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Ignacy Kaliszewski
Janusz Miroforidis
Dmitry Podkopaev

Multiple Criteria Decision Making by Multiobjective Optimization

A Toolbox

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To all
who struggle with decisions

Contents

List of Figures xi

List of Tables xiii

Preface xv

Conventions xix

About the Authors xxi

1 Introduction **1**

1.1 Chapter Content 1
1.2 Basic Notions 2
1.3 Sum-Up 13
1.4 Creative Assignment 13

2 Solving Decision Problems **15**

2.1 Chapter Content 15
2.2 The Decision Process Scheme 16
2.3 Sum-Up 18

3 Decision Problem: Selection of a Single Variant **19**

3.1 Chapter Content 19
3.2 Problem Formulation 20
3.3 Problem Modeling 20
3.4 Variant Selection 22
3.5 Problem Verification 23

3.6 Sum-Up 23
3.7 Creative Assignment 23

4 Derivation of Efficient Variants 25
4.1 Chapter Content 25
4.2 Algorithms to Derive Efficient Variants 26
4.3 Sum-Up 35
4.4 Creative Assignment 35

5 Decision Problem: Selection of a Variant Portfolio—The Discrete Case 37
5.1 Chapter Content 37
5.2 Problem Formulation 39
5.3 Problem Modeling 39
5.4 Variant Selection 41
5.5 Problem Verification 41
5.6 Sum-Up 42
5.7 Creative Assignment 42

6 Decision Problem: Selection of a Variant Portfolio—The Continuous Case 43
6.1 Chapter Content 43
6.2 Decision Problem: Selection of Mixture Composition 44
6.3 Problem Formulation 44
6.4 Problem Modeling 45
6.5 Variant Selection 47
6.6 Problem Verification 48
6.7 Sum-Up 48
6.8 Creative Assignment 48

7 Derivation of Efficient Portfolios 51
7.1 Chapter Content 51
7.2 Characterization of Efficient Outcomes: Characterization A 52
7.3 Characterization of Efficient Outcomes: Characterization B 54

7.4 Efficiency Test 56
7.5 Derivation of Efficient Variants and Variant Ranking 57
7.6 Weight Normalization 57
7.7 Scalarizing Functions and Value Functions 58
7.8 Sum-Up 58
7.9 Creative Assignment 59

8 Supporting the Process of the Most Preferred Variant Selection 61
8.1 Chapter Content 61
8.2 Searching Over the Set of Efficient Variants . . . 62
8.3 Searching for Efficient Outcomes with the Compromise Half Line as a Guideline 68
8.4 Decision Process Support 69
8.5 Sum-Up 70
8.6 Creative Assignment 71

9 Decision Problems, Continuation 73
9.1 Chapter Content 73
9.2 Selection of a Single Variant, Continuation 74
9.3 Selection of a Variant Portfolio—The Discrete Case, Continuation 78
9.4 Selection of a Variant Portfolio—The Continuous Case, Continuation 83
9.5 Sum-Up 95
9.6 Creative Assignment 96

10 Decision Problem: Selection of a Stock Portfolio 97
10.1 Chapter Content 97
10.2 Problem Formulation 98
10.3 Problem Modeling 99
10.4 Variant Selection 100
10.5 Problem Verification 101
10.6 Sum-Up 101
10.7 Creative Assignment 102

11 Relations **103**
11.1 Chapter Content 103
11.2 Definition of Relation 103
11.3 Properties of Relations 104
11.4 Ordering Relations 105
11.5 Ordered Sets 106
11.6 Maximal Elements and Greatest Elements 106
11.7 Relations and Preference Relations 108
11.8 Sum-up 109
11.9 Creative Assignment 109

Electronic Companion **111**

References **113**

Index **115**

Glossary **117**

List of Figures

1.1 An example of the decision problem with two decision variants represented by numerical criteria, one of them is ideal 3
1.2 An example of the decision problem with two decision variants represented by numerical criteria, none of them is ideal 4
1.3 Pairs of decision variants in the Pareto dominance relation, represented by numerical criteria—case I 7
1.4 Pairs of decision variants in the Pareto dominance relation, represented by numerical criteria—case II 9
1.5 Pairs of decision variants in the Pareto dominance relation, represented by numerical criteria—case III 9

2.1 Four phases of the decision making process 16

4.1 A graphical interpretation of relation (4.1). 27

7.1 A graphical interpretation of deriving efficient outcomes by optimization problem (7.1). 53
7.2 A graphical interpretation of deriving efficient outcomes in set Z which is not convex, by optimization problem (7.2). 55
7.3 A graphical interpretation of deriving efficient outcomes in convex set Z, by optimization problem (7.2). 56

8.1 Four phases of decision making problems with efficient variant derivation in phase 3 62
8.2 Vector of concessions τ and the corresponding compromise half line 63

8.3 A graphical interpretation of deriving the most preferred outcome among outcomes located on the compromise half line 64
8.4 The case, where the compromise half line contains no outcome 65
8.5 The case, where the outcome on the compromise half line, which is closest to the element y^*, is not efficient 66
8.6 A graphical interpretation of deriving an efficient outcome by optimization problem (8.3) ((7.1))—case I. 67
8.7 A graphical interpretation of deriving an efficient outcome by optimization problem (8.3) ((7.1))—case II. 68

10.1 An example of the set of outcomes for a four stock portfolio 98

11.1 A graphical interpretation of the relation from Example 11.2.2 104
11.2 Hasse diagrams of ordered sets 105
11.3 Hasse diagrams of ordered sets with and without greatest elements 106

List of Tables

1.1 Two decision variants, one of them is ideal 4
1.2 Two decision variants, none of them is ideal . . . 4
1.3 Data to Example 1.2.5 7
1.4 Data to Example 1.2.5—an extension 10

3.1 The table representation of the river crossing problem 22

4.1 Values of criteria in Example 4.2.3 34

9.1 Data for project portfolio selection 78
9.2 Table of food prices and ingredient content in products 85
9.3 Table of ingredient content in products, adjusted for wastes 86
9.4 Table of safe and recommended ingredient content in daily diets for girls in the age [. . .] 87
9.5 Table of ingredient content in products, adjusted for wastes—supplement I 89
9.6 Table of ingredient content in products, adjusted for wastes—supplement II 90

Preface

Science is fun
Authors' conviction

Multiple criteria decision making is a field researched vastly and deeply over the last 40 years. As the result, a multitude of approaches and methods have been proposed, and those are well represented in books and journal papers. A researcher in this field, actual or perspective, has many choices.

On the other hand, any other person, in want to start quickly with decision making in a multiple criteria context, would face a serious problem. Where should one find a set of notions and prescriptions, simple but comprehensive—a *toolbox*? A *universal toolbox*, domain and application unspecific?

This textbook comes to such persons in assistance.

The messages we convey in the text are as follows:

– Multiple criteria decision making is a right framework to handle decision making problems whenever more than one criterion comes into play.

– Multiple criteria decision making offers a toolbox, meant as above, to handle such problems, and this toolbox sets a relatively low cognitive barrier for its potential users.

However, there are prerequisites for an easy reading of the textbook and the effective use of the toolbox. It is assumed that the reader is already acquainted with the notions of *optimal value* (*maximal value*, *minimal value*) of *function* under *constraints* on *arguments* of that function.

An *optimization model*, i.e., the set of three elements,

– a *criterion function*,

– a *maximization operator* or *minimization operator*,

– a *set of constraints* on *arguments* of the criterion function,

is a formal framework for modeling many economic, technical, and social phenomena, taking the form of *decision making problems*. By filling data into an optimization model, we get an optimization problem. The values of arguments which yield the optimal value of the criterion function, i.e., the *optimal solution* to the optimization problem, represent the *most preferred decision variant*. Considerations of that sort are the subject of *operations research*.

Optimization models are too often oversimplifications of decision problems met in practice. For instance, modeling company performance by an optimization model, in which the criterion function is short-term profit to be maximized, does not fully reflect the essence of business management. The company managing staff is accountable not only for operational decisions but also for actions which shall result in the company's ability to generate a decent profit in the future. This calls for management decisions and actions which ensure short-term profitability but also maintaining long-term relations with clients, introducing innovative products, financing long-term investments, etc. Each of those indispensable actions and effects they produce can be modeled separately, case by case, by an optimization model with a criterion function adequately selected. However, in each case, the same set of constraints represents the range of company admissible actions. The aim and the scope of this textbook is to present methodologies and methods enabling modeling of such actions jointly.

This textbook is primarily intended for students of a PhD program in a field related to management science, operational research, or industrial mathematics, but it can also serve as a base for a graduate course. However, for a PhD program it is recommended to add some illustrative examples and/or problems from the specific students' domain.

The textbook is self-contained; to follow it the reader does not need to refer to additional sources. However, the references given at the end can be a starting point for further research into methodologies and applications of decision making in the *multiple criteria setting*.

Acknowledgments

The content of this textbook draws to the large extent from the authors' research conducted in the Systems Research Institute of the Polish Academy of Sciences and also from the first author's research and lecturing activity at the Warsaw School of Information Technology (WIT).

The continuous assistance and support of these institutions are kindly acknowledged.

Warszawa, Poland — Ignacy Kaliszewski
Wrocław, Poland — Janusz Miroforidis
Warszawa, Poland — Dmitry Podkopaev

Conventions

Notation

$\mathcal{R}^k$	set of k-component vectors with real values ($\mathcal{R}$ for $k = 1$)
$A, \ldots, Z$	sets
N	set of natural numbers
R^k_+	nonnegative orthant of $\mathcal{R}^k$
x, y	elements (vectors) of a set
x_i, y_i	i-th component of x, y
$\{x^1, x^2, \ldots\}$	set composed of elements $x^1, x^2, \ldots$
$\emptyset$	empty set
$\in$	membership
$\subseteq$	set inclusion
$\cup$	union of sets
$\setminus$	difference of sets
$\lvert\cdot\rvert$	cardinality of a set or absolute value of a number
$\succ$	relation
$\square$	end of proof
$:=$	assignment
$/$	(in combination with relation sign, e.g., $\not\subseteq$) negation

Number Rounding Off

We adopt the following convention for representing and rounding off numbers:

– Data (hypothetical or actual) are given without rounding.

– Numbers calculated from formulas given in the textbook are rounded off to three decimal digits.

– Numbers calculated by an optimization package are rounded off accounting for their physical interpretation.

All optimization calculations in the textbook are done in *Microsoft Excel* spreadsheets using the add-in *Solver*.

About the Authors

Ignacy Kaliszewski, Full Professor at the Systems Research Institute of the Polish Academy of Sciences, graduated from the Technical University of Warsaw. He got his Ph.D. and habilitation degrees from the Systems Research Institute of the Polish Academy of Science for his research in quantitative management science and operations research. He has published over 40 scientific papers in journals and books of international circulation and two monographs: Quantitative Pareto Analysis by Cone Separation Technique (Kluwer Academic Publ., Dordrecht, 1994), Soft Computing for Complex Multiple Criteria Decision Making (Springer, 2006). His current field of research is decision making in multicriteria environment.

Janusz Miroforidis, Assistant Professor at the Systems Research Institute of the Polish Academy of Sciences, received his M.S. degree in computer science from the University of Wrocław, and his Ph.D. from the Systems Research Institute of the Polish Academy of Science for his research in soft computing and MCDM methods for management needs. His major research interests include computer aided multiple criteria decision making, particularly as applied to complex decision problems and evolutionary multiobjective optimization. He is a co-founder of Treeffect, a consulting company.

Dmitry Podkopaev, Assistant Professor at the Systems Research Institute of the Polish Academy of Sciences, graduated from the Belarusian State University and got his Ph.D. degree in mathematics from the National Academy of Sciences of Belarus. He has published over 25 journal articles in the fields of discrete and multiobjective optimization, and their applications in industry, economy and environmental sciences. His current field of research is multiobjective preference modeling and its application in decision making.

Chapter 1

Introduction

Trurl let the machine warm up first, kept the power low, ran up the metal stairs several times to take readings... .[1]

1.1 Chapter Content

This chapter contains the definition of the decision problem and a collection of notions necessary to analyze the decision making problem in the multiple criteria setting.

[1] All mottos in this textbook are excerpts from Stainsław Lem, "The Cyberiad", Harvest/HBJ Books, 1985.

Electronic supplementary material The online version of this chapter (doi: 10.1007/978-3-319-32756-3_1) contains supplementary material, which is available to authorized users.

I. Kaliszewski et al., *Multiple Criteria Decision Making by Multiobjective Optimization*, International Series in Operations Research & Management Science 242, DOI 10.1007/978-3-319-32756-3_1

1.2 Basic Notions

In a somewhat simplified but general manner, the *decision making problem* is defined as

for a given set of decision variants
select a decision variant which
in a given decision context
is the most preferred one. (1.1)

Decision variants are compared with the use of some selected *criterion*,[2] (or, as we do in the textbook, with the use of some selected criteria) which provide for decision *variant valuations*.

The sequence of actions which lead to the selection of the most preferred decision variant is called the *decision process.*

Example 1.2.1 *Examples of decision variants*
- (choice of university) Warsaw University, Jagiellonian University, Helsinki University of Technology, London School of Economics;
- (choice of technology) wind farm, hydro power station, biomass, solar energy;
- (choice of route) Warsaw-Vienna via Brno, Warsaw-Vienna via Munich.

Example 1.2.2 *Examples of criteria*
- (numerical or qualitative) tuition, cost, time of traversing, sustainability, landscape attractiveness.

Since we have assumed that we shall deal with more than one criterion, the selection of the most preferred decision variant, as shown in this chapter, is not trivial.

1.2.1 Ideal Variants

Let us observe that even in the case of two or more criteria there are situations where among the decision variants, one decision

[2] Variants have *attributes.* Attributes selected to compare decision variants become *criteria.*

variant is more preferred than any other. Such decision variants are called *ideal*. Obviously, the ideal variant, if exists, is the most preferred variant.

Table 1.1 and the corresponding Fig. 1.1 represent an example of the decision problem with just two decision variants, one of them is ideal (this is variant x^1).

Table 1.2 and the corresponding Fig. 1.2 represent an example of the decision problem with just two decision variants, none of them is ideal (because there does not exist a decision variant with the most preferred criterion values represented by element $\hat{y}$ of $\mathcal{R}^2$).

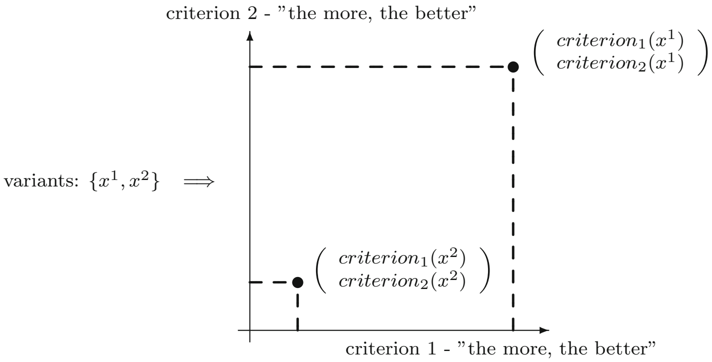

Figure 1.1 An example of the decision problem with two decision variants represented by numerical criteria, one of them is ideal

1.2.2 Dominance

In the sequel, we assume that decision problems are non-trivial, i.e., they do not include ideal variants. However, this assumption does not exclude cases where in a pair of variants, one decision variant is preferred over the other with respect to all criteria. For such cases the notion of *dominance* applies, as defined below.

For example, if decision variants are compared with respect to the cost criterion (think of buying a house), then it is natural to agree that the cheapest house is the most preferred variant.

But what if there is another criterion? When buying a house we certainly do not forget about its square footage. In the popular belief, the larger house, the better (in a sensible range, of course).

Table 1.1 Two decision variants, one of them is ideal

	Criterion type	Criterion value Decision variant x^1	Decision variant x^2
Criterion 1	"the more, the better"	75	25
Criterion 2	"the more, the better"	75	25

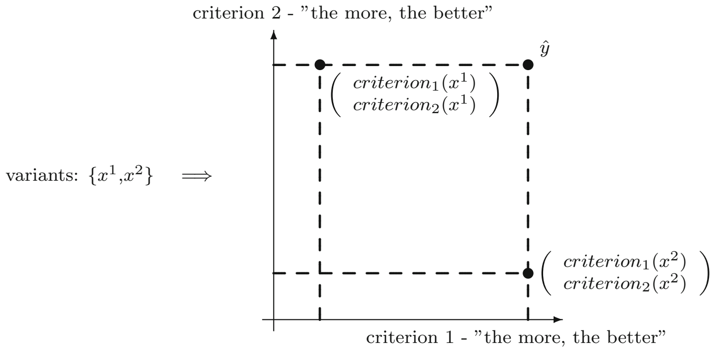

Figure 1.2 An example of the decision problem with two decision variants represented by numerical criteria, none of them is ideal

Table 1.2 Two decision variants, none of them is ideal

	Criterion type	Criterion value Decision variant x^1	Decision variant x^2
Criterion 1	"the more, the better"	25	75
Criterion 2	"the more, the better"	75	25

But how to proceed if both criteria are considered jointly? Could the cheapest house among many be the largest offered? If this were the case, such a house would be a real bargain, but on real estate markets usually the bigger house, the higher price. However, sometimes it may happen that a bigger house is cheaper than a smaller one. Such a situation is formalized by the already mentioned notion of dominance.

Given a set of decision variants, decision variant x is called *dominated*, if in this set there exists another decision variant, say decision variant x', such that

– decision variant x' is preferred at least as much as decision variant x with respect to all criteria,

and

– decision variant x' is more preferred than decision variant x with respect to at least one criterion.

If this is the case, we say that decision variant x' *dominates* decision variant x.

We say also that two decision variants, where one dominates another, are in *Pareto dominance relation*.[3]

Obviously, in a set containing more than two decision variants, the same decision variant can dominate one decision variant or a number of decision variants, and at the same time it can be dominated by another decision variant or a number of decision variants.

Example 1.2.3 *A trip by bike from Warsaw to Paris certainly takes more time than a trip by train. But the trip by bike can also be more expensive if we take into account costs of lodging and food. If the trip by bike takes more time and is more expensive than the trip by train, decision variant "train" dominates decision variant "bike". On the other hand, decision variant "plane" for the same reasons (cheap airlines!), can dominate decision variant "train".*

In the scope of this textbook, we assume that all criteria are numerical. Moreover, we assume that all criteria are of "the more, the better" type.

With these assumption in force, the definition of dominated decision variant takes the form as follows.

Given a set of decision variants, decision variant x is called dominated, if in this set there exists another decision variant, say decision variant x', such that

[3] Named after Vilfredo Federigo Damaso Pareto, 1848–1923, Italian economist and sociologist.

– the values of all criteria for decision variant x' are not smaller than for decision variant x,
and
– the value of at least one criterion for decision variant x' is greater than for decision variant x.

If a criterion is of the type "the less, the better", in all further considerations we shall transform it to the type "the more, the better" by multiplying all its values by -1. By this, we will be able to interpret the interplay of criteria in a uniform manner.

Example 1.2.4 *To illustrate the effect of transforming a criterion of the type "the less, the better" to the type "the more, the better", one probably cannot find a better example than the pair of two notions: "loss" and "profit". Certainly, the smaller loss, the better. Let us interpret loss as negative profit, which is very common among accountants. As we all agree, profit (irrelevant of its sign) is the higher, the better. Hence, minimizing loss and maximizing negative profit are equivalent actions.*

In terms of loss, among three decision variants bringing losses, respectively loss of $2, 4$ *and* 7 *units, the best is the first one. This is equivalent to the statement that, in terms of profit, among three decision variants bringing profits, respectively* $-2, -4$ *and* -7 *units, the best decision variant is the first one.*

1.2.3 Efficiency

The notion of dominance leads us directly to the notion of *efficiency.* This notion, like the notion of dominance, comes out naturally when one compares decision variants with respect to more than one criterion.

Given a set of decision variants, a decision variant which is not dominated by any other decision variant of this set, is called *efficient.*

Example 1.2.5 *Among three decision variants* x^1, x^2, x^3*, valuated with respect to two criteria as in Table 1.3, decision variant* x^1 *is not dominated by any other decision variant, therefore it* *is an efficient variant. The same holds for decision variant x^2. Decision variant x^3 is not efficient because it is dominated by decision variant x^2.*

In other words, given a set of decision variants, decision variant x is efficient, if in this set there does not exist other decision variant, say decision variant x', such that

– the values of all criteria for decision variant x' are at least as great as for decision variant x,

and

– the value of at least one criterion for decision variant x' is greater than for decision variant x.

Table 1.3 Data to Example 1.2.5

	x^1	x^2	x^3
Criterion 1	7	4	3
Criterion 2	5	6	6

Conditions for a decision variant to be efficient are weaker than to be ideal. Therefore, efficient variants are more common than ideal variants.

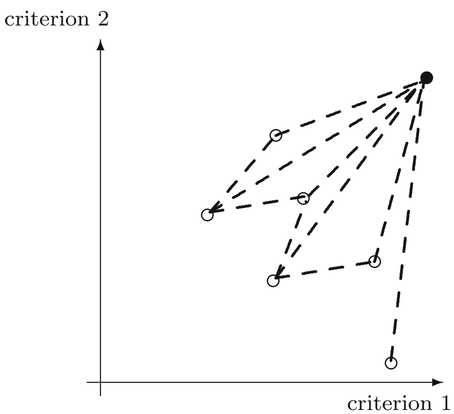

Figure 1.3 Pairs of decision variants in the Pareto dominance relation, represented by numerical criteria—case I

An ideal variant is efficient. The opposite does not hold, i.e., an efficient variant is not, in general, ideal.

With the convention that discs represent valuations of efficient variants and circles represent valuations of dominated decision variants, Figs. 1.3 and 1.4 present decision variant valuations with respect to two criteria, both of the type “the more, the better”, for a couple of variants. Dashed lines indicate pairs of decision variants in the Pareto dominance relation. In Fig. 1.3, one decision variant is clearly ideal (and therefore efficient). By definition, this decision variant dominates any other decision variant from the set.

Figure 1.4 shows the valuations of decision variants from Fig. 1.3 after removing the ideal variant from the set. In this case, there are several efficient variants.

The illustrations given above apply for the cases where sets of decision variants are finite. If sets of decision variants are given implicitly by conditions (constraints), then they can be infinite. If a set of decision variants is infinite, then all dominance relations cannot be sensibly presented graphically, however this can be done for selected pairs of decision variants, as it is shown in Fig. 1.5. The figure presents decision variant valuations with respect to two criteria for an infinite set of variants, where the set of valuations has the form of a polygon. In this case, efficient variants are those which valuations form the part of the polygon border drawn in thick line.

In decision making processes, decision variants which are dominated (i.e., are not efficient) are as a rule ignored, because clearly they are not rational (commonsense) candidates for the most preferred variant. In consequence, the only candidates for the most preferred variant remain efficient variants (In the house buying example given above, if one can have more for less, why doing the opposite? Why not to take advantage of a clear opportunity?).

To summarize, the most preferred variant is always selected from the efficient variants.

However, it is not recommended to remove dominated decision variants permanently from the set of variants. In a dynamic decision making environment (e.g., when the set of criteria may change), dominated decision variants can become not dominated (i.e., efficient). This is illustrated by the following example.

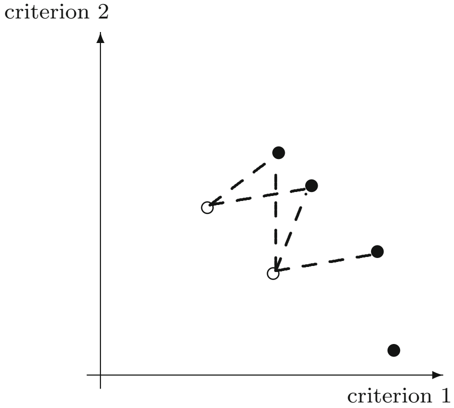

Figure 1.4 Pairs of decision variants in the Pareto dominance relation, represented by numerical criteria—case II

Example 1.2.6 *Consider three decision variants $x^1, x^2,\ x^3$, valuated with respect to two criteria, as in Example 1.2.5. After adding the third criterion, taking values as in Table 1.4, no decision variant is dominated by another decision variant, hence with respect to three criteria all three decision variants are efficient.*

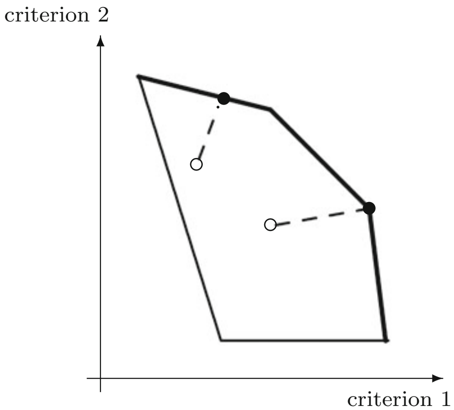

Figure 1.5 Pairs of decision variants in the Pareto dominance relation, represented by numerical criteria—case III

Table 1.4 Data to Example 1.2.5—an extension

	x^1	x^2	x^3
Criterion 1	7	4	3
Criterion 2	5	6	6
Criterion 3	1	5	8

1.2.4 Multiple Criteria Decision Making

To keep our considerations precise, we need a dose of formalism and to this aim we define the Multiple Criteria Decision Making (MCDM) problem in terms of decision variants, criteria, and multiple criteria variant valuations.

The formal (in the sense: mathematical) model of the MCDM problem has the form

$$\textit{select a variant } x \in X_0 \subseteq X \textit{ for which } f(x)$$
$$\textit{is the most preferred multiple criteria variant valuation}, \tag{1.2}$$

where

– X is the *decision space*,

– X_0 is the set of decision variants,[4,5]

– $f : X \to \mathcal{R}^k$ is the *criteria mapping*, $f = (f_1, \ldots, f_k)$,

– $f_l : X \to \mathcal{R}$ are the *criteria functions*, $l = 1, \ldots, k$.[6]

The decision space X contains all conceivable variants, but only variants from X_0—decision variants—are considered. As we are only interested in decision variants, in the sequel we will often abbreviate this term to just *variants*. When we will occasionally speak about variants which belong to X but not to X_0, we will stress this fact explicitly.

[4] Set X_0 can be defined by a set of constraints, e.g., by a set of inequalities and/or equations (see the material in subsequent chapters), or given explicitly.

[5] In the optimization domain (i.e., when $k = 1$), decision variants, i.e., elements of set X_0, are called *feasible solutions*.

[6] In the optimization domain, the criterion function is customarily called the *objective function*. Throughout the textbook we shall use the first term.

As already said, we assume that all criteria are of "the more, the better" type.

In the MCDM model, variant x for which $f(x)$ is the most preferred multiple criteria variant valuation is the *most preferred variant.*

The model (1.2) is not operational (in the sense—practical). Indeed, as long as the notion "the most preferred" is not precisely defined, we are not able to propose any procedure to derive the most preferred variant.

The knowledge of what the notion "the most preferred" in given circumstances actually means is almost always in the sole possession of the person accountable for the final decision—the *decision maker* (DM). The fundamental (and having firm ground in the surrounding world) paradigm, underlying all methodologies we are concerned here with, is that knowledge cannot be fully elicited from the decision maker before the decision process starts. This paradigm significantly complicates formalization of decision making processes.

In the sequel, we shall be concerned with construction of tools for decision making process support, which despite of the above paradigm, enable the DM to identify the most preferred variant.

The problem

$$\text{vmax } f(x)$$

$$\text{subject to } x \in X_0 \subseteq X\,, \tag{1.3}$$

where vmax denotes the operator of derivation of all efficient variants, is called *multiobjective*[7] *optimization* problem (MO).

We often exploit problem (1.3) for decision process modeling since it is almost always well-defined. By "well-defined" we mean that under weak assumptions on properties of functions f_l, $l = 1, \ldots, k$, and set X_0, satisfied in the vast majority of practical applications, problem (1.3) can be solved. Hence, the set of efficient variants for that problem can be derived by a formal, not necessarily finite, procedure.

[7] It is named so by tradition, but we mean by this that multiple criteria are involved.

For the sake of presentation clarity, below, whenever convenient, we use notation y and Z, where

$$y = f(x),\ Z = f(X_0) = \{f(x) \mid x \in X_0\}\,.$$

Clearly, $Z \subseteq \mathcal{R}^k$. Elements of set Z (multiple criteria variant valuations) in the MCDM domain are customarily called *outcomes*, and the space $\mathcal{R}^k$—the *outcome space*.

With this convention, given decision variant x, $y_i = f_i(x)$ denotes the value of i-th component of its outcome $y = f(x)$. In other words, y_i is the value of i-th criterion for variant x.

Outcomes of efficient variants are called *efficient outcomes*.

The outcome of the most preferred variants (there could be multiple variants with the same outcome) is called the *most preferred outcome*.

In the sequel, we make use of two distinguished elements of outcome space $\mathcal{R}^k$, namely element $\hat{y}$ and element y^*.

Element $\hat{y}$ of $\mathcal{R}^k$, called the *ideal element*, is defined as

$$\hat{y}_l = \max_{y \in Z} y_l,\ l = 1, \ldots, k.$$

We assume that all the maxima exist.

As said at the beginning of this chapter, it can happen that element $\hat{y}$ represents no element of X_0, i.e., there does not exist variant $x \in X_0$, such that $f(x) = \hat{y}$.

But it can also happen that element $\hat{y}$ represents no element of decision space X, i.e., there does not exist variant $x \in X$, such that $f(x) = \hat{y}$.

Element y^* of outcome space $\mathcal{R}^k$ is found as

$$y^*_l > \hat{y}_l\,,\ l = 1, \ldots, k. \tag{1.4}$$

In particular, it can be defined as

$$y^*_l = \hat{y}_l + \varepsilon,\ l = 1, \ldots, k,$$

where $\varepsilon > 0$.

1.3 Sum-Up

This chapter is the key to master the material of the textbook. The notions introduced thus far are used in all remaining chapters. Therefore, a thorough understanding of this chapter is absolutely necessary for absorbing the remaining part of the material.

1.4 Creative Assignment

Formulate your own multiple criteria decision making problem MY_PROBLEM_1.

Have the number of factors (later: variables) not too small but still being easily manageable.

Keep the problem for the future use in connection with the textbook.

At no point you will be asked to present your problem, just keep it for yourself as your private field for your own hands-on experiments.

As the textbook evolves, you will be supplied with some hints on what further actions you can take, making use of your knowledge accumulated thus far.

Chapter 2

Solving Decision Problems

"So I deliberated, read up on the problem, went methodically through several libraries, pored over all sorts of ancient tomes, until one day I found the answer... ."

2.1 Chapter Content

This chapter presents the decision process scheme, its principal phases and also introduces the generic idea of decision problem solving. The idea of the scheme is to repeat the principal phases of the process in cycles, till the DM concludes that among variants identified in the decision process, one variant can be regarded, in his/her opinion, as the most preferred variant.

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2.2 The Decision Process Scheme

As said before, the decision process ends up when the most preferred variant is selected.

The following four generic phases can be distinguished in any decision process[1]:

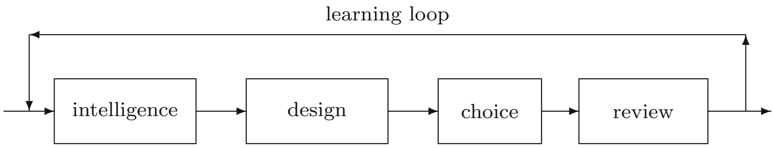

Figure 2.1 Four phases of the decision making process

– intelligence (1),
– design (2),
– choice (3),
– review (4).

These four phases are closed in the loop called the *learning loop*.

The scheme of the decision process is graphically represented in Fig. 2.1. In contrast to algorithms, the scheme does not have a clearly defined stopping rule. The decision process is stopped when the DM finds it is expedient. We refer here to decision makers which are free to make their choices without any external ties, as it is the case where the decision process is not formalized by any set of rules.

A good example of a non-formalized process is planning one's professional career. On the other extreme, the process of selecting the most preferred offers in public tenders is, as a rule, highly formalized.

Below, we shall be concerned only with processes for which no formal rules or ties are imposed.

In the intelligence phase (1), the DM specifies the aims and scope of the decision problem.

[1] The scheme presented here draws from works of Herbert Simon, the American economist and sociologist, the laureate of the Turing Prize in 1975 and the Nobel Prize in 1978.

In the design phase (2), the DM (occasionally supported by some analytical staff) specifies a model of the problem consisting of: the decision space X, the set of decision variants X_0, and the list of criteria $f_1, \ldots, f_k$.

By variant enumeration (partial or complete) in the choice phase (3), a variant which is regarded by the DM as the most preferred is selected. Enumeration is performed within the model specified in the design phase (2).

The selection of the most preferred variant in the choice phase (3) can be made automatically, i.e., by an algorithm, or by the DM in a sequence of $DM \Leftrightarrow model$ interactions.

Until now no universal method for automatic selection of the most preferred variant has been proposed. Nowadays, in the majority of cases, the most preferred variant is selected in the interactive manner which works as follows. The DM valuates a sequence of variants in turn. By this, some of his/her partial preferences become explicit. Mechanisms for the derivation of subsequent variants account for those explicit preferences and, presumably, more and more preferred variants are derived.

Variant derivation mechanisms make use of the model specified in the design phase (2). The selection process in the choice phase (3) ends up when the DM is convinced that one of valuated variants satisfies him/her more than any other. That variant is considered the most preferred variant.

The adequacy of the most preferred variant selected in the choice phase (3) to the decision problem under consideration is verified in the review phase (4). In other words, this variant is confronted with the reality of the decision making context. That variant can turn out to be inadequate (nonrealistic, not admissible) because not all circumstances (limits, bounds, conditions, constraints) have been recognized or taken into account in the intelligence phase (1) and in consequence, the model specified in the design phase (2) inadequately represents the problem. This is the stage of the decision making process, where the DM can recognize the existence or significance of such circumstances (the DM *learns*).

If the variant selected in the choice phase (3) is inadequate, the whole sequence of phases (1)–(4) is to be repeated.

2.3 Sum-Up

In this chapter, we have introduced the decision process scheme which is used in subsequent chapters.

The choice phase (3) of the decision process is computationally the most intensive. Usually, in this phase it is necessary to employ some formal (algorithmic, mathematical) tools, and those in turn call for the use of computers and specific software. Such tools are introduced in Chaps. 4 and 8.

Chapter 3

Decision Problem: Selection of a Single Variant

The tremendous success of their application of the Gargantius Effect gave both constructors such an appetite for adventure, that they resolved to sally forth once again to parts unknown.

3.1 Chapter Content

In this chapter, we are concerned with decision problems in which a single variant has to be selected from a set of variants given explicitly (e.g., in the form of a list of variants). To illustrate our considerations, we use the problem which frequently appears in practice: selection of an investment variant. Data and the problem setting presented in this chapter are hypothetical.

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In subsequent sections, from Sects. 3.2 to 3.5, we follow the four-phased decision scheme presented in Chap. 2.

3.2 Problem Formulation

Problem formulation—the intelligence phase (1).

Consider the following illustrative problem: a town council has to decide where to build a crossing over the river (a bridge and a network of connecting roads) which flows through the city. This new river crossing is indispensable for efficient urban transport.

Two years ago the town council ordered a study on this problem. The aim of the study was to identify advantages to the community from the investment, possible negative impacts, potential harmful consequences, and costs. The expected result of the study was a list of variants satisfying technical parameters and legal regulations.

The final report of the study recommended seven variants of the river crossing. The report was presented to the town council. The recommendations have been recently consulted with town residents, all parties potentially involved and all sides interested.

3.3 Problem Modeling

Problem modeling—the design phase (2).

The community consultations, indecisive about what variant to select, had not excluded any variant from further considerations. Even the strong local ecological lobby admitted that none of seven variants proposed could have any significant negative impact on the environment.

The town council formed then a task group with the aim to prepare and support the process of selecting the most preferred variant. However, the final decision would be made by the town general assembly.

Here are the findings of the task group used to specify a model of the problem.

It is planned that the river crossing will be financed from the city own funds. This will be a significant budget spending in the period of 3 years from the start of the construction. Only a small part of funds is planned to be financed from external sources (European Union structural funds).

After analyzing the recommendations of the study, the task group has concluded that:

– all seven river crossing variants satisfy all the requirements for undertakings of that sort; these variants form the set of *decision variants* (see Chap. 1);

– the significant factors which differentiate the variants are *investment cost* (in millions of PLN[1]) and *investment completion time* (in year quarters);

– since the city, in the scope of financing secured from a European Union structural fund, has a lot of elasticity in pledging for subsequent partial payments (it is enough to document the need of funds by reporting the advance of the works), cash flow differences between variants are negligible; hence, the net present value of variant costs is an adequate and the only important measure of financial aspects of the planned investment.

In its final recommendation, the task group proposed that all seven variants should be considered, and the selection should be made with respect to two aspects: the investment cost (net present value) and the investment completion time. In other words, these two aspects should be used as selection criteria.

Values of criteria functions: $f_1(x)$—the investment completion time (in year quarters), $f_2(x)$—the investment cost (in millions of PLN), for variants $\{x^1, \ldots, x^7\}$ are

$$f(x^1) = (9,1)\,, \quad f(x^2) = (7,4)\,, \quad f(x^3) = (8,2)\,,$$

$$f(x^4) = (8,3)\,, \quad f(x^5) = (6,3)\,, \quad f(x^6) = (9,2)\,,$$

$$f(x^7) = (1,8)\,.$$

[1] PLN is the Polish currency unit, called “złoty”.

Summing up, the MCDM model (1.2) for the problem considered here has the form

$$\textit{select a variant } x \in X_0 \subseteq X \textit{ for which } f(x)$$

$$\textit{is the most prefered bi-criteria valuation},$$

where
– $X_0 = \{x^1, x^2, \ldots, x^7\}$ is the set of decision variants,
– $f_1,\ f_2$ are criteria functions defined by the values of outcome components as above (e.g., $f_1(x^1) = 9,\ f_2(x^1) = 1$).

In the class of decision problems considered in this chapter, the model can be represented as a table.[2] For example, the model of the problem considered above can have the form as Table 3.1. This table has size 2×7, the columns correspond to the variants, the rows correspond to the criteria functions, and the table elements represent the value of the corresponding criterion function for the corresponding variant.

Table 3.1 The table representation of the river crossing problem

9	7	8	8	6	9	1
1	4	2	3	3	2	8

3.4 Variant Selection

Variant selection—the choice phase (3).

Following considerations of Chap. 1, the most preferred variant is to be selected from efficient variants.

For the moment, we do not know how to select the most preferred variant. An approach to this task will be proposed in Chap. 8.

We do not even know yet how to identify efficient variants in a formal manner. Approaches to this task, for problems like that considered in this chapter, are present in Chaps. 4 and 7.

[2] Such a table is often called the *decision matrix*.

For the reasons listed above, we return later to the choice phase of the decision process for the problem considered here, namely in Chap. 9.

3.5 Problem Verification

Problem verification—the review phase (4).

For the reasons listed in the previous section, we return later to the review phase of the decision process for the problem considered here, namely in Chap. 9.

3.6 Sum-Up

The problem presented in this chapter is very simple. Indeed, the number of criteria functions is just two, which is minimum to consider a decision problem as an MCDM case. Also the number of variants in the problem is very limited. But one can easily imagine a practical problem of importance with just two variants! Moreover, the model specified can be extended to include more variants and more criteria.

3.7 Creative Assignment

Model MY_PROBLEM_1 as the problem of multiple criteria variant selection.

Find/propose all the required data (preferably real ones, but if no such data is available, propose hypothetical data).

Chapter 4

Derivation of Efficient Variants

"In which case who could say and to whom could it be said that the order was carried out and I am an efficient and capable machine?"

4.1 Chapter Content

The subject of this chapter are methods for derivation of efficient variants in problems, in which variants are explicitly given as a list of variants.

The methods will be presented in the form of algorithms.

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4.2 Algorithms to Derive Efficient Variants

We denote that two variants x and x', where $x, x' \in X_0$, are in the Pareto dominance relation (as defined in Sect. 1.2.2) and variant x dominates variant x', by symbol $\overset{P}{\succ}$, i.e.,

$$x \overset{P}{\succ} x' , \quad \text{if } x \text{ dominates } x'.$$

Let us consider the following problem. For a given MCDM model: set X_0 and criteria functions $f_l,\ l = 1, \ldots, k$, derive an efficient variant.

We assume here that $|X_0| = n$ (X_0 is a finite set composed of n elements) and $X_0 \neq \emptyset$ (X_0 is not empty).

The following algorithm derives an efficient variant from X_0.

Step 0. Select variant x from X_0.
$\quad$ $candidate := x$.
$\quad$ $X_0' := X_0 \setminus \{x\}$.
Step 1. Check $X_0' = \emptyset$. If yes, STOP.
Step 2. Select variant x from X_0'.
$\quad$ $X_0' := X_0' \setminus \{x\}$.
Step 3. Check $x \overset{P}{\succ} candidate$.
$\quad$ If yes, then $candidate := x$.
$\quad$ Go to Step 1.

When the algorithm terminates, variant *candidate* is an efficient variant.

Algorithm E_1 verifies, variant by variant, whether the Pareto dominance relation holds between the given variant, called *candidate*, and one selected variant. If variant *candidate* is dominated by the selected variant, then the latter becomes *candidate*.

Algorithm E_1 makes use of an auxiliary set X_0'. This set contains variants for which it has not been verified yet whether variant x from this set dominates variant *candidate* (Step 3). After selecting a variant *candidate* from set X_0, set X_0' contains all the remaining elements of set X_0 (Step 0). By removing variants from set X_0' (Step 2) it is guaranteed that in Step 3, no variant meets variant *candidate* twice.

It is easy to observe that in order to derive an efficient variant in set X_0, the verification of the Pareto dominance relation between two variants must be done $n-1$ times, which amounts to $(n-1)\times k$ comparisons of numbers.

Example 4.2.1 *Let $X_0 = \{x^1, \ldots, x^7\}$. Assume that criteria functions are such that the following relationships hold:*

$$\begin{array}{ccc} x^2 \overset{P}{\succ} x^5, & x^4 \overset{P}{\succ} x^5, & x^4 \overset{P}{\succ} x^3, \\ x^6 \overset{P}{\succ} x^7, & x^3 \overset{P}{\succ} x^7, & x^7 \overset{P}{\succ} x^1 \\ x^4 \overset{P}{\succ} x^7, & x^3 \overset{P}{\succ} x^1, & x^4 \overset{P}{\succ} x^1, \\ & x^6 \overset{P}{\succ} x^1 \,. & \end{array} \tag{4.1}$$

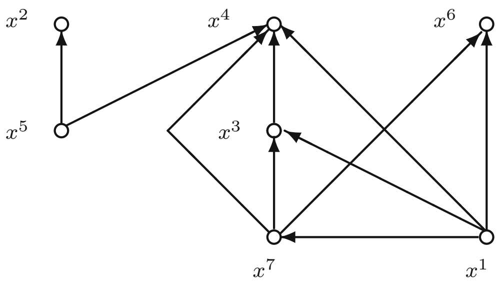

Figure 4.1 A graphical interpretation of relation (4.1), arrows go towards dominating variants

Relation (4.1) is represented graphically in Fig. 4.1.
We derive one efficient variant of X_0 by Algorithm E_1.

Step 0. *Select variant* x^7.
candidate $:= x^7$.
$X'_0 := \{x^1, x^2, x^3, x^4, x^5, x^6\}$.
Step 1. $X'_0 \neq \emptyset$.
Step 2. *Select variant* x^5.
$X'_0 := \{x^1, x^2, x^3, x^4, x^6\}$.
Step 3. $x^5 \overset{P}{\not\succ} x^7$.
Step 1. $X'_0 \neq \emptyset$.
Step 2. *Select variant* x^2.
$X'_0 := \{x^1, x^3, x^4, x^6\}$.

Step 3. $x^2 \overset{P}{\not\succ} x^7$.
Step 1. $X'_0 \neq \emptyset$.
Step 2. *Select variant* x^4.
$X'_0 := \{x^1, x^3, x^6\}$.
Step 3. $x^4 \overset{P}{\succ} x^7$.
candidate $:= x^4$.
Step 1. $X'_0 \neq \emptyset$.
Step 2. *Select variant* x^1.
$X'_0 := \{x^3, x^6\}$.

Step 3. $x^1 \overset{P}{\not\succ} x^4$.
Step 1. $X'_0 \neq \emptyset$.
Step 2. *Select variant* x^3.
$X'_0 := \{x^6\}$.
Step 3. $x^3 \overset{P}{\not\succ} x^4$.
Step 1. $X'_0 \neq \emptyset$.
Step 2. *Select variant* x^6.
$X'_0 := \emptyset$.
Step 3. $x^6 \overset{P}{\not\succ} x^4$.
Step 1. $X'_0 = \emptyset$. *STOP.*

The efficient variant is the current *candidate*, i.e., variant x^4.

Algorithm E_1 in one run derives only one efficient variant from a set. To derive another efficient variant, Algorithm E_1 has to be run again, this time on set

$$X_0 := X_0 \setminus (\{x\} \cup \{\textit{variants dominated by } x\}),$$

where x is the efficient variant derived in the previous run. Indeed, it is necessary to remove from set X_0 variants dominated by the derived efficient variant. Removing only the derived efficient variant could result in derivation in the next run a dominated variant instead of an efficient one.

To determine all efficient variants in set X_0, Algorithm E_1 has to be run m times (each time on a subset of X_0), where m is the number of efficient variants in the original set X_0.

The actions described above to derive all efficient variants can be certainly set as one algorithm.

Algorithm E_m presented below derives all efficient variants in finite set X_0. When the algorithm terminates, all (and only) efficient variants are elements of set X_E.

Algorithm E_m

Step 0. $X_E := \emptyset$.
Step 1. Check $X_0 = \emptyset$.
If yes, STOP .
Select variant x from X_0 .
$\textit{candidate} := x$.
$X_0' := X_0 \setminus x$.
Step 2. Check $X_0' = \emptyset$.
If yes, then:
a. $X_E := X_E \cup \{\textit{candidate}\}$.
b. $X_0 := X_0 \setminus \{\textit{candidate}\}$.
c. Go to Step 1 .
Step 3. Select variant x from X_0' .
$X_0' := X_0' \setminus \{x\}$.

Step 4. Check $x \overset{P}{\succ} candidate$.
If yes, then:
a. $X_0 := X_0 \setminus \{candidate\}$.
b. $candidate := x$.
c. Go to Step 2 .

Step 5. Check $candidate \overset{P}{\succ} x$.
If yes, then:
$X_0 := X_0 \setminus \{x\}$.
Go to Step 2 .

Similarly to Algorithm E_1, Algorithm E_m makes use of an auxiliary set X_0'. This set contains variants for which it has not been verified yet whether:

– variant x from this set dominates variant *candidate* (Step 4),
– variant *candidate* dominates variant x from this set (Step 5).

After selecting a variant *candidate* from set X_0, set X_0' contains all the remaining elements of set X_0 (Step 1). By removing variants from set X_0' (Step 3) it is guaranteed that in Step 4 and in Step 5 no variant meets the variant *candidate* twice.

Example 4.2.2 *For data from Example 4.2.1 we derive now all efficient variants by Algorithm E_m.*

Step 0. $X_E := \emptyset$.
Step 1. $X_0 \neq \emptyset$.
We select variant x^7.
$candidate := x^7$.
$X_0' := \{x^1, x^2, x^3, x^4, x^5, x^6\}$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^5.
$X_0' := \{x^1, x^2, x^3, x^4, x^6\}$.
Step 4. $x^5 \overset{P}{\not\succ} x^7$.
Step 5. $x^7 \overset{P}{\not\succ} x^5$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^2.
$X_0' := \{x^1, x^3, x^4, x^6\}$.
Step 4. $x^2 \overset{P}{\not\succ} x^7$.

Step 5. $x^7 \overset{P}{\not\succ} x^2$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^4.
$X_0' := \{x^1, x^3, x^6\}$.
Step 4. $x^4 \overset{P}{\succ} x^7$.
a. $X_0 := \{x^1, x^2, x^3, x^4, x^5, x^6\}$.
b. $candidate := x^4$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^1.
$X_0' := \{x^3, x^6\}$.
Step 4. $x^4 \overset{P}{\succ} x^7$.
a. $X_0 := \{x^1, x^2, x^3, x^4, x^5, x^6\}$.
b. $candidate := x^4$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^1.
$X_0' := \{x^3, x^6\}$.
Step 4. $x^1 \overset{P}{\not\succ} x^4$.
Step 5. $x^4 \overset{P}{\succ} x^1$.
$X_0 := \{x^2, x^3, x^4, x^5, x^6\}$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^3.
$X_0' := \{x^6\}$.
Step 4. $x^3 \overset{P}{\not\succ} x^4$.
Step 5. $x^4 \overset{P}{\succ} x^3$.
$X_0 := \{x^2, x^4, x^5, x^6\}$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^6.
$X_0' = \emptyset$.
Step 4. $x^6 \overset{P}{\not\succ} x^4$.
Step 5. $x^4 \overset{P}{\not\succ} x^6$.
Step 2. $X_0' = \emptyset$.
a. $X_E := \{x^4\}$.
b. $X_0 := \{x^2, x^5, x^6\}$.
c. *Go to Step 1*.

Step 1. $X_0 \neq \emptyset$.
We select variant x^6.
candidate $:= x^6$.
$X_0' := \{x^2, x^5\}$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^5.
$X_0' := \{x^2\}$.
Step 4. $x^5 \overset{P}{\not\succ} x^6$.
Step 5. $x^6 \overset{P}{\not\succ} x^5$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^2.
$X_0' = \emptyset$.

Step 4. $x^2 \overset{P}{\not\succ} x^6$.
Step 2. $X_0' = \emptyset$.
a. $X_E := \{x^4, x^6\}$.
b. $X_0 := \{x^2, x^5\}$.
c. Go to Step 1 .
Step 1. $X_0 \neq \emptyset$.
We select variant x^5.
candidate $:= x^5$.
$X_0' := \{x^2\}$.
Step 2. $X_0' \neq \emptyset$.
Step 3. *We select variant* x^2.
$X_0' := \emptyset$.
Step 4. $x^2 \overset{P}{\succ} x^5$.
a. $X_0 := \{x^2\}$.
b. candidate $:= x^2$.
Step 2. $X_0' = \emptyset$.
a. $X_E := \{x^2, x^4, x^6\}$.
b. $X_0 := \emptyset$.
c. Go to Step 1 .
Step 1. $X_0 = \emptyset$, *STOP.*

The following variants are efficient: x^2, x^4, x^6.

Example 4.2.3 *Data presented below comes from a public tender for replacing windows in five nurseries in a local community. There are three criteria: cost (the smaller cost, the better), evaluation of technical quality (the higher evaluation, the better), and the guarantee period (the longer period, the better, with the saturation point*[1] *5 years*[2]*). The criteria are defined as follows.*

All three criteria are represented on cardinal scales as scores. The maximal score for the cost criterion is set to 280*, for the evaluation of technical quality criterion to* 80*, and for the criterion of the guarantee period to* 20*. The maximal scores in criteria represent opinions (preferences) of the tender organizers (the DM) on relative criteria importance, being in this case* $(280, 80, 20)$*, or in relative terms,* $(1, \frac{80}{280}, \frac{20}{280})$*.*

The best offer with respect to cost gets the maximal score and the other offers get scores prorated to the maximum. For example, the offer twice as expensive as the cheapest offer (which gets score 280*) gets score* 140*. In that manner, the cost criterion, which by its nature is of "the less, the better" type, becomes of "the more, the better" type.*

The evaluation of technical quality criterion is made with respect to four subcriteria. Variants are evaluated with respect to all subcriteria by experts. The maximal score for each subcriterion is 20*. The final score with respect to the evaluation of technical quality criterion is the sum of scores with respect to four subcriteria. In the tender considered, no offer scored in that criterion reached the maximal score* 80*.*

With respect to the guarantee period criterion, the best offer (i.e., the offer for which the guarantee period is 5 years of longer) gets the maximal score (i.e., 20), and all other offers get scores prorated to the maximum.

[1] Which means that guarantee periods longer than 5 years are regarded equally good as the guarantee period 5 years.

[2] At the time of the tender, 5 year guarantee for windows was regarded just "fair". This was the cause why the guarantee period criterion played no role in the winner selection, see data in Table 4.1; all parties participating in the tender offered 5 years guarantee of higher.

With the values of the criteria derived as above, for each pair of offers (variants) it can be checked whether the dominance relation holds (see the definition of dominance relation in Sect. 1.2.2).

The three criteria have been used to select the most preferred variant.

From 23 offers submitted, 7 offers have been discarded because of some formal deficiencies. Data for the problem are given in Table 4.1.

This problem has been presented here to illustrate a wide range of practical decision problems in which Algorithm E_1 and Algorithm E_m are of use and importance.

Table 4.1 Values of criteria in Example 4.2.3

No	Price	Technical quality	Guarantee period
1	236.99	70.94	20.00
2	280.00	75.54	20.00
3	207.31	70.82	20.00
4	229.24	73.96	20.00
5	256.34	72.04	20.00
6	216.86	68.73	20.00
7	211.76	72.08	20.00
8	242.12	68.39	20.00
9	245.76	64.62	20.00
10	240.16	73.96	20.00
11	260.04	67.66	20.00
12	273.90	67.66	20.00
13	172.59	64.50	20.00
14	214.09	71.83	20.00
15	198.21	75.79	20.00
16	237.02	71.85	20.00
max	280.00	75.79	20.00

4.3 Sum-Up

As we can conclude from the above considerations, derivation of efficient variants for problems with just several variants is not very complex. However, for several tens of variants such a task becomes too intricate for a human to cope. In this case, algorithms like E_1 or E_m should be used to profit on the computers calculating power.

As stated in Chap. 1, the most preferred variant should be sought among efficient variants. With algorithms presented in this chapter in place, we know how to derive efficient variants in problems where variants are explicitly given, as it is in the problem defined in Chap. 3. So we could now return to solving that problem. But we still do not know how to search for the most preferred variants. An approach (an approach rather than a formally rigid procedure) to that task is proposed in Chap. 8.

And therefore, we return to the selection of the most preferred variant in the problem defined in Chap. 3 not earlier than in Chap. 9.

4.4 Creative Assignment

Derive all efficient variants to MY_PROBLEM_1 model by any of the algorithms presented in this chapter.

Chapter 5

Decision Problem: Selection of a Variant Portfolio—The Discrete Case

The Adviser did not deny that the letter could be read in a variety of ways if one rearranged the letters of the letter; it had itself discovered an additional hundred thousand variants...

5.1 Chapter Content

In Chap. 3, we have been concerned with the problem of selecting a single decision variant from a set of decision variants.

In this chapter, we consider problems in which variants in decision space X are not just single variants, but also collections of

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single variants. Hence, from now on we regard single variants as elementary objects of variant collections.

Each single variant brings to a collection of variants its own valuation as a contribution to the collective valuation of this collection.

Collections of variants are called in this textbook *variant portfolios*, and if they belong to set X_0—*decision variant portfolios*. Here we adopt the convention as in Sect. 1.2.4 that the decision space X contains all conceivable variant portfolios, but only variant portfolios from X_0—decision variant portfolios—are considered. As we are only interested in decision variant portfolios, in the sequel we will often abbreviate this term to just *variant portfolio*. When we will occasionally speak about variant portfolios which belong to X but not to X_0, we will stress this fact explicitly.

A special case of a (decision) variant portfolio is a single (decision) variant.

In the full analogy to single variants, collective valuations of decision variant portfolios (i.e., we recall, variant portfolios which belong to X_0) are called *outcomes*.

In this chapter, we assume that a single variant is either included into a collection on the whole, or is not included in that collection at all. Hence, we deal in this chapter with the so-called discrete case (where variables representing variants can take some specific values only), in contrast to the so-called continuous case (where variables representing variants can take any values). The latter case is considered in Chap. 6.

As shown below, the decision problem of selection of a variant portfolio is technically more complex than the decision problem of selection of a single variant. In the case of selection of a variant portfolio, it is often practically impossible to list all feasible variant portfolios explicitly because there are too many of them. However, as we show in Chap. 7, also in such cases it is possible to derive efficient variant portfolios, and in consequence, the most preferred variant portfolio.

Data and the problem setting presented in this chapter are hypothetical.

5.2 Problem Formulation

Problem formulation—the intelligence phase (1).

Consider the following problem.

A pharmaceutical company invests in research and development (R&D). Every year the company qualifies R&D projects submitted by its research laboratories for financing. The qualification is made by a selection committee. All projects are to be completed within a year.

Each project is assessed in two categories: expenditures on the project and the project profitability. The total budget for R&D projects is limited. Project profitability is measured by two indicators: *net present value of the company (forecasted) profit increase* in the period of 5 years (the average time in which a company realizes gains from innovations), and *the company (forecasted) market share increase* in the 5 year span.[1]

The selection committee has the task to select from the submitted projects a collection of projects (a portfolio) which will be financed in the next year.

5.3 Problem Modeling

Problem modeling—the design phase (2).

Each project to be realized needs financing. The company has a budget for R&D, which is, at least at the time of project selection, fixed.

Let j be project index, $j = 1, \ldots, n$. Denote by c_j (forecasted) company profit increase, and by d_j (forecasted) company market share increase resulting from the successful completion of project j. Denote by a_j the value of financing necessary for the completion of project j and by a_0 the value of the budget for R&D projects in the considered year.

To construct a formula defining variant portfolios we make use of *binary variables*, i.e., variables assuming two specific values only, namely 0 and 1. We interpret variable x_j as follows:

[1] A significant market share stabilizes the company profit in long term, but as a rule it is in conflict with the postulate to increase company profits in short and medium terms.

$$x_j = \begin{cases} 1, & \text{if project of index } j \\ & \text{is qualified for financing,} \\ \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

With this convention, a portfolio can be represented by a vector of n binary variables $(x_1, \ldots, x_n)$. For example, if $n = 4$, then $(0, 1, 0, 1)$ denotes the variant portfolio in which the only projects selected for financing (and therefore for realization) are project 2 and project 4.

Selecting project j entails a_j spending, not selecting it entails no spending on that project (in other words, there is 0 spending on a nonselected project). All this can be expressed by formula $a_j x_j$.

In the decision problem, only decision variant portfolios (i.e., variant portfolios for which the total spending is not greater than the budget) are considered and they form set X_0 (see Sect. 1.2.4). Formally, the condition that $x = (x_1, \ldots, x_n)$ belongs to X_0 has the form

$$a_1 x_1 + \ldots + a_n x_n \leq a_0 ,$$

or more compactly

$$\sum_{j=1}^{n} a_j x_j \leq a_0 .$$

The above condition constitutes the rule for decision variant portfolio construction. With such a condition in place, we avoid the necessity to list all the variant portfolios explicitly.

By the same principle, the formula for the profit increase resulting from realization of all projects in decision variant portfolio x has the form

$$\sum_{j=1}^{n} c_j x_j ,$$

and the formula for the market share increase has the form

$$\sum_{j=1}^{n} d_j x_j .$$

The multiobjective optimization model (1.3) takes in this case the form

$$\begin{aligned} &\text{vmax } f(x) = \left(\textstyle\sum_{j=1}^{n} c_j x_j,\ \sum_{j=1}^{n} d_j x_j\right) \\ &\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{n} a_j x_j \leq a_0\,, \\[2ex] x_j = 0 \text{ or } 1\,,\ j = 1, \ldots, n, \end{array} \right\}, \end{aligned} \tag{5.2}$$

where vmax, we recall, denotes the operator of derivation of all efficient variants (in the considered case: efficient variant portfolios).

5.4 Variant Selection

Variant selection—the choice phase (3).

Following considerations of Chap. 1, the most preferred variant portfolio is to be selected from efficient variant portfolios.

For the moment, we do not know how to select the most preferred variant portfolios. An approach to this task is proposed in Chap. 8.

We even do not know yet how to identify efficient variant portfolios in a formal manner. An approach to this task is presented in Chap. 7.

For the reasons listed above, we will return later to the choice phase of the decision process for the problem considered here, namely in Chap. 9.

5.5 Problem Verification

Problem verification—the review phase (4).

For the reasons listed in the previous section we return later to the review phase of the decision process for the problem considered here, namely in Chap. 9.

5.6 Sum-Up

In this chapter, we have shown how to construct variants composed of single variants—variant portfolios.

Theoretically, one can always attempt to list all conceivable variant portfolios for the problem (5.2). The number of all conceivable variant portfolios with n projects (i.e., the number of all possible n-element vectors with binary components) is equal to 2^n. The exponential increase of this number with n (and in general also the exponential increase of the number of decision variant portfolios) precludes such a procedure even for moderate sizes of n.

Formal model (5.2) sets lines for algorithmic approaches to constructing decision variant portfolios. Such mechanisms can be built into optimization packages, e.g., the *Microsoft Excel* add-in *Solver*.

In the problem considered above, it is assumed that a single variant is used either on the whole or is not used at all. This assumption can be relaxed by admitting multiples of single variants. For example, two teams execute the same project to reduce the failure risk or enhance competition.

5.7 Creative Assignment

Modify MY_PROBLEM_1 model to a portfolio selection problem/model MY_PROBLEM_2, by admitting variant portfolios to be composed of up to two variants. To this aim, add the constraint

$$x_1 + \ldots + x_n \leq 2\,,$$

which ensures this condition to hold, provided that variables are binary.

Propose a real-world interpretation for the modified problem, different from that given in this chapter.

Chapter 6

Decision Problem: Selection of a Variant Portfolio—The Continuous Case

And Trurl went home, threw six heaping teaspoons of transistors into a big pot, added again as many condensers and resistors, poured electrolyte over it, stirred well and covered tightly with a lid, then went to bed…

6.1 Chapter Content

As in Chap. 5, also in this chapter we are concerned with problems in which variants are variant portfolios, i.e., collections of single variants. However, in contrast to problems considered in Chap. 5, here we admit that a single variant can be used for portfolio

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construction in any quantity. We refer to this situation, as already mentioned in Chap. 5, the *continuous case*.

Each single decision variant brings to a variant portfolio its own outcome as a contribution to the collective outcome of this variant portfolio.

In the full analogy to the discrete case, in the continuous case a variant portfolio consisting of a single variant only, is also, as the special case, a variant portfolio.

Some data and the problem setting presented in this chapter are hypothetical, but a part of data is real.

6.2 Decision Problem: Selection of Mixture Composition

One of the most often quoted examples of applications of optimization to practice is the problem of optimal mixture composition. The problem consists in finding a percentage share of components (i.e., a portfolio of components) which ensure the extremal (minimal or maximal) value of some criterion referring to mixture properties.

We consider here this problem in the multiple criteria setting, with an application to the problem of selecting diets for humans.

6.3 Problem Formulation

Problem formulation—the intelligence phase (1).[1]

In the table *Diet Standards (Normy Żywienia)*, we find recommendations for diets—recommended energy consumption and recommended quantities of diet ingredients, such as: proteins, fats, minerals and vitamins, for specific groups of population.[2]

[1] All the data and sources used for the diet problem considered here represent the state-of-the-art in the dietary domain around the year 2010. The knowledge and recommendations in this domain change fast and therefore the data presented in this textbook can serve only as an illustration for the problem and should not be regarded as a base for any binding diet hints.

[2] *Normy Żywienia*, Ś. Ziemlański, Ed., the home page of The National Food and Nutrition Institute, Poland, as by 2008; revised recommendations in 2012, http://www.izz.waw.pl/wwzz/normy.html.

For example (see Table 9.4), for girls in the age 16–18, body weight 60 kg, with moderate physical activity, the recommended daily energy consumption is 2200 kcal. All considerations below refer to persons of that sex, age and weight.

In the book "Tables of content and nutrition properties of food" ("Tabele składu i wartości odżywczej żywności"),[3] we find data for the, so-called, energy equivalents (content of energy) and content of ingredients in unprocessed food.

With that data in place, we can formulate the following problem: propose a scheme to determine a daily diet taking into account the diet cost as well as the recommendations for daily energy consumption.

We assume that diet cost is calculated on the basis of prices from a local wholesale market.[4]

6.4 Problem Modeling

Problem modeling—the design phase (2)

6.4.1 Linear Programming Problem

The starting point for modeling the considered problem is the most publicized optimization model among the problems in the scope of Operations Research, namely the *linear programming problem*.[5]

In the linear programming problem, the maximum or the minimum (whatever is needed) of a linear criterion function is sought over a set defined by linear conditions on nonnegative values of variables. This problem, in the case of criterion function maximization, takes the form

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[3] Kunachowicz H., Nadolna I., Przygoda B., Iwanow K., "Tabele składu i wartości odżywczej żywności". Wydawnictwo Lekarskie PZWL, Warszawa, 2005.

[4] E.g., The Prag Food Wholesale Market of Warsaw (in Polish: *Praska Giełda Spożywcza*) http://www.praskagieldaspozywcza.pl.

[5] According to our convention, as long as we do not consider specific data we should refer to (linear programming) model, but the name *linear programming problem*, referring both to problems and models, is deeply rooted in the literature of the subject.

$$\max f(x) = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{ll} \sum_{j=1}^{n} a_{i,j} x_j \leq a_{i,0}, & i = 1, \ldots, m, \\ x_j \geq 0, & j = 1, \ldots, n, \end{array} \right\}. \tag{6.1}$$

In the above formulation, in each individual structural constraint (i.e., constraints other than the variable nonnegativity constraints) sign $\leq$ can be replaced, if needed, by sign $=$ or sign $\geq$.

6.4.2 Modeling the Diet Problem

In the diet problem, we make use of variables x_j to represent the quantities of specific products.

We denote price of product j by c_j. Hence, the cost of product j in a diet is equal to $c_j x_j$.

The diet cost is the cost of all products, namely, $f_1(x) = \sum_{j=1}^{n} c_j x_j$. Function $f_1(x)$ is the first criterion in the model we build.

Product j is the source of ingredient i in quantity $a_{i,j}$ for one unit of the product, hence product j used in quantity x_j is the source of ingredient i in quantity $a_{i,j} x_j$. With all its products, the diet is the source of ingredient i in quantity $\sum_{j=1}^{n} a_{i,j} x_j$, and this ingredient should be present in the diet in the quantity not less than (hence the signs in constraints (6.1) should be reverted from $\leq$ to $\geq$) recommended quantity $a_{i,0}$.

Because of their physical interpretation, variables x_j in the diet problem can assume only nonnegative values.

In addition to ingredients, product j is the source of the energy equivalent in quantity d_j units (here: kilocalories) for one unit of the product. Hence, product j used in quantity x_j is the source of $d_j x_j$ units of energy. With all its products, the diet is the source of the energy equivalent in quantity $f_2(x) = \sum_{j=1}^{n} d_j x_j$. Function $f_2(x)$ is the second criterion function in the model.

According to the present tendencies in dietetics, of interest are diets which satisfy recommendations (specific for a population group) for ingredient content and energy consumption. Assessing a diet, one has also to pay attention to its cost, where obviously the lower cost, the better. Recommendations for energy consumption are not regarded here categorically but only indicatively, and this allows us to investigate the whole range of compromises between the diet cost and the diet energy equivalent.

Assume that the DM is interested in diets which satisfy recommendations on ingredient content and at the same time are highly energetic and not expensive. Hence, the cost criterion $f_1(x)$ is of the type "the less, the better" and the criterion of the energy equivalent is of the type "the more, the better". According to the assumption made throughout this textbook that all criteria functions are of the form "the more, the better", instead of searching for the minimal value of $\sum_{j=1}^{n} c_j x_j$ we search for the maximal value of $f_1'(x) = -\sum_{j=1}^{n} c_j x_j$.

Conditions on minimal content of ingredients in a diet preclude "the zero option", i.e., the diet defined by $x_j = 0,\ j = 1, \ldots, n$, (which defines the diet which is the cheapest and of the least energy equivalent).

The MO formulation of the problem considered here takes form of the multiobjective model 1.3 (see Chap. 1)

$$
\begin{aligned}
&\text{vmax} \quad (f_1'(x), f_2(x)) \\
&\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{ll} \sum_{j=1}^{n} a_{i,j} x_j \geq a_{i,0}, & i = 1, \ldots, m, \\ x_j \geq 0, & j = 1, \ldots, n, \end{array} \right\}.
\end{aligned}
\tag{6.2}
$$

6.5 Variant Selection

Variant selection—the choice phase (3).

Following considerations of Chap. 1, the most preferred variant portfolio is to be selected from efficient variant portfolios (here diets).

For the moment, we do not know how to select the most preferred variant portfolios. An approach to this task is proposed in Chap. 8.

We even do not know yet how to identify efficient variant portfolios in a formal manner. An approach to this task is proposed in Chap. 7.

For the reasons listed above, we return later to the choice phase of the decision process for the problem considered here, namely in Chap. 9.

6.6 Problem Verification

Problem verification—the review phase (4).

For the reasons listed in the previous section, we return later to the review phase of the decision process for the problem considered here, namely in Chap. 9.

6.7 Sum-Up

In this chapter, we have shown how to construct variants composed of single variants—variant portfolios. In contrast to the previous chapter, where single variants are included in a portfolio only as the whole, here we have considered problems in which single variants can be used for portfolio construction in any quantity.

Formal model (6.2) sets lines for algorithmic approaches to constructing feasible variant portfolios. Such mechanisms can be built into optimization packages, e.g., the *Microsoft Excel* add-in *Solver*.

6.8 Creative Assignment

Modify MY_PROBLEM_2 to a portfolio selection problem/model MY_PROBLEM_3, by admitting variant portfolios to be composed of any multiple of single variants. Replace the constraint

$$x_1 + \ldots + x_n \leq 2$$

by a constraint

$$a_1x_1 + \ldots + a_nx_n <= a_0$$

(or a set of m constraints

$$a_{i,1}x_{i,1} + \ldots + a_{i,n}x_{i,n} <= a_{i,0}, \quad i = 1, \ldots, m).$$

At this point, you have to propose more data, namely data for coefficients $a_1, \ldots, a_n, a_0$ (or for coefficients $a_{i,1}, \ldots, a_{i,n}, a_{i,0}, i = 1, \ldots, m$).

Propose a real-world interpretation for the modified problem, different than that given in this chapter.

Chapter 7

Derivation of Efficient Portfolios

Towards the end of his second audience with the King, Klapaucius inquired if perhaps Trurl were on the planet and gave a detailed description of his comrade.

7.1 Chapter Content

In this chapter, we are getting acquainted with the methods for efficient portfolio derivation.

Since that task, except for trivial cases, excesses human capabilities, we need precise procedures which would allow to entrust the task to a computer. The very principle of such procedures is simple—an appropriate *scalar* (number) measure of variant portfolio "fitness" is set, and next a single criterion optimization problem is solved in which a portfolio of the highest "fitness" is derived.

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It is necessary then to have measures which ensure that portfolios with the highest value of such measures are efficient portfolios. That property can be enforced only if a measure of that sort depends on all criteria.

In MCDM, measures with such properties are functions, called *scalarizing functions*. The intensity of dependence of a scalarizing function on a single criterion can be varied, and this is achieved by assigning to each criterion function a coefficient. In MCDM, such coefficients are customarily called *weights*.

By weight manipulations (changing weights and by this changing proportions between weights) one gets different scalarizing function instances (a set of scalarizing functions) which, when applied to an optimization problem, deliver different, in general, efficient variant portfolios.

Of particular importance are scalarizing functions with the property that all (or almost all) efficient variant portfolios can be derived. Below two scalarizing functions with that property are presented. Each of those scalarizing functions provides for a *characterization of efficient outcomes*, and hence, also for a *characterization of efficient variants*. By a characterization we understand the *necessary and sufficient conditions*, here—conditions for outcome efficiency. We call those characterizations, respectively, *Characterization A* and *Characterization B*.

7.2 Characterization of Efficient Outcomes: Characterization A

For the sake of presentation clarity, in this chapter we use the notation introduced in Chap. 1: y and Z, where $y = f(x)$, $Z = f(X_0) = \{f(x) \mid x \in X_0\}$.

Characterization A: *the sufficient condition.*

An outcome y which solves optimization problem

$$\min_{y \in Z} \max_{l} \lambda_l (y_l^* - y_l), \tag{7.1}$$

where $\lambda_l > 0,\ l = 1, \ldots, k,\ y^*$ *satisfies condition* (1.4), *is efficient.*[1]

Characterization A: *the necessary condition.*

Every efficient outcome y *solves optimization problem* (7.1) *for some* $\lambda_l > 0,\ l = 1, \ldots, k.$

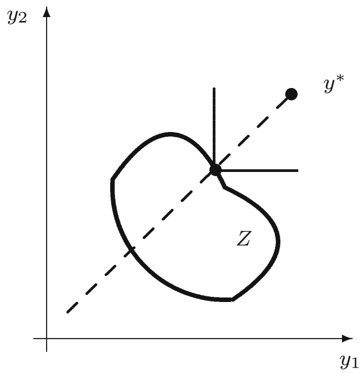

Figure 7.1 A graphical interpretation of deriving efficient outcomes (and thus efficient variants) by optimization problem (7.1). The *dashed line* is the locus of apexes of the criterion function contours

Figure 7.1 represents the contour of the criterion function in optimization problem (7.1), corresponding to the minimal value of this function on set Z. By Characterization A, the minimum of this criterion function is attained at an efficient outcome.

[1] In general, this statement is valid for a certain superset of the set of efficient outcomes, namely for the set of *weakly efficient outcomes*. However, here we assume that the set of weakly efficient outcomes coincides with the set of efficient outcomes. The definitions of weakly efficient outcomes and weakly efficient variants can be found, e.g., in Kaliszewski [2], Ehrgott [1], and Miettinen [3]. One can also find there how to ensure by a slight modification of optimization problem (7.1) that only efficient outcomes are derived.

Scalarizing function $\max_l \lambda_l(y_l^* - y_l)$ is called the *weighted Tchebycheff function.*

Proof of the necessary condition.

Let $\bar{y}$ be an efficient outcome and weights $\bar{\lambda}_l$ be defined as $\bar{\lambda}_l = (y_l^* - \bar{y}_l)^{-1},\ l = 1, \dots, k$.

By the definition of efficiency, there is no outcome $y \in Z$ such that $y_l \geq \bar{y}_l$, for $l = 1, \dots, k$, and $y_l > \bar{y}_l$ for some l. Hence, for every $y \in Z,\ y \neq \bar{y}$, there exists index l such that

$$y_l < \bar{y}_l\,.$$

For index l the following also holds:

$$y_l^* - y_l > y_l^* - \bar{y}_l$$

and

$$\bar{\lambda}_l(y_l^* - y_l) > \bar{\lambda}_l(y_l^* - \bar{y}_l).$$

Since for each $l,\ l = 1, \dots, k$, $\bar{\lambda}_l(y_l^* - \bar{y}_l) = 1$ holds, we get

$$\max_l \bar{\lambda}_l(y_l^* - y_l) > \max_l \bar{\lambda}_l(y_l^* - \bar{y}_l),$$

which means that $\bar{y}$ solves (7.1).

If there is no $y \in Z,\ y \neq \bar{y}$, such that $y_l < \bar{y}_l$ for some l, then $Z = \{\bar{y}\}$ (Z is a singleton), and $\bar{y}$ clearly solves (7.1). □

7.3 Characterization of Efficient Outcomes: Characterization B

Characterization B: *the sufficient condition.*

An outcome which solves optimization problem

$$\max_{y \in Z} \sum_{l=1}^{k} \lambda_l y_l\,, \tag{7.2}$$

where $\lambda_l > 0,\ l = 1, \dots, k$, is efficient.[2]

Characterization B: *the necessary condition.*

[2] In general, this statement is valid for a certain subset of the set of efficient outcomes, namely for the set of *properly efficient outcomes*. However, in practice the difference between the set of efficient outcomes and the set of properly efficient outcomes is either nonexistent or negligible. The definitions of properly efficient outcomes and properly efficient variants can be found e.g., in Kaliszewski [2], Ehrgott [1], and Miettinen [3].

Assume that set Z is convex.[3] *Every efficient outcome* (see footnote 2) *solves optimization problem* (7.2) *for some* $\lambda_l > 0$, $l = 1, \dots, k$.

Figure 7.2 represents the contour of the criterion function in optimization problem (7.2), corresponding to the maximal value of this function on set Z which is not convex. By Characterization B, the maximum of this criterion function is attained at an efficient outcome.

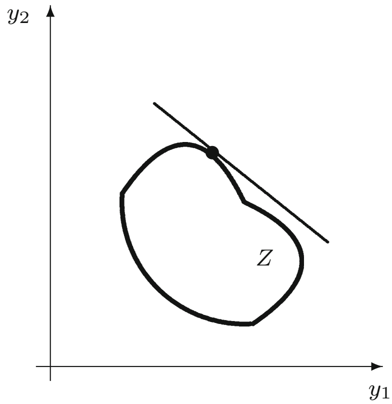

Figure 7.2 A graphical interpretation of deriving efficient outcomes in set Z which is not convex, by optimization problem (7.2)

Figure 7.3 represents the contour of the criterion function in optimization problem (7.2), corresponding to the maximal value of this function on convex set Z. By Characterization B, the maximum of this criterion function is attained at an efficient outcome.

Scalarizing function $\sum_{l=1}^{k} \lambda_l y_l$ is called the *weighted linear function.*

[3] Set A is convex if $y \in A$ and $y' \in A$ implies $y\lambda + y'(1-\lambda) \in A$ for every $0 \le \lambda \le 1$.

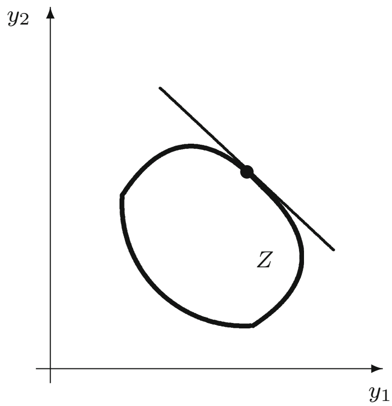

Figure 7.3 A graphical interpretation of deriving efficient outcomes (and thus efficient variants) in convex set Z, by optimization problem (7.2)

7.4 Efficiency Test

The proof of the necessary condition of Characterization A is constructive, i.e., it provides a formula to calculate coefficients λ_l, $l = 1, \ldots, k$, for which an efficient outcome (and thus an efficient variant or some efficient variants) solves optimization problem (7.1).

In consequence, the following procedure tests whether a given outcome $\bar{y}$ is efficient.[4]

Efficiency Test Procedure

1. Calculate $\bar{\lambda}_l$, $l = 1, \ldots, k$, by the formula from the proof of the necessary condition for Characterization A, namely $\bar{\lambda}_l = (y_i^* - \bar{y}_i)^{-1}$, $l = 1, \ldots, k$.

2. Solve optimization problem (7.1) with $\bar{\lambda}_l$, $l = 1, \ldots, k$; if $\bar{y}$ solves the optimization problem, then $\bar{y}$ is an efficient outcome; otherwise $\bar{y}$ is not efficient.

[4] If more than one outcome pass this test for the same $\bar{\lambda}_l$, $l = 1, \ldots, k$, then all such outcomes are weakly efficient (see footnote 2 of this chapter), but only one of them is efficient.

The above procedure at the same time verifies if variant $\bar{x}$ is efficient, where $\bar{y} = f(\bar{x})$.

7.5 Derivation of Efficient Variants and Variant Ranking

It is worth observing that both characterizations, Characterization A and Characterization B, assign to each variant a score, i.e., a value of the corresponding scalarizing function. These scores establish *rankings of variants* (see Sect. 11.5), since variants can be ranked in decreasing or increasing order of the assigned scores.

7.6 Weight Normalization

The set of weights $\lambda_l > 0,\ l = 1, \ldots, k$, is unbounded. However, in numerical computations it is more convenient to deal with a bounded set.

Let us observe that the "intensity" of dependence of a scalarizing function on criteria is related to proportions of weights rather than to their absolute values. For instance, multiplication of all weights by a positive number does not change the properties of the scalarizing functions used in Characterization A and Characterization B, in the sense that the original and the modified scalarizing functions yield the same efficient outcomes (and thus the same efficient variants) and the same variant ranking. This observation allows us to deal with bounded sets of weights.

For any vector of weights $\lambda_l > 0,\ l = 1, \ldots, k$, we can multiply each weight by the reciprocal of their sum, i.e., by $(\sum_{l=1}^{k} \lambda_l)^{-1}$, obtaining in this way a new vector of weights $\lambda'_l,\ l = 1, \ldots, k$. According to the above argument, each of two instances of the scalarizing function, as in Characterization A or in Characterization B, one instance with $\lambda_l > 0,\ l = 1, \ldots, k$, another with $\lambda'_l > 0,\ l = 1, \ldots, k$, yields the same efficient outcomes (and thus an efficient variants). However, in the latter case the additional condition $\sum_{l=1}^{k} \lambda_l' = 1$ holds. Indeed,

$$\sum_{l=1}^{k} \lambda_l' = \sum_{l=1}^{k} \frac{\lambda_l}{\sum_{l=1}^{k} \lambda_l} = \frac{1}{\sum_{l=1}^{k} \lambda_l} \sum_{l=1}^{k} \lambda_l = 1.$$

Hence, we always can confine ourselves to vectors of weights which belong to the bounded set

$$\{\lambda \,|\, \lambda_l > 0,\ l = 1, \ldots, k,\ \sum_{l=1}^{k} \lambda_l = 1\}.$$

Such a transformation is called *normalization* of weight vectors.

7.7 Scalarizing Functions and Value Functions

Scalarizing functions used in both characterizations are composed of criteria functions, about which no assumption is made. In particular, we do not assume that criteria functions are given in the same units. This in turn implies that all operations we perform on those functions, such as multiplications and taking their sum or maxima have no physical or economic interpretation. Therefore, one should remember that scalarizing functions are only a technical tool to derive efficient outcomes (and thus efficient variants).

To interpret scalarizing functions in terms of aggregated partial *utilities* represented by individual criteria functions, a number of additional assumptions is to be made. However, this topic, belonging entirely to the scope of economic theories, is not covered here.

7.8 Sum-Up

It is easy to observe (see Fig. 7.2) that optimization problem (7.2) cannot derive an efficient outcome if set Z cannot be supported at that outcome by a hyperplane $\lambda_1 y_1 + \ldots + \lambda_k y_k$ with all positive coefficients. In other words, if set Z is not convex, it is not guaranteed (opposite to the case of optimization problem (7.1), see Fig. 7.1), that every efficient variant can be derived by solving optimization problem (7.2).[5] Because of this, from now on we exclusively make use of optimization problem (7.1), which is free of this flaw.

In Characterization A as well as in Characterization B, it makes no difference whether we search for an efficient variant portfolio or for a single efficient variant. Hence, there is no impediment to use these characterizations to derive single efficient variants instead of algorithms presented in Chap. 4, if advisable (e.g., because of high number of variants).

7.9 Creative Assignment

For all three MY_PROBLEM_i models, derive a handful of efficient variants or efficient variant portfolios, as appropriate, by the methods presented in this chapter.

[5] A consequence of this fact can be, e.g., as follows. Suppose that in a tender with two criteria, the weighted linear function is used to provide scores. Suppose also that there are "non-balanced" offers with outcomes $(u, 0)$ and $(0, u)$, where $u = t + t + \varepsilon$, $\varepsilon > 0$ is an arbitrarily positive number. Then, for no combination of positive weights, the "balanced" offer with outcome (t, t) can be the winning offer, even if outcome (t, t) is efficient. An example: $(49, 49), (100, 0), (0, 100)$.

Chapter 8

Supporting the Process of the Most Preferred Variant Selection

Several police divisions rushed here and there, searched the grounds, every bush, every weed, and both x-rays and laboratory samples were diligently taken of everything imaginable.

8.1 Chapter Content

In this chapter, we outline a general scheme how to select a decision variant in the third phase of the decision making process—the choice phase—a decision variant which the Decision Maker regards as the most preferred.

Electronic supplementary material The online version of this chapter (doi: 10.1007/978-3-319-32756-3_8) contains supplementary material, which is available to authorized users.

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8.2 Searching Over the Set of Efficient Variants

Every rational methodology of decision making has to build on the bare fact that the state of problem awareness and understanding, allowing the DM to select a decision variant which he/she regards as the most preferred one, is attained by him/her only gradually. Derogations from this general rule can occur in practice only in specific, and rather rare, cases. In reference to the decision process scheme introduced in Chap. 2, this means that in general one cannot expect that the most preferred variant can be identified

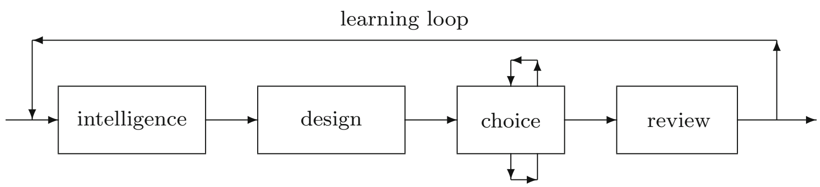

Figure 8.1 Four phases of decision making problems with efficient variant derivation in phase 3

by the DM in one pass of the four phases—*intelligence* → *design* → *choice* → *review* of the decision making process.

The third phase of the decision making scheme can be formalized to a much higher extent than the other three phases. This explains why in the literature devoted to the computer supported multiple criteria decision making, the most focus is on that phase.

With the notion of efficiency defined, it is possible to partition the set of decision variants into the set of efficient variants and the set of remaining (dominated) variants. As argued in Sect. 1.2.3, only efficient variants are rational candidates for the most preferred variant.

Since, in general, the set of efficient variants is not a singleton, to identify the most preferred variant a form of enumeration (rather partial than complete) is necessary. Therefore, the scheme of the decision making process, as represented in Fig. 2.1, has to be extended to the form represented in Fig. 8.1. In the latter figure, the arrows indicate the loop related to the interactive enumeration of the set of efficient variants.

As yet, despite many efforts, the process of selecting the most preferred variant has not been successfully automated. No algorithm of the general acceptance, especially by practitioners, has been proposed. Even attempts to give a precise, formal version of the vague and non-constructive notion of the most preferred variant given in Chap. 1, have all failed. The DM remains a sole sovereign actor, with all his/her knowledge, prejudice and hesitations, to decide what "the most preferred" means for him/her. Therefore, the choice of the most preferred variant is nowadays made, as a rule, in *interactive mode*. That mode relies on successive calling by the DM on the (computer based) model (of a decision making problem) and successive responses computed by that model.

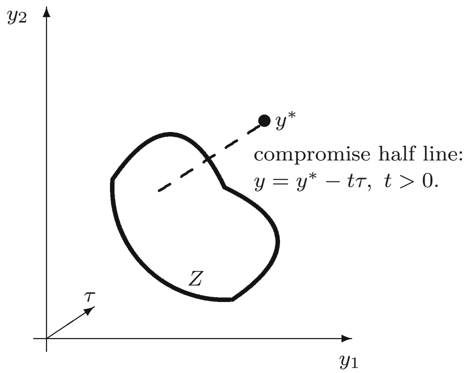

Figure 8.2 Vector of concessions τ and the corresponding compromise half line

Searching for the most preferred outcome (and hence also the most preferred variant or variants, there can be more than one variant corresponding to the most preferred outcome) in interactive mode reduces, by the very principle, to a partial enumeration of the set of efficient outcomes. At each iteration (a *DM* $\Leftrightarrow$ *model* interaction), the DM, on the base of his/her vague and incomplete knowledge about the decision problem he/she tries to solve, expresses his/her preferences with respect to outcomes which have been derived in previous iterations. Those preferences are used to define "regions of the DM interest" in which the search for the most preferred outcome should be continued. If the DM does not want to continue the process, then the incumbent, i.e., the most preferred variant derived thus far, becomes the problem solution (the most preferred variant).

We now undertake the task to formalize the process of searching for the most preferred decision (variant, outcome).

Let us start from the observation that the ideal element $\hat{y}$ carries the following information for the DM:

– the component $\hat{y}_l,\ l = 1, \ldots, k$, represents the maximal value of criterion l, which is attainable on the set of decision variants,

– any outcome $y \in Z$ represents a collection of concessions with respect to the maximal values $\hat{y}_l,\ l = 1, \ldots, k$, of respective criteria, the quantities of such concessions are $\hat{y}_l - y_l,\ l = 1, \ldots, k$.

Since, in general, a decision variant x such that $f(x) = \hat{y}$ is not available, the DM has to compromise on the component values of $\hat{y}$.

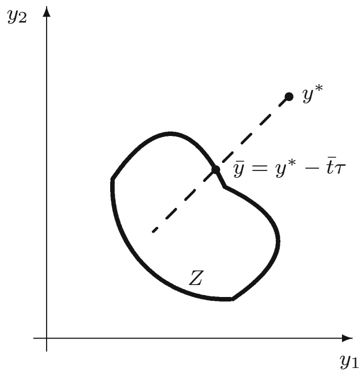

Figure 8.3 A graphical interpretation of deriving the most preferred outcome among outcomes located on the compromise half line

"If I cannot get $\hat{y}$ (i.e., if I cannot get the maximal values of criteria simultaneously), then I admit (I have to make) concessions; but let those concessions be made according to proportions set by myself".

For example: *"Let a deterioration (decrease) of the value of a selected criterion by one unit be accompanied by a deterioration of the remaining criteria by two units."*

A vector in R^k_+ with positive components, representing proportions (thus, the length of the vector can be any) of concessions, is called the *vector of concessions*; we denote it by τ.

In the sequel, we consider concessions made with respect to element y^* instead of to element $\hat{y}$, where, we recall, $y^*_l = \hat{y}_l + \varepsilon$, $l = 1, \ldots, k$, $\varepsilon > 0$. As seen below, such a replacement is essential for some formal reasons (see formula (8.5)). With arbitrary small values of ε, the difference between y^* and $\hat{y}$ is, especially in practical applications, negligible.

We call the half line defined as

$$y = y^* - t\tau,\ t > 0, \tag{8.1}$$

the *compromise half line*. The compromise half line is a collection of elements of $\mathcal{R}^k$, which all have the same proportions of concessions with respect to element y^*, and the proportions are set by the DM in the form of a vector of concessions τ. Figure 8.2 presents an instance of the vector of concessions τ and the corresponding compromise half line.

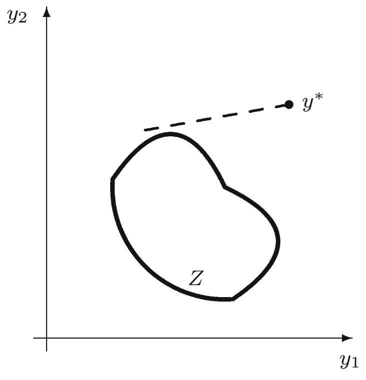

Figure 8.4 The case, where the compromise half line contains no outcome

There exist two methods of presenting the vector of concession: explicit, by providing vector τ directly, and implicit, by providing a *base element*. A base element can be any element y of the outcome space (not necessarily from set Z), which satisfies $y_l < y^*_l$, $l = 1, \ldots, k$. Then $\tau = y^* - y$.

These two methods of presenting vectors of concessions represent two different approaches to express DM's preferences.

When vector of concessions τ is presented explicitly (directly, by vector τ), the preferences are expressed in the form of proportions of concessions accepted (or set) by the DM. In this case, we say that preferences are expressed in the *elementary form*.

When vector of concessions τ is presented implicitly (indirectly, by a base element), we say that preferences are expressed in the *holistic form*.

These two methods of presenting vectors of concessions underly two main methods of interactively enumerating the set of efficient outcomes, represented in the literature of the MCDM domain: the *weight method* and the *reference point method*.[1]

Irrespective of how a vector of concessions is presented, once it is known, it defines the compromise half line.

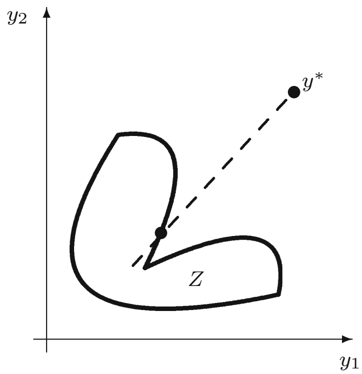

Figure 8.5 The case, where the outcome on the compromise half line, which is closest to the element y^*, is not efficient

Assume that vector of concessions τ is given. Assume also that we are interested only in outcomes which are located on the compromise half line $\{y \mid y = y^* - t\tau,\ t > 0\}$. This assumption is quite artificial, but we make it for a while, just to illustrate our reasoning. Below we shall relax this assumption. But as long as this assumption holds, the outcome which is the most preferred with

[1] This statement is a slight simplification, in each case one should rather speak of a class of methods which differ in formal and technical details. However, here we do not elaborate on this.

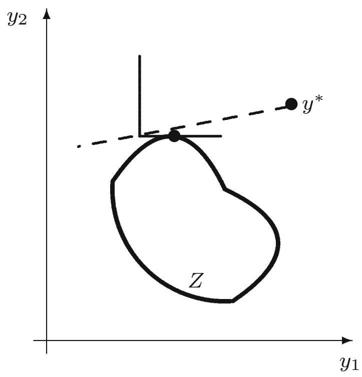

Figure 8.6 A graphical interpretation of deriving an efficient outcome by optimization problem (8.3) ((7.1))—case I. The compromise half line does not contain any outcome, the solution of the problem is an efficient outcome which is not on the compromise half line

respect to preferences represented by τ, is the outcome closest to the element y^*. Such an outcome can be derived in an obvious manner—it is the outcome

$$\bar{y} = y^* - \bar{t}\tau, \tag{8.2}$$

where

$$\bar{t} = \min\{\, t \,|\, y^* - t\tau \in Z,\ t > 0 \,\}\,.$$

A graphical interpretation of how outcome $\bar{y}$, the outcome closest to y^* among all outcomes on the compromise half line, can be derived is presented in Fig. 8.3.

To be consistent with the assumption that the most preferred outcome (variant) is to be sought among efficient outcomes (variants), searching for efficient outcomes on a compromise half line has to satisfy some additional requirements. Namely,

– the compromise half line has to contain at least one outcome,

– the outcome derived has to be efficient.

The first of the conditions is not fulfilled, e.g., in the case presented in Fig. 8.4.

The second of the conditions is not fulfilled, e.g., in the case presented in Fig. 8.5.

To deal with the above two cases, we have to proceed as in the next section.

8.3 Searching for Efficient Outcomes with the Compromise Half Line as a Guideline

To derive the most preferred outcome (and hence also the most preferred variant) we make use of optimization problem (7.1), i.e., we recall, the problem

$$\min_{y \in Z} \max_{l} \lambda_l (y_l^* - y_l), \tag{8.3}$$

where $\lambda_l > 0,\ l = 1, \ldots, k$. According to Characterization A, a solution of this problem is an efficient outcome. This outcome is the closest to element y^*, with the weighted Tchebycheff function

$$\max_{l} \lambda_l (y_l^* - y_l), \tag{8.4}$$

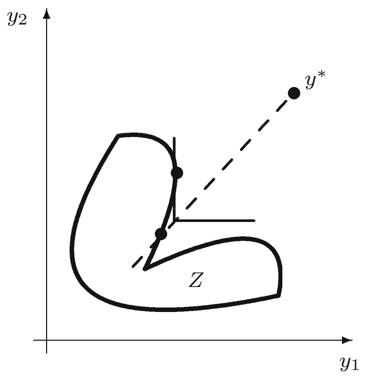

Figure 8.7 A graphical interpretation of deriving an efficient outcome by optimization problem (8.3) ((7.1))—case II. The compromise half line contains outcomes but no efficient outcome; the solution of the problem is an efficient outcome which is not on the compromise half line

introduced in Chap. 7, as a measure of distance.

Three cases are possible here.

If the compromise half line $\{y \mid y = y^* - t\tau,\ t > 0\}$, contains an efficient outcome, then that outcome solves optimization problem (8.3) (7.1) with

$$\lambda_l = \tau_l^{-1},\ l = 1, \ldots, k. \tag{8.5}$$

Among all outcomes on the compromise half line, this outcome is the closest one to element y^*. This case is graphically interpreted in Fig. 7.1.

If the compromise half line $y = y^* - t\tau,\ t > 0$, contains no outcome, then some other efficient outcome solves optimization problem (8.3) (7.1) with $\lambda_l,\ l = 1, \ldots, k$, set by formula (8.5). This case is graphically interpreted in Fig. 8.6.

If the outcome of the compromise half line which is closest to element y^* is not efficient, then an efficient outcome solves optimization problem (8.3) (7.1) with $\lambda_l,\ l = 1, \ldots, k$, set by formula (8.5). This case is graphically interpreted in Fig. 8.7.

In all three cases, the compromise half line serves as a searching guideline, but only in the first case the derived efficient outcome actually lies on it.

The selection of y^*, namely $y_l^* = \hat{y}_l + \varepsilon,\ l = 1, \ldots, k,\ \varepsilon > 0$, ensures that $\tau_l > 0,\ l = 1, \ldots, k$, irrespective of choice of a base element, thus formula (8.5) is always well-defined.

The plausible properties of optimization problem (8.3) (7.1) (see Sects. 7.7 and 7.8) are the reason, why in the sequel we use exclusively this problem to derive efficient outcomes (and the corresponding efficient variants), in relation to the DM preferences expressed in terms of vectors of concessions and compromise half lines.

8.4 Decision Process Support

The method of deriving efficient outcomes by selecting different vectors of concessions, to valuate outcomes in the decision process, does not require any assumption except the existence of element $\hat{y}$. It is hard to imagine a practical decision making problem where this element does not exist.

In contrast, other methods, represented in the literature of the MCDM domain, require that various assumptions on preference forms and preference properties presented by the DM are to be adopted. A barrier for practical applications of those methods is the necessity to verify if in each particular case such assumptions are fulfilled. Despite the fact that, on top of the decision problem under consideration, this is another problem by itself, such a verification in practice is hardly possible. This is a consequence of quite obvious and understandable unwillingness of the DM to undergo a procedure of that sort. In fact, it is unreal to think that a politician or a company CEO would declare his/her readiness to be verified on the account of consistency with some adopted assumptions.

Embedding the method for the implicit enumeration of the set of efficient outcomes, as presented above, into a user friendly interface with graphical tools is purely a technical matter. However, such tools for computer assisted MCDM decision processes are, as yet, not widely in use.

8.5 Sum-Up

Representing vectors of concessions either in terms of "element y^*, vector of concessions τ" or in terms of "element y^*, base element y", has rational grounds. In that manner, DM preferences are expressed by proportions of concessions necessary to depart from unattainable y^* to reach an efficient outcome. The notion of vector of concessions emerges here quite naturally. There are research results in the field of mathematical psychology[2] which show that the DM assigns more importance to potential losses than to potential gains, hence he/she is more careful in thinking of losses. Therefore, the notion of direction of concessions can be the proper framework for preference expressing.

As shown above, the nature of the modeled decision processes can be such that efficient outcomes, derived with the help of the

[2] See *Prospect Theory* by David Kahneman and Amos Tversky; David Kahneman, the American psychologist, the Nobel Price laureate in 2002; Amos Tversky, the American psychologist.

notion of vector of concessions and the corresponding compromise half line, are actually on that half line. However, the adopted method of deriving efficient outcomes, based on Characterization A, ensures that this happens always, whenever the nature of the problem permits.

8.6 Creative Assignment

For all three MY_PROBLEM_i models, propose a handful of vectors of concessions τ and derive a handful of efficient variants or efficient variant portfolios, as appropriate, corresponding to those vectors.

For all three MY_PROBLEM_i models, propose a handful of (decision) variants or (decision) variant portfolios, as appropriate, and perform the efficiency test for them, as presented in Chap. 7.

Select a vector of concessions τ and for MY_PROBLEM_1 rank variants with respect to increasing values of the Tchebycheff function corresponding to that τ.

Chapter 9

Decision Problems, Continuation

Without further ado I stocked my ship with necessary provisions, took off and, after numerous adventures we need not go into here, finally spotted in a great swarm of stars one that differed from all the rest, since it was a perfect cube.

9.1 Chapter Content

We return now to three decision problems formulated earlier: selection of an investment variant (Chap. 3), selection of a variant portfolio—the discrete case (Chap. 5) and selection of a variant portfolio—the continuous case (Chap. 6). For all these problems the decision process has already undergone the intelligence phase (1) and the design phase (2).

As we are now acquainted with the content of Chaps. 7 and 8, we have in our disposal notions, knowledge and tools which

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allow us to pass to the choice phase (3) and the review phase (4). All together, we are now in the position to solve all three decision problems following the decision process scheme presented in Chap. 1.

9.2 Selection of a Single Variant, Continuation

9.2.1 Problem Formulation

Problem formulation—the intelligence phase (1).

The problem has been formulated in Sect. 3.2.

9.2.2 Problem Modeling

Problem modeling—the design phase (2).

The problem has been modeled in Sect. 3.3 and in Chap. 5.

9.2.3 Variant Selection

Variant selection—the choice phase (3).

For data for the problem of selection of an investment variant (selection of a river crossing), specified in Chap. 3, we could derive all efficient variants by Algorithm E_m, presented in Chap. 4. But even with all efficient variants determined, the decision problem (selecting the most preferred variant) remains to be solved. Therefore, here we make use of the method presented in Chap. 7.

To change criteria functions "investment completion time" and "total investment cost" to the type "the more, the better" we introduce new criterion function f_1': "$-1 \times$ investment completion time" and new criterion function f_2': "$-1 \times$ total investment cost". With these new functions the ideal element is $\hat{y} = (-1, -1)$. Assume, arbitrarily, $\varepsilon = 0.10$. Then $y^* = (-0.90, -0.90)$.

Suppose that the DM (e.g., the town council, a team set by the town council or a council member), in order to prepare himself/herself for the town general assembly called to select an investment variant, simulates the decision process. Knowing that no ideal variant (a variant with outcome $(-1, -1)$) does not exist, he/she proposes the following vector of concessions:

'let the deterioration (decrease) of the value of the "investment completion time" criterion function by one unit (in year quarters) be accompanied by the deterioration of the "total investment cost" criterion function by one unit (in millions of PLN), i.e., $\tau = (1,\ 1)$.'

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 1.000,\ \ \lambda_2 = 1.000.$$

Normalizing weights (see Sect. 7.5) we get $\lambda_1 = 0.500, \lambda_2 = 0.500$.

Since the investment variants are in this case given explicitly, to solve optimization problem (8.3) it is enough to calculate for each variant $x^j,\ j = 1, \dots, 7$, the value of the weighted Tchebycheff function (8.4) (the criterion function in the optimization problem (8.3)),

$$\max_l \lambda_l (y_l^* - f_l'(x^j)),$$

and to select a variant with the minimal value of this function.

In this case, the solution to optimization problem (8.3) is outcome $y = (-6,\ -3)$ which corresponds to variant x^5. According to Characterization A, this outcome is efficient, hence the corresponding variant x^5 is efficient.

Assume that after analyzing variant x^5—a potential candidate for the most preferred variant—the DM changes the vector of concessions to

$$\tau = y^* - y,$$

where y is base element $(-1.90, -8.90)$ selected by the DM himself/herself. Selection of such an element can be dictated by the DM's experience. For example, the DM may know that in a partnership town, somewhere in the world, an almost identical investment has just been completed with cost and completion time as represented by the selected base element. Hence,

$$\tau = (-0.90, -0.90) - (-1.90, -8.90) = (1.00, 8.00).$$

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 1.000,\ \ \lambda_2 = 0.125\,.$$

Normalizing weights we get $\lambda_1 = 0.889, \lambda_2 = 0.111$. In this case, the solution to optimization problem (8.3) is outcome $(-1,\ -8)$ which corresponds to variant x^7. According to Characterization A, this outcome is efficient, hence the corresponding variant x^7 is efficient.

Assume that the DM selects variant x^7 as the most preferred variant.

9.2.4 Problem Verification

Problem verification—the review phase (4).

Assume that because of protests of ecological lobbies, which conducted an independent research on the river crossing impact on rare bird nesting habitat along the river, variant x^7 has been dropped. This means that the whole decision cycle has to be repeated.

9.2.5 Problem Formulation

Problem formulation—the intelligence phase (1).

Assume that except dropping variant x^7 no other corrections to the problem formulation are made.

9.2.6 Problem Modeling

Problem modeling—the design phase (2).

The model can be adapted to the new circumstances (dropping of variant x^7) in two ways. The first way is to remove from the model all data related to variant x^7. The second way is to keep those data but neglect them in computations. The second way is more practical—it permits to keep all the original data without changing the data structure.

Here, we follow the second way.

9.2.7 Variant Selection

Variant selection—the choice phase (3).

Since set X_0 has been changed, the ideal element calculated for this new set is $\hat{y} = (-6, -1)$. Hence, with $\varepsilon = 0.10$, $y^* = (-5.9, -0.9)$.

Assume that the DM does not change the vector of concessions. Hence, the weights remain unchanged: $\lambda_1 = 0.889, \lambda_2 = 0.111$. In this case, the solution to optimization problem (8.3) is again outcome $y = (-6,\ -3)$ which corresponds to variant x^5. According to Characterization A, this outcome is efficient, hence the corresponding variant x^5 is efficient.

Assume that after analyzing (for the second time) variant x^5—a potential candidate for the most preferred variant—the DM changes the vector of concessions to:

'let the deterioration (decrease) of the value of the "investment completion time" criterion function by four units (in year quarters) be accompanied by the deterioration of the "total investment cost" criterion function by one unit (in millions of PLN), i.e., $\tau = (4,\ 1)$.'

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 0.250,\ \ \lambda_2 = 1.000\,.$$

Normalizing weights we get $\lambda_1 = 0.200, \lambda_2 = 0.800$. In this case, the solution to optimization problem (8.3) is outcome $y = (-8,\ -2)$ which corresponds to variant x^3. According to Characterization A, this outcome is efficient, hence the corresponding variant x^3 is efficient.[1]

Assume that the DM selects variant x^3 as the most preferred variant.

[1] Actually, this is the case where the set of weakly efficient variants does not coincide with the set of efficient variants (see footnote 2 of Chap. 7). Indeed, besides outcome $(-8, -2)$ also outcome $(-9, -2)$ solves optimization problem (8.3). From those two outcomes outcome $(-9, -2)$ is clearly dominated by outcome $(-8, -2)$. In consequence, only outcome $(-8, -2)$ is efficient.

9.2.8 Problem Verification

Problem verification—the review phase (4).

Assume that the DM, after confronting variant x^3 against all circumstances of the decision process not represented in the formal model, decides that variant x^3 is indeed the most preferred variant.

At this the decision process terminates.

9.3 Selection of a Variant Portfolio—The Discrete Case, Continuation

9.3.1 Problem Formulation

Problem formulation—the intelligence phase (1).

The problem has been formulated in Sect. 5.2.

Table 9.1 Data for project portfolio selection

Project	x^1	x^2	x^3	x^4	x^5	x^6	x^7
c_i	9	7	8	8	6	9	1
d_i	1	4	2	3	3	2	8
a_i	70	12	33	40	65	75	45

9.3.2 Problem Modeling

Problem modeling—the design phase (2).

The problem has been modeled in Sect. 5.3.

9.3.3 Variant Selection

Variant selection—the choice phase (3).

The intelligence phase (1) and the design phase (2) for this problem have been conducted without any reference to data. The choice phase (3) cannot be conducted until all the necessary data for the formal model built in the design phase (2) are in place. Assume now that all the necessary data has been collected.

Assume further that 11 projects have been submitted, from which only 7 satisfy formal requirements. For those 7 projects values of coefficients c_i—(forecasted) company profit increase (net present value in millions of PLN), d_i—(forecasted) company market share increase (in percent), a_i—the necessary financing for the project (in thousands of PLN), are given in Table 9.1. The other 4 projects have been disqualified.

The total amount of funds to finance R&D projects is 155 thousands PLN.

To derive ideal element $\hat{y}$, we have to solve two optimization problems:

$$\max f_1(x) = \textstyle\sum_{j=1}^{7} c_j x_j$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{7} a_j x_j \le a_0\,, \\ \\ x_j = 0 \text{ or } 1, \;\; j = 1, \ldots, 7, \end{array} \right\}, \tag{9.1}$$

and

$$\max f_2(x) = \textstyle\sum_{j=1}^{7} d_j x_j$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{7} a_j x_j \le a_0\,, \\ \\ x_j = 0 \text{ or } 1, \;\; j = 1, \ldots, 7, \end{array} \right\}. \tag{9.2}$$

In this case, there are $2^7 = 128$ vectors with components 0 or 1—they represent all conceivable project portfolios. One can list all project portfolios and next check which of them satisfy the model constraint inequality. Those which satisfy the constraint are decision project portfolios. For each decision project portfolio the value of the respective criterion function is to be calculated, and the decision project portfolio of the largest value defines the respective component of $\hat{y}$.

It is certainly not the approach to be recommended in general. Instead, to solve those problems one can use an optimization package (such as the *Microsoft Excel* add-in *Solver*, or any other).

The solution to the first optimization problem is (decision) project portfolio $x = (1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0)$, for which the value of criterion function $f_1(x)$ is 32.00 (and the value of criterion function $f_2(x)$ is 10.00), and the solution to the second optimization problem is (decision) project portfolio $x = (0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 1)$ for which the value of criterion function $f_2(x)$ is 17.00 (and the value of criterion function $f_1(x)$ is 24.00). Hence, $\hat{y} = (32.00, 17.00)$. Assuming (arbitrarily) $\varepsilon = 0.10$, we get $y^* = (32.10, 17.10)$.

To verify whether there exists the ideal project portfolio, i.e., the project portfolio x for which $f_1(x) = 32.00$ and $f_2(x) = 17.00$, we have to solve the following optimization problem:

$$\max f_1(x) = \textstyle\sum_{j=1}^{7} c_j x_j$$

$$\text{subject to } x \in X_0' = \left\{ x \;\middle|\; \begin{array}{l} (f_2(x) =)\ \sum_{j=1}^{7} d_j x_j \geq 17.00\ (= \hat{y}_2)\,, \\[2ex] \sum_{j=1}^{7} a_j x_j \leq a_0\,, \\[2ex] x_j = 0 \text{ or } 1,\ \ j = 1, \ldots, 7, \end{array} \right\},$$

or the optimization problem

$$\max f_2(x) = \textstyle\sum_{j=1}^{7} d_j x_j$$

$$\text{subject to } x \in X_0'' = \left\{ x \;\middle|\; \begin{array}{l} (f_1(x) =)\ \sum_{j=1}^{7} c_j x_j \geq 32.00\ (= \hat{y}_1)\,, \\[2ex] \sum_{j=1}^{7} a_j x_j \leq a_0\,, \\[2ex] x_j = 0 \text{ or } 1,\ \ j = 1, \ldots, 7, \end{array} \right\},$$

In the first case, we get the maximal value of criterion function $f_1(x)$ equal to 24.00, and in the second case, we get the maximal value of criterion function $f_2(x)$ equal to 10.00. From this, we infer that the ideal project portfolio does not exist.

Assume that the DM (in this case: the selection committee), knowing that the ideal project portfolio does not exist, selects the following vector of concessions:

'let the deterioration (decrease) of the value of the "company profit increase" criterion function by one unit (net present value in millions of PLN) be accompanied by the deterioration of the "company market share increase" (in percent) criterion function by one unit, i.e., $\tau = (1,\ 1)$.'

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 1.000,\ \ \lambda_2 = 1.000\,.$$

Normalizing weights we get $\lambda_1 = 0.500$, $\lambda_2 = 0.500$. In this case, the solution to optimization problem (8.3) with X_0 defined as in (9.1) or (9.2), is outcome $y = (29.00, 12.00)$. According to Characterization A, this outcome is efficient, hence the corresponding project portfolio $x = (0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0)$ is efficient.

Assume that after analyzing this project portfolio—a potential candidate for the most preferred project portfolio—the DM changes the vector of concessions to

$$\tau = y^* - y,$$

where y is base element $(20.00, 13.00)$ selected by himself/herself. Selection of such an element can be dictated by the DM's experience. For example, the DM may use as the base element the outcome of the last year winner. Hence,

$$\tau = (32.10, 17.10) - (20.00, 13.00) = (12.10, 4.10).$$

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 0.083,\ \ \lambda_2 = 0.244\,.$$

Normalizing weights we get $\lambda_1 = 0.253$, $\lambda_2 = 0.747$. In this case, the solution to optimization problem (8.3) with X_0 defined as in (9.1) or (9.2), is outcome $y = (24.00,\ 17.00)$ which corresponds to project portfolio $x = (0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 1)$. According to Characterization A, this outcome and the corresponding project portfolio are efficient.

Assume that the DM selects project portfolio $x = (0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 1)$ as the most preferred portfolio.

9.3.4 Problem Verification

Problem verification—the review phase (4).

Assume that the company management, acting with some delay, does not approve the recommendation of the DM (the selection committee) and excludes form the competition (without giving specific reasons) project x^7. This means that the whole decision cycle has to be repeated.

9.3.5 Problem Formulation

Problem formulation—the intelligence phase (1).

Assume that except excluding project x^7, no other corrections to the problem formulation are made.

9.3.6 Problem Modeling

Problem modeling—the design phase (2).

The model can be adapted to the new circumstances (exclusion of project x^7) in two ways. The first way is to remove from the model all data related to project x^7. The second way is to keep those data and to add the additional constraint $x^7 = 0$, which guarantees that no portfolio derived includes project 7 . The second way is more practical—it permits to keep all the original data without changing the data structure.

Here, we follow the second way.

9.3.7 Variant Selection

Variant selection—the choice phase (3).

Since set X_0 has been changed, the ideal element calculated for this new set is $\hat{y} = (32.00, 12.00)$. Hence, with $\varepsilon = 0.10$, $y^* = (32.10, 12.10)$.

Assume that the DM does not change the vector of concessions. Hence, the weights remain unchanged: $\lambda_1 = 0.238$, $\lambda_2 = 0.762$. In this case, the solution to optimization problem (8.3) with X_0 defined as in (9.1) or (9.2), is outcome $y = (29.00, 12.00)$ which corresponds to project portfolio $x = (0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0)$. Observe that this project portfolio has been already derived at the previous iteration. According to Characterization A, this outcome is efficient, hence the corresponding variant x is efficient.

Assume that after analyzing this project portfolio—a potential candidate for the most preferred project portfolio—the DM selects the following vector of concessions:

'let the deterioration (decrease) of the value of the "company profit increase" criterion function by one unit (net present value in millions of PLN) be accompanied by the deterioration of the "company market share increase" (in percent) criterion function by three units, i.e., $\tau = (1,\ 3)$*.'*

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 1.000,\ \ \lambda_2 = 0.333\,.$$

Normalizing weights we get $\lambda_1 = 0.750, \lambda_2 = 0.250$. In this case, the solution to optimization problem (8.3) with X_0 defined as in (9.1) or (9.2), is outcome $y = (32.00, 10.00)$ which corresponds to project portfolio $x = (1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0)$. According to Characterization A, this outcome is efficient, hence the corresponding variant x is efficient.

Assume that the DM selects project portfolio $x = (1, 1, 1, 1, 0, 0, 0)$ as the most preferred portfolio.

9.3.8 Problem Verification

Problem verification—the review phase (4).

Assume that the DM, after confronting the project portfolio selected in the choice phase (3) against all circumstances of the decision process not represented in the formal model, decides that this project portfolio is indeed the most preferred variant.

At this the decision process terminates.

9.4 Selection of a Variant Portfolio—The Continuous Case, Continuation

9.4.1 Problem Formulation

Problem formulation—the intelligence phase (1).

The problem has been formulated in Sect. 6.2.

9.4.2 Problem Modeling

Problem modeling—the design phase (2).

The problem has been modeled in Sect. 6.3.

The intelligence phase (1) and the design phase (2) for this problem have been conducted without any reference to data. The choice phase (3) cannot be conducted until all the necessary data for the formal model, built in the design phase (2), are in place. Assume now that all the necessary data has been collected.

Assume that, as a first approximation to solving the problem on a practical scale, we consider diets composed of just seven products:

> cottage (high-fat) cheese, bananas, apples, red beets, potatoes (winter or spring), pork butt, pork loin (bone in).

The requirements for diets refer to energy equivalent (kilocalorie) and twelve ingredients:

> proteins, fats, carbohydrates, calcium, phosphorus, magnesium, iron, vitamin A, vitamin D, vitamin B_1 (thiamine), vitamin B_2 (riboflavin), vitamin C (ascorbic acid).

Prices of products (in PLN per 100 g), energy equivalents (in the eatable part of the product) and the content of twelve ingredients in the selected products are presented in Table 9.2. Product prices are as quoted on The Prag Food Wholesale Market, as of January 9, 2009, and they are the maximal prices quoted.

By convenience, in computations we use the energy unit equal to 100 kcal.

The ingredient content in products are taken from the book referred to in Sect. 6.3. Since that data refer to 100 g of a product, below we assume 100 g as the unit of products in all our considerations and calculations. Product ingredients are expressed in the following units:

> proteins—grams, fats—grams, carbohydrates—grams, calcium—milligrams, phosphorus—milligrams,

Table 9.2 Table of food prices and ingredient content in products

	Cott. cheese	Bananas	Apples	Red beets	Potatoes	Pork butt	Pork lion
Price	1.056	0.400	0.220	0.100	0.080	1.080	1.309
Energy equivalent	1.75	0.95	0.46	0.38	0.85	2.57	1.74
Proteins	17.7	1.0	0.4	1.8	1.9	16.0	21.0
Fats	10.1	0.3	0.4	0.1	0.1	21.7	10
Carbohydrates	37.0	23.5	12.1	9.5	20.5	0.0	0.0
Calcium	88	6	4	41	4	5	15
Phosphorus	216	20	9	17	61	159	208
Magnesium	9	33	3	17	23	19	24
Iron	0.2	0.4	0.3	1.7	0.6	1.1	1.0
Vitamin A	83	8	4	2	1	0	0
Vitamin D	62.0	0.0	0.0	0.0	0.0	0.7	0.6
Vitamin B_1	0.031	0.040	0.034	0.020	0.096	0.559	0.989
Vitamin B_2	0.358	0.100	0.026	0.050	0.041	0.276	0.186
Vitamin C	0.0	9.0	9.2	10.0	11.0	0.0	0.0

magnesium—milligrams, iron—milligrams, vitamin A—micrograms, vitamin D—micrograms, vitamin B_1—milligrams, vitamin B_2—milligrams, vitamin C—milligrams.

The data for ingredients given in Table 9.2 refer to unprocessed products. However, when preparing products for consumption, some parts must be removed and constitute waste. For example, when we eat bananas we eat only the actual fruit from inside, the skin is the waste. Since the data refer to ingredient content in unprocessed food, we have to account for wastes and the difference between the amount of unprocessed food and the amount of food actually consumed. Table 9.3 presents fractions of waste in the unit of products and ingredient content in the eatable part of a product, calculated according to those fractions.

We forgo losses of ingredients when preparing dishes (e.g., by cooking); it is the second simplification we make here. The first one is limiting the number of products in a diet to seven. However, the considerations presented here have just the aim to illustrate the idea of decision process support. To make them practically applicable, a full-fledged model and a lot more reliable and actual (diet recommendations change!) data would be required.

In Table *Diet Standards* mentioned in Chap. 6, we find data for safe and recommended levels for twelve food ingredients in a daily diet. We regard all those twelve levels as the constraints on diets.

Table 9.3 Table of ingredient content in products, adjusted for wastes

	Cott. cheese	Bananas	Apples	Red beets	Potatoes	Pork butt	Pork lion
Wastes	0.0	0.37	0.27	0.25	0.24	0.18	0.32
Price	1.056	0.400	0.220	0.100	0.080	1.080	1.309
Energy equivalent	1.75	0.95	0.46	0.38	0.85	2.57	1.74
Proteins	17.7	0.6	0.3	1.4	1.4	13.1	14.3
Fats	10.1	0.19	0.29	0.08	0.08	17.79	6.80
Carbohydrates	37.0	14.8	8.8	7.1	15.6	0.0	0.0
Calcium	88	4	3	31	3	4	10
Phosphorus	216.00	12.60	6.57	12.75	46.36	130.38	141.44
Magnesium	9.00	20.79	2.19	12.75	17.48	15.58	16.32
Iron	0.2	0.3	0.2	1.3	0.5	0.9	0.7
Vitamin A	83.00	5.04	2.92	1.5	0.76	0	0
Vitamin D	62.0	0.0	0.0	0.0	0.0	0.6	0.4
Vitamin B_1	0.031	0.025	0.025	0.015	0.073	0.491	0.673
Vitamin B_2	0.358	0.063	0.019	0.038	0.031	0.226	0.126
Vitamin C	0.0	5.7	6.7	7.5	8.4	0.0	0.0

An admissible diet (a decision variant) is a diet which contains at least the safe levels of ingredients (in case of lack of data for safe levels, we use recommended levels).

We select data referring to the population group we are interested in, namely girls in the age 16–18, body weight 60 kg, with moderate physical activity. The data are collected in Table 9.4 (the units are the same as in tables of ingredient content—Tables 9.2 and 9.3).

Table *Diet Standards* contains neither safe nor recommended daily consumption levels for carbohydrates. However, on the web page of Scientific Circle for Human Feeding of the Medical Academy of Gdynia (now the Medical University)[2] we find the following excerpt referring to carbohydrates.

> **Recommended daily consumption of carbohydrates.** As consumption of fats should be limited, the main source of energy in the daily diet should be carbohydrates in quantities to meet human daily needs for energy not covered by fats and proteins. According to the standards [which this text refers to] for adults, fats should provide 25–30 % and proteins

[2] Scientific Circle for Human Feeding of the Medical Academy of Gdynia, http://kn.am.gdynia.pl/nkzc/kz/kz_norm.html , as of the end of 2008.

Table 9.4 Table of safe and recommended ingredient content in daily diets for girls in the age 16–18, body weight 60 kg, with moderate physical activity

	Safe level	Recommended level
Energy equivalent	–	2200
Proteins	54	80
Fats	–	81
Calcium	1100	1200
Phosphorus	800	900
Magnesium	320	340
Iron	15	17
Vitamin A	600	800
Vitamin D	–	10
Vitamin B_1	1.4	1.6
Vitamin B_2	1.9	2.0
Vitamin C	60	70

> 12–14 % of energy content in daily diets, which leaves room of 56–63 % for carbohydrates. Even higher level of carbohydrates, up to 70 % of energy content, is recommended on the expense of fats. These should be mainly the complex carbohydrates (as opposed to simple carbohydrates, mainly sugars). It should be noticed that some carbohydrate products are also the source of minerals (magnesium, iron, zinc) and of vitamins (B1 and B2). In the said standards [which this text refers to] levels of carbohydrate consumption are not given in numbers but only in the form of percentages. Keeping this as a reference, we can infer that [. . .]. Male youth in age of 13–15 and 15–20 should consume daily, respectively: 420–470 g and 450–545 g of carbohydrates per person. Female youth in the same age should consume daily, respectively: 365–400 g and 355–390 g per person.

Following the above quoted source, we assume that for the group of persons considered here (girls in the age 16–18, body weight 60 kg, with moderate physical activity) the safe and the recommended levels of daily carbohydrate consumption are, respectively, 355 and 390 g.

To derive ideal element $\hat{y}$ we have to solve two optimization problems. Since originally the first criterion function is of the type "the less, the better" and since according to the convention adopted in this textbook we assume that all criteria functions are of the type "the more, the better", we have to introduce new criterion function $f_1'(x) = -f_1(x)$.

The optimization problems have the following form:

$$\max f_1'(x)$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{7} a_{i,j}x_j \geq a_{i,0},\ i = 1, \ldots, 12, \\ \\ x_j \geq 0,\ j = 1, \ldots, 7, \end{array} \right\}, \tag{9.3}$$

and

$$\max f_2(x)$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{7} a_{i,j}x_j \geq a_{i,0},\ i = 1, \ldots, 12, \\ \\ x_j \geq 0,\ j = 1, \ldots, 7, \end{array} \right\}. \tag{9.4}$$

To solve these problems we have to use an optimization package (such as the *Microsoft Excel* add-in *Solver*, or any other).

The solution to optimization problem (9.3) is daily diet (product portfolio)

$$x = (6.87, 0.00, 0.00, 15.10, 9.47, 0.55, 0.00)$$

(recall: 1 product unit = 100 g), for which the value of criterion function $f_1'(x)$ is -10.08 (PLN) (and the value of criterion function $f_2(x)$ is 20.85 (kilocalorie $\times 100$)).

Analyzing this solution we clearly see that we have to correct the model, since daily consumption of 700 g of cottage cheese, 1500 g of red beets and 950 g of potatoes is not acceptable. To adapt the model to the reality, we limit daily consumption of any product to 250 g.

To this aim, we introduce additional constraints (keeping in mind that the adopted product unit is 100 g)

$$x_j \leq 2.50\,,\ j = 1, \ldots, 7\,.$$

These are quite arbitrary limits, which on the subsequent stages of the decision process can be changed individually for each product and hence adapted to individual tastes and habits of the DM.

However, for the revised model the corresponding optimization problems (9.3) and (9.4), modified accordingly, have no solution (the corresponding set X_0 is empty). In other words, with daily consumption of products limited to 250 g, no combination (mix) of quantities of seven selected products yields the required quantity of ingredients in a daily diet.

Table 9.5 Table of ingredient content in products, adjusted for wastes—supplement I

	Milk 2 %	Butter
Wastes	0	0
Price	0.185	0.98
Energy equivalent	0.470	7.350
Proteins	3.4	0.7
Fats	1.5	82.5
Carbohydrates	5.0	0.7
Calcium	120	16
Phosphorus	86	12
Magnesium	12	1
Iron	0.1	0.1
Vitamin A	25	814
Vitamin D	0.0	0.8
Vitamin B_1	0.037	0.007
Vitamin B_2	0.170	0.035
Vitamin C	1.0	0.0

It is necessary then to extend the list of products. Analyzing information provided by the optimization package (in our case it was the *Microsoft Excel* add-in *Solver*) we observe that the required levels of ingredients are not met for: carbohydrates, calcium, magnesium, iron and vitamin A. Therefore, the list of products should be extended by products which are relatively rich in those ingredients. Such products are, e.g., milk and butter. We extent the list of products by these two products to nine products ($n = 9$) total. We also increase the maximal admissible daily consumption of milk to 500 g (ca. 0.5 l) and limit the maximal admissible daily consumption of butter to 100 g.

Table 9.5 presents the supplement to the table of prices and ingredients (Table 9.3) for milk and butter.

Even for the extended list of products ($n = 9$) optimization problems (9.3) and (9.4) still have no solution—with the imposed limits on the daily consumption of products, no combination (mix) of quantities of nine selected products yields the required quantities of ingredients in a daily diet.

Table 9.6 Table of ingredient content in products, adjusted for wastes—supplement II

	Gouda cheese	Pumpkin seeds	Corn flakes	Rye bread whole grain
Wastes	0.09	0	0	0
Price	1.2	0.35	0.45	0.65
Energy equivalent	2.88	5.56	3.63	2.25
Proteins	25.4	24.5	6.9	6.8
Fats	20.8	45.8	2.5	1.8
Carbohydrates	0.1	18.0	83.6	53.8
Calcium	734	43	8	66
Phosphorus	470	1170	40	245
Magnesium	28	540	6	71
Iron	0.6	15.0	0.8	2.5
Vitamin A	251	38	0	0
Vitamin D	0.22	0.00	0.00	0.00
Vitamin B_1	0.024	0.210	0.007	0.192
Vitamin B_2	0.357	0.320	0.048	0.172
Vitamin C	0.0	0.0	0.0	0.1

It is necessary then to further extend the list of products. Analyzing information provided by the optimization package (in our case it was the *Microsoft Excel* add-in *Solver*) we infer that the required levels of ingredients are not met for: carbohydrates, calcium, magnesium and iron. Therefore, the list of products should be extended by products which are relatively rich in those ingredients. Such products are, e.g., Gouda cheese (high content of calcium and magnesium), pumpkin seeds (high content of carbohydrates and magnesium), corn flakes (high content of carbohydrates) and rye bread (high content of carbohydrates and magnesium). We extend the list of products by these four products to thirteen products ($n = 13$) total. We also limit the maximal admissible daily consumption of the first three products (Gouda cheese, pumpkin seeds, corn flakes) to 100 g and of rye bread to 200 g.

Table 9.6 presents the supplement to the table of prices and ingredients (Table 9.3) for Gouda cheese, pumpkin seeds, corn flakes and rye bread. The prices of pumpkin seeds, corn flakes and rye bread, which on January 9, 2009 were not listed on The Prag Food Wholesale Market, have been taken equal to prices in one of hypermarkets in Warsaw on the same day.

For the list of products extended for the second time, optimization problems (9.3) and (9.4), modified accordingly, have solutions.

The solution to optimization problem (9.3) is daily diet (product portfolio)

$$x = (1.52, 0.00, 2.50, 2.50, 2.50, 0.42, 0.36, 5.00, 0.36, 0.16, 0.18, 1.00, 2.00),$$

for which the value of criterion function $f_1'(x)$ is -7.85 (PLN) (and the value of criterion function $f_2(x)$ is 21.71 (kilocalorie $\times 100$)).

The solution to optimization problem (9.4) is daily diet (product portfolio)

$$x = (2.50, 2.50, 2.50, 2.50, 2.50, 2.50, 2.50, 5.00, 1.00, 1.00, 1.00, 1.00, 2.00),$$

for which the value of criterion function $f_2(x)$ is 43.53 (kilocalorie $\times 100$) (and the value of criterion function $f_1'(x)$ is -18.05 (PLN)).

Hence, $\hat{y} = (-7.85, 43.53)$. Assuming (arbitrarily) $\varepsilon = 0.10$ we have $y^* = (-7.75,\ 43.63)$.

To verify whether the ideal diet exists, i.e., daily diet x for which $f_1'(x) = -7.85$ and $f_2(x) = 43.53$, we solve the following optimization problem:

$$\max f_1'(x)$$

$$\text{subject to } x \in X_0' = \left\{ x \;\middle|\; \begin{array}{l} f_2(x) \geq 43.53, \\ \\ \sum_{j=1}^{13} a_j x_j \geq a_{i,0}\,,\ \ i = 1, \ldots, 12, \\ \\ x_j \geq 0,\ \ j = 1, \ldots, 13, \end{array} \right\},$$

or the optimization problem

$$\max f_2(x)$$

$$\text{subject to } x \in X_0'' = \left\{ x \;\middle|\; \begin{array}{l} f_1'(x) \geq -7.85, \\ \\ \sum_{j=1}^{13} a_j x_j \geq a_{i,0}\,,\ \ i = 1, \ldots, 12, \\ \\ x_j \geq 0,\ \ j = 1, \ldots, 13, \end{array} \right\}.$$

In the first case, we get the maximal value of criterion function $f_1'(x)$ equal to -18.05, and in the second case, we get the maximal value of criterion function $f_2(x)$ equal to 21.71. From this, we infer that the ideal diet does not exist.

These proceedings complete the design phase (2) of the decision process. The model obtained can provide decision variants (daily diets, product portfolios) according to DM's preferences.

9.4.3 Variant Selection

Variant selection—the choice phase (3).

Assume that the DM (in this case: a dietitian, a physician, any interested person), knowing that the ideal diet does not exist, selects the following vector of concessions:

'let the deterioration (increase) of the value of the "diet cost" criterion function by one unit (PLN) be accompanied by the deterioration (decrease) of the "energy equivalent" criterion function by one unit (recall that the adopted energy unit is 100 *kcal), i.e., $\tau = (1,\ 1)$.'*

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 1.000,\ \ \lambda_2 = 1.000.$$

Normalizing weights, we get $\lambda_1 = 0.5$, $\lambda_2 = 0.5$. In this case, the solution to optimization problem (8.3) with X_0 defined as in (9.3), (9.4) and modified accordingly to account for all thirteen products, is outcome $y = (-13.52, 37.86)$. According to Characterization A, this outcome is efficient, and hence the corresponding daily diet

$$x = (2.47, 2.50, 2.50, 2.50, 2.50, 1.43, 0.00, 5.00, 1.00, 0.95, 0.97, 0.98, 2.00)$$

is efficient.

Assume that after analyzing this diet—a potential candidate for the most preferred diet—the DM changes vector of concessions to

$$\tau = y^* - y,$$

where y is base element $(-13.00, 25.00)$ selected by himself/herself. Selection of such an element can be dictated by the DM's experience. For example, the DM may use as the base element the outcome of some specific, widely publicized type of diet. Hence,

$$\tau = (-7.75, 43.63) - (-13.00, 25.00) = (5.25, 18.63).$$

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 5.25^{-1}, \ \lambda_2 = 18.63^{-1} .$$

Normalizing weights we get $\lambda_1 = 0.780$, $\lambda_2 = 0.220$. In the considered case the solution to optimization problem (8.3) with X_0 defined as in (9.3), (9.4) and modified accordingly to account for all thirteen products, is outcome $y = (-10.88, 32.54)$. According to Characterization A, this outcome is efficient, hence the corresponding daily diet

$$x = (1.31, 2.50, 2.50, 2.50, 2.50, 0.74, 0.00, 5.00, 1.00, 0.50, 0.93, 0.91, 1.96)$$

is efficient.

Assume that the DM selects this daily diet as the most preferred daily diet.

9.4.4 Problem Verification

Problem verification—the review phase (4).

Assume that the DM, after consulting a dietitian, decides that daily diets should not contain meat (meat-free diet). This means that the whole decision cycle has to be repeated.

9.4.5 Problem Formulation

Problem formulation—the intelligence phase (1).

Assume that except excluding meat from daily diets, no other corrections to the problem formulations are made.

9.4.6 Problem Modeling

Problem modeling—the design phase (2).

The model can be adapted to the new circumstances (exclusion of meat from daily diets) in two ways. The first way is to remove from the model all data related to pork butt (variable x_6) and pork lion (variable x_7). The second way is to keep those data and to add the additional constraints $x_6 = 0$ and $x_7 = 0$, which guarantees that no daily diet derived includes products indexed by 6 and 7 (i.e., pork butt and pork lion). The second way is more practical—it permits to keep all the original data without changing the data structure.

Here, we follow the second way.

9.4.7 Variant Selection

Variant selection—the choice phase (3).

Assume that the DM does not change the vector of concessions. Hence, the weights $\lambda_1 = 0.780$, $\lambda_2 = 0.220$ remain unchanged. In this case, optimization problem (8.3) with X_0 defined as in (9.3) and (9.4) and modified accordingly to account for all thirteen products and conditions $x_6 = 0$ and $x_7 = 0$, has no solution. Analyzing information provided by the optimization package (in our case it was the *Microsoft Excel* add-in *Solver*) we observe that the required ingredient level is not met only for vitamin B_1 by 0.2 mg. After consulting a physician, the DM decided that, if not perpetuated, such a small difference between the actually consumed and the required level of that vitamin in the daily diet is insignificant, and the level 1.2 mg is acceptable. Hence, the required level of vitamin B_1 is set to 1.2 mg and the optimization problem is solved again with the values of weights unchanged. Since set X_0 has changed, the ideal element recalculated for this new set is

$\hat{y} = (-9.03, 35.30)$. Hence, with $\varepsilon = 0.10$, $y^* = (-8.93, 35.40)$.

The solution to optimization problem (8.3) with X_0 defined as in (9.3) and (9.4) and modified accordingly to account for all thirteen products, conditions $x_6 = 0$ and $x_7 = 0$ and the lowered required level of vitamin B_1, is outcome $y = (-10.28, 30.61)$. According to Characterization A, this outcome is efficient, hence the corresponding daily diet

$$x = (1.97, 2.35, 2.46, 2.50, 2.50, 0.00, 0.00, 5.00, 1.00, 0.44, 1.00, 0.44, 2.00)$$

is efficient.

Assume that the DM selects this diet as the most preferred daily diet.

9.4.8 Problem Verification

Problem verification—the review phase (4).

Assume that the daily diet selected in the choice phase (3) is satisfactory, as perceived by the DM, in all aspects and perspectives. So, he/she decides that this daily diet is the most preferred diet.

At this the decision process terminates.

9.5 Sum-Up

In this chapter, we have used our skills acquired in the former chapters to derive single efficient variants and efficient portfolios of variants, to support decision making in problems which can be framed as the MCDM model (1.2) and the underlying multiobjective optimization problem (1.3).

We have considered three specific forms of that model, which are widely used in the managerial practice. Another form of the model (1.2), accounting for risk, is presented in the next chapter.

9.6 Creative Assignment

For each of three MY_PROBLEM_i models, propose decision making scenarios leading to the selection of the most preferred variant or the most preferred variant portfolio, as appropriate (try to traverse the whole learning loop at least once).

Chapter 10

Decision Problem: Selection of a Stock Portfolio

"Yes, that's gold, but I'm too big to go running around like that after atoms." "No problem, we'll give you a suitable machine!" coaxed Trurl.

10.1 Chapter Content

The problem considered in this chapter, namely the problem of stock portfolio selection, is similar to the problem considered in Chap. 6 and in Sect. 9.4—the optimal mixture composition. However, in the problem of stock portfolio selection, risk related to investing in stock (and more generally—in securities) is directly included in the model.

Electronic supplementary material The online version of this chapter (doi: 10.1007/978-3-319-32756-3_10) contains supplementary material, which is available to authorized users.

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10.2 Problem Formulation

Problem formulation—the intelligence phase (1).

We consider here the problem of stock portfolio selection in terms of investment return and investment risk.

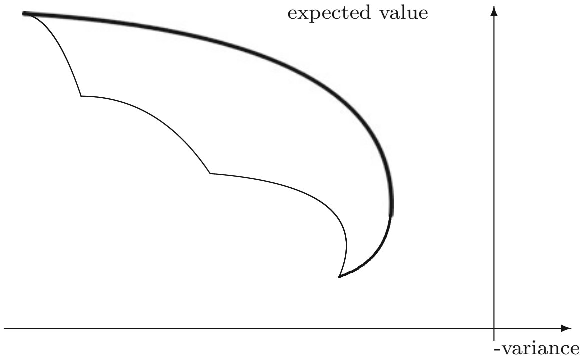

Figure 10.1 An example of the set of outcomes for a four stock portfolio. The *thick line* shows efficient portfolio outcomes

Selection of the most preferred stock portfolio is made from available stock (single decision variants), with two criteria: portfolio expected return (as a measure of profit, to be maximized) and portfolio variance (as a measure of risk, to be minimized). In the formal model of the problem, there is one linear criterion and one quadratic criterion, defined on an infinite, bounded set of combinations of available stock. The set of efficient outcomes (efficient stock portfolios) has the form as shown in Fig. 10.1 by thick line (to keep our adopted convention that all criteria are of the type "the more, the better", the figure is drawn for the case where the negative of portfolio variance is maximized).

It is assumed that stocks selected to a stock portfolio should make use of all capital available. The problem is modeled with the capital normalized to 1 (one unit), and in consequence, stock shares in a portfolio are represented by proper fractions which sum up to 1.

10.3 Problem Modeling

Problem modeling—the design phase (2).

The problem formulated verbally in the former phase is framed here as the so-called *Markowitz model*.[1] This model has the following form:

$$\max -\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{i,j}x_i x_j \quad \text{(maximize negative of variance)}$$

$$\max \sum_{j=1}^{n} e_j x_j \quad \text{(maximize expected return)}$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{n} x_j = 1, \ \text{(all capital to be consumed)}, \\ x_j \geq 0, \ j = 1, \ldots, n, \end{array} \right\},$$

where $\beta_{i,j}$ denotes the element of the covariance matrix corresponding to stock i and j, and e_j denotes the expected return of stock j.[2]

Here we make use of data from the original Markowitz example for three company stocks denoted ATT, GMC, USX, characterized by the following covariance matrix and the expected returns:

	ATT	*GMC*	*USX*	
ATT	0.01080754	0.01240721	0.01307513	
GMC	0.01240721	0.05839170	0.05542639	covariance matrix
USX	0.01307513	0.05542639	0.09422681	
	0.0890833	0.213667	0.234583	expected returns

Following our adopted notation, for given decision variant x we have outcome $y = (y_1, y_2)$, $y_1 = f_1(x) = -\sum_{i=1}^{3}\sum_{j=1}^{3}\beta_{i,j}x_i x_j$ and $y_2 = f_2(x) = \sum_{j=1}^{3} e_j x_j$.

We derive the value of $\hat{y}_2$ ($\hat{y}$ is the ideal element) by solving the optimization problem

$$\max \sum_{j=1}^{3} e_j x_j$$

[1] Harry Max Markowitz, the American economist, the Nobel Price laureate in 1990. The model is normative, i.e., it provides recommendations how the DM should compose his/her portfolio. However, since it ignores actual stock prices, it does not explain the DM's actual investment behavior.

[2] Coefficients $\beta_{i,j}$ and e_j are derived by statistical analysis of stock return time series.

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{3} x_j = 1\,, \\ \\ x_j \geq 0,\ \ j = 1, 2, 3, \end{array} \right\}.$$

We get $\hat{y}_2 = 0.235$ for stock portfolio $x = (0.000, 0.000, 1.000)$.

Because of the linearity of the criterion function representing the expected return from a stock portfolio, and because of the specific form of the single constraint in the problem, in this special case we did not actually need to solve the optimization problem to derive value $\hat{y}_2$. Value $\hat{y}_2$ is obviously given by allocating all capital into the stock of the highest return.

We get $\hat{y}_1$ by solving the optimization problem

$$\max - \sum_{i=1}^{3} \sum_{j=1}^{3} \beta_{i,j} x_i x_j$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{3} x_j = 1\,, \\ \\ x_j \geq 0,\ \ j = 1, 2, 3, \end{array} \right\}.$$

This problem yields $\hat{y}_1 = -0.011$ for stock portfolio $x = (1.000, 0.000, 0.000)$.

Assuming (arbitrarily) $\varepsilon = 0.001$ we have $y^* = (\hat{y}_1 + \varepsilon,\ \hat{y}_2 + \varepsilon) = (-0.010,\ 0.236)$.

10.4 Variant Selection

Variant selection—the choice phase (3).

Assume that the DM (an investor, a financial analyst) having checked that the problem does not admit the ideal stock portfolio (i.e., stock portfolio corresponding to $\hat{y}$, to see how this can be done one should refer to Chap. 9), selects the following vector of concessions:

‘let the deterioration (increase) of the value of the “negative of variance” (dimensionless quantity) criterion by 0.1 be accompanied by the deterioration of the “expected return” (in percent) criterion by 0.1 *, i.e., τ = (0.1, 0.1).’*

By formula (8.5), we calculate weights for optimization problem (8.3),

$$\lambda_1 = 10.000,\ \ \lambda_2 = 10.000,$$

In this instance, problem (8.3) has the form

$$\min \max \left\{ \lambda_1 (y_1^* + \sum_{i=1}^{3} \sum_{j=1}^{3} \beta_{i,j} x_i x_j); \lambda_2 (y_2^* - \sum_{j=1}^{3} e_j x_j) \right\}$$

$$\text{subject to } x \in X_0 = \left\{ x \;\middle|\; \begin{array}{l} \sum_{j=1}^{3} x_j = 1\,, \\ \\ x_j \geq 0,\ \ j = 1, 2, 3, \end{array} \right\}.$$

Normalizing weights we get $\lambda_1 = 0.500$, $\lambda_2 = 0.500$. In this case, the solution to the optimization problem is outcome $y = (-0.047, 0.199)$ which corresponds to stock portfolio $x = (0.164, 0.545, 0.291)$. According to Characterization A, this outcome and the corresponding stock portfolio are efficient.

Assume that the DM selects stock portfolio $x = (0.164, 0.545, 0.291)$ as the most preferred stock portfolio.

10.5 Problem Verification

Problem verification—the review phase (4).

Assume that the DM, after confronting the stock portfolio selected in the choice phase (3) against all circumstances of the decision process not represented in the formal model (current and forecasted economy condition, global trends, legislation uncertainty), decides that this stock portfolio is indeed the most preferred variant.

At this the decision process terminates.

10.6 Sum-Up

The variant of model (1.2) presented in this chapter represents one possible way to account for risk in decision processes. There are alternative ways, however this topic is beyond the scope of this textbook.

10.7 Creative Assignment

For each of three MY_PROBLEM_i models, propose a modification involving at least one nonlinear criterion function.

Apply the respective scenarios proposed in Chap. 9 to select the most preferred variant or the most preferred variant portfolio, as appropriate.

Chapter 11

Relations

Trurl had set the apparatus on his head for purposes of demonstration and was explaining the procedure to the King...

11.1 Chapter Content

In the previous chapters, we have made use of the notions "preference" and "preferred", as they are commonly understood and interpreted. In this chapter, we consider these notions in a formal manner. To this aim we need another notion, fairly grounded in mathematics, namely the notion of *relation*.

11.2 Definition of Relation

Let set Ω be given.

Definition 11.2.1 *The set of all pairs of elements* Ω *is called the Cartesian product of* Ω *and denoted* $\Omega \times \Omega$.

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Example 11.2.1 *The Cartesian product of set* $\{a, b, c\}$ *is set* $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$.

Definition 11.2.2 *A subset of the Cartesian product of* Ω *is called the* relation *in* Ω.

In other words, a relation is a subset of ordered pairs of elements of Ω.

We denote the fact that $(a, b) \in A \subseteq \Omega \times \Omega$ by $a \succ b$, and we say that element a is in relation A with element b. Therefore, instead of saying "relation A" one can say "relation $\succ$".

Example 11.2.2 *Figure 11.1 presents a graphical interpretation of relation* $A = \{(a, a), (b, a), (c, a), (a, b)\}$, *where* $\Omega = \{a, b, c\}$.

		the second elements in a pair		
		a	b	c
	a	○	○	
the first elements in a pair	b	○		
	c	○		

Figure 11.1 A graphical interpretation of the relation from Example 11.2.2

11.3 Properties of Relations

Let set A, $A \subseteq \Omega \times \Omega$, be given. In other words, relation $\succ$ is given.

Definition 11.3.1 *Relation* $\succ$ *is*

1. reflexive, *if for all* $a \in \Omega$, $a \succ a$,
2. irreflexive, *if for all* $a \in \Omega$, $a \not\succ a$,
3. symmetric, *if for all* $a, b \in \Omega$, $a \succ b$ *implies* $b \succ a$,
4. asymmetric, *if for all* $a, b \in \Omega$, $a \succ b$ *implies* $b \not\succ a$,
5. antisymmetric, *if for all* $a, b \in \Omega$, $a \succ b$ *and* $b \succ a$ *implies* $a = b$,

6. transitive, *if for all* $a, b, c \in \Omega$, $a \succ b$ *and* $b \succ c$ *implies* $a \succ c$,
7. total, *if for all* $a, b \in \Omega$, $a \succ b$ *or* $b \succ a$.

Example 11.3.1 *In the set of real numbers, relation*

$\geq$ *is reflexive,*
$>$ *is irreflexive,*
$<$ *is asymmetric,*
$=$ *is antisymmetric,*
$\geq$ *is transitive,*
$\leq$ *is total.*

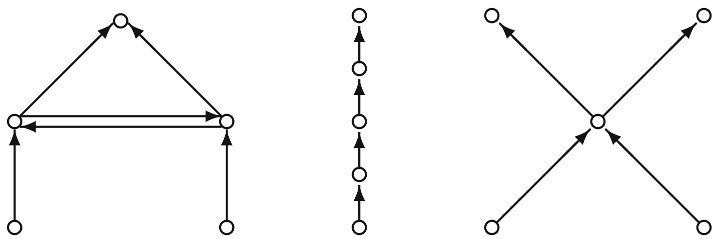

Figure 11.2 Hasse diagrams of ordered sets: quasi ordered (*left*), linearly ordered (*middle*), partially ordered (*right*)

11.4 Ordering Relations

Definition 11.4.1 *A relation which is reflexive (1) and transitive (6) is called* the quasi-ordering relation.

Definition 11.4.2 *A relation which is reflexive (1), antisymmetric (5) and transitive (6) is called* the partially ordering relation.

Definition 11.4.3 *A relation which is reflexive (1), antisymmetric (5), transitive (6) and total (7) is called* the linearly ordering relation.

11.5 Ordered Sets

Definition 11.5.1 *A set on which a quasi-ordering relation is defined is called the quasi-ordered set.*

Definition 11.5.2 *A set on which a partially ordering relation is defined is called the partially ordered set.*

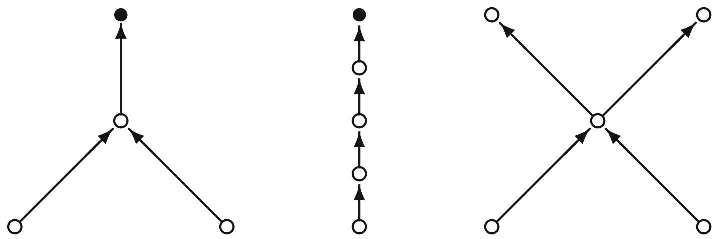

Figure 11.3 Hasse diagrams of ordered sets with and without greatest elements

Definition 11.5.3 *A set on which a linearly ordering relation is defined is called* the linearly ordered set.

For graphical interpretations of relations we adopt the following convention: by $\circ$ we represent elements and by arrows we represent the fact that a pair of elements belongs to the relation with the rule that $b \rightarrow a$ if and only if $a \succ b$. A standard graphical representation of relations is *Hasse diagram* in which $a \succ a$ (reflexivity), and $a \succ c$ whenever $a \succ b$ and $b \succ c$ (transitivity), are not shown in the graph.

Figure 11.2 presents examples of ordered sets.

11.6 Maximal Elements and Greatest Elements

Definition 11.6.1 The maximal element *in partially ordered set* Ω *is element* $a \in \Omega$ *for which there does not exists element* $b \in \Omega,\ b \neq a$*, such that* $b \succ a$*.*

Definition 11.6.2 The greatest element *in partially ordered set* Ω *is element* $a \in \Omega$ *such that* $a \succ b$ *for all elements* $b \in \Omega$.

We represent the greatest elements by $\bullet$. Figure 11.3 shows examples of ordered sets with and without greatest elements.

Remark 11.6.1 *The greatest element is the maximal element.*

Remark 11.6.2 *In a partially ordered* finite *set the maximal element always exists.*

Remark 11.6.3 *In a linearly ordered* finite *set the greatest element always exists.*

Remark 11.6.4 *In a linearly ordered set there exist at most one greatest element.*

Let us come back to Pareto dominance relation $\overset{P}{\succ}$ introduced in Chap. 1 (the notation for this relation has been introduced in Chap. 4). This is the relation defined in a set of vectors, each vector composed of k components. When interpreted in multiobjective optimization (MO) terms (see Chap. 1), each component represents the value of the corresponding criterion.

Example 11.6.1 *For set of variants* $A = \{1,2,3,4\}$ *and two criteria functions* f_1 *and* f_2 *taking values*

$$\begin{array}{ccc} \textit{variant} & f_1 & f_2 \\ x^1 & 2 & 2 \\ x^2 & 3 & 3 \\ x^3 & 4 & 3 \\ x^4 & 3 & 4 \\ x^5 & 1 & 2 \\ x^6 & 2 & 1 \end{array} ,$$

the following relationships hold:

$$x^1 \overset{P}{\nsucc} x^2,$$
$$x^2 \overset{P}{\nsucc} x^2,$$
$$x^1 \overset{P}{\nsucc} x^3,$$
$$x^3 \overset{P}{\succ} x^1,$$
$$x^3 \overset{P}{\nsucc} x^4,$$
$$x^4 \overset{P}{\nsucc} x^3.$$

11.7 Relations and Preference Relations

In terms of decision problems, some relations are called *preference relations*. If the DM prefers variant a to variant b, then this is customarily denoted $a \succ b$, as in the language of relations.

For a relation to be a preference relation it is required (it is a postulate which can be interpreted as *the minimum of rationality*) to be irreflexive and asymmetric. If this holds, we can regard the notions of *relation* and *preference* as equivalent.

In particular, irreflexivity of a relation precludes cycles, i.e., sequences of relations of the form

$$a \succ b \ldots \succ \ldots \succ a.$$

The partially ordering relation and the linearly ordering relation, both being reflexive, are not preference relations.

Pareto dominance relation $\overset{P}{\succ}$ is clearly irreflexive and asymmetric, thus it is a preference relation.

Preferences can emerge from *holistic* variant valuations, as in Example 11.7.1, or from valuations of variant attributes, as done in this textbook (see also Example 11.7.2).

Example 11.7.1 *Given is a set of candidates (for a mission, for a job, etc.):* $\Omega = \{1, 2, 3, 4\}$.

The Cartesian product for the set of the candidates is

$$\Omega \times \Omega = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),\\ (3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}.$$

Let, for example, candidate #1 be more preferred than candidate #2 and let it be the only preference in the set of candidate pairs $\Omega \times \Omega$. *Then* $A = \{(1, 2)\}$, $A \subseteq \Omega \times \Omega$, *and this fact is denoted as:* $1 \succ 2$*—candidate #1 and candidate #2 are in the preference relation.*

If $A = \{(1, 2), (1, 4), (3, 1)\}$, *then, for example,* $3 \succ 1$,

If $A = \{(3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$, *then, for example,* $4 \succ 3$.

Example 11.7.2 $\Omega = \{$*fruit 1, fruit 2, fruit 3, fruit 4*$\}$.

Outcomes (with respect to a single attribute—juiciness): fruit 1—juicy, fruit 2—little juicy, fruit 3—dry, fruit 4—dehydrated.

An example of a preference relation:

$fruit\ 1 \succ fruit\ 2$, $fruit\ 2 \succ fruit\ 3$, $fruit\ 3 \succ fruit\ 4$, $fruit\ 2 \succ fruit\ 4$.

11.8 Sum-Up

The basic notions related to relations, ordering relations, and ordered sets, constitute the conceptual base for formulating and analyzing decision problems. An example of such a notion is the *dominance relation*, introduced in Chap. 1, serving throughout the course as the pivot idea.

11.9 Creative Assignment

For each of three MY_PROBLEM_i models, construct the Hasse diagram for variants or variant portfolios, as appropriate, derived in previous Creative Assignments.

Electronic Companion

There are six *Microsoft Excel 2013* workbooks which complete the material of this textbook. They refer to three problems considered in the textbook, namely for the River Crossing Selection problem, Project Portfolio Selection problem and the Diet Selection problem. The workbooks (*.xlsx files) can be downloaded from http://www.ibspan.waw.pl/~kaliszew/main_page.html (once there, click the textbook icon).

1. The workbooks:
River_Crossing_Selection.xlsx,
Project_Portfolio_Selection.xlsx,
Diet_Selection.xlsx,
contain data and formulas for calculating the Tchebycheff function values for respective models for values of λ_1 and λ_2 set by the user.

2. The workbooks:
River_Crossing_Selection_Interfaced.xlsx,
Project_Portfolio_Selection_Interfaced.xlsx,
Diet_Selection_Interfaced.xlsx
contain data and formulas for calculating the Tchebycheff function values for respective models for values of τ_1 and τ_2 set by the user.

In the spreadsheets *DM_Interface* of each of the workbooks it is possible to select the method of defining vector τ; it is also possible to define there vector λ directly.

I. Kaliszewski et al., *Multiple Criteria Decision Making by Multiobjective Optimization*, International Series in Operations Research
& Management Science 242, DOI 10.1007/978-3-319-32756-3

Workbooks which refer to Project Portfolio Selection problem and the Diet Selection problem need to call for an optimization package, which in the *Microsoft Excel* is the add-in *Solver*.

To use the add-in *Solver*, it has to be loaded to the *Microsoft Excel* first. For example, in the *Microsoft Excel 2013*, one can do that in the *Add-Ins* pane (*File* → *Options* → *Add-Ins*), and then in the *Manage* box select *Excel Add-Ins*, and in the *Add-Ins available* box select the *Solver Add-in* check box and then click *OK*. After doing this, *Solver* can be invoked from the *Data* tab in any spreadsheet of a workbook.

The optimization models are to be introduced in the *Solver* pane which pops-up after *Solver* is invoked. In the spreadsheets *Project_Portfolio* and *Diet_Selection* of the corresponding workbooks, the respective models are already there. The option available in the *Solver* pane “Make Unconstrained Variables Non-Negative” is not used, as these constraints are set in respective models explicitly.

In the case of Project Portfolio Selection problem, from the *Solver* pane go to *Options* and make sure that the box *Ignore integer conditions* is unchecked. Back in the *Solver* pane, select *Nonlinear GRG* or *Evolutionary* as the solution method.

In the case of Diet Selection problem, which is constrained by linear constraints, the best practice is to transform the optimization model to its linear equivalent. This can be done by replacing the Tchebycheff function by a single variable u to be minimized and by adding two additional constraints

$$\begin{aligned} u &\geq \lambda_1(y_1^* - f_1(x))\,, \\ u &\geq \lambda_2(y_2^* - f_2(x))\,. \end{aligned}$$

The *Solver* model in *Diet_Selection.xlsx* workbook has this linear form, likewise the *Solver* model in *Diet_Selection_Interfaced .xlsx* workbook. Therefore, for both models it is advisable to select in the *Solver* pane *LP simplex* as the solution method.

Solver is an illustrative tool rather than a reliable optimization package. As such, solutions it delivers should be used with caution.

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Additional Readings

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A good source of other texts on MCDM and MO is the web page of the International Society on Multiple Criteria Decision Making:

http://www.mcdmsociety.org (select *PUBLICATIONS* tab).

Index

argument, xvi

base element, 65
binary variable, 39

compromise half line, 65
condition
  necessary, 52
  sufficient, 52
constraint, xvi
criteria mapping, 10
criterion, 2
  function, xvi, 10
    linear, 45

decision maker, 11
decision making problem, xvi, 2
decision process, 2
  phases, 16
dominance, 3

efficient outcome
  characterization, 52
efficient variant
  characterization, 52
element
  base, 65
  ideal, 12

function
  criterion, xvi
  objective, 10
  scalarizing, 52

Hasse diagram, 106

interactive mode, 63
investment
  return, 98
  risk, 98

learning loop, 16

method
  reference point, 66
  weight, 66
model
  linear programming, 45
Multiple Criteria Decision Making, 10

normalization, 58

operator
  maximization, xvi
  minimization, xvi
optimal solution, xvi

I. Kaliszewski et al., *Multiple Criteria Decision Making by Multiobjective Optimization*, International Series in Operations Research & Management Science 242, DOI 10.1007/978-3-319-32756-3

optimization
  model, xvi
optimization problem, xvi
outcome
  efficient, 12
  most preferred, 12
  properly efficient, 54
  weakly efficient, 53

preference
  relation, 108
problem
  multiobjective
    optimization, 11
product
  Cartesian, 103

relation, 103, 104
  antisimmetric, 104
  asymmetric, 104
  dominace, 5
  irreflexive, 104
  reflexive, 104
  symmetric, 104
  total, 105
  transitive, 105

scalarizing function
  weighted linear, 55
  weighted Tchebycheff, 54
set
  convex, 55
  finite, 8
  infinite, 8
  of constraints, xvi
space
  decision, 10
  outome, 12
stopping rule, 16

valuation
  holistic, 108
  most preferred, 10
  of variant, 2
value
  maximal, xvi
  minimal, xvi
  optimal, xvi
variant, 10
  decision, 2
  dominated, 5
  efficient, 6
  ideal, 3
  most preferred, xvi, 2, 11
  ranking, 57
variant portfolio, 38, 44
vector
  of concessions, 65

weight, 52

Glossary

base element an element of the outcome space specified by the decision maker to define the vector of concessions. 65

Cartesian product the set of all pairs of elements of a set. 103

characterization of efficient variants necessary and sufficient conditions for a variant to be efficient. 52

characterization of efficient outcomes necessary and sufficient conditions for an outcome to be efficient. 52

compromise half line a half line starting from y^* towards the set of efficient outcomes; points on this half line represent the same pattern of concessions with respect to unattainable (in general) y^*. 65

constraints a set of rules which decide whether variants are decision variants (feasible solutions). xv, 10

criteria functions functions relating variants to components of outcomes; criteria functions form criteria mapping. 10

criteria mapping a set of criteria functions relating variants with variant valuations. 10

decision maker the person accountable for the final decision. 11

I. Kaliszewski et al., *Multiple Criteria Decision Making by Multiobjective Optimization*, International Series in Operations Research
& Management Science 242, DOI 10.1007/978-3-319-32756-3

decision space a set of all conceivable variants irrespective if they are decision variants, (i.e., variants considered in a decision problem) or not. 10

decision process a sequence of actions leading to selecting a decision variant – the *most preferred variant* – which suits the Decision Maker best. 2

decision making problem a problem which consists in selecting the most preferred variant. 1

dominance the situation, when one variant outperforms another with respect to a given set of criteria. 3

learning loop a repetitive sequence of the decision process phases. 16

normalization a transformation on a set of numbers which results in the transformed numbers summing to one. 58

outcome space a set of all variant valuations by a criteria mapping. 12

preference relation a relation with some specific properties. 108

relation a subset of the Cartesian product of the elements of a set. 103

scalarizing function a function which aggregates criteria functions, whose optimal value over decision variants is attained at an efficient variant. 52

variant valuation the process of assigning a value, qualitative or numerical. 2, 10, 108

vector of concessions a vector (with positive components) representing the Decision Maker preferences about (consent on) how to compromise on unattainable y^*. 65

weight a parameter of a scalarizing function. 52