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Damiano Rotondo

Advances in Gain-Scheduling and Fault Tolerant Control Techniques

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Damiano Rotondo

Advances in Gain-Scheduling and Fault Tolerant Control Techniques

Doctoral Thesis accepted by
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Supervisors' Foreword

It is a great pleasure to introduce Dr. Damiano Rotondo's thesis work, accepted for publication within Springer Theses and awarded with a prize for an outstanding original work. Dr. Rotondo joined the Research Center for Supervision, Safety and Automatic Control (CS2AC) of UPC at Campus of Terrassa (Spain) as an Erasmus student in 2009 and carried out the Master thesis under our supervision in 2010. At that time, he was a master student at the University of Pisa in Italy. The master thesis was also of a very good quality and was published as a book in Italy. After finishing his master thesis in 2011, he started the doctoral studies at UPC doing the research in the context of FP7 European project i-Sense where CS2AC research center was participating as a partner. A bit later he got a 4-year Ph.D. scholarship from the Catalonia regional government. Finally, he received the UPC Ph.D. degree (excellent cum laude) with the International Mention in Automatic Control and Robotics in April 2016.

Dr. Rotondo's thesis includes a significant amount of original scientific research in the field of gain-scheduling theory and its application to Fault Tolerant Control (FTC) using the Linear Parameter Varying (LPV) and Takagi–Sugeno (TS) paradigms in a unified manner. LPV/TS system theory allows dealing with the nonlinear and time-varying behavior of some family of systems using linear-like techniques. This thesis presents several relevant contributions to the LPV/TS gain-scheduling theory, with an emphasis on their application to FTC. In particular, the thesis:

- provides an up-to-date review of the state of the art of the gain-scheduling and fault tolerant control, with particular emphasis on LPV and TS systems;
- explores the strong similarities between polytopic LPV and TS models, showing how techniques developed for the former framework can be easily extended to be applied to the latter, and vice versa;
- proposes approaches to address the problem of designing an LPV state-feedback controller for uncertain LPV systems and to consider shifting specifications as well as its application to FTC;

- proposes several FTC strategies for LPV systems subject to actuator faults based on model reference control and virtual actuators, including actuator saturations and fault isolation delays;
- all the proposed methods are illustrated using application examples and real case studies based on an omnidirectional robot.

A significant part of Dr. Rotondo's thesis has been published in top journals and international well-recognized conferences; some of them are fruit of joint collaborations with the research group of Prof. Marcin Witczak from the Zielona-Gora University in Poland and Prof. Didier Theilliol and Prof. Jean-Christophe Ponsart from CRAN—Nancy at the University of Lorraine in France. Moreover, Dr. Rotondo's research has also been enriched during the Ph.D. thesis by his research stays at the KIOS Research Center (Cyprus) under the supervision of Prof. Marios Polycarpou and Dr. Vasso Reppa, and later on at AMOS Research Center (Norway), where actually he is doing his postdoc, under the supervision of Prof. Tor Arne Johansen and Dr. Andrea Cristofaro. The thesis has also been improved by the valuable comments of the external reviewers Prof. Christopher Edwards and Prof. Olivier Sename during the Ph.D. thesis evaluation phase.

Terrassa
January 2017

Prof. Fatiha Nejjari Akhi-elarab
Prof. Vicenç Puig Cayuela

Foreword by CEA

It is a great pleasure for us to introduce here the work of Damiano Rotondo, awarded for the best Ph.D. thesis in Control Engineering during the 2016 edition of the award call organized by the control engineering group of Comité Español de Automática (CEA), the Spanish Committee of Automatic Control. This yearly award is aimed at recognizing outstanding Ph.D. research carried out within the Control Engineering field. At least one of the supervisors must be partner of the Spanish Committee of Automatic Control and member of the Control Engineering group. The jury is composed of three well-known doctors in the control engineering field: two of them are partners of the CEA, while the third one is a foreign professor. The first edition counted submissions from Ph.D. students examined between July 2015 and July 2016. Notably, the scientific production of the candidates consisted of a total of more than 20 publications in international indexed journals, where 15 of them are ranked in first quartile journals. This shows the high scientific quality of the submitted Ph.D.s. All the thesis presented:

- have been under the international doctorate mention (the candidate must have at least one international research stay out of Spain, write part of the thesis in English, and have foreign members in the Ph.D. jury);
- have been mostly oriented to academic research than to industrial applications;
- were presented at international conferences. Among them, IFAC and IEEE are considered as preferred conferences.

As a result, Damiano Rotondo's Ph.D. thesis was selected as the best one among an excellent group of candidates. His Ph.D. thesis deserves the label "the best of the best", Springer Theses' main motto. Not only his nomination as winner of the Spanish control engineering context has been our great pleasure, but also to recommend his thesis for publication within the Springer Theses collection. On the behalf of CEA, we wish Damiano to continue his outstanding scientific career, keeping his genuine enthusiasm. We also hope that his work may be of inspiration for other students working in the control engineering field.

Prof. Dr. Joseba Quevedo
President of CEA
Dr. Ramon Vilanova
Coordinators of the Control Engineering group
Dr. Jose Luis Guzman
Coordinators of the Control Engineering group

Parts of this thesis have been published in the following articles:

Journals

D. Rotondo, F. Nejjari, and V. Puig. Robust state-feedback control of uncertain LPV systems: an LMI-based approach. *Journal of the Franklin Institute*, 351:2781–2803, 2014.

D. Rotondo, F. Nejjari, and V. Puig. A virtual actuator and sensor approach for fault tolerant control of LPV systems. *Journal of Process Control*, 24(3):203–222, 2014.

D. Rotondo, V. Puig, F. Nejjari, and M. Witczak. Automated generation and comparison of Takagi-Sugeno and polytopic quasi-LPV models. *Fuzzy Sets and Systems*, 277:44–64, 2015.

D. Rotondo, F. Nejjari, and V. Puig. Design of parameter-scheduled state-feedback controllers using shifting specifications. *Journal of the Franklin Institute*, 352:93–116, 2015.

D. Rotondo, F. Nejjari, V. Puig, and J. Blesa. Model reference FTC for LPV systems using virtual actuator and set-membership fault estimation. *International Journal of Robust and Nonlinear Control*, 25(5):735–760, 2015.

D. Rotondo, F. Nejjari, and V. Puig. Robust quasi-LPV model reference FTC of a quadrotor UAV subject to actuator faults. *International Journal of Applied Mathematics and Computer Science*, 25(1):7–22, 2015.

D. Rotondo, J.-C. Ponsart, D. Theilliol, F. Nejjari, and V. Puig. A virtual actuator approach for the fault tolerant control of unstable linear systems subject to actuator saturation and fault isolation delay. *Annual Reviews in Control*, 39:68–80, 2015

D. Rotondo, V. Puig, F. Nejjari, and J. Romera. A fault-hiding approach for the switching quasi-LPV fault tolerant control of a four-wheeled omnidirectional mobile robot. *IEEE Transactions on Industrial Electronics*, 62(6):3932–3944, 2015.

D. Rotondo, F. Nejjari, and V. Puig. Fault tolerant control of a PEM fuel cell using Takagi-Sugeno virtual actuators. *Journal of Process Control*, 45:12–29, 2016.

Conferences

D. Rotondo, F. Nejjari, and V. Puig. A shifting pole placement approach for the design of parameter-scheduled state-feedback controllers. In *Proceedings of the 12th European Control Conference (ECC)*, pages 1829–1834, 2013.

D. Rotondo, F. Nejjari, and V. Puig. Passive and active FTC comparison for polytopic LPV systems. In *Proceedings of the 12th European Control Conference (ECC)*, pages 2951–2956, 2013.

D. Rotondo, F. Nejjari, and V. Puig. Fault tolerant control design for polytopic uncertain LPV systems. In *Proceedings of the 21st Mediterranean Conference on Control and Automation (MED)*, pages 66–72, 2013.

D. Rotondo, F. Nejjari, A. Torren, and V. Puig. Fault tolerant control design for polytopic uncertain LPV systems: application to a quadrotor. In *Proceedings of the 2nd International Conference on Control and Fault-Tolerant Systems (SYSTOL)*, pages 643–648, 2013.

D. Rotondo, F. Nejjari, and V. Puig. Shifting finite time stability and boundedness design for continuous-time LPV systems. In *Proceedings of the 32nd American Control Conference (ACC)*, pages 838–843, 2015.

D. Rotondo, J.-C. Ponsart, D. Theilliol, F. Nejjari, and V. Puig. Fault tolerant control of unstable LPV systems subject to actuator saturations using virtual actuators. In *Proceedings of the 9th IFAC Symposium on Fault Detection, Supervision and Safety for Technical Processes (SAFEPROCESS)*, pages 18–23, 2015.

D. Rotondo, J.-C. Ponsart, F. Nejjari, D. Theilliol, V. Puig. Virtual actuator-based FTC for LPV systems with saturating actuators and FDI delays. In *Proceedings of the 3rd Conference on Control and Fault-Tolerant Systems*, pages 831–837, 2016.

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This thesis concludes an important chapter in my life, in which I have grown in many ways, both professionally and personally. Before presenting the scientific results obtained in these years, I would like to thank all the people who have contributed to such growth.

First of all, I would like to thank my supervisors and Ph.D. directors: Prof. Fatiha Nejjari Akhi-Elarab and Prof. Vicenç Puig Cayuela. This thesis would not have been possible without their support. I thank them for providing me a wonderful environment for working, learning, and expressing my creativity, for believing in me and for giving me ideas, suggestions, and words of encouragement whenever I needed them. They have been, and will always be, examples of good workers, good colleagues, good researchers, and good persons. Thank you.

I would like to thank Prof. Christopher Edwards and Prof. Olivier Sename for the valuable time spent reading and reviewing my thesis. I would also like to thank them, together with Prof. Joseba Quevedo Casin, Prof. Mickael Rodrigues, and Prof. Carlos Ocampo-Martínez for accepting to be part of the thesis examination panel.

In these years, I had the opportunity to enjoy three stays abroad. I would like to thank Prof. Marios Polycarpou and Dr. Vasso Reppa for receiving me in the KIOS Research Center (Cyprus), Prof. Didier Theilliol and Prof. Jean-Christophe Ponsart for welcoming me in the CRAN Research Center (France), and Prof. Tor Arne Johansen and Dr. Andrea Cristofaro for receiving me in the AMOS Research Center (Norway). Additionally, I would like to express my thanks to Prof. Marcin Witczak from the University of Zielona Góra (Poland), with whom I have collaborated in several occasions, and who has been an important source of knowledge and inspiration. I am grateful for all the fruitful scientific discussions, which have been strong stimuli for developing new ideas.

I would like to thank my colleagues from UPC, who have contributed in making these years pleasant. In particular, I would like to thank Andreu Quesada Gomez and Judit Esteve Almirall for their administrative support, which made my life as a Ph.D. student easier in many occasions. Also, special thanks to Dr. Joaquin Blesa Izquierdo, with whom I have collaborated often during these years.

I would like to thank all the wonderful people that I have met in these years and who have become my friends. I have great memories of the time spent together, and I think that a lot of my personal growth during these years would have not been possible without them. Thank you.

Finally, a special thanks to my family, which is the most solid pillar of my life. Thanks to my father Nicola, my mother Claudia, and my brother Massimiliano, who have always believed in me, have supported my choices, and have provided all the help they could provide. I could not desire a better family than the one I already have. *Grazie di cuore*.

Contents

Part II Advances in Fault Tolerant Control Techniques

About the Author

Dr. Damiano Rotondo was born in Francavilla Fontana, Italy, in 1987. He received the B.S. degree (with honors) from the Second University of Naples, Italy, the M.S. degree (with honors) from the University of Pisa, Italy, and the Ph.D. degree (with honors) from the Universitat Politècnica de Catalunya, Spain in 2008, 2011, and 2016, respectively. He has visited the KIOS Research Center for Intelligent Systems and Networks, University of Cyprus, Nicosia, Cyprus, from January to March 2013, the Research Center for Automatic Control of Nancy (CRAN) at the University of Lorraine, Nancy, France, from March 2014 to July 2014 and the Centre for Autonomous Marine Operations and Systems (AMOS) at the Norwegian University of Science and Technology (NTNU) from March 2015 to July 2015. From May 2016 until April 2017, he has been an ERCIM post-doctoral researcher at AMOS/NTNU (Norway). Since May 2017, he is a postdoctoral researcher at the Research Centre for Supervision, Safety and Automatic Control (CS2AC) of the UPC. His main research interests include gain-scheduled control systems, Fault Detection and Isolation (FDI), and Fault Tolerant Control (FTC) of dynamic systems. He has published several papers in international conference proceedings and scientific journals.

Acronyms

AC	Adaptive Control
BMI	Bilinear Matrix Inequality
BRL	Bounded Real Lemma
CR	Controller Reconfiguration
CT	Continuous-Time
DI	Dynamic Inversion
DLMI	Differential Linear Matrix Inequality
DT	Discrete-Time
EA	Eigenstructure Assignment
FDI	Fault Detection and Isolation
FL	Feedback Linearization
FTB	Finite-Time Bounded/Boundedness
FTC	Fault Tolerant Control
FTCS	Fault Tolerant Control System
FTS	Finite-Time Stable/Stability
GS	Gain-Scheduling
IC	Intelligent Control
IDC	Integrated Diagnostic and Control
IO	Input–Output
KYP	Kalman–Yabukovich–Popov
LFT	Linear Fractional Transformation
LMI	Linear Matrix Inequality
LPV	Linear Parameter Varying
LQ	Linear Quadratic
LTI	Linear Time Invariant
LTV	Linear Time Varying
MF	Model Following
MIMO	Multi-Input Multi-Output
MM	Multiple Model
MPC	Model Predictive Control

PIM	Pseudo-Inverse Method
PSM	Parameter Set Mapping
QFT	Quantitative Feedback Theory
SMC	Sliding Mode Control
SMILE	State-space Model Interpolation Local Estimates
SS	State-Space
TRMS	Twin Rotor MIMO System
TS	Takagi–Sugeno
UAV	Unmanned Aerial Vehicle
VSC	Variable Structure Control

Symbols

A	State matrix
A_c	Controller state matrix
A_f	Faulty state matrix
B	Input matrix
B_c	Controller input matrix
B_f	Faulty input matrix
C	Output matrix
C_c	Controller output matrix
D	Feedthrough matrix
D_c	Controller feedthrough matrix
K	Controller matrix
I	Identity matrix
L	Observer matrix
M	Virtual actuator matrix
N	Number of vertex systems/subsystems
O	Zero matrix
P	Lyapunov matrix
Q	Lyapunov matrix ($Q = P^{-1}$)
R_A	Region of attraction
T_s	Sampling time
V	Lyapunov function
c	Known exogenous input vector
e	Error vector
$\hat{e}$	Estimated error vector
f	Fault vector
$\hat{f}$	Estimated fault vector
f_a	Additive fault vector
$\hat{f}_a$	Estimated additive fault vector
g	Nonlinear state function
h	Nonlinear output function
k	Time sample (discrete-time)

s	Continuous-time complex variable
sat	Saturation function
t	Time variable (continuous-time)
t_f	Fault occurrence time
t_I	Fault isolation time
x	State vector
$\hat{x}$	Estimated state vector
x_c	Controller state vector
x_{ref}	Reference state vector
x_{v}	Virtual actuator state vector
u	Input vector
u_c	Controller output vector
u_{ref}	Reference input vector
u^{MAX}	Saturation vector
w	Disturbance vector
y	Output vector
y_{ref}	Reference output vector
z	Discrete-time complex variable
z_∞	$\mathcal{H}_\infty$ performance output vector
z_2	$\mathcal{H}_2$ performance output vector
Γ	Auxiliary variable to convert BMIs into LMIs
γ_∞	$\mathcal{H}_\infty$ performance
γ_2	$\mathcal{H}_2$ performance
μ_i	i-th coefficient of a polytopic decomposition
ρ_i	Level of activation of the i-th local model
ε	Output error vector
ς_i	i-th saturation scheduling parameter
σ	Generic complex variable
θ	Vector of varying parameters
ϑ	Vector of premise variables
φ	Roll angle
ψ	Yaw angle
τ	Generic time variable
τ_D	Generic fault detection time
τ_{I}	Generic fault isolation time
ϱ	Pitch angle
Φ	Actuator fault distribution matrix
Θ	Domain of variation of θ
Ω	Rotor speed
$\mathcal{D}$	Subset of the complex plane
$\mathcal{E}$	Ellipsoid
$\mathcal{L}$	Region of the state-space in which the actuators do not saturate
$\mathcal{S}$	Domain of attraction of the origin

Chapter 1
Introduction

1.1 Context of the Thesis

The results presented in this thesis have been developed at the Research Center for Supervision, Safety and Automatic Control (CS2AC) of the Universitat Politècnica de Catalunya (UPC) in Terrassa, Spain. The research was jointly supervised by Dr. Fatiha Nejjari and Dr. Vicenç Puig, and was sponsored partly by UPC through an FPI-UPC grant and by the Agència de Gestió d'Ajuts Universitaris i de Recerca (AGAUR) through contracts FI-DGR. The supports are gratefully acknowledged.

1.2 Motivations

The development of gain scheduling and fault tolerant control (FTC) techniques has attracted a lot of attention in the last decades, as testified by the increasing number of publications dealing with these topics.

In the first case, the interest of the research community has been attracted by the possibility of dealing with nonlinear control problems. In particular, the linear parameter varying (LPV) [1] and the fuzzy Takagi-Sugeno (TS) [2] paradigms have provided an elegant way to apply linear techniques to nonlinear systems with theoretical guarantees of stability and performance. Some recent works have presented some clues about the existence of a close connection between the LPV theory and the fuzzy TS paradigms [3–6]. However, even if from theoretical analysis and design points of view it is difficult to find clear differences between the two paradigms, they are still considered different and their equality is dubious [7].

Most of the available results about the design of controllers for LPV systems make the assumption that the model is perfectly known. Only a few works, e.g. [8, 9] have stated the importance of considering robustness against uncertainties. Hence, designing controllers for uncertain LPV systems that can guarantee some desired performances in spite of the uncertainties is still an open problem. Furthermore, an interesting twist on the application of LPV/TS theory, that has never been considered before, is designing the controller in such a way that different values of the varying parameters imply different performances of the closed-loop system.

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On the other hand, the increasing need for safety and reliability has motivated the development of FTC techniques, which are able to maintain the overall system stability and acceptable performance in presence of faults [10]. The LPV paradigm has been successfully applied in the FTC field, due to the time-varying nature of systems affected by faults and the need of dealing not only with linear plants, but also with nonlinear ones. In this case, open issues that motivate further research consist in how to take into account effectively the uncertainties in the fault estimation, and how to improve the behavior of fault tolerant control systems subject to constraints on the actuator action. This last problem is of particular importance in the case of open-loop unstable systems, because neglecting it could lead to instability under fault occurrence [11].

1.3 Thesis Objectives

The objectives of this thesis are the following:

- to state clearly the analogies and connections between LPV and TS systems;
- to show how methods developed for the LPV representation could be easily extended in order to be applied to the TS one, and vice versa;
- to propose measures in order to compare different LPV/TS models and choose which one can be considered *the best one*;
- to propose an approach for the design of robust LPV state-feedback controllers for uncertain LPV systems that can guarantee some desired performances in spite of the uncertainties;
- to propose an approach for the design of LPV state-feedback controllers such that different values of the varying parameters imply different performances of the closed-loop system;
- to use the robust LPV controller design method for FTC, giving rise to different strategies (passive/active/hybrid) depending on the available information about the faults;
- to take into account the presence of actuator saturations in the FTC scheme, such that guarantees of convergence to zero of the state trajectory are obtained, even in presence of delays between the fault occurrence and its isolation.

1.4 Outline of the Thesis

The thesis is organized in two parts:

Part I presents the results that constitute a contribution to the state-of-the-art of gain-scheduling. It is made up of four chapters:

- Chapter 2 recalls some background on gain-scheduling, with particular emphasis on LPV and TS systems. Known results about modeling, analysis and control of LPV/TS systems are presented and discussed. In particular, it is shown how LPV/TS representations can be obtained starting from a given nonlinear system, using different approaches, i.e. Jacobian linearization, state transformation, function substitution, sector nonlinearity and local approximation in fuzzy partition spaces. Then, the analysis and design of LPV control systems using the quadratic framework is reviewed, discussing several possible specifications, as stability, $\mathcal{D}$-stability (pole clustering in a subset of the complex plane), $\mathcal{H}_\infty/\mathcal{H}_2$ performance and finite time stability/boundedness.

- Chapter 3 addresses the presence of strong analogies between LPV and TS models. In particular, the connections between LPV and TS systems are clearly stated. It is shown that the method for the automated generation of LPV models by nonlinear embedding [12] can be easily extended to solve the corresponding problem for TS models. Similarly, it is shown that the method for the generation of a TS model for a given nonlinear multivariable function based on the sector nonlinearity approach [13] can be extended to the problem of generating a polytopic LPV model for a given nonlinear system. Finally, two measures, the first based on the notion of *overboundedness*, while the second based on *region of attraction estimates*, are proposed in order to compare different models and choose which one can be considered *the best one*. The chapter is concluded by a mathematical example that shows an application of the proposed methodologies.
- Chapter 4 considers the problem of designing a robust LPV state-feedback controller for uncertain LPV systems that can guarantee some desired performances. In the proposed approach, the vector of varying parameters is used to schedule between uncertain linear time invariant (LTI) systems. The resulting idea consists in using a double-layer polytopic description so as to take into account both the variability due to the parameter vector and the uncertainty. The first polytopic layer manages the varying parameters and is used to obtain the vertex uncertain systems, where the vertex controllers are designed. The second polytopic layer is built at each vertex system so as to take into account the model uncertainties and add robustness into the design step. Under some assumptions, the problem reduces to finding a solution to a finite number of linear matrix inequalities (LMIs), a problem for which efficient solvers are available nowadays. The proposed technique is applied to numerical examples, showing that it achieves the desired performances, whereas the traditional LPV gain-scheduling technique fails.
- In Chap. 5, by taking advantage of the properties of polytopes and linear matrix inequalities (LMIs), new problems that can be seen as extensions of the more classical $\mathcal{D}$-stability, $\mathcal{H}_\infty$ performance, $\mathcal{H}_2$ performance, finite time boundedness and finite time stability specifications are solved. In these new problems, referred to as *shifting $\mathcal{D}$-stability*, *shifting $\mathcal{H}_\infty$ performance, shifting $\mathcal{H}_2$ performance*, *shifting finite time stability* and *shifting finite time boundedness*, by introducing some varying parameters, or using the existing ones, the controller is designed in such a way that different values of these parameters imply different performances of the closed-loop system. The results obtained with an academic example are used to demonstrate the effectiveness and some characteristics of the proposed approach.

Part II presents the results that constitute a contribution to the state-of-the-art of fault tolerant control. It is made up of four chapters:

- Chapter 6 recalls some background on fault tolerant control. Different approaches are resumed, following the well-established distinction between *hardware redundancy* and *analytical redundancy* techniques and, with regards to the latter, the additional distinction between *passive* and *active* approaches. The last part of the chapter resumes recent developments of fault tolerant control theory, highlighting some open issues that motivate further investigation in this topic.
- Chapter 7 shows how the framework proposed in Chap. 4 for the design of robust LPV controllers can be used for FTC, with the advantage that, depending on how much information is available, it gives rise to different strategies. If the faults are considered as perturbations, a *passive FTC* would arise. On the other hand, if the faults are used as additional scheduling parameters, an *active FTC* would be obtained. Finally, if the fault estimation uncertainty is taken into account explicitly during the design step, the robust LPV polytopic technique would lead to *hybrid FTC*. The different controllers are obtained using LMIs, in order to achieve regional pole placement and $\mathcal{H}_\infty$ performance constraints. Results obtained using a quadrotor unmanned aerial vehicle (UAV) simulator are used to show the effectiveness of the proposed approach.
- Chapter 8 concerns the development of an FTC strategy for LPV systems involving a reconfigured reference model and virtual actuators. The use of the reference model framework allows assuring that the desired tracking performances are kept despite the fault occurrence, thanks to the action brought by the virtual actuator. By including the saturations in the reference model equations, it is shown that it is possible to design a model reference FTC system that automatically retunes the reference states whenever the system input is affected by saturation nonlinearities. Hence, another contribution of this chapter is to take into account the saturations as scheduling parameters, such that their inclusion in both the reference model and the system provides an elegant way to incorporate a graceful performance degradation in presence of actuator saturations. The potential and performance of the proposed approach are demonstrated with two different examples: a twin rotor multiple-input multiple-output (MIMO) system (TRMS) and a four wheeled omnidirectional mobile robot.
- Chapter 9 deals with the design of an active FTC strategy for unstable LPV systems subject to actuator saturation. Under the assumption that a nominal controller has been already designed, a block is added to the control loop for achieving fault tolerance against a predefined set of possible faults. In particular, faults affecting the actuators and causing a change in the system input matrix are considered. The design of this block is performed in such a way that, if at the fault isolation time the closed-loop system state is inside a region defined by a value of the Lyapunov function, the state trajectory will converge to zero despite the appearance of the faults. Also, it is shown that it is possible to obtain some guarantees about the tolerated delay between the fault occurrence and its isolation. Moreover, the design

of the nominal controller can be performed so as to maximize the tolerated delay. A numerical example is used to show the effectiveness of the proposed approach.

Finally, the thesis is concluded by:

- Chapter 10, which summarizes the main conclusions and suggests possible lines of future research;
- Appendix A, which provides new characterizations for the analysis of finite time boundedness and finite time stability. This new characterization allows considering parameter-dependent Lyapunov functions easily, thus decreasing the conservativeness with respect to other approaches available in the literature;
- Appendix B, which completes the results presented in Chap. 8, by demonstrating that a particular matrix is independent from the values of the faults.

1.5 Notation

Following the notation used by [8], σ stands for the Laplace variable s in the continuous-time (CT) case and for the $\mathcal{Z}$-transform variable z in the discrete-time (DT) case. Similarly, τ will stand for the time $t \in \mathbb{R}^+$ in the CT case and for the time samples $k \in \mathbb{Z}^+$ in the DT case. The notation $\sigma.x(\tau)$ stands for $\dot{x}(t)$ for CT systems and for $x(k+1)$ for DT systems.

For a complex number σ, its complex conjugate will be denoted by σ^*.

Given a vector $v \in \mathbb{R}^{n_v}$, its ith element will be denoted as v_i. For a given matrix $M = [m_{kl}]_{k\in\{1,\dots,n_r\},l\in\{1,\dots,n_c\}} \in \mathbb{R}^{n_r \times n_c}$, the ith row will be denoted as M_i, and the element located in its ith row and jth column as m_{kl}. The notation M^{T} will indicate the transpose operation, and M^H will denote the Hermitian transpose operation. For brevity, symmetric elements in a matrix are denoted by $*$ and $M + M^{\mathrm{T}}$ will be indicated as $He\{M\}$. If a matrix $M \in \mathbb{R}^{n\times n}$ is symmetric, then $M \in \mathbb{S}^{n\times n}$. A matrix $M \in \mathbb{S}^{n\times n}$ is said *positive definite* ($M \succ 0$) if all its eigenvalues are positive, and *negative definite* ($M \prec 0$) if all its eigenvalues are negative. Moreover, the symbol $\otimes$ denotes the Kronecker product, $\dagger$ denotes the Moore–Penrose pseudoinverse and Tr the trace of a matrix.

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Part I
Advances in Gain-Scheduling Techniques

Chapter 2
Background on Gain-Scheduling

2.1 Gain-Scheduling: LPV Systems and TS Systems

After World War II, the development of advanced jet aircrafts, the advent of guided missiles and the need of stability and performance requirements for a wide set of operating conditions pushed towards a rapid adoption of *gain scheduled* autopilot systems [1]. As examples of first proposed solutions, the B-52 autopilot, developed around 1951, incorporated an airspeed-based mechanism to compensate for changes in the aero-surface effectiveness [1]. The autopilot of the Talos missile, developed in the early 1950s, adjusted the gains to compensate for changes in altitude and speed, thus exhibiting a rudimentary form of gain-scheduling [2]. Since then, gain-scheduling began to play an important role not only in military applications, but in commercial ones too. For example, in response to the dual imperatives of improved fuel economy and reduction of exhaust emissions, gain scheduling began to be used in automobile engine controllers for electronic fuel control [1], starting from [3], in which a closed-loop electronic fuel injection control with a gain influenced by measured variables was described.

The first gain scheduled controller design approach involved selecting several operating points, covering the range of the plant's working conditions, where linear time invariant (LTI) controllers were designed. Then, between these operating points, the parameters (gains) of the controller were interpolated (scheduled) [4]. However, this approach lacked in providing stability and performance guarantees for all the possible operating conditions and, moreover, it needed the assumption of slow variation in time of the parameters [5]. For this reason, the necessity for systematic analysis and design tools for gain-scheduled controllers arised. Among the most successful approaches, there are the linear parameter varying (LPV) and the Takagi-Sugeno (TS) paradigms.

LPV systems were introduced by Shamma [6] to distinguish such systems from LTI and linear time varying (LTV) ones [7]. More specifically, LPV systems are a particular class of LTV systems, where the time-varying elements depend on measurable parameters that can vary over time [8]. The LPV framework has proved to

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be suitable for controlling nonlinear systems by embedding the nonlinearities in the varying parameters, that will depend on some endogenous signals, e.g. states, inputs or outputs. In this case, the system is referred to as *quasi-LPV*, to make a further distinction with respect to *pure* LPV systems, where the varying parameters only depend on exogenous signals [9].

Since the introduction of this paradigm, a lot of research has concerned the development of design techniques for LPV systems. At first, the small gain theorem was applied to LPV systems with a linear fractional transformation (LFT) form [10, 11]. However, this approach took into account complex varying parameters, that did not appear in real plants, thus introducing a strong source of conservatism [8]. For this reason, Lyapunov-based approaches were developed, allowing to take into account not only arbitrarily fast parameter variations [12], but also known bounds on the rate of parameter variation [13–15]. A unified scheme combining the small gain theorem and the Lyapunov-based approach was developed by [16].

The LPV paradigm has evolved rapidly in the last two decades and has been applied successfully to a big number of applications, e.g. active vision systems [17], airplanes [18, 19], bioreactors [20], canals [21], CD players [22], container crane load swing [23], control moment gyroscopes [24], electromagnetic actuators [25], engines [26], flexible ball screw drives [27], fuel cells [28, 29], glycemic regulation [30], induction motors [31], internet web servers [32], inverted pendula [33], ionic polymer-metal composites [34], magneto-rheological dampers [35], robots [36], unmanned aerial vehicles (UAVs) [37, 38], vehicle suspensions [39–41], wafer scanners [42], wind turbines [43] and winding machines [44]. Recently, the LPV paradigm has also been applied to time delay systems with time varying delays [45–47].

On the other hand, TS systems, introduced by [48], basically provide an effective way of representing nonlinear systems with the aid of fuzzy sets, fuzzy rules and a set of local linear models which are smoothly connected by fuzzy membership functions [49]. TS fuzzy models are universal approximators, since they can approximate any smooth nonlinear function to any degree of accuracy [50–54], such that they can represent complex nonlinear systems.

The design approaches for TS systems can be classified into six categories [49]: (i) local controller design, where feedback controllers are designed for each local model and combined to obtain the global controller, and some stability criteria is used to check stability [55, 56]; (ii) stabilization based on a nominal linear model with nonlinearities considered as uncertainties [57, 58]; (iii) stabilization based on a common quadratic Lyapunov function [59–67]; (iv) stabilization based on a piecewise quadratic Lyapunov function [57, 68–70]; (v) stabilization based on a fuzzy Lyapunov function [71, 72]; (vi) adaptive control, when the parameters of the TS fuzzy models are unknown [73–75].

Also the TS paradigm has been successfully applied in several fields, among which active suspension of vehicles [76], aircrafts [77], electromechanical systems [78], energy production systems [79], missiles [80], robotic systems [81], spark ignition engines [82], transmission systems [83] and time delay systems [84].

2.2 Modeling of LPV Systems

In this section, some basic concepts about modeling of LPV systems are recalled. Since the thesis deals with methods developed for LPV models with polytopic parameter dependence, the case of LFT parameter dependence [10, 11] will not be considered. This does not cause a loss of generality, since [8] has demonstrated that an LPV model with LFT parameter dependence can be converted into an LPV model with polytopic parameter dependence. Also, the thesis will focus on LPV state-space (SS) representations, even though LPV input-output (IO) models have been proposed too [85]. Reference [86] has suggested practically applicable approaches for the conversion of an LPV IO model in a discrete-time LPV SS representation; thus, considering SS models does not cause a loss of generality. Finally, the methods recalled hereafter provide an LPV model starting from a nonlinear model that is assumed to be available. In cases different from this, LPV models can be identified from IO data [87, 88].

An LPV system is defined as a finite-dimensional LTV system whose state space matrices are fixed functions of some varying parameters $\theta(\tau) \in \mathbb{R}^{n_\theta}$, assumed to be unknown a priori, but measured or estimated in real-time [89]:

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + B\left(\theta(\tau)\right)u(\tau) \tag{2.1}$$

$$y(\tau) = C\left(\theta(\tau)\right)x(\tau) + D\left(\theta(\tau)\right)u(\tau) \tag{2.2}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are the state, the input, and the output vector, respectively, and $A\left(\theta(\tau)\right)$, $B\left(\theta(\tau)\right)$, $C\left(\theta(\tau)\right)$ and $D\left(\theta(\tau)\right)$ are varying matrices of appropriate dimensions.

Among the available analysis/synthesis approaches, the most popular, at least taking into account the number of publications, is the polytopic approach [4]. An LPV system is called *polytopic* when it can be represented by matrices $A\left(\theta(\tau)\right)$, $B\left(\theta(\tau)\right)$, $C\left(\theta(\tau)\right)$ and $D\left(\theta(\tau)\right)$, where the parameter vector $\theta(\tau)$ ranges over a fixed polytope Θ, and the dependence of the matrices on θ is affine [12], resulting in the following representation:

$$\sigma.x(\tau) = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right)\left(A_i x(\tau) + B_i u(\tau)\right) \tag{2.3}$$

$$y(\tau) = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right)\left(C_i x(\tau) + D_i u(\tau)\right) \tag{2.4}$$

where the quadruples (A_i, B_i, C_i, D_i) define the so-called *vertex systems*, and μ_i are the coefficients of the polytopic decomposition, such that:

$$\sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right) = 1, \quad \mu_i\left(\theta(\tau)\right) \geq 0, \ \forall i = 1, \ldots, N, \ \forall \theta \in \Theta \tag{2.5}$$

In the following, some methods for obtaining an LPV model starting from an available nonlinear SS model are recalled. For sake of simplicity, only continuous-time (CT) nonlinear systems in the form:

$$\dot{x}(t) = g\left(x(t), u(t)\right) \tag{2.6}$$

$$y(t) = h\left(x(t), u(t)\right) \tag{2.7}$$

are considered. Notice that most of the physical systems of interest for control purposes are CT, and if discrete-time (DT) LPV representations are desired for digital implementation, such models can be obtained from CT LPV models using discretization techniques, such as Euler or more sophisticated ones [90, 91].

2.2.1 Jacobian Linearization

The Jacobian linearization approach is the simplest technique that can be applied for obtaining LPV models. It assumes that the nonlinear system can be linearized around some equilibrium points of interest [9]. The basis of the method is to use a first-order Taylor-series approximation of (2.6)–(2.7), and then an interpolation of the obtained LTI models, when the system is working in operating points different from the equilibrium ones.

Despite its simplicity, the behavior of the obtained LPV model could diverge from the behavior of the nonlinear model [9]. The use of higher-order Taylor expansions could alleviate this issue, but would lead to impractical implementations [92]. Also, it is essentially impossible to capture the transient behavior of the nonlinear plant using this method [93].

Hereafter, an example of the application of the Jacobian linearization technique, taken from [94], is shown.

Consider the nonlinear system [95]:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) \\ x_1(t) - |x_2(t)|\, x_2(t) - 10 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \tag{2.8}$$

$$y(t) = x_2(t) \tag{2.9}$$

The set of linearized models obtained from (2.8)–(2.9) is:

$$\begin{pmatrix} \delta\dot{x}_1(t) \\ \delta\dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -2\left|x_2^{eq}(t)\right| \end{pmatrix} \begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta u(t) \tag{2.10}$$

$$\delta y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix} \tag{2.11}$$

Then, by considering the scheduling parameter $\theta(t) = \left|x_2^{eq}(t)\right|$, the model (2.10)–(2.11) would appear in the form (2.1)–(2.2). The resulting system would be referred to as *quasi-LPV*, due to the dependence of $\theta(t)$ on $x_2^{eq}(t)$.

2.2.2 State Transformation

In the state transformation approach, a coordinate change is performed with the aim of removing any nonlinear term not dependent on the scheduling parameters [89]. This method assumes that the nonlinear system is in the following form:

$$\begin{pmatrix} \dot{z}(t) \\ \dot{l}(t) \end{pmatrix} = g\left(z(t)\right) + A\left(z(t)\right)\begin{pmatrix} z(t) \\ l(t) \end{pmatrix} + B\left(z(t)\right)u(t) \tag{2.12}$$

where $z(t) \in \mathbb{R}^{n_z}$ are the scheduling states, and $l(t) \in \mathbb{R}^{n_h}$ are the non-scheduling ones, with $n_z = n_u$. Under the assumptions that there exists a family of equilibrium states parameterized by $z(t)$, such that:

$$0 = g\left(z(t)\right) + A\left(z(t)\right)\begin{pmatrix} z(t) \\ l_{eq}\left(z(t)\right) \end{pmatrix} + B\left(z(t)\right)u_{eq}\left(z(t)\right) \tag{2.13}$$

with $l_{eq}\left(z(t)\right)$ and $u_{eq}\left(z(t)\right)$ continuously differentiable functions, and that $A\left(z(t)\right)$ and $B\left(z(t)\right)$ are partitioned as:

$$A\left(z(t)\right) = \begin{pmatrix} A_{11}\left(z(t)\right) & A_{12}\left(z(t)\right) \\ A_{21}\left(z(t)\right) & A_{22}\left(z(t)\right) \end{pmatrix} \tag{2.14}$$

$$B\left(z(t)\right) = \begin{pmatrix} B_1\left(z(t)\right) \\ B_2\left(z(t)\right) \end{pmatrix} \tag{2.15}$$

it is possible to rewrite the state dynamics as:

$$\begin{aligned} \begin{pmatrix} \dot{z}(t) \\ \dot{l}(t) - \dot{l}_{eq}\left(z(t)\right) \end{pmatrix} &= \begin{pmatrix} 0 & A_{12}\left(z(t)\right) \\ 0 & A_{22}\left(z(t)\right) - \left.\frac{\partial l_{eq}(z)}{\partial z}\right|_{z(t)} A_{12}\left(z(t)\right) \end{pmatrix}\begin{pmatrix} z(t) \\ l(t) - l_{eq}(t) \end{pmatrix} \\ &+ \begin{pmatrix} B_1\left(z(t)\right) \\ B_2\left(z(t)\right) - \left.\frac{\partial l_{eq}(z)}{\partial z}\right|_{z(t)} B_1\left(z(t)\right) \end{pmatrix}\left(u(t) - u_{eq}\left(z(t)\right)\right) \end{aligned} \tag{2.16}$$

thus obtaining a quasi-LPV form different from the one obtained by performing the Jacobian linearization, and exactly representing the original nonlinear system. However, the presence of an inner-loop feedback due to the term $u_{eq}\left(z(t)\right)$ can deteriorate the properties of the system by adversely exciting flexible mode dynamics [5, 6]. Hence, special care should be taken when applying this technique.

For the example (2.8)–(2.9), the quasi-LPV model:

$$\begin{pmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{pmatrix} = \begin{pmatrix} -1 - 2\,|\tilde{x}_2(t)| & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{u}(t) \tag{2.17}$$

would be generated by changing the state coordinates as [94]:

$$\tilde{x}_1(t) = x_1(t) - x_1^{eq}\,(x_2(t)) \tag{2.18}$$

$$\tilde{x}_2(t) = x_2(t) \tag{2.19}$$

$$\tilde{u}(t) = u(t) - u_{eq}\,(x_2(t)) \tag{2.20}$$

with:

$$u_{eq}(t) = x_1^{eq}\,(x_2(t)) = |x_2(t)|\,x_2(t) + 10 \tag{2.21}$$

2.2.3 Function Substitution

An alternative approach to obtain a quasi-LPV model is the *function substitution* approach [96, 97], which consists in replacing the so-called *decomposition function* with functions that are linear with respect to the scheduling parameters. This decomposition function is formed by combining all the terms of the nonlinear system that are not both affine with respect to the non-scheduling states and control inputs, and function of the scheduling parameters alone (after a coordinate change with respect to a single equilibrium point has been performed) [9]. The decomposition is carried out through a minimization procedure, which leads to numerical optimization problems [88].

For the example (2.8)–(2.9), the nonlinear system is rewritten as [94]:

$$\begin{pmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{u}(t) + \begin{pmatrix} -x_1^{eq} + u_{eq} \\ x_1^{eq} - \left|\tilde{x}_2(t) + x_2^{eq}\right| \left(\tilde{x}_2(t) + x_2^{eq}\right) - 10 \end{pmatrix} \tag{2.22}$$

where:

$$\tilde{x}_1(t) = x_1(t) - x_1^{eq} \tag{2.23}$$

$$\tilde{x}_2(t) = x_2(t) - x_2^{eq} \tag{2.24}$$

$$\tilde{u}(t) = u(t) - u_{eq} \tag{2.25}$$

with trim point $(x_1^{eq}, x_2^{eq}) = (11, 1)$. Then, by replacing the nonlinearity in (2.22) with:

$$g\left(\tilde{x}_2(t)\right) = \begin{cases} \frac{\left[\left|x_2^{eq}\right|x_2^{eq} - \left|\tilde{x}_2(t)+x_2^{eq}\right|\left(\tilde{x}_2(t)+x_2^{eq}\right)\right]}{\tilde{x}_2(t)} & \tilde{x}_2(t) \neq 0 \\ 0 & \tilde{x}_2(t) = 0 \end{cases} \tag{2.26}$$

the following quasi-LPV model is obtained:

$$\begin{pmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & g\left(\tilde{x}_2(t)\right) \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{u}(t) \tag{2.27}$$

2.2.4 Other Approaches and Current Directions of Research

The problem of modeling a nonlinear system as a quasi-LPV model is still a hot topic of research. For example, [98] have suggested that linearization and local controller design should be carried out not only at equilibrium states, but also in transient operating regimes.

In [99], a method for automated generation of LPV models, to be used when affine representations of polytopic models are desired, has been presented. The affine LPV representations are generated from a general nonlinear model by *hiding* the nonlinearities in the scheduling parameters. These LPV representations are not unique and different models have different properties that may facilitate, complicate, or even make impossible, the controller synthesis. For instance, two representations of the same system may differ in the number of parameters, in the property of stabilizability, or in the degree of overbounding of the admissible parameter set. Hence, [99] also proposed a heuristic measure for the quality of different LPV models.

In the case of overbounding, i.e. when the obtained quasi-LPV model displays more behaviors than the underlying nonlinear model, it is possible to use the method proposed in [100]. This method is based on parameter set mapping (PSM) [101] and leads to the generation of less conservative representations.

A SS model interpolation of local estimates (SMILE) technique has been presented in [102] for estimating LPV SS models, based on the interpolation of LTI models estimated for constant values of the scheduling parameters. The interpolation is based on the formulation of a linear least-squares problem that can be efficiently solved, yielding homogeneous polynomial LPV models that are numerically well-conditioned and therefore suitable for LPV control synthesis.

In [103], inspired by the feedback linearization theory, a systematic procedure is proposed to convert control affine nonlinear SS representation into state minimal LPV SS representations in an observable canonical form, where the scheduling parameter depends on the derivatives of the inputs and outputs of the system. In addition, if the states of the nonlinear model can be measured or estimated, then the procedure can be modified to provide LPV models scheduled by these states.

2.3 Modeling of TS Systems

In this section, some basic concepts about modeling of TS systems are recalled. Also in this case, as in the LPV modeling, the approach that constructs a TS fuzzy model using an identification procedure applied to IO data is not considered. The interested reader may find some details about this approach, suitable for plants that cannot or are too difficult to be represented by means of analytical/physical models, in [104, 105].

TS systems, as proposed by Takagi and Sugeno [48], are described by local models merged together using fuzzy IF-THEN rules [54], as follows:

$$
\begin{aligned}
& IF\ \vartheta_1(\tau)\ is\ M_{i1}\ AND\ \cdots\ AND\ \vartheta_p(\tau)\ is\ M_{ip} \\
& THEN \begin{cases} \sigma.x_i(\tau) = A_i x(\tau) + B_i u(\tau) \\ y_i(\tau) = C_i x(\tau) + D_i u(\tau) \end{cases} \quad i = 1, \ldots, N
\end{aligned}
\tag{2.28}
$$

where $\vartheta_1(\tau), \ldots, \vartheta_p(\tau)$ are the *premise variables*, that can be functions of the state variables, controlled inputs, external disturbances and/or time. Each linear consequent equation represented by $A_i x(\tau) + B_i u(\tau)$ is called a *subsystem*. Given a pair $(x(\tau), u(\tau))$, the state and output of the TS system can be inferred easily as:

$$
\sigma.x(\tau) = \frac{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)\left(A_i x(\tau) + B_i u(\tau)\right)}{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)} = \sum_{i=1}^{N} \rho_i\left(\vartheta(\tau)\right)\left(A_i x(\tau) + B_i u(\tau)\right) \tag{2.29}
$$

$$
y(\tau) = \frac{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)\left(C_i x(\tau) + D_i u(\tau)\right)}{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)} = \sum_{i=1}^{N} \rho_i\left(\vartheta(\tau)\right)\left(C_i x(\tau) + D_i u(\tau)\right) \tag{2.30}
$$

where $\vartheta(\tau) = \left(\vartheta_1(\tau), \ldots, \vartheta_p(\tau)\right)^{\mathrm{T}}$ is the vector containing the premise variables, and $\varpi_i\left(\vartheta(\tau)\right)$ and $\rho_i\left(\vartheta(\tau)\right)$ are defined as follows:

$$
\varpi_i\left(\vartheta(\tau)\right) = \prod_{j=1}^{p} M_{ij}\left(\vartheta_j(\tau)\right) \tag{2.31}
$$

$$
\rho_i\left(\vartheta(\tau)\right) = \frac{\varpi_i\left(\vartheta(\tau)\right)}{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)} \tag{2.32}
$$

where $M_{ij}\left(\vartheta_j(\tau)\right)$ is the grade of membership of $\vartheta_j(\tau)$ in M_{ij} and $\rho_i\left(\vartheta(\tau)\right)$ is such that:

$$\sum_{i=1}^{N} \rho_i\left(\vartheta(\tau)\right) = 1, \quad \rho_i\left(\vartheta(\tau)\right) \geq 0, \ \forall i = 1, \ldots, N \tag{2.33}$$

In the following, some methods for the CT TS modeling will be recalled.

2.3.1 Sector Nonlinearity

The main idea behind this method appeared for the first time in [106]. Given a nonlinear system $\dot{x}(t) = g\left(x(t)\right)$ with $g(0) = 0$, this approach aims at finding a global sector such that $\dot{x}(t) \in [a_1 \ a_2]x(t)$. This approach guarantees an exact model construction [54], but in some cases it is hard to apply, and local sector nonlinearities should be considered instead.

The following example, taken from [54], shows an application of this approach.

Consider the nonlinear system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) + x_1(t)x_2^3(t) \\ -x_2(t) + \left(3 + x_2(t)\right)x_1^3(t) \end{pmatrix} \quad \begin{matrix} x_1(t) \in [-1, 1] \\ x_2(t) \in [-1, 1] \end{matrix} \tag{2.34}$$

Equation (2.34) can be rewritten as:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & x_1(t)x_2^2(t) \\ \left(3 + x_2(t)\right)x_1^2(t) & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{2.35}$$

By choosing the premise variables $\vartheta_1(t) = x_1(t)x_2^2(t)$ and $\vartheta_2(t) = \left(3 + x_2(t)\right)x_1^2(t)$, and calculating the minimum and maximum values of $\vartheta_1(t)$ and $\vartheta_2(t)$ over the considered intervals, i.e. $\vartheta_1(t) \in [-1, 1]$ and $\vartheta_2(t) \in [0, 4]$, the fuzzy model (2.28) is obtained, with:

$$M_{11} = M_{21} = \frac{z_1(t) + 1}{2} \tag{2.36}$$

$$M_{31} = M_{41} = \frac{1 - z_1(t)}{2} \tag{2.37}$$

$$M_{12} = M_{32} = \frac{z_2(t)}{4} \tag{2.38}$$

$$M_{22} = M_{42} = \frac{4 - z_2(t)}{4} \tag{2.39}$$

and:

$$A_1 = \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} \ A_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix} \; A_4 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \tag{2.40}$$

It is worth mentioning that the choice of the premise variables is not unique, and different TS representations of the same nonlinear system are possible. This fact will be further investigated in the next chapter.

2.3.2 Local Approximation in Fuzzy Partition Spaces

The spirit of this approach is to approximate nonlinear terms by judiciously choosing linear terms, thus reducing the number of fuzzy rules, being this particularly important at the control system design step [54]. However, since the obtained model does not represent exactly the original nonlinear system, the designed control system could fail in guaranteeing the stability of the original nonlinear system.

The following example, taken from [54], shows an application of this approach.

Let us consider the equations of motion for an inverted pendulum [107]:

$$\dot{x}_1(t) = x_2(t) \tag{2.41}$$

$$\dot{x}_2(t) = \frac{g \sin(x_1(t)) - amlx_2^2(t) \sin(2x_1(t))/2 - a\cos(x_1(t))\, u(t)}{4l/3 - aml\cos^2(x_1(t))} \tag{2.42}$$

where $x_1(t)$ denotes the angle of the pendulum from the vertical, and $x_2(t)$ is the angular velocity; g is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, u is the force applied to the cart, and $a = 1/(m + M)$.

When $x_1(t)$ is near zero, (2.42) can be simplified as:

$$\dot{x}_2(t) = \frac{gx_1(t) - au(t)}{4l/3 - aml} \tag{2.43}$$

On the other hand, when $x_1(t)$ is near $\pm\pi/2$, (2.42) can be simplified as:

$$\dot{x}_2(t) = \frac{2gx_1(t)/\pi - a\beta u(t)}{4l/3 - aml\beta^2} \tag{2.44}$$

with $\beta = \cos(88°)$.

Then, a TS fuzzy model with two subsystems can be obtained:

$$A_1 = \begin{pmatrix} 0 & 1 \\ \frac{g}{4l/3-aml} & 0 \end{pmatrix} \; B_1 = \begin{pmatrix} 0 \\ -\frac{a}{4l/3-aml} \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3-aml\beta^2)} & 0 \end{pmatrix} \; B_2 = \begin{pmatrix} 0 \\ -\frac{a\beta}{4l/3-aml\beta^2} \end{pmatrix} \tag{2.45}$$

Notice that by applying the sector nonlinearity approach described in Sect. 2.3.1, sixteen subsystems would have been obtained. Hence, the reduction of fuzzy rules is considerable.

2.4 Analysis of LPV and TS Systems

This section recalls some of the most popular approaches for analyzing an LPV or a TS system. As it will be shown in the next chapter, there are strong analogies between the two frameworks, and the tools developed for a class of system usually apply to the other one too. For this reason, the definitions and theorems recalled in this section are shown for the LPV framework only.

First of all, let us recall some definitions.

Definition 2.1 (*Poles of an LPV system* [108]) Given an autonomous LPV system:

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) \tag{2.46}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $\theta(\tau) \in \Theta \subset \mathbb{R}^{n_\theta}$ is the varying parameter vector, $A\left(\theta(\tau)\right)$ is a varying matrix of appropriate dimensions, the poles of (2.46) are the set of all the poles of the LTI systems obtained by freezing $\theta(\tau)$ to all its possible values $\theta \in \Theta$.

Definition 2.2 (*$\mathcal{D}$-stability of an LPV system*) Given a subset $\mathcal{D}$ of the complex plane, the autonomous LPV system (2.46) is said to be $\mathcal{D}$-stable if all its poles lie in $\mathcal{D}$.

Notice that, unlike the LTI case, in general the notions of stability and $\mathcal{D}$-stability are not related. In fact, a $\mathcal{D}$-stable system could be unstable even if $\mathcal{D}$ is contained within the left-hand semiplane $Re(s) < 0$ in the CT case or the unit circle in the DT case [109]. Also, an LPV system could have some unstable poles, and yet be stable [110].

Definition 2.3 (*LMI regions* [111]) A subset $\mathcal{D}$ of the complex plane is called a linear matrix inequality (LMI) region if there exist matrices $\alpha = [\alpha_{kl}]_{k,l\in\{1,\dots,m\}} \in \mathbb{S}^{m\times m}$ and $\beta = [\beta_{k,l}]_{k,l\in\{1,\dots,m\}} \in \mathbb{R}^{m\times m}$ such that:

$$\mathcal{D} = \{\sigma \in \mathbb{C} : f_{\mathcal{D}}(\sigma) \prec O\} \tag{2.47}$$

where $f_{\mathcal{D}}(\sigma)$ is the *characteristic function* defined as:

$$f_{\mathcal{D}}(\sigma) = \alpha + \beta\sigma + \beta^{\mathrm{T}}\sigma^* = \left[\alpha_{kl} + \beta_{kl}\sigma + \beta_{lk}\sigma^*\right]_{k,l\in\{1,\dots,m\}} \tag{2.48}$$

In other words, LMI regions are subsets of the complex plane that are represented by an LMI in σ and σ^*. In [111], it was shown that LMI regions do not only include a wide variety of typical clustering regions, but also form a dense subset of the convex regions that are symmetric with respect to the real axis. Among the regions that are representable as LMI regions, there are:

- Left-hand semiplanes $Re(\sigma) < \lambda$

$$\alpha = -2\lambda \; \beta = 1$$

- Right-hand semiplanes $Re(\sigma) > \lambda$

$$\alpha = 2\lambda \; \beta = -1$$

- Disks of radius r and center $(-q, 0)$

$$\alpha = \begin{pmatrix} -r & q \\ q & -r \end{pmatrix} \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- Horizontal strips $-\omega < Im(\sigma) < \omega$

$$\alpha = \begin{pmatrix} -2\omega & 0 \\ 0 & -2\omega \end{pmatrix} \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Definition 2.4 ($\mathcal{H}_\infty$ *norm* [112]) For a stable real-rational transfer matrix $T(\sigma)$, the $\mathcal{H}_\infty$ norm is defined as:

$$\begin{aligned} \|T(s)\|_\infty &= \sup_{\omega \in \mathbb{R}} \sigma_{\max}\left(T(j\omega)\right) && \text{CT systems} \\ \|T(z)\|_\infty &= \sup_{\omega \in [-\pi,\pi]} \sigma_{\max}\left(T(e^{j\omega})\right) && \text{DT systems} \end{aligned} \tag{2.49}$$

where $\sigma_{max}(M)$ denotes the largest singular value of the matrix M.

The $\mathcal{H}_\infty$ norm measures the system input-output gain for finite energy signals across all input/output channels.

Definition 2.5 ($\mathcal{H}_\infty$ *performance of an LPV system*) The LPV system:

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + B_w\left(\theta(\tau)\right)w(\tau) \tag{2.50}$$

$$z_\infty(\tau) = C_{z_\infty}\left(\theta(\tau)\right)x(\tau) + D_{z_\infty w}\left(\theta(\tau)\right)w(\tau) \tag{2.51}$$

has $\mathcal{H}_\infty$ performance γ_∞ if $\left\|T_{z_\infty w}(\sigma, \theta)\right\|_\infty < \gamma_\infty \; \forall\theta \in \Theta$, where $T_{z_\infty w}(\sigma, \theta)$ denotes the closed-loop transfer function from $w(\tau)$ to $z_\infty(\tau)$.

The $\mathcal{H}_\infty$ performance can be interpreted as a disturbance rejection performance, and is convenient to enforce robustness against model uncertainty, and to express frequency-domain specifications such as bandwidth, low-frequency gain, and roll-off [113].

Definition 2.6 (*$\mathcal{H}_2$ norm* [113]) For a stable real-rational transfer matrix $T(\sigma)$, the $\mathcal{H}_2$ norm is defined as:

$$\begin{aligned} \|T(s)\|_2 &= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} Tr\left(T(j\omega)^H T(j\omega)\right) d\omega} \text{ CT systems} \\ \|T(z)\|_2 &= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} Tr\left(T(e^{j\omega})^H T(e^{j\omega})\right) d\omega} \text{ DT systems} \end{aligned} \tag{2.52}$$

where $Tr(M)$ denotes the trace of the matrix M.

The $\mathcal{H}_2$ norm is equal to the root-mean-square of the impulse response of the system. It measures the steady-state covariance (or power) of the output response $z_2 = T(\sigma)w$ to unit white noise inputs w.

Definition 2.7 (*$\mathcal{H}_2$ performance of an LPV system*) The LPV system (2.50) and:

$$z_2(\tau) = C_{z_2}(\theta(\tau))\, x(\tau) \tag{2.53}$$

has $\mathcal{H}_2$ performance γ_2 if $\left\|T_{z_2 w}(\sigma, \theta)\right\|_2 < \gamma_2\ \forall\theta \in \Theta$, where $T_{z_2 w}(\sigma, \theta)$ denotes the closed-loop transfer function from $w(\tau)$ to $z_2(\tau)$.

The $\mathcal{H}_2$ performance is useful to handle stochastic aspects such as measurement noise and random disturbances [113].

Definition 2.8 (*Finite time stability* [114, 115]) The autonomous LPV system (2.46) is said to be *finite time stable* (FTS) with respect to (c_1, c_2, T, R) with $c_2 > c_1 > 0$ and $R \succ 0$ if:

$$x(0)^{\mathrm{T}} R x(0) \leq c_1 \ \Rightarrow\ x(\tau)^{\mathrm{T}} R x(\tau) < c_2 \quad \begin{array}{ll} \forall t \in [0, T] & \text{CT systems} \\ \forall k \in \{1, \ldots, T\} & \text{DT systems} \end{array} \tag{2.54}$$

The idea of finite time stability, originally formulated by [116], concerns the boundedness of the state of a system over a finite time interval for given initial conditions. Notice that this definition of finite time stability is different from the one provided in other works, e.g. [117], where the property of a given system to be driven to the equilibrium point in finite time is considered instead.

Definition 2.9 (*Finite time boundedness* [114, 115]) The CT LPV system:

$$\dot{x}(t) = A(\theta(t))\, x(t) + B_w(\theta(t))\, w(t) \tag{2.55}$$

and the DT LPV system:

$$\begin{cases} x(k+1) = A(\theta(k))\, x(k) + B_w(\theta(k))\, w(k) \\ w(k+1) = W(\theta(k))\, w(k) \end{cases} \tag{2.56}$$

are said to be *finite time bounded* (FTB) with respect to (c_1, c_2, T, R, d), with $c_2 > c_1 > 0$, $R \succ 0$ and $d > 0$ if:

$$\begin{cases} x(0)^{\mathrm{T}} R x(0) \leq c_1 \\ w(t)^{\mathrm{T}} w(t) \leq d \end{cases} \Rightarrow x(\tau)^{\mathrm{T}} R x(\tau) < c_2 \quad \begin{matrix} \forall t \in [0, T] & \text{CT systems} \\ \forall k \in \{1, \ldots, T\} & \text{DT systems} \end{matrix} \tag{2.57}$$

The idea of state boundedness is more general and concerns the behavior of the state in presence of external disturbances. Notice that FTS can be recovered as a special case of FTB when $w = 0$.

2.4.1 Analysis Based on a Common Quadratic Lyapunov Function

The simplest approach for the analysis of LPV/TS systems is the one based on a common quadratic Lyapunov function. In this case, the Lyapunov candidate function used to assess the chosen specification is:

$$V(x(\tau)) = x(\tau)^{\mathrm{T}} P x(\tau) \tag{2.58}$$

where $P \succ O$.

Theorem 2.1 (Quadratic stability of CT LPV systems) *The autonomous LPV system (2.46) with $t = \tau$ is quadratically stable:*

1. *if there exists $P \succ O$ such that [118]:*

$$A(\theta)^{\mathrm{T}} P + P A(\theta) \prec O \qquad \forall \theta \in \Theta \tag{2.59}$$

2. *if there exists $Q \succ 0$ such that [119]:*

$$Q A(\theta)^{\mathrm{T}} + A(\theta) Q \prec O \qquad \forall \theta \in \Theta \tag{2.60}$$

Proof It is straightforward to obtain (2.59) by calculating $\dot{V}(x(t))$, replacing $\dot{x}(t)$ with (2.46), and imposing the condition $\dot{V}(x(t)) < 0$. Then, (2.60) can be obtained from (2.59) with $Q = P^{-1}$ [119]. A relevant consequence is that the stability of the dual system:

$$\dot{x}(t) = A(\theta(t))^{\mathrm{T}} x(t) \tag{2.61}$$

is also characterized by (2.59)–(2.60). ■

Theorem 2.2 (Quadratic stability of DT LPV systems) *The autonomous LPV system (2.46) with $\tau = k$ is quadratically stable:*

1. *if there exists $P \succ O$ such that [120]:*

$$A(\theta)^{\mathrm{T}} P A(\theta) - P \prec O \qquad \forall \theta \in \Theta \tag{2.62}$$

2. *if there exists* $P \succ O$ *such that:*

$$\begin{pmatrix} -P & PA(\theta) \\ A(\theta)^{\mathrm{T}}P & -P \end{pmatrix} \prec O \qquad \forall \theta \in \Theta \tag{2.63}$$

3. *if there exists* $Q \succ O$ *such that [49]:*

$$\begin{pmatrix} -Q & A(\theta)Q \\ QA(\theta)^{\mathrm{T}} & -Q \end{pmatrix} \prec O \qquad \forall \theta \in \Theta \tag{2.64}$$

Proof It is straightforward to obtain (2.62) by calculating $\Delta V(x(k))$, replacing $x(k+1)$ with (2.46) and imposing the condition $\Delta V(x(k)) < 0$. Then, (2.63) can be easily obtained from (2.62) by using the Schur complement. Finally, (2.64) can be obtained from (2.63) with $P = Q^{-1}$. ■

Theorem 2.3 (Quadratic $\mathcal{D}$-stability of LPV systems) *Given an LMI region* $\mathcal{D}$ *defined as in (2.47), the autonomous LPV system (2.46) is quadratically* $\mathcal{D}$*-stable:*

1. *if there exists* $P \succ O$ *such that [121]:*

$$\begin{aligned} &\alpha \otimes P + \beta \otimes PA(\theta) + \beta^{\mathrm{T}} \otimes A(\theta)^{\mathrm{T}}P \\ &\quad = \left[\alpha_{kl}P + \beta_{kl}PA(\theta) + \beta_{lk}A(\theta)^{\mathrm{T}}P\right]_{k,l\in\{1,\ldots,m\}} \prec O \qquad \forall \theta \in \Theta \end{aligned} \tag{2.65}$$

2. *if there exists* $Q \succ 0$ *such that:*

$$\begin{aligned} &\alpha \otimes Q + \beta \otimes A(\theta)Q + \beta^{\mathrm{T}} \otimes QA(\theta)^{\mathrm{T}} \\ &\quad = \left[\alpha_{kl}Q + \beta_{kl}A(\theta)Q + \beta_{lk}QA(\theta)^{\mathrm{T}}\right]_{k,l\in\{1,\ldots,m\}} \prec O \qquad \forall \theta \in \Theta \end{aligned} \tag{2.66}$$

Proof The proof follows from the reasoning provided in [111], and (2.66) can be obtained as the dual matrix inequality of (2.65) [119]. ■

For an LPV system quadratically $\mathcal{D}$-stable, it is assured that its poles are in $\mathcal{D}$. As shown by [108], the quadratic $\mathcal{D}$-stability also affects the dynamical behavior of the system, justifying from the engineering point of view the definition of LPV poles given in Definition 2.1. It should be highlighted that in the case of CT systems, it can be proved that some transient properties, usually defined in terms of pole location in the case of LTI systems, hold for the LPV case too. This fact has been shown by [121], taking into account the reasoning in [122].

Corollary 2.1 (Exponential decay/growth of LPV systems) *Let* $V(x(t))$ *be defined as in (2.58), and let the autonomous LPV system (2.46) be quadratically* $\mathcal{D}$ *stable, i.e. (2.65) holds. Then, the Lyapunov function* $V(x(t))$ *satisfies, for all* $x(t) \neq 0$*:*

$$\frac{1}{2}\frac{\dot{V}(x(t))}{V(x(t))} \in \mathcal{D} \cap \mathbb{R} \tag{2.67}$$

Proof Pre-multiplying (2.65) by $I \otimes x(t)^{\mathrm{T}}$, and post-multiplying it by $I \otimes x(t)$, respectively, the following is obtained for all $x(t) \neq 0$:

$$\alpha \otimes x(t)^{\mathrm{T}} P x(t) + \beta \otimes x(t)^{\mathrm{T}} P A\left(\theta(t)\right) x(t) + \beta^{\mathrm{T}} \otimes x(t)^{\mathrm{T}} A\left(\theta(t)\right)^{\mathrm{T}} P x(t) \prec O \tag{2.68}$$

Recalling that:

$$\frac{1}{2}\dot{V}\left(x(t)\right) = x(t)^{\mathrm{T}} P A\left(\theta(t)\right) x(t) = x(t)^{\mathrm{T}} A\left(\theta(t)\right)^{\mathrm{T}} P x(t) \tag{2.69}$$

and dividing (2.68) by $V\left(x(t)\right)$, this process leads to:

$$\alpha \otimes 1 + \beta \otimes \frac{1}{2}\frac{\dot{V}\left(x(t)\right)}{V\left(x(t)\right)} + \beta^{\mathrm{T}} \otimes \frac{1}{2}\frac{\dot{V}\left(x(t)\right)}{V\left(x(t)\right)} \prec O \tag{2.70}$$

which implies (2.67). ■

As a consequence of Corollary 2.1, the system's decay/growth rate lies within the LMI region $\mathcal{D}$. Reference [121] have shown that the concept of $\mathcal{D}$-stability can also be used for imposing limits on the energy of the rate of state change, thus imposing a limit on the system's oscillatory behaviors.

Theorem 2.4 (Quadratic $\mathcal{H}_\infty$ performance of CT LPV systems) *The LPV system (2.50)–(2.51) with $\tau = t$ has quadratic $\mathcal{H}_\infty$ performance γ_∞ [111]:*

1. *if there exists $P \succ O$ such that:*

$$\begin{pmatrix} A(\theta)^{\mathrm{T}} P + P A(\theta) & P B_w(\theta) & C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} P & -I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ C_{z_\infty}(\theta) & D_{z_\infty w}(\theta) & -\gamma_\infty^2 I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.71}$$

2. *if there exists $Q \succ O$ such that:*

$$\begin{pmatrix} A(\theta) Q + Q A(\theta)^{\mathrm{T}} & B_w(\theta) & Q C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} & -I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ C_{z_\infty}(\theta) Q & D_{z_\infty w}(\theta) & -\gamma_\infty^2 I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.72}$$

Proof See [123]. ■

It is worth recalling that (2.71)–(2.72) can be replaced with [12]:

$$\begin{pmatrix} A(\theta)^{\mathrm{T}} P + P A(\theta) & P B_w(\theta) & C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} P & -\gamma_\infty I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ C_{z_\infty}(\theta) & D_{z_\infty w}(\theta) & -\gamma_\infty I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.73}$$

$$\begin{pmatrix} A(\theta) Q + Q A(\theta)^{\mathrm{T}} & B_w(\theta) & Q C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} & -\gamma_\infty I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ C_{z_\infty}(\theta) Q & D_{z_\infty w}(\theta) & -\gamma_\infty I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.74}$$

Theorem 2.5 (Quadratic $\mathcal{H}_\infty$ performance of DT LPV systems) *The LPV system (2.50)–(2.51) with $\tau = k$ has quadratic $\mathcal{H}_\infty$ performance γ_∞ [124]:*

1. if there exists $P \succ O$ such that:

$$\begin{pmatrix} P & PA(\theta) & PB_w(\theta) & O \\ A(\theta)^{\mathrm{T}}P & P & O & C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}}P & O & I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ O & C_{z_\infty}(\theta) & D_{z_\infty w}(\theta) & \gamma_\infty^2 I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.75}$$

2. if there exists $Q \succ O$ such that:

$$\begin{pmatrix} Q & A(\theta)Q & B_w(\theta) & O \\ QA(\theta)^{\mathrm{T}} & Q & O & QC_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} & O & I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ O & C_{z_\infty}(\theta)Q & D_{z_\infty w}(\theta) & \gamma_\infty^2 I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.76}$$

Proof See [123]. ■

Also in this case, (2.75)–(2.76) can be rewritten as [12]:

$$\begin{pmatrix} P & PA(\theta) & PB_w(\theta) & O \\ A(\theta)^{\mathrm{T}}P & P & O & C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}}P & O & \gamma_\infty I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ O & C_{z_\infty}(\theta) & D_{z_\infty w}(\theta) & \gamma_\infty I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.77}$$

$$\begin{pmatrix} Q & A(\theta)Q & B_w(\theta) & O \\ QA(\theta)^{\mathrm{T}} & Q & O & QC_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} & O & \gamma_\infty I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ O & C_{z_\infty}(\theta)Q & D_{z_\infty w}(\theta) & \gamma_\infty I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.78}$$

The results provided in Theorems 2.4 and 2.5 are also known as the *bounded real lemma* (BRL). Several results developed throughout this thesis based on (2.71)–(2.72) and (2.75)–(2.76) can be easily extended to the alternative formulations given by (2.73)–(2.74) and (2.77)–(2.78).

Theorem 2.6 (Quadratic $\mathcal{H}_2$ performance of CT LPV systems) *The LPV system (2.50) and (2.53) with $\tau = t$ has quadratic $\mathcal{H}_2$ performance γ_2 [111]:*

1. if there exist $P \succ O$ and $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr\,(Y(\theta)) < \gamma_2^2$ $\forall\theta \in \Theta$ and:

$$\begin{pmatrix} A(\theta)^{\mathrm{T}}P + PA(\theta) & B_w(\theta) \\ B_w(\theta)^{\mathrm{T}} & -I \end{pmatrix} \prec O \quad \forall\theta \in \Theta \tag{2.79}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta) \\ C_{z_2}(\theta)^{\mathrm{T}} & P \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.80}$$

2. *if there exist* $Q \succ O$ *and* $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr\,(Y(\theta)) < \gamma_2^2\ \forall\theta \in \Theta$ *and:*

$$\begin{pmatrix} A(\theta)Q + QA(\theta)^{\mathrm{T}} & B_w(\theta) \\ B_w(\theta)^{\mathrm{T}} & -I \end{pmatrix} \prec O \quad \forall\theta \in \Theta \tag{2.81}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q \\ QC_{z_2}(\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.82}$$

Proof See [111]. ■

Theorem 2.7 (Quadratic $\mathcal{H}_2$ performance of DT LPV systems) *The LPV system (2.50) and (2.53) with* $\tau = k$ *has quadratic* $\mathcal{H}_2$ *performance* γ_2 *[124]:*

1. *if there exist* $P \succ O$ *and* $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr\,(Y(\theta)) < \gamma_2^2\ \forall\theta \in \Theta$ *and:*

$$\begin{pmatrix} P & PA(\theta) & PB_w(\theta) \\ A(\theta)^{\mathrm{T}}P & P & O \\ B_w(\theta)^{\mathrm{T}}P & O & I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.83}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta) \\ C_{z_2}(\theta)^{\mathrm{T}} & P \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.84}$$

2. *if there exist* $Q \succ O$ *and* $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr\,(Y(\theta)) < \gamma_2^2\ \forall\theta \in \Theta$ *and:*

$$\begin{pmatrix} Q & A(\theta)Q & B_w(\theta) \\ QA(\theta)^{\mathrm{T}} & Q & O \\ B_w(\theta)^{\mathrm{T}} & O & I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.85}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q \\ QC_{z_2}(\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.86}$$

Proof See [124]. ■

Notice that, according to Schur's complements [125], in case a multiobjective specification is considered, some of the provided conditions are redundant. For example, the stability conditions provided in Theorems 2.1–2.2 can be found in the upper-left parts of (2.71)–(2.79), (2.81), (2.83) and (2.85). Also, if both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performances are considered at the same time, (2.79), (2.81), (2.83) and (2.85) are not needed, since they are already included in (2.71)–(2.76).

Theorem 2.8 (Quadratic FTB of CT LPV systems) *The LPV system (2.55) is quadratically FTB with respect to* (c_1, c_2, T, R, d) *if, letting* $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, *there exist positive scalars* a, λ_1, λ_2, λ_3 *and two positive definite matrices* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and* $Q_2 \in \mathbb{S}^{n_w \times n_w}$ *such that:*

$$\begin{pmatrix} A(\theta)\,\tilde{Q}_1 + \tilde{Q}_1 A(\theta)^{\mathrm{T}} - a\tilde{Q}_1 & B_w(\theta)\,Q_2 \\ Q_2 B_w(\theta)^{\mathrm{T}} & -aQ_2 \end{pmatrix} \prec O \quad \forall\theta \in \Theta \tag{2.87}$$

$$\lambda_1 I \prec Q_1 \prec I \tag{2.88}$$

$$\lambda_2 I \prec Q_2 \prec \lambda_3 I \tag{2.89}$$

$$\begin{pmatrix} c_2 e^{-aT} & \sqrt{c_1} & \sqrt{d} \\ \sqrt{c_1} & \lambda_1 & 0 \\ \sqrt{d} & 0 & \lambda_2 \end{pmatrix} \succ O \tag{2.90}$$

Proof It is obtained straightforwardly, taking into account that the conditions presented in Lemma 6 of [114] should hold for every possible value of θ. ■

Theorem 2.9 (Quadratic FTB of DT LPV systems) *The discrete-time LPV system (2.56) is quadratically FTB with respect to* (c_1, c_2, T, R, d) *if there exist positive scalars* a, λ_1, λ_2 *with* $a \geq 1$ *and two positive definite matrices* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and* $Q_2 \in \mathbb{S}^{n_w \times n_w}$ *such that:*

$$\begin{pmatrix} -aQ_1 & Q_1 A(\theta)^{\mathrm{T}} & O & O \\ A(\theta) Q_1 & -Q_1 & B_w(\theta) & O \\ O & B_w(\theta)^{\mathrm{T}} & -aQ_2 & W(\theta)^{\mathrm{T}} Q_2 \\ O & O & Q_2 W(\theta) & -Q_2 \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.91}$$

$$\lambda_1 R^{-1} \prec Q_1 \prec R^{-1} \tag{2.92}$$

$$O \prec Q_2 \prec \lambda_2 I \tag{2.93}$$

$$\begin{pmatrix} \frac{c_2}{a^T} - \lambda_2 d & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{2.94}$$

Proof It is obtained straightforwardly, taking into account that the conditions presented in Lemma 1 of [115] should hold for every possible value of θ. ■

Theorem 2.10 (Quadratic FTS of CT LPV systems) *The autonomous LPV system (2.46) with* $\tau = t$ *is quadratically FTS with respect to* (c_1, c_2, T, R) *if, letting* $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, *there exist positive scalars* a, λ_1 *and a positive definite matrix* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *such that (2.88) and:*

$$A(\theta) \tilde{Q}_1 + \tilde{Q}_1 A(\theta)^{\mathrm{T}} - a\tilde{Q}_1 \prec O \quad \forall \theta \in \Theta \tag{2.95}$$

$$\begin{pmatrix} c_2 e^{-aT} & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{2.96}$$

hold.

Proof It is a direct consequence of Theorem 2.8, when $B_w(\theta(t)) = O$ and $d = 0$. ■

Theorem 2.11 (Quadratic FTS of DT LPV systems) *The autonomous LPV system (2.46) with $\tau = k$ is quadratically FTS with respect to (c_1, c_2, T, R) if there exist positive scalars a, λ_1 with $a \geq 1$ and a positive definite matrix $Q_1 \in \mathbb{S}^{n_x \times n_x}$ such that:*

$$\begin{pmatrix} -a\mathrm{Q}_1 & \mathrm{Q}_1 A(\theta)^{\mathrm{T}} \\ A(\theta)\, Q_1 & -Q_1 \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.97}$$

$$\begin{pmatrix} \frac{c_2}{a^T} & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{2.98}$$

$$\lambda_1 R^{-1} \prec Q_1 \prec R^{-1} \tag{2.99}$$

Proof It is a direct consequence of Theorem 2.9, when $W(\theta(k)) = B_w(\theta(k)) = O$ and $d = 0$. ■

The problem with the conditions provided in Theorems 2.1–2.11 is that they rely on the satisfaction of infinite constraints. However, this difficulty can be overcome by considering the polytopic approach, as recalled in Sect. 2.2. In the following, for each theorem, an appropriate corollary is obtained. A mathematical proof is provided for Corollary 2.2 only, while it is omitted for the other corollaries, due to the similarity of the reasoning behind their proofs with the provided one.

Corollary 2.2 (Quadratic stability of CT LPV systems, polytopic version) *The autonomous polytopic CT LPV system:*

$$\dot{x}(t) = \sum_{i=1}^{N} \mu_i(\theta(t)) A_i x(t) \tag{2.100}$$

with coefficients μ_i such that (2.5) holds, is quadratically stable:

1. if there exists $P \succ O$ such that:

$$A_i^{\mathrm{T}} P + P A_i \prec O \qquad \forall i = 1, \ldots, N \tag{2.101}$$

2. if there exists $Q \succ O$ such that:

$$Q A_i^{\mathrm{T}} + A_i Q \prec O \qquad \forall i = 1, \ldots, N \tag{2.102}$$

Proof Due to a basic property of matrices [126], any linear combination of (2.101) and (2.102) with non-negative coefficients, of which at least one different from zero, is negative definite. Hence, using the coefficients $\mu_i(\theta(t))$, and taking into account (2.5), (2.59) and (2.60) are obtained. ■

Corollary 2.3 (Quadratic stability of DT LPV systems, polytopic version) *The autonomous polytopic DT LPV system:*

$$x(k+1) = \sum_{i=1}^{N} \mu_i\,(\theta(k)) A_i x(k) \tag{2.103}$$

with coefficients μ_i such that (2.5) holds, is quadratically stable:

1. *if there exists $P \succ O$ such that:*

$$\begin{pmatrix} -P & PA_i \\ A_i^{\mathrm{T}} P & -P \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.104}$$

2. *if there exists $Q \succ O$ such that:*

$$\begin{pmatrix} -Q & A_i Q \\ Q A_i^{\mathrm{T}} & -Q \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.105}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.4 (Quadratic $\mathcal{D}$-stability of LPV systems, polytopic version) *Given an LMI region $\mathcal{D}$ defined as in (2.47), the autonomous polytopic LPV system:*

$$\sigma.x(\tau) = \sum_{i=1}^{N} \mu_i\,(\theta(\tau)) A_i x(\tau) \tag{2.106}$$

with coefficients μ_i such that (2.5) holds, is quadratically $\mathcal{D}$-stable:

1. *if there exists $P \succ O$ such that:*

$$\begin{aligned} &\alpha \otimes P + \beta \otimes PA_i + \beta^{\mathrm{T}} \otimes A_i^{\mathrm{T}} P \\ &\quad = \left[\alpha_{kl} P + \beta_{kl} PA_i + \beta_{lk} A_i^{\mathrm{T}} P\right]_{k,l \in \{1,\ldots,m\}} \prec O \quad \forall i = 1, \ldots, N \end{aligned} \tag{2.107}$$

2. *if there exists $Q \succ O$ such that:*

$$\begin{aligned} &\alpha \otimes Q + \beta \otimes A_i Q + \beta^{\mathrm{T}} \otimes Q A_i^{\mathrm{T}} \\ &\quad = \left[\alpha_{kl} Q + \beta_{kl} A_i Q + \beta_{lk} Q A_i^{\mathrm{T}}\right]_{k,l \in \{1,\ldots,m\}} \prec O \quad \forall i = 1, \ldots, N \end{aligned} \tag{2.108}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.5 (Quadratic $\mathcal{H}_\infty$ performance of CT LPV systems, polytopic version) *The polytopic CT LPV system:*

$$\dot{x}(t) = \sum_{i=1}^{N} \mu_i\,(\theta(t)) \left[A_i x(t) + B_{w,i} w(t)\right] \tag{2.109}$$

$$z_\infty(t) = \sum_{i=1}^{N} \mu_i\left(\theta(t)\right)\left[C_{z_\infty,i}x(t) + D_{z_\infty w,i}w(t)\right] \tag{2.110}$$

with coefficients μ_i *such that* (2.5) *holds, has quadratic* $\mathcal{H}_\infty$ *performance* γ_∞*:*

1. *if there exists* $P \succ O$ *such that:*

$$\begin{pmatrix} A_i^{\mathrm{T}}P + PA_i & PB_{w,i} & C_{z_\infty,i}^{\mathrm{T}} \\ B_{w,i}^{\mathrm{T}}P & -I & D_{z_\infty w,i}^{\mathrm{T}} \\ C_{z_\infty,i} & D_{z_\infty w,i} & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall i = 1,\ldots,N \tag{2.111}$$

2. *if there exists* $Q \succ O$ *such that:*

$$\begin{pmatrix} A_iQ + QA_i^{\mathrm{T}} & B_{w,i} & QC_{z_\infty,i}^{\mathrm{T}} \\ B_{w,i}^{\mathrm{T}} & -I & D_{z_\infty w,i}^{\mathrm{T}} \\ C_{z_\infty,i}Q & D_{z_\infty w,i} & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall i = 1,\ldots,N \tag{2.112}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.6 (Quadratic $\mathcal{H}_\infty$ performance of DT LPV systems, polytopic version) *The polytopic DT LPV system:*

$$x(k+1) = \sum_{i=1}^{N} \mu_i\left(\theta(k)\right)\left[A_ix(k) + B_{w,i}w(k)\right] \tag{2.113}$$

$$z_\infty(k) = \sum_{i=1}^{N} \mu_i\left(\theta(k)\right)\left[C_{z_\infty,i}x(k) + D_{z_\infty w,i}w(k)\right] \tag{2.114}$$

with coefficients μ_i *such that* (2.5) *holds, has quadratic* $\mathcal{H}_\infty$ *performance* γ_∞*:*

1. *if there exists* $P \succ O$ *such that:*

$$\begin{pmatrix} P & PA_i & PB_{w,i} & O \\ A_i^{\mathrm{T}}P & P & O & C_{z_\infty,i}^{\mathrm{T}} \\ B_{w,i}^{\mathrm{T}}P & O & I & D_{z_\infty w,i}^{\mathrm{T}} \\ O & C_{z_\infty,i} & D_{z_\infty w,i} & \gamma_\infty^2 I \end{pmatrix} \succ O \qquad \forall i = 1,\ldots,N \tag{2.115}$$

2. *if there exists* $Q \succ O$ *such that:*

$$\begin{pmatrix} Q & A_iQ & B_{w,i} & O \\ QA_i^{\mathrm{T}} & Q & O & QC_{z_\infty,i}^{\mathrm{T}} \\ B_{w,i}^{\mathrm{T}} & O & I & D_{z_\infty w,i}^{\mathrm{T}} \\ O & C_{z_\infty,i}Q & D_{z_\infty w,i} & \gamma_\infty^2 I \end{pmatrix} \succ O \qquad \forall i = 1,\ldots,N \tag{2.116}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.7 (Quadratic $\mathcal{H}_2$ performance of CT LPV systems, polytopic version) *The polytopic CT LPV system (2.109) and:*

$$z_2(t) = \sum_{i=1}^{N} \mu_i\left(\theta(t)\right) C_{z_2,i} x(t) \tag{2.117}$$

with coefficients μ_i such that (2.5) holds, has quadratic $\mathcal{H}_2$ performance γ_2:

1. *if there exist $P \succ O$ and N matrices $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr(Y_i) < \gamma_2^2$ $\forall i = 1, \ldots, N$ and:*

$$\begin{pmatrix} A_i^{\mathrm{T}} P + P A_i & B_{w,i} \\ B_{w,i}^{\mathrm{T}} & -I \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.118}$$

$$\begin{pmatrix} Y_i & C_{z_2,i} \\ C_{z_2,i}^{\mathrm{T}} & P \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.119}$$

2. *if there exist $Q \succ O$ and N matrices $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr(Y_i) < \gamma_2^2$ $\forall i = 1, \ldots, N$ and:*

$$\begin{pmatrix} A_i Q + Q A_i^{\mathrm{T}} & B_{w,i} \\ B_{w,i}^{\mathrm{T}} & -I \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.120}$$

$$\begin{pmatrix} Y_i & C_{z_2,i} Q \\ Q C_{z_2,i}^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.121}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.8 (Quadratic $\mathcal{H}_2$ performance of DT LPV systems, polytopic version) *The polytopic DT LPV system (2.113) and:*

$$z_2(k) = \sum_{i=1}^{N} \mu_i\left(\theta(k)\right) C_{z_2,i} x(k) \tag{2.122}$$

with coefficients μ_i such that (2.5) holds, has quadratic $\mathcal{H}_2$ performance γ_2:

1. *if there exist $P \succ O$ and N matrices $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr(Y_i) < \gamma_2^2$ $\forall i = 1, \ldots, N$ and:*

$$\begin{pmatrix} P & P A_i & P B_{w,i} \\ A_i^{\mathrm{T}} P & P & O \\ B_{w,i}^{\mathrm{T}} P & O & I \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.123}$$

$$\begin{pmatrix} Y_i & C_{z_2,i} \\ C_{z_2,i}^{\mathrm{T}} & P \end{pmatrix} \succ O \qquad \forall i = 1, \dots, N \tag{2.124}$$

2. *if there exist* $Q \succ O$ *and* N *matrices* $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr(Y_i) < \gamma_2^2$ $\forall i = 1, \dots, N$ *and:*

$$\begin{pmatrix} Q & A_i Q & B_{w,i} \\ Q A_i^{\mathrm{T}} & Q & O \\ B_{w,i}^{\mathrm{T}} & O & I \end{pmatrix} \succ O \qquad \forall i = 1, \dots, N \tag{2.125}$$

$$\begin{pmatrix} Y_i & C_{z_2,i} Q \\ Q C_{z_2,i}^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad \forall i = 1, \dots, N \tag{2.126}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.9 (Quadratic FTB of CT LPV systems, polytopic version) *The polytopic CT LPV system (2.109), with coefficients* μ_i *such that (2.5) holds, is quadratically FTB with respect to* (c_1, c_2, T, R, d) *if, letting* $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$*, there exist positive scalars* a, λ_1, λ_2, λ_3 *and two positive definite matrices* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and* $Q_2 \in \mathbb{S}^{n_w \times n_w}$ *such that:*

$$\begin{pmatrix} A_i \tilde{Q}_1 + \tilde{Q}_1 A_i^{\mathrm{T}} - a \tilde{Q}_1 & B_{w,i} Q_2 \\ Q_2 B_{w,i}^{\mathrm{T}} & -a Q_2 \end{pmatrix} \prec O \quad \forall i = 1, \dots, N \tag{2.127}$$

and (2.88)–(2.90) hold.

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.10 (Quadratic FTB of DT LPV systems, polytopic version) *The polytopic DT LPV system (2.113) and:*

$$w(k+1) = \sum_{i=1}^{N} \mu_i \left(\theta(k)\right) W_i w(k) \tag{2.128}$$

with coefficients μ_i *such that (2.5) holds, is quadratically FTB with respect to* (c_1, c_2, T, R, d) *if there exist positive scalars* a, λ_1, λ_2*, with* $a \geq 1$ *and two positive definite matrices* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and* $Q_2 \in \mathbb{S}^{n_w \times n_w}$ *such that:*

$$\begin{pmatrix} -a Q_1 & Q_1 A_i^{\mathrm{T}} & O & O \\ A_i Q_1 & -Q_1 & B_{w,i} & O \\ O & B_{w,i}^{\mathrm{T}} & -a Q_2 & W_i^{\mathrm{T}} Q_2 \\ O & O & Q_2 W_i & -Q_2 \end{pmatrix} \prec O \quad \forall i = 1, \dots, N \tag{2.129}$$

and (2.92)–(2.94) hold.

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.11 (Quadratic FTS of CT LPV systems, polytopic version) *The autonomous polytopic CT LPV system (2.100), with coefficients μ_i such that (2.5) holds, is quadratically FTS with respect to (c_1, c_2, T, R) if, letting $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, there exist positive scalars a, λ_1 and a positive definite matrix $Q_1 \in \mathbb{S}^{n_x \times n_x}$ such that:*

$$A_i \tilde{Q}_1 + \tilde{Q}_1 A_i^{\mathrm{T}} - a\tilde{Q}_1 \prec O \quad \forall i = 1, \ldots, N \tag{2.130}$$

(2.88) and (2.96) hold.

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.12 (Quadratic FTS of DT LPV systems, polytopic version) *The autonomous polytopic DT LPV system (2.103), with coefficients μ_i such that (2.5) holds, is quadratically FTS with respect to (c_1, c_2, T, R) if there exist positive scalars a, λ_1 with $a \geq 1$ and a positive definite matrix $Q_1 \in \mathbb{S}^{n_x \times n_x}$ such that:*

$$\begin{pmatrix} -a\mathrm{Q}_1 & \mathrm{Q}_1 A_i^{\mathrm{T}} \\ A_i Q_1 & -Q_1 \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.131}$$

and (2.98)–(2.99) hold.

Proof Similar to that of Corollary 2.2, thus omitted. ■

2.4.2 Analysis Based on Other Lyapunov Functions

In some situations, using a common quadratic Lyapunov function, as shown in Sect. 2.4.1, could not be enough, due to the introduction of conservativeness of these functions. In these cases, other types of Lyapunov functions could be used, even though at the expense of increasing the complexity of the analysis. This section reviews some of the results in this field. Mathematical details will not be provided, but the interested reader could find easily further informations in the references provided throughout this section.

The main weakness of quadratic stability is that it considers arbitrarily fast parameter variations. As a consequence, the analysis performed using the conditions presented in Sect. 2.4.1 can be very conservative for constant or slowly-varying parameters. In order to reduce the conservatism, [127] proposed extending the class of Lyapunov functions to include parameter-dependent Lyapunov functions:

$$V\left(x(\tau)\right) = x(\tau)^{\mathrm{T}} P\left(\theta(\tau)\right) x(\tau) \tag{2.132}$$

Also, [128] showed that robust stability of a time-varying system is equivalent to the existence of a parameter-dependent Lyapunov function (2.132) for some augmented system. However, the approaches proposed in [127, 128] are non-convex, and thus hardly tractable from a computational point of view. For this reason, [13] proposed a way to convexify the problem by imposing additional constraints on the parameter-dependent Lyapunov functions, obtaining a numerically tractable LMI feasibility problem. Since the bounds on the derivatives of the scheduling parameters are explicitly taken into account, the approach proposed in [13] provides a smooth transition between time invariant parameters and arbitrarily fast parameter variations. Further development of this approach can be found in [15], where $\mathcal{H}_\infty$ control synthesis was considered, in [129], where $\mathcal{H}_2$ control synthesis was considered, and in [130], where an extended characterization of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms was provided, allowing to further decrease the conservatism when using parameter-dependent Lyapunov functions. Homogeneous polynomially parameter-dependent quadratic Lyapunov functions were proposed by [131], demonstrating their effectiveness with respect to linearly parameter-dependent Lyapunov functions. A systematic procedure for constructing a family of LMI conditions of increasing precision is given in [132]. At each step, a set of LMIs provides sufficient conditions for the existence of an affine parameter-dependent Lyapunov function. Necessity is asymptotically attained through a relaxation based on a generalization of Pólya's theorem. A robust stability approach based on a Lyapunov function which depends quadratically both on the system state and the varying parameters (biquadratic stability) has been proposed by [133]. References [134–136] have shown that, by employing Lyapunov functions associated with higher-order time-derivatives of the state, simpler inequalities in a higher-dimensional space can be obtained, leading to not only simple and tractable, but also less conservative LMI conditions.

Notice that the use of parameter-dependent Lyapunov functions in the case of LPV systems is akin to the use of fuzzy Lyapunov functions in TS systems, as proposed in [72, 137].

It is worth recalling an additional line of research, that tries to enhance the concept of LMI region provided in Definition 2.3. For example, [138] have introduced $\mathcal{D}_R$ regions, obtained modifying the characteristic function (2.48), as follows:

$$f_{D_R}(\sigma) = \alpha + \beta\sigma + \beta^{\mathrm{T}}\sigma^* + \chi\sigma\sigma^* = \left[\alpha_{kl} + \beta_{kl}\sigma + \beta_{lk}\sigma^* + \chi_{kl}\sigma\sigma^*\right]_{k,l\in\{1,\dots,m\}} \tag{2.133}$$

with $\chi = [\chi_{k,l}]_{k,l\in\{1,\dots,m\}} \in \mathbb{S}^{m\times m}$. Without any assumption on the matrix χ, $\mathcal{D}_R$ regions are not convex, but when χ is positive semidefinite, $\mathcal{D}_R$ are only a slight modification of LMI regions [138], that allow applying parameter-dependent Lyapunov functions for assessing the pole clustering property. On the other hand, [139] have developed an approach that allows specifying not only a simple convex region, but also a non-convex region, defined as a number of convex subregions. The introduction of extra variables and the use of additional LMIs have been considered by [140], requiring greater computational effort, but providing sufficient conditions that are much more close to necessity. A Kalman-Yakubovich-Popov (KYP) lemma for LMI regions, to guarantee the satisfaction of a frequency domain inequality, has been discussed in [141].

2.5 Control of LPV and TS Systems

Taking into account the analysis conditions presented in Sect. 2.4, the problem of designing a control law such that the resulting closed-loop system has some desired properties will be analyzed hereafter.

For the sake of simplicity, only the case of a state-feedback control law of the form:

$$u(\tau) = K(\theta(\tau))\, x(\tau) \tag{2.134}$$

will be considered. Even though in many situations the state is not available, in most of them the system is observable, thus it is possible to add a state observer to the control loop. Then, the state observer would provide an estimation of the state to be fed back to the controller [142]. In cases where this would not be possible, other approaches may be viable, e.g. output-feedback controller synthesis [143–145] or IO controller synthesis [85, 146, 147].

The following theorems can be easily obtained taking into account the results presented in the previous section.

Theorem 2.12 (Quadratic stabilization of CT LPV systems) *The LPV system (2.1) with control law (2.134) and $\tau = t$ is quadratically stabilizable if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$He\{A(\theta)Q + B(\theta)K(\theta)Q\} \prec O \qquad \forall \theta \in \Theta \tag{2.135}$$

Proof It is obtained straightforwardly from Theorem 2.1, by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the autonomous state matrix $A(\theta)$. ■

Theorem 2.13 (Quadratic stabilization of DT LPV systems) *The LPV system (2.1) with control law (2.134) and $\tau = k$ is quadratically stabilizable if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\begin{pmatrix} -Q & A(\theta)Q + B(\theta)K(\theta)Q \\ * & -Q \end{pmatrix} \prec O \qquad \forall \theta \in \Theta \tag{2.136}$$

Proof It is obtained straightforwardly from Theorem 2.2, by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the autonomous state matrix $A(\theta)$. ■

Theorem 2.14 (Quadratic $\mathcal{D}$-stabilizability of LPV systems) *Given an LMI region $\mathcal{D}$ defined as in (2.47), the LPV system (2.1) with control law (2.134) is quadratically $\mathcal{D}$-stabilizable if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\alpha \otimes Q + He\{\beta \otimes [A(\theta)Q + B(\theta)K(\theta)Q]\} \prec O \qquad \forall \theta \in \Theta \tag{2.137}$$

Proof It is obtained straightforwardly from Theorem 2.3, by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the autonomous state matrix $A(\theta)$. ■

Theorem 2.15 (Quadratic $\mathcal{H}_\infty$ state-feedback for CT LPV systems) *The CT LPV system:*

$$\dot{x}(t) = A\left(\theta(t)\right)x(t) + B\left(\theta(t)\right)u(t) + B_w\left(\theta(t)\right)w(t) \tag{2.138}$$

$$z_\infty(t) = C_{z_\infty}\left(\theta(t)\right)x(t) + D_{z_\infty u}\left(\theta(t)\right)u(t) + D_{z_\infty w}\left(\theta(t)\right)w(t) \tag{2.139}$$

with control law (2.134) has quadratic $\mathcal{H}_\infty$ performance γ_∞ if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:

$$\begin{pmatrix} He\left\{A(\theta)Q + B(\theta)K(\theta)Q\right\} & * & * \\ B_w(\theta)^{\mathrm{T}} & -I & * \\ C_{z_\infty}(\theta)Q + D_{z_\infty u}(\theta)K(\theta)Q & D_{z_\infty w}(\theta) & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.140}$$

Proof It is obtained straightforwardly from Theorem 2.4 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$, and the closed-loop z_∞ output matrix $C_{z_\infty}(\theta) + D_{z_\infty u}(\theta)K(\theta)$ instead of the z_∞ output matrix $C_{z_\infty}(\theta)$. ■

Theorem 2.16 (Quadratic $\mathcal{H}_\infty$ state-feedback for DT LPV systems) *The DT LPV system:*

$$x(k+1) = A\left(\theta(k)\right)x(k) + B\left(\theta(k)\right)u(k) + B_w\left(\theta(k)\right)w(k) \tag{2.141}$$
$$z_\infty(k) = C_{z_\infty}\left(\theta(k)\right)x(k) + D_{z_\infty u}\left(\theta(k)\right)u(k) + D_{z_\infty w}\left(\theta(k)\right)w(k) \tag{2.142}$$

with control law (2.134) has quadratic $\mathcal{H}_\infty$ performance γ_∞ if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:

$$\begin{pmatrix} Q & A(\theta)Q + B(\theta)K(\theta)Q & B_w(\theta) & O \\ * & Q & O & QC_{z_\infty}(\theta)^{\mathrm{T}} + QK(\theta)^{\mathrm{T}}D_{z_\infty u}(\theta)^{\mathrm{T}} \\ * & * & I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ * & * & * & \gamma_\infty^2 I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.143}$$

Proof It is obtained straightforwardly from Theorem 2.5 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$, and the closed-loop z_∞ output matrix $C_{z_\infty}(\theta) + D_{z_\infty u}(\theta)K(\theta)$ instead of the z_∞ output matrix $C_{z_\infty}(\theta)$. ■

Theorem 2.17 (Quadratic $\mathcal{H}_2$ state-feedback for CT LPV systems) *The CT LPV system (2.138) and:*

$$z_2(t) = C_{z_2}\left(\theta(t)\right)x(t) + D_{z_2 u}\left(\theta(t)\right)u(t) \tag{2.144}$$

with control law (2.134) has quadratic $\mathcal{H}_2$ performance γ_2 if there exist $Q \succ O$ and matrix functions $K(\theta) \in \mathbb{R}^{n_u \times n_x}$, $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr\,(Y(\theta)) < \gamma_2^2 \;\forall\theta \in \Theta$ and:

$$\begin{pmatrix} He\,\{A(\theta)Q + B(\theta)K(\theta)Q\} & B_w(\theta) \\ * & -I \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.145}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q + D_{z_2u}(\theta)K(\theta)Q \\ * & Q \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{2.146}$$

Proof It is obtained straightforwardly from Theorem 2.6 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$, and the closed-loop z_2 output matrix $C_{z_2}(\theta) + D_{z_2u}(\theta)K(\theta)$ instead of the z_2 output matrix $C_{z_2}(\theta)$. ■

Theorem 2.18 (Quadratic $\mathcal{H}_2$ state-feedback for DT LPV systems) *The DT LPV system (2.141) and:*

$$z_2(k+1) = C_{z_2}\,(\theta(k))\,x(k) + D_{z_2u}\,(\theta(k))\,u(k) \tag{2.147}$$

with control law (2.134) has quadratic $\mathcal{H}_2$ performance γ_2 if there exist $Q \succ O$ and matrix functions $K(\theta) \in \mathbb{R}^{n_u \times n_x}$, $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr\,(Y(\theta)) < \gamma_2^2 \;\forall\theta \in \Theta$ and:

$$\begin{pmatrix} Q & A(\theta)Q + B(\theta)K(\theta)Q & B_w(\theta) \\ * & Q & O \\ * & * & I \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{2.148}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q + D_{z_2u}(\theta)K(\theta)Q \\ * & Q \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{2.149}$$

Proof It is obtained straightforwardly from Theorem 2.7 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$, and the closed-loop z_2 output matrix $C_{z_2}(\theta) + D_{z_2u}(\theta)K(\theta)$ instead of the z_2 output matrix $C_{z_2}(\theta)$. ■

Theorem 2.19 (Quadratic FTB state-feedback for CT LPV systems) *The CT LPV system (2.138) with control law (2.134) is quadratically FTB with respect to (c_1, c_2, T, R, d) if, letting $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, there exist positive scalars a, λ_1, λ_2, λ_3, two positive definite matrices $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and $Q_2 \in \mathbb{S}^{n_w \times n_w}$, and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\begin{pmatrix} He\left\{A(\theta)\tilde{Q}_1 + B(\theta)K(\theta)\tilde{Q}_1\right\} - a\tilde{Q}_1 & B_w(\theta)Q_2 \\ * & -aQ_2 \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.150}$$

and (2.88)–(2.90) hold.

Proof It is obtained straightforwardly from Theorem 2.8 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$. ■

Theorem 2.20 (Quadratic FTB state-feedback for DT LPV systems) *The DT LPV system (2.141) and:*

$$w(k+1) = W(\theta(k))\, w(k) \tag{2.151}$$

with control law (2.134) is quadratically FTB with respect to (c_1, c_2, T, R, d) *if there exist positive scalars* a, λ_1, λ_2 *with* $a \geq 1$, *two positive definite matrices* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and* $Q_2 \in \mathbb{S}^{n_w \times n_w}$ *and a matrix function* $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ *such that:*

$$\begin{pmatrix} -aQ_1 & * & * & * \\ A(\theta)Q_1 + B(\theta)K(\theta)Q_1 & -Q_1 & * & * \\ O & B_w(\theta)^{\mathrm{T}} & -aQ_2 & * \\ O & O & Q_2W(\theta) & -Q_2 \end{pmatrix} \prec O \qquad \forall \theta \in \Theta \tag{2.152}$$

and (2.92)–(2.94) hold.

Proof It is obtained straightforwardly from Theorem 2.9 by considering $A(\theta) + B(\theta)K(\theta)$ instead of $A(\theta)$. ■

Theorem 2.21 (Quadratic finite time stabilization of CT LPV systems) *The LPV system (2.1) with control law (2.134) and* $\tau = t$ *is quadratically finite time stabilizable with respect to* (c_1, c_2, T, R) *if, letting* $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, *there exist positive scalars* a, λ_1, *a positive definite matrix* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and a matrix function* $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ *such that:*

$$He\left\{A(\theta)\tilde{Q}_1 + B(\theta)K(\theta)\tilde{Q}_1\right\} - a\tilde{Q}_1 \prec O \qquad \forall \theta \in \Theta \tag{2.153}$$

(2.88) and (2.96) hold.

Proof It is obtained straightforwardly from Theorem 2.10 by considering $A(\theta) + B(\theta)K(\theta)$ instead of $A(\theta)$. ■

Theorem 2.22 (Quadratic finite time stabilization of DT LPV systems) *The LPV system (2.1) with control law (2.134) and* $\tau = k$ *is quadratically finite time stabilizable with respect to* (c_1, c_2, T, R) *if there exist positive scalars* a, λ_1 *with* $a \geq 1$, *a positive definite matrix* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and a matrix function* $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ *such that:*

$$\begin{pmatrix} -\alpha Q_1 & * \\ A(\theta)Q_1 + B(\theta)K(\theta)Q_1 & -Q_1 \end{pmatrix} \prec O \qquad \forall \theta \in \Theta \tag{2.154}$$

and (2.98)–(2.99) hold.

Proof It is obtained straightforwardly from Theorem 2.11 by considering $A(\theta) + B(\theta)K(\theta)$ instead of $A(\theta)$. ■

However, similarly to the previous section, Theorems 2.12–2.22 imply infinite constraints to be checked, that can be reduced to a finite number using the polytopic approach described in Sect. 2.2. In this case, the matrices $A\left(\theta(\tau)\right)$, $B_w\left(\theta(\tau)\right)$, $C_{z_\infty}\left(\theta(\tau)\right)$, $D_{z_\infty w}\left(\theta(\tau)\right)$, $C_{z_2}\left(\theta(\tau)\right)$, $W\left(\theta(k)\right)$ are assumed to be polytopic, as follows:

$$\begin{pmatrix} A\left(\theta(\tau)\right) \\ B_w\left(\theta(\tau)\right) \\ C_{z_\infty}\left(\theta(\tau)\right) \\ D_{z_\infty w}\left(\theta(\tau)\right) \\ C_{z_2}\left(\theta(\tau)\right) \\ W\left(\theta(k)\right) \end{pmatrix} = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right) \begin{pmatrix} A_i \\ B_{w,i} \\ C_{z_\infty,i} \\ D_{z_\infty w,i} \\ C_{z_2,i} \\ W_i \end{pmatrix} \tag{2.155}$$

where the coefficients μ_i satisfy the property (2.5). On the other hand, the matrices B, $D_{z_\infty u}$ and $D_{z_2 u}$ are assumed to be constant. This assumption is not restrictive, since in the case of varying matrices $B\left(\theta(\tau)\right)$, $D_{z_\infty u}\left(\theta(\tau)\right)$ and $D_{z_2 u}\left(\theta(\tau)\right)$, a prefiltering of the input $u(\tau)$ would lead to obtain a new system with constant matrices $\tilde{B}$, $\tilde{D}_{z_\infty u}$ and $\tilde{D}_{z_2 u}$ [12]. More specifically, for the system:

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + B\left(\theta(\tau)\right)u(\tau) + B_w\left(\theta(\tau)\right)w(\tau) \tag{2.156}$$

$$z_\infty(\tau) = C_{z_\infty}\left(\theta(\tau)\right)x(\tau) + D_{z_\infty u}\left(\theta(\tau)\right)u(\tau) + D_{z_\infty w}\left(\theta(\tau)\right)w(\tau) \tag{2.157}$$

$$z_2(\tau) = C_{z_2}\left(\theta(\tau)\right)x(\tau) + D_{z_2 u}\left(\theta(\tau)\right)u(\tau) \tag{2.158}$$

let us define a new control input $\tilde{u}(\tau)$ such that:

$$\sigma.x_u(\tau) = A_u\left(\theta(\tau)\right)x_u(\tau) + B_u\tilde{u}(\tau) \tag{2.159}$$

$$u(\tau) = C_u x_u(\tau) \tag{2.160}$$

with $A_u\left(\theta(\tau)\right)$ stable. Then, the resulting LPV system would be:

$$\begin{pmatrix} \sigma.x(\tau) \\ \sigma.x_u(\tau) \end{pmatrix} = \begin{pmatrix} A\left(\theta(\tau)\right) & B\left(\theta(\tau)\right)C_u \\ O & A_u\left(\theta(\tau)\right) \end{pmatrix} \begin{pmatrix} x(\tau) \\ x_u(\tau) \end{pmatrix} + \begin{pmatrix} O \\ B_u \end{pmatrix} \tilde{u}(\tau) + \begin{pmatrix} B_w\left(\theta(\tau)\right) \\ O \end{pmatrix} w(\tau) \tag{2.161}$$

$$z_\infty(\tau) = \begin{pmatrix} C_{z_\infty}\left(\theta(\tau)\right) & D_{z_\infty u}\left(\theta(\tau)\right)C_u \end{pmatrix} \begin{pmatrix} x(\tau) \\ x_u(\tau) \end{pmatrix} + D_{z_\infty w}\left(\theta(\tau)\right)w(\tau) \tag{2.162}$$

$$z_2(\tau) = \begin{pmatrix} C_{z_2}\left(\theta(\tau)\right) & D_{z_2 u}\left(\theta(\tau)\right)C_u \end{pmatrix} \begin{pmatrix} x(\tau) \\ x_u(\tau) \end{pmatrix} \tag{2.163}$$

that are in the desired form.

It is worth recalling that some recent research has developed design conditions that would work in the case where the matrices $B\left(\theta(\tau)\right)$, $D_{z_\infty u}\left(\theta(\tau)\right)$ and $D_{z_2 u}\left(\theta(\tau)\right)$

are varying, without the need of resorting to the input prefiltering [148]. Since these conditions are in some way conservative, many works try to reduce their pessimism. Among these works, [149] is recognized to lead to a good compromise between complexity and conservatism.

The following corollaries are obtained from Theorems 2.12–2.22, and consider a polytopic state-feedback control law (2.134), as follows:

$$u(\tau) = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right) K_i x(\tau) \tag{2.164}$$

The mathematical proof is provided only for Corollary 2.13, since the proofs of the remaining ones can be presented by a similar reasoning.

Corollary 2.13 (Design of a quadratically stabilizing polytopic state-feedback controller for CT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$ be such that:*

$$He\left\{A_i Q + B\Gamma_i\right\} \prec O \qquad \forall i = 1, \ldots, N \tag{2.165}$$

Then, the closed-loop system made up by the LPV system (2.1), with $\tau = t$, $B\left(\theta(t)\right) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, is quadratically stable.

Proof By considering that $K_i = \Gamma_i Q^{-1}$ is equivalent to $\Gamma_i = K_i Q$, (2.165) can be rewritten as:

$$He\left\{A_i Q + BK_i Q\right\} \prec O \qquad \forall i = 1, \ldots, N \tag{2.166}$$

Then, taking into account the basic property of matrices [126] that any linear combination of (2.166) with non-negative coefficients, of which at least one different from zero, is negative definite, using the coefficients $\mu_i\left(\theta(\tau)\right)$, and taking into account (2.155) and (2.164), (2.135) is obtained. ■

Corollary 2.14 (Design of a quadratically stabilizing polytopic state-feedback controller for DT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} -Q & A_i Q + B\Gamma_i \\ * & -Q \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.167}$$

Then, the closed-loop system made up by the LPV system (2.1), with $\tau = k$, $B\left(\theta(k)\right) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, is quadratically stable.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.15 (Design of a quadratically $\mathcal{D}$-stabilizing polytopic state-feedback controller for LPV systems) *Given an LMI region $\mathcal{D}$ defined as in (2.47), let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\alpha \otimes Q + He\{\beta \otimes [A_i Q + B\Gamma_i]\} \prec O \qquad \forall i = 1, \ldots, N \tag{2.168}$$

Then, the closed-loop system made up by the LPV system (2.1), with $B(\theta(\tau)) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, is quadratically $\mathcal{D}$-stable.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.16 (Design of a quadratic $\mathcal{H}_\infty$ polytopic state-feedback controller for CT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} He\{A_i Q + B\Gamma_i\} & * & * \\ B_{w,i}^{\mathrm{T}} & -I & * \\ C_{z_\infty,i} Q + D_{z_\infty u}\Gamma_i & D_{z_\infty w,i} & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.169}$$

Then, the closed-loop system made up by the LPV system (2.138)–(2.139), with $B(\theta(t)) = B$, $D_{z_\infty u}(\theta(t)) = D_{z_\infty u}$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, has quadratic $\mathcal{H}_\infty$ performance γ_∞.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.17 (Design of a quadratic $\mathcal{H}_\infty$ polytopic state-feedback controller for DT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} Q & A_i Q + B\Gamma_i & B_{w,i} & O \\ * & Q & O & QC_{z_\infty,i}^{\mathrm{T}} \\ * & * & I & D_{z_\infty w,i}^{\mathrm{T}} \\ * & * & * & \gamma_\infty^2 \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.170}$$

Then, the closed-loop system made up by the LPV system (2.141)–(2.142), with $B(\theta(k)) = B$, $D_{z_\infty u}(\theta(k)) = D_{z_\infty u}$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, has quadratic $\mathcal{H}_\infty$ performance γ_∞.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.18 (Design of a quadratic $\mathcal{H}_2$ polytopic state-feedback controller for CT LPV systems) *Let $Q \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$ and $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$, $i = 1, \ldots, N$, be such that:*

$$Tr(Y_i) < \gamma_2^2 \qquad \forall i = 1, \ldots, N \tag{2.171}$$

$$\begin{pmatrix} He\{A_i Q + B\Gamma_i\} & B_{w,i} \\ * & -I \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.172}$$

$$\begin{pmatrix} Y_i & C_{z_2,i}Q + D_{z_2u}\Gamma_i \\ * & Q \end{pmatrix} \succ O \qquad \forall i = 1, \dots, N \tag{2.173}$$

Then, the closed-loop system made up by the CT LPV system (2.138) and (2.144), with $B(\theta(t)) = B$, $D_{z_2u}(\theta(t)) = D_{z_2u}$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \dots, N$, has quadratic $\mathcal{H}_2$ performance γ_2.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.19 (Design of a quadratic $\mathcal{H}_2$ polytopic state-feedback controller for DT LPV systems) *Let* $Q \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$ *and* $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$, $i = 1, \dots, N$, *be such that:*

$$Tr(Y_i) < \gamma_2^2 \qquad \forall i = 1, \dots, N \tag{2.174}$$

$$\begin{pmatrix} Q & A_iQ + B\Gamma_i & B_{w,i} \\ * & Q & O \\ * & * & I \end{pmatrix} \succ O \qquad \forall i = 1, \dots, N \tag{2.175}$$

$$\begin{pmatrix} Y_i & C_{z_2,i}Q + D_{z_2u}\Gamma_i \\ * & Q \end{pmatrix} \succ O \qquad \forall i = 1, \dots, N \tag{2.176}$$

Then, the closed-loop system made up by the DT LPV system (2.141) and (2.147), with $B(\theta(k)) = B$, $D_{z_2u}(\theta(k)) = D_{z_2u}$, *and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as* $K_i = \Gamma_i Q^{-1}$, $i = 1, \dots, N$, *has quadratic* $\mathcal{H}_2$ *performance* γ_2.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.20 (Design of a quadratic FTB polytopic state-feedback controller for CT LPV systems) *Fix* $a > O$, *and let* $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $Q_1 \succ O$, $Q_2 \succ O$, *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, N$, *be such that:*

$$\begin{pmatrix} He\left\{A_i\tilde{Q}_1 + B\Gamma_i\right\} - a\tilde{Q}_1 & B_{w,i}Q_2 \\ * & -aQ_2 \end{pmatrix} \prec O \qquad \forall i = 1, \dots, N \tag{2.177}$$

and (2.88)–(2.90) hold, where $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$. *Then, the closed-loop system made up by the CT LPV system (2.138), with* $B(\theta(t)) = B$, *and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as* $K_i = \Gamma_i\tilde{Q}_1^{-1}$, $i = 1, \dots, N$, *is quadratically FTB with respect to* (c_1, c_2, T, R, d).

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.21 (Design of a quadratic FTB polytopic state-feedback controller for DT LPV systems) *Fix* $a \geq 1$, *and let* $\lambda_1 > 0$, $\lambda_2 > 0$, $Q_1 \succ O$, $Q_2 \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, N$, *be such that:*

$$\begin{pmatrix} -aQ_1 & * & * & * \\ A_iQ_1 + B\Gamma_i & -Q_1 & * & * \\ O & B_{w,i}^{\mathrm{T}} & -aQ_2 & * \\ O & O & Q_2W_i & -Q_2 \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.178}$$

and (2.92)–(2.94) hold. Then, the closed-loop system made up by the DT LPV system (2.141) and (2.151), with $B(\theta(k)) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q_1^{-1}$, $i = 1, \ldots, N$, is quadratically FTB with respect to (c_1, c_2, T, R, d).

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.22 (Design of a quadratically finite time stabilizing polytopic state-feedback controller for CT LPV systems) *Fix $a > 0$, and let $\lambda_1 > 0$, $Q_1 \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$He\left\{A_i\tilde{Q}_1 + B\Gamma_i\right\} - a\tilde{Q}_1 \prec O \qquad \forall i = 1, \ldots, N \tag{2.179}$$

Equation (2.88) and (2.96) hold, where $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$. Then, the closed-loop system made up by the CT LPV system (2.1), with $\tau = t$, $B(\theta(t)) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i\tilde{Q}_1^{-1}$, $i = 1, \ldots, N$, is quadratically FTS with respect to (c_1, c_2, T, R).

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.23 (Design of a quadratically finite time stabilizing polytopic state-feedback controller for DT LPV systems) *Fix $a \geq 1$, and let $Q_1 \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} -aQ_1 & * \\ A_iQ_1 + B\Gamma_i & -Q_1 \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.180}$$

and (2.98)–(2.99) hold. Then, the closed-loop system made up by the DT LPV system (2.1), with $\tau = k$, $B(\theta(k)) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q_1^{-1}$, $i = 1, \ldots, N$, is quadratically FTS with respect to (c_1, c_2, T, R).

Proof Similar to that of Corollary 2.13, thus omitted. ■

2.6 Conclusions

This chapter has presented some background on gain-scheduling. Some basic concepts about modeling of LPV and TS systems have been recalled, and different methods for obtaining such models starting from an available nonlinear state-space model have been illustrated using some examples. For LPV systems, the following methods have been recalled: (a) the Jacobian linearization approach, based on the interpolation of LTI models obtained as first-order Taylor-series approximations of the nonlinear systems around some equilibrium points of interest; (b) the state transformation approach, where a coordinate change is performed with the aim of removing any nonlinear term not dependent on the scheduling parameters; and (c) the function substitution approach, that replaces a *decomposition function* with functions that are linear with respect to the scheduling parameters. For TS systems, the following methods have been recalled: (d) the sector nonlinearity approach, that aims at finding global sectors through which an exact model representation is guaranteed; and (e) the local approximation in fuzzy partition spaces, where nonlinear terms are approximated by judiciously choosing linear terms, with the effect of reducing the number of fuzzy rules.

Afterwards, the problem of analyzing whether or not some properties hold for a given LPV system has been considered. The definitions in the case of LPV systems of poles, LMI regions, $\mathcal{H}_\infty$ norm, $\mathcal{H}_\infty$ performance, $\mathcal{H}_2$ norm, $\mathcal{H}_2$ performance, finite time stability and finite time boundedness have been provided. Detailed conditions to perform the analysis based on a common quadratic Lyapunov function have been listed, and it has been shown that a finite number of LMIs can be obtained by considering the polytopic approach.

Finally, it has been shown how the analysis conditions can be taken into account for designing a state-feedback control law such that the resulting closed-loop system has some desired properties.

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Chapter 3
Automated Generation and Comparison of Takagi-Sugeno and Polytopic Quasi-LPV Models

The content of this chapter is based on the following work:

- [1] D. Rotondo, V. Puig, F. Nejjari, M. Witczak. Automated generation and comparison of Takagi-Sugeno and polytopic quasi-LPV models. *Fuzzy Sets and Systems*, 277:44–64, 2015.

3.1 Introduction

Despite the strong similarities of the two paradigms, LPV and TS systems have nearly always been treated as though as they belonged to two different worlds. In fact, the research for each of them has been performed in an independent way and, as a result, cross-references between papers dealing with the LPV theory and those dealing with the TS theory are quite uncommon. As a consequence, some theoretical results that could be useful for both types of systems have been applied only to one type.

However, in some recent works, some clues that there is a close connection between the LPV theory and the fuzzy TS paradigms have been presented [2, 3]. In [4], Rong and Irwin have pointed out that LPV systems can describe TS fuzzy models if the scheduling functions of the former paradigm are treated as membership functions of the latter one. Bergsten and his co-workers [5] have pointed out that, since it has been proved that a TS fuzzy system, where the local affine dynamic models are off-equilibrium local linearizations, leads to an arbitrarily close approximation of an LTV dynamical system about an arbitrary trajectory [6], the results concerning observers for TS fuzzy systems are also relevant to LPV systems. In [7], Collins has commented that, even though the results in [8] seem to be very related to existing results on LPV control, they are not put in perspective with those existing for LPV systems. He also claimed that it is apparent that the fuzzy TS model is a special case of an LPV model. However, even if from theoretical analysis and design points of view it is difficult to find clear differences between the two paradigms [9], LPV and TS systems are still considered different and their equality is dubious [10].

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This chapter openly addresses the presence of strong analogies between LPV and TS models, in an attempt to establish a bridge between these two worlds, so far considered to be different. In particular, this chapter considers the modeling problem, with the following important contributions:

- the analogies and connections between LPV and TS systems are clearly stated;
- it is shown that the method for the automated generation of LPV models by nonlinear embedding presented in [11] can be easily extended to solve the corresponding problem for TS models;
- it is shown that the method for the generation of a TS model for a given nonlinear multivariable function based on the sector nonlinearity concept [12], can be extended to the problem of generating a polytopic LPV model for a given nonlinear dynamical system;
- two measures are proposed in order to compare the obtained models and choose which one can be considered *the best one*. The first measure is based on the notion of *overboundedness*. The second measure is based on *region of attraction estimates*;

Notice that the resulting method for automated generation of TS models by nonlinear embedding has been already used by the fuzzy community in an intuitive way. For example, one can verify that the TS models obtained by Tanaka and Wang in [8] are contained within the set of TS models obtained through the method proposed in this chapter. Hereafter, the method used in [8] is *automated* adapting a technique developed by the LPV community that had never been used for TS systems until now.

3.2 Analogies Between Polytopic LPV and TS Systems

There are strong analogies between polytopic LPV and TS systems. In fact, the only remarkable difference between the two frameworks is the set of mathematical tools that are used for obtaining the system description. In the LPV case, these tools belong to the standard mathematics; on the other hand, in the TS case, they belong to the fuzzy theory. In particular, the correspondences between polytopic LPV and TS systems are as follows:

- the scheduling parameter θ of LPV systems correspond to the premise variables ϑ of TS systems;
- the coefficients of the polytopic decomposition μ_i correspond to the coefficients ρ_i that describe the level of activation of each local model;
- the *vertex systems* in the polytopic LPV case correspond to the *subsystems* in the TS case.

These analogies can be strongly exploited for extending techniques and results that have been developed for polytopic LPV systems to the TS case, and viceversa.

3.3 Measures for Comparison Between LPV and TS Models

3.3.1 Overboundedness-Based Measure

Given a nonlinear system of the form:

$$\sigma.x(\tau) = g\left(x(\tau), u(\tau), w(\tau)\right) \tag{3.1}$$

$$y(\tau) = h\left(x(\tau), u(\tau), w(\tau)\right) \tag{3.2}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is some exogenous signal and $y \in \mathbb{R}^{n_y}$ is the output, the approaches for automated generation of polytopic LPV and TS models proposed in this chapter provide a systematic methodology for building a whole set of LPV/TS models representing the nonlinear system (3.1)–(3.2). Hence, it is interesting to compare the obtained models in order to choose which one is *the best*.

Hereafter, a measure based on the notion of *overboundedness* is proposed, similar to the one presented in [11]. The idea is to calculate the volume of the (hyper)region contained between the vertices/subsystems (hyper)planes: the smaller is this volume, the better is the approximation offered by the polytopic LPV/TS model. To obtain the measure, subsets $S_1, \ldots, S_{n_x}$ of $\{\mathbb{X}, \mathbb{U}, \mathbb{W}, \mathbb{F}_1\}, \ldots, \left\{\mathbb{X}, \mathbb{U}, \mathbb{W}, \mathbb{F}_{n_x}\right\}$ must be chosen, where $\mathbb{X}$, $\mathbb{U}$, $\mathbb{W}$ and $\mathbb{F}_i$, $i = 1, \ldots, n_x$ are the state space, the input space, the exogenous signal space and the ith state variable derivative space, respectively. Then, if $V_1^{(S)}, \ldots, V_{n_x}^{(S)}$ are the volumes of the subsets $S_1, \ldots, S_{n_x}$, and $V_1, \ldots, V_{n_x}$ are the volumes of the (hyper)regions contained between the vertices/subsystems (hyper)planes in $S_1, \ldots, S_{n_x}$, a measure of the goodness of the polytopic LPV/TS model is given by:

$$M = \frac{V_1 V_2 \cdots V_{n_x}}{V_1^{(S)} V_2^{(S)} \cdots V_{n_x}^{(S)}} \tag{3.3}$$

where the smaller is this measure, the better is the model.[1]

Notice that in some situations, calculating the volumes $V_1, \ldots, V_{n_x}$ can be a hard task. Then, an approximate measure can be used as follows:

$$\tilde{M} = \frac{\tilde{V}_1 \tilde{V}_2 \cdots \tilde{V}_{n_x}}{V_1^{(S)} V_2^{(S)} \cdots V_{n_x}^{(S)}} \tag{3.4}$$

where $\tilde{V}_i$ is an approximation of V_i. In particular, in this chapter, each factor $\tilde{V}_i / V_i^{(S)}$ is obtained generating randomly a certain number N of points inside the subset S_i,

[1] The measure M usually decreases when the number of vertex systems/subsystems used in the considered polytopic LPV/TS model increases. In some cases, e.g. controller synthesis, this could lead to an increase in the computational effort that is not taken into account by the proposed measure M. If it is desired to include such an effect in the evaluation of the goodness of the model, a slight modification of M should be done.

and then calculating the ratio between the points that can be described by a polytopic combination through the model taken into consideration, and the total number of points. Obviously, $\tilde{M}$ approaches M in the limit as $N \to \infty$. However, it is impossible to set $N = \infty$. Thus, the problem becomes the one of selecting N in such a way that $\tilde{M}$, i.e. the estimation of M, has some desired properties. In order to do this, notice that the process of generating points in the subset S_i and checking whether or not they can be described by the model taken into consideration is a Bernoulli process [13] with a limited number N of Bernoulli trials. Hence, the estimator $\tilde{M}$ can be analyzed using the results coming from the theory of statistics and probability [14].

3.3.2 Region of Attraction Estimates-Based Measure

It is often believed that a closed-loop quasi-LPV/TS system, obtained from a nonlinear system using an exact transformation procedure, that satisfies stability (or some other goal) for all parameters varying in a convex region, e.g. a bounding box, implies that stability is satisfied for the underlying nonlinear system. This is not always true, as shown in [15], where a Van der Pol equation with reversed vector field example was used to demonstrate that the LPV/TS analysis of the nonlinear system does not guarantee local asymptotic stability. However, [15] also shows that the LPV/TS analysis can be used to estimate the region of attraction for the underlying nonlinear system. In fact, even though finding the exact region of attraction analytically might be difficult or even impossible [16], the Lyapunov functions can be used to estimate the region of attraction.

Assume that the autonomous LPV system with $\theta(\tau)$ dependent on the state $x(\tau)$:

$$\sigma.x(\tau) = A\left(\theta\left(x(\tau)\right)\right)x(\tau) \tag{3.5}$$

satisfies some stability and performance conditions, as the ones proposed in Sect. 2.4.1 $\forall\theta \in \Theta$, in the sense of decreasing the Lyapunov function (2.58):

$$V\left(x(\tau)\right) = x(\tau)^{\mathrm{T}} P x(\tau) \tag{3.6}$$

with $P \succ O$. Moreover, let us define the following sets:

$$\mathcal{X} = \{x \in D | \theta(x) \in \Theta\} \tag{3.7}$$

$$\Gamma_\beta = \{x \in D | V(x) \le \beta\} \tag{3.8}$$

where $D \subset \mathbb{R}^{n_x}$ is a given domain containing $x = 0$, and, for the nonlinear system:

$$\sigma.x(\tau) = g(x(\tau)) \tag{3.9}$$

with the origin being an equilibrium point, let us define the *region of attraction* as the set:

$$R_A = \left\{ x(0) | \lim_{\tau \to \infty} \phi(\tau; x(0)) = 0 \right\} \tag{3.10}$$

where $\phi(\tau; x(0))$ denotes the solution that starts at initial state $x(0)$ at time $\tau = 0$.

Then, the following theorem holds:

Theorem 3.1 *Consider the nonlinear system* (3.9)*, with the exact quasi-LPV representation* (3.5)*. If* $\Gamma_\beta \subseteq \mathcal{X}$ *then* $\Gamma_\beta \subseteq R_A$*, where* R_A *is the region of attraction.*

Proof See [15]. □

A consequence of this theorem is that an approximation of the region of attraction is given by the (hyper)ellipsoid provided by the positive definite matrix P of the Lyapunov function (3.6). Hereafter, a measure based on the approximation of the region of attraction is proposed in order to compare quasi-LPV and TS models obtained from the same nonlinear system, as follows:

$$M_\beta = \frac{V_\beta}{V_\Theta} \tag{3.11}$$

where V_β is the volume of Γ_β, and V_Θ is the volume of the polytopic region Θ within which the parameter vector θ (or the premise variables ϑ in the case of TS representation) can take values.

3.4 Generation of TS Models via Nonlinear Embedding

A method for the automated generation of LPV models, when affine or polytopic models are desired, has been presented in [11]. These models are generated from a general nonlinear model by *hiding* the nonlinearities in the scheduling parameters. In this section, it is shown that this method can be used for generating a TS model from a given nonlinear model.

Consider the nonlinear state[2] equation (3.1). The automated generation of TS models via nonlinear embedding consists of the following five steps:

- In the first step, (3.1) is rewritten in a standard form, that is, each of its rows is expanded into its summands g_{ij}:

$$\sigma.x_i = \sum_{j=1}^{T_i} g_{ij}(x, u, w), \quad i = 1, \ldots, n_x \tag{3.12}$$

[2]The method can be applied to the output equation (3.2) without significant differences.

where T_i is the total number of summands of that row. Then, each summand is decomposed into its numerator α_{ij}, denominator β_{ij} and constant factor κ_{ij}:

$$\sigma.x_i = \sum_{j=1}^{T_i} \kappa_{ij} \frac{\alpha_{ij}(x, u, w)}{\beta_{ij}(x, u, w)}, \quad i = 1, \ldots, n_x \tag{3.13}$$

Finally, the numerator is factored as the product of non-factorisable terms l_{ij} and integer powers of the states x_q, $q = 1, \ldots, n_x$ and the inputs u_r, $r = 1, \ldots, n_u$:

$$\alpha_{ij} = \prod_{q=1}^{n_x} \prod_{r=1}^{n_u} l_{ij}(x, u, w) x_q^{\mu_{ijq}} u_r^{\nu_{ijr}} \tag{3.14}$$

- In the second step, two classes of summands are distinguished: (a) *constant or non-factorisable numerator*, $\mathcal{K}_0$, when neither a power of the state x_i nor of an input u_i is a factor of the numerator; and (b) *arbitrary positive power of factor*, $\mathcal{K}_P$, when the summand has a numerator with positive integer powers of a state variable x_i or input u_i;
- In the third step, according to the classification of each summand, components ϑ^a_{ijk} and ϑ^b_{ijk} that link the summand to the entries of the state and input matrices A and B are chosen. If the summand g_{ij} belongs to $\mathcal{K}_0$, one can obtain n_x possible assignments to the state matrix A and n_u possible assignments to the input matrix B, with ϑ^a_{ijk} and ϑ^b_{ijk} defined as follows:

$$\vartheta^a_{ijk} = \kappa_{ij} \frac{\alpha_{ij}(x, u, w)}{\beta_{ij}(x, u, w) x_k}, \quad k = 1, \ldots, n_x \tag{3.15}$$

$$\vartheta^b_{ijk} = \kappa_{ij} \frac{\alpha_{ij}(x, u, w)}{\beta_{ij}(x, u, w) u_k}, \quad k = 1, \ldots, n_u \tag{3.16}$$

  Otherwise, if the summand g_{ij} belongs to $\mathcal{K}_p$, one can choose to assign the summand to an element of the state or input matrix, as long as the element is a factor of the numerator, i.e. if there exists a k for which $\mu_{ijk} \neq 0$ or $\nu_{ijk} \neq 0$;
- In the fourth step, the premise variables ϑ are derived from ϑ^a_{ijk} and ϑ^b_{ijk}. This can be done either by *direct assignment* or by *superposition*. In the *direct assignment* case, the premise variables are directly chosen as ϑ^a_{ijk} and ϑ^b_{ijk}, such that:

$$a_{ik} = \sum_{j=1}^{\zeta_a} \vartheta^a_{ijk} \quad b_{ik} = \sum_{j=1}^{\zeta_b} \vartheta^b_{ijk} \tag{3.17}$$

  where ζ_a and ζ_b are the number of components of the same equation $\sigma.x_i$ that are assigned to the same state x_k or input u_k, respectively, but have been obtained from different summands. In the *superposition* case, the premise variables, denoted by

ϑ^a_{ik} and ϑ^b_{ik}, are obtained through a sum of all the contributions of a summand to the same element of A or B:

$$\vartheta^a_{ik} = \sum_{j=1}^{\zeta_a} \vartheta^a_{ijk} \quad \vartheta^b_{ik} = \sum_{j=1}^{\zeta_b} \vartheta^b_{ijk} \tag{3.18}$$

such that the premise variables correspond to the elements of the state space matrices:

$$a_{ik} = \vartheta^a_{ik} \quad b_{ik} = \vartheta^b_{ik} \tag{3.19}$$

In both cases, the premise variables need to be renumbered in order to be coherent with the numbering presented in (2.28):

$$\begin{array}{l} IF\ \vartheta_1(\tau)\ is\ M_{i1}\ AND\ \cdots\ AND\ \vartheta_p(\tau)\ is\ M_{ip} \\ THEN \begin{cases} \sigma.x_i(\tau) = A_i x(\tau) + B_i u(\tau) \\ y_i(\tau) = C_i x(\tau) + D_i u(\tau) \end{cases} \quad i = 1, \ldots, N \end{array} \tag{3.20}$$

- In the final step, an adaptation of the technique used in [17] for obtaining polytopic LPV models, often referred to as *bounding box method*, is used to complete the generation of the TS model. The minimum and maximum values of each premise variable ϑ_i over the possible values of x, u and w, are obtained as follows:

$$\underline{\vartheta}_i = \min_{x,u,w} \vartheta_i \quad \overline{\vartheta}_i = \max_{x,u,w} \vartheta_i \tag{3.21}$$

From the maximum and minimum values, ϑ_i can be represented as:

$$\vartheta_i = M_{1i}(\vartheta_i)\underline{\vartheta}_i + M_{2i}(\vartheta_i)\overline{\vartheta}_i \tag{3.22}$$

with the additional constraint:

$$M_{1i}(\vartheta_i) + M_{2i}(\vartheta_i) = 1 \tag{3.23}$$

such that the membership functions are calculated as:

$$M_{1i}(\vartheta_i) = \frac{\overline{\vartheta}_i - \vartheta_i}{\overline{\vartheta}_i - \underline{\vartheta}_i} \quad \text{and} \quad M_{2i}(\vartheta_i) = \frac{\vartheta_i - \underline{\vartheta}_i}{\overline{\vartheta}_i - \underline{\vartheta}_i} \tag{3.24}$$

Finally, the *subsystems* are obtained by considering each possible combination of membership functions in the IF clauses of the TS model.

3.5 Generation of Polytopic LPV Models via Sector Nonlinearity

The idea of using sector nonlinearity in TS model construction first appeared in [18], where the single variable system case was considered, and extended to the multivariable case in [12]. In this section, it is shown that this method can also be used for generating a polytopic LPV model from a given nonlinear model.

Consider the nonlinear state equation (3.1), under the hypothesis that the function $g(x, u, w)$ is differentiable everywhere (as in the previous method, the application to the output equation (3.2) can be performed without significant differences). The automated generation of polytopic LPV models via sector nonlinearity concept consists of the following steps:

- In the first step, the space $\{\mathbb{X}, \mathbb{U}, \mathbb{W}\}$ is partitioned into its $2^{n_x+n_u+n_w}$ quadrants. Each quadrant is denoted by:

$$R\left(s_1^{(x)}, \ldots, s_{n_x}^{(x)}, s_1^{(u)}, \ldots, s_{n_u}^{(u)}, s_1^{(w)}, \ldots, s_{n_w}^{(w)}\right) \tag{3.25}$$

where:

$$\begin{cases} s_j^{(x)} = 1 \Leftrightarrow x_j \geq 0 \\ s_j^{(x)} = 0 \Leftrightarrow x_j \leq 0 \end{cases} \tag{3.26}$$

$$\begin{cases} s_j^{(u)} = 1 \Leftrightarrow u_j \geq 0 \\ s_j^{(u)} = 0 \Leftrightarrow u_j \leq 0 \end{cases} \tag{3.27}$$

$$\begin{cases} s_j^{(w)} = 1 \Leftrightarrow w_j \geq 0 \\ s_j^{(w)} = 0 \Leftrightarrow w_j \leq 0 \end{cases} \tag{3.28}$$

Then, each quadrant R is associated to its symmetric quadrant R^* to obtain $Q = 2^{n_x+n_u+n_w-1}$ regions:

$$R_q\left(s_1^{(x)}, \ldots, s_j^{(u)}, \ldots, s_{n_w}^{(w)}\right) \cup R_q^*\left(\neg s_1^{(x)}, \ldots, \neg s_j^{(u)}, \ldots, \neg s_{n_w}^{(w)}\right) \tag{3.29}$$

where $\neg$ denotes the negation operator and $q = 1, \ldots, Q$.
- In the second step, for each of the regions $R_q \cup R_q^*$, $q = 1, \ldots, Q$ defined in (3.29), after partially differentiating each row f_i of (3.1) with respect to $x_1, \ldots, x_{n_x}$, $u_1, \ldots, u_{n_u}$, the minimum and maximum values in the region $R_q \cup R_q^*$ are found:

$$\overline{a}_{ij}^{(q)} = \max_{x,u,w \in R_q \cup R_q^*} \frac{\partial g_i(x, u, w)}{\partial x_j} \quad \begin{array}{l} i = 1, \ldots, n_x \\ j = 1, \ldots, n_x \end{array} \tag{3.30}$$

$$\underline{a}_{ij}^{(q)} = \min_{x,u,w \in R_q \cup R_q^*} \frac{\partial g_i(x, u, w)}{\partial x_j} \quad \begin{array}{l} i = 1, \ldots, n_x \\ j = 1, \ldots, n_x \end{array} \tag{3.31}$$

$$\overline{b}_{ij}^{(q)} = \max_{x,u,w \in R_q \cup R_q^*} \frac{\partial g_i(x,u,w)}{\partial u_j} \quad \begin{array}{l} i = 1, \ldots, n_x \\ j = 1, \ldots, n_u \end{array} \tag{3.32}$$

$$\underline{b}_{ij}^{(q)} = \min_{x,u,w \in R_q \cup R_q^*} \frac{\partial g_i(x,u,w)}{\partial u_j} \quad \begin{array}{l} i = 1, \ldots, n_x \\ j = 1, \ldots, n_u \end{array} \tag{3.33}$$

- In the third step, the vertex matrices $\left(A_j^{(q)}, B_j^{(q)}\right)$ are obtained by taking into consideration all the possible combinations of the row vectors $\left[\overset{\frown}{a}_i^{(q)}, \overset{\frown}{b}_i^{(q)}\right]$ and $\left[\overset{\smile}{a}_i^{(q)}, \overset{\smile}{b}_i^{(q)}\right]$, as follows:

$$A_j^{(q)}\left(t_1^{(j)}, \ldots, t_i^{(j)}, \ldots, t_{n_x}^{(j)}\right) = \begin{pmatrix} \tilde{a}_1^{(q)} \\ \vdots \\ \tilde{a}_i^{(q)} \\ \vdots \\ \tilde{a}_{n_x}^{(q)} \end{pmatrix} \tag{3.34}$$

$$B_j^{(q)}\left(t_1^{(j)}, \ldots, t_i^{(j)}, \ldots, t_{n_x}^{(j)}\right) = \begin{pmatrix} \tilde{b}_1^{(q)} \\ \vdots \\ \tilde{b}_i^{(q)} \\ \vdots \\ \tilde{b}_{n_x}^{(q)} \end{pmatrix} \tag{3.35}$$

where:

$$\tilde{a}_i^{(q)} = \begin{cases} \overset{\frown}{a}_i^{(q)} = \left[\overset{\frown}{a}_{i1}^{(q)}\ \overset{\frown}{a}_{i2}^{(q)}\ \ldots\ \overset{\frown}{a}_{in_x}^{(q)}\right] & if \quad t_i^{(j)} = 1 \\ \overset{\smile}{a}_i^{(q)} = \left[\overset{\smile}{a}_{i1}^{(q)}\ \overset{\smile}{a}_{i2}^{(q)}\ \ldots\ \overset{\smile}{a}_{in_x}^{(q)}\right] & if \quad t_i^{(j)} = 0 \end{cases} \tag{3.36}$$

$$\tilde{b}_i^{(q)} = \begin{cases} \overset{\frown}{b}_i^{(q)} = \left[\overset{\frown}{b}_{i1}^{(q)}\ \overset{\frown}{b}_{i2}^{(q)}\ \ldots\ \overset{\frown}{b}_{in_x}^{(q)}\right] & if \quad t_i^{(j)} = 1 \\ \overset{\smile}{b}_i^{(q)} = \left[\overset{\smile}{b}_{i1}^{(q)}\ \overset{\smile}{b}_{i2}^{(q)}\ \ldots\ \overset{\smile}{b}_{in_x}^{(q)}\right] & if \quad t_i^{(j)} = 0 \end{cases} \tag{3.37}$$

and:

$$\overset{\frown}{a}_{ij}^{(q)} = \begin{cases} \overline{a}_{ij}^{(q)} & if \quad s_j^{(x)}(q) = 1 \\ \underline{a}_{ij}^{(q)} & if \quad s_j^{(x)}(q) = 0 \end{cases} \tag{3.38}$$

$$\overset{\smile}{a}_{ij}^{(q)} = \begin{cases} \underline{a}_{ij}^{(q)} & if \quad s_j^{(x)}(q) = 1 \\ \overline{a}_{ij}^{(q)} & if \quad s_j^{(x)}(q) = 0 \end{cases} \tag{3.39}$$

$$\overset{\frown}{b}{}_{ij}^{(q)} = \begin{cases} \overline{b}_{ij}^{(q)} & if \quad s_j^{(u)}(q) = 1 \\ \underline{b}_{ij}^{(q)} & if \quad s_j^{(u)}(q) = 0 \end{cases} \tag{3.40}$$

$$\overset{\frown}{b}{}_{ij}^{(q)} = \begin{cases} \underline{b}_{ij}^{(q)} & if \quad s_j^{(u)}(q) = 1 \\ \overline{b}_{ij}^{(q)} & if \quad s_j^{(u)}(q) = 0 \end{cases} \tag{3.41}$$

Then, (3.1) can be reconstructed from $\left(A_j^{(q)}, B_j^{(q)}\right)$ as follows:

$$\sigma.x = g(x, u, w) = \sum_{q=1}^{Q} \sum_{j=1}^{2^{n_x}} \alpha_j^{(q)}(x, u, w) \left(A_j^{(q)} x + B_j^{(q)} u\right) \tag{3.42}$$

where:

$$\alpha_j^{(q)}(x, u, w) = \prod_{i=1}^{n_x} \left[t_i^{(j)} \overset{\frown}{\alpha}{}_i^{(q)}(x, u, w) + \left(1 - t_i^{(j)}\right) \overset{\smile}{\alpha}{}_i^{(q)}(x, u, w) \right] \tag{3.43}$$

with:

$$\overset{\frown}{\alpha}{}_i^{(q)}(x, u, w) = \frac{g_i(x, u, w) - \overset{\smile}{a}{}_i^{(q)} x - \overset{\smile}{b}{}_i^{(q)} u}{\overset{\frown}{a}{}_i^{(q)} x + \overset{\frown}{b}{}_i^{(q)} u - \overset{\smile}{a}{}_i^{(q)} x - \overset{\smile}{a}{}_i^{(q)} u} R_q^{\in}(x, u, w) \tag{3.44}$$

$$\overset{\smile}{\alpha}{}_i^{(q)}(x, u, w) = \frac{\overset{\frown}{a}{}_i^{(q)} x + \overset{\frown}{b}{}_i^{(q)} u - g_i(x, u, w)}{\overset{\frown}{a}{}_i^{(q)} x + \overset{\frown}{b}{}_i^{(q)} u - \overset{\smile}{a}{}_i^{(q)} x - \overset{\smile}{b}{}_i^{(q)} u} R_q^{\in}(x, u, w) \tag{3.45}$$

where $R_q^{\in}(x, u, w)$ is an operator that returns 1 if (x, u, w) belongs to the region $R_q \cup R_q^*$ and 0 otherwise.

Remark: Notice that the polytopic system (3.42) is equivalent to the following quasi-LPV system:

$$\sigma.x = A(x, u, w)x + B(x, u, w)u \tag{3.46}$$

with:

$$A(x, u, w) = \sum_{q=1}^{Q} \sum_{j=1}^{2^{n_x}} \alpha_j^{(q)}(x, u, w) A_j^{(q)} \tag{3.47}$$

$$B(x, u, w) = \sum_{q=1}^{Q} \sum_{j=1}^{2^{n_x}} \alpha_j^{(q)}(x, u, w) B_j^{(q)} \tag{3.48}$$

Remark: The obtained polytopic system exhibits discontinuities in the polytopic decomposition coefficients $\alpha_j^{(q)}(x, u, w)$ at the region boundaries, i.e. along the axes that define the quadrants. In order to avoid this phenomenon, [12] suggests to add some compatibility conditions. In particular, this is obtained by replacing $\overline{a}_{ij}^{(q)}$, $\underline{a}_{ij}^{(q)}$, $\overline{b}_{ij}^{(q)}$ and $\underline{b}_{ij}^{(q)}$ in (3.38)–(3.41) with $\overline{a}_{ij}$, $\underline{a}_{ij}$, $\overline{b}_{ij}$, $\underline{b}_{ij}$, defined as follows:

$$\overline{a}_{ij} = \max_{q=1,\ldots,Q} \overline{a}_{ij}^{(q)} \quad \begin{matrix} i = 1, \ldots, n_x \\ j = 1, \ldots, n_x \end{matrix} \tag{3.49}$$

$$\underline{a}_{ij} = \min_{q=1,\ldots,Q} \underline{a}_{ij}^{(q)} \quad \begin{matrix} i = 1, \ldots, n_x \\ j = 1, \ldots, n_x \end{matrix} \tag{3.50}$$

$$\overline{b}_{ij} = \max_{q=1,\ldots,Q} \overline{b}_{ij}^{(q)} \quad \begin{matrix} i = 1, \ldots, n_x \\ j = 1, \ldots, n_u \end{matrix} \tag{3.51}$$

$$\underline{b}_{ij} = \min_{q=1,\ldots,Q} \underline{b}_{ij}^{(q)} \quad \begin{matrix} i = 1, \ldots, n_x \\ j = 1, \ldots, n_u \end{matrix} \tag{3.52}$$

3.6 Application Example

Consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = x_1 + 3\sin x_1 + x_2 - 2\sin x_2 + u_1 \\ \dot{x}_2 = x_1^2\sqrt{1 + x_2^2} + x_1 x_2 + u_2 \\ \dot{x}_3 = x_1 + x_2 - x_3 \end{cases} \tag{3.53}$$

with:

$$x_1, x_2, x_3 \in \mathcal{P} = [-\pi, \pi] \times [-\pi, \pi] \times [-\pi, \pi]$$

Hereafter, the methods described in Sects. 3.4 and 3.5 will be used to obtain TS and quasi-LPV representations of (3.53), and the measures introduced in Sect. 3.3 will be used to compare the obtained models.

3.6.1 Generation of TS Models via Nonlinear Embedding

The TS representations are obtained applying the nonlinear embedding method described in Sect. 3.4, where the final step is done by *superposition*, such that eight different TS models are generated. The general form for each TS model is the following:

$$
\begin{aligned}
&IF\ \vartheta_{11}^{(j)}\ is\ M_{i11}^{(j)}\ AND\ \vartheta_{12}^{(j)}\ is\ M_{i12}^{(j)}\ AND\ \vartheta_{21}^{(j)}\ is\ M_{i21}^{(j)}\ AND\ \vartheta_{22}^{(j)}\ is\ M_{i22}^{(j)} \\
&\quad THEN \quad \dot{x}(t) = A_i^{(j)} x(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u(t) \qquad \begin{matrix} i = 1, \dots, N_j \\ j = 1, \dots, 8 \end{matrix}
\end{aligned}
\tag{3.54}
$$

where for the jth TS model, the $N_j \in \{4, 8, 16\}$ linear models are obtained taking into consideration all possible combinations of minimum and maximum values of the premise variables $\vartheta_{11}^{(j)}$, $\vartheta_{12}^{(j)}$, $\vartheta_{21}^{(j)}$ and $\vartheta_{22}^{(j)}$.

In particular, the premise variables are defined as follows[3]:

$$
\begin{aligned}
\vartheta_{11}^{(1)}(x_1, x_2) &= \vartheta_{11}^{(2)}(x_1, x_2) = 1 + 3\frac{\sin x_1}{x_1} - 2\frac{\sin x_2}{x_1} \\
\vartheta_{11}^{(3)}(x_1) &= \vartheta_{11}^{(4)}(x_1) = 1 + 3\frac{\sin x_1}{x_1} \\
\vartheta_{11}^{(5)}(x_1, x_2) &= \vartheta_{11}^{(6)}(x_1, x_2) = 1 - 2\frac{\sin x_2}{x_1} \\
\vartheta_{12}^{(3)}(x_2) &= \vartheta_{12}^{(4)}(x_2) = 1 - 2\frac{\sin x_2}{x_2} \\
\vartheta_{12}^{(5)}(x_1, x_2) &= \vartheta_{12}^{(6)}(x_1, x_2) = 1 + 3\frac{\sin x_1}{x_2} \\
\vartheta_{12}^{(7)}(x_1, x_2) &= \vartheta_{12}^{(8)}(x_1, x_2) = 1 + 3\frac{\sin x_1}{x_2} - 2\frac{\sin x_2}{x_2} \\
\vartheta_{21}^{(1)}(x_1, x_2) &= \vartheta_{21}^{(3)}(x_1, x_2) = \vartheta_{21}^{(5)}(x_1, x_2) = \vartheta_{21}^{(7)}(x_1, x_2) = x_1\sqrt{1 + x_2^2} + x_2 \\
\vartheta_{21}^{(2)}(x_1, x_2) &= \vartheta_{21}^{(4)}(x_1, x_2) = \vartheta_{21}^{(6)}(x_1, x_2) = \vartheta_{21}^{(8)}(x_1, x_2) = x_1\sqrt{1 + x_2^2} \\
\vartheta_{22}^{(2)}(x_1) &= \vartheta_{22}^{(4)}(x_1) = \vartheta_{22}^{(6)}(x_1) = \vartheta_{22}^{(8)}(x_1) = x_1
\end{aligned}
$$

Among the obtained models, the ones that are considered to be more suitable for representing the original nonlinear system (3.53) are those given by $j = 3$ and $j = 4$. This is motivated by the fact that in the remaining six TS models, i.e. $j \in \{1, 2, 5, 6, 7, 8\}$, terms of the type $\sin x_1/x_2$ or $\sin x_2/x_1$ appear, which are not defined in some subsets of the region $\mathcal{P}$.

For the models obtained with $j = 3$ and $j = 4$, the *subsystems* in (3.54) are defined by the following state matrices (see Figs. 3.1, 3.2 and 3.3 for a graphical representation of the nonlinear system equations and their subsystem counterparts):

$$
A_1^{(3)} = \begin{pmatrix} 4 & 1 & 0 \\ k_\pi & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_2^{(3)} = \begin{pmatrix} 4 & -1 & 0 \\ k_\pi & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_3^{(3)} = \begin{pmatrix} 4 & 1 & 0 \\ -k_\pi & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}
$$

[3]Notice that the real premise variables can be a subset of those listed in (3.54), when some of them are constants, i.e. $\vartheta_{12}^{(1)} = \vartheta_{12}^{(2)} = \vartheta_{11}^{(7)} = \vartheta_{11}^{(8)} = 1$, $\vartheta_{22}^{(1)} = \vartheta_{22}^{(3)} = \vartheta_{22}^{(5)} = \vartheta_{22}^{(7)} = 0$.

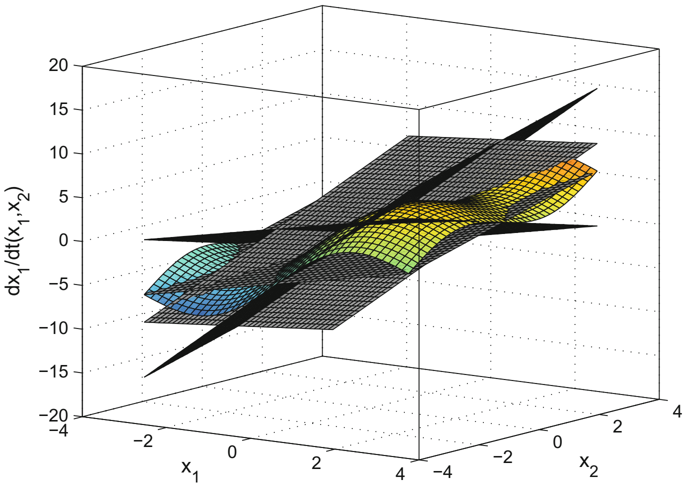

Fig. 3.1 Representation of the nonlinear equation $\dot{x}_1 = x_1 + 3\sin x_1 + x_2 - 2\sin x_2$ in $\mathcal{P}$ and its approximation using the subsystems described by $A_i^{(3)}$ or $A_i^{(4)}$. After [1]

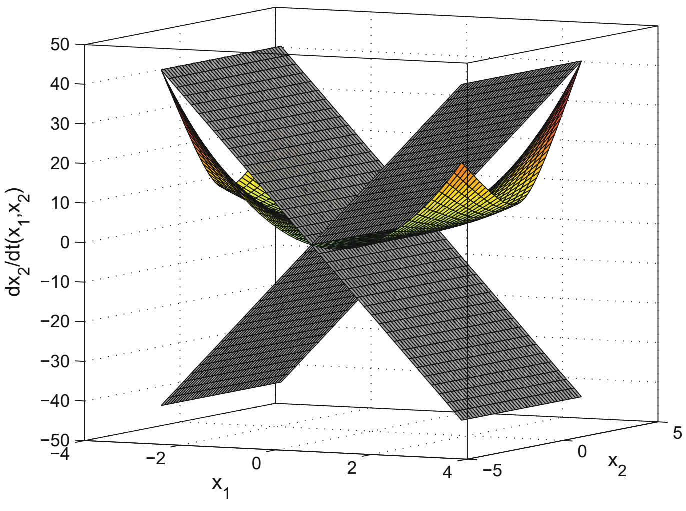

Fig. 3.2 Representation of the nonlinear equation $\dot{x}_2 = x_1^2\sqrt{1+x_2^2} + x_1x_2$ in $\mathcal{P}$ and its approximation using the subsystems described by $A_i^{(3)}$. After [1]

$$A_4^{(3)} = \begin{pmatrix} 4 & -1 & 0 \\ -k_\pi & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_5^{(3)} = \begin{pmatrix} 1 & 1 & 0 \\ k_\pi & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_6^{(3)} = \begin{pmatrix} 1 & -1 & 0 \\ k_\pi & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

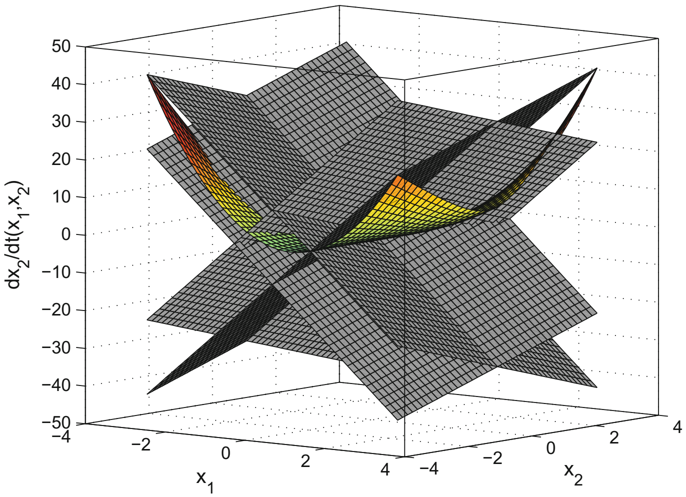

Fig. 3.3 Representation of the nonlinear equation $\dot{x}_2 = x_1^2\sqrt{1+x_2^2} + x_1 x_2$ in $\mathcal{P}$ and its approximation using the subsystems described by $A_i^{(4)}$. After [1]

$$A_7^{(3)} = \begin{pmatrix} 1 & 1 & 0 \\ -k_\pi & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_8^{(3)} = \begin{pmatrix} 1 & -1 & 0 \\ -k_\pi & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_1^{(4)} = \begin{pmatrix} 4 & 1 & 0 \\ q_\pi & \pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A_2^{(4)} = \begin{pmatrix} 4 & 1 & 0 \\ q_\pi & -\pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_3^{(4)} = \begin{pmatrix} 4 & 1 & 0 \\ -q_\pi & \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_4^{(4)} = \begin{pmatrix} 4 & 1 & 0 \\ -q_\pi & -\pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A_5^{(4)} = \begin{pmatrix} 4 & -1 & 0 \\ q_\pi & \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_6^{(4)} = \begin{pmatrix} 4 & -1 & 0 \\ q_\pi & -\pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_7^{(4)} = \begin{pmatrix} 4 & -1 & 0 \\ -q_\pi & \pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A_8^{(4)} = \begin{pmatrix} 4 & -1 & 0 \\ -q_\pi & -\pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_9^{(4)} = \begin{pmatrix} 1 & 1 & 0 \\ q_\pi & \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_{10}^{(4)} = \begin{pmatrix} 1 & 1 & 0 \\ q_\pi & -\pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A_{11}^{(4)} = \begin{pmatrix} 1 & 1 & 0 \\ -q_\pi & \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_{12}^{(4)} = \begin{pmatrix} 1 & 1 & 0 \\ -q_\pi & -\pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_{13}^{(4)} = \begin{pmatrix} 1 & -1 & 0 \\ q_\pi & \pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A_{14}^{(4)} = \begin{pmatrix} 1 & -1 & 0 \\ q_\pi & -\pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_{15}^{(4)} = \begin{pmatrix} 1 & -1 & 0 \\ -q_\pi & \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_{16}^{(4)} = \begin{pmatrix} 1 & -1 & 0 \\ -q_\pi & -\pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

where k_π and q_π are constants defined as:

$$k_\pi = \pi\sqrt{1+\pi^2} + \pi \; q_\pi = \pi\sqrt{1+\pi^2}$$

The membership functions $M_{i11}^{(3)}$, $M_{i12}^{(3)}$, $M_{i21}^{(3)}$, $M_{i11}^{(4)}$, $M_{i12}^{(4)}$, $M_{i21}^{(4)}$, $M_{i22}^{(4)}$ are defined using (3.24):

$$M_{i11}^{(3)}\left(\vartheta_{11}^{(3)}(x_1)\right) = \begin{cases} \sin x_1/x_1 & i = 1, 2, 3, 4 \\ 1 - \sin x_1/x_1 & i = 5, 6, 7, 8 \end{cases}$$

$$M_{i12}^{(3)}\left(\vartheta_{12}^{(3)}(x_2)\right) = \begin{cases} 1 - \sin x_2/x_2 & i = 1, 3, 5, 7 \\ \sin x_2/x_2 & i = 2, 4, 6, 8 \end{cases}$$

$$M_{i21}^{(3)}\left(\vartheta_{21}^{(3)}(x_1, x_2)\right) = \begin{cases} \frac{x_1\sqrt{1+x_2^2}+x_2+k_\pi}{2k_\pi} & i = 1, 2, 5, 6 \\ \frac{k_\pi - x_1\sqrt{1+x_2^2}-x_2}{2k_\pi} & i = 3, 4, 7, 8 \end{cases}$$

$$M_{i11}^{(4)}\left(\vartheta_{11}^{(4)}(x_1)\right) = \begin{cases} \sin x_1/x_1 & i = 1, 2, 3, 4, 5, 6, 7, 8 \\ 1 - \sin x_1/x_1 & i = 9, 10, 11, 12, 13, 14, 15, 16 \end{cases}$$

$$M_{i12}^{(4)}\left(\vartheta_{12}^{(4)}(x_2)\right) = \begin{cases} 1 - \sin x_2/x_2 & i = 1, 2, 3, 4, 9, 10, 11, 12 \\ \sin x_2/x_2 & i = 5, 6, 7, 8, 13, 14, 15, 16 \end{cases}$$

$$M_{i21}^{(4)}\left(\vartheta_{21}^{(4)}(x_1, x_2)\right) = \begin{cases} \frac{x_1\sqrt{1+x_2^2}+q_\pi}{2q_\pi} & i = 1, 2, 5, 6, 9, 10, 13, 14 \\ \frac{q_\pi - x_1\sqrt{1+x_2^2}}{2q_\pi} & i = 3, 4, 7, 8, 11, 12, 15, 16 \end{cases}$$

$$M_{i22}^{(4)}\left(\vartheta_{22}^{(4)}(x_1)\right) = \begin{cases} \frac{x_1+\pi}{2\pi} & i = 1, 3, 5, 7, 9, 11, 13, 15 \\ \frac{\pi - x_1}{2\pi} & i = 2, 4, 6, 8, 10, 12, 14, 16 \end{cases}$$

Finally, the coefficients that describe the level of activation of each local model are obtained using (2.32):

$$\rho_i\left(\vartheta(\tau)\right) = \frac{w_i\left(\vartheta(\tau)\right)}{\sum_{i=1}^{N} w_i\left(\vartheta(\tau)\right)} \tag{3.55}$$

as:

$$\rho_i^{(3)}(x_1, x_2) = \frac{M_{i11}^{(3)} M_{i12}^{(3)} M_{i21}^{(3)}}{\sum_{i=1}^{8} M_{i11}^{(3)} M_{i12}^{(3)} M_{i21}^{(3)}}$$

$$\rho_i^{(4)}(x_1, x_2) = \frac{M_{i11}^{(4)} M_{i12}^{(4)} M_{i21}^{(4)} M_{i22}^{(4)}}{\sum_{i=1}^{16} M_{i11}^{(4)} M_{i12}^{(4)} M_{i21}^{(4)} M_{i22}^{(4)}}$$

Remark: Notice that the obtained TS models can be interpreted as if they were polytopic quasi-LPV systems as follows:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = A_3\left(x_1, x_2\right)\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$
$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = A_4\left(x_1, x_2\right)\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

where:

$$A_3\left(x_1, x_2\right) = \begin{pmatrix}\vartheta_{11}^{(3)}(x_1) & \vartheta_{12}^{(3)}(x_2) & 0\\ \vartheta_{21}^{(3)}(x_1, x_2) & 0 & 0\\ 1 & 1 & -1\end{pmatrix} = \sum_{i=1}^{8}\rho_i^{(3)}(x_1, x_2)A_i^{(3)}$$
$$A_4\left(x_1, x_2\right) = \begin{pmatrix}\vartheta_{11}^{(4)}(x_1) & \vartheta_{12}^{(4)}(x_2) & 0\\ \vartheta_{21}^{(4)}(x_1, x_2) & \vartheta_{22}^{(4)}(x_1) & 0\\ 1 & 1 & -1\end{pmatrix} = \sum_{i=1}^{16}\rho_i^{(4)}(x_1, x_2)A_i^{(4)}$$

where $\rho_i^{(3)}(x_1, x_2)$ and $\rho_i^{(4)}(x_1, x_2)$ can be interpreted as coefficients of a polytopic decomposition.

3.6.2 Generation of Polytopic LPV Models via Sector Nonlinearity

Hereafter, a polytopic representation for (3.53) is obtained applying the method described in Sect. 3.5.

The space $\{\mathbb{X}_1, \mathbb{X}_2\}$ is partitioned into 4 quadrants, that give rise to the following 2 regions as described by (3.29):

$$R_1 : [-\pi, 0] \times [-\pi, 0] \cup [0, \pi] \times [0, \pi]$$
$$R_2 : [-\pi, 0] \times [0, \pi] \cup [0, \pi] \times [-\pi, 0]$$

Then, the partial derivatives of (3.53) are calculated:

$$\frac{\partial g_1}{\partial x_1} = 1 + 3\cos x_1 \quad \frac{\partial g_1}{\partial x_2} = 1 - 2\cos x_2$$
$$\frac{\partial g_2}{\partial x_1} = 2x_1\sqrt{1 + x_2^2} + x_2 \quad \frac{\partial g_2}{\partial x_2} = \frac{x_1^2 x_2}{\sqrt{1 + x_2^2}} + x_1$$

and their minimum and maximum values in R_1 and R_2 are found:

$$\begin{aligned}
\overline{a}_{11}^{(1)} &= \max_{R_1} \frac{\partial g_1}{\partial x_1} = 4 \quad \underline{a}_{11}^{(1)} = \min_{R_1} \frac{\partial g_1}{\partial x_1} = -2 \\
\overline{a}_{11}^{(2)} &= \max_{R_2} \frac{\partial g_1}{\partial x_1} = 4 \quad \underline{a}_{11}^{(2)} = \min_{R_2} \frac{\partial g_1}{\partial x_1} = -2 \\
\overline{a}_{12}^{(1)} &= \max_{R_1} \frac{\partial g_1}{\partial x_2} = 3 \quad \underline{a}_{12}^{(1)} = \min_{R_1} \frac{\partial g_1}{\partial x_2} = -1 \\
\overline{a}_{12}^{(2)} &= \max_{R_2} \frac{\partial g_1}{\partial x_2} = 3 \quad \underline{a}_{12}^{(2)} = \min_{R_2} \frac{\partial g_1}{\partial x_2} = -1 \\
\overline{a}_{21}^{(1)} &= \max_{R_1} \frac{\partial g_2}{\partial x_1} = r_\pi + \pi \quad \underline{a}_{21}^{(1)} = \min_{R_1} \frac{\partial g_2}{\partial x_1} = -\left(r_\pi + \pi\right) \\
\overline{a}_{21}^{(2)} &= \max_{R_2} \frac{\partial g_2}{\partial x_1} = r_\pi - \pi \quad \underline{a}_{21}^{(2)} = \min_{R_2} \frac{\partial g_2}{\partial x_1} = -r_\pi + \pi \\
\overline{a}_{22}^{(1)} &= \max_{R_1} \frac{\partial g_2}{\partial x_2} = w_\pi + \pi \quad \underline{a}_{22}^{(1)} = \min_{R_1} \frac{\partial g_2}{\partial x_2} = -\left(w_\pi + \pi\right) \\
\overline{a}_{22}^{(2)} &= \max_{R_2} \frac{\partial g_2}{\partial x_2} = w_\pi - \pi \quad \underline{a}_{22}^{(2)} = \min_{R_2} \frac{\partial g_2}{\partial x_2} = -w_\pi + \pi
\end{aligned}$$

where:

$$r_\pi = 2\pi\sqrt{1+\pi^2} \; w_\pi = \tfrac{\pi^3}{\sqrt{1+\pi^2}}$$

Afterwards, using (3.34)–(3.41), the state matrices of the vertex systems are calculated, resulting in the following eight matrices (see Figs. 3.4 and 3.5 for a graphical representation):

$$A_1^{(1)} = \begin{pmatrix} -2 & -1 & 0 \\ -\left(r_\pi + \pi\right) & -\left(w_\pi + \pi\right) & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_2^{(1)} = \begin{pmatrix} -2 & -1 & 0 \\ r_\pi + \pi & w_\pi + \pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A_3^{(1)} = \begin{pmatrix} 4 & 3 & 0 \\ -\left(r_\pi + \pi\right) & -\left(w_\pi + \pi\right) & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_4^{(1)} = \begin{pmatrix} 4 & 3 & 0 \\ r_\pi + \pi & w_\pi + \pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A_1^{(2)} = \begin{pmatrix} -2 & -1 & 0 \\ -r_\pi + \pi & w_\pi - \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_2^{(2)} = \begin{pmatrix} -2 & -1 & 0 \\ r_\pi - \pi & -w_\pi + \pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A_3^{(2)} = \begin{pmatrix} 4 & 3 & 0 \\ -r_\pi + \pi & w_\pi - \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_4^{(2)} = \begin{pmatrix} 4 & 3 & 0 \\ r_\pi - \pi & -w_\pi + \pi & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

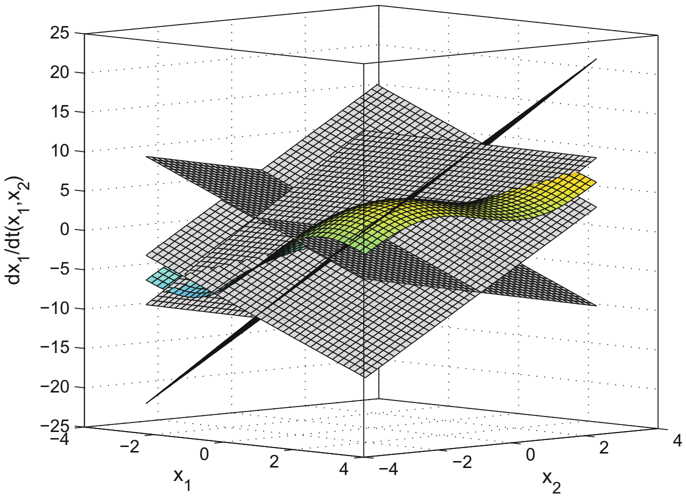

Fig. 3.4 Representation of the nonlinear equation $\dot{x}_1 = x_1 + 3\sin x_1 + x_2 - 2\sin x_2$ in $\mathcal{P}$ and its approximation using the vertex systems of (3.56). After [1]

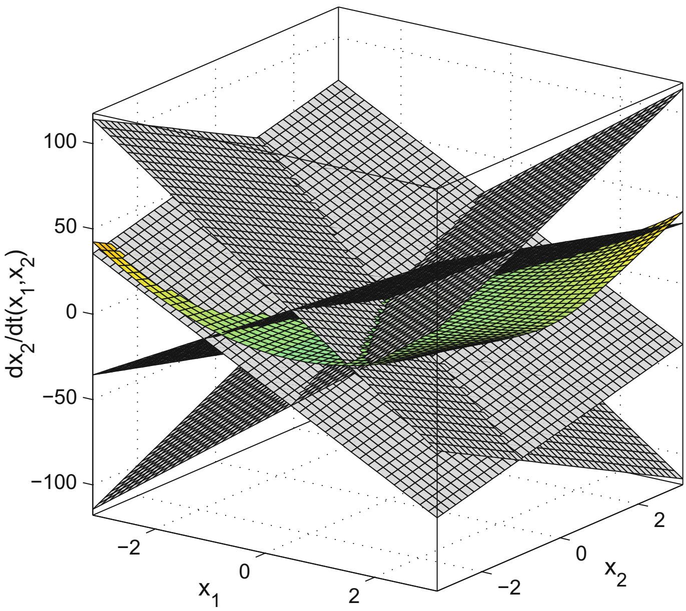

Fig. 3.5 Representation of the nonlinear equation $\dot{x}_2 = x_1^2\sqrt{1 + x_2^2} + x_1 x_2$ in $\mathcal{P}$ and its approximation using the vertex systems of (3.56). After [1]

such that (3.53) results expressed in the following polytopic LPV form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \sum_{j=1}^{4} \alpha_j^{(1)}(x_1, x_2) A_j^{(1)} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \sum_{j=1}^{4} \alpha_j^{(2)}(x_1, x_2) A_j^{(2)} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.56}$$

where the coefficients of the polytopic decomposition are obtained using (3.43)–(3.45), as follows:

$$\begin{aligned}
\alpha_1^{(1)}(x_1, x_2) &= \overset{\frown}{\alpha}{}_1^{(1)}(x_1, x_2)\overset{\frown}{\alpha}{}_2^{(1)}(x_1, x_2) R_1^{\in}(x_1, x_2) \\
\alpha_2^{(1)}(x_1, x_2) &= \overset{\frown}{\alpha}{}_1^{(1)}(x_1, x_2)\overset{\smile}{\alpha}{}_2^{(1)}(x_1, x_2) R_1^{\in}(x_1, x_2) \\
\alpha_3^{(1)}(x_1, x_2) &= \overset{\smile}{\alpha}{}_1^{(1)}(x_1, x_2)\overset{\frown}{\alpha}{}_2^{(1)}(x_1, x_2) R_1^{\in}(x_1, x_2) \\
\alpha_4^{(1)}(x_1, x_2) &= \overset{\smile}{\alpha}{}_1^{(1)}(x_1, x_2)\overset{\smile}{\alpha}{}_2^{(1)}(x_1, x_2) R_1^{\in}(x_1, x_2) \\
\alpha_1^{(2)}(x_1, x_2) &= \overset{\frown}{\alpha}{}_1^{(2)}(x_1, x_2)\overset{\frown}{\alpha}{}_2^{(2)}(x_1, x_2) R_2^{\in}(x_1, x_2) \\
\alpha_2^{(2)}(x_1, x_2) &= \overset{\frown}{\alpha}{}_1^{(2)}(x_1, x_2)\overset{\smile}{\alpha}{}_2^{(2)}(x_1, x_2) R_2^{\in}(x_1, x_2) \\
\alpha_3^{(2)}(x_1, x_2) &= \overset{\smile}{\alpha}{}_1^{(2)}(x_1, x_2)\overset{\frown}{\alpha}{}_2^{(2)}(x_1, x_2) R_2^{\in}(x_1, x_2) \\
\alpha_4^{(2)}(x_1, x_2) &= \overset{\smile}{\alpha}{}_1^{(2)}(x_1, x_2)\overset{\smile}{\alpha}{}_2^{(2)}(x_1, x_2) R_2^{\in}(x_1, x_2)
\end{aligned}$$

with:

$$\begin{aligned}
\overset{\frown}{\alpha}{}_1^{(1)}(x_1, x_2) &= \frac{3x_1 + 2x_2 - 3\sin x_1 + 2\sin x_2}{6x_1 + 4x_2} \\
\overset{\smile}{\alpha}{}_1^{(1)}(x_1, x_2) &= \frac{3x_1 + 2x_2 + 3\sin x_1 - 2\sin x_2}{6x_1 + 4x_2} \\
\overset{\frown}{\alpha}{}_1^{(2)}(x_1, x_2) &= \frac{3x_1 - 2x_2 - 3\sin x_1 + 2\sin x_2}{6x_1 - 4x_2} \\
\overset{\smile}{\alpha}{}_1^{(2)}(x_1, x_2) &= \frac{3x_1 - 2x_2 + 3\sin x_1 - 2\sin x_2}{6x_1 - 4x_2} \\
\overset{\frown}{\alpha}{}_2^{(1)}(x_1, x_2) &= \frac{(r_\pi + \pi)\, x_1 + (w_\pi + \pi)\, x_2 - x_1^2\sqrt{1 + x_2^2} - x_1 x_2}{2\left[(r_\pi + \pi)\, x_1 + (w_\pi + \pi)\, x_2\right]} \\
\overset{\smile}{\alpha}{}_2^{(1)}(x_1, x_2) &= \frac{(r_\pi + \pi)\, x_1 + (w_\pi + \pi)\, x_2 + x_1^2\sqrt{1 + x_2^2} + x_1 x_2}{2\left[(r_\pi + \pi)\, x_1 + (w_\pi + \pi)\, x_2\right]} \\
\overset{\frown}{\alpha}{}_2^{(2)}(x_1, x_2) &= \frac{(r_\pi - \pi)x_1 + (\pi - w_\pi)x_2 - x_1^2\sqrt{1 + x_2^2} - x_1 x_2}{2\left[(r_\pi - \pi)x_1 + (\pi - w_\pi)x_2\right]}
\end{aligned}$$

$$\overset{\smile}{\alpha}{}_2^{(2)}(x_1, x_2) = \frac{(r_\pi - \pi)x_1 + (\pi - w_\pi)x_2 + x_1^2\sqrt{1 + x_2^2} + x_1 x_2}{2\left[(r_\pi - \pi)x_1 + (\pi - w_\pi)x_2\right]}$$
$$R_1^{\in}(x_1, x_2) = \max\left(0, sgn(x_1)sgn(x_2)\right)$$
$$R_2^{\in}(x_1, x_2) = \max\left(0, -sgn(x_1)sgn(x_2)\right)$$

where sgn denotes the sign function.

Remark: Notice that the quasi-LPV representation of (3.53) obtained using this method has the following structure:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} a_{11}(x_1, x_2) & a_{12}(x_1, x_2) & 0 \\ a_{21}(x_1, x_2) & a_{22}(x_1, x_2) & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.57}$$

Remark: The obtained quasi-LPV system can be interpreted as a TS model, if $a_{11}(x_1, x_2)$, $a_{12}(x_1, x_2)$, $a_{21}(x_1, x_2)$, $a_{22}(x_1, x_2)$ in (3.57) and $sgn(x_1)sgn(x_2)$ are considered to be the premise variables, and $\overset{\frown}{\alpha}{}_1^{(1)}, \overset{\smile}{\alpha}{}_1^{(1)}, \overset{\frown}{\alpha}{}_1^{(2)}, \overset{\smile}{\alpha}{}_1^{(2)}, \overset{\frown}{\alpha}{}_2^{(1)}, \overset{\smile}{\alpha}{}_2^{(1)}, \overset{\frown}{\alpha}{}_2^{(2)}, \overset{\smile}{\alpha}{}_2^{(2)}$, $R_1^{\in}$, $R_2^{\in}$ the membership functions.

Remark: If the conditions (3.49)–(3.52) are used in order to avoid the discontinuity phenomenon, as described in Sect. 3.5, the matrices $A_1^{(2)}$, $A_2^{(2)}$, $A_3^{(2)}$ and $A_4^{(2)}$ change as follows:

$$A_1^{(2)} = \begin{pmatrix} -2 & -1 & 0 \\ -(r_\pi + \pi) & w_\pi + \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_2^{(2)} = \begin{pmatrix} -2 & -1 & 0 \\ r_\pi + \pi & -(w_\pi + \pi) & 0 \\ 1 & 1 & -1 \end{pmatrix}$$
$$A_3^{(2)} = \begin{pmatrix} 4 & 3 & 0 \\ -(r_\pi + \pi) & w_\pi + \pi & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad A_4^{(2)} = \begin{pmatrix} 4 & 3 & 0 \\ r_\pi + \pi & -(w_\pi + \pi) & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

3.6.3 Comparison

Hereafter, the comparison criteria between the models described in Sect. 3.3 are applied to the proposed example.

The subsets $S_1 \subset \mathbb{X}_1 \times \mathbb{X}_2 \times \dot{\mathbb{X}}_1$ and $S_2 \subset \mathbb{X}_1 \times \mathbb{X}_2 \times \dot{\mathbb{X}}_2$ are chosen as follows:

$$S_1 = [-\pi, \pi] \times [-\pi, \pi] \times [-7\pi, 7\pi] \tag{3.58}$$
$$S_2 = [-\pi, \pi] \times [-\pi, \pi] \times [-h_\pi, h_\pi] \tag{3.59}$$

with:

$$h_\pi = \pi^2\left(2 + 2\sqrt{1 + \pi^2} + \frac{\pi^2}{\sqrt{1 + \pi^2}}\right)$$

so that:

$$V_1^{(S)} = 56\pi^3 \quad V_2^{(S)} = 8\pi^2 h_\pi$$

The volumes $\tilde{V}_i$ have been calculated using (3.4) on the basis of $N = 16588$ points,[4] generated randomly using a uniform distribution:
Model generated via nonlinear embedding $A_i^{(3)}$:

$$\frac{\tilde{V}_1}{V_1^{(S)}} = \frac{5856 + 3.3179}{16588 + 6.6358} = 0.35 \quad \frac{\tilde{V}_2}{V_2^{(S)}} = \frac{6178 + 3.3179}{16588 + 6.6358} = 0.37 \; \tilde{M} = \frac{\tilde{V}_1 \tilde{V}_2}{V_1^{(S)} V_2^{(S)}} = 0.13$$

Model generated via nonlinear embedding $A_i^{(4)}$:

$$\frac{\tilde{V}_1}{V_1^{(S)}} = \frac{5856 + 3.3179}{16588 + 6.6358} = 0.35 \quad \frac{\tilde{V}_2}{V_2^{(S)}} = \frac{4866 + 3.3179}{16588 + 6.6358} = 0.29 \; \tilde{M} = \frac{\tilde{V}_1 \tilde{V}_2}{V_1^{(S)} V_2^{(S)}} = 0.10$$

Model generated via sector nonlinearity concept:

$$\frac{\tilde{V}_1}{V_1^{(S)}} = \frac{7546 + 3.3179}{16588 + 6.6358} = 0.45 \quad \frac{\tilde{V}_2}{V_2^{(S)}} = \frac{9844 + 3.3179}{16588 + 6.6358} = 0.59 \; \tilde{M} = \frac{\tilde{V}_1 \tilde{V}_2}{V_1^{(S)} V_2^{(S)}} = 0.27$$

Model generated via sector nonlinearity concept (conservative):

$$\frac{\tilde{V}_1}{V_1^{(S)}} = \frac{7546 + 3.3179}{16588 + 6.6358} = 0.45 \quad \frac{\tilde{V}_2}{V_2^{(S)}} = \frac{11197 + 3.3179}{16588 + 6.6358} = 0.67 \; \tilde{M} = \frac{\tilde{V}_1 \tilde{V}_2}{V_1^{(S)} V_2^{(S)}} = 0.30$$

Hence, according to the measure of overboundedness (3.4), the best model is the one generated via nonlinear embedding and described by the matrices $A_i^{(4)}$. In general, models obtained via nonlinear embedding tend to be less conservative than the ones obtained via sector nonlinearity concept. This is probably due to the fact that the nonlinear embedding method tries to find the maximum and minimum value of $g(x)/x_i$, whereas the other method finds the maximum and minimum value of $\partial g(x)/\partial x_i$. Then, according to the mean-value theorem, $g(x)/x_i$ is bounded by $\partial g(x)/\partial x_i$, so that the extreme values of the former are smaller than those of the latter.

To conclude the comparison between the models, let us consider the measure based on the region of attraction as introduced in Sect. 3.3.2, with controllers designed in order to achieve quadratic $\mathcal{D}$-stability in the following LMI region:

$$\mathcal{D} = \{z \in \mathbb{C} : \text{Re}(z) < -1\} \tag{3.60}$$

The measure M_β defined in (3.11) has been calculated for each model, giving the following results:

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[4]This particular value of N is chosen using statistical reasoning, in order to guarantee that the semi-length of the 99% Agresti-Coull confidence interval will be less than 0.01 [19].

Model generated via non-linear embedding $A_i^{(3)}$:

$$M_\beta = \frac{V_\beta}{V_\Theta} = \frac{122.7323}{248.0502} = 0.4948$$

Model generated via non-linear embedding $A_i^{(4)}$:

$$M_\beta = \frac{V_\beta}{V_\Theta} = \frac{122.8700}{248.0502} = 0.4953$$

Model generated via sector non-linearity concept:

$$M_\beta = \frac{V_\beta}{V_\Theta} = \frac{122.7605}{248.0502} = 0.4949$$

Model generated via sector non-linearity concept (conservative):

$$M_\beta = \frac{V_\beta}{V_\Theta} = \frac{122.6771}{248.0502} = 0.4946$$

It can be seen that, also in this case, the model generated via nonlinear embedding with matrices $A_i^{(4)}$ performs slightly better than the others, thus confirming to be the *best* obtained model.

3.7 Conclusions

In this chapter, the presence of strong analogies between polytopic LPV and TS systems and the automated generation of polytopic LPV and TS models have been addressed. In particular, it has been shown that the method for the automated generation of LPV models by nonlinear embedding can be easily extended to generate automatically TS models from a given nonlinear system. Similarly, a method already used in the TS framework for finding a model that describes in a fuzzy way a given nonlinear function has been extended to the case of polytopic LPV description of nonlinear systems.

Results obtained with a mathematical example have been presented and it has been shown, using an overboundedness measure, that the automated generation via nonlinear embedding provides less conservative models than the automated generation via sector nonlinearity concept. Also, a measure based on the region of attraction estimates has been introduced for comparing the closed-loop performances of the different models.

The overboundedness measure has shown to be an objective criterion that can be used to select which model can be considered *the best one*. However, in the general case, which model is *the best one* also depends on the context in which the model is used, i.e. whether it is used for stabilization or observation, and which structure of controller/observer is used for achieving the desired goal. Some information in this sense has been provided by the measure based on the region of attraction estimates, that allows comparing the closed-loop performances obtained with the different models. The proposed measure could be easily extended to the observation case. However, it has the limit of providing an indication of which model is the best only *a posteriori*. It seems clear that an important path for future research is the development of a procedure that automatically selects the best model during the design step, taking into account what the model is used for and the used controller/observer structure.

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Chapter 4
Robust State-Feedback Control of Uncertain LPV Systems

The content of this chapter is based on the following work:

- [1] D. Rotondo, F. Nejjari, V. Puig. Robust state-feedback control of uncertain LPV systems: an LMI-based approach. *Journal of the Franklin Institute*, 351(5):2781–2803, 2014.

4.1 Introduction

LMI-based results have been used to cope with both uncertain LTI systems and certain LPV systems throughout the last two decades. However, the design of controllers for LPV systems has been usually performed under the assumption that there was no model uncertainty. Only a few papers have stated the importance of considering robustness against uncertainty [2–11]. In recent years, works dealing with *inexactly measured* parameters have been an important field of research. The realistic case, where only some of the parameters are measured and therefore available for feedback and the remaining parameters are treated as uncertainty, was analyzed by [12], where an affine dependence on the measurable parameters and an LFT dependence on the uncertain parameters were assumed. A solution in the convex programming framework with the use of LMI solvers in the case of polytopic parameter dependence was proposed by [13]. In this case, the measurement errors were modeled by imposing an a priori bound on the distance between the real and the measured parameters. The uncertainty was modeled as a hypersphere of a certain radius, and the analysis/design conditions were given in function of this radius. The same problem was analyzed in the works of Sato and his coworkers [14–17], where an additive uncertainty on the scheduling parameter was considered. Hence, both the real scheduling parameter and the uncertainty were assumed to lie in two hyper-rectangles, and the analysis/design conditions were given at the vertices of these hyper-rectangles. Some recent results

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in this area were presented in [18], where an a posteriori analysis is used to verify that the closed-loop system is robust against deviations within known bounds in the scheduling signals, and in [19], where the design technique has been performed taking into account a stochastic description of the parameter uncertainty. Finally, it is worth recalling the work [20], where a general framework for the systematic synthesis of robust gain-scheduling controllers by convex optimization techniques for uncertain dynamical systems in LFT form has been presented.

In this chapter, the problem of designing an LPV state-feedback controller for uncertain LPV systems that can guarantee some desired performances is considered. In the proposed approach, the vector of varying parameters is used to schedule between uncertain LTI systems. The resulting idea consists in using a double-layer polytopic description so as to take into account both the variability due to the parameter vector and the uncertainty. The first polytopic layer manages the varying parameters and is used to obtain the vertex uncertain systems, where the vertex controllers are designed. The second polytopic layer is built at each vertex system so as to take into account the model uncertainties and add robustness into the design step. Under some assumptions, the problem reduces to finding a solution to a finite number of LMIs, a problem for which efficient solvers are available nowadays [21, 22]. It is worth highlighting that the proposed approach allows to cope with the problem of inexactly measured scheduling parameters as long as the vertex uncertain systems are obtained taking into account the uncertainty in the measurement of the varying parameters.

The solution proposed in this chapter differs from [3, 12] in not assuming an LFT dependence, but a polytopic description of the system matrices dependence on the scheduling parameters and the uncertainties. In contrast with [2, 7, 8, 18], the robustness in the proposed approach is guaranteed a priori during the design phase. This is different from [6] because it does not use weighting transfer functions; from [11] because the matrices obtained for different values of the uncertainty are not set to zero, but assume constant values; from [13] where the uncertainty is expressed as a hypersphere; from [14–17] in that the proposed method copes with the general case of uncertain matrices while the works by Sato and coworkers consider only the case of uncertain scheduling parameters. It is also different from [19] in not assuming a stochastic description of the parameter uncertainty.

4.2 Problem Formulation

Consider the following uncertain LPV system:

$$\sigma.x(\tau) = \tilde{A}\left(\theta(\tau)\right)x(\tau) + \tilde{B}u(\tau) + \tilde{B}_w\left(\theta(\tau)\right)w(\tau) \tag{4.1}$$

$$z_\infty(\tau) = \tilde{C}_{z_\infty}\left(\theta(\tau)\right)x(\tau) + \tilde{D}_{z_\infty u}u(\tau) + \tilde{D}_{z_\infty w}\left(\theta(\tau)\right)w(\tau) \tag{4.2}$$

$$z_2(\tau) = \tilde{C}_{z_2}\left(\theta(\tau)\right)x(\tau) + \tilde{D}_{z_2 u}u(\tau) \tag{4.3}$$

where $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is a vector of exogenous inputs (such as reference signals, disturbance signals, sensor noise), $z_\infty \in \mathbb{R}^{n_{z_\infty}}$ is a vector of output signals related to the $\mathcal{H}_\infty$ performance of the control system (see Definition 2.5), $z_2 \in \mathbb{R}^{n_{z_2}}$ is a vector of output signals related to the $\mathcal{H}_2$ performance of the control system (see Definition 2.7), and $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ is the vector of varying parameters.

Remark: In cases of LPV systems with varying input matrices $\tilde{B}\,(\theta(\tau))$, $\tilde{D}_{z_\infty u}\,(\theta(\tau))$, $\tilde{D}_{z_2 u}\,(\theta(\tau))$, it is possible to obtain a system in the form (4.1)–(4.3) by prefiltering the inputs $u(\tau)$ as proposed in [2], and recalled in (2.156)–(2.163).

The system state-space matrices take values inside a polytope, as follows:

$$\begin{pmatrix} \tilde{A}\,(\theta(\tau)) & \tilde{B} & \tilde{B}_w\,(\theta(\tau)) \\ \tilde{C}_{z_\infty}\,(\theta(\tau)) & \tilde{D}_{z_\infty u} & \tilde{D}_{z_\infty w}\,(\theta(\tau)) \\ \tilde{C}_{z_2}\,(\theta(\tau)) & \tilde{D}_{z_2 u} & \end{pmatrix} = \sum_{i=1}^{N} \mu_i\,(\theta(\tau)) \begin{pmatrix} \tilde{A}_i & \tilde{B} & \tilde{B}_{w,i} \\ \tilde{C}_{z_\infty,i} & \tilde{D}_{z_\infty u} & \tilde{D}_{z_\infty w,i} \\ \tilde{C}_{z_2,i} & \tilde{D}_{z_2 u} & \end{pmatrix} \tag{4.4}$$

with the coefficients μ_i satisfying (2.5):

$$\sum_{i=1}^{N} \mu_i\,(\theta(\tau)) = 1, \quad \mu_i\,(\theta(\tau)) \geq 0, \;\; \forall i = 1, \ldots, N, \;\; \forall \theta \in \Theta \tag{4.5}$$

The matrices $\tilde{A}_i$, $\tilde{B}_{w,i}$, $\tilde{C}_{z_\infty,i}$, $\tilde{C}_{z_2,i}$, $\tilde{D}_{z_\infty w,i}$ denote the values of $\tilde{A}\,(\theta(\tau))$, $\tilde{B}_w\,(\theta(\tau))$, $\tilde{C}_{z_\infty}\,(\theta(\tau))$, $\tilde{C}_{z_2}\,(\theta(\tau))$, $\tilde{D}_{z_\infty w}\,(\theta(\tau))$ at the ith vertex of the polytope. Each of these matrices, together with $\tilde{B}$, $\tilde{D}_{z_\infty,u}$ and $\tilde{D}_{z_2 u}$, is uncertain, with an uncertainty that can be described as well in a polytopic way by M_i LTI systems, as follows:

$$\begin{pmatrix} \tilde{A}_i & \tilde{B} & \tilde{B}_{w,i} \\ \tilde{C}_{z_\infty,i} & \tilde{D}_{z_\infty u} & \tilde{D}_{z_\infty w,i} \\ \tilde{C}_{z_2,i} & \tilde{D}_{z_2 u} & \end{pmatrix} = \sum_{j=1}^{M_i} \eta_{ij} \begin{pmatrix} A_{ij} & B_j & B_{w,ij} \\ C_{z_\infty,ij} & D_{z_\infty u,j} & D_{z_\infty w,ij} \\ C_{z_2,ij} & D_{z_2 u,j} & \end{pmatrix} \tag{4.6}$$

The goal is to compute an LPV static state-feedback control law of the form (2.134):

$$u(\tau) = K\,(\theta(\tau))\,x(\tau) \tag{4.7}$$

that meets one (or more) of the following specifications on the closed-loop behavior in the *robust LPV* sense, i.e. for each possible value that the parameter θ and the uncertain matrices $\tilde{A}, \ldots, \tilde{D}_{z_2 u}$ in (4.1)–(4.3) can take:

- stability
- $\mathcal{D}$-stability
- $\mathcal{H}_\infty$ performance
- $\mathcal{H}_2$ performance
- finite time boundedness
- finite time stability

4.3 Design Using a Common Quadratic Lyapunov Function

The design conditions presented in Sect. 2.5 can be extended so as to cope with uncertain LPV systems and solve the problem formulated in Sect. 4.2 with the use of a common quadratic Lyapunov function, as in (2.58):

$$V\left(x(\tau)\right) = x(\tau)^{\mathrm{T}} P x(\tau) \tag{4.8}$$

Indeed, an LPV state-feedback gain $K\left(\theta(\tau)\right)$ that meets the desired specifications for each possible value taken by the scheduling parameters θ and in spite of the uncertainty in the matrices $\tilde{A}\left(\theta(\tau)\right), \tilde{B}_w\left(\theta(\tau)\right), \ldots, \tilde{D}_{z_2u}\left(\theta(\tau)\right)$, should satisfy the conditions presented in the Theorems 2.12–2.22 with the following changes:

$$A(\theta) \to \tilde{A}(\theta)$$

$$B(\theta) \to \tilde{B}$$

$$B_w(\theta) \to \tilde{B}_w(\theta)$$

$$C_{z_\infty}(\theta) \to \tilde{C}_{z_\infty}(\theta)$$

$$D_{z_\infty u}(\theta) \to \tilde{D}_{z_\infty u}$$

$$D_{z_\infty w}(\theta) \to \tilde{D}_{z_\infty w}(\theta)$$

$$C_{z_2}(\theta) \to \tilde{C}_{z_2}(\theta)$$

$$D_{z_2u}(\theta) \to \tilde{D}_{z_2u}$$

Then, by choosing $K\left(\theta(\tau)\right)$ in (4.7) to be polytopic, as in (2.164):

$$u(\tau) = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right) K_i x(\tau) \tag{4.9}$$

it is possible to obtain the following theorems, that are based on a finite number of LMIs.

Theorem 4.1 (Design of a robust quadratically stabilizing polytopic state-feedback controller for uncertain CT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$He\left\{A_{ij} Q + B_j \Gamma_i\right\} \prec O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.10}$$

Then, the closed-loop system made up by the uncertain CT LPV system (4.1)*, with* $\tau = t$*,* $\tilde{B}_w(\theta(t)) = O$*, and matrices* $\tilde{A}(\theta(t))$ *and* $\tilde{B}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$*,* $i = 1, \ldots, N$*, is quadratically stable in the robust LPV sense.*

Proof The uncertain CT LPV system (4.1), with $\tau = t$ and $\tilde{B}_w(\theta(t)) = O$, is quadratically stable in the robust LPV sense if the following condition, derived from (2.135) with the changes $A(\theta) \to \tilde{A}(\theta)$ and $B(\theta) \to \tilde{B}$, holds:

$$He\left\{\tilde{A}(\theta)Q + \tilde{B}K(\theta)Q\right\} \prec O \qquad \forall \theta \in \Theta \tag{4.11}$$

Taking into account (4.4), (4.6) and (4.9), (4.11) can be rewritten as:

$$He\left\{\sum_{i=1}^{N}\mu_i(\theta)\sum_{j=1}^{M_i}\eta_{ij}A_{ij}Q + \sum_{j=1}^{M_i}\eta_{ij}B_j\sum_{i=1}^{N}\mu_i(\theta)\Gamma_i\right\} \prec O \tag{4.12}$$

with $\Gamma_i = K_i Q$.

Then, from a basic property of matrices [23], which states that any linear combination of negative definite matrices with non-negative coefficients, whose sum is positive, is negative definite, (4.10) is obtained, completing the proof. ■

Theorem 4.2 (Design of a robust quadratically stabilizing polytopic state-feedback controller for uncertain DT LPV systems) *Let* $Q \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, N$*, be such that:*

$$\begin{pmatrix} -Q & A_{ij}Q + B_j\Gamma_i \\ * & -Q \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.13}$$

Then, the closed-loop system made up by the uncertain DT LPV system (4.1)*, with* $\tau = k$*,* $\tilde{B}_w(\theta(k)) = O$*, and matrices* $\tilde{A}(\theta(k))$ *and* $\tilde{B}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$*,* $i = 1, \ldots, N$*, is quadratically stable in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.3 (Design of a robust quadratically $\mathcal{D}$-stabilizing polytopic state-feedback controller for uncertain LPV systems) *Given an LMI region* $\mathcal{D}$ *defined as in (2.47):*

$$\mathcal{D} = \{\sigma \in \mathbb{C} : f_{\mathcal{D}}(\sigma) \prec 0\} \tag{4.14}$$

with the characteristic function $f_{\mathcal{D}}(\sigma)$ *given by (2.48):*

$$f_{\mathcal{D}}(\sigma) = \alpha + \beta\sigma + \beta^{\mathrm{T}}\sigma^* = \left[\alpha_{kl} + \beta_{kl}\sigma + \beta_{lk}\sigma^*\right]_{k,l\in\{1,\ldots,m\}} \tag{4.15}$$

where $\alpha = [\alpha_{kl}]_{k,l\in\{1,\ldots,m\}} \in \mathbb{S}^{m\times m}$ *and* $\beta = [\beta_{k,l}]_{k,l\in\{1,\ldots,m\}} \in \mathbb{R}^{m\times m}$*, let* $Q \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, N$*, be such that:*

$$\alpha \otimes Q + He\left\{\beta \otimes \left[A_{ij}Q + B_j\Gamma_i\right]\right\} \prec O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.16}$$

Then, the closed-loop system made up by the uncertain LPV system (4.1)*, with* $\tilde{B}_w(\theta(\tau)) = O$*, and matrices* $\tilde{A}(\theta(\tau))$ *and* $\tilde{B}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$*,* $i = 1, \ldots, N$*, is quadratically* $\mathcal{D}$*-stable in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.4 (Design of a robust quadratic $\mathcal{H}_\infty$ polytopic state-feedback controller for uncertain CT LPV systems) *Let* $Q \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, N$*, be such that:*

$$\begin{pmatrix} He\left\{A_{ij}Q + B_j\Gamma_i\right\} & * & * \\ B_{w,ij}^{\mathrm{T}} & -I & * \\ C_{z_\infty,ij}Q + D_{z_\infty u}\Gamma_i & D_{z_\infty w,ij} & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.17}$$

Then, the closed-loop system made up by the uncertain CT LPV system (4.1)–(4.2) *with* $\tau = t$ *and matrices* $\tilde{A}(\theta(t))$*,* $\tilde{B}$*,* $\tilde{B}_w(\theta(t))$*,* $\tilde{C}_{z_\infty}(\theta(t))$*,* $\tilde{D}_{z_\infty u}$ *and* $\tilde{D}_{z_\infty w}(\theta(t))$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$*,* $i = 1, \ldots, N$*, has quadratic* $\mathcal{H}_\infty$ *performance* γ_∞ *in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.5 (Design of a robust quadratic $\mathcal{H}_\infty$ polytopic state-feedback controller for uncertain DT LPV systems) *Let* $Q \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, N$*, be such that:*

$$\begin{pmatrix} Q & A_{ij}Q + B_j\Gamma_i & B_{w,ij} & O \\ * & Q & O & QC_{z_\infty,ij}^{\mathrm{T}} \\ * & * & I & D_{z_\infty w,ij}^{\mathrm{T}} \\ * & * & * & \gamma_\infty^2 \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.18}$$

Then, the closed-loop system made up by the uncertain DT LPV system (4.1)–(4.2) *with* $\tau = k$ *and matrices* $\tilde{A}(\theta(k))$*,* $\tilde{B}$*,* $\tilde{B}_w(\theta(k))$*,* $\tilde{C}_{z_\infty}(\theta(k))$*,* $\tilde{D}_{z_\infty u}$ *and* $\tilde{D}_{z_\infty w}(\theta(k))$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$*,* $i = 1, \ldots, N$*, has quadratic* $\mathcal{H}_\infty$ *performance* γ_∞ *in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.6 (Design of a robust quadratic $\mathcal{H}_2$ polytopic state-feedback controller for uncertain CT LPV systems) *Let* $Q \succ O$*,* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$ *and* $Y_{ij} \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$*,* $i = 1, \ldots, N$*,* $j = 1, \ldots, M_i$*, be such that:*

$$Tr(Y_{ij}) < \gamma_2^2 \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.19}$$

$$\begin{pmatrix} He\left\{A_{ij}Q + B_j\Gamma_i\right\} & B_{w,ij} \\ * & -I \end{pmatrix} \prec O \quad \forall i = 1, \ldots, N \;\; \forall j = 1, \ldots, M_i \tag{4.20}$$

$$\begin{pmatrix} Y_{ij} & C_{z_2,ij}Q + D_{z_2u}\Gamma_i \\ * & Q \end{pmatrix} \succ O \quad \forall i = 1, \ldots, N \;\; \forall j = 1, \ldots, M_i \tag{4.21}$$

Then, the closed-loop system made up by the uncertain CT LPV system (4.1) *and* (4.3) *with* $\tau = t$ *and matrices* $\tilde{A}\,(\theta(t))$, $\tilde{B}$, $\tilde{B}_w\,(\theta(t))$, $\tilde{C}_{z_2}\,(\theta(t))$ *and* $\tilde{D}_{z_2u}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$*, has quadratic* $\mathcal{H}_2$ *performance* γ_2 *in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.7 (Design of a robust quadratic $\mathcal{H}_2$ polytopic state-feedback controller for uncertain DT LPV systems) *Let* $Q \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$ *and* $Y_{ij} \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$, $i = 1, \ldots, N$, $j = 1, \ldots, M_i$*, be such that:*

$$Tr(Y_{ij}) < \gamma_2^2 \quad \forall i = 1, \ldots, N \;\; \forall j = 1, \ldots, M_i \tag{4.22}$$

$$\begin{pmatrix} Q & A_{ij}Q + B_j\Gamma_i & B_{w,ij} \\ * & Q & O \\ * & * & I \end{pmatrix} \succ O \quad \forall i = 1, \ldots, N \;\; \forall j = 1, \ldots, M_i \tag{4.23}$$

$$\begin{pmatrix} Y_{ij} & C_{z_2,ij}Q + D_{z_2u}\Gamma_i \\ * & Q \end{pmatrix} \succ O \quad \forall i = 1, \ldots, N \;\; \forall j = 1, \ldots, M_i \tag{4.24}$$

Then, the closed-loop system made up by the uncertain DT LPV system (4.1) *and* (4.3) *with* $\tau = k$ *and matrices* $\tilde{A}\,(\theta(k))$, $\tilde{B}$, $\tilde{B}_w\,(\theta(k))$, $\tilde{C}_{z_2}\,(\theta(k))$ *and* $\tilde{D}_{z_2u}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$*, has quadratic* $\mathcal{H}_2$ *performance* γ_2 *in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.8 (Design of a robust quadratic FTB polytopic state-feedback controller for uncertain CT LPV systems) *Fix* $\alpha > 0$*, and let* $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $Q_1 \succ O$, $Q_2 \succ O$*, and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$*, be such that:*

$$\begin{pmatrix} He\left\{A_{ij}\tilde{Q}_1 + B_j\Gamma_i\right\} - \alpha\tilde{Q}_1 & B_{w,ij}Q_2 \\ * & -\alpha Q_2 \end{pmatrix} \prec O \quad \forall i = 1, \ldots, N \;\; \forall j = 1, \ldots, M_i \tag{4.25}$$

and (2.88)–(2.90):

$$\lambda_1 I \prec Q_1 \prec I \tag{4.26}$$

$$\lambda_2 I \prec Q_2 \prec \lambda_3 I \tag{4.27}$$

$$\begin{pmatrix} c_2 e^{-\alpha T} & \sqrt{c_1} & \sqrt{d} \\ \sqrt{c_1} & \lambda_1 & 0 \\ \sqrt{d} & 0 & \lambda_2 \end{pmatrix} \succ O \tag{4.28}$$

hold, where $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$*. Then, the closed-loop system made up by the uncertain CT LPV system* (4.1)*, with* $\tau = t$ *and matrices* $\tilde{A}\,(\theta(t))$*,* $\tilde{B}$ *and* $\tilde{B}_w\,(\theta(t))$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i \tilde{Q}^{-1}$*,* $i = 1, \ldots, N$*, is FTB with respect to* (c_1, c_2, T, R, d) *in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.9 (Design of a robust quadratic FTB polytopic state-feedback controller for uncertain DT LPV systems) *Fix* $\alpha \geq 1$*, and let* $\lambda_1 > 0$*,* $\lambda_2 > 0$*,* $Q_1 \succ O$*,* $Q_2 \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, N$*, be such that:*

$$\begin{pmatrix} -\alpha Q_1 & * & * & * \\ A_{ij} Q_1 + B_j \Gamma_i & -Q_1 & * & * \\ O & B_{w,ij}^{\mathrm{T}} & -\alpha Q_2 & * \\ O & O & Q_2 W_{ij} & -Q_2 \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.29}$$

and (2.92)–(2.94):

$$\lambda_1 R^{-1} \prec Q_1 \prec R^{-1} \tag{4.30}$$

$$O \prec Q_2 \prec \lambda_2 I \tag{4.31}$$

$$\begin{pmatrix} \frac{c_2}{\alpha^T} - \lambda_2 d & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{4.32}$$

hold. Then, the closed-loop system made up by the uncertain DT LPV system (4.1)*, with* $\tau = k$*, matrices* $\tilde{A}\,(\theta(k))$*,* $\tilde{B}$ *and* $\tilde{B}_w\,(\theta(k))$ *satisfying* (4.4) *and* (4.6)*, and input* $w(k)$ *given by:*

$$w(k+1) = \sum_{i=1}^{N} \mu_i\,(\theta(k)) \sum_{j=1}^{M_i} \eta_{ij} W_{ij} w(k) \tag{4.33}$$

and the polytopic state-feedback control law (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$*,* $i = 1, \ldots, N$*, is FTB with respect to* (c_1, c_2, T, R, d) *in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.10 (Design of a robust quadratically finite time stabilizing polytopic state-feedback controller for uncertain CT LPV systems) *Fix* $\alpha > 0$*, and let* $\lambda_1 > 0$*,* $Q_1 \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, N$*, be such that* (4.26) *holds together with:*

$$He\left\{A_{ij}\tilde{Q}_1 + B_j\Gamma_i\right\} - \alpha\tilde{Q}_1 \prec O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.34}$$

and (2.96):

$$\begin{pmatrix} c_2 e^{-\alpha T} & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{4.35}$$

where $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$*. Then, the closed-loop system made up by the uncertain CT LPV system* (4.1)*, with* $\tau = t$*,* $\tilde{B}_w(\theta(t)) = O$*, and matrices* $\tilde{A}(\theta(t))$ *and* $\tilde{B}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i\tilde{Q}^{-1}$*,* $i = 1, \ldots, N$*, is FTS with respect to* (c_1, c_2, T, R) *in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

Theorem 4.11 (Design of a robust quadratically finite time stabilizing polytopic state-feedback controller for uncertain DT LPV systems) *Fix* $\alpha \geq 1$*, and let* $\lambda_1 > 0$*,* $Q_1 \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, N$*, be such that* (4.30) *holds together with:*

$$\begin{pmatrix} -\alpha Q_1 & * \\ A_{ij}Q_1 + B_j\Gamma_i & -Q_1 \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.36}$$

and (2.98):

$$\begin{pmatrix} \frac{c_2}{\alpha^T} & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{4.37}$$

Then, the closed-loop system made up by the uncertain DT LPV system (4.1)*, with* $\tau = k$*,* $\tilde{B}_w(\theta(k)) = O$*, and matrices* $\tilde{A}(\theta(k))$ *and* $\tilde{B}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i Q^{-1}$*,* $i = 1, \ldots, N$*, is FTS with respect to* (c_1, c_2, T, R) *in the robust LPV sense.*

Proof Similar to that of Theorem 4.1, thus omitted. ■

The idea that lies behind Theorems 4.1–4.11 is to use a double-layer polytopic description so as to take into account both the variability due to the parameter vector θ and the variability due to the uncertainty. The first polytopic layer manages the parameter θ and is used to obtain the vertex uncertain systems, where the vertex controllers are designed. The second polytopic layer is built at each vertex system so as to take into account the model uncertainties and add robustness in the design step.

4.4 Design Using a Parameter-Dependent Lyapunov Function

The solution proposed in the previous section, despite the advantage of simplicity, has the drawback of the conservativeness due to matrix P being constant in (4.8). This source of conservativeness can be eliminated by using a parameter-dependent Lyapunov function as in (4.38):

$$V\left(x(\tau)\right) = x(\tau)^{\mathrm{T}} P\left(\theta(\tau)\right) x(\tau) \tag{4.38}$$

At the end of the last century, the authors of [24, 25] showed that, in a DT setting, the dilation of the matrix inequality characterizations and the introduction of auxiliary variables allowed to achieve decoupling between the Lyapunov variables and the controller variables. In this way, some technical issues that had hindered the use of parameter-dependent Lyapunov functions up to that point were overcome. Similar results for CT systems were obtained in [26, 27] via a particular application of the Schur complement technique. Hereafter, these results are extended to the case of uncertain LPV systems, to solve the problem formulated in Sect. 4.2 with the use of a parameter-dependent quadratic Lyapunov function, as in (4.38).

To the best of our knowledge, the cases of FTB and FTS were never treated using the dilation approach. Appendix A shows how new dilated LMIs for the FTB and the FTS analysis can be obtained in the case of DT systems, thus allowing the use of a parameter-dependent quadratic Lyapunov function for solving the problem of robust finite time state-feedback control of uncertain LPV systems. The obtention of dilated LMIs for the FTB/FTS analysis of CT systems is still under investigation, and will be addressed in future work.

Theorem 4.12 (Design of a robust stabilizing polytopic state-feedback controller for uncertain CT LPV systems) *Let* $Q_{ij} \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, $j = 1, \ldots, M_i$, *and* $S \in \mathbb{R}^{n_x \times n_x}$ *be such that:*

$$\begin{pmatrix} O & -Q_{ij} \\ -Q_{ij} & O \end{pmatrix} + He\left\{\begin{pmatrix} A_{ij} S + B_j \Gamma_i \\ S \end{pmatrix}\right\} \prec O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.39}$$

Then, the closed-loop system made up by the uncertain CT LPV system (4.1)*, with* $\tau = t$, $\tilde{B}_w\left(\theta(t)\right) = O$, *and matrices* $\tilde{A}\left(\theta(t)\right)$ *and* $\tilde{B}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i S^{-1}$, $i = 1, \ldots, N$, *is stable in the robust LPV sense.*

Proof The uncertain CT LPV system (4.1), with $\tau = t$ and $\tilde{B}_w\left(\theta(t)\right) = O$, is stable in the robust LPV sense if the following condition, derived from (2.135) with the changes $A(\theta) \to \tilde{A}(\theta)$, $B(\theta) \to \tilde{B}$ and $Q \to Q(\theta)$, holds:

$$He\left\{\tilde{A}(\theta) Q(\theta) + \tilde{B} K(\theta) Q(\theta)\right\} \prec O \qquad \forall \theta \in \Theta \tag{4.40}$$

Following [26], (4.40) is equivalent to:

$$\begin{pmatrix} O & -Q(\theta) \\ -Q(\theta) & O \end{pmatrix} + He\left\{\begin{pmatrix} \tilde{A}(\theta) + \tilde{B}K(\theta) \\ I \end{pmatrix} S\right\} \prec O \qquad \forall\theta \in \Theta \tag{4.41}$$

that, through the change of variables $\Gamma(\theta) = K(\theta)S$, becomes:

$$\begin{pmatrix} O & -Q(\theta) \\ -Q(\theta) & O \end{pmatrix} + He\left\{\begin{pmatrix} \tilde{A}(\theta)S + \tilde{B}\Gamma(\theta) \\ S \end{pmatrix}\right\} \prec O \qquad \forall\theta \in \Theta \tag{4.42}$$

By choosing:

$$Q(\theta) = \sum_{i=1}^{N} \mu_i(\theta) \sum_{j=1}^{M_i} \eta_{ij} Q_{ij} \tag{4.43}$$

and taking into account (4.4), (4.6) and (4.9), (4.40) can be rewritten as:

$$\sum_{i=1}^{N} \mu_i(\theta) \sum_{j=1}^{M_i} \eta_{ij} \left\{\begin{pmatrix} O & -Q_{ij} \\ -Q_{ij} & O \end{pmatrix} + He\left\{\begin{pmatrix} A_{ij}S + B_j\Gamma_i \\ S \end{pmatrix}\right\}\right\} \prec O \tag{4.44}$$

with $\Gamma_i = K_i S$.

Then, from a basic property of matrices [23], which states that any linear combination of negative definite matrices with non-negative coefficients, whose sum is positive, is negative definite, (4.39) is obtained, completing the proof. ■

Theorem 4.13 (Design of a robust stabilizing polytopic state-feedback controller for uncertain DT LPV systems) *Let* $Q_{ij} \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, $j = 1, \ldots, M_i$, *and* $S \in \mathbb{R}^{n_x \times n_x}$ *be such that:*

$$\begin{pmatrix} Q_{ij} & A_{ij}S + B_j\Gamma_i \\ * & S + S^{\mathrm{T}} - Q_{ij} \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, M_i \tag{4.45}$$

Then, the closed-loop system made up by the uncertain DT LPV system (4.1)*, with* $\tau = k$, $\tilde{B}_w(\theta(k)) = O$*, and matrices* $\tilde{A}(\theta(k))$ *and* $\tilde{B}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i S^{-1}$, $i = 1, \ldots, N$*, is stable in the robust LPV sense.*

Proof The uncertain DT LPV system (4.1), with $\tau = k$ and $\tilde{B}_w(\theta(k)) = O$, is stable in the robust LPV sense if the following condition, derived from (2.136), holds:

$$\begin{pmatrix} -Q(\theta) & \tilde{A}(\theta)Q(\theta) + \tilde{B}K(\theta)Q(\theta) \\ * & -Q(\theta) \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{4.46}$$

Following [24], (4.46) is equivalent to:

$$\begin{pmatrix} Q(\theta) & \tilde{A}(\theta)S + \tilde{B}K(\theta)S \\ * & S + S^{\mathrm{T}} - Q(\theta) \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{4.47}$$

The remaining of the proof follows a reasoning similar to the one of Theorem 4.12, and thus is omitted. ■

In order to apply efficiently the parameter-dependent Lyapunov framework to the problem of pole clustering, the concept of LMI regions (see Definition 2.3) has been slightly revised by [28], as follows:

Definition 4.1 ($\mathcal{D}_R$ *regions* [28]) A subset $\mathcal{D}_R$ of the complex plane is called a $\mathcal{D}_R$ region if there exist matrices $\alpha = [\alpha_{kl}]_{k,l\in\{1,\ldots,m\}} \in \mathbb{S}^{m\times m}$, $\beta = [\beta_{k,l}]_{k,l\in\{1,\ldots,m\}} \in \mathbb{R}^{m\times m}$ and $\chi = [\chi_{k,l}]_{k,l\in\{1,\ldots,m\}} \in \mathbb{S}^{m\times m}$ such that:

$$\mathcal{D}_R = \left\{\sigma \in \mathbb{C} : f_{\mathcal{D}_R}(\sigma) \prec 0\right\} \tag{4.48}$$

where $f_{\mathcal{D}_R}(\sigma)$ is the *characteristic function* defined as:

$$f_{\mathcal{D}_R}(\sigma) = \alpha + \beta\sigma + \beta^{\mathrm{T}}\sigma^* + \chi\sigma\sigma^* = \left[\alpha_{kl} + \beta_{kl}\sigma + \beta_{lk}\sigma^* + \chi_{kl}\sigma\sigma^*\right]_{k,l\in\{1,\ldots,m\}} \tag{4.49}$$

Without any assumption on the matrix χ, $\mathcal{D}_R$ regions are not convex, but with the assumption $\chi \geq 0$, $\mathcal{D}_R$ regions are a slight modification of the characterization provided by LMI regions.

Theorem 4.14 (Design of a robust quadratically $\mathcal{D}_R$-stabilizing polytopic state-feedback controller for uncertain LPV systems) *Given a $\mathcal{D}_R$-region defined as in* (4.48)*, let $Q_{ij} \succ O$, $\Gamma_i \in \mathbb{R}^{n_u\times n_x}$, $i = 1,\ldots,N$, $j = 1,\ldots,M_i$, and $S \in \mathbb{R}^{n_x\times n_x}$ be such that:*

$$\begin{pmatrix} \left[\alpha_{kl}Q_{ij} + \beta_{kl}U_{ij}\left(S,\Gamma_i\right) + \beta_{lk}U_{ij}\left(S,\Gamma_i\right)^{\mathrm{T}}\right]_{k,l\in\{1,\ldots,m\}} & * \\ \left[\beta_{kl}\left(Q_{ij} - S\right) + \chi_{kl}U_{ij}\left(S,\Gamma_i\right)^{\mathrm{T}}\right]_{k,l\in\{1,\ldots,m\}} & \left[\chi_{kl}\left(Q_{ij} - S - S^{\mathrm{T}}\right)\right]_{k,l\in\{1,\ldots,m\}} \end{pmatrix} \prec O \tag{4.50}$$

$\forall i = 1,\ldots,N$, $\forall j = 1,\ldots,M_i$, with:

$$U_{ij}\left(S,\Gamma_i\right) = A_{ij}S + B_j\Gamma_i \tag{4.51}$$

Then, the closed-loop system made up by the uncertain LPV system (4.1)*, with $\tilde{B}_w\left(\theta(\tau)\right) = O$, and matrices $\tilde{A}\left(\theta(\tau)\right)$ and $\tilde{B}$ satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as $K_i = \Gamma_i S^{-1}$, $i = 1,\ldots,N$, is $\mathcal{D}_R$-stable in the robust LPV sense.*

Proof The proof uses the results obtained by [28], and is similar to the one of Theorem 4.12, thus omitted. ■

Theorem 4.15 (Design of a robust $\mathcal{H}_\infty$ polytopic state-feedback controller for uncertain CT LPV systems) *Let* $Q_{ij} \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, $j = 1, \ldots, M_i$, *and* $S \in \mathbb{R}^{n_x \times n_x}$ *be such that:*

$$\begin{pmatrix} He\left\{U_{ij}\left(S, \Gamma_i\right)\right\} & -Q_{ij} + S^{\mathrm{T}} - U_{ij}\left(S, \Gamma_i\right) & B_{w,ij} & V_{ij}\left(S, \Gamma_i\right)^{\mathrm{T}} \\ * & -He\left\{S\right\} & O & -V_{ij}\left(S, \Gamma_i\right)^{\mathrm{T}} \\ * & * & -I & D_{z_\infty w,ij}^{\mathrm{T}} \\ * & * & * & -\gamma_\infty^2 I \end{pmatrix} \prec O \quad (4.52)$$

$\forall i = 1, \ldots, N$, $\forall j = 1, \ldots, M_i$, *with* $U_{ij}(S, \Gamma_i)$ *defined as in* (4.51)*, and:*

$$V_{ij}\left(S, \Gamma_i\right) = C_{z_\infty,ij} S + D_{z_\infty u,j} \Gamma_i \quad (4.53)$$

Then, the closed-loop system made up by the uncertain CT LPV system (4.1)–(4.2) *with* $\tau = t$ *and matrices* $\tilde{A}\left(\theta(t)\right)$, $\tilde{B}$, $\tilde{B}_w\left(\theta(t)\right)$, $\tilde{C}_{z_\infty}\left(\theta(t)\right)$, $\tilde{D}_{z_\infty u}$ *and* $\tilde{D}_{z_\infty w}\left(\theta(t)\right)$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i S^{-1}$, $i = 1, \ldots, N$, *has* $\mathcal{H}_\infty$ *performance* γ_∞ *in the robust LPV sense.*

Proof The proof uses the results obtained by [26], and is similar to the one of Theorem 4.12, thus omitted. ■

Theorem 4.16 (Design of a robust $\mathcal{H}_\infty$ polytopic state-feedback controller for uncertain DT LPV systems) *Let* $Q_{ij} \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, $j = 1, \ldots, M_i$, *and* $S \in \mathbb{R}^{n_x \times n_x}$ *be such that:*

$$\begin{pmatrix} Q_{ij} & U_{ij}\left(S, \Gamma_i\right) & B_{w,ij} & O \\ * & S + S^{\mathrm{T}} - Q_{ij} & O & V_{ij}\left(S, \Gamma_i\right)^{\mathrm{T}} \\ * & * & I & D_{z_\infty w,ij}^{\mathrm{T}} \\ * & * & * & \gamma_\infty^2 I \end{pmatrix} \succ O \quad (4.54)$$

$\forall i = 1, \ldots, N$, $\forall j = 1, \ldots, M_i$, *with* $U_{ij}(S, \Gamma_i)$ *defined as in* (4.51)*, and* $V_{ij}\left(S, \Gamma_i\right)$ *defined as in* (4.53)*. Then, the closed-loop system made up by the uncertain DT LPV system* (4.1)–(4.2) *with* $\tau = k$ *and matrices* $\tilde{A}\left(\theta(k)\right)$, $\tilde{B}$, $\tilde{B}_w\left(\theta(k)\right)$, $\tilde{C}_{z_\infty}\left(\theta(k)\right)$, $\tilde{D}_{z_\infty u}$ *and* $\tilde{D}_{z_\infty w}\left(\theta(k)\right)$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i S^{-1}$, $i = 1, \ldots, N$, *has* $\mathcal{H}_\infty$ *performance* γ_∞ *in the robust LPV sense.*

Proof The proof uses the results obtained by [25], and is similar to the one of Theorem 4.12, thus omitted. ■

Theorem 4.17 (Design of a robust $\mathcal{H}_2$ polytopic state-feedback controller for uncertain CT LPV systems) *Let* $Q_{ij} \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $Y_{ij} \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$, $i = 1, \ldots, N$, $j = 1, \ldots, M_i$, *and* $S \in \mathbb{R}^{n_x \times n_x}$ *be such that* (4.19) *and:*

$$\begin{pmatrix} Y_{ij} & W_{ij}(S,\Gamma_i) \\ * & S+S^{\mathrm{T}}-Q_{ij} \end{pmatrix} \succ O \tag{4.55}$$

$$\begin{pmatrix} He\left\{U_{ij}(S,\Gamma_i)\right\} & -Q_{ij}+S^{\mathrm{T}}-U_{ij}(S,\Gamma_i) & B_{w,ij} \\ * & -He\{S\} & O \\ * & * & -I \end{pmatrix} \prec O \tag{4.56}$$

hold $\forall i=1,\ldots,N$, $\forall j=1,\ldots,M_i$, *with* $U_{ij}(S,\Gamma_i)$ *defined as in* (4.51)*, and:*

$$W_{ij}(S,\Gamma_i) = C_{z_2,ij}S + D_{z_2u,j}\Gamma_i \tag{4.57}$$

Then, the closed-loop system made up by the uncertain CT LPV system (4.1) *and* (4.3) *with* $\tau=t$ *and matrices* $\tilde{A}(\theta(t))$, $\tilde{B}$, $\tilde{B}_w(\theta(t))$, $\tilde{C}_{z_2}(\theta(t))$ *and* $\tilde{D}_{z_2u}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i=\Gamma_i S^{-1}$, $i=1,\ldots,N$, *has* $\mathcal{H}_2$ *performance* γ_2 *in the robust LPV sense.*

Proof The proof uses the results obtained by [27], and is similar to the one of Theorem 4.12, thus omitted. ■

Theorem 4.18 (Design of a robust $\mathcal{H}_2$ polytopic state-feedback controller for uncertain DT LPV systems) *Let* $Q_{ij}\succ O$, $\Gamma_i\in\mathbb{R}^{n_u\times n_x}$, $Y_{ij}\in\mathbb{S}^{n_{z_2}\times n_{z_2}}$, $i=1,\ldots,N$, $j=1,\ldots,M_i$, *and* $S\in\mathbb{R}^{n_x\times n_x}$ *be such that* (4.19), (4.55) *and:*

$$\begin{pmatrix} Q_{ij} & U_{ij}(S,\Gamma_i) & B_{w,ij} \\ * & He\{S\}-Q_{ij} & O \\ * & * & I \end{pmatrix} \succ O \tag{4.58}$$

hold $\forall i=1,\ldots,N$, $\forall j=1,\ldots,M_i$, *with* $U_{ij}(S,\Gamma_i)$ *defined as in* (4.51)*, and* $W_{ij}(S,\Gamma_i)$ *defined as in* (4.57)*. Then, the closed-loop system made up by the uncertain DT LPV system* (4.1) *and* (4.3) *with* $\tau=k$ *and matrices* $\tilde{A}(\theta(k))$, $\tilde{B}$, $\tilde{B}_w(\theta(k))$, $\tilde{C}_{z_2}(\theta(k))$ *and* $\tilde{D}_{z_2u}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i=\Gamma_i S^{-1}$, $i=1,\ldots,N$, *has* $\mathcal{H}_2$ *performance* γ_2 *in the robust LPV sense.*

Proof The proof uses the results obtained by [25], and is similar to the one of Theorem 4.12, thus omitted. ■

Theorem 4.19 (Design of a robust FTB polytopic state-feedback controller for uncertain DT LPV systems) *Fix* $\alpha\geq 1$*, and let* $\lambda_{1,ij}>0$, $\lambda_{2,ij}>0$, $Q_{1,ij}\succ O$, $Q_{2,ij}\succ O$, $\Gamma_i\in\mathbb{R}^{n_u\times n_x}$, $i=1,\ldots,N$, $j=1,\ldots,M_i$, $S_1\in\mathbb{R}^{n_x\times n_x}$ *and* $S_2\in\mathbb{R}^{n_w\times n_w}$ *be such that:*

$$\begin{pmatrix} -\alpha\left(S_1+S_1^{\mathrm{T}}-Q_{1,ij}\right) & * & * & * \\ A_{ij}S_1+B_j\Gamma_i & -Q_{1,ij} & * & * \\ O & B_{w,ij}^{\mathrm{T}} & -Q_{2,ij} & * \\ O & O & S_2^{\mathrm{T}}W_{ij} & Q_{2,ij}-S-S^{\mathrm{T}} \end{pmatrix} \prec O \tag{4.59}$$

$$\lambda_{1,ij} R^{-1} \prec Q_{1,ij} \prec R^{-1} \tag{4.60}$$

$$O \prec Q_{2,ij} \prec \lambda_{2,ij} I \tag{4.61}$$

$$\begin{pmatrix} \frac{c_2}{\alpha^T} - \lambda_{2,ij} d & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_{1,ij} \end{pmatrix} \succ O \tag{4.62}$$

hold $\forall i = 1, \dots, N$, $\forall j = 1, \dots, M_i$. *Then, the closed-loop system made up by the uncertain DT LPV system* (4.1)*, with* $\tau = k$*, matrices* $\tilde{A}(\theta(k))$*,* $\tilde{B}$ *and* $\tilde{B}_w(\theta(k))$ *satisfying* (4.4) *and* (4.6)*, and input* $w(k)$ *given by* (4.33)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i S_1^{-1}$*,* $i = 1, \dots, N$*, is FTB with respect to* (c_1, c_2, T, R, d) *in the robust LPV sense.*

Proof Similar to the proof of Theorem 4.12, taking into account a modified version of Theorem A.1 (see Appendix A), where the open-loop state matrix A is replaced with the closed-loop state matrix $A + BK$, and the change of variable $KS_1 = \Gamma$ is applied. ■

Theorem 4.20 (Design of a robust finite time stabilizing polytopic state-feedback controller for uncertain DT LPV systems) *Fix* $\alpha \geq 1$*, and let* $\lambda_{ij} > 0$*,* $Q_{ij} \succ O$*,* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \dots, N$*,* $j = 1, \dots, M_i$*, and* $S \in \mathbb{R}^{n_x \times n_x}$ *be such that:*

$$\begin{pmatrix} -\alpha\left(S + S^{\mathrm{T}} - Q_{ij}\right) & * \\ A_{ij} S + B_j \Gamma_i & -Q_{ij} \end{pmatrix} \prec O \tag{4.63}$$

$$\lambda_{ij} R^{-1} \prec Q_{ij} \prec R^{-1} \tag{4.64}$$

$$\begin{pmatrix} \frac{c_2}{\alpha^T} & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_{ij} \end{pmatrix} \succ O \tag{4.65}$$

hold $\forall i = 1, \dots, N$, $\forall j = 1, \dots, M_i$. *Then, the closed-loop system made up by the uncertain DT LPV system* (4.1)*, with* $\tau = k$*,* $\tilde{B}_w(\theta(k)) = O$*, and matrices* $\tilde{A}(\theta(k))$ *and* $\tilde{B}$ *satisfying* (4.4) *and* (4.6)*, and the polytopic state-feedback control law* (4.9) *with gains calculated as* $K_i = \Gamma_i S^{-1}$*,* $i = 1, \dots, N$*, is FTS with respect to* (c_1, c_2, T, R) *in the robust LPV sense.*

Proof Similar to the proof of Theorem 4.12, taking into account a modified version of Theorem A.2 (see Appendix A), where the open-loop state matrix A is replaced with the closed-loop state matrix $A + BK$, and the change of variable $KS = \Gamma$ is applied. ■

4.5 Examples

4.5.1 Example 1: $\mathcal{D}$-stability

Consider an uncertain CT LPV system described by (4.1) with:

$$\tilde{A}\left(\theta(t)\right) = \begin{pmatrix} \nu_1 & \nu_2\theta(t) \\ -2 & -4\nu_2 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & \nu_1 \end{pmatrix}$$

and $\tilde{B}_w\left(\theta(t)\right) = O$, with the varying parameter $\theta(t) \in [2, 4]$ and the uncertainty given by $\nu_1 \in [0.9, 1.1]$ and $\nu_2 \in [0.9, 1.1]$. This system can be described as a polytopic combination of uncertain LTI systems as in (4.4), as follows:

$$\tilde{A}\left(\theta(t)\right) = \mu_1\left(\theta(t)\right)\tilde{A}_1 + \mu_2\left(\theta(t)\right)\tilde{A}_2$$

with:

$$\tilde{A}_1 = \begin{pmatrix} \nu_1 & 2\nu_2 \\ -2 & -4\nu_2 \end{pmatrix} \qquad \tilde{A}_2 = \begin{pmatrix} \nu_1 & 4\nu_2 \\ -2 & -4\nu_2 \end{pmatrix}$$

$$\mu_1\left(\theta(t)\right) = \frac{4-\theta(t)}{2} \qquad \mu_2\left(\theta(t)\right) = \frac{\theta(t)-2}{2}$$

The pair $(\tilde{A}_1, \tilde{B})$ can be described in a polytopic way by four LTI systems, as in (4.6):

$$\begin{pmatrix} \tilde{A}_1 \\ \tilde{B} \end{pmatrix} = \eta_{11}\begin{pmatrix} A_{11} \\ B_1 \end{pmatrix} + \eta_{12}\begin{pmatrix} A_{12} \\ B_2 \end{pmatrix} + \eta_{13}\begin{pmatrix} A_{13} \\ B_3 \end{pmatrix} + \eta_{14}\begin{pmatrix} A_{14} \\ B_4 \end{pmatrix}$$

with:

$$A_{11} = \begin{pmatrix} 0.9 & 1.8 \\ -2 & -3.6 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0.9 & 2.2 \\ -2 & -4.4 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 1.1 & 1.8 \\ -2 & -3.6 \end{pmatrix} \qquad A_{14} = \begin{pmatrix} 1.1 & 2.2 \\ -2 & -4.4 \end{pmatrix}$$

$$B_1 = B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0.9 \end{pmatrix} \qquad B_3 = B_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1.1 \end{pmatrix}$$

In the same manner, the pair $(\tilde{A}_2, \tilde{B})$ can be described in the polytopic form (4.6), as follows:

$$\begin{pmatrix} \tilde{A}_2 \\ \tilde{B} \end{pmatrix} = \eta_{21}\begin{pmatrix} A_{21} \\ B_1 \end{pmatrix} + \eta_{22}\begin{pmatrix} A_{22} \\ B_2 \end{pmatrix} + \eta_{23}\begin{pmatrix} A_{23} \\ B_3 \end{pmatrix} + \eta_{24}\begin{pmatrix} A_{24} \\ B_4 \end{pmatrix}$$

with:

$$A_{21} = \begin{pmatrix} 0.9 & 3.6 \\ -2 & -3.6 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 0.9 & 4.4 \\ -2 & -4.4 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 1.1 & 3.6 \\ -2 & -3.6 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1.1 & 4.4 \\ -2 & -4.4 \end{pmatrix}$$

Let us solve the design problem of finding a state-feedback gain:

$$u(t) = K\left(\theta(t)\right) x(t) = \mu_1\left(\theta(t)\right) K_1 x(t) + \mu_2\left(\theta(t)\right) K_2 x(t)$$

that places the closed-loop poles in a disk of radius r and center $(-q, 0)$.

In this case, the problem can be solved either using Theorem 4.3, i.e. a common quadratic Lyapunov function, or using Theorem 4.14, i.e. a parameter-dependent Lyapunov function. In the first case, the LMIs (4.16) take the following form:

$$\begin{pmatrix} -rQ & qQ + A_{ij}Q + B_j\Gamma_i \\ * & -rQ \end{pmatrix} \prec O$$

with $i = 1, 2$, $j = 1, 2, 3, 4$, while in the second case, the LMIs (4.50) become:

$$\begin{pmatrix} \left(q^2 - r^2\right) Q_{ij} + qHe\left\{A_{ij}S + B_j\Gamma_i\right\} & q\left(Q_{ij} - S^{\mathrm{T}}\right) + A_{ij}S + B_j\Gamma_i \\ * & Q_{ij} - S - S^{\mathrm{T}} \end{pmatrix} \prec O$$

with $i = 1, 2$, $j = 1, 2, 3, 4$.

In this example, the desired circle has been chosen as the one with center $(-10, 0)$ and radius $r = 10$. By applying Theorem 4.14, the robust controller vertex gains are obtained using YALMIP toolbox [21] with SeDuMi solver [22]:

$$K_1 = \begin{pmatrix} -1.4754 & 3.8629 \\ 0.7617 & 2.7419 \end{pmatrix} \qquad K_2 = \begin{pmatrix} -1.5504 & 2.9062 \\ 0.5752 & 2.6576 \end{pmatrix}$$

with:

$$S = \begin{pmatrix} 0.1421 & -0.0116 \\ 0.0081 & 0.0293 \end{pmatrix}$$

$$Q_{11} = \begin{pmatrix} 0.1442 & -0.0008 \\ -0.0008 & 0.0301 \end{pmatrix} \qquad Q_{12} = \begin{pmatrix} 0.1435 & -0.0012 \\ -0.0012 & 0.0312 \end{pmatrix}$$

$$Q_{13} = \begin{pmatrix} 0.1422 & -0.0013 \\ -0.0013 & 0.0296 \end{pmatrix} \qquad Q_{14} = \begin{pmatrix} 0.1420 & -0.0011 \\ -0.0011 & 0.0305 \end{pmatrix}$$

$$Q_{21} = \begin{pmatrix} 0.1437 & -0.0014 \\ -0.0014 & 0.0304 \end{pmatrix} \qquad Q_{22} = \begin{pmatrix} 0.1438 & -0.0025 \\ -0.0025 & 0.0313 \end{pmatrix}$$

$$Q_{23} = \begin{pmatrix} 0.1424 & -0.0016 \\ -0.0016 & 0.0294 \end{pmatrix} \qquad Q_{24} = \begin{pmatrix} 0.1427 & -0.0029 \\ -0.0029 & 0.0306 \end{pmatrix}$$

In the following, a comparison with a controller obtained applying a classical LPV technique that does not take into account the uncertainty (denoted as *fragile*) is done. The vertex gains of the fragile controller are:

$$K_1 = \begin{pmatrix} -1.3170 & 0.6347 \\ 0.1779 & 3.6717 \end{pmatrix} \qquad K_2 = \begin{pmatrix} -1.4257 & -0.3826 \\ -0.5024 & 3.5638 \end{pmatrix}$$

Figure 4.1 shows the results of the pole placement when the proposed approach is used. The closed-loop poles of the vertex systems are depicted as blue dots, and it can be seen that they are always inside the desired region, thus demonstrating that the proposed technique can guarantee pole clustering in presence of model uncertainties. When the problem is solved using the classical approach that takes into account during the design phase only the varying parameters (but not the uncertainties), the pole placement specification is satisfied only for the nominal system (cyan dots in Fig. 4.1), and fails in achieving the goal as soon as the uncertainties are taken into consideration (red dots in Fig. 4.1).

Figure 4.1 shows that for some realizations of the uncertainty the robust controller would be stable, whereas the fragile controller would be unstable. This fact can be seen from simulation analysis, for example by considering the time response obtained with $\nu_1 = 1.1$, $\nu_2 = 0.9$ and $\theta(t) = 3 + \sin t$, starting from the initial condition $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}$. The evolutions of the states x_1 and x_2 are depicted in Figs. 4.2

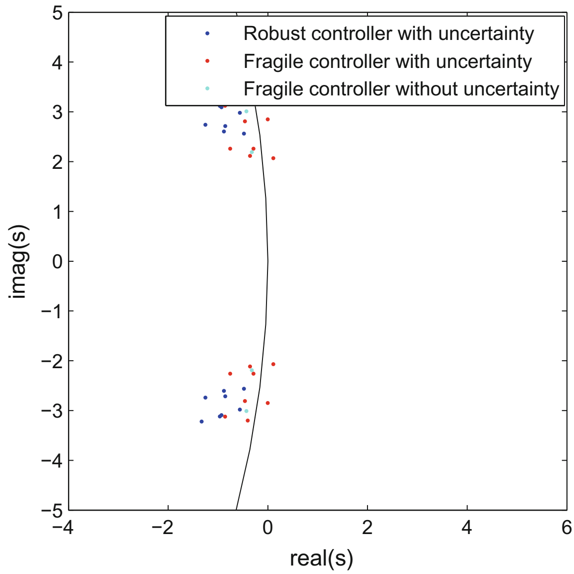

Fig. 4.1 Robust $\mathcal{D}$-stability: comparison between robust and fragile controller (zoom)

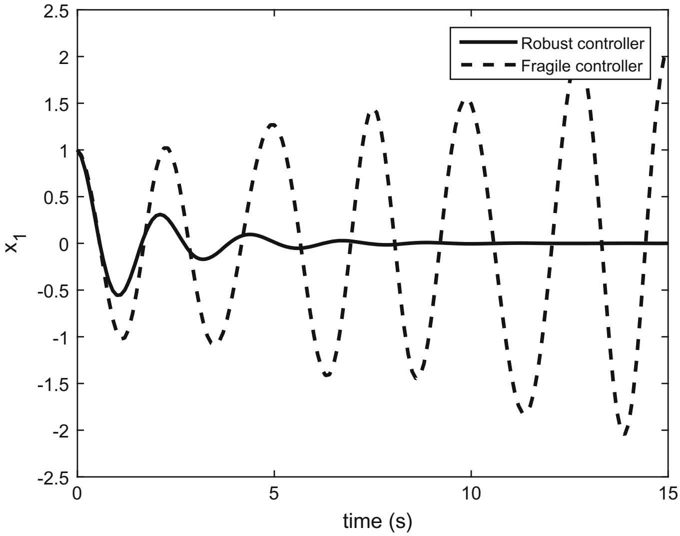

Fig. 4.2 Robust $\mathcal{D}$-stability: closed-loop response of $x_1(t)$

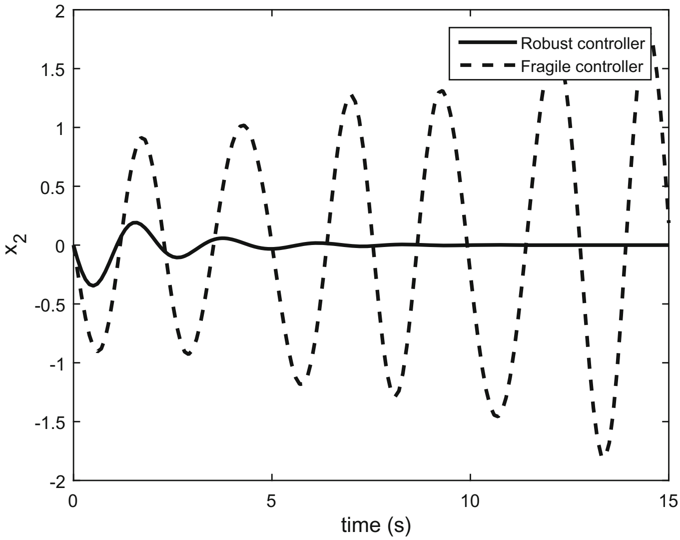

Fig. 4.3 Robust $\mathcal{D}$-stability: closed-loop response of $x_2(t)$

and 4.3, respectively, in both the cases when a robust controller is applied and when the fragile controller is applied. While in the first case the closed-loop state trajectory converges to zero, in the second case the closed-loop state trajectory exhibits divergence.

4.5.2 Example 2: $\mathcal{H}_\infty$ Performance

Consider an uncertain CT LPV system described by (4.1) with:

$$\tilde{A}\left(\theta(t)\right) = \begin{pmatrix} \nu_1 & \nu_2\theta(t) \\ -2 & -4\nu_2 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & \nu_1 \end{pmatrix} \qquad \tilde{B}_w\left(\theta(t)\right) = \begin{pmatrix} \nu_1\theta(t) \\ 0 \end{pmatrix}$$

with the varying parameter $\theta(t) \in [2, 4]$ and the uncertainty given by $\nu_1 \in [0.6, 1.4]$ and $\nu_2 \in [0.6, 1.4]$. This system can be described as a polytopic combination of uncertain LTI systems as in (4.4), as follows:

$$\begin{pmatrix} \tilde{A}\left(\theta(t)\right) \\ \tilde{B}_w\left(\theta(t)\right) \end{pmatrix} = \mu_1\left(\theta(t)\right)\begin{pmatrix} \tilde{A}_1 \\ \tilde{B}_{w,1} \end{pmatrix} + \mu_2\left(\theta(t)\right)\begin{pmatrix} \tilde{A}_2 \\ \tilde{B}_{w,2} \end{pmatrix}$$

with:

$$\tilde{A}_1 = \begin{pmatrix} \nu_1 & 2\nu_2 \\ -2 & -4\nu_2 \end{pmatrix} \quad \tilde{B}_{w,1} = \begin{pmatrix} 2\nu_1 \\ 0 \end{pmatrix} \quad \tilde{A}_2 = \begin{pmatrix} \nu_1 & 4\nu_2 \\ -2 & -4\nu_2 \end{pmatrix} \quad \tilde{B}_{w,2} = \begin{pmatrix} 4\nu_1 \\ 0 \end{pmatrix}$$

$$\mu_1\left(\theta(t)\right) = \frac{4-\theta(t)}{2} \qquad \mu_2\left(\theta(t)\right) = \frac{\theta(t)-2}{2}$$

The triplet $(\tilde{A}_1, \tilde{B}, \tilde{B}_{w,1})$ can be described in a polytopic way by four LTI systems, as in (4.6):

$$\begin{pmatrix} \tilde{A}_1 \\ \tilde{B} \\ \tilde{B}_{w,1} \end{pmatrix} = \eta_{11}\begin{pmatrix} A_{11} \\ B_1 \\ B_{w,11} \end{pmatrix} + \eta_{12}\begin{pmatrix} A_{12} \\ B_2 \\ B_{w,12} \end{pmatrix} + \eta_{13}\begin{pmatrix} A_{13} \\ B_3 \\ B_{w,13} \end{pmatrix} + \eta_{14}\begin{pmatrix} A_{14} \\ B_4 \\ B_{w,14} \end{pmatrix}$$

with:

$$A_{11} = \begin{pmatrix} 0.6 & 1.2 \\ -2 & -2.4 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0.6 & 2.8 \\ -2 & -5.6 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 1.4 & 1.2 \\ -2 & -2.4 \end{pmatrix} \qquad A_{14} = \begin{pmatrix} 1.4 & 2.8 \\ -2 & -5.6 \end{pmatrix}$$

$$B_1 = B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0.6 \end{pmatrix} \qquad B_3 = B_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1.4 \end{pmatrix}$$

$$B_{w,11} = B_{w,12} = \begin{pmatrix} 1.2 \\ 0 \end{pmatrix} \qquad B_{w,13} = B_{w,14} = \begin{pmatrix} 2.8 \\ 0 \end{pmatrix}$$

In the same manner, the triplet $(\tilde{A}_2, \tilde{B}, \tilde{B}_{w,2})$ can be described in the polytopic form (4.6), as follows:

$$\begin{pmatrix} \tilde{A}_2 \\ \tilde{B} \\ \tilde{B}_{w,2} \end{pmatrix} = \eta_{21} \begin{pmatrix} A_{21} \\ B_1 \\ B_{w,21} \end{pmatrix} + \eta_{22} \begin{pmatrix} A_{22} \\ B_2 \\ B_{w,22} \end{pmatrix} + \eta_{23} \begin{pmatrix} A_{23} \\ B_3 \\ B_{w,23} \end{pmatrix} + \eta_{24} \begin{pmatrix} A_{24} \\ B_4 \\ B_{w,24} \end{pmatrix}$$

with:

$$A_{21} = \begin{pmatrix} 0.6 & 2.4 \\ -2 & -2.4 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 0.6 & 5.6 \\ -2 & -5.6 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 1.4 & 2.4 \\ -2 & -2.4 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1.4 & 5.6 \\ -2 & -5.6 \end{pmatrix}$$

$$B_{w,21} = B_{w,22} = \begin{pmatrix} 2.4 \\ 0 \end{pmatrix} \qquad B_{w,23} = B_{w,24} = \begin{pmatrix} 5.6 \\ 0 \end{pmatrix}$$

Let us solve the design problem of finding a state-feedback gain:

$$u(t) = K(\theta(t))\, x(t) = \mu_1(\theta(t))\, K_1 x(t) + \mu_2(\theta(t))\, K_2 x(t)$$

such that the closed-loop system has $\mathcal{H}_\infty$ performance less than 1 in the robust LPV sense (this specification corresponds to the attenuation of the exogenous input across all frequencies).

In this case, the problem can be solved either using Theorem 4.4 (common quadratic Lyapunov function) or using Theorem 4.15 (parameter-dependent Lyapunov function). By applying the latter, the robust controller vertex gains are obtained using YALMIP toolbox [21] with SeDuMi solver [22]:

$$K_1 = \begin{pmatrix} -12.6028 & -3.3688 \\ -1.8928 & -1.8585 \end{pmatrix} \qquad K_2 = \begin{pmatrix} -28.7550 & -7.3552 \\ -3.2000 & -3.1281 \end{pmatrix}$$

with:

$$S = \begin{pmatrix} 0.7395 & -0.2440 \\ -0.4165 & 1.0987 \end{pmatrix}$$

$$Q_{11} = \begin{pmatrix} 7.9460 & 0.4642 \\ 0.4642 & 4.6821 \end{pmatrix} \qquad Q_{12} = \begin{pmatrix} 8.6724 & -0.9988 \\ -0.9988 & 7.6659 \end{pmatrix}$$

$$Q_{13} = \begin{pmatrix} 7.2706 & 0.5011 \\ 0.5011 & 6.1806 \end{pmatrix} \qquad Q_{14} = \begin{pmatrix} 8.0755 & -1.1003 \\ -1.1003 & 8.9602 \end{pmatrix}$$

$$Q_{21} = \begin{pmatrix} 17.6562 & 0.7993 \\ 0.7993 & 5.2005 \end{pmatrix} \qquad Q_{22} = \begin{pmatrix} 19.1482 & -0.8789 \\ -0.8789 & 7.8580 \end{pmatrix}$$

$$Q_{23} = \begin{pmatrix} 17.8220 & 0.3148 \\ 0.3148 & 7.8933 \end{pmatrix} \qquad Q_{24} = \begin{pmatrix} 19.4641 & -2.1891 \\ -2.1891 & 9.9042 \end{pmatrix}$$

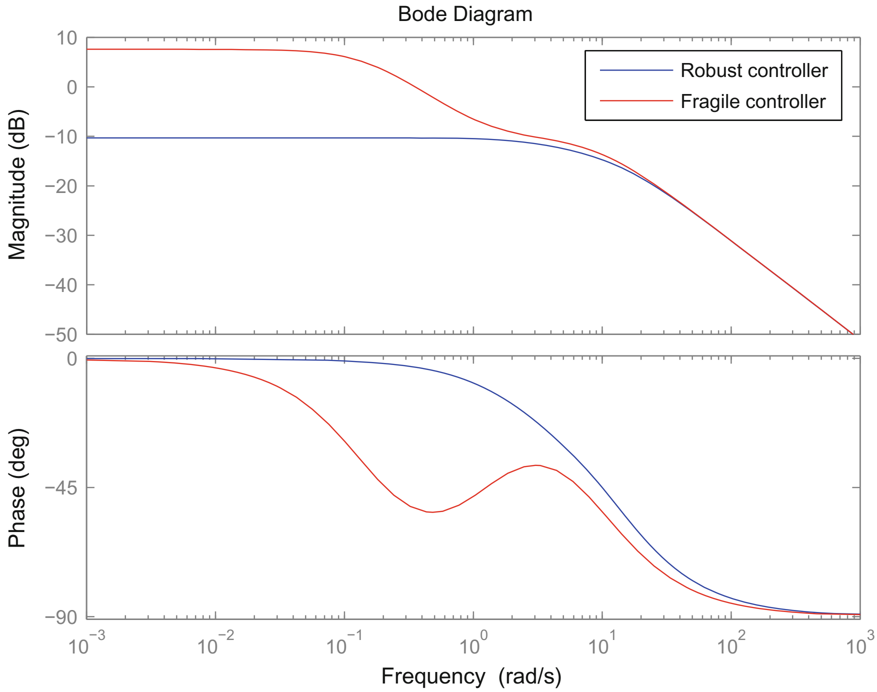

Fig. 4.4 Robust $\mathcal{H}_\infty$ performance: Bode plot

In the following, a comparison with a fragile controller, obtained applying a classical LPV technique without taking into account the uncertainty, is done. The vertex gains of the fragile controller are:

$$K_1 = \begin{pmatrix} -9.5607 & -3.0468 \\ -1.8819 & 0.8398 \end{pmatrix} \qquad K_2 = \begin{pmatrix} -19.1879 & -5.7774 \\ -4.7960 & 0.1083 \end{pmatrix}$$

Figure 4.4 shows the Bode diagrams of the closed-loop system obtained with $\nu_1 = 1.4$, $\nu_2 = 0.6$ and $\theta = 2$. It can be seen that for low frequencies, the desired specification is not attained by the fragile controller. On the other hand, the robust controller successfully achieves the performance (the magnitude plot is always below the threshold of $0\,dB$).

Finally, to complete the analysis, a simulation with $\nu_1 = 1.4$, $\nu_2 = 0.6$, $\theta(t) = 3 + \sin(t)$, $w(t) = \sin(0.01t)$, and initial condition $x(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^{\mathrm{T}}$ is considered. The evolution of the output related to the $\mathcal{H}_\infty$ ($z_\infty = x_1$) is plotted in Fig. 4.5. It is confirmed that the robust controller successfully achieves the attenuation of the exogenous disturbance $w(t)$, whereas the fragile controller does not.

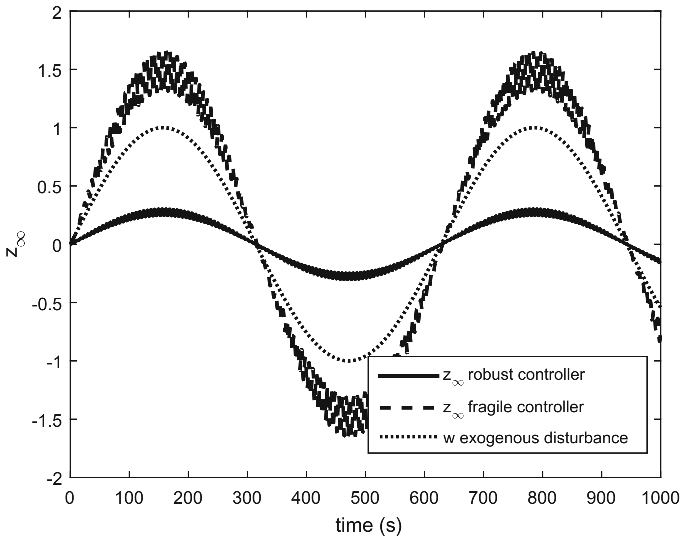

Fig. 4.5 Robust $\mathcal{H}_\infty$ performance: closed-loop response of $z_\infty(t)$

4.6 Conclusions

In this chapter, the problem of designing an LPV state-feedback controller for uncertain LPV systems has been considered. The controller has been designed such that some desired performances are achieved in the robust LPV sense, i.e. for each possible value that the scheduling parameters and the uncertainty can take.

Some well-known results obtained in the last decades in the robust and in the LPV control fields have been extended to obtain conditions that can be used to solve this problem. The provided solution relies on a double-layer polytopic description that takes into account both the variability due to the scheduling parameter vector and the uncertainty. The first polytopic layer manages the varying parameters and is used to obtain the vertex uncertain systems, where the vertex controllers are designed. The second polytopic layer is built at each vertex system so as to take into account the model uncertainties and add robustness into the design step.

The problem has been tackled using both a common quadratic Lyapunov function and a parameter-dependent Lyapunov function. In both cases, under some assumptions, a finite number of LMIs, that can be solved efficiently using available solvers, is obtained.

The proposed technique has been applied to numerical examples, showing that it achieves correctly the desired performances, i.e. robust $\mathcal{D}$-stability and robust $\mathcal{H}_\infty$ performance, whereas the traditional LPV gain-scheduling technique fails.

An open issue that requires further investigation is the obtention of dilated LMIs for the FTB/FTS analysis of CT systems. This step is necessary in order to obtain conditions for the design of robust FTB/FTS polytopic state-feedback controllers for uncertain CT LPV systems.

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Chapter 5
Shifting State-Feedback Control of LPV Systems

The content of this chapter is based on the following works:

- [1] D. Rotondo, F. Nejjari, V. Puig. A shifting pole placement approach for the design of parameter-scheduled state-feedback controllers. In *Proceedings of the 12th European Control Conference (ECC)*, pages 1829–1834, 2013;
- [2] D. Rotondo, F. Nejjari, V. Puig. Design of parameter-scheduled state-feedback controllers using shifting specifications. *Journal of the Franklin Institute*, 352(1): 93–116, 2015;
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5.1 Introduction

In this chapter, the problem of designing a parameter-scheduled state-feedback controller is investigated. In particular, this chapter takes advantage of the properties of polytopes and LMIs to solve new problems, that can be seen as extensions of the more classical $\mathcal{D}$-stability, $\mathcal{H}_\infty$ performance, $\mathcal{H}_2$ performance, finite time boundedness and finite time stability specifications, that will be referred to as *shifting $\mathcal{D}$-stability*, *shifting $\mathcal{H}_\infty$ performance*, *shifting $\mathcal{H}_2$ performance*, *shifting finite time stability* and *shifting finite time boundedness*. In these new problems, by introducing some parameters, or using the existing ones, the controller can be designed in such a way that different values of these parameters imply different performances. Notice that this is akin to the approach described in [4], where a methodology for designing a sampling period dependent controller with performance adaptation has been proposed.

D. Rotondo, *Advances in Gain-Scheduling and Fault Tolerant Control Techniques*,
Springer Theses, https://doi.org/10.1007/978-3-319-62902-5_5

From a practical point of view, reasons for which such a problem can be of interest include all situations where some performance degradation could be desirable, e.g. high-/low-gain control, control of systems with saturation nonlinearities [5], graceful performance degradation for active fault tolerant control [6] and actuator health degradation avoidance [7].

5.2 Problem Formulation

Consider the LPV system given by (2.156)–(2.158):

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + B\left(\theta(\tau)\right)u(\tau) + B_w\left(\theta(\tau)\right)w(\tau) \tag{5.1}$$

$$z_\infty(\tau) = C_{z_\infty}\left(\theta(\tau)\right)x(\tau) + D_{z_\infty u}\left(\theta(\tau)\right)u(\tau) + D_{z_\infty w}\left(\theta(\tau)\right)w(\tau) \tag{5.2}$$

$$z_2(\tau) = C_{z_2}\left(\theta(\tau)\right)x(\tau) + D_{z_2 u}\left(\theta(\tau)\right)u(\tau) \tag{5.3}$$

and divide the n_θ-dimensional set Θ into three subsets, i.e. an n_{θ_s}-dimensional set Θ_s, an n_{θ_r}-dimensional set Θ_r, and an n_{θ_p}-dimensional set Θ_p, such that:

$$\Theta = \Theta_s \times \Theta_r \times \Theta_p \tag{5.4}$$

where:

- $\theta_s(\tau)$ are varying parameters used to schedule the controller (they would correspond to variables that can be either measured or estimated);
- $\theta_r(\tau)$ are parameters that are not used to schedule the controller, and robustness must be guaranteed against their variations (these parameters would correspond to unmeasurable variables that cannot be estimated, but also to unknown but bounded uncertainties affecting the system, e.g. the ones arising from noise or estimation errors);
- $\theta_p(\tau)$ are varying parameters used to schedule not only the controller as in the case of $\theta_s(\tau)$, but also the shifting specifications $\mathcal{D}\left(\theta_p(\tau)\right)$, $\gamma_\infty\left(\theta_p(\tau)\right)$, $\gamma_2\left(\theta_p(\tau)\right)$, $c_1\left(\theta_p(\tau)\right)$ and $c_2\left(\theta_p(\tau)\right)$, defined formally in the following (see Definitions 5.1–5.7).

Θ_s, Θ_r and Θ_p are assumed to be polytopes, such that:

$$\theta_s(\tau) = \sum_{i=1}^{S} s_i\left(\theta_s(\tau)\right)\theta_{s,i}\,, \quad \sum_{i=1}^{S} s_i\left(\theta_s(\tau)\right) = 1\,, \quad s_i\left(\theta_s(\tau)\right) \geq 0\,,\ i = 1, \ldots, S \tag{5.5}$$

$$\theta_r(\tau) = \sum_{j=1}^{R} r_j\left(\theta_r(\tau)\right)\theta_{r,j}\,, \quad \sum_{j=1}^{R} r_j\left(\theta_r(\tau)\right) = 1\,, \quad r_j\left(\theta_r(\tau)\right) \geq 0\,,\ j = 1, \ldots, R \tag{5.6}$$

$$\theta_p(\tau) = \sum_{h=1}^{P} \pi_h\left(\theta_p(\tau)\right)\theta_{p,h}\,, \quad \sum_{h=1}^{P} \pi_h\left(\theta_p(\tau)\right) = 1\,, \quad \pi_h\left(\theta_p(\tau)\right) \geq 0\,,\ h = 1, \ldots, P \tag{5.7}$$

with S, R and P the numbers of vertices, denoted by $\theta_{s,i}$, $\theta_{r,j}$ and $\theta_{p,h}$ of Θ_s, Θ_r and Θ_p, respectively. Then, Θ is a Cartesian product of polytopes [8], such that:

$$\theta(\tau) = \sum_{i=1}^{S} s_i\left(\theta_s(\tau)\right) \sum_{j=1}^{R} r_j\left(\theta_r(\tau)\right) \sum_{h=1}^{P} \pi_h\left(\theta_p(\tau)\right)\theta_{ijh} \tag{5.8}$$

where θ_{ijh} is defined as:

$$\theta_{ijh} = \left[\,\theta_{s,i}\ \theta_{r,j}\ \theta_{p,h}\,\right]^{\mathrm{T}} \tag{5.9}$$

In this chapter, the problem of designing the controller:

$$u(\tau) = K\left(\theta_s(\tau), \theta_p(\tau)\right) x(\tau) \tag{5.10}$$

so as to satisfy one of the following specifications:

- shifting $\mathcal{D}$-stability
- shifting $\mathcal{H}_\infty$ performance
- shifting $\mathcal{H}_2$ performance
- shifting finite time stability
- shifting finite time boundedness

is considered. These specifications are defined in the following.

Definition 5.1 (*Shifting $\mathcal{D}$-stability of an LPV system*) Given the following scheduled subset of the complex plane:

$$\mathcal{D}(\theta_p) = \left\{\sigma \in C : f_{\mathcal{D}(\theta_p)}(\sigma, \theta_p) < O\right\} \tag{5.11}$$

where $f_{\mathcal{D}(\theta_p)}(\sigma, \theta_p)$ is the *shifting characteristic function* defined as:

$$f_{\mathcal{D}(\theta_p)}(\sigma, \theta_p) = \alpha(\theta_p) + \beta(\theta_p)\sigma + \beta(\theta_p)^{\mathrm{T}}\sigma^* = \left[\alpha_{kl}(\theta_p) + \beta_{kl}(\theta_p)\sigma + \beta_{lk}(\theta_p)\sigma^*\right]_{k,l\in\{1,\ldots,m\}} \tag{5.12}$$

where $\alpha(\theta_p) = [\alpha_{kl}(\theta_p)]_{k,l\in\{1,\ldots,m\}} \in \mathbb{S}^{m\times m}$ and $\beta(\theta_p) = [\beta_{k,l}(\theta_p)]_{k,l\in\{1,\ldots,m\}} \in \mathbb{R}^{m\times m}$, the autonomous LPV system (2.46):

$$\sigma.x(\tau) = A\left(\theta(\tau)\right) x(\tau) \tag{5.13}$$

with $\theta \in \Theta$, and Θ as in (5.4), is shifting $\mathcal{D}$-stable with respect to $\mathcal{D}(\theta_p)$ if, for every possible $\theta \in \Theta$, the poles of (5.13) are inside $\mathcal{D}(\theta_p)$.

Definition 5.2 (*Shifting $\mathcal{H}_\infty$ performance of an LPV system*) The LPV system (2.50)–(2.51):

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + B_w\left(\theta(\tau)\right)w(\tau) \tag{5.14}$$

$$z_\infty(\tau) = C_{z_\infty}\left(\theta(\tau)\right)x(\tau) + D_{z_\infty w}\left(\theta(\tau)\right)w(\tau) \tag{5.15}$$

has shifting $\mathcal{H}_\infty$ performance $\gamma_\infty(\theta_p)$ if $\left\|T_{z_\infty w}(\sigma, \theta)\right\|_\infty < \gamma_\infty(\theta_p)\ \forall\theta \in \Theta$, with Θ as in (5.4), and $T_{z_\infty w}(\sigma, \theta)$ denoting the closed-loop transfer function from $w(\tau)$ to $z_\infty(\tau)$.

Definition 5.3 (*Shifting $\mathcal{H}_2$ performance of an LPV system*) The LPV system (5.14) and (2.53):

$$z_2(\tau) = C_{z_2}\left(\theta(\tau)\right)x(\tau) \tag{5.16}$$

has shifting $\mathcal{H}_2$ performance $\gamma_2(\theta_p)$ if $\left\|T_{z_2 w}(\sigma, \theta)\right\|_2 < \gamma_2(\theta_p)\ \forall\theta \in \Theta$, with Θ as in (5.4), and $T_{z_2 w}(\sigma, \theta)$ denoting the closed-loop transfer function from $w(\tau)$ to $z_2(\tau)$.

Definition 5.4 (*Shifting finite time stability of CT LPV systems*) The autonomous LPV system (5.13), with $\tau = t$, is said to be *shifting finite time stable* (SFTS) with respect to $\left(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R\right)$ with $c_2(\theta_p) > c_1(\theta_p) > 0\ \forall\theta_p \in \Theta_p$ and $R \succ O$ if:

$$\begin{cases} x(t_0)^{\mathrm{T}}Rx(t_0) \le c_1(\theta_{p_0}) \\ \theta_p(t) = \theta_{p_0} \quad \forall t \in \left[t_0, t_0 + T(\theta_{p_0})\right] \end{cases} \Rightarrow \quad \begin{matrix} x(t)^{\mathrm{T}}Rx(t) \le c_2(\theta_{p_0}) \\ \forall t \in \left[t_0, t_0 + T(\theta_{p_0})\right] \end{matrix} \tag{5.17}$$

Definition 5.5 (*Shifting finite time stability of DT LPV systems*) The autonomous LPV system (5.13), with $\tau = k$, is said to be *shifting finite time stable* (SFTS) with respect to $\left(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R\right)$ with $c_2(\theta_p) > c_1(\theta_p) > 0\ \forall\theta_p \in \Theta_p$ and $R \succ O$ if:

$$\begin{cases} x(k_0)^{\mathrm{T}}Rx(k_0) \le c_1(\theta_{p_0}) \\ \theta_p(k) = \theta_{p_0} \quad \forall k \in \left\{k_0, \ldots, k_0 + T(\theta_{p_0}) - 1\right\} \end{cases} \Rightarrow \quad \begin{matrix} x(k)^{\mathrm{T}}Rx(k) \le c_2(\theta_{p_0}) \\ \forall k \in \left\{k_0 + 1, \ldots, k_0 + T(\theta_{p_0})\right\} \end{matrix} \tag{5.18}$$

Definition 5.6 (*Shifting finite time boundedness of CT LPV systems*) The CT LPV system (2.55):

$$\dot{x}(t) = A\left(\theta(t)\right)x(t) + B_w\left(\theta(t)\right)w(t) \tag{5.19}$$

is said to be *shifting finite time bounded* (SFTB) with respect to $\left(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R, d(\theta_p)\right)$ with $c_2(\theta_p) > c_1(\theta_p) > 0\ \forall\theta_p \in \Theta_p$ and $R \succ O$ if:

$$\begin{cases} x(t_0)^{\mathrm{T}}Rx(t_0) \le c_1(\theta_{p_0}) \\ w(t)^{\mathrm{T}}w(t) \le d(\theta_{p_0}) \quad \forall t \in \left[t_0, t_0 + T(\theta_{p_0})\right] \\ \theta_p(t) = \theta_{p_0} \end{cases} \Rightarrow \quad \begin{matrix} x(t)^{\mathrm{T}}Rx(t) \le c_2(\theta_{p_0}) \\ \forall t \in \left[t_0, t_0 + T(\theta_{p_0})\right] \end{matrix} \tag{5.20}$$

Definition 5.7 (*Shifting finite time boundedness of DT LPV systems*) The DT LPV system (2.56):

$$\begin{cases} x(k+1) = A\left(\theta(k)\right)x(k) + B_w\left(\theta(k)\right)w(k) \\ w(k+1) = W\left(\theta(k)\right)w(k) \end{cases} \tag{5.21}$$

is said to be SFTB with respect to $\left(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R, d(\theta_p)\right)$ with $c_2(\theta_p) > c_1(\theta_p) > 0\ \forall\theta_p \in \Theta_p$ and $R \succ O$ if:

$$\begin{cases} x(k_0)^{\mathrm{T}}Rx(k_0) \leq c_1(\theta_{p0}) \\ w(k)^{\mathrm{T}}w(k) \leq d(\theta_{p0}) \quad \forall k \in \left\{k_0, \ldots, k_0 + T(\theta_{p0}) - 1\right\} \\ \theta_p(k) = \theta_{p0} \end{cases} \Rightarrow \begin{array}{c} x(k)^{\mathrm{T}}Rx(k) \leq c_2(\theta_{p0}) \\ \forall k \in \left\{k_0 + 1, \ldots, k_0 + T(\theta_{p0})\right\} \end{array} \tag{5.22}$$

Despite the problem of design using shifting specifications is being considered for the case of LPV systems, the proposed method is useful for LTI systems too. In this case, a vector $\theta_p(t)$, exogenous with respect to the system to be controlled, is introduced, and used to schedule the controller, such that, even though the plant to be controlled is LTI, the overall system is LPV and the mathematical reasoning developed hereafter can be applied. The reason behind doing so is that in this way the performance of the closed-loop system can be varied in time according to some criterium, e.g. energetic issues. The introduction of an exogenous $\theta_p(t)$ can also be done in the case of LPV systems, when it is desired to vary the performance according to criteria that are not connected with the intrinsic varying parameters of the LPV system.

5.3 Design Using a Common Quadratic Lyapunov Function

The following theorems provide some conditions for designing the controller (5.10) in order to satisfy the shifting specifications introduced in Definitions 5.1–5.7. For the sake of simplicity, only the case where a common quadratic Lyapunov function, as in (2.58):

$$V\left(x(\tau)\right) = x(\tau)^{\mathrm{T}}Px(\tau) \tag{5.23}$$

will be considered.

Theorem 5.1 (Quadratic shifting $\mathcal{D}$-stabilizability of LPV systems) *Given an LMI region scheduled by θ_p, as in (5.11), the LPV system (5.1) with $B_w\left(\theta(\tau)\right) = O$ and control law (5.10) is quadratically shifting $\mathcal{D}$-stable with respect to $\mathcal{D}(\theta_p)$ if there exist $Q \succ O$ and $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\alpha(\theta_p) \otimes Q + He\left\{\beta(\theta_p) \otimes \left[A(\theta)Q + B(\theta)K(\theta_s, \theta_p)Q\right]\right\} \prec O \qquad \forall\theta \in \Theta \tag{5.24}$$

Proof The condition (5.24) is obtained from (2.137), by considering that α and β are not constant, but functions of θ_p, and that K depends only on θ_s and θ_p. ■

Theorem 5.2 (Quadratic shifting $\mathcal{H}_\infty$ state feedback for CT LPV systems) *The CT LPV system* (5.1)–(5.2) *with* $\tau = t$ *and control law* (5.10) *has quadratic shifting* $\mathcal{H}_\infty$ *performance* $\gamma_\infty(\theta_p)$ *if there exist* $Q \succ O$ *and* $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ *such that:*

$$\begin{pmatrix} He\left\{A(\theta)Q + B(\theta)K(\theta_s,\theta_p)Q\right\} & * & * \\ B_w(\theta)^{\mathrm{T}} & -I & * \\ C_{z_\infty}(\theta)Q + D_{z_\infty u}(\theta)K(\theta_s,\theta_p)Q & D_{z_\infty w}(\theta) & -\gamma_\infty(\theta_p)^2 I \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{5.25}$$

Proof The condition (5.25) is obtained from (2.140), by considering that γ_∞ is not constant, but function of θ_p, and that K depends only on θ_s and θ_p. ■

Theorem 5.3 (Quadratic shifting $\mathcal{H}_\infty$ state feedback for DT LPV systems) *The DT LPV system* (5.1)–(5.2) *with* $\tau = k$ *and control law* (5.10) *has quadratic shifting* $\mathcal{H}_\infty$ *performance* $\gamma_\infty(\theta_p)$ *if there exist* $Q \succ O$ *and* $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ *such that:*

$$\begin{pmatrix} Q & A(\theta)Q + B(\theta)K(\theta_s,\theta_p)Q & B_w(\theta) & O \\ * & Q & O & QC_{z_\infty}(\theta)^{\mathrm{T}} + QK(\theta_s,\theta_p)^{\mathrm{T}} D_{z_\infty u}(\theta)^{\mathrm{T}} \\ * & * & I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ * & * & * & \gamma_\infty(\theta_p)^2 I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{5.26}$$

Proof The condition (5.26) is obtained from (2.143), by considering that γ_∞ is not constant, but function of θ_p, and that K depends only on θ_s and θ_p. ■

Theorem 5.4 (Quadratic shifting $\mathcal{H}_2$ state feedback for CT LPV systems) *The CT LPV system* (5.1) *and (2.158) with* $\tau = t$ *and control law* (5.10) *has quadratic shifting* $\mathcal{H}_2$ *performance* $\gamma_2(\theta_p)$ *if there exist* $Q \succ O$, $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ *and* $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr\,(Y(\theta)) < \gamma_2(\theta_p)^2$ $\forall\theta \in \Theta$ *and:*

$$\begin{pmatrix} He\left\{A(\theta)Q + B(\theta)K(\theta_s,\theta_p)Q\right\} & B_w(\theta) \\ * & -I \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{5.27}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q + D_{z_2 u}(\theta)K(\theta_s,\theta_p)Q \\ * & Q \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{5.28}$$

Proof It follows from the conditions of Theorem 2.17, by considering that γ_2 is not constant, but function of θ_p, and that K only depends on θ_s and θ_p. ■

Theorem 5.5 (Quadratic shifting $\mathcal{H}_2$ state feedback for DT LPV systems) *The DT LPV system* (5.1) *and* (5.3) *with* $\tau = k$ *and control law* (5.10) *has quadratic shifting* $\mathcal{H}_2$ *performance* $\gamma_2(\theta_p)$ *if there exist* $Q \succ O$, $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ *and* $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr\,(Y(\theta)) < \gamma_2(\theta_p)^2$ $\forall\theta \in \Theta$ *and:*

$$\begin{pmatrix} Q & A(\theta)Q + B(\theta)K(\theta_s,\theta_p)Q & B_w(\theta) \\ * & Q & O \\ * & * & I \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{5.29}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q + D_{z_2u}(\theta)K(\theta_s, \theta_p)Q \\ * & Q \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{5.30}$$

Proof It follows from the conditions of Theorem 2.18, by considering that γ_2 is not constant, but function of θ_p, and that K depends only on θ_s and θ_p. ■

Theorem 5.6 (Quadratic SFTB state feedback for CT LPV systems) *The CT LPV system* (5.1) *with $\tau = t$ and control law* (5.10) *is quadratically SFTB with respect to $\big(c_1(\theta_p), c_2(\theta_p), \ T(\theta_p), R, d(\theta_p)\big)$ if, letting $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, there exist a positive scalar α, positive functions $\lambda_1(\theta_p)$, $\lambda_2(\theta_p)$, $\lambda_3(\theta_p)$, positive definite matrices $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and $Q_2 \in \mathbb{S}^{n_w \times n_w}$, and a matrix function $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\begin{pmatrix} He\left\{A(\theta)\tilde{Q}_1 + B(\theta)K(\theta_s, \theta_p)\tilde{Q}_1\right\} - \alpha\tilde{Q}_1 & B_w(\theta)Q_2 \\ * & -\alpha Q_2 \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{5.31}$$

$$\lambda_1(\theta_p)I \prec Q_1 \prec I \qquad \forall\theta_p \in \Theta_p \tag{5.32}$$

$$\lambda_2(\theta_p)I \prec Q_2 \prec \lambda_3(\theta_p)I \qquad \forall\theta_p \in \Theta_p \tag{5.33}$$

$$\begin{pmatrix} c_2(\theta_p)e^{-\alpha T(\theta_p)} & \sqrt{c_1(\theta_p)} & \sqrt{d(\theta_p)} \\ \sqrt{c_1(\theta_p)} & \lambda_1(\theta_p) & 0 \\ \sqrt{d(\theta_p)} & 0 & \lambda_2(\theta_p) \end{pmatrix} \succ O \qquad \forall\theta_p \in \Theta_p \tag{5.34}$$

Proof From Definition 5.6, by introducing the new time variable $\tilde{t} = t - t_0$, (5.20) becomes:

$$\begin{cases} x(0)^{\mathrm{T}}Rx(0) \le c_1(\theta_{p_0}) \\ w(\tilde{t})^{\mathrm{T}}w(\tilde{t}) \le d(\theta_{p_0}) & \forall\tilde{t} \in \left[0, T(\theta_{p_0})\right] \\ \theta_p(\tilde{t}) = \theta_{p_0} \end{cases} \Rightarrow \quad \begin{array}{c} x(\tilde{t})^{\mathrm{T}}Rx(\tilde{t}) \le c_2(\theta_{p_0}) \\ \forall\tilde{t} \in \left[0, T(\theta_{p_0})\right] \end{array} \tag{5.35}$$

Since $\theta_p(t)$ is constant during the considered time interval, it follows that in order to obtain (5.35), the property of finite time boundedness, as defined in Definition 2.9, should hold $\forall\theta_p \in \Theta_p$. The remaining of the proof follows from the conditions of Theorem 2.19, taking into account that K depends only on θ_s and θ_p. ■

Theorem 5.7 (Quadratic SFTB state feedback for DT LPV systems) *The DT LPV system* (5.1) *with $\tau = k$ and (2.151):*

$$w(k+1) = W\left(\theta(k)\right)w(k) \tag{5.36}$$

with control law (5.10) *is quadratically SFTB with respect to $\big(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R, d(\theta_p)\big)$ if there exist a positive scalar $\alpha \ge 1$, positive functions $\lambda_1(\theta_p)$, $\lambda_2(\theta_p)$, positive definite matrices $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and $Q_2 \in \mathbb{S}^{n_w \times n_w}$ and a matrix function $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\begin{pmatrix} -\alpha Q_1 & * & * & * \\ A(\theta)Q_1 + B(\theta)K(\theta_s, \theta_p)Q_1 & -Q_1 & * & * \\ O & B_w(\theta)^{\mathrm{T}} & -\alpha Q_2 & * \\ O & O & Q_2 W(\theta) & -Q_2 \end{pmatrix} \prec O \qquad \forall \theta \in \Theta \tag{5.37}$$

$$\lambda_1(\theta_p)R^{-1} \prec Q_1 \prec R^{-1} \qquad \forall \theta_p \in \Theta_p \tag{5.38}$$

$$O \prec Q_2 \prec \lambda_2(\theta_p)I \qquad \forall \theta_p \in \Theta_p \tag{5.39}$$

$$\begin{pmatrix} \frac{c_2(\theta_p)}{\alpha^{T(\theta_p)}} - \lambda_2(\theta_p)d(\theta_p) & \sqrt{c_1(\theta_p)} \\ \sqrt{c_1(\theta_p)} & \lambda_1(\theta_p) \end{pmatrix} \succ O \qquad \forall \theta_p \in \Theta_p \tag{5.40}$$

Proof From Definition 5.7, by introducing the new time variable $\tilde{k} = k - k_0$, (5.22) becomes:

$$\begin{cases} x(0)^{\mathrm{T}} R x(0) \le c_1(\theta_{p0}) \\ w(\tilde{k})^{\mathrm{T}} w(\tilde{k}) \le d(\theta_{p0}) & \forall \tilde{k} \in \{0, \dots, T(\theta_{p0}) - 1\} \\ \theta_p(\tilde{k}) = \theta_{p0} \end{cases} \Rightarrow \begin{matrix} x(\tilde{k})^{\mathrm{T}} R x(\tilde{k}) \le c_2(\theta_{p0}) \\ \forall \tilde{k} \in \{1, \dots, T(\theta_{p0})\} \end{matrix} \tag{5.41}$$

Since $\theta_p(k)$ is constant during the considered time interval, it follows that in order to obtain (5.41), the property of finite time boundedness, as defined in Definition 2.9, should hold $\forall \theta_p \in \Theta_p$. The remaining of the proof follows from the conditions of Theorem 2.20, taking into account that K only depends on θ_s and θ_p. ■

Theorem 5.8 (Quadratic shifting finite time stabilization of CT LPV systems) *The CT LPV system* (5.1)*, with* $\tau = t$*,* $B_w(\theta(t)) = O$*, and control law* (5.10)*, is quadratically shifting finite time stabilizable with respect to* $\big(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R\big)$ *if, letting* $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$*, there exist a positive scalar* α*, a positive function* $\lambda_1(\theta_p)$*, a positive definite matrix* $Q_1 \in \mathbb{S}^{n_x \times n_x}$*, and a matrix function* $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ *such that:*

$$He\left\{A(\theta)\tilde{Q}_1 + B(\theta)K(\theta_s, \theta_p)\tilde{Q}_1\right\} - \alpha\tilde{Q}_1 \prec O \qquad \forall \theta \in \Theta \tag{5.42}$$

$$\begin{pmatrix} c_2(\theta_p)e^{-\alpha T(\theta_p)} & \sqrt{c_1(\theta_p)} \\ \sqrt{c_1(\theta_p)} & \lambda_1(\theta_p) \end{pmatrix} \succ O \qquad \forall \theta_p \in \Theta_p \tag{5.43}$$

and (5.32) *hold.*

Proof It is a direct consequence of Theorem 5.6 when $B_w(\theta(t)) = O$ and $d(\theta_p) = 0$. ■

Theorem 5.9 (Quadratic shifting finite time stabilization of DT LPV systems) *The DT LPV system* (5.1)*, with* $\tau = k$*,* $B_w(\theta(k)) = O$*, and control law* (5.10)*, is quadratically shifting finite time stabilizable with respect to* $\big(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R\big)$ *if there exist a positive scalar* $\alpha \ge 1$*, a positive function* $\lambda_1(\theta_p)$*, a positive definite matrix* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and a matrix function* $K(\theta_s, \theta_p) \in \mathbb{R}^{n_u \times n_x}$ *such that:*

$$\begin{pmatrix} -\alpha Q_1 & * \\ A(\theta)Q_1 + B(\theta)K(\theta_s, \theta_p)Q_1 & -Q_1 \end{pmatrix} \prec O \qquad \forall \theta_p \in \Theta_p \tag{5.44}$$

$$\begin{pmatrix} \frac{c_2(\theta_p)}{\alpha^{T(\theta_p)}} & \sqrt{c_1(\theta_p)} \\ \sqrt{c_1(\theta_p)} & \lambda_1(\theta_p) \end{pmatrix} \succ O \qquad \forall \theta_p \in \Theta_p \tag{5.45}$$

and (5.38) *hold.*

Proof It is a direct consequence of Theorem 5.7 when $W(\theta(k)) = B_w(\theta(k)) = O$ and $d(\theta_p) = 0$. ■

In the CT case, it can be proved that the quadratic shifting $\mathcal{D}$-stability specification allows varying in time the transient performance of the closed-loop system, i.e. its decay or growth rate. This is stated by the following corollary, that is based on Corollary 2.1 [9, 10].

Corollary 5.1 *Let $V(x(t))$ be defined as in* (5.23)*, and let the autonomous LPV system* (5.13) *be quadratically shifting $\mathcal{D}$-stable with respect to $\mathcal{D}(\theta_p)$, i.e.* (5.24) *holds. Then, the Lyapunov function $V(x(t))$ satisfies, for all $x(t) \neq 0$:*

$$\frac{1}{2}\frac{\dot{V}(x(t))}{V(x(t))} \in \mathcal{D}(\theta_p) \cap \mathbb{R} \tag{5.46}$$

Proof It follows the reasoning of the proof of Corollary 2.1, thus it is omitted. ■

Looking at Corollary 5.1, it can be seen that using a shifting $\mathcal{D}$-stability specification, it is possible to modify online the constraint on the minimum decay rate (if $\mathcal{D}\left(\theta_p(t)\right)$ is contained in the left half plane), or the maximum possible growth rate of the Lyapunov function used to assess the shifting $\mathcal{D}$-stability. From a practical point of view, it is possible to vary online other transient performances using the proposed shifting specifications, also in the case of DT systems.

However, the conditions provided in Theorems 5.1–5.9 cannot be used for the controller design, since they impose an infinite number of constraints. This difficulty can be alleviated under the following assumptions:

- $\beta\left(\theta_p(\tau)\right)$ is a constant matrix and:

$$\begin{pmatrix} \alpha\left(\theta_p(\tau)\right) \\ \gamma_\infty\left(\theta_p(\tau)\right)^2 \\ \gamma_2\left(\theta_p(\tau)\right)^2 \\ \sqrt{c_1\left(\theta_p(\tau)\right)} \end{pmatrix} = \sum_{h=1}^{P} \pi_h\left(\theta_p(\tau)\right) \begin{pmatrix} \kappa_{1,h} \\ \kappa_{2,h} \\ \kappa_{3,h} \\ \kappa_{4,h} \end{pmatrix} \tag{5.47}$$

$$\begin{pmatrix} c_2\left(\theta_p(t)\right) e^{-\alpha T\left(\theta_p(t)\right)} \\ \sqrt{d\left(\theta_p(t)\right)} \end{pmatrix} = \sum_{h=1}^{P} \pi_h\left(\theta_p(t)\right) \begin{pmatrix} \kappa_{5,h} \\ \kappa_{6,h} \end{pmatrix} \qquad \text{CT systems} \tag{5.48}$$

$$\begin{pmatrix} \frac{c_2(\theta_p(k))}{\alpha^T(\theta_p(k))} \\ d\left(\theta_p(t)\right) \end{pmatrix} = \sum_{h=1}^{P} \pi_h \left(\theta_p(k)\right) \begin{pmatrix} \kappa_{5,h} \\ \kappa_{6,h} \end{pmatrix} \qquad \text{DT systems} \tag{5.49}$$

Looking at the examples of LMI regions provided in Chap. 2, it can be seen that the assumption of a constant β matrix corresponds to fixing the shape of the shifting LMI region.

- The matrices $B\,(\theta(\tau))$, $D_{z_\infty u}\,(\theta(\tau))$ and $D_{z_2 u}\,(\theta(\tau))$ only depend on $\theta_r(\tau)$ and:

$$\begin{pmatrix} A\,(\theta(\tau)) \\ B_w\,(\theta(\tau)) \\ C_{z_\infty}\,(\theta(\tau)) \\ D_{z_\infty w}\,(\theta(\tau)) \\ C_{z_2}\,(\theta(\tau)) \\ W\,(\theta(\tau)) \\ Y\,(\theta(\tau)) \end{pmatrix} = \sum_{i=1}^{S} s_i\,(\theta_s(\tau)) \sum_{j=1}^{R} r_j\,(\theta_r(\tau)) \sum_{h=1}^{P} \pi_h\left(\theta_p(\tau)\right) \begin{pmatrix} A_{ijh} \\ B_{w,ijh} \\ C_{z_\infty,ijh} \\ D_{z_\infty w,ijh} \\ C_{z_2,ijh} \\ W_{ijh} \\ Y_{ijh} \end{pmatrix} \tag{5.50}$$

$$\begin{pmatrix} B\,(\theta_r(\tau)) \\ D_{z_\infty u}\,(\theta_r(\tau)) \\ D_{z_2 u}\,(\theta_r(\tau)) \end{pmatrix} = \sum_{j=1}^{R} r_j\,(\theta_r(\tau)) \begin{pmatrix} B_j \\ D_{z_\infty u,j} \\ D_{z_2 u,j} \end{pmatrix} \tag{5.51}$$

Notice that when the assumption that $B\,(\theta(\tau))$, $D_{z_\infty u}\,(\theta(\tau))$ and $D_{z_2 u}\,(\theta(\tau))$ only depend on $\theta_r(\tau)$ introduces too much conservativeness, it is possible to relax this assumption by filtering the inputs, as proposed by [11].

Then, it is possible to consider the following control law:

$$u(\tau) = \sum_{i=1}^{S} s_i\,(\theta_s(\tau)) \sum_{h=1}^{P} \pi_h\left(\theta_p(\tau)\right) K_{ih}\, x(\tau) \tag{5.52}$$

and reduce the conditions provided by Theorems 5.1–5.9 to a finite number of matrix inequalities, by rewriting them at the $S \cdot R \cdot P$ vertices of Θ, as stated by the following corollaries.

Corollary 5.2 (Design of a quadratically shifting $\mathcal{D}$-stabilizing polytopic state-feedback controller for LPV systems) *Given an LMI region scheduled by θ_p, as in (5.11), with $\beta\left(\theta_p(\tau)\right) = \beta$, let $Q \succ O$ and $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, S$, $h = 1, \ldots, P$, be such that:*

$$\kappa_{1,h} \otimes Q + He\left\{\beta \otimes \left[A_{ijh} Q + B_j \Gamma_{ih}\right]\right\} \prec O \tag{5.53}$$

$\forall i = 1, \ldots, S$, $\forall j = 1, \ldots, R$, $\forall h = 1, \ldots, P$. Then, the closed-loop system made up by the LPV system (5.1), with $B_w\,(\theta(\tau)) = O$ and polytopic matrices as in (5.50)–(5.51), and the polytopic state-feedback control law (5.52) with gains calculated as $K_{ih} = \Gamma_{ih} Q^{-1}$, $i = 1, \ldots, S$, $h = 1, \ldots, P$, is quadratically shifting $\mathcal{D}$-stable with respect to $\mathcal{D}(\theta_p)$.

Proof It follows from the basic property of matrices [12] that any linear combination of negative definite matrices with non-negative coefficients, whose sum is positive, is negative definite, and it uses a reasoning similar to the one used in previous theorems, thus it is omitted. ■

Corollary 5.3 (Design of a quadratic shifting $\mathcal{H}_\infty$ polytopic state-feedback controller for CT LPV systems) *Let $Q \succ O$ and $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, S$, $h = 1, \ldots, P$, be such that:*

$$\begin{pmatrix} He\left\{A_{ijh}Q + B_j\Gamma_{ih}\right\} & * & * \\ B_{w,ijh}^{\mathrm{T}} & -I & * \\ C_{z_\infty,ijh}Q + D_{z_\infty u,j}\Gamma_{ih} & D_{z_\infty w,ijh} & -\kappa_{2,h}I \end{pmatrix} \prec O \qquad \begin{array}{l} \forall i = 1, \ldots, S \\ \forall j = 1, \ldots, R \\ \forall h = 1, \ldots, P \end{array} \tag{5.54}$$

Then, the closed-loop system made up by the CT LPV system (5.1)–(5.2), with $\tau = t$, and polytopic matrices as in (5.50)–(5.51), and the polytopic state-feedback control law (5.52) with gains calculated as $K_{ih} = \Gamma_{ih}Q^{-1}$, $i = 1, \ldots, S$, $h = 1, \ldots, P$, has quadratic shifting $\mathcal{H}_\infty$ performance $\gamma_\infty(\theta_p)$.

Proof Similar to that of Corollary 5.2, thus omitted. ■

Corollary 5.4 (Design of a quadratic shifting $\mathcal{H}_\infty$ polytopic state-feedback controller for DT LPV systems) *Let $Q \succ O$ and $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, S$, $h = 1, \ldots, P$, be such that:*

$$\begin{pmatrix} Q & A_{ijh}Q + B_j\Gamma_{ih} & B_{w,ijh} & O \\ * & Q & O & QC_{z_\infty,ijh}^{\mathrm{T}} + \Gamma_{ih}^{\mathrm{T}}D_{z_\infty u,j}^{\mathrm{T}} \\ * & * & I & D_{z_\infty w,ijh}^{\mathrm{T}} \\ * & * & * & \kappa_{2,h}I \end{pmatrix} \succ O \qquad \begin{array}{l} \forall i = 1, \ldots, S \\ \forall j = 1, \ldots, R \\ \forall h = 1, \ldots, P \end{array} \tag{5.55}$$

Then, the closed-loop system made up by the DT LPV system (5.1)–(5.2), with $\tau = k$, and polytopic matrices as in (5.50)–(5.51), and the polytopic state-feedback control law (5.52) with gains calculated as $K_{ih} = \Gamma_{ih}Q^{-1}$, $i = 1, \ldots, S$, $h = 1, \ldots, P$, has quadratic shifting $\mathcal{H}_\infty$ performance $\gamma_\infty(\theta_p)$.

Proof Similar to that of Corollary 5.2, thus omitted. ■

Corollary 5.5 (Design of a quadratic shifting $\mathcal{H}_2$ polytopic state-feedback controller for CT LPV systems) *Let $Q \succ O$, $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$ and $Y_{ijh} \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$, $i = 1, \ldots, S$, $j = 1, \ldots, R$, $h = 1, \ldots, P$, be such that:*

$$Tr\left(Y_{ijh}\right) < \kappa_{3,h} \tag{5.56}$$

$$\begin{pmatrix} He\left\{A_{ijh}Q + B_j\Gamma_{ih}\right\} & B_{w,ijh} \\ * & -I \end{pmatrix} \prec O \tag{5.57}$$

$$\begin{pmatrix} Y_{ijh} & C_{z_2,ijh}Q + D_{z_2u,j}\Gamma_{ih} \\ * & Q \end{pmatrix} \succ O \tag{5.58}$$

Then, the closed-loop system made up by the CT LPV system (5.1) *and* (5.3)*, with* $\tau = t$*, and polytopic matrices as in* (5.50)–(5.51)*, and the polytopic state-feedback control law* (5.52) *with gains calculated as* $K_{ih} = \Gamma_{ih}Q^{-1}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, has quadratic shifting* $\mathcal{H}_2$ *performance* $\gamma_2(\theta_p)$*.*

Proof Similar to that of Corollary 5.2, thus omitted. ■

Corollary 5.6 (Design of a quadratic shifting $\mathcal{H}_2$ polytopic state-feedback controller for DT LPV systems) *Let* $Q \succ O$*,* $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$ *and* $Y_{ijh} \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$*,* $i = 1, \ldots, S$*,* $j = 1, \ldots, R$*,* $h = 1, \ldots, P$*, be such that* (5.56)*,* (5.58) *and:*

$$\begin{pmatrix} Q & A_{ijh}Q + B_j\Gamma_{ih} & B_{w,ijh} \\ * & Q & O \\ * & * & I \end{pmatrix} \succ O \tag{5.59}$$

hold $\forall i = 1, \ldots, S$*,* $\forall j = 1, \ldots, R$*,* $\forall h = 1, \ldots, P$*. Then, the closed-loop system made up by the DT LPV system* (5.1) *and* (5.3)*, with* $\tau = k$*, and polytopic matrices as in* (5.50)–(5.51)*, and the polytopic state-feedback control law* (5.52) *with gains calculated as* $K_{ih} = \Gamma_{ih}Q^{-1}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, has quadratic shifting* $\mathcal{H}_2$ *performance* $\gamma_2(\theta_p)$*.*

Proof Similar to that of Corollary 5.2, thus omitted. ■

Corollary 5.7 (Design of a quadratic SFTB polytopic state-feedback controller for CT LPV systems) *Fix* $\alpha > 0$*, and let* $\lambda_{1,h} > 0$*,* $\lambda_{2,h} > 0$*,* $\lambda_{3,h} > 0$*,* $Q_1 \succ O$*,* $Q_2 \succ O$*, and* $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, be such that:*

$$\begin{pmatrix} He\left\{A_{ijh}\tilde{Q}_1 + B_j\Gamma_{ih}\right\} - \alpha\tilde{Q}_1 & B_{w,ijh}Q_2 \\ * & -\alpha Q_2 \end{pmatrix} \prec O \tag{5.60}$$

$$\lambda_{1,h}I \prec Q_1 \prec I \tag{5.61}$$

$$\lambda_{2,h}I \prec Q_2 \prec \lambda_{3,h}I \tag{5.62}$$

$$\begin{pmatrix} \kappa_{5,h} & \kappa_{4,h} & \kappa_{6,h} \\ \kappa_{4,h} & \lambda_{1,h} & 0 \\ \kappa_{6,h} & 0 & \lambda_{2,h} \end{pmatrix} \succ O \tag{5.63}$$

hold $\forall i = 1, \ldots, S$*,* $\forall j = 1, \ldots, R$*,* $\forall h = 1, \ldots, P$*, where* $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$*. Then, the closed-loop system made up by the CT LPV system* (5.1)*, with* $\tau = t$*, and polytopic matrices as in* (5.50)–(5.51)*, and the polytopic state-feedback control law* (5.52) *with gains calculated as* $K_{ih} = \Gamma_{ih}\tilde{Q}_1^{-1}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, is quadratically SFTB with respect to* $\left(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R, d(\theta_p)\right)$*.*

Proof Similar to that of Corollary 5.2, thus omitted. ■

Corollary 5.8 (Design of a quadratic SFTB polytopic state-feedback controller for DT LPV systems) *Fix* $\alpha \geq 1$*, and let* $\lambda_{1,h} > 0$*,* $\lambda_2 > 0$*,* $Q_1 \succ O$*,* $Q_2 \succ O$ *and* $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, be such that:*

$$\begin{pmatrix} -\alpha Q_1 & * & * & * \\ A_{ijh} Q_1 + B_j \Gamma_{ih} & -Q_1 & * & * \\ O & B_{w,ijh}^{\mathrm{T}} & -\alpha Q_2 & * \\ O & O & Q_2 W_{ijh} & -Q_2 \end{pmatrix} \prec O \tag{5.64}$$

$$\lambda_{1,h} R^{-1} \prec Q_1 \prec R^{-1} \tag{5.65}$$

$$O \prec Q_2 \prec \lambda_2 I \tag{5.66}$$

$$\begin{pmatrix} \kappa_{5,h} - \lambda_2 \kappa_{6,h} & \kappa_{4,h} \\ \kappa_{4,h} & \lambda_{1,h} \end{pmatrix} \succ O \tag{5.67}$$

hold $\forall i = 1, \ldots, S$*,* $\forall j = 1, \ldots, R$*,* $\forall h = 1, \ldots, P$*. Then, the closed-loop system made up by the DT LPV system* (5.1) *and* (5.36)*, with* $\tau = k$*, and polytopic matrices as in* (5.50)–(5.51)*, and the polytopic state-feedback control law* (5.52) *with gains calculated as* $K_{ih} = \Gamma_{ih} Q_1^{-1}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, is quadratically SFTB with respect to* $\big(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R, d(\theta_p)\big)$*.*

Proof Similar to that of Corollary 5.2, thus omitted. ■

Corollary 5.9 (Design of a quadratically shifting finite time stabilizing polytopic state-feedback controller for CT LPV systems) *Fix* $\alpha > 0$*, and let* $\lambda_{1,h} > 0$*,* $Q_1 \succ O$*, and* $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, be such that:*

$$He\left\{A_{ijh} \tilde{Q}_1 + B_j \Gamma_{ih}\right\} - \alpha \tilde{Q}_1 \prec O \tag{5.68}$$

$$\begin{pmatrix} \kappa_{5,h} & \kappa_{4,h} \\ \kappa_{4,h} & \lambda_{1,h} \end{pmatrix} \succ O \tag{5.69}$$

and (5.61) *hold* $\forall i = 1, \ldots, S$*,* $\forall j = 1, \ldots, R$*,* $\forall h = 1, \ldots, P$*, where* $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$*. Then, the closed-loop system made up by the CT LPV system* (5.1)*, with* $\tau = t$*,* $B_w\,(\theta(t))$*, and polytopic matrices as in* (5.50)–(5.51)*, and the polytopic state-feedback control law* (5.52) *with gains calculated as* $K_{ih} = \Gamma_{ih} \tilde{Q}_1^{-1}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, is quadratically SFTS with respect to* $\big(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R\big)$*.*

Proof It is a direct consequence of Corollary 5.7 when $B_{w,ijh} = O$ and $\kappa_{6,h} = 0$. ■

Corollary 5.10 (Design of a quadratically shifting finite time stabilizing polytopic state-feedback controller for DT LPV systems) *Fix* $\alpha \geq 1$*, and let* $\lambda_{1,h} > 0$*,* $Q_1 \succ O$ *and* $\Gamma_{ih} \in \mathbb{R}^{n_u \times n_x}$*,* $i = 1, \ldots, S$*,* $h = 1, \ldots, P$*, be such that* (5.65)*,* (5.69) *and:*

$$\begin{pmatrix} -\alpha Q_1 & * \\ A_{ijh} Q_1 + B_j \Gamma_{ih} & -Q_1 \end{pmatrix} \prec O \tag{5.70}$$

hold $\forall i = 1, \ldots, S$, $\forall j = 1, \ldots, R$, $\forall h = 1, \ldots, P$. *Then, the closed-loop system made up by the DT LPV system* (5.1)*, with* $\tau = k$, $B_w\left(\theta(k)\right)$*, and polytopic matrices as in* (5.50)–(5.51)*, and the polytopic state-feedback control law* (5.52) *with gains calculated as* $K_{ih} = \Gamma_{ih} Q_1^{-1}$, $i = 1, \ldots, S$, $h = 1, \ldots, P$*, is quadratically SFTS with respect to* $\left(c_1(\theta_p), c_2(\theta_p), T(\theta_p), R\right)$.

Proof It is a direct consequence of Corollary 5.8 when $W_{ijh} = B_{w,ijh} = O$ and $\kappa_{6,h} = 0$. ■

5.4 Examples

5.4.1 Example 1: Shifting $\mathcal{D}$-Stability

Let us consider a CT LPV system described by (5.1) with:

$$A\left(\theta(t)\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\theta_p(t) & -\theta_s(t) & \theta_p(t) & 0 \\ 0 & 0 & 0 & 1 \\ \theta_p(t) & 0 & -\theta_p(t) & -\theta_r(t) \end{pmatrix} \quad B\left(\theta_r(t)\right) = \begin{pmatrix} 0 & 0 \\ \theta_r(t) & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad B_w\left(\theta(t)\right) = O$$

with the varying parameters $\theta_s \in [2, 3]$, $\theta_r \in [2, 3]$ and $\theta_p \in [1, 2]$ (in this example, the subscripts s, r and p are used following the notation explained in Sect. 5.3). Notice that the assumption about the matrix B depending only on the subset of varying parameters $\theta_r(t)$ is verified.

The LPV system matrices can be described as polytopic combinations of LTI system matrices as in (5.50)–(5.51):

$$\begin{aligned} A\left(\theta(t)\right) = {} & s_1\left(\theta_s(t)\right) r_1\left(\theta_r(t)\right) \pi_1\left(\theta_p(t)\right) A_{111} + s_1\left(\theta_s(t)\right) r_1\left(\theta_r(t)\right) \pi_2\left(\theta_p(t)\right) A_{112} \\ & + s_1\left(\theta_s(t)\right) r_2\left(\theta_r(t)\right) \pi_1\left(\theta_p(t)\right) A_{121} + s_1\left(\theta_s(t)\right) r_2\left(\theta_r(t)\right) \pi_2\left(\theta_p(t)\right) A_{122} \\ & + s_2\left(\theta_s(t)\right) r_1\left(\theta_r(t)\right) \pi_1\left(\theta_p(t)\right) A_{211} + s_2\left(\theta_s(t)\right) r_1\left(\theta_r(t)\right) \pi_2\left(\theta_p(t)\right) A_{212} \\ & + s_2\left(\theta_s(t)\right) r_2\left(\theta_r(t)\right) \pi_1\left(\theta_p(t)\right) A_{221} + s_2\left(\theta_s(t)\right) r_2\left(\theta_r(t)\right) \pi_2\left(\theta_p(t)\right) A_{222} \end{aligned}$$

$$B\left(\theta_r(t)\right) = r_1\left(\theta_r(t)\right) B_1 + r_2\left(\theta_r(t)\right) B_2$$

with:

$$A_{111} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{pmatrix} \qquad A_{112} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix}$$

$$A_{121} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -3 \end{pmatrix} \qquad A_{122} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -3 \end{pmatrix}$$

$$A_{211} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{pmatrix} \qquad A_{212} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -2 \end{pmatrix}$$

$$A_{221} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -3 \end{pmatrix} \qquad A_{222} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & -3 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let us solve the design problem of finding a state-feedback gain:

$$\begin{aligned} K\left(\theta_s(t), \theta_p(t)\right) &= s_1\left(\theta_s(t)\right)\pi_1\left(\theta_p(t)\right)K_{11} + s_1\left(\theta_s(t)\right)\pi_2\left(\theta_p(t)\right)K_{12} \\ &\quad + s_2\left(\theta_s(t)\right)\pi_1\left(\theta_p(t)\right)K_{21} + s_2\left(\theta_s(t)\right)\pi_2\left(\theta_p(t)\right)K_{22} \end{aligned}$$

that places the closed-loop poles in a disk of radius $r\left(\theta_p(t)\right)$ and center $\left(-q\left(\theta_p(t)\right), 0\right)$, described by the characteristic function:

$$f_{D(\theta_p)}\left(z, \theta_p(t)\right) = \begin{pmatrix} -r\left(\theta_p(t)\right) & q\left(\theta_p(t)\right) + z \\ q\left(\theta_p(t)\right) + z^* & -r\left(\theta_p(t)\right) \end{pmatrix}$$

with $r\left(\theta_p(t)\right)$ and $q\left(\theta_p(t)\right)$ defined as:

$$r\left(\theta_p(t)\right) = 1 + \theta_p(t) \qquad q\left(\theta_p(t)\right) = -1 + 3\theta_p(t)$$

The design is done using Corollary 5.2, such that (5.53) becomes a set of eight LMIs with variables Q ($Q \succ O$ being the ninth LMI), K_{11}, K_{12}, K_{21}, K_{22}:

$$\begin{pmatrix} -2Q & 2Q + A_{111}Q + B_1\Gamma_{11} \\ 2Q + QA_{111}^{\mathrm{T}} + \Gamma_{11}^{\mathrm{T}}B_1^{\mathrm{T}} & -2Q \end{pmatrix} \prec O$$

$$\begin{pmatrix} -3Q & 5Q + A_{112}Q + B_1\Gamma_{12} \\ 5Q + QA_{112}^{\mathrm{T}} + \Gamma_{12}^{\mathrm{T}}B_1^{\mathrm{T}} & -3Q \end{pmatrix} \prec O$$

$$\begin{pmatrix} -2Q & 2Q + A_{121}Q + B_2\Gamma_{11} \\ 2Q + QA_{121}^{\mathrm{T}} + \Gamma_{11}^{\mathrm{T}}B_2^{\mathrm{T}} & -2Q \end{pmatrix} \prec O$$

$$\begin{pmatrix} -3Q & 5Q + A_{122}Q + B_2\Gamma_{12} \\ 5Q + QA_{122}^{\mathrm{T}} + \Gamma_{12}^{\mathrm{T}}B_2^{\mathrm{T}} & -3Q \end{pmatrix} \prec O$$

$$\begin{pmatrix} -2Q & 2Q + A_{211}Q + B_1\Gamma_{21} \\ 2Q + QA_{211}^{\mathrm{T}} + \Gamma_{21}^{\mathrm{T}}B_1^{\mathrm{T}} & -2Q \end{pmatrix} \prec O$$

$$\begin{pmatrix} -3Q & 5Q + A_{212}Q + B_1\Gamma_{22} \\ 5Q + QA_{212}^{\mathrm{T}} + \Gamma_{22}^{\mathrm{T}}B_1^{\mathrm{T}} & -3Q \end{pmatrix} \prec O$$

$$\begin{pmatrix} -2Q & 2Q + A_{221}Q + B_2\Gamma_{21} \\ 2Q + QA_{221}^{\mathrm{T}} + \Gamma_{21}^{\mathrm{T}}B_2^{\mathrm{T}} & -2Q \end{pmatrix} \prec O$$

$$\begin{pmatrix} -3Q & 5Q + A_{222}Q + B_2\Gamma_{22} \\ 5Q + QA_{222}^{\mathrm{T}} + \Gamma_{22}^{\mathrm{T}}B_2^{\mathrm{T}} & -3Q \end{pmatrix} \prec O$$

Then, using the YALMIP toolbox [13] with SeDuMi solver [14], the controller vertex gains are obtained:

$$K_{11} = \begin{pmatrix} -1.5769 & -0.9760 & -0.4146 & -0.0109 \\ -0.9610 & 0.0224 & -4.2662 & -2.0773 \end{pmatrix}$$

$$K_{12} = \begin{pmatrix} -4.1989 & -2.1903 & -0.8467 & -0.0156 \\ -1.8680 & 0.0333 & -11.0760 & -5.1844 \end{pmatrix}$$

$$K_{21} = \begin{pmatrix} -1.5692 & -0.5853 & -0.4143 & -0.0107 \\ -0.9637 & 0.0206 & -4.2667 & -2.0774 \end{pmatrix}$$

$$K_{22} = \begin{pmatrix} -4.1541 & -1.7798 & -0.8423 & -0.0148 \\ -1.8749 & 0.0225 & -11.0776 & -5.1848 \end{pmatrix}$$

with:

$$Q = \begin{pmatrix} 0.0949 & -0.2352 & -0.0020 & 0.0072 \\ -0.2352 & 0.7420 & 0.0026 & -0.0214 \\ -0.0020 & 0.0026 & 0.0875 & -0.2230 \\ 0.0072 & -0.0214 & -0.2230 & 0.7340 \end{pmatrix}$$

Table 5.1 Shifting $\mathcal{D}$-stability: closed-loop eigenvalues of the matrices $A_{ijh} + B_j K_{ih}$

$A_{ijh} + B_j K_{ih}$	Eig. 1	Eig. 2	Eig. 3	Eig. 4
$A_{111} + B_1 K_{11}$	$-1.973 + 0.500i$	$-1.973 - 0.500i$	$-2.042 + 1.054i$	$-2.042 - 1.054i$
$A_{112} + B_1 K_{12}$	$-3.216 + 0.442i$	$-3.216 - 0.442i$	$-3.567 + 0.422i$	$-3.567 - 0.422i$
$A_{121} + B_2 K_{11}$	-1.4539	-1.8811	-3.0436	-3.6267
$A_{122} + B_2 K_{12}$	-2.1872	-2.3338	-6.0023	-6.2321
$A_{211} + B_1 K_{21}$	-1.6266	$-2.040 + 1.053i$	$-2.040 - 1.053i$	-2.5405
$A_{212} + B_1 K_{22}$	-2.5932	$-3.584 + 0.449i$	$-3.584 - 0.449i$	-3.9844
$A_{221} + B_2 K_E 21$	-1.4538	$-2.377 + 0.235i$	$-2.377 - 0.235i$	-3.626
$A_{222} + B_2 K_{22}$	-2.1850	-2.4509	-5.8787	-6.0097

Table 5.1 lists the eigenvalues of the vertex closed-loop matrices $A_{ijh} + B_j K_{ih}$. Also, Fig. 5.1 shows how the closed-loop poles shift according to different values of the scheduling parameter θ_p, proving that the shifting $\mathcal{D}$-stability specification is correctly satisfied. In particular, it can be seen that in contrast with the classical $\mathcal{D}$-stability approach, where a region is selected and all the poles of the closed-loop system are forced to be in such region, the shifting $\mathcal{D}$-stability approach allows to select different regions for different values of the scheduling parameter θ_p. According to Corollary 5.1, the decay rate of the Lyapunov function varies with the value of θ_p. In fact, by taking a look at the dominant poles of the vertex systems (denoted as Fig. 1 in Table 5.1), one can see that the range of the real parts of the dominant poles for $\theta_p = 1$ (index $h = 1$) is $[-1.973, -1.4538]$, while when $\theta_p = 2$ (index $h = 2$) such range is $[-3.216, -2.1850]$.

The effect of the shifting pole placement specification on the transient dynamics of the closed-loop system can be effectively seen by taking a look at the free responses of the state variables, shown in Figs. 5.2, 5.3, 5.4 and 5.5. These free responses have been obtained starting from the initial state $x(0) = [\,2\;1\;2\;1\,]^{\mathrm{T}}$ in four different cases, three of which with constant values of the scheduling parameter $\theta_p(t)$ ($\theta_p = 1$, $\theta_p = 1.5$ and $\theta_p = 2$, corresponding to the solid, dashed and dotted lines, respectively), and one with a varying scheduling parameter $\theta_p(t) = 1.5 + 0.5\sin(\pi t/2)$ (corresponding to the dash-dotted line). The remaining scheduling parameters have been chosen as $\theta_s(t) = 2.5 + 0.5\cos t$ and $\theta_r(t) = 2.5 + 0.5\sin t$.

It can be seen from these figures that the closed-loop system behaves as expected: a big value of θ_p corresponds to faster dynamics of the closed-loop system. In the fourth case, that is, with a time-varying θ_p, the dynamics of the closed-loop system around $t = 0s$ is the same as the one of the closed-loop system scheduled by the constant $\theta_p = 1.5$. As the time increases, so does the value of θ_p and the system gets faster until $t = 1s$ when θ_p begins to decrease and the trend reverses, with the system getting slower. However, this last effect and the increasing of speed from time $t = 3s$ are not appreciable because the steady-state has almost been reached. The input signals u_1 and u_2 are shown in Figs. 5.6 and 5.7. It can be seen that the bigger

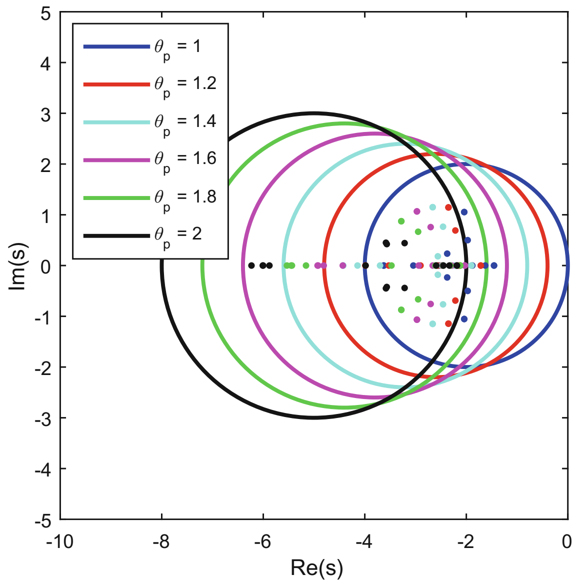

Fig. 5.1 Shifting $\mathcal{D}$-stability: closed-loop poles. After [2]

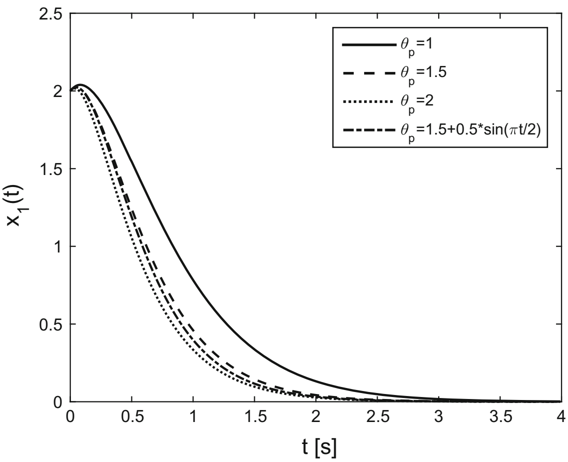

Fig. 5.2 Shifting $\mathcal{D}$-stability: closed-loop response of $x_1(t)$. After [2]

is θ_p, the bigger are the control signals, and vice versa. This is consistent with the fact that strong control actions are required to make the controlled system faster.

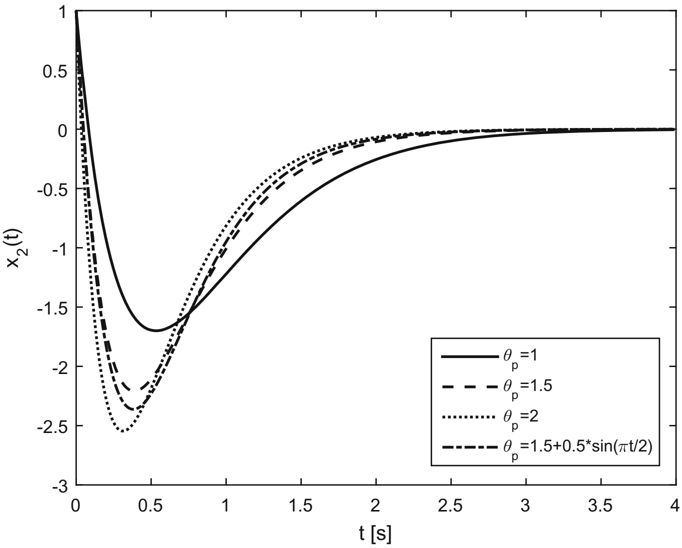

Fig. 5.3 Shifting $\mathcal{D}$-stability: closed-loop response of $x_2(t)$. After [2]

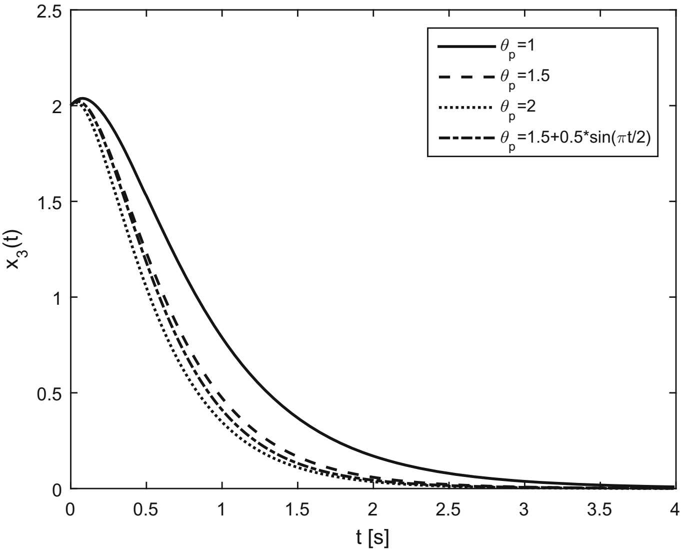

Fig. 5.4 Shifting $\mathcal{D}$-stability: closed-loop response of $x_3(t)$. After [2]

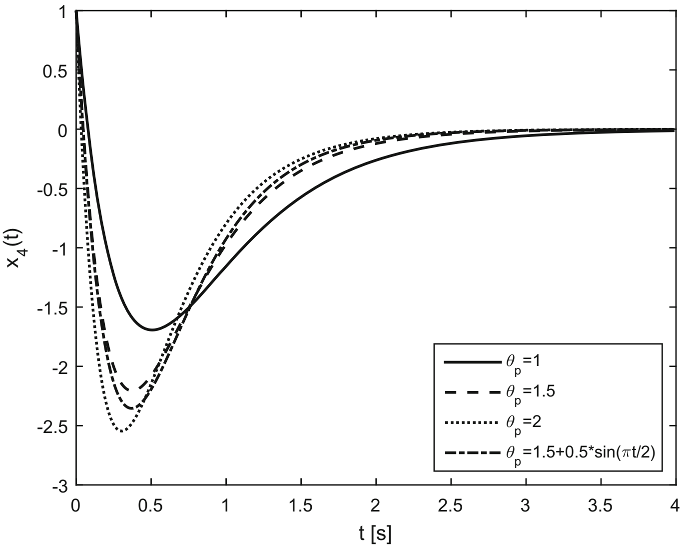

Fig. 5.5 Shifting $\mathcal{D}$-stability: closed-loop response of $x_4(t)$. After [2]

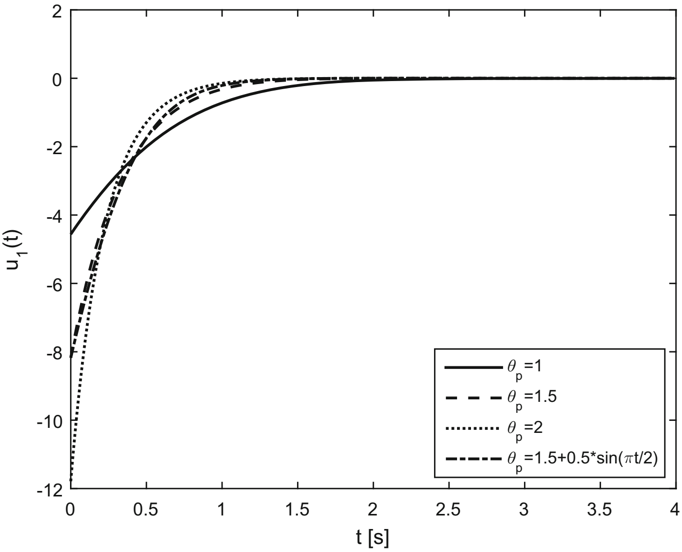

Fig. 5.6 Shifting $\mathcal{D}$-stability: input signal $u_1(t)$. After [2]

5.4.2 Example 2: Shifting $\mathcal{H}_\infty$ Performance

Let us consider a CT LPV system described by (5.1)–(5.2) with matrices $A(\theta(t))$ and $B(\theta_r(t))$ defined as in the previous example, $D_{z_\infty u}(\theta_r(t)) = O$, $D_{z_\infty w}(\theta(t)) = O$ and:

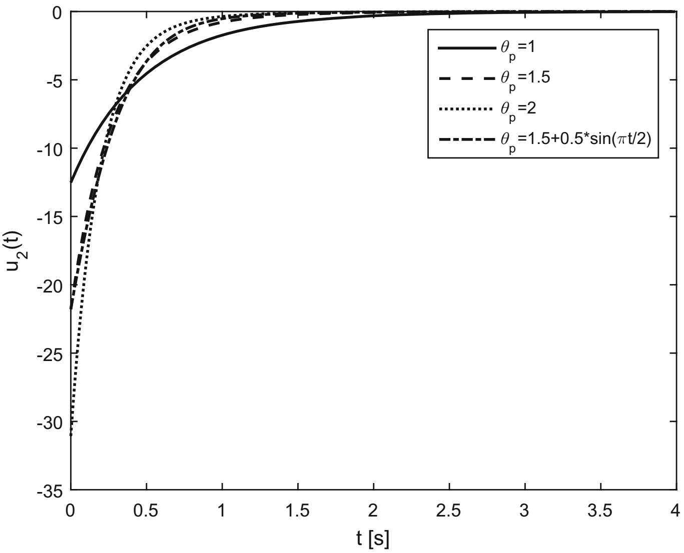

Fig. 5.7 Shifting $\mathcal{D}$-stability: input signal $u_2(t)$. After [2]

$$B_w\left(\theta(t)\right) = B_w = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_{z_\infty}\left(\theta(t)\right) = C_{z_\infty} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

and let us solve the problem of finding a state-feedback gain:

$$\begin{aligned} K\left(\theta_s(t), \theta_p(t)\right) = {} & s_1\left(\theta_s(t)\right)\pi_1\left(\theta_p(t)\right)K_{11} + s_1\left(\theta_s(t)\right)\pi_2\left(\theta_p(t)\right)K_{12} \\ & + s_2\left(\theta_s(t)\right)\pi_1\left(\theta_p(t)\right)K_{21} + s_2\left(\theta_s(t)\right)\pi_2\left(\theta_p(t)\right)K_{22} \end{aligned}$$

such that the transfer function from w to z_∞ satisfies the following desired bound on the $\mathcal{H}_\infty$ norm:

$$\gamma_\infty\left(\theta_p(t)\right) = \sqrt{0.01 + 0.24\left(\theta_p(t) - 1\right)} \qquad \theta_p \in [1, 2]$$

Notice that the particular structure of the shifting bound allows to obtain:

$$\gamma_\infty\left(\theta_p(t)\right)^2 = \pi_1\left(\theta_p(t)\right)\kappa_{2,1} + \pi_2\left(\theta_p(t)\right)\kappa_{2,2}$$

as in (5.47), with $\kappa_{2,1} = 0.01$ and $\kappa_{2,2} = 0.25$ and:

$$\pi_1\left(\theta_p(t)\right) = 2 - \theta_p(t) \qquad \pi_2\left(\theta_p(t)\right) = \theta_p(t) - 1$$

The design is done using Corollary 5.3, solving the LMIs (5.54) using the YALMIP toolbox [13] with SeDuMi solver [14], obtaining the following controller vertex gains:

$$K_{11} = \begin{pmatrix} -1588.9 & -107.2 & -2.5 & -2.2 \\ 155.7 & 11.0 & -1.0 & -0.9 \end{pmatrix}$$

$$K_{12} = \begin{pmatrix} -12.8 & -5.2 & -2.4 & -2.4 \\ 4.9 & 3.8 & -0.5 & -1.4 \end{pmatrix}$$

$$K_{21} = \begin{pmatrix} -1570.2 & -105.6 & -2.4 & -2.2 \\ 154.6 & 10.9 & -1.0 & -0.9 \end{pmatrix}$$

$$K_{22} = \begin{pmatrix} -12.6 & -4.8 & -2.4 & -2.5 \\ 5.8 & 4.2 & -0.5 & -1.4 \end{pmatrix}$$

with:

$$Q = \begin{pmatrix} 0.0562 & -0.7985 & 0.0120 & 0.0042 \\ -0.7985 & 11.9183 & -0.2890 & -0.0497 \\ 0.0120 & -0.2890 & 12.1721 & -6.0386 \\ 0.0042 & -0.0497 & -6.0386 & 6.1784 \end{pmatrix} \quad (5.71)$$

It can be seen from the figures that the closed-loop system behaves as expected. The shifting $\mathcal{H}_\infty$ performance specification results satisfied for each possible value of the scheduling parameter θ_p, as depicted in Fig. 5.8. It can be seen that the relevant feature of the proposed approach with respect to the classical $\mathcal{H}_\infty$ design is that it allows to specify different bounds for the $\mathcal{H}_\infty$ norm corresponding to different values of $\theta_p(t)$, thus allowing to vary online the exogenous input rejection characteristics.

The effect of the shifting $\mathcal{H}_\infty$ specification on the closed-loop system can be seen from the plots of $T_{z_\infty w}$ for different values of the parameter θ_p (see Fig. 5.9). In particular, it can be seen that the higher the value of θ_p is, the higher is the peak of the magnitude of $T_{z_\infty w}$. This result is consistent with the definition of the $\mathcal{H}_\infty$ norm and its shifting counterpart.

Finally, to conclude this analysis, let us consider a sinusoidal exogenous input $w = \sin(t)$, and let us analyse the response of the closed-loop system to this input, starting from zero initial condition. As in the previous example, the simulations are performed with $\theta_s(t) = 2.5 + 0.5\cos(t)$ and $\theta_r(t) = 2.5 + 0.5\sin(t)$, and different values of $\theta_p(t)$ have been considered for comparison purpose. The obtained results are plotted in Fig. 5.10. As expected, a stronger rejection of the exogenous input corresponds to a small value of θ_p (solid line, corresponding to $\theta_p = 1$). By increasing θ_p, e.g. to values of $\theta_p = 1.5$ (dashed line) and $\theta_p = 2$ (dotted line), the rejection performance of the control system decreases. When considering a varying $\theta_p(t) = 1.5 + 0.5\sin(0.5\pi t)$, the rejection characteristics vary in time: at the beginning of the simulation, when $\theta_p(t)$ is approximately 1.5, the response of the system with the

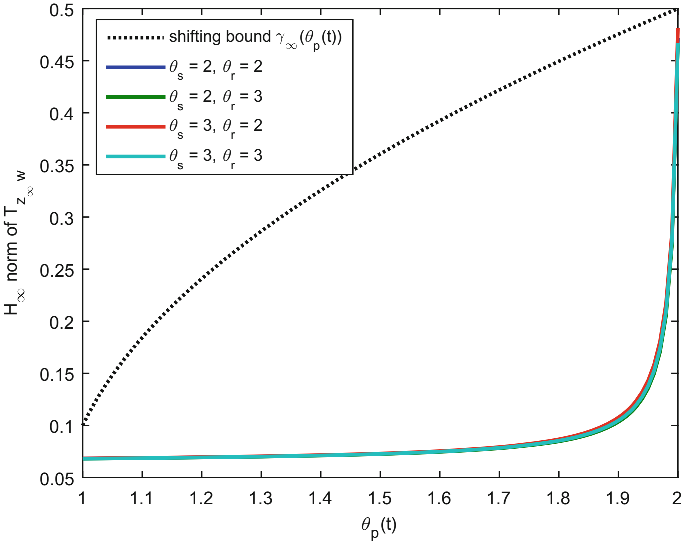

Fig. 5.8 Shifting $\mathcal{H}_\infty$ performance: bound on the $\mathcal{H}_\infty$ norm. After [2]

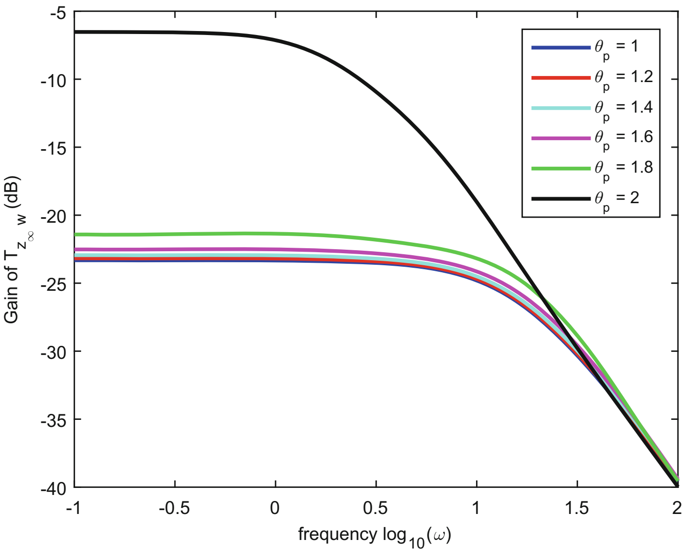

Fig. 5.9 Shifting $\mathcal{H}_\infty$ performance: variation of the Bode plot according to changes in θ_p, obtained for $\theta_s = \theta_r = 2$. After [2]

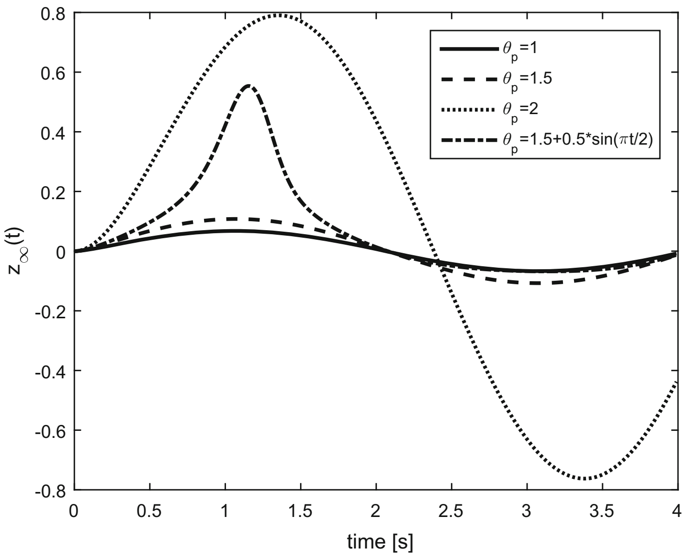

Fig. 5.10 Shifting $\mathcal{H}_\infty$ performance: response of the closed-loop system to an exogenous input $w = \sin(t)$. After [2]

varying θ_p (dash-dotted line) equals the one with a constant $\theta_p = 1.5$ (dashed line). As the time increases, so does $\theta_p(t)$, and the gain of the transfer function from w to z_∞ increases. Hence, the effect of the sinusoidal signal on z_∞ becomes stronger until time $t = 1\,s$, when the varying parameter $\theta_p(t)$ starts decreasing again, and the trend reverses.

5.5 Conclusions

In this chapter, the problem of designing a parameter-scheduled state-feedback controller that satisfies a new kind of specifications, referred to as *shifting specifications*, has been investigated. In particular, the concepts of $\mathcal{D}$-stability, $\mathcal{H}_\infty$ performance, $\mathcal{H}_2$ performance, finite time boundedness and finite time stability have been extended in a *shifting* sense, introducing the *shifting* $\mathcal{D}$*-stability*, *shifting* $\mathcal{H}_\infty$ *performance*, *shifting* $\mathcal{H}_2$ *performance*, *shifting finite time boundedness* and *shifting finite time stability* specifications. The problem has been analyzed in the LPV case, and the solution, expressed as LMIs for which a feasible solution should be found, has been obtained using a common quadratic Lyapunov function.

The results obtained with CT LPV academic examples have demonstrated the effectiveness and some characteristics of the proposed approach. In particular, in contrast with the classical specifications, the design using the shifting ones allows to select different performances for different values of the scheduling parameter θ_p, thus allowing to vary online the control system performance.

As a future work, since the use of a common quadratic Lyapunov function is potentially conservative, the application of other types of Lyapunov functions, e.g. parameter-dependent ones, can be investigated. Also, a future comparison of the proposed approach with the use of parameter-dependent weighting functions could be interesting.

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Part II
Advances in Fault Tolerant Control Techniques

Chapter 6
Background on Fault Tolerant Control

6.1 Motivation

Fault tolerant control (FTC) is the name given to all those techniques that are capable of maintaining the overall system stability and acceptable performance in the presence of faults. In other words, a closed-loop system which can tolerate component malfunctions, while maintaining desirable performance and stability properties is said to be a *fault tolerant control system* (FTCS). Starting from the 80s, FTC applications began to appear in the scientific literature, mainly motivated by aircraft flight control system designs. The goal was to provide self-repairing capability in order to ensure a safe landing in the event of severe faults in the aircraft [1]. Since then, a lot of effort has been put in developing FTC schemes. Interests in diagnostics and fault tolerant control have been intensified since the Three Mile Island incident on March 28, 1979 and the tragedy at the Chernobyl nuclear power plant on April 26, 1986. The FTC problem has begun to draw more and more attention in a wider range of industrial and academic communities, due to the increased safety and reliability demands beyond what a conventional control system can offer. FTC applications include aerospace, nuclear power, automotive, manufacturing and other process industries.

The existing FTC techniques can be divided into three categories:

- Hardware redundancy techniques
- Analytical redundancy techniques: passive fault tolerant control (PFTC)
- Analytical redundancy techniques: active fault tolerant control (AFTC).

The *hardware redundancy techniques* try to achieve fault tolerance by exploiting hardware redundancy in the system. Its main advantage is simplicity, but at the cost of an increased hardware and maintenance cost that can be avoided using analytical redundancy techniques. The *passive FTC techniques* are control laws that take into account the fault appearance as a system perturbation. Thus, within certain margins, the control law has inherent fault tolerant capabilities, allowing the system to cope with the fault presence, thanks to its robustness against a class of faults. This approach has the advantage of needing neither fault diagnosis nor controller reconfiguration, but it has limited fault tolerance capabilities (for example, it needs to take all possible faults of a system in consideration during the design stage, thus it cannot

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be guaranteed that unanticipated failures are handled). Moreover, there is a loss of performance with respect to the nominal case. On the other hand, the *active FTC techniques* adapt the control law using the information given by the fault diagnosis. With this information, some automatic adjustments in the control loop are done after the fault appearance, trying to satisfy the control objectives with minimum performance degradation.

Several reviews on FTCS have appeared since the 80s. A good recent review can be found in [2], where a comparison of different approaches according to different criteria such as design methodologies and applications is carried out and 376 references, dating back to 1971, are compiled to provide an overall picture of historical, current, and future developments in this area. A few books on this subject have been published in recent years too, e.g. [3].

Some of the existing techniques will be briefly discussed in this chapter.

6.2 Hardware Redundancy Techniques

In principle, the tolerance to control system failures can be improved if two or more sensors/actuators, each separately capable of satisfactory control, are implemented in parallel. This approach is referred to as *hardware redundancy*. A voting scheme is used for the redundancy management, comparing control signals to detect and overcome failures. With two identical channels, a comparator can determine whether or not the control signals are identical; hence, it can detect a failure but cannot identify which component has failed. Using three identical channels, the control signal with the middle value can be selected (or voted), assuring that a single failed channel never controls the plant. A two-channel system is considered *fail-safe* because the presence of a failure can be determined, but it is left to additional logic to select the unfailed channel for control. The three-channel system is *fail-operational*, as the task can be completed following a single failure. Systems with four identical control channels can tolerate two failures and still yield nominal performance. Problems encountered in implementing hardware redundancy include: selection logic, reliability of voting, increased hardware and maintenance costs.

6.3 Passive Fault Tolerant Control Techniques

The passive FTC techniques synthesize a controller so that the closed-loop system is stable, or has some desired performance, for some combinations of failure elements defined a priori. This is done by using the results in the robust control area, considering faults as if they were uncertainties or system perturbations. In particular, the term *passive* indicates that no actions are required by the FTCS after the prescribed faults have occurred during the system's operation.

This approach has the advantage of needing neither fault diagnosis schemes nor controller reconfiguration, but it has limited fault tolerant capabilities and the price to pay for its simplicity is a loss of performance with respect to the nominal case. Also, in passive FTC no time delay exists between the fault occurrence and the corresponding action. For these reasons, the design of passive FTCS has attracted a lot of attention from the academic community [4].

A good historical overview about development and research of passive FTC techniques can be found in [5]. Some of the passive FTC approaches found in the literature are: reliable linear quadratic (LQ), $\mathcal{H}_\infty$ robust control and passive FTC using LMIs.

6.3.1 Reliable Linear Quadratic (LQ) Approach

The LQ approach is one of the most established passive FTC techniques, and relies on the robustness of the LQ theory. This approach was first used to accommodate requirements for robustness against sensor failures by [6]. Later, [7] developed a methodology for the design of reliable centralized and decentralized control systems that provided guaranteed stability and performance not only when all control components are operational, but also for sensor or actuator outages in the centralized case, or for control-channel outages in the decentralized case. In [8], the LQ regulator technique has been exploited to design a reliable controller against a class of actuator outages using Riccati equations. A method based on robust pole region assignment and a pre-compensator that modifies the dynamic characteristics of the redundant actuator control channels has been presented by [9] to synthesize a reliable controller for dynamic systems possessing actuator redundancies. The results obtained by [8] were extended to deal with nonlinear systems in [10], where Hamilton-Jacobian inequalities were employed instead of the Riccati equations. Despite being a well-established technique, reliable LQ control is still a subject of investigation, e.g. [11]. The main problem with this technique is that the stability is not guaranteed for faults outside the pre-selected ones, and since there is no reconfiguration of the controller, the nominal performance is not optimal.

6.3.2 $\mathcal{H}_\infty$ Robust Control

Another control approach widely used for passive FTC is $\mathcal{H}_\infty$ robust control, that uses the results developed by [12]. Important results were obtained by [7], where the $\mathcal{H}_\infty$ performance has been guaranteed not only in the nominal case, but also in presence of control component outages, as long as these results do not affect the observability and controllability of the system. The results obtained by [7] were extended by [13], by considering not only outages but also loss of effectiveness in sensors and actuators. Recent research has also dealt with the same problem for networked control systems [14]. However, when the magnitude of the fault goes beyond the range considered for the design, the $\mathcal{H}_\infty$ norm performance can no longer be guaranteed. Also, since the reference signal is assumed to be arbitrary, the controller is conservative because it takes into account the worst case reference signal [15].

6.3.3 Passive FTC Using Linear Matrix Inequalities (LMIs)

The goal of this approach is to achieve optimal performance in the nominal situation and acceptable level of performance under occurrence of faults in the control components. In [16], the robust FTC problem has been formulated in an LMI setting, in which satisfactory performance and stability robustness are introduced. In particular, a multi-objective approach is used to establish a matrix inequality formulation for FTCS design. Later, in [17], a reliable output-feedback controller has been designed using an iterative LMI approach. In this case, the design goal is to find an internally stabilizing controller such that the nominal performance of a closed-loop transfer matrix is optimized. The designed controller also satisfies the reliability constraint, in terms of stability and performance, under the actuator/sensor faults condition.

6.4 Active Fault Tolerant Control Techniques

As pointed out in [2], an active FTC system can be typically divided into four subsystems:

- a reconfigurable controller;
- a fault diagnosis scheme;
- a controller reconfiguration mechanism;
- a command/reference governor.

The inclusion of both a fault diagnoser and a reconfigurable controller within the overall control system scheme is the main feature distinguishing active FTC from passive FTC. Key issues in active FTC consist in how to design:

- a controller that can be reconfigured;
- a fault diagnosis scheme with high sensitivity to faults and robustness against model uncertainties, variations of the operating conditions, and external disturbances;
- a reconfiguration mechanism that allows recovering the pre-fault system performance as much as possible, in presence of uncertainties and time delays in the fault diagnosis, as well as constraints on the control inputs and the allowed system states.

Based on the online information of the post-fault system, the reconfigurable controller should be designed to maintain stability, desired dynamic performance and steady-state performance. In addition, in order to ensure that the closed-loop system can track a desired trajectory under fault occurrence, a reconfigurable feedforward controller often needs to be synthesized. Also, a command/reference governor that adjusts the reference trajectory automatically should be added to avoid potential actuator saturation and to take into consideration the degraded performance after fault occurrence.

Some of the existing active FTC techniques that can be found in the literature are the following: linear quadratic (LQ), pseudo-inverse method (PIM), intelligent control (IC), gain-scheduling (GS), model following (MF), adaptive control (AC), multiple model (MM), integrated diagnostic and control (IDC), eigenstructure assignment (EA), feedback linearization (FL)/dynamic inversion (DI), model predictive control (MPC), quantitative feedback theory (QFT) and variable structure control (VSC)/sliding mode control (SMC).

Anyway, even if each individual control design method has been developed separately, in practice a combination of several of these methods may be more appropriate to achieve the best performance.

6.4.1 Linear Quadratic (LQ) Approach

The LQ approach has been used also for active FTC in several papers. For example, [18] has presented an LQ-based approach for the automatic redesign of flight control systems for aircrafts that have suffered control element failures. Reference [19] have developed a self-repairing flight control system concept in which the control law is reconfigured after actuator and/or control surface damage to preserve stability and pilot command tracking. In [20], the use of integral control achieves reconfiguration and acceptable performance in the presence of several simultaneous control actuator failures and exogenous disturbances.

6.4.2 Pseudo-inverse Method (PIM)

The main idea of the PIM is to modify the feedback gain so that the reconfigured system approximates the nominal system in some sense. It is an attractive approach because of its simplicity in computation and implementation. The main drawback of the PIM is that the stability of the reconfigured system is not guaranteed. As a result, if applied without appropriate care, the PIM can lead to instability. The theoretical basis for this approach have been developed by [21], where the lack of stability guarantees has been pointed out and an approach that provides stability constraints for the solution of the PIM has been proposed. The PIM has later been revisited by Staroswiecki [22], where the use of a set of admissible models, rather than searching for an optimal one which does not provide any stability guarantee, has been proposed.

6.4.3 Intelligent Control (IC)

IC uses expert systems, fuzzy logic, neural networks and similar tools to detect and accommodate faults. Its advantage is the possibility to use easily heuristic knowledge for achieving fault tolerance, but it also requires a high computational power and a very precise knowledge of the fault. Reference [23] presents a controller that uses a rule-based expert system approach to transform the task of failure accommodation into a problem of search, with the advantage of enhancing the existing redundancy. Reference [24] presented a methodology that accommodates unanticipated faults using learning techniques. In [25], the fuzzy model reference learning controller has been used to reconfigure the nominal controller in an F-16 aircraft to compensate for various actuator failures without using explicit failure information. An expert supervision strategy is also applied, such that the performance of the control reconfiguration is increased. Reference [26] have proposed the use of online learning neural network controllers that have the capability of bringing a system affected by substantial damage back to an equilibrium condition. This goal has been achieved through the use of a specific training algorithm and proper collection of the architectures for the neural network controllers. Reference [27] has presented a learning methodology for failure accommodation that uses online approximators, i.e. generic function approximators with adjustable parameters, such as polynomials, splines and neural network topologies, e.g. sigmoidal multilayer networks and radial basis function networks. Reference [28] has presented an approach that integrates a fuzzy TS model-based adaptive control with the reconfiguration concept. Reference [29] have proposed a neural network-based FTC for unknown nonlinear systems that introduces an extra neural network-based fault compensation loop under fault occurrence. The learning capabilities of neural networks and fuzzy systems have been exploited for FTC in [30], where online approximation-based stable adaptive neural/fuzzy control has been studied for a class of input-output feedback linearizable time-varying nonlinear systems. In this work, a fault diagnosis unit designed by interfacing multiple models with an expert supervisory scheme is also used for improving the fault tolerance ability of the adaptive controller. Reference [31] have presented the architecture and synthesis of a damage-mitigating control system where the objective is to achieve high performance, with increased reliability, availability, component durability, and maintainability. Such an objective is accomplished using a fuzzy controller that makes a trade-off between system dynamic performance and structural durability in critical components.

6.4.4 Gain-Scheduling (GS)

The idea of this method is to generate a control law that depends on varying parameters that include the fault signals generated by the fault diagnosis unit. For example, in [19], a self-repairing flight control system concept, where the scheduled gain stabilizes a collection of models representing the aircraft in various control failure modes, has been described. A similar approach has been used later by [32] for the reconfigurable LPV control of a Boeing 747-100/200, where the controller was scheduled by three parameters: flight altitude, velocity, and a fault identification signal. The gain scheduling approach has been coupled to an adaptive Kalman filter estimation in [33]. A static output feedback synthesis in presence of multiple actuator failures is developed by [34], such that the closed-loop stability can be maintained for any combination of multiple actuator failures.

6.4.5 Model Following (MF)

The basic idea of the MF approach is to design a control system that makes the output or the state vector trajectories of the real plant follow the ones of a reference system as closely as possible [35]. If the MF is achieved even in the presence of faults, the control system is fault tolerant. This idea has been first exploited by [36], where a restructurable control using proportional-integral implicit MF has been presented. Frequency domain necessary and sufficient conditions for perfect MF are developed and used by [37] to design reconfigurable control systems. The case of output feedback control, both using the implicit MF and the explicit MF principles, has been used for aircraft FTC in [38].

6.4.6 Adaptive Control (AC)

An AC scheme is able to deal with a time-varying uncertain system. Thus, it can be efficiently used to deal with faults. AC has been used in [39] to accommodate failures in the F-16 aircraft. In [40], three adaptive algorithms for reconfigurable flight control are compared, and their advantages and disadvantages concerning the complexity and the assumptions that they require are discussed. The direct adaptive reconfigurable flight control approach described by [41] uses a mix of dynamic inversion controller in an explicit MF architecture, neural network, control allocation scheme and system identification module to achieve fault tolerance of a tailless fighter aircraft. The case where some inputs are stuck at some fixed or varying values which cannot be influenced by the control action has been analysed in [42, 43] for the state feedback and the output feedback case, respectively.

6.4.7 Multiple Model (MM)

The idea of this approach is to compute a bank of models offline, each of which describes the system behavior in the presence of a particular fault, and to calculate the corresponding control law for each of them. When a fault is detected, the most suitable model is selected, and the corresponding control law becomes active, allowing to increase the performance under faulty situation. This idea has been considered first by [44] for designing an aircraft flight control system with reconfigurable capabilities. In [45], an MM Kalman filtering approach has been introduced for the estimation of the model of a damaged aircraft, and used as the basis for the reconfiguration of the flight control system. MM adaptive estimation methods have been incorporated into the design of a flight control system for an F-16 aircraft in [46], providing it with the capability to detect and compensate for sensor and control surface/actuator failures. In particular, the algorithm consists of an estimator, composed of a bank of parallel Kalman filters, each matched to a specific hypothesis about the failure status of the system, a means of blending the filter outputs through a probability-weighted average, and an algorithm that redistributes the control commands, that would normally be sent to the detected failed surfaces, to the non-failed surfaces, accomplishing the same control action on the aircraft. An integrated fault detection, diagnosis, and reconfigurable control scheme based on interacting MM approach, with the relevant feature of being able to deal not only with actuator and sensor faults, but also with failures in the system components, has been proposed by [47]. A combination between MM and adaptive reconfiguration control has been developed by [48] to compensate for the effect of actuator faults and asymptotically track a reference model.

6.4.8 Integrated Diagnostics and Control (IDC)

Another possible approach for achieving fault tolerance is to design together the controller and the diagnostic module, instead of designing them independently, thus accounting for the interactions which occur between these two components [49]. In [50], it has been shown how a combined module for control and diagnosis can be designed, such that references are tracked, disturbances are robustly rejected, undetected faults do not have disastrous effects, the number of false alarms are reduced and the faults which have occurred are identified. Demonstrations of the applicability of this approach to valve fault accommodation on rocket engines, heat exchangers and autonomous underwater vehicles have been provided in [28, 51, 52], respectively.

6.4.9 Eigenstructure Assignment (EA)

EA is a technique used to control multiple input multiple output (MIMO) systems, that has been applied to FTC systems with the aim of designing the eigenstructure of the reconfigured system to be as similar as possible to the nominal one, as shown in [53]. The application of EA to FTC has been further developed in [54], where the problem of robust reconfigurable controller design, which makes the after-fault closed-loop insensitive as much as possible to the parameter uncertainties of the after-fault model, has been considered.

6.4.10 Feedback Linearization (FL)/Dynamic Inversion (DI)

The idea of applying FL in FTC dates back to [55], that presented a restructurable flight control system design method based on this technique. In a subsequent work, the idea has been extended to DT systems and applied to aircraft failures occurring with the control effectors [56]. A DI-based adaptive/reconfigurable control system has been designed to provide fault and damage tolerance for an X-33 on the ascent flight phase in [57].

6.4.11 Model Predictive Control (MPC)

In MPC, a model of the system is used to predict its behavior over a future time interval. Then, based on these predictions, the sequence of inputs is calculated by minimizing a cost function. The first input of the sequence is applied and, at the following time sample, the process is repeated over a shifted time interval [58]. The idea of applying MPC to FTC dates back to [59], where it was used for maximizing aircraft tracking performance before and after control surface failure, preventing instability. In [60], the approach proposed for reconfiguring control systems in the event of major failures makes use of a combination of constrained MPC and other technologies, such as high-fidelity nonlinear simulation models, effective approximation and identification algorithms, and fault detection and isolation (FDI) capability. Formulations and experimental evaluations of various MPC schemes applied to a realistic full envelope nonlinear model of a fighter aircraft are presented in [61]. In [62], two FTC strategies based on MPC are proposed and compared: passive fault tolerant MPC, that takes advantage of natural tolerance of MPC, and active fault tolerant MPC, that uses active fault tolerance techniques in combination with MPC. The comparison is performed through an application over a portion of the Barcelona sewage network.

6.4.12 Quantitative Feedback Theory (QFT)

QFT, developed by [63] in the early 1970s, is a frequency domain based design technique where the controllers can be designed to achieve a set of performance and stability objectives over a specific range of plant parameter uncertainty. The QFT method takes into account quantitative information on the variability of the plant, the robust performance requirements, the tracking performance specifications, the expected disturbance amplitude and its attenuation requirements [64]. The feasibility of utilizing a robust QFT controller that meets flying quality specifications for an aircraft subject to control surface failures has been investigated by [65]. A reconfigurable flight control system that uses the series of a robust QFT controller and an adaptive filter has been presented in [66]. The design and experimental evaluation of a fault tolerant controller for an electrohydraulic servo positioning system subject to sensor failures or faults in the servovalve and supply pump has been performed in [67].

6.4.13 Variable Structure Control (VSC)/Sliding Mode Control (SMC)

VSC systems are characterized by a suite of feedback control laws and a decision rule [68]. The decision rule, named *switching function*, decides which feedback controller should be used at a given instant, based on the system state. A VSC system can be regarded as a combination of subsystems, where each subsystem has a fixed control structure and is valid for a specific subset of system states. SMC is a particular type of VSC, where the state of the system is driven and constrained to lie in a neighbourhood of the switching function, with the advantage of making the system insensitive to a particular class of uncertainty. VSC/SMC has been used successfully in many applications of FTC [69, 70]. A reconfigurable SMC that achieves robust tracking after damage in an aircraft has been designed in [71, 72]. A MF scheme, based on VSC, that possesses a fault tolerance property has been proposed by [73]. The combination of integral SMC methodology and observers with hypothesis testing has been used for FTC of a spark ignition engine in [74].

6.5 Recent Developments of Fault Tolerant Control

This section resumes some recent developments of FTC theory [2], highlighting some open issues that motivate further investigation in this topic.

- **Redundancy**: since the introduction of the concept of *analytical redundancy*, i.e. the use of a mathematical model of the system for FDI/FTC, important research efforts on how to efficiently utilize this concept have been made [75, 76]. The following challenging issues regarding redundancy can be detected [2]: (i) the design of the overall fault tolerant and redundant system architecture; (ii) the optimal configuration of redundancy, achieving a tradeoff between specifications and cost; (iii) the design and implementation of a fault tolerant controller that utilizes at best both the hardware and the analytical redundancy to achieve the control objectives; and (iv) the introduction of quantitative measures of the degree of redundancy [9, 77, 78].

- **Integrated design of fault diagnosis/fault tolerant control**: as remarked by [2], in order to obtain a functional active FTCS, it is important to ensure an adequate cooperation between the fault diagnosis and the FTC algorithms. In fact, an incorrect information provided by the fault diagnoser can lead potentially to undesired behaviors and overall degradation of the FTCS performance. For this reason, the way how to integrate both the subsystems is an important topic of research, and recent papers have addressed this issue [79–82]. Also, another important issue worth investigating is how to mitigate the adverse interactions between each subsystem [83]. For further discussion about the integration of fault diagnosis and fault tolerant control, the reader is referred to the survey paper in [84].
- **Design for graceful performance degradation**: recently, some techniques have tried to achieve fault tolerance without aiming at recovering the original system performance, but accepting some performance degradation instead. These techniques are of particular interest in the case of actuator faults. In fact, once an actuator is affected by a fault, maintaining the original performance will typically increase the effort distributed on the remaining actuators, which is highly undesirable in practice, due to physical constraints on the actuators. Therefore, recent works have considered the design of FTCS with graceful performance degradation, e.g. [85, 86].
- **Stability and stability robustness**: in the case of active FTCS, stability requirements are specified under different situations: (i) fault-free operation; (ii) transient during reconfiguration; (iii) steady-state after reconfiguration. In all these situations, it is important to investigate the stability robustness [87]. As remarked by [2], despite much work has been done, e.g. [88], stability analysis and stability robustness for real-time reconfigurable control systems in practical environments still need further investigation.
- **FTC design for nonlinear systems**: several strategies have been proposed to deal with nonlinear systems, such as feedback linearization [89], nonlinear dynamic inversion [57], backstepping [90] and neural networks [91] among others. However, as stated by [2], the development of effective design methods for dealing with nonlinear FTCS issues is still an open research problem.
- **FTC of constrained systems**: the design of FTCS subject to actuator amplitude and rate saturation constraints has been investigated by a few works, such as [59, 92]. However, there are still many open problems in the framework of MIMO systems [2].
- **Dealing with fault diagnosis uncertainties/delays**: the presence of errors in the fault diagnosis process are inevitable [93]. Also, time delays and false alarms are associated with fault diagnosis decisions [94]. Hence, it is important to take into account these undesired effects to reduce their impact on the FTCS, and the development of new and practical approaches to accomplish this goal is an investigation hot topic [2].
- **Self-designing FTC**: recent research activities have focused on FTC laws which rely on online estimation of plant parameters [95–97]. There are still some challenges in this field, e.g. dealing with poor input excitation and the adverse interactions between the identification and the control in a closed-loop setting [2].

- **Control allocation**: in presence of actuator redundancy, control allocation techniques aim at choosing how to use the available actuators in order to achieve a specified objective. In case of actuator faults, these techniques try to make the best use of the remaining healthy actuators [98, 99]. As remarked by [100], there are still computational issues in the application of control allocation to nonlinear systems.
- **Transient management**: undesired transients during the reconfiguration process may be harmful to the safe operation of an FTCS, and lead to undesired consequences, such as saturations in the actuators, and damage to the components. For this reason, these transients should be minimized as much as possible which, in spite of a few results available in the literature, e.g. [101], is still an open problem.
- **Real-time issues**: all the subsystems in an AFTCS should operate in real-time, and for this reason there should be hard deadlines for controller reconfiguration, in order to avoid risky situations. This issue, despite its criticality, has not been dealt with satisfactorily, and there are only a few works addressing it, e.g. [76].
- **Fault-tolerant networks**: FTC in networked control systems is a challenging problem due to timing issues, network-induced delays and packet losses, whose effects should be carefully taken into consideration [102, 103].

6.6 Conclusions

In this chapter, a review of the available FTC approaches has been performed. The ability to maintain stability and acceptable performance in spite of faults has motivated a lot of effort in developing FTC strategies. FTC techniques can be divided into three categories. *Hardware redundancy* approaches achieve fault tolerance by exploiting extra components, providing a simple but expensive solution. Different *passive FTC* approaches have been recalled (reliable linear quadratic control, $\mathcal{H}_\infty$ control, and LMI-based design). They all increase the robustness of the nominal controller against some predefined set of faults, and share the advantage of needing neither fault diagnosis nor controller reconfiguration, but at the expense of limited fault tolerance capabilities. Finally, the *active FTC* approaches (reconfigured linear quadratic control, pseudo-inverse method, intelligent control, gain-scheduling, model following, adaptive control, multiple model, integrated diagnostics and control, eigenstructure assignment, feedback linearization/dynamic inversion, model predictive control, quantitative feedback theory, variable structure control/sliding mode control) use the information given by the fault diagnosis to perform some automatic adjustments after the fault appearance, in order to achieve fault tolerance.

Despite the strong development of FTC theory in the last decades, a lot of open issues, resumed in the last part of the chapter, motivate further investigation in this topic.

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Chapter 7
Fault Tolerant Control of LPV Systems Using Robust State-Feedback Control

The content of this chapter is based on the following works:

- [1] D. Rotondo, F. Nejjari, V. Puig. Passive and active FTC comparison for polytopic LPV systems. In *Proceedings of the 12th European Control Conference (ECC)*, p. 2951–2956, 2013.
- [2] D. Rotondo, F. Nejjari, V. Puig. Fault tolerant control design for polytopic uncertain LPV systems. In *Proceedings of the 21st Mediterranean Control Conference (MED)*, p. 66–72, 2013.
- [3] D. Rotondo, F. Nejjari, A. Torren, V. Puig. Fault tolerant control design for polytopic uncertain LPV systems: application to a quadrotor. In *Proceedings of the 2nd International Conference on Control and Fault-Tolerant Systems (SysTol)*, p. 643–648, 2013.
- [4] D. Rotondo, F. Nejjari, V. Puig. Robust quasi-LPV model reference FTC of a quadrotor UAV subject to actuator faults. *International Journal of Applied Mathematics and Computer Science*, 25(1):7–22, 2015.

7.1 Introduction

In Chap. 4, the idea of the robust LPV polytopic technique, obtained merging known results from the robust polytopic control area and the traditional LPV polytopic control area, has been introduced. In the proposed technique, the vector of varying parameters is used to schedule between uncertain LTI systems. The resulting approach consists in using a double-layer polytopic description to take into account both the variability due to the varying parameter vector and the uncertainty. The first polytopic layer manages the varying parameters and is used to obtain the vertex uncertain systems, where the vertex controllers are designed. The second polytopic layer is built at each vertex system to take into account the model uncertainties and add robustness in the design step.

In this chapter, it is shown that the proposed framework can be used for FTC, with the advantage that, depending on how much information is available, it gives rise to different strategies. If the faults are considered as though as they were perturbations, a *passive FTC* would arise. On the other hand, if the faults are used as additional

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scheduling parameters, an *active FTC* would be obtained. Finally, if the fault estimation uncertainty is taken into account explicitly during the design step, the robust LPV polytopic technique would lead to a *hybrid FTC*. The different controllers are obtained using LMIs, in order to achieve regional pole placement and $\mathcal{H}_\infty$ performance constraints. Results obtained using a quadrotor UAV simulator are used to show the effectiveness of the proposed approach.

7.2 Problem Formulation

Consider the following slight modification of the LPV system (2.1):

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + B\left(\theta(\tau)\right)u(\tau) + c(\tau) \tag{7.1}$$

where $c(\tau)$ is a known exogenous input, and let us consider the problem of designing a control scheme in order to achieve the goal of tracking a desired trajectory. The conditions provided in Chap. 2 for the analysis and state-feedback controller design for LPV systems cannot be directly applied, since they refer to the stability of the origin. In order to assure the convergence of the system trajectory to the desired one, the idea of LPV model reference control is considered. Originally developed for the LTI case [5], this technique has been successfully extended to the LPV case [6, 7] and relies on the use of a reference model, as follows:

$$\sigma.x_{ref}(\tau) = A\left(\theta(\tau)\right)x_{ref}(\tau) + B\left(\theta(\tau)\right)u_{ref}(\tau) + c(\tau) \tag{7.2}$$

where $x_{ref} \in \mathbb{R}^{n_x}$ is the reference state vector and $u_{ref} \in \mathbb{R}^{n_u}$ is the reference input vector. The reference model gives the trajectories to be followed by the real system. Thus, considering the tracking error, defined as $e(\tau) \triangleq x_{ref}(\tau) - x(\tau)$, the following *error system* is obtained:

$$\sigma.e(\tau) = A\left(\theta(\tau)\right)e(\tau) + B\left(\theta(\tau)\right)\Delta u(\tau) \tag{7.3}$$

with $\Delta u(\tau) = u_{ref}(\tau) - u(\tau)$.

Then, the results presented in Chap. 2 can be applied for the design of an error-feedback control law of the form:

$$\Delta u(\tau) = K\left(\theta(\tau)\right)e(\tau) \tag{7.4}$$

that constitutes a slight modification of (2.134):

$$u(\tau) = K\left(\theta(\tau)\right)x(\tau) \tag{7.5}$$

Hence, the control action to be applied to the system will be made up by the sum of two components, the feedforward one $u_{ref}(\tau)$, and the feedback one $\Delta u(\tau)$.

However, in presence of faults, the Eq. (7.1), in the following denoted as *nominal system*, does not describe correctly the system dynamics anymore, and the obtained results in terms of stability and performance could not hold anymore. In particular, in this chapter, two types of faults are considered: (i) parametric faults, affecting the matrix $A(\theta(\tau))$ and changing it into $A_f(\theta(\tau), f(\tau))$; and (ii) actuator faults, affecting the matrix $B(\theta(\tau))$ and changing it into $B_f(\theta(\tau), f(\tau))$. Hence, under fault occurrence, the Eq. (7.1) becomes:

$$\sigma.x(\tau) = A_f(\theta(\tau), f(\tau))\, x(\tau) + B_f(\theta(\tau), f(\tau))\, u(\tau) + c(\tau) \tag{7.6}$$

that will be referred to as the *faulty system*, and the error system changes into:

$$\sigma.e(\tau) = A(\theta(\tau))\, x_{ref}(\tau) + B(\theta(\tau))\, u_{ref}(\tau) - A_f(\theta(\tau), f(\tau))\, x(\tau) - B_f(\theta(\tau), f(\tau))\, u(\tau) \tag{7.7}$$

from which is evident that a controller in the form (7.4), designed to behave in some desired way when applied to (7.1), could lead to a very different behavior when applied to the error faulty system (7.7).

In the following section, the problem to be solved is the one of adding fault tolerance to the control scheme. This will be done by redefining the reference model and by designing the error-feedback controller using some results about the robust feedback control of uncertain LPV systems presented in Chap. 4.

7.3 Fault Tolerant Control

7.3.1 Passive FTC Reference Model

In passive FTC, no information about the fault is available on-line. Hence, the same reference model used for the nominal case, i.e. (7.2), should be used.

7.3.2 Active FTC Reference Model

In active FTC, an estimation of the faults, denoted in the following by $\hat{f}(\tau)$, is available. This information is added to the reference model by changing $A(\theta(\tau))$ into $A_f\left(\theta(\tau), \hat{f}(\tau)\right)$ and $B(\theta(\tau))$ into $B_f\left(\theta(\tau), \hat{f}(\tau)\right)$, such that (7.2) becomes:

$$\sigma.x_{ref}(\tau) = A_f\left(\theta(\tau), \hat{f}(\tau)\right) x_{ref}(\tau) + B_f\left(\theta(\tau), \hat{f}(\tau)\right) u_{ref}(\tau) + c(\tau) \tag{7.8}$$

7.3.3 Passive FTC Error Model

In order to obtain the passive FTC error model, some manipulation is performed on (7.7), in order to obtain a structure suitable for applying the design technique presented in Chap. 4. The typical way to do so would be to rewrite $A_f(\theta(\tau), f(\tau))$ and $B_f(\theta(\tau), f(\tau))$, as follows:

$$A_f(\theta(\tau), f(\tau)) = A(\theta(\tau)) + \Delta A(\theta(\tau), f(\tau)) \tag{7.9}$$

$$B_f(\theta(\tau), f(\tau)) = B(\theta(\tau)) + \Delta B(\theta(\tau), f(\tau)) \tag{7.10}$$

where $\Delta A(\theta(\tau), f(\tau))$ and $\Delta B(\theta(\tau), f(\tau))$ contain the changes to the state space matrices brought by the faults. Hence, (7.7) can be rewritten as:

$$\sigma.e(\tau) = A(\theta(\tau))\, e(\tau) + B(\theta(\tau))\, \Delta u(\tau) - \Delta A(\theta(\tau), f(\tau))\, x(\tau) - \Delta B(\theta(\tau), f(\tau))\, u(\tau) \tag{7.11}$$

Then, by doing some manipulations and rewriting $f(\tau)$ as $\Delta f(\tau) = f(\tau) - 1$, (7.11) can be brought to the following form:

$$\sigma.e(\tau) = A(\theta(\tau))\, e(\tau) + B(\theta(\tau))\, \Delta u(\tau) + D(\theta(\tau), x(\tau), u(\tau), \Delta f(\tau))\, \Delta f(\tau) \tag{7.12}$$

where $D(\theta(\tau), x(\tau), u(\tau), \Delta f(\tau))$ is the matrix such that:

$$D(\theta(\tau), x(\tau), u(\tau), \Delta f(\tau))\, \Delta f(\tau) = -\Delta A(\theta(\tau), f(\tau))\, x(\tau) - \Delta B(\theta(\tau), f(\tau))\, u(\tau) \tag{7.13}$$

Through the introduction of a new varying parameter vector $\theta_p(\tau)$, containing $\theta(\tau)$, $x(\tau)$ and $u(\tau)$, and by taking into account the uncertainty due to $\Delta f(\tau)$, (7.12) can be reshaped as follows:

$$\sigma.e(\tau) = A\left(\theta_p(\tau)\right) e(\tau) + B\left(\theta_p(\tau)\right) \Delta u(\tau) + \tilde{D}\left(\theta_p(\tau)\right) \Delta f(\tau) \tag{7.14}$$

where $\tilde{D}\left(\theta_p(\tau)\right)$ is obtained by rewriting $D(\theta(\tau), x(\tau), u(\tau), \Delta f(\tau))$ as a function of $\theta_p(\tau)$.

Notice that (7.14) is in a form similar to (4.1):

$$\sigma.x(\tau) = \tilde{A}(\theta(\tau))\, x(\tau) + \tilde{B}u(\tau) + \tilde{B}_w(\theta(\tau))\, w(\tau) \tag{7.15}$$

where A and B are known matrices[1] (a constant input matrix that does not depend on $\theta_p(\tau)$ can be obtained easily by prefiltering $\Delta u(\tau)$ as proposed in [8], see (2.156)–(2.163)) and $\tilde{D}$ is uncertain due to the fact that the exact value of $\Delta f(\tau)$ at a given

[1] Notice that the reasoning can be easily generalized to uncertain LPV systems subject to actuator faults, i.e. to the case where A and B are replaced by uncertain $\tilde{A}$ and $\tilde{B}$. However, this has not been done in order to keep the formulation simple.

moment is not known. Hence, if $\Delta f(\tau)$ is considered a disturbance that should be rejected, fault tolerance can be achieved through the robust LPV $\mathcal{H}_\infty$ (or $\mathcal{H}_2$) performance approach proposed in Chap. 4.

7.3.4 Active FTC Error Model

In order to obtain the active FTC error model, the assumption that $\hat{f}(\tau) = f(\tau)$ is done, such that subtracting (7.6) from (7.8) leads to:

$$\sigma.e(\tau) = A_f\left(\theta(\tau), \hat{f}(\tau)\right) e(\tau) + B_f\left(\theta(\tau), \hat{f}(\tau)\right) \Delta u(\tau) \tag{7.16}$$

Then, by introducing a new parameter vector $\theta_a(\tau)$, containing $\theta(\tau)$ and $\hat{f}(\tau)$, the following is obtained:

$$\sigma.e(\tau) = A_f\left(\theta_a(\tau)\right) e(\tau) + B_f\left(\theta_a(\tau)\right) \Delta u(\tau) \tag{7.17}$$

that is in a quite standard form for applying the LPV framework for designing an error-feedback controller scheduled by both the varying parameters $\theta(\tau)$ and the fault estimation $\hat{f}(\tau)$, as follows:

$$\Delta u(\tau) = K\left(\theta_a(\tau)\right) e(\tau) \tag{7.18}$$

7.3.5 Hybrid FTC Error Model

Fault estimation algorithms are affected by uncertainties that will cause the estimated value given by the algorithm to differ from the real fault value. Among the causes of uncertainty, there are the presence of external disturbances, the mismatch between the real and modeled dynamics, due to unmodeled nonlinearities and errors in the calibration of the model parameters during the identification phase, and the noise affecting the measurements given by the sensors. The presence of these uncertainties in the fault estimation, if not properly taken into account, can degrade the FTC system performances and give rise to undesired behaviors. This fact motivates combining the benefits of the passive and the active FTC strategies in order to obtain a hybrid passive/active FTC.

In order to obtain the hybrid FTC error model, let us rewrite $f(\tau) = \hat{f}(\tau) + \Delta f(\tau)$, where $\Delta f(\tau)$ is the unknown error on the fault estimation,[2] such that (7.6) is rewritten as:

[2]Notice that the $\Delta f(\tau)$ used in the passive FTC error model is different from the $\Delta f(\tau)$ used in the hybrid FTC error model.

$$\sigma.x(\tau) = A_f\left(\theta(\tau),\, \hat{f}(\tau) + \Delta f(\tau)\right) x(\tau) + B_f\left(\theta(\tau),\, \hat{f}(\tau) + \Delta f(\tau)\right) u(\tau). \tag{7.19}$$

Then, some manipulation is performed on (7.19), in order to obtain a structure suitable for applying the design technique presented in Chap. 4, similar to what has already been done in Sect. 7.3.3 for obtaining the passive FTC error model. In particular, $A_f\left(\theta(\tau),\, \hat{f}(\tau) + \Delta f(\tau)\right)$ and $B_f\left(\theta(\tau),\, \hat{f}(\tau) + \Delta f(\tau)\right)$ are rewritten as follows:

$$A_f\left(\theta(\tau),\, \hat{f}(\tau) + \Delta f(\tau)\right) = A_f\left(\theta(\tau),\, \hat{f}(\tau)\right) + \Delta A_f\left(\theta(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right) \tag{7.20}$$

$$B_f\left(\theta(\tau),\, \hat{f}(\tau) + \Delta f(\tau)\right) = B_f\left(\theta(\tau),\, \hat{f}(\tau)\right) + \Delta B_f\left(\theta(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right) \tag{7.21}$$

where $\Delta A_f\left(\theta(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right)$ and $\Delta B_f\left(\theta(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right)$ contain the effects that the fault estimation uncertainty $\Delta f(\tau)$ has on the faulty state space matrices. Hence, by subtracting (7.19) from the active FTC reference model (7.8), and taking into account (7.20)–(7.21), the following is obtained:

$$\begin{aligned}\sigma.e(\tau) = {} & A_f\left(\theta(\tau),\, \hat{f}(\tau)\right) e(\tau) + B_f\left(\theta(\tau),\, \hat{f}(\tau)\right) \Delta u(\tau)\\ & -\Delta A\left(\theta(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right) x(\tau) - \Delta B\left(\theta(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right) u(\tau)\end{aligned} \tag{7.22}$$

that, through some manipulations, can be reshaped as:

$$\begin{aligned}\sigma.e(\tau) = {} & A_f\left(\theta(\tau),\, \hat{f}(\tau)\right) e(\tau) + B_f\left(\theta(\tau),\, \hat{f}(\tau)\right) \Delta u(\tau)\\ & +D_f\left(\theta(\tau), x(\tau), u(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right) \Delta f(\tau)\end{aligned} \tag{7.23}$$

where $D_f\left(\theta(\tau), x(\tau), u(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right)$ is the matrix such that:

$$\begin{aligned} & D_f\left(\theta(\tau), x(\tau), u(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right) \Delta f(\tau) = \\ & -\Delta A\left(\theta(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right) x(\tau) - \Delta B\left(\theta(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right) u(\tau)\end{aligned} \tag{7.24}$$

Through the introduction of a new parameter vector $\theta_h(\tau)$, containing $\theta(\tau)$, $x(\tau)$, $u(\tau)$, $\hat{f}(\tau)$, $\Delta f(\tau)$ and possibly their powers and/or some combinations of them, the following is obtained:

$$\sigma.e(\tau) = A_f\left(\theta_h(\tau)\right) e(\tau) + B_f\left(\theta_h(\tau)\right) \Delta u(\tau) + \tilde{D}_f\left(\theta_h(\tau)\right) \Delta f(\tau) \tag{7.25}$$

where $\tilde{D}_f\left(\theta_h(\tau)\right)$ is obtained by rewriting $D_f\left(\theta(\tau), x(\tau), u(\tau),\, \hat{f}(\tau),\, \Delta f(\tau)\right)$ as a function of $\theta_h(\tau)$.

Similar to the passive FTC case described in Sect. 7.3.3, if $\Delta f(\tau)$ is considered as a disturbance that should be rejected, the robustness of the FTC against fault estimation uncertainties can be achieved through the robust LPV $\mathcal{H}_\infty$ (or $\mathcal{H}_2$) performance approach proposed in Chap. 4.

Notice that, differently from the passive FTC case, the hybrid FTC error model is scheduled also by the fault estimation $\hat{f}(\tau)$, embedded into the new scheduling vector $\theta_h(\tau)$. In other words, the proposed hybrid FTC approach adds the rejection characteristic of the passive FTC method to the active FTC strategy.

7.4 Reconfigurable Controller Strategy

In this section, it is shown how the FTC strategy proposed in Sect. 7.3 can be used for the implementation of a bank of controllers, such that the signal provided by the fault diagnosis unit determines which controller should be active at a given moment.

It is assumed that the faults belong to a set $\mathcal{F}$, that can be expressed as:

$$\mathcal{F} = \left\{f_1, \ldots, f_{n_x}\right\} = \left[\underline{f_1}, \overline{f_1}\right] \times \left[\underline{f_2}, \overline{f_2}\right] \times \cdots \times \left[\underline{f_{n_x}}, \overline{f_{n_x}}\right] \tag{7.26}$$

7.4.1 Passive FTC

In the passive FTC approach, it is assumed that no information about the faults is available. Hence, tolerance against faults can only be achieved by considering faults as if they were uncertainties. A single controller is designed in such a way that it exhibits some robustness properties. More specifically, a single controller K is designed so as to be scheduled by the parameter vector $\theta(\tau)$ and to be robust against the faults, that are considered as if they were additional uncertainties, as shown in Sect. 7.3.3. This strategy has the advantage of not needing a fault diagnosis algorithm but, on the other hand, the controller has the highest possible conservativeness.

7.4.2 Active FTC Without Controller Reconfiguration

The conservativeness of the passive approach can be overcome by considering that some information available about the faults can be used to schedule accordingly the controller. In this case, the faults are considered to be varying parameters $\theta_f(\tau)$, whose values are known or can be estimated through the information coming from a fault estimation module, and can be used to schedule accordingly a single controller $K\left(\theta(\tau),\ \theta_f(\tau)\right)$, designed as shown either in Sect. 7.3.4 or in Sect. 7.3.5. Notice that the controller is not reconfigured, as it is the same in both the nominal and the faulty case.

7.4.3 Reconfigured FTC with Fault Detection

In this case, the faults are considered as uncertainty, similar to what has been described in Sect. 7.4.1, but a fault detection algorithm can detect the fault occurrence at time instant τ_D. Then, two controllers are designed and switched according to the following law:

$$K = \begin{cases} K_0\left(\theta(\tau)\right) & if\ \tau < \tau_D \\ K_D\left(\theta(\tau)\right) & if\ \tau \geq \tau_D \end{cases} \tag{7.27}$$

where:

- $K_0\left(\theta(\tau)\right)$ is the nominal controller, designed without taking into account the uncertainty introduced by the faults;
- $K_D\left(\theta(\tau)\right)$ is the reconfigured controller, designed to be robust against all the possible faults.

This approach is less conservative in the nominal case than approaches without controller reconfiguration. Notice that the case when $K_0\left(\theta(\tau)\right) = K_D\left(\theta(\tau)\right)$ corresponds to the active FTC without controller reconfiguration.

7.4.4 Reconfigured FTC with FDI

In this case, the faults are considered as uncertainty, and an FDI algorithm can detect the fault occurrence at time instant τ_D and isolate the fault at time instant τ_I. Then, $n_f + 2$ controllers are designed and switched according to the following law:

$$K = \begin{cases} K_0\left(\theta(\tau)\right) & if\ \tau < \tau_D \\ K_D\left(\theta(\tau)\right) & if\ \tau_D \leq \tau < \tau_I \\ K_I^i\left(\theta(\tau)\right) & if\ \tau \geq \tau_I \end{cases} \tag{7.28}$$

where:

- $K_0\left(\theta(\tau)\right)$ is the nominal controller, designed without taking into account the uncertainty introduced by the faults;
- $K_D\left(\theta(\tau)\right)$ is the reconfigured post-detection controller, designed to be robust against all the possible faults;
- the n_f controllers $K_I^1\left(\theta(\tau)\right), \ldots, K_I^{n_f}\left(\theta(\tau)\right)$ are the reconfigured post-isolation controllers, each one designed to be robust against a specific fault f_i;

Notice that the case when $K_0\left(\theta(\tau)\right) = K_D\left(\theta(\tau)\right) = K_I^i\left(\theta(\tau)\right)$ corresponds to the active FTC without controller reconfiguration.

7.4.5 Reconfigured FTC with Fault Detection, Isolation and Estimation

In this case, an estimation of the fault f_i is provided by a fault estimation algorithm, and this estimation can be used as a scheduling parameter, denoted as θ^i_f. Moreover, it is assumed that the FDI algorithm can detect the fault occurrence at time instant τ_D and isolate the fault at time instant τ_I. Then, $n_f + 2$ controllers are designed and switched according to the following law:

$$K = \begin{cases} K_0\left(\theta(\tau)\right) & if\ \tau < \tau_D \\ K_D\left(\theta(\tau)\right) & if\ \tau_D \leq \tau < \tau_I \\ K_I^i\left(\theta(\tau), \theta^i_f(\tau)\right) & if\ \tau \geq \tau_I \end{cases} \tag{7.29}$$

where:

- $K_0\left(\theta(\tau)\right)$ is the nominal controller, designed without taking into account the uncertainty introduced by the faults;
- $K_D\left(\theta(\tau)\right)$ is the reconfigured post-detection controller, designed to be robust against all the possible faults;
- the n_f controllers $K_I^1\left(\theta(\tau), \theta^i_f(\tau)\right), \ldots, K_I^{n_f}\left(\theta(\tau), \theta^i_f(\tau)\right)$ are the reconfigured post-isolation controllers, each one scheduled not only by the vector of varying parameters $\theta(\tau)$ but by the estimation of the specific fault f_i too, through $\theta^i_f(\tau)$. This controller would be designed following either Sect. 7.3.4 or Sect. 7.3.5.

It is evident that the advantage of the reconfigured controllers with respect to the non-reconfigured ones lies in that the formers have to cope only with specific faults and allow to improve the performances in the non-faulty case using the nominal controller, whose design does not take into account the possibility of fault occurrence.

7.5 Application to a Quadrotor System

In this application, the problem of fault tolerant tracking for a quadrotor UAV will be solved using the robust state-feedback control technique, as shown in the previous sections of this chapter.

7.5.1 Quadrotor Modeling

A quadrotor is a vehicle that has four propellers in a cross configuration. Two propellers can rotate in a clockwise direction, while the remaining two can rotate anticlockwisely. The quadrotor is moved by changing the rotor speeds. For example, increasing or decreasing together the four propeller speeds, vertical motion is achieved. Changing only the speeds of the propellers situated oppositely produces either roll or pitch motion. Finally, yaw rotation results from the difference in the counter-torque between each pair of propellers.

Let us consider a body fixed frame $\{x_b, y_b, z_b\}$ with origin in the quadrotor center of mass. Under the assumptions that the body is rigid and symmetrical, and the propellers are rigid, i.e. no blade flapping occurs, the quadrotor dynamic model is described by the following equations [9]:

$$\ddot{x}_b(t) = (\cos\varphi(t)\sin\varrho(t)\cos\psi(t) + \sin\varphi(t)\sin\psi(t))\,\frac{U_1(t)}{m} \tag{7.30}$$

$$\ddot{y}_b(t) = (\cos\varphi(t)\sin\varrho(t)\sin\psi(t) + \sin\varphi(t)\cos\psi(t))\,\frac{U_1(t)}{m} \tag{7.31}$$

$$\ddot{z}_b(t) = -g + \cos\varphi(t)\cos\varrho(t)\frac{U_1(t)}{m} \tag{7.32}$$

$$\ddot{\varphi}(t) = \dot{\varrho}(t)\dot{\psi}(t)\frac{I_y - I_z}{I_x} - \frac{J_{TP}}{I_x}\dot{\varrho}(t)\Omega(t) + \frac{lU_2(t)}{I_x} \tag{7.33}$$

$$\ddot{\varrho}(t) = \dot{\varphi}(t)\dot{\psi}(t)\frac{I_z - I_x}{I_y} + \frac{J_{TP}}{I_y}\dot{\varphi}(t)\Omega(t) + \frac{lU_3(t)}{I_y} \tag{7.34}$$

$$\ddot{\psi}(t) = \dot{\varphi}(t)\dot{\varrho}(t)\frac{I_x - I_y}{I_z} + \frac{U_4(t)}{I_z} \tag{7.35}$$

where $\varphi(t)$ is the roll angle, $\varrho(t)$ is the pitch angle, $\psi(t)$ is the yaw angle and the inputs $U_1(t)$, $U_2(t)$, $U_3(t)$, $U_4(t)$, $\Omega(t)$ are defined as follows:

$$U_1(t) = b\left(\Omega_1(t)^2 + \Omega_2(t)^2 + \Omega_3(t)^2 + \Omega_4(t)^2\right) \tag{7.36}$$

$$U_2(t) = b\left(\Omega_4(t)^2 - \Omega_2(t)^2\right) \tag{7.37}$$

$$U_3(t) = b\left(\Omega_3(t)^2 - \Omega_1(t)^2\right) \tag{7.38}$$

$$U_4(t) = d\left(\Omega_2(t)^2 + \Omega_4(t)^2 - \Omega_1(t)^2 - \Omega_3(t)^2\right) \tag{7.39}$$

$$\Omega(t) = \Omega_2(t) + \Omega_4(t) - \Omega_1(t) - \Omega_3(t) \tag{7.40}$$

where $\Omega_i(t)$ denotes the ith rotor speed. For a description of the system parameters, as well as the values used in the simulations taken from [10], see Table 7.1.

Table 7.1 Quadrotor parameters description and values

Param	Description	Value
I_x	Body moment of inertia around the x-axis	8.1×10^{-3} [Nms^2]
I_y	Body moment of inertia around the y-axis	8.1×10^{-3} [Nms^2]
I_z	Body moment of inertia around the z-axis	14.2×10^{-3} [Nms^2]
m	Mass of the quadrotor	1 [kg]
g	Acceleration due to gravity	9.81 [ms^{-2}]
J_{TP}	Total rotational moment of inertia around the propeller axis	104×10^{-6} [Nms^2]
l	Center of quadrotor to center of propeller distance	0.24 [m]
b	Thrust factor	54.2×10^{-6} [Ns^2]
d	Drag factor	1.1×10^{-6} [Nms^2]

Hereafter, only the problem of attitude/altitude tracking control will be addressed. Hence, the dynamics of the system along the x_b and y_b axes, i.e. Eqs. (7.30)–(7.31), will be neglected.

By introducing the state $x(t)$, the input $u(t)$ and the varying parameter vector $\theta(t)$, as follows:

$$x(t) = \begin{pmatrix} \varphi(t) \\ \dot{\varphi}(t) \\ \varrho(t) \\ \dot{\varrho}(t) \\ \psi(t) \\ \dot{\psi}(t) \\ z_b(t) \\ \dot{z}_b(t) \end{pmatrix} \qquad u(t) = \begin{pmatrix} \Omega_1(t) \\ \Omega_2(t) \\ \Omega_3(t) \\ \Omega_4(t) \end{pmatrix} \qquad \theta(t) = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \\ \theta_4(t) \\ \theta_5(t) \\ \theta_6(t) \\ \theta_7(t) \\ \theta_8(t) \end{pmatrix} = \begin{pmatrix} \dot{\varphi}(t) \\ \dot{\varrho}(t) \\ \dot{\psi}(t) \\ \Omega_1(t) \\ \Omega_2(t) \\ \Omega_3(t) \\ \Omega_4(t) \\ \cos\varphi(t)\cos\varrho(t) \end{pmatrix}$$

the system (7.30)–(7.40) can be brought to the form (7.1) with:

$$A\left(\theta(t)\right) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(I_y - I_z)}{2I_x}\theta_3(t) & 0 & \frac{(I_y - I_z)}{2I_x}\theta_2(t) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{(I_z - I_x)}{2I_y}\theta_3(t) & 0 & 0 & 0 & \frac{(I_z - I_x)}{2I_y}\theta_1(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{(I_x - I_y)}{2I_z}\theta_2(t) & 0 & \frac{(I_x - I_z)}{2I_z}\theta_1(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.41}$$

$$B\left(\theta(t)\right) = \left(\begin{array}{cc} 0 & 0 \\ \frac{J_{TP}}{I_x}\theta_2(t) & -\frac{J_{TP}}{I_x}\theta_2(t) - \frac{lb}{I_x}\theta_5(t) \\ 0 & 0 \\ -\frac{J_{TP}}{I_y}\theta_1(t) - \frac{lb}{I_y}\theta_4(t) & \frac{J_{TP}}{I_y}\theta_1(t) \\ 0 & 0 \\ -\frac{d}{I_z}\theta_4(t) & \frac{d}{I_z}\theta_5(t) \\ 0 & 0 \\ \frac{b}{m}\theta_4(t)\theta_8(t) & \frac{b}{m}\theta_5(t)\theta_8(t) \end{array} \right. \cdots$$

$$\cdots \left. \begin{array}{cc} 0 & 0 \\ \frac{J_{TP}}{I_x}\theta_2(t) & -\frac{J_{TP}}{I_x}\theta_2(t) + \frac{lb}{I_x}\theta_7(t) \\ 0 & 0 \\ -\frac{J_{TP}}{I_y}\theta_1(t) + \frac{lb}{I_y}\theta_6(t) & \frac{J_{TP}}{I_y}\theta_1(t) \\ 0 & 0 \\ -\frac{d}{I_z}\theta_6(t) & -\frac{d}{I_z}\theta_7(t) \\ 0 & 0 \\ \frac{b}{m}\theta_6(t)\theta_8(t) & \frac{b}{m}\theta_7(t)\theta_8(t) \end{array} \right) \tag{7.42}$$

$$c(t) = -g \tag{7.43}$$

By considering multiplicative faults in the actuators, that change $\Omega_i \rightarrow f_i\Omega_i$, $i = 1, \ldots, 4$, in (7.36)–(7.40), the faulty system (7.6) is obtained, with $A_f\left(\theta(t), f(t)\right) = A\left(\theta(t)\right)$, and:

$$B_f\left(\theta(t), f(t)\right) = \left(\begin{array}{cc} 0 & 0 \\ \frac{J_{TP}}{I_x}\theta_2(t)f_1(t) & -\frac{J_{TP}}{I_x}\theta_2(t)f_2(t) - \frac{lb}{I_x}\theta_5(t)f_2(t)^2 \\ 0 & 0 \\ -\frac{J_{TP}}{I_y}\theta_1(t)f_1(t) - \frac{lb}{I_y}\theta_4(t)f_1(t)^2 & \frac{J_{TP}}{I_y}\theta_1(t)f_2(t) \\ 0 & 0 \\ -\frac{d}{I_z}\theta_4(t)f_1(t)^2 & \frac{d}{I_z}\theta_5(t)f_2(t)^2 \\ 0 & 0 \\ \frac{b}{m}\theta_4(t)\theta_8(t)f_1(t)^2 & \frac{b}{m}\theta_5(t)\theta_8(t)f_2(t)^2 \end{array} \right. \cdots$$

$$\cdots \left. \begin{array}{cc} 0 & 0 \\ \frac{J_{TP}}{I_x}\theta_2(t)f_3(t) & -\frac{J_{TP}}{I_x}\theta_2(t)f_4(t) + \frac{lb}{I_x}\theta_7(t)f_4(t)^2 \\ 0 & 0 \\ -\frac{J_{TP}}{I_y}\theta_1(t)f_3(t) + \frac{lb}{I_y}\theta_6(t)f_3(t)^2 & \frac{J_{TP}}{I_y}\theta_1(t)f_4(t) \\ 0 & 0 \\ -\frac{d}{I_z}\theta_6(t)f_3(t)^2 & -\frac{d}{I_z}\theta_7(t)f_4(t)^2 \\ 0 & 0 \\ \frac{b}{m}\theta_6(t)\theta_8(t)f_3(t)^2 & \frac{b}{m}\theta_7(t)\theta_8(t)f_4(t)^2 \end{array} \right) \tag{7.44}$$

7.5.1.1 Passive FTC Error Model of the Quadrotor

By following the reasoning provided in Sect. 7.3.3, through the introduction of the new varying parameter vector:

$$\theta_p(t) = \begin{pmatrix} \theta(t) \\ \theta_9(t) \\ \theta_{10}(t) \\ \theta_{11}(t) \\ \theta_{12}(t) \end{pmatrix} = \begin{pmatrix} \theta(t) \\ \Omega_1(t)^2 \\ \Omega_2(t)^2 \\ \Omega_3(t)^2 \\ \Omega_4(t)^2 \end{pmatrix}$$

the passive FTC error model of the quadrotor can be brought to the form (7.14) with:

$$\tilde{D}\left(\theta_p(t)\right) = \left(\begin{matrix} 0 & 0 \\ -\frac{J_{TP}}{I_x}\theta_2(t)\theta_4(t) & \frac{J_{TP}}{I_x}\theta_2(t)\theta_5(t) + \frac{lb(2+\Delta f_2(t))}{I_x}\theta_{10}(t) \\ 0 & 0 \\ \frac{J_{TP}}{I_y}\theta_1(t)\theta_4(t) + \frac{lb(2+\Delta f_1(t))}{I_y}\theta_9(t) & -\frac{J_{TP}}{I_y}\theta_1(t)\theta_5(t) \\ 0 & 0 \\ \frac{d(2+\Delta f_1(t))}{I_z}\theta_9(t) & -\frac{d(2+\Delta f_2(t))}{I_z}\theta_{10}(t) \\ 0 & 0 \\ -\frac{b(2+\Delta f_1(t))}{m}\theta_8(t)\theta_9(t) & -\frac{b(2+\Delta f_2(t))}{m}\theta_8(t)\theta_{10}(t) \end{matrix} \right. \quad \ldots$$

$$\ldots \quad \left. \begin{matrix} 0 & 0 \\ -\frac{J_{TP}}{I_x}\theta_2(t)\theta_6(t) & \frac{J_{TP}}{I_x}\theta_2(t)\theta_7(t) - \frac{lb(2+\Delta f_4(t))}{I_x}\theta_{12}(t) \\ 0 & 0 \\ \frac{J_{TP}}{I_y}\theta_1(t)\theta_6(t) - \frac{lb(2+\Delta f_3(t))}{I_y}\theta_{11}(t) & \frac{J_{TP}}{I_y}\theta_1(t)\theta_7(t) \\ 0 & 0 \\ \frac{d(2+\Delta f_3(t))}{I_z}\theta_{11}(t) & -\frac{d(2+\Delta f_4(t))}{I_z}\theta_{12}(t) \\ 0 & 0 \\ -\frac{b(2+\Delta f_3(t))}{m}\theta_8(t)\theta_{11}(t) & -\frac{b(2+\Delta f_4(t))}{m}\theta_8(t)\theta_{12}(t) \end{matrix} \right) \tag{7.45}$$

7.5.1.2 Active FTC Error Model of the Quadrotor

By following the reasoning provided in Sect. 7.3.4, through the introduction of the new varying parameter:

$$\theta_a(t) = \begin{pmatrix} \theta(t) \\ \theta_{13}(t) \\ \theta_{14}(t) \\ \theta_{15}(t) \\ \theta_{16}(t) \\ \theta_{17}(t) \\ \theta_{18}(t) \\ \theta_{19}(t) \\ \theta_{20}(t) \end{pmatrix} = \begin{pmatrix} \theta(t) \\ \hat{f}_1(t)^2 \\ \hat{f}_1(t) \\ \hat{f}_2(t)^2 \\ \hat{f}_2(t) \\ \hat{f}_3(t)^2 \\ \hat{f}_3(t) \\ \hat{f}_4(t)^2 \\ \hat{f}_4(t) \end{pmatrix}$$

the active FTC error model of the quadrotor can be brought to the form (7.17) with $A_f(\theta_a(t)) = A(\theta(t))$ and:

$$B_f(\theta_a(t)) = \begin{pmatrix} 0 & 0 \\ \frac{J_{TP}}{I_x}\theta_2(t)\theta_{14}(t) & -\frac{J_{TP}}{I_x}\theta_2(t)\theta_{16}(t) - \frac{lb}{I_x}\theta_5(t)\theta_{15}(t) \\ 0 & 0 \\ -\frac{J_{TP}}{I_y}\theta_1(t)\theta_{14}(t) - \frac{lb}{I_y}\theta_4(t)\theta_{13}(t) & \frac{J_{TP}}{I_y}\theta_1(t)\theta_{16}(t) \\ 0 & 0 \\ -\frac{d}{I_z}\theta_4(t)\theta_{13}(t) & \frac{d}{I_z}\theta_5(t)\theta_{15}(t) \\ 0 & 0 \\ \frac{b}{m}\theta_4(t)\theta_8(t)\theta_{13}(t) & \frac{b}{m}\theta_5(t)\theta_8(t)\theta_{15}(t) \end{pmatrix} \cdots$$

$$\cdots \begin{pmatrix} 0 & 0 \\ \frac{J_{TP}}{I_x}\theta_2(t)\theta_{18}(t) & -\frac{J_{TP}}{I_x}\theta_2(t)\theta_{20}(t) + \frac{lb}{I_x}\theta_7(t)\theta_{19}(t) \\ 0 & 0 \\ -\frac{J_{TP}}{I_y}\theta_1(t)\theta_{18}(t) + \frac{lb}{I_y}\theta_6(t)\theta_{17}(t) & \frac{J_{TP}}{I_y}\theta_1(t)\theta_{20}(t) \\ 0 & 0 \\ -\frac{d}{I_z}\theta_6(t)\theta_{17}(t) & -\frac{d}{I_z}\theta_7(t)\theta_{19}(t) \\ 0 & 0 \\ \frac{b}{m}\theta_6(t)\theta_8(t)\theta_{17}(t) & \frac{b}{m}\theta_7(t)\theta_8(t)\theta_{19}(t) \end{pmatrix} \tag{7.46}$$

7.5.1.3 Hybrid FTC Error Model of the Quadrotor

By following the reasoning provided in Sect. 7.3.5, through the introduction of the new parameter vector:

$$\theta_h(t) = \begin{pmatrix} \theta_p(t) \\ \theta_{13}(t) \\ \vdots \\ \theta_{20}(t) \end{pmatrix}$$

the hybrid FTC error model of the quadrotor can be brought to the form (7.25) with $A_f(\theta_h(t)) = A(\theta(t))$, $B_f(\theta_h(t)) = B_f(\theta_a(t))$ and:

$$\tilde{D}_f(\theta_h(t)) = \left(\begin{array}{cc} 0 & 0 \\ -\frac{J_{TP}}{I_x}\theta_2(t)\theta_4(t) & \frac{J_{TP}}{I_x}\theta_2(t)\theta_5(t) + \frac{lb(2\theta_{16}(t)+\Delta f_2(t))}{I_x}\theta_{10}(t) \\ 0 & 0 \\ \frac{J_{TP}}{I_y}\theta_1(t)\theta_4(t) + \frac{lb(2\theta_{14}(t)+\Delta f_1(t))}{I_y}\theta_9(t) & -\frac{J_{TP}}{I_y}\theta_1(t)\theta_5(t) \\ 0 & 0 \\ \frac{d(2\theta_{14}(t)+\Delta f_1(t))}{I_z}\theta_9(t) & -\frac{d(2\theta_{16}(t)+\Delta f_2(t))}{I_z}\theta_{10}(t) \\ 0 & 0 \\ -\frac{b(2\theta_{14}(t)+\Delta f_1(t))}{m}\theta_8(t)\theta_9(t) & -\frac{b(2\theta_{16}(t)+\Delta f_2(t))}{m}\theta_8(t)\theta_{10}(t) \end{array}\right.$$

$$\left.\cdots \begin{array}{cc} 0 & 0 \\ -\frac{J_{TP}}{I_x}\theta_2(t)\theta_6(t) & \frac{J_{TP}}{I_x}\theta_2(t)\theta_7(t) - \frac{lb(2\theta_{20}(t)+\Delta f_4(t))}{I_x}\theta_{12}(t) \\ 0 & 0 \\ \frac{J_{TP}}{I_y}\theta_1(t)\theta_6(t) - \frac{lb(2\theta_{18}(t)+\Delta f_3(t))}{I_y}\theta_{11}(t) & -\frac{J_{TP}}{I_y}\theta_1(t)\theta_7(t) \\ 0 & 0 \\ \frac{d(2\theta_{18}(t)+\Delta f_3(t))}{I_z}\theta_{11}(t) & -\frac{d(2\theta_{20}(t)+\Delta f_4(t))}{I_z}\theta_{12}(t) \\ 0 & 0 \\ -\frac{b(2\theta_{18}(t)+\Delta f_3(t))}{m}\theta_8(t)\theta_{11}(t) & -\frac{b(2\theta_{20}(t)+\Delta f_4(t))}{m}\theta_8(t)\theta_{12}(t) \end{array}\right)$$

7.5.2 Reference Inputs Calculation for Trajectory Tracking

To make the quadrotor track a desired trajectory, proper values of the reference inputs, in the following denoted as $\Omega_{1,ref}$, $\Omega_{2,ref}$, $\Omega_{3,ref}$ and $\Omega_{4,ref}$, respectively, should be fed to the reference model, such that its state equals the one corresponding to the desired trajectory.

Here, for illustrative purpose, the case of sinusoidal trajectories is considered, as follows:

$$\varphi_{ref}(t) = \Phi \sin\left(\frac{2\pi t}{N_\varphi}\right) \tag{7.47}$$

$$\varrho_{ref}(t) = \mathrm{P} \sin\left(\frac{2\pi t}{N_\varrho}\right) \tag{7.48}$$

$$\psi_{ref}(t) = \Psi \sin\left(\frac{2\pi t}{N_\psi}\right) \tag{7.49}$$

$$z_{ref}(t) = Z \sin\left(\frac{2\pi t}{N_z}\right) \tag{7.50}$$

where Φ, P, Ψ, Z are the amplitudes, and N_φ, N_ϱ, N_ψ, N_z are the periods. Taking the derivatives of (7.47)–(7.50) and considering the reference model equivalent of (7.32)–(7.35), i.e.:

$$\dot{\varphi}_{ref}(t) = v_\varphi^{ref}(t) \tag{7.51}$$

$$\dot{v}_{\varphi}^{ref}(t) = \dot{\varrho}(t)\dot{v}_{\psi}^{ref}(t)\frac{I_y - I_z}{2I_x} + \dot{v}_{\varrho}^{ref}(t)\dot{\psi}(t)\frac{I_y - I_z}{2I_x} - \frac{J_{TP}}{I_x}\dot{\varrho}(t)\Omega_{ref}(t) + \frac{lU_2^{ref}(t)}{I_x} \tag{7.52}$$

$$\dot{\varrho}_{ref}(t) = v_{\varrho}^{ref}(t) \tag{7.53}$$

$$\dot{v}_{\varrho}^{ref}(t) = \dot{\varphi}(t)\dot{v}_{\psi}^{ref}(t)\frac{I_z - I_x}{2I_y} + \dot{v}_{\varphi}^{ref}(t)\dot{\psi}(t)\frac{I_z - I_x}{2I_y} + \frac{J_{TP}}{I_y}\dot{\varphi}(t)\Omega_{ref}(t) + \frac{lU_3^{ref}(t)}{I_y} \tag{7.54}$$

$$\dot{\psi}_{ref}(t) = v_{\psi}^{ref}(t) \tag{7.55}$$

$$\dot{v}_{\psi}^{ref}(t) = v_{\varphi}^{ref}(t)\dot{\varrho}(t)\frac{I_x - I_y}{2I_z} + \dot{\varphi}(t)v_{\varrho}^{ref}(t)\frac{I_x - I_y}{2I_z} + \frac{dU_4^{ref}(t)}{I_z} \tag{7.56}$$

$$\dot{z}_{ref}(t) = v_z^{ref}(t) \tag{7.57}$$

$$\dot{v}_z^{ref}(t) = -g + \cos\varphi(t)\cos\varrho(t)\frac{U_1^{ref}(t)}{m} \tag{7.58}$$

with:

$$U_1^{ref}(t) = b\Big(\hat{f}_1(t)^2\Omega_1(t)\Omega_{1,ref}(t) + \hat{f}_2(t)^2\Omega_2(t)\Omega_{2,ref}(t) + \hat{f}_3(t)^2\Omega_3(t)\Omega_{3,ref}(t) + \hat{f}_4(t)^2\Omega_4(t)\Omega_{4,ref}(t)\Big) \tag{7.59}$$

$$U_2^{ref}(t) = b\Big(\hat{f}_4(t)^2\Omega_4(t)\Omega_{4,ref}(t) - \hat{f}_2(t)^2\Omega_2(t)\Omega_{2,ref}(t)\Big) \tag{7.60}$$

$$U_3^{ref}(t) = b\Big(\hat{f}_3(t)^2\Omega_3(t)\Omega_{3,ref}(t) - \hat{f}_1(t)^2\Omega_1(t)\Omega_{1,ref}(t)\Big) \tag{7.61}$$

$$U_4^{ref}(t) = d\Big(\hat{f}_2(t)^2\Omega_2(t)\Omega_{2,ref}(t) + \hat{f}_4(t)^2\Omega_4(t)\Omega_{4,ref}(t) - \hat{f}_1(t)^2\Omega_1(t)\Omega_{1,ref}(t) - \hat{f}_3(t)^2\Omega_3(t)\Omega_{3,ref}(t)\Big) \tag{7.62}$$

$$\Omega_{ref} = \hat{f}_2(t)\Omega_{2,ref}(t) + \hat{f}_4(t)\Omega_{4,ref}(t) - \hat{f}_1(t)\Omega_{1,ref}(t) - \hat{f}_3(t)\Omega_{3,ref}(t) \tag{7.63}$$

where $\Omega_{i,ref}$ denotes the ith reference rotor speed, then the following is obtained:

$$\dot{\varphi}_{ref}(t) = v_{\varphi}^{ref}(t) = \Phi\cos\left(\frac{2\pi t}{N_{\varphi}}\right)\frac{2\pi}{N_{\varphi}} \tag{7.64}$$

$$\dot{\varrho}_{ref}(t) = v_{\varrho}^{ref}(t) = \mathrm{P}\cos\left(\frac{2\text{ßt}}{\mathrm{N}_{\%}}\right)\frac{2\text{ß}}{\mathrm{N}_{\%}} \tag{7.65}$$

$$\dot{\psi}_{ref}(t) = v_{\psi}^{ref}(t) = \Psi\cos\left(\frac{2\pi t}{N_{\psi}}\right)\frac{2\pi}{N_{\psi}} \tag{7.66}$$

$$\dot{z}_{ref}(t) = v_{z}^{ref}(t) = Z\cos\left(\frac{2\pi t}{N_{z}}\right)\frac{2\pi}{N_{z}} \tag{7.67}$$

Then, another differentiation of (7.64)–(7.67) leads to:

$$\ddot{\varphi}_{ref}(t) = \dot{v}_{\varphi}^{ref}(t) = -\Phi\left(\frac{2\pi}{N_{\varphi}}\right)^2\sin\left(\frac{2\pi t}{N_{\varphi}}\right) \tag{7.68}$$

$$\ddot{\varrho}_{ref}(t) = \dot{v}_{\varrho}^{ref}(t) = -\mathrm{P}\left(\frac{2\text{ß}}{\mathrm{N}_{\%}}\right)^2\sin\left(\frac{2\text{ßt}}{\mathrm{N}_{\%}}\right) \tag{7.69}$$

$$\ddot{\psi}_{ref}(t) = \dot{v}_{\psi}^{ref}(t) = -\Psi\left(\frac{2\pi}{N_{\psi}}\right)^2\sin\left(\frac{2\pi t}{N_{\psi}}\right) \tag{7.70}$$

$$\ddot{z}_{ref}(t) = \dot{v}_{z}^{ref}(t) = -Z\left(\frac{2\pi}{N_{z}}\right)^2\sin\left(\frac{2\pi t}{N_{z}}\right) \tag{7.71}$$

and, by properly replacing (7.64)–(7.71) into (7.52), (7.54), (7.56) and (7.58), and taking into account (7.59)–(7.63), we obtain:

$$\begin{aligned}
&\dot{\varrho}(t)\Psi\cos\left(\tfrac{2\pi t}{N_{\psi}}\right)\tfrac{2\pi}{N_{\psi}}\tfrac{I_y-I_z}{2I_x} + \dot{\psi}(t)\mathrm{P}\cos\left(\tfrac{2\text{ßt}}{\mathrm{N}_{\%}}\right)\tfrac{2\text{ß}}{\mathrm{N}_{\%}}\tfrac{\mathrm{I_y-I_z}}{\mathrm{2I_x}} + \Phi\left(\tfrac{2\text{ß}}{\mathrm{N}'}\right)^2\sin\left(\tfrac{2\text{ßt}}{\mathrm{N}'}\right)\\
&-\tfrac{J_{TP}}{I_x}\dot{\varrho}(t)\left(\hat{f}_2(t)\Omega_{2,ref}(t)+\hat{f}_4(t)\Omega_{4,ref}(t)-\hat{f}_1(t)\Omega_{1,ref}(t)-\hat{f}_3(t)\Omega_{3,ref}(t)\right)\\
&+\tfrac{lb}{I_x}\left[\hat{f}_4(t)^2\left(\Omega_{4,ref}(t)-\Delta u_4(t)\right)\Omega_{4,ref}-\hat{f}_2(t)^2\left(\Omega_{2,ref}(t)-\Delta u_2(t)\right)\Omega_{2,ref}(t)\right]=0
\end{aligned} \tag{7.72}$$

$$\begin{aligned}
&\dot{\varphi}(t)\Psi\cos\left(\tfrac{2\pi t}{N_{\psi}}\right)\tfrac{2\pi}{N_{\psi}}\tfrac{I_z-I_x}{2I_y} + \dot{\psi}(t)\Phi\cos\left(\tfrac{2\pi t}{N_{\varphi}}\right)\tfrac{2\pi}{N_{\varphi}}\tfrac{I_z-I_x}{2I_y} + \mathrm{P}\left(\tfrac{2\text{ß}}{\mathrm{N}_{\%}}\right)^2\sin\left(\tfrac{2\text{ßt}}{\mathrm{N}_{\%}}\right)\\
&+\tfrac{J_{TP}}{I_y}\dot{\varphi}\left(\hat{f}_2(t)\Omega_{2,ref}(t)+\hat{f}_4(t)\Omega_{4,ref}(t)-\hat{f}_1(t)\Omega_{1,ref}(t)-\hat{f}_3(t)\Omega_{3,ref}(t)\right)\\
&+\tfrac{lb}{I_y}\left[\hat{f}_3(t)^2\left(\Omega_{3,ref}(t)-\Delta u_3(t)\right)\Omega_{3,ref}(t)-\hat{f}_1(t)^2\left(\Omega_{1,ref}(t)-\Delta u_1(t)\right)\Omega_{1,ref}(t)\right]=0
\end{aligned} \tag{7.73}$$

$$\begin{aligned}
&\dot{\varrho}(t)\Phi\cos\left(\tfrac{2\pi t}{N_{\varphi}}\right)\tfrac{2\pi}{N_{\varphi}}\tfrac{I_x-I_y}{2I_z} + \dot{\varphi}(t)\mathrm{P}\cos\left(\tfrac{2\text{ßt}}{\mathrm{N}_{\%}}\right)\tfrac{2\text{ß}}{\mathrm{N}_{\%}}\tfrac{\mathrm{I_x-I_y}}{\mathrm{2I_z}} + \Psi\left(\tfrac{2\text{ß}}{\mathrm{N}}\right)^2\sin\left(\tfrac{2\text{ßt}}{\mathrm{N}}\right)\\
&+\tfrac{d}{I_z}\left[\hat{f}_2(t)^2\left(\Omega_{2,ref}(t)-\Delta u_2(t)\right)\Omega_{2,ref}(t)+\hat{f}_4(t)^2\left(\Omega_{4,ref}(t)-\Delta u_4(t)\right)\Omega_{4,ref}(t)\right]\\
&-\tfrac{d}{I_z}\left[\hat{f}_1(t)^2\left(\Omega_{1,ref}(t)-\Delta u_1(t)\right)\Omega_{1,ref}(t)+\hat{f}_3(t)^2\left(\Omega_{3,ref}(t)-\Delta u_3(t)\right)\Omega_{3,ref}(t)\right]=0
\end{aligned} \tag{7.74}$$

$$\frac{b\cos\varphi(t)\cos\varrho(t)}{m}\left[\hat{f}_1(t)^2\left(\Omega_{1,ref}(t)-\Delta u_1(t)\right)\Omega_{1,ref}(t)+\hat{f}_2(t)^2\left(\Omega_{2,ref}(t)-\Delta u_2(t)\right)\Omega_{2,ref}(t)\right]$$
$$+\frac{b\cos\varphi(t)\cos\varrho(t)}{m}\left[\hat{f}_3(t)^2\left(\Omega_{3,ref}(t)-\Delta u_3(t)\right)\Omega_{3,ref}(t)+\hat{f}_4(t)^2\left(\Omega_{4,ref}(t)-\Delta u_4(t)\right)\Omega_{4,ref}(t)\right]$$
$$-g+Z\left(\frac{2\pi}{N_z}\right)^2\sin\left(\frac{2\pi t}{N_z}\right)=0 \tag{7.75}$$

7.5.3 Results

The results presented hereafter compare the proposed FTC strategies. Since the input matrices $B(\theta(t))$ and $B_f(\theta_a(t))$ are not constant, a prefiltering of the inputs is needed in order to obtain constant input matrices [8]. This is done by adding the states x_{u1}, x_{u2}, x_{u3} and x_{u4} to the error vector, such that $\Delta u_i(t) = x_{ui}(t)$, $i = 1, \ldots, 4$, together with the state equations:

$$\dot{x}_{ui}(t) = -\omega_i x_{ui}(t) + \omega_i \Delta\tilde{u}_i(t) \tag{7.76}$$

where $\Delta\tilde{u}_i(t)$, $i = 1, \ldots, 4$, are the new inputs, and ω_i has been chosen as $\omega_i = 100$, $i = 1, \ldots, 4$.

The polytopic approximation of the quadrotor quasi-LPV passive FTC error model (7.14), with matrices $A\left(\theta_p(t)\right)$, $B\left(\theta_p(t)\right)$ and $\tilde{D}\left(\theta_p(t)\right)$ defined as in (7.41), (7.42) and (7.45), respectively, has been obtained by considering:

$$\theta_1 \in [\min(\dot{\varphi}), \max(\dot{\varphi})] = [-0.25, 0.25]$$

$$\theta_2 \in [\min(\dot{\varrho}), \max(\dot{\varrho})] = [-0.25, 0.25]$$

$$\theta_3 \in [\min(\dot{\psi}), \max(\dot{\psi})] = [-0.25, 0.25]$$

$$\begin{pmatrix}\theta_{i+3}\\ \theta_{i+8}\end{pmatrix} \in Tr\left\{\begin{pmatrix}\min(\Omega_i)\\ \min(\Omega_i)^2\end{pmatrix}, \begin{pmatrix}\max(\Omega_i)\\ \min(\Omega_i)^2\end{pmatrix}, \begin{pmatrix}\max(\Omega_i)\\ \max(\Omega_i)^2\end{pmatrix}\right\}$$

with $\min(\Omega_i) = 100$, $\max(\Omega_i) = 500$, $i = 1, 2, 3, 4$ and Tr denoting a triangular polytopic approximation, that has been preferred to a bounding box one in order to reduce the conservativeness. Finally, $\theta_8 \in [0.5, 1]$, that corresponds to the interval of possible values of θ_8 when $\varphi \in [-\pi/4, \pi/4]$ and $\varrho \in [-\pi/4, \pi/4]$.

The polytopic approximation of the quadrotor quasi-LPV active FTC error model (7.17) with $A_f(\theta_a(t)) = A(\theta(t))$ and $B_f(\theta_a(t))$ defined as in (7.41) and (7.46), respectively, has been obtained by considering:

$$\theta_1 \in [\min(\dot{\varphi}), \max(\dot{\varphi})] = [-0.25, 0.25]$$

$$\theta_2 \in [\min(\dot{\varrho}), \max(\dot{\varrho})] = [-0.25, 0.25]$$

$$\theta_3 \in [\min(\dot{\psi}), \max(\dot{\psi})] = [-0.25, 0.25]$$

$$\theta_{i+3} \in [\min(\Omega_i), \max(\Omega_i)] = [100, 500] \qquad i = 1, 2, 3, 4$$

$$\theta_8 \in [0.5, 1]$$

$$\begin{pmatrix} \theta_{2i+11} \\ \theta_{2i+12} \end{pmatrix} \in Tr \left\{ \begin{pmatrix} \min(f_i)^2 \\ \min(f_i) \end{pmatrix}, \begin{pmatrix} \min(f_i)^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Similar considerations have been applied to the quadrotor quasi-LPV hybrid FTC error model for obtaining its polytopic approximation. In particular, the results presented hereafter have been obtained considering $\min(f_i) = 0.7$.

The passive/active/hybrid controllers have been designed using Theorems 4.3 and 4.4, to assure pole clustering in:

$$\mathcal{D} = \left\{ z \in \mathbb{C} : \mathrm{Re}(z) < -0.5, \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 < 10000, \tan(0.3)\mathrm{Re}(z) < -|\mathrm{Im}(z)| \right\}$$

and a $\mathcal{H}_\infty$ performance bound $\gamma_\infty = 1000$, considering:

$$z_\infty(t) = \begin{pmatrix} \varphi(t) \\ \varrho(t) \\ \psi(t) \\ z_b(t) \end{pmatrix}$$

It must be remarked that, due to the exponential growth of the vertices with the number of faults taken into consideration ($2^8 \cdot 3^i$ vertices in the passive and active FTC cases, $2^{8-i} \cdot 3^{2i}$ vertices in the hybrid FTC case, where i is the number of considered faults), the time needed to solve the LMIs grows exponentially too. However, the strong calculating capacity available nowadays, and the fact that the controller design is performed offline and only the coefficients of the polytopic decomposition must be calculated online, make this issue less critical.

The results shown hereafter refer to simulations which last 30 s, where the quadrotor is driven from the initial state:

$$\begin{array}{llll} \varphi(0) = \pi/6 & \varrho(0) = \pi/6 & \psi(0) = \pi/6 & z_b(0) = 0 \\ \dot{\varphi}(0) = 0 & \dot{\varrho}(0) = 0 & \dot{\psi}(0) = 0 & \dot{z}_b(0) = 0 \end{array}$$

to the desired trajectory defined as in (7.47)–(7.50) with $\Phi = \mathrm{P} = \Psi = 0.1$, $Z = 0$, $N_\varphi = N_\varrho = N_\psi = N_z = 10$. The desired trajectory has been generated by the reference model (7.51)–(7.58) starting from the initial reference state:

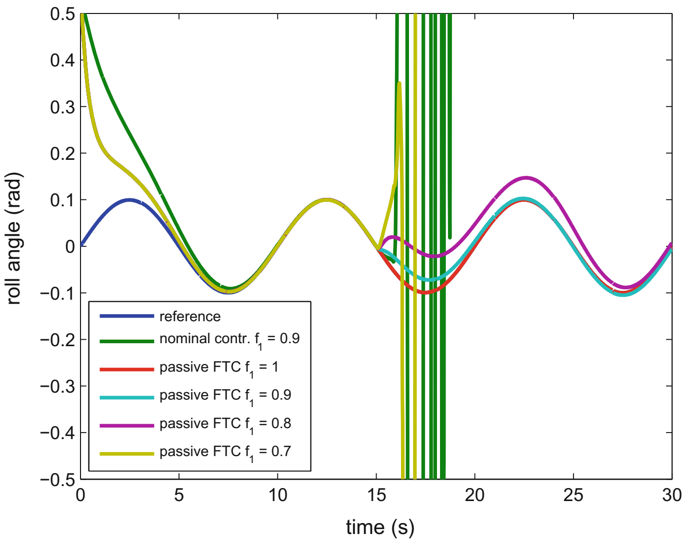

Fig. 7.1 Roll angle response (comparison between the nominal controller and the passive FTC). After [4]

$$\begin{pmatrix} \varphi_{ref}(0) \\ v_{\varphi}^{ref}(0) \\ \varrho_{ref}(0) \\ v_{\varrho}^{ref}(0) \\ \psi_{ref}(0) \\ v_{\psi}^{ref}(0) \\ z_{ref}(0) \\ v_{z}^{ref}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2\pi\Phi/N_{\varphi} \\ 0 \\ 2\pi \mathrm{P}/N_{\varrho} \\ 0 \\ 2\pi\Psi/N_{\psi} \\ 0 \\ 2\pi Z/N_{z} \end{pmatrix}$$

Figures 7.1, 7.2, 7.3 and 7.4 present a comparison between the responses obtained with a nominal controller and the ones obtained with the proposed FTC approach. A fault in the first actuator acts starting from the time instant $t = 15\,s$. It can be seen that even a small fault, e.g. $f_1 = 0.9$, is enough to drive the system to instability if the nominal controller is used (see green lines). On the other hand, the passive FTC shows some tolerance capability since, for $f_1 = 0.8$ and $f_1 = 0.9$ (purple and cyan lines, respectively), the stability is preserved, even though with a steady-state error due to the effect of the fault.[3]

On the other hand, the proposed active FTC technique can achieve a perfect fault tolerance as long as the fault is correctly estimated, as shown in Figs. 7.5, 7.6, 7.7 and 7.8 (green lines), where a fault $f_1 = 0.7$ acting from $t = 15$ s is considered. However,

[3]Adding an integral action to the controller could eliminate the steady-state error, even though at the expense of worsening the dynamical transient performance of the closed-loop system.

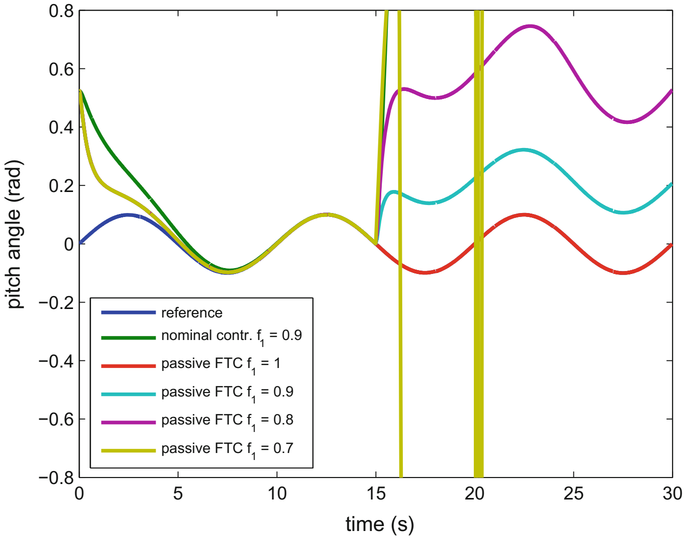

Fig. 7.2 Pitch angle response (comparison between the nominal controller and the passive FTC). After [4]

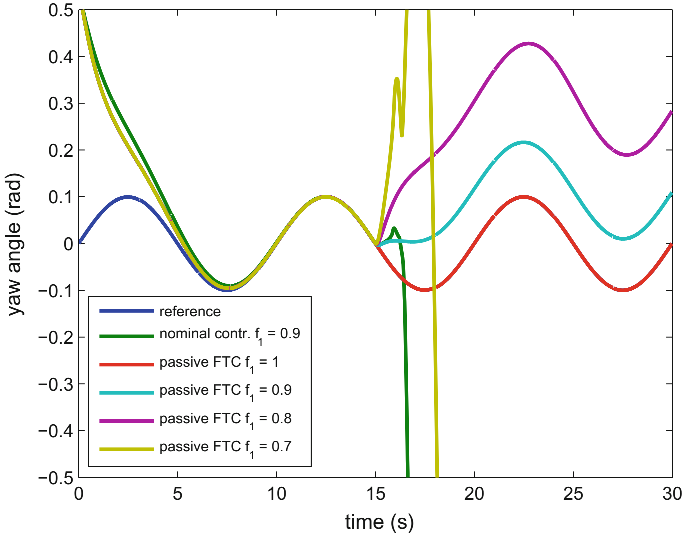

Fig. 7.3 Yaw angle response (comparison between the nominal controller and the passive FTC). After [4]

as the uncertainty in the fault estimation, in this work modeled as uniformly bounded noise, increases, so does the error between the real trajectory and the reference one.

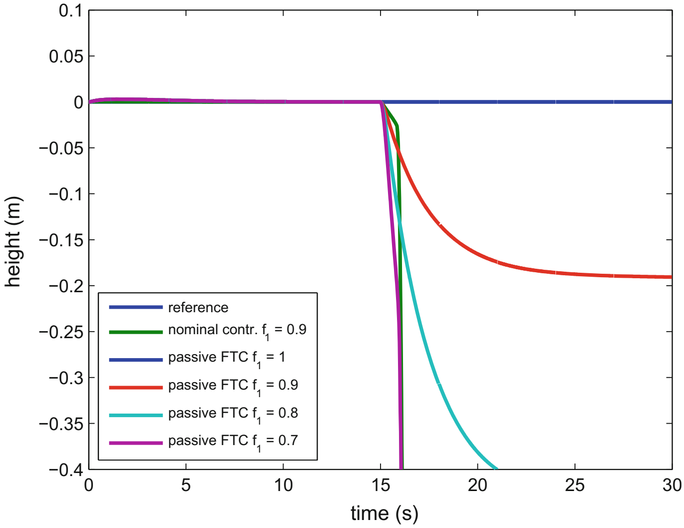

Fig. 7.4 Height response (comparison between the nominal controller and the passive FTC). After [4]

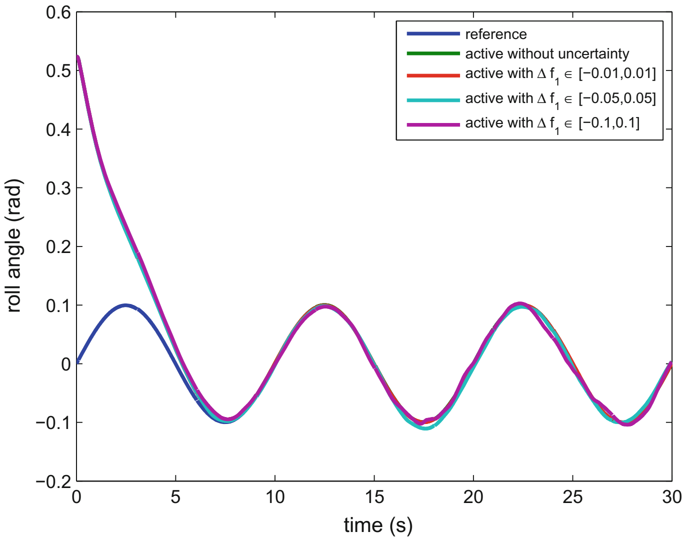

Fig. 7.5 Roll angle response (active FTC without and with uncertainty, $f_1 = 0.7$). After [4]

By applying the proposed hybrid FTC method, the overall performance can be improved, thus reducing the effect that the fault estimation error has on the closed-loop response, as shown in Figs. 7.9, 7.10, 7.11 and 7.12.

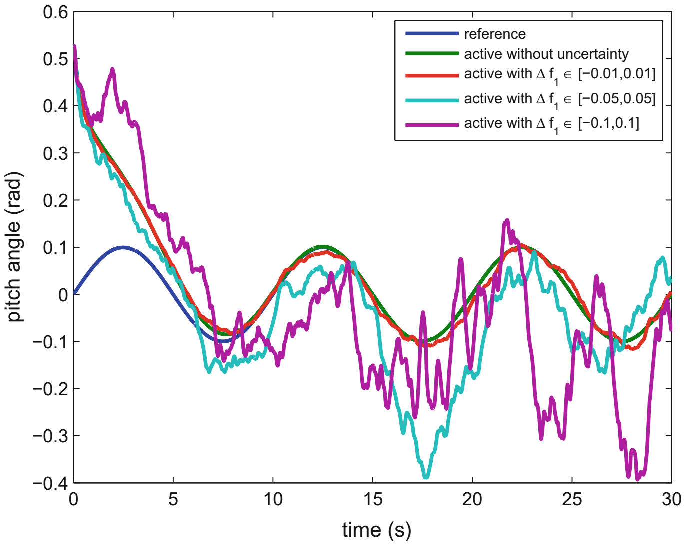

Fig. 7.6 Pitch angle response (active FTC without and with uncertainty, $f_1 = 0.7$). After [4]

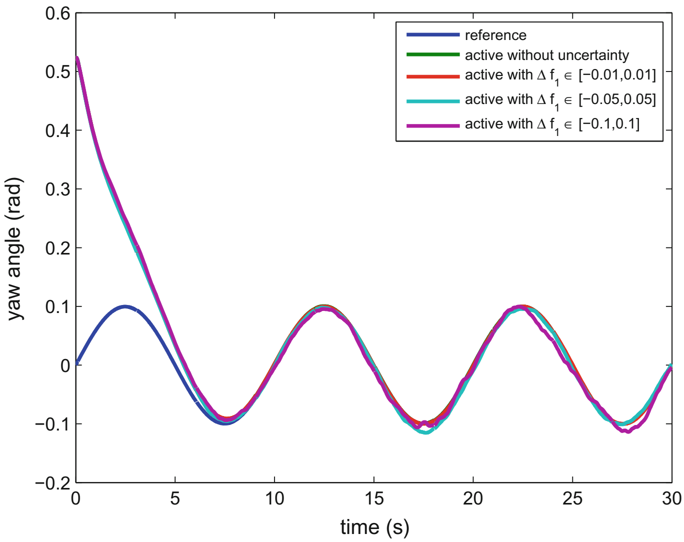

Fig. 7.7 Yaw angle response (active FTC without and with uncertainty, $f_1 = 0.7$). After [4]

In order to quantify numerically the improvement brought by the considered FTC strategies, let us introduce the following performance measures:

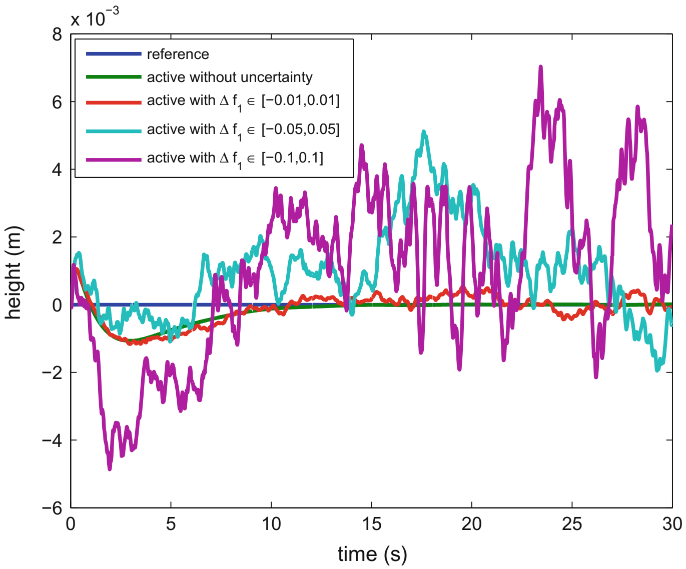

Fig. 7.8 Height response (active FTC without and with uncertainty, $f_1 = 0.7$). After [4]

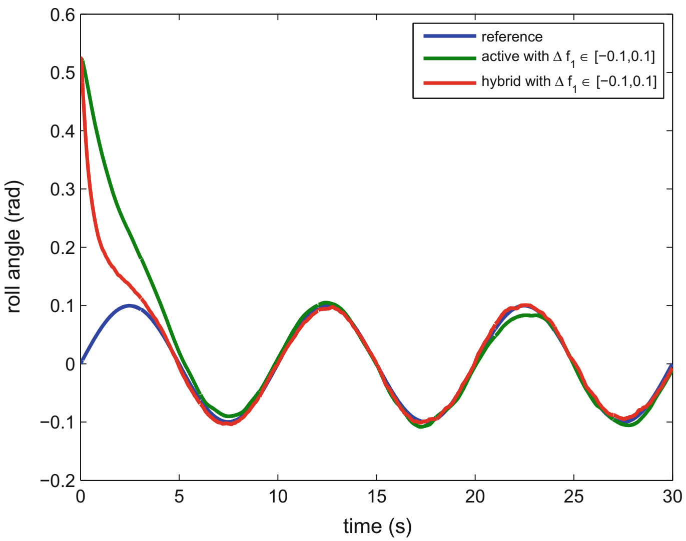

Fig. 7.9 Roll angle response (comparison between the active FTC and the hybrid FTC, $f_1 = 0.7$). After [4]

$$J_{\varphi} = \frac{\sum_{k=1500}^{3000} \left(\varphi_r\left(\frac{k}{100}\right) - \varphi\left(\frac{k}{100}\right)\right)^2}{1500} \tag{7.77}$$

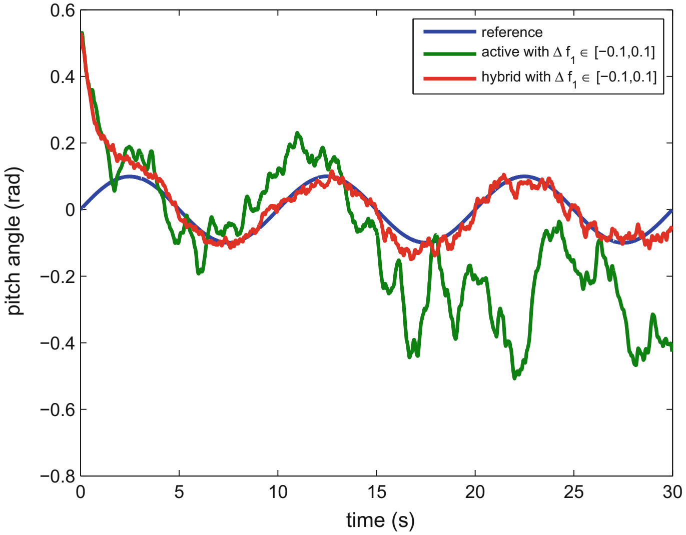

Fig. 7.10 Pitch angle response (comparison between the active FTC and the hybrid FTC, $f_1 = 0.7$). After [4]

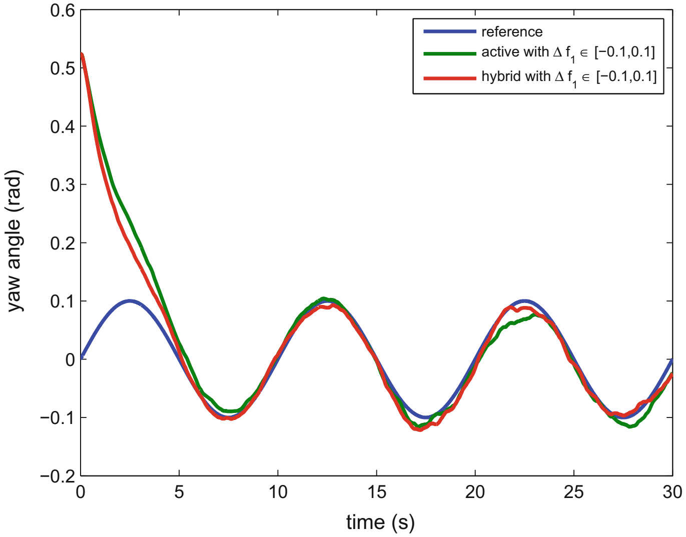

Fig. 7.11 Yaw angle response (comparison between the active FTC and the hybrid FTC, $f_1 = 0.7$). After [4]

$$J_\varrho = \frac{\sum_{k=1500}^{3000} \left(\varrho_r\left(\frac{k}{100}\right) - \varrho\left(\frac{k}{100}\right)\right)^2}{1500} \tag{7.78}$$

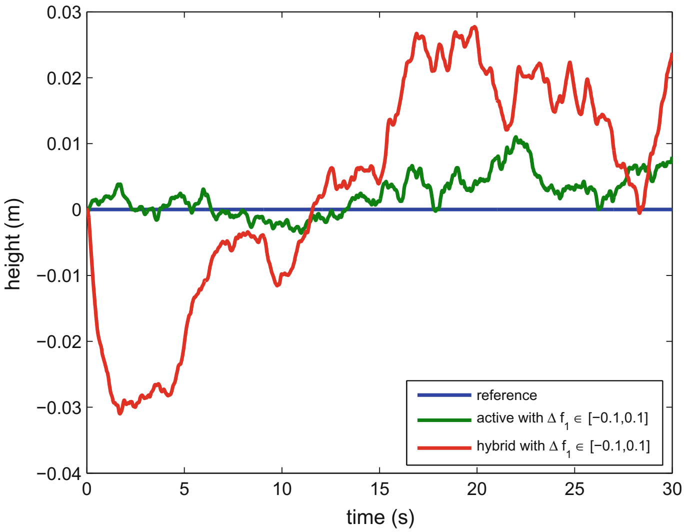

Fig. 7.12 Height response (comparison between the active FTC and the hybrid FTC, $f_1 = 0.7$). After [4]

Table 7.2 Comparison of nominal controller with passive/active/hybrid FTC

Type of FTC strategy	Fault/uncertainty magnitude	J_φ	J_ϱ	J_ψ	J_z	J
Nominal	$f_1 = 1$	$3.9 \cdot 10^{-9}$	$3.4 \cdot 10^{-9}$	$4.1 \cdot 10^{-9}$	$1.6 \cdot 10^{-11}$	$1.1 \cdot 10^{-8}$
	$f_1 = 0.9$	105.2	593.6	388.5	$1.6 \cdot 10^{3}$	$2.7 \cdot 10^{3}$
Passive	$f_1 = 1$	$1.4 \cdot 10^{-10}$	$1.4 \cdot 10^{-10}$	$1.2 \cdot 10^{-10}$	$2.7 \cdot 10^{-10}$	$6.7 \cdot 10^{-10}$
	$f_1 = 0.9$	$2.0 \cdot 10^{-4}$	0.047	0.012	0.027	0.087
	$f_1 = 0.8$	0.002	0.329	0.084	0.143	0.558
	$f_1 = 0.7$	$7.0 \cdot 10^{4}$	$2.3 \cdot 10^{3}$	$2.3 \cdot 10^{4}$	$1.5 \cdot 10^{5}$	$2.5 \cdot 10^{5}$
Active	$f_1 = 0.7$					
	$\Delta f_1 = 0$	$4.9 \cdot 10^{-10}$	$1.2 \cdot 10^{-8}$	$9.4 \cdot 10^{-10}$	$7.3 \cdot 10^{-11}$	$1.3 \cdot 10^{-8}$
	$\Delta f_1 \in [-.01, .01]$	$5.5 \cdot 10^{-7}$	$1.8 \cdot 10^{-4}$	$5.7 \cdot 10^{-7}$	$5.5 \cdot 10^{-8}$	$1.8 \cdot 10^{-4}$
	$\Delta f_1 \in [-.05, .05]$	$3.2 \cdot 10^{-5}$	0.015	$5.0 \cdot 10^{-5}$	$4.8 \cdot 10^{-6}$	0.015
	$\Delta f_1 \in [-.10, .10]$	$3.8 \cdot 10^{-5}$	0.027	$8.5 \cdot 10^{-5}$	$8.9 \cdot 10^{-6}$	0.027
Hybrid	$f_1 = 0.7$ $\Delta f_1 \in [-.10, .10]$	$2.0 \cdot 10^{-5}$	$8.0 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$3.4 \cdot 10^{-4}$	0.001

$$J_\psi = \frac{\sum_{k=1500}^{3000} \left(\psi_r\left(\frac{k}{100}\right) - \psi\left(\frac{k}{100}\right)\right)^2}{1500} \tag{7.79}$$

$$J_z = \frac{\sum\limits_{k=1500}^{3000} \left(z_r\left(\frac{k}{100}\right) - z\left(\frac{k}{100}\right)\right)^2}{1500} \tag{7.80}$$

$$J = J_\varphi + J_\varrho + J_\psi + J_z \tag{7.81}$$

A comparison of the performance measures obtained in the different cases, as resumed in Table 7.2, shows the improvement brought by the proposed FTC strategies with respect to the nominal one, as well as the one brought by the hybrid FTC with respect to the passive and active FTC strategies.

7.6 Conclusions

In this chapter, the idea of the robust LPV polytopic technique, introduced in Chap. 4, has been applied to FTC, giving rise to different strategies. A passive FTC strategy has been obtained by considering the faults as exogenous perturbations that should be rejected. An active FTC strategy has been obtained by considering the faults as additional scheduling parameters. Finally, a hybrid FTC strategy has been obtained by taking into account explicitly the fault estimation uncertainty during the design step.

It has also been shown how the proposed FTC strategy can be used for the implementation of a bank of controllers, such that the signal provided by the fault diagnosis unit determines which controller should be active at a given moment. The advantage of the reconfigured controllers with respect to the non-reconfigured ones lies in that the formers have to cope with specific faults and allow to improve the performances in the non-faulty case using the nominal controller, whose design does not take into account the possibility of fault occurrence.

The proposed method has been applied to solve the FTC problem for a quadrotor UAV. The results presented have shown the relevant features of the proposed FTC strategy, that is able to improve the performances under fault occurrence. In particular, whereas the passive FTC shows some limited tolerance capability, due to the appearance of steady-state errors due to the fault effect, the active FTC technique can achieve a perfect fault tolerance as long as the fault is correctly estimated. However, as the uncertainty in the fault estimation increases, so does the error between the real trajectory and the reference one. By applying the proposed hybrid FTC method, the overall performance can be improved, thus reducing the effect that the fault estimation error has on the closed-loop response. The introduction and comparison of some performance measures have allowed confirming numerically such analysis.

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Chapter 8
Fault Tolerant Control of LPV Systems Using Reconfigured Reference Model and Virtual Actuators

The content of this chapter is based on the following works:

- [1] D. Rotondo, F. Nejjari, V. Puig. A virtual actuator and sensor approach for fault tolerant control of LPV systems. *Journal of Process Control*, 24(3):203–222, 2014.
- [2] D. Rotondo, F. Nejjari, V. Puig, J. Blesa. Model reference FTC for LPV systems using virtual actuator and set-membership fault estimation. *International Journal of Robust and Nonlinear Control*, 25(5):735–760, 2015.
- [3] D. Rotondo, V. Puig, F. Nejjari, J. Romera. A fault-hiding approach for the switching quasi-LPV fault tolerant control of a four wheeled omnidirectional mobile robot. *IEEE Transactions on Industrial Electronics*, 62(6):3932–3944, 2015.
- [4] D. Rotondo, F. Nejjari, V. Puig. Fault tolerant control of a PEM fuel cell using Takagi-Sugeno virtual actuators. *Journal of Process Control*, 45:12–29, 2016.

8.1 Introduction

In recent years, the *fault-hiding* paradigm has been proposed as an active strategy to obtain fault tolerance [5]. In this paradigm, the controller reconfiguration (CR) unit reconfigures the faulty plant instead of the controller/observer. The nominal controller is kept in the loop by inserting a reconfiguration block between the faulty plant and the nominal controller/observer when a fault occurs. The reconfiguration block is chosen so as to hide the fault from the controller point of view, allowing it to see the same plant as before the fault. In case of actuator faults, as the ones considered in this chapter, the reconfiguration block is named *virtual actuator*. Initially proposed in a state space formulation for LTI systems [6], this active FTC strategy has been extended successfully to many classes of systems, e.g. LPV [7], TS [8], piecewise affine [9], Lipschitz [10] and Hammerstein-Weiner [11] systems.

D. Rotondo, *Advances in Gain-Scheduling and Fault Tolerant Control Techniques*, Springer Theses, https://doi.org/10.1007/978-3-319-62902-5_8

The work presented in this chapter is concerned with the development of an FTC strategy for LPV systems involving a reconfigured reference model and virtual actuators. The use of the reference model framework allows to assure that the desired tracking performances are kept despite the fault occurrence, thanks to the action brought by the virtual actuator.

In all controlled systems, the actuator capacity is limited by physical constraints and limitations of the actuators. The effects of saturations on the control loop could be performance degradation, large overshoot and possible instability in spite of satisfactory performances predicted from the linear design [12, 13]. While the system analysis, including saturated actuators, is relatively easy, the controller synthesis problem in presence of input nonlinearities is a much more involved task. In [14], a systematic anti-windup control synthesis approach for systems with actuator saturation is provided within an LPV design framework. The advantage of this approach is that it directly utilizes saturation indicator parameters to schedule accordingly the parameter-varying controller.

In this chapter, by including the saturations in the reference model equations, it is shown that it is possible to design a model reference FTC system that automatically retunes the reference states whenever the system input is affected by saturation nonlinearities. Hence, another contribution of this chapter is to take into account the saturations as scheduling parameters, such that their inclusion in both the reference model and the system provides an elegant way to incorporate a graceful performance degradation in presence of actuator saturations.

8.2 FTC Using Reconfigured Reference Model and Virtual Actuators

8.2.1 Model Reference Control

Let us consider an LPV system in state-space form, described by (2.1)–(2.2) (for the sake of simplicity, $D(\theta(\tau)) = O$ in further deliberations):

$$\sigma.x(\tau) = A(\theta(\tau))\,x(\tau) + B(\theta(\tau))\,u(\tau) + c(\tau) \tag{8.1}$$

$$y(\tau) = C(\theta(\tau))\,x(\tau) \tag{8.2}$$

where $c(\tau)$ is a known exogenous input. Similar to what has been done in Chap. 7, the idea of LPV reference model control is considered to assure the convergence of the system trajectory to the desired one [15, 16]. Hence, the following reference model is considered for the synthesis of the LPV controller:

$$\sigma.x_{ref}(\tau) = A(\theta(\tau))\,x_{ref}(\tau) + B(\theta(\tau))\,u^{c}_{ref}(\tau) + c(\tau) \tag{8.3}$$

$$y_{ref}(\tau) = C(\theta(\tau))\, x_{ref}(\tau) \tag{8.4}$$

where $x_{ref} \in \mathbb{R}^{n_x}$ is the reference state vector, $u^c_{ref} \in \mathbb{R}^{n_u}$ is the nominal reference input vector, and $y_{ref} \in \mathbb{R}^{n_y}$ is the reference output vector.

Thus, considering the tracking error, defined as $e(\tau) \triangleq x_{ref}(\tau) - x(\tau)$, the following *error system* is obtained:

$$\sigma.e(\tau) = A(\theta(\tau))\, e(\tau) + B(\theta(\tau))\, \Delta u_c(\tau) \tag{8.5}$$

$$\varepsilon_c(\tau) = C(\theta(\tau))\, e(\tau) \tag{8.6}$$

with $\Delta u_c(\tau) \triangleq u^c_{ref}(\tau) - u(\tau)$ and $\varepsilon_c(\tau) \triangleq y_{ref}(\tau) - y(\tau)$.

The LPV error system (8.5)–(8.6) is controlled by an error-feedback control law:

$$\Delta u_c(\tau) = K(\theta(\tau))\, \hat{e}(\tau) \tag{8.7}$$

where $\hat{e}(\tau)$ is an estimation of the error $e(\tau)$, provided by the following LPV error observer:

$$\sigma.\hat{e}(\tau) = A(\theta(\tau))\, \hat{e}(\tau) + B(\theta(\tau))\, \Delta u_c(\tau) + L(\theta(\tau)) \left[C(\theta(\tau))\, \hat{e}(\tau) - \varepsilon_c(\tau) \right] \tag{8.8}$$

where the gain $L(\theta(\tau)) \in \mathbb{R}^{n_x \times n_y}$ is a design parameter. Notice that the information given by the error observer can be used directly to obtain the state estimation, since $\hat{x}(\tau) = x_{ref}(\tau) - \hat{e}(\tau)$.

8.2.2 Fault Definition

In this chapter, the considered actuator faults change the nominal state equation of the system (8.1), as follows:

$$\sigma.x(\tau) = A(\theta(\tau))\, x(\tau) + B_f(\theta(\tau), f(\tau))\, u(\tau) + \Phi(\theta(\tau), f(\tau)) f_a(\tau) + c(\tau) \tag{8.9}$$

where $f_a(\tau) \in \mathbb{R}^{n_u}$ denotes the additive actuator faults, being $\Phi(\theta(\tau), f(\tau)) \in \mathbb{R}^{n_u \times n_x}$ the actuator fault distribution matrix. The multiplicative actuator faults are embedded in the matrix $B_f(\theta(\tau), f(\tau))$, as follows:

$$B_f(\theta(\tau), f(\tau)) = B(\theta(\tau))\, F(f(\tau)) \tag{8.10}$$

with:

$$F\left(f(\tau)\right) = \begin{pmatrix} f_1(\tau) & 0 & \cdots & 0 \\ 0 & f_2(\tau) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{n_u}(\tau) \end{pmatrix} \tag{8.11}$$

where $f_i(\tau) \in [0, 1]$, $i = 1, \ldots, n_u$, represents the effectiveness of the ith actuator, such that the extreme values $f_i = 0$ and $f_i = 1$ represent a total failure of the ith actuator and the healthy ith actuator, respectively.

8.2.3 Fault Tolerant Control Strategy

The fault tolerant control strategy proposed in this chapter is based on a reconfiguration of the reference model (8.3)–(8.4), and the addition of a virtual actuator block. At first, the reference model state equation (8.3) is slightly modified to take into account the actuator faults, as follows:

$$\sigma.x_{ref}(\tau) = A\left(\theta(\tau)\right) x_{ref}(\tau) + B_f\left(\theta(\tau), \hat{f}(\tau)\right) u_{ref}(\tau) + \Phi\left(\theta(\tau), \hat{f}(\tau)\right) \hat{f}_a(\tau) + c(\tau) \tag{8.12}$$

where $\hat{f}(\tau)$ and $\hat{f}_a(\tau)$ are estimations of $f(\tau)$ and $f_a(\tau)$, respectively, and $u_{ref}(\tau) \in \mathbb{R}^{n_u}$ is the reconfigured reference input vector. Hence, under the assumption that $\hat{f}(\tau) \cong f(\tau)$ and $\hat{f}_a(\tau) \cong f_a(\tau)$, the error system equation (8.5) becomes:

$$\sigma.e(\tau) = A\left(\theta(\tau)\right) e(\tau) + B_f\left(\theta(\tau), \hat{f}(\tau)\right) \Delta u(\tau) \tag{8.13}$$

with $\Delta u(\tau) \triangleq u_{ref}(\tau) - u(\tau)$.

Then, the concept of virtual actuator introduced in [6] for LTI systems is extended to LPV systems, such that it can be applied to the error model (8.13). The main idea of this FTC method is to reconfigure the faulty plant such that the nominal controller could still be used without need of retuning it. The plant with the faulty actuators is modified adding the virtual actuator block that masks the fault and allows the controller to see the same plant as before the fault.

The virtual actuator can be either a static or a dynamic block, depending on the satisfaction of the following rank condition:

$$rank\left(B_f\left(\theta(\tau), f(\tau)\right)\right) = rank\left(B\left(\theta(\tau)\right) \quad B_f\left(\theta(\tau), f(\tau)\right)\right) \tag{8.14}$$

If (8.14) holds (e.g. when the fault has only changed the actuator gain, but it has not completely broken it), the reconfiguration structure is static and can be expressed as:

$$\Delta u(\tau) = N\left(\theta(\tau), \hat{f}(\tau)\right) \Delta u_c(\tau) \tag{8.15}$$

where $\Delta u_c(\tau)$ is the controller output, and the matrix $N\left(\theta(\tau), \hat{f}(\tau)\right)$ is given by:

$$N\left(\theta(\tau), \hat{f}(\tau)\right) = B_f\left(\theta(\tau), \hat{f}(\tau)\right)^{\dagger} B\left(\theta(\tau)\right) \tag{8.16}$$

Cases where (8.14) does not hold should be described through values of the matrix $B^*\left(\theta(\tau)\right)$, such that the following condition holds:

$$B^*\left(\theta(\tau)\right) = B_f\left(\theta(\tau), f(\tau)\right) N\left(\theta(\tau), \hat{f}(\tau)\right) \tag{8.17}$$

Notice that the matrix $B^*\left(\theta(\tau)\right)$ does not depend on $f(\tau)$ because the matrix $N\left(\theta(\tau), \hat{f}(\tau)\right)$ eliminates the effects of partial faults, as discussed in Appendix B.

In such cases, the reconfiguration structure is expressed by:

$$\Delta u(\tau) = N\left(\theta(\tau), \hat{f}(\tau)\right)\left(\Delta u_c(\tau) - M\left(\theta(\tau)\right) x_v(\tau)\right) \tag{8.18}$$

where $M\left(\theta(\tau)\right)$ is the gain of the LPV virtual actuator, while the virtual actuator state $x_v(\tau)$ is calculated as:

$$\sigma.x_v(\tau) = \left(A\left(\theta(\tau)\right) + B^*\left(\theta(\tau)\right) M\left(\theta(\tau)\right)\right) x_v(\tau) + \left(B\left(\theta(\tau)\right) - B^*\left(\theta(\tau)\right)\right) \Delta u_c(\tau) \tag{8.19}$$

In these cases, the LPV error observer (8.8) is also modified, as follows:

$$\sigma.\hat{e}(\tau) = A\left(\theta(\tau)\right) \hat{e}(\tau) + B\left(\theta(\tau)\right) \Delta u_c(\tau) + L\left(\theta(\tau)\right)\left(C\left(\theta(\tau)\right) \hat{e}(\tau) - \varepsilon(\tau)\right) \tag{8.20}$$

where:

$$\varepsilon(\tau) = \varepsilon_c(\tau) + C\left(\theta(\tau)\right) x_v(\tau) \tag{8.21}$$

Thanks to the introduction of the virtual actuator block, the *separation principle* holds for the augmented system made up by the LPV error system, the LPV virtual actuator, the LPV error-feedback controller and the LPV error observer, i.e. the augmented system can be brought to a block-triangular form, as shown by the following theorem.

Theorem 8.1 (Separation principle for the augmented system) *Consider the augmented model that includes the faulty LPV error system state* (8.13) *and output* (8.6) *equations, the LPV virtual actuator* (8.18)–(8.19)*, the LPV error-feedback controller* (8.7) *and the LPV error observer* (8.20)*, as shown*[1] *in Fig. 8.1:*

[1] In the remaining of the theorem, and in its proof, the dependence of the matrices on the vector of scheduling parameters $\theta(\tau)$ and the multiplicative faults $f(\tau)$, or their estimation $\hat{f}(\tau)$, will be omitted.

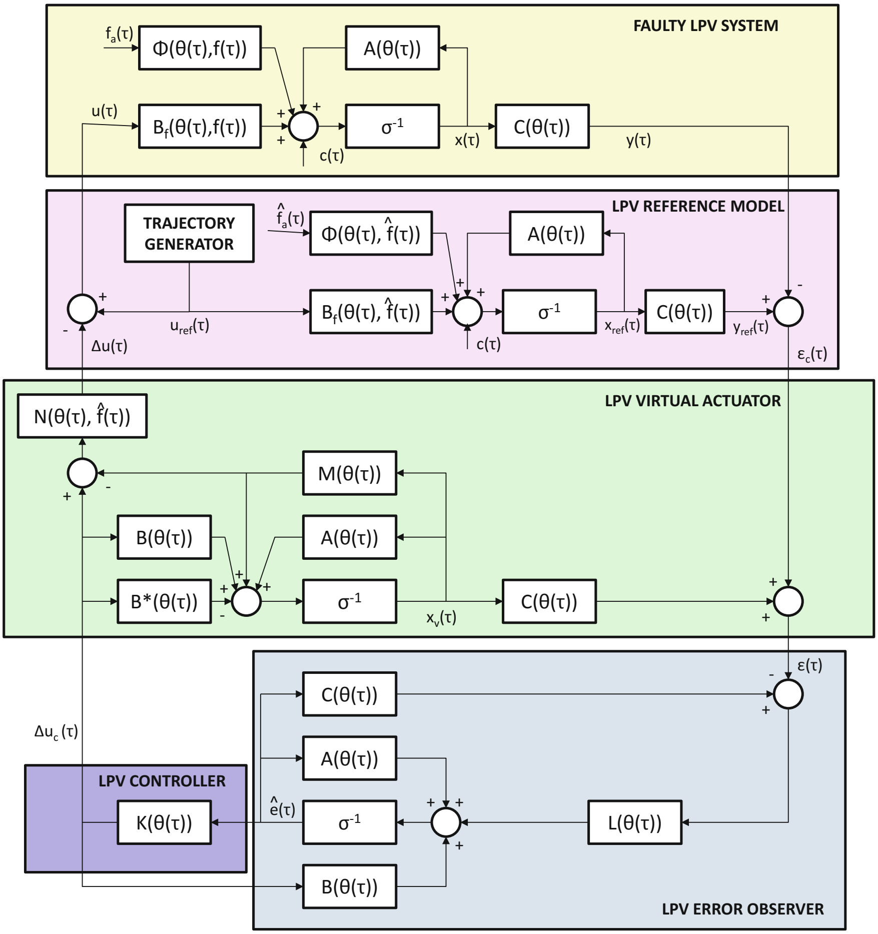


Fig. 8.1 Virtual actuator FTC scheme. After [2]

$$\begin{pmatrix} \sigma.\hat{e}(\tau) \\ \sigma.e(\tau) \\ \sigma.x_v(\tau) \end{pmatrix} = \begin{pmatrix} A+BK+LC & -LC & -LC \\ B^*K & A & -B^*M \\ (B-B^*)K & O & A+B^*M \end{pmatrix} \begin{pmatrix} \hat{e}(\tau) \\ e(\tau) \\ x_v(\tau) \end{pmatrix} \tag{8.22}$$

Then, there exists a similarity transformation such that the state matrix of the augmented system in the new state variables is block-triangular, as follows:

$$A_{aug}\left(\theta(\tau)\right) = \begin{pmatrix} A+LC & O & O \\ BK & A+BK & O \\ (B-B^*)K & (B-B^*)K & A+B^*M \end{pmatrix} \tag{8.23}$$

Proof The proof is straightforward, and comes from introducing the new state variables:

$$x_1(\tau) = \hat{e}(\tau) - x_v(\tau) - e(\tau) \tag{8.24}$$

$$x_2(\tau) = e(\tau) + x_v(\tau) \tag{8.25}$$

$$x_3(\tau) = x_v(\tau) \tag{8.26}$$

that correspond to a similarity transformation using the following change of basis matrix:

$$T = \begin{pmatrix} I & -I & -I \\ O & I & I \\ O & O & I \end{pmatrix}. \tag{8.27}$$

■

Looking at (8.23), it can be seen that the state $x_1(\tau)$ is affected by the matrix $L(\theta(\tau))$ through the matrix $A(\theta(\tau)) + L(\theta(\tau))\, C(\theta(\tau))$; the state $x_2(\tau)$ is influenced by the matrix $A(\theta(\tau)) + B(\theta(\tau))\, K(\theta(\tau))$; finally, the matrix $M(\theta(\tau))$ affects the behavior of the state $x_3(\tau)$ through the matrix $A(\theta(\tau)) + B^*(\theta(\tau))\, M(\theta(\tau))$. This means that, thanks to the reconfiguration of the reference model and the introduction of the virtual actuator, the nominal location of the poles of the closed-loop system and the error observer are not modified by the fault occurrence. Hence, the gains $K(\theta(\tau))$ and $L(\theta(\tau))$ do not need to be retuned, and the overall system is modified only by the additional poles introduced by the virtual actuator.

Remark: The location of the virtual actuator poles will have certain effects on the performance of the reconfigured system. In general, it is wished the virtual actuator to be faster than the controller. However, this specification is limited by the problem of actuator saturations and by the observer poles, that should be faster than the virtual actuator ones.

8.2.4 *Graceful Performance Degradation in Presence of Actuator Saturations*

Physical systems have maximum and minimum limits or saturations on their control signals and, as a consequence, the system input is different from the controller output. This difference is usually referred to as *controller windup* [17] and can result in a significant performance degradation, large overshoots and even instability, if saturations are not taken into account properly [12, 13].

An advantage of the model reference control strategy proposed in this chapter is that, by including the saturations in the reference model equations, it is possible to design an FTC system that automatically retunes the reference states whenever $u_{ref}(\tau)$ is such that the saturation nonlinearities become active. In fact, $u_{ref}(\tau)$ (or $u^c_{ref}(\tau)$

in nominal conditions) is usually calculated such that the reference model shows some desired behavior, e.g. some subset of the reference model states are driven to some desired steady-state values. In general, $u^c_{ref}(\tau)$ should be such that it does not activate the saturation nonlinearities when the system is working in nominal conditions. However, under fault occurrence, the reconfigured $u_{ref}(\tau)$ could be such that some saturation nonlinearities are activated. In this case, the desired performance is not achievable. The inclusion of the saturation nonlinearities in the reference model equations provides an elegant way to incorporate a graceful performance degradation in presence of actuator saturations.

More specifically, let $sat : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ be a saturation function that specifies the limited actuator capacity on the control input $u(\tau)$ in (8.1). The saturation is assumed to be a decoupled, sector-bounded, static actuator nonlinearity with a constant saturation limit u_i^{MAX} in the ith input, such that:

$$sat(u) = \begin{pmatrix} sat_1(u_1) \\ \vdots \\ sat_i(u_i) \\ \vdots \\ sat_{n_u}(u_{n_u}) \end{pmatrix} \quad sat_i(u_i) = \begin{cases} u_i^{MAX} & if\ u_i > u_i^{MAX} \\ u_i & if\ |u_i| \le u_i^{MAX} \\ -u_i^{MAX} & if\ u_i < -u_i^{MAX} \end{cases} \tag{8.28}$$

for $i = 1, \ldots, n_u$, where $u^{MAX} = \left(u_1^{MAX}, \ldots, u_{n_u}^{MAX}\right)^{\mathrm{T}} \in \mathbb{R}^{n_u}$ is a given vector with positive entries. Consequently, (8.9) is changed to:

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + B_f\left(\theta(\tau), f(\tau)\right)sat\left(u(\tau)\right) + \Phi\left(\theta(\tau), f(\tau)\right)f_a(\tau) + c(\tau) \tag{8.29}$$

Then, the reference model (8.12) is changed accordingly, as follows:

$$\sigma.x_{ref}(\tau) = A\left(\theta(\tau)\right)x_{ref}(\tau) + B_f\left(\theta(\tau), \hat{f}(\tau)\right)sat\left(u_{ref}(\tau)\right) + \Phi\left(\theta(\tau), \hat{f}(\tau)\right)\hat{f}_a(\tau) + c(\tau) \tag{8.30}$$

This modification of the reference model equation, under the assumption that $\hat{f}(\tau) \cong f(\tau)$ and $\hat{f}_a(\tau) \cong f_a(\tau)$, allows to write the error model as follows:

$$\sigma.e(\tau) = A\left(\theta(\tau)\right)e(\tau) + B_f\left(\theta(\tau), f(\tau)\right)\left(sat\left(u_{ref}(\tau)\right) - sat\left(u(\tau)\right)\right) \tag{8.31}$$

In order to assess stability or performance using LPV techniques, it is possible to apply a slight modification of the anti-windup control design approach proposed in [14], where the actuator saturation nonlinearities are directly taken into account by representing the status of each saturated actuator as a varying parameter. In particular, through the introduction of the following saturation scheduling parameter:

$$\varsigma_i\left(u_{i,ref}(t), u_i(t)\right) = \frac{sat_i\left(u_{i,ref}(\tau)\right) - sat_i\left(u_i(\tau)\right)}{u_{i,ref}(\tau) - u_i(\tau)} \tag{8.32}$$

the error model (8.31) becomes:

$$\sigma.e(\tau) = A\left(\theta_{\Sigma}(\tau)\right) e(\tau) + B_{\Sigma f}\left(\theta_{\Sigma}(\tau), f(\tau)\right) \Delta u(\tau) \tag{8.33}$$

where:

$$B_{\Sigma f}\left(\theta_{\Sigma}(\tau), f(\tau)\right) = B_{\Sigma}\left(\theta_{\Sigma}(\tau)\right) F\left(f(\tau)\right) \tag{8.34}$$

$$B_{\Sigma}\left(\theta_{\Sigma}(\tau)\right) = B\left(\theta(\tau)\right) diag\left(\varsigma_1\left(u_{1,ref}(\tau), u_1(\tau)\right), \ldots, \varsigma_{n_u}\left(u_{n_u,ref}(\tau), u_{n_u}(\tau)\right)\right) \tag{8.35}$$

with:

$$\theta_{\Sigma}(\tau) = \begin{pmatrix} \theta(\tau) \\ \varsigma_1\left(u_{1,ref}(\tau), u_1(\tau)\right) \\ \vdots \\ \varsigma_{n_u}\left(u_{n_u,ref}(\tau), u_{n_u}(\tau)\right) \end{pmatrix} \tag{8.36}$$

and:

$$\varsigma_i\left(u_{i,ref}(\tau), u_i(\tau)\right) = \begin{cases} 1 & if\ \left|u_{i,ref}(\tau)\right| < u_i^{MAX},\ \left|u_i(\tau)\right| < u_i^{MAX} \\ \frac{u_{i,ref}(\tau) - sign(u_i(\tau))u_i^{MAX}}{u_{i,ref}(\tau) - u_i(\tau)} & if\ \left|u_{i,ref}(\tau)\right| < u_i^{MAX},\ \left|u_i(\tau)\right| \geq u_i^{MAX} \\ \frac{sign\left(u_{i,ref}(\tau)\right)u_i^{MAX} - u_i(\tau)}{u_{i,ref}(\tau) - u_i(\tau)} & if\ \left|u_{i,ref}(\tau)\right| \geq u_i^{MAX},\ \left|u_i(\tau)\right| < u_i^{MAX} \\ \frac{\left[sign\left(u_{i,ref}(\tau)\right) - sign(u_i(\tau))\right]u_i^{MAX}}{u_{i,ref}(\tau) - u_i(\tau)} & if\ \left|u_{i,ref}(\tau)\right| \geq u_i^{MAX},\ \left|u_i(\tau)\right| \geq u_i^{MAX} \end{cases} \tag{8.37}$$

ς_i can take values between 0 and 1, where 1 corresponds to the case where both the system and the reference model work in the linear zone, and 0 corresponds to the case that both the system and the reference model are in the saturation zone with $u_i(\tau)$ and $u_{i,ref}(\tau)$ of the same sign.

8.2.5 Effects of the Fault Estimation Errors

Hereafter, the effects of the fault estimation errors over the FTC system, i.e. the more realistic case where $\hat{f}(\tau) \neq f(\tau)$ and $\hat{f}_a(\tau) \neq f_a(\tau)$, will be briefly discussed. By considering $f(\tau) = \hat{f}(\tau) + \Delta f(\tau)$ and $f_a(\tau) = \hat{f}_a(\tau) + \Delta f_a(\tau)$, where $\Delta f(\tau)$ and $\Delta f_a(\tau)$ are the uncertainties in the estimation of the multiplicative fault and the additive fault, respectively, then the faulty system (8.29) can be rewritten as:

$$\begin{aligned}\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + B_f\left(\theta(\tau), \hat{f}(\tau) + \Delta f(\tau)\right) sat\left(u(\tau)\right) \\ +\Phi\left(\theta(\tau), \hat{f}(\tau) + \Delta f(\tau)\right)\left(\hat{f}_a(\tau) + \Delta f_a(\tau)\right) + c(\tau)\end{aligned} \tag{8.38}$$

that, taking into account the reference model (8.30), under the assumption that:

$$\Phi\left(\theta(\tau), f(\tau)\right) = \Psi\left(\theta(\tau)\right)F\left(f(\tau)\right) \tag{8.39}$$

and by neglecting the terms arising of the type $\Delta f_i(\tau)\Delta f_{a,i}(\tau)$, can be brought to the following error model:

$$\begin{aligned}\sigma.e(\tau) = A\left(\theta_\Sigma(\tau)\right)e(\tau) + B_{\Sigma f}\left(\theta_\Sigma(\tau), f(\tau)\right)\Delta u(\tau) - \Phi\left(\theta_\Sigma(\tau), \hat{f}(\tau)\right)\Delta f_a(\tau) \\ -\left[B_{sat}\left(\theta_\Sigma(\tau), u(\tau)\right) + \Psi_f\left(\theta_\Sigma(\tau), \hat{f}_a(\tau)\right)\right]\Delta f(\tau)\end{aligned} \tag{8.40}$$

where $B_{\Sigma f}\left(\theta_\Sigma(\tau), f(\tau)\right)$ and $\theta_\Sigma(\tau)$ are defined as in (8.34)–(8.36) and:

$$B_{sat}\left(\theta_\Sigma(\tau), u(\tau)\right) = B\left(\theta_\Sigma(\tau)\right) diag\left(sat_1\left(u_1(\tau)\right), \ldots, sat_{n_u}\left(u_{n_u}(\tau)\right)\right) \tag{8.41}$$

$$\Psi_f\left(\theta_\Sigma(\tau), \hat{f}_a(\tau)\right) = \Psi\left(\theta_\Sigma(\tau)\right) diag\left(\hat{f}_{a1}(\tau), \ldots, \hat{f}_{an_u}(\tau)\right) \tag{8.42}$$

By considering the output equation (8.6), the LPV virtual actuator (8.18)–(8.19), the LPV error-feedback controller (8.7) and the LPV error observer (8.20), and by using the similarity transformation of Theorem 8.1, the following is obtained:

$$\begin{aligned}\begin{pmatrix}\sigma.x_1(\tau)\\ \sigma.x_2(\tau)\\ \sigma.x_3(\tau)\end{pmatrix} = \begin{pmatrix}A+LC & O & O\\ BK & A+BK & O\\ (B-B^*)K & (B-B^*)K & A+B^*M\end{pmatrix}\begin{pmatrix}x_1(\tau)\\ x_2(\tau)\\ x_3(\tau)\end{pmatrix} \\ +\begin{pmatrix}B_{sat}\left(\theta_\Sigma(\tau), u(\tau)\right) + \Psi_f\left(\theta_\Sigma(\tau), \hat{f}_a(\tau)\right) & \Phi\left(\theta_\Sigma(\tau), \hat{f}(\tau)\right)\\ -\left[B_{sat}\left(\theta_\Sigma(\tau), u(\tau)\right) + \Psi_f\left(\theta_\Sigma(\tau), \hat{f}_a(\tau)\right)\right] & -\Phi\left(\theta_\Sigma(\tau), \hat{f}(\tau)\right)\\ O & O\end{pmatrix}\begin{pmatrix}\Delta f(\tau)\\ \Delta f_a(\tau)\end{pmatrix}\end{aligned} \tag{8.43}$$

It can be seen that, taking advantage of the boundedness of $u(\tau)$, it would be possible to improve the robustness of the FTC system against the uncertainties in the multiplicative and additive fault estimations using perturbation rejection techniques, such as $\mathcal{H}_2 \backslash \mathcal{H}_\infty$ norm optimization.

8.3 Design Using LMIs

8.3.1 Properties of Block-Triangular LPV Systems

Let us consider the following block-triangular LPV system:

$$\begin{pmatrix} \sigma.x_1(\tau) \\ \sigma.x_2(\tau) \end{pmatrix} = \begin{pmatrix} A_{11}\,(\theta(\tau)) & O \\ A_{21}\,(\theta(\tau)) & A_{22}\,(\theta(\tau)) \end{pmatrix} \begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix} = A_{triang}\,(\theta(\tau)) \begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix} \tag{8.44}$$

and assume that:

- the subsystem:

$$\sigma.x_1(\tau) = A_{11}\,(\theta(\tau))\,x_1(\tau) \tag{8.45}$$

  is quadratically stable, that is, there exists $Q_1 \succ O$ such that (see Theorems 2.1–2.2):

$$Q_1 A_{11}(\theta)^{\mathrm{T}} + A_{11}(\theta) Q_1 \prec O \qquad \forall \theta \in \Theta \tag{8.46}$$

  or:

$$\begin{pmatrix} -Q_1 & A_{11}(\theta) Q_1 \\ Q_1 A_{11}(\theta)^{\mathrm{T}} & -Q_1 \end{pmatrix} \prec O \qquad \forall \theta \in \Theta \tag{8.47}$$

  for CT and DT systems, respectively.
- the subsystem obtained from (8.44) when $x_1(\tau) = 0$:

$$\sigma.x_2(\tau) = A_{22}\,(\theta(\tau))\,x_2(\tau) \tag{8.48}$$

  is quadratically stable, i.e. there exists $Q_2 \succ O$ such that:

$$Q_2 A_{22}(\theta)^{\mathrm{T}} + A_{22}(\theta) Q_2 \prec O \qquad \forall \theta \in \Theta \tag{8.49}$$

  or:

$$\begin{pmatrix} -Q_2 & A_{22}(\theta) Q_2 \\ Q_2 A_{22}(\theta)^{\mathrm{T}} & -Q_2 \end{pmatrix} \prec O \qquad \forall \theta \in \Theta \tag{8.50}$$

  for the CT and the DT case, respectively.

Then, an interesting question is whether the system (8.44) is quadratically stable too. In fact, even though its asymptotic stability is guaranteed by well-known results in the control systems theory [18], the quadratic stability is a stronger requirement, because it implies the existence of a common matrix $Q \succ O$ such that:

$$Q A_{triang}(\theta)^{\mathrm{T}} + A_{triang}(\theta) Q \prec O \qquad \forall \theta \in \Theta \tag{8.51}$$

or:

$$\begin{pmatrix} -Q & A_{triang}(\theta)Q \\ QA_{triang}(\theta)^{\mathrm{T}} & -Q \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{8.52}$$

In the remainder of this section, it is shown that if (8.46) and (8.49) hold, then there exists $Q \succ O$ such that (8.51) hold. Similarly, if (8.47) and (8.50) hold, then there exists $Q \succ O$ such that (8.52) hold. The proofs make use of the following lemma.

Lemma 8.1 *Given $Z \succ O$ and a matrix W of the same order, there exists $\kappa > 0$ such that $\kappa Z - W \succ O$.*

Proof Z has some minimum singular value σ_Z such that $\sigma_Z > 0$ and W has some maximum singular value σ_W. Also, for any non-zero vector v:

$$v^{\mathrm{T}}Zv \geq \|v\|^2 \sigma_Z \tag{8.53}$$

$$v^{\mathrm{T}}Wv \leq \|v\|^2 \sigma_W \tag{8.54}$$

So $v^{\mathrm{T}}(\kappa Z - W)v \geq \|v\|^2 (\kappa\sigma_Z - \sigma_W)$ and $\|v\|^2 (\kappa\sigma_Z - \sigma_W) > 0$ whenever $\kappa\sigma_Z \succ W$. Hence, from the definition of *positive definite matrix* results that $\kappa Z - W$ is positive definite. ■

Hence, the following theorems are true.

Theorem 8.2 (Quadratic stability of a block-triangular CT LPV system) *Given the block-triangular CT LPV system* (8.44) *with $\tau = t$, assume that there exist $Q_1 \succ O$ and $Q_2 \succ O$ such that* (8.46) *and* (8.49) *hold. Then, there exists $\kappa > 0$ such that* (8.51) *holds with:*

$$Q = \begin{pmatrix} Q_1 & O \\ O & \kappa Q_2 \end{pmatrix} \tag{8.55}$$

Proof Replacing (8.55) into (8.51) leads to the following condition:

$$\begin{pmatrix} Q_1A_{11}(\theta)^{\mathrm{T}} + A_{11}(\theta)Q_1 & Q_1A_{21}(\theta)^{\mathrm{T}} \\ A_{21}(\theta)Q_1 & \kappa\left(Q_2A_{22}(\theta)^{\mathrm{T}} + A_{22}(\theta)Q_2\right) \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{8.56}$$

Using Schur complements [19], (8.56) is equivalent to $Q_1A_{11}(\theta)^{\mathrm{T}} + A_{11}(\theta)Q_1 \prec O$, that holds due to (8.46), and:

$$\kappa\left(-Q_2A_{22}(\theta)^{\mathrm{T}} - A_{22}(\theta)Q_2\right) - A_{21}(\theta)Q_1\,\Xi_{CT}(\theta)Q_1A_{21}(\theta)^{\mathrm{T}} \succ O \qquad \forall\theta \in \Theta \tag{8.57}$$

with:

$$\Xi_{CT}(\theta) = \left(-Q_1A_{11}(\theta)^{\mathrm{T}} - A_{11}(\theta)Q_1\right)^{-1} \tag{8.58}$$

The application of Lemma 8.1, taking into account that $-Q_2A_{22}(\theta)^{\mathrm{T}} - A_{22}(\theta)Q_2 \succ O$ due to (8.49), completes the proof. ■

Theorem 8.3 (Quadratic stability of a block-triangular DT LPV system) *Given the block-triangular DT LPV system* (8.44) *with* $\tau = k$*, assume that there exist* $Q_1 \succ O$ *and* $Q_2 \succ O$ *such that* (8.47) *and* (8.50) *hold. Then, there exists* $\kappa > 0$ *such that* (8.52) *holds, with* Q *defined as in* (8.55).

Proof Replacing (8.55) into (8.52) leads to the following condition:

$$\begin{pmatrix} Q_1 & O & -A_{11}(\theta)Q_1 & O \\ O & \kappa Q_2 & -A_{21}(\theta)Q_1 & -\kappa A_{22}(\theta)Q_2 \\ -Q_1 A_{11}(\theta)^{\mathrm{T}} & -Q_1 A_{21}(\theta)^{\mathrm{T}} & Q_1 & O \\ \mathrm{O} & -\kappa Q_2 A_{22}(\theta)^{\mathrm{T}} & O & \kappa Q_2 \end{pmatrix} \succ O \qquad \forall \theta \in \Theta \tag{8.59}$$

Using Schur complements [19], (8.59) is equivalent to $Q \succ O$, which holds due to the positiveness of Q_1, Q_2 and κ, and:

$$\begin{pmatrix} Q_1 - A_{11}(\theta)Q_1 A_{11}(\theta)^{\mathrm{T}} & -A_{11}(\theta)Q_1 \mathrm{A}_{21}(\theta)^{\mathrm{T}} \\ -\mathrm{A}_{21}(\theta)Q_1 A_{11}(\theta)^{\mathrm{T}} & \Psi_{22}(\theta) \end{pmatrix} \succ O \qquad \forall \theta \in \Theta \tag{8.60}$$

with:

$$\Psi_{22}(\theta) = \kappa \left(Q_2 - A_{22}(\theta) Q_2 A_{22}(\theta)^{\mathrm{T}} \right) - A_{21}(\theta) Q_1 \mathrm{A}_{21}(\theta)^{\mathrm{T}} \tag{8.61}$$

Using again Schur complements [19], (8.60) is equivalent to (8.47) and:

$$\kappa \left(Q_2 - A_{22}(\theta) Q_2 A_{22}(\theta)^{\mathrm{T}} \right) - \mathrm{A}_{21}(\theta)\, \Xi_{DT}(\theta) \mathrm{A}_{21}(\theta)^{\mathrm{T}} \succ O \qquad \forall \theta \in \Theta \tag{8.62}$$

with:

$$\Xi_{DT}(\theta) = Q_1 A_{11}(\theta)^{\mathrm{T}} \left(Q_1 - A_{11}(\theta) Q_1 A_{11}(\theta)^{\mathrm{T}} \right)^{-1} A_{11}(\theta) + Q_1 \tag{8.63}$$

The application of Lemma 8.1, taking into account (8.50), completes the proof. ■

Remark: Notice that similar versions of Theorems 8.2–8.3 hold for LPV systems in an upper block-triangular form.

The results shown in Theorems 8.2–8.3 justify the separate design of the LPV error observer, the LPV error-feedback controller and the LPV virtual actuator, because the augmented model can be brought to a block-triangular form, as shown by Theorem 8.1. In fact, the quadratic stability of the augmented system (8.23) can be obtained from the quadratic stability of each subsystem. It is worth remarking that the separate design of each subsystem is more conservative than the design of the augmented system as a whole, due to the block-diagonality of Q in (8.55). However, the separate design has the indisputable advantage of simplicity, e.g. due to the possibility of reducing the design conditions into LMIs.

8.3.2 Overall FTC Scheme Design

The design of the overall FTC scheme implies the following:

- finding the N_Σ controller vertex gains K_i such that:

$$K\left(\theta_\Sigma(\tau)\right) = \sum_{i=1}^{N_\Sigma} \mu_i\left(\theta_\Sigma(\tau)\right) K_i \tag{8.64}$$

  guarantees the quadratic stability of $A\left(\theta_\Sigma(\tau)\right) + B_\Sigma\left(\theta_\Sigma(\tau)\right) K\left(\theta_\Sigma(\tau)\right)$ and some desired performance, under the assumption that the pair:

$$\begin{pmatrix} A\left(\theta_\Sigma(\tau)\right) \\ B_\Sigma\left(\theta_\Sigma(\tau)\right) \end{pmatrix} = \sum_{i=1}^{N_\Sigma} \mu_i\left(\theta_\Sigma(\tau)\right) \begin{pmatrix} A_i \\ B_{\Sigma,i} \end{pmatrix} \tag{8.65}$$

  is quadratically stabilizable in Θ_Σ;
- finding the N_Σ virtual actuator vertex gains M_i such that:

$$M\left(\theta_\Sigma(\tau)\right) = \sum_{i=1}^{N_\Sigma} \mu_i\left(\theta_\Sigma(\tau)\right) M_i \tag{8.66}$$

  guarantees the quadratic stability of $A\left(\theta_\Sigma(\tau)\right) + B^*_\Sigma\left(\theta_\Sigma(\tau)\right) M\left(\theta_\Sigma(\tau)\right)$ and some desired performance, under the assumption that the pair:

$$\begin{pmatrix} A\left(\theta_\Sigma(\tau)\right) \\ B^*_\Sigma\left(\theta_\Sigma(\tau)\right) \end{pmatrix} = \sum_{i=1}^{N_\Sigma} \mu_i\left(\theta_\Sigma(\tau)\right) \begin{pmatrix} A_i \\ B^*_{\Sigma,i} \end{pmatrix} \tag{8.67}$$

  is quadratically stabilizable in Θ_Σ;
- finding the N observer vertex gains L_i such that:

$$L\left(\theta(\tau)\right) = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right) L_i \tag{8.68}$$

  guarantees the quadratic stability of $A\left(\theta(\tau)\right) + L\left(\theta(\tau)\right) C\left(\theta(\tau)\right)$ and some desired performance, under the assumption that the pair:

$$\begin{pmatrix} A\left(\theta(\tau)\right) \\ C\left(\theta(\tau)\right) \end{pmatrix} = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right) \begin{pmatrix} A_i \\ C_i \end{pmatrix} \tag{8.69}$$

  is quadratically detectable in Θ. Notice that the quadratic detectability of a given system is equivalent to the quadratic stabilizability of the dual system.

Taking into account the design conditions presented in Sect. 2.5, it is possible to design the matrices K_i, M_i and L_i separately.

8.4 Application Examples

8.4.1 Application to a Twin Rotor MIMO System

8.4.1.1 Description of the Twin Rotor MIMO System

The twin rotor MIMO system (TRMS) is a laboratory aeromechanical system, developed by Feedback Instruments Ltd. for control experiments. The system is considered a challenging engineering problem due to its high nonlinearity, the presence of cross-coupling between its axes and inaccessibility of some of its states for measurements. In order to achieve satisfactory control objectives, an accurate model of the system is needed [20].

The TRMS is similar in its behavior to a helicopter. At both ends of its beam, there are two propellers driven by DC motors, each perpendicular to the other one. The beam can rotate freely in the horizontal and vertical planes, in such a way that its ends move on spherical surfaces. The joined beam can be moved by changing the motor supply voltages, thus controlling the rotational speed of the propellers. A counter-weight fixed to the beam is used for balancing the angular momentum in a stable equilibrium position. The rotor generating the vertical movement is called the main rotor. It enables the TRMS to pitch, which is a rotation around the horizontal axis. The rotor generating the horizontal movement is called the tail rotor. It enables the TRMS to yaw, which is a rotation in the horizontal plane around the vertical axis.

An accurate nonlinear model for the TRMS has been obtained by Rahideh and Shaheed [20] and further improved in [21], resulting in the following set of differential equations:

$$\dot{\omega}_h(t) = \frac{k_a k_1}{J_{tr} R_a} u_h(t) - \left(\frac{B_{tr}}{J_{tr}} + \frac{k_a^2}{J_{tr} R_a} \right) \omega_h(t) - \frac{f_1\left(\omega_h(t)\right)}{J_{tr}} \tag{8.70}$$

$$\begin{aligned}
\dot{\Omega}_h(t) = & \frac{l_t f_2\left(\omega_h(t)\right) \cos \alpha_v(t) - k_{oh} \Omega_h(t) - f_3\left(\alpha_h(t)\right) + f_6\left(\alpha_v(t)\right)}{K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F} \\
& + \frac{k_m \cos \alpha_v(t) \left[k_a k_2 u_v(t)/R_a - \left(B_{mr} + k_a^2/R_a \right) \omega_v(t) - f_4\left(\omega_v(t)\right) \right]}{J_{mr} \left(K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F \right)} \\
& + \frac{k_m \omega_v(t) \sin \alpha_v(t) \Omega_v(t) \left(K_D \cos^2 \alpha_v(t) - K_E \sin^2 \alpha_v(t) - K_F - 2K_E \cos^2 \alpha_v(t) \right)}{\left(K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F \right)^2}
\end{aligned} \tag{8.71}$$

$$\dot{\alpha}_h(t) = \Omega_h(t) \tag{8.72}$$

$$\dot{\omega}_v(t) = \frac{k_a k_2}{J_{mr} R_a} u_v(t) - \left(\frac{B_{mr}}{J_{mr}} + \frac{k_a^2}{J_{mr} R_a} \right) \omega_v(t) - \frac{f_4\left(\omega_v(t)\right)}{J_{mr}} \tag{8.73}$$

$$\begin{aligned}\dot{\Omega}_v(t) = &\frac{l_m f_5\left(\omega_v(t)\right) + k_g \Omega_h(t) f_5\left(\omega_v(t)\right) \cos\alpha_v(t) - k_{ov}\Omega_v(t)}{J_v} \\ &+ \frac{g\left[(K_A - K_B)\cos\alpha_v(t) - K_C \sin\alpha_v(t)\right] - \Omega_h(t)^2 K_H \sin\alpha_v(t)\cos\alpha_v(t)}{J_v} \\ &+ \frac{k_t\left[k_a k_1 u_h(t)/R_a - \left(B_{tr} + k_a^2/R_a\right)\omega_h(t) - f_1\left(\omega_h(t)\right)\right]}{J_v J_{tr}}\end{aligned} \tag{8.74}$$

$$\dot{\alpha}_v(t) = \Omega_v(t) \tag{8.75}$$

where u_h and u_v are the input voltage of the tail and main motor, respectively, ω_h and ω_v are the rotational velocity of the tail and main rotor, respectively, and Ω_h and Ω_v are the angular velocity of the TRMS for the yaw and the pitch angle, respectively. Finally, α_h is the yaw angle of the beam, and α_v is the pitch angle of the beam.

The nonlinear functions $f_i(\cdot)$ that take into account the frictions and coupling effects between the horizontal and the vertical dynamics, are defined as:

$$f_1\left(\omega_h(t)\right) = \begin{cases} k_{thp}\omega_h(t)^2 & if\ \omega_h(t) \geq 0 \\ -k_{thn}\omega_h(t)^2 & if\ \omega_h(t) < 0 \end{cases}$$

$$f_2\left(\omega_h(t)\right) = \begin{cases} k_{fhp}\omega_h(t)^2 & if\ \omega_h(t) \geq 0 \\ -k_{fhn}\omega_h(t)^2 & if\ \omega_h(t) < 0 \end{cases}$$

$$f_3\left(\alpha_h(t)\right) = \begin{cases} k_{chp}\alpha_h(t) & if\ \alpha_h(t) \geq 0 \\ k_{chn}\alpha_h(t) & if\ \alpha_h(t) < 0 \end{cases}$$

$$f_4\left(\omega_v(t)\right) = \begin{cases} k_{tvp}\omega_v(t)^2 & if\ \omega_v(t) \geq 0 \\ -k_{tvn}\omega_v(t)^2 & if\ \omega_v(t) < 0 \end{cases}$$

$$f_5\left(\omega_v(t)\right) = \begin{cases} k_{fvp}\omega_v(t)^2 & if\ \omega_v(t) \geq 0 \\ -k_{fvn}\omega_v(t)^2 & if\ \omega_v(t) < 0 \end{cases}$$

$$f_6\left(\alpha_v(t)\right) = \begin{cases} k_{cvp}\left(\alpha_v(t) - \alpha_v^0\right)^2 & if\ \alpha_v(t) \geq \alpha_v^0 \\ k_{cvn}\left(\alpha_v(t) - \alpha_v^0\right)^2 & if\ \alpha_v(t) < \alpha_v^0 \end{cases}$$

where α_v^0 is the equilibrium point for the pitch angle, corresponding to $u_v = 0$. For a complete description of the TRMS parameters, and their values, see Table 8.1.

Table 8.1 TRMS parameters description and values

Param.	Description	Value
B_{mr}	Viscous friction coefficient of the main propeller	0.0026 [Ω^{-1}]
B_{tr}	Viscous friction coefficient of the tail propeller	0.0086 [Ω^{-1}]
g	Gravitational acceleration at sea level	9.81 [ms^{-2}]
J_{mr}	Moment of inertia of the main propeller	0.0254 [$\mathrm{kg\,m}^2$]
J_{tr}	Moment of inertia of the tail propeller	0.0059 [$\mathrm{kg\ m}^2$]
J_v	Vertical moment of inertia	0.0643 [$\mathrm{kg\,m}^2$]
K_A	Physical constant	0.0980 [kg m]
K_B	Physical constant	0.1137 [kg m]
K_C	Physical constant	0.0220 [kg m]
K_D	Physical constant	0.0553 [$\mathrm{kg\,m}^2$]
K_E	Physical constant	0.0058 [$\mathrm{kg\,m}^2$]
K_F	Physical constant	0.0059 [$\mathrm{kg\,m}^2$]
K_H	Physical constant	0.0591 [$\mathrm{kg\,m}^2$]
k_1	Input constant of the tail motor	6.5
k_2	Input constant of the main motor	8.5
k_a	Torque constant of the DC motors	0.0202
k_{chn}	Cable force coefficient for $\alpha_h < 0$	0.0111 [$\mathrm{kg\,m}^{-2}\mathrm{s}^{-2}$]
k_{chp}	Cable force coefficient for $\alpha_h \geq 0$	0.0158 [$\mathrm{kg\,m}^{-2}\mathrm{s}^{-2}$]
k_{cvn}	Coupling coefficient for $\alpha_v < \alpha_v^0$	0.0563 [$\mathrm{kg\,m}^{-2}\mathrm{s}^{-2}$]
k_{cvp}	Coupling coefficient for $\alpha_v \geq \alpha_v^0$	0.0623 [$\mathrm{kg\,m}^{-2}\mathrm{s}^{-2}$]
k_{fhn}	Aerodynamic force coefficient of the tail rotor for $\omega_h < 0$	0.0660 [$\mathrm{kg\,m\,s}^{-2}\mathrm{V}^{-2}$]
k_{fhp}	Aerodynamic force coefficient of the tail rotor for $\omega_h \geq 0$	0.0566 [$\mathrm{kg\,m\,s}^{-2}\mathrm{V}^{-2}$]
k_{fvn}	Aerodynamic force coefficient of the main rotor for $\omega_v < 0$	0.2197 [$\mathrm{kg\,m\,s}^{-2}\mathrm{V}^{-2}$]
k_{fvp}	Aerodynamic force coefficient of the main rotor for $\omega_v \geq 0$	0.3819 [$\mathrm{kg\,m\,s}^{-2}\mathrm{V}^{-2}$]
k_g	Gyroscopic constant	0.2 [m s]
k_m	Physical constant	0.0017 [$\mathrm{kg\,m}^2\mathrm{s}^{-1}\mathrm{V}^{-1}$]
k_{oh}	Horizontal friction coefficient of the beam subsystem	0.0185 [$\mathrm{kg\,m}^2\mathrm{s}^{-1}$]
k_{ov}	Vertical friction coefficient of the beam subsystem	0.1026 [$\mathrm{kg\,m}^2\mathrm{s}^{-1}$]
k_t	Physical constant	0 [$\mathrm{kg\,m}^2\mathrm{s}^{-1}\mathrm{V}^{-1}$]
k_{thn}	Drag friction coefficient of the tail propeller for $\omega_h < 0$	0.0028 [$\mathrm{V}^{-1}\Omega^{-1}$]
k_{thp}	Drag friction coefficient of the tail propeller for $\omega_h \geq 0$	0.0027 [$\mathrm{V}^{-1}\Omega^{-1}$]
k_{tvn}	Drag friction coefficient of the main propeller for $\omega_v < 0$	0.0155 [$\mathrm{V}^{-1}\Omega^{-1}$]
k_{tvp}	Drag friction coefficient of the main propeller for $\omega_v \geq 0$	0.0168 [$\mathrm{V}^{-1}\Omega^{-1}$]
l_m	Length of the main part of the beam	0.246 [m]
l_t	Length of the tail part of the beam	0.282 [m]
R_a	Armature resistance of the DC motors	8 [Ω]

8.4.1.2 Quasi-LPV Error Model

In the following, only the problem of controlling the yaw angle α_h will be considered. The reason for this choice is that there exists a coupling between the main motor and the yaw angle of the beam ($k_m \neq 0$), so that the yaw angle can be controlled to some desired value despite the complete loss of one of the two motors. On the other hand, the same is not true in the case of the pitch angle α_v, because it is driven only by the main motor ($k_t = 0$).

As a consequence, the problem of the FTC of the yaw angle of the TRMS allows showing some features of the proposed methodology that could not be shown if the control of the pitch angle was also considered. In particular, it will be shown that the proposed methodology can tolerate complete losses of actuators, as long as there is sufficient actuator redundancy in the controlled system.

In order to obtain a quasi-LPV error model for the TRMS, the first step is to reshape the nonlinear equations (8.70)–(8.73) into the quasi-LPV form (8.1), as follows:

$$\begin{pmatrix} \dot{\omega}_h(t) \\ \dot{\Omega}_h(t) \\ \dot{\alpha}_h(t) \\ \dot{\omega}_v(t) \end{pmatrix} = \begin{pmatrix} \theta_1(t) & 0 & 0 & 0 \\ \theta_2(t) & \theta_3(t) & \theta_4(t) & \theta_5(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \theta_6(t) \end{pmatrix} \begin{pmatrix} \omega_h(t) \\ \Omega_h(t) \\ \alpha_h(t) \\ \omega_v(t) \end{pmatrix} + \begin{pmatrix} b_{11} & 0 \\ 0 & \theta_7(t) \\ 0 & 0 \\ 0 & b_{42} \end{pmatrix} \begin{pmatrix} u_h(t) \\ u_v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ c_2(t) \\ 0 \\ 0 \end{pmatrix}$$

where $\theta(t) = (\theta_1(t), \ldots, \theta_7(t))^{\mathrm{T}}$ is the vector of varying parameters, scheduled by the state variables $\omega_h(t)$, $\alpha_h(t)$, $\omega_v(t)$ and the exogenous variable (with respect to the considered control problem) $\alpha_v(t)$, with:

$$\theta_1(t) = -\frac{k_a^2/R_a + B_{tr} + g_1(\omega_h(t))}{J_{tr}}$$

$$\theta_2(t) = \frac{l_t g_2(\omega_h(t)) \cos \alpha_v(t)}{K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F}$$

$$\theta_3(t) = -\frac{k_{oh}}{K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F}$$

$$\theta_4(t) = \frac{g_3(\alpha_h(t))}{K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F}$$

$$\theta_5(t) = \frac{k_m \cos \alpha_v(t) \left(k_a^2/R_a + B_{mr} + g_4(\omega_v(t))\right)}{J_{mr} \left(K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F\right)}$$

$$\theta_6(t) = -\frac{k_a^2/R_a + B_{mr} + g_4(\omega_v(t))}{J_{mr}}$$

$$\theta_7(t) = \frac{k_m \cos \alpha_v(t) k_a k_2}{R_a J_{mr} \left(K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F \right)}$$

$$b_{11} = \frac{k_a k_1}{J_{tr} R_a}$$

$$b_{42} = \frac{k_a k_2}{J_{mr} R_a}$$

$$c_2(t) = \frac{f_6\left(\alpha_v(t)\right)}{K_D \cos^2 \alpha_v(t) + K_E \sin^2 \alpha_v(t) + K_F}$$

where the functions $g_i(\cdot)$ are given by:

$$g_1\left(\omega_h(t)\right) = \begin{cases} k_{thp}\omega_h(t) & \text{if } \omega_h(t) \geq 0 \\ -k_{thn}\omega_h(t) & \text{if } \omega_h(t) < 0 \end{cases}$$

$$g_2\left(\omega_h(t)\right) = \begin{cases} k_{fhp}\omega_h(t) & \text{if } \omega_h(t) \geq 0 \\ -k_{fhn}\omega_h(t) & \text{if } \omega_h(t) < 0 \end{cases}$$

$$g_3\left(\alpha_h(t)\right) = \begin{cases} k_{chp} & \text{if } \alpha_h(t) \geq 0 \\ k_{chn} & \text{if } \alpha_h(t) < 0 \end{cases}$$

$$g_4\left(\omega_v(t)\right) = \begin{cases} k_{tvp}\omega_v(t) & \text{if } \omega_v(t) \geq 0 \\ -k_{tvn}\omega_v(t) & \text{if } \omega_v(t) < 0 \end{cases}$$

$$g_6\left(\alpha_v(t)\right) = \begin{cases} k_{cvp}\left(\alpha_v(t) - \alpha_v^0\right) & \text{if } \alpha_v(t) \geq \alpha_v^0 \\ k_{cvn}\left(\alpha_v(t) - \alpha_v^0\right) & \text{if } \alpha_v(t) < \alpha_v^0 \end{cases}$$

Then, using the reference model (8.5), and by defining the tracking errors $e_h(t) \triangleq \omega_h^{ref}(t) - \omega_h(t)$, $e_\Omega(t) \triangleq \Omega_h^{ref}(t) - \Omega_h(t)$, $e_\alpha(t) \triangleq \alpha_h^{ref}(t) - \alpha_h(t)$ and $e_v(t) \triangleq \omega_v^{ref}(t) - \omega_v(t)$, and the new inputs $\Delta u_h(t) \triangleq u_h^{ref}(t) - u_h(t)$, $\Delta u_v(t) \triangleq u_v^{ref}(t) - u_v(t)$, the following DT quasi-LPV error model can be obtained through an Euler approximation [22] with a sampling time $T_s = 0.01\,s$:

$$\begin{pmatrix} e_h(k+1) \\ e_\Omega(k+1) \\ e_\alpha(k+1) \\ e_v(k+1) \end{pmatrix} = \begin{pmatrix} \theta_1^d(k) & 0 & 0 & 0 \\ \theta_2^d(k) & \theta_3^d(k) & \theta_4^d(k) & \theta_5^d(k) \\ 0 & T_s & 1 & 0 \\ 0 & 0 & 0 & \theta_6^d(k) \end{pmatrix} \begin{pmatrix} e_h(k) \\ e_\Omega(k) \\ e_\alpha(k) \\ e_v(k) \end{pmatrix} + \begin{pmatrix} b_{11}^d & 0 \\ 0 & \theta_7^d(k) \\ 0 & 0 \\ 0 & b_{42}^d \end{pmatrix} \begin{pmatrix} \Delta u_h(t) \\ \Delta u_v(t) \end{pmatrix}$$

where $\theta_1^d(k) = 1 + T_s\theta_1(k)$, $\theta_2^d(k) = T_s\theta_2(k)$, $\theta_3^d(k) = 1 + T_s\theta_3(k)$, $\theta_4^d(k) = T_s\theta_4(k)$, $\theta_5^d(k) = T_s\theta_5(k)$, $\theta_6^d(k) = 1 + T_s\theta_6(k)$, $\theta_7^d(k) = T_s\theta_7(k)$, $b_{11}^d = T_s b_{11}$ and $b_{42}^d = T_s b_{42}$.

Finally, the saturations have been taken into account following the approach proposed in Sect. 8.2.4, and a polytopic model with 512 vertices has been obtained using the nonlinear embedding approach [23] (see Chap. 3).

8.4.1.3 Reference Input Calculation and Design of the FTC Scheme

In the following, actuator faults in the tail and the main motors are considered. These faults cause the following changes in (8.70), (8.71), (8.73) and (8.74):

$$u_h(t) \rightarrow f_h(t)\left(u_h(t) + f_{ah}(t)\right)$$
$$u_v(t) \rightarrow f_v(t)\left(u_v(t) + f_{av}(t)\right)$$

where f_h, f_{ah}, f_v, f_{av} are the multiplicative and additive faults in the tail and the main motor, respectively.

Consequently, the reference model is changed as described in Sect. 8.2.3, using the fault estimations $\hat{f}_h(t), \hat{f}_{ah}(t), \hat{f}_v(t)$ and $\hat{f}_{av}(t)$. Under the assumption that $\hat{f}_h(t) \cong f_h(t)$, $\hat{f}_{ah}(t) \cong f_{ah}(t)$, $\hat{f}_v(t) \cong f_v(t)$ and $\hat{f}_{av}(t) \cong f_{av}(t)$, the error model input matrix becomes:

$$B_f\left(\theta_7^d(k), f_h(k), f_v(k)\right) = \begin{pmatrix} f_h(k)b_{11}^d & 0 \\ 0 & f_v(k)\theta_7^d(k) \\ 0 & 0 \\ 0 & f_v(k)b_{42}^d \end{pmatrix}$$

In order to drive the TRMS to a desired yaw angle $\alpha_h^{des}(t)$, it is required to choose properly $u_h^{ref}(t)$ and $u_v^{ref}(t)$. These values can be obtained from the TRMS nonlinear model (8.70)–(8.75) by imposing all the derivatives equal to zero and $\alpha_h(t) = \alpha_h^{des}(t)$. This leads to the following solution:

$$f_2^{des}\left(\alpha_h^{des}(t), \alpha_v(t)\right) = \frac{f_3\left(\alpha_h^{des}(t)\right) - f_6\left(\alpha_v(t)\right)}{l_t \cos\alpha_v(t)}$$

$$f_5^{des}\left(\alpha_v(t)\right) = \frac{g\left[(K_B - K_A)\cos\alpha_v(t) + K_C \sin\alpha_v(t)\right]}{l_m}$$

$$\omega_h^{des}\left(\alpha_h^{des}(t), \alpha_v(t)\right) = \begin{cases} \sqrt{\frac{f_2^{des}\left(\alpha_h^{des}(t), \alpha_v(t)\right)}{k_{fhp}}} & if\ f_2^{des}\left(\alpha_h^{des}(t), \alpha_v(t)\right) \geq 0 \\ -\sqrt{\frac{-f_2^{des}\left(\alpha_h^{des}(t), \alpha_v(t)\right)}{k_{fhn}}} & if\ f_2^{des}\left(\alpha_h^{des}(t), \alpha_v(t)\right) < 0 \end{cases}$$

$$\omega_v^{des}(\alpha_v(t)) = \begin{cases} \sqrt{\frac{f_5^{des}(\alpha_v(t))}{k_{fvp}}} & if\ f_5^{des}(\alpha_v(t)) \geq 0 \\ -\sqrt{\frac{-f_5^{des}(\alpha_v(t))}{k_{fvn}}} & if\ f_5^{des}(\alpha_v(t)) < 0 \end{cases}$$

$$u_h^{ref}\left(\alpha_h^{des}(t), \alpha_v(t), \hat{f}_h, \hat{f}_{ah}(t)\right) = \frac{1}{\hat{f}_h}\frac{\left(B_{tr}R_a + k_a^2\right)\omega_h^{des}\left(\alpha_h^{des}(t), \alpha_v(t)\right)}{k_a k_1} + \frac{1}{\hat{f}_h}\frac{f_1\left(\omega_h^{des}\left(\alpha_h^{des}(t), \alpha_v(t)\right)\right)R_a}{k_a k_1} - \hat{f}_{ah}(t)$$

$$u_v^{ref}\left(\alpha_v(t), \hat{f}_v, \hat{f}_{av}(t)\right) = \frac{1}{\hat{f}_v}\frac{\left(B_{mr}R_a + k_a^2\right)\omega_v^{des}(\alpha_v(t)) + f_4\left(\omega_v^{des}(\alpha_v(t))\right)R_a}{k_a k_2} - \hat{f}_{av}(t)$$

Notice that for a given desired yaw angle $\alpha_h^{des}(t)$, infinite couples $\left(u_h^{ref}(t), u_v^{ref}(t)\right)$ could be obtained. In particular, the correspondence between $\alpha_v(t)$ and the couples $\left(u_h^{ref}(t), u_v^{ref}(t)\right)$ is a bijection. In the following, among all the possible couples $\left(u_h^{ref}(t), u_v^{ref}(t)\right)$, the one that corresponds to the minimum Euclidean norm is chosen, as follows[2]:

$$\bar{u}_h^{ref}, \bar{u}_v^{ref} : \min_{\alpha_v \in [\underline{\alpha}_v, \overline{\alpha}_v]} \|u_{ref}\|_2^2 = \min_{\alpha_v \in [\underline{\alpha}_v, \overline{\alpha}_v]} \left[\left(u_h^{ref}\right)^2 + \left(u_v^{ref}\right)^2\right]$$

Remark: In case of a complete loss of the tail rotor, $u_h = 0$, and u_h^{ref} is chosen as the value corresponding to $\alpha_v = \alpha_v^0$.

Remark: A normally distributed noise with zero mean and standard deviation 0.02 has been added to u_h^{ref} and u_v^{ref} in order to assure the excitation needed by the fault estimation algorithm.

The controller and the virtual actuators (one for each possible complete loss of actuator) have been designed considering the specifications of quadratic stability (Corollary 2.13) and quadratic $\mathcal{D}$-stability (Corollary 2.15), considering as LMI region, the half-plane with minimum abscissa $\lambda = 0.7$, such that (2.168) is particularized as follows:

$$1.4Q - He\{A_i Q + B\Gamma_i\} \prec 0 \qquad i = 1, \ldots, N$$

On the other hand, the observer design has been performed considering the disk of radius $r = 0.3$ and center $(-q, 0) = (0.5, 0)$, such that (2.168) becomes:

[2]The pitch angle interval for searching the reference input values has been chosen with $\underline{\alpha}_v = \alpha_v^0 - 0.2$ and $\overline{\alpha}_v = \alpha_v^0 + 0.3$.

$$\begin{pmatrix} -0.3Q & -0.5Q + A_i^{\mathrm{T}}Q + C^{\mathrm{T}}\Gamma_i \\ -0.5Q + QA_i + \Gamma_i^{\mathrm{T}}C & -0.3Q \end{pmatrix} \prec 0 \qquad i = 1, \ldots, N$$

The LMIs have been solved using the YALMIP toolbox [24] and the SeDuMi solver [25].

8.4.1.4 Results

The results shown in this section refer to simulations that last 120 s, in which the TRMS should reach and maintain the desired yaw angle $\alpha_h^{des} = 0.5$ rad despite the initial error, due to the difference between the TRMS initial state:

$$\begin{pmatrix} x(0) \\ \Omega_v(0) \\ \alpha_v(0) \end{pmatrix} = \begin{pmatrix} \omega_h(0) \\ \Omega_h(0) \\ \alpha_h(0) \\ \omega_v(0) \\ \Omega_v(0) \\ \alpha_v(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \alpha_v^0 + 0.8 \end{pmatrix}$$

and the reference model initial state:

$$x_{ref}(0) = \begin{pmatrix} \omega_h^{ref}(0) \\ \Omega_h^{ref}(0) \\ \alpha_h^{ref}(0) \\ \omega_v^{ref}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the faults. In particular, a fault scenario with the values of $f_h(t), f_{ah}(t), f_v(t)$ and $f_{av}(t)$ resumed in Table 8.2 is considered.

The nominal closed-loop system response is shown in Figs. 8.2, 8.3, 8.4 and 8.5, where the TRMS states are compared with the reference model ones. At steady-state, the error is zero, and the system reaches the desired yaw angle $\alpha_h^{des} = 0.5$ rad, as expected. Figure 8.6 depicts the response of the pitch angle α_v, and Fig. 8.7 shows the control inputs. It can be seen that after a short transient, the control inputs converge to the reference ones.

Table 8.2 Fault scenario description

	0–30 s	30–60 s	60–90 s	90–120 s
$f_h(t)$	1	0.5	0.5	0
$f_{ah}(t)$	0	0	0	0
$f_v(t)$	1	1	0.7	0.6
$f_{av}(t)$	0	0	0.1	0

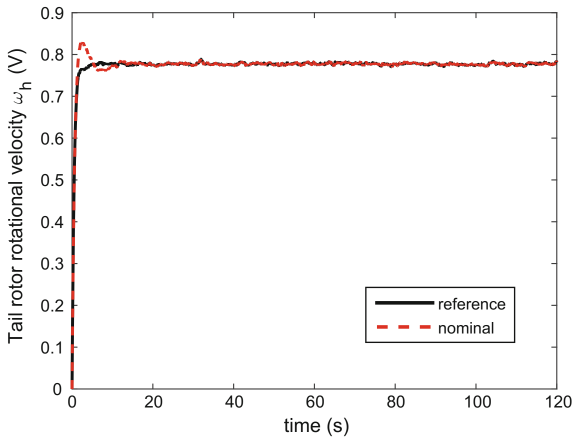


Fig. 8.2 Nominal tail rotor response (comparison between TRMS and reference model states). After [2]

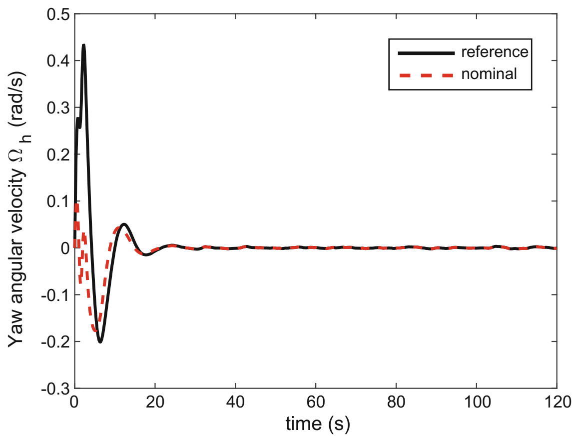


Fig. 8.3 Nominal yaw angular velocity response (comparison between TRMS and reference model states). After [2]

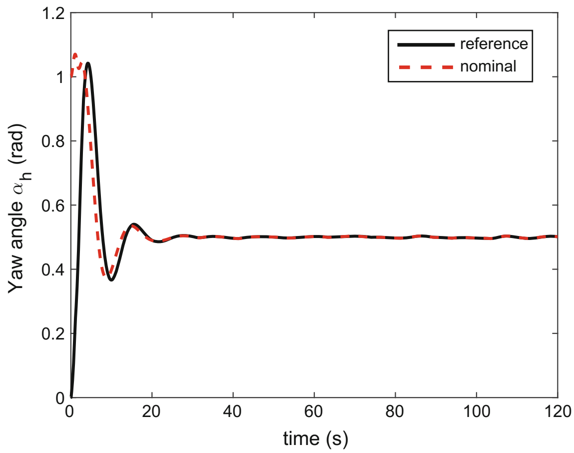


Fig. 8.4 Nominal yaw angle response (comparison between TRMS and reference model states). After [2]

Fig. 8.5 Nominal main rotor response (comparison between TRMS and reference model states). After [2]

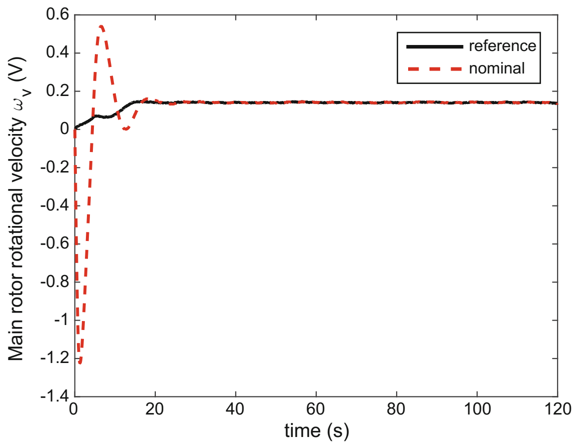


Fig. 8.6 Nominal pitch angle response. After [2]

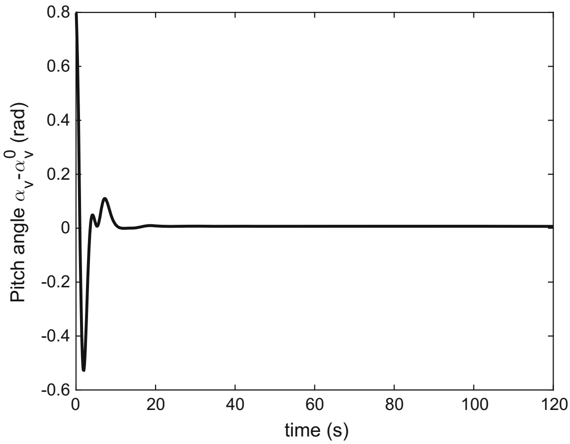
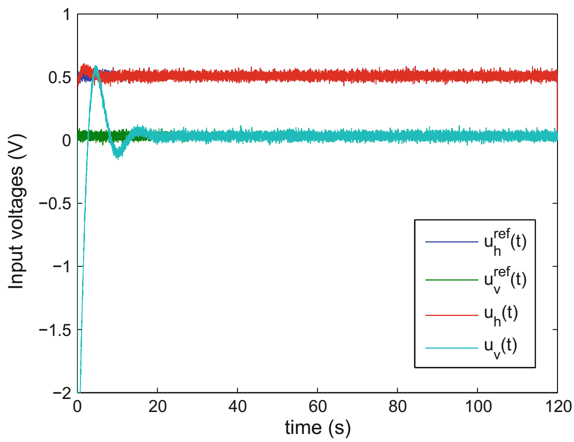


Fig. 8.7 Nominal control inputs. After [2]



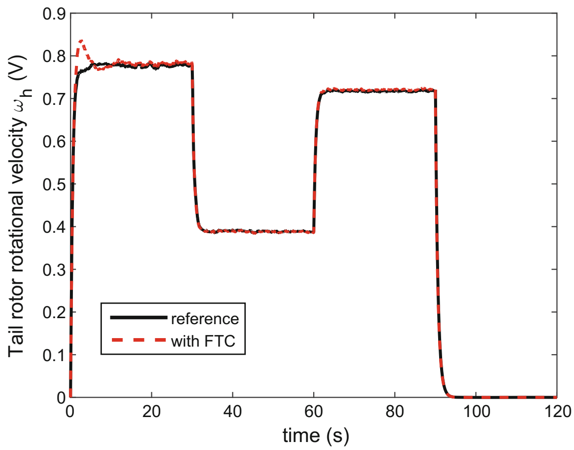


Fig. 8.8 Faulty tail rotor response (comparison between TRMS and reference model states, with FTC). After [2]

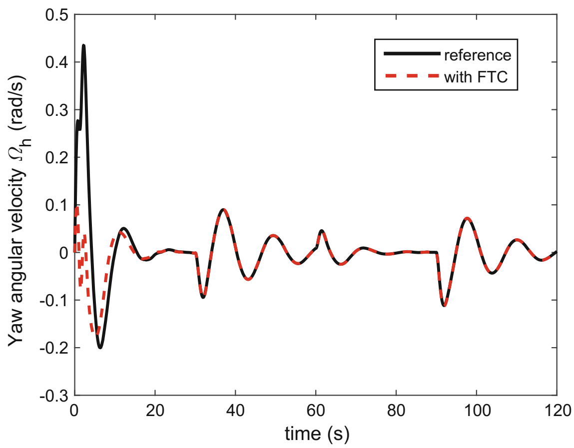


Fig. 8.9 Faulty yaw angular velocity response (comparison between TRMS and reference model states, with FTC). After [2]

The proposed FTC strategy allows the system to reach the desired yaw angle $\alpha_h^{des} = 0.5$ rad in all cases except the one of complete loss of the tail actuator (see Figs. 8.8, 8.9, 8.10 and 8.11). In this case, the reference is automatically changed to the biggest value of α_h achievable within the following range of variation of the pitch angle, $\alpha_v \in [\alpha_v^0 - 0.2\,\text{rad}, \alpha_v^0 + 0.3\,\text{rad}]$, as shown in Fig. 8.10.

It should be pointed out that the oscillations that appear in the yaw angle response are mainly due to the changes in the working point of the pitch angle, as shown in Fig. 8.12. In order to conclude the presentation of the results with FTC, the control inputs are shown in Fig. 8.13.

The responses obtained in the case where the proposed FTC strategy is not applied are illustrated in Figs. 8.14, 8.15, 8.16 and 8.17. It can be seen that if the faults are not taken into account properly, some errors between the reference states and the system

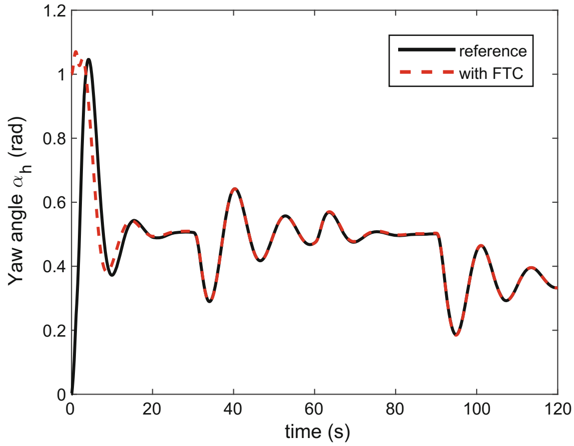


Fig. 8.10 Faulty yaw angle response (comparison between TRMS and reference model states, with FTC). After [2]

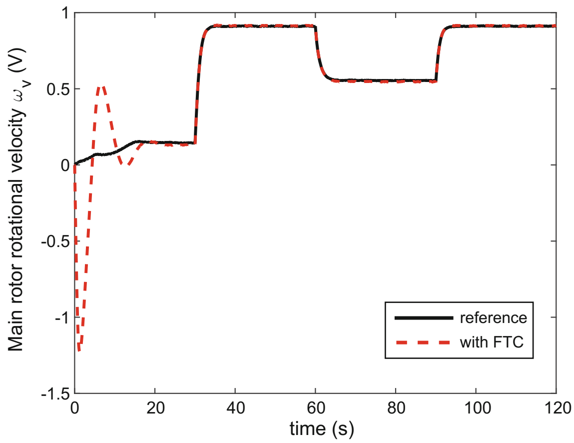


Fig. 8.11 Faulty main rotor response (comparison between TRMS and reference model states, with FTC). After [2]

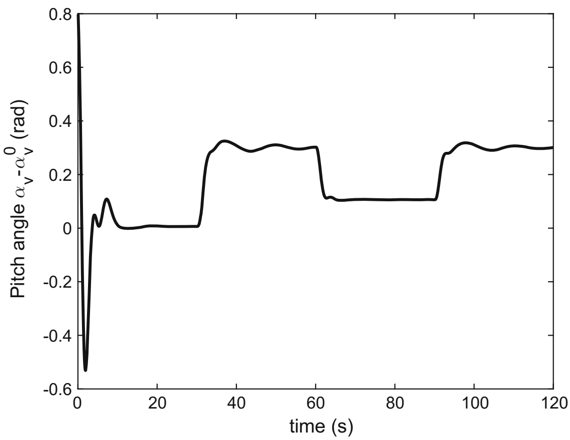


Fig. 8.12 Faulty pitch angle response (with FTC). After [2]

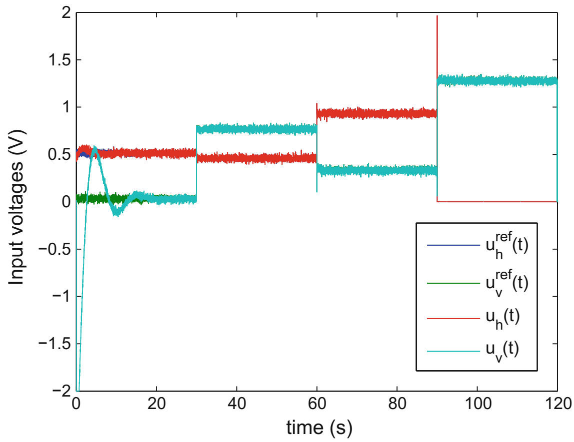


Fig. 8.13 Faulty control inputs (with FTC). After [2]

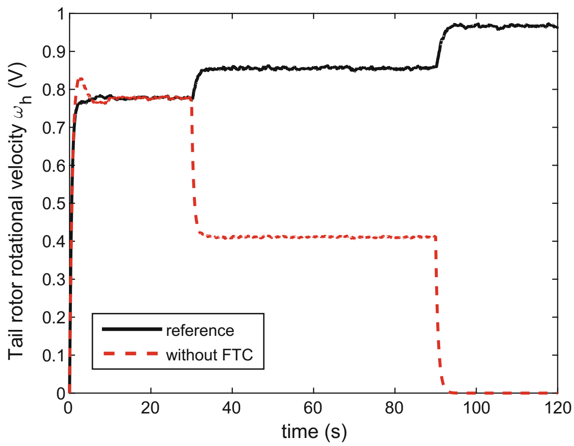


Fig. 8.14 Faulty tail rotor response (comparison between TRMS and reference model states, without FTC). After [2]

states appear. Moreover, the same reference model pitch angle α_h^{ref} is affected by the fault occurrence, such that α_h^{ref} does not converge to $\alpha_h^{des} = 0.5$ rad in steady-state.

8.4.2 Application to a Four Wheeled Omnidirectional Mobile Robot

8.4.2.1 Description of the Four Wheeled Omnidirectional Mobile Robot

Omnidirectional mobile robots are gaining popularity due to their enhanced mobility with respect to traditional robots [26]. The omnidirectional feature provides a great

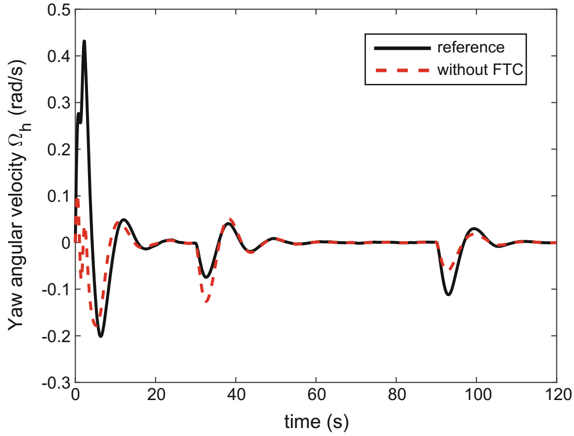


Fig. 8.15 Faulty yaw angular velocity response (comparison between TRMS and reference model states, without FTC). After [2]

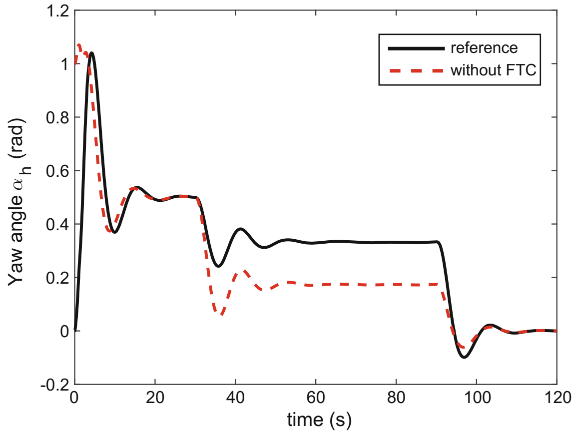


Fig. 8.16 Faulty yaw angle response (comparison between TRMS and reference model states, without FTC). After [2]

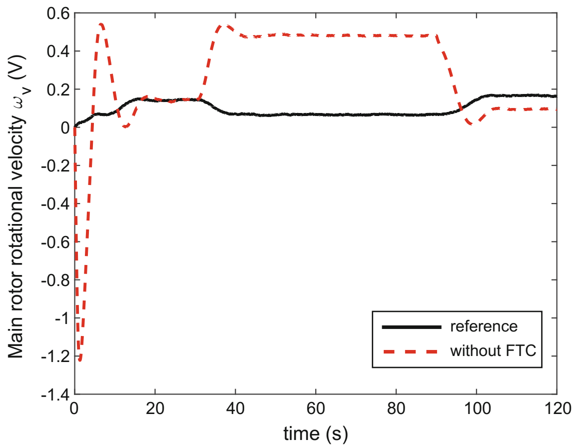


Fig. 8.17 Faulty main rotor response (comparison between TRMS and reference model states, without FTC). After [2]

maneuverability and effectiveness, and is obtained thanks to the characteristics of the wheels, which roll forward like normal wheels, but can also slide sideways at the same time.

The dynamic model of the four wheeled omnidirectional mobile robot relates the wheel inputs and robot velocities with the corresponding accelerations, taking into account the traction, viscous friction and Coulomb friction forces. The model is given by the following set of differential equations, obtained from the ones presented in [26] by considering the linear velocities on the static axis instead of the ones on the robot's axis:

$$\dot{x}(t) = v_x(t) \tag{8.76}$$

$$\begin{aligned}\dot{v}_x(t) = &\left(A_{11}\cos^2\varphi(t) + A_{22}\sin^2\varphi(t)\right)v_x(t) + \left[(A_{11} - A_{22})\sin\varphi(t)\cos\varphi(t) - \omega(t)\right]v_y(t) \\ &+K_{11}\cos\varphi(t) sign\left(v_x(t)\cos\varphi(t) + v_y(t)\sin\varphi(t)\right) - B_{21}\sin\varphi(t)u_0(t) + B_{12}\cos\varphi(t)u_1(t) \\ &-K_{22}\sin\varphi(t) sign\left(-v_x(t)\sin\varphi(t) + v_y(t)\cos\varphi(t)\right) - B_{23}\sin\varphi(t)u_2(t) + B_{14}\cos\varphi(t)u_3(t)\end{aligned} \tag{8.77}$$

$$\dot{y}(t) = v_y(t) \tag{8.78}$$

$$\begin{aligned}\dot{v}_y(t) = &\left[(A_{11} - A_{22})\sin\varphi(t)\cos\varphi(t) + \omega(t)\right]v_x(t) + \left(A_{11}\sin^2\varphi(t) + A_{22}\cos^2\varphi(t)\right)v_y(t) \\ &+K_{11}\sin\varphi(t) sign\left(v_x(t)\cos\varphi(t) + v_y(t)\sin\varphi(t)\right) + B_{21}\cos\varphi(t)u_0(t) + B_{12}\sin\varphi(t)u_1(t) \\ &+K_{22}\cos\varphi(t) sign\left(-v_x(t)\sin\varphi(t) + v_y(t)\cos\varphi(t)\right) + B_{23}\cos\varphi(t)u_2(t) + B_{14}\sin\varphi(t)u_3(t)\end{aligned} \tag{8.79}$$

$$\dot{\varphi}(t) = \omega(t) \tag{8.80}$$

$$\dot{\omega}(t) = A_{33}\left(\omega(t)\right)\omega(t) + B_{31}u_0(t) + B_{32}u_1(t) + B_{33}u_2(t) + B_{34}u_3(t) + K_{33} sign\left(\omega(t)\right) \tag{8.81}$$

The values of the robot parameters, identified from data obtained with a real setup [3], are provided in Table 8.3. Also, $A_{33}\left(\omega(t)\right)$ is defined as follows:

$$A_{33}\left(\omega(t)\right) = -0.0062\omega(t)^2 + 0.0028\omega(t) - 0.4406. \tag{8.82}$$

8.4.2.2 Quasi-LPV Error Model

In order to obtain a quasi-LPV error model for the four wheeled omnidirectional mobile robot, the first step is to reshape the nonlinear equations (8.76)–(8.81) into

Table 8.3 Robot parameters values

Param	Value	Param	Value	Param	Value
A_{11}	−1.4904	B_{21}	0.0089	B_{31}	0.05
A_{22}	−1.4904	B_{12}	−0.0089	B_{32}	0.05
K_{11}	−0.5340	B_{23}	−0.0089	B_{33}	0.05
K_{22}	−0.5340	B_{24}	0.0089	B_{34}	0.05

the quasi-LPV form (8.1). Taking into account that $B_3 \triangleq B_{31} = B_{32} = B_{33} = B_{34}$, $A_l \triangleq A_{11} = A_{22}$, $B_l \triangleq B_{21} = -B_{12} = -B_{23} = B_{14}$, and $K_l \triangleq K_{11} = K_{22}$, the quasi-LPV model is obtained as follows:

$$\begin{pmatrix} \dot{\mathrm{x}}(t) \\ \dot{v}_x(t) \\ \dot{\mathrm{y}}(t) \\ \dot{v}_y(t) \\ \dot{\varphi}(t) \\ \dot{\omega}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & A_l & 0 & -\theta_1(t) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \theta_1(t) & 0 & A_l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \theta_2(t) \end{pmatrix} \begin{pmatrix} \mathrm{x}(t) \\ v_x(t) \\ \mathrm{y}(t) \\ v_y(t) \\ \varphi(t) \\ \omega(t) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ -B_l\theta_3(t) & -B_l\theta_4(t) & B_l\theta_3(t) & B_l\theta_4(t) \\ 0 & 0 & 0 & 0 \\ B_l\theta_4(t) & -B_l\theta_3(t) & -B_l\theta_4(t) & B_l\theta_3(t) \\ 0 & 0 & 0 & 0 \\ B_3 & B_3 & B_3 & B_3 \end{pmatrix} \begin{pmatrix} u_0(t) \\ u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ c_x(t) \\ 0 \\ c_y(t) \\ 0 \\ c_\varphi(t) \end{pmatrix}$$

where $\theta(t) = (\theta_1(t), \ldots, \theta_4(t))^{\mathrm{T}}$ is the vector of varying parameters, scheduled by the state variables $\varphi(t)$ and $\omega(t)$, with:

$$\theta_1(t) = \omega(t)$$

$$\theta_2(t) = A_{33}(\omega(t))$$

$$\theta_3(t) = \sin\varphi(t)$$

$$\theta_4(t) = \cos\varphi(t)$$

where:

$$\begin{aligned} c_x(t) = {} & K_l \cos\varphi(t) sign\left(v_x(t)\cos\varphi(t) + v_y(t)\sin\varphi(t)\right) \\ & -K_l \sin\varphi(t) sign\left(-v_x(t)\sin\varphi(t) + v_y(t)\cos\varphi(t)\right) \end{aligned}$$

$$\begin{aligned} c_y(t) = {} & K_l \sin\varphi(t) sign\left(v_x(t)\cos\varphi(t) + v_y(t)\sin\varphi(t)\right) \\ & +K_l \cos\varphi(t) sign\left(-v_x(t)\sin\varphi(t) + v_y(t)\cos\varphi(t)\right) \end{aligned}$$

$$c_\varphi(t) = K_{33} sign\left(\omega(t)\right)$$

Then, using the reference model (8.3), and by defining the tracking errors $e_1(t) \triangleq \mathrm{x}_{ref}(t) - \mathrm{x}(t)$, $e_2(t) \triangleq v_x^{ref}(t) - v_x(t)$, $e_3(t) \triangleq \mathrm{y}_{ref}(t) - y(t)$, $e_4(t) \triangleq v_y^{ref}(t) - v_y(t)$, $e_5(t) \triangleq \varphi_{ref}(t) - \varphi(t)$, $e_6(t) \triangleq \omega_{ref}(t) - \omega(t)$, and the new inputs $\Delta u_i(t) \triangleq u_i^{ref}(t) - u_i(t)$, $i = 0, 1, 2, 3$, the following DT quasi-LPV error model can be obtained through an Euler approximation [22] with a sampling time $T_s = 0.04\,s$:

$$\begin{pmatrix} e_1(k+1) \\ e_2(k+1) \\ e_3(k+1) \\ e_4(k+1) \\ e_5(k+1) \\ e_6(k+1) \end{pmatrix} = \begin{pmatrix} 1 & T_s & 0 & 0 & 0 & 0 \\ 0 & A_l^d & 0 & -\theta_1^d(k) & 0 & 0 \\ 0 & 0 & 1 & T_s & 0 & 0 \\ 0 & \theta_1^d(k) & 0 & A_l^d & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 0 & 0 & \theta_2^d(k) \end{pmatrix} \begin{pmatrix} e_1(k) \\ e_2(k) \\ e_3(k) \\ e_4(k) \\ e_5(k) \\ e_6(k) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ -B_l\theta_3^d(k) & -B_l\theta_4^d(k) & B_l\theta_3^d(k) & B_l\theta_4^d(k) \\ 0 & 0 & 0 & 0 \\ B_l\theta_4^d(k) & -B_l\theta_3^d(k) & -B_l\theta_4^d(k) & B_l\theta_3^d(k) \\ 0 & 0 & 0 & 0 \\ B_3T_s & B_3T_s & B_3T_s & B_3T_s \end{pmatrix} \begin{pmatrix} \Delta u_0(k) \\ \Delta u_1(k) \\ \Delta u_2(k) \\ \Delta u_3(k) \end{pmatrix}. \tag{8.83}$$

8.4.2.3 Reference Input Calculation

In the following, multiplicative actuator faults in the motors are considered. These faults cause the change $u_i(t) \rightarrow f_i(t)u_i(t)$, $i = 0, 1, 2, 3$, in (8.76)–(8.81).

Consequently, the reference model is changed as described in Sect. 8.2.3, using the fault estimations $\hat{f}_i(t)$, $i = 0, 1, 2, 3$. Under the assumption that $\hat{f}_i(t) \cong f_i(t)$, $i = 0, 1, 2, 3$, the error model input matrix becomes:

$$B_f\left(\theta_3^d(k), \theta_4^d(k), f(k)\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -f_0(k)B_l\theta_3^d(k) & -f_1(k)B_l\theta_4^d(k) & f_2(k)B_l\theta_3^d(k) & -f_3(k)B_l\theta_4^d(k) \\ 0 & 0 & 0 & 0 \\ f_0(k)B_l\theta_4^d(k) & -f_1(k)B_l\theta_3^d(k) & -f_2(k)B_l\theta_4^d(k) & f_3(k)B_l\theta_3^d(k) \\ 0 & 0 & 0 & 0 \\ f_0(k)B_3T_s & f_1(k)B_3T_s & f_2(k)B_3T_s & f_3(k)B_3T_s \end{pmatrix}$$

To make the robot track a desired trajectory, proper values of u_i^{ref}, $i = 0, 1, 2, 3$, should be fed to the reference model, such that its state equals the one corresponding to the desired trajectory. In the following, a circular trajectory is chosen and defined as follows:

$$\mathrm{x}_{\mathrm{ref}}(t) = \rho\cos\left(\varphi_{\mathrm{ref}}(t)\right) \tag{8.84}$$

$$\mathrm{y}_{\mathrm{ref}}(t) = \rho\sin\left(\varphi_{\mathrm{ref}}(t)\right) \tag{8.85}$$

$$\varphi_{\mathrm{ref}}(t) = \frac{2\pi t}{T} \tag{8.86}$$

where ρ is the circle radius and T is the desired revolution period around the circle center.

Taking the derivatives and second derivatives of (8.84)–(8.86), considering (8.76), (8.78) and (8.80), and replacing into the faulty versions of (8.76)–(8.81), the following is obtained:

$$A_{ref}(t)\begin{pmatrix} u_0^{ref}(t)\\ u_1^{ref}(t)\\ u_2^{ref}(t)\\ u_3^{ref}(t)\end{pmatrix} = B_{ref}(t)$$

with:

$$A_{ref}(t) = \begin{pmatrix} -\hat{f}_0(t)B_l\sin\varphi(t) & -\hat{f}_1(t)B_l\cos\varphi(t) & \hat{f}_2(t)B_l\sin\varphi(t) & \hat{f}_3(t)B_l\cos\varphi(t)\\ \hat{f}_0(t)B_l\cos\varphi(t) & -\hat{f}_1(t)B_l\sin\varphi(t) & -\hat{f}_2(t)B_l\cos\varphi(t) & \hat{f}_3(t)B_l\sin\varphi(t)\\ \hat{f}_0(t)B_3 & \hat{f}_1(t)B_3 & \hat{f}_2(t)B_3 & \hat{f}_3(t)B_3 \end{pmatrix}$$

$$B_{ref}(t) = \begin{pmatrix} \rho\frac{2\pi}{T}\left[A_l\sin\frac{2\pi t}{T} + \left(\omega(t) - \frac{2\pi}{T}\right)\cos\frac{2\pi t}{T}\right] - c_x(t)\\ \rho\frac{2\pi}{T}\left[\left(\omega(t) - \frac{2\pi}{T}\right)\sin\frac{2\pi t}{T} - A_l\cos\frac{2\pi t}{T}\right] - c_y(t)\\ -A_{33}\left(\omega(t)\right)\frac{2\pi}{T} - c_\varphi(t) \end{pmatrix}$$

Finally, the reference model inputs $u_i^{ref}(t)$, $i = 0, 1, 2, 3$, are obtained as:

$$\begin{pmatrix} u_0^{ref}(t)\\ u_1^{ref}(t)\\ u_2^{ref}(t)\\ u_3^{ref}(t)\end{pmatrix} = A_{ref}(t)^{\dagger}B_{ref}(t)$$

Remark: The obtained values $u_i^{ref}(t)$, $i = 0, 1, 2, 3$, depend on the specifications, defined by the radius ρ and revolution period T of the desired circular trajectory (8.84)–(8.86). Special care should be taken in choosing ρ and T, such that the resulting reference inputs do not cause the motors to work near/in their saturation region.

Remark: The reference input calculation presented in this section can be applied to obtain the tracking of a wider class of trajectories. In particular, if $\mathrm{x}_{ref}(t)$, $\mathrm{y}_{ref}(t)$, $\varphi_{ref}(t) \in \mathcal{C}^2$ in some time interval $[t_0, t_f]$, then $B_{ref}(t)$ takes the following form for $t \in [t_0, t_f]$:

$$B_{ref}(t) = \begin{pmatrix} \ddot{\mathrm{x}}_{ref}(t) - A_l\dot{\mathrm{x}}_{ref}(t) + \omega(t)\dot{\mathrm{y}}_{ref}(t) - c_x(t)\\ \ddot{\mathrm{y}}_{ref}(t) - A_l\dot{\mathrm{y}}_{ref}(t) - \omega(t)\dot{\mathrm{x}}_{ref}(t) - c_y(t)\\ \ddot{\varphi}_{ref} - A_{33}\left(\omega(t)\right)\dot{\varphi}_{ref}(t) - c_\varphi(t) \end{pmatrix}$$

In this way, most of the trajectories that are of interest in mobile robot applications can be obtained, e.g. polynomials, conic and polygonal trajectories.

8.4.2.4 Design of the FTC Scheme Using a Switching Framework

When the polytopic conditions presented in Sect. 2.5 are applied to some polytopic approximation of the four wheeled omnidirectional mobile robot quasi-LPV model

(8.83), it is found that a solution does not exist due to the loss of controllability occurring for $\theta_3^d = \theta_4^d = 0$, values for which the input matrix becomes:

$$B_{\theta_3^d=\theta_4^d=0} = \begin{pmatrix} O_{5\times 1} & O_{5\times 1} & O_{5\times 1} & O_{5\times 1} \\ B_3 T_s & B_3 T_s & B_3 T_s & B_3 T_s \end{pmatrix} \tag{8.87}$$

Due to the fact that the sets described by the polytopic approximations are convex, it is straightforward that any polytopic approximation of the admissible values for $\theta_3^d(k) = \sin(\varphi(k))\, T_s$ and $\theta_4^d(k) = \cos(\varphi(k))\, T_s$ will contain the origin, i.e. the singularity (8.87) of the input matrix.

However, this problem can be avoided using a switching LPV framework, by splitting the subset of the parameter space generated by θ_3^d and θ_4^d in more regions, such that in each region the resulting polytopic approximation does not include the origin.

More specifically, the (quasi-)LPV error system (8.5) is modified by including a switching part, as follows:

$$\sigma.e(\tau) = A_\xi(\theta(\tau))\, e(\tau) + B_\xi(\theta(\tau))\, \Delta u_c(\tau)$$

with $\xi = 1\ \forall\theta \in \Theta_1$, $\xi = 2\ \forall\theta \in \Theta_2$, ..., $\xi = Z\ \forall\theta \in \Theta_Z$, where $\Theta_1, \ldots, \Theta_z$ are subsets of the varying parameter space Θ, such that $\Theta = \Theta_1 \cup \Theta_2 \cup \cdot \cup \Theta_Z$. In each subset Θ_ξ, $\xi = 1, \ldots, Z$, the system is described by a polytopic combination of vertex. Then, the error-feedback control law is chosen to be:

$$\Delta u_c(\tau) = K_\xi(\theta(\tau))\, e(\tau)$$

and the virtual actuator reconfiguration structure is expressed as:

$$\Delta u(\tau) = N_\xi\left(\theta(\tau), \hat{f}(\tau)\right)\left(\Delta u_c(\tau) - M_\xi(\theta(\tau))\, x_v(\tau)\right)$$

$$\sigma.x_v(\tau) = \left(A_\xi(\theta(\tau)) + B_\xi^*(\theta(\tau))\, M_\xi(\theta(\tau))\right) x_v(\tau) + \left(B_\xi(\theta(\tau)) - B_\xi^*(\theta(\tau))\right)\Delta u_c(\tau)$$

with:

$$N_\xi\left(\theta(\tau), \hat{f}(\tau)\right) = B_{\xi f}\left(\theta(\tau), \hat{f}(\tau)\right) B(\theta(\tau))$$

$$B_{\xi f}(\theta(\tau), f(\tau)) = B_\xi(\theta(\tau))\, F(f(\tau))$$

$$B_\xi^*(\theta(\tau)) = B_{\xi f}\left(\theta(\tau), \hat{f}(\tau)\right) N_\xi\left(\theta(\tau), \hat{f}(\tau)\right)$$

Then, by using a common fixed Lyapunov function, the design conditions appear to be only a slight modification of the ones provided in Sect. 2.5.

Remark: In general, other Lyapunov functions, e.g. parameter-dependent ones [27], could be used for control design of switched LPV systems. However, in the application to the four wheeled omnidirectional mobile robot, a common fixed Lyapunov function has proved to be enough for stabilization and pole clustering in the desired LMI region $\mathcal{D}$, and it has been preferred due to its simplicity.

In the case of the four wheeled omnidirectional mobile robot, the quadrants have been considered as regions, with $\varphi = i\pi/2, i \in \mathbb{Z}$ being the switching condition such that:

$$\xi = \begin{cases} 1 & \textit{if} \quad \cos\varphi \geq 0 \textit{ AND } \sin\varphi \geq 0 \\ 2 & \textit{if} \quad \cos\varphi \geq 0 \textit{ AND } \sin\varphi < 0 \\ 3 & \textit{if} \quad \cos\varphi < 0 \textit{ AND } \sin\varphi < 0 \\ 4 & \textit{if} \quad \cos\varphi < 0 \textit{ AND } \sin\varphi \geq 0 \end{cases}$$

A triangular approximation has been used in each region, for the pair $\{\theta_3^d, \theta_4^d\}$, with the following structure:

$$\begin{pmatrix} \theta_3^d \\ \theta_4^d \end{pmatrix} \in Co\left\{\begin{pmatrix} \pm T_s \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm T_s \end{pmatrix}, \begin{pmatrix} \pm T_s \\ \pm T_s \end{pmatrix}\right\}$$

where Co denotes the convex set, and whether $\pm$ is $+$ or $-$ depends, for each varying parameter, on the region that is being considered.

The polytopic approximation of the four wheeled omnidirectional mobile robot error model (8.83) has been obtained by considering $T_s = 0.04\,s$ and $\omega \in [-2.5\,\text{rad}/s, 2.5\,\text{rad/s}]$.

The controller and the virtual actuators, one for each wheel, have been designed[3] to assure quadratic stability and quadratic $\mathcal{D}$-stability in:

$$\mathcal{D} = \left\{z \in \mathbb{C} : \text{Re}(z) > 0.40, \text{Re}(z)^2 + \text{Im}(z)^2 < 0.9997^2\right\} \tag{8.88}$$

The LMIs have been solved using the YALMIP toolbox [24] and the SeDuMi solver [25].

8.4.2.5 Results

Three control experiments have been considered, where the robot started from different initial states:

- Experiment 1

$$\begin{pmatrix} \text{x}(0) & v_x(0) & \text{y}(0) & v_y(0) & \varphi(0) & \omega(0) \end{pmatrix}^{\text{T}} = \begin{pmatrix} 1.5 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{\text{T}}$$

---

[3]The state can be directly estimated from the available sensors, thus no observer has been designed.

- Experiment 2

$$\begin{pmatrix} \mathrm{x}(0) & v_x(0) & \mathrm{y}(0) & v_y(0) & \varphi(0) & \omega(0) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$$

- Experiment 3

$$\begin{pmatrix} \mathrm{x}(0) & v_x(0) & \mathrm{y}(0) & v_y(0) & \varphi(0) & \omega(0) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$$

and tracked the desired trajectory, a circle centered in the origin of the $(\mathrm{x}-\mathrm{y})$ plane with a radius of 1 m and a revolution period of 20 s, generated from the initial reference state:

$$\begin{pmatrix} \mathrm{x}_{\mathrm{ref}}(0) & v_x^{ref}(0) & \mathrm{y}_{\mathrm{ref}}(0) & v_y^{ref}(0) & \varphi_{ref}(0) & \omega_{ref}(0) \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 & \frac{\pi}{10} & 0 & \frac{\pi}{10} \end{pmatrix}^{\mathrm{T}}$$

The considered fault scenario is a total loss of the first wheel motor starting from time $t = 20\,s$:

$$f_0(t) = \begin{cases} 1 & if\ t < 20\,s \\ 0 & if\ t \geq 20\,s \end{cases}$$

Figure 8.18 shows the tracking of the desired circular trajectory in the $(\mathrm{x}-\mathrm{y})$ plane for Experiment 1, obtained in a simulation environment. It can be seen that, in the case where the proposed FTC technique is not applied, the robot trajectory deviates from the reference trajectory after the fault appears. On the other hand, adding the virtual actuator to the control loop increases the tracking performance of the robot. Table 8.4 resumes the mean squared errors for the trajectory tracking in all the three considered experiments, obtained in a simulation environment. The improvement brought by the proposed FTC strategy on the tracking performance can be seen clearly in all the considered experiments.

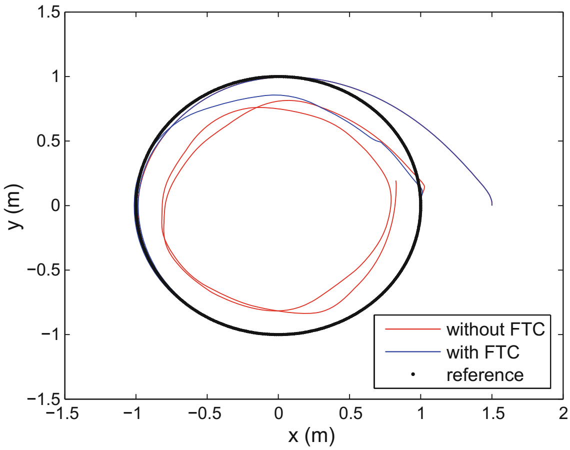


Fig. 8.18 Tracking of the desired circular trajectory: $(x-y)$ plane (Simulation 1). © 2014 IEEE. Reprinted, with permission, from: D. Rotondo, V. Puig, F. Nejjari, J. Romera. A fault-hiding approach for the switching quasi-LPV fault tolerant control of a four wheeled omnidirectional mobile robot. *IEEE Transactions on Industrial Electronics*, 62(6):3932–3944, 2015

Table 8.4 Mean squared errors without and with FTC (simulation). ©2014 IEEE. Reprinted, with permission, from: D. Rotondo, V. Puig, F. Nejjari, J. Romera. A fault-hiding approach for the switching quasi-LPV fault tolerant control of a four wheeled omnidirectional mobile robot. *IEEE Transactions on Industrial Electronics*, 62(6):3932-3944, 2015

	e_1^2	e_2^2	e_3^2	e_4^2	e_5^2	e_6^2
Sim.1 without FTC	0.024	0.004	0.022	0.003	1.438	0.025
Sim.1 with FTC	0.007	0.001	0.002	0.001	0.297	0.016
Sim.2 without FTC	0.018	0.003	0.022	0.003	1.440	0.029
Sim.2 with FTC	0.001	0.000	0.001	0.001	0.291	0.020
Sim.3 without FTC	0.037	0.007	0.020	0.003	1.440	0.027
Sim.3 with FTC	0.021	0.004	0.002	0.001	0.295	0.023

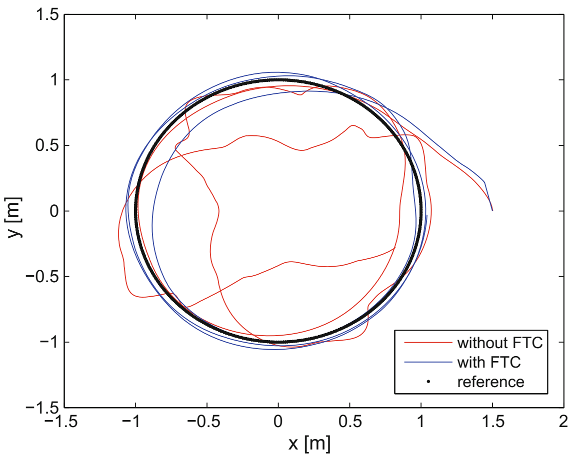


Fig. 8.19 Tracking of the desired circular trajectory: $(x - y)$ plane (Experiment 1). ©2014 IEEE. Reprinted, with permission, from: D. Rotondo, V. Puig, F. Nejjari, J. Romera. A fault-hiding approach for the switching quasi-LPV fault tolerant control of a four wheeled omnidirectional mobile robot. *IEEE Transactions on Industrial Electronics*, 62(6):3932–3944, 2015

Figures 8.19, 8.20 and 8.21 show experimental results for Experiment 1, while Table 8.5 resumes the mean squared errors for the trajectory tracking in all the three considered experiments. The results demonstrate that the omnidirectional mobile robot is able to operate under a severe fault occurrence, e.g. the total loss of one motor, if an appropriate fault-hiding strategy is implemented.

Figures 8.19 and 8.20 show the tracking of the desired circular trajectory in the (x − y) plane and a comparison between the system states and the reference ones. When the proposed FTC strategy is applied, all the system states go to the reference ones, i.e. the tracking errors go to zero, except for a steady-state error in the φ angle. The addition of an integral action could eliminate such error, even though at the expense of a probable decrease in the system performance, as well as the appearance of the need to introduce anti-windup mechanisms to avoid undesired effects due to the motor saturation nonlinearities. Finally, in Fig. 8.21, the control inputs are presented.

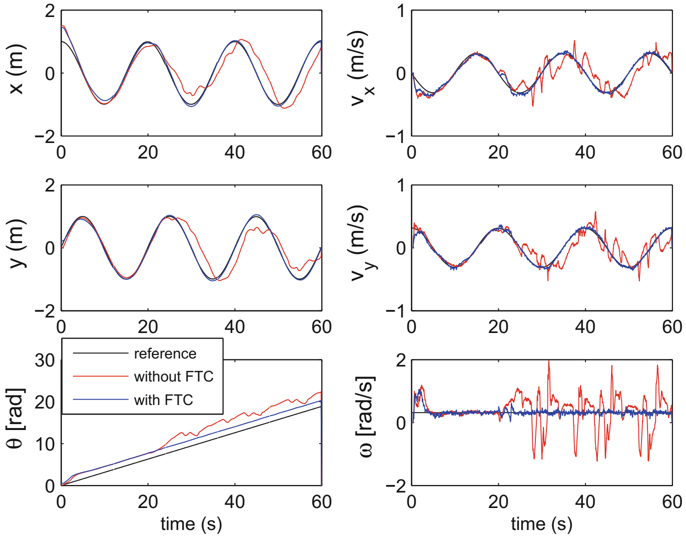


Fig. 8.20 Tracking of the desired circular trajectory: states (Experiment 1). ©2014 IEEE. Reprinted, with permission, from: D. Rotondo, V. Puig, F. Nejjari, J. Romera. A fault-hiding approach for the switching quasi-LPV fault tolerant control of a four wheeled omnidirectional mobile robot. *IEEE Transactions on Industrial Electronics*, 62(6):3932–3944, 2015

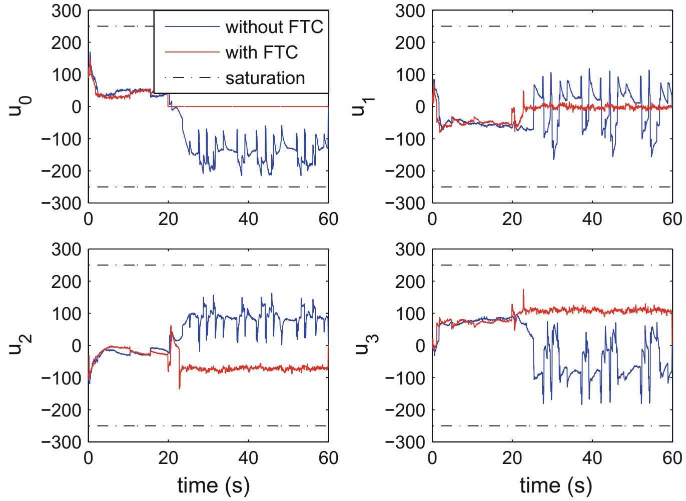


Fig. 8.21 Tracking of the desired circular trajectory: inputs (Experiment 1). ©2014 IEEE. Reprinted, with permission, from: D. Rotondo, V. Puig, F. Nejjari, J. Romera. A fault-hiding approach for the switching quasi-LPV fault tolerant control of a four wheeled omnidirectional mobile robot. *IEEE Transactions on Industrial Electronics*, 62(6):3932–3944, 2015

Table 8.5 Mean squared errors without and with FTC (experimental). ©2014 IEEE. Reprinted, with permission, from: D. Rotondo, V. Puig, F. Nejjari, J. Romera. A fault-hiding approach for the switching quasi-LPV fault tolerant control of a four wheeled omnidirectional mobile robot. *IEEE Transactions on Industrial Electronics*, 62(6):3932–3944, 2015

	e_1^2	e_2^2	e_3^2	e_4^2	e_5^2	e_6^2
Exp.1 without FTC	0.110	0.017	0.081	0.016	7.284	0.259
Exp.1 with FTC	0.009	0.001	0.002	0.001	2.023	0.014
Exp.2 without FTC	0.048	0.006	0.038	0.010	1.814	0.158
Exp.2 with FTC	0.006	0.001	0.004	0.002	3.630	0.024
Exp.3 without FTC	0.085	0.015	0.051	0.012	1.757	0.153
Exp.3 with FTC	0.024	0.004	0.003	0.002	3.417	0.026

It can be seen that the control inputs are such that all the motors are working in their linear regions. Moreover, under fault occurrence, the effect of the first wheel on the system is redistributed among the remaining wheels to achieve fault tolerance.

8.5 Conclusions

In this chapter, an FTC strategy based on model reference control and virtual actuators has been proposed for LPV systems subject to actuator faults. The proposed FTC strategy adapts the reference model to the faults and utilizes the virtual actuator technique in order to recover the nominal stability and behavior of the error model, with some minimum or graceful performance degradation.

The overall control loop is made up by an LPV error feedback controller, an LPV error observer and the LPV virtual actuator. It has been shown that the principle of separation holds, since there exists a similarity transformation that brings the augmented model to a block-triangular form. Hence, the stability and the satisfaction of the desired specifications can be assessed separately.

The potential and performance of the proposed approach have been demonstrated with two different examples: a twin rotor MIMO system and a four wheeled omnidirectional mobile robot, showing promising results.

Future research on this topic will aim at improving the robustness of the proposed FTC strategy against model uncertainties and errors in the fault estimation.

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Chapter 9
Fault Tolerant Control of Unstable LPV Systems Subject to Actuator Saturations and Fault Isolation Delay

The content of this chapter is based on the following works:

- [1] D. Rotondo, J.-C. Ponsart, D. Theilliol, F. Nejjari, V. Puig. A virtual actuator approach for the fault tolerant control of unstable linear systems subject to actuator saturation and fault isolation delay. *Annual Reviews in Control*, 39:68–80, 2015.
- [2] D. Rotondo, J.-C. Ponsart, D. Theilliol, F. Nejjari, V. Puig. Fault tolerant control of unstable LPV systems subject to actuator saturations using virtual actuators. In *Proceedings of the 9th IFAC Symposium SAFEPROCESS-2015: Fault Detection, Supervision and Safety for Technical Processes*, pages 18–23, 2015.
- [3] D. Rotondo, J.-C. Ponsart, F. Nejjari, D. Theilliol, V. Puig. Virtual actuator-based FTC for LPV systems with saturating actuators and FDI delays. In *Proceedings of the 3rd Conference on Control and Fault-Tolerant Systems*, pages 831–837, 2016.

9.1 Introduction

Real-world actuators are always subject to limits in the magnitude of the manipulated input. The control techniques that ignore these actuator limits can be affected by degraded performance, and may even lead to instability of the closed-loop system. Hence, recent research has focused on the analysis and synthesis of control systems with saturating actuators [4, 5]. The developed solutions mainly use two approaches: the *two-step* paradigm, also called *anti-windup compensation* [6, 7], where a controller which does not explicitly take into account the saturation is designed, and then a compensator is added to handle the saturation constraints; and the *one-step* paradigm, also called *direct control design* [8, 9], where the input constraints are taken into account at the controller design stage.

D. Rotondo, *Advances in Gain-Scheduling and Fault Tolerant Control Techniques*,
Springer Theses, https://doi.org/10.1007/978-3-319-62902-5_9

It is important to consider the actuator saturation constraints in the application of an FTC strategy, especially when actuator faults are considered. In fact, fault tolerance against actuator faults is usually achieved redistributing, in some way, the control effort corresponding to the faulty actuators among the remaining healthy ones. This redistribution may lead to saturation of both the faulty and the healthy actuators. Thus, if this fact is neglected in the FTC system design, severe performance degradation or instability may occur [10]. Some recent works have considered the problem of FTC systems subject to actuator saturations. [11] show that failures resulting from loss of actuator effectiveness in systems with input saturations can be dealt with in the context of absolute stability theory framework. [12] present two kinds of fault tolerant controllers (fixed-gain and adaptive) for singular systems subject to actuator saturation. Both of these two controllers are in the form of saturation avoidance feedback. [13] develop a fault tolerant control scheme that can achieve attitude tracking control objective for a flexible spacecraft in the presence of partial loss of actuator effectiveness fault and actuator saturation using sliding mode control. The solution proposed by [14] avoids to use the failed control actuators in the presence of a fault. Also, concepts such as *graceful performance degradation* [15, 16] and *reference reconfiguration* [17, 18] have been introduced in the context of FTC of systems subject to actuator saturations. In Chap. 8, it has been shown that, by embedding the saturations in the varying parameter vector, the LPV paradigm can be used to deal with them. However, the proposed approach fails when applied to open-loop unstable systems, for which special care should be taken. In spite of the importance of developing a valid FTC strategy for unstable systems subject to actuator saturations, this problem has been considered only by a few works. [19] has proposed an LTV fault tolerant compensator, using the relevant ability of LTV compensators to achieve simultaneous stabilization of several systems. An active FTC scheme based on gain-scheduled $\mathcal{H}_\infty$ control and neural network for unstable systems has been proposed by [20]. Finally, [21] have developed a robust fault tolerant scheme based on variable structure control for an orbiting spacecraft with a combination of unknown actuator failures and input saturation.

However, even though an active FTC system can react to faults more effectively than a passive FTC system, passive FTC techniques are preferred to the active ones when dealing with unstable systems [10, 19, 21]. In fact, the active FTC strategies require an FDI module, and when unstable systems are considered, the time delay between the appearance of the fault and the moment in which the active strategy is activated (at the fault detection or isolation time) may destabilize the system. According to the author's knowledge, [20] is the only work dealing with active FTC for unstable systems. However, in this reference, the issues arising from the FDI time delay were not considered. Also, another issue that has not been considered is the fact that, when dealing with unstable systems, the stability properties guaranteed by the control design are *regional*, i.e. hold only for inputs up to some size or for initial states inside a region of the state space [22]. The fault appearance, and the subsequent control system reconfiguration brought by the active FTC strategies change the regional stability properties of the control system, so it is necessary to take into account this fact explicitly when the system is subject to actuator saturations.

The main contribution of this chapter consists in the design of an active FTC strategy for unstable LPV systems subject to actuator saturation. Under the assumption that a nominal controller has been already designed using the direct control design paradigm to take into account the saturations, virtual actuators are added to the control loop for achieving fault tolerance against a predefined set of possible faults. In particular, faults affecting the actuators and causing a change in the system input matrix are considered. The design of the virtual actuators is performed in such a way that, if at the fault isolation time the closed-loop system state is inside a region defined by a value of the Lyapunov function, the state trajectory will converge to zero despite the appearance of the faults. Also, it is shown that it is possible to obtain some guarantees about the tolerated delay between the fault occurrence and its isolation. Moreover, the design of the nominal controller can be performed so as to maximize the tolerated delay.

It should be pointed out that, although the Hammerstein-Wiener formulation of the virtual actuators can be used to deal with the saturations, the approach proposed in this chapter can be distinguished from the one introduced in [23] since less restrictive assumptions are required. In particular, some delay in the fault isolation is accepted, and the system matrix could be non-Hurwitz. In fact, although applicable to stable systems, the approach proposed hereafter focuses on the unstable ones.

9.2 Preliminaries

Consider the autonomous nonlinear system:

$$\sigma.x(\tau) = g\left(x(\tau)\right) \tag{9.1}$$

where $x \in \mathbb{R}^{n_x}$ is the state and g denotes a nonlinear function. For $x(0) = x_0 \in \mathbb{R}^{n_x}$, let us denote the trajectory of the system (9.1) as $\psi(\tau, x_0)$. Then, the *domain of attraction* of the origin is:

$$\mathcal{S} := \left\{ x_0 \in \mathbb{R}^{n_x} : \lim_{\tau \to +\infty} \psi(\tau, x_0) = 0 \right\} \tag{9.2}$$

Let $P \succ 0$ and denote:

$$\mathcal{E}(P, \rho) = \left\{ x_0 \in \mathbb{R}^{n_x} : x^{\mathrm{T}} P x \le \rho \right\} \tag{9.3}$$

and let $V\left(x(\tau)\right) = x(\tau)^{\mathrm{T}} P x(\tau)$ be a candidate Lyapunov function. The ellipsoid $\mathcal{E}(P, \rho)$ is said to be *contractively invariant* if $\sigma.V\left(x(\tau)\right) < 0$ for all $x \in \mathcal{E}(P, \rho) \backslash \{0\}$. Clearly, if $\mathcal{E}(P, \rho)$ is contractively invariant, it is inside the domain of attraction $\mathcal{S}$ [24].

Remark: As stated in [25], there is a tradeoff between the degree of approximation of the domain of attraction and the simplicity of the representation. In the literature, several shapes for determining contractively invariant regions have been considered, e.g. polytopes, but ellipsoids are widely used due to their simplicity. For this reason, ellipsoids have been considered in this chapter, even though the general idea behind the developed theory could be adapted to more complex shapes, at the expense of increasing the complexity of the approach.

Now, let us consider the following LPV system subject to actuator saturations:

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) + Bsat\left(u(\tau)\right) \tag{9.4}$$

$$y(\tau) = Cx(\tau) \tag{9.5}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output, $A\left(\theta(\tau)\right) \in \mathbb{R}^{n_x \times n_x}$ is the parameter varying state matrix, whose values depend on the vector $\theta(\tau) \in \Theta \subset \mathbb{R}^{n_\theta}$, $B \in \mathbb{R}^{n_x \times n_u}$ is the input matrix, $C \in \mathbb{R}^{n_y \times n_x}$ is the output matrix, and $sat : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is the saturation function, defined as in (8.28):

$$sat(u) = \begin{pmatrix} sat_1(u_1) \\ \vdots \\ sat_j(u_j) \\ \vdots \\ sat_{n_u}(u_{n_u}) \end{pmatrix} \quad sat_j(u_j) = \begin{cases} u_j^{MAX} & if\ u_j > u_j^{MAX} \\ u_j & if\ \left|u_j\right| \leq u_j^{MAX} \\ -u_j^{MAX} & if\ u_j < -u_j^{MAX} \end{cases} \tag{9.6}$$

For an output feedback law $u(\tau) = h\left(y(\tau)\right) = h\left(Cx(\tau)\right)$, let us define $\mathcal{L}(u, u^{MAX})$ the region of the state space in which the actuators are not saturated.

Let us consider the preliminary problem of designing an LPV dynamic output feedback controller for the system (9.4)–(9.5):

$$\sigma.x_c(\tau) = A_c\left(\theta(\tau)\right)x_c(\tau) + B_c\left(\theta(\tau)\right)y(\tau) \tag{9.7}$$

$$u_c(\tau) = C_c\left(\theta(\tau)\right)x_c(\tau) + D_c\left(\theta(\tau)\right)y(\tau) \tag{9.8}$$

where $x_c \in \mathbb{R}^{n_x}$ is the controller state and $u_c \in \mathbb{R}^{n_u}$ is the controller output, such that if $u(\tau) = u_c(\tau)$, then $\mathcal{E}(P, 1) \subseteq \mathcal{S}$ and $\mathcal{E}(P, 1) \subseteq \mathcal{L}(u, u^{MAX})$, i.e. the controller will be such that for any initial closed-loop state vector satisfying:

$$\left(x(0)^{\mathrm{T}}\ x_c(0)^{\mathrm{T}}\right) P \begin{pmatrix} x(0) \\ x_c(0) \end{pmatrix} \leq 1 \tag{9.9}$$

the control input never saturates, and the closed-loop state trajectory converges to the origin. For the sake of simplicity, only the CT case will be considered in the following.

In order to achieve this objective, the following theorem is proposed, obtained as an extension of a similar theorem presented for the LTI case in [26].

Theorem 9.1 (Design of a non-saturating stabilizing LPV output feedback controller) *Let* $X, Y \in \mathbb{S}^{n_x \times n_x}$, $F(\theta) \in \mathbb{R}^{n_x \times n_y}$, $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ *and* $L(\theta) \in \mathbb{R}^{n_u \times n_y}$ *be such that:*

$$He\{XA(\theta) + F(\theta)C\} \prec O \tag{9.10}$$

$$He\{A(\theta)Y + BK(\theta)\} \prec O \tag{9.11}$$

$$\begin{pmatrix} X & I & CL_{(j)}(\theta)^{\mathrm{T}} \\ I & Y & K_{(j)}(\theta)^{\mathrm{T}} \\ L_{(j)}(\theta)C & K_{(j)}(\theta) & \left(u_j^{MAX}\right)^2 \end{pmatrix} \succ O \qquad \begin{matrix} \forall j = 1, \ldots, n_u \\ \forall \theta \in \Theta \end{matrix} \tag{9.12}$$

Then, the controller (9.7)–(9.8), with $\tau = t$ and matrices calculated as:

$$\begin{aligned} \begin{pmatrix} A_c(\theta) & B_c(\theta) \\ C_c(\theta) & D_c(\theta) \end{pmatrix} &= \begin{pmatrix} Z & XB \\ O & I \end{pmatrix}^{-1} \cdots \\ \cdots &\begin{pmatrix} -(A(\theta) + BL(\theta)C)^{\mathrm{T}} - XA(\theta)Y & F(\theta) \\ K(\theta) & L(\theta) \end{pmatrix} \begin{pmatrix} -Y & O \\ CY & I \end{pmatrix}^{-1} \end{aligned} \tag{9.13}$$

$$Z = X - Y^{-1} \tag{9.14}$$

is such that, for the closed-loop system obtained with $u(t) = u_c(t)$, $\mathcal{E}(P, 1) \subseteq \mathcal{S}$ and $\mathcal{E}(P, 1) \subseteq \mathcal{L}(u, \alpha)$ where:

$$P = \begin{pmatrix} X & Z \\ Z & Z \end{pmatrix} \tag{9.15}$$

Proof The proof follows the reasoning developed in [26] in the case of LTI systems, and is based on demonstrating that if (9.10)–(9.12) hold and the controller matrices are calculated as in (9.13), then $\mathcal{E}(P, 1)$ is *contractively invariant*, i.e. by defining the quadratic Lyapunov function $V(x(t)) = x(t)^{\mathrm{T}} P x(t)$, it is obtained that $\dot{V}(x(t)) < 0$ for all $x \in \mathcal{E}(P, \rho) \backslash \{0\}$. Since $\mathcal{E}(P, 1)$ is contractively invariant, it is inside the domain of attraction $\mathcal{S}$ [24] such that the stability is guaranteed over the whole set of possible values of θ. ■

The conditions provided by Theorem 9.1 rely on the satisfaction of infinite constraints, due to the fact that (9.10)–(9.12) should hold for all the possible values of θ. However, by considering a polytopic approach, as already described in Chap. 2, (9.10)–(9.12) can be transformed in a finite number of LMIs, as shown by the following corollary.

Corollary 9.1 (Design of a non-saturating stabilizing polytopic LPV output feedback controller) *Assume that the LPV system (9.4) is polytopic, i.e. the matrix $A(\theta(t))$ can be written as:*

$$A(\theta(t)) = \sum_{i=1}^{N} \mu_i(\theta(t)) A_i \tag{9.16}$$

with coefficients $\mu_i(\theta(t))$ *such that (2.5) holds:*

$$\sum_{i=1}^{N} \mu_i(\theta(\tau)) = 1, \quad \mu_i(\theta(\tau)) \geq 0, \ \forall i = 1, \ldots, N, \ \forall \theta \in \Theta \tag{9.17}$$

and let $X, Y \in \mathbb{S}^{n_x \times n_x}$, $F_i \in \mathbb{R}^{n_x \times n_y}$, $K_i \in \mathbb{R}^{n_u \times n_x}$ *and* $L_i \in \mathbb{R}^{n_u \times n_y}$, $i = 1, \ldots, N$, *be such that:*

$$He\{XA_i + F_i C\} \prec O \qquad \forall i = 1, \ldots, N \tag{9.18}$$

$$He\{A_i Y + BK_i\} \prec O \qquad \forall i = 1, \ldots, N \tag{9.19}$$

$$\begin{pmatrix} X & I & CL_{i(j)}^{\mathrm{T}} \\ I & Y & K_{i(j)}^{\mathrm{T}} \\ L_{i(j)}C & K_{i(j)} & \left(u_j^{MAX}\right)^2 \end{pmatrix} \succ O \qquad \begin{matrix} \forall i = 1, \ldots, N \\ \forall j = 1, \ldots, n_u \end{matrix} \tag{9.20}$$

Then, the controller (9.7)–(9.8), with:

$$\begin{pmatrix} A_c(\theta(t)) & B_c(\theta(t)) \\ C_c(\theta(t)) & D_c(\theta(t)) \end{pmatrix} = \sum_{i=1}^{N} \mu_i(\theta(t)) \begin{pmatrix} A_{c,i} & B_{c,i} \\ C_{c,i} & D_{c,i} \end{pmatrix} \tag{9.21}$$

and vertex controller gains calculated as:

$$\begin{pmatrix} A_{c,i} & B_{c,i} \\ C_{c,i} & D_{c,i} \end{pmatrix} = \begin{pmatrix} Z & XB \\ O & I \end{pmatrix}^{-1} \begin{pmatrix} -(A_i + BL_i C)^{\mathrm{T}} - XA_i Y & F_i \\ K_i & L_i \end{pmatrix} \begin{pmatrix} -Y & O \\ CY & I \end{pmatrix} \tag{9.22}$$

with Z *defined as in (9.14) is such that, for the closed-loop system obtained with* $u(t) = u_c(t)$, $\mathcal{E}(P, 1) \subseteq \mathcal{S}$ *and* $\mathcal{E}(P, 1) \subseteq \mathcal{L}(u, u^{MAX})$ *with* P *defined as in (9.15).*

Proof It follows from the basic property of matrices [27] that any linear combination of negative (positive) definite matrices with non-negative coefficients, whose sum is positive, is negative (positive) definite. Hence, using the coefficients $\mu_i(\theta(t))$, taking into account (9.17), (9.10)–(9.12) follow directly from (9.18)–(9.20). ■

Remark: The shape of the ellipsoidal invariant set can be fixed, or forced to be optimized in some desired sense, e.g. optimizing $det(X)$ or $trace(X)$, during the application of Theorem 9.1/Corollary 9.1. However, this optimization could lead to an ellipsoid that favors some state variables more than the others, which may be undesired in some situations.

Also, the following lemma gives a constraint on the scalar product of two vectors [28].

Lemma 9.1 (Magnitude of the scalar product of two vectors) *Given two vectors m and x, the existence of $Q \succ O$ such that:*

$$\begin{pmatrix} Q^{-1} & Q^{-1}m \\ m^{\mathrm{T}}Q^{-1} & \gamma^2 \end{pmatrix} \succ O \tag{9.23}$$

implies that $|m^{\mathrm{T}}x| \leq \gamma \; \forall x \in \mathcal{E}(Q, 1)$.

Proof This lemma is a direct consequence of applying Schur complements [29] to (9.23). ■

9.3 Problem Statement

Let us consider the following LPV system subject to actuator saturations:

$$\dot{x}(t) = A\left(\theta(t)\right)x(t) + \mathfrak{B}(t)sat\left(u(t)\right) \tag{9.24}$$

$$y(t) = Cx(t) \tag{9.25}$$

with:

$$\mathfrak{B}(t) = \begin{cases} B & t < t_f \\ B_f \in B_f^{(1)}, \ldots, B_f^{(n_f)} & t \geq t_f \end{cases} \tag{9.26}$$

where $B \in \mathbb{R}^{n_x \times n_u}$ and the corresponding LPV system obtained from (9.24)–(9.25), that corresponds to (9.4)–(9.5) will be referred to as *nominal input matrix* and *nominal system*, respectively, $B_f \in \mathbb{R}^{n_x \times n_u}$ and the corresponding LPV system obtained from (9.24)–(9.25) will be referred to as *faulty input matrix* and *faulty system*, respectively, $t_f \in \mathbb{R}^+$ is the *fault occurrence time* and the function $sat\left(u(t)\right)$ is defined as in (9.6). The n_f matrices $B_f^{(1)}, \ldots, B_f^{(n_f)} \in \mathbb{R}^{n_x \times n_u}$ are such that:

$$rank\left(B_f^{(h)}\right) < rank\left(B\right) \tag{9.27}$$

and the pairs:

$$\left(A(\theta), B_f^{(h)}\left(B_f^{(h)}\right)^{\dagger} B\right) \tag{9.28}$$

are stabilizable, $\forall \theta \in \Theta$ and $\forall h = 1, \ldots, n_f$.

Problem 1 Assume that an output feedback controller (9.7)–(9.8) has been designed for the nominal system using Theorem 9.1, such that $\mathcal{E}(P, 1) \subseteq \mathcal{S}$ and $\mathcal{E}(P, 1) \subseteq \mathcal{L}(u, u^{MAX})$, and let us consider the control law:

$$u(t) = \begin{cases} u_c(t) & t < t_I \\ u_f^{(1)}(t) & t \geq t_I, \mathfrak{B}(t) = B_f^{(1)} \\ \vdots & \vdots \\ u_f^{(n_f)}(t) & t \geq t_I, \mathfrak{B}(t) = B_f^{(n_f)} \end{cases} \tag{9.29}$$

where $t_I \in \mathbb{R}^+$, $t_I \geq t_f$ is the *fault isolation time*, that is assumed to be provided by an FDI module. Design $u_f^{(1)}(t), \ldots, u_f^{(n_f)}(t)$ and maximize $\nu_f \in]0, 1]$ such that, for all $t \geq t_I$, $\mathcal{E}(P, \nu_f)$ is contractively invariant for the system (9.24)–(9.25) with the control law (9.29), and $\mathcal{E}(P, \nu_f) \subseteq \mathcal{L}(u, u^{MAX})$. □

In other words, in Problem 1, it is wished to design $u_f^{(1)}(t), \ldots, u_f^{n_f}(t)$ and maximize the value of ν_f such that it is guaranteed that if at the fault isolation time t_I:

$$\left(x(t_I)^{\mathrm{T}} \; x_c(t_I)^{\mathrm{T}} \right) P \begin{pmatrix} x(t_I) \\ x_c(t_I) \end{pmatrix} \leq \nu_f \tag{9.30}$$

then the control input will not saturate for all $t \geq t_I$, and the state trajectory will converge to the origin.

It is clear that under the assumption of *instantaneous fault isolation*, if the closed-loop state trajectory has reached $\mathcal{E}(P, \nu_f)$, the solution of Problem 1 guarantees the state trajectory convergence under fault occurrence. However, this is not the case when there is a delay in the fault isolation, i.e. $t_I - t_f > 0$. In fact, between the occurrence of the fault, that changes the system input matrix from B to some $B_f^{(h)}$, and the fault isolation time, when the appropriate control $u_f^{(h)}$ begins to be applied, there is a time interval where the system is driven by the nominal control $u_c(t)$. During this period, there is no guarantee that, if the system has reached $\mathcal{E}(P, \nu_f)$ at time t_f, it will stay inside this region until t_I. This fact can lead to severe consequences, because if the state trajectory leaves $\mathcal{E}(P, \nu_f)$ before t_I, the system could be destabilized [20].

Given $\left[x(t_f)^{\mathrm{T}} \; x_c(t_f)^{\mathrm{T}} \right]^{\mathrm{T}} \in \mathcal{E}(P, \nu_f)$, let us define, for the faulty system (9.24)–(9.25) with $\mathfrak{B}(t) = B_f^{(h)}$, $h = 1, \ldots, n_f$, under control law $u(t) = u_c(t)$, the *critical fault isolation time* $\widehat{t}_I^{(h)} \left(x(t_f), x_c(t_f) \right) \geq t_f$ as the time instant such that:

$$\begin{bmatrix} x\left(\widehat{t}_I^{(h)}\right) \\ x_c\left(\widehat{t}_I^{(h)}\right) \end{bmatrix} \in \mathcal{E}(P, \nu_f) \tag{9.31}$$

but:

$$\begin{bmatrix} x\left(\overset{\frown}{t}_I^{(h)} + t_\varepsilon\right) \\ x_c\left(\overset{\frown}{t}_I^{(h)} + t_\varepsilon\right) \end{bmatrix} \notin \mathcal{E}(P, \nu_f) \tag{9.32}$$

for all $t_\varepsilon > 0$.

Hence, it is interesting to solve the following problem, that improves the overall system robustness against the time isolation delay.

Problem 2 Find, among the output feedback controllers (9.7)–(9.8) that can be obtained from Theorem 9.1, the one that maximizes $\min_{h=1,\ldots,n_f} \hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ for all $\left[x(t_f)^{\mathrm{T}}\ x_c(t_f)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathcal{E}(P, \nu_f)$, where $\hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ is an estimation of $\overset{\frown}{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$.

Remark: The *critical fault isolation time* indicates that the guarantees of non-saturating control input and state trajectory convergence to the origin given by the solution of Problem 1 are lost if $t_I > \overset{\frown}{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$. It is worth highlighting that the conditions given in this chapter are sufficient, as always happens when using Lyapunov theory results. Hence, it is possible that the system exhibits state trajectory convergence to zero with non-saturating control input even if $t_I > \overset{\frown}{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$.

9.4 Design of Non-saturating Stabilizing LPV Virtual Actuators

The solution to Problem 1 proposed in this chapter relies on LPV virtual actuators, with a structure similar to the one presented in Chap. 8, but with the remarkable difference that the reference model is not used. In particular, the considered LPV virtual actuators are as follows:

$$\dot{x}_v^{(h)}(t) = \left(A\left(\theta(t)\right) + B_*^{(h)} M^{(h)}\left(\theta(t)\right)\right) x_v^{(h)}(t) + \left(B - B_*^{(h)}\right) u_c(t) \tag{9.33}$$

$$u_f^{(h)}(t) = N^{(h)}\left(u_c(t) - M^{(h)}\left(\theta(t)\right) x_v^{(h)}(t)\right) \tag{9.34}$$

where $h = 1, \ldots, n_f$, $x_v^{(h)}$ are the virtual actuators states with $x_v^{(h)}(t_I) = 0$, $M^{(h)}$ $(\theta(t)) \in \mathbb{R}^{n_u \times n_x}$ are the virtual actuators gains to be designed, and the matrices $N^{(h)}$ and $B_*^{(h)}$ are given by:

$$N^{(h)} = \left(B_f^{(h)}\right)^{\dagger} B \tag{9.35}$$

$$B_*^{(h)} = B_f^{(h)} N^{(h)} = B_f^{(h)} \left(B_f^{(h)}\right)^{\dagger} B \tag{9.36}$$

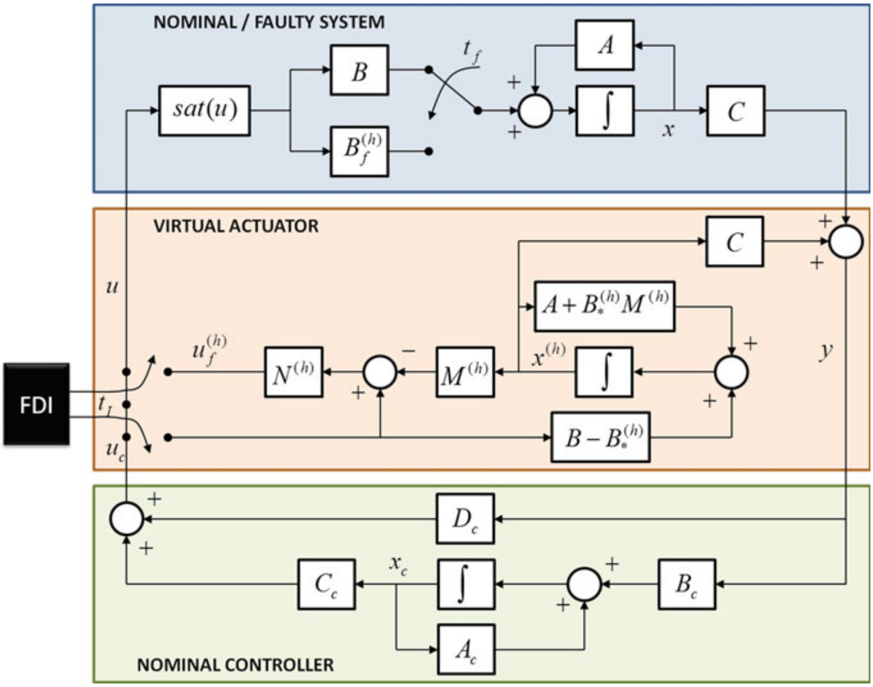


Fig. 9.1 Overall fault tolerant control scheme. After [1]

Also, in order to obtain the *fault-hiding* characteristic, the output equation (9.25) is slightly changed after t_I, as follows:

$$y(t) = C\left(x(t) + x_v^{(h)}(t)\right) \quad t \geq t_I,\ \mathfrak{B}(t) = B_f^{(h)} \tag{9.37}$$

The overall fault tolerant control scheme, made up by the system (9.24) with output equation (9.37) and control law (9.29), the output feedback controller (9.7)–(9.8) and the virtual actuators (9.33)–(9.34), is shown in Fig. 9.1 (the dependence of some matrices on the vector of varying parameters $\theta(t)$ has been omitted).

Then, the following theorem provides the conditions to design the virtual actuators with guarantees that, if at the fault isolation time t_I, the closed-loop system state is inside $\mathcal{E}(P, \nu_f)$, the state trajectory will converge to zero despite the change of the input matrix from B to $B_f^{(h)}$ due to the fault.

Theorem 9.2 (Design of non-saturating stabilizing LPV virtual actuators) *Let* $X_{va}^{-1} \in \mathbb{S}^{n_x \times n_x}$ *and* $\Gamma^{(h)}(\theta) \in \mathbb{R}^{n_u \times n_x}$, $h = 1, \ldots, n_f$ *be such that:*

$$He\left\{\begin{pmatrix} \nu_f A_{cl}(\theta) P^{-1} & O_{2n_x \times n_x} \\ \nu_f A_*^{(h)}(\theta) P^{-1} & A(\theta) X_{va}^{-1} + B_*^{(h)} \Gamma^{(h)}(\theta) \end{pmatrix}\right\} \prec O \tag{9.38}$$

$$\begin{pmatrix} X_{va}^{-1} & \Gamma_{(k)}^{(h)}(\theta)^{\mathrm{T}} \\ \Gamma_{(k)}^{(h)}(\theta) & \frac{\left(\frac{\alpha_j}{\left\|N_{(j)}^{(h)}\right\|}-\mu_f\right)^2}{n_{\tilde{u}}^{(h)}} \end{pmatrix} \succ O \qquad \begin{array}{c} j = 1, \dots, n_u \\ \left\|N_{(j)}^{(h)}\right\| \neq 0 \end{array} \tag{9.39}$$

hold $\forall\theta \in \Theta$, *where:*

$$A_{cl}(\theta) = \begin{pmatrix} A(\theta) + BD_c(\theta)C & BC_c(\theta) \\ B_c(\theta)C & A_c(\theta) \end{pmatrix} \tag{9.40}$$

$$A_*^{(h)}(\theta) = \left(\left(B - B_*^{(h)}\right) D_c(\theta)C \;\; \left(B - B_*^{(h)}\right) C_c(\theta) \right) \tag{9.41}$$

$$\mu_f = \max_{\mathcal{E}(P,\nu_f)} \|u_c\| \tag{9.42}$$

$n_{\tilde{u}}^{(h)}$ *is the number of non-zero elements in* $N_{(j)}^{(h)}$*, and* k *in (9.39) takes values corresponding to the indices of the non-zero elements in* $N_{(j)}^{(h)}$*. Then, if the virtual actuators gains* $M^{(h)}(\theta)$ *in (9.33)–(9.34) are calculated as* $M^{(h)}(\theta) = \Gamma^{(h)}(\theta)X_{va}$*,* $\mathcal{E}(P, \nu_f)$ *is contractively invariant for the system (9.24)–(9.25) with the control law (9.29), and* $\mathcal{E}(P, \nu_f) \subseteq \mathcal{L}(u, \alpha)$*,* $\forall t \geq t_I$*.*

Proof When the system is working in the region $\mathcal{L}(u, \alpha)$, there exists a similarity transformation that transforms the closed-loop system made up by the system (9.24), with output equation (9.37) and control law (9.29), the nominal controller (9.7)–(9.8), and the virtual actuator (9.33)–(9.34), in an equivalent block-triangular form:

$$\begin{pmatrix} \dot{x}_{cl}^{(h)}(t) \\ \dot{x}_v^{(h)}(t) \end{pmatrix} = \begin{pmatrix} A_{cl}(\theta) & O_{2n_x \times n_x} \\ A_*^{(h)}(\theta) & A_v^{(h)}(\theta) \end{pmatrix} \begin{pmatrix} x_{cl}^{(h)}(t) \\ x_v^{(h)}(t) \end{pmatrix} \tag{9.43}$$

where:

$$x_{cl}^{(h)}(t) = \begin{pmatrix} x_w^{(h)}(t) \\ x_c(t) \end{pmatrix} \tag{9.44}$$

$$x_w^{(h)}(t) = x(t) + x_v^{(h)}(t) \tag{9.45}$$

$$A_v^{(h)}(\theta) = A(\theta) + B_*^{(h)} M^{(h)}(\theta) \tag{9.46}$$

By considering:

$$V_2(t) = \begin{pmatrix} x_{cl}^{(h)}(t) \\ x_v^{(h)}(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \frac{P}{\nu_f} & O_{2n_x \times n_x} \\ O_{n_x \times 2n_x} & X_{va} \end{pmatrix} \begin{pmatrix} x_{cl}^{(h)}(t) \\ x_v^{(h)}(t) \end{pmatrix} \tag{9.47}$$

with $X_{va} \succ 0$ to assess the stability of (9.43), the following Lyapunov inequality is obtained:

$$He\left\{\begin{pmatrix} \frac{P}{\nu_f} & O_{2n_x\times n_x} \\ O_{n_x\times 2n_x} & X_{va} \end{pmatrix}\begin{pmatrix} A_{cl}(\theta) & O_{2n_x\times n_x} \\ A_*^{(h)}(\theta) & A_v^{(h)}(\theta) \end{pmatrix}\right\} \prec O \tag{9.48}$$

that is equivalent to its dual version [30]:

$$He\left\{\begin{pmatrix} A_{cl}(\theta) & O_{2n_x\times n_x} \\ A_*^{(h)}(\theta) & A_v^{(h)}(\theta) \end{pmatrix}\begin{pmatrix} \nu_f P^{-1} & O_{2n_x\times n_x} \\ O_{n_x\times 2n_x} & X_{va}^{-1} \end{pmatrix}\right\} \prec O \tag{9.49}$$

that can be brought to the LMI form (9.38) by considering $\Gamma^{(h)}(\theta) = M^{(h)}(\theta)X_{va}^{-1}$.

Provided that, if (9.38) holds, then the convergence of the closed-loop trajectories of (9.43) to zero is assured as long as the inputs u do not saturate, the remaining of the proof will demonstrate that, if the LMIs (9.39) hold and:

$$\left(x(t_I)^{\mathrm{T}}\; x_c(t_I)^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathcal{E}\left(P, \nu_f\right) \tag{9.50}$$

then the additional effort brought by the virtual actuator will not cause the saturation of the control inputs u.

To this aim, since (9.34) is equivalent to:

$$u_{f,j}^{(h)}(t) = N_{(j)}^{(h)}\left(u_c(t) - M^{(h)}(\theta)x_v^{(h)}(t)\right) \tag{9.51}$$

where $u_{f,j}^{(h)}$, $j = 1, \ldots, n_u$, denotes the j-th input, the condition of non-saturation can be written as:

$$\begin{aligned}\left|u_{f,j}^{(h)}(t)\right| &= \left|N_{(j)}^{(h)}\left(u_c(t) - M^{(h)}(\theta)x_v^{(h)}(t)\right)\right| \le \left\|N_{(j)}^{(h)}\right\|\left(\|u_c(t)\| + \left\|M^{(h)}(\theta)x_v^{(h)}(t)\right\|\right) \\ &\le \left\|N_{(j)}^{(h)}\right\|\left(\mu_f + \left\|M^{(h)}(\theta)x_v^{(h)}(t)\right\|\right) \le \alpha_i\end{aligned} \tag{9.52}$$

that leads to:

$$\left\|N_{(j)}^{(h)}\right\|\left(\mu_f + \left\|M^{(h)}(\theta)x_v^{(h)}(t)\right\|\right) \le \alpha_j \tag{9.53}$$

For values of j such that $\left\|N_{(j)}^{(h)}\right\| = 0$, (9.53) is obviously satisfied. On the other hand, when $\left\|N_{(j)}^{(h)}\right\| \neq 0$, (9.53) becomes:

$$\left\|M^{(h)}(\theta)x_v^{(h)}\right\| \le \frac{\alpha_j}{\left\|N_{(j)}^{(h)}\right\|} - \mu_f \tag{9.54}$$

At the expense of introducing conservativeness, it is possible to transform (9.54), whose left-hand side concerns the norm of a vector, into a condition about the norms of scalars. This is done by taking advantage of the fact that only the rows of $M^{(h)}(\theta)$

corresponding to non-zero elements of $N_{(j)}^{(h)}$ will contribute to $u_{f,j}^{(h)}$ in (9.51). By denoting these rows as $M_k^{(h)}(\theta)$, and the number of non-zero elements of $N_{(j)}^{(h)}$ as $n_{\tilde{u}}^{(h)}$, (9.54) can be replaced by:

$$\sqrt{\sum_{k=1, N_{(j)k}^{(h)} \neq 0}^{n_u} \left| M_k^{(h)}(\theta) x_v^{(h)} \right|^2} \leq \frac{\alpha_j}{\left\| N_{(j)}^{(h)} \right\|} - \mu_f \tag{9.55}$$

that holds if:

$$\left| M_k^{(h)}(\theta) x_v^{(h)} \right| \leq \frac{\left(\frac{\alpha_j}{\left\| N_{(j)}^{(h)} \right\|} - \mu_f \right)}{\sqrt{n_{\tilde{u}}^{(h)}}} \qquad \begin{array}{c} k = 1, \ldots, n_u \\ N_{(j)k}^{(h)} \neq 0 \end{array} \tag{9.56}$$

Applying Lemma 9.1, it is obtained that the existence of $Q \succ O$ such that:

$$\begin{pmatrix} Q^{-1} & Q^{-1} \left(M_{(k)}^{(h)}(\theta) \right)^{\mathrm{T}} \\ M_{(k)}^{(h)}(\theta) Q^{-1} & \frac{\left(\frac{\alpha_j}{\left\| N_{(j)}^{(h)} \right\|} - \mu_f \right)^2}{n_{\tilde{u}}^{(h)}} \end{pmatrix} \succ O \qquad \begin{array}{c} k = 1, \ldots, n_u \\ N_{(j)k}^{(h)} \neq 0 \end{array} \tag{9.57}$$

implies that (9.56) holds $\forall x_v^{(h)} \in \mathcal{E}(Q, 1)$. By choosing $Q = X_{va}$, and applying the change of variable $\Gamma^{(h)}(\theta) = M^{(h)}(\theta) X_{va}^{-1}$ the LMIs (9.39) are obtained.

Finally, it is needed to demonstrate that if (9.50) holds, then $x_v^{(h)}(t) \in \mathcal{E}(X_{va}, 1)$ $\forall t \geq t_I$. This is straightforward, since (9.50) corresponds to (9.30), that is equivalent to

$$\left(x(t_I)^{\mathrm{T}} \; x_c(t_I)^{\mathrm{T}} \right) \frac{P}{\nu_f} \begin{pmatrix} x(t_I) \\ x_c(t_I) \end{pmatrix} \leq 1 \tag{9.58}$$

and, since $x_v^{(h)}(t_I) = 0$ (see Eq. (9.33)), $x(t_I)$ in (9.58) can be replaced with $x_w^{(h)}(t_I)$, thus obtaining that $V_2(t_I) \leq 1$, where $V_2(t)$ is defined in (9.47). Due to the fact that $\dot{V}_2(t) < 0 \, \forall t \geq t_I$, it follows that $x_v^{(h)}(t) \in \mathcal{E}(X_{va}, 1)$. ■

By relying on a polytopic representation, it is possible to transform the infinite number of conditions provided by Theorem 9.2 in a finite number of conditions, as stated by the following corollary.

Corollary 9.2 (Design of non-saturating stabilizing polytopic LPV virtual actuators) *Assume that the matrices* $A(\theta(t))$, $A_c(\theta(t))$, $B_c(\theta(t))$, $C_c(\theta(t))$, $D_c(\theta(t))$ *are polytopic as in* (9.16) *and* (9.21), *with the coefficients* $\mu_i(\theta(t))$ *satisfying* (9.17), *and that the virtual actuator gain is chosen as:*

$$M^{(h)}\left(\theta(t)\right)=\sum_{i=1}^{N}\mu_i\left(\theta(t)\right)M_i^{(h)} \tag{9.59}$$

and let $X_{va}^{-1}\in\mathbb{S}^{n_x\times n_x}$ *and* $\Gamma_i^{(h)}\in\mathbb{R}^{n_u\times n_x}$, $h=1,\ldots,n_f$, $i=1,\ldots,N$, *be such that:*

$$He\left\{\begin{pmatrix}\nu_f A_{cl,i}P^{-1} & O_{2n_x\times n_x}\\ \nu_f A_{*,i}^{(h)}P^{-1} & A_iX_{va}^{-1}+B_*^{(h)}\Gamma_i^{(h)}\end{pmatrix}\right\}\prec O \tag{9.60}$$

$$\begin{pmatrix}X_{va}^{-1} & \left(\Gamma_{i(k)}^{(h)}\right)^{\mathrm{T}}\\ \Gamma_{i(k)}^{(h)} & \frac{\left(\frac{\alpha_j}{\left\|N_j^{(h)}\right\|}-\mu_f\right)^2}{n_{\tilde{u}}^{(h)}}\end{pmatrix}\succ O \quad \begin{array}{c} j=1,\ldots,n_u\\ \left\|N_j^{(h)}\right\|\neq 0\end{array} \tag{9.61}$$

hold $\forall i=1,\ldots,N$, *where:*

$$A_{cl,i}=\begin{pmatrix}A_i+BD_{c,i}C & BC_{c,i}\\ B_{c,i}C & A_{c,i}\end{pmatrix} \tag{9.62}$$

$$A_{*,i}^{(h)}=\left(\left(B-B_*^{(h)}\right)D_{c,i}C\ \left(B-B_*^{(h)}\right)C_{c,i}\right) \tag{9.63}$$

μ_f *is defined as in* (9.42), $n_{\tilde{u}}^{(h)}$ *is the number of non-zero elements in* $N_{(j)}^{(h)}$, *and k in* (9.61) *takes values corresponding to the indices of the non-zero elements in* $N_{(j)}^{(h)}$. *Then, if the vertex virtual actuators gains in* (9.59) *are calculated as* $M_i^{(h)}=\Gamma_i^{(h)}X_{va}$, $\mathcal{E}(P,\nu_f)$ *is contractively invariant for the system (9.24)–(9.25) with control law* (9.29), *and* $\mathcal{E}(P,\nu_f)\subseteq\mathcal{L}(u,\alpha)$, $\forall t\geq t_I$.

Proof It follows the reasoning provided in Corollary 9.1 and thus it is omitted. ■

Remark: The feasibility of the conditions (9.38)–(9.39) provided by Theorem 9.2 depends on the value of ν_f. The smaller is ν_f, the more likely is the feasibility of (9.38)–(9.39). Notice that, due to the block-triangularity of the state matrix in (9.43), a necessary condition for the closed-loop stability is that $A_v^{(h)}(\theta)$, defined as in (9.46), is stable. A necessary condition for the existence of $M^{(h)}(\theta)$ such that $A_v^{(h)}(\theta)$ is stable is the stabilizability of the pair $\left(A(\theta),\,B_*^{(h)}\right)$, which explains why the stabilizability of the pairs (9.28) $\forall\theta\in\Theta$ and $\forall h=1,\ldots,n_f$ was requested.

Remark: Solving Problem 1 using Theorem 9.2/Corollary 9.2 involves finding μ_f as in (9.42). μ_f can be found using optimization algorithms, e.g. the *fmincon* function in the Matlab Optimization Toolbox [31]. Due to the linearity of the control input u_c with respect to the states x and x_c (see Eq. (9.8)), it is possible to reduce the inequality constraint given by $\mathcal{E}(P,\nu_f)$ to an equality constraint, by searching the maximum of u_c on the frontier of $\mathcal{E}(P,\nu_f)$.

9.5 Robustness of the Controller Against Fault Isolation Delays

As a first step to solve Problem 2, let us consider the following theorem, that provides $\hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ for all $\left[\, x(t_f)^{\mathrm{T}} \; x_c(t_f)^{\mathrm{T}} \,\right]^{\mathrm{T}} \in \mathcal{E}(P, \nu_f)$.

Theorem 9.3 (Estimation of the critical fault isolation time) *Let* $\lambda^{(h)} \in \mathbb{R}^+$ *be such that:*

$$-2\lambda^{(h)} P + He\left\{ P \begin{pmatrix} A(\theta) + B_f^{(h)} D_c(\theta) C & B_f^{(h)} C_c(\theta) \\ B_c(\theta) C & A_c(\theta) \end{pmatrix} \right\} \prec 0 \qquad \forall \theta \in \Theta \tag{9.64}$$

and let $\left[\, x(t_f)^{\mathrm{T}} \; x_c(t_f)^{\mathrm{T}} \,\right]^{\mathrm{T}} = x_f \in \mathcal{E}(P, \nu_f)$. *Then:*

$$\left[\, x(t)^{\mathrm{T}} \; x_c(t)^{\mathrm{T}} \,\right]^{\mathrm{T}} \in \mathcal{E}(P, \nu_f) \quad \forall t \in \left[t_f, \hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right) \right] \tag{9.65}$$

with:

$$\hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right) = t_f + \frac{1}{2\lambda^{(h)}} \ln\left(\frac{\nu_f}{x_f^{\mathrm{T}} P x_f} \right) \tag{9.66}$$

Proof The faulty system (9.24)–(9.25), with $\mathfrak{B}(t) = B_f^{(h)}$, together with the output feedback controller (9.7)–(9.8), can be rewritten in the closed-loop autonomous form as:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{pmatrix} = \begin{pmatrix} A\left(\theta(t)\right) + B_f^{(h)} D_c\left(\theta(t)\right) C & B_f^{(h)} C_c\left(\theta(t)\right) \\ B_c\left(\theta(t)\right) C & A_c\left(\theta(t)\right) \end{pmatrix} \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix} \tag{9.67}$$

Let us apply Corollary 2.1 to (9.67) using the region $Re(z) < \lambda^{(h)}$, that corresponds to (2.48):

$$f_{\mathcal{D}}(z) = \alpha + \beta z + \beta^{\mathrm{T}} z^* = \left[\alpha_{kl} + \beta_{kl} z + \beta_{lk} z^* \right]_{k,l \in \{1,\ldots,m\}} \tag{9.68}$$

with $\alpha = -2\lambda^{(h)}$ and $\beta = 1$, such that (2.65):

$$\begin{aligned} &\alpha \otimes P + \beta \otimes P A(\theta) + \beta^{\mathrm{T}} \otimes A(\theta)^{\mathrm{T}} P \\ &\quad = \left[\alpha_{kl} P + \beta_{kl} P A(\theta) + \beta_{lk} A(\theta)^{\mathrm{T}} P \right]_{k,l \in \{1,\ldots,m\}} \prec O \qquad \forall \theta \in \Theta \end{aligned} \tag{9.69}$$

reads as:

$$-2\lambda^{(h)} P + He\left\{ P \begin{pmatrix} A(\theta) + B_f^{(h)} D_c(\theta) C & B_f^{(h)} C_c(\theta) \\ B_c(\theta) C & A_c(\theta) \end{pmatrix} \right\} \prec 0 \tag{9.70}$$

Hence, if (9.70) holds, (2.67):

$$\frac{1}{2}\frac{\dot{V}\left(x(t)\right)}{V\left(x(t)\right)} \in \mathcal{D} \cap \mathbb{R} \tag{9.71}$$

is true for the quadratic function:

$$V\left(x(t), x_c(t)\right) = \left(x(t)^{\mathrm{T}}\ x_c(t)^{\mathrm{T}}\right) P \left(x(t)^{\mathrm{T}}\ x_c(t)^{\mathrm{T}}\right)^{\mathrm{T}} \tag{9.72}$$

that implies:

$$V\left(x(t), x_c(t)\right) \leq V(x_f)e^{2\lambda^{(h)}(t-t_f)} = x_f^{\mathrm{T}} P x_f e^{2\lambda^{(h)}(t-t_f)} \tag{9.73}$$

By considering the condition $V\left(x(t), x_c(t)\right) \leq \nu_f$, that defines $\mathcal{E}(P, \nu_f)$, it is straightforward to obtain (9.65). ■

From (9.66) it can be seen that, to attain a solution to Problem 2, it is necessary to minimize $\lambda = \max\limits_{h=1,\ldots,n_f} \lambda^{(h)}$. Hence, this solution is given by the following corollary, which is obtained combining Theorems 9.1 and 9.3.

Corollary 9.3 (Design of a robust against fault isolation delay output feedback controller) *Let* $X, Y \in \mathbb{S}^{n_x \times n_x}$, $F(\theta) \in \mathbb{R}^{n_x \times n_y}$, $K(\theta) \in \mathbb{R}^{n_u \times n_x}$, $L(\theta) \in \mathbb{R}^{n_u \times n_y}$, $F^{(h)}(\theta) \in \mathbb{R}^{n_x \times n_y}$ *and* $N^{(h)}(\theta) \in \mathbb{R}^{n_x \times n_x}$, $h = 1, \ldots, n_f$, *correspond to the solution to the following constrained minimization problem:*

$$\min \lambda \tag{9.74}$$

subject to (9.10)–(9.12), $\lambda \geq 0$ *and:*

$$-2\lambda \begin{pmatrix} X & I \\ I & Y \end{pmatrix} + He\left\{\begin{pmatrix} XA(\theta) + F^{(h)}(\theta)C & N^{(h)}(\theta) \\ A(\theta) + B_f^{(h)}L(\theta)C & A(\theta)Y + B_f^{(h)}K(\theta) \end{pmatrix}\right\} \prec 0 \quad \begin{matrix} \forall h = 1, \ldots, n_f \\ \forall \theta \in \Theta \end{matrix} \tag{9.75}$$

Then, the output feedback controller (9.7)–(9.8), with matrices calculated as (9.13)–(9.14) maximizes $\min\limits_{h=1,\ldots,n_f} \hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ for all $\left[x(t_f)^{\mathrm{T}}\ x_c(t_f)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathcal{E}\left(P, \nu_f\right)$, where $\hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ is the estimation of $\widehat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ obtained as (9.66) with P defined as in (9.15).

Proof The design condition (9.75) corresponds to the analysis condition (9.64) with $\lambda = \lambda^{(h)}$. In fact, by applying a congruent transformation to (9.64) with:

$$\Gamma = \begin{pmatrix} I & 0 \\ Y & -Y \end{pmatrix} \tag{9.76}$$

and $\lambda = \lambda^{(h)}$, the following is obtained:

$$-2\lambda\begin{pmatrix} X & I \\ I & Y \end{pmatrix} + He\left\{\begin{pmatrix} XA(\theta) + XB_f^{(h)}C_c(\theta) & \Upsilon^{(h)}(\theta) \\ A(\theta) + B_f^{(h)}D_c(\theta)C & A(\theta)Y + B_f^{(h)}D_c(\theta)CY - B_f^{(h)}C_c(\theta)Y \end{pmatrix}\right\} < 0 \tag{9.77}$$

with:

$$\Upsilon^{(h)}(\theta) = XA(\theta)Y + XB_f^{(h)}D_c(\theta)CY - XB_f^{(h)}C_c(\theta)Y + ZB_c(\theta)CY - ZA_c(\theta)Y \tag{9.78}$$

From (9.77), (9.75) can be obtained using the following change of variables:

$$\begin{pmatrix} N^{(h)}(\theta) & F^{(h)}(\theta) \\ K(\theta) & L(\theta) \end{pmatrix} = \begin{pmatrix} XA(\theta)Y & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Z & XB_f^{(h)} \\ 0 & I \end{pmatrix}\begin{pmatrix} A_c(\theta) & B_c(\theta) \\ C_c(\theta) & D_c(\theta) \end{pmatrix}\begin{pmatrix} -Y & 0 \\ CY & I \end{pmatrix} \tag{9.79}$$

Since a common λ is being used, it is clear that $\lambda = \max_{h=1,\ldots,n_f} \lambda^{(h)}$, and by minimizing λ, we are maximizing $\min_{h=1,\ldots,n_f} \hat{t}_I^{(h)}$, defined as in (9.66). ■

Also in this case, by relying on a polytopic representation, it is possible to obtain conditions that can be applied for the design, as stated by the following corollary.

Corollary 9.4 (Design of a robust against fault isolation delay polytopic output feedback controller) *Assume that the LPV system (9.24)–(9.25) and the output feedback controller (9.7)–(9.8) are polytopic, i.e. the matrices* $A(\theta(t))$, $A_c(\theta(t))$, $B_c(\theta(t))$, $C_c(\theta(t))$, $D_c(\theta(t))$ *can be written as in* (9.16) *and* (9.21)*, with the coefficients* $\mu_i(\theta(t))$ *satisfying* (9.17)*, and let* $X, Y \in \mathbb{S}^{n_x \times n_x}$, $F_i \in \mathbb{R}^{n_x \times n_y}$, $K_i \in \mathbb{R}^{n_u \times n_x}$, $L_i \in \mathbb{R}^{n_u \times n_y}$, $F_i^{(h)} \in \mathbb{R}^{n_x \times n_y}$ *and* $N_i^{(h)} \in \mathbb{R}^{n_x \times n_x}$, $h = 1, \ldots, n_f$, $i = 1, \ldots, N$, *correspond to the solution of the constrained minimization problem* (9.74)*:*

$$\min \lambda \tag{9.80}$$

subject to (9.18)–(9.20), $\lambda \geq 0$ *and:*

$$-2\lambda\begin{pmatrix} X & I \\ I & Y \end{pmatrix} + He\left\{\begin{pmatrix} XA_i + F_i^{(h)}C & N_i^{(h)} \\ A_i + B_f^{(h)}L_iC & A(\theta)Y + B_f^{(h)}K_i \end{pmatrix}\right\} \prec 0 \quad \begin{array}{l} \forall h = 1, \ldots, n_f \\ \forall i = 1, \ldots, N \end{array} \tag{9.81}$$

Then, the controller (9.7)–(9.8), with matrices calculated as in (9.21)–(9.22) maximizes $\min_{h=1,\ldots,n_f} \hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ *for all* $\left[x(t_f)^{\mathrm{T}}\ x_c(t_f)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathcal{E}\left(P, \nu_f\right)$*, where* $\hat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ *is the estimation of* $\widehat{t}_I^{(h)}\left(x(t_f), x_c(t_f)\right)$ *obtained as* (9.66) *with* P *defined as in* (9.15).

Proof It follows the reasoning provided in Corollary 9.1 and thus it is omitted. ■

Finally, in case that the design of an FTC system that solves Problems 1 and 2 at the same time is desired, the following algorithm summarizes the necessary steps to do so.

Step 1: Find $X, Y, F_i, K_i, L_i, F_i^{(h)}, N_i^{(h)}, h = 1, \ldots, n_f, i = 1, \ldots, N$ that minimize $\lambda \geq 0$ subject to (9.18)–(9.20) and (9.81).
Step 2: Calculate the controller matrix functions $A_c(\theta(t)), B_c(\theta(t))$, $C_c(\theta(t)), D_c(\theta(t))$ using (9.21)–(9.22).
Step 3: Find $X_{va}^{-1}, \Gamma_i^{(h)}, h = 1, \ldots, n_f, i = 1, \ldots, N$ that maximize μ_f subject to (9.60)–(9.61).
Step 4: Calculate the virtual actuator vertex gains $M_i^{(h)} = \Gamma_i^{(h)} X_{va}$.

Algorithm 1: Algorithm for solving Problems 1 and 2

9.6 Example

Let us consider an open-loop unstable LPV system subject to actuator saturations as in (9.24)–(9.25), with:

$$A(\theta(t)) = \begin{pmatrix} 2+\theta(t) & 0 \\ 1 & 1.5 \end{pmatrix} \quad \theta \in [-1, 1]$$

$$B(t) = \begin{cases} B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} & t < t_f \\ B_f = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} & t \geq t_f \end{cases}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $sat(u)$ defined as in (9.6), with $u_i^{MAX} = 10, i = 1, 2$.

The matrix $A(\theta(t))$ can be easily expressed in the polytopic form (9.16) with:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1.5 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 3 & 0 \\ 1 & 1.5 \end{pmatrix}$$

By choosing $X = I$ in order to guarantee that, if $x_c(0) = 0$, the input will never saturate, and the state trajectory will converge to the origin if $x_1(0)^2 + x_2(0)^2 \leq 1$, Corollary 9.1 is applied, providing the following solution to the LMIs (9.18)–(9.20):

$$Y = \begin{pmatrix} 20.7646 & -8.5567 \\ -8.5567 & 23.2966 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} -16.9244 & 0 \\ -1.0000 & -14.6153 \end{pmatrix} \; F_2 = \begin{pmatrix} -23.9527 & 0 \\ -1.0000 & -14.6153 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} -17.3400 & -2.1823 \\ 7.0320 & -34.9140 \end{pmatrix} \; K_2 = \begin{pmatrix} -33.9554 & 4.6475 \\ 8.6895 & -33.2737 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} -0.9980 & -0.3504 \\ -0.4759 & -1.4632 \end{pmatrix} \; L_2 = \begin{pmatrix} -1.7132 & -0.5390 \\ 0.0046 & -1.2606 \end{pmatrix}$$

Then, the controller matrices are calculated using (9.22), as follows:

$$A_{c,1} = \begin{pmatrix} -17.0033 & -0.5609 \\ -0.2593 & -14.0042 \end{pmatrix} \; A_{c,2} = \begin{pmatrix} -22.4814 & -0.1261 \\ -0.7687 & -14.0818 \end{pmatrix}$$

$$B_{c,1} = \begin{pmatrix} -15.8464 & 0.4370 \\ -0.8999 & -13.8432 \end{pmatrix} \; B_{c,2} = \begin{pmatrix} -21.7951 & 0.8325 \\ -1.5366 & -14.0479 \end{pmatrix}$$

$$C_{c,1} = \begin{pmatrix} 0.0315 & 0.1215 \\ -0.1473 & 0.1562 \end{pmatrix} \; C_{c,2} = \begin{pmatrix} 0.1168 & -0.0663 \\ 0.2050 & 0.2413 \end{pmatrix}$$

$$D_{c,1} = \begin{pmatrix} -0.9980 & -0.3504 \\ -0.4759 & -1.4632 \end{pmatrix} \; D_{c,2} = \begin{pmatrix} -1.7132 & -0.5390 \\ 0.0046 & -1.2606 \end{pmatrix}$$

and the Lyapunov matrix is given by (9.15):

$$P = \begin{pmatrix} 1.0000 & 0 & 0.9433 & -0.0208 \\ 0 & 1.0000 & -0.0208 & 0.9494 \\ 0.9433 & -0.0208 & 0.9433 & -0.0208 \\ -0.0208 & 0.9494 & -0.0208 & 0.9494 \end{pmatrix}$$

Problem 1, as described in Sect. 9.4, is solved applying an iterative optimization algorithm, obtaining a maximum value $\nu_f = 0.04$, that corresponds to a value of μ_f, defined as in (9.42), equal to 1.5040.

Then, by applying Corollary 9.2, the matrix X_{va} and the virtual actuator gains M_1, M_2 are given by

$$X_{va} = \begin{pmatrix} 0.2841 & 0.5611 \\ 0.5611 & 1.3431 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} -3.0973 & -5.9092 \\ 0 & 0 \end{pmatrix} \; M_2 = \begin{pmatrix} -3.6264 & -4.9828 \\ 0 & 0 \end{pmatrix}$$

Let us consider a simulation that lasts 20 s with $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}$, $x_c(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^{\mathrm{T}}$, $\theta(t) = \sin(5t)$, and $t_f = 0.5\,s$. At first, the assumption of *instantaneous fault isolation* is done, i.e. $t_I = t_f$. Since $\begin{pmatrix} x(0)^{\mathrm{T}} & x_c(0)^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \in \mathcal{E}(P, 1)$, the state trajectory will

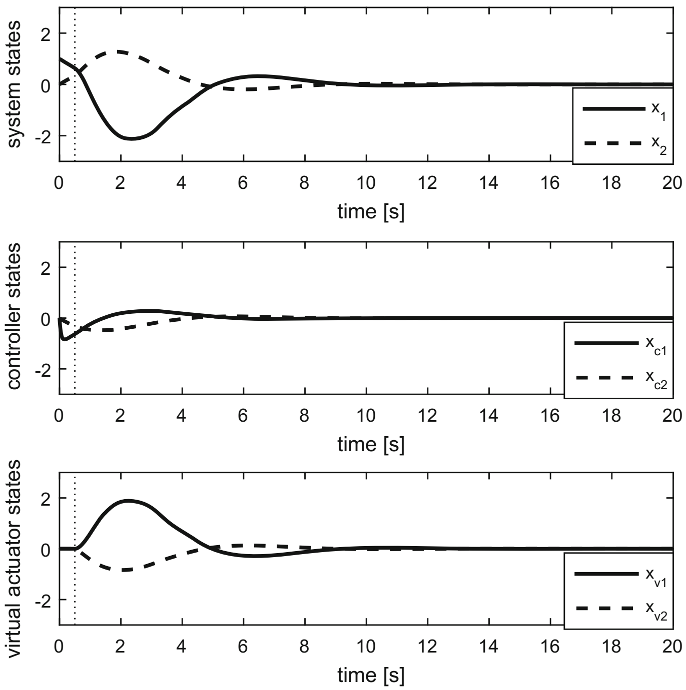
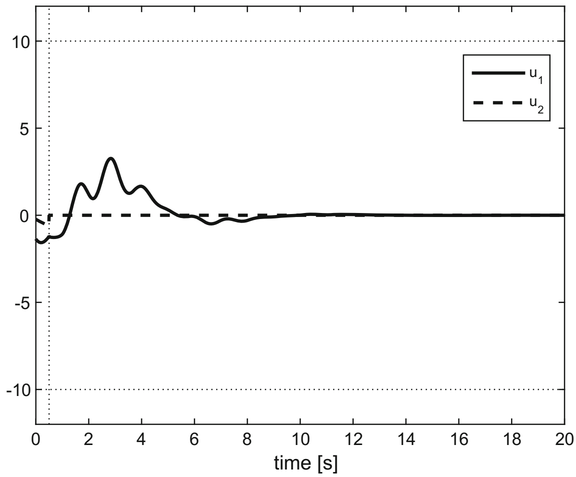


Fig. 9.2 State trajectory, $t_f = t_I = 0.5\,s$

Fig. 9.3 Control inputs, $t_f = t_I = 0.5\,s$



converge towards the origin and the control input will not saturate in the time interval $[0, t_f]$, as shown in Figs. 9.2 and 9.3, respectively.

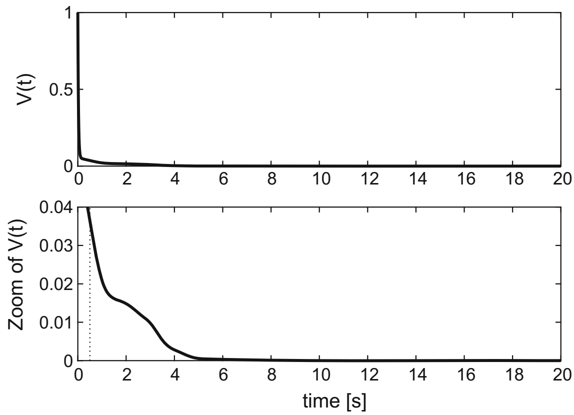


Fig. 9.4 Lyapunov function $V(t)$, $t_f = t_I = 0.5\,s$

Also, as shown in Fig. 9.4, the evolution of the Lyapunov function

$$V(t) = \begin{pmatrix} x_{cl}(t) \\ x_v(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P & O_{4\times 2} \\ O_{2\times 4} & \nu_f X_{va} \end{pmatrix} \begin{pmatrix} x_{cl}(t) \\ x_v(t) \end{pmatrix}$$

is such that $V(t_f) = 0.0364 < \nu_f = 0.04$. Hence, according to Corollary 9.2, the activation of the virtual actuator at time $t_I = t_f$ guarantees that the system trajectory will converge to the origin with non-saturating control inputs despite the change in the input matrix from B to B_f. This is shown in Fig. 9.2, where it can be seen clearly that, due to the activation of the virtual actuator, the states x_{v1} and x_{v2} take values different from zero, and in Fig. 9.3, where the reconfiguration of the control inputs brought by the change in the control law from $u_c(t)$ to $u_f(t)$ is depicted. Also, as expected, the Lyapunov function $V(t)$ takes decreasing-in-time values despite the fault occurrence, as shown in Fig. 9.4.

To conclude the analysis of the results, let us analyze the trajectories in the phase planes, shown in Figs. 9.5, 9.6 and 9.7. It can be seen that the evolution of $x(t)$, $x_c(t)$ and $x_v(t)$ is such that at time t_I all the states are inside $\mathcal{E}(P, \nu_f)$, whose projections in the considered phase plane are depicted in magenta color. After t_I, the state $x_w(t) = x(t) + x_v(t)$ continues smoothly the evolution of the state $x(t)$ before t_I; on the other hand, $x(t)$ will eventually converge to the origin because both $x_w(t)$ and $x_v(t)$ will do so.

Let us now consider the more realistic case where $t_I > t_f$, using the results obtained in Sect. 9.5. The application of Theorem 9.3 gives a value $\lambda = 1.9197$, that corresponds to:

$$\hat{t}_I = t_f + \frac{1}{3.8394} \ln\left(\frac{0.04}{x_f^{\mathrm{T}} P x_f}\right)$$

At time $t_f = 0.5\,s$, $x_f^{\mathrm{T}} P x_f = 0.0364$, such that $\hat{t}_I = 0.025\,s$, i.e. if the fault isolation is performed within $0.025\,s$, the system state is guaranteed to be inside $\mathcal{E}(P, \nu_f)$

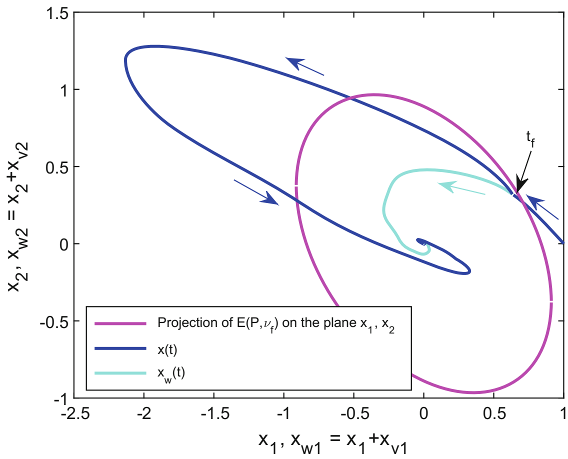


Fig. 9.5 Phase plane of $x(t)$ and $x_w(t)$, $t_f = t_I = 0.5\,s$

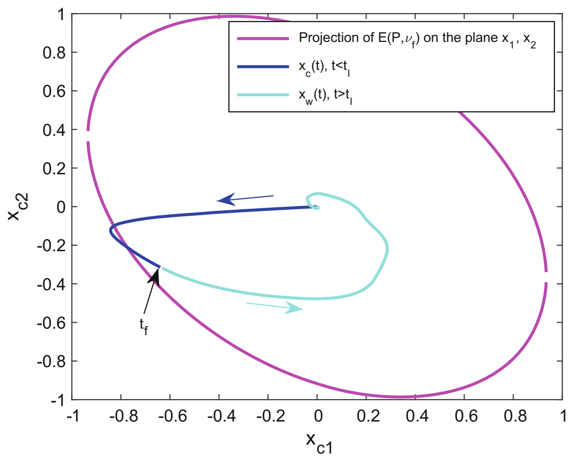


Fig. 9.6 Phase plane of $x_c(t)$, $t_f = t_I = 0.5\,s$

for $\nu_f = 0.04$ when the control $u_f(t)$ begins to be used instead of $u_c(t)$. This is confirmed by the simulation, as shown in Fig. 9.8. It is worth remarking that $\hat{t}_I$ is only an estimation of the critical fault isolation time that, for the considered example, can be determined by various simulations as $\widehat{t}_I = 0.748\,s$.

By applying Corollary 9.4, a value $\lambda = 0$ is achieved with the controller matrices:

$$A_{c,1} = \begin{pmatrix} -15.0033 & -3.8448 \\ -4.6729 & -23.0056 \end{pmatrix} A_{c,2} = \begin{pmatrix} -14.4869 & -3.9639 \\ -4.3637 & -23.0946 \end{pmatrix}$$

$$B_{c,1} = \begin{pmatrix} -11.6573 & 4.2273 \\ -3.1405 & -17.6345 \end{pmatrix} B_{c,2} = \begin{pmatrix} -11.7806 & 2.8664 \\ -2.7281 & -17.5742 \end{pmatrix}$$

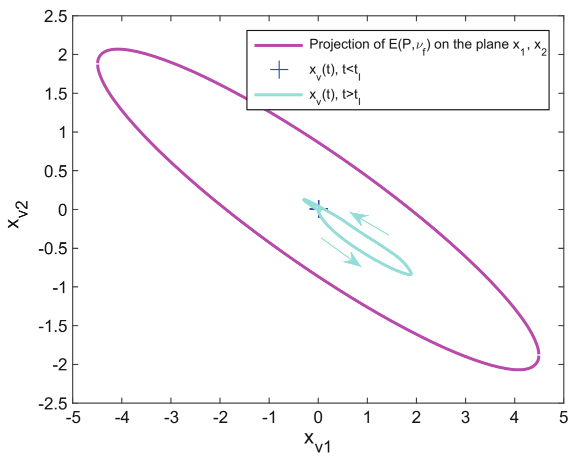


Fig. 9.7 Phase plane of $x_v(t)$, $t_f = t_I = 0.5\,s$

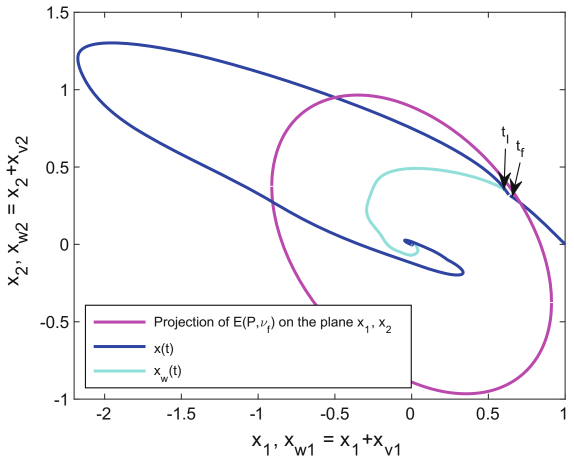


Fig. 9.8 Phase plane of $x(t)$ and $x_w(t)$, $t_f = 0.5\,s$, $t_I = 0.525\,s$

$$C_{c,1} = \begin{pmatrix} 0.2476 & 0.0168 \\ -0.0335 & -0.1618 \end{pmatrix} \; C_{c,2} = \begin{pmatrix} -0.0125 & 0.0227 \\ -0.0633 & -0.0698 \end{pmatrix}$$

$$D_{c,1} = \begin{pmatrix} -1.5938 & -3.1773 \\ -0.8091 & -2.5079 \end{pmatrix} \; D_{c,2} = \begin{pmatrix} -2.5406 & -2.5509 \\ -1.0639 & -2.7606 \end{pmatrix}$$

and Lyapunov matrix:

$$P = \begin{pmatrix} 1.0000 & 0 & 0.9143 & -0.1412 \\ 0 & 1.0000 & -0.1412 & 0.6099 \\ 0.9143 & -0.1412 & 0.9143 & -0.1412 \\ -0.1412 & 0.6099 & -0.1412 & 0.6099 \end{pmatrix}$$

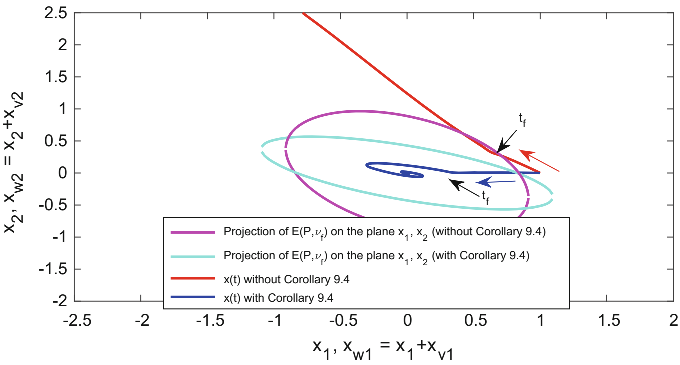


Fig. 9.9 Comparison between the state trajectories obtained with the controllers designed using Corollaries 9.1 and 9.4 ($\lambda = 0, t_f = 0.5\,s$), respectively, when no fault isolation is performed during the simulation

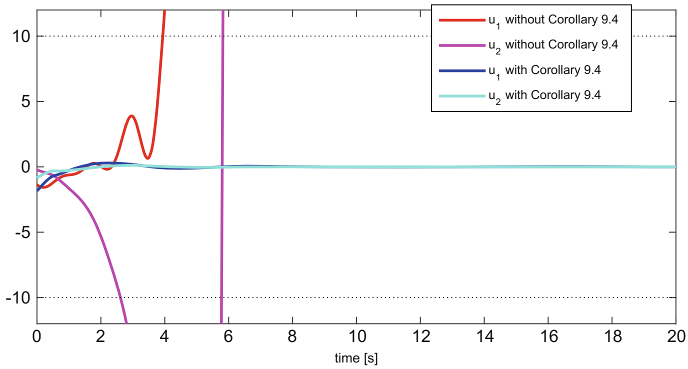


Fig. 9.10 Control inputs with and without applying Corollary 9.4, $t_f = 0.5\,s$, and no fault isolation during the simulation ($t_I > 20\,s$)

Notice that achieving the case $\lambda = 0$, that would correspond to $\hat{t}_I = \infty$ using (9.66), is equivalent to the existence of a nominal controller that is robust against the considered fault. In fact, by repeating the simulation with this controller, assuming that the fault is not isolated during the simulation, it can be seen that the state trajectory with the nominal controller will still converge to zero despite the fault occurrence (see blue line in Fig. 9.9). On the other hand, the closed-loop system with the controller that had been designed without applying Corollary 9.4, i.e. using Corollary 9.1, is such that the state trajectory diverges if no fault isolation is performed (see red line in Figs. 9.9 and 9.10).

9.7 Conclusions

In this chapter, the problem of FTC of unstable LPV systems subject to actuator saturation and fault isolation delay has been considered. The adopted solution relies on virtual actuators, a fault-hiding active FTC strategy that reconfigures the faulty plant instead of the controller. Some conditions have been obtained for designing the virtual actuators in such a way that it is guaranteed that, if at the fault isolation time the closed-loop system state is inside a region defined by a value of the Lyapunov function, the state trajectory will converge to zero despite the fault and, moreover, the inputs will not saturate at any time. Afterwards, the problem of delays in the fault isolation has been considered by showing that an estimation of the allowed fault isolation delay can be obtained by analyzing the Lyapunov function using the notion of LMI regions. Moreover, the nominal controller can be designed so as to maximize the allowed fault isolation delay.

A numerical example has shown the effectiveness of the proposed strategy. In particular, it has been demonstrated that the proposed design strategy enhances the performances of the control system against fault isolation delay. As a special result of the design conditions, it has been obtained a controller that is robust against the considered fault, such that no fault isolation is needed for the system to keep its stability, and for the state trajectory to converge asymptotically to zero under fault occurrence.

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Chapter 10
Conclusions and Future Work

This thesis has proposed some contributions to the field of LPV systems, with an emphasis to their application to FTC. This chapter summarizes the work presented in this thesis, in order to review the main conclusions and explore the possibilities of further research.

10.1 Conclusions

LPV and TS systems can incorporate the nonlinear and time-varying behavior of some systems, allowing to deal with them using linear-like techniques. They have been investigated throughout the last decades, and several theoretical results have been presented in the literature. Nevertheless, there is still space for further investigation, and this thesis has contributed to the advancement of the state-of-the-art of this field.

- Chapter 3 has addressed the strong similarities between polytopic LPV and TS models. It has been shown how techniques developed for the former framework can be easily extended in order to be applied to the latter, and vice versa. In particular, the method for automated generation of LPV models by nonlinear embedding has been extended to generate automatically TS models from a given nonlinear system. Similarly, the method for the generation of a TS model based on the sector nonlinearity concept has been extended to the problem of generating a polytopic LPV model for a given nonlinear dynamical system. With these many alternatives for the generation of LPV/TS models, it becomes relevant to compare the obtained models in order to choose which one could be considered *the best one*. To this end, two measures have been proposed, the first one based on the notion of *overboundedness*, and the second one based on *region of attraction estimates*. Results obtained with a mathematical example have shown that the automated

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generation via nonlinear embedding provides less conservative models than the automated generation via sector nonlinearity, which has been justified using the mean-value theorem.

- Chapter 4 has considered the problem of designing an LPV state-feedback controller for uncertain LPV systems. The controller has been designed such that some desired performances are achieved in the robust LPV sense, i.e. for each possible value that the scheduling parameters and the uncertainty can take. Some well-known results obtained in the last decades in the robust and in the LPV control fields have been extended to obtain conditions that can be used to solve this problem. The provided solution relies on a double-layer polytopic description that takes into account both the variability due to the scheduling parameter vector and the uncertainty. The first polytopic layer manages the varying parameters and is used to obtain the vertex uncertain systems, where the vertex controllers are designed. The second polytopic layer is built at each vertex system so as to take into account the model uncertainties and add robustness into the design step. The problem has been tackled using both a common quadratic Lyapunov function and a parameter-dependent quadratic Lyapunov function. In both cases, under some assumptions, a finite number of LMIs, that can be solved efficiently using available solvers, has been obtained. The proposed technique has been applied to numerical examples, showing that it achieves correctly the desired performances, i.e. robust $\mathcal{D}$-stability and robust $\mathcal{H}_\infty$ performance, whereas the traditional LPV gain-scheduling technique fails.
- Chapter 5 has considered the problem of designing a parameter-scheduled state-feedback controller that satisfies a new kind of specifications, referred to as *shifting specifications*. In particular, the concepts of $\mathcal{D}$-stability, $\mathcal{H}_\infty$ performance, $\mathcal{H}_2$ performance, finite time boundedness and finite time stability have been extended in a *shifting* sense, introducing the *shifting $\mathcal{D}$-stability*, *shifting $\mathcal{H}_\infty$ performance*, *shifting $\mathcal{H}_2$ performance*, *shifting finite time boundedness* and *shifting finite time stability* specifications. The main idea behind these new specifications is to introduce some varying parameters, or using the existing ones, to design the controller in such a way that different values of these parameters imply different performances. The solution to the design problem, expressed in the form of LMIs for which a feasible solution should be found, has been obtained using a common quadratic Lyapunov function. The results obtained with academic examples have demonstrated the effectiveness and some characteristics of the proposed approach. In particular, in contrast with the classical specifications, the design using the shifting ones has allowed selecting different performances for different values of the scheduling parameters, thus allowing the online variation of the control system performance.
- In Chap. 7, the idea of the robust LPV polytopic technique has been applied to FTC, giving rise to different strategies. A passive FTC strategy has been obtained by considering the faults as exogenous perturbations that should be rejected. An active FTC strategy has been obtained by considering the faults as additional scheduling parameters. Finally, a hybrid FTC strategy has been obtained by taking into account explicitly the fault estimation uncertainty during the design step. It has also been

shown how the proposed FTC strategy can be used for the implementation of a bank of controllers, such that the signal provided by the fault diagnosis unit determines which controller should be active at a given moment. The advantage of the reconfigured controllers with respect to the non-reconfigured ones lies in that the formers have to cope with specific faults and allow to improve the performances in the non-faulty case using the nominal controller, whose design does not take into account the possibility of fault occurrence. The proposed method has been applied to solve the FTC problem for a quadrotor UAV. The results presented have shown the relevant features of the proposed FTC strategy, that is able to improve the performances under fault occurrence. In particular, whereas the passive FTC shows some limited tolerance capability, because of the appearance of steady-state errors due to the fault effect, the active FTC technique can achieve a perfect fault tolerance as long as the fault is correctly estimated. However, as the uncertainty in the fault estimation increases, so does the error between the real trajectory and the reference one. By applying the proposed hybrid FTC method, the overall performance can be improved, thus reducing the effect that the fault estimation error has on the closed-loop response. The introduction and comparison of some performance measures have allowed confirming numerically such analysis.

- Chapter 8 has proposed an FTC strategy for LPV systems subject to actuator faults based on model reference control and virtual actuators. The proposed FTC strategy adapts the reference model to the faults and utilizes the virtual actuator technique in order to recover the nominal stability and behavior of the error model, with some minimum or graceful performance degradation. The overall control loop is made up by an LPV error feedback controller, an LPV error observer and the LPV virtual actuator. It has been shown that the principle of separation holds, since there exists a similarity transformation that brings the augmented model to a block-triangular form. Hence, the stability and the satisfaction of the desired specifications can be assessed separately. The potential and performance of the proposed approach have been demonstrated with two different examples: a twin rotor MIMO system and a four wheeled omnidirectional mobile robot, showing promising results.
- In Chap. 9, a solution to the problem of FTC of unstable LPV systems subject to actuator saturation and fault isolation delay based on virtual actuators has been proposed. Some conditions have been obtained for designing the virtual actuators in such a way that it is guaranteed that, if at the fault isolation time the closed-loop system state is inside a region defined by a value of the Lyapunov function, the state trajectory will converge to zero despite the fault and, moreover, the inputs will not saturate at any time. Afterwards, the problem of delays in the fault isolation has been considered by showing that an estimation of the allowed fault isolation delay can be obtained by analyzing the Lyapunov function using the notion of LMI regions. Moreover, the nominal controller can be designed so as to maximize the allowed fault isolation delay. A numerical example has shown the effectiveness of the proposed strategy. In particular, it has been demonstrated that the proposed design strategy enhances the performances of the control system against fault isolation delay. As a special result of the design conditions, a controller that is

robust against the considered fault has been obtained, such that no fault isolation is needed for the system to keep its stability, and for the state trajectory to converge asymptotically to zero under fault occurrence.

10.2 Perspectives and Future Work

This section resumes the open issues that could be addressed in future work.

- The measures proposed in Chap. 3 have shown to be objective criteria that can be used to select which model can be considered *the best one*. However, in the general case, which model is *the best one* also depends on the context in which the model is used, i.e. whether it is used for stabilization or observation, and which structure of controller/observer is used to achieve the desired goal. It seems clear that an important issue to be addressed in future research is the development of an automatic procedure that selects the best model during the design step, taking into account what the model is used for, and the chosen structure for the controller/observer.
- The dilation of the matrix inequality characterizations and the introduction of auxiliary variables allow using parameter-dependent Lyapunov functions in order to assess stability or other desired specifications. Appendix A has shown how new dilated LMIs for the FTB and the FTS analysis can be obtained in the case of DT systems. In Chap. 4, these results allowed using a parameter-dependent quadratic Lyapunov function for solving the problem of robust finite time state-feedback control of uncertain LPV systems. However, the obtention of dilated LMIs for the FTB/FTS analysis of CT systems is still an open issue that requires further investigation. This step is necessary in order to obtain conditions for the design of robust FTB/FTS polytopic state-feedback controllers for uncertain CT LPV systems.
- The examples in Chap. 5 have demonstrated how the design using shifting specifications allows varying online the control system performance. However, the LMIs to be solved in order to design the controller, have been obtained using a common quadratic Lyapunov function, which is potentially conservative. An interesting line for future research would be investigating the application of other types of Lyapunov functions, e.g. parameter-dependent ones, in order to decrease the conservativeness of the solution. Also, a future comparison of the proposed approach with the use of parameter-dependent weighting functions could be interesting.
- The theory developed in Chap. 7 has been applied successfully to a quadrotor UAV simulator. Future research will be aimed at applying the proposed FTC strategy to a real set-up. This goal brings additional challenges, due to the presence of many sources of uncertainties that must be taken into account in order to enforce the robustness of the FTC strategy. Moreover, since the inclusion of an FDI module can potentially allow increasing the obtainable performance, future research could investigate FDI and fault estimation algorithms that can be applied successfully to quadrotor UAVs.

- The technique developed in Chap. 8 has achieved fault tolerance using a mix of reference model reconfiguration and virtual actuators. However, the theory has been developed under the assumption of perfect knowledge of the system model and perfect fault estimation. Future research on this topic will aim at improving the robustness of the proposed FTC strategy against model uncertainties and errors in the fault estimation.
- The approach developed in Chap. 9 for the FTC of unstable LPV systems subject to actuator saturations and fault isolation delays has been devoted to regulation, i.e. convergence of the state trajectory to zero. However, in many control applications, it is desired that the state trajectory tracks a desired reference trajectory. Future research will extend the proposed technique to solve the problem of fault tolerant tracking of open-loop unstable LPV systems subject to actuator saturations and fault isolation delays.

Appendix A
Dilated LMIs for the Finite Time Boundedness and Stability Analysis of Discrete-Time Systems

The content of this appendix is based on the following work:

- [1] D. Rotondo, F. Nejjari, V. Puig. Dilated LMI characterization for the robust finite time control of discrete-time uncertain linear systems. *Automatica*, 63:16–20, 2016.

When the robust finite time control of uncertain linear systems is considered, the existing works either use a common Lyapunov function [2] or rely on differential linear matrix inequalities (DLMIs) [3]. Recently, [4] showed that, by *dilating* the matrix inequality characterizations and introducing auxiliary variables, the technical restriction to a common Lyapunov variable could be overcome. The suggested approach led to a new set of matrix inequalities that included the original ones as a particular case, and that had a structure such that parameter-dependent Lyapunov functions could be easily applied in the case of real polytopic uncertainty, with the consequence of reducing the conservatism. This idea has been successfully applied to the case of pole clustering [5], $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control [6]. However, the cases of FTS and FTB have never been tackled before using the aforementioned approach.

In this appendix, we provide new dilated LMI characterizations for the FTB and the FTS analysis. The dilated LMIs have the relevant feature of decoupling between the Lyapunov and the system matrices. This fact allows considering parameter-dependent Lyapunov functions easily, thus decreasing the conservativeness with respect to the classical quadratic results.

The following lemma, known as *Schur complement* will be used [7].

Lemma A.1 *Let a matrix* $\Phi = \Phi^{\mathrm{T}}$ *be such that:*

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^{\mathrm{T}} & \Phi_{22} \end{pmatrix} \tag{A.1}$$

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Then, the following three conditions are equivalent:

1. $\Phi \prec O$
2. $\Phi_{11} \prec O, \quad \Phi_{22} - \Phi_{12}^{\mathrm{T}} \Phi_{11}^{-1} \Phi_{12} \prec O$
3. $\Phi_{22} \prec O, \quad \Phi_{11} - \Phi_{12} \Phi_{22}^{-1} \Phi_{12}^{\mathrm{T}} \prec O$

Proof See [7]. ∎

Also, let us recall the following result, known as *Elimination Lemma* [8, 9].

Lemma A.2 *Let matrices* $E \in \mathbb{R}^{n_Y \times n_E}$*,* $D \in \mathbb{R}^{n_D \times n_Y}$ *and* $Y \in \mathbb{S}^{n_Y \times n_Y}$ *be given. Then, the following two conditions are equivalent:*

1. *The following two conditions hold:*

$$\begin{cases} E^{\perp} Y \left(E^{\perp}\right)^{\mathrm{T}} \prec O & if\ n_Y > n_E \\ E E^{\mathrm{T}} \succ O & if\ n_Y \leq n_E \end{cases} \tag{A.2}$$

$$\begin{cases} \left(D^{\mathrm{T}}\right)^{\perp} Y D^{\perp} \prec O & if\ n_Y > n_D \\ D^{\mathrm{T}} D \succ O & if\ n_Y \leq n_D \end{cases} \tag{A.3}$$

2. *There exists a matrix* $F \in \mathbb{R}^{n_E \times n_D}$ *such that:*

$$Y + He\{EFD\} \prec O \tag{A.4}$$

Proof See [8]. ∎

Hereafter, new dilated LMIs for analyzing the finite time boundedness and the finite time stability properties of discrete-time LTI systems are obtained, as stated by the following theorems.

Theorem A.1 (Extended FTB of DT LTI systems) *The DT LTI system:*

$$\begin{cases} x(k+1) = Ax(k) + B_w w(k) \\ w(k+1) = W w(k) \end{cases} \tag{A.5}$$

is FTB with respect to (c_1, c_2, T, R, d) *if there exist positive scalars* α*,* λ_1*,* λ_2 *with* $\alpha \geq 1$*, two positive definite matrices* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and* $Q_2 \in \mathbb{S}^{n_w \times n_w}$*, and two matrices* $S_1 \in \mathbb{R}^{n_x \times n_x}$ *and* $S_2 \in \mathbb{R}^{n_w \times n_w}$ *such that:*

$$\begin{pmatrix} -\alpha\left(S_1 + S_1^{\mathrm{T}} - Q_1\right) & S_1 A^{\mathrm{T}} & O & O \\ A S_1 & -Q_1 & B_w & O \\ O & B_w^{\mathrm{T}} & -\alpha Q_2 & W^{\mathrm{T}} S_2 \\ O & O & S_2^{\mathrm{T}} W & Q_2 - S_2 - S_2^{\mathrm{T}} \end{pmatrix} \prec O \tag{A.6}$$

and (2.92)–(2.94):

$$\lambda_1 R^{-1} \prec Q_1 \prec R^{-1} \tag{A.7}$$

$$O \prec Q_2 \prec \lambda_2 I \tag{A.8}$$

$$\begin{pmatrix} \frac{c_2}{\alpha^T} - \lambda_2 d & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{A.9}$$

hold.

Proof The proof is inspired by the results obtained in [4]. We must show that (2.91) with $A(\theta) = A$, $B_w(\theta) = B_w$ and $W(\theta) = W$:

$$\begin{pmatrix} -\alpha Q_1 & Q_1 A^{\mathrm{T}} & O & O \\ A Q_1 & -Q_1 & B_w & O \\ O & B_w^{\mathrm{T}} & -\alpha Q_2 & W^{\mathrm{T}} Q_2 \\ O & O & Q_2 W & -Q_2 \end{pmatrix} \prec O \tag{A.10}$$

is equivalent to (A.6).

We first show that (A.10) implies (A.6). In fact, if (A.10) holds, we can choose $S_1 = S_1^{\mathrm{T}} = Q_1$ and $S_2 = S_2^{\mathrm{T}} = Q_2$ in (A.6) and recover (A.10).

It remains to show that (A.6) implies (A.10). To do so, let us assume that (A.6) holds, and let us notice that it can be rewritten as:

$$\begin{pmatrix} \alpha Q_1 & O & O & O \\ O & -Q_1 & B_w & O \\ O & B_w^{\mathrm{T}} & -\alpha Q_2 & O \\ O & O & O & Q_2 \end{pmatrix} + He\left\{\begin{pmatrix} -\alpha I & O \\ A & O \\ O & W^{\mathrm{T}} \\ O & -I \end{pmatrix}\begin{pmatrix} S_1 & O \\ O & S_2 \end{pmatrix}\begin{pmatrix} I & O & O & O \\ O & O & O & I \end{pmatrix}\right\} \prec O \tag{A.11}$$

such that Lemma A.2 can be applied with:

$$Y = \begin{pmatrix} \alpha Q_1 & O & O & O \\ O & -Q_1 & B_w & O \\ O & B_w^{\mathrm{T}} & -\alpha Q_2 & O \\ O & O & O & Q_2 \end{pmatrix} \tag{A.12}$$

$$E = \begin{pmatrix} -\alpha I & O \\ A & O \\ O & W^{\mathrm{T}} \\ O & -I \end{pmatrix} \tag{A.13}$$

$$E^{\perp} = \begin{pmatrix} A & \alpha I & O & O \\ O & O & I & W^{\mathrm{T}} \end{pmatrix} \tag{A.14}$$

$$F = \begin{pmatrix} S_1 & O \\ O & S_2 \end{pmatrix} \tag{A.15}$$

$$D = \begin{pmatrix} I & O & O & O \\ O & O & O & I \end{pmatrix} \tag{A.16}$$

Hence, (A.2) becomes:

$$\begin{pmatrix} \alpha A Q_1 A^{\mathrm{T}} - \alpha^2 Q_1 & \alpha B_w \\ \alpha B_w^{\mathrm{T}} & W^{\mathrm{T}} Q_2 W - \alpha Q_2 \end{pmatrix} \prec O \tag{A.17}$$

that, using a congruence transformation with $diag(\alpha^{-1} I, I)$, is equivalent to:

$$\begin{pmatrix} \alpha^{-1} A Q_1 A^{\mathrm{T}} - Q_1 & B_w \\ B_w^{\mathrm{T}} & W^{\mathrm{T}} Q_2 W - \alpha Q_2 \end{pmatrix} \prec O \tag{A.18}$$

According to Lemma A.1, (A.10) is equivalent to (A.18), thus completing the proof. ■

Theorem A.2 (Extended FTS of DT LTI systems) *The DT LTI system:*

$$x(k+1) = Ax(k) \tag{A.19}$$

is FTS with respect to (c_1, c_2, T, R) *if there exist positive scalars* α, λ, *with* $\alpha \geq 1$, *a positive definite matrix* $Q \in \mathbb{S}^{n_x \times n_x}$ *and a matrix* $S \in \mathbb{R}^{n_x \times n_x}$ *such that:*

$$\begin{pmatrix} -\alpha \left(S + S^{\mathrm{T}} - Q \right) & S^{\mathrm{T}} A^{\mathrm{T}} \\ AS & -Q \end{pmatrix} \prec O \tag{A.20}$$

and (2.98)–(2.99) hold:

$$\begin{pmatrix} \frac{c_2}{\alpha^T} & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{A.21}$$

$$\lambda_1 R^{-1} \prec Q_1 \prec R^{-1} \tag{A.22}$$

Proof It is a direct consequence of Theorem A.1, when $B_w = O$, $W = O$ and $d = 0$. ■

Appendix B
Proof of the Independence of the Matrix $B^*(\theta(\tau))$ from $f(\tau)$

In this appendix, it is proved that the matrix $B^*(\theta(\tau))$, introduced in (8.17), as follows:

$$B^*(\theta(\tau)) = B_f(\theta(\tau), f(\tau))\, N\left(\theta(\tau), \hat{f}(\tau)\right) \tag{B.1}$$

with:

$$N\left(\theta(\tau), \hat{f}(\tau)\right) = B_f\left(\theta(\tau), \hat{f}(\tau)\right)^{\dagger} B(\theta(\tau)) \tag{B.2}$$

is independent from $f(\tau)$.

Let us assume that the nominal input matrix $B(\theta(\tau))$ is full, as follows:

$$B(\theta(\tau)) = \begin{pmatrix} b_{11}(\theta(\tau)) & b_{21}(\theta(\tau)) & \cdots & b_{1n_u}(\theta(\tau)) \\ b_{12}(\theta(\tau)) & b_{22}(\theta(\tau)) & \cdots & b_{2n_u}(\theta(\tau)) \\ \vdots & \vdots & \ddots & \vdots \\ b_{n_x1}(\theta(\tau)) & b_{n_x2}(\theta(\tau)) & \cdots & b_{n_xn_u}(\theta(\tau)) \end{pmatrix} \tag{B.3}$$

Without loss of generality, the case where the first n_f actuators are completely lost, i.e. where the matrix $F(f(\tau))$ in (8.11) takes the following form:

$$F(f(\tau)) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{n_f+1}(\tau) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & f_{n_f+2}(\tau) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & f_{n_u}(\tau) \end{pmatrix} \tag{B.4}$$

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is considered. Hence, the matrix $B_f\left(\theta(\tau), f(\tau)\right)$, calculated following (8.10), is[1]:

$$B_f\left(\theta(\tau), f(\tau)\right) = \begin{pmatrix} 0 & 0 & \cdots & 0 & b_{1(n_f+1)} f_{n_f+1} & b_{1(n_f+2)} f_{n_f+2} & \cdots & b_{1n_u} f_{n_u} \\ 0 & 0 & \cdots & 0 & b_{2(n_f+1)} f_{n_f+1} & b_{2(n_f+2)} f_{n_f+2} & \cdots & b_{2n_u} f_{n_u} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & b_{n_x(n_f+1)} f_{n_f+1} & b_{n_x(n_f+2)} f_{n_f+2} & \cdots & b_{n_x n_u} f_{n_u} \end{pmatrix} \tag{B.5}$$

that can be rewritten as:

$$B_f\left(\theta(\tau), f(\tau)\right) = \Lambda\left(\theta(\tau)\right) \Upsilon\left(f(\tau)\right) \tag{B.6}$$

with:

$$\Lambda\left(\theta(\tau)\right) = \begin{pmatrix} b_{1(n_f+1)}\left(\theta(\tau)\right) & b_{1(n_f+2)}\left(\theta(\tau)\right) & \cdots & b_{1n_u}\left(\theta(\tau)\right) \\ b_{2(n_f+1)}\left(\theta(\tau)\right) & b_{2(n_f+2)}\left(\theta(\tau)\right) & \cdots & b_{2n_u}\left(\theta(\tau)\right) \\ \vdots & \vdots & \ddots & \vdots \\ b_{n_x(n_f+1)}\left(\theta(\tau)\right) & b_{n_x(n_f+2)}\left(\theta(\tau)\right) & \cdots & b_{n_x n_u}\left(\theta(\tau)\right) \end{pmatrix} \tag{B.7}$$

$$\Upsilon\left(f(\tau)\right) = \begin{pmatrix} 0 & 0 & \cdots & 0 & f_{n_f+1}(\tau) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & f_{n_f+2}(\tau) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & f_{n_u}(\tau) \end{pmatrix} \tag{B.8}$$

Then, $B_f\left(\theta(\tau), \hat{f}(\tau)\right)^{\dagger}$ can be calculated as follows [10]:

$$B_f\left(\theta(\tau), \hat{f}(\tau)\right)^{\dagger} = \Upsilon\left(\hat{f}(\tau)\right)^{\mathrm{T}} \left(\Upsilon\left(\hat{f}(\tau)\right) \Upsilon\left(\hat{f}(\tau)\right)^{\mathrm{T}}\right)^{-1} \left(\Lambda\left(\theta(\tau)\right)^{\mathrm{T}} \Lambda\left(\theta(\tau)\right)\right)^{-1} \Lambda\left(\theta(\tau)\right)^{\mathrm{T}} \tag{B.9}$$

It is quite straightforward to show that:

$$\Upsilon\left(\hat{f}(\tau)\right)^{\mathrm{T}} \left(\Upsilon\left(\hat{f}(\tau)\right) \Upsilon\left(\hat{f}(\tau)\right)^{\mathrm{T}}\right)^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \frac{1}{\hat{f}_{n_f+1}(\tau)} & 0 & \cdots & 0 \\ 0 & \frac{1}{\hat{f}_{n_f+2}(\tau)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\hat{f}_{n_u}(\tau)} \end{pmatrix} \tag{B.10}$$

---

[1]Dependence of the elements b_{ij} and f_j, $i = 1, \ldots, n_x$, $j = n_f + 1, \ldots, n_u$, on $\theta(\tau)$ and τ, respectively, is skipped to ease the notation.

such that, under the assumption that $\hat{f}(\tau) \cong f(\tau)$:

$$
\begin{aligned}
B^*(\theta(\tau)) &= B_f(\theta(\tau), f(\tau))\, B_f(\theta(\tau), f(\tau))^\dagger\, B(\theta(\tau)) \\
&= B(\theta(\tau)) \begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & f_{n_f+1}(\tau) & 0 & \cdots & 0 \\
0 & \cdots & 0 & f_{n_f+2}(\tau) & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & f_{n_u}(\tau)
\end{pmatrix} \cdots \\
&\cdots \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\frac{1}{f_{n_f+1}(\tau)} & 0 & \cdots & 0 \\
0 & \frac{1}{f_{n_f+2}(\tau)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{f_{n_u}(\tau)}
\end{pmatrix} \left(\Lambda(\theta(\tau))^{\mathrm{T}} \Lambda(\theta(\tau))\right)^{-1} \Lambda(\theta(\tau))^{\mathrm{T}} \qquad \text{(B.11)} \\
&= B(\theta(\tau)) \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix} \left(\Lambda(\theta(\tau))^{\mathrm{T}} \Lambda(\theta(\tau))\right)^{-1} \Lambda(\theta(\tau))^{\mathrm{T}}
\end{aligned}
$$

that does not depend on $f(\tau)$. This completes the proof.

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