state packets fixed. Taking into account the normalization factor in eq. (3.20), we find

$$\frac{|A_{i \rightarrow f}|^2}{(\sqrt{4\pi}\sigma_f)^{3n}} \xrightarrow{\sigma_f \rightarrow 0} W, \quad (3.22)$$

where σ_f is the width of the final-state packets. Equation (3.22) gives the mathematical definition of the momentum-space probability density. For practical purposes, it is sufficient to assume that the momentum spread of the final packets is much smaller than $\hbar$ over the space resolution of particle counters. We are going to see in a moment that an analogous condition must be satisfied by the incoming wave packets; thus, putting apart the mathematical form of (3.22), we can keep the prescription of a common spread for all the wave packets.

In all cases of practical interest, the initial-state wave packets do not spread significantly; hence, the time evolution of the corresponding probability densities is simply a rigid translation:

$$\rho_i(\mathbf{r}, t) = \rho_i(\mathbf{r} - \mathbf{v}_i t), \quad i = 1, 2 \quad (3.23)$$

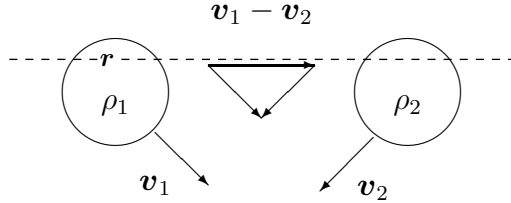
where

$$\mathbf{v}_i = \frac{\mathbf{p}_i}{E_{p_i}} = \frac{\mathbf{p}_i}{\sqrt{|\mathbf{p}_i|^2 + m_i^2}}. \quad (3.24)$$

We define the *integrated luminosity* L of the prepared system as

$$L = \lim_{d \rightarrow 0} \frac{P(d_{12} < d)}{\pi d^2}, \quad (3.25)$$

where $P(d_{12} < d)$ is the probability that, in the absence of interactions, the two colliding particles come to a relative distance less than d .¹



The integrated luminosity is easily computed in terms of the probability densities ρ_1, ρ_2 of the initial state particles. Consider the above picture: the probability that a particle of beam 1, initially at $\mathbf{r}$, come closer than d to one of the particles of beam 2 is given, in the small- d limit, by πd^2 times the integral of ρ_2 along the dashed line:

$$\pi d^2 \int_{-\infty}^{+\infty} dt |\mathbf{v}_1 - \mathbf{v}_2| \rho_2(\mathbf{r} + (\mathbf{v}_1 - \mathbf{v}_2)t). \quad (3.26)$$

Then, the probability $P(d_{12} < d)$ is obtained by integrating this quantity over all possible choices of $\mathbf{r}$ with weight $\rho_1(\mathbf{r}, t)$. Hence

$$\begin{aligned} L &= |\mathbf{v}_1 - \mathbf{v}_2| \int d\mathbf{r} dt \rho_1(\mathbf{r}) \rho_2(\mathbf{r} + (\mathbf{v}_1 - \mathbf{v}_2)t) \\ &= |\mathbf{v}_1 - \mathbf{v}_2| \int d\mathbf{r} dt \rho_1(\mathbf{r} - \mathbf{v}_1 t) \rho_2(\mathbf{r} - \mathbf{v}_2 t) \\ &= |\mathbf{v}_1 - \mathbf{v}_2| \int d\mathbf{r} dt \rho_1(\mathbf{r}, t) \rho_2(\mathbf{r}, t). \end{aligned} \quad (3.27)$$

¹ This definition of integrated luminosity is the same as the one currently adopted in experimental particle physics, referred to the case of a single collision.

In terms of initial-state wave functions, we have

$$\begin{aligned}
L &= \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{(2\pi)^6} \int d\mathbf{r} dt d\mathbf{Q}_1 d\mathbf{Q}'_1 d\mathbf{Q}_2 d\mathbf{Q}'_2 \psi_{\mathbf{p}_1}^*(\mathbf{Q}'_1) \psi_{\mathbf{p}_1}(\mathbf{Q}_1) \psi_{\mathbf{p}_2}^*(\mathbf{Q}'_2) \psi_{\mathbf{p}_2}(\mathbf{Q}_2) \\
&\quad e^{i(\mathbf{Q}_1 + \mathbf{Q}_2 - \mathbf{Q}'_1 - \mathbf{Q}'_2) \cdot \mathbf{r}} e^{-i(E_{Q_1} + E_{Q_2} - E_{Q'_1} - E_{Q'_2})t} \\
&= \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{(2\pi)^2} \int d\mathbf{Q}_1 d\mathbf{Q}'_1 d\mathbf{Q}_2 d\mathbf{Q}'_2 \psi_{\mathbf{p}_1}^*(\mathbf{Q}'_1) \psi_{\mathbf{p}_1}(\mathbf{Q}_1) \psi_{\mathbf{p}_2}^*(\mathbf{Q}'_2) \psi_{\mathbf{p}_2}(\mathbf{Q}_2) \\
&\quad \delta(\mathbf{Q}_1 + \mathbf{Q}_2 - \mathbf{Q}'_1 - \mathbf{Q}'_2) \delta(E_{Q_1} + E_{Q_2} - E_{Q'_1} - E_{Q'_2}). \quad (3.28)
\end{aligned}$$

It is interesting to compute as an example the integrated luminosity for two Gaussian packets, eq. (3.19), centered around the momenta $\mathbf{p}_i$, $i = 1, 2$, in the limit $\sigma \rightarrow 0$. Defining $\mathbf{q}_i \equiv \mathbf{Q}_i - \mathbf{p}_i$ e $\mathbf{q}'_i \equiv \mathbf{Q}'_i - \mathbf{p}_i$, we have, in this limit,

$$E_{Q_i} = E_{p_i} + \mathbf{v}_i \cdot \mathbf{q}_i + O(|\mathbf{q}_i|^2); \quad E_{Q'_i} = E_{p_i} + \mathbf{v}_i \cdot \mathbf{q}'_i + O(|\mathbf{q}'_i|^2). \quad (3.29)$$

Equation (3.28) then gives

$$\begin{aligned}
L &= \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{4\pi^5 \sigma^6} \int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}'_1 d\mathbf{q}'_2 \exp\left(-\frac{|\mathbf{q}_1|^2 + |\mathbf{q}'_1|^2 + |\mathbf{q}_2|^2 + |\mathbf{q}'_2|^2}{2\sigma^2}\right) \\
&\quad \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}'_1 - \mathbf{q}'_2) \delta(\mathbf{v}_1 \cdot (\mathbf{q}_1 - \mathbf{q}'_1) - \mathbf{v}_2 \cdot (\mathbf{q}_2 - \mathbf{q}'_2)). \quad (3.30)
\end{aligned}$$

We now define $\mathbf{q}_{\pm} = \mathbf{q}_1 \pm \mathbf{q}'_1$; after some algebra, we get

$$\begin{aligned}
L &= \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{32\pi^5 \sigma^6} \int d\mathbf{q}_+ d\mathbf{q}_2 d\mathbf{q}_- \delta((\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{q}_-) e^{-\frac{q_+^2 + q_-^2 + 2q_2^2 + 2(q_2 + q_-)^2}{4\sigma^2}} \\
&= \frac{1}{32\pi^5 \sigma^6} \int d\mathbf{q}_+ d\mathbf{q}_2 d\mathbf{q}_-^T e^{-\frac{q_+^2 + 3q_-^T{}^2 + 4q_2^2 + 4q_2 \cdot \mathbf{q}_-^T}{4\sigma^2}} = \frac{\sigma^2}{2\pi}, \quad (3.31)
\end{aligned}$$

where the suffix T is for the transverse projection with respect to $\mathbf{v}_1 - \mathbf{v}_2$.

We now go back to the transition probability density eq. (3.18). Taking eq. (3.21) into account, we obtain

$$\begin{aligned}
W(\mathbf{k}_1, \dots, \mathbf{k}_n) &= (2\pi)^2 \int d\mathbf{Q}_1 d\mathbf{Q}'_1 d\mathbf{Q}_2 d\mathbf{Q}'_2 \psi_{\mathbf{p}_1}^*(\mathbf{Q}'_1) \psi_{\mathbf{p}_1}(\mathbf{Q}_1) \psi_{\mathbf{p}_2}^*(\mathbf{Q}'_2) \psi_{\mathbf{p}_2}(\mathbf{Q}_2) \\
&\quad \delta(E_{Q_1} + E_{Q_2} - E_{Q'_1} - E_{Q'_2}) \delta(\mathbf{Q}_1 + \mathbf{Q}_2 - \mathbf{Q}'_1 - \mathbf{Q}'_2) \\
&\quad \delta(E_f - E_{Q_1} - E_{Q_2}) \delta(\mathbf{k}_f - \mathbf{Q}_1 - \mathbf{Q}_2) \\
&\quad T(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{Q}_1, \mathbf{Q}_2) T^*(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{Q}'_1, \mathbf{Q}'_2), \quad (3.32)
\end{aligned}$$

where

$$E_f = \sum_j E_{k_j}; \quad \mathbf{k}_f = \sum_j \mathbf{k}_j. \quad (3.33)$$

This expression will eventually be integrated over the final-state momenta $\mathbf{k}_1, \dots, \mathbf{k}_n$ selected by the detectors. The energy and momentum resolutions of the analysis process must necessarily be worse than those of the preparation

process; equivalently, the parameter σ for the incoming packets is assumed to be infinitesimal. In this limit, we may replace

$$\delta(E_f - E_{Q_1} - E_{Q_2}) \rightarrow \delta(E_f - E_i) \quad (3.34)$$

and

$$\delta(\mathbf{k}_f - \mathbf{Q}_1 - \mathbf{Q}_2) \rightarrow \delta(\mathbf{k}_f - \mathbf{p}_i), \quad (3.35)$$

where

$$E_i = E_{p_1} + E_{p_2}; \quad \mathbf{p}_i = \mathbf{p}_1 + \mathbf{p}_2. \quad (3.36)$$

Furthermore, we must assume that the scattering amplitude T is a slowly varying function of the initial state momenta; in this case, we may replace

$$T(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{Q}_1, \mathbf{Q}_2) T^*(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{Q}'_1, \mathbf{Q}'_2) \rightarrow |T(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{p}_1, \mathbf{p}_2)|^2. \quad (3.37)$$

With these assumptions, we find that the probability density W is proportional to the integrated luminosity:

$$W(\mathbf{k}_1, \dots, \mathbf{k}_n) = L \frac{(2\pi)^4}{|\mathbf{v}_1 - \mathbf{v}_2|} |T(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{p}_1, \mathbf{p}_2)|^2 \delta(E_f - E_i) \delta(\mathbf{k}_i - \mathbf{p}_i). \quad (3.38)$$

The proportionality coefficient is called the *cross section*. More precisely, we define the *differential cross section* $d\sigma$ through

$$W(\mathbf{k}_1, \dots, \mathbf{k}_n) d\mathbf{k}_1 \dots d\mathbf{k}_n = L d\sigma, \quad (3.39)$$

or

$$d\sigma = \frac{(2\pi)^4}{|\mathbf{v}_1 - \mathbf{v}_2|} |T|^2 \delta^{(4)}(P_f - P_i) d\mathbf{k}_1 \dots d\mathbf{k}_n, \quad (3.40)$$

where we have defined the initial and final four-momenta

$$P_i \equiv p_1 + p_2; \quad P_f \equiv \sum_{j=1}^n k_j. \quad (3.41)$$

A remarkable amount of information on the standard model comes from the study of the decay properties of unstable particles. The best known example is the width of the Z boson, that gives information on the number of light neutrinos. In field theory, unstable particles play a delicate role since, strictly speaking, they are not true particles, in the sense that they do not appear in the space of scattering asymptotic states. The reason for this is that radiative corrections introduce a complex correction to the mass of an unstable particle, whose imaginary part is related to the particle decay probability per unit time. In most cases of interest, the interaction that induces the particle decay can be considered as weak, and hence the decay process has a perturbative origin. Under these circumstances, one can use the well-known Fermi golden rule to obtain the decay probability per unit time from an initial state of wave

function Ψ_i to a final scattering state $\Psi_f^{(-)}$ due to the interaction Hamiltonian H_W :

$$\frac{dP_f(t)}{dt} = 2\pi \left| \left(\Psi_f^{(-)}, H_W \Psi_i \right) \right|^2 \rho(E_i), \quad (3.42)$$

where $\rho(E)$ is the density of final states with energy E . This formula is a direct consequence of first order, time dependent perturbation theory.

In the case of a relativistic theory, the initial state corresponds to a particle state $\int d\mathbf{p} f(\mathbf{p}) \Psi_{\mathbf{p}}$, which would be stable in the absence of the small perturbation H_W . The final state is a scattering state, eq. (3.1), of n particles with momenta $\mathbf{k}_1, \dots, \mathbf{k}_n$. Due to translation invariance one has

$$\left(\Psi_{\mathbf{k}_1, \dots, \mathbf{k}_n}^{(-)}, H_W \Psi_{\mathbf{p}} \right) \equiv \delta \left(\mathbf{p} - \sum_{i=1}^n \mathbf{k}_i \right) T(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{p}). \quad (3.43)$$

If the final states are selected in a region Ω of the momentum space, eq. (3.42) becomes

$$\begin{aligned} \frac{dP_{\Omega}(t)}{dt} &= 2\pi \int_{\Omega} d\mathbf{k}_1 \dots d\mathbf{k}_n d\mathbf{p} |f(\mathbf{p})|^2 |T(\mathbf{k}_1, \dots, \mathbf{k}_n; \mathbf{p})|^2 \\ &\quad \delta \left(\mathbf{p} - \sum_{i=1}^n \mathbf{k}_i \right) \delta \left(E_{\mathbf{p}} - \sum_{i=1}^n E_{\mathbf{k}_i} \right), \end{aligned} \quad (3.44)$$

where $E_{\mathbf{p}}$ is the energy of the initial particle with momentum $\mathbf{p}$ and $E_{\mathbf{k}_i}$ those of the n particles in the final state. Under the same assumptions that lead to eq. (3.40) for the differential cross section, we can define a differential decay probability per unit time as

$$d\Gamma = 2\pi d\mathbf{k}_1 \dots d\mathbf{k}_n |T|^2 \delta^{(4)} \left(p - \sum_{i=1}^n k_i \right). \quad (3.45)$$

Equation (3.44) is completely analogous to the expression eq. (3.40) for the differential cross section in the case of two-particle scattering.

3.2 Semi-classical approximation and asymptotic conditions

We have seen in the previous Section that the computation of cross sections and decay rates requires, according to eqs. (3.40) and (3.45), the evaluation of the amplitude T . In this Section, we will perform this calculation in the semi-classical approximation.

It is well known from ordinary quantum mechanics that wave functions in the semi-classical limit $\hbar \rightarrow 0$ are proportional to $\psi \sim e^{\frac{iS}{\hbar}}$, where S is the Hamilton-Jacobi function. Similarly, the probability amplitude for the transition from an initial state i to a final state f is approximately given by

$$\Psi_{i \rightarrow f} = e^{\frac{i}{\hbar} S_{i \rightarrow f}}, \quad (3.46)$$

where $S_{i \rightarrow f}$ is the action computed on the trajectory (path) that describes the transition in the classical limit. A well-known example is provided by a single free, non-relativistic particle going from $\mathbf{r}_i$ to $\mathbf{r}_f$ in the time interval $t_f - t_i$; the classical motion takes place with velocity $\mathbf{v} = \frac{\mathbf{r}_f - \mathbf{r}_i}{t_f - t_i}$, and the action is

$$S_{i \rightarrow f} = \frac{1}{2} m \mathbf{v}^2 (t_f - t_i) = \frac{m(\mathbf{r}_f - \mathbf{r}_i)^2}{2(t_f - t_i)}. \quad (3.47)$$

The corresponding amplitude in the semi-classical limit is therefore

$$\Psi_{i \rightarrow f} \sim \exp \left(\frac{i}{\hbar} \frac{m(\mathbf{r}_f - \mathbf{r}_i)^2}{2(t_f - t_i)} \right), \quad (3.48)$$

that gives rise to the well-known interference phenomenon.

We now consider the case of a scattering process characterized by relativistic energies E . The duration of the full process, from the preparation of the initial state to the detection of final state particles, is essentially infinite with respect to the characteristic time scale of the interaction, of order $\frac{\hbar}{E}$. Hence, the amplitude must be computed by evaluating the action integral for a solution of the field equation (2.8) that interpolates between an initial field configuration at $t \rightarrow -\infty$ and a final configuration at $t \rightarrow +\infty$. These, in turn, are solutions of the free-field equation, eq. (2.9), consistently with the formulation of scattering theory. The relationship between the asymptotic field configurations and the wave functions of initial and final states is fixed by the decomposition of the asymptotic field in terms of creation and annihilation operator, eq. (2.52), that are in correspondence with the wave functions as explained in the previous Chapter.

In order to complete this program, for a given process we determine a field configuration $\phi_i(\mathbf{r})$ at a large negative time $-T$, another field configuration $\phi_f(\mathbf{r})$ at time $+T$, and we take the limit $T \rightarrow +\infty$. This procedure is not entirely straightforward, as we now show in the simple case of free fields. It will be convenient to take the three-dimensional Fourier transform of the field:

$$\tilde{\phi}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \int d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \phi(\mathbf{r}, t). \quad (3.49)$$

To simplify notations, we choose the so-called natural units, where $\hbar = c = 1$. In these units, we have

$$m = \mu; \quad E_k = \sqrt{|\mathbf{k}|^2 + m^2}. \quad (3.50)$$

The boundary conditions for the Fourier-transformed field are then

$$\tilde{\phi}(\mathbf{k}, +T) = \tilde{\phi}_f(\mathbf{k}), \quad \tilde{\phi}(\mathbf{k}, -T) = \tilde{\phi}_i(\mathbf{k}). \quad (3.51)$$

On the other hand, the generic solution of eq. (2.9) has the form

$$\tilde{\phi}(\mathbf{k}, t) = \tilde{\phi}_+(\mathbf{k}) e^{iE_k t} + \tilde{\phi}_-(\mathbf{k}) e^{-iE_k t}. \quad (3.52)$$

The free-field trajectory is determined by the two functions $\tilde{\phi}_+(\mathbf{k})$ and $\tilde{\phi}_-(\mathbf{k})$, related to initial and final field configurations through

$$\begin{pmatrix} \tilde{\phi}_f(\mathbf{k}) \\ \tilde{\phi}_i(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} e^{iE_k T} & e^{-iE_k T} \\ e^{-iE_k T} & e^{iE_k T} \end{pmatrix} \begin{pmatrix} \tilde{\phi}_+(\mathbf{k}) \\ \tilde{\phi}_-(\mathbf{k}) \end{pmatrix}. \quad (3.53)$$

One should therefore solve the system eq. (3.53) for $\tilde{\phi}_+$ and $\tilde{\phi}_-$, evaluate the action functional over the time interval $(-T, T)$, and take the limit $T \rightarrow +\infty$. This is however not possible; the matrix of coefficients in eq. (3.53) is manifestly not uniformly invertible as a function of T . The boundary conditions must therefore be imposed in a different way.

For this purpose, let us recall two points that we have stressed in the previous Section. The first is that annihilation (creation) operators contribute to a second-quantized matrix element through the (complex conjugate) wave functions of initial (final) particles. The second point is that in scattering theory the initial state corresponds to asymptotic times in the past, while the final state corresponds to the far future. In both cases, the field operator has a free limit in which it decomposes into creation and annihilation operators according to eq. (2.59).

The above remarks imply that the annihilation (negative frequency) part of the field should tend, as $t \rightarrow -\infty$, to the wave function of some particle in the initial state, and that the creation (positive frequency) part should tend, as $t \rightarrow \infty$, to the complex conjugate wave function of a particle in the final state. Since initial and final states are multi-particle states, and the particles are identical, the asymptotic limit at $t \rightarrow -\infty$ of the annihilation part must be the sum of the initial particle wave functions, and the creation part at $t \rightarrow \infty$ must tend to the sum of the final particle complex-conjugate wave functions.

The point which remains to be clarified is how positive and negative frequency parts are selected in the asymptotic limit. We note that eq. (3.52) implies

$$e^{\pm iE_k t} \tilde{\phi}(\mathbf{k}, t) \xrightarrow{|t| \rightarrow +\infty} \tilde{\phi}_{\mp}(\mathbf{k}), \quad (3.54)$$

where the limits must be understood in the weak sense, i.e. up to oscillating factors that vanish if one replaces $e^{\pm iE_k t} \tilde{\phi}(\mathbf{k}, t)$ by its average over a small volume in momentum space. More precisely, we introduce the averaged field

$$\tilde{\phi}_{f,\pm}(t) \equiv \int d\mathbf{k} f(\mathbf{k}) e^{\mp iE_k t} \tilde{\phi}(\mathbf{k}, t),$$

where f is a suitable regular function rapidly vanishing outside a small volume in momentum space. From the result given in Appendix A, eq. (A.7) for f Gaussian, one sees that

$$\tilde{\phi}_{f,\pm}(t) \xrightarrow{|t| \rightarrow +\infty} \int d\mathbf{k} f(\mathbf{k}) \tilde{\phi}_{\pm}(\mathbf{k}), \quad (3.55)$$

which is a rigorous definition of the limit in eq. (3.54). Therefore we impose the following weak boundary conditions for the field:

$$e^{+iE_k t} \tilde{\phi}(\mathbf{k}, t) \xrightarrow{t \rightarrow -\infty} \tilde{\phi}_i(\mathbf{k}) \quad (3.56)$$

$$e^{-iE_k t} \tilde{\phi}(\mathbf{k}, t) \xrightarrow{t \rightarrow +\infty} \tilde{\phi}_f(\mathbf{k}), \quad (3.57)$$

where

$$\tilde{\phi}_i(\mathbf{k}) = \frac{1}{\sqrt{(2\pi)^3 2E_k}} \sum_{i=1}^2 \psi_{\mathbf{p}_i}(\mathbf{k}) \quad (3.58)$$

$$\tilde{\phi}_f(\mathbf{k}) = \frac{1}{\sqrt{(2\pi)^3 2E_k}} \sum_{j=1}^n \psi_{\mathbf{k}_j}^*(\mathbf{k}). \quad (3.59)$$

Here, $\mathbf{p}_i$ and $\mathbf{k}_j$ are the average momenta of initial and final state particles, respectively.

The two asymptotic conditions can be summarized in a single one; indeed, the free-field solution

$$\tilde{\phi}^{(\text{as})}(\mathbf{k}, t) = \frac{1}{\sqrt{(2\pi)^3 2E_k}} \left[\sum_{i=1}^2 \psi_{\mathbf{p}_i}(\mathbf{k}) e^{-iE_k t} + \sum_{j=1}^n \psi_{\mathbf{k}_j}^*(\mathbf{k}) e^{iE_k t} \right] \quad (3.60)$$

obeys both eqs. (3.56, 3.57). Thus, we will require that the interacting field evolve, in the weak sense, toward the asymptotic field eq. (3.60). The inverse Fourier transform of eq. (3.60) is

$$\phi^{(\text{as})}(\mathbf{x}, t) = \frac{1}{\sqrt{(2\pi)^3}} \int d\mathbf{k} \frac{1}{\sqrt{2E_k}} \left[\sum_{i=1}^2 \psi_{\mathbf{p}_i}(\mathbf{k}) e^{-ik \cdot x} + \sum_{j=1}^n \psi_{\mathbf{k}_j}^*(\mathbf{k}) e^{ik \cdot x} \right]. \quad (3.61)$$

Using Gaussian wave packets in the limit $\sigma \rightarrow 0$ the asymptotic conditions become

$$e^{iE_k t} \tilde{\phi}(\mathbf{k}, t) \xrightarrow{t \rightarrow -\infty} \frac{(\sqrt{4\pi}\sigma)^{3/2}}{(2\pi)^{\frac{3}{2}}} \sum_{i=1}^2 \frac{\delta(\mathbf{k} - \mathbf{p}_i)}{\sqrt{2E_{p_i}}} \quad (3.62)$$

$$e^{-iE_k t} \tilde{\phi}(\mathbf{k}, t) \xrightarrow{t \rightarrow +\infty} \frac{(\sqrt{4\pi}\sigma)^{3/2}}{(2\pi)^{\frac{3}{2}}} \sum_{j=1}^n \frac{\delta(\mathbf{k} + \mathbf{k}_j)}{\sqrt{2E_{k_j}}}, \quad (3.63)$$

and eq. (3.61) becomes

$$\phi^{(\text{as})}(\mathbf{x}, t) = \frac{(\sqrt{4\pi}\sigma)^{3/2}}{\sqrt{(2\pi)^3}} \left[\sum_{i=1}^2 \frac{e^{-ip_i \cdot x}}{\sqrt{2E_{p_i}}} + \sum_{j=1}^n \frac{e^{ik_j \cdot x}}{\sqrt{2E_{k_j}}} \right]. \quad (3.64)$$

This method is employed in Appendix B in a simple case, where the particles interact with an external time-dependent potential.

3.3 Solution of the field equation

The next step toward the computation of the scattering amplitude in the semi-classical approximation is the calculation of a solution of the field equation (2.8) with the asymptotic conditions eq. (3.64). With the scalar potential in eq. (2.10), eq. (2.8) takes the form

$$(\partial^2 + m^2) \phi = -\frac{\lambda}{3!} \phi^3 \equiv J, \quad (3.65)$$

or, in terms of Fourier transforms,

$$(\partial_t^2 + \mathbf{k}^2 + m^2) \tilde{\phi}(\mathbf{k}, t) = (\partial_t^2 + E_k^2) \tilde{\phi}(\mathbf{k}, t) = \tilde{J}(\mathbf{k}, t). \quad (3.66)$$

We shall find the requested solution by the method of Green's functions. We define the *Green function* $\tilde{\Delta}(\mathbf{k}, t)$ as the solution of the inhomogeneous equation

$$(\partial_t^2 + E_k^2) \tilde{\Delta}(\mathbf{k}, t) = \delta(t) \quad (3.67)$$

with the conditions

$$\tilde{\Delta}(\mathbf{k}, t) \sim e^{iE_k t} \quad \text{for } t < 0 \quad (3.68)$$

$$\tilde{\Delta}(\mathbf{k}, t) \sim e^{-iE_k t} \quad \text{for } t > 0. \quad (3.69)$$

One can check that

$$\tilde{\Delta}(\mathbf{k}, t) = \frac{i}{2E_k} [\theta(t)e^{-iE_k t} + \theta(-t)e^{iE_k t}] \quad (3.70)$$

by an explicit calculation. It is easy to see that

$$\begin{aligned} \tilde{\phi}(\mathbf{k}, t) &= \tilde{\phi}^{(\text{as})}(\mathbf{k}, t) + \int_{-\infty}^{+\infty} dt' \tilde{\Delta}(\mathbf{k}, t - t') \tilde{J}(\mathbf{k}, t') \\ &= \tilde{\phi}^{(\text{as})}(\mathbf{k}, t) \\ &\quad + \frac{i}{2E_k} \left[e^{-iE_k t} \int_{-\infty}^t dt' e^{iE_k t'} \tilde{J}(\mathbf{k}, t') + e^{iE_k t} \int_t^{\infty} dt' e^{-iE_k t'} \tilde{J}(\mathbf{k}, t') \right] \end{aligned} \quad (3.71)$$

is a solution of eq. (3.66), and approaches eq. (3.64) asymptotically: for example, for $t \rightarrow -\infty$ the first term in the squared bracket vanishes, and the second one tends to the time Fourier transform of $\tilde{J}$ times a rapidly oscillating phase factor $e^{iE_k t}$, that does not contribute in the weak limit. Similarly, the squared bracket vanishes as $t \rightarrow +\infty$, and in both cases the asymptotic limit is $\tilde{\phi}^{(\text{as})}$.

It will be useful to write the solution eq. (3.71) in an explicitly covariant form. To this purpose, we take its inverse Fourier transform:

$$\begin{aligned} \phi(\mathbf{r}, t) &= \phi^{(\text{as})}(\mathbf{r}, t) + \int dt' \int d\mathbf{k} e^{-i\mathbf{k} \cdot \mathbf{r}} \tilde{\Delta}(\mathbf{k}, t - t') \frac{1}{(2\pi)^3} \int d\mathbf{r}' e^{i\mathbf{k} \cdot \mathbf{r}'} J(\mathbf{r}', t') \\ &= \phi^{(\text{as})}(x) + \int d^4 x' J(x') \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \tilde{\Delta}(\mathbf{k}, t - t'), \end{aligned} \quad (3.72)$$

where $x = (t, \mathbf{r})$ and $x' = (t', \mathbf{r}')$. Using the explicit expression of the Green function $\tilde{\Delta}(\mathbf{k}, t)$, eq. (3.70), and the integral representation of the step function

$$\theta(t) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega t}}{\omega + i\epsilon}, \quad (3.73)$$

we find

$$\begin{aligned} & \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{-i\mathbf{k} \cdot \mathbf{r}} \tilde{\Delta}(\mathbf{k}, t) \\ &= - \int \frac{d\mathbf{k}}{(2\pi)^4} e^{-i\mathbf{k} \cdot \mathbf{r}} \frac{1}{2E_k} \int_{-\infty}^{+\infty} d\omega \left[\frac{e^{-i(E_k + \omega)t}}{\omega + i\epsilon} + \frac{e^{i(E_k + \omega)t}}{\omega + i\epsilon} \right] \\ &= \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot x} \frac{1}{2E_k} \left[\frac{1}{E_k + \omega - i\epsilon} + \frac{1}{E_k - \omega - i\epsilon} \right], \end{aligned} \quad (3.74)$$

where we have shifted $\omega \rightarrow -E_k - \omega$ in the first term, and $\omega \rightarrow -E_k + \omega$ in the second one, and we have defined $k = (\omega, \mathbf{k})$. This gives

$$\frac{1}{(2\pi)^3} \int d\mathbf{k} e^{-i\mathbf{k} \cdot \mathbf{r}} \tilde{\Delta}(\mathbf{k}, t) = \frac{1}{(2\pi)^4} \int d^4 k e^{ik \cdot x} \frac{1}{m^2 - k^2 - i\epsilon} \equiv \Delta(x) \quad (3.75)$$

(the positive infinitesimal ϵ can be redefined at each step, provided its sign is kept unchanged.) The function $\Delta(x)$ is a Lorentz scalar; hence, the covariant form of eq. (3.71) is

$$\phi(x) = \phi^{(\text{as})}(x) + \int d^4 y \Delta(x - y) J(y) \equiv (\phi^{(\text{as})} + \Delta \circ J)(x), \quad (3.76)$$

with $\phi^{(\text{as})}$ given in eq. (3.61). Equation (3.76) is an implicit solution of the field equation, since J depends on ϕ ; the explicit solution can be found using eq. (3.76) recursively. In practice, we have transformed the differential equation (2.8) into an integral equation, that embodies the boundary conditions.

Finally, we note that

$$(\partial^2 + m^2) \Delta(x) = \delta(x). \quad (3.77)$$

This relation will be useful in the computation of the Green function for fields with different transformation properties with respect to Lorentz transformations.

Our next step will be the computation of the action, and hence of the transition amplitude.

3.4 Calculation of the scattering amplitude

The calculation of the scattering amplitude involves, as already mentioned, the evaluation of the action integral over an infinite time interval:

$$\int_{-\infty}^{+\infty} dt \int d\mathbf{r} \mathcal{L}(\mathbf{r}, t). \quad (3.78)$$

In view of the oscillating behaviour of the asymptotic solution, the integral in not necessarily convergent; we therefore regularize the integrand by replacing $\phi^{(\text{as})}(x)$ with

$$\phi^{(\text{as}, \eta)}(x) \equiv e^{-\eta|t|} \phi^{(\text{as})}(x), \quad (3.79)$$

and we take the limit $\eta \rightarrow 0+$ at the end of the calculation. Clearly, $\phi^{(\text{as}, \eta)}$ is no longer a solution of the free-field equation; correspondingly,

$$\phi = \phi^{(\text{as}, \eta)} + \Delta \circ J \quad (3.80)$$

is not a stationary field configuration for the original action, but for the modified action obtained from the modified Lagrangian density

$$\mathcal{L} \rightarrow \mathcal{L} + \phi (\partial^2 + m^2) \phi^{(\text{as}, \eta)}. \quad (3.81)$$

The added term vanishes as $\eta \rightarrow 0$, but its integral is not necessarily zero in the same limit.

Taking eq. (3.81) into account, we should evaluate

$$S_{i \rightarrow f}^{(\eta)} = - \int_{-\infty}^{+\infty} dt \int d\mathbf{r} \left[\frac{1}{2} \phi (\partial^2 + m^2) (\phi - 2\phi^{(\text{as}, \eta)}) + \frac{\lambda}{4!} \phi^4 \right]. \quad (3.82)$$

We obtain

$$\begin{aligned} S_{i \rightarrow f}^{(\eta)} &= - \int_{-\infty}^{+\infty} dt \int d\mathbf{r} \left[\frac{1}{2} (\phi^{(\text{as}, \eta)} + \Delta \circ J) (\partial^2 + m^2) (\Delta \circ J - \phi^{(\text{as}, \eta)}) \right. \\ &\quad \left. + \frac{\lambda}{4!} (\phi^{(\text{as}, \eta)} + \Delta \circ J)^4 \right] \\ &= \int_{-\infty}^{+\infty} dt \int d\mathbf{r} \left[\frac{1}{2} \phi^{(\text{as}, \eta)} (\partial^2 + m^2) \phi^{(\text{as}, \eta)} - \frac{1}{2} (\Delta \circ J) (\partial^2 + m^2) \Delta \circ J \right. \\ &\quad \left. - \frac{\lambda}{4!} (\phi^{(\text{as}, \eta)} + \Delta \circ J)^4 \right] \\ &= \int_{-\infty}^{+\infty} dt \int d\mathbf{r} \left[\frac{1}{2} \phi^{(\text{as}, \eta)} (\partial^2 + m^2) \phi^{(\text{as}, \eta)} - \frac{1}{2} J \Delta \circ J \right. \\ &\quad \left. - \frac{\lambda}{4!} (\phi^{(\text{as}, \eta)} + \Delta \circ J)^4 \right], \end{aligned} \quad (3.83)$$

where we have performed a partial integration in the second step, and we have used $(\partial^2 + m^2) \Delta \circ J = J$ in the last one. The contribution of the first term in brackets,

$$\begin{aligned}
& \frac{1}{2} \int_{-\infty}^{+\infty} dt \int d\mathbf{r} \phi^{(\text{as}, \eta)} (\partial^2 + m^2) \phi^{(\text{as}, \eta)} \\
&= \frac{(2\pi)^3}{2} \int_{-\infty}^{+\infty} dt \int d\mathbf{k} \tilde{\phi}^{(\text{as}, \eta)}(-\mathbf{k}, t) (\partial_t^2 + E_k^2) \tilde{\phi}^{(\text{as}, \eta)}(\mathbf{k}, t) \quad (3.84)
\end{aligned}$$

vanishes as $\eta \rightarrow 0$. Indeed, using

$$\tilde{\phi}^{(\text{as}, \eta)}(\mathbf{k}, t) = e^{-\eta|t|} [\phi_+(\mathbf{k})e^{iE_k t} + \phi_-(\mathbf{k})e^{-iE_k t}], \quad (3.85)$$

we find

$$\begin{aligned}
(\partial_t^2 + E_k^2) \tilde{\phi}^{(\text{as}, \eta)}(\mathbf{k}, t) &= (\eta^2 - 2\eta\delta(t)) \tilde{\phi}^{(\text{as}, \eta)}(\mathbf{k}, t) \\
&\quad - 2i\eta \text{sign}(t) E_k e^{-\eta|t|} (\phi_+(\mathbf{k})e^{iE_k t} - \phi_-(\mathbf{k})e^{-iE_k t}). \quad (3.86)
\end{aligned}$$

The term proportional to η^2 does not contribute to eq. (3.84) as $\eta \rightarrow 0$, because the time integration is proportional to $1/\eta$, and the term proportional to $\eta\delta(t)$ vanishes, because the delta functions makes the integral convergent. The remaining term is

$$\begin{aligned}
& -i\eta(2\pi)^3 \int_{-\infty}^{+\infty} dt \text{sign}(t) e^{-2\eta|t|} \int d\mathbf{k} E_k [\phi_+(-\mathbf{k})e^{iE_k t} + \phi_-(-\mathbf{k})e^{-iE_k t}] \\
&\quad \times [\phi_+(\mathbf{k})e^{iE_k t} - \phi_-(\mathbf{k})e^{-iE_k t}] + O(\eta) \\
&= -i\eta(2\pi)^3 \int d\mathbf{k} E_k \int_{-\infty}^{+\infty} dt \text{sign}(t) e^{-2\eta|t|} \\
&\quad [e^{2iE_k t} \phi_+(-\mathbf{k})\phi_+(\mathbf{k}) - e^{-2iE_k t} \phi_-(-\mathbf{k})\phi_-(\mathbf{k})] + O(\eta). \quad (3.87)
\end{aligned}$$

The time integral can be performed, with the result

$$2\eta(2\pi)^3 \int d\mathbf{k} \frac{E_k^2}{\eta^2 + E_k^2} [\phi_+(-\mathbf{k})\phi_+(\mathbf{k}) + \phi_-(-\mathbf{k})\phi_-(\mathbf{k})] + O(\eta), \quad (3.88)$$

which is also vanishing for $\eta \rightarrow 0$. Hence, we obtain in this limit

$$S_{i \rightarrow f} = - \int_{-\infty}^{+\infty} dt \int d\mathbf{r} \left[\frac{1}{2} J \Delta \circ J + \frac{\lambda}{4!} (\phi^{(\text{as})} + \Delta \circ J)^4 \right], \quad (3.89)$$

where, taking eqs. (3.65) and (3.80) into account,

$$J = -\frac{\lambda}{3!} (\phi^{(\text{as})} + \Delta \circ J)^3, \quad (3.90)$$

and $\phi^{(\text{as})}$ is given by

$$\phi^{(\text{as})}(x) = \frac{(\sqrt{4\pi}\sigma)^{3/2}}{(2\pi)^{3/2}} \left[\sum_{i=1}^2 \frac{e^{-ip_i \cdot x}}{\sqrt{2E_{p_i}}} + \sum_{j=1}^n \frac{e^{ik_j \cdot x}}{\sqrt{2E_{k_j}}} \right]. \quad (3.91)$$

It can be shown that in this case the result eq. (3.89) is a sum of terms proportional to $\lambda^k (\phi^{(\text{as})})^{2k+2}$, $k = 1, 2, \dots$ ²

Once the semi-classical action $S_{i \rightarrow f}$ has been computed, one has immediately

$$\Psi_{i \rightarrow f} \simeq \exp(i S_{i \rightarrow f}). \quad (3.92)$$

In the framework of the Lippman-Schwinger formulation presented in Section 3.1, eq. (3.92) is identified with

$$\lim_{t \rightarrow \infty} \int d\mathbf{q} g^*(\mathbf{q}) \left(\phi_{\mathbf{q}}(t), \Psi_{\mathbf{k}}^{(+)}(t) \right) \quad (3.93)$$

where g is a packet function for the final state. In the same framework, eq. (3.7) can be rewritten as

$$\begin{aligned} & -2\pi i \delta(E_k - E_p) \delta(\mathbf{k} - \mathbf{p}) T(\mathbf{p}, \mathbf{k}) \\ &= \lim_{t \rightarrow \infty} \left[\left(\phi_{\mathbf{p}}(t), \Psi_{\mathbf{k}}^{(+)}(t) \right) - \left(\phi_{\mathbf{p}}(t), \Psi_{\mathbf{k}}^{(+)}(t) \right) \Big|_{V=0} \right] \end{aligned} \quad (3.94)$$

It is clear from eqs. (3.89,3.90) and (3.92) that the second term in the r.h.s. of eq. (3.94) is equal to 1 in the semi-classical approximation, since $S_{i \rightarrow f}$ vanishes with λ .

Comparing this result with eqs. (3.17,3.18), we get

$$A_{i \rightarrow f} = \Psi_{i \rightarrow f} - \Psi_{i \rightarrow f} \Big|_{V=0} = \Psi_{i \rightarrow f} - 1 = i S_{i \rightarrow f} + \frac{1}{2} (i S_{i \rightarrow f}) (i S_{i \rightarrow f}) + \dots \quad (3.95)$$

The first term is simply proportional to the action, given by eqs. (3.89,3.90) and (3.91).

The amplitude for a process with n particles in the final state arises from terms of degree $n+2$ in $\phi^{(\text{as})}$ in the expansion of the semi-classical amplitude, eq. (3.92), since it must be linear (or anti-linear) in each particle wave function. Such terms are selected in the expansion of $A_{i \rightarrow f}$, eq. (3.95), by means of the following mechanism. Terms of lower degree are excluded because energy-momentum conservation cannot be simultaneously satisfied in a transition $2 \rightarrow n$ and in a transition involving a subset of the $n+2$ particles. Terms of higher degree can only be of the form $(\phi^{(\text{as})})^{(n+2)m}$, again because of energy and momentum conservation, and arise both in the expansion of $S_{i \rightarrow f}$ and in contributions to $A_{i \rightarrow f}$ of higher order in $S_{i \rightarrow f}$. As shown in Section 3.1, differential cross sections are determined by taking the limit $\sigma \rightarrow 0$ of perfect resolution in the wave packets. It can also be shown that terms of degree $(n+2)m$ in $\phi^{(\text{as})}$ in $S_{i \rightarrow f}$ vanish³ as $\sigma^{(\frac{3}{2}n+1)m-2}$. Hence, terms with $m > 1$

² Expanding the r.h.s. of eq. (3.90) one can show that J is a sum of terms proportional to $\lambda^k (\phi^{(\text{as})})^{2k+1}$; replacing this expansion in eq. (3.89) yields the announced result.

³ The degree of homogeneity in σ is obtained by rescaling all momentum differences $\mathbf{p} - \mathbf{p}_0$ (see eq. (3.19)) by σ . Each wave packet contributes a factor of $\sigma^{3/2}$, while

vanish more rapidly than those with $m = 1$. Similarly, contributions to $A_{i \rightarrow f}$ of higher degree in $S_{i \rightarrow f}$ are higher-order infinitesimal with respect to those of first degree. For this reason, as $\sigma \rightarrow 0$ we find

$$A_{i \rightarrow f} \simeq i S_{i \rightarrow f}. \quad (3.96)$$

The leading contribution to $A_{i \rightarrow f}$ is therefore proportional to $\sigma^{\frac{3}{2}n-1}$. The calculations performed in Section 3.1 show how this leads to a σ -independent differential cross section. In the following, we will omit the wave packets in explicit calculations, and simply neglect terms that are not linear or anti-linear in each wave function.

We conclude this Section by giving a generalized version of eq. (3.89). Indeed, using eq. (3.76), it can be written as

$$S_{i \rightarrow f} = - \int d^4x \frac{\lambda}{4!} \phi^4(x) - \frac{1}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y). \quad (3.97)$$

This expression readily generalizes to any theory, in the form

$$S_{i \rightarrow f} = \int d^4x \mathcal{L}_I - \frac{1}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y). \quad (3.98)$$

When more fields are involved, ϕ_i , $i = 1, \dots, N$ (not necessarily scalar fields), there will be a source J_i for each of them, given by

$$J_i = \frac{\partial}{\partial \phi_i} \mathcal{L}_I - \partial_\mu \frac{\partial}{\partial (\partial_\mu \phi_i)} \mathcal{L}_I; \quad i = 1, \dots, N, \quad (3.99)$$

as one can see by inspection of the Euler-Lagrange equations. Equation (3.89) can now be rewritten by defining the functional

$$S \left[\phi^{(\text{as})} \right] = - \int_{-\infty}^{+\infty} dt \int d\mathbf{r} \left[\frac{1}{2} J_i \Delta_{ij} \circ J_j - \mathcal{L}_I \right]_{\phi_i = \phi_i^{(\text{as})} + (\Delta \circ J)_i}, \quad (3.100)$$

where we have taken into account the fact that, in the general case, the propagator Δ is a $N \times N$ matrix. This formula is immediately applicable to all cases of physical interest.

This concludes our construction of field theory scattering amplitudes in the semi-classical approximation. It is convenient at this point to state how our results are related to those of the general relativistic scattering theory. [1] This starts from the assumption of some basic properties of the Hilbert space of asymptotic states, which is identified with a free-particle Fock space, and of the interacting field operators. Among these assumptions, locality plays a crucial role. Using these properties, R. Haag has shown that interacting

the global four-momentum conservation delta function carries a factor of σ^{-4} . Furthermore, one must take into account $2(m-1)$ mass-shell singularities, each carrying a factor $1/\sigma$.

field operators tend, for asymptotic times and in a suitable weak topology, to asymptotic free fields, which are built with the creation and annihilation operators of the particles in the asymptotic states. This result justifies the construction of relativistic scattering amplitudes in terms of Fourier transforms of suitable Green functions through the LSZ reduction formulae. In the most common approach, Green functions are identified with vacuum expectation values of time-ordered products of interacting fields. The collection of all Green functions can be cast into a functional generator that, in principle, can be computed using the Feynman functional integral formula. Our results are recovered by computing the Feynman functional integral in the saddle-point approximation. Here one finds again the result that characterizes the semi-classical approximation in path integral formulations of Quantum Mechanics. Radiative corrections correspond to corrections to the saddle point approximation.

Feynman diagrams

4.1 The method of Feynman diagrams

The calculation of the scattering amplitude for a given process is greatly simplified by a graphical technique, originally introduced by R. Feynman in the context of quantum electrodynamics. We define the following symbols:

$$\phi^{(\text{as})}(x) \equiv \text{---}\bullet \quad J(x) \equiv \bullet\bigcirc \quad \Delta(x-y) \equiv \bullet\text{---}\bullet \quad (4.1)$$

Equation (3.89) is represented as

$$S_{i \rightarrow f} = -\frac{1}{2} \bigcirc\text{---}\bigcirc - \frac{\lambda}{4!} \left[\begin{array}{c} | \\ | \\ | \\ | \end{array} + 4 \begin{array}{c} | \\ | \\ | \\ \bigcirc \end{array} + 6 \begin{array}{c} \bigcirc \\ | \\ | \\ | \\ \bigcirc \end{array} + 4 \begin{array}{c} \bigcirc \\ | \\ | \\ \bigcirc \end{array} + \begin{array}{c} \bigcirc \\ | \\ \bigcirc \end{array} + \begin{array}{c} \bigcirc \\ | \\ \bigcirc \end{array} \right] \quad (4.2)$$

while the graphical form of eq. (3.90) is

$$\bullet\bigcirc = -\frac{\lambda}{3!} \left[\begin{array}{c} \diagup \\ \diagdown \end{array} + 3 \begin{array}{c} \diagup \\ \diagdown \end{array} \bigcirc + 3 \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \bigcirc \\ \diagup \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \bigcirc \\ \diagup \\ \diagdown \end{array} \begin{array}{c} \bigcirc \\ \diagup \\ \diagdown \end{array} \right]. \quad (4.3)$$

It is understood that each internal vertex corresponds to a space-time integration.

Equation (4.3) can be solved iteratively:

$$\bullet\bigcirc = -\frac{\lambda}{3!} \begin{array}{c} \diagup \\ \diagdown \end{array} + \frac{\lambda^2}{12} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} - \frac{\lambda^3}{72} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} - \frac{\lambda^3}{24} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} + \dots \quad (4.4)$$

Using the iterative solution of eq. (4.3) in eq. (4.2) we find

$$S_{i \rightarrow f} = -\frac{\lambda}{4!} \text{---} + \frac{\lambda^2}{72} \text{---} - \frac{\lambda^3}{144} \text{---} + \dots \quad (4.5)$$

The crossing points of the lines in each diagram of this expansion are called *vertices*; lines connecting two vertices are called *internal lines*, and correspond to a factor of Δ (usually called the *propagator*) in the amplitude. The remaining lines are *external lines*; each of them corresponds to a factor $\phi^{(\text{as})}$. The numerical coefficients in eq. (4.5) are combinatorial factors, that can be written as

$$\frac{1}{k \prod_v n_v!}, \quad (4.6)$$

where v is a generic vertex in the diagram, n_v is the number of external lines of the vertex v , and k is the symmetry class of the diagram, i.e. the number of rigid transformations that leave the diagram unchanged: we find $k = 1$ for the first diagram, and $k = 2$ for the others. Finally, each diagram carries one power of $-\lambda$ for each vertex.

We now proceed to the computation of the amplitude. It will be useful to introduce four-dimensional Fourier transforms, defined as

$$\hat{F}(k) \equiv \frac{1}{(2\pi)^4} \int d^4x e^{ik \cdot x} F(x) \quad F(x) = \int d^4k e^{-ik \cdot x} \hat{F}(k) \quad (4.7)$$

for a generic $F(x)$. We find

$$\hat{\phi}(q) = \hat{\phi}^{(\text{as})}(q) + \frac{1}{m^2 - q^2 - i\epsilon} \hat{J}(q) \quad (4.8)$$

and therefore

$$\begin{aligned} S_{i \rightarrow f} = & -\frac{1}{2} \int d^4k d^4k' (2\pi)^4 \delta^{(4)}(k + k') \hat{J}(k) \frac{1}{m^2 - k^2 - i\epsilon} \hat{J}(k') \\ & - \frac{\lambda}{4!} \int \prod_{l=1}^4 d^4q_l \left[\hat{\phi}^{(\text{as})}(q_l) + \frac{1}{m^2 - q_l^2 - i\epsilon} \hat{J}(q_l) \right] (2\pi)^4 \delta^{(4)} \left(\sum_{l=1}^4 q_l \right). \end{aligned} \quad (4.9)$$

Similarly, eq. (3.90) becomes

$$\begin{aligned} \hat{J}(k) = & -\frac{\lambda}{3!} \int \frac{d^4x}{(2\pi)^4} e^{ik \cdot x} \prod_{l=1}^3 \int d^4q_l e^{-iq_l \cdot x} \left[\hat{\phi}^{(\text{as})}(q_l) + \frac{\hat{J}(q_l)}{m^2 - q_l^2 - i\epsilon} \right] \\ = & -\frac{\lambda}{3!} \int \prod_{l=1}^3 d^4q_l \left[\hat{\phi}^{(\text{as})}(q_l) + \frac{\hat{J}(q_l)}{m^2 - q_l^2 - i\epsilon} \right] \delta^{(4)} \left(k - \sum_{l=1}^3 q_l \right). \end{aligned} \quad (4.10)$$

Finally, the Fourier-transformed asymptotic field is given by

$$\hat{\phi}^{(\text{as})}(k) = \frac{1}{(2\pi)^{\frac{3}{2}}} \left[\sum_{i=1}^2 \frac{\delta^{(4)}(k - p_i)}{\sqrt{2E_{p_i}}} + \sum_{j=1}^n \frac{\delta^{(4)}(k + k_j)}{\sqrt{2E_{k_j}}} \right]. \quad (4.11)$$

The iterative expression of eq. (4.5) in terms of Fourier transforms of the various quantities then reads

$$\begin{aligned}
 S_{i \rightarrow f} = & -\frac{\lambda}{4!} \int \left[\prod_{l=1}^4 d^4 q_l \hat{\phi}^{(\text{as})}(q_l) \right] (2\pi)^4 \delta^{(4)} \left(\sum_{l=1}^4 q_l \right) \\
 & + \frac{\lambda^2}{72} \int \left[\prod_{l=1}^6 d^4 q_l \hat{\phi}^{(\text{as})}(q_l) \right] (2\pi)^4 \delta^{(4)} \left(\sum_{l=1}^6 q_l \right) \frac{1}{m^2 - \left(\sum_{l=1}^3 q_l \right)^2 - i\epsilon} \\
 & + \dots
 \end{aligned} \tag{4.12}$$

By comparing eq. (4.12) with its graphical form, eq. (4.5), and taking eq. (4.11) into account, we note that each line in the diagrams corresponds to a four-momentum integration variable; these momenta flow from initial to final external lines, and are constrained by momentum conservation at each vertex. The momenta carried by external lines are identified with the momenta of asymptotic initial (p_i) and final (k_j) state particles, while momentum conservation fixes the momenta of internal lines. The contribution of a given diagram to the amplitude is therefore the product of factors

$$\frac{1}{m^2 - q_l^2 - i\epsilon} \tag{4.13}$$

corresponding to internal lines, of a factor $-\lambda$ for each vertex, and a factor

$$\frac{1}{\sqrt{(2\pi)^3 2E}} \tag{4.14}$$

for each external line. Finally, the amplitude is proportional to the factor

$$(2\pi)^4 \delta^{(4)} \left(\sum_{j=1}^n k_j - \sum_{i=1}^2 p_i \right) \tag{4.15}$$

that appears explicitly in eq. (4.12). Summing over all possible assignments of asymptotic particles to external lines, one can check that every diagram appears with a multiplicity which is given by the number of permutations of the n_v asymptotic particles entering a given vertex v , multiplied by the number k of symmetry transformations of the diagram into itself. This gives a factor $k \prod_v n_v!$, that compensates exactly the denominator that appears explicitly in each term of eq. (4.5).

Let us consider, as an example, the case of two particles in the final state, usually called elastic scattering:

$$\phi(p_1) + \phi(p_2) \rightarrow \phi(k_1) + \phi(k_2). \tag{4.16}$$

The relevant term is the first one in eq. (4.12). We get

$$\begin{aligned}
A_{2 \rightarrow 2} &= -i \frac{\lambda}{4!} \int \left[\prod_{l=1}^4 d^4 q_l \hat{\phi}^{(\text{as})}(q_l) \right] (2\pi)^4 \delta^{(4)} \left(\sum_{l=1}^4 q_l \right) \\
&= -i \frac{\lambda}{4!} \frac{4! (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2)}{\sqrt{\prod_{i=1}^2 [(2\pi)^3 2E_{p_i}] \prod_{j=1}^2 [(2\pi)^3 2E_{k_j}]} } .
\end{aligned} \tag{4.17}$$

The factor $4!$ accounts for the number of ways one can extract terms linear in each plane wave from $\hat{\phi}^{(\text{as})}(q_1) \dots \hat{\phi}^{(\text{as})}(q_4)$.

The function T , defined in eq. (3.18), that appears in the general expression for the cross section, is now easily identified: since

$$A_{2 \rightarrow 2} = -2\pi i T(k_1, k_2; p_1, p_2) \delta^{(4)}(k_1 + k_2 - p_1 - p_2), \tag{4.18}$$

comparing with eq. (4.17) we find

$$T(k_1, k_2; p_1, p_2) = \frac{\lambda}{\sqrt{4E_{p_1}E_{p_2}}} \frac{1}{\sqrt{(2\pi)^3 2E_{k_2}} \sqrt{(2\pi)^3 2E_{k_2}}}. \tag{4.19}$$

The differential cross section is given by

$$\begin{aligned}
d\sigma_2 &= \frac{|\lambda|^2}{4E_{p_1}E_{p_2}} \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \frac{d\mathbf{k}_1}{(2\pi)^3 2E_{k_1}} \frac{d\mathbf{k}_2}{(2\pi)^3 2E_{k_2}} \\
&= \frac{|\lambda|^2}{4|\mathbf{p}_1 E_{p_2} - \mathbf{p}_2 E_{p_1}|} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \frac{d\mathbf{k}_1}{(2\pi)^3 2E_{k_1}} \frac{d\mathbf{k}_2}{(2\pi)^3 2E_{k_2}} .
\end{aligned} \tag{4.20}$$

The integration measure

$$d\phi_2 = (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \frac{d\mathbf{k}_1}{(2\pi)^3 2E_{k_1}} \frac{d\mathbf{k}_2}{(2\pi)^3 2E_{k_2}} \tag{4.21}$$

is usually called the *invariant phase space*; it appears in the expression of the differential cross section for any process with two particles in the final state. Thanks to its transformation properties under Lorentz transformations, the invariant phase space can be computed in any reference frame. In many cases, a convenient choice is the rest frame of the center of mass of the particles in the initial state, where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. In this frame,

$$\begin{aligned}
d\phi_2 &= \frac{1}{(2\pi)^2} \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \delta(\sqrt{s} - E_{k_1} - E_{k_2}) \frac{d\mathbf{k}_1}{2E_{k_1}} \frac{d\mathbf{k}_2}{2E_{k_2}} \\
&= \frac{1}{(2\pi)^2} \delta(\sqrt{s} - E_{k_1} - E_{k_2}) \frac{d\mathbf{k}_1}{4E_{k_1}E_{k_2}},
\end{aligned} \tag{4.22}$$

where we have defined $s = (p_1 + p_2)^2 = (E_{p_1} + E_{p_2})^2$. In the second step we have used the spatial momentum conservation factor $\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2)$ to perform the integral over $\mathbf{k}_2$, therefore implicitly setting $\mathbf{k}_2 = -\mathbf{k}_1$ in the integrand. We

may further simplify this expression using polar coordinates $\mathbf{k}_1 = (|\mathbf{k}_1|, \theta, \phi)$ and orienting the z axis in the direction of the momenta of incoming particles. Because of the symmetry of the whole process under rotations around the z axis, the squared amplitude cannot depend on the azimuthal angle ϕ . Hence,

$$d\phi_2 = \frac{1}{2\pi} \delta(\sqrt{s} - E_{k_1} - E_{k_2}) \frac{|\mathbf{k}_1|^2 d|\mathbf{k}_1|}{4E_{k_1}E_{k_2}} d\cos\theta. \quad (4.23)$$

The integral over $|\mathbf{k}_1|$ can now be performed using the delta function

$$\delta(\sqrt{s} - E_{k_1} - E_{k_2}) = \frac{E_{k_1}E_{k_2}}{\sqrt{s}|\mathbf{k}_1|} \delta\left(|\mathbf{k}_1| - \frac{\sqrt{s}}{2} \sqrt{1 - \frac{4m^2}{s}}\right). \quad (4.24)$$

We get

$$d\phi_2 = \frac{1}{8\pi} \frac{|\mathbf{k}_1|}{\sqrt{s}} d\cos\theta, \quad (4.25)$$

where

$$|\mathbf{k}_1| = \frac{\sqrt{s}}{2} \sqrt{1 - \frac{4m^2}{s}}. \quad (4.26)$$

The angular differential cross section is therefore given by

$$d\sigma_2 = \frac{|\lambda|^2}{32\pi s} \sqrt{1 - \frac{4m^2}{s}} d\cos\theta. \quad (4.27)$$

The total cross section is obtained by integrating $d\sigma_2$ in $\cos\theta$, recalling that in this case only one half of the total solid angle contributes, because of the identity of the particles in the final state.

4.2 The invariant amplitude

The calculations of the previous Section can be immediately generalized to the case of a generic process with n particles in the final state. It is convenient to define the *invariant amplitude* $\mathcal{M}_{fi}$ through

$$A_{i \rightarrow f} = i(2\pi)^4 \delta^{(4)}\left(\sum_{j=1}^n k_j - \sum_{i=1}^2 p_i\right) \frac{\mathcal{M}_{fi}}{\sqrt{\prod_{i=1}^2 [(2\pi)^3 2E_{p_i}] \prod_{j=1}^n [(2\pi)^3 2E_{k_j}]}}. \quad (4.28)$$

In the case of elastic scattering, the explicit calculation gives $\mathcal{M}_{2 \rightarrow 2} = -\lambda$.

The function T is related to the invariant amplitude by comparison with its definition:

$$A_{i \rightarrow f} = -2\pi i T(k_1, \dots, k_n; p_1, p_2) \delta^{(4)}(p_1 + p_2 - k_1 - \dots - k_n). \quad (4.29)$$

We find

$$T(k_1, \dots, k_n; p_1, p_2) = -\frac{\mathcal{M}_{fi}}{\sqrt{4E_{p_1}E_{p_2}}} \frac{1}{\prod_{j=1}^n \sqrt{(2\pi)^3 2E_{k_j}}}. \quad (4.30)$$

The differential cross section is therefore given by

$$d\sigma_n = \frac{|\mathcal{M}_{fi}|^2}{4|\mathbf{p}_1 E_{p_2} - \mathbf{p}_2 E_{p_1}|} d\phi_n(p_1, p_2; k_j), \quad (4.31)$$

where

$$d\phi_n(p_1, p_2; k_j) = (2\pi)^4 \delta^{(4)}\left(\sum_{j=1}^n k_j - \sum_{i=1}^2 p_i\right) \prod_{j=1}^n \frac{d\mathbf{k}_j}{(2\pi)^3 2E_{k_j}} \quad (4.32)$$

is the invariant phase space for n particles in the final state. Finally, we observe that in all reference frames in which $\mathbf{p}_1$ and $\mathbf{p}_2$ have the same direction, we have

$$|\mathbf{p}_1 E_{p_2} - \mathbf{p}_2 E_{p_1}| = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \quad \text{when } \mathbf{p}_1 \parallel \mathbf{p}_2, \quad (4.33)$$

where m_1 and m_2 are the masses of initial state particles.

The same reduction to an invariant amplitude can be performed in the case of particle decays. Notice, first of all, that the matrix element eq. (3.43) corresponds, up to a factor $-2\pi i \delta(E_f - E_i)$, to the linear part in H_W of the transition amplitude from a single-particle initial state to an n -particle final state. This amplitude can be computed in the semi-classical approximation using the method of Feynman diagrams.

Therefore, the general result has the form

$$T(k_1, \dots, k_n; p) = -\sqrt{(2\pi)^3} \frac{\mathcal{M}_{fi}}{\sqrt{2E_p}} \frac{1}{\prod_{j=1}^n \sqrt{(2\pi)^3 2E_{k_j}}}. \quad (4.34)$$

where $\mathcal{M}_{fi}$ corresponds to the sum of Feynman diagrams with n particles in the final state and a single heavy particle in the initial state. Inserting eq. (4.34) into eq. (3.45) yields

$$d\Gamma = \frac{|\mathcal{M}_{fi}|^2}{2E_p} d\phi_n(p; k_1, \dots, k_n). \quad (4.35)$$

Needless to say, E_p equals the mass of the decaying particle in its rest frame, which is usually the most convenient choice for the computation of decay rates.

4.3 Feynman rules for the scalar theory

The results obtained so far can be summarized as follows. In the context of the scalar theory defined by the interaction Lagrangian

$$\mathcal{L}_I = -\frac{\lambda}{4!} \phi^4 \quad (4.36)$$

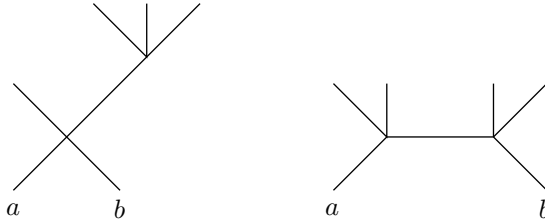
the invariant amplitude $\mathcal{M}_{fi}$ for a generic process with two particles in the initial state and $2n$ particles in the final state is obtained, in the semi-classical approximation, by

1. selecting all connected diagrams with $2n + 2$ external lines that can be built with four-line vertices (note that $\mathcal{L}_I$ in eq. (4.36) is proportional to the fourth power of ϕ);
2. assigning the asymptotic particles to the external lines of each diagram in all possible ways, and determining the momentum flux by means of four-momentum conservation at each vertex;
3. summing over all diagrams and over all possible assignments the product of $-\lambda$ to the power n_v , which is the number of vertices (equal to n in the present case), times a factor $\frac{1}{m^2 - q_l^2 - i\epsilon}$ for each internal line, where the four-momenta q_l are fixed by four-momentum conservation at each vertex.

We see that the essential ingredients of the calculation are propagators and vertex factors; these are characteristic of the particular theory one is considering. In particular, the propagators are the Green function of the free field equations, while the vertex factors can be extracted from the interaction Lagrangian (eq. (4.36) in our example) by computing the amplitude for a virtual process in which the lines entering a given vertex are considered as external lines. It is easy to see that this general rule gives the correct result in the case of eq. (4.36): each vertex has four lines, and the corresponding virtual amplitude is given by $-\lambda/4!$ times the number of possible assignments of external lines, $4!$. This gives the correct factor of $-\lambda$. In the following, we will encounter more complicated cases, where the vertex factor may depend on particle momenta.

In order to illustrate the above procedure, let us compute the invariant amplitude for the process with two particles in the initial state and four particles in the final state ($n = 2$), that corresponds to the second term in eq. (4.5).

Inequivalent assignments of asymptotic particles to external lines are usually expressed by oriented diagrams, with the external lines corresponding to the initial particles (labelled by a and b) incoming from below (or from the left), and final state particles (labelled by 1, 2, 3 and 4) outgoing above (or to the right). In the case at hand, we have two diagrams:



In the case of scalar particles, the orientation of internal lines is irrelevant. The diagram on the left corresponds to four inequivalent contributions, one for each choice $j = 1, \dots, 4$ of the final state particle outgoing from the lower vertex; the momentum carried by the internal line is in this case $p_1 + p_2 - k_j$. The diagram on the right corresponds to six inequivalent contributions, that correspond to the six ways of choosing the two particles outgoing from the left vertex ($i < j = 1, \dots, 4$); the momentum of the internal line is in this case $p_1 - k_i - k_j$. Summing all contributions according to the rules given above, we find

$$\mathcal{M}_{2 \rightarrow 4} = \lambda^2 \left[\sum_{j=1}^4 \frac{1}{m^2 - (p_1 + p_2 - k_j)^2} + \sum_{j=2}^4 \sum_{i=1}^{j-1} \frac{1}{m^2 - (p_1 - k_i - k_j)^2} \right]. \quad (4.37)$$

Note that we have not included the $i\epsilon$ terms in the internal line propagators; this is because for this particular amplitude the denominators never vanish for physical values of external particle four-momenta. For example, it is easy to show that four-momentum conservation implies $(p_1 + p_2 - k_j)^2 > 9m^2$. In general, this is always the case for theories that do not involve unstable particles.

In order to provide a sample calculation of a differential decay rate, we should consider a theory that describes particles with different masses. To this purpose, we extend our simple scalar model to include a second scalar field Φ , associated with particles of mass $M \gg m$, coupled to ϕ through an interaction term of the form

$$\mathcal{L}_I = -\frac{gM}{2} \Phi \phi^2, \quad (4.38)$$

where g is a new coupling constant (assumed small) and the factor of M was introduced to keep g dimensionless. To first order in g , the invariant amplitude for the two-body decay

$$\Phi(p) \rightarrow \phi(k_1) + \phi(k_2) \quad (4.39)$$

is simply given by

$$\mathcal{M}_{1 \rightarrow 2} = -gM \quad (4.40)$$

while the invariant amplitude for

$$\Phi(p) \rightarrow \phi(k_1) + \phi(k_2) + \phi(k_3) + \phi(k_4) \quad (4.41)$$

gets contributions from diagrams with one internal line, and has therefore a non-trivial dependence on external particle momenta:

$$\mathcal{M}_{1 \rightarrow 4} = g\lambda M \sum_{j=1}^4 \frac{1}{m^2 - (p - k_j)^2 - i\epsilon}. \quad (4.42)$$

4.4 Unitarity, radiative corrections and renormalizability

The constraint in eq. (3.14), that follows from unitarity of the scattering matrix, can be formulated in terms of invariant amplitudes, defined as in eq. (4.30). We find

$$\mathcal{M}_{ij} - \mathcal{M}_{ji}^* = -i \sum_f \int d\phi_{n_f}(P_i; k_1^f, \dots, k_{n_f}^f) \mathcal{M}_{if} \mathcal{M}_{jf}^*, \quad (4.43)$$

with $d\phi_{n_f}$ given in eq. (4.32). The above constraint is relevant to our analysis, because it is systematically violated in the semi-classical approximation, thus indicating that corrections to this approximation are needed. In the semi-classical approximation the amplitudes are real and symmetric under the exchange of initial and final states;¹ hence, the left-hand side of eq. (4.43) vanishes, while the right-hand side does not. This can be seen explicitly in the simple case of two-particle elastic scattering in the context of the real scalar field theory. We have shown in Section 4.1 that the invariant amplitude in this case is

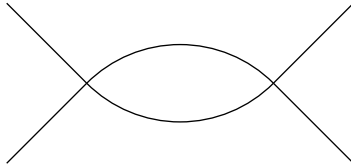
$$\mathcal{M}_{2 \rightarrow 2} = -\lambda. \quad (4.44)$$

The left-hand side of the unitarity constraint eq. (4.43) vanishes in the semi-classical approximation, since $\mathcal{M}_{2 \rightarrow 2}$ is real. The right-hand side of eq. (4.43) is easily computed if we further choose the center-of-mass energy to be smaller than $4m$, so that the production of more than two scalars is kinematically forbidden. We get

$$-i\lambda^2 \int d\phi_2(P_i; k_1, k_2) = -\frac{i\lambda^2 |\mathbf{k}_1|}{16\pi E_{k_1}}, \quad (4.45)$$

where we have used eq. (4.25) for the invariant two-particle phase space, and the integral has been restricted to one half of the total solid angle due to the identity of scalar particles. Equation (4.45) is manifestly nonzero, and the unitarity constraint is therefore violated.

Unitarity is recursively restored taking radiative corrections into account. In particular, the one-loop contribution corresponds to the diagram with two internal lines and two vertices, and hence one loop:



¹ Because of time-reversal invariance.

The amplitudes corresponding to loop diagrams can be computed using the rules given in the previous Section, with two further instructions: first, the integral over the loop momentum q , which is not fixed by momentum conservation at the vertices, must be performed with the measure

$$i \frac{d^4 q}{(2\pi)^4}, \quad (4.46)$$

and second, the amplitude carries a combinatorial factor, given by the inverse of the number of symmetry transformation of the diagram. In the present case this factor is $1/2$, because the diagram is symmetric under permutation of the internal lines. The contribution to the invariant amplitude is therefore given by

$$\begin{aligned} \mathcal{M}_{2 \rightarrow 2}^{\text{1loop}} = i \sum_{j=1}^3 \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} & \left[\frac{1}{m^2 - q^2 - i\epsilon} \frac{1}{m^2 - (P_j - q)^2 - i\epsilon} \right. \\ & \left. - \frac{1}{M^2 - q^2 - i\epsilon} \frac{1}{M^2 - (P_j - q)^2 - i\epsilon} \right] - C, \end{aligned} \quad (4.47)$$

where $P_1 = p_1 + p_2$, $P_2 = p_1 - k_1$, $P_3 = p_1 - k_2$. Note that we have introduced in the integrand of eq. (4.47) a term that depends on a large mass parameter M in order to regularize the loop integral, otherwise divergent in the large-momentum region. This procedure is called *Pauli-Villars regularization*; the parameter M is eventually taken to infinity. It is apparent that, for any finite value of M , the integral of the difference of the two terms in eq. (4.47) is convergent, while that of each single term diverges. The introduction of the regularizing term must be compensated by the real constant C , whose contribution to the scattering amplitude appears as a correction (counter-term) to λ . The constant C must be tuned in such a way that the invariant scattering amplitude is equal to $-\lambda$ for some fixed value of the external momenta (for example, at the scattering threshold.) This is called a *renormalization prescription*.

Let us first compute the term with $j = 1$. Choosing the center-of-mass frame of the particles in the initial state, we find

$$\begin{aligned} \mathcal{M}_{2 \rightarrow 2}^{\text{1loop}} = i \frac{\lambda^2}{2(2\pi)^4} \int d\mathbf{q} \int dq^0 & \left[\frac{1}{E_q - q^0 - i\epsilon} \frac{1}{E_q + q^0 - i\epsilon} \right. \\ & \left. \times \frac{1}{E_q + 2E_{k_1} - q^0 - i\epsilon} \frac{1}{E_q - 2E_{k_1} + q^0 - i\epsilon} - \text{PV} \right], \end{aligned} \quad (4.48)$$

where $E_q = \sqrt{m^2 + \mathbf{q}^2}$ and PV stands for an analogous fraction with E_r replaced by $\sqrt{M^2 + \mathbf{q}^2}$. Integrating with respect to q^0 we get

$$\mathcal{M}_{2 \rightarrow 2}^{\text{1loop}} = \frac{\lambda^2}{16(2\pi)^3 E_{k_1}} \int \frac{d\mathbf{q}}{E_q} \left[\frac{1}{E_q + E_{k_1}} - \frac{1}{E_q - E_{k_1} - i\epsilon} - \text{PV} \right]. \quad (4.49)$$

The $i\epsilon$ term is immaterial in the Pauli-Villars term, provided that M is large enough ($M > 2E_{k_1}$); hence,

$$\mathcal{M}_{2\rightarrow 2}^{\text{1loop}} - \mathcal{M}_{2\rightarrow 2}^{*\text{1loop}} = -\frac{i\lambda^2 |\mathbf{k}_1|}{16\pi E_{k_1}}. \quad (4.50)$$

It is apparent that this contribution to the left-hand side of eq. (4.43) equals exactly eq. (4.45). The remaining two terms of the sum in eq. (4.47) contribute only to the real part of the amplitude. This can be checked by rotating the q^0 integration path from the real to the imaginary axis of the complex plane, which is allowed since the rotating path does not cross any singularity of the integrand. Thus, we have seen that the semi-classical approximation violates $\mathcal{S}$ -matrix unitarity, which however is recursively recovered by taking into account the contributions of radiative corrections, i.e. of Feynman diagrams with closed loops.

As we have seen explicitly in the example of elastic scattering, the four-momentum integration variables of loop diagrams are no longer completely determined by four-momentum conservation, and therefore the amplitudes contain integrations over undetermined momenta. Such integrals are in general divergent in the region of large integration four-momenta (the so-called *ultraviolet* region.) As shown in the example, divergent integrals must be made convergent by a regularization procedure, which must be accompanied by the introduction of counter-terms. These are physically acceptable if, as in the example, they can be considered as corrections to physical parameters. If this is not the case, the introduction of counter-terms automatically implies the appearance of further physical parameters. If this process does not come to an end, the theory fails to be predictive, because the number of input parameters diverges. Such theories are called non-renormalizable.

There is a strict connection between the properties of the interaction terms in the Lagrangian density and the strength of divergences of loop integrals. This can be seen using pure dimensional analysis. In natural units, there is only one independent scale, say the scale of energies or masses. Since the Lagrangian density scales as the fourth power of mass, bosonic fields have the dimensionality of a mass, so that $(\partial\phi)^2$ has the correct dimensionality. For the same reason, an interaction term of degree D in the fields and field derivatives is proportional to a coefficient with the dimension of a mass to the power $4 - D$. Thus, for example, an interaction term proportional ϕ^6 appears in the Lagrangian density with a coefficient of dimension $(\text{mass})^{-2}$. For a given amplitude, diagrams with loops must have the same dimensionality as diagrams in the semi-classical approximation; the presence of coefficients with the dimension of negative powers of an energy must be compensated by positive powers of integration four-momenta, therefore making the ultraviolet behaviour of the integral worse. It is possible to show that a field theory whose Lagrangian density is a generic linear combination of all possible monomials in the fields and their derivatives of dimensionality $D \leq 4$ is renormalizable in the sense that the problem of divergent integrals in the ultraviolet region can

be solved by suitable redefinitions of the parameters in the action. This result is sufficient to guarantee the physical predictivity of field theories involving scalars fields, and it is immediately extended to fermions. When vector fields are present, further difficulties arise, as we shall see in Chapter 8.

Spinor fields

5.1 Spinor representations of the Lorentz group

We have discussed in Section 2.1 the importance of relativistic invariance in the formulation of theories of fundamental interactions. We have studied in detail the case of scalar fields, transforming as

$$\phi'(x) = \phi(\Lambda^{-1}x) \quad (5.1)$$

under a Lorentz transformation Λ . A second well-known example (which will be discussed at length in Chapters 7 and 8) is provided by vector fields:

$$A'^{\mu}(x) = \Lambda^{\mu}_{\nu} A^{\nu}(\Lambda^{-1}x). \quad (5.2)$$

In the general case, we have a system of complex fields ϕ_{α} , $\alpha = 1, \dots, n$, with the transformation law

$$\phi'_{\alpha}(x) = \sum_{\beta=1}^n S(\Lambda)_{\alpha}^{\beta} \phi_{\beta}(\Lambda^{-1}x), \quad (5.3)$$

where the matrix $S(\Lambda)$ obeys the condition

$$S(\Lambda\Lambda') = S(\Lambda)S(\Lambda') \quad (5.4)$$

for any two Lorentz transformations Λ, Λ' . The vector field is an obvious example; in that case, we have simply $S(\Lambda) \equiv \Lambda$.

In this Section, we shall discuss a less trivial example of a realization of eq. (5.3). We observe that 2×2 hermitian matrices are in one-to-one correspondence with four vectors. Indeed, a generic 2×2 hermitian matrix x can be parametrized as

$$x = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \equiv x^{\mu} \sigma_{\mu} \quad (5.5)$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.6)$$

We have

$$\det x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \equiv x^\mu g_{\mu\nu} x^\nu = x^\mu x_\mu. \quad (5.7)$$

Let us consider a generic 2×2 complex matrix L with unit determinant ($L \in SL(2C)$):

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \det L = ad - bc = 1. \quad (5.8)$$

Such a matrix is completely specified by six real parameters, the same number as the parameters in Λ . The linear transformation

$$x \rightarrow x' = LxL^\dagger \quad (5.9)$$

leaves both $\det x$ and the sign of x^0 unchanged. Furthermore, x' is also hermitian. Thus, this transformation defines a Lorentz transformation

$$x^\mu \rightarrow x'^\mu = \Lambda_\nu^\mu(L) x^\nu. \quad (5.10)$$

In order to compute the matrix Λ that corresponds to a given L , we observe that

$$LxL^\dagger = L\sigma_\nu L^\dagger x^\nu = \sigma_\mu x'^\mu = \sigma_\mu \Lambda_\nu^\mu(L) x^\nu \quad (5.11)$$

or

$$L\sigma_\nu L^\dagger = \sigma_\mu \Lambda_\nu^\mu. \quad (5.12)$$

It will be convenient to introduce matrices $\bar{\sigma}_\mu$ defined as

$$\bar{\sigma}_0 = \sigma_0; \quad \bar{\sigma}_i = -\sigma_i. \quad (5.13)$$

It is easy to check that

$$\text{Tr}(\bar{\sigma}_\mu \sigma_\nu) = 2g_{\mu\nu}. \quad (5.14)$$

Therefore

$$\frac{1}{2} \text{Tr}(\bar{\sigma}_\mu L \sigma_\nu L^\dagger) = g_{\mu\rho} \Lambda_\nu^\rho, \quad (5.15)$$

or equivalently

$$\frac{1}{2} \text{Tr}(\bar{\sigma}^\mu L \sigma_\nu L^\dagger) = \Lambda_\nu^\mu. \quad (5.16)$$

Consider as an example the matrix

$$L = \begin{pmatrix} e^{\frac{\chi+i\theta}{2}} & 0 \\ 0 & e^{-\frac{\chi+i\theta}{2}} \end{pmatrix}. \quad (5.17)$$

By explicit calculation, we find that the corresponding $\Lambda(L)$ is given by

$$\Lambda(L) = \begin{pmatrix} \cosh \chi & 0 & 0 & \sinh \chi \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ \sinh \chi & 0 & 0 & \cosh \chi \end{pmatrix}, \quad (5.18)$$

which represents a rotation by an angle θ about the z axis, and a boost with velocity $\beta = \tanh \chi$ along the z axis.

The reader has certainly recognized in eq. (5.17) with $\chi = 0$ the matrix that represents a rotation by an angle θ around the z axis for an ordinary spinor, the vector representing the quantum states of a spin-1/2 particle in ordinary quantum mechanics.

In a natural way, one would invert the relation eq. (5.16) between Λ and L , and introduce a new kind of fields, called the *spinor* fields, transforming as

$$\xi'_\alpha(x) = \sum_{\beta=1}^2 L(\Lambda)_\alpha^\beta \xi_\beta(\Lambda^{-1}x). \quad (5.19)$$

The relation eq. (5.16), however, is not invertible, since

$$\Lambda(L) = \Lambda(-L), \quad (5.20)$$

and eq. (5.19) is ill defined. There is an obvious physical reason behind this situation, that arises from the fact that spin rotation matrices are two-valued functions of the rotation angle. There is also a mathematical reason, that provides a deeper understanding of this phenomenon. We will limit our discussion to rotations, since they are responsible for the double-valuedness of $L(\Lambda)$. The crucial point is that the Lorentz group and the rotation group are continuous groups, which means that the action of a rotation on a physical system or reference frame is a continuous operation, resulting from the combination of a sequence of infinitesimal transformations. On general grounds, we should therefore expect that $L(\Lambda)$ depends on the particular sequence that takes the space-time coordinate x to Λx :

$$\Lambda(t), \quad 0 \leq t \leq 1; \quad \Lambda(0) = I, \quad \Lambda(1) = \Lambda. \quad (5.21)$$

This correspondence between L and Λ has an obvious limitation: the family of sequences $\Lambda(t)$ with fixed boundary values depends on an infinity of parameters, while L , as observed above, has the same number of parameters as the boundary value Λ . As a consequence, L cannot vary for small deformations of the path $\Lambda(t)$ with fixed boundary values. If all possible paths that take from I to Λ are equivalent in the sense that they can be continuously deformed into each other, then L depends only on the boundary value Λ .

There is however the possibility that different inequivalent paths with the same boundary values exist; in this case, each subset of equivalent paths can lead to a different dependence of L on the boundary value Λ . As a matter of facts, this is the case for rotations: consider e.g. a rotation by an angle π around the z axis performed in two different ways:

$$\Lambda_{\pm}(t) = \Lambda(\theta_{\pm}(t)); \quad \theta_{\pm}(t) = \pm\pi t, \quad (5.22)$$

with $\Lambda(\theta)$ as in eq. (5.18). The two paths are manifestly inequivalent. We conclude that in the most general formulation of the symmetry action $L(\Lambda_+(t))$ can differ from $L(\Lambda_-(t))$, and indeed $L(\Lambda_+(t)) = -L(\Lambda_-(t))$. This discussion shows that the very existence of particles with half-integer spin requires the use of the most general interpretation of the symmetry action on quantum systems corresponding to the dependence of the space-time symmetry transformation, not only on the new and old coordinate systems, but also on the equivalence class of the path leading from the old to the new system. Therefore, the space-time transformation symmetry does not correspond to the Lorentz group, but rather to the group of the L matrices, $SL(2C)$. Finally, we note that the above considerations do not affect the theory of bosonic fields, since in those cases the action of the Lorentz group and of $SL(2C)$ coincide.

Let us therefore introduce a complex two-component spinor field $\xi_R(x)$ that under a change of reference frame transforms into

$$\xi_R(x) \rightarrow \xi'_R(x) = L \xi_R(\Lambda^{-1}(L)x), \quad (5.23)$$

(the meaning of the suffix R will be clear soon), and let us build Lorentz-invariant Lagrangian densities out of the new fields and their derivatives. These must contain even powers of the spinor fields, otherwise they would change sign under a rotation by an angle 2π around any axis. We observe that

$$L^T \epsilon L = \epsilon, \quad (5.24)$$

where ϵ is the antisymmetric matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (5.25)$$

and T denotes the usual operation of matrix transposition. Indeed,

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (5.26)$$

and $ad - bc = \det L = 1$. Using eqs. (5.12) and (5.24), one can prove that

$$\xi_R^\dagger \bar{\sigma}_\mu \xi_R \quad (5.27)$$

has the same transformation properties as a covariant four-vector:

$$\begin{aligned} \xi_R^\dagger L^\dagger \bar{\sigma}_\mu L \xi_R &= -\xi_R^\dagger L^\dagger \epsilon \sigma_\mu^T \epsilon L \xi_R \\ &= -\xi_R^\dagger \epsilon (L^{-1})^* \sigma_\mu^T (L^{-1})^T \epsilon \xi_R \\ &= -(\Lambda^{-1})_\mu^\nu \xi_R^\dagger \epsilon \sigma_\nu^T \epsilon \xi_R \\ &= (\Lambda^{-1})_\mu^\nu \xi_R^\dagger \bar{\sigma}_\nu \xi_R. \end{aligned} \quad (5.28)$$

where we have used

$$\epsilon \sigma_\mu \epsilon = -\bar{\sigma}_\mu^T. \quad (5.29)$$

Therefore, the bilinear form

$$\xi_R^\dagger \bar{\sigma}_\mu \partial^\mu \xi_R \quad (5.30)$$

transforms as a scalar under Lorentz transformations. This result allows us to build a Lagrangian density for ξ_R :

$$\mathcal{L}_{\text{Weyl}}^{(R)} = i \xi_R^\dagger \bar{\sigma}_\mu \partial^\mu \xi_R. \quad (5.31)$$

It is easy to check that $\mathcal{L}^{(R)}$ is hermitian, up to a total derivative, and that it transforms as a scalar field. The corresponding equation of motion for ξ_R is

$$\bar{\sigma}_\mu \partial^\mu \xi_R = 0. \quad (5.32)$$

This equation has negative-frequency plane-wave solutions

$$\xi_R^{(-)} = \frac{u(p)}{(2\pi)^{\frac{3}{2}}} e^{-i(|\mathbf{p}|t - \mathbf{p} \cdot \mathbf{r})}, \quad (5.33)$$

and positive-frequency solutions

$$\xi_R^{(+)} = \frac{v(p)}{(2\pi)^{\frac{3}{2}}} e^{i(|\mathbf{p}|t - \mathbf{p} \cdot \mathbf{r})}, \quad (5.34)$$

where $u(p)$ e $v(p)$ are such that

$$\boldsymbol{\sigma} \cdot \mathbf{p} u(p) = |\mathbf{p}| u(p) \quad (5.35)$$

$$\boldsymbol{\sigma} \cdot \mathbf{p} v(p) = |\mathbf{p}| v(p). \quad (5.36)$$

The free theory can be quantized in analogy with the case of the scalar field, with one important difference: the total energy in this case is found to be unbounded from below, because states with negative frequency correspond to negative energy. The system would therefore be intrinsically unstable, unless one assumes that negative-energy states are all occupied and inaccessible because of the Pauli exclusion principle. For this reason, in the semi-classical limit spinor fields acquire an anti-commutativity property, so that any exchange operation on a given amplitude induces a sign flip. As a consequence, amplitudes are antisymmetric functions of momenta and spins of identical spinor particles.

The particles associated to $\mathcal{L}_{\text{Weyl}}^{(R)}$, corresponding to negative-frequency solutions (5.33), have therefore zero mass and *helicity* (the projection of spin along the direction of motion) equal to $+\frac{1}{2}$, as one can read off eq. (5.35).

We have seen in Section 2.4 that positive-frequency solutions correspond to antiparticles in the final state. Recalling eq. (3.59) and the related comments, the spinor v is to be considered as a complex conjugate spinor. From eq. (5.36) we get

$$\boldsymbol{\sigma}^* \cdot \boldsymbol{p} \, v^* = \boldsymbol{\sigma}^T \cdot \boldsymbol{p} \, v^* = |\boldsymbol{p}| v^*. \quad (5.37)$$

In order to obtain the effect of the helicity operator, we multiply eq. (5.37) on the left by the antisymmetric matrix ϵ . We obtain

$$\boldsymbol{\sigma} \cdot \boldsymbol{p} (\epsilon v^*) = -|\boldsymbol{p}| (\epsilon v^*), \quad (5.38)$$

where we have taken into account that

$$\epsilon \boldsymbol{\sigma}^T = -\boldsymbol{\sigma} \epsilon \quad (5.39)$$

as a consequence of eq. (5.29). This shows that the antiparticles associated to ξ_R have helicity $-\frac{1}{2}$. We conclude that particles associated to ξ_R have helicity $+\frac{1}{2}$, and are therefore polarized according to the right-hand rule; this explains the use of the suffix R . The corresponding antiparticles have opposite helicity. It follows that the Lagrangian density $\mathcal{L}_{\text{Weyl}}^{(R)}$ is not invariant under the effect of space inversion.

Finally, we observe that the Lagrangian density eq. (5.31) is manifestly invariant under phase transformations of the fields, similar to eq. (2.36):

$$\xi_R(x) \rightarrow \xi'_R(x) = e^{i\varphi} \xi_R(x). \quad (5.40)$$

According to eq. (2.32), the corresponding conserved current is given by

$$J_R^\mu = -\xi_R^\dagger \bar{\sigma}^\mu \xi_R. \quad (5.41)$$

We have studied the properties of the spinor field ξ_R under the action of Lorentz transformations. We now discuss its behaviour under the effect of *parity inversion*, that is, the change of axis orientation in ordinary three-dimensional space. This operation is usually denoted by P . Let us consider the matrix in eq. (5.17); under parity inversion, the parameter χ , which is related to the relative velocity of the two reference frames, changes sign, while the angle θ is unchanged. It is easy to see that such a transformation is achieved by transforming L as follows:

$$L \rightarrow L_P = \epsilon L^* \epsilon^T. \quad (5.42)$$

It is also easy to show that L_P is not equivalent to L , in the sense that it cannot be reduced to L by a transformation

$$L \rightarrow L' = z L z^{-1} \quad (5.43)$$

with z independent of θ and χ . Hence, the P -reflected image of ξ_R is a new spinor field ξ_L :

$$P : \xi_R(x) \rightarrow \xi_L(x_P); \quad x_P^0 = x^0, \quad \boldsymbol{x}_P = -\boldsymbol{x}. \quad (5.44)$$

that transforms as

$$\xi_L(x) \rightarrow \xi'_L(x) = \epsilon L^* \epsilon^T \xi_L(\Lambda^{-1}(L)x). \quad (5.45)$$

Note that ξ_L transforms as $\epsilon \xi_R^*$:

$$\epsilon \xi_R^* \rightarrow \epsilon L^* \xi_R^* = \epsilon L^* \epsilon^T (\epsilon \xi_R^*). \quad (5.46)$$

It is easy to check that $\xi_L^\dagger \sigma_\mu \xi_L$ transforms as a covariant four-vector; therefore,

$$\mathcal{L}_{\text{Weyl}}^{(L)} = i \xi_L^\dagger \sigma_\mu \partial^\mu \xi_L \quad (5.47)$$

is a good Lagrangian density for ξ_L . By the same argument employed for ξ_R , one can show that the particles associated to ξ_L have negative helicity.

The sum

$$\mathcal{L}_{\text{Weyl}} = \mathcal{L}_{\text{Weyl}}^{(R)} + \mathcal{L}_{\text{Weyl}}^{(L)} \quad (5.48)$$

is invariant under parity inversion:

$$P : i \xi_R^\dagger \bar{\sigma}_\mu \partial^\mu \xi_R \leftrightarrow i \xi_L^\dagger \sigma_\mu \partial^\mu \xi_L. \quad (5.49)$$

A further transformation that leaves eq. (5.48) unchanged is

$$\xi_L(x) \rightarrow \epsilon \xi_R^*(x), \quad \xi_R(x) \rightarrow \epsilon \xi_L^*(x). \quad (5.50)$$

Indeed, under the transformations eq. (5.50) $\mathcal{L}_{\text{Weyl}}$ transforms as

$$\begin{aligned} & i \xi_L^T \epsilon^T \bar{\sigma}_\mu \epsilon \partial^\mu \xi_R^* + i \xi_R^T \epsilon^T \sigma_\mu \epsilon \partial^\mu \xi_L^* \\ &= -i \partial^\mu \xi_L^\dagger \epsilon^T \bar{\sigma}_\mu^T \epsilon \xi_L - i \partial^\mu \xi_R^\dagger \epsilon^T \sigma_\mu^T \epsilon \xi_R \\ &= -i \xi_L^\dagger \epsilon \bar{\sigma}_\mu^T \epsilon \partial^\mu \xi_L - i \xi_R^\dagger \epsilon \sigma_\mu^T \epsilon \partial^\mu \xi_R, \end{aligned} \quad (5.51)$$

where we have omitted total derivatives and we have taken into account the anti-commuting character of the spinor fields. Using eq. (5.29) we get

$$\mathcal{L}_{\text{Weyl}} \rightarrow i(\xi_L^\dagger \bar{\sigma}_\mu \partial^\mu \xi_L + \xi_R^\dagger \sigma_\mu \partial^\mu \xi_R) = \mathcal{L}_{\text{Weyl}} \quad (5.52)$$

as announced. Note that, much in the same way as the transformation in eq. (5.44), eq. (5.50) is defined up to two arbitrary phase factors. This is obvious since the Weyl theory is left invariant by independent phase transformations of the spinor fields. Adding further terms to the Lagrangian density, e.g. a mass term, this phase freedom will be reduced together with the invariance of the Lagrangian.

Now, considering the physical content of the transformations eq. (5.50), we notice that a transformation of a field into a Hermitian conjugate one corresponds to a particle-antiparticle transformation. In the present case the left-handed particle transforms into the anti-right-handed particle, which is left-handed, as it should, since eq. (5.50) does not act on space-time. Thus we can call the action of transformation (5.50) particle-antiparticle conjugation, or *charge conjugation*, usually denoted by C . It is left as an exercise to the

reader to verify that the conserved current $-(\xi_L^\dagger \bar{\sigma}_\mu \xi_L + \xi_R^\dagger \sigma_\mu \xi_R)$ changes sign under C conjugation. One can also check that both $\mathcal{L}_{\text{Weyl}}^{(R)}$ and $\mathcal{L}_{\text{Weyl}}^{(L)}$ are individually invariant under the combined action of P and C conjugations.

A third discrete transformation of great importance in particle physics is time reversal T. This reflection is anti-linear, and exchanges final and initial states of any given process. Its action on spinor fields is

$$\xi_{L/R}(t, \mathbf{r}) \rightarrow \xi_{L/R}^\dagger(-t, \mathbf{r}) \epsilon. \quad (5.53)$$

Hence, spinor bilinear forms transform as

$$\xi_R^\dagger \bar{\sigma}^\mu \eta_R(t, \mathbf{r}) \rightarrow \eta_R^\dagger \bar{\sigma}_\mu \xi_R(-t, \mathbf{r}) \quad (5.54)$$

$$\xi_L^\dagger \sigma^\mu \eta_L(t, \mathbf{r}) \rightarrow \eta_L^\dagger \sigma_\mu \xi_L(-t, \mathbf{r}) \quad (5.55)$$

$$\xi_{R/L}^T \epsilon \eta_{R/L}(t, \mathbf{r}) \rightarrow \eta_{R/L}^\dagger \epsilon \xi_{R/L}(-t, \mathbf{r}). \quad (5.56)$$

The invariance under T conjugation is often considered as equivalent to the symmetry under CP, since a general theorem of Quantum Field Theory asserts that any covariant, dynamically stable and local theory is invariant under the combined action of P, C, and T conjugations.

5.2 Mass terms and coupling to scalars

In this Section we study the possibility of building mass terms (i.e., bilinears in the fields without derivatives) for spinor fields. A straightforward extension is the formation of interaction terms among spinor and scalar fields.

First of all, we note that the conventional scalar product between two spinors

$$\xi_R^\dagger \eta_R = \sum_{\alpha=1}^2 (\xi_R^*)_\alpha (\eta_R)_\alpha \quad (5.57)$$

is not Lorentz-invariant, because the transformation matrix L is not unitary. Instead, the quantity $\xi_R^T \epsilon \eta_R$ is invariant: using eq. (5.24) we find

$$\xi_R^T \epsilon \eta_R \rightarrow \xi_R^T L^T \epsilon L \eta_R = \xi_R^T \epsilon \eta_R. \quad (5.58)$$

Similarly, one can check that $\xi_L^T \epsilon \eta_L$ is also invariant. Consider now a right-handed spinor ξ_R and a left-handed one, ξ_L . Since ξ_L transforms as $\epsilon \xi_R^*$, it can be written as $\xi_L = \epsilon \eta_R^*$ for some right-handed spinor η_R . Thus,

$$\xi_L^\dagger \xi_R = -\eta_R^T \epsilon \xi_R \quad (5.59)$$

is also invariant.

In a generic theory, built out of the spinor and scalar fields

$$\xi_R^{(r)}, \quad r = 1, \dots, n_r \quad (5.60)$$

$$\xi_L^{(l)}, \quad l = 1, \dots, n_l \quad (5.61)$$

$$\phi^{(s)}, \quad s = 1, \dots, n_s \quad (5.62)$$

the following terms can be included in the Lagrangian density:

$$\begin{aligned} \mathcal{L}_M = & - \sum_{rl} \left[m_{rl} \xi_R^{(r)\dagger} \xi_L^{(l)} + m_{rl}^* \xi_L^{(l)\dagger} \xi_R^{(r)} \right] \\ & - \frac{1}{2} \sum_{rr'} \left[M_{rr'}^{(R)} \xi_R^{(r)T} \epsilon \xi_R^{(r')} - M_{rr'}^{(R)*} \xi_R^{(r')\dagger} \epsilon \xi_R^{(r)*} \right] \\ & - \frac{1}{2} \sum_{ll'} \left[M_{ll'}^{(L)} \xi_L^{(l)T} \epsilon \xi_L^{(l')} - M_{ll'}^{(L)*} \xi_L^{(l')\dagger} \epsilon \xi_L^{(l)*} \right] \end{aligned} \quad (5.63)$$

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} = & - \sum_{rls} \left[g_{rls} \phi^{(s)} \xi_R^{(r)\dagger} \xi_L^{(l)} + g_{rls}^* \phi^{(s)\dagger} \xi_L^{(l)\dagger} \xi_R^{(r)} \right] \\ & - \sum_{rr's} \left[G_{rr's}^{(R)} \phi^{(s)} \xi_R^{(r)T} \epsilon \xi_R^{(r')} - G_{rr's}^{(R)*} \phi^{(s)\dagger} \xi_R^{(r')\dagger} \epsilon \xi_R^{(r)*} \right] \\ & - \sum_{ll's} \left[G_{ll's}^{(L)} \phi^{(s)} \xi_L^{(l)T} \epsilon \xi_L^{(l')} - G_{ll's}^{(L)*} \phi^{(s)\dagger} \xi_L^{(l')\dagger} \epsilon \xi_L^{(l)*} \right]. \end{aligned} \quad (5.64)$$

Spinor fields have the dimension of an energy to the power $3/2$, while scalar fields have the dimension of energy. Hence, since the Lagrangian density has the dimension of an energy to the fourth power, the constants $M^{(R)}$, $M^{(L)}$, m have the dimension of an energy, and $G^{(R)}$, $G^{(L)}$, g are dimensionless.

The term $\mathcal{L}_M$ contains mass terms for the spinor fields. The first one is invariant under multiplication of all spinor fields by a common phase factor, and is usually called a *Dirac mass term*. The remaining two terms do not share the same property, and are called *Majorana mass terms*. The mass matrices $M^{(R)}$ and $M^{(L)}$ are symmetric, as one can check taking into account the anti-commutation relations of spinor fields mentioned in Section 5.1 (which is related to Fermi-Dirac statistics) and the antisymmetry of ϵ . Similar considerations hold for $\mathcal{L}_{\text{Yukawa}}$; spinor-spinor-scalar interaction terms are usually referred to as *Yukawa couplings*.

Let us concentrate on mass terms. To simplify notations, we collect all left-handed spinors $\xi_L^{(l)}$ and $\epsilon \xi_R^{(r)*}$ in an $(n_l + n_r)$ -component spinor Ξ . Then, we may write eq. (5.63) in the form

$$\mathcal{L}_M = -\frac{1}{2} \sum_{\alpha, \beta} \left[\mathcal{M}_{\alpha\beta} \Xi_\alpha^T \epsilon \Xi_\beta - \mathcal{M}_{\beta\alpha}^* \Xi_\alpha^\dagger \epsilon \Xi_\beta^* \right], \quad (5.65)$$

where $\mathcal{M}$ is a complex symmetric matrix, with

$$\mathcal{M}_{ll'} = M_{ll'}^{(L)} \quad \mathcal{M}_{rr'} = -M_{rr'}^{(R)*} \quad \mathcal{M}_{lr} = m_{lr}. \quad (5.66)$$

A complex symmetric matrix can always be written in the form

$$\mathcal{M} = \mathcal{U}^T \hat{\mathcal{M}} \mathcal{U}, \quad (5.67)$$

with $\mathcal{U}$ unitary and $\hat{\mathcal{M}}$ real and diagonal (the number of real parameters that determine an $N \times N$ matrix is $N(N+1)$ if the matrix is symmetric, N^2 if it is unitary, N if it is real and diagonal.) On the other hand, a unitary transformation on Ξ ,

$$\Xi \rightarrow \mathcal{U} \Xi; \quad \mathcal{U}^\dagger \mathcal{U} = I \quad (5.68)$$

does not affect kinetic terms in the canonical form, and brings $\mathcal{L}_M$ in diagonal form:

$$\mathcal{L}_M = -\frac{1}{2} \sum_{\alpha} \hat{\mathcal{M}}_{\alpha\alpha} (\Xi_{\alpha}^T \epsilon \Xi_{\alpha} - \Xi_{\alpha}^{\dagger} \epsilon \Xi_{\alpha}^*) . \quad (5.69)$$

We have shown that $\mathcal{L}_M$ can always be written as a sum of Majorana mass terms, with real and positive coefficients, without any Dirac mass terms. At first sight, this is a surprising result, since we know that only Dirac mass terms are compatible with invariance under phase rotations. However, it is easy to show that each Dirac mass term in the original Lagrangian corresponds to a pair of degenerate eigenvalues in $\mathcal{L}_M$ written in the canonical Majorana form, eq. (5.69); a phase transformation then appears as a rotation in the subspace of the two degenerate spinor fields.

Let us consider the case when Ξ has only two components $\xi_L, \epsilon \xi_R^*$. The mass matrix is given by

$$\mathcal{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (5.70)$$

with a, b, c complex. The squared masses are given by the eigenvalues of

$$\mathcal{M}^\dagger \mathcal{M} = \begin{pmatrix} a^* & b^* \\ b^* & c^* \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & a^*b + b^*c \\ ab^* + bc^* & |b|^2 + |c|^2 \end{pmatrix}, \quad (5.71)$$

that is,

$$m_{\pm}^2 = \frac{1}{2} \left[|a|^2 + |c|^2 + 2|b|^2 \pm \sqrt{(|a|^2 - |c|^2)^2 + 4|b|^2(|a|^2 + |c|^2 + 2\text{Re}(a^*c^*b^2))} \right]. \quad (5.72)$$

An interesting limit of eq. (5.72) is the case $c = 0, |b| \ll |a|$. In this case we find

$$m_{\pm}^2 = \frac{1}{2} \left[|a|^2 + 2|b|^2 \pm \sqrt{|a|^4 + 4|b|^2|a|^2} \right]. \quad (5.73)$$

and therefore, to first order in $|b/a|^2$,

$$m_+^2 \simeq |a|^2; \quad m_-^2 \simeq |b|^2 \left| \frac{b}{a} \right|^2. \quad (5.74)$$

This is the so called *see-saw* mechanism, which is employed to explain the small values of neutrino masses, as we shall see in Chapter 12.

Under CP conjugation,

$$\Xi_\alpha(x) \rightarrow i\epsilon \Xi_\alpha^*(x_P). \quad (5.75)$$

This leaves eqs. (5.69) and (5.48) invariant (up to total derivatives). Coupling terms are left invariant provided a transformation as in eq. (5.68) exists, such that all parameters are made real. Parity invariance, on the other hand, requires the presence of a pair of spinors with opposite chiralities and equal masses for each fermion species.

Gauge symmetries

6.1 Electrodynamics

The simplest example of a phenomenological application of the theory of spinor fields is electrodynamics, originally formulated by P.A.M. Dirac to provide a relativistic description of electrons and photons. The theory is based on a pair of spinor fields of opposite chiralities, ξ_R and ξ_L ; invariance under phase multiplication of the spinor fields is assumed, and the corresponding conserved current is identified with the electromagnetic current. No scalar field is present. Under these assumptions, the free-field Lagrangian is given by

$$\mathcal{L}_0 = i\xi_R^\dagger \bar{\sigma}_\mu \partial^\mu \xi_R + i\xi_L^\dagger \sigma_\mu \partial^\mu \xi_L - m \left(\xi_R^\dagger \xi_L + \xi_L^\dagger \xi_R \right). \quad (6.1)$$

$\mathcal{L}_0$ is manifestly invariant under the phase transformations

$$\xi_R(x) \rightarrow e^{ie\Lambda} \xi_R(x) \quad (6.2)$$

$$\xi_L(x) \rightarrow e^{ie\Lambda} \xi_L(x), \quad (6.3)$$

where e, Λ are real constants. Notice that we have inserted a Dirac mass term, which is allowed by the assumed invariance properties, while Majorana mass terms are not, as mentioned in Section 5.2. The conserved current is given by

$$J_\mu = -e \left(\xi_R^\dagger \bar{\sigma}_\mu \xi_R + \xi_L^\dagger \sigma_\mu \xi_L \right); \quad \partial^\mu J_\mu = 0, \quad (6.4)$$

and the constant e plays the role of elementary charge.

It is known from classical physics that electromagnetic fields interact with matter through the electric current density $e\mathbf{J}$ and the charge density $e\rho$; the interaction energy has the form

$$H_I = \frac{e}{c} \int d\mathbf{r} \left(\rho \Phi_{\text{em}} - \mathbf{J} \cdot \mathbf{A} \right), \quad (6.5)$$

where e is the elementary charge, Φ_{em} the electrostatic potential, $\mathbf{A}$ the vector potential, ρ the position density of charged particles, or the charge density

divided by the elementary charge, and $\mathbf{J}$ the current density. In the covariant formalism, one recognizes that $e(c\rho, \mathbf{J})$ form a current four-vector J_μ , and $(\frac{\Phi_{\text{em}}}{c}, \mathbf{A})$ a potential four-vector A^μ ; hence, in our units,

$$H_I = \int d\mathbf{r} A^\mu J_\mu. \quad (6.6)$$

We will therefore take

$$\mathcal{L}_I = -A^\mu J_\mu = e A^\mu \left(\xi_R^\dagger \bar{\sigma}_\mu \xi_R + \xi_L^\dagger \sigma_\mu \xi_L \right) \quad (6.7)$$

as the Lagrangian density for the electromagnetic interactions. Thus,

$$\mathcal{L}_{\text{QED}} = i\xi_R^\dagger \bar{\sigma}_\mu (\partial^\mu - ieA^\mu) \xi_R + i\xi_L^\dagger \sigma_\mu (\partial^\mu - ieA^\mu) \xi_L - m \left(\xi_R^\dagger \xi_L + \xi_L^\dagger \xi_R \right). \quad (6.8)$$

The theory is invariant with respect to both parity inversion P and charge conjugation C, assuming that the vector field A_μ changes sign under charge conjugation. Note that the condition of parity invariance requires that the hidden phases in eq. (5.44) be the same for both ξ_R and ξ_L , while C invariance is achieved if the two phases differ by π .

Notations are remarkably simplified by the following, universally adopted definitions. The two spinors are paired into a single four-component spinor

$$\psi(x) \equiv \begin{pmatrix} \xi_R(x) \\ \xi_L(x) \end{pmatrix} \equiv \begin{pmatrix} \xi_{R1}(x) \\ \xi_{R2}(x) \\ \xi_{L1}(x) \\ \xi_{L2}(x) \end{pmatrix}. \quad (6.9)$$

Next, one defines the 4×4 matrices

$$\gamma_\mu \equiv \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma_5 \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = -i\gamma_0\gamma_1\gamma_2\gamma_3 \quad (6.10)$$

and the Dirac-conjugate spinor

$$\bar{\psi}(x) \equiv \left(\xi_L^\dagger(x) \quad \xi_R^\dagger(x) \right) = \psi^\dagger \gamma_0. \quad (6.11)$$

Through the matrix γ_5 , it is possible to define projection operators, that project the left and right components out of the four-spinor ψ :

$$\frac{1 + \gamma_5}{2} \psi = \begin{pmatrix} \xi_R \\ 0 \end{pmatrix}; \quad \frac{1 - \gamma_5}{2} \psi = \begin{pmatrix} 0 \\ \xi_L \end{pmatrix}. \quad (6.12)$$

The notation

$$\not{q} \equiv \gamma_\mu q^\mu \quad (6.13)$$

is often used. With these definitions, the Lagrangian density of electrodynamics becomes

$$\mathcal{L}_{\text{QED}} = i\bar{\psi} \gamma_\mu (\partial^\mu - ieA^\mu) \psi - m\bar{\psi}\psi. \quad (6.14)$$

An even simpler expression is obtained introducing the *covariant derivative* of the field ψ as

$$D_\mu \psi(x) \equiv \partial_\mu \psi(x) - ieA_\mu(x)\psi(x). \quad (6.15)$$

The new, important fact that arises after introduction of the interaction with the electromagnetic field is that the Lagrangian density

$$\mathcal{L}_{\text{QED}} = i\bar{\psi} \gamma_\mu D^\mu \psi - m\bar{\psi}\psi \quad (6.16)$$

is now invariant with respect to the transformations

$$\psi(x) \rightarrow e^{+ie\Lambda(x)} \psi(x) \quad (6.17)$$

$$\bar{\psi}(x) \rightarrow e^{-ie\Lambda(x)} \bar{\psi}(x) \quad (6.18)$$

$$A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \Lambda(x), \quad (6.19)$$

that are a generalization of the transformations in eq. (2.36) to the case $\Lambda = \Lambda(x)$. For historical reasons, the transformations (6.17-6.19) are called *gauge transformations*.

It is now clear why D^μ is called a covariant derivative: from eq. (6.17) we see that the transformation of the ordinary partial derivative of ψ ,

$$\partial_\mu \psi(x) \rightarrow \partial_\mu \left(e^{ie\Lambda(x)} \psi(x) \right) = e^{ie\Lambda(x)} \partial_\mu \psi(x) + ie\partial_\mu \Lambda(x) e^{ie\Lambda(x)} \psi(x) \quad (6.20)$$

is not just a multiplication by a phase factor. This is instead true for $D_\mu \psi$, since the second term in eq. (6.15) transforms as

$$-ieA_\mu(x)\psi(x) \rightarrow -ieA_\mu(x)e^{ie\Lambda(x)}\psi(x) - ie\partial_\mu \Lambda(x)e^{ie\Lambda(x)}\psi(x), \quad (6.21)$$

thus compensating the term proportional to $\partial_\mu \Lambda$ in eq. (6.20). We have therefore established that

$$D_\mu \psi(x) \rightarrow e^{ie\Lambda(x)} D_\mu \psi(x). \quad (6.22)$$

It is immediate to show that the same result extends to multiple covariant derivatives:

$$D_{\mu_1} \dots D_{\mu_n} \psi(x) \rightarrow e^{ie\Lambda(x)} D_{\mu_1} \dots D_{\mu_n} \psi(x) \quad (6.23)$$

for any value of n .

The above construction is the simplest application of the *principle of gauge invariance*. It can be shown on general grounds that any field theory that involves four-vector fields must possess a gauge invariance under transformations analogous to eqs. (6.17-6.19). Gauge invariance is needed in order to eliminate the effect of those components of the vector field that do not correspond to physical degrees of freedom. Such components are necessarily present, since the vector field has four components, while the associated spin-1 particles have at most three degrees of freedom. An explicit example of this cancellation will be shown in Chapter 8 in the context of the Higgs model.

In order to complete the Lagrangian density of electrodynamics, we should add a term for the electromagnetic field. The energy density of electric and magnetic fields is given by $\frac{\mathbf{E}^2 + \mathbf{B}^2}{2}$; the term proportional to $\mathbf{E}^2$ can be interpreted as a 'kinetic' term, because the electric field $\mathbf{E}$ is linear in the time derivative of the vector potential $\mathbf{A}$, while the magnetic field $\mathbf{B}$ is given by spatial derivatives (the curl of $\mathbf{A}$). Hence, we conclude that the electromagnetic Lagrangian density is

$$\mathcal{L}_{\text{em}} = \frac{\mathbf{E}^2 - \mathbf{B}^2}{2}. \quad (6.24)$$

This can be written in an explicitly covariant form by means of the *field tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (6.25)$$

We find

$$\mathcal{L}_{\text{em}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \quad (6.26)$$

The tensor $F_{\mu\nu}$ is invariant under gauge transformations, as one can verify directly using eq. (6.19). Alternatively, one may observe that

$$F_{\mu\nu}(x)\psi(x) = -\frac{i}{e}(D_\mu D_\nu - D_\nu D_\mu)\psi(x) \equiv -\frac{i}{e}[D_\mu, D_\nu]\psi(x), \quad (6.27)$$

from which it is immediately clear that $F_{\mu\nu}$, and therefore $\mathcal{L}_{\text{em}}$, are gauge-invariant. We should stress that $\mathcal{L}_{\text{em}}$ is the only possible term that depends on A^μ and its first derivatives, compatible with gauge and Lorentz invariance, C and P invariance, and renormalizability. In particular, gauge invariance in the form of eqs. (6.17), (6.18) and (6.19) excludes the possibility of introducing a term $\mu^2 A_\mu A^\mu$, that would correspond to a non-zero value for the photon mass.¹

The electromagnetic Lagrangian density is therefore

$$\mathcal{L}_{\text{QED}} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \quad (6.28)$$

The equations of motion are immediately derived. We find

$$i\gamma^\mu\partial_\mu\psi - m\psi = -e\gamma^\mu A_\mu\psi \quad (6.29)$$

$$\partial^2 A^\mu - \partial^\mu\partial A = -J^\mu. \quad (6.30)$$

Next, we must determine the Green functions for both equations. The Green function $S(x)$ for the spinor field is defined by

¹ As a matter of fact, there is an alternative formulation of gauge invariance, associated with the Higgs mechanism, which we shall describe in Section 8.2. As remarked at the end of this Section, in a particular limit, which is only allowed in an abelian gauge theory, such as QED, the Higgs mechanism corresponds to the introduction of a mass term for the photon. The resulting theory is Stueckelberg's massive QED.

$$(i\gamma^\mu \partial_\mu - m) S(x) = \delta(x). \quad (6.31)$$

The calculation of $S(x)$ is greatly simplified by the algebraic properties of the γ matrices,

$$\{\gamma^\mu, \gamma^\nu\} = 2I g^{\mu\nu}, \quad \text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}. \quad (6.32)$$

We multiply both sides of eq. (6.31) with $i\gamma^\mu \partial_\mu + m$ on the left:

$$(i\gamma^\mu \partial_\mu + m) (i\gamma^\nu \partial_\nu - m) S(x) = (i\gamma^\mu \partial_\mu + m) \delta(x), \quad (6.33)$$

and we observe that, because of eq. (6.32),

$$(i\gamma^\mu \partial_\mu + m) (i\gamma^\nu \partial_\nu - m) = -(\partial^2 + m^2). \quad (6.34)$$

Hence,

$$-(\partial^2 + m^2) S(x) = (i\gamma^\mu \partial_\mu + m) \delta(x), \quad (6.35)$$

and comparing with eqs. (3.75,3.77) we find

$$S(x) = -(i\gamma^\mu \partial_\mu + m) \Delta(x) = \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma_\mu k^\mu + m}{k^2 - m^2 + i\epsilon} e^{-ik \cdot x}, \quad (6.36)$$

where $\Delta(x)$ is the Green function for the scalar field, eq. (3.75).

The natural definition for the Green function for A^μ would be

$$\partial^2 \Delta^{\mu\nu}(x) - \partial^\mu \partial_\lambda \Delta^{\lambda\nu}(x) = -g^{\mu\nu} \delta^4(x). \quad (6.37)$$

This equation, however, does not have a unique solution: $\Delta^{\mu\nu}(x) = \partial^\mu X^\nu(x)$ is a solution of the homogeneous equation associated with (6.37) for any choice of the four-vector function X^ν . This difficulty is related to the fact that Maxwell's equations do not determine the vector potential: they must therefore be solved with a supplementary prescription. The choice of this prescription has been one of the central problems in theoretical physics during the last century; it is connected with the axioms of quantum mechanics, and in particular with the conservation of probability. However, in the semi-classical approximation of electrodynamics this problem has a simple solution, that will be presented here in a further simplified version. We introduce an extra (*gauge fixing*) term in the Lagrangian density, namely

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2} (\partial_\mu A^\mu)^2. \quad (6.38)$$

The Lagrangian density becomes

$$\mathcal{L}_{\text{QED}} = i\bar{\psi} \gamma^\mu D_\mu \psi - m\bar{\psi}\psi - \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2} (\partial_\mu A^\mu)^2. \quad (6.39)$$

The last two terms can be rewritten as

$$\frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2} (\partial_\mu A^\mu)^2 = \frac{1}{2} \partial_\mu (A_\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu), \quad (6.40)$$

which is a four-divergence. By the Gauss-Green theorem, this contributes to the action functional only through surface terms, and therefore has no effect on the equations of motion. We may therefore omit it altogether, and write the QED Lagrangian in its final form,

$$\mathcal{L}_{\text{QED}} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu. \quad (6.41)$$

The equations of motion for the vector field A^μ are now

$$\partial^2 A^\mu = -J^\mu. \quad (6.42)$$

This implies that ∂A is a free field, whenever the current J^μ is conserved. The Green function corresponding to eq. (6.42) is the solution of

$$\partial^2 \Delta^{\mu\nu}(x) = -g^{\mu\nu} \delta(x), \quad (6.43)$$

and is immediately computed using eq. (3.75) with $m = 0$; we find

$$\Delta^{\mu\nu}(x) = g^{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{k^2 + i\epsilon}. \quad (6.44)$$

The presence of $g^{\mu\nu}$ in the photon propagator implies that the sign of the propagator of the time-like component is opposite to that of the space-like components, which is the same as for a massless scalar field. This corresponds to the fact that time-like polarization states have negative norm, as we shall see in Section 8.3. This is physically relevant, since the norm of states has a probabilistic interpretation: the presence of negative norms violates the conservation of probability, or equivalently the unitarity of the $\mathcal{S}$ -matrix. We are going to see in a moment that time-like polarization states do not participate in the scattering process in the semi-classical approximation.

We are now able to work out the Feynman rules for electrodynamics. The Dirac field ψ is a charged field, and the corresponding lines have an orientation. The propagator $S(x-y)$, eq. (6.36), is a matrix in the space of Dirac indices; the line index corresponds to the field $\psi(x)$, and the column index to $\bar{\psi}(y)$. The photon propagator is given in eq. (6.44). Finally, there is only one interaction term in the Lagrangian density (6.41), namely

$$eA_\mu\bar{\psi}\gamma^\mu\psi. \quad (6.45)$$

The commonly adopted graphical symbols are

$$\alpha \longrightarrow \longleftarrow \beta \quad \leftrightarrow \quad \frac{(\gamma^\mu q_\mu + m)_\beta^\alpha}{q^2 - m^2 + i\epsilon} \quad (6.46)$$

for the spinor propagator, with the momentum q flowing in the direction of the arrow;

$$\mu \text{ ~~~~~ } \nu \quad \leftrightarrow \quad \frac{g^{\mu\nu}}{k^2 + i\epsilon} \quad (6.47)$$

for the photon propagator, and

$$\begin{array}{c} \mu \\ \text{~~~~~} \\ \alpha \text{ --- } \text{---} \beta \end{array} \quad \leftrightarrow \quad e (\gamma^\mu)^\alpha_\beta \quad (6.48)$$

for the vertex factor. Note that each endpoint of a fermion line carries a spinor index (α and β in the figure), and that each endpoint of a photon line carries a space-time index (μ and ν in the figure.)

Finally, we need the expressions of the asymptotic vector and spinor fields for given initial and final states of a transition. The asymptotic vector field can be obtained from the asymptotic expression for the scalar field, eq. (3.91), modified in order to account for the polarization states of vector particles. For each plane wave with momentum p_μ we introduce a complex polarization four-vector ϵ_μ , that depends on the helicity state λ of the asymptotic particle. The asymptotic vector field turns out to be

$$A_\mu^{(\text{as})}(x) = \frac{(\sqrt{4\pi}\sigma)^{3/2}}{(2\pi)^{3/2}} \left[\sum_i \epsilon_\mu^{(i)}(p_i, \lambda_i) \frac{e^{-ip_i \cdot x}}{\sqrt{2E_{p_i}}} + \sum_f \epsilon_\mu^{(f)*}(k_f, \lambda_f) \frac{e^{ik_f \cdot x}}{\sqrt{2E_{k_f}}} \right]. \quad (6.49)$$

A generic polarization vector $\epsilon^\mu(p, \lambda)$ can be decomposed as

$$\epsilon^\mu = \epsilon_T^\mu + ap^\mu + b\bar{p}^\mu, \quad (6.50)$$

where

$$\bar{p} = (p^0, -\mathbf{p}); \quad \epsilon_T = (0, \boldsymbol{\epsilon}_T); \quad \mathbf{p} \cdot \boldsymbol{\epsilon}_T = 0. \quad (6.51)$$

The asymptotic value of ∂A is therefore proportional to b . Since, as shown above, ∂A is a free field, and hence it does not participate in the scattering process, one can choose $b = 0$ without loss of generality. The longitudinal polarization component ap^μ , on the other hand, may be eliminated by a suitable gauge transformation: this is possible, because even after the introduction of the gauge fixing term $\mathcal{L}_{\text{gf}}$ the Lagrangian density has a residual gauge invariance with respect to transformations $A^\mu \rightarrow A^\mu + \partial^\mu \Lambda$ with $\partial^2 \Lambda = 0$. We conclude that the particles associated with the field A^μ can only have transverse polarization states.

The transverse polarization vectors must be normalized as

$$\epsilon_{T\mu}^*(p, \lambda) \epsilon_T^\mu(p, \lambda) = -1, \quad (6.52)$$

and it is easy to show that the sum over physical photon polarization can be performed using

$$\sum_{\lambda} \epsilon_{\mu}(p, \lambda) \epsilon_{\nu}^*(p, \lambda) = -g_{\mu\nu} + \frac{p_{\mu} \bar{p}_{\nu} + \bar{p}_{\mu} p_{\nu}}{p \cdot \bar{p}}. \quad (6.53)$$

Similarly, the asymptotic spinor field is written in terms of coefficients that describe the spin state of particle in the initial and final states:

$$\psi^{(\text{as})}(x) = \frac{(\sqrt{4\pi}\sigma)^{3/2}}{(2\pi)^{3/2}} \left[\sum_i \frac{u(p_i, \lambda_i)}{\sqrt{2E_{p_i}}} e^{-ip_i \cdot x} + \sum_f \frac{v(k'_f, \lambda'_f)}{\sqrt{2E_{k'_f}}} e^{ik'_f \cdot x} \right] \quad (6.54)$$

$$\bar{\psi}^{(\text{as})}(x) = \frac{(\sqrt{4\pi}\sigma)^{3/2}}{(2\pi)^{3/2}} \left[\sum_i \frac{\bar{v}(p'_i, \lambda'_i)}{\sqrt{2E_{p'_i}}} e^{-ip'_i \cdot x} + \sum_f \frac{\bar{u}(k_f, \lambda_f)}{\sqrt{2E_{k_f}}} e^{ik_f \cdot x} \right] \quad (6.55)$$

where

- p_i, λ_i are momenta and helicities of particles in the initial state
- k'_f, λ'_f are momenta and helicities of antiparticles in the final state
- p'_i, λ'_i are momenta and helicities of antiparticles in the initial state
- k_f, λ_f are momenta and helicities of particles in the final state.

Note that $\bar{\psi}^{(\text{as})}$ is *not* the Dirac conjugate of $\psi^{(\text{as})}$, because the weak limit defined in Section 3.2 does not preserve the hermiticity properties of the fields. The spinors u and v obey the constraints

$$(\gamma_{\mu} p^{\mu} - m) u(p, \lambda) = 0 \quad (6.56)$$

$$(\gamma_{\mu} p^{\mu} + m) v(p, \lambda) = 0 \quad (6.57)$$

as a consequence of the equations of motion for the free Dirac field.

Due to the factor $1/\sqrt{2E_p}$ in eqs. (6.54,6.55), u and v are normalized as

$$u^{\dagger}(p, \lambda) u(p, \lambda') = v^{\dagger}(p, \lambda) v(p, \lambda') = 2E_p \delta_{\lambda\lambda'}; \quad v^{\dagger}(\bar{p}, \lambda) u(p, \lambda') = 0, \quad (6.58)$$

where $\bar{p} = (p^0, -\mathbf{p})$. In the case $m \neq 0$, the above normalization can be translated into the invariant form

$$\bar{u}(p, \lambda) u(p, \lambda') = 2m \delta_{\lambda\lambda'} \quad (6.59)$$

$$\bar{v}(p, \lambda) v(p, \lambda') = -2m \delta_{\lambda\lambda'} \quad (6.60)$$

$$\bar{v}(p, \lambda) u(p, \lambda') = 0. \quad (6.61)$$

These relations imply

$$\sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda) = \gamma_{\mu} p^{\mu} + m \quad (6.62)$$

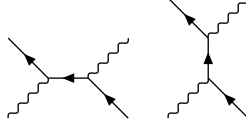
$$\sum_{\lambda} v(p, \lambda) \bar{v}(p, \lambda) = \gamma_{\mu} p^{\mu} - m. \quad (6.63)$$

6.2 A sample calculation: Compton scattering

As an example, we compute the cross section for the process

$$e^-(p) + \gamma(q, \epsilon) \rightarrow e^-(p') + \gamma(q', \epsilon'), \quad (6.64)$$

usually called Compton scattering (the four-momenta of the particles involved are indicated in brackets.) The invariant amplitude is obtained from the diagrams


(6.65)

We find

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_{\alpha\beta} \epsilon^\alpha \epsilon'^{* \beta} \\ \mathcal{M}_{\alpha\beta} &= e^2 \bar{u}(p') \left[\gamma_\alpha \frac{\not{p}' - \not{q} + m}{(q - p')^2 - m^2} \gamma_\beta + \gamma_\beta \frac{\not{p}' + \not{q}' + m}{(q' + p')^2 - m^2} \gamma_\alpha \right] u(p), \end{aligned} \quad (6.66)$$

where ϵ ϵ' are the polarization vectors of the photons in the initial and final states:

$$\epsilon \cdot q = \epsilon' \cdot q' = 0. \quad (6.67)$$

The second term in the amplitude is obtained from the first one by *crossing*, that is, by replacing $q \rightarrow -q'$ and $\epsilon \rightarrow \epsilon'^*$. Observe that

$$q^\alpha \mathcal{M}_{\alpha\beta} = q'^\beta \mathcal{M}_{\alpha\beta} = 0 \quad (6.68)$$

as a consequence of the equations of motion for $u(p)$ and $u(p')$. The relations in eq. (6.68) are particular cases of a class of identities obeyed by physical amplitudes as a consequence of gauge invariance.

The complex conjugate amplitude is easily computed with the help of the results of Appendix C; we find

$$\mathcal{M}_{\alpha\beta}^* = e^2 \bar{u}(p) \left[\gamma_\beta \frac{\not{p}' - \not{q} + m}{(q - p')^2 - m^2} \gamma_\alpha + \gamma_\alpha \frac{\not{p}' + \not{q}' + m}{(q' + p')^2 - m^2} \gamma_\beta \right] u(p'). \quad (6.69)$$

The calculation simplifies considerably in the high-energy limit, in which the electron mass can be neglected. In this limit, we find

$$\begin{aligned} \sum_{\text{pol}_\gamma} \mathcal{M}^* \mathcal{M} &= e^4 \bar{u}(p') \left[\gamma_\alpha \frac{\not{p}' - \not{q}}{(q - p')^2} \gamma_\beta + \gamma_\beta \frac{\not{p}' + \not{q}'}{(q' + p')^2} \gamma_\alpha \right] u(p) \\ &\quad \times \bar{u}(p) \left[\gamma^\beta \frac{\not{p}' - \not{q}}{(q - p')^2} \gamma^\alpha + \gamma^\alpha \frac{\not{p}' + \not{q}'}{(q' + p')^2} \gamma^\beta \right] u(p'), \end{aligned} \quad (6.70)$$

where we have used eq. (6.53) for the sum over photon helicity states, and eq. (6.68). Equation (6.70) can be written in the form of a trace over spinor indices by means of eqs. (6.62,6.63):

$$\sum_{\text{pol}_\gamma, \text{pol}_e} \mathcal{M}^* \mathcal{M} = e^4 \text{Tr} \left[\gamma_\alpha \frac{\not{p}' - \not{q}}{(q - p')^2} \gamma_\beta + \gamma_\beta \frac{\not{p}' + \not{q}'}{(q' + p')^2} \gamma_\alpha \right] \not{p} \left[\gamma_\beta \frac{\not{p}' - \not{q}}{(q - p')^2} \gamma^\alpha + \gamma^\alpha \frac{\not{p}' + \not{q}'}{(q' + p')^2} \gamma^\beta \right] \not{p}'. \quad (6.71)$$

The trace can be computed by means of the results of Appendix C, eqs. (C.3-C.5) and (C.8). For example,

$$\begin{aligned} \text{Tr} \gamma_\alpha (\not{p}' - \not{q}) \gamma_\beta \not{p} \gamma^\beta (\not{p}' - \not{q}) \gamma^\alpha \not{p}' &= 4 (p' - q)_\mu p_\rho (p' - q)_\nu p'_\sigma \text{Tr} \gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma \\ &= 32 p \cdot q' (p \cdot p' + p \cdot q'), \end{aligned} \quad (6.72)$$

where we have repeatedly used the mass-shell conditions and the momentum conservation relation. A straightforward calculation leads to

$$\sum_{\text{pol}} \mathcal{M}^* \mathcal{M} = -8e^4 \left(\frac{t}{s} + \frac{s}{t} \right), \quad (6.73)$$

where we have defined

$$s = (p + q)^2; \quad t = (p - q')^2. \quad (6.74)$$

In the center-of-mass frame we have

$$t = -\frac{s}{2}(1 - \cos \theta), \quad (6.75)$$

where θ is the photon scattering angle. The unpolarized differential cross section, averaged over the four initial-state polarizations, is now immediately obtained:

$$\begin{aligned} d\sigma &= \frac{1}{4} \frac{1}{2s} \sum_{\text{pol}} \mathcal{M}^* \mathcal{M} d\phi_2(p + q; p', q') \\ &= \frac{e^4}{16\pi s} \frac{1 + 2 \sin^4 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} d\cos \theta, \end{aligned} \quad (6.76)$$

where we have used the expression eq. (4.25) for the two-body invariant phase space.

6.3 Non-commutative charges: the Yang-Mills theory

As an obvious generalization of the symmetry transformations that characterize quantum electrodynamics, we may consider a group of transformations

that act on fields in a multi-dimensional isotopic space ϕ_i , $i = 1, \dots, N$. For definiteness, we consider the transformations

$$\phi_i \rightarrow U_{ij}(\alpha)\phi_j, \quad (6.77)$$

where U is an $N \times N$ unitary matrix with unit determinant. The matrices U do not commute with one another; this is why eq. (6.77) is called a *non-abelian* set of transformations, as opposed to the case of multiplication by phase factors, which is an *abelian* (or commutative) operation. The set of matrices U is a particular representation (called the *fundamental representation*) of the group $SU(N)$. Any such matrix can be written as

$$U = U(\alpha) = \exp(ig\alpha^A t^A), \quad (6.78)$$

where the t^A , $A = 1, \dots, N^2 - 1$ are a basis in the linear space of hermitian traceless matrices, the *generators*, and α^A are real constants. The generators can always be chosen so that

$$\text{Tr } t^A t^B = T_F \delta^{AB}, \quad (6.79)$$

with T_F a constant. Then

$$[t^A, t^B] = if^{ABC} t^C; \quad A, B, C = 1, \dots, N^2 - 1, \quad (6.80)$$

where f^{ABC} is a set of constants (the *structure constants* of the group) completely antisymmetric in the three indices. We have inserted in eq. (6.78) a coupling constant g in analogy with the phase transformations in electrodynamics. Let us assume that the Lagrangian density is invariant under the transformations (6.77), or, in infinitesimal form,

$$\delta\phi_i = ig\alpha^A t_{ij}^A \phi_j, \quad (6.81)$$

where summation on both indices A and j is understood. As we have seen in Section 2.2, the invariance of the Lagrangian density under the transformations (6.81) implies the existence of a set of conserved currents,

$$J_A^\mu = ig \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i(x)} t_{ij}^A \phi_j(x). \quad (6.82)$$

Symmetry under local transformations, $\alpha^A = \alpha^A(x)$, is achieved by replacing ordinary derivatives by covariant derivatives, defined as

$$D^\mu = \partial^\mu I - igA^\mu, \quad (6.83)$$

where I is the unity matrix in the representation space, and the vector field A^μ is now a traceless hermitian matrix

$$A^\mu = A_A^\mu t^A. \quad (6.84)$$

It is easy to show, in analogy with the abelian case, that the transformation law

$$A^\mu \rightarrow A'^\mu = U A^\mu U^{-1} + \frac{i}{g} U \partial^\mu U^{-1} \quad (6.85)$$

ensures that

$$D^\mu \rightarrow U D^\mu U^{-1}. \quad (6.86)$$

To first order in the parameters α^A , eq. (6.85) becomes

$$\begin{aligned} A'^\mu &= A^\mu + ig[\alpha^A t^A, A^\mu] - \frac{i}{g} ig \partial^\mu \alpha^A t^A \\ &= A_C^\mu t^C - g \alpha^A A_B^\mu f^{ABC} t^C + \partial^\mu \alpha^C t^C, \end{aligned} \quad (6.87)$$

or

$$A'^\mu_C = A_C^\mu - g \alpha^A A_B^\mu f^{ABC} + \partial^\mu \alpha^C. \quad (6.88)$$

A Lagrangian density for the vector fields A_A^μ can be built in analogy with the abelian case. Recalling eq. (6.27), we define a field tensor $F^{\mu\nu}$ through

$$(D^\mu D^\nu - D^\nu D^\mu)\phi = -ig F^{\mu\nu} \phi, \quad (6.89)$$

where ϕ is a multiplet of some $SU(N)$ representation, and $F^{\mu\nu} = F_A^{\mu\nu} t^A$. We find

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu], \\ F_A^{\mu\nu} &= \partial^\mu A_A^\nu - \partial^\nu A_A^\mu + gf^{ABC} A_B^\mu A_C^\nu. \end{aligned} \quad (6.90)$$

The Lagrangian density for the vector fields is then given by

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F_A^{\mu\nu} F_{\mu\nu}^A, \quad (6.91)$$

where the suffix YM stands for Yang and Mills. Since $F_A^{\mu\nu}$ contains terms quadratic in the vector fields, $\mathcal{L}_{\text{YM}}$ contains self-interaction terms. This is related to the fact that, contrary to the abelian case, the field strength $F^{\mu\nu}$ transforms non-trivially under a gauge transformation:

$$F^{\mu\nu} \rightarrow F'^{\mu\nu} = U F^{\mu\nu} U^{-1}. \quad (6.92)$$

To first order in the parameters α^A ,

$$F'^{\mu\nu}_A = F_A^{\mu\nu} - gf^{ABC} \alpha^B F_C^{\mu\nu}, \quad (6.93)$$

which means that the components $F_A^{\mu\nu}$ form a multiplet in the *adjoint representation* of the gauge group, whose generators are

$$T_{BC}^A = -if^{ABC}. \quad (6.94)$$

A simple calculation gives

Strong interactions are now known to be described by a field theory that possesses a non-abelian gauge invariance under the group $SU(3)$. The associated non-commutative charges are called *colour charges*. This theory is obtained by coupling a pure $SU(3)$ Yang-Mills theory, whose vector bosons are usually called *gluons*, to a system of fermions, called *quarks*, that are assumed to transform according to the fundamental (dimension 3) representation of the invariance group. The Lagrangian density is built in analogy with electrodynamics:

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{q}_f (i \not{D} - m_f) q_f + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{gf}}, \quad (6.101)$$

where the index f labels different quark species (usually called *flavours*), and the covariant derivative D is defined as

$$(D^\mu q_f)_i = \left[\partial^\mu \delta_i^j - ig_s A_A^\mu (t^A)_i^j \right] (q_f)_i, \quad (6.102)$$

where i, j are colour indices. The generators t^A of $SU(3)$ in the fundamental representation are usually chosen as one half the familiar Gell-Mann matrices λ^A ; in this case, $T_F = 1/2$ (see [4].) It is clear from eq. (6.101) that strong interactions do not make distinctions among the different flavours. The nature of the flavour quantum number, and the origin of mass terms, is related to the fact that quarks also participate in the electroweak interactions, and will be described in Chapters 7 and 9.

The above theory of strong interactions is usually called Quantum Chromodynamics (QCD), due to its close analogy with QED. The analogy extends to the invariance with respect to the T, C and P discrete transformations.

For completeness, we mention that non-abelian vector bosons have the same unphysical components as photons. However, here there is the further difficulty that, contrary to what happens in QED, $\partial_\mu A^\mu$ is not a free field. Thus the quantization of non-abelian gauge theories requires the introduction of a further set of unphysical fields, called *Faddeev-Popov ghosts*, with the role of compensating the effects of unphysical gluon components in the computation of probabilities.

The standard model

7.1 A gauge theory of weak interactions

Weak interaction processes such as nucleon β decay, or μ decay, are correctly described by an effective theory, usually referred to as the Fermi theory of weak interactions. The Fermi Lagrangian density for β and μ decays is given by ¹

$$\mathcal{L} = -\frac{G^{(\beta)}}{\sqrt{2}}\bar{p}\gamma^\alpha(1-a\gamma_5)n\bar{e}\gamma_\alpha(1-\gamma_5)\nu_e - \frac{G^{(\mu)}}{\sqrt{2}}\bar{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu\bar{e}\gamma_\alpha(1-\gamma_5)\nu_e. \quad (7.1)$$

From the measured values of muon and neutron lifetimes,

$$\tau(N) = 885.7 \pm 0.8 \text{ s} \quad \tau(\mu) = 2.19703 \pm 0.00004 \text{ s}, \quad (7.2)$$

one obtains

$$G^{(\mu)} \simeq 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}; \quad G^{(\beta)} \simeq G^{(\mu)} \equiv G_F, \quad (7.3)$$

while the value

$$a = 1.2695 \pm 0.0029 \quad (7.4)$$

can be extracted from the measurement of baryon semi-leptonic decay rates. The most striking feature of weak interactions, correctly taken into account by the Fermi theory, is the violation of parity invariance, that arises from the measurement of neutrino helicities in weak decay processes. [5] Note also that the Fermi constant sets a natural mass scale for weak interactions:

$$\Lambda \sim \frac{1}{\sqrt{G_F}} \sim 300 \text{ GeV}. \quad (7.5)$$

We note that the field theory defined by the interaction in eq. (7.1) is non-renormalizable, since it contains operators with mass dimension 6; it is an

¹ Particle fields will be denoted by the symbols usually adopted for the corresponding particles: e for the electron, ν_e for the electron neutrino, and so on.

effective theory, in the sense that it can only be used to compute amplitudes in the semi-classical approximation. Furthermore, we expect the results obtained by this effective theory to be accurate only when the energies of the particles involved are smaller than Λ . This condition is manifestly fulfilled by β and μ decays.

It can be shown (see Appendix D) that some $2 \rightarrow 2$ amplitudes computed with eq. (7.1) grow quadratically with the total energy, and therefore eventually violate the unitarity bound. This confirms that the Fermi theory is only reliable at sufficiently low energies. This is a typical example of a general phenomenon. The presence of coefficients with negative mass dimension in the interaction induces, for dimensional reasons, the dominant terms in the amplitudes at large energies. This in turn originates a violation of the unitarity constraint eq. (4.43). In the case of $2 \rightarrow 2$ particle amplitudes, if these grow as a positive power α of the energy E , the right-hand side of eq. (4.43) grows as $E^{2\alpha}$, since the two-body phase space is asymptotically constant in energy, while the left-hand side grows at most as E^α .

Note that this source of violation of unitarity is not related to the semi-classical approximation: if one tries to recover unitarity by means of radiative corrections, as in the example of elastic scattering of scalar particles presented in Section 4.4, a series of terms is generated, that grow as a power of the energy which increases with the iterative order. At the same time, the divergence of Feynman diagram integrals grows order by order, and the theory is not renormalizable.

For these reasons, it is clear that a renormalizable and unitary theory of weak interactions is needed in order to perform precision calculations, to be compared with accurate measurements performed at large energies. We shall see that the Fermi theory contains the physical information needed to build such a theory.

The idea is that of building a theory with local invariance under the action of some group of transformations, a gauge theory, in analogy with quantum electrodynamics. The local four-fermion interaction of the Fermi Lagrangian will be interpreted as the interaction vertex that arises from the exchange of a massive vector boson with momentum much smaller than its mass. In this way, both problems of renormalizability and unitarity will be solved, since gauge theories are known to be renormalizable, and the mass of the intermediate vector boson will act as a cut-off that stops the growth of cross sections with energy, thus ensuring unitarity of the scattering matrix.

In order to complete this program, we must choose the group of local invariance, and then assign particle fields to representations of the chosen group. Both steps can be performed with the help of the information contained in the Fermi Lagrangian. Let us first consider the electron and the electron neutrino. They participate in the weak interaction via the current

$$J_\mu = \bar{\nu}_e \gamma_\mu \frac{1}{2}(1 - \gamma_5) e. \quad (7.6)$$

We would like to rewrite J_μ in the form of a Noether current,

$$\bar{\psi}_i \gamma_\mu T_{ij}^A \psi_j, \quad (7.7)$$

where ψ_i are the components of a multiplet in some representation of the (as yet unknown) gauge group, and T_{ij}^A are the corresponding generators. This can be done in the following way. We observe that the current J_μ can be written as

$$J_\mu = \bar{\ell}_L \gamma_\mu \tau^+ \ell_L, \quad (7.8)$$

where

$$\ell_L = \begin{pmatrix} \frac{1}{2}(1 - \gamma_5) \nu_e \\ \frac{1}{2}(1 - \gamma_5) e \end{pmatrix} \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}; \quad \tau^+ = \frac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (7.9)$$

and τ_i are the usual Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7.10)$$

The hermitian conjugate current

$$J_\mu^\dagger = \bar{\ell}_L \gamma_\mu \tau^- \ell_L; \quad \tau^- = \frac{1}{2}(\tau_1 - i\tau_2) \quad (7.11)$$

will also participate in the interaction. As we have seen in Section 6.3, the currents are in one-to-one correspondence with the generators of the symmetry group, which, in turn, form a closed set with respect to the commutation operation. Therefore, the current

$$J_3^\mu = \bar{\ell}_L \gamma^\mu [\tau^+, \tau^-] \ell_L = \bar{\ell}_L \gamma^\mu \tau_3 \ell_L \quad (7.12)$$

will also be present. No other current must be introduced, since

$$[\tau_3, \tau^\pm] = 2\tau^\pm. \quad (7.13)$$

We have thus interpreted the current J_μ as one of the three conserved currents that arise from invariance under transformations of $SU(2)$ (it is known from ordinary quantum mechanics that the Pauli matrices are the generators of $SU(2)$ in the fundamental representation.) Furthermore, we have assigned the left-handed neutrino and electron fields to the fundamental representation of $SU(2)$, a *doublet*:

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \rightarrow \exp\left(ig\alpha_i \frac{\tau_i}{2}\right) \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}. \quad (7.14)$$

The right-handed neutrino and electron components, ν_{eR} and e_R , do not take part in the weak-interaction phenomena described by the Fermi Lagrangian, so they must be assigned to the singlet (or scalar) representation of $SU(2)$.

For this reason, the invariance group $SU(2)$ relevant for weak interactions is sometimes referred to as $SU(2)_L$. Of course, this is not the only possible choice, but it is the simplest possibility since it does not require the introduction of fermion fields other than the observed ones.

The current J_3^μ is a *neutral* current: it is bilinear in creation and annihilation operators of particles with the same charge (actually, of the same particle.) Neutral currents do not appear in the Fermi Lagrangian, because no neutral current weak interaction phenomena were observed at the time of its formulation. As we shall see, the experimental observation of phenomena induced by weak neutral currents is a crucial test of the validity of the standard model. Notice also that the neutral current J_3^μ cannot be identified with the only other neutral current we know of, the electromagnetic current. There are two reasons for this: first, the electromagnetic current involves both left-handed and right-handed fermion fields with the same weight; and second, the electromagnetic current does not contain a neutrino term, the neutrino being chargeless. We will come back later to the problem of neutral currents, that will end up with the inclusion of the electromagnetic current in the theory. For the moment, we go on with the construction of our gauge theory based on $SU(2)$ invariance. We must introduce vector meson fields W_i^μ , one for each of the three $SU(2)$ generators, and build a covariant derivative

$$D^\mu = \partial^\mu - igW_i^\mu T_i, \quad (7.15)$$

where we have introduced a coupling constant g . The matrices T_i are generators of $SU(2)$ in the representation of the multiplet the covariant derivative is acting on. For example, when D^μ acts on the doublet ℓ_L , we have $T_i \equiv \tau_i/2$, and when it acts on the $SU(2)$ singlet e_R we have $T_i \equiv 0$. We are now ready to write the gauge-invariant Lagrangian for the fermion fields (which we assume massless for the time being):

$$\begin{aligned} \mathcal{L} &= i\bar{\ell}_L \not{D} \ell + i\bar{\nu}_{eR} \not{D} \nu_{eR} + i\bar{e}_R \not{D} e_R \\ &= \mathcal{L}_0 + \mathcal{L}_c + \mathcal{L}_n, \end{aligned} \quad (7.16)$$

where $\not{D} = \gamma_\mu D^\mu$. The Lagrangian $\mathcal{L}$ contains free terms for massless fermions,

$$\mathcal{L}_0 = i\bar{\ell}_L \not{\partial} \ell_L + i\bar{\nu}_{eR} \not{\partial} \nu_{eR} + i\bar{e}_R \not{\partial} e_R, \quad (7.17)$$

and an interaction term $\mathcal{L}_c + \mathcal{L}_n$, where

$$\mathcal{L}_c = gW_1^\mu \bar{\ell}_L \gamma_\mu \frac{\tau_1}{2} \ell_L + gW_2^\mu \bar{\ell}_L \gamma_\mu \frac{\tau_2}{2} \ell_L \quad (7.18)$$

corresponds to charged-current interactions, and

$$\mathcal{L}_n = gW_3^\mu \bar{\ell}_L \gamma_\mu \frac{\tau_3}{2} \ell_L = \frac{g}{2} W_3^\mu (\bar{\nu}_{eL} \gamma_\mu \nu_{eL} - \bar{e}_L \gamma_\mu e_L) \quad (7.19)$$

to neutral-current interactions. The charged-current term $\mathcal{L}_c$ is usually expressed in terms of the complex vector fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2). \quad (7.20)$$

We find

$$\mathcal{L}_c = \frac{g}{\sqrt{2}} \bar{\ell}_L \gamma^\mu \tau^+ \ell_L W_\mu^+ + \frac{g}{\sqrt{2}} \bar{\ell}_L \gamma^\mu \tau^- \ell_L W_\mu^-. \quad (7.21)$$

We have already observed that the neutral current $J_3^\mu = \bar{\ell}_L \gamma^\mu \tau_3 \ell_L$ cannot be identified with the electromagnetic current, and correspondingly that the gauge vector boson W_3^μ cannot be interpreted as the photon field. The construction of the model can therefore proceed in two different directions: either we modify the multiplet structure of the theory, in order to make J_3^μ equal to the electromagnetic current; or we admit the possibility of the existence of weak neutral currents, and we extend the gauge group in order to accommodate also the electromagnetic current in addition to J_3^μ . In the next Section we proceed to describe the second possibility, which turns out to be the correct one, after the discovery of weak processes induced by neutral currents. Nevertheless, it should be kept in mind that this was not at all obvious to physicists before the observation of weak neutral-current effects.

7.2 Electroweak unification

The simplest way of extending the gauge group $SU(2)$ to include a second neutral generator is to include an abelian factor $U(1)$:

$$SU(2) \rightarrow SU(2) \times U(1). \quad (7.22)$$

We will require that the Lagrangian density be invariant also under the $U(1)$ gauge transformations

$$\psi \rightarrow \psi' = \exp \left(ig' \alpha \frac{Y(\psi)}{2} \right) \psi, \quad (7.23)$$

where ψ is a generic field in the theory, g' is the coupling constant associated to the $U(1)$ factor of the gauge group, and $Y(\psi)$ is a quantum number, usually called the *weak hypercharge*, to be specified for each field ψ . Since the $SU(2)$ factor of the gauge group acts in a different way on left-handed and right-handed fermions (it is a *chiral* group), it is natural to allow for the possibility of assigning different hypercharge quantum numbers to left and right components of the same fermion field.

A new gauge vector field B^μ must be introduced, and the covariant derivative becomes

$$D^\mu = \partial^\mu - igW_i^\mu T_i - ig' \frac{Y}{2} B^\mu, \quad (7.24)$$

where Y is a diagonal matrix with the hypercharges in its diagonal entries. Since Y is diagonal, the introduction of the B^μ term in the covariant derivative only affects the neutral-current interaction $\mathcal{L}_n$. We have now

$$\begin{aligned}\mathcal{L}_n = & \frac{g}{2} W_3^\mu (\bar{\nu}_{eL} \gamma_\mu \nu_{eL} - \bar{e}_L \gamma_\mu e_L) \\ & + \frac{g'}{2} B^\mu [Y(\ell_L) (\bar{\nu}_{eL} \gamma_\mu \nu_{eL} + \bar{e}_L \gamma_\mu e_L) + Y(\nu_{eR}) \bar{\nu}_{eR} \gamma_\mu \nu_{eR} + Y(e_R) \bar{e}_R \gamma_\mu e_R].\end{aligned}\quad (7.25)$$

This can be written as

$$\mathcal{L}_n = \bar{\Psi} \gamma_\mu \left[g T_3 W_3^\mu + g' \frac{Y}{2} B^\mu \right] \Psi, \quad (7.26)$$

where Ψ is a column vector formed with all left-handed and right-handed fermion fields in the theory, and $T_3 = \pm 1/2$ for ν_{eL} and e_L respectively, and $T_3 = 0$ for ν_{eR} and e_R .

We can now assign the quantum numbers Y in such a way that the electromagnetic interaction term appear in eq. (7.26). To do this, we perform a rotation by an angle θ_w in the space of the two neutral gauge fields W_3^μ, B^μ :

$$B^\mu = A^\mu \cos \theta_w - Z^\mu \sin \theta_w \quad (7.27)$$

$$W_3^\mu = A^\mu \sin \theta_w + Z^\mu \cos \theta_w. \quad (7.28)$$

In terms of the new vector fields A^μ, Z^μ , eq. (7.26) takes the form

$$\begin{aligned}\mathcal{L}_n = & \bar{\Psi} \gamma_\mu \left(g \sin \theta_w T_3 + \frac{Y}{2} g' \cos \theta_w \right) \Psi A^\mu \\ & + \bar{\Psi} \gamma_\mu \left(g \cos \theta_w T_3 - \frac{Y}{2} g' \sin \theta_w \right) \Psi Z^\mu.\end{aligned}\quad (7.29)$$

In order to identify one of the two neutral vector fields, say A^μ , with the photon field, we must choose $Y(\ell_L), Y(\nu_{eR})$ and $Y(e_R)$ so that A^μ couples to the electromagnetic current

$$J_{\text{em}}^\mu = -e (\bar{e}_R \gamma^\mu e_R + \bar{e}_L \gamma^\mu e_L) \equiv e \bar{\Psi} \gamma^\mu Q \Psi, \quad (7.30)$$

where Q is the diagonal matrix of electromagnetic charges in units of the positron charge e . In other words, we require that

$$T_3 g \sin \theta_w + \frac{Y}{2} g' \cos \theta_w = e Q. \quad (7.31)$$

The weak hypercharges Y appear in eq. (7.31) only through the combination $Y g'$: thus, we have the freedom of rescaling the hypercharges by a common factor K , provided we rescale g' by $1/K$. This freedom can be used to fix arbitrarily the value of one of the three hypercharges $Y(\ell_L), Y(\nu_{eR}), Y(e_R)$. The conventionally adopted choice is

$$Y(\ell_L) = -1. \quad (7.32)$$

With this choice, eq. (7.31) restricted to the doublet of left-handed leptons reads

$$+\frac{1}{2}g \sin \theta_w - \frac{1}{2}g' \cos \theta_w = 0 \quad (7.33)$$

$$-\frac{1}{2}g \sin \theta_w - \frac{1}{2}g' \cos \theta_w = -e, \quad (7.34)$$

which gives

$$g \sin \theta_w = g' \cos \theta_w = e. \quad (7.35)$$

(For a generic doublet of fields with charges Q_1 and Q_2 , the r.h.s. of eq. (7.35) becomes $e(Q_1 - Q_2)$, but charge conservation requires $Q_1 - Q_2 = 1$.) Equation (7.31) then reduces to

$$T_3 + \frac{Y}{2} = Q. \quad (7.36)$$

Therefore, we find

$$Y(\nu_{eR}) = 0; \quad Y(e_R) = -2. \quad (7.37)$$

This completes the assignments of weak hypercharges to all fermion fields. Note that the right-handed neutrino is an $SU(2)$ singlet with zero hypercharge: it does not take part in electroweak interactions, and can be omitted altogether. We shall consider the possible effects of right-handed neutrinos in Chapter 12.

The second term in eq. (7.29) defines the weak neutral current coupled to the weak neutral vector boson Z_μ . It can be written as

$$e \bar{\Psi} \gamma_\mu Q_Z \Psi Z^\mu, \quad (7.38)$$

where

$$Q_Z = \frac{1}{\cos \theta_w \sin \theta_w} (T_3 - Q \sin^2 \theta_w). \quad (7.39)$$

This structure can be replicated for an arbitrary numbers of lepton families, assigned to the same representations of $SU(2) \times U(1)$ as the electron and the electron neutrino. At present, the existence of two more lepton families is experimentally established: the lepton μ^- with its neutrino ν_μ , and lepton τ^- with its neutrino ν_τ .

7.3 Hadrons

We must now include hadrons in the theory. We will do this in terms of quark fields, taking as a starting point the hadronic current responsible for β decay and strange particle decays:

$$J_{\text{hadr}}^\mu = \cos \theta_c \bar{u} \gamma^\mu \frac{1}{2}(1 - \gamma_5)d + \sin \theta_c \bar{u} \gamma^\mu \frac{1}{2}(1 - \gamma_5)s, \quad (7.40)$$

where θ_c is the Cabibbo angle ($\theta_c \sim 13^\circ$) and u, d, s are the up, down and strange quark fields respectively (for simplicity, we have not displayed the colour indices carried by quarks.) As noted in Section 6.3, different quark

flavours are not distinguished by strong interactions. The corresponding particles have however different masses, which are of the Dirac type because of different symmetry reasons, the simplest of them being parity invariance of strong interactions. In eq. (7.40), the fields u, d, s represent definite mass eigenfields. Lepton and quark masses will be discussed in detail in Chapter 9.

We are tempted to proceed as in the case of leptons: we define

$$q_L = \frac{1}{2}(1 - \gamma_5) \begin{bmatrix} u \\ d \\ s \end{bmatrix} \equiv \begin{bmatrix} u_L \\ d_L \\ s_L \end{bmatrix} \quad (7.41)$$

and

$$T^+ = \begin{bmatrix} 0 & \cos \theta_c & \sin \theta_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (7.42)$$

so that

$$J_{\text{hadr}}^\mu = \bar{q}_L \gamma^\mu T^+ q_L. \quad (7.43)$$

This procedure would lead to a system of currents which is in contrast with experimental observations. Indeed, we find

$$T_3 = [T^+, T^-] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos^2 \theta_c & -\cos \theta_c \sin \theta_c \\ 0 & -\cos \theta_c \sin \theta_c & -\sin^2 \theta_c \end{bmatrix}. \quad (7.44)$$

The corresponding neutral current contains flavour-changing terms, such as for example $\bar{d}_L \gamma^\mu s_L$, with a weight of the same order of magnitude of flavour-conserving ones. These terms induce processes at a rate which is not compatible with data. For example, the ratio of the decay rates for the processes

$$K^+ \rightarrow \pi^0 e^+ \nu_e \quad (7.45)$$

$$K^+ \rightarrow \pi^+ e^+ e^- \quad (7.46)$$

would be approximately

$$r = \left[\frac{\sin \theta_c}{\sin \theta_c \cos \theta_c} \right]^2 = \frac{1}{\cos^2 \theta_c} \simeq 1.1, \quad (7.47)$$

while observations give

$$r_{\text{exp}} \simeq 1.3 \times 10^5, \quad (7.48)$$

that is, the charged-current process ($s \rightarrow u$) is enhanced by five orders of magnitude with respect to the neutral-current process ($s \rightarrow d$). Our theory should therefore be modified in order to avoid the introduction of flavour-changing neutral currents. The solution to this puzzle was found by S. Glashow, J. Iliopoulos and L. Maiani. They suggested to introduce a fourth quark c (for *charm*) with charge $2/3$ like the up quark, and to assume that its couplings to down and strange quarks are given by

$$J_{\text{hadr}}^\mu = \cos \theta_c \bar{u} \gamma^\mu \frac{1}{2}(1 - \gamma_5)d + \sin \theta_c \bar{u} \gamma^\mu \frac{1}{2}(1 - \gamma_5)s \\ - \sin \theta_c \bar{c} \gamma^\mu \frac{1}{2}(1 - \gamma_5)d + \cos \theta_c \bar{c} \gamma^\mu \frac{1}{2}(1 - \gamma_5)s. \quad (7.49)$$

The c quark being not observed at the time, they had to assume that its mass was much larger than those of u , d and s quarks, and therefore outside the energy range of available experimental devices. The current J_{hadr}^μ can still be cast in the form of eq. (7.43), where now

$$q_L = \begin{bmatrix} u_L \\ c_L \\ d_L \\ s_L \end{bmatrix}; \quad T^+ = \begin{bmatrix} 0 & 0 & \cos \theta_c & \sin \theta_c \\ 0 & 0 & -\sin \theta_c & \cos \theta_c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (7.50)$$

No flavour-changing neutral current is now present, since

$$[T^+, T^-] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (7.51)$$

as a consequence of the fact that the upper right 2×2 block of T^+ was cleverly chosen to be an orthogonal matrix. The existence of the quark c was later confirmed by the discovery of the J/ψ particle. The current J_{hadr}^μ is usually written in the following form, analogous to the corresponding leptonic current:

$$J_{\text{hadr}}^\mu = (\bar{u}_L \vec{d}'_L) \gamma^\mu \tau^+ \begin{pmatrix} u_L \\ d'_L \end{pmatrix} + (\bar{c}_L \vec{s}'_L) \gamma^\mu \tau^+ \begin{pmatrix} c_L \\ s'_L \end{pmatrix}, \quad (7.52)$$

where

$$\begin{pmatrix} d'_L \\ s'_L \end{pmatrix} = V \begin{pmatrix} d_L \\ s_L \end{pmatrix}, \quad V = \begin{pmatrix} \cos \theta_c & \sin \theta_c \\ -\sin \theta_c & \cos \theta_c \end{pmatrix}. \quad (7.53)$$

The pairs (u, d) , (c, s) are called *quark families*. The structure outlined above can be extended to an arbitrary number of quark families. With n families, V becomes an $n \times n$ matrix, and it must be unitary in order to ensure the absence of flavour-changing neutral currents. One more quark family, in addition to (u, d) and (c, s) , has been detected, namely the one formed by the *top* and the *bottom* quarks (t, b) . In the following, quark families will be denoted by (u^f, d^f) , with the index f running over the families.

The final form of the standard model Lagrangian with n families of leptons and quarks is then

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_c + \mathcal{L}_n \quad (7.54)$$

$$\mathcal{L}_0 = i\bar{\ell}_L^f \not{\partial} \ell_L^f + i\bar{\nu}_R^f \not{\partial} \nu_R^f + i\bar{e}_R^f \not{\partial} e_R^f \\ + i\bar{q}_L^f \not{\partial} q_L^f + i\bar{u}_R^f \not{\partial} u_R^f + i\bar{d}_R^f \not{\partial} d_R^f \quad (7.55)$$

$$\mathcal{L}_n = e \bar{\Psi} \gamma_\mu Q \Psi A^\mu + e \bar{\Psi} \gamma_\mu Q_Z \Psi Z^\mu \quad (7.56)$$

$$\mathcal{L}_c = \frac{g}{\sqrt{2}} \sum_{f=1}^n \left(\bar{\ell}_L^f \gamma^\mu \tau^+ \ell_L^f + \bar{q}_L^f \gamma^\mu \tau^+ q_L^f \right) W_\mu^+ + \text{h.c.}, \quad (7.57)$$

where

$$\ell_L^f = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \dots \quad (7.58)$$

$$q_L^f = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \dots \quad (7.59)$$

The vector Ψ now also includes left-handed and right-handed quark fields. Right-handed quark fields are assigned to $SU(2)$ singlets, with hypercharges given by eq. (7.36) (we recall that $Q = 2/3$ for $u, c, t, \dots$ and $Q = -1/3$ for $d, s, b, \dots$) and Q_Z is given by the same expression eq. (7.39), with $T_3 = +1/2$ for u_L , $T_3 = -1/2$ for d_L , $T_3 = 0$ for right-handed quarks.

An equivalent (and often more useful) form of eq. (7.57) is

$$\mathcal{L}_c = \frac{g}{\sqrt{2}} \left(\sum_{f=1}^n \bar{\nu}_L^f \gamma^\mu e_L^f + \sum_{f,g=1}^n \bar{u}_L^f \gamma^\mu V_{fg} d_L^g \right) W_\mu^+ + \text{h.c.} \quad (7.60)$$

To conclude the construction of the standard model, we must include the vector boson Lagrangian density

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_{\mu\nu}^i W_i^{\mu\nu}, \quad (7.61)$$

where

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$

$$W_i^{\mu\nu} = \partial^\mu W_i^\nu - \partial^\nu W_i^\mu + g \epsilon_{ijk} W_j^\mu W_k^\nu. \quad (7.62)$$

The corresponding expressions in terms of $W_\mu^\pm$, Z_μ and A_μ can be easily worked out with the help of eqs. (7.20), (7.27) and (7.28), which we repeat here:

$$W_\mu^1 = \frac{1}{\sqrt{2}} (W_\mu^+ + W_\mu^-) \quad (7.63)$$

$$W_\mu^2 = \frac{i}{\sqrt{2}} (W_\mu^+ - W_\mu^-) \quad (7.64)$$

$$W_\mu^3 = A_\mu \sin \theta_w + Z_\mu \cos \theta_w \quad (7.65)$$

$$B_\mu = A_\mu \cos \theta_w - Z_\mu \sin \theta_w. \quad (7.66)$$

We get

$$\begin{aligned}
W_{\mu\nu}^1 &= \frac{1}{\sqrt{2}} [W_{\mu\nu}^+ + ig \sin \theta_w (W_\mu^+ A_\nu - W_\nu^+ A_\mu) \\
&\quad + ig \cos \theta_w (W_\mu^+ Z_\nu - W_\nu^+ Z_\mu)] + \text{h.c.} \\
W_{\mu\nu}^2 &= \frac{i}{\sqrt{2}} [W_{\mu\nu}^+ + ig \sin \theta_w (W_\mu^+ A_\nu - W_\nu^+ A_\mu) \\
&\quad + ig \cos \theta_w (W_\mu^+ Z_\nu - W_\nu^+ Z_\mu)] + \text{h.c.} \\
W_{\mu\nu}^3 &= F_{\mu\nu} \sin \theta_w + Z_{\mu\nu} \cos \theta_w - ig (W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+) \\
B_{\mu\nu} &= F_{\mu\nu} \cos \theta_w - Z_{\mu\nu} \sin \theta_w,
\end{aligned} \tag{7.67}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{7.68}$$

$$Z^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu \tag{7.69}$$

$$W_\pm^{\mu\nu} = \partial^\mu W_\pm^\nu - \partial^\nu W_\pm^\mu. \tag{7.70}$$

It follows that

$$\begin{aligned}
\mathcal{L}_{\text{YM}} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{2} W_{\mu\nu}^+ W_-^{\mu\nu} \\
&\quad + ig \sin \theta_w (W_{\mu\nu}^+ W_-^\mu A^\nu - W_{\mu\nu}^- W_+^\mu A^\nu + F_{\mu\nu} W_+^\mu W_-^\nu) \\
&\quad + ig \cos \theta_w (W_{\mu\nu}^+ W_-^\mu Z^\nu - W_{\mu\nu}^- W_+^\mu Z^\nu + Z_{\mu\nu} W_+^\mu W_-^\nu) \\
&\quad + \frac{g^2}{2} (2g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \left[\frac{1}{2} W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- \right. \\
&\quad \left. - W_\mu^+ W_\nu^- (A_\rho A_\sigma \sin^2 \theta_w + Z_\rho Z_\sigma \cos^2 \theta_w + 2A_\rho Z_\sigma \sin \theta_w \cos \theta_w) \right].
\end{aligned} \tag{7.71}$$

Spontaneous breaking of the gauge symmetry

8.1 Masses for vector bosons

We have seen in Chapter 7 that the neutral vector boson fields coupled to the $SU(2)$ and $U(1)$ charges appear as linear combinations of the massless photon field and of the Z^0 field; similarly, left-handed quarks with charge $-1/3$ enter the weak interaction term as linear combinations of the corresponding mass eigenfields. We show here that these mixing phenomena are consequences of the mechanism of mass generation in the model.

We first show that, in order to make contact with the Fermi theory, which is known to describe correctly weak interactions at low energies, the gauge vector bosons of weak interactions must have non-vanishing masses. On the same basis, we will also be able to set a lower bound to the mass of the W boson. Let us consider the amplitude for down-quark β decay. In the Fermi theory, it is simply given by

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} \bar{u} \gamma^\mu (1 - \gamma_5) d \bar{e} \gamma_\mu (1 - \gamma_5) \nu_e. \quad (8.1)$$

In the context of the standard model, the same process is induced by the exchange of a W boson, with amplitude

$$\mathcal{M}^{\text{SM}} = \left(\frac{g}{\sqrt{2}} \bar{u}_L \gamma^\mu d_L \right) \frac{1}{q^2 - m_W^2} \left(\frac{g}{\sqrt{2}} \bar{e}_L \gamma_\mu \nu_{eL} \right), \quad (8.2)$$

(we are neglecting the Cabibbo angle for simplicity). The virtuality q^2 of the exchanged vector boson is bounded from above by the square of the neutron-proton mass difference, $q^2 \leq (m_N - m_P)^2 \sim (1.3 \text{ MeV})^2$. Equation (8.2) reduces to the Fermi amplitude provided $m_W^2 \gg q^2$, and

$$\frac{G_F}{\sqrt{2}} = \left(\frac{g}{2\sqrt{2}} \right)^2 \frac{1}{m_W^2}. \quad (8.3)$$

Recalling that $g = e/\sin \theta_W$, eq. (8.3) gives the lower bound

$$m_w \geq 37.3 \text{ GeV}, \quad (8.4)$$

which is quite a large value, compared to the nucleon mass, and an enormous number, compared to the present upper bound on the photon mass,

$$m_\gamma \leq 2 \cdot 10^{-16} \text{ eV}. \quad (8.5)$$

We conclude that if weak interactions are to be mediated by vector bosons, these must be very heavy. On the other hand, we also know that gauge theories are incompatible with mass terms for the vector bosons.

One possible way out is breaking gauge invariance explicitly; this, however, as already observed in Section 6.1, leads to a non-renormalizable and non-unitary theory. Let us investigate this point in more detail. One may formulate a theory for a massive gauge boson with the propagator

$$\Delta^{\mu\nu}(k) = \frac{1}{k^2 - \mu^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{\mu^2} \right), \quad (8.6)$$

so that non-physical components of the vector field do not contribute. For large values of the momentum k , the term proportional to $k^\mu k^\nu$ in the propagator eq. (8.6) dominates; it is therefore clear that the behaviour of this propagator at large k is much worse than that of the scalar propagator, that vanishes at infinity as $1/k^2$. This suggests that the propagator eq. (8.6) leads to a non-renormalizable theory.

A related problem of a massive vector boson theory without gauge invariance is unitarity of the scattering matrix. The amplitude for a generic physical process which involves the emission or the absorption of a vector boson with four-momentum k and polarization vector $\epsilon(k)$ has the form

$$\mathcal{M} = \mathcal{M}^\mu \epsilon_\mu(k). \quad (8.7)$$

A massive vector (contrary to a massless one) may be polarized longitudinally. In this case, choosing the z axis along the direction of the 3-momentum of the vector boson, the polarization is given by

$$\epsilon_L = (|\mathbf{k}|/\mu, 0, 0, E/\mu) = k/\mu + \mathcal{O}(\mu^2/E^2), \quad (8.8)$$

where we have imposed the transversity condition $k \cdot \epsilon = 0$ and the normalization condition $\epsilon^2 = -1$. Clearly, the amplitude $\mathcal{M}$ will grow indefinitely with the energy E , thus eventually violating the unitarity bound.

Both sources of power-counting violation are rendered harmless if the vector particles are coupled to conserved currents, thus confirming the need of gauge invariance.

8.2 Scalar electrodynamics and the abelian Higgs model

In order to see how one can introduce a mass term for a gauge vector boson without spoiling renormalizability and unitarity, we first consider a simple example, and then we generalize our considerations to the standard model. The

simple theory we consider is scalar electrodynamics, that is, a $U(1)$ gauge theory for one complex scalar field Φ with charge e . The requirement of invariance under the gauge transformations

$$\Phi(x) \rightarrow e^{ie\Lambda(x)}\Phi(x) \quad (8.9)$$

$$\Phi^*(x) \rightarrow e^{-ie\Lambda(x)}\Phi^*(x) \quad (8.10)$$

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\Lambda(x) \quad (8.11)$$

and the gauge choice eq. (6.38) lead uniquely to the Lagrangian density

$$\mathcal{L} = D_\mu\Phi^* D^\mu\Phi - m^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2 - \frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu, \quad (8.12)$$

where

$$D_\mu\Phi(x) = \partial_\mu\Phi(x) - ieA_\mu(x)\Phi(x). \quad (8.13)$$

Observe that scalar QED involves one more coupling constant (λ) than spin-1/2 QED.

The field equations are given by

$$(\partial^2 + m^2)\Phi = ieA^\mu\partial_\mu\Phi + ie\partial_\mu(A^\mu\Phi) + e^2A^2\Phi - 2\lambda\Phi^*\Phi^2 \equiv J_\Phi \quad (8.14)$$

$$\partial^2 A^\mu = -ie(\Phi^*D^\mu\Phi - \Phi D^\mu\Phi^*) \equiv -J_A^\mu. \quad (8.15)$$

The Green function for the scalar field is $\Delta(x)$, eq. (3.75), while the Green function for the vector field A_μ is the same as in Section 6.1.

The gauge transformations eqs. (8.9,8.10) have the property that they leave the origin of the isotopic space unchanged. There is however the possibility of defining gauge transformations on scalar fields that do not have this feature; let us explore this possibility. We require invariance under the transformations

$$\Phi(x) \rightarrow e^{ie\Lambda(x)}\left(\Phi(x) + \frac{v}{\sqrt{2}}\right) - \frac{v}{\sqrt{2}} \quad (8.16)$$

$$\Phi^*(x) \rightarrow e^{-ie\Lambda(x)}\left(\Phi^*(x) + \frac{v}{\sqrt{2}}\right) - \frac{v}{\sqrt{2}} \quad (8.17)$$

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\Lambda(x), \quad (8.18)$$

where v is a real constant. The covariant derivative term must be consistently modified. We observe that the quantity

$$D_\mu\left(\Phi(x) + \frac{v}{\sqrt{2}}\right) \equiv \partial_\mu\Phi(x) - ieA_\mu(x)\left(\Phi(x) + \frac{v}{\sqrt{2}}\right) \quad (8.19)$$

has the same transformation property as $\Phi + \frac{v}{\sqrt{2}}$; thus, it can be taken as the definition of the covariant derivative when the gauge transformation is defined as in eqs. (8.16-8.18). The self-interaction of the field Φ is entirely determined by the requirements of renormalizability and invariance under the gauge transformations eqs. (8.16-8.18). The further condition, discussed in

Section 2.1, that the scalar potential have a minimum at the origin of the isotopic space, fixes the form of the scalar potential up to an overall constant:

$$V(\Phi) = \lambda \left[\left(\Phi(x) + \frac{v}{\sqrt{2}} \right) \left(\Phi^*(x) + \frac{v}{\sqrt{2}} \right) - \frac{v^2}{2} \right]^2. \quad (8.20)$$

In this context, it is convenient to replace the gauge-fixing term $\mathcal{L}_{\text{gf}}$ of eq. (6.38) with

$$\mathcal{L}_{\text{gfh}} = -\frac{1}{2} \left(\partial_\mu A^\mu - ie \frac{v}{\sqrt{2}} (\Phi - \Phi^*) \right)^2, \quad (8.21)$$

first introduced by G. 't Hooft. We recall that the gauge-fixing term provides a gauge condition as a consequence of the field equations; the choice of one particular gauge condition is not related to physical considerations. The gauge-fixing term completes the formulation of a generalization of scalar QED, which is called the *abelian Higgs model*. The Lagrangian density is

$$\begin{aligned} \mathcal{L}_H = & \left[D_\mu \left(\Phi + \frac{v}{\sqrt{2}} \right) \right]^* D^\mu \left(\Phi + \frac{v}{\sqrt{2}} \right) - V(\Phi) \\ & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left(\partial_\mu A^\mu - ie \frac{v}{\sqrt{2}} (\Phi - \Phi^*) \right)^2. \end{aligned} \quad (8.22)$$

The particle content of the theory is immediately determined by inspection of the free Lagrangian density

$$\begin{aligned} \mathcal{L}_0 = & \partial_\mu \Phi^* \partial^\mu \Phi + ie \frac{v}{\sqrt{2}} A^\mu \partial_\mu (\Phi - \Phi^*) + \frac{e^2 v^2}{2} A^2 - \frac{\lambda}{2} v^2 (\Phi + \Phi^*)^2 \\ & - \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2} (\partial A)^2 + \frac{e^2 v^2}{4} (\Phi - \Phi^*)^2 \\ & + ie \frac{v}{\sqrt{2}} \partial_\mu A^\mu (\Phi - \Phi^*) \\ = & \partial_\mu \Phi^* \partial^\mu \Phi - \frac{\lambda}{2} v^2 (\Phi + \Phi^*)^2 + \frac{e^2 v^2}{4} (\Phi - \Phi^*)^2 - \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{e^2 v^2}{2} A^2. \end{aligned} \quad (8.23)$$

In the second step we have performed a partial integration on the last term; we see that the gauge-fixing term was chosen so that bilinear terms proportional to $A^\mu \partial_\mu (\Phi - \Phi^*)$ cancel. We have also omitted a term proportional to a four-divergence, in analogy with ordinary electrodynamics (see eq. (6.40)). It is convenient to decompose the field Φ into its real and imaginary parts:

$$\Phi(x) = \frac{H(x) + iG(x)}{\sqrt{2}}. \quad (8.24)$$

The real scalar field H is usually called the *Higgs field*, while G is a *Goldstone field*. We find

$$\mathcal{L}_0 = \frac{1}{2} \left[(\partial H)^2 - 2\lambda v^2 H^2 + (\partial G)^2 - e^2 v^2 G^2 - \partial_\mu A_\nu \partial^\mu A^\nu + e^2 v^2 A^2 \right]. \quad (8.25)$$

We see that the model contains spin 1 and spin 0 bosons with mass ev , associated with the field A , and two different kinds of scalars: the Goldstone boson, with the same mass, and the Higgs boson, with mass $\sqrt{2\lambda}v$.

The phenomenon we have just described is usually called *Higgs mechanism for the spontaneous breaking of the gauge symmetry*, even though the symmetry is not actually broken. In fact, the Lagrangian density is still gauge invariant, up to the gauge-fixing term, which however does not influence physical quantities. Hence, all physical properties connected with gauge invariance (such as, for example, current conservation) are still there. It is important to stress this point, because at the quantum level this is essentially what guarantees the renormalizability of the theory, which would instead be lost in the case of an explicit breaking of the gauge symmetry.

The interaction term is given by

$$\begin{aligned} \mathcal{L}_I = & -eA^\mu (H\partial_\mu G - G\partial_\mu H) + e^2 v A^2 H - \lambda v H (H^2 + G^2) \\ & + \frac{e^2}{2} A^2 H^2 + \frac{e^2}{2} A^2 G^2 - \frac{\lambda}{4} (H^4 + G^4 + 2H^2 G^2). \end{aligned} \quad (8.26)$$

The field equations for A^μ , H and G can be written in full analogy with eqs. (3.65, 8.15):

$$(\partial^2 + 2\lambda v^2) H = J_H \quad (8.27)$$

$$(\partial^2 + e^2 v^2) G = J_G \quad (8.28)$$

$$(\partial^2 + e^2 v^2) A^\mu = -J^\mu, \quad (8.29)$$

where the r.h.s. can be directly computed from the interaction Lagrangian.

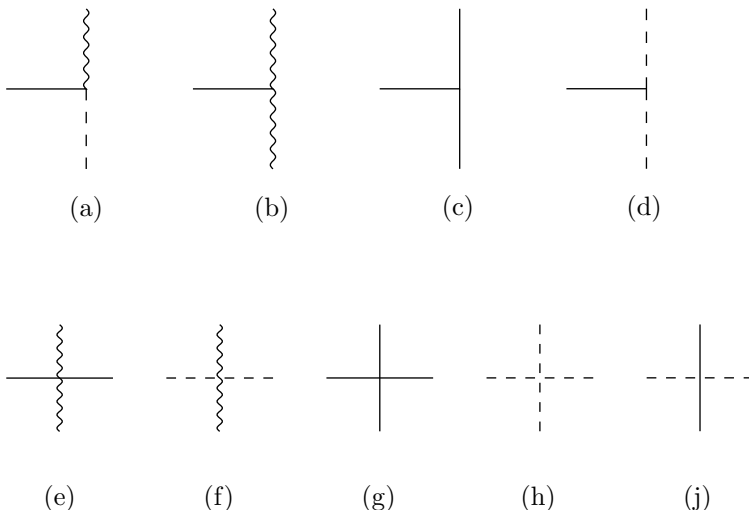
The Feynman rules for the abelian Higgs model can be worked out by the same procedure adopted in the case of the scalar theory. The propagators for scalar fields are obtained from eq. (3.75) replacing the appropriate values of the masses; that of the vector field is obtained from eq. (6.44) inserting the mass squared $e^2 v^2$ in the denominator. The correspondence between line factors and symbols is therefore:

$$\text{—————} \leftrightarrow \frac{1}{2\lambda v^2 - q^2 - i\epsilon} \quad \text{for the field } H \quad (8.30)$$

$$\text{-----} \leftrightarrow \frac{1}{e^2 v^2 - q^2 - i\epsilon} \quad \text{for the field } G \quad (8.31)$$

$$\text{~~~~~} \leftrightarrow \frac{g^{\mu\nu}}{q^2 - e^2 v^2 + i\epsilon} \quad \text{for the field } A^\mu. \quad (8.32)$$

Using the same symbols for lines, the interaction vertices induced by the interaction terms eq. (8.26) are given by



The vertex (a) is originated by the first term in eq. (8.26); it corresponds to an interaction term that involves derivative of the fields. Following the usual procedure, we obtain

$$ie (p_G^\mu - p_H^\mu), \quad (8.33)$$

where the four-momenta p_H of H and p_G of G are taken to be incoming in the vertex. Their sum is zero, according to the energy-momentum conservation constraint at each vertex. All other interaction vertex factors are simply given by the corresponding coefficient in the interaction Lagrangian, multiplied with the combinatorial factor $\prod_c n_c!$, where n_c is the power of the field of species c in the vertex. The vertex factors are listed in table 8.1.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(j)
$ie (p_G^\mu - p_H^\mu)$	$2e^2 v g^{\mu\nu}$	$-6\lambda v$	$-2\lambda v$	$2e^2 g^{\mu\nu}$	$2e^2 g^{\mu\nu}$	-6λ	-6λ	-2λ

Table 8.1. Feynman rules for the interaction vertices in the abelian Higgs model.

Notice that in the limit $e \rightarrow 0$, $\lambda \rightarrow 0$ and $v \rightarrow \infty$ with $\lambda v^2 \equiv M^2$ and $ev \equiv m$ fixed the abelian Higgs model becomes a free theory involving two scalar and one vector field, all of them massive. The vector field can now be coupled to some other matter fields, such as e.g. fermions as in the case of spinor QED, through a charge $\hat{e}$ which is kept fixed in the above limit; the resulting theory is a massive version of QED, which was originally discovered by Stueckelberg. Both scalar fields remain free and can be omitted. This

shows that, in practice, one can simply add a mass term to the Lagrangian of QED, without actually spoiling the internal consistency of the theory.

However, the same procedure cannot be applied to non-abelian gauge theories (based e.g. on $SU(N)$ invariance.) In that case, the gauge fields are coupled to matter fields through a coupling constant g , which is the same that appears in self-interaction terms. Therefore, non-abelian gauge-invariance is lost in the limit $g \rightarrow 0$.

8.3 Physical content of the abelian Higgs model

It is interesting to study the physical content of the abelian Higgs model described in the previous Section, since many of its features are shared by its extension to cases of practical interest, to be discussed later in this Chapter.

The spin of the particles involved is related to the transformation properties of the corresponding fields under spatial rotations: invariants correspond to spin 0, three-vectors to spin 1. Thus, the Higgs and Goldstone fields correspond to spinless particles. The four-vector field contains four degrees of freedom, three of which correspond to a spin-1 particle, and one to spin 0. To see this, we observe that from the free-field Lagrangian eq. (8.25) one obtains an expression of the free field in terms of creation and annihilation operators, that allows quantization by analogy with the simple harmonic oscillator:

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E_k}} \sum_{\lambda=0}^3 \left[a_k^{(\lambda)} \epsilon_\mu^{(\lambda)}(k) e^{-ikx} + a_k^{\dagger(\lambda)} \epsilon_\mu^{(\lambda)*}(k) e^{ikx} \right], \quad (8.34)$$

where

$$E_k = \sqrt{|\mathbf{k}|^2 + e^2 v^2}. \quad (8.35)$$

For each value of k , the four-vectors $\epsilon^{(\lambda)}(k)$ can be defined in the rest frame, where $k = (ev, 0)$, as the unit vectors in the four space-time directions:

$$\epsilon^{(0)}(k) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(1)}(k) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(2)}(k) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \epsilon^{(3)}(k) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (8.36)$$

The corresponding expressions in a generic frame are obtained by suitable Lorentz transformations. In particular, we find $\epsilon_\mu^{(0)}(k) = k_\mu/(ev)$; for this reason, the vector field component that corresponds to $\lambda = 0$ is proportional to the four-divergence of a scalar field, and therefore describes a spin-0 degree of freedom.

It should be noted, however, that the Lagrangian density for A^0 has the opposite sign of those of the scalar fields H and G and of the space components $A^i, i = 1, 2, 3$. This is reflected in the sign of the propagator eq. (8.32) for

$\mu = \nu = 0$. This means that the spin-0 component of A does not correspond to physical quantum states, because it induces negative transition probabilities. Indeed, it can be shown that the creation and annihilation operators obey the commutation rules

$$\left[a_k^{(\lambda)}, a_q^{\dagger(\lambda')} \right] = -g^{\lambda\lambda'} \delta^{(3)}(\mathbf{k} - \mathbf{q}) \quad (8.37)$$

$$\left[a_k^{(\lambda)}, a_q^{(\lambda')} \right] = \left[a_k^{\dagger(\lambda)}, a_q^{\dagger(\lambda')} \right] = 0. \quad (8.38)$$

Note that the commutation rules for $a_k^{(\lambda)}$; $\lambda = 1, 2, 3$ are the same as in the case of the real scalar field, while for $\lambda = 0$ the sign of the commutator is reversed, as a consequence of the fact that the Lagrangian density for A^0 has the opposite sign of the Lagrangian for A^i , $i = 1, 2, 3$. This is easy to understand by analogy with the simple harmonic oscillator in ordinary quantum mechanics. In that case, the Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \quad (8.39)$$

and the creation and annihilation operators are related to p, q by

$$a = \sqrt{\frac{m\omega}{2}} \left(q + i \frac{1}{m\omega} p \right); \quad a^\dagger = \sqrt{\frac{m\omega}{2}} \left(q - i \frac{1}{m\omega} p \right). \quad (8.40)$$

If the sign of the Lagrangian is reversed, $L \rightarrow -L$, then $p = dL/d\dot{q} \rightarrow -p$, and a and $a^\dagger$ are interchanged.

Let us now consider a one-particle state with polarization $\lambda = 0$:

$$|\mathbf{k}\rangle = \int d^3p f_k(p) a_p^{\dagger(0)} |0\rangle, \quad (8.41)$$

where $|0\rangle$ is the vacuum state, and $f_k(p)$ is a packet function, centered around $\mathbf{p} = \mathbf{k}$. Using eq. (8.37), we find

$$\begin{aligned} \langle \mathbf{k} | \mathbf{k} \rangle &= \int d^3p d^3p' f_k(p) f_k^*(p') \langle 0 | a_{p'}^{(0)} a_p^{\dagger(0)} | 0 \rangle \\ &= -\langle 0 | 0 \rangle \int d^3p |f_k(p)|^2. \end{aligned} \quad (8.42)$$

We see that the states generated by $a_p^{\dagger(0)}$ have negative norm, and must therefore be removed from the physical spectrum. This can be achieved requiring

$$\epsilon^{(i)}(k) \cdot k = 0; \quad \epsilon^{(i)}(k) \cdot \epsilon^{(i)*}(k) = -1 \quad (8.43)$$

for physical states. These conditions identify, for each value of the momentum, three independent polarization states, that correspond to the three different helicities of a spin-1 particle. In the rest frame of the vector boson, only the

time component of k is non-zero, and ϵ must be a purely spatial vector, by eqs. (8.43). This excludes $\epsilon^{(0)}$ from the physical spectrum. In practice, one is often interested in unpolarized cross sections, that are proportional to the tensor

$$P_{\mu\nu}(k) = \sum_{i=1}^3 \epsilon_{\mu}^{(i)}(k) \epsilon_{\nu}^{(i)*}(k). \quad (8.44)$$

It is easy to compute $P_{\mu\nu}$ in the rest frame of the vector boson: from eq. (8.36) we get

$$P_{\mu\nu} = \begin{cases} 1 & \mu = \nu \neq 0 \\ 0 & \text{otherwise} \end{cases}, \quad (8.45)$$

or equivalently

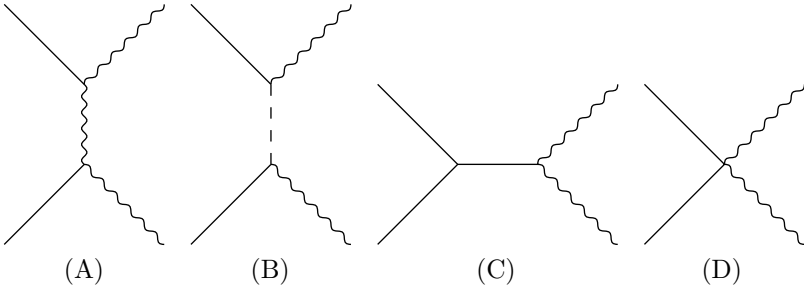
$$P_{\mu\nu}(k) = -g_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{e^2 v^2}, \quad (8.46)$$

since $k = (ev, \mathbf{0})$ in the rest frame. Equation (8.46) is in covariant form, and therefore holds in any reference frame.

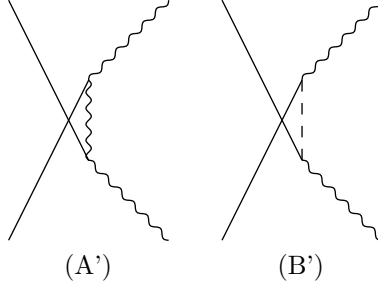
The problem of the internal quantum-mechanical consistency of the theory is not completely solved by imposing the constraints eq. (8.43) on asymptotic states, because the scalar component of A^{μ} may still contribute to physical amplitudes as an intermediate state through the propagator, eq. (8.32). However, it can be shown that such contributions are exactly cancelled by the intermediate states that correspond to the Goldstone field G , provided that one only considers processes in which the asymptotic states are either Higgs bosons, or spin-1 bosons. We will not prove this statement in full generality: we will illustrate how this cancellation takes place in a simple example. We consider the scattering process

$$H + \gamma \rightarrow H + \gamma. \quad (8.47)$$

The invariant amplitude can be computed by identifying the relevant Feynman diagrams, with two H and two A external lines. Recalling the Feynman rules given in the previous Section, it is immediate to recognize that the relevant diagrams are



Furthermore, one should include the diagrams obtained from (A) and (B) by permutation of the two external H lines:



Diagrams (A) and (A') contain a four-vector internal line, while diagrams (B) and (B') have a Goldstone internal line.

We now proceed to compute the amplitude. We assign momenta p and p' to initial and final H lines respectively, and momenta q and q' to initial and final state vector bosons. We denote by ϵ and ϵ' the corresponding polarization vectors, with the conditions

$$\epsilon \cdot q = \epsilon' \cdot q' = 0. \quad (8.48)$$

The invariant amplitude for diagrams (A) and (B) are given by

$$\mathcal{M}_A = \frac{4e^4 v^2 \epsilon \cdot \epsilon'}{(p+q)^2 - e^2 v^2} \quad (8.49)$$

$$\mathcal{M}_B = \frac{e^2 \epsilon \cdot (2p+q) \epsilon' \cdot (p+q+p')}{e^2 v^2 - (p+q)^2}. \quad (8.50)$$

Using eq. (8.48) and four-momentum conservation $p+q = p'+q'$, we may replace $\epsilon \cdot (2p+q) \rightarrow 2\epsilon \cdot (p+q)$ and $\epsilon' \cdot p' \rightarrow \epsilon' \cdot (p+q)$. Thus,

$$\mathcal{M}_{A+B} = \frac{4e^4 v^2}{(p+q)^2 - e^2 v^2} \left[\epsilon \cdot \epsilon' - \frac{\epsilon \cdot (p+q) \epsilon' \cdot (p+q)}{e^2 v^2} \right]. \quad (8.51)$$

The amplitudes for diagrams (A') and (B') are obtained from $\mathcal{M}_{A+B}$ by the replacement $p \leftrightarrow -p'$:

$$\mathcal{M}_{A'+B'} = \frac{4e^4 v^2}{(p'-q)^2 - e^2 v^2} \left[\epsilon \cdot \epsilon' - \frac{\epsilon \cdot (p'-q) \epsilon' \cdot (p'-q)}{e^2 v^2} \right], \quad (8.52)$$

The expressions for $\mathcal{M}_{A+B}$ and $\mathcal{M}_{A'+B'}$, eqs. (8.51,8.52), are what we would obtain from diagrams (A) and (A') with the vector boson propagator replaced as follows:

$$\frac{g^{\mu\nu}}{q^2 - e^2 v^2 + i\epsilon} \rightarrow \frac{g^{\mu\nu} - \frac{q^\mu q^\nu}{e^2 v^2}}{q^2 - e^2 v^2 + i\epsilon}. \quad (8.53)$$

This corresponds to replacing $g^{\mu\nu}$ with the projection operator

$$g^{\mu\nu} - \frac{q^\mu q^\nu}{e^2 v^2}, \quad (8.54)$$

that projects a generic polarization vector on the physical subspace, defined by eq. (8.43).

Finally, the amplitudes for diagrams (C) e (D) are given by

$$\mathcal{M}_C = -\frac{12\lambda e^2 v^2 \epsilon \cdot \epsilon'}{2\lambda v^2 - (p - p')^2}; \quad \mathcal{M}_D = 2 e^2 \epsilon \cdot \epsilon'. \quad (8.55)$$

We observe that in this example the contributions of the scalar component of A and of the Goldstone degree of freedom G as intermediate states cancel exactly. It can be shown that this is a general feature of the theory: the abelian Higgs model is internally consistent, provided the asymptotic particles are either Higgs scalars, or spin-1 vectors. This result generalizes to more complicated Higgs models of relevance in the study of weak interactions.

This example shows that gauge invariance is needed in order to decouple non-physical from physical degrees of freedom, thus preserving unitarity. It is therefore necessary to prove that the counter-terms needed to make the theory finite when loop corrections are included do not violate gauge-invariance. It turns out that this is indeed the case, the only exception being the presence of the anomaly of the axial current.

In the semi-classical limit, non-physical degrees of freedom can be eliminated from the very beginning. We consider the first three terms of the Lagrangian density eq. (8.22), that are invariant under gauge transformations, and we parametrize the scalar field as

$$\Phi(x) + \frac{v}{\sqrt{2}} = e^{ie\Theta(x)} \frac{H(x) + v}{\sqrt{2}}. \quad (8.56)$$

(instead of eq. (8.24)). The field $\Theta(x)$ can be eliminated by a suitable gauge transformation:

$$\mathcal{A}_\mu(x) = A_\mu(x) - \partial_\mu \Theta(x). \quad (8.57)$$

In this way, we obtain an expression of the Lagrangian density that depends only on H e $\mathcal{A}$:

$$\begin{aligned} \mathcal{L}'_H = & \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{e^2}{2} \mathcal{A}^2 (H + v)^2 - \frac{\lambda}{4} \left[(H + v)^2 - v^2 \right]^2 \\ & - \frac{1}{2} \partial_\mu \mathcal{A}_\nu (\partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu). \end{aligned} \quad (8.58)$$

The free Lagrangian density is now

$$\mathcal{L}_0 = \frac{1}{2} \left[(\partial H)^2 - 2\lambda v^2 H^2 - \partial_\mu \mathcal{A}_\nu (\partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu) + e^2 v^2 \mathcal{A}^2 \right], \quad (8.59)$$

and the field equations become

$$(\partial^2 + 2\lambda v^2) H = J_H \quad (8.60)$$

$$(\partial^2 + e^2 v^2) \mathcal{A}^\mu - \partial^\mu \partial_\nu \mathcal{A}^\nu = -J^\mu. \quad (8.61)$$

It is easy to show that the Green function for eq. (8.61) corresponds to the propagator in the r.h.s. of eq. (8.53).

8.4 The Higgs mechanism in the standard model

The procedure described in Section 8.3 can be extended to the standard model, with few modifications. To this purpose, we introduce a scalar field that transforms non-trivially under that subset of gauge transformations that we want to undergo spontaneous breaking. We should keep in mind that the subgroup $U(1)_{\text{em}}$ of the gauge group, that corresponds to electrodynamics, should not be spontaneously broken. In other words, we should perform the spontaneous breaking of the gauge symmetry in such a way that a mass term for the photon is not generated. This means that spontaneous symmetry breaking must take place in three of the four “directions” of the $SU(2) \times U(1)$ gauge group, the fourth one being that corresponding to electric charge. The simplest way to do this is to assign the scalar field ϕ to a doublet representation of $SU(2)$:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad (8.62)$$

with the $SU(2)$ transformation property

$$\phi \rightarrow e^{ig\alpha_i \tau_i/2} \left(\phi + \frac{v}{\sqrt{2}} \right) - \frac{v}{\sqrt{2}}, \quad (8.63)$$

where

$$v = \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad (8.64)$$

is a constant $SU(2)$ doublet. The value of the hypercharge of the scalar doublet ϕ is fixed by the requirement that the zero-field configuration is left unchanged by electromagnetic gauge transformations that correspond to the subgroup $U(1)_{\text{em}}$, or equivalently that it be electrically neutral. This amounts to requiring that

$$e^{ie\alpha Q} \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (8.65)$$

or equivalently

$$\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1/2 + Y/2 & 0 \\ 0 & -1/2 + Y/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (8.66)$$

where Q_1, Q_2 are the electric charges of ϕ_1, ϕ_2 , and we have used eq. (7.36). The non-trivial solutions of eq. (8.66) are

$$1) \quad v_1 = 0, \quad |v_2| = v, \quad Y = +1 \quad (8.67)$$

$$2) \quad v_2 = 0, \quad |v_1| = v, \quad Y = -1. \quad (8.68)$$

Without loss of generality, we will adopt the first choice, with $Y = +1$ and therefore $Q_1 = 1, Q_2 = 0$. We shall further assume that v_2 is real and positive.

The Higgs mechanism takes place in analogy with scalar electrodynamics. The most general scalar potential consistent with gauge invariance and renormalizability which is minimum at the origin of the isotopic space is

$$V(\phi) = \lambda \left[\left(\phi + \frac{v}{\sqrt{2}} \right)^\dagger \left(\phi + \frac{v}{\sqrt{2}} \right) - \frac{v^2}{2} \right]^2. \quad (8.69)$$

We can reparametrize ϕ in the following way:

$$\phi + \frac{v}{\sqrt{2}} = \frac{1}{\sqrt{2}} e^{i\tau^i \theta^i(x)/v} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \quad (8.70)$$

with $\theta^i(x)$ and $H(x)$ real. With this parametrization it is apparent that the fields θ_i can be transformed away by an $SU(2)$ gauge transformation. as in the case of the abelian Higgs model. This choice is perfectly adequate for calculation in the semi-classical limit. We will set $\theta_i = 0$ henceforth; this choice is called the *unitary gauge*.

The scalar potential takes the form

$$V = \frac{1}{2}(2\lambda v^2)H^2 + \lambda v H^3 + \frac{1}{4}\lambda H^4; \quad (8.71)$$

we see that the Higgs scalar H has a squared mass $m_H^2 = 2\lambda v^2$. Using eq. (8.70) with $\theta^i = 0$ we get

$$\begin{aligned} D^\mu \left(\phi + \frac{v}{\sqrt{2}} \right) &= \left(\partial^\mu - i\frac{g}{2}\tau^i W_\mu^i - i\frac{g'}{2}B_\mu \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ H(x) + v \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} - \frac{i}{2}(H+v) \frac{1}{\sqrt{2}} \begin{pmatrix} g(W_1^\mu - iW_2^\mu) \\ -gW_3^\mu + g'B^\mu \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} - \frac{i}{2}(H+v) \begin{pmatrix} gW^{\mu+} \\ -\sqrt{(g^2 + g'^2)/2} Z^\mu \end{pmatrix}, \end{aligned} \quad (8.72)$$

where in the last step we have used eqs. (7.20), (7.27), (7.28) and (7.35). We have therefore

$$\begin{aligned} &\left[D^\mu \left(\phi + \frac{v}{\sqrt{2}} \right) \right]^\dagger D_\mu \left(\phi + \frac{v}{\sqrt{2}} \right) \\ &= \frac{1}{2} \partial^\mu H \partial_\mu H + \left[\frac{1}{4} g^2 W^{\mu+} W_\mu^- + \frac{1}{8} (g^2 + g'^2) Z^\mu Z_\mu \right] (H+v)^2. \end{aligned} \quad (8.73)$$

We see that the W and Z bosons have acquired masses

$$m_w^2 = \frac{1}{4}g^2v^2 \quad (8.74)$$

$$m_z^2 = \frac{1}{4}(g^2 + g'^2)v^2. \quad (8.75)$$

Note that the photon remains massless. With the scalar field ϕ transforming as a doublet of $SU(2)$, there is always a linear combination of B^μ and W_3^μ that does not receive a mass term, but only if $Y(\phi) = 1$ (or -1) does this linear combination coincide with the one in eq. (7.27).

The value of v , the *vacuum expectation value* of the neutral component of the Higgs doublet, can be obtained combining eqs. (8.3) and (8.74), and using the measured value of the Fermi constant. We get

$$v = \sqrt{\frac{1}{G_F\sqrt{2}}} \simeq 246.22 \text{ GeV}. \quad (8.76)$$

The value of the Higgs quartic coupling λ (or equivalently the Higgs mass) is not fixed by our present knowledge.

Breaking of accidental symmetries

9.1 Quark masses and flavour-mixing

Fermion mass terms are forbidden by the gauge symmetry of the standard model. Indeed, a Dirac mass term for a fermion field ψ

$$-m \bar{\psi}\psi = -m (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \quad (9.1)$$

is not invariant under a chiral transformation, i.e. a transformation that acts differently on left-handed and right-handed components. The gauge transformations of the standard model are precisely of this kind. Again, this difficulty can be circumvented by means of the Higgs mechanism.

We first consider the hadronic sector. We have seen in Section 7.3 that the interaction Lagrangian is not diagonal in terms of quark fields with definite flavours. Let u'^f and d'^f be the fields that bring the interaction terms in diagonal form (the index f runs over the n fermion generations); in principle, there is no reason why only down-type quark fields should be rotated. We also define

$$q'^f_L = \begin{pmatrix} u'^f_L \\ d'^f_L \end{pmatrix}. \quad (9.2)$$

A Yukawa interaction term can be added to the Lagrangian density:

$$\begin{aligned} \mathcal{L}_Y^{\text{hadr}} = & - \left[\bar{q}'_L \left(\phi + \frac{v}{\sqrt{2}} \right) h'_D d'_R + \bar{d}'_R \left(\phi + \frac{v}{\sqrt{2}} \right)^\dagger h'^\dagger_D q'_L \right] \\ & - \left[\bar{q}'_L \epsilon \left(\phi + \frac{v}{\sqrt{2}} \right)^* h'_U u'_R - \bar{u}'_R \left(\phi + \frac{v}{\sqrt{2}} \right)^T \epsilon h'^\dagger_U q'_L \right], \quad (9.3) \end{aligned}$$

where h'_U and h'_D are generic $n \times n$ constant complex matrices in the generation space. It is easy to check that $\mathcal{L}_Y^{\text{hadr}}$ is Lorentz-invariant, gauge-invariant¹

¹ If ϕ transforms as an $SU(2)$ doublet, so does $\phi_c = \epsilon \phi^*$, where ϵ is the antisymmetric tensor in two dimension.

and renormalizable, and therefore it can be included in the Lagrangian. The matrices h'_U and h'_D can be diagonalized by means of bi-unitary transformations:

$$h_U \equiv V_L^{U\dagger} h'_U V_R^U \quad (9.4)$$

$$h_D \equiv V_L^{D\dagger} h'_D V_R^D, \quad (9.5)$$

where $V_{L,R}^{U,D}$ are unitary matrices, chosen so that h_U and h_D are diagonal with real, non-negative entries. Now, we define new quark fields u and d by

$$u'_L = V_L^U u_L, \quad u'_R = V_R^U u_R \quad (9.6)$$

$$d'_L = V_L^D d_L, \quad d'_R = V_R^D d_R, \quad (9.7)$$

In the unitary gauge, eq. (9.3) becomes

$$\mathcal{L}_Y^{\text{hadr}} = -\frac{1}{\sqrt{2}}(v + H) \sum_{f=1}^n (h_D^f \bar{d}^f d^f + h_U^f \bar{u}^f u^f), \quad (9.8)$$

where $h_{U,D}^f$ are the diagonal entries of the matrices $h_{U,D}$. We can now identify the quark masses with

$$m_U^f = \frac{v h_U^f}{\sqrt{2}}, \quad m_D^f = \frac{v h_D^f}{\sqrt{2}}. \quad (9.9)$$

Since the matrices $V_{L,R}^{U,D}$ are constant in space-time, eqs. (9.6,9.7) obviously define global symmetry transformations of the free quark Lagrangian. They also leave unchanged the neutral-current interaction term, because of the universality of the couplings of fermions of different families to the photon and the Z . The only term in the Lagrangian which is affected by the transformations in eqs. (9.6,9.7) is the charged-current interaction, because the up and down components of the same left-handed doublet are transformed in different ways. Indeed, we find

$$\mathcal{L}_c^{\text{hadr}} = \frac{g}{\sqrt{2}} W_\mu^+ \sum_{f=1}^n \bar{q}_L'^f \gamma^\mu \tau^+ q_L'^f + \text{h.c.} = \frac{g}{\sqrt{2}} W_\mu^+ \sum_{f,g=1}^n \bar{u}_L^f \gamma^\mu V_{fg} d_L^g + \text{h.c.}, \quad (9.10)$$

where

$$V = V_L^{U\dagger} V_L^D. \quad (9.11)$$

The matrix V is usually called the *Cabibbo-Kobayashi-Maskawa* (CKM) matrix. It is a unitary matrix, and its unitarity guarantees the suppression of flavour-changing neutral currents, as we already discussed in Section 7.1 in the case of two fermion families. The elements of V are fundamental parameters of the standard model Lagrangian, on the same footing as masses and gauge couplings, and must be extracted from experiments.

To conclude this Section, we determine the number of independent parameters in the CKM matrix. A generic $n \times n$ unitary matrix depends on n^2 independent real parameters. Some (n_A) of them can be thought of as rotation angles in the n -dimensional space of generations, and they are as many as the coordinate planes in n dimensions:

$$n_A = \binom{n}{2} = \frac{1}{2}n(n-1). \quad (9.12)$$

The remaining

$$\hat{n}_P = n^2 - n_A = \frac{1}{2}n(n+1) \quad (9.13)$$

parameters are complex phases. Some of them can be eliminated from the Lagrangian density by redefining the left-handed quark fields as

$$u_L^f \rightarrow e^{i\alpha_f} u_L^f; \quad d_L^g \rightarrow e^{i\beta_g} d_L^g, \quad (9.14)$$

with α_f, β_g real constants. Indeed, the transformations eq. (9.14) are symmetry transformations for the whole standard model Lagrangian except $\mathcal{L}_c^{\text{had}r}$, and therefore amount to a redefinition of the CKM matrix:

$$V_{fg} \rightarrow e^{i(\beta_g - \alpha_f)} V_{fg}. \quad (9.15)$$

The $2n$ constants α_f, β_g can be chosen so that $2n-1$ phases are eliminated from the matrix V , since there are $2n-1$ independent differences $\beta_g - \alpha_f$. The number of really independent complex phases in V is therefore

$$n_P = \hat{n}_P - (2n-1) = \frac{1}{2}(n-1)(n-2). \quad (9.16)$$

Observe that, with one or two fermion families, the CKM matrix can be made real. The first case with non-trivial phases is $n=3$, which corresponds to $n_P=1$. In the standard model with three fermion families, the CKM matrix has four independent parameters: three rotation angles and one complex phase. In the general case, the total number of independent parameters in the CKM matrix is

$$n_A + n_P = (n-1)^2. \quad (9.17)$$

The presence of complex coupling constants implies violation of the CP symmetry. CP violation phenomena in weak interactions were first observed around 1964 in the $K^0 - \bar{K}^0$ system (see [5] and [7]); the existence of a third quark generation may therefore be considered as a prediction of the standard model, confirmed by the discovery of the b and t quarks.

9.2 Lepton masses

The same procedure can be applied to the leptonic sector. Everything is formally unchanged: up-quarks are replaced by neutrinos and down-quarks are

replaced by charged leptons (e^- , μ^- and τ^- .) There is however an important difference, which leads to considerable simplifications: as we have seen, right-handed neutrinos have no interactions, and hence can be omitted. Therefore, there is no Yukawa coupling involving the conjugate scalar field ϕ_c :

$$\mathcal{L}_Y^{\text{lept}} = - \left[\bar{\ell}'_L \left(\phi + \frac{v}{\sqrt{2}} \right) h'_E e_R + \bar{e}_R \left(\phi + \frac{v}{\sqrt{2}} \right)^\dagger h'^{\dagger}_E \ell'_L \right]. \quad (9.18)$$

The matrix of Yukawa couplings h'_E can be diagonalized by means of a bi-unitary transformation

$$h_E = V_L^E \dagger h'_E V_R^E. \quad (9.19)$$

The difference with respect to the case of quarks is that now we have the freedom of redefining the left-handed neutrino fields using *the same* matrix V_L^E that rotates charged leptons:

$$\nu'_L = V_L^E \nu_L \quad (9.20)$$

$$e'_L = V_L^E e_L, \quad e'_R = V_R^E e_R. \quad (9.21)$$

This brings the lepton Yukawa interaction in diagonal form,

$$\mathcal{L}_Y^{\text{lept}} = - \sum_{f=1}^n h_E^f \left[\bar{\ell}_L^f \left(\phi + \frac{v}{\sqrt{2}} \right) e_R^f + \bar{e}_R^f \left(\phi + \frac{v}{\sqrt{2}} \right)^\dagger \ell_L^f \right], \quad (9.22)$$

but, contrary to what happens in the quark sector, leaves the charged-current interaction term unchanged:

$$\mathcal{L}_c^{\text{lept}} = \frac{g}{\sqrt{2}} W_\mu^+ \sum_{f=1}^n \bar{\ell}_L^f \gamma^\mu \tau^+ \ell'^f_L + \text{h.c.} = \frac{g}{\sqrt{2}} W_\mu^+ \sum_{f=1}^n \bar{\ell}_L^f \gamma^\mu \tau^+ \ell_L^f + \text{h.c.} \quad (9.23)$$

In other words, in the leptonic sector there is no mixing among different generations, because the Yukawa coupling matrix can be diagonalized by a global transformation under which the full Lagrangian is invariant. As a consequence, not only the overall leptonic number, but also individual leptonic flavours are conserved. This is due to the absence of right-handed neutrinos.

The values of the Yukawa couplings h_E^f are determined by the values of the observed lepton masses. In fact, using eq. (8.70), we find

$$\mathcal{L}_Y^{\text{lept}} = - \sum_{f=1}^n \frac{h_E^f}{\sqrt{2}} (v + H) \bar{e}_f e_f, \quad (9.24)$$

thus allowing the identifications

$$m_E^f = \frac{v h_E^f}{\sqrt{2}}. \quad (9.25)$$

9.3 Accidental symmetries

The need for Yukawa interaction terms of fermion fields with scalar fields can be motivated in a different way. Consider the standard model with only one generation of quarks and leptons, and no scalar fields. The Lagrangian for fermion fields can be written in the following compact form:

$$\mathcal{L} = \sum_{k=1}^5 \bar{\psi}_k i \not{D} \psi_k, \quad (9.26)$$

where the sum runs over the five different irreducible representations of $SU(2) \times U(1)$ of fermions within a generation:

$$\begin{aligned} \psi_1 &= e_R \sim (\mathbf{1}, -2) \\ \psi_2 &= \ell_L \sim (\mathbf{2}, -1) \\ \psi_3 &= u_R \sim (\mathbf{1}, 4/3) \\ \psi_4 &= d_R \sim (\mathbf{1}, -2/3) \\ \psi_5 &= q_L \sim (\mathbf{2}, 1/3). \end{aligned}$$

Here, the symbol $\sim$ means “transforms as”, and the two numbers in brackets stand for the $SU(2)$ representation ($\mathbf{2}$ for the doublet, $\mathbf{1}$ for the scalar) and for the hypercharge quantum number, respectively. Mass terms are forbidden by the gauge symmetry.

In addition to the assumed gauge symmetry, the Lagrangian in eq. (9.26) is manifestly invariant under a large class of global transformations: namely, the fermion fields within each representation can be multiplied by an arbitrary constant phase

$$\psi_k \rightarrow e^{i\phi_k} \psi_k \quad (9.27)$$

without affecting $\mathcal{L}$. This $[U(1)]^5$ global symmetry was not imposed at the beginning: it is just a consequence of the assumed gauge symmetry and of the renormalizability condition. It is therefore called an *accidental* symmetry.

Let us take a closer look at the accidental symmetry. The five conserved currents corresponding to the global transformations (9.27) are

$$\begin{aligned} J_1^\mu &= \bar{e}_R \gamma^\mu e_R \\ J_2^\mu &= \bar{\nu}_L \gamma^\mu \nu_L + \bar{e}_L \gamma^\mu e_L \\ J_3^\mu &= \bar{u}_R \gamma^\mu u_R \\ J_4^\mu &= \bar{d}_R \gamma^\mu d_R \\ J_5^\mu &= \bar{u}_L \gamma^\mu u_L + \bar{d}_L \gamma^\mu d_L. \end{aligned}$$

A deeper insight in the meaning of these conserved currents is achieved by replacing $J_1^\mu, \dots, J_5^\mu$ by the following independent linear combinations of them:

$$\begin{aligned}
J_Y^\mu &= \sum_{k=1}^5 \frac{Y_k}{2} J_k^\mu \\
J_\ell^\mu &= J_1^\mu + J_2^\mu = \bar{\nu} \gamma^\mu \nu + \bar{e} \gamma^\mu e \\
J_{\ell 5}^\mu &= J_1^\mu - J_2^\mu = \bar{\nu} \gamma^\mu \gamma_5 \nu + \bar{e} \gamma^\mu \gamma_5 e \\
J_b^\mu &= \frac{1}{3} (J_3^\mu + J_4^\mu + J_5^\mu) = \frac{1}{3} (\bar{u} \gamma^\mu u + \bar{d} \gamma^\mu d) \\
J_{b5}^\mu &= J_3^\mu + J_4^\mu - J_5^\mu = \bar{u} \gamma^\mu \gamma_5 u + \bar{d} \gamma^\mu \gamma_5 d.
\end{aligned}$$

The current J_Y is the hypercharge current, which corresponds to a local invariance of the theory. The true accidental symmetry is therefore $[U(1)]^4$, rather than $[U(1)]^5$.

The currents J_ℓ and J_b are immediately recognized to be the leptonic and baryonic number currents, respectively. The invariance of the Lagrangian under the corresponding global symmetries is certainly good news, since baryonic and leptonic number are known to be conserved to an extremely high accuracy. On the other hand, experiments show no sign of the conservation of $J_{\ell 5}$ and J_{b5} ; in a realistic theory, the corresponding symmetries should be broken. In fact, they are incompatible with mass terms, and they are broken by the Yukawa interaction terms that generate fermion masses via the Higgs mechanism. When the theory is extended to include more fermion generations, the accidental symmetry gets much larger, since also mixing among different generations is allowed. The Yukawa interaction terms of the previous Sections break this larger accidental symmetry too, leaving however baryonic and leptonic numbers conserved. Individual leptonic numbers are separately conserved, while only the total baryonic number is conserved, because of flavour mixing in the quark sector.

It should be noted that, because of accidental symmetries, Yukawa interactions cannot be generated by radiative corrections. This is the mechanism that keeps neutrinos massless, and that protects fermion masses from receiving large radiative corrections.

We conclude this Section by reviewing the most important experimental evidences of baryon and lepton number conservation. The most obvious test of baryon number conservation is proton stability. The experimental lower bound on the proton lifetime is at present

$$\tau_p > 2.1 \cdot 10^{29} \text{ y}. \quad (9.28)$$

The most accurate tests of lepton number conservation are provided by the following observables:

$$B(\mu \rightarrow e \gamma) \leq 1.2 \cdot 10^{-11}; \quad B(\tau \rightarrow \mu \gamma) \leq 2.7 \cdot 10^{-6} \quad (9.29)$$

$$B(\mu \rightarrow 3e) \leq 1 \cdot 10^{-12} \quad (9.30)$$

$$\frac{\Gamma(\mu \text{ Ti} \rightarrow e \text{ Ti})}{\Gamma(\mu \text{ Ti} \rightarrow \text{all})} \leq 4 \cdot 10^{-12}. \quad (9.31)$$

Summary

We present here the full Lagrangian density of the standard model with one Higgs doublet. We have

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_0 + \mathcal{L}_{\text{em}} + \mathcal{L}_c + \mathcal{L}_n + \mathcal{L}_V + \mathcal{L}_{\text{Higgs}} \quad (10.1)$$

where

- $\mathcal{L}_0$ is the quadratic part of the Lagrangian density:

$$\begin{aligned} \mathcal{L}_0 = & \sum_{f=1}^n \left[\bar{\nu}^f i \not{\partial} \nu^f + \bar{e}^f (i \not{\partial} - m_{\text{e}}^f) e^f + \bar{u}^f (i \not{\partial} - m_{\text{u}}^f) u^f + \bar{d}^f (i \not{\partial} - m_{\text{d}}^f) d^f \right] \\ & - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \frac{1}{2} m_Z^2 Z^\mu Z_\mu - \frac{1}{2} W_{\mu\nu}^+ W_{-}^{\mu\nu} + m_W^2 W^{\mu+} W_{\mu}^{-} \\ & - \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial^\mu H \partial_\mu H - \frac{1}{2} m_H^2 H^2, \end{aligned} \quad (10.2)$$

where

$$Z^{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu; \quad W_{\pm}^{\mu\nu} = \partial^\mu W_{\pm}^\nu - \partial^\nu W_{\pm}^\mu. \quad (10.3)$$

The index f labels the n fermion families.

We have adopted the gauge fixing eq. (6.38) for electrodynamics, so that the photon propagator is

$$\Delta_{\gamma}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 + i\epsilon}, \quad (10.4)$$

and the unitary gauge for weak interactions, so that the W and Z boson propagators are

$$\Delta_V^{\mu\nu}(k) = \frac{1}{k^2 - m_V^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_V^2} \right); \quad V = W, Z. \quad (10.5)$$

- $\mathcal{L}_{\text{em}}$ is the electromagnetic coupling:

$$\mathcal{L}_{\text{em}} = e \sum_{f=1}^n \left(-\bar{e}^f \gamma_\mu e^f + \frac{2}{3} \bar{u}^f \gamma_\mu u^f - \frac{1}{3} \bar{d}^f \gamma_\mu d^f \right) A^\mu. \quad (10.6)$$

- $\mathcal{L}_c$ is the charged-current interaction term:

$$\begin{aligned} \mathcal{L}_c = & \frac{g}{2\sqrt{2}} \left[\sum_{f=1}^n \bar{\nu}^f \gamma^\mu (1 - \gamma_5) e^f + \sum_{f,g=1}^n \bar{u}^f \gamma^\mu (1 - \gamma_5) V_{fg} d^g \right] W_\mu^+ \\ & + \frac{g}{2\sqrt{2}} \left[\sum_{f=1}^n \bar{e}^f \gamma^\mu (1 - \gamma_5) \nu^f + \sum_{f,g=1}^n \bar{d}^g \gamma^\mu (1 - \gamma_5) V_{fg}^* u^f \right] W_\mu^-. \end{aligned} \quad (10.7)$$

- $\mathcal{L}_n$ is the neutral-current interaction term:

$$\begin{aligned} \mathcal{L}_n = & \frac{e}{4 \cos \theta_w \sin \theta_w} \sum_{f=1}^n \left[\bar{\nu}^f \gamma_\mu (1 - \gamma_5) \nu^f + \bar{e}^f \gamma_\mu (-1 + 4 \sin^2 \theta_w + \gamma_5) e^f \right. \\ & \left. + \bar{u}^f \gamma_\mu \left(1 - \frac{8}{3} \sin^2 \theta_w - \gamma_5 \right) u^f + \bar{d}^f \gamma_\mu \left(-1 + \frac{4}{3} \sin^2 \theta_w + \gamma_5 \right) d^f \right] Z^\mu. \end{aligned} \quad (10.8)$$

- $\mathcal{L}_V$ contains vector boson interactions among themselves:

$$\begin{aligned} \mathcal{L}_{\text{YM}} = & +ig \sin \theta_w (W_{\mu\nu}^+ W_-^\mu A^\nu - W_{\mu\nu}^- W_+^\mu A^\nu + F_{\mu\nu} W_+^\mu W_-^\nu) \\ & +ig \cos \theta_w (W_{\mu\nu}^+ W_-^\mu Z^\nu - W_{\mu\nu}^- W_+^\mu Z^\nu + Z_{\mu\nu} W_+^\mu W_-^\nu) \\ & + \frac{g^2}{2} (2g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \left[\frac{1}{2} W_\mu^+ W_\nu^+ W_\rho^- W_\sigma^- \right. \\ & \left. - W_\mu^+ W_\nu^- (A_\rho A_\sigma \sin^2 \theta_w + Z_\rho Z_\sigma \cos^2 \theta_w + 2A_\rho Z_\sigma \sin \theta_w \cos \theta_w) \right], \end{aligned}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (10.9)$$

- The Higgs interaction Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} = & \left(m_w^2 W_\mu^+ W_\mu^- + \frac{1}{2} m_z^2 Z^\mu Z_\mu \right) \left(\frac{H^2}{v^2} + \frac{2H}{v} \right) \\ & - \frac{H}{v} \sum_{f=1}^n (m_D^f \bar{d}^f d^f + m_U^f \bar{u}^f u^f + m_E^f \bar{e}^f e^f) \\ & - \lambda v H^3 - \frac{1}{4} \lambda H^4. \end{aligned} \quad (10.10)$$

The parameters appearing in $\mathcal{L}_{\text{SM}}$ are not all independent. The gauge-Higgs sector is entirely specified by the four parameters

$$g, \quad g', \quad v, \quad m_H, \quad (10.11)$$

since

$$m_W^2 = \frac{1}{4}g^2v^2, \quad m_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2, \quad \lambda = \frac{m_H^2}{2v^2}, \quad \tan \theta_w = \frac{g'}{g} \quad (10.12)$$

and $g \sin \theta_w = g' \cos \theta_w = e$. However, v, g, g' are often eliminated in favour of the Fermi constant G_F , the electromagnetic coupling α_{em} and the Z^0 mass m_Z , which are measured with high accuracy. We have

$$G_F = \frac{1}{\sqrt{2}v^2}, \quad \alpha_{\text{em}} = \frac{g^2 g'^2}{4\pi(g^2 + g'^2)}, \quad m_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2, \quad (10.13)$$

and therefore

$$v^2 = \frac{1}{\sqrt{2}G_F} \quad (10.14)$$

$$g^2 = 2\sqrt{2}m_Z^2 G_F \left(1 + \sqrt{1 - \frac{4\pi\alpha_{\text{em}}}{\sqrt{2}m_Z^2 G_F}} \right) \quad (10.15)$$

$$g'^2 = 2\sqrt{2}m_Z^2 G_F \left(1 - \sqrt{1 - \frac{4\pi\alpha_{\text{em}}}{\sqrt{2}m_Z^2 G_F}} \right) \quad (10.16)$$

since $\tan \theta_w < 1$. The free parameters in the fermionic sector are the $3n$ masses m_t^f, m_b^f, m_e^f , and the $(n-1)^2$ independent parameters in the Cabibbo-Kobayashi-Maskawa matrix V . Including the QCD coupling constant g_s , this gives a total of $4 + 3n + (n-1)^2 + 1 = 18$ free parameters for the standard model with three fermion generations.

Applications

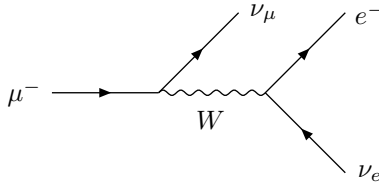
11.1 Muon decay

The μ^- decays predominantly into an electron, a muon neutrino and an electron anti-neutrino:

$$\mu^-(p) \rightarrow e^-(k) + \bar{\nu}_e(k_1) + \nu_\mu(k_2). \quad (11.1)$$

This is a process of great phenomenological importance, since the measurement of the muon lifetime provides the most precise determination of the Fermi constant G_F , and one of the most precise measurements in the whole field of elementary particle physics. In this Section, we describe the calculation of the corresponding decay rate in some detail.

The invariant amplitude arises, in the semi-classical approximation, from the single diagram



The amplitude is easily computed:

$$\mathcal{M} = \frac{g}{2\sqrt{2}} \bar{u}(k_2) \gamma^\mu (1-\gamma_5) u(p) \frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2}}{q^2 - m_W^2} \frac{g}{2\sqrt{2}} \bar{u}(k) \gamma^\nu (1-\gamma_5) v(k_1). \quad (11.2)$$

The squared momentum $q^2 = (p - k_2)^2 = (k + k_1)^2$ flowing in the internal W line can be safely neglected with respect to the W mass. This is an excellent approximation, since $q^2 \leq m_\mu^2 \simeq (105.6 \text{ MeV})^2$, while $m_W \simeq 80.4 \text{ GeV}$.

Furthermore, the term $q_\mu q_\nu / m_W^2$ in the numerator of the W propagator gives zero contribution in the limit $m_e = m_\nu = 0$, since

$$(k + k_1)^\mu \bar{u}(k) \gamma_\mu (1 - \gamma_5) v(k_1) = \bar{u}(k) \not{k} (1 - \gamma_5) v(k_1) \\ + \bar{u}(k) (1 + \gamma_5) \not{k}_1 v(k_1) = 0 \quad (11.3)$$

by the Dirac equation. Hence, using eq. (8.3) to relate the constant g to the Fermi constant G_F ,

$$\mathcal{M} \simeq -\frac{G_F}{\sqrt{2}} \bar{u}(k_2) \gamma^\mu (1 - \gamma_5) u(p) \bar{u}(k) \gamma_\mu (1 - \gamma_5) v(k_1). \quad (11.4)$$

The squared amplitude, summed over all spin states of the particles involved, can be computed by reducing spinor products to traces over spinor indices, as we have seen in Section 6.2 in the case of the Compton scattering cross section, and using eqs. (6.62,6.63) for the sum over fermion spin states. We find

$$\sum_{\text{pol}} |\mathcal{M}|^2 = \frac{G_F^2}{2} \text{Tr} [\gamma^\mu (1 - \gamma_5) (\not{p} + m_\mu) \gamma^\nu (1 - \gamma_5) \not{k}_2] \\ \times \text{Tr} [\gamma_\mu (1 - \gamma_5) \not{k}_1 \gamma_\nu (1 - \gamma_5) \not{k}]. \quad (11.5)$$

The term proportional to m_μ vanishes, because $(1 + \gamma_5)(1 - \gamma_5) = 0$, and we are left with

$$\sum_{\text{pol}} |\mathcal{M}|^2 = 2G_F^2 \text{Tr} [\gamma^\mu \not{p} \gamma^\nu \not{k}_2 (1 + \gamma_5)] \text{Tr} [\gamma_\mu \not{k}_1 \gamma_\nu \not{k} (1 + \gamma_5)]. \quad (11.6)$$

The traces over spinor indices are computed with the help of eqs. (C.8,C.9); we find

$$\text{Tr} [\gamma^\mu \not{p} \gamma^\nu \not{k}_2 (1 + \gamma_5)] = 4 (p^\mu k_2^\nu - g^{\mu\nu} p \cdot k_2 + p^\nu k_2^\mu + i\epsilon^{\mu\alpha\nu\beta} p_\alpha k_{2\beta}), \quad (11.7)$$

and a similar expression for $\text{Tr} [\gamma^\mu \not{k}_1 \gamma^\nu \not{k} (1 + \gamma_5)]$. The calculation then proceeds in a tedious but straightforward way. The result is remarkably simple:

$$|\mathcal{M}|^2 = 128 G_F^2 p \cdot k_1 k \cdot k_2, \quad (11.8)$$

where we have used the identity

$$\epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu\nu\gamma\delta} = -2 (\delta_\gamma^\alpha \delta_\delta^\beta - \delta_\delta^\alpha \delta_\gamma^\beta). \quad (11.9)$$

The differential decay width is given by eq. (4.35):

$$d\Gamma = \frac{1}{2m_\mu} \frac{1}{2} \sum_{\text{pol}} |\mathcal{M}|^2 d\phi_3(p; k, k_1, k_2), \quad (11.10)$$

where we have averaged over the two spin states of the decaying muon, and

$$d\phi_3(p; k, k_1, k_2) = \frac{d\mathbf{k}}{(2\pi)^3 2k^0} \frac{d\mathbf{k}_1}{(2\pi)^3 2k_1^0} \frac{d\mathbf{k}_2}{(2\pi)^3 2k_2^0} (2\pi)^4 \delta^{(4)}(p - k - k_1 - k_2). \quad (11.11)$$

An interesting quantity from the phenomenological point of view is the electron energy spectrum $d\Gamma/dE$. This can be computed by integrating eq. (11.10) over neutrino momenta. Recalling the expression eq. (11.8) of the squared amplitude, we see that in order to obtain the electron energy spectrum we must compute

$$I^{\alpha\beta}(K) = \int \frac{d\mathbf{k}_1}{k_1^0} \frac{d\mathbf{k}_2}{k_2^0} k_1^\alpha k_2^\beta \delta^{(4)}(K - k_1 - k_2), \quad (11.12)$$

where we have defined $K = p - k = k_1 + k_2$. Because the integration measure is Lorentz invariant, $I^{\alpha\beta}(K)$ is a tensor, and it only depends on the four-momentum K . Hence, it has necessarily the form

$$I^{\alpha\beta}(K) = A K^\alpha K^\beta + B K^2 g^{\alpha\beta}. \quad (11.13)$$

The constants A and B can be determined as follows. We observe that, for massless neutrinos,

$$k_1 \cdot k_2 = \frac{1}{2}(k_1 + k_2)^2 = \frac{K^2}{2}; \quad K \cdot k_1 K \cdot k_2 = (k_1 \cdot k_2)^2 = \frac{K^4}{4} \quad (11.14)$$

and we multiply eq. (11.13) by $g_{\alpha\beta}$ and $K_\alpha K_\beta / K^2$. We obtain the system of equations

$$A + 4B = \frac{I}{2} \quad (11.15)$$

$$A + B = \frac{I}{4} \quad (11.16)$$

where

$$I = \int \frac{d\mathbf{k}_1}{k_1^0} \frac{d\mathbf{k}_2}{k_2^0} \delta^{(4)}(K - k_1 - k_2) = \int \frac{d\mathbf{k}_1}{|\mathbf{k}_1|^2} \delta(K^0 - 2|\mathbf{k}_1|) = 2\pi, \quad (11.17)$$

and therefore

$$A = \frac{\pi}{3}; \quad B = \frac{\pi}{6}. \quad (11.18)$$

Replacing this result in eq. (11.13), and then in eq. (11.10), we find

$$d\Gamma = \frac{G_F^2}{12\pi^3} (3m_\mu^2 - 4m_\mu E) E^2 dE, \quad (11.19)$$

where E is the energy of the emitted electron in the rest frame of the decaying muon.

The total rate is now easily computed. The maximum value of E is achieved when the two neutrinos are emitted in the same direction; in this case, $E = E_{\max} = m_\mu/2$. Hence,

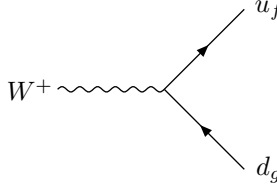
$$\Gamma = \frac{G_F^2}{12\pi^3} \int_0^{m_\mu/2} dE E^2 (3m_\mu^2 - 4m_\mu E) = \frac{G_F^2 m_\mu^5}{192\pi^3}. \quad (11.20)$$

11.2 Width of the W boson

A simple and interesting application of what we have learned about relativistic processes and the standard model is the computation of the decay rate of the W boson into a quark-antiquark pair:

$$W^+(p, \epsilon) \rightarrow u_f(k_1) + \bar{d}_g(k_2), \quad (11.21)$$

where f, g are generation indices, and particle four-momenta are displayed in brackets. In the semi-classical approximation, the only relevant diagram is



that corresponds to the invariant amplitude

$$\mathcal{M} = \frac{g}{2\sqrt{2}} V_{fg} \bar{u}(k_1) \gamma^\mu (1 - \gamma_5) v(k_2) \epsilon_\mu(p), \quad (11.22)$$

where ϵ is the polarization vector of the W , and $p = k_1 + k_2$. We get

$$\sum_{\text{pol}} |\mathcal{M}|^2 = \frac{3g^2 |V_{fg}|^2}{4} \text{Tr} [\gamma^\mu (\not{p}_2 - m_d) \gamma^\nu (1 - \gamma_5) (\not{p}_1 + m_u)] \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_W^2} \right), \quad (11.23)$$

where we have summed over the three polarization states of the W boson using eq. (8.46), and we have inserted a factor of 3 that accounts for the three different colour states of the quark-antiquark pair. The term proportional to γ_5 gives zero contribution, because of the symmetry of the sum over W polarization. Thus, dropping terms proportional to an odd number of γ matrices,

$$\sum_{\text{pol}} |\mathcal{M}|^2 = \frac{3g^2}{4} |V_{fg}|^2 \text{Tr} [\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1 - m_u m_d \gamma^\mu \gamma^\nu] \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_W^2} \right). \quad (11.24)$$

The calculation is now straightforward; using the formulae obtained in Appendix C for the trace of products of two and four γ matrices we obtain

$$\sum_{\text{pol}} |\mathcal{M}|^2 = 3g^2 |V_{fg}|^2 \left[k_1 \cdot k_2 + 3m_u m_d + \frac{2p \cdot k_1 p \cdot k_2}{m_W^2} \right]. \quad (11.25)$$

The values of the scalar products can be obtained by taking the square of the momentum conservation relation, and using the mass-shell conditions $p^2 = m_w^2, k_1^2 = m_u^2, k_2^2 = m_d^2$. We find

$$k_1 \cdot k_2 = \frac{m_w^2 - m_u^2 - m_d^2}{2} \quad (11.26)$$

$$p \cdot k_1 = \frac{m_w^2 + m_u^2 - m_d^2}{2} \quad (11.27)$$

$$p \cdot k_2 = \frac{m_w^2 - m_u^2 + m_d^2}{2}. \quad (11.28)$$

The kinematically-allowed channels for W decay into a quark-antiquark pair include all combinations $u_f \bar{d}_g$, except those containing the top quark, whose mass is around 175 GeV, more than twice the W mass. All other quark masses are much smaller than m_w , the largest of them being $m_b \simeq 5$ GeV. We may therefore neglect fermion masses in eq. (11.25) to an excellent degree of accuracy. In this limit,

$$k_1 \cdot k_2 \simeq p \cdot k_1 \simeq p \cdot k_2 \simeq \frac{m_w^2}{2}, \quad (11.29)$$

and we find

$$\sum_{\text{pol}} |\mathcal{M}|^2 \simeq 3g^2 m_w^2 |V_{fg}|^2. \quad (11.30)$$

The differential decay rate is now immediately computed using eq. (4.35) and the expression eq. (4.25) for the two-body phase space. After averaging eq. (11.30) over the three polarization states of the W boson, we get

$$d\Gamma(W \rightarrow u_f \bar{d}_g) = \frac{3g^2 m_w}{96\pi} \frac{|V_{fg}|^2}{d \cos \theta} = \frac{3G_F m_w^3}{12\sqrt{2}\pi} \frac{|V_{fg}|^2}{d \cos \theta}, \quad (11.31)$$

where θ is the angle formed by the direction of the decay products and the z axis in the rest frame of the decaying W .

The total decay rate of the W boson is obtained by performing the (trivial) angular integration, summing over the allowed quark-antiquark channels, and adding the contributions of the three leptonic channels, given by eq. (11.31) with the replacement $3|V_{fg}|^2 \rightarrow 1$. The final result is

$$\Gamma_W = \frac{G_F m_w^3}{6\sqrt{2}\pi} \left[3 \sum_{f=u,c} \sum_{g=d,s,b} |V_{fg}|^2 + 3 \right]. \quad (11.32)$$

Neglecting, to a first approximation, the mixing of the third generation, we have

$$\sum_{f=u,c} \sum_{g=d,s,b} |V_{fg}|^2 \simeq \sum_{f=u,c} \sum_{g=d,s} |V_{fg}|^2 \simeq 2(\cos^2 \theta_c + \sin^2 \theta_c) = 2, \quad (11.33)$$

and therefore

$$\Gamma_W = \frac{3G_F m_W^3}{2\sqrt{2}\pi} \simeq 2.05 \text{ GeV}, \quad (11.34)$$

in good agreement with the measured value

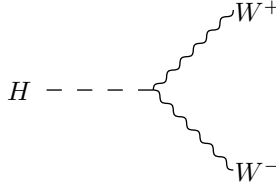
$$\Gamma_W^{\text{exp}} = 2.124 \pm 0.041 \text{ GeV}. \quad (11.35)$$

11.3 Higgs decay into a vector boson pair

In this Section, we consider the decay of a Higgs boson into a W^+W^- pair:

$$H(p) \rightarrow W^+(k_1, \epsilon_1) + W^-(k_2, \epsilon_2). \quad (11.36)$$

The invariant amplitude is given by the single diagram



and from eq. (10.10) we obtain

$$\mathcal{M} = \frac{2m_W^2}{v} \epsilon_1^* \cdot \epsilon_2^*. \quad (11.37)$$

Observe that the coupling of the Higgs boson to W bosons is proportional to the W mass, which is a general feature of Higgs couplings.

In a reference frame in which the decaying Higgs is at rest, the two W bosons are emitted with three-momenta $\pm \mathbf{k}$, in the same direction:

$$p = (m_H, \mathbf{0}); \quad k_1 = \left(\frac{m_H}{2}, \mathbf{k}\right); \quad k_2 = \left(\frac{m_H}{2}, -\mathbf{k}\right) \quad (11.38)$$

with

$$|\mathbf{k}| = \frac{m_H}{2} \sqrt{1 - \frac{4m_W^2}{m_H^2}}. \quad (11.39)$$

Since the Higgs boson is a spin-0 particle, the amplitude is invariant under rotations, and we may choose to orient the z axis in the direction of the three-momenta of the decay products. The polarization vectors ϵ_1, ϵ_2 in the rest frame of each of the W bosons are given in eq. (8.36). After a Lorentz boost to the Higgs rest frame, $\epsilon^{(1)}$ and $\epsilon^{(2)}$ are unchanged, while

$$\epsilon^{(3)}(k_1) = \frac{1}{m_W} \begin{pmatrix} -|\mathbf{k}| \\ 0 \\ 0 \\ m_H/2 \end{pmatrix}; \quad \epsilon^{(3)}(k_2) = \frac{1}{m_W} \begin{pmatrix} |\mathbf{k}| \\ 0 \\ 0 \\ m_H/2 \end{pmatrix}. \quad (11.40)$$

It follows that the amplitude $\mathcal{M}$ is nonzero only when $\epsilon_1 = \epsilon^{(i)}(k_1)$, $\epsilon_2 = \epsilon^{(i)}(k_2)$, $i = 1, 2, 3$. The corresponding amplitudes are

$$\mathcal{M}^{(11)} = \mathcal{M}^{(22)} = -\frac{2m_W^2}{v} \quad (11.41)$$

$$\mathcal{M}^{(33)} = -\frac{2m_W^2}{v} \frac{1}{m_W^2} \left(|\mathbf{k}|^2 + \frac{m_H^2}{4} \right) = -\frac{1}{v} (m_H^2 - 2m_W^2). \quad (11.42)$$

It is interesting to observe that $\mathcal{M}^{(11)}$ and $\mathcal{M}^{(22)}$ vanish for $m_W \rightarrow 0$ with v fixed (or equivalently in the limit $g \rightarrow 0$, in which gauge interactions are switched off), as one would expect on the basis that the HWW coupling is proportional to m_W , while $\mathcal{M}^{(33)}$ is nonzero.

The calculation of the decay rate proceeds in the usual way; we find

$$d\Gamma^{(i)} = \frac{1}{2m_H} \left| \mathcal{M}^{(ii)} \right|^2 \sqrt{1 - \frac{4m_W^2}{m_H^2}} \frac{1}{16\pi} d\cos\theta \quad (11.43)$$

and

$$\begin{aligned} \Gamma(H \rightarrow WW) &= \int \sum_{i=1}^3 d\Gamma^{(i)} \\ &= \frac{G_F m_H^3}{8\sqrt{2}\pi} \sqrt{1 - \frac{4m_W^2}{m_H^2}} \left[1 + \frac{12m_W^4}{m_H^4} - \frac{4m_W^2}{m_H^2} \right], \end{aligned} \quad (11.44)$$

where we have used $G_F = \frac{1}{\sqrt{2}v^2}$, eq. (10.13).

We would have obtained exactly the same result, up to a factor $1/2$ due to the identity of final particles, if we had computed the decay rate $\Gamma(H \rightarrow \gamma\gamma)$ of the scalar H into a pair of massive photons in the context of the abelian Higgs model of Section 8.2. One can check that, in the limit $m_\gamma = ev \rightarrow 0$ with v fixed, $\Gamma(H \rightarrow \gamma\gamma)$ is equal to the decay rate $\Gamma(H \rightarrow GG)$ of the scalar H into a pair of Goldstone bosons. This is a particular case of a general result, sometimes called the *equivalence theorem*: at energies much larger than the vector boson mass, the couplings of the longitudinal components of vector bosons are the same as those of the corresponding Goldstone scalars (recall that the decay rate into pairs of transversally polarized W 's vanishes as $m_W \rightarrow 0$.)

11.4 Weak neutral currents

The first experimental confirmation of the standard model was the observation of the effects of weak neutral currents, that allowed a measurement of $\sin\theta_W$

and therefore an estimate of the masses of the intermediate vector bosons W and Z . The relevant process is inclusive neutrino scattering off nucleon targets,

$$\text{NC : } \nu_\mu(p_1) + H(p) \rightarrow \nu_\mu(k_1) + X \quad (11.45)$$

$$\text{CC : } \nu_\mu(p_1) + H(p) \rightarrow \mu^-(k_1) + X, \quad (11.46)$$

where H is a nuclear target with the same number of protons and neutrons and X any hadronic final state. In the parton model, the cross section for a process with a hadron H of momentum p in the initial state is given by

$$\sigma(p) = \sum_q \int_0^1 dz f_q^H(z) \sigma_q(zp), \quad (11.47)$$

where q is a generic parton, $\sigma_q(zp)$ is the cross section for the same process with parton q with momentum zp in the initial state, and $f_q^H(z)$ are universal functions, that characterize the structure of the hadron H : $f_q^H(z) dz$ is the probability that parton q carries a fraction of the hadron momentum between z and $z + dz$. The cross section for the process eq. (11.45) is therefore given by

$$\begin{aligned} \sigma_\nu^{(\text{NC})}(p) &= \frac{1}{2} \left[\sigma_{\nu P}^{(\text{NC})}(p) + \sigma_{\nu N}^{(\text{NC})}(p) \right] \\ &= \int_0^1 dz \left[\frac{f_u^P(z) + f_u^N(z)}{2} \sigma_{\nu u}^{(\text{NC})}(zp) + \frac{f_d^P(z) + f_d^N(z)}{2} \sigma_{\nu d}^{(\text{NC})}(zp) \right] \\ &= \int_0^1 dz \frac{f_u(z) + f_d(z)}{2} \left[\sigma_{\nu u}^{(\text{NC})}(zp) + \sigma_{\nu d}^{(\text{NC})}(zp) \right], \end{aligned} \quad (11.48)$$

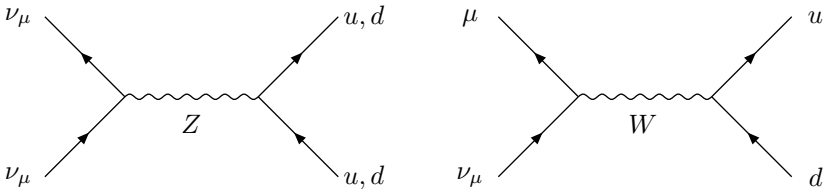
where $f_{u,d}(z) = f_{u,d}^P(z) = f_{u,d}^N(z)$. A similar expression is found for $\sigma_\nu^{(\text{CC})}(p)$.

In the case of the processes in eqs. (11.45,11.46), the relevant parton subprocesses are

$$\text{NC : } \nu_\mu(p_1) + q(p_2) \rightarrow \nu_\mu(k_1) + q(k_2); \quad q = u, d \quad (11.49)$$

$$\text{CC : } \nu_\mu(p_1) + d(p_2) \rightarrow \mu^-(k_1) + u(k_2) \quad (11.50)$$

The relevant lowest-order diagrams are



and the relevant Feynman rules are read off the following terms of the standard model Lagrangian:

$$\begin{aligned}\mathcal{L} = & \frac{g}{4 \cos \theta_w} \left[\bar{\nu}_\mu \gamma_\mu (1 - \gamma_5) \nu_\mu + \bar{u} \gamma_\mu (V_u - A_u \gamma_5) u + \bar{d} \gamma_\mu (V_d - A_d \gamma_5) d \right] Z^\mu \\ & + \frac{g}{2\sqrt{2}} [\bar{\nu}_\mu \gamma^\mu (1 - \gamma_5) e + \bar{u} \gamma^\mu (1 - \gamma_5) V_{ud} d] W_\mu^+ \\ & + \frac{g}{2\sqrt{2}} [\bar{e} \gamma^\mu (1 - \gamma_5) \nu_\mu + \bar{d} \gamma^\mu (1 - \gamma_5) V_{ud}^* u] W_\mu^-, \quad (11.51)\end{aligned}$$

where

$$V_u = 1 - \frac{8}{3} \sin^2 \theta_w; \quad A_u = 1. \quad (11.52)$$

$$V_d = -1 + \frac{4}{3} \sin^2 \theta_w; \quad A_d = -1. \quad (11.53)$$

Neglecting all fermion masses we find

$$\begin{aligned}\mathcal{M}(\nu q \rightarrow \nu q) = & \frac{g^2}{16 \cos^2 \theta_w} \frac{1}{s - m_Z^2} \\ & \bar{u}(k_1) \gamma^\mu (1 - \gamma_5) u(p_1) \bar{u}(k_2) \gamma_\mu (V_q - A_q \gamma_5) u(p_2), \quad (11.54)\end{aligned}$$

where $s = 2p_1 \cdot p_2$, and therefore

$$\begin{aligned}\sum_{\text{pol}} |\mathcal{M}(\nu q \rightarrow \nu q)|^2 = & \left(\frac{g^2}{16 \cos^2 \theta_w} \right)^2 \frac{1}{(s - m_Z^2)^2} \\ & 2 \text{Tr} [\gamma^\mu \not{p}_1 \gamma^\nu \not{k}_1 (1 + \gamma_5)] \text{Tr} [\gamma_\mu \not{p}_2 \gamma_\nu \not{k}_2 (V_q^2 + A_q^2 + 2V_q A_q \gamma_5)]. \quad (11.55)\end{aligned}$$

The amplitude for the conjugated process is given by

$$\begin{aligned}\mathcal{M}(\bar{\nu} q \rightarrow \bar{\nu} q) = & \frac{g^2}{16 \cos^2 \theta_w} \frac{1}{s - m_Z^2} \\ & \bar{v}(p_1) \gamma^\mu (1 - \gamma_5) v(k_1) \bar{u}(k_2) \gamma_\mu (V_q - A_q \gamma_5) u(p_2), \quad (11.56)\end{aligned}$$

whose modulus squared is obtained from eq. (11.55) by the replacement $p_1 \leftrightarrow k_1$, that amounts to reversing the sign of γ_5 in the first trace:

$$\begin{aligned}\sum_{\text{pol}} |\mathcal{M}(\bar{\nu} q \rightarrow \bar{\nu} q)|^2 = & \left(\frac{g^2}{16 \cos^2 \theta_w} \right)^2 \frac{1}{(s - m_Z^2)^2} \\ & 2 \text{Tr} [\gamma^\mu \not{p}_1 \gamma^\nu \not{k}_1 (1 - \gamma_5)] \text{Tr} [\gamma_\mu \not{p}_2 \gamma_\nu \not{k}_2 (V_q^2 + A_q^2 + 2V_q A_q \gamma_5)]. \quad (11.57)\end{aligned}$$

An analogous calculation yields the result

$$\sum_{\text{pol}} |\mathcal{M}(\nu d \rightarrow \mu u)|^2 = \left(\frac{g^2 |V_{ud}|}{8} \right)^2 \frac{1}{(s - m_W^2)^2} \quad (11.58)$$

$$\begin{aligned}
& 4\text{Tr} [\gamma^\mu \not{p}_1 \gamma^\nu \not{k}_1 (1 + \gamma_5)] \text{Tr} [\gamma_\mu \not{p}_2 \gamma_\nu \not{k}_2 (1 + \gamma_5)] \cdot \\
\sum_{\text{pol}} |\mathcal{M}(\bar{\nu}u \rightarrow \mu d)|^2 &= \left(\frac{g^2 |V_{ud}|}{8} \right)^2 \frac{1}{(s - m_w^2)^2} \\
& 4\text{Tr} [\gamma^\mu \not{p}_1 \gamma^\nu \not{k}_1 (1 - \gamma_5)] \text{Tr} [\gamma_\mu \not{p}_2 \gamma_\nu \not{k}_2 (1 + \gamma_5)] \cdot
\end{aligned} \tag{11.59}$$

The parton cross sections are given by the general formula eq. (4.31); in our cases,

$$\sigma = \frac{1}{8s} \int d\phi_2(p_1 + p_2; k_1, k_2) |\mathcal{M}|^2, \tag{11.60}$$

where we have inserted a factor of $1/4$ to average over initial spin states, and we have assumed fermions to be massless. When $s \ll m_z^2, m_w^2$, the squared matrix elements are linear in the initial parton momentum p_2 ; since s is also linear in p_2 , it follows that the parton cross sections are independent of the momentum fraction z carried by the initial parton, $p_2 = zp$. Thus, using eq. (11.48) we find

$$\begin{aligned}
\sigma_{\bar{\nu}}^{(\text{NC})} - \sigma_{\nu}^{(\text{NC})} &= \int_0^1 dz \frac{f_u(z) + f_d(z)}{2} \left[\sigma_{\bar{\nu}u}^{(\text{NC})} - \sigma_{\nu u}^{(\text{NC})} + \sigma_{\bar{\nu}d}^{(\text{NC})} - \sigma_{\nu d}^{(\text{NC})} \right] \\
&= \int_0^1 dz \frac{f_u(z) + f_d(z)}{2} \frac{1}{S} \left(\frac{g^2}{16m_z^2 \cos^2 \theta_w} \right)^2 (-V_u A_u - V_d A_d) \\
&\times \int d\phi_2 \text{Tr} [\gamma^\mu \not{p}_1 \gamma^\nu \not{k}_1 \gamma_5] \text{Tr} [\gamma_\mu \not{p}_2 \gamma_\nu \not{k}_2 \gamma_5]
\end{aligned} \tag{11.61}$$

where $S = 2p_1 \cdot p$, and similarly

$$\begin{aligned}
\sigma_{\bar{\nu}}^{(\text{CC})} - \sigma_{\nu}^{(\text{CC})} &= \int_0^1 dz \frac{f_u(z) + f_d(z)}{2} \left[\sigma_{\bar{\nu}u}^{(\text{CC})} - \sigma_{\nu d}^{(\text{CC})} \right] \\
&= \int_0^1 dz \frac{f_u(z) + f_d(z)}{2} \frac{1}{S} \left(\frac{g^2 |V_{ud}|}{8m_w^2} \right)^2 \\
&\times \int d\phi_2 \text{Tr} [\gamma^\mu \not{p}_1 \gamma^\nu \not{k}_1 \gamma_5] \text{Tr} [\gamma_\mu \not{p}_2 \gamma_\nu \not{k}_2 \gamma_5].
\end{aligned} \tag{11.62}$$

Most factors, including those that depend on the parton distribution functions, cancel in the ratio

$$R = \frac{\sigma_{\bar{\nu}}^{(\text{NC})} - \sigma_{\nu}^{(\text{NC})}}{\sigma_{\bar{\nu}}^{(\text{CC})} - \sigma_{\nu}^{(\text{CC})}}, \tag{11.63}$$

which therefore provides a direct measure of $\sin \theta_w$. We find

$$R = -\frac{V_u A_u + V_d A_d}{4|V_{ud}|^2} = \frac{1 - 2\sin^2 \theta_w}{2|V_{ud}|^2}, \tag{11.64}$$

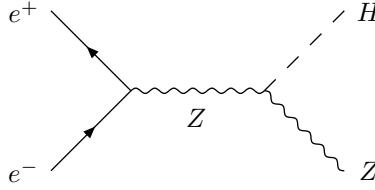
which is the so-called Paschos-Wolfenstein relation.

11.5 Higgs production in e^+e^- collisions

The present experimental lower bound on the mass of the standard model Higgs boson, about 115 GeV, is obtained from non-observation of Higgs bosons in electron-positron collisions in a range of center-of-mass energies between about 90 and 210 GeV. In this kind of experiments, Higgs bosons are produced in the process

$$e^-(p_1) + e^+(p_2) \rightarrow H(k_1) + Z^0(k_2, \epsilon), \quad (11.65)$$

that takes place in the semi-classical approximation through the diagram



The relevant terms in the interaction Lagrangian are the weak neutral-current coupling of the electron with the Z boson, and the HZZ coupling, which we rewrite here:

$$\mathcal{L} = \frac{g}{4 \cos \theta_w} \bar{e} \gamma_\mu (-1 + 4 \sin^2 \theta_w + \gamma_5) e Z^\mu + \frac{m_z^2}{v} H Z^\mu Z_\mu. \quad (11.66)$$

This gives an invariant amplitude

$$\mathcal{M} = \frac{g}{\cos \theta_w} \frac{m_z^2}{2v} \epsilon^{*\mu}(k_2) \frac{g_{\mu\nu} - \frac{q_\mu q_\nu}{m_z^2}}{q^2 - m_z^2} \bar{v}(p_2) \gamma^\nu (V - A\gamma_5) u(p_1), \quad (11.67)$$

where $q = p_1 + p_2$ and

$$V = -1 + 4 \sin^2 \theta_w; \quad A = -1. \quad (11.68)$$

It will be convenient to express the amplitude in terms of the Fermi constant G_F ; to this purpose, we observe that

$$\frac{g}{\cos \theta_w} \frac{m_z}{2v} = \sqrt{2} G_F m_z^2. \quad (11.69)$$

Neglecting the electron mass, which is perfectly adequate in this energy regime, we can ignore the $q_\mu q_\nu$ term in the internal propagator, and compute the unpolarized squared amplitude in the usual way:

$$\sum_{\text{pol}} |\mathcal{M}|^2 = \frac{2 G_F^2 m_z^6}{(s - m_z^2)^2} \left(-g^{\mu\nu} + \frac{k_2^\mu k_2^\nu}{m_z^2} \right) \text{Tr} [\gamma_\mu (V - A\gamma_5) \not{p}_1 \gamma_\nu (V - A\gamma_5) \not{p}_2] \quad (11.70)$$

where $s = (p_1 + p_2)^2$. Terms proportional to γ_5 are proportional to the product of an antisymmetric ϵ tensor with the sum over Z polarization, which is symmetric; hence, they do not contribute to the unpolarized cross section, and eq. (11.70) becomes

$$\begin{aligned} \sum_{\text{pol}} |\mathcal{M}|^2 &= \frac{2 G_F^2 m_Z^6 (V^2 + A^2)}{(s - m_Z^2)^2} \left(-g^{\mu\nu} + \frac{k_2^\mu k_2^\nu}{m_Z^2} \right) \text{Tr} [\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2] \\ &= \frac{4 G_F^2 m_Z^6 (V^2 + A^2)}{(s - m_Z^2)^2} \left[s + \frac{(t - m_Z^2)(u - m_Z^2)}{m_Z^2} \right], \end{aligned} \quad (11.71)$$

where we have used eq. (C.8) and we have defined

$$t = (p_1 - k_2)^2; \quad u = (p_2 - k_2)^2. \quad (11.72)$$

The invariants s, t, u are related by

$$s + t + u = m_Z^2 + m_H^2 \quad (11.73)$$

through energy-momentum conservation.

In the center-of-mass frame of the colliding leptons we have

$$t = m_Z^2 - \sqrt{s} (E_{k_2} - |\mathbf{k}_2| \cos \theta) \quad (11.74)$$

$$u = m_Z^2 - \sqrt{s} (E_{k_2} + |\mathbf{k}_2| \cos \theta), \quad (11.75)$$

where θ is the scattering angle of the Z boson in the final state, and the value of $|\mathbf{k}_2|$ is found by solving the energy conservation constraint

$$\sqrt{s} = \sqrt{|\mathbf{k}_2|^2 + m_Z^2} + \sqrt{|\mathbf{k}_2|^2 + m_H^2}; \quad (11.76)$$

we find

$$\begin{aligned} |\mathbf{k}_2|^2 &= \frac{\lambda(s, m_Z^2, m_H^2)}{4s}; \quad \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \\ E_{k_2} &= \frac{s + m_Z^2 - m_H^2}{2\sqrt{s}}. \end{aligned} \quad (11.77)$$

Thus,

$$\sum_{\text{pol}} |\mathcal{M}|^2 = \frac{4 G_F^2 m_Z^4 (V^2 + A^2) s}{(s - m_Z^2)^2} \left(m_Z^2 + E_{k_2}^2 - |\mathbf{k}_2|^2 \cos^2 \theta \right). \quad (11.78)$$

In terms of center-of-mass quantities, the two body phase-space measure is given by

$$d\phi_2(p_1 + p_2; k_1, k_2) = \frac{1}{8\pi} \frac{|\mathbf{k}_2|}{\sqrt{s}} d\cos\theta. \quad (11.79)$$

The differential cross section is therefore given by

$$d\sigma = \frac{G_F^2 m_z^4 (V^2 + A^2)}{16\pi} \frac{|\mathbf{k}_2|}{\sqrt{s}} \frac{m_z^2 + E_{k_2}^2 - |\mathbf{k}_2|^2 \cos^2 \theta}{(s - m_z^2)^2} d\cos \theta \quad (11.80)$$

and the total cross section by

$$\sigma(e^+e^- \rightarrow HZ) = \frac{G_F^2 m_z^4 (V^2 + A^2)}{12\pi} \frac{|\mathbf{k}_2|}{\sqrt{s}} \frac{3m_z^2 + |\mathbf{k}_2|^2}{(s - m_z^2)^2}. \quad (11.81)$$

Neutrino masses and mixing

In the original formulation of the standard model, presented in Chapters 7, 8 and 9, neutrinos are massless particles. This feature is well motivated by direct experimental upper bounds on neutrino masses:

$$m_{\nu_e} \leq 3 \text{ eV}; \quad m_{\nu_\mu} \leq 0.19 \text{ MeV}; \quad m_{\nu_\tau} \leq 18.2 \text{ MeV}, \quad (12.1)$$

and even if observations indicate that neutrino masses are in fact non-zero, the approximation $m_\nu \ll m_f$, where f is any fermion in the standard model spectrum, is extremely good for most applications. In view of experimental results, however, it is interesting to study the possible ways neutrino mass terms can be consistently introduced.

The absence of neutrino mass terms in the standard model is related to the absence of right-handed components for the neutrino fields; these would be assigned (as any other right-handed fermion) to the singlet representation of $SU(2)$, and would have zero charge and hypercharge. Therefore, the corresponding particles do not participate in electroweak interactions, and can be simply omitted in the context of the standard model.

One may nevertheless assume that right-handed neutrinos do exist, possibly participating in interactions that are not observed at the energies of present-time experiments. This assumption brings us outside the standard model, and has far-reaching consequences. Assuming for the moment the existence of only one lepton generation, we introduce a covariant derivative term

$$\bar{\nu}_R i \not{D} \nu_R \equiv \bar{\nu}_R i \not{\partial} \nu_R. \quad (12.2)$$

In the presence of a right-handed neutrino field, a Dirac mass term is generated through the Higgs mechanism by a Yukawa coupling similar to that of up-type quarks:

$$-h_N \left[\bar{\ell}_L \epsilon \left(\phi + \frac{v}{\sqrt{2}} \right)^* \nu_R - \bar{\nu}_R \left(\phi + \frac{v}{\sqrt{2}} \right)^T \epsilon \ell_L \right], \quad (12.3)$$

that contains a term

$$-m (\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L); \quad m = \frac{h_N v}{\sqrt{2}} \quad (12.4)$$

in full analogy with the case of quarks. As observed in Section 9.3, the Yukawa coupling in eq. (12.3) cannot be generated by radiative corrections, because it breaks explicitly the global accidental symmetry

$$\nu_R \rightarrow e^{i\phi} \nu_R \quad (12.5)$$

of the kinetic term eq. (12.2), which is the only other place in the Lagrangian density where ν_R appears.

If eq. (12.4) were the only possible neutrino mass term, then the constant h_N should be smaller than the corresponding constants for charged leptons by several orders of magnitude, in order to obey the severe bounds eq. (12.1); for example,

$$\frac{h_N}{h_e} = \frac{m}{m_e} \lesssim 10^{-6}. \quad (12.6)$$

In general, however, this is not the case. Because of its transformation properties with respect to gauge transformations, right-handed neutrinos also admit a Majorana mass term:

$$-\frac{1}{2} M (\bar{\nu}_R^c \nu_R + \bar{\nu}_R \nu_R^c) \quad (12.7)$$

where ν_R^c is the charge-conjugated spinor.¹ As already observed in Section 5.2, this interesting feature is not shared by any other fermion field in the standard model, because of the limitations imposed by gauge invariance. In particular, it is not possible to build a Majorana mass term for left-handed neutrinos by means of renormalizable terms in the Lagrangian density. The Majorana mass parameter M , contrary to the Dirac mass m , can assume arbitrarily large values, since no extra symmetry is recovered in the limit $M = 0$. Furthermore, Majorana mass terms violate lepton number conservation; thus, we must assume that M is large enough, in order that lepton number violation effects, typically suppressed by inverse powers of M , are compatible with observations. It is natural to assume that M is of the order of the energy scale characteristic of the unknown phenomena (e.g. the effects of grand unification) experienced by right-handed neutrinos.

The most general neutrino mass term can therefore be written in the form

$$\mathcal{L}_{\nu \text{ mass}} = -\frac{1}{2} (\bar{\nu}_L^c \ \bar{\nu}_R) \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{h.c.}, \quad (12.8)$$

¹ It is easy to check that, with the definitions given in eqs. (6.9-6.11), the effect of charge-conjugation as defined in eq. (5.50) on a Dirac spinor ψ is $\psi \rightarrow \psi^c = \gamma_1 \gamma_3 \bar{\psi}^T$. If the arbitrary phases are constrained as in electrodynamics, then one has $\psi \rightarrow \psi^c = \gamma_0 \gamma_2 \bar{\psi}^T$.

where we have used $\bar{\nu}_L^c \nu_R^c = \bar{\nu}_R \nu_L$. We have seen in Section 5.2 that the mass matrix that appears in eq. (12.8) can be diagonalized by a linear transformation

$$\begin{pmatrix} 0 & m \\ m & M \end{pmatrix} = \mathcal{U}^T \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \mathcal{U} \quad (12.9)$$

with $\mathcal{U}$ unitary, and m_1, m_2 real and positive. In our case, we find

$$\mathcal{U} = \begin{pmatrix} i \cos \theta & -i \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}; \quad \tan 2\theta = \frac{2m}{M} \quad (12.10)$$

and

$$m_1 = \frac{1}{2} \left(M + \sqrt{M^2 + 4m^2} \right); \quad m_2 = \frac{1}{2} \left(M - \sqrt{M^2 + 4m^2} \right). \quad (12.11)$$

For $m \ll M$, $\theta \simeq m/M$, and

$$m_1 \simeq M; \quad m_2 \simeq \frac{m^2}{M}. \quad (12.12)$$

This mechanism, usually called the *see-saw* mechanism, provides a natural explanation of the observed smallness of neutrino masses: one of the two mass eigenstates in the neutrino sector is extremely heavy, and has no observable effects on physics at the weak scale, while the other has a mass which is suppressed with respect to typical fermion masses m by a factor m/M . In this way, light neutrinos arise without the need of assuming unnaturally small values of the Yukawa couplings.

The mass term eq. (12.8) takes the form

$$\mathcal{L}_{\nu \text{ mass}} = -\frac{1}{2} m_1 (\bar{\nu}_1^c \nu_1 + \bar{\nu}_1 \nu_1^c) - \frac{1}{2} m_2 (\bar{\nu}_2^c \nu_2 + \bar{\nu}_2 \nu_2^c), \quad (12.13)$$

where

$$\nu_1 = \nu_L \sin \theta + \nu_R^c \cos \theta \quad (12.14)$$

$$\nu_2 = -i\nu_L \cos \theta + i\nu_R^c \sin \theta. \quad (12.15)$$

For $m \ll M$, the heavy eigenvector ν_1 is predominantly ν_R^c , while the light eigenvector ν_2 approximately coincides with the ordinary left-handed neutrino.

In the general case, there are n different species of left-handed neutrinos, with $n = 3$ according to present knowledge, and an undetermined number k of right-handed neutrinos; correspondingly, m is in general a $k \times n$ matrix, and M a $k \times k$ matrix. After diagonalization in the generation space, the phenomenon of flavour mixing takes place in a similar way as in the quark sector. For simplicity, we will consider the case $k = n$, when there are as many right-handed as left-handed neutrinos. In this case, the Yukawa interaction eq. (12.3) should be modified to account for the generation structure. Specifically, we replace eq. (9.18) by

$$\begin{aligned} \mathcal{L}_Y^{\text{lept}} = & - \left[\bar{\ell}'_L \left(\phi + \frac{v}{\sqrt{2}} \right) h'_e e'_R + \bar{e}'_R \left(\phi + \frac{v}{\sqrt{2}} \right)^\dagger h'^\dagger_e \ell'_L \right] \\ & - \left[\bar{\ell}'_L \epsilon \left(\phi + \frac{v}{\sqrt{2}} \right)^* h'_N \nu'_R - \bar{\nu}'_R \left(\phi + \frac{v}{\sqrt{2}} \right)^T \epsilon h'^\dagger_N \ell'_L \right], \end{aligned} \quad (12.16)$$

where we have introduced an array ν'^f_R ; $f = 1, \dots, n$ of right-handed neutrinos, and h'_N is a generic complex constant matrix. Similarly, the Majorana mass terms for right-handed neutrinos take the form

$$-\frac{1}{2} \left(\bar{\nu}'^c_R M \nu'_R + \bar{\nu}'_R M^\dagger \nu'^c_R \right), \quad (12.17)$$

where M is a matrix in lepton flavour space. Lepton flavour mixing (and, possibly, CP violation) therefore originates both from the diagonalization of the Yukawa coupling h'_N , which gives rise to the leptonic analogous of the CKM matrix, and from the diagonalization of the Majorana mass M .

In general, single-particle neutrino quantum states with definite lepton flavour are linear combinations of mass eigenstates. Neutrinos are produced in weak-interaction processes with a definite flavour: for example, β decays of nuclei in the Sun produce electron neutrinos. Consider a neutrino beam of definite flavour, produced at some initial time $t = 0$; denoting flavour eigenstates by Greek indices, and mass eigenstates by Latin indices, we have

$$|\nu_\alpha\rangle = \sum_{i=1}^n U_{\alpha i}^* |\nu_i\rangle, \quad (12.18)$$

with U a unitary matrix. Each definite-mass component of such neutrino states has a time evolution, which in the rest frame is described by

$$|\nu_i(\tau)\rangle = e^{-im_i\tau} |\nu_i(0)\rangle, \quad (12.19)$$

or, in the laboratory frame,

$$|\nu_i(t)\rangle = e^{-i(E_i t - p_i L)} |\nu_i(0)\rangle, \quad (12.20)$$

where L is the distance travelled in the time interval t . We now exploit the fact that neutrinos are almost massless. This gives

$$L \simeq t; \quad E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2E}, \quad (12.21)$$

where $E \simeq p_i \simeq p_j$. Hence,

$$|\nu_\alpha(L)\rangle \simeq \sum_{i=1}^n U_{\alpha i}^* \exp\left(-i\frac{m_i^2}{2E}L\right) |\nu_i(0)\rangle. \quad (12.22)$$

The probability amplitude of observing the flavour β at distance L is given by

$$\begin{aligned}\langle \nu_\beta | \nu_\alpha(L) \rangle &= \sum_{i=1}^n U_{\alpha i}^* \exp\left(-i \frac{m_i^2}{2E} L\right) \sum_{j=1}^n U_{\beta j} \langle \nu_j | \nu_i \rangle \\ &= \sum_{i=1}^n \xi_i^{\alpha\beta} \exp(-i \epsilon_i L),\end{aligned}\quad (12.23)$$

where we have used the unitarity of U , and we have defined

$$\xi_i^{\alpha\beta} = U_{\alpha i}^* U_{\beta i}; \quad \epsilon_i = \frac{m_i^2}{2E}.\quad (12.24)$$

The corresponding probability is given by

$$\begin{aligned}P_{\alpha\beta}(L) &= |\langle \nu_\beta | \nu_\alpha(L) \rangle|^2 = \delta_{\alpha\beta} - 4 \sum_{i=1}^n \sum_{j=i+1}^n \operatorname{Re} \left(\xi_i^{\alpha\beta} \xi_j^{*\alpha\beta} \right) \sin^2 \frac{1}{2} (\epsilon_j - \epsilon_i) L \\ &\quad - 2 \sum_{i=1}^n \sum_{j=i+1}^n \operatorname{Im} \left(\xi_i^{\alpha\beta} \xi_j^{*\alpha\beta} \right) \sin(\epsilon_j - \epsilon_i) L.\end{aligned}\quad (12.25)$$

Observe that $P_{\alpha\beta}$ is unchanged if one replaces $U \rightarrow U^*$ and $\alpha \leftrightarrow \beta$:

$$P(\nu_\alpha \rightarrow \nu_\beta; U^*) = P(\nu_\beta \rightarrow \nu_\alpha; U).\quad (12.26)$$

On the other hand, CPT invariance implies

$$P(\nu_\beta \rightarrow \nu_\alpha; U) = P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta; U).\quad (12.27)$$

Hence,

$$P(\nu_\alpha \rightarrow \nu_\beta; U^*) = P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta; U),\quad (12.28)$$

or in other words neutrino oscillation probabilities can only differ from anti-neutrino oscillation probabilities if $U \neq U^*$, which is also a condition for CP violation.

The above computation of $P_{\alpha\beta}$ is unsatisfactory on many respects. Quantum states with definite momentum have an infinite uncertainty in position, and therefore it makes no sense to talk about observation at distance L . A rigorous treatment requires defining neutrino states as wave packets, which is equivalent to studying neutrino beams with a finite energy spread. The initial neutrino wave packet decomposes into mass-eigenstate wave packets, each travelling with a different velocity, given by the average over the packet of the ratio of $\mathbf{p}_i/E$, and the oscillations due to interference among the different components is lost at a distance at which the wave packets cease to overlap. As long as this is not the case, the oscillation probability is correctly given by eq. (12.25).

Equation (12.25) takes a simple form if one assumes CP invariance in the leptonic sector, and that only two flavours are involved. In this case, the last term in eq. (12.25) vanishes, and the matrix U is a real orthogonal matrix, depending on a single mixing angle θ_{12} . We have

$$\xi_1^{12} = \xi_1^{21} = -\cos\theta_{12} \sin\theta_{12} \quad (12.29)$$

$$\xi_2^{12} = \xi_2^{21} = +\cos\theta_{12} \sin\theta_{12}. \quad (12.30)$$

and therefore

$$P_{12}(L) = P_{21}(L) = \sin^2 2\theta_{12} \sin^2 \frac{L\Delta m_{12}^2}{4E}. \quad (12.31)$$

The results of neutrino oscillation experiments are usually displayed in the form of allowed regions in the $(\Delta m_{12}^2, \theta_{12})$ plane. The following units are often adopted:

$$\frac{L\Delta m^2}{4E} \simeq 1.27 \frac{\Delta m^2(\text{eV}^2) L(\text{km})}{4E(\text{GeV})}. \quad (12.32)$$

A

Large-time evolution of the free field

Let us consider a negative-frequency free-field solution of the equation of motion for a real scalar field:

$$F(\mathbf{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\mathbf{p} \frac{\phi(\mathbf{p})}{\sqrt{2E_p}} e^{i(\mathbf{p} \cdot \mathbf{r} - E_p t)} \equiv \int d\mathbf{p} \psi(\mathbf{p}) e^{i(\mathbf{p} \cdot \mathbf{r} - E_p t)}, \quad (\text{A.1})$$

where $\phi(\mathbf{p})$ is a linear combination of wave packets, assumed to be sufficiently regular (typically, of gaussian shape.) For $t \rightarrow \pm\infty$ and fixed $\mathbf{r}$, the integral vanishes faster than any power of t , because the integral over momenta of the phase factor averages to zero. If however the position $\mathbf{r}$ is not held fixed, there is in general a region in the space of momenta where the phase factor is stationary; this region gives the dominant contribution to the integral in the large-time limit. Thus, we take $\mathbf{r} = \mathbf{v}t$ and we consider the limit of eq. (A.1) for large t at constant $\mathbf{v}$:

$$\lim_{t \rightarrow \infty} \int d\mathbf{p} \psi(\mathbf{p}) e^{i(\mathbf{p} \cdot \mathbf{v} - E_p)t}. \quad (\text{A.2})$$

The condition of stationary phase

$$\frac{\partial}{\partial \mathbf{p}} (\mathbf{p} \cdot \mathbf{v} - E_p) = 0 \rightarrow \mathbf{v} = \frac{\partial}{\partial \mathbf{p}} E_p = \frac{\mathbf{p}}{E_p} \quad (\text{A.3})$$

gives

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-v^2}} \quad , \quad E = \frac{m}{\sqrt{1-v^2}}. \quad (\text{A.4})$$

and expanding the phase in powers of $\mathbf{q} \equiv \mathbf{p} - \frac{m\mathbf{v}}{\sqrt{1-v^2}}$ up to second order we get

$$\mathbf{p} \cdot \mathbf{v} - E_p = -\sqrt{1-v^2} \left[m - \frac{1}{2m} (q^2 - (\mathbf{q} \cdot \mathbf{v})^2) \right] + O(q^3). \quad (\text{A.5})$$

The asymptotic limit of the integral is therefore

$$\begin{aligned}
\int d\mathbf{p} \, \psi(\mathbf{p}) e^{i(\mathbf{p} \cdot \mathbf{v} - E_p)t} &\underset{t \rightarrow \infty}{\sim} e^{-im\sqrt{1-v^2}t} \psi\left(\frac{m\mathbf{v}}{\sqrt{1-v^2}}\right) \int d\mathbf{q} e^{-i\sqrt{1-v^2} \frac{q^2 - (\mathbf{q} \cdot \mathbf{v})^2}{2m} t} \\
&= e^{-im\sqrt{1-v^2}t} \psi\left(\frac{m\mathbf{v}}{\sqrt{1-v^2}}\right) \left(\frac{2\pi m}{it}\right)^{\frac{3}{2}} \frac{1}{(1-v^2)^{\frac{5}{4}}}.
\end{aligned} \tag{A.6}$$

With the same technique, we can obtain the following interesting result:

$$\int d\mathbf{p} e^{-\frac{(\mathbf{p}-\mathbf{k})^2}{\sigma^2}} e^{\pm iE_p t} \underset{t \rightarrow \infty}{\sim} e^{\pm imt - \frac{k^2}{2\sigma^2}} \left(\pm \frac{2\pi im}{t}\right)^{\frac{3}{2}}. \tag{A.7}$$

B

Scattering from an external density

A simple and interesting application of our formulae eqs. (3.89,3.100) is the computation of scattering amplitudes in a model with interaction given by

$$\mathcal{L}_I = \frac{g(x)}{2} \phi^2(x). \quad (\text{B.1})$$

In this model, the field equations are linear, and the function $g(x)$ plays a role which is similar to that of a potential in the Schrödinger equation. The solution of eq. (3.100) is

$$\begin{aligned} S \left[\phi^{(\text{as})} \right] &= -\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \int d^4x g(x) \phi^{(\text{as})}(x) \left((\Delta * g)^n \phi^{(\text{as})} \right)(x) \\ &\equiv -\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \int d^4x \prod_{i=1}^n dy_i \phi^{(\text{as})}(x) g(x) \Delta(x - y_1) g(y_1) \\ &\quad \Delta(y_1 - y_2) g(y_2) \cdots \Delta(y_{n-1} - y_n) g(y_n) \phi^{(\text{as})}(y_n). \end{aligned} \quad (\text{B.2})$$

Thus, S is quadratic in $\phi^{(\text{as})}(x)$, and describes three possible processes: two-particle annihilation, pair production, and single-particle scattering. To first order in g , one has

$$S \left[\phi^{(\text{as})} \right] = -\frac{1}{2} \int dx g(x) \left(\phi^{(\text{as})}(x) \right)^2, \quad (\text{B.3})$$

and therefore the scattering amplitude

$$S_{\mathbf{p} \rightarrow \mathbf{k}}[g] = - \int dx g(x) \frac{e^{i(\mathbf{k}-\mathbf{p}) \cdot \mathbf{x}}}{2(2\pi)^3 \sqrt{E_p E_k}} \equiv -\pi \frac{\tilde{g}(\mathbf{p} - \mathbf{k})}{\sqrt{E_p E_k}} \quad (\text{B.4})$$

while the annihilation amplitude for a pair of particles with momenta p_1 and p_2 is

$$S_{\mathbf{p}_1, \mathbf{p}_2 \rightarrow 0}[g] = -\pi \frac{\tilde{g}(\mathbf{p}_1 + \mathbf{p}_2)}{\sqrt{E(p_1)E(p_2)}}, \quad (\text{B.5})$$

and finally the pair production amplitude is

$$S_{0 \rightarrow \mathbf{k}_1, \mathbf{k}_2}[g] = -\pi \frac{\tilde{g}(-\mathbf{k}_1 - \mathbf{k}_2)}{\sqrt{E_{\mathbf{k}_1} E_{\mathbf{k}_2}}}. \quad (\text{B.6})$$

We now consider the case when

$$g(x) = g_1(x) + g_2(x), \quad (\text{B.7})$$

with

$$\theta(x_2^0 - x_1^0) g_1(x_1) g_2(x_2) = g_1(x_1) g_2(x_2), \quad (\text{B.8})$$

that is, g_1 acts before g_2 . Selecting in the scattering amplitude due to $g(x)$ the first order terms in g_1 and g_2 , one has

$$\begin{aligned} S_{1,2} &= \int d^4x g_1(x) \phi^{(\text{as})}(x) \left(\Delta \circ g_2 \phi^{(\text{as})} \right)(x) \\ &= \int dx_1 dx_2 g_1(x_1) \phi^{(\text{as})}(x_1) \Delta(x_1 - x_2) g_2(x_2) \phi^{(\text{as})}(x_2). \end{aligned} \quad (\text{B.9})$$

Replacing

$$\phi^{(\text{as})}(x) = \frac{e^{ik \cdot x}}{\sqrt{2E_k(2\pi)^3}} + \frac{e^{-ip \cdot x}}{\sqrt{2E_p(2\pi)^3}}, \quad (\text{B.10})$$

and recalling eqs. (B.8) and (3.70), we find

$$\begin{aligned} S_{1,2} &= \frac{\pi^2}{\sqrt{E_p E_k}} \int \frac{d\mathbf{q}}{E_q} [\tilde{g}_1(\mathbf{p} - \mathbf{q}) \tilde{g}_2(\mathbf{q} - \mathbf{k}) + \tilde{g}_1(-\mathbf{k} - \mathbf{q}) \tilde{g}_2(\mathbf{q} + \mathbf{p})] \\ &= \int d\mathbf{q} [S_{\mathbf{p} \rightarrow \mathbf{q}}[g_1] S_{\mathbf{q} \rightarrow \mathbf{k}}[g_2] + S_{0 \rightarrow \mathbf{q}, \mathbf{k}}[g_1] S_{\mathbf{p}, \mathbf{q} \rightarrow 0}[g_2]]. \end{aligned} \quad (\text{B.11})$$

This shows that the scattering amplitude appears as the sum of two terms. The first describes the scattering due to g_1 from $\mathbf{p}$ to $\mathbf{q}$, followed by the scattering due to g_2 from $\mathbf{q}$ to $\mathbf{k}$. The second term describes creation from the vacuum of a pair with momenta $\mathbf{k}$ and $\mathbf{q}$, followed by the annihilation of the initial particle with momentum $\mathbf{p}$ with that with momentum $\mathbf{q}$. Thus, the scattering process factorizes into the contributions due to g_1 and g_2 consistently with the causal order.

C

Dirac matrices

Most calculations involving Dirac matrices can be performed using only their general properties,

$$\begin{aligned}\{\gamma_\mu, \gamma_\nu\} &= 2I g_{\mu\nu}; & \gamma_5 &= -i\gamma_0\gamma_1\gamma_2\gamma_3 = -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma, \\ \gamma_\mu^\dagger &= \gamma_0 \gamma_\mu \gamma_0,\end{aligned}\tag{C.1}$$

without making reference to a specific representation. Some immediate consequences of eq. (C.1) are

$$\text{Tr } \gamma_\mu \gamma_\nu = 4 g_{\mu\nu}; \quad \{\gamma_\mu, \gamma_5\} = 0; \quad \gamma_5^2 = I,\tag{C.2}$$

and the identities

$$\gamma^\mu \gamma^\alpha \gamma_\mu = -2\gamma^\alpha\tag{C.3}$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 4\gamma^{\alpha\beta}\tag{C.4}$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma_\mu = -2\gamma^\gamma \gamma^\beta \gamma^\alpha.\tag{C.5}$$

The trace of the product of an odd number of γ vanishes. Indeed,

$$\gamma_\mu = -\gamma_5 \gamma_\mu \gamma_5,\tag{C.6}$$

and therefore

$$\begin{aligned}\text{Tr } \gamma_{\mu_1} \dots \gamma_{\mu_{2n+1}} &= (-1)^{2n+1} \text{Tr } (\gamma_5 \gamma_{\mu_1} \gamma_5) \dots (\gamma_5 \gamma_{\mu_{2n+1}} \gamma_5) \\ &= -\text{Tr } \gamma_{\mu_1} \dots \gamma_{\mu_{2n+1}},\end{aligned}\tag{C.7}$$

where we have used the circular property of the trace. It is easy to prove that

$$\text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})\tag{C.8}$$

$$\text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 = 4i\epsilon^{\mu\nu\rho\sigma}.\tag{C.9}$$

In Section 6.1 we have introduced a particular representation of the Dirac matrices:

$$\gamma_\mu \equiv \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}. \quad (\text{C.10})$$

Different representations of the γ matrices are related by similarity transformations on spinor fields. A representation which is often used (especially in application that involve the non-relativistic limit) is the so-called standard representation:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad , \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (\text{C.11})$$

D

Violation of unitarity in the Fermi theory

In this Appendix, we show that unitarity of the $\mathcal{S}$ matrix is violated in the Fermi theory of weak interactions. We rewrite the unitarity constraint, eq. (4.43), for $i = j$:

$$\sum_f \int d\phi_{n_f}(P_i; k_1^f, \dots, k_{n_f}^f) |\mathcal{M}_{if}|^2 = -2 \operatorname{Im} \mathcal{M}_{ii}, \quad (\text{D.1})$$

which is the so-called optical theorem: the total cross section for the process $i \rightarrow f$, summed over all possible final states f , is proportional to the imaginary part of the forward invariant amplitude $\mathcal{M}_{ii}$.

Let us now assume that i is a state of two massless particles with momenta p_1, p_2 ; furthermore, let us assume that only $2 \rightarrow 2$ processes are allowed. Under these conditions, the states f are also two-particle states, and the amplitudes $\mathcal{M}_{if}$ depend on the initial and final states through the two independent Mandelstam variables s, t :

$$\mathcal{M}_{if} \equiv \mathcal{M}(s, t), \quad (\text{D.2})$$

where

$$s = (p_1 + p_2)^2, \quad t = (p_1 - k_1)^2. \quad (\text{D.3})$$

In the center-of-mass frame,

$$t = -\frac{s}{2}(1 - \cos \theta) \rightarrow \cos \theta = 1 + \frac{2t}{s}, \quad (\text{D.4})$$

where θ is the scattering angle. Thus, for a given value of the center-of mass squared energy s , the amplitude $\mathcal{M}(s, t)$ is a function of $\cos \theta$ only, and can be expanded on the basis of the Legendre polynomials

$$P_J(z) = \frac{1}{J!2^J} \frac{d^J}{dz^J} (z^2 - 1)^J. \quad (\text{D.5})$$

The Legendre polynomials obey the orthogonality conditions

$$\int_{-1}^1 dz P_J(z) P_K(z) = \frac{2}{2J+1} \delta_{JK} \quad (\text{D.6})$$

and the normalization conditions

$$P_J(1) = 1. \quad (\text{D.7})$$

We find

$$\mathcal{M}(s, t) = 16\pi \sum_J (2J+1) a_J(s) P_J(\cos \theta), \quad (\text{D.8})$$

where the partial-wave amplitudes a_J are given by

$$a_J(s) = \frac{1}{32\pi} \int_{-1}^1 d\cos \theta P_J(\cos \theta) \mathcal{M}(s, t). \quad (\text{D.9})$$

Replacing eq. (D.8) in the l.h.s. of eq. (D.1) we get

$$\begin{aligned} & \int \frac{d\mathbf{k}_1}{(2\pi)^3 2E_{k_1}} \frac{d\mathbf{k}_2}{(2\pi)^3 2E_{k_2}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) |\mathcal{M}(s, t)|^2 \\ &= \frac{1}{16\pi} \int_{-1}^1 d\cos \theta \\ & \quad \left[16\pi \sum_J (2J+1) a_J(s) P_J(\cos \theta) \right] \left[16\pi \sum_K (2K+1) a_K^*(s) P_K(\cos \theta) \right] \\ &= 32\pi \sum_J (2J+1) |a_J(s)|^2, \end{aligned} \quad (\text{D.10})$$

while the r.h.s. is given by

$$-2 \operatorname{Im} \mathcal{M}(s, 0) = -32\pi \sum_J (2J+1) \operatorname{Im} a_J(s), \quad (\text{D.11})$$

where we have set $t = 0$, or equivalently $\cos \theta = 1$, as appropriate for a forward amplitude, and we have used the normalization condition (D.7). Therefore, the unitarity constraint eq. (D.1) requires

$$|a_J(s)|^2 = -\operatorname{Im} a_J(s) \quad (\text{D.12})$$

for all partial amplitudes. Equation (D.12) provides the unitarity bound

$$|a_J(s)| \leq 1. \quad (\text{D.13})$$

Let us now consider a specific process, namely

$$e^-(p_1) + \nu_\mu(p_2) \rightarrow \mu^-(k_1) + \nu_e(k_2) \quad (\text{D.14})$$

within the Fermi theory. The relevant amplitude is

$$\mathcal{M}(s, t) = -\frac{G_F}{\sqrt{2}} \bar{u}(k_2) \gamma^\alpha (1 - \gamma_5) u(p_1) \bar{u}(k_1) \gamma_\alpha (1 - \gamma_5) u(p_2), \quad (\text{D.15})$$

where all lepton masses have been neglected. This gives

$$\begin{aligned} \sum_{\text{pol}} |\mathcal{M}(s, t)|^2 &= 2G_F^2 \text{Tr} [\gamma^\alpha \not{p}_1 \gamma^\beta (1 - \gamma_5) \not{k}_2] \text{Tr} [\gamma_\alpha \not{p}_2 \gamma_\beta (1 - \gamma_5) \not{k}_1] \\ &= 32G_F^2 s^2. \end{aligned} \quad (\text{D.16})$$

We see that only the partial amplitude $a_0(s)$ is nonzero, since there is no t dependence at all. Using the definition eq. (D.9) we obtain

$$|a_0(s)| = \frac{G_F s}{2\sqrt{2}\pi}. \quad (\text{D.17})$$

The unitarity bound eq. (D.13) is therefore violated at

$$\sqrt{s} = \sqrt{\frac{2\sqrt{2}\pi}{G_F}} \simeq 875 \text{ GeV}. \quad (\text{D.18})$$

The total cross section obtained from eq. (D.16),

$$\sigma = \frac{G_F^2 s}{2\pi}, \quad (\text{D.19})$$

grows linearly with the squared center-of-mass energy s .

In the standard model, the same amplitude involves the exchange of a virtual W boson with mass m_W and coupling $g/(2\sqrt{2})$ to left-handed fermions. The standard model squared amplitude is obtained from the result in eq. (D.16) by the replacement

$$-\frac{G_F}{\sqrt{2}} \rightarrow \frac{g^2}{8} \frac{1}{t - m_W^2} = \frac{G_F}{\sqrt{2}} \frac{m_W^2}{t - m_W^2}. \quad (\text{D.20})$$

We get

$$\sum_{\text{pol}} |\mathcal{M}^{\text{SM}}(s, t)|^2 = 64G_F^2 s^2 \left(\frac{m_W^2}{t - m_W^2} \right)^2. \quad (\text{D.21})$$

The total cross section is now given by

$$\sigma^{\text{SM}} = \frac{G_F^2 s}{2\pi} \frac{m_W^2}{s + m_W^2}, \quad (\text{D.22})$$

that reduces to the result obtained in the Fermi theory, eq. (D.19), for $s \ll m_W^2$. In this case, however, the linear growth of the cross section with s is cut off at $s \sim m_W^2$. At very large energy,

$$\sigma^{\text{SM}} \rightarrow \frac{G_F^2 m_W^2}{2\pi}. \quad (\text{D.23})$$

The value of m_w is related to the size of the coupling g through $g_F/\sqrt{2} = g^2/(8m_w^2)$. If m_w were close to the energy at which the Fermi theory breaks down, about 900 GeV, then g would take a value close to 10, far from the perturbative domain. The fact that the measured value m_w is instead much smaller, $m_w \simeq 80$ GeV, is a signal of the fact that a theory of weak interactions with an intermediate vector boson can be treated perturbatively: indeed, in this case we get $g \sim 0.7$.

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