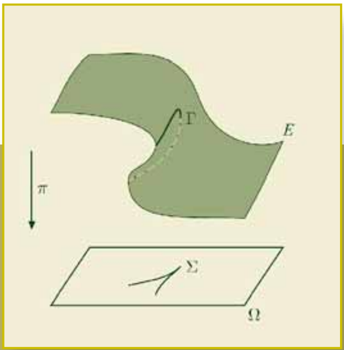
Yves Balasko

The Equilibrium Manifold

Postmodern Developments in the Theory of General Economic Equilibrium



The Equilibrium Manifold

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About the Arne Ryde Foundation

Arne Ryde was an exceptionally promising student in the Ph.D. program at the Department of Economics, Lund University. He was tragically killed in a car accident in 1968 at the age of twenty-three.

The Arne Ryde Foundation was established by his parents, the pharmacist Sven Ryde and his wife, Valborg, in commemoration of Arne Ryde. The aim of the foundation, as given in the deed from 1971, is to foster and promote advanced economic research in cooperation with the Department of Economics at Lund University. The foundation acts by lending support to conferences, symposia, lecture series, and publications that are initiated by faculty members of the department.

Preface

This book provides a panorama of developments in our understanding of some general equilibrium models in the past three decades. These properties go far beyond the existence of equilibrium or the welfare theorems to which many current textbooks seem to limit microeconomics. The equilibrium manifold and the way that set is projected into the parameter space play a fundamental role in these developments.

Now a few words to justify the term *postmodern general equilibrium theory*. There is widespread consensus to associate modern mathematics with van der Waerden's *Moderne Algebra* (1931) and the Bourbaki series, *Elements of Mathematics*. Debreu's pivotal *Theory of Value* (1959) is the Bourbaki-style formulation of modern general equilibrium theory. Jacques Drèze once described that book as the "last definitive posthumous edition" of Walras's *Elements of Political Economy* (1874). This reminds us that, according to Weil (1991, 104), the initial goal of the Bourbaki group was to reformulate Goursat's *Cours d'analyse mathématique* (1902) in the style of modern mathematics. The main feature of modern general equilibrium theory is, in addition to the abstract and axiomatic style, the focus placed on properties that are satisfied by all, or almost all, equilibria and economies, in other words, absolute properties.

But many problems have no answer within the setup of modern general equilibrium theory. An example is the stability of competitive equilibrium. Properties like stability are not satisfied by all equilibria. Only some equilibria can be stable, and only some economies have stable equilibria. More generally, only some equilibria and economies are going to satisfy an economically meaningful property. This is where the equilibrium manifold takes the central role because it enables us to characterize and study sets of equilibria that satisfy any given property as subsets of the equilibrium manifold. From absolute, properties become relative. This change in perspective is sufficiently important to differentiate the relativistic postmodern approach from Debreu's absolute (modern) approach.

Postmodern general equilibrium theory has nothing to do with the particular tone and political agenda of the postmodernism of political science and philosophy. Nevertheless, if postmodern theories in science and art seem to have little in common, they all stress to varying extents the idea that properties are relative rather than absolute. Postmodern general equilibrium theory thus fits very well into this setup.

The first two chapters of this book are devoted to a short presentation of the evolution of the theory of general equilibrium from a theory of rational economics (Divisia 1928) into the style epitomized by Debreu's *Theory of Value* (1959) and finally into a theory where the main goal is the characterization through the equilibrium manifold approach of sets of economies and equilibria that satisfy specific economic properties.

Chapters 3 and 4 deal with the core of the equilibrium manifold approach applied to the study of the equilibrium equation of the pure exchange Arrow-Debreu model.

Chapter 5 describes a dual formulation of the equilibrium manifold approach. That approach generalizes to the case of many consumers the indirect utility of Hotelling (1938) and Roy (1942) considered in many textbooks. The interest of this dual formulation comes from its superiority in dealing with rather complex properties of the equilibrium equation, properties that prove crucial in later chapters of the book.

Chapter 6 illustrates the versatility of the equilibrium manifold approach by applying it to a version of the Arrow-Debreu model where preferences are price-dependent. That model is shown to have the same properties as the standard Arrow-Debreu model with price-independent preferences, provided total resources are variable. That model plays an important role in the analysis of the temporary equilibrium model, which is taken up in chapter 9.

Chapter 7 addresses the definition of a realistic dynamic adjustment process for nonequilibrium prices. Stability is typical of those properties that cannot be satisfied by all equilibria. Samuelson's (1941) definition of what he calls Walras tatonnement suffers from a speed of adjustment of the price dynamics that is totally arbitrary. The problem with such a definition is that the stability properties of equilibria depend, among other things, on the speed of adjustment. In this chapter, the speed of adjustment is derived from the structure of the exchange process, out of and at equilibrium. Stability is then studied for this intrinsically defined speed of adjustment.

Chapter 8 deals with an extension of the Arrow-Debreu model, namely, the fully stationary intertemporal Arrow-Debreu model where some consumers face restrictions in their ability to transfer wealth between time periods. This extension represents one of the first steps into a genuinely general equilibrium analysis of economic fluctuations. The question here becomes whether a fully stationary model can feature equilibrium solutions that are not stationary. It turns out that if there are no restrictions on intertemporal transfers, then all equilibrium solutions are asymptotically stationary, with little room left for fluctuations. The picture becomes totally different, however, if some restrictions exist on individual intertemporal transfers. Nonstationary equilibrium allocations can then exist. The equilibrium manifold approach is used to gain insight on these economies and to assess the role of restrictions in intertemporal wealth transfers in creating or amplifying economic fluctuations.

Chapter 9 deals with the two-period temporary equilibrium model with financial assets and arbitrary forecast functions of future prices, an approach in the tradition of Lindahl (1939) and Hicks (1946). At variance with earlier treatments of the temporary equilibrium model, future prices are included in the states of nature. In other words, economic agents forecast probability distributions of future prices. This approach enables us to give to the temporary equilibrium model with assets a reduced form that is the Arrow-Debreu model with the price-dependent preferences of chapter 6.

An appendix is devoted to a detailed and rigorous analysis of the properties of the set of Pareto optima in the Arrow-Debreu model. Most of these properties are well-known though rarely treated with sufficient rigor. Since their role is crucial in several parts of the book, they deserve a presentation on a par with the rigor of the rest of the book.

The mathematical prerequisites for reading this book have been keptto a minimum. Readers are expected to have a basic knowledge of point-set topology. They are also expected to be comfortable with properties of smooth maps from open sets of Euclidean spaces into Euclidean space. These properties include the inverse function theorem and the implicit function theorem. A very neat and accessible presentation of this material can be found in Spivak's *Calculus on Manifolds* (1965). As a reminder, it is worth recalling how the implicit function theorem is a workhorse of Samuelson's *Foundations of Economic Analysis* (1947).

The theory of smooth manifolds and smooth mappings provides the mathematical setup for the study of the properties of general equilibrium models through the equilibrium manifold and the natural projection approach. But only the most elementary properties of smooth manifolds are needed to go through chapters 2–5, and these properties are not even strictly necessary in the later chapters. By skipping the sections in chapters 2–4 that deal with the equilibrium manifold without using an explicit set of coordinates, no knowledge of the theory of smooth manifolds is even strictly necessary. These mathematics can be found in the first chapters of Milnor's marvelous little book *Topology from the Differentiable Viewpoint* (1997). Guillemin and Pollack's more lengthy treatment of the same subjects in *Differential Topology* (1974) nicely complements Milnor's book with the bonus of pictures that help develop a strong geometric intuition. A mathematically more advanced presentation that goes far beyond the needs of the current book can be found in, for example, Hirsch's *Differential Topology* (1976).

An introductory knowledge of microeconomics at the graduate level is also sufficient but not even strictly necessary provided readers work through some of the easy exercises that complement several sections of the book.

Acknowledgments

This book grew out of the Arne Ryde Memorial Lectures that I gave in September 2006 in the wonderful location run by the foundation near Lund. I would like to thank the Arne Ryde Foundation through Anders Borglin and Bo Larson for giving me the honor to speak in their distinguished series and for their warm hospitality.

Writing a book is a big undertaking. It requires much encouragement and support. I have never been short on them. I am grateful to the many participants in the lectures for the quality of their questions and comments. Special thanks are due to Anders Borglin, Hildegard and Egbert Dierker, Sjur Flåm, Monique Florenzano, Christian Ghiglino, Ali Kahn, Hans Keiding, Bo Larson, Salim Rachid, Donald Saari, Karl Shell, David Schmeidler, Gabriel Talmain, and Mich Tvede. None of them are responsible for any remaining errors and inaccuracies, of course.

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I am particularly indebted to Mateo de Tina and Mich Tvede, who read and commented on a first version of the manuscript, and to my son, Gregoire, whose insistent comments led me to rewrite and restructure the first chapters of the book.

No words would be strong enough to express my feelings to Martine Orelio, her patience and understanding.

I thank Alice Cheyer for the outstanding editing of the manuscript and MIT Press staff, especially John Covell, Deborah Cantor-Adams, and Katie Helke, for their unfailing support during all phases of publication of this book.

The Equilibrium Manifold

1 The First Two Phases of General Equilibrium Theory

1.1 Introduction

The starting point of the economic theory of general equilibrium is the mathematical expression of the equality between supply and demand in an exchange economy. The first general formulation of this equation is due to Walras (1874). Three years earlier, Jevons (1871) had independently described that equation but in the special case of only two goods. Since then, the goal of the theory of general equilibrium has been and still is the study of the properties of this equation. This is essentially a mathematical problem. Quite a few remarkable properties of this equation have been discovered since its formulation by Jevons and Walras. The theory of general equilibrium is an account of these properties and their economic interpretation.

This chapter surveys the first hundred years of the theory of general equilibrium, the period from 1870 to 1970. It does not provide a historical account, which can be found in Chipman (1965) and Weintraub (1985; 2002), for example. The focus here is on how the theory has evolved from the problems studied by Walras, Jevons, Pareto, and their contemporaries to the emergence of the Arrow-Debreu model and the successes and sometimes failures of its study. The evolution of the theory of general equilibrium has followed the progress of mathematics very closely. New ways of studying the equilibrium equation made possible by new mathematical approaches have led to the discovery of properties that considerably enrich the economic message of the theory.

The insight provided by the evolutionary perspective is also crucial if one wants to understand why an important problem like the existence of equilibrium, one of the major achievements of the study of the Arrow-Debreu model in the 1950s, was not perceived as being a problem by Walras or Pareto. The existence problem is not alone.

Several properties of the equilibrium equation thought to be of mediocre interest were forgotten during that period. They were rediscovered much later, having acquired a new significance with the evolution of the theory.

The central object of the theory of general equilibrium—its equilibrium equation—has been remarkably constant through time. Apart from minor variations in the notation, the equilibrium equation of a pure exchange economy is already found in very general form in Walras (1874). I take advantage of this time invariance to employ the standard formulation of the Arrow-Debreu model throughout, even though, historically speaking, it did not exist before the 1950s. This presentation not being a historical account, I use hindsight to assume that consumers' utility functions are differentiable up to any arbitrary order, even though Arrow and Debreu did not need nor use differentiability to prove their famous existence theorem.

This chapter therefore starts with a brief presentation of the smooth version of the Arrow-Debreu model, which provides us with a well-defined and unique setup to describe the evolution of the theory of general equilibrium from its beginnings to the 1970s and later. I do not recall in this book the most basic properties of utility functions, indifference surfaces, and individual demand functions. They can be found in any good microeconomics textbook. Nevertheless, in order to make this book as self-contained as possible, all useful properties and some others that are less useful are stated in a set of simple exercises that readers are encouraged to solve as a complement to the main text.

1.2 The Arrow-Debreu Model: Assumptions and Notation

For simplicity's sake, the assumptions considered here are less general than those considered in Debreu (1959), for example. These more restrictive assumptions have become standard in the current development of general equilibrium theory. In addition, only exchange economies are considered.

1.2.1 Goods and Prices

There are ℓ goods. The consumption space is the strictly positive orthant

$$X = \mathbb{R}^{\ell}_{++} = \{x_i = (x_i^k) \in \mathbb{R}^{\ell} \mid x_i^k > 0 \text{ for } k = 1, \dots, \ell.\}.$$

The ℓth commodity is chosen as numeraire, i.e., $p_\ell = 1$. Let

$$S = \{p = (p_j) \in \mathbb{R}^{\ell}_{++} \mid p_\ell = 1\}$$

denote the set of *numeraire normalized* prices.

1.2.2 Consumers

Let m denote the (finite) number of consumers. Consumer i's preferences are represented by a utility function $u_i : X \to \mathbb{R}$, which satisfies the following assumptions:

i. The map $u_i : X \to \mathbb{R}$ is surjective and differentiable up to any order.

ii. The gradient vector $Du_i(x_i)$ has > 0 coordinates, i.e., $Du_i(x_i) \in X$. (*Smooth monotonicity*)

iii. The restriction of the quadratic form defined by the Hessian matrix of second-order partial derivatives $D^2u_i(x_i)$ to the tangent hyperplane at $x_i \in X$ to the indifference surface through x_i is negative definite. (*Smooth quasi-concavity*. This condition is equivalent to the only solution to $X^T D^2 u_i(x_i) X \geq 0$ and $Du_i(x_i)^T X = 0$ being the $1 \times \ell$ column matrix $X = 0$.)

iv. For any $u_i \in \mathbb{R}$, the indifference surface $\{x_i \in X \mid u_i(x_i) = u_i\}$ is closed in $\mathbb{R}^\ell$. (*Necessity of every commodity*)

The *demand* $f_i(p, w_i)$ of consumer i maximizes the utility $u_i(x_i)$ under the budget constraint $p \cdot x_i \leq w_i$. The *demand function* $f_i : S \times \mathbb{R}_{++} \to \mathbb{R}^{\ell}_{++}$ is smooth and satisfies *Walras' law* $p \cdot f_i(p, w_i) = w_i$.

Consumer i is *endowed* with the commodity bundle $\omega_i \in X$. Assuming that preferences do not vary, the economy is parameterized by the endowment vector $\omega = (\omega_i) \in X^m$. We denote by $\Omega = X^m$ this parameter space. Other parameter spaces can be and have been considered in the literature. For example, sign constraints can be relaxed, in which case the parameter space becomes $(\mathbb{R}^\ell)^m$. Another example is when total resources are held constant and equal to some vector $r \in X$, in which case the parameter space is $\Omega(r) = \{\omega = (\omega_i) \in \Omega \mid \sum_i \omega_i = r\}$. In the first four chapters, the parameter space is $\Omega = X^m$ unless otherwise specified.

1.2.3 The Equilibrium Equation

Given the price vector $p \in S$, consumer i's demand is equal to $f_i(p, p \cdot \omega_i)$. The *total* or *aggregate demand* is the sum of individual demands. It is equal to $\sum_i f_i(p, p \cdot \omega_i)$.

Aggregate excess demand is the difference between aggregate demand and supply. It is equal to

$$z(p,\omega) = \sum_i (f_i(p, p \cdot \omega_i)) - \sum_i \omega_i. \tag{1.1}$$

The aggregate excess demand satisfies the identity

$$p \cdot z(p,\omega) = 0. \tag{1.2}$$

This identity is also known in the literature as Walras' law. This is *Walras' law for the aggregate excess demand*, and it is not to be confused with *Walras' law for individual demand functions*, even though the former is a direct consequence of the latter.

An *equilibrium price vector* of the economy $\omega = (\omega_i) \in \Omega$ is a price vector $p \in S$ such that total supply and demand are equal, or aggregate excess demand is equal to zero:

$$z(p,\omega) = \sum_i \left(f_i(p, p \cdot \omega_i) - \sum_i \omega_i \right) = 0.$$

This equation is known as the *equilibrium equation*.

Let $W(\omega)$ denote the set of equilibrium price vectors associated with the endowment vector $\omega = (\omega_i) \in \Omega$. At this stage, the set $W(\omega)$ can be any subset of the price set S. It will follow from general theorems that the equilibrium equation always has a solution for $\omega \in \Omega$, but this solution is not necessarily unique. The equilibrium set $W(\omega)$ is therefore nonempty for $\omega \in \Omega$ and may contain more than one element. The relation that associates with $\omega \in \Omega$ the set of Walrasian equilibria $W(\omega)$ is a *correspondence* known in the literature under the two names of *Walras correspondence* and *equilibrium set correspondence*.

Exercises

1.1. Let

$$Du_i(x_i) = \left(\frac{\partial u_i}{\partial x^1}(x_i), \ldots, \frac{\partial u_i}{\partial x^\ell}(x_i) \right)$$

denote the gradient vector of u_i at $x_i \in X$. The normalized gradient of u_i at $x_i \in X$ is defined by

$$D_n u_i(x_i) = \frac{1}{\frac{\partial u_i}{\partial x^\ell}(x_i)} Du_i(x_i).$$

Show that the map $D_n u_i : X \to S$ is differentiable up to any order.

1.2. In the problem of maximizing $u_i(x_i)$ with $x_i \in \mathbb{R}^{\ell}_{++}$ subject to the budget constraint $p \cdot x_i \leq w_i$, where $p \in S$ and $w_i \in \mathbb{R}_{++}$ are given, show that the budget constraint is binding.

1.3. Show that the set $\{x_i \in X \,|\, u_i(x_i) \geq u_i^*\}$ is closed in $\mathbb{R}^{\ell}$. (*Hint:* Let $x_i^* = \lim_{n\to\infty} x_i^n$ with $u_i(x_i^n) \geq u_i^*$. Define y_i^n by $y_i^n = \lambda^n x_i^n$, with $0 < \lambda^n \leq 1$ and $u_i(y_i^n) = u_i^*$. Using property (iv) of the utility functions, show that the sequence y_i^n has a subsequence that converges to some $y_i^* \in X$. Using the convergent subsequence, show that $y_i^* \leq x_i^*$.)

1.4. Let $x_i^* = \lim_{n\to\infty} x_i^n$ with $x_i^n \in X$. Show that if x_i^* does not belong to X (i.e., some coordinates of x_i^* are equal to zero), then $\lim_{n\to\infty} u_i(x_i^n) = -\infty$. (Recall that we assume $u_i(X) = \mathbb{R} = (-\infty, +\infty)$.)

1.5. Show that the problem of maximizing $u_i(x_i)$ with $x_i \in \mathbb{R}^{\ell}_{++}$ subject to the budget constraint $p \cdot x_i \leq w_i$ has a solution. (*Hint:* Reduce this problem to one of maximizing $u_i(x_i)$ on some suitably defined compact set, and exploit continuity of u_i.) Show that this solution is unique.

1.6.

a. Show that the first-order conditions for the problem of maximizing $u_i(x_i)$, with $x_i \in \mathbb{R}^{\ell}_{++}$ subject to the constraint $p \cdot x_i = w_i$, and $p \in S$ and $w_i \in \mathbb{R}_{++}$ given, take the form

$$D_n u_i(x_i) = p, \qquad p \cdot x_i = w_i,$$

where the normalized gradient $D_n u_i(x_i)$ is defined in exercise 1.1. Prove that these conditions are necessary and sufficient.

b. Prove that the demand function $f_i : S \times \mathbb{R}_{++} \to X$ is differentiable up to any order. (*Hint:* Apply the implicit function theorem to the first-order conditions.)

1.7. Define $\lambda_i(p, w_i)$ by the formula $Du_i(f_i(p, w_i)) = \lambda_i(p, w_i)p$. Show that the map $(p, w_i) \to \lambda_i(p, w_i)$ is indefinitely differentiable.

1.8. Show the following:

$$p \cdot \frac{\partial f_i}{\partial p_j}(p, w_i) = -f_i^j(p, w_i), \qquad p \cdot \frac{\partial f_i}{\partial w_i}(p, w_i) = 1.$$

(*Hint:* Derive the identity $p \cdot f_i(p, w_i) = w_i$ with respect to p_j and w_i.)

1.9. Let $\omega_i \in X$ be given. Show that the derivative of the map $p \to f_i(p, p \cdot \omega_i)$ with respect to p_j is equal to

$$\frac{df_i^k}{dp_j} = \frac{\partial f_i^k}{\partial p_j}(p, p \cdot \omega_i) + \frac{\partial f_i^k}{\partial w_i}(p, p \cdot \omega_i)\omega_i^j.$$

1.10.

a. Let $g_i : X \to S \times \mathbb{R}_{++}$ be the map defined by

$$g_i(x_i) = (D_n u_i(x_i), D_n u_i(x_i) \cdot x_i).$$

(See exercise 1.1 for the definition of the normalized gradient.) Show that the map g_i is differentiable up to any order.

b. Prove the following:

$$f_i \circ g_i = \mathrm{id}_X, \qquad g_i \circ f_i = \mathrm{id}_{S \times \mathbb{R}_{++}}.$$

(The maps f_i and g_i, being differentiable up to any order and inverse to each other, are said to be *diffeomorphisms*.)

1.11. Show that the matrix

$$Df_i(p, w_i) = \begin{bmatrix} \frac{\partial f_i^1}{\partial p_1} & \frac{\partial f_i^1}{\partial p_2} & \cdots & \frac{\partial f_i^1}{\partial p_{\ell-1}} & \frac{\partial f_i^1}{\partial w_i} \\ \frac{\partial f_i^2}{\partial p_1} & \frac{\partial f_i^2}{\partial p_2} & \cdots & \frac{\partial f_i^2}{\partial p_{\ell-1}} & \frac{\partial f_i^2}{\partial w_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_i^\ell}{\partial p_1} & \frac{\partial f_i^\ell}{\partial p_2} & \cdots & \frac{\partial f_i^\ell}{\partial p_{\ell-1}} & \frac{\partial f_i^\ell}{\partial w_i} \end{bmatrix}$$

computed at $(p, w_i) \in S \times \mathbb{R}_{++}$ is invertible. (*Hint:* Apply exercise 1.10b.)

1.12. Let $\omega_i \in X$ be given. Let $p^* = D_n u_i(\omega_i)$. (See exercise 1.1 for the definition of the normalized gradient.) Prove that the function $S \to \mathbb{R}$ defined by $p \to u_i(f_i(p, p \cdot \omega_i))$ has a unique maximum at $p = p^*$.

1.13.

a. Compute the first derivatives of the function $p \to u_i(f_i(p, p \cdot \omega_i))$ of exercise 1.12. Check that they are equal to zero for $p = p^*$.

b. Prove the equality

$$\left. \frac{d^2 u_i(f_i(p, p \cdot \omega_i))}{dp_h \, dp_k} \right|_{p=p^*} = -\lambda_i(p^*, p^* \cdot \omega_i) \frac{df_i^h}{dp_k}(p^*, p^* \cdot \omega_i)$$

relating the second derivatives of the above function and the first derivatives of the map $p \to f_i(p, p \cdot \omega_i)$. (The function $\lambda_i(p, w_i)$ is defined in exercise 1.7.)

1.14. The Slutsky matrix $M_i(p, w_i)$ associated with the demand function $f_i : S \times \mathbb{R}_{++} \to X$ at $(p, w_i) \in S \times \mathbb{R}_{++}$ is the $\ell - 1 \times \ell - 1$ matrix whose (h, k) coefficient is equal to

$$m_{hk}(p, w_i) = \frac{df_i^h}{dp_k}(p, p \cdot \omega_i) \quad \text{for } \omega_i = f_i(p, w_i).$$

a. Show that the Slutsky matrix is symmetric.
b. Using exercises 1.12 and 1.13, prove that the quadratic form associated with the symmetric Slutsky matrix $M_i(p, w_i)$ is semidefinite negative.

1.15. Let $f_i : S \times \mathbb{R}_{++} \to X$ be the demand function of some consumer i. Let $(p, w_i) \neq (p', w_i') \in S \times \mathbb{R}_{++}$. Prove that if the inequality $p \cdot f_i(p', w_i') \leq w_i$ is satisfied, then necessarily the strict inequality $p' \cdot f_i(p, w_i) > w_i'$ is satisfied. (This property is known in the literature as the weak *revealed preference property*.)

1.16. Prove that the Slutsky matrix $M_i(p, w_i)$ is invertible. (*Hint:* Multiply the first row of matrix $Df_i(p, w_i)$ in exercise 1.11 by p_1, the second row by p_2, and so on up to the ℓth row by $p_\ell = 1$, add them up, and substitute the resulting new row for the ℓth row of $Df_i(p, w_i)$. Apply exercise 1.8. Then add to the first column of the new matrix the last column multiplied by $f_i^1(p, w_i)$, add to the second column the last column multiplied by $f_i^2(p, w_i)$, and so on up to the $(\ell - 1)$th column to which is added the last column multiplied by $f_i^{\ell-1}(p, w_i)$.)

1.17. Using exercise 1.16, prove that the quadratic form associated with the Slutsky matrix (see exercise 1.14) is in fact negative definite.

1.18. The bordered Hessian matrix of consumer i's utility function u_i at $x_i \in X$ is equal by definition to the matrix

$$H_i(x_i) = \begin{bmatrix} D^2 u_i(x_i) & Du_i(x_i) \\ Du_i(x_i)^T & 0 \end{bmatrix},$$

where $D^2 u_i(x_i)$ is the Hessian matrix consisting of the second-order partial derivatives of u_i at x_i; $Du_i(x_i)$ is the column matrix defined by the first-order derivatives of u_i at x_i; and $Du_i(x_i)^T$ is its transpose.

Show $\det H_i(x_i) \neq 0$. (*Hint:* Assume $\det H_i(x_i) = 0$. There exists a column matrix $\ell + 1 \times 1 \begin{bmatrix} X \\ y \end{bmatrix} \neq 0$ (with X a $\ell \times 1$ matrix) such that the matrix product $H_i(x_i)\begin{bmatrix} X \\ y \end{bmatrix}$ is equal to zero. Show that this implies $\begin{bmatrix} X \\ y \end{bmatrix} = 0$; hence, a contradiction.)

1.19. Let X be a $\ell \times 1$ column matrix. Show that the equalities $Du_i(x_i)^T X = 0$ and $D^2 u_i(x_i) X = 0$ imply $X = 0$.

1.3 First Phase, or Rational General Equilibrium Theory

The first phase of general equilibrium theory lasted for about 80 years. It started with the publications of Jevons (1871) and Walras (1874), continued with the works of Edgeworth (1881), Pareto (1909), Bowley (1924), and Divisia (1928), and ended with the contributions of Hicks (1937; 1946), Allais (1943), and Samuelson (1947), to name just a limited list of writers in a particularly long and rich period.

The issues considered during that period deal with the *determinacy* of the solutions of the equilibrium equation $z(p, \omega) = 0$ (for given $\omega \in \Omega$) and the *Pareto optimality* of the corresponding equilibrium allocation $x = (f_i(p, p \cdot \omega_i))$.

1.3.1 Determinacy of Equilibrium

Determinacy is solved by computing the *degree of freedom* of the solution of the equilibrium equation, a concept borrowed from rational mechanics, which boils down to counting equations and unknowns, the degree of freedom being simply the difference between the number of unknowns and the number of (independent) equations. The degree of freedom of the solutions of the equilibrium equation was shown by Walras (1874) to be equal to zero.

1.3.2 Pareto Optimality or Efficiency

The rigorous definition of Pareto optimality first appeared in 1909 in the mathematical appendix of the French edition of Pareto's *Manual of Political Economy*. This definition opened up the way to the two theorems of welfare economics, in short, the welfare theorems: every equilibrium allocation is a Pareto optimum, and conversely, every Pareto optimum can be realized as the equilibrium allocation of some economy. The proofs of the welfare theorems then exploit the first-order conditions of the theory of constrained optimization, which was introduced in mathematics by Lagrange in the early nineteenth century for the specific needs of rational mechanics.

1.3.3 Parallel with Rational Mechanics

The parallel between the first phase of the theory of general equilibrium and nineteenth-century rational mechanics involves more than the use of the degree of freedom and constrained optimization. Rational mechanics tends to focus on an idealized world without frictions and other perturbations, for example, air resistance. For Walras and Pareto, general equilibrium theory, or "pure economics," was similarly concerned with a fictitious world that had been purified from the many "imperfections" of real economies. The proximity to rational mechanics is openly claimed by the title *Economique Rationelle* given by Divisia (1928) to a book later credited by Debreu with having stimulated his interest in economics.

1.3.4 Transition to Modern General Equilibrium Theory

The transition from rational to modern general equilibrium theory occurred with the study of the existence issue. This problem was not addressed seriously before the mid 1930s. Von Neumann (1937) opened up a new line of research in a path-breaking article that used Brouwer's fixed-point theorem to prove the existence of equilibrium in a model that was sufficiently close to the general equilibrium model to underscore the potential for the theory of general equilibrium of fixed-point theorems.

The two issues of existence and uniqueness were addressed within the strict setup of the theory of general equilibrium by Wald (1936). Wald assumed two rather restrictive properties: (1) gross substitutability at all prices, and (2) a property now known as aggregate weak revealed preference.[1] Under these assumptions, he was able to give elementary proofs of the existence and uniqueness of equilibrium. Unfortunately, his assumptions are not satisfied by economies where consumers' utility functions satisfy the assumptions of standard smooth consumer theory (see section 1.2.2). The important novelties in form and substance brought by the Von Neumann and Wald publications mark the next phase of general equilibrium theory.

Exercises

1.20. A quasi-linear economy consists of m consumers whose preferences are defined by log-linear utility functions. Assume that prices are

1. For a definition of gross substitutability, see section 4.8.2 in chapter 4. The aggregate weak revealed preference property means that, for the equilibrium price vector $p^* \in S$, the strict inequality $p^* \cdot z(p, \omega) > 0$ is satisfied for every $p \neq p^*$.

not normalized. Show that the partial derivative $\frac{\partial z^j}{\partial p_k}(p, \omega)$ of aggregate demand for commodity j with respect to the price p_k of commodity k, with $1 \le j \ne k \le \ell$, is > 0. (Property of gross substitutability of all goods for all prices)

1.21. Consider a quasi-linear economy with m consumers and not normalized price vectors.
a. Write the equilibrium equation of that economy.
b. Show that the equilibrium equation can be given the form $Ap = p$, where A is some $\ell \times \ell$ square matrix.
c. Show that $\lambda = 1$ is an eigenvalue of matrix A.
d. Prove that there exists an eigenvector $p \in \mathbb{R}^\ell$ with strictly positive coordinates associated with the eigenvalue $\lambda = 1$. (*Hint:* Apply the Perron-Frobenius theorem for positive matrices.)

1.22. Consider an economy where the only assumption about consumer i, with $i = 1, 2, \ldots, m$, is that the demand function $f_i : S \times \mathbb{R}_{++} \to X$ is smooth and satisfies Walras' law, namely, the identity $p \cdot f_i(p, w_i) = w_i$. The demand functions are not supposed to result from utility maximization subject to a budget constraint.

The pair $(p, \omega) \in S \times \Omega$ is said to be a *demand-oriented equilibrium* if the vector inequality

$$\sum_i f_i(p, p \cdot \omega_i) \le \sum_i \omega_i$$

is satisfied.
a. Give an economic interpretation of this equilibrium concept.
b. Show that a demand-oriented equilibrium is a Walrasian equilibrium.
c. Assume now that Walras' law is not satisfied by the demand functions of some consumers. Does the identity between demand-oriented and Walrasian equilibria still hold? Give an example.

1.23. Consider an economy defined by demand functions, as in exercise 1.22. By definition, the pair $(p, \omega) \in S \times \Omega$ is an *equilibrium with fully employed resources* if the vector inequality

$$\sum_i f_i(p, p \cdot \omega_i) \ge \sum_i \omega_i$$

is satisfied.

a. Give an economic interpretation of this equilibrium concept.
b. Show that an equilibrium with fully employed resources is a Walrasian equilibrium.
c. Assume now that Walras' law is not satisfied by the demand functions of some consumers. Does the identity between equilibria with fully employed resources and Walrasian equilibria still hold? Give an example.

1.24. Consider an economy defined by demand functions, as in exercise 1.22.
a. Show that there exists a pair $(p, \omega) \in S \times \Omega$ that satisfies the equilibrium equation

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i.$$

b. Prove by way of example that there exist economies $\omega \in \Omega$ that have no equilibria. (*Hint:* Consider demand functions that are bounded for some goods.)

1.4 Second Phase, or Modern General Equilibrium Theory

1.4.1 The Axiomatic Approach

The need for carefully stated assumptions, a necessary condition for any rigorous solution of the existence problem, explains the adoption by general equilibrium theory of the abstract axiomatic approach by the end of the 1940s. That approach gradually emerged in pure mathematics during the second half of the nineteenth century through the work of major figures like Artin, Cantor, Hilbert, Peano, Weber, and Weierstrass. One of the best illustrations of the new axiomatic style is van der Waerden's *Moderne Algebra* (1931). Analysis rapidly follows suit under the influence of the Bourbaki group. Dieudonné's outstanding *Foundations of Modern Analysis* (1960) is to analysis the equivalent of van der Waerden's book.

The axiomatic approach yields the mathematical model of modern general equilibrium theory, a model now known as the *Arrow-Debreu model.* Utility functions lose their differentiability. Individual demands become multivalued correspondences. Aggregate demand is also a correspondence. The definition of the equilibrium equation is adjusted accordingly.

1.4.2 Existence and Efficiency of Equilibria

The first successful proofs of the existence of equilibrium under reasonably general assumptions were achieved independently by McKenzie (1954) and Arrow and Debreu (1954). They used Brouwer's and Kakutani's fixed-point theorems, respectively. These proofs do not require differentiability. The advance with regard to existence followed on the heels of another equally remarkable achievement: new proofs of the Pareto optimality or efficiency of equilibrium allocations (the two welfare theorems) by Arrow (1951) and Debreu (1951). These proofs apply the separation theorems of functional analysis and are perfectly suited to the nondifferentiable formulation of modern general equilibrium theory. These new proofs of the existence of equilibrium and of the welfare theorems were resounding successes. Debreu's *Theory of Value* (1959) became the manifesto of modern general equilibrium theory.

1.4.3 Equilibrium Allocations and the Core

Building on a theme introduced by Edgeworth (1881), another dazzling success of modern general equilibrium theory was the clarification of properties of equilibrium allocations with respect to coalition formation. The first welfare theorem states that equilibrium allocations are Pareto-efficient. The extension of the first welfare theorem to the more general property that no coalition can improve on any equilibrium allocation is almost obvious. Far more difficult is analyzing the relations between the set of allocations that cannot be improved by any coalition (a set also known as the *core* of the economy) and the set of equilibrium allocations. These two sets were shown to be identical by Debreu (1963) and Debreu and Scarf (1963) for economies with a countable infinity of consumers; by Aumann (1964) in the case of a continuum of consumers—consumers identified to the points of the interval $[0, 1]$; and by Vind (1964) in the case of a nonatomic measure space of consumers. These results were later unified by Hildenbrand (1974) into a general theory.

1.4.4 Uniqueness and Stability of Equilibria

In the wake of the existence and welfare theorems, intensive research was devoted during the 1950s and 1960s to the issues of uniqueness and (tatonnement) stability of the solutions of the equilibrium equation.

Wald's (1936) work on uniqueness was generalized and extended to (tatonnement) stability in the late 1950s by Arrow, Block, Hahn, Hurwicz, Negishi, and Uzawa, among others.

The progress made on uniqueness and stability was surveyed by Negishi (1962) and, less than a decade later, by Arrow and Hahn in their monumental *General Competitive Analysis* (1971), whose chapters 9 and 11 became the new references on uniqueness and tatonnement stability.

1.5 Limitations of Modern General Equilibrium Theory

The high quality of the works surveyed by Negishi and by Arrow and Hahn cannot hide the fact that approaches that were so successful in dealing with existence, the welfare theorems, and the core were less so in dealing with uniqueness and stability. Two disturbing examples discovered by Scarf (1960) and Gale (1963) of economies with equilibria that were simultaneously unique and tatonnement unstable cast serious doubts on the chances of finding general conditions on preferences or utility functions that would be compatible with or imply the uniqueness and (tatonnement) stability of equilibrium.

These doubts become certainty with the discovery in the late 1960s and early 1970s of two new properties of the Arrow-Debreu model. One is the Debreu-Mantel-Sonnenschein (DMS) theorem. The other is the continuity (more accurately, the generic continuity) of Walras correspondence and the related generic finiteness of the number of solutions of the equilibrium equation.

The DMS theorem widely amplifies the disturbing impact of the two early examples of Gale and Scarf. Sonnenschein's first version appeared in print in 1973 and 1974. It was improved by Mantel (1974) and by Debreu (1974). In its most general form, the theorem states that, given any continuous function $\zeta : K \to \mathbb{R}^\ell$, where K is some compact subset of the normalized price set $S = \mathbb{R}^{\ell-1}_{++} \times \{1\}$, a function that satisfies Walras' law (i.e., the identity $p \cdot \zeta(p) = 0$), there exists an Arrow-Debreu exchange economy with an arbitrary number of consumers provided the latter is at least equal to the number of goods such that the aggregate excess demand function $z(., \omega) : S \to \mathbb{R}^\ell$, a function whose domain is the full price set S, coincides with the function ζ on the compact set K, a *subset* of the price set S, not the price set itself.

DMS goes a long way beyond the Scarf and Gale examples in providing examples of badly behaved economies. Nevertheless, the real meaning of DMS is grossly misunderstood by the economic profession. Many nonmathematically oriented economists (and even some with a good understanding of mathematics) confuse the compact subset K

with the price set S and conclude that the aggregate excess demand function in the Arrow-Debreu model is purely arbitrary. If true, this property would imply that the Arrow-Debreu model does not have enough structure to carry truly interesting economic properties. Economists who for various reasons do not like the general equilibrium approach find in this interpretation of DMS the justification for their views.

In fact, the aggregate excess demand function of an economy, the function $z(.,\omega) : S \to \mathbb{R}^\ell$ with domain the full price set S, is very far from being arbitrary. By suitably extending the definition of the (topological) degree of a map to the setup of the aggregate excess demand map, I showed that the degree of the aggregate demand function of an economy is necessarily equal to 1, and the degree of an arbitrary function can be any integer (Balasko 1986; 1988, sec. 5.5). Loosely speaking, not many functions $\zeta : S \to \mathbb{R}^\ell$ satisfying Walras' law can be identified with the aggregate excess demand function $z(.,\omega) : S \to \mathbb{R}^\ell$ of an economy.

The aggregate excess demand function of an economy is therefore far from being totally arbitrary even if its restriction to any compact subset of the price set can be arbitrary.[2] But this does not prevent DMS from being a powerful tool for producing examples of badly behaved economies over compact subsets of the price set S in the vein of Gale's and Scarf's examples. Another consequence of DMS is that the equilibrium price set $W(\omega)$ of the economy ω can be any compact subset of the price set S, as shown by Mas-Colell (1977).

The existence of badly behaved economies can be quite disturbing, but the theoretical and practical effects of these examples depend on how large the sets of these economies are. If these sets are small, then the probability that an economy is badly behaved is also small. In the extreme case where the sets have measure zero, the probability of a randomly selected badly behaved economy is zero. To achieve full impact, DMS requires the characterization of the set of economies for which the aggregate excess demand function satisfies a given property. This kind of problem is never considered by modern general equilibrium theory.

1.5.1 Comparative Statics

The behavior of the equilibrium prices and allocations when the endowment parameter $\omega \in \Omega$ is varied, a subject known as comparative

2. This shows the importance of the behavior of aggregate excess demand when prices tend to the boundary of the price set S, i.e., to 0 or to ∞.

statics, is at least as important as the existence and efficiency of equilibrium allocations. Blaug (1985), the historian of economic ideas, criticized the theory of general equilibrium for its lack of results in comparative statics. This may have been true until the late 1960s, but comparative statics came to the forefront of the research agenda at the time DMS was in gestation.

Comparative statics has qualitative and quantitative aspects. The accurate determination of the amount of the price increase resulting from a given variation of the total supply is quantitative. The much vaguer question of whether some prices will increase for a given decrease of the total supply of some goods is qualitative. Quantitative issues cannot be answered without complete understanding of the global qualitative picture.

1.5.2 Continuity of the Walras Correspondence

The most basic qualitative property is continuity. Roughly speaking, the question is whether sufficiently small variations of the fundamentals of the economy translate into small variations of the equilibrium price system. Mathematically, this question takes the form of the *continuity* of the *Walras correspondence* $W : \Omega \to S$.

Continuity is a well-known mathematical concept for functions. It involves two different ideas, upper and lower semicontinuity. The combination of the two properties yields continuity. Upper and lower semicontinuity have been extended to correspondences under the names of upper and lower hemicontinuity (Debreu 1959; Hildenbrand 1974). (See exercises 2.22 and 2.23 in chapter 2.) Like functions, a correspondence is continuous if it is both upper and lower hemicontinuous.

1.5.3 Upper Hemicontinuity

Upper hemicontinuity of the Walras correspondence was proved by Hildenbrand and Mertens (1972). They considered the standard setup of the Arrow-Debreu model typical of modern general equilibrium theory, which includes as a special case the pure exchange model considered here, and they proved upper hemicontinuity.

1.5.4 Lower Hemicontinuity at Regular Economies

Lower hemicontinuity of the Walras correspondence cannot be established under the assumptions that suffice to establish its upper hemicontinuity. The solution given by Debreu (1970) in a landmark article rehabilitates differentiability as a major tool of equilibrium analysis.

The Walras correspondence $W : \Omega \to S$ turns out not to be lower hemicontinuous over its domain Ω. But Debreu showed that there exists an open subset $\mathcal{R}$ of Ω that has full measure (its complement $\Sigma = \Omega \backslash \mathcal{R}$ has Lebesgue measure zero), a set with the property that the *restriction* of the Walras correspondence to this subset is lower hemicontinuous. Debreu called an element of the set $\mathcal{R}$ a *regular economy.*

1.5.5 Finiteness of Equilibria at Regular Economies

Debreu also showed that for every regular economy $\omega \in \mathcal{R}$, the equilibrium price set $W(\omega)$ is finite. In addition, there exist an open set U contained in $\mathcal{R}$ and containing ω, and a finite number n of smooth functions $s_k : U \to S$, with k varying from 1 to n, such that for $\omega' \in U$, the equilibrium price set $W(\omega')$ consists of the n elements $s_1(\omega')$, $s_2(\omega'), \dots, s_n(\omega')$. These equilibria depend continuously (in fact differentiably) on the endowment vector $\omega' \in U$. Not only is the number of equilibria finite for every regular economy $\omega \in \mathcal{R}$, but these equilibria are also *isolated.* In other words, every equilibrium price vector associated with the regular economy $\omega \in \mathcal{R}$ is locally unique, i.e., unique in some neighborhood of the equilibrium price. By proving local uniqueness for regular economies, Debreu gave a partial solution to the longstanding uniqueness problem of modern general equilibrium theory.

The importance of this solution was recognized immediately in the economic profession. Arrow and Hahn (1971, 244) expressed an opinion widely held among economists when they wrote of the finiteness of equilibria that it is “the best possible result short of ... much more restrictive assumptions.”

1.6 Limitations of the Absolute Perspective

The approach followed by modern general equilibrium theory is absolute in the sense that its aim is to establish properties that are true for all economies satisfying reasonable assumptions. This program has been successful in dealing with existence, the two welfare theorems, and the inclusion of the set of equilibrium allocations in the core.

The generic finiteness of the number of equilibria and the generic continuity of the Walras correspondence—*generic* meaning that these properties are satisfied for an open and dense set of economies—are the first breach in the absolute perspective. The necessity of a relative perspective going way beyond genericity is made even more compelling by DMS.

1.7 Notes and Comments

The first formulation of the equilibrium equation is due to Jevons (1871). But Jevons considered only the case of two goods. Walras formulated essentially the same equation as Jevons but for an arbitrary number of goods and consumers. Walras's research had already gone on for a few years when he got news of Jevons's results. This led him to speed up publication. The part of his book already completed (Walras 1874) was published immediately, and the full content appeared in the second edition, released three years later.

Existence of solutions of systems consisting of a finite number of equations and unknowns was not considered an important issue at the time of Walras. Existence of solutions of the equation systems of rational mechanics that are similar to the equilibrium equation of economics was handled at that time by the explicit computation of the solutions. Along this line, Walras saw in tatonnement an algorithm that could compute the solutions of the equilibrium equation, which, by the same token, solved the theoretical existence problem.

The first really general existence theorems available in the mathematics of finite equation systems were Brouwer's fixed-point theorem (1912) and the equivalent Poincaré-Miranda theorem, which was first stated without proof by Poincaré in 1883. Poincaré had to solve a problem of celestial mechanics, namely, finding a periodic orbit, a problem that turned out to resist explicit computations (Poincaré 1883; 1884). That theorem of Poincaré was rediscovered several decades later by Miranda (1941), who also proved its equivalence with Brouwer's fixed-point theorem. The rediscovery of these theorems occurred many years after the first publication of Walras's (1874) book and illustrated the lack of interest in the existence issues at the time.

The concept of Pareto efficiency first appeared in the mathematical appendix of Pareto's *Manuel d'Economie Politique* in 1909.

The first explicit criticism of a mistaken interpretation of DMS holding that aggregate excess demand functions do not face restrictions other than Walras' law is due to Balasko (1986).

The realization that multiple equilibria are more than a mathematical curiosity and may have deep economic implications came very gradually. Walras understood that there could be more than one equilibrium in the two-good case but thought that this possibility did not extend to three goods and more. In other words, he strongly believed that uniqueness is the rule when there are more than two goods.

Auspitz and Lieben (1889) were the first economists to suspect the importance of the number of equilibria. Though they understood that the multiplicity of equilibria raised many new issues, they realized that this problem was beyond their mathematical expertise. In retrospect, this problem was beyond the possibilities of their time. Auspitz and Lieben chose the strategy of ignoring the problems raised by the multiplicity of equilibria by postulating that equilibrium is unique. This assumption, however, is logically inconsistent with their other assumptions. Their continued interest in the multiplicity issue is demonstrated by Lieben's 1907 letter to Walras (Jaffe 1965, let. 1654) and by Auspitz and Lieben's 1908 article.

Progress on the uniqueness and multiplicity issues was slow. Bowley (1924) acknowledged, a couple of decades after Auspitz and Lieben, that "there is nothing in the nature of the case to prevent multiple solutions." He concluded that "in practice if we had any numerical values there is not likely to be difficulty in knowing which set is appropriate."

Schumpeter seems to be the first to fully realize in the late 1940s the importance of the multiplicity issue. Schumpeter (1954) writes, "Multiple equilibria are not necessarily useless but ... the existence of a 'uniquely determined equilibrium ...' is, of course, of the utmost importance" (969). He predated the literature on chaos and nonlinear dynamics by more than two decades: "Without any possibility of proving the existence of uniquely determined equilibrium—or at all events, of a small number of possible equilibria—at however high a level of abstraction, a field of phenomena is really a chaos that is not under analytic control."

Local uniqueness is a weaker property than global uniqueness. Global uniqueness implies local uniqueness, whereas the converse is not true. The importance of the concept of local uniqueness was first identified in the study of the solutions of partial differential equations. Hadamard (1932) defined a partial differential equation (with boundary conditions) as defining a "well-posed problem" if the solutions of the partial differential equation are locally determined (i.e., locally unique). Courant and Hilbert (1962, 226–228) argued that the interest of the concept of "well-posedness" is not limited to partial differential equations. At this point, Walras is again a forerunner when he counts equations and unknowns in order to establish the determinateness of equilibrium. It is only with Debreu's (1970) paper that the issue of the local determinateness of equilibria was addressed in modern terms within the setup of the Arrow-Debreu model. Interestingly, the motivations for the study of local uniqueness given by Debreu (1970) reproduce almost word for word those stated by Courant and Hilbert (1962).

The mathematical ideas of regularity, transversality, and genericity, and their application to exchange economies made by Debreu, became easily accessible to economists thanks to a book by Dierker (1974). The influence of this book in spreading the idea of regularity was enormous. The flexibility of the concept of regular economies is illustrated by its extension to a variety of setups. Of particular interest is the extension made by Smale (1974a; 1974b) to the case where utility functions are not necessarily strictly quasi-concave.

An idea of the questions that Debreu (1970) stimulated in the group around him in Berkeley can be found in a dissertation turned into a book by de Montbrial (1971), one of Debreu's doctoral students. This book listed the following three problems: (1) Does the measure of the set of economies with more than n equilibria tend to zero as n tends to ∞? (2) Are there restrictions on the parity of the number of equilibria? (3) What assumptions on preferences imply the smoothness of individual demand functions? The third question was quickly solved by Debreu (1972), who showed that smooth quasi-concave utility functions having indifference surfaces with everywhere nonzero Gaussian curvature yield smooth demand functions. The second question was answered by Dierker (1972) with a proof that the number of equilibria of regular economies is odd. (See corollary 4.6.4 in chapter 4.) The first question was solved by Balasko (1979c), who gave an upper bound in the form c/n, where c is some constant, for the measure of the set of economies with more than n equilibria that belong to an arbitrary compact set. (See proposition 2.8.1 in chapter 2.)

2 The Equilibrium Manifold and the Natural Projection

2.1 Introduction

In this chapter I introduce the concept of the equilibrium manifold as a tool for the study of properties that might not be satisfied by all equilibria. The equilibrium manifold leads to the related concept of natural projection, a map from the equilibrium manifold into the space of economies. I then show that the natural projection is smooth and proper, which suffices for this map to define a structure that is known in mathematics as a ramified covering. The results of Debreu (1970) then become straightforward consequences of this structure.

2.2 Equilibrium Manifold

2.2.1 Definition

Now that we know that a given property is not necessarily satisfied by all economies $\omega \in \Omega$, the issue is to characterize the subset of Ω that consists of the economies for which that property is satisfied. But properties of economies $\omega \in \Omega$ are of two kinds: properties of each equilibrium taken in isolation, and properties of the equilibrium set as a whole, in other words, of all the equilibria of an economy. The relative perspective leads us to focus in a first stage on the properties that are satisfied by the pairs $(p, \omega) \in S \times \Omega$, where the price vector $p \in S$ is an equilibrium price vector of the economy $\omega \in \Omega$, i.e., $p \in W(\omega)$.

Walras Correspondence

We recognize in this setup the Walras correspondence and, more particularly, its graph, the set of pairs $(p, \omega) \in S \times \Omega$ such that $p \in W(\omega)$. This new importance of the *graph of the Walras correspondence* justifies an adjustment of the terminology. By definition, the pair $(p, \omega) \in S \times \Omega$

is an equilibrium if it satisfies the equation $z(p, \omega) = 0$. This is equivalent to the price vector $p \in S$ being an *equilibrium price vector* associated with the economy $\omega \in \Omega$, i.e., $p \in W(\omega)$.

The change in terminology from the Walras correspondence to the equilibrium manifold is not purely semantic. With the equilibrium manifold, the emphasis shifts from comparative statics, unquestionably an important subject, to the wider subject of the properties of equilibria regardless of whether these properties come under the heading of comparative statics.

Equilibrium Manifold

The *equilibrium manifold* E is the subset of $S \times \Omega$ defined by the equation $z(p, \omega) = 0$. At this stage, the equilibrium manifold E has no more structure than just being a subset of the Cartesian product $S \times \Omega$. The name equilibrium manifold will be justified by the fact that E is indeed a *smooth submanifold* of the Cartesian product $S \times \Omega$, a property that is insufficiently obvious without proof. The following property is far easier to prove.

Proposition 2.2.1 The equilibrium manifold E is a closed subset of $S \times \Omega$.

Proof The equilibrium manifold E is closed in $S \times \Omega$ as the preimage of the point (or vector) $0 \in \mathbb{R}^\ell$ by the continuous map $(p, \omega) \to z(p, \omega)$. ∎

Note that proposition 2.2.1 requires only the continuity of the aggregate demand function, itself a consequence of the continuity of the individual demand functions.

2.2.2 The Relative Perspective

Let $\mathfrak{P}(p, \omega)$ denote some property that may be satisfied at the equilibrium $(p, \omega) \in E$. Let $E(\mathfrak{P})$ be the subset of the equilibrium manifold E that consists of the equilibria for which the property $\mathfrak{P}(p, \omega)$ is true. The study of $\mathfrak{P}(p, \omega)$ then reduces to the study of the set $E(\mathfrak{P})$ as a subset of the equilibrium manifold E.

This program requires a thorough understanding of the equilibrium manifold. An important issue in that regard is the existence of practical coordinate systems. Their existence would greatly simplify the study of the subsets that would now be defined by systems of equations and inequations. Hopefully, all the subsets we are interested in will belong to that category.

2.2.3 Local Structure

The problem of the existence and nature of coordinate systems for the equilibrium manifold E has two versions. The *local version* deals with the structure of sufficiently small open sets of the equilibrium manifold. The most relevant mathematical property in that direction is provided by the structure of *smooth manifold*, where sufficiently small open sets are homeomorphic to Euclidean spaces. These open subsets can then be parameterized by a finite number of real numbers that act as local coordinates. With these local coordinates, it is possible to give sense to the concept of *smooth maps.* Differentiability makes it possible to apply the powerful methods of differential topology. Before undertaking the study of the smooth manifold structure of the equilibrium manifold, I insert a short mathematical parenthesis.

2.3 Mathematical Parenthesis: Smooth Maps and Related Concepts

2.3.1 Regular and Critical Points

Let $\rho : X \to Y$ be a smooth map (a map differentiable up to any order) between the smooth manifolds X and Y. Let the vector spaces $T_x X$ and $T_{\rho(x)} Y$ be the tangent spaces to the smooth manifolds X and Y at the points x and $\rho(x)$, respectively.

The point $x \in X$ is a *critical point* of the map ρ if the tangent map (also known as the derivative) $(d\rho)_x : T_x X \to T_{\rho(x)} Y$ is not onto. We denote by Γ the set of critical points.

The point $x \in X$ is *regular* if the tangent map $(d\rho)_x : T_x X \to T_{\rho(x)} Y$ is onto.

Proposition 2.3.1 The set Γ of critical points of the smooth map $\rho : X \to Y$ is closed.

Proof Let us use local coordinates for the manifolds X and Y at x and $y = \rho(x)$. This means that we have open neighborhoods U of $x \in X$ and V of $y = \rho(x) \in Y$ that are homeomorphic to $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. We can use the coordinates $(\xi_1, \ldots, \xi_p)$ and $(\rho_1, \ldots, \rho_q)$ of $\mathbb{R}^p$ and $\mathbb{R}^q$ to represent the elements of U and V, respectively. Then, the map ρ is represented by its coordinate functions $\rho_1(\xi_1, \ldots, \xi_p), \ldots, \rho_q(\xi_1, \ldots, \xi_p)$. The *tangent map* $(d\rho)_x$ becomes in the local coordinate system the linear map represented by the Jacobian matrix of ρ. This matrix is defined by the first-order partial derivatives of the q coordinate functions with respect to the p variables. These derivatives are also smooth functions of the coordinates.

The condition that the tangent map $(d\rho)_x$ is not onto is equivalent to the property that the rank of the Jacobian matrix is less than the dimension q of the tangent space $T_{\rho(x)}Y$. This is equivalent to having all the determinants of the square matrices of order q that can be extracted from the Jacobian matrix equal to zero. These determinants are polynomial functions of their coefficients, which are themselves smooth functions of the coordinates. The set of critical points of the smooth map ρ is therefore the set of zeros of a collection of continuous maps. Its complement, the set of regular points, is therefore open in U. Now, the manifold X can be made the countable union of open sets like U. The union of all the open sets made of regular points is therefore an open subset of X. Its complement, the set of critical points of the map $\rho : X \to Y$, is therefore closed in X. ■

2.3.2 Singular and Regular Values

By definition, the image $\rho(x)$ of the critical point $x \in X$ is a *singular value* of the map $\rho : X \to Y$. Let Σ denote the set of singular values. We then have $\Sigma = \rho(\Gamma)$.

By definition, the element $y \in Y$ is a *regular value* of the map $\rho : X \to Y$ if it does not belong to Σ. In other words, a regular value is not the image of any critical point. Let $\mathcal{R}$ be the set of regular values. We have $\mathcal{R} = Y \backslash \Sigma$, the complement of Σ in Y.

It follows from this definition that an element $y \in Y$ that does not belong to the image $\rho(X)$ of the map ρ is a regular value, although it is not a value of that map. This observation underlines the importance of being accurate when specifying the domain and range of maps.

2.3.3 Regular Value Theorem

The following proposition is very useful for proving that a set defined by an equation system is actually a smooth manifold. It is in fact an extension to the setup of smooth manifolds of the *implicit function theorem*. It also illustrates the importance of the concept of regular value.

Proposition 2.3.2 (Regular value theorem) The preimage $\rho^{-1}(y)$ of the regular value $y \in \mathcal{R} \subset Y$ for the smooth map $\rho : X \to Y$ is a smooth submanifold of X whose dimension is equal to $\dim X - \dim Y$.

For mathematical references, see, for example, Guillemin and Pollack (1974, 21), Hirsch (1976, 22), and Milnor (1997, 11).

2.3.4 Sard's Theorem

A particularly important property of smooth maps is Sard's theorem. Since a smooth manifold is the union of a countable collection of open sets each diffeomorphic to an open set of a Euclidean space, it is possible to define sets of measure zero in smooth manifolds: a set has measure zero if its intersection with each open set of the collection has measure zero. Then, Sard's theorem states the following:

Proposition 2.3.3 (Sard's theorem) The set Σ of singular values of the smooth map $\rho : X \to Y$ has measure zero in Y.

An alternative formulation is to say that the complement of Σ, the set $\mathcal{R}$ of regular values, has full measure in Y. For proofs, see, for example, Dubrovkin, Fomenko, and Novikov (1985, 79), Guillemin and Pollack (1974, 39), Hirsch (1976, 69), or Milnor (1997, 10).

Exercises

2.1. Let E be a Euclidean space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in E that converges to the element $x^* \in E$. By using the characterization of compact subsets of a Euclidean space as closed and bounded, show that the set $K = \bigcup_{n \in \mathbb{N}} \{x_n\} \cup \{x^*\}$ is compact.

2.2. Let E be a metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in E that converges to the element $x^* \in E$. Show that the set $K = \bigcup_{n \in \mathbb{N}} \{x_n\} \cup \{x^*\}$ is compact. (*Hint:* Use the property that every open covering of a compact set has a finite subcovering.)

2.3. Let E and F be two metric spaces. Let $f : E \to F$ be a proper map, i.e., a map such that the preimage $f^{-1}(K)$ of every compact set of F is compact in E. Prove that the direct image by f of every closed set of E is closed in F.

2.4. Let K be a set equipped with the discrete topology. What are the open sets for this topology? Show that K compact is equivalent to K finite.

2.5. Let E be a metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in E that converges to the element $x^* \in E$. Assume $x^* \neq x_n$ for all $n \in \mathbb{N}$. The set K is equipped with the topology induced by the topology of E. Prove that the subset $\{x^*\}$ of K cannot be open in K. Can the induced topology on K be the discrete topology?

2.6. Let E and F be two metric spaces. The map $f : E \to F$ is said to be *continuous at the point* $x \in E$ if, for every sequence $(x_n)_{n \in \mathbb{N}}$ converging to

x, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is also convergent and its limit is $f(x)$. The map $f : E \to F$ is said to be continuous if it is continuous for all $x \in E$.

a. Show that the preimage of every closed subset of F is closed in E if f is continuous.

b. Show that the preimage of every closed subset of F is closed in E if and only if the preimage of every open subset of F is open in E. (*Hint:* Compare the preimage of the complement $F\backslash U$ with the complement $E\backslash f^{-1}(U)$.)

c. Show that the map $f : E \to F$ is continuous if the preimage of every open subset of F is open in E.

2.7. Let E and F be two metric spaces. Let $f : E \to F$ be a continuous map. Show that the image of a connected set is connected. (*Hint:* Assume the contrary and get a contradiction.)

2.8. Let $\mathbb{N}$ be the set of natural integers equipped with the discrete topology. Show that the only connected subsets are reduced to a point.

The following exercises assume some knowledge of the properties of the Lebesgue measure.

2.9. The set V has measure zero in the Euclidean space $E = \mathbb{R}^n$ if, for every $\varepsilon > 0$, there exists a countable set of cubes that cover V and such that the sum of the volumes (or Lebesgue measures) of the cubes that make up the covering is less than ε. Prove that the Lebesgue measure of a point in a Euclidean space is zero.

2.10. Prove that the Lebesgue measure of any finite subset of a Euclidean space is zero.

2.11. Prove that the Lebesgue measure of the subset of a Euclidean space defined by a sequence of elements is equal to zero.

2.12. Show that the Lebesgue measure of the subset $\mathbb{Q} \cap [0,1]$ consisting of the rational numbers of the interval $[0,1]$ is equal to zero.

2.13. Show that the closure $\overline{\mathbb{Q} \cap [0,1]}$ of the set $\mathbb{Q} \cap [0,1]$ is equal to $[0,1]$.

2.14. Let $E = \mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q$ and $F = \mathbb{R}^p \times \{0\}$ with $0 \in \mathbb{R}^q$ and $q \geq 1$. Show that the Lebesgue measure in E of the subset F of E is equal to zero.

2.15. Let E be the Euclidean space $\mathbb{R}^n$. Let V be a submanifold of E of dimension $m < n$. Show that the Lebesgue measure of V in E is equal to zero.

2.16. Let E be the Euclidean space $\mathbb{R}^n$. Let W be the union of a finite number of smooth submanifolds of E of dimension strictly less than n. Show that the Lebesgue measure of W in E is equal to zero. (*Hint:* Apply exercise 2.15.)

2.17. Let E and F be two Euclidean spaces. Let $f : E \to F$ be a smooth map. Show that the direct image of a set of measure zero by f has measure zero.

2.18. Let Y be the subset of $Z = \mathbb{R}^2$ defined by equation $x^2 + y^2 = 1$. Let X be the subset of $Z \times \mathbb{R}$ equal to $Y \times \mathbb{R}$. Let $\pi : X \to Z$ be the map defined by formula $\pi(x, y, z) = (x, y)$. Interpret geometrically the sets Y and X and the map π. What is the domain of the map π? Show that every point of the domain of π is critical. Show that Y is the image of the map π. Show that Y is also the set of singular values of the map π. Does this contradict Sard's theorem?

2.19. Define the sets X, Y, and Z as in exercise 2.18. Consider the map $\pi : X \to Y$ by the same formula $\pi(x, y, z) = (x, y)$ except that now the range is Y instead of Z. Show that the map π has no critical point.

2.20. Let X and Y be two smooth manifolds. Let $\alpha : X \to Y$ and $\tilde{\beta} : Y \to X$ be two smooth maps. The composition $\alpha \circ \tilde{\beta} : Y \to Y$ is the identity map id_Y.
a. Let $Z = \tilde{\beta}(Y)$ be the image of the map $\tilde{\beta}$. Define $\beta : Y \to Z$ by $\beta(y) = \tilde{\beta}(y)$. Show that $\beta : Y \to Z$ is a bijection whose inverse is the map $\alpha \,|\, Z : Z \to Y$, the restriction of the map α to Z.
b. Show that the maps $\alpha \,|\, Z : Z \to Y$ and $\beta : Y \to Z$ are continuous for Z equipped with the topology induced by the topology of X.
c. Let $y \in Y$. Let $x = \beta(y) \in Z$. Let $T_y(Y)$ and $T_x(X)$ be the tangent spaces to Y at y and to X at x, respectively. Let $d\tilde{\beta}_y : T_y(Y) \to T_x(X)$ be the tangent (or derivative) map to $\tilde{\beta} : Y \to X$. Show that the linear map $d\tilde{\beta}_y$ is an injection. (*Hint:* Use the fact that the relation $\alpha \circ \tilde{\beta} = \mathrm{id}_Y$ implies for the tangent maps the relation $d\alpha_x \circ d\tilde{\beta}_y = id_{T_y(Y)}$.)
d. Conclude that the map $\tilde{\beta} : Y \to X$ is an immersion that defines a homeomorphism between its domain Y and its image $Z = \tilde{\beta}(Y)$. (Such a map $\tilde{\beta}$ is known as an *embedding*.)

2.21. Let the smooth map $\beta : Y \to X$ be an embedding, i.e., an immersion that is a homeomorphism between the domain Y and the image $Z = \beta(Y)$. Show that the subset Z is a smooth submanifold of X diffeomorphic to Y.

2.4 Smooth Manifold Structure of the Equilibrium Manifold

The question is whether the equilibrium manifold E is actually a smooth manifold or, even better, a smooth submanifold of $S \times \Omega$. The answer turns out to be positive, as follows from proposition 2.4.1.

Proposition 2.4.1 The equilibrium manifold E is a smooth submanifold of dimension ℓm of $S \times \Omega$.

We are going to prove this property as a consequence of the regular value theorem. The idea is to apply the *regular value theorem* to the map $(p, \omega) \to \bar{z}(p, \omega)$ defined by the first $\ell - 1$ coordinates of the aggregate excess demand $z(p, \omega)$. (Recall that the ℓ coordinates of the map $(p, \omega) \to z(p, \omega)$ are not independent, because of Walras' law.)

2.4.1 Application of the Regular Value Theorem

The equilibrium manifold E is defined by equation $\bar{z}(p, \omega) = 0$. It is therefore the preimage of $0 \in \mathbb{R}^{\ell-1}$ by the map $\bar{z} : S \times \Omega \to \mathbb{R}^{\ell-1}$. The *regular value theorem* tells us that a sufficient condition for E to be a smooth submanifold of $S \times \Omega$ is that the element $0 \in \mathbb{R}^{\ell-1}$ is a regular value of the map $\bar{z}$. This is equivalent to the map $\bar{z}$ having no critical point that is also an equilibrium. In fact, it will be shown that the map $\bar{z}$ has no critical point, which is equivalent to showing that the Jacobian matrix of $\bar{z}$ at $(p, \omega) \in S \times \Omega$ has rank $\ell - 1$, this matrix having $\ell - 1$ rows and $m\ell + \ell - 1$ columns.

2.4.2 The Rank Property

To prove the rank property, it suffices to extract from the Jacobian matrix a submatrix that still has rank $\ell - 1$. Pick arbitrarily some consumer i. Let us look at the block made of the ℓ columns (and $\ell - 1$ rows) made of the derivatives of $\bar{z}$ with respect to the coordinates $\omega_i^1, \ldots, \omega_i^\ell$ of ω_i, the endowment of consumer i. In the computation, we apply the chain rule. Given the fact that consumer i's demand does not depend on consumer j's wealth, with $j \neq i$, this yields for the Jacobian matrix the rather simple expression

$$\begin{bmatrix} \dfrac{\partial z^1}{\partial \omega_i^1} & \cdots & \dfrac{\partial z^1}{\partial \omega_i^\ell} \\ \dfrac{\partial z^2}{\partial \omega_i^1} & \cdots & \dfrac{\partial z^2}{\partial \omega_i^\ell} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial z^{\ell-1}}{\partial \omega_i^1} & \cdots & \dfrac{\partial z^{\ell-1}}{\partial \omega_i^\ell} \end{bmatrix}$$

$$= \begin{bmatrix} p_1 \dfrac{\partial f_i^1}{\partial w_i} - 1 & p_2 \dfrac{\partial f_i^1}{\partial w_i} & \cdots & p_{\ell-1} \dfrac{\partial f_i^1}{\partial w_i} & \dfrac{\partial f_i^1}{\partial w_i} \\ p_1 \dfrac{\partial f_i^2}{\partial w_i} & p_2 \dfrac{\partial f_i^2}{\partial w_i} - 1 & \cdots & p_{\ell-1} \dfrac{\partial f_i^2}{\partial w_i} & \dfrac{\partial f_i^2}{\partial w_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_1 \dfrac{\partial f_i^{\ell-1}}{\partial w_i} & p_2 \dfrac{\partial f_i^{\ell-1}}{\partial w_i} & \cdots & p_{\ell-1} \dfrac{\partial f_i^{\ell-1}}{\partial w_i} - 1 & \dfrac{\partial f_i^{\ell-1}}{\partial w_i} \end{bmatrix}.$$

In the right-hand matrix, multiply the last column by p_1 and subtract from the first column, multiply again the last column by p_2 and subtract from the second column, and so on until multiplication of the last column by $p_{\ell-1}$ and subtraction from the $(\ell - 1)$th column. This yields the $\ell - 1 \times \ell$ matrix

$$\begin{bmatrix} -1 & 0 & \cdots & 0 & \dfrac{\partial f_i^1}{\partial w_i} \\ 0 & -1 & \cdots & 0 & \dfrac{\partial f_i^2}{\partial w_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \dfrac{\partial f_i^{\ell-1}}{\partial w_i} \end{bmatrix}$$

that has same rank. The rank of this new matrix is equal to $\ell - 1$ because the block made of its first $\ell - 1$ columns has obviously rank $\ell - 1$.

2.4.3 Dimension of the Equilibrium Manifold

It also follows from the regular value theorem that the dimension of the equilibrium manifold E is equal to the dimension of $S \times \Omega$ minus the dimension of $\mathbb{R}^{\ell-1}$, hence to $\ell - 1 + m\ell - (\ell - 1) = m\ell$.

2.4.4 Smoothness of the Embedding Map

The following consequence of E being a smooth submanifold of $S \times \Omega$ is going to be particularly useful.

Proposition 2.4.2 The embedding map $E \to S \times \Omega$ is smooth.

Proof The embedding map from E into $S \times \Omega$ is the identity map from $S \times \Omega$ restricted to the subset E. The proposition then follows readily from the definition of a smooth submanifold. ∎

The property conveyed by the proposition may seem to be of limited value. However, it is very convenient when it comes to proving the differentiability of maps that are the restrictions to the equilibrium manifold E of maps defined on the Cartesian product $S \times \Omega$. An immediate application of that observation is to the projection map $S \times \Omega \to \Omega$ restricted to the equilibrium manifold E.

2.5 Natural Projection

The properties of the solutions of the equilibrium equation $z(p, \omega) = 0$ when the parameter ω is varied in the parameter space Ω depend not only on the structure of the equilibrium manifold E but also on how this manifold is embedded in the Cartesian product $S \times \Omega$.

The latter aspect is conveyed by the map that is the *restriction* of the projection map $S \times \Omega \to \Omega$ to the equilibrium manifold E. This map $\pi : E \to \Omega$, called from now on the *natural projection*, is related to the Walras correspondence by the equality $\pi^{-1}(\omega) = W(\omega) \times \{\omega\}$. The equilibrium manifold E is the *domain* of the natural projection.

We now establish two important properties of the natural projection, *smoothness* and *properness*. Smoothness enables us to apply to the natural projection important properties of differential topology. Properness is an additional property that also plays an important role. Properness is usually defined by the property that the preimage of every compact set is compact. This property then implies that the direct image of every closed set is closed.

2.5.1 Smoothness

Proposition 2.5.1 The natural projection $\pi : E \to \Omega$ is smooth.

Proof The natural projection $\pi : E \to \Omega$ is the composition of two maps: the natural embedding $E \to S \times \Omega$, which is smooth because E is a smooth submanifold of the Cartesian product $S \times \Omega$, and the projection map $S \times \Omega \to \Omega$, which is also smooth because the coordinate functions of this map are smooth. ■

2.5.2 Properness

Proposition 2.5.2 The natural projection $\pi : E \to \Omega$ is a proper map.

Proof A map is *proper* if the preimage of every compact set is compact. Let K be a compact subset of the parameter space Ω. Let H be the image of K by the continuous map $\omega = (\omega_1, \omega_2, \dots, \omega_m) \to \omega_1 + \omega_2 + \cdots + \omega_m$ from Ω into X. The set H is compact and therefore *bounded from above:* there exists some $r^* \in X$ such that $\omega_1 + \omega_2 + \cdots + \omega_m \leq r^*$ for all $\omega \in K$. It follows from the equilibrium equality

$$f_1(p, p \cdot \omega_1) + \cdots + f_m(p, p \cdot \omega_m) = \omega_1 + \cdots + \omega_m$$

that

$$f_1(p, p \cdot \omega_1) + \cdots + f_m(p, p \cdot \omega_m) \leq r^* \quad \text{for } (p, \omega) \in \pi^{-1}(K).$$

Pick now some consumer i arbitrarily. It follows from the previous inequality combined with the fact that the consumption space is the strictly positive orthant X that

$$f_i(p, p \cdot \omega_i) \leq r^* \quad \text{for } (p, \omega) \in \pi^{-1}(K).$$

Let K_i be the image of K by the projection map

$$\omega = (\omega_1, \dots, \omega_i, \dots, \omega_m) \to \omega_i.$$

That map being continuous, its image K_i is a compact subset of $X = \mathbb{R}^{\ell}_{++}$. There exists a lower bound A such that the coordinates ω_i^k satisfy the inequalities $0 < A \leq \omega_i^k$ for $k = 1, 2, \dots, \ell$ for $\omega \in K$. Let $\omega_i^* = (A, A, \dots, A)$.

The utility $u_i(f_i(p, p \cdot \omega_i))$ is greater than or equal to $u_i(\omega_i)$, itself greater than or equal to $u_i(\omega_i^*)$. We therefore have that, for every

$(p, \omega) \in \pi^{-1}(K)$, the demand $x_i = f_i(p, p \cdot \omega_i)$ is such that $x_i \leq r^*$ and $u_i(x_i) \geq u_i(\omega_i^*)$. We can assume without loss of generality the strict inequality $\omega_i^* < r^*$.

Lemma 2.5.3 The nonempty set $L_i = \{x_i \in X \mid u_i(x_i) \geq u_i(\omega_i^*)$ and $x_i \leq r^*\}$ is compact.

Proof Let us show that L_i is a closed and bounded subset of $\mathbb{R}^\ell$. Closedness in $\mathbb{R}^\ell$ of the set $\{x_i \in X \mid u_i(x_i) \geq u_i(\omega_i^*)\}$ follows from property (iv) of the utility functions considered in section 1.2.2. (See exercise 1.3 in chapter 1.) The set $\{x_i \in \mathbb{R}^\ell \mid x_i \leq r^*\}$ is also closed in $\mathbb{R}^\ell$. The intersection of these two sets is therefore closed in $\mathbb{R}^\ell$.

The set L_i is bounded from above by r^* and from below by 0. This proves the compactness of L_i. ∎

Lemma 2.5.4 The image $D_n u_i(L_i)$ is a compact subset of the price set S.

Proof The normalized gradient map $D_n u_i : X \to S$ is continuous. (See exercise 1.1 in chapter 1 for the definition of the normalized gradient of utility.) The set L_i is compact. The image $D_n u_i(L_i)$ is then compact as the image of a compact set by a continuous map. ∎

It follows from the continuity of the natural projection that the restriction of π to $D_n u_i(L_i) \times K$ is continuous. Therefore, $\pi^{-1}(K)$ is a closed subset of the set $D_n u_i(L_i) \times K$ by the continuity of π. The Cartesian product of the two compact sets K and $D_n u_i(L_i)$ is compact. The combination of the two properties implies that $\pi^{-1}(K)$ is compact, which proves the properness of the natural projection $\pi : E \to \Omega$. ∎

2.6 Structure of the Equilibrium Manifold over the Set of Regular Economies

2.6.1 Critical and Regular Equilibria

Let $\Re(p, \omega)$ denote the property for the equilibrium $(p, \omega) \in E$ to be a regular point of the natural projection $\pi : E \to \Omega$. An equilibrium $(p, \omega) \in E$ that satisfies the property $\Re(p, \omega)$ is known as a *regular equilibrium.* Let the set $E(\Re)$ consist of the regular equilibria. We denote by Γ the set of critical equilibria in E. The two sets $E(\Re)$ and Γ are complementary to each other.

It follows from proposition 2.3.1 that the set of critical equilibria Γ is closed in E. Its complement, the set of regular equilibria $E(\Re)$, is therefore open in E.

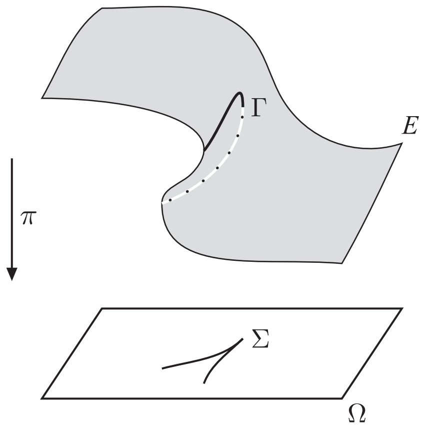

Figure 2.1
The equilibrium manifold and the natural projection

2.6.2 Openness of the Set of Regular Economies $\mathcal{R}$

Proposition 2.6.1 The set of regular economies $\mathcal{R}$ is open in Ω.

Proof The set of regular economies $\mathcal{R}$ is the complement in Ω of the set Σ, the set of singular values of the map $\pi : E \to \Omega$. By definition, singular values are images of critical points: $\Sigma = \pi(\Gamma)$. The set Γ of critical points of the map $\pi : E \to \Omega$ is *closed* in E. In general, the direct image of a closed set by a continuous map is not closed. But the map $\pi : E \to \Omega$ is also proper. The images of closed sets by proper maps are closed. This implies that Σ is closed and its complement, the set of regular economies $\mathcal{R}$, is therefore *open* in Ω (figure 2.1). ■

2.6.3 Finiteness of the Equilibrium Set of Regular Economies

Proposition 2.6.2 For every $\omega \in \mathcal{R}$, the set $\pi^{-1}(\omega)$ is finite.

Proof The proposition follows from the compactness and discreteness of the set $\pi^{-1}(\omega)$.

Compactness The set $\pi^{-1}(\omega)$ is compact as the preimage of the compact set $\{\omega\}$ by the proper map $\pi : E \to \Omega$.

Discreteness Let us show that the topological space $\pi^{-1}(\omega)$ is *discrete,* which means that it is equipped with the discrete topology. The discrete topology is the topology where each subset is open. For a topological space to be discrete, it suffices that each set consisting of one element is open.

Let $\{x\}$ be the subset of $\pi^{-1}(\omega)$ consisting of the unique element x. Clearly, x cannot be a critical point of the map π because ω is a regular value. Therefore, the tangent map $(d\pi)_x$ is a bijection and, by the inverse function theorem, the natural projection $\pi : E \to \Omega$ is a local diffeomorphism at the point $x \in E$. This means that there exist open neighborhoods U of ω and V of x such that the restriction $\pi \,|\, V : V \to U$ is a diffeomorphism. This map being one-to-one, the intersection $\pi^{-1}(\omega) \cap V$ contains the element x and no other element. Therefore, we have $\{x\} = \pi^{-1}(\omega) \cap V$, where V is open in E. It follows from the definition of the open sets of $\pi^{-1}(x)$ as the intersection with $\pi^{-1}(\omega)$ of open sets of E that the subset $\{x\}$ is open in $\pi^{-1}(x)$.

Open Covering of $\pi^{-1}(\omega)$ The topological space $\pi^{-1}(\omega)$ being equipped with the discrete topology, the subsets consisting of a unique elements of $\pi^{-1}(\omega)$, the subsets $\{x\}$ with $x \in \pi^{-1}(\omega)$, are open. Their union is obviously equal to X. They define an open covering of the set $\pi^{-1}(\omega)$.

Finiteness of $\pi^{-1}(\omega)$ It follows from the compactness of $\pi^{-1}(\omega)$ that the open covering of the set $\pi^{-1}(\omega)$ by sets with a unique element (the sets defined by the elements of $\pi^{-1}(\omega)$) has a finite subcovering. The set $\pi^{-1}(\omega)$ is therefore the union of only a finite number of its elements. This set is therefore finite. ∎

2.6.4 Finite Covering Property

The following proposition gives us a fairly accurate image of the equilibrium manifold E over the set of regular economies. This image is only partial, however, because the statement of the proposition is only local, i.e., it holds true only for sufficiently small open subsets of $\mathcal{R}$.

Proposition 2.6.3 For every $\omega \in \mathcal{R}$, there exists an open neighborhood U of ω with $U \subset \mathcal{R}$ and the property that, if $\pi^{-1}(\omega)$ is nonempty, the preimage $\pi^{-1}(U)$ is the union of a finite number of pairwise disjoint open sets $V_1, V_2, \ldots, V_k, \ldots, V_n$ and the restriction $\pi_k : V_k \to U$ of the map π is a diffeomorphism for $k = 1, 2, \ldots, n$.

Proof Let $n \geq 1$ be the (finite) number of elements of $\pi^{-1}(\omega)$.

Let $x_1, \ldots, x_n$ be all the elements of $\pi^{-1}(\omega)$. Provided the open sets are small enough, it is always possible to consider open pairwise disjoint neighborhoods $U'_1, U'_2, \ldots, U'_k, \ldots, U'_n$ in E of $x_1, x_2, \ldots, x_k, \ldots, x_n$

such that the restriction of π to U'_k is a diffeomorphism with $U_k = \pi(U'_k)$.

The set $E \backslash (U'_1 \cup \cdots \cup U'_k \cup \cdots \cup U'_n)$ is closed in E. Its image by the natural projection π is closed in Ω because π is proper. Let us define the set U as $U = (U_1 \cap U_2 \cap \cdots \cap U_k \cap \cdots \cap U_n) \backslash \pi(E \backslash (U'_1 \cup \cdots \cup U'_k \cup \cdots \cup U'_n))$, i.e., U is the intersection of the sets U_k for k varying from 1 to n and of the complement in Ω of the set $\pi(E \backslash (U'_1 \cup \cdots \cup U'_k \cup \cdots \cup U'_n))$. Clearly, U is open in Ω. Let us show that ω belongs to U. All we have to check is that ω does not belong to $\pi(E \backslash (U'_1 \cup \cdots \cup U'_k \cup \cdots \cup U'_n))$, which follows from the inclusion

$$\pi^{-1}(\omega) \subset U'_1 \cup \cdots \cup U'_k \cup \cdots \cup U'_n.$$

Let $V_k = U'_k \cap \pi^{-1}(U)$. The restriction $\pi_k = \pi \mid V_k$ defines a diffeomorphism between V_k, and $\pi(V_k)$.

Let us check that $\pi^{-1}(U)$ is the union of $V_1 \cup \cdots \cup V_k \cup \cdots \cup V_n$. Let $x \in \pi^{-1}(U)$. Assume that x does not belong to any V_k. Then it belongs to the set $E \backslash (U'_1 \cup \cdots \cup U'_k \cup \cdots \cup U'_n)$, which implies

$$\omega = \pi(x) \in \pi(E \backslash (U'_1 \cup \cdots \cup U'_k \cup \cdots U'_n)).$$

Therefore, ω belongs to the open set U, hence a contradiction. This ends the proof of the proposition. (See figure 2.2.) ∎

The mathematical structure described by proposition 2.6.3 is known as the property that the restriction of the map π to $\pi^{-1}(\mathcal{R})$ is an open finite covering of the set of regular economies $\mathcal{R}$.

Selections of Equilibrium Prices

Proposition 2.6.4 Let $\omega \in \mathcal{R}$. There exists an open neighborhood U of ω with $U \subset \mathcal{R}$ and a finite number n of smooth maps $s_k : U \to S$ such that the union $\bigcup_{1 \le k \le n} s_k(\omega')$ is identical to the set $W(\omega')$ of equilibrium price vectors associated with every $\omega' \in U$.

In other words,

$$\pi^{-1}(\omega') = \bigcup_{1 \le k \le n} \{(s_k(\omega'), \omega')\} \quad \text{for } \omega' \in U.$$

Proof To prove this proposition, let us go back to proposition 2.6.3. It suffices to compose the map $\pi_k^{-1} : U \to V_k$ with the projection $S \times \Omega \to S$ to define the map $s_k : U \to S$, which ends the proof of the proposition. ∎

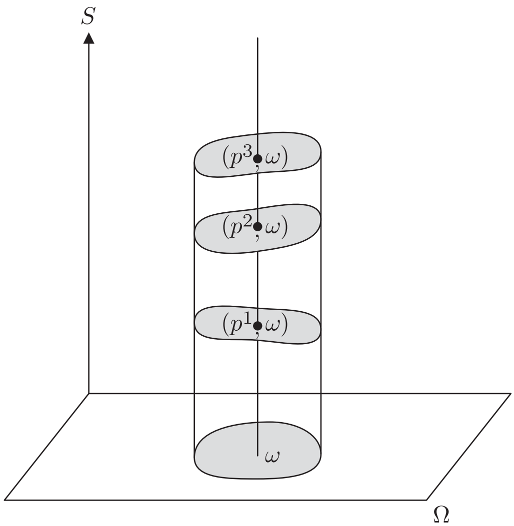

Figure 2.2
Structure of the equilibrium manifold over the neighborhood U of the regular economy $\omega \in \mathcal{R}$

Proposition 2.6.4 enables us to express the equilibrium prices associated with a regular economy as a function of the parameter ω describing these economies.

This property also implies the lower hemicontinuity of the Walras correspondence over the set of regular economies $\mathcal{R}$.

Remark Smooth selections of equilibria are defined only for open neighborhoods of regular economies. It is important for both practical and theoretical reasons to have some more information about the domains of these selection maps. In particular, does there exist some largest domain for the smooth selection maps instead of these open sets?

A complete treatment of this question is beyond the scope of this chapter. This problem is easier in the context of chapter 5 in the general case of an arbitrary number of goods and consumers.

Local Constancy of the Number of Equilibria at Regular Economies

Proposition 2.6.5 Let $\omega \in \mathcal{R}$ be a regular economy. There exists an open neighborhood U of ω such that $U \subset \mathcal{R}$ and the number of equilibria is constant all over U.

Proof Let $\omega \in \mathcal{R}$ be a regular economy. It then suffices to pick U as in proposition 2.6.4 to prove the proposition. ∎

Proposition 2.6.5 is often stated as the function number of equilibria $N : \omega \to \#\pi^{-1}(\omega)$ being *locally constant.*

Constancy of the Number of Equilibria over the Connected Components of $\mathcal{R}$

By definition, the *connected component of a point* in a topological space is the largest connected set containing that point (see, e.g., Dieudonné 1960, sec. 3.1.9). The *connected components of a set* are the various connected components of the points of this set. It follows from this definition that the set of regular economies $\mathcal{R}$ is partitioned into its connected components. By *partitioned* is meant that the various connected components are pairwise disjoint and that their union is equal to the full set $\mathcal{R}$.

Proposition 2.6.6 The number of equilibria is constant over each connected component of the set of regular economies $\mathcal{R}$.

Proof Let $N(\omega) = \#\pi^{-1}(\omega)$ denote the number of equilibria of the regular economy $\omega \in \mathcal{R}$. This defines a function $N : \mathcal{R} \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural integers. Let us equip this set with the discrete topology, the topology where each subset is open (and also closed).

Proposition 2.6.5 tells us that, for $\omega \in \mathcal{R}$, there exists an open neighborhood U of ω contained in $\mathcal{R}$ over which the number of equilibria $N(\omega)$ is constant. The function N is said to be *locally constant.*

Let us show that a locally constant function is necessarily constant on every connected component of its domain. First, let us show that the function number of equilibria N is continuous.

Continuity is established if we can show that the preimage by the function N of every open subset of $\mathbb{N}$ is open in $\mathcal{R}$. Since the topology of $\mathbb{N}$ is discrete, open subsets of $\mathbb{N}$ are the union of open subsets reduced to just one point. The preimage of a union of sets being also the union of the preimages of the sets, and the union of open sets being open, it therefore suffices to show that the preimage $N^{-1}(k)$ of the set reduced to the element k by the map $N : \mathcal{R} \to \mathbb{N}$ is open. This set consists of the economies $\omega \in \mathcal{R}$ that have k equilibria. If this set is empty (i.e., there are no such economies), then it is open, because an empty set is open by definition. If $N^{-1}(k)$ is nonempty, let $\omega \in N^{-1}(k)$. It follows from proposition 2.6.5 that there exists an open set U where the number of equilibria is equal to k. This is equivalent to the inclusion $U \subset N^{-1}(k)$. This shows that the set $N^{-1}(k)$ is an open neighborhood of each of its elements. This property is characteristic of an open set and implies that $N^{-1}(k)$ is open. This proves the continuity of the map $N : \mathcal{R} \to \mathbb{N}$.

Let C be a connected component of $\mathcal{R}$. The image of a connected set by a continuous map is connected. Therefore, the image $N(C)$ is a connected subset of the set $\mathbb{N}$ (equipped with the discrete topology). It follows from the definition of a connected set combined with the definition of the discrete topology that the only connected sets of $\mathbb{N}$ equipped with the discrete topology are the sets that consist of a unique element. This implies that the set $N(C)$ consists of just one element, which is another way of saying that the map N is constant on C. This ends the proof that the number of equilibria is constant on every connected component of the set of regular economies. ∎

2.7 Genericity of Regular Economies

We have seen that regular economies and their equilibria enjoy relatively nice properties. It is therefore important to have some information about the size of the set of regular economies $\mathcal{R}$.

2.7.1 Full Measure

We already know that the set $\mathcal{R}$ is open. The following proposition tells us that this set is really large in the sense that its complement has Lebesgue measure zero in Ω.

Proposition 2.7.1 The set of regular economies $\mathcal{R}$ is open with full measure in Ω.

Proof Sard's theorem, proposition 2.3.3, tells us that the set of singular values of a smooth map between two smooth manifolds has Lebesgue measure zero. Let us apply this to the set $\Sigma = \pi(\Gamma)$, the set of singular values of the natural projection $\pi : E \to \Omega$. The set Σ has therefore measure zero in Ω and its complement $\mathcal{R} = \Omega \backslash \Sigma$ has full measure. ∎

2.7.2 Density

The full measure property implies an interesting topological property, density, as follows from corollary 2.7.2.

Corollary 2.7.2 The set of regular economies $\mathcal{R}$ is open and dense in Ω.

Proof All we have to show is that $\mathcal{R}$ is dense. Assume the contrary. Then there exists a nonempty open cube U such that the intersection $\mathcal{R} \cap U$ is empty. This means that the nonempty open set U is contained in Σ. Therefore, the measure of Σ must be larger than, or at least equal to, the measure of U. But the measure of the nonempty cube U is the product of the lengths of its sides and, as such, is strictly positive. This yields a contradiction. ■

Remark Density is considered to mean "large" from a topological perspective. Note, however, that the set of rational numbers $\mathbb{Q}$ is dense in the set of real numbers $\mathbb{R}$. Nevertheless, its Lebesgue measure is equal to zero as the measure of a countable set. (Remember, the set of rational numbers is countable.) Here, we have much more than just density because the set $\mathcal{R}$ is also open.

Exercises

2.22. The correspondence $W : \Omega \to S$ is upper hemicontinuous (u.h.c.) at $\omega^* \in \Omega$ if $W(\omega^*) \neq \emptyset$ and if for every neighborhood U of $W(\omega^*)$ there exists a neighborhood V of ω^* such that $W(\omega) \subset U$ for every $\omega \in V$. The correspondence W is u.h.c. if it is u.h.c. at every $\omega^* \in \Omega$ (Hildenbrand 1974, 21, def. 1). Show that properness of the projection map $\pi : E \to \Omega$ implies that the Walras correspondence $W : \Omega \to S$ is u.h.c. (*Hint:* See Hildenbrand 1974, 24, theorem 1.)

2.23. The correspondence $W : \mathcal{U} \to S$ (where $\mathcal{U}$ is an open subset of Ω) is lower hemicontinuous (l.h.c.) at $\omega^* \in \mathcal{U}$ if $W(\omega^*) \neq \emptyset$ and if, for every open set G of S with $W(\omega^*) \cap G \neq \emptyset$, there exists a neighborhood V of ω^* such that $W(\omega) \cap G \neq \emptyset$ for every $\omega \in V$. The correspondence is l.h.c. if it is l.h.c. at every $\omega^* \in \mathcal{U}$ (Hildenbrand 1974, 26, def. 3). Show that the restriction of the Walras correspondence to the set of regular economies $\mathcal{R}$ is l.h.c. (*Hint:* See Hildenbrand 1974, 27, prop. 7.)

2.8 Economies with a Large Number of Equilibria

I conclude this chapter with another property that requires nothing more than the smoothness and properness of the natural projection, a property that complements nicely the fact that the set of economies with an infinite number of equilibria is contained in a closed set with measure zero.

The only reservation I have about including it at this early stage is that its proof is mathematically demanding in terms of covering maps and Riemannian geometry. The proof needs mathematical concepts that are not used anywhere else in the book. The reader is therefore encouraged to focus on the general ideas, which are quite simple, and to skip the technical details in a first reading.

2.8.1 Upper Bound of the Measure of the Set of Economies with More Than a Given Number of Equilibria

Proposition 2.6.2 gives no upper bound on the number of equilibria. In fact, DMS tells us that there is no upper bound. Nevertheless, there exists an upper bound to the size of the set of economies with more than a given number of equilibria.

Let us consider a compact subset K of Ω. Let $\Omega_n(K)$ denote the set of economies $\omega \in K$ having at least n equilibria, and let $\mu(\Omega_n(K))$ denote the Lebesgue measure of this set.

Proposition 2.8.1 There exists a constant $c(K)$ such that the inequality

$$\mu(\Omega_n(K)) \leq c(K)/n$$

is satisfied for every $n \geq 1$.

2.8.2 Sketch of the Proof

Assume to fix ideas that Ω is two-dimensional and S one-dimensional. Then, $S \times \Omega$ can be identified with the (strictly positive orthant of the) ordinary Euclidean space $\mathbb{R}^3$, and the equilibrium manifold E is a surface in that space. This surface comes with a notion of area (e.g., the definition of the area of the sphere). The area in Ω coincides with the Lebesgue measure of $\mathbb{R}^2$.

The next important step is to observe that the area does not increase through orthogonal projection. More specifically, let H be some subset of the surface E. Then the area of H is bigger than the area of its orthogonal projection $\pi(H)$ in the plane Ω. This property is just the generalization to surfaces of a well-known property of solid geometry, namely, that the area of the orthogonal projection of a triangle is less than or equal to the area of the triangle that is projected (figure 2.3).

Let now K be some compact subset of Ω. The preimage $\pi^{-1}(K)$ is compact in the surface E. It has therefore a finite area $c(\pi^{-1}(K))$.

The set of singular economies Σ having measure zero, the intersection $\Omega_n(K) \cap \mathcal{R}$ has the same measure as $\Omega_n(K)$.

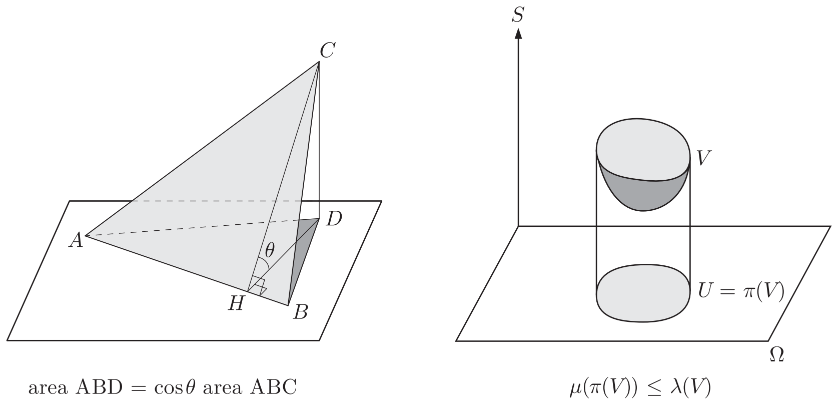

Figure 2.3
Area of an orthogonal projection

Let now U be some subset of $\Omega_n(K) \cap \mathcal{R}$ such that $\pi^{-1}(U)$ consists of a finite number of layers diffeomorphic to U. Not only is the number of layers $\geq n$ but the area of each layer is $\geq$ to the area of U onto which each layer projects orthogonally. Therefore, the area of $\pi^{-1}(U)$ is $\geq n$ times the area of U. It is shown in a following section that the set $\Omega_n(K) \cap \mathcal{R}$ can be partitioned into a countable collection of such open sets U. The inequalities for each open set U then add up to the inequality

$$c(\pi^{-1}(K)) \geq \text{area}(\pi^{-1}(\Omega_n(K)) \geq n \times \text{area}(\Omega_n(K)),$$

which ends the proof of the proposition. Let us now develop the more technical aspects of this proof.

Area Element for the Equilibrium Manifold

The equilibrium manifold E is embedded in $S \times \Omega$, which is itself an open subset of the Euclidean space $\mathbb{R}^{\ell-1} \times \mathbb{R}^{\ell m}$. The restriction of the scalar product to the tangent spaces to E defines a Riemannian structure on E. (See, e.g., Hicks 1965 for these elementary notions of Riemannian geometry.) This Riemannian structure leads to a concept of ℓm dimensional area on E, which is the extension to arbitrary dimensions of the usual area concept for two-dimensional surfaces.

Let λ denote the measure defined by the Riemannian structure on the equilibrium manifold E. The orthogonal projection does not increase the measure. More specifically, let μ denote the Lebesgue measure of Ω. We then have $\mu(\pi(V)) \leq \lambda(V)$ for every measurable subset V of E.

Finite Area of $\pi^{-1}(K)$

The set K is compact and the natural projection π proper. The set $\pi^{-1}(K)$ is therefore compact, hence measurable, i.e., with a finite measure $c(K) = \lambda(\pi^{-1}(K))$.

Reduction to $\Omega_n(K) \cap \mathcal{R}$

The set of regular economies $\mathcal{R}$ having full measure, we have

$$\mu(\Omega_n(K)) = \mu(\Omega_n(K) \cap \mathcal{R}).$$

Existence of a Suitable Countable Partition

Every open subset of a Euclidean space can be decomposed into a countable union of pairwise disjoint open cubes and a set of measure zero. (For a proof, see, e.g., Rudin 1966, 52.) We apply this property to the open set $\mathcal{R}$ and neglect from now on the set of measure zero.

A cube is connected and simply connected. The connectedness property implies that each cube of the decomposition of $\mathcal{R}$ is contained in one connected component of $\mathcal{R}$. Therefore, the number of equilibria of economies belonging to the same cube is constant.

Simple connectedness implies that the covering of each cube of the collection by the map π is trivial. (See, e.g., Dieudonné 1973, sec. 16.28.6.) This means that the preimage is the disjoint union of a finite number of open sets all diffeomorphic to the cube through the projection map π, a map that we identify with the orthogonal projection.

Let us consider in the countable collection of cubes those that have at least n equilibria. Let U_j be the collection of these cubes. This collection is at most countable. In addition, each economy $\omega \in U_j$ has at least n equilibria. The set $\Omega_n(K)$ is equal, up to a set of measure zero, to the union of those cubes U_j. The set $\pi^{-1}(U_j)$ consists of at least n layers, each one having a λ-measure at least equal to $\mu(U_j)$. Adding up all these inequalities yields

$$n\mu(U_j) \leq \lambda(\pi^{-1}(U_j)).$$

Summing up these inequalities over the cubes U_j gives

$$n\mu(\Omega_n(K)) \leq \lambda(\pi^{-1}(\Omega_n(K)) \leq \lambda(\pi^{-1}(K)) = c(K). \quad \blacksquare$$

Proposition 2.8.1 tells us that the probability of observing economies with more than n equilibria, although not zero, tends to zero as $1/n$ and is therefore small for n large.

Proposition 2.8.1 is an asymptotic version of the property that the set of economies with an infinite number of equilibria has Lebesgue measure zero in Ω.

2.9 Conclusion

The natural projection $\pi : E \to \Omega$ is therefore an open finite covering of its set of regular values $\mathcal{R}$. In addition, the complement $\Omega \backslash \mathcal{R}$, the set of singular values Σ, is closed with measure zero. Such a map $\pi : E \to \Omega$ is known in mathematics as defining a *ramified covering*, the ramifications taking place over the set of singular values Σ. The genericity of regular values follows directly from Sard's theorem. The other properties in Debreu (1970) are just reformulations of the finite covering property of the set of regular values $\mathcal{R}$.

A first outcome of the relative postmodern perspective is therefore to put the Debreu (1970) results into the wider setup of the properties of the solutions of an equation system that depends on some parameter. The equation system is here the equilibrium equation, the parameter the vector of individual endowments, with variable total resources. Note that the results depend on the choice of the parameter space and would not necessarily hold with parameter spaces different from $\Omega = X^m$.

The two key properties for the results of this chapter are the smoothness and properness of the projection map. Smoothness is a fairly general property in this kind of approach. It is in fact satisfied for a large class of equations in addition to the equilibrium equation of the Arrow-Debreu model. Properness is more specific to the economic model. Nevertheless, all it requires is that there be at least one consumer whose demand tends to infinity when (normalized) prices tend to the boundary of the price set, i.e., either to zero or to infinity. Interestingly, properness is the mathematical version of a property that economists have discussed for a long time in relation to the existence of free goods.

Smoothness and properness of the natural projection require only a small fraction of the assumptions made on utility functions. The next chapters are therefore devoted to the properties of the Arrow-

Debreu model that follow from a more intensive exploitation of these assumptions.

2.10 Notes and Comments

The proof that the equilibrium manifold is indeed a smooth manifold for the parameter space $\Omega = X^m$, i.e., for total resources that are variable, is given by Delbaen (1971). A set closely related to the equilibrium manifold is also considered by Smale (1974a).

The name catastrophe theory is often given to the application of the theory of singularities of smooth maps to the study of the solutions of finite equation systems. Most models of catastrophe theory involve smooth maps similar to the natural projection. The main differences between these models are in the interpretation of the singularities and their importance. In the natural projection, the regular values (regular economies) are as important as the singular values (singular economies) or the critical equilibria. This is not the case for all the models of catastrophe theory. The most remarkable feature of catastrophe theory is its ability to generate discontinuities in otherwise differentiable models. Structural stability is a concept also introduced by catastrophe theory, and it corresponds to the lack of discontinuities. Typically, structural stability is observed at regular values.

Thom (1975) and Zeeman (1977) tend to reserve the name catastrophe theory to the methods that consist in deriving mathematical models of real-world phenomena through the identification of the phenomenon's qualitative features with appropriate singularities of smooth maps. That method makes sense only when the mathematical model is unknown, which is not the case for economic theory with its Arrow-Debreu model. For various applications of catastrophe theory, see Arnold (1992), Thom (1975), and Zeeman (1977).

The results of sections 2.6 and 2.7 are in Debreu (1970). The introduction of the natural projection $\pi : E \to \Omega$ and the proof of the ramified finite covering property of the natural projection are due to Balasko (1975a). It is the natural projection approach that Debreu (1976) presented in an address to the American Economic Association meeting of 1975, devoted to the theory of regular economies.

The bound on the size of the set of economies with at least n equilibria is given by Balasko (1979c).

3 The Set of No-Trade Equilibria

In this chapter I exploit the fact that the Arrow-Debreu model describes a process that reallocates the (initial) endowment $\omega = (\omega_i) \in \Omega$ into some (final) allocation $x = (x_i) \in \Omega$. The importance of these final allocations rests on two grounds: (1) they are easy to observe, and (2) they determine the final utility level of each consumer. The set of final allocations is therefore an important aspect of any allocation process, of the competitive process in particular. This leads to consideration of the vector of net trades at equilibrium.

3.1 No-Trade Equilibria

3.1.1 Net Trade Vector

Given the price vector $p \in S$ and the endowment vector $\omega_i \in \mathbb{R}^{\ell}_{++}$, consumer i's consumption bundle is equal to $f_i(p, p \cdot \omega_i) \in \mathbb{R}^{\ell}_{++}$. The amount of goods to be traded by consumer i to go from the endowment vector ω_i to the consumption bundle $f_i(p, p \cdot \omega_i)$ is equal to $f_i(p, p \cdot \omega_i) - \omega_i$. This is consumer i's net trade vector.

The *net trade vector* for all the consumers present in the economy is then defined by the collection of all individual net trade vectors:

$$t(p, \omega) = (f_i(p, p \cdot \omega_i) - \omega_i) \in (\mathbb{R}^{\ell})^m.$$

3.1.2 Feasibility of the Net Trade Vector and Equilibrium

The pair $(p, \omega) \in S \times \Omega$ is an equilibrium if and only if the corresponding net trade vector $t(p, \omega)$ is *feasible*. This condition is equivalent to the equality

$$\sum_{i=1}^{m} (f_i(p, p \cdot \omega_i) - \omega_i) = 0.$$

3.1.3 No-Trade Equilibrium

Let us denote by $\mathfrak{T}(p, \omega)$ the property that the net trade vector $t(p, \omega)$ is equal to zero for the equilibrium $(p, \omega) \in E$. Property $\mathfrak{T}(p, \omega)$ is equivalent to the property that there is no trade at the equilibrium (p, ω).

Property $\mathfrak{T}(p, \omega)$ therefore amounts to the equality $f_i(p, p \cdot \omega_i) = \omega_i$ for i varying from 1 to m.

3.1.4 Set of No-Trade Equilibria

The *set of no-trade equilibria* T consists of the equilibria that satisfy property $\mathfrak{T}(p, \omega)$. It is a subset of the equilibrium manifold E.

A common reaction to the definition of no-trade equilibria is to question their very existence. After all, there is some kind of contradiction between the concept of no-trade equilibrium, an equilibrium where no trade takes place, and the idea of the market as a place dedicated to the exchange of goods. In the real world, if an economy is to be at a no-trade equilibrium, the absence of trade will quickly lead to the dismantlement of all the costly facilities that make trade possible. The existence of no-trade equilibria is therefore not that obvious after all. Existence of the no-trade equilibria follows from the analysis of the structure of the set of no-trade equilibria.

3.2 Structure of the Set of No-Trade Equilibria

3.2.1 Diffeomorphism between *T* and *B*

The following proposition describes the global and local structure of the set of no-trade equilibria T as a subset of the equilibrium manifold E.

Proposition 3.2.1 The set of no-trade equilibria T is a smooth submanifold of the equilibrium manifold E that is diffeomorphic to the set of price-income vectors $B = S \times \mathbb{R}^m_{++}$.

First, a few words about the proof. One strategy is to start by proving the smooth submanifold structure with the regular value theorem (see exercise 3.2). Once the smooth manifold structure is established, it then suffices to find two smooth maps that are inverse to each other.

Here, I follow a different route because it will be useful in a number of other situations. For this, I exploit a remarkable property of *smooth embeddings.*

By definition, an *embedding* is a smooth map that, in addition to being an *immersion*, is a homeomorphism with its image. An immersion is a smooth map whose derivative (or tangent map) is an injection. Not all immersions are embeddings. An embedding has the remarkable property that its image (which is homeomorphic to the domain of the embedding) is then a smooth submanifold of its range and the homeomorphism between the domain and the image of the map is actually a diffeomorphism. Showing that a map is an embedding is therefore a very neat method for proving that the image of that embedding is actually a smooth submanifold of the range in addition to having the diffeomorphism property. Another modest though real advantage of the embedding map approach compared to the more elementary one through the regular value theorem is that it also works for a setup of demand functions that are just continuous instead of being differentiable.

The main difficulty with the embedding map approach is in proving that a suitable map is an embedding. The following lemma comes in handy.

Lemma 3.2.2 Let $\phi : X \to Y$ and $\psi : Y \to X$ be two smooth mappings between smooth manifolds such that the composition $\phi \circ \psi : Y \to Y$ is the identity map of Y. Then the map $\psi : Y \to X$ is an embedding. The image $Z = \psi(Y)$ is a smooth submanifold of X that is diffeomorphic to Y.

Proof See exercise 3.1. ■

Let us now prove the smooth submanifold structure of the set of no-trade equilibria T and the diffeomorphism with $B = S \times \mathbb{R}^m_{++}$ by exhibiting an embedding from B into E whose image is the set T.

3.2.2 Proof of the Diffeomorphism

Let $(p, w_1, w_2, \dots, w_m) \in B = S \times \mathbb{R}^m_{++}$ be a price-income vector. Let the map $f : B \to S \times \Omega$ be defined by the formula

$$f(p, w_1, w_2, \dots, w_m) = (p, f_1(p, w_1), f_2(p, w_2), \dots, f_m(p, w_m)).$$

The individual demand functions $f_1, f_2, \dots, f_m$ are smooth functions of $p \in S$ and $w_1, w_2, \dots, w_m$, respectively. This implies that the map f is also smooth.

Let $\varphi : S \times \Omega \to B = S \times \mathbb{R}^m_{++}$ be the map defined by the formula

$$\varphi(p, \omega_1, \omega_2, \dots, \omega_m) = (p, p \cdot \omega_1, p \cdot \omega_2, \dots, p \cdot \omega_m).$$

This map is also smooth because all its coordinates are smooth functions of p and $\omega_1, \omega_2, \ldots, \omega_m$.

Let us show that (1) the set of no-trade equilibria is the image of the map f, i.e., $f(B) = T$, and (2) the map $f : B \to S \times \Omega$ is an embedding.

Inclusion $f(B) \subset T$

Let $(p, f_1(p, w_1), \ldots, f_m(p, w_m))$ be an element of $f(B)$. Let $\omega_1 = f_1(p, w_1), \ldots, \omega_i = f_i(p, w_i), \ldots, \omega_m = f_m(p, w_m)$. Then, for i varying from 1 to m, we have

$$f_i(p, p \cdot \omega_i) = f_i(p, p \cdot f_i(p, w_i)).$$

By Walras' law, we have $p \cdot f_i(p, w_i) = w_i$, so that the right-hand term is equal to $f_i(p, w_i)$, hence to ω_i. This proves the equality $\omega_i = f_i(p, p \cdot \omega_i)$ for $i = 1, 2, \ldots, m$. In other words, $(p, \omega_1, \ldots, \omega_m)$ is a no-trade equilibrium, which establishes the inclusion $f(B) \subset T$.

Inclusion $T \subset f(B)$

Let $(p, \omega_1, \ldots, \omega_m)$ be a no-trade equilibrium such that $\omega_i = f_i(p, p \cdot \omega_i)$ for i varying from 1 to m. Define $w_i = p \cdot \omega_i$ for $1 \leq i \leq m$. We then have $\omega_i = f_i(p, w_i)$, from which it follows that we have $(p, \omega_1, \ldots, \omega_m) = f(p, w_1, \ldots, w_m)$, hence the inclusion $T \subset f(B)$.

Composition $\varphi \circ f$

Let $b = (p, w_1, \ldots, w_m) \in B = S \times \mathbb{R}^m_{++}$. By definition, we have

$$f(b) = (p, f_1(p, w_1), \ldots, f_m(p, w_m)).$$

Then

$$\varphi(f(b)) = (p, p \cdot f_1(p, w_1), \ldots, p \cdot f_m(p, w_m)).$$

The right-hand term is equal to $b = (p, w_1, \ldots, w_m)$, as follows from Walras' law applied to each individual demand function. This proves the identity $\varphi \circ f = \mathrm{id}_B$, where id_B is the identity map of $B = S \times \mathbb{R}^m_{++}$.

We conclude that the map $f : B \to S \times \Omega$ is an embedding, as an application of lemma 3.2.2.

The set of no-trade equilibria T is therefore a smooth submanifold of $S \times \Omega$ that is diffeomorphic to $B = S \times \mathbb{R}^m_{++}$. Its dimension is equal to $\ell + m - 1$. This ends the proof of proposition 3.2.1. ∎

A particularly obvious consequence of proposition 3.2.1 is that the set of no-trade equilibria is nonempty. But this set is not really a large subset of the equilibrium manifold. The equilibrium manifold E has dimension ℓm, whereas the submanifold of no-trade equilibria T has dimension $\ell + m - 1$, which is strictly less than ℓm. The measure of T is therefore zero in E. In particular, the interior of T is empty.

The measure zero property implies that an equilibrium picked at random has a probability equal to zero of being a no-trade equilibrium. This property is consistent with the intuition that no-trade equilibria are unlikely to be observed in the real world.

The concept of no-trade equilibrium corresponds indeed to an extreme situation. Extreme situations are hardly observable in practice. Their importance lies in the fact that real-world situations can be sufficiently close to the extreme situations to have many of their properties. For example, the perfect vacuum is a very useful concept in physics. It has nice properties in the sense that many laws of physics have simple formulations in the case of the perfect vacuum. However, the perfect vacuum does not really exist. At best, the density of the medium can be sufficiently small for the resistance to motion to be negligible. Such medium then satisfies the same properties as the perfect vacuum, at least as a first approximation. The situation is the same in economics. No-trade equilibria may never be observed in practice. Nevertheless, their properties are shared by the equilibria where the intensity of trade is sufficiently small for the approximation to work.

3.2.3 Relation between *T* and the Set of Equilibrium Allocations

Another reason for being interested in the set of no-trade equilibria is its relation with the set of equilibrium allocations, as follows from proposition 3.2.3.

Proposition 3.2.3 The set of equilibrium allocations is the image $\pi(T)$ of the set of no-trade equilibria by the natural projection.

Proof The vector $x = (x_i) \in \Omega$ is an equilibrium allocation if there exists an endowment vector $\omega = (\omega_i)$ and a price vector $p \in S$ such that the corresponding equilibrium allocation is equal to x. This is equivalent to having $x_i = f_i(p, p \cdot \omega_i)$ for $i = 1, 2, \ldots, m$. But then it follows from Walras' law that we have $p \cdot \omega_i = p \cdot x_i$, which implies that the equality $x_i = f_i(p, p \cdot x_i)$ is satisfied for $i = 1, 2, \ldots, m$. Conversely, if (p, x) is a no-trade equilibrium, then obviously x is the equilibrium allocation associated with the price vector $p \in S$ and the endowment vector $x \in \Omega$. ∎

Exercises

3.1. Let $\phi : X \to Y$ and $\psi : Y \to X$ be two smooth mappings between smooth manifolds such that the composition $\phi \circ \psi : Y \to Y$ is the identity map of Y.
a. Show that the map $\psi : Y \to X$ defines a homeomorphism between its domain Y and its image $Z = \psi(Y)$. (*Hint:* Show that the restriction $\phi \,|\, Z$ is a bijection.)
b. Show that the map $\psi : Y \to X$ is an immersion, i.e., the derivative $d\psi_y : T_y(Y) \to T_{\psi(y)}(X)$ is an injection. (*Hint:* The relation $\phi \circ \psi = id_Y$ yields for the tangent maps (or derivatives) of ϕ and ψ the equality $d\phi_{\psi(y)} \circ d\psi_y = \mathrm{id}_{T_y(Y)}$ (chain rule), where $T_y(Y)$ denotes the tangent space to the manifold Y at y.)

3.2. The set of no-trade equilibria T is the subset of the equilibrium manifold E defined by the equations $\omega_i = f_i(p, p \cdot \omega_i)$ for $i = 1, \ldots, m$. Using the regular value theorem, give an alternative proof that T is a smooth submanifold of $S \times \Omega$ and of E.

3.3. Apply the smooth submanifold structure of the set of no-trade equilibria T proved in exercise 3.2 to simplify the proof of proposition 3.2.1 that T is diffeomorphic to $B = S \times \mathbb{R}^m_{++}$. Check that the individual demand functions $f_i : S \times \mathbb{R}_{++} \to X$ (for $i = 1, \ldots, m$) have only to be smooth and satisfy Walras' law.

3.4.
a. Show that the set of Pareto optima P is the subset of Ω defined by the equation system

$$D_n u_1(x_1) = D_n u_2(x_2) = \cdots = D_n u_m(x_m),$$

where $D_n u_i(x_i)$ denotes the normalized gradient of function u_i.
b. Apply the regular value theorem to prove that P is a smooth submanifold of Ω in the case $(\ell, m) = (2, 2)$.
c. Apply now the regular value theorem to the general case of (m, ℓ) arbitrary. (This approach yields a proof based on the regular value theorem that the set of Pareto optima P is a smooth submanifold of Ω of dimension $\ell + m - 1$.)

3.5. Define in $\Omega \times \mathbb{R}^{m-1}$ the set Q of elements $(x_1, x_2, \ldots, x_m, \lambda_2, \ldots, \lambda_m)$ satisfying the equation system

$$Du_1(x_1) = \lambda_2 Du_2(x_2) = \cdots = \lambda_m Du_m(x_m).$$

a. Prove using the regular value theorem that Q is a smooth submanifold of $\Omega \times \mathbb{R}^{m-1}$ of dimension $\ell + m - 1$.

b. Show that the set of Pareto optima P is the image of Q by the projection map $\Omega \times \mathbb{R}^{m-1} \to \Omega$. Let $\rho : Q \to \Omega$ denote the restriction of this projection map to Q.
c. Show that the map $\Omega \to \Omega \times \mathbb{R}^{m-1}$ defined by

$$(x_1, x_2, \dots, x_m) \to (x_1, x_2, \dots, x_m, \lambda_2, \dots, \lambda_m),$$

where

$$\lambda_2 = \frac{\dfrac{\partial u_2}{\partial x^\ell}(x_2)}{\dfrac{\partial u_1}{\partial x^\ell}(x_1)}, \dots, \lambda_m = \frac{\dfrac{\partial u_m}{\partial x^\ell}(x_m)}{\dfrac{\partial u_1}{\partial x^\ell}(x_1)},$$

is smooth. Using this map and the map ρ of exercise 3.5b, show that the set of Pareto optima P is a smooth submanifold of Ω that is diffeomorphic to Q. (This approach gives an alternative proof based on the property that the map $\rho : Q \to \Omega$ is an embedding.)

3.3 The Two Theorems of Welfare Economics Revisited

The existence of a subset of the equilibrium manifold E like the set of no-trade equilibria T adds a distinctive feature to the natural projection $\pi : E \to \Omega$, a feature that is absent from all the other models of catastrophe theory. In addition, the economic interpretation of the image $\pi(T)$ of the set of no-trade equilibria T is of the utmost importance because this set coincides with the set of equilibrium allocations. The two theorems of welfare economics give another interpretation of the set of equilibrium allocations by identifying it with the set of Pareto optima.

3.3.1 Set of Pareto Optima

By definition, the allocation $x = (x_i) \in \Omega$ is a *Pareto optimum* if it is not *Pareto-dominated* by some $x' = (x'_i) \in \Omega$, in other words, if there exists no $x' = (x'_i) \in \Omega$ that satisfies the inequalities $u_i(x_i) \leq u_i(x'_i)$ for every i, one inequality at least being strict, and $\sum_i x'_i = \sum_i x_i$. Let P denote the set of Pareto optima. It is a subset of Ω.

3.3.2 The Two Theorems of Welfare Economics

The relations between the set of equilibrium allocations $\pi(T)$ and the set of Pareto optima P are encapsulated in the two *theorems of welfare economics.* The *first theorem* states that every equilibrium allocation is a Pareto optimum. This is equivalent to the inclusion $\pi(T) \subset P$.

The *second theorem* states that every Pareto optimum is the equilibrium allocation of some economy. This is equivalent to the inclusion $P \subset \pi(T)$.

The combination of the two welfare theorems therefore amounts to the equality $\pi(T) = P$.

3.3.3 Diffeomorphism between *T* and *P*

The equality $\pi(T) = P$ is one important aspect, but one aspect only, of the following more general property of the natural projection $\pi : E \to \Omega$.

Proposition 3.3.1 The restriction of the natural projection $\pi : E \to \Omega$ to the submanifold T of no-trade equilibria defines a diffeomorphism between T and the set of Pareto optima P.

Proof We already know that the set of no-trade equilibria is a smooth submanifold of E. Therefore, the restriction $(\pi \mid T) : T \to \Omega$ is also a smooth map.

Let $x = (x_1, \ldots, x_m) \in P$. Let $D_n u_1(x_1)$ denote the numeraire normalized gradient vector associated with the utility function u_1 at $x_1 \in X$. (See exercise 1.1 in chapter 1 for the definition of the normalized gradient.) It follows from $x = (x_1, \ldots, x_m)$ being a Pareto optimum that the price vector $p = D_n u_1(x_1)$ supports the allocations $x_1, x_2, \ldots, x_m$, in other words, we have $x_1 = f_1(p, p \cdot x_1), x_2 = f_2(p, p \cdot x_2), \ldots, x_m = f_m(p, p \cdot x_m)$.

Let $\psi : \Omega \to S \times \Omega$ be the map

$$\psi(x_1, x_2, \ldots, x_m) = (D_n u_1(x_1), x_1, x_2, \ldots, x_m).$$

This map is clearly smooth. Its restriction to the set of Pareto optima is continuous. In addition, the image of the Pareto optimum $x = (x_1, x_2, \ldots, x_m)$ by ψ is the no-trade equilibrium $(p, x_1, x_2, \ldots, x_m)$ with $p = D_n u_1(x_1)$. The composition $\pi \circ (\psi \mid P)$ is the identity map of P, and the composition $\psi \circ (\pi \mid T)$ is the identity map of T. Therefore, the smooth proper map $(\pi \mid T)$ defines a homeomorphism between T and its image $\pi(T) = P$. This proves that the set of Pareto optima P is a smooth submanifold of Ω, a submanifold that is diffeomorphic to T (figure 3.1). ∎

The following corollary is obvious.

Corollary 3.3.2 The set of Pareto optima P is a submanifold of Ω diffeomorphic to $B = S \times \mathbb{R}^m_{++}$.

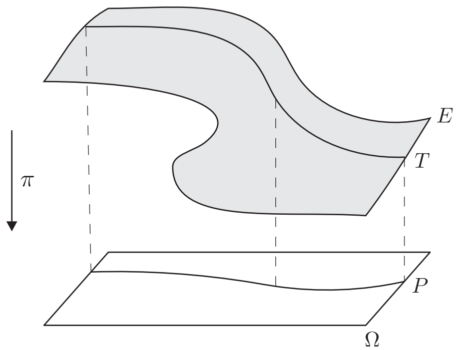

Figure 3.1
Diffeomorphism between T and P

It follows from the corollary that the set of Pareto optima is also diffeomorphic to $\mathbb{R}^{\ell+m-1}$.

The map $(\pi \mid T) : T \to P$ is a diffeomorphism. Let $\psi : \Omega \to S$ be the map $\psi(\omega) = D_n u_1(\omega_1)$. This map is smooth. Its restriction to the submanifold P of Ω is therefore smooth. (Incidentally, this map associates with every Pareto optimum its supporting price vector.) The map $\omega \to (\psi(\omega), \omega)$ is then the inverse map $(\pi \mid T)^{-1} : P \to S$.

The inclusion $T \subset \pi^{-1}(P)$ follows from the equality $P = \pi(T)$. In order for the equality $T = \pi^{-1}(P)$ to be satisfied, it is necessary that the economy $\omega \in P$ have a unique equilibrium. This property (see chapter 4) is *not* a direct consequence of proposition 3.3.1.

3.4 Notes and Comments

The first mention in the literature of an equilibrium without trade is in Arrow and Hurwicz (1958). Despite their importance to a number of issues that go beyond the diffeomorphism between the sets of no-trade equilibria and Pareto optima (see chapter 4), no further mention of no-trade equilibria is found in the literature until Balasko (1973; 1975a; 1975b) showed that the structure of the equilibrium manifold depends heavily on the structure of the set of these no-trade equilibria.

The proof given in this chapter that the set of no-trade equilibria is diffeomorphic to $\mathbb{R}^{\ell+m-1}$ in the case of variable total resources is due to Balasko (1973; 1975b). This proof requires only that individual demand functions exist, be smooth, and satisfy Walras' law. These functions do not have to result from utility maximization subject to budget constraints. This feature is useful in several extensions of the Arrow-Debreu model (see, e.g., chapter 6, which deals with price-dependent preferences).

The application of the structure of the set of no-trade equilibria to the structure of the set of Pareto optima in the case of variable total resources is also due to Balasko (1975a; 1978b). The case where total resources are fixed needs a wholly different treatment (see chapter 5).

4 The Global Structure of the Equilibrium Manifold

We have seen that the equilibrium manifold E comes equipped with local coordinate systems because it is a smooth submanifold of the Cartesian product $S \times \Omega$. The existence of a global coordinate system boils down to whether the equilibrium manifold E is homeomorphic or, even better, diffeomorphic to some Euclidian space. Short of proving such diffeomorphism, several global topological properties like pathconnectedness, simple connectedness, and contractibility are important steps in that direction. Though weaker than the homeomorphism or diffeomorphism properties, these topological properties are also interesting for their own sake. This is particularly true of pathconnectedness and simple connectedness. The question is therefore whether the equilibrium manifold E is pathconnected, simply connected, contractible, homeomorphic, and diffeomorphic to some Euclidean space, the latter property implying all the others but at the price of a much more complex proof.

This chapter is divided into roughly three parts. In the first part, I give some intuition for the mathematical concepts of pathconnectedness, simple connectedness, and contractibility. The interest of these properties is not limited to the equilibrium manifold. It extends to subsets of the equilibrium manifold and other economically interesting sets that satisfy one or several of these properties. The second part of this chapter is devoted to proving these global properties of the equilibrium manifold. This part exploits the partition of the equilibrium manifold into linear fibers that are “glued” together by the no-trade equilibria. It ends with the specification of two coordinate systems for the equilibrium manifold. The third part of this chapter discusses the application of these coordinate systems to the study of the natural projection and to the study of subsets of the equilibrium manifold that are associated with properties of equilibria such as tatonnement stability or gross substitutability.

4.1 Some Mathematical Concepts

4.1.1 Pathconnectedness

A topological space is *pathconnected* if it is always possible to link two points of that space by a continuous path. For example, every convex set is pathconnected. Indeed, the segment defined by two points is contained in the set and therefore defines a continuous path linking the two points.

Can there be an economic interpretation for pathconnectedness? Consider two equilibria (p, ω) and (p^*, ω^*). Assume, for example, that (p, ω) describes a current state of the economy and (p^*, ω^*) is a state to be achieved at some point in the future.

The problem then becomes one of having the economy, represented by the endowment vector and its associated equilibrium price vector, evolve from the initial state (p, ω) to the final one (p^*, ω^*). If discontinuities are allowed, there are many ways of moving from (p, ω) to (p^*, ω^*). But discontinuities do not have a good reputation, for many reasons. For example, they are incompatible with any reasonnable way of forecasting future prices. Often discontinuities are associated in economics with wars and revolutions, whereas reforms imply a more gradual or continuous evolution of the economy. Therefore, we take for granted that any economic policy that translates into a continuous evolution is preferred to any policy that can generate discontinuities.

Then there is the question of choosing which path to follow in order to move from (p, ω) to (p^*, ω^*). A natural idea is to choose a path with the shortest length. Mathematically, this implies that the equilibrium manifold comes equipped with a suitable notion of length. Manifolds with such structure are known as *Riemannian manifolds*. Paths with minimal length are called the *geodesics* of the Riemannian manifold. Incidentally, every submanifold of a Euclidean space is equipped with a Riemannian metric defined by the Euclidean distance of the Euclidean space. In our case, this distance would be the Euclidean distance on $S \times \Omega$. The only problem—but this is not a small problem—is that this distance does not make much economic sense. The structure of the Arrow-Debreu model is not rich enough to feature a distance concept that would be sufficiently relevant for the purpose of economic policy and planning.

Instead of seeking the shortest path from one point to another in a given metric, we can at least investigate whether there exist continuous paths linking the two points. This amounts to investigating the path-connectedness properties of the equilibrium manifold E.

Proposition 4.1.1 The equilibrium manifold E is pathconnected.

A by-product of the pathconnectedness of the equilibrium manifold is a new contribution to the centuries-old debate of revolution versus reform. Let us understand reform as the property of controlling equilibria continuously, which implies the potential of following any continuous path on the equilibrium manifold. Let us understand revolution as the imposition of discontinuous changes in the economy, translating into jumps from one equilibrium to another. Revolution is therefore unavoidable if the end point (the equilibrium (p^*, ω^*)) cannot be attained by following a continuous path from the initial point (the equilibrium (p, ω)), or in mathematical terms, if the initial and end points do not belong to the same pathconnected component of the equilibrium manifold. This argument disappears with pathconnectedness. Of course, the scope of this observation is limited to the setup of the Arrow-Debreu model. A less superficial analysis would require more complex models than the Arrow-Debreu model. But, even more for such models, the global structure of their equilibrium manifold would convey many important properties.

4.1.2 Simple Connectedness

It follows from pathconnectedness that there exists at least one continuous path linking one point of E to any other. But more than one such path exists in general, and it is natural to try to compare these different paths. In particular, is it possible to deform continuously the first path into the second one? Having this property satisfied for all paths linking any two points is known in mathematics as *simple connectedness*.

Again, is it possible to give an economic interpretation of simple connectedness? A path in the equilibrium manifold can be identified with an economic policy. The choice of a policy is often the outcome of arbitrations and compromises starting from options that initially are quite different. It is natural to model this process of compromises and arbitration as small steps that are represented mathematically by continuous changes of the paths corresponding to the policies.

Proposition 4.1.2 The equilibrium manifold E is simply connected.

Simple connectedness also has a purely geometric interpretation: a connected manifold is simply connected if it has no “holes.” For example, the sphere is simply connected, whereas the torus is not.

4.1.3 Contractibility

The two properties of pathconnectedness and simple connectedness are found in an important class of topological spaces, namely, the contractible spaces. A topological space is contractible if it can be continuously deformed to just one point. Let a and b be two arbitrarily chosen points. These points follow continuous paths ending in the same point under the deformation process. Therefore, a suitable combination of these two paths yields a continuous path from a to b, which proves pathconnectedness. It is also intuitively obvious that a contractible space cannot have any “hole,” hence is simply connected. An important class of contractible spaces consists of star-shaped sets, a class that includes the convex sets. Therefore, every Euclidean space $\mathbb{R}^n$ is contractible.

Proposition 4.1.3 The equilibrium manifold E is contractible.

4.1.4 Homeomorphism with $\mathbb{R}^{\ell m}$

The Euclidean space $\mathbb{R}^n$ is pathconnected, simply connected, and contractible. Therefore, all these properties of the equilibrium manifold are in fact special cases of the following proposition.

Proposition 4.1.4 The equilibrium manifold E is homeomorphic to $\mathbb{R}^{\ell m}$.

4.1.5 Diffeomorphism with $\mathbb{R}^{\ell m}$

We now know that the equilibrium manifold E is a smooth manifold that is homeomorphic to the Euclidean space $\mathbb{R}^{\ell m}$. Can this homeomorphism with the Euclidean space $\mathbb{R}^{\ell m}$ be smoothed to make it a diffeomorphism?

Proposition 4.1.5 The equilibrium manifold E is diffeomorphic to $\mathbb{R}^{\ell m}$.

This proposition tells us that there exists a global system of (smooth) coordinates for the equilibrium manifold. This may sound rather abstract at this stage. But, with such a coordinate system, the equilibrium manifold E can be identified to $\mathbb{R}^{\ell m}$, and the natural projection π becomes a map from $\mathbb{R}^{\ell m}$ into Ω, an open subset of $\mathbb{R}^{\ell m}$. In other words, we can express the natural projection by ℓm real-valued functions depending on ℓm variables.

The good news is that, courtesy of this global coordinate system, we can forget all the discussion in the previous sections about smooth manifolds and start with the study of the natural projection as a smooth map from some open subset of $\mathbb{R}^{\ell m}$ into $\mathbb{R}^{\ell m}$. The mathematical interest of this simplification is obvious. But this approach may also be economically interesting if the global coordinate system can have some simple economic interpretation.

The bad news is that there is no smooth global coordinate system having an economic interpretation for the equilibrium manifold E associated with the endowment space $\Omega = X^m$. But there is a way out of this difficulty. It consists in extending the equilibrium concept by allowing endowments to have some negative coordinates. The collection of extended equilibria yields the *extended equilibrium manifold* $\tilde{E}$, of which the equilibrium manifold E is an open subset. The remarkable fact is that the extended equilibrium manifold $\tilde{E}$ has, like the equilibrium manifold E, a global structure that comes with a global (smooth) coordinate system, in fact, two of them. These coordinate systems have really simple economic interpretations. They can be used for the points of the extended equilibrium manifold $\tilde{E}$ and also for the points of the equilibrium manifold E that is an open subset of R, and of course also for the natural projection.

4.2 Linear Fibers of the Equilibrium Manifold

4.2.1 Linear Fiber Associated with a Price-Income Vector

Let $b = (p, w_1, \dots, w_m) \in B = S \times \mathbb{R}^m_{++}$ be some price-income distribution. The *linear fiber* $F(b)$ is the subset of $S \times \Omega$ defined as

$$F(b) = \left\{ (p, \omega) \in S \times \Omega \,\middle|\, \begin{array}{ll} p \cdot \omega_i = w_i, & i = 1, 2, \dots, m \\ \sum_i \omega_i = \sum_i f_i(p, w_i), & \end{array} \right\}. \quad (4.1)$$

It is obvious from the equations defining the linear fiber $F(b)$ that all its elements are equilibria. The linear fiber $F(b)$ is therefore a subset of the equilibrium manifold E for all $b \in B = S \times \mathbb{R}^m_{++}$.

4.2.2 Partition of the Equilibrium Manifold into Its Linear Fibers

Let $(p, \omega) \in E$ be an equilibrium. Then there is a unique linear fiber through that equilibrium (p, ω). It is the linear fiber $F(b)$ for $b = \varphi(p, \omega) = (p, p \cdot \omega_1, \dots, p \cdot \omega_m)$.

The union $\bigcup_{b \in B} F(b)$ is equal to the equilibrium manifold E because every equilibrium belongs to some linear fiber. In addition, for $b \neq b'$, the intersection $F(b) \cap F(b')$ is empty. This implies that the equilibrium manifold is partitioned into its linear fibers.

4.2.3 Uniqueness of the No-Trade Equilibrium in a Linear Fiber

Let $F(b)$ be the linear fiber associated with the price-income vector $b = (p, w_1, \ldots, w_m) \in B$. We then have for the intersection of the linear fiber $F(b)$ with T the equality

$$F(b) \cap T = \{f(b)\}.$$

The argument goes as follows. Let $(p', x') \in T \cap F(b)$. Necessarily, we have $p' = p$ and $p' \cdot x'_i = p \cdot x'_i = w_i$ for $i = 1, \ldots, m$. Since (p', x') is a no-trade equilibrium, we therefore have $x'_i = f_i(p', p' \cdot x'_i) = f_i(p, w_i)$ for $i = 1, \ldots, m$. This implies $(p', x') = (p, f_1(p, w_1), \ldots, f_m(p, w_m)) = f(b)$. Conversely, $f(b)$ is a no-trade equilibrium and belongs to the linear fiber $F(b)$.

4.2.4 Linear Fibers and the Set of No-Trade Equilibria

The linear fibers $F(b)$ that make up the equilibrium manifold E are parameterized by the price-income vector $b \in B = S \times \mathbb{R}^m_{++}$. In the same way, the set of no-trade equilibria T consists of the elements $f(b)$, where b is varied in $B = S \times \mathbb{R}^m_{++}$. Since the no-trade equilibrium $f(b)$ is the unique no-trade equilibrium of the linear fiber $F(b)$, there is one and only one fiber through every element of the set of no-trade equilibria. These fibers can also be viewed as parameterized by the elements

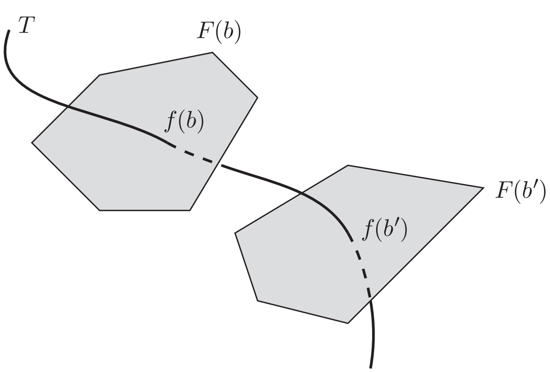

Figure 4.1
No-trade equilibria and linear fibers

of the set of no-trade equilibria T. They appear to be "glued" together by the no-trade equilibria that belong to the set T (figure 4.1).

Unsurprisingly, the structural properties of the equilibrium manifold E are going to reflect those of the set of no-trade equilibria T and those of the linear fibers $F(b)$, fibers that have more in common than their linearity.

4.2.5 Convexity and Boundedness of the Linear Fiber $F(b)$

The linear fiber $F(b)$ is defined by the set of linear equalities corresponding to the total resource and individual wealth constraints and inequalities for the positivity requirement imposed on the endowment vector $\omega = (\omega_1, \ldots, \omega_m) \in \Omega = X^m$. It follows from $p \cdot \omega_i = w_i$ for $i = 1, \ldots, m$ combined with $\omega_i \in X = \mathbb{R}^\ell_{++}$ that the set $F(b)$ is bounded.

By definition, a *polytope* is a bounded convex subset of a Euclidean space defined by a set of linear (or better, affine) inequalities. The linear fiber $F(b)$ is therefore a *convex polytope* for every $b \in B = S \times \mathbb{R}^m_{++}$. (Note that the fiber $F(b)$ is closed in $S \times \Omega$ but not in $S \times \mathbb{R}^{\ell m}$ because the inequalities $\omega_i^j > 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, \ell$ are strict.)

The linearity and convexity of the typical fiber is the second key factor in the global structure of the equilibrium manifold, the first one being the diffeomorphism of the set of no-trade equilibria to a Euclidean space. Let us recall some properties of convex polytopes: the open convex polytope W is pathconnected, simply connected, contractible, homeomorphic, and diffeomorphic to $\mathbb{R}^n$.

Pathconnectedness

Pathconnectedness is almost obvious: the map $t \in [0,1] \to (1-t)x + ty$ defines a continuous path (whose image is the segment $[x, y]$) linking the two elements x and y of the polytope W.

Simple Connectedness and Contractibility

Simple connectedness is a direct consequence of contractibility, the latter being equivalent to showing that the polytope W can be contracted continuously into a point. Pick some $x^* \in W$. The map $(t, x) \to h(t, x) = (1-t)x + tx^*$, where $h(0, .)$ is the identity map of W and $h(1, x)$ the constant map with value x^*, proves the contractibility property for W.

Homeomorphism with a Euclidean Space

The proof of the homeomorphism with $\mathbb{R}^n$, with n such that W is open in $\mathbb{R}^n$, can go as follows. Let again x^* in W. Let $B(x^*; r)$ be the open ball centered at x^* with radius $r > 0$ (figure 4.2).

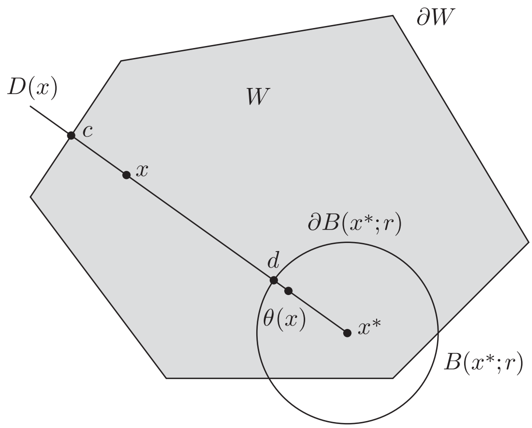

Figure 4.2
Homeomorphism between W and $B(x^*; r)$

Let $x \in W$, with $x \neq x^*$. Denote by $D(x)$ the half-line defined by the two points x and x^* with origin x^*. Let c and d be the intersection points of $D(x)$ with the boundaries ∂W of W and $\partial B(x^*; r) = S(x^*; r)$, the latter set being the sphere of radius r centered at x^*. The map $\theta : W \to B(x^*; r)$, defined by the formula

$$\theta(x) = x^* + \frac{\|d - x^*\|}{\|c - x^*\|}(x - x^*), \tag{4.2}$$

is easily seen to be a homeomorphism between W and $B(x^*; r)$. Note that the same map also defines a homeomorphism between the compact set $\overline{W} = W \cup \partial W$, which is the closure of W, and the closed ball $\overline{B(x; r)}$.

If, instead of being a convex polytope, the boundary of the convex set W is a smooth hypersurface like a sphere, then the map θ is a diffeomorphism between W and any open ball of $\mathbb{R}^n$ and therefore $\mathbb{R}^n$ itself.

Diffeomorphism with a Euclidean Space

There seems to be no elementary constructive proof of the diffeomorphism property between W and $\mathbb{R}^n$ when the boundary of the convex set W is not a smooth hypersurface. Unfortunately, this is the case with convex polytopes. Diffeomorphism, however, can be proved by an approximation argument, an approach that is nonconstructive, which makes it rather impractical for an explicit coordinate system.

4.3 Global Properties of the Equilibrium Manifold

We are now ready to address the global properties of the equilibrium manifold E stated in propositions 4.1.1–4.1.5. Intuitively, the proofs consist in showing that the properties of the fibers as convex polytopes can be extended to the full equilibrium manifold through the parameterization by the no-trade equilibria. In addition, these extensions are rather straightforward.

4.3.1 Pathconnectedness

Let (p, ω) and (p', ω') be two equilibria. Let us construct a continuous path linking these two equilibria and contained in the equilibrium manifold E. Let $b = \phi(p, \omega)$ and $b' = \phi(p', \omega')$, and let $f(b)$ and $f(b')$ be the corresponding no-trade equilibria. The equilibria (p, ω) and $f(b)$ belong to the same fiber $F(b)$. The segment $[(p, \omega), f(b)]$ is contained in the convex fiber $F(b)$. Similarly, the segment $[(p', \omega'), f(b')]$ is contained in the convex fiber $F(b')$. To construct a continuous path linking the no-trade equilibria $f(b)$ and $f(b')$, it suffices to consider the segment $[b, b']$ linking b and b' in $B = S \times \mathbb{R}^m_{++}$. The image $f([b, b'])$ of this segment by the continuous map f then defines a continuous path linking $f(b)$ and $f(b')$ in the set of no-trade equilibria T. Piecing together the three continuous paths $[(p, \omega), f(b)]$, $f([b, b'])$ and $[f(b'), (p', \omega')]$ defines a continuous path contained in the equilibrium manifold E and linking the equilibria (p, ω) and (p', ω') (figure 4.3).

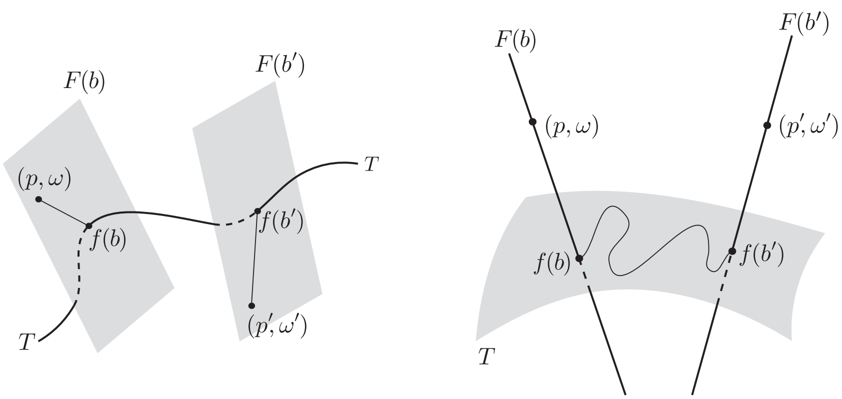

Figure 4.3
Pathconnectedness of the equilibrium manifold

4.3.2 Simple Connectedness and Contractibility

We prove simple connectedness as a consequence of contractibility, and we prove contractibility by showing that the equilibrium manifold E can be continuously deformed into the set of no-trade equilibria T. In fact, we are going to show that the subset T is a *deformation retract* of the set E, which is equivalent to the existence of a continuous map $h : E \times [0,1] \to E$ having the following properties:

(i) The partial map $h(.,0) : E \to E$ is the identity map.

(ii) The partial map $h(.,1) : E \to E$ takes its values in the set T.

(iii) The restriction of the map $h(.,t) : E \to E$ for $t \in [0,1]$ to the subset T is the identity of T.

Let $x = (p, \omega_1, \ldots, \omega_m)$ be an element of E. The maps (already defined) $\varphi : E \to B = S \times \mathbb{R}^m_{++}$ and $f : B \to E$ are continuous. In addition, we have $f(B) = T$.

Let $h(x,t)$ for $x \in E$ and $t \in [0,1]$ be defined by the expression

$$h(x,t) = (1-t)x + tf \circ \varphi(x).$$

This map is continuous as a linear combination of continuous maps. We have $h(x,0) = x$ and $h(x,1) = f(\varphi(x)) \in T$. Last, let $x \in T$. Then there exists some $b = (p, w_1, \ldots, w_m) \in B$ with $x = f(b)$. Then

$$\begin{aligned} h(x,t) &= h(f(b),t) \\ &= (1-t)f(b) + tf(\varphi(f(b))), \end{aligned}$$

from which follows by $\varphi \circ f = \mathrm{id}_B$

$$\begin{aligned} h(x,t) &= (1-t)f(b) + tf(b) \\ &= f(b) = x. \end{aligned}$$

This proof shows us that the equilibrium manifold E can be continuously deformed into the set of no-trade equilibria T. Note that the contraction map $h(.,t)$, for $t \in [0,1]$ given, contracts every fiber in the direction of the no-trade equilibrium of that fiber.

Now it suffices to observe that the set of no-trade equilibria T itself is contractible, i.e., can be continuously deformed into a single point. The combination of these two contractions yields a continuous deformation of the equilibrium manifold E into a point.

4.3.3 Homeomorphism with a Euclidean Space

Let $r > 0$ be given. We extend the definition of the map θ used to prove the homeomorphism of the open convex set W with the ball $B(x^*; r)$ to the case of the equilibrium manifold. For $x = (p, \omega) \in E$ and $b = \varphi(x)$, let $x^* = f(b)$. Then x and x^* belong to the same fiber $F(b)$. Define the map $\theta_E : E \to B \times B(0; r^*)$ by the formula

$$\theta_E(x) = \left(\varphi(x), \frac{\|d - x^*\|}{\|c - x^*\|}(x - x^*) \right). \tag{4.3}$$

This map is easily seen to be a homeomorphism. It then suffices to observe that both B and $B(0; r^*)$ are homeomorphic to Euclidean spaces, and so is their Cartesian product.

4.3.4 Diffeomorphism with a Euclidean Space

The idea of parameterizing the homeomorphisms of every fiber with the open ball $B(0; r^*)$ in a continuous way so as to get a homeomorphism of the equilibrium manifold E with the Cartesian product $B \times B(0; r^*)$ works also for the case of diffeomorphisms. For a proof along these lines, including the construction of the diffeomorphisms of every fiber with the open ball $B(0; r^*)$, see Schecter (1979). This line of approach is not pursued here because, despite its interest, it does not give a practical coordinate system for the equilibrium manifold.

4.4 Coordinates for the Equilibrium Manifold

What we want is a coordinate system that can be practical. For example, the coordinate system must enable us to work with an explicit formulation of the natural projection π as a map from some open subset of $\mathbb{R}^{\ell m}$ into $\mathbb{R}^{\ell m}$. Such formulation will also enable us to compute the Jacobian matrices that characterize critical equilibria and economies.

4.4.1 Partition in Fibers of the Equilibrium Manifold and the Coordinate System

Elements of the equilibrium manifold E can be identified by (1) the linear fiber to which they belong, and (2) their location within the fiber. More specifically, the equilibrium $(p, \omega) \in E$ belongs to the fiber $F(b)$, where $b = \varphi(p, \omega)$ is the price-income vector associated with $(p, \omega) \in E$. All we then need is a way to parameterize the fiber $F(b)$.

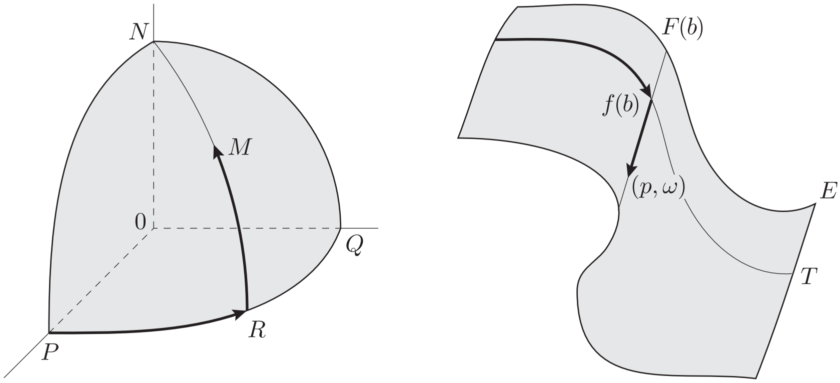

Figure 4.4
Coordinates for the sphere and the equilibrium manifold

The solution is going to be no different from the way points on the surface of the Earth (identified with a sphere) are represented by their longitude and latitude. The longitude determines the great circle that goes through the North and South poles, the great circle to which the point belongs. The fiber $F(b)$ is the analogue of this great circle, and b of the longitude (figure 4.4). The position of the point in the great circle is then given by the latitude. The next goal is therefore to define the equivalent of the latitude for every fiber of the equilibrium manifold. For this, we could use the diffeomorphism of every fiber with $\mathbb{R}^{(\ell-1)(m-1)}$. Unfortunately, the nonconstructive nature of this diffeomorphism prevents its use in any real computation. The solution is to define coordinates for a larger set than the fiber $F(b)$, which extends the equilibrium concept by enabling the parameter ω to belong to a larger set than $\Omega = X^m$.

4.4.2 Extended Equilibrium Manifold

We now drop all sign restrictions on the endowment vector $\omega = (\omega_1, \ldots, \omega_m)$. Let $\tilde{\Omega} = (\mathbb{R}^\ell)^m$ denote the *extended parameter space*.

We define an *extended equilibrium* $(p, \omega) \in S \times \tilde{\Omega}$ as a pair such that consumer i's wealth $w_i = p \cdot \omega_i$ is > 0 for $i = 1, 2, \ldots, m$ and such that $\sum \omega_i = \sum_i f_i(p, p \cdot \omega_i)$.

The *extended equilibrium manifold* $\tilde{E}$ is the subset of $S \times \tilde{\Omega}$ consisting of the extended equilibria.

The *extended fiber* $\widetilde{F(b)}$ consists of the extended equilibria compatible with the price-income vector $b = (p, w_1, \ldots, w_m) \in B = S \times \mathbb{R}^m_{++}$. It is the affine subspace of $S \times \tilde{\Omega}$ determined by the fiber $F(b)$. This amounts to extending the fiber $F(b)$ in all possible directions. The extended fiber $\widetilde{F(b)}$ can be identified with $\mathbb{R}^{(\ell-1)(m-1)}$. The way this identification is made will provide coordinates for the extended equilibrium manifold $\tilde{E}$ and, as a by-product, for the equilibrium manifold E, which is a subset of $\tilde{E}$.

Let $\bar{\omega}_i = (\omega_i^1, \ldots, \omega_i^{\ell-1}) \in \mathbb{R}^{\ell-1}$ denote the projection of ω_i on the subspace generated by the first $\ell - 1$ coordinates of $\mathbb{R}^\ell$.

Proposition 4.4.1 The extended equilibrium manifold $\tilde{E}$ is diffeomorphic to $B \times \mathbb{R}^{(\ell-1)(m-1)}$ by the map

$$\Theta(p, \omega_1, \ldots, \omega_m) \to (\varphi(p, \omega), \bar{\omega}_1, \ldots, \bar{\omega}_{m-1}).$$

Let us also denote by $\Theta : S \times \tilde{\Omega} \to S \times \mathbb{R}^m \times \mathbb{R}^{(\ell-1)(m-1)}$ the map with the same formula as in the proposition.

Let $\Psi : S \times \mathbb{R}^m_{++} \times \mathbb{R}^{(\ell-1)(m-1)}$ be the map where $p \in S$, ω_i^j are left unchanged for $1 \le i \le m-1$ and $1 \le j \le \ell - 1$ and the coordinates ω_i^ℓ for $1 \le i \le m-1$ are defined by the formula

$$\left.\begin{aligned}
\omega_1^\ell &= w_1 - p_1\omega_1^1 - \cdots - p_{\ell-1}\omega_1^{\ell-1} \\
\omega_2^\ell &= w_2 - p_1\omega_2^1 - \cdots - p_{\ell-1}\omega_2^{\ell-1} \\
\vdots &= \qquad \vdots \\
\omega_{m-1}^\ell &= w_{m-1} - p_1\omega_{m-1}^1 - \cdots - p_{\ell-1}\omega_{m-1}^{\ell-1}
\end{aligned}\right\},$$

the coordinates $\omega_m^1, \ldots, \omega_m^{\ell-1}$ by the formula

$$\left.\begin{aligned}
\omega_m^1 &= r^1 - \omega_1^1 - \omega_2^1 - \cdots - \omega_{m-1}^1 \\
\omega_m^2 &= r^2 - \omega_1^2 - \omega_2^1 - \cdots - \omega_{m-1}^2 \\
\vdots &= \qquad \vdots \\
\omega_m^{\ell-1} &= r^{\ell-1} - \omega_1^{\ell-1} - \omega_2^{\ell-1} - \cdots - \omega_{m-1}^{\ell-1}
\end{aligned}\right\},$$

and finally ω_m^ℓ by

$$\omega_m^\ell = w_m - p_1\omega_m^1 - \cdots - p_{\ell-1}\omega_m^{\ell-1}.$$

It is then straightforward to check that the extended equilibrium manifold $\tilde{E}$ is also the image $\Psi(S \times \mathbb{R}^m_{++} \times \mathbb{R}^{(\ell-1)(m-1)})$ and that the composition $\Theta \circ \Psi$ is the identity map of $S \times \mathbb{R}^m_{++} \times \mathbb{R}^{(\ell-1)(m-1)}$. This proves the diffeomorphism property. (Incidentally, this also yields as a by-product an alternative proof that the equilibrium manifold E is a smooth submanifold of $S \times \Omega$.)

4.4.3 Coordinate System A

Coordinate system A is simply the coordinate system of the extended equilibrium manifold $\tilde{E}$ defined by the diffeomorphism of proposition 4.4.1. This means that the equilibrium $(p, \omega) \in \tilde{E}$ is represented by the price-income vector $b = \varphi(p, \omega) = (p, p \cdot \omega_1, \ldots, p \cdot \omega_m)$ and the location of the equilibrium in the extended fiber $\widetilde{F(b)}$ by the coordinates $(\bar{\omega}_1, \ldots, \bar{\omega}_{m-1})$.

Coordinate system A has the advantage of simplicity. This will be particularly useful when writing the coordinates of the natural projection map.

4.4.4 Coordinate System B

The drawback of coordinate system A is that it does not lend itself to an easy identification of the no-trade equilibria and, by extension, of the proximity of a given equilibrium to the no-trade equilibrium of its fiber. This leads to consideration of an alternative coordinate system.

Let $-\bar{y}_i = \bar{f}_i(p, w_i) - \bar{\omega}_i \in \mathbb{R}^{\ell-1}$ denote the first $\ell - 1$ coordinates of the net trade vector of consumer i, with i varying from 1 to $m-1$. The matrix Y^T (the transpose of matrix Y) is then defined by

$$Y^T = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{m-1}).$$

The $(m-1) \times (\ell - 1)$ matrix Y parameterizes the extended fiber $\widetilde{F(b)}$.

To sum up, it is equivalent to represent the extended equilibrium (p, ω) by coordinate system A or by the pair (b, Y), with $b \in B$ and Y a $(m-1) \times (\ell - 1)$ matrix. This defines coordinate system B.

A nice feature of coordinate system B is that the coordinate $(b, 0)$ corresponds to the no-trade equilibrium $f(b)$ of the fiber $F(b)$.

Exercises

4.1. The topological space M is said to be connected if the only subsets that are both open and closed are the empty set $\emptyset$ and the full set M.

a. Show that the set M is connected if there are no two nonempty open subsets U and V of M with empty intersection $U \cap V = \emptyset$ and union $M = U \cup V$.

b. Show that the intervals of $\mathbb{R}$ are the only connected subsets of $\mathbb{R}$.
c. Show that $\mathbb{R}^n$ is connected.
d. Show that if the topological space M is pathconnected, then it is also connected.
e. Show that a smooth manifold that is connected is also pathconnected. (*Hint:* Use the fact that every point of the smooth manifold has a neighborhood that is diffeomorphic to a Euclidean space.)

4.2. Let M be a bounded open convex subset of $\mathbb{R}^n$. Let $x \in M$. For $t \in [0,1]$ and $y \in M$, define $h(y,t) = (1-t)y + tx$.
a. Show that this formula defines a continuous map $h(\cdot,t) : M \to M$.
b. Show that the map $h : M \times [0,1] \to M$ is continuous.
c. Check that $h(\cdot,0) = \mathrm{id}_M$ and $h(\cdot,1)$ is the constant map with value the element $x \in M$.

4.3. Let M be a bounded open convex subset of $\mathbb{R}^n$. Let $x \in M$.
a. Show that there exists some $r > 0$ such that the open ball $B = B(x;r)$ centered at $x \in M$ with radius r is contained in M.
b. Let $y \in M$ with $y \neq x$. Consider the half-line defined by the two points x and y with origin x. Show that this half-line intersects the sphere $S(x;r)$ of points of $\mathbb{R}^n$ at the distance r from x at a unique point, denoted $b(y)$. Prove that $b(y)$ is a continuous function of $y \in M$.
c. Show that the half-line considered in exercise 4.3b also intersects the boundary ∂M at a unique point $a(y)$. Prove that $a(y)$ is also a continuous function of $y \in M$.
d. Define z by the expression

$$z - x = \frac{\|b(y)\|}{\|a(y)\|}(y - x).$$

Show that z is in B if y is in M. Define the map $\theta : M \to B$ by $\theta(y) = z$. Show that the map θ is continuous.
e. Show that the map θ has an inverse $\theta^{-1} : B \to M$. Show that this inverse is continuous. (*Hint:* Give an explicit expression to the map θ^{-1} and reproduce the line of reasoning of exercise 4.3d.)
f. What can be concluded from exercises 4.3d and 4.3e regarding the topological spaces M and B?

4.4. Same assumptions and notation as in exercise 4.3.
a. Assume that the boundary ∂M of the open convex set M is a smooth hypersurface of $\mathbb{R}^n$ (i.e., a smooth submanifold of $\mathbb{R}^n$ of dimension $n-1$). Show that the map θ and its inverse θ^{-1} are smooth. What can be concluded of the topological spaces M and B?

b. Assume that the open convex set M is defined by a collection of strict linear inequalities, with $n \geq 2$. Is the map θ differentiable?

4.5 Applications of the Coordinate Systems

4.5.1 Natural Projection in Coordinate System A

Let us use coordinate system A for the equilibrium manifold E. For $\Omega = X^m \subset (\mathbb{R}^\ell)^m$, the use of coordinates $\omega_1, \omega_2, \ldots, \omega_{m-1}$, and r (instead of ω_m) turns out to simplify some computations. (We have $\omega_m = r - \omega_1 - \omega_2 - \cdots - \omega_{m-1}$.)

Proposition 4.5.1 The analytic expression of the natural projection π takes the form

$$r^1 = \sum_{i=1}^{m} f_i^1(p, w_i);$$

$$\vdots$$

$$r^{\ell-1} = \sum_{i=1}^{m} f_i^{\ell-1}(p, w_i);$$

$$\omega_1^\ell = w_1 - p_1\omega_1^1 - \cdots - p_{\ell-1}\omega_1^{\ell-1};$$

$$\vdots$$

$$\omega_{m-1}^\ell = w_{m-1} - p_1\omega_{m-1}^1 - \cdots - p_{\ell-1}\omega_{m-1}^{\ell-1};$$

$$r^\ell = w_1 + w_2 + \cdots + w_m - p_1 r^1 - p_2 r^2 - \cdots - p_{\ell-1} r^{\ell-1};$$

$$\omega_i^j = \omega_i^j \qquad \text{for } 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq \ell - 1.$$

This expression follows readily from the definition of the map $\pi : E \to \Omega$.

4.5.2 Characterization of Critical and Regular Equilibria

This section cannot avoid being computational, but details can be skipped in a first reading.

The equilibrium $(p, \omega) \in E$ is *critical* if it is a critical point of the natural projection $\pi : E \to \Omega$. This is equivalent to saying that the tangent map $T_{(p,\omega)}\pi$ is not onto. With coordinate systems for E and Ω, the matrix associated with the tangent map $T_{(p,\omega)}\pi$ is simply the Jacobian matrix of the map π expressed in these coordinates.

Similarly, the equilibrium $(p,\omega) \in E$ is *regular* if it is a regular point of the natural projection $\pi : E \to \Omega$. This is equivalent to the tangent map $T_{(p,\omega)}\pi$ being onto or to the corresponding Jacobian matrix having a rank equal to the dimension of Ω, that is, $m\ell$.

Let us define matrices $M(p,\omega)$ and $N(p,\omega)$ as follows:

$$M(p,\omega) =$$

$$\begin{bmatrix} \sum_j \dfrac{\partial f_j^1(p,w_j)}{\partial p_1} & \cdots & \sum_j \dfrac{\partial f_j^1(p,w_j)}{\partial p_{\ell-1}} & \dfrac{\partial f_1^1(p,w_1)}{\partial w_1} & \cdots & \dfrac{\partial f_m^1(p,w_m)}{\partial w_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_j \dfrac{\partial f_j^{\ell-1}(p,w_j)}{\partial p_1} & \cdots & \sum_j \dfrac{\partial f_j^{\ell-1}(p,w_j)}{\partial p_{\ell-1}} & \dfrac{\partial f_1^{\ell-1}(p,w_1)}{\partial w_1} & \cdots & \dfrac{\partial f_m^{\ell-1}(p,w_m)}{\partial w_m} \\ -\omega_1^1 & \cdots & -\,\omega_1^{\ell-1} & 1 & \cdots & 0 \\ -\omega_2^1 & \cdots & -\,\omega_2^{\ell-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\omega_{m-1}^1 & \cdots & -\,\omega_{m-1}^{\ell-1} & 0 & \cdots & 0 \\ -\omega_m^1 & \cdots & -\,\omega_m^{\ell-1} & 0 & \cdots & 1 \end{bmatrix}.$$

Matrix $N(p,\omega)$ is identical to $M(p,\omega)$ except for its last row, which is equal to

$$(-r^l, \dots, -r^{\ell-1}, 1, \dots, 1, 1).$$

Recall that $J(p,\omega)$ is the Jacobian matrix of the aggregate excess demand map $\bar{z}(.,\omega) : S \to \mathbb{R}^{\ell-1}$.

Lemma 4.5.2 $\det N(p,\omega) = \det M(p,\omega) = \det J(p,\omega)$.

Proof In $N(p,\omega)$, subtract the rows ℓ to $\ell + m - 1$ from the last row. We thus obtain the last row of $M(p,\omega)$. This proves $\det N(p,\omega) = \det M(p,\omega)$.

Matrix $M(p,\omega)$ can be given the following particular form

$$M(p,\omega) = \begin{bmatrix} A & B \\ C & I \end{bmatrix},$$

where I denotes the identity matrix. The coefficient $-\omega_i^j$ belonging to row i and column j of matrix C is “killed” by multiplying column

$((\ell-1)+i)$ of $M(p,\omega)$ by ω_i^j and adding the result to column j of $M(p,\omega)$. Performing this operation for every element of C yields matrix

$$\begin{bmatrix} J(p,\omega) & B \\ 0 & I \end{bmatrix},$$

whose determinant is equal to $\det J(p,\omega)$. ■

Proposition 4.5.3 The equilibrium (p,ω) is critical (resp. regular) if and only if $\det N(p,\omega)=0$ (resp. $\neq 0$).

Proof With the analytical expression of the map π given in proposition 4.5.1, the Jacobian matrix $T_{(p,\omega)}\pi$ of π at (p,ω) is seen to be equal to

$$\begin{bmatrix} N(p,\omega) & * \\ 0 & I \end{bmatrix},$$

where I is the $(\ell-1)(m-1)$ identity matrix. We therefore have $\det T_{(p,\omega)}\pi = \det N(p,\omega)$. ■

Proposition 4.5.4 The equilibrium $(p,\omega)\in E$ is critical (resp. regular) if and only if $\det J(p,\omega)=0$ (resp. $\neq 0$).

Proof The equilibrium $(p,\omega)\in E$ is critical if and only if $\det N(p,\omega)$ is equal to 0. It then suffices to apply the equality $\det N(p,\omega) = \det M(p,\omega) = \det J(p,\omega)$ of lemma 4.5.2. ■

Corollary 4.5.5 The economy $\omega\in\Omega$ is regular if and only if the vector $0\in\mathbb{R}^{\ell-1}$ is a regular value of the aggregate excess demand map $\bar{z}(.,\omega): S\to\mathbb{R}^{\ell-1}$.

Proof By definition, the vector $0\in\mathbb{R}^{\ell-1}$ is a regular value of the excess demand map $\bar{z}(.,\omega): S\to\mathbb{R}^{\ell-1}$ associated with the economy $\omega\in\Omega$ if and only if for every equilibrium $(p,\omega)\in E$, $\det J(p,\omega)$ is not equal to zero. This condition is equivalent to having (p,ω) not a critical point of the mapping π, which also means that ω is a regular value of the natural projection. ■

4.5.3 Jacobian Matrix of Aggregate Excess Demand $J(p,\omega)$ and Coordinate System B

Matrix $J(p,\omega)$ takes a remarkably simple form with coordinate system B, where the equilibrium (p,ω) is represented by its coordinates (b,Y).

Let $M(b) = \sum_{i=1}^{m} M_i(p, w_i)$ be the sum of the Slutsky matrices $M_i(p, w_i)$ for the m consumers, and $(\ell - 1) \times (m - 1)$ matrix $K(b)$ be with (h, i) coefficient equal to

$$k_{(h,i)} = \frac{\partial f_i^h(p, w_i)}{\partial w_i} - \frac{\partial f_m^h(p, w_m)}{\partial w_m} .$$

Proposition 4.5.6 $J(b, Y) = M(b) + K(b)Y$.

This formula follows readily from the chain rule applied to the aggregate excess demand map $p \to \bar{z}(p, \omega)$.

Corollary 4.5.7 Matrix $M(b)$ with $b = (p, w_1, \ldots, w_m) \in B$ is symmetric negative definite.

Proof Matrix $M(b)$ is the sum of the Slutsky matrices $M_1(p, w_1), \ldots,$ $M_m(p, w_m)$, which are symmetric and negative definite. Their sum is therefore symmetric and defines a quadratic form. That quadratic form is the sum of negative definite forms and is therefore negative definite. ■

Corollary 4.5.8 $\det M(b) \neq 0$.

The second corollary follows readily from the property that the matrix defining a definite quadratic form has necessarily a nonzero determinant.

4.5.4 Regularity of the No-Trade Equilibria

Proposition 4.5.6 and its corollaries give us precious information on the set of critical equilibria Γ and on its complement, the set $E(\mathfrak{R})$ of regular equilibria.

Proposition 4.5.9 Inclusion $T \subset E(\mathfrak{R})$.

In other words, every no-trade equilibrium is *regular*. The proposition is just a restatement of corollary 4.5.8.

4.6 Application to the Natural Projection

The goal is to exploit the regularity of the no-trade equilibria to obtain several additional properties of the natural projection $\pi : E \to \Omega$. We first compute the degree of that map, whether it is the degree modulo 2 (a measure of the parity of the number of equilibria at a regular economy) or the topological degree whose computation requires defining orientations for the equilibrium manifold and for Ω. The computation of these degrees requires good candidates for regular values of the natural projection.

4.6.1 Inclusion $P \subset \mathcal{R}$

We are going to show that every Pareto optimum $\omega \in P$ is a regular economy, i.e., a regular value of the natural projection.

Proposition 4.6.1 Let $\omega \in P$ and $p \in S$ its supporting price vector. Then $\pi^{-1}(\omega) = \{(p, \omega)\}$.

Proof By definition, the price vector $p \in S$ supports the Pareto optimum $\omega = (\omega_1, \ldots, \omega_m) \in P$ if the equalities $\omega_1 = f_1(p, p \cdot \omega_1), \ldots, \omega_m = f_m(p, p \cdot \omega_m)$ are all satisfied. They are equivalent to the pair (p, ω) being a no-trade equilibrium. Therefore, we have the inclusion $\{(p, \omega)\} \subset \pi^{-1}(\omega)$.

Assume now that there exists some equilibrium $(p', \omega) \in \pi^{-1}(\omega)$ with $p' \neq p$. Let $x' = (x'_i) = (f_i(p', p' \cdot \omega_i))$ denote the corresponding equilibrium allocation. It follows from the definition of consumer i's demand that the utility $u_i(x'_i)$ is greater than or equal to $u_i(\omega_i)$ for every i. In addition, since p' is different from p and does not support the allocation ω_i, the strict inequality $u_i(x'_i) > u_i(\omega_i)$ is satisfied for every i. The price vector $p' \in S$ is also an equilibrium price vector associated with ω, so $\sum_i f_i(p', p' \cdot \omega_i) = \sum \omega_i$. This proves that $x' = (x'_i)$ Pareto-dominates $\omega = (\omega_i)$, a contradiction.

This proves the equality $\pi^{-1}(\omega) = \{(p, \omega)\}$. ∎

Proposition 4.6.2 $P \subset \mathcal{R}$.

Proof The proposition follows readily from the definition of a regular value, combined with the property that the no-trade equilibria are regular. ∎

4.6.2 Modulo 2 Degree

Proposition 4.6.3 The modulo 2 degree of the natural projection is equal to 1.

Proof It suffices to pick any $\omega \in P$. This is a regular value of the map π, and the preimage $\pi^{-1}(\omega)$ contains just one element. This ends the proof of the proposition. ∎

Oddness of the Number of Equilibria of Regular Economies

Corollary 4.6.4 The number of equilibria of a regular economy $\omega \in \mathcal{R}$ is odd.

Proof The number of elements of $\pi^{-1}(\omega)$ being equal to 1 when ω is Pareto-optimal, the number of elements of $\pi^{-1}(\omega)$ is then odd for any regular $\omega \in \mathcal{R}$. ∎

Surjectivity of the Natural Projection, or the Existence of Equilibrium

Proposition 4.6.5 The natural projection $\pi : E \to \Omega$ is onto.

Proof Assume that $\pi : E \to \Omega$ is not surjective. Let $\omega' \in \Omega$ but not in the image $\pi(E)$. This ω' is a regular value of the map π because it is not the image of a critical point. Therefore, the number of elements of the set $\pi^{-1}(\omega')$ is odd, by the definition of the modulo 2 degree. This contradicts the fact that the integer 0 is even. ∎

The surjectivity property of the natural projection means that there exists an equilibrium (p, ω) in E for every $\omega \in \Omega$. This is just a complicated way of saying that the equilibrium equation $z(p, \omega) = 0$ has a solution $p \in S$ for every $\omega \in \Omega$.

4.6.3 Uniqueness Domain

Proposition 4.6.6 The set of Pareto optima P is contained in only one of the connected components of the set of regular economies $\mathcal{R}$.

Proof The set of Pareto optima P is a subset of the set of regular economies $\mathcal{R}$. The set of Pareto optima P is diffeomorphic to $B = S \times \mathbb{R}^m_{++}$ and therefore pathconnected. The set of Pareto optima P, being pathconnected, is contained in *one connected component* of the set of regular economies $\mathcal{R}$. ∎

Let us denote by $\mathcal{R}_1$ this connected component of $\mathcal{R}$.

Proposition 4.6.7 Equilibrium is unique all over the connected component $\mathcal{R}_1$.

Proof The proposition follows from the uniqueness of equilibrium at a Pareto optimum combined with the constancy of the number of equilibria over the connected components of $\mathcal{R}$. ∎

The uniqueness domain $\mathcal{R}_1$ contains the set of Pareto optima in its interior and, in this topological sense, is a neighborhood of the set of Pareto optima. Nevertheless, the uniqueness domain may be quite a large set. Uniqueness of equilibrium for all economies $\omega \in \Omega$ is equivalent to the equality $\mathcal{R}_1 = \Omega$. This is also equivalent to the emptyness of the set of singular economies Σ.

4.7 Regular Equilibria and Their Genericity

Debreu's genericity theorem for regular economies tells us that the set of regular economies $\mathcal{R}$ is open with full measure in Ω. We have seen that this property is nothing more than Sard's theorem applied to the natural projection. An important consequence for $\omega \in \mathcal{R}$ of being a regular economy is the existence of a sufficiently small open neighborhood U of ω such that $\pi^{-1}(U)$ is a disjoint union of open sets, each one diffeomorphic to U. If (p, ω) is one of the elements of $\pi^{-1}(\omega)$, there exists an open neighborhood V of (p, ω) such that the restriction $\pi \mid V : V \to U$ is a diffeomorphism. Let $(s(\omega'), \omega') = (\pi \mid V)^{-1}(\omega')$. The map $s : U \to S$ is known as an *equilibrium price selection map.* It follows from the existence of the equilibrium price selection map $s : U \to S$ that, for any continuous small change of the endowment parameter ω' (*small* meaning that ω' belongs to the open set U), there always exists an equilibrium price vector $s(\omega')$ associated with ω' that varies continuously with ω'.

An important feature of regular economies is the property that all equilibria define smooth price selection maps defined in neighborhoods of the regular economies. No discontinuities of the equilibrium price vector as a function of the fundamentals can be triggered by smooth changes of the endowment parameter ω when the latter is a regular economy.

4.7.1 Equilibrium Price Selection Map at a Regular Equilibrium

It is worth noting that the local continuity of the equilibrium price selection map that associates with $\omega \in \mathcal{R}$ the equilibrium price vector $p \in S$ requires only that the equilibrium (p, ω) be regular. In other words, the selection property may be satisfied at the regular equilibrium (p, ω) even if there exists a price vector $p' \in S$ with $p' \neq p$ such that the pair (p', ω) is a critical equilibrium, in which case ω is a singular economy. Therefore, understanding of the discontinuities of the equilibrium selection maps would be much improved with a better knowledge of the set of regular equilibria $E(\Re)$.

4.7.2 Full Measure of the Set of Regular Equilibria

The next question is whether there are enough regular equilibria in the equilibrium manifold. The answer is provided by the following proposition.

Proposition 4.7.1 The set of regular equilibria $E(\Re)$ is open with full measure in the equilibrium manifold E.

Proof The difficulty in proving this property is that there is no general theorem similar to Sard's theorem regarding the measure or even the density of the set of critical points of a smooth map. Nothing prevents such sets from having a nonempty interior or even from being equal to the full domain of the smooth map.

The solution consists in a careful analysis of the set of critical equilibria in every fiber of the equilibrium manifold E.

Critical Equilibria in a Fiber

Let us use for the equilibrium manifold E the coordinate system B of section (4.4.4), with $b = (p, w_1, \ldots, w_m) \in B = S \times \mathbb{R}^m_{++}$, and Y is a $(\ell - 1) \times (m - 1)$ real matrix. The equilibrium with coordinates (b, Y) is critical if the determinant of the Jacobian matrix $J(b, Y)$ is equal to zero. The set of critical equilibria Γ is defined by equation $\det J(b, Y) = 0$.

Let the price-income vector $b \in B$ be fixed. It follows from

$$\det J(b, Y) = \det(M(b) + K(b)Y)$$

that the restriction of the function $\det J(b, Y)$ to the fiber $F(b)$ is a polynomial function in the coefficients of the $(\ell - 1) \times (m - 1)$ matrix Y. The set of critical equilibria $\Gamma(b)$ in the fiber $F(b)$ is defined by equation $\det J(b, Y) = 0$, which is a polynomial equation in the coefficients of Y. The set $\Gamma(b)$ is therefore a real algebraic subset of the fiber $F(b)$.

Stratifiability of Real Algebraic Sets

A rather deep property of real algebraic sets (and more generally of semialgebraic sets, i.e., sets defined by polynomial equalities and inequalities) is that such sets are *stratifiable.* This implies that these sets are the union of a finite number of smooth submanifolds of the ambiant space. (For mathematical references on algebraic sets, see Whitney (1957); on semialgebraic sets, see Benedetti and Risler (1990, sec. 2.6.12.) The semialgebraic set $\Gamma(b)$ is therefore the union of a finite number of smooth submanifolds of the fiber $F(b)$. The fiber having dimension $(\ell - 1)(m - 1)$, the dimension of these submanifolds is $\leq (\ell - 1)(m - 1)$.

Can some of these submanifolds that make up the set $\Gamma(b)$ have the maximum dimension $(\ell - 1)(m - 1)$? If so, this would imply that the set $\Gamma(b)$ would have a nonempty interior. In that case, the polynomial function $Y \to \det(M(b) + K(b)\, Y)$ would take the value zero on some nonempty open subset of the fiber $F(b)$. This property is sufficient to imply that the polynomial function is identically equal to zero. (The latter property is easily proved by an induction argument on the number of variables.) This function therefore takes the value zero for all $(\ell - 1) \times (m - 1)$ matrices Y. This obviously applies to the matrix $Y = 0$. This implies that $\det M(b) = 0$, which contradicts the property that $\det M(b) \neq 0$, i.e., the property that no-trade equilibria are regular equilibria.

The algebraic set $\Gamma(b)$ is therefore a finite union in $\mathbb{R}^{(\ell-1)(m-1)}$ of smooth manifolds of dimension strictly less than $(\ell - 1)(m - 1)$. Each one of these submanifolds has measure zero in the fiber $F(b)$. Their union $\Gamma(b)$ has therefore measure zero. This property holds true for every $b \in B$.

The set of critical equilibria Γ is a subset of $S \times \Omega$ such that the "slice" $\Gamma(b)$ has measure zero for every $b \in B = S \times \mathbb{R}^m_{++}$. We then conclude that the set Γ has Lebesgue measure zero by applying Fubini's theorem about the integral of functions of several variables. (All that is needed here is the special case of Fubini's theorem that deals with sets of measure zero.) ■

Alternative Proof of the Genericity of Regular Economies

The set of singular economies Σ is the image of the set of critical equilibria Γ by the natural projection $\pi : E \to \Omega$. The image of a set of measure zero by a smooth map has also measure zero. It then follows from $\Sigma = \pi(\Gamma)$, combined with the property that Γ has measure zero, that the set Σ has also measure zero.

Exercises

4.5. Let $\widetilde{E(\mathfrak{R})}$ denote the set of regular extended equilibria. Prove that this set is not pathconnected if there exists at least some $b \in B$ such that matrix $K(b) \neq 0$. Interpret the condition $K(b) = 0$ for all $b \in B = S \times \mathbb{R}^m_{++}$. Prove that the set $E(\mathfrak{R})$ is not necessarily pathconnected.

4.6. The equilibrium (resp. extended equilibrium) $(p, \omega) \in E$ (resp. $\tilde{E}$) is said to be positive regular if it is regular and $\det J(p, \omega)$ has the same sign as $(-1)^{\ell-1}$.

Let $E(\mathfrak{R}_+)$ (resp. $\widetilde{E(\mathfrak{R}_+)}$) denote the set of positive regular equilibria (resp. positive regular extended equilibria).

a. Give an economic justification to the concept of positive regularity.

b. Prove the inclusion $T \subset E(\mathfrak{R}_+)$.

c. Prove that the set $\widetilde{E(\mathfrak{R}_+)}$ is pathconnected. (*Hint:* Use the results of exercise 4.6a to reduce the problem to the equilibria belonging to an extended fiber.)

4.7. The extended equilibrium $(p, \omega) \in \tilde{E}$ is said to be negative regular if it is regular and $\det J(p, \omega)$ has the opposite sign of $(-1)^{\ell-1}$. Let $\widetilde{E(\mathfrak{R}_-)}$ denote the set of negative regular extended equilibria. Show by way of examples for the case $(\ell, m) = (2, 2)$ and fixed total resources that the set $\widetilde{E(\mathfrak{R}_-)}$ can have an arbitrary number of pathconnected components.

4.8 Application to Some Properties of Equilibria

4.8.1 The Law of Demand

The *law of demand*, originally stated for a partial equilibrium setup, says that demand is a decreasing function of price. Initially, this law was thought to be always true. In particular, it was thought to be satisfied at all equilibria. It is not. In addition, though defined for a partial equilibrium setup, it is possible to extend the law of demand to a general equilibrium setup provided prices are not normalized. (In that case, the aggregate demand function $p \to z(p, \omega)$ is a homogeneous function (of degree 0) of the non-normalized price vector $p = (p_1, p_2, \ldots, p_\ell)$.)

The equilibrium $(p, \omega) \in E$ is said to be compatible with the *local law of demand*, or to satisfy property $\mathfrak{D}(p, \omega)$, if the partial derivative $\dfrac{\partial z^j}{\partial p_j}(p, \omega)$ evaluated at p is < 0 for $j = 1, \ldots, \ell$. In other words, the equilibrium (p, ω) satisfies property $\mathfrak{D}(p, \omega)$ if the diagonal coefficients of the Jacobian matrix $\tilde{J}(p, \omega)$ of the homogeneous aggregate excess demand with respect to non-normalized prices are all < 0.

Let $E(\mathfrak{D})$ denote the set of equilibria that satisfy the local law of demand.

Proposition 4.8.1 The following properties are satisfied by the set $E(\mathfrak{D})$: (1) inclusion $T \subset E(\mathfrak{D})$, and (2) the fibers of $E(\mathfrak{D})$ are open convex.

Proof Let $(p, \omega) \in T$. The matrix $\tilde{J}(p, \omega)$ is negative semidefinite, and all its $\ell - 1$ principal submatrices are negative definite. (The $\ell - 1 \times \ell - 1$ matrix $J(p, \omega)$ is the principal submatrix of $\tilde{J}(p, \omega)$, obtained by deleting the ℓth row and column, and $J(p, \omega)$ is negative definite; the latter property is independent of the choice of a numeraire.) It follows from the negative definiteness of $J(p, \omega)$ that all the diagonal coefficients of $J(p, \omega)$ are < 0. Changing the numeraire commodity is sufficient to see that the ℓth diagonal coefficient of $\tilde{J}(p, \omega)$ is also < 0.

Let $b \in S \times \mathbb{R}^m_{++}$ be fixed. The equilibrium (p, ω) in the fiber $F(b)$ satisfies $\mathfrak{D}(p, \omega)$ if the ℓ inequalities $\dfrac{\partial z^j}{\partial p_j}(p, \omega) < 0$ are satisfied in addition to the sign conditions $\omega_i^j > 0$.

The coefficients $\dfrac{\partial z^j}{\partial p_j}(p, \omega)$ are linear functions of the coordinates ω_i^j. This proves that the set of equilibria in the fiber $F(b)$ that satisfy the property $\mathfrak{D}(p, \omega)$ is a nonempty open convex set. This set is path-connected, simply connected, contractible, homeomorphic, and diffeomorphic to $\mathbb{R}^{(\ell-1)(m-1)}$.

It then suffices to have b vary in $B = S \times \mathbb{R}^m_{++}$ and to reproduce the methods used in the study of global properties of the equilibrium manifold to have the same properties here for the set $E(\mathfrak{D})$. ■

Exercises

4.8. Prove that the local law of demand is satisfied if the vector of net trade is sufficiently small.

4.9. Assume that at least two consumers have different preferences. Prove that, for sufficiently large vectors of total resources $r \in X$, there exist equilibria that do not satisfy the local law of demand.

4.8.2 Gross Substitutability

The property of gross substitutability is of long standing in economic theory. As for the law of demand, only a local version is considered here.

The equilibrium $(p, \omega) \in E$ satisfies *local gross substitutability*, or property $\mathfrak{S}(p, \omega)$, if the derivative of the aggregate excess demand for some commodity, derivative taken with respect to the non-normalized prices of any of the other commodities, is > 0:

$$\frac{\delta z^j}{\delta p_k}(p, \omega) > 0, \qquad j \neq k.$$

This is equivalent to the Jacobian matrix of aggregate excess demand $\tilde{J}(p, \omega)$ (for non-normalized price vectors) having strictly positive off-diagonal terms.

Let us denote by $E(\mathfrak{S})$ the set of equilibria satisfying the gross substitutability property $\mathfrak{S}(p, \omega)$.

Proposition 4.8.2 The following properties are satisfied by the set $E(\mathfrak{S})$: (1) inclusion $E(\mathfrak{S}) \subset E(\mathfrak{D})$, and (2) the fibers of $E(\mathfrak{S})$ are open convex.

Proof The inequalities defining the equilibria satisfying property $\mathfrak{S}$ are linear with respect to the coordinates ω_i^j of the equilibrium (p, ω) in a given fiber, which proves the second part of the proposition.

Walras' law (the identity $p \cdot z(p, \omega) = 0$) is satisfied for all non-normalized price vectors $p = (p_1, p_2, \ldots . p_\ell)$. Taking the derivative with respect to p_j yields the equality

$$z^j(p, \omega) + p \cdot \frac{\delta z}{\delta p_j}(p, \omega) = 0.$$

If follows from the equilibrium condition $z^j(p, \omega) = 0$ that

$$p_j \frac{\delta z^j}{\delta p_j}(p, \omega) = -\sum_{h \neq j} p_h \frac{\delta z^h}{\delta p_j}.$$

Combined with local gross substitutability, this equality yields the strict inequality $\frac{\delta z^j}{\delta p_j}(p, \omega) < 0$, which proves that the local law of demand is satisfied, hence the inclusion stated in the first part of the proposition. ■

Remark It is worth noting that the set of no-trade equilibria T is not necessarily a subset of $E(\mathfrak{S})$.

4.8.3 Tatonnement Stability

A Digression on the Idea of Stability

Walras's original definition of stability concept is nothing more than the property that the law of demand is satisfied. Walras's formulation is made in a partial equilibrium setup, which is very surprising coming from the inventor of the general equilibrium model. Walras's view of stability developed into two different approaches: a static one by Hicks (1946) and a dynamic one by Samuelson (1941).

Hicks's concept of stability consists in a reformulation of the law of demand where the indirect effects of price changes on the demands and supplies of the other goods are taken into account. It is the law of demand reformulated for the general equilibrium setup. At variance with Walras's approach, the prices of the other goods are adjusted to maintain the equality of supply and demand for those goods.

Hicksian stability is known to be equivalent to the principal minors of the Jacobian matrix of aggregate excess demand $J(p, \omega)$ being of alternate signs, the first one being < 0 (Hicks 1946).

The definition of gross substitutability shows that this property is a strengthening of the law of demand and is related to the idea of the stability of equilibrium. This intuition can be made rigorous by checking that *local gross susbstitutability* implies *Hicksian stability.* This is equivalent to showing that a $\ell \times \ell$ off-diagonal strictly positive Jacobian matrix $\tilde{J}$ (with respect to non-normalized prices) is such that all its principal minors of order $k < \ell$ have the same sign as $(-1)^k$. This follows readily from Horn and Johnson (1985, theorem 6.1.10).

Samuelson's approach consists in formulating a differential equation that expresses the evolution through time of the price system. Following Samuelson (1941), this differential equation has been known ever since as Walras tatonnement.

Tatonnement's Differential Equation

In this section, prices are normalized by the numeraire convention $p_\ell = 1$. *Walras tatonnement* is defined by the differential equation where $p^* = (\bar{p}^*, 1) \in S$ denotes the initial condition at time $t = 0$ and $p(t) = (\bar{p}(t), 1) \in S$ denotes the price vector at time $t \geq 0$:

$$\begin{cases} \dot{\bar{p}}(t) = \bar{z}(p(t), \omega), & t \geq 0, \\ \bar{p}(0) = \bar{p}^*. \end{cases} \tag{4.4}$$

By definition, the equilibrium (resp. extended equilibrium) $(p, \omega) \in E$ (resp. $\tilde{E}$) is *tatonnement-stable* if the price vector $p = (\bar{p}, 1) \in S$ is locally asymptotically stable for Walras tatonnement. The linearized version of this differential equation takes the form

$$\begin{cases} \dot{\bar{p}}(t) = J(p, \omega)(\bar{p}(t) - \bar{p}), & t \geq 0, \\ \bar{p}(0) = \bar{p}^*. \end{cases} \tag{4.5}$$

By definition, the *regular equilibrium* (p, ω) is *hyperbolic for Walras tatonnement*, or *tatonnement-hyperbolic*, if no eigenvalue of the Jacobian matrix $J(p, \omega)$ is purely imaginary.

A straightforward adaptation of the proof used earlier to show that the subset of critical equilibria Γ of the equilibrium manifold E is closed with measure zero shows that the complement of the subset of the equilibrium manifold E consisting of the *tatonnement-hyperbolic equilibria* is also closed with measure zero. The same property is true for the extended equilibria.

Hyperbolic Tatonnement-Stable Equilibria

It follows from the characterization of local asymptotic stability for the hyperbolic stationary points of a dynamic system (see, e.g., Hirsch and Smale 1974, ch. 9) that the *tatonnement-hyperbolic equilibrium* (p, ω) is *tatonnement-stable* if and only if the nonzero eigenvalues of the Jacobian matrix of aggregate excess demand $J(p, \omega)$ have strictly negative real parts.

Let $E(\mathfrak{T})$ (resp. $\widetilde{E(\mathfrak{T})}$) denote the set of equilibria (resp. extended equilibria) (p, ω) such that the eigenvalues of the Jacobian matrix of aggregate excess demand $J(p, \omega)$ have strictly negative real parts. The study of tatonnement stability is now reduced to the study of the set $E(\mathfrak{T})$ and its extended version $\widetilde{E(\mathfrak{T})}$.

Proposition 4.8.3 The following properties are satisfied by the sets $E(\mathfrak{T})$ and $\widetilde{E(\mathfrak{T})}$: (1) inclusion $T \subset E(\mathfrak{T})$, (2) inclusion $E(\mathfrak{S}) \subset E(\mathfrak{T})$, and (3) the set $\widetilde{E(\mathfrak{T})}$ is pathconnected.

Proof At the no-trade equilibrium $(p, \omega) \in T$, the Jacobian matrix of aggregate excess demand $J(p, \omega)$ is symmetric negative definite. Therefore, its eigenvalues are all strictly negative. This implies that the no-trade equilibrium (p, ω) is *tatonnement-hyperbolic* and also *tatonnement-stable*.

Let (p, ω) be a *GS*-equilibrium. The inclusion is equivalent to showing that every $\ell \times \ell$ extended Jacobian matrix $\tilde{J}(p, \omega)$ (with respect to non-normalized prices) with off-diagonal strictly positive coefficients is such that its $\ell - 1 \times \ell - 1$ principal submatrix $J(p, \omega)$ is stable in the sense that all its eigenvalues have strictly negative real parts. This follows, for example, from Horn and Johnson (1991, prop. 2.5.3.13) applied to matrix $J(p, \omega)$.

The main result of this section is the pathconnectedness of the set of tatonnement-stable extended equilibria $\tilde{E}(\mathfrak{T})$.

As in previous examples, the idea of the proof is as follows: (1) show that the set of no-trade T (which we know to be pathconnected) is a subset of the set $\widetilde{E(\mathfrak{T})}$, and (2) show that the intersection of the set $\widetilde{E(\mathfrak{T})}$ with the extended fiber $\widetilde{F(b)}$, with $b \in B = S \times \mathbb{R}^m_{++}$, is pathconnected.

We already know that the first property is true. In order to prove the second property, we use the expression of the Jacobian matrix $J(\widetilde{b, Y})$ as a function of matrix Y that parameterizes the extended fiber $\widetilde{F(b)}$. It suffices then to show that the set of matrices of that form is pathconnected. This is a problem in linear algebra. For a detailed proof, see Balasko (1978a, theorems 2, 4). ■

4.9 Conclusion

The properties of the equilibrium manifold E depend on the parameter space Ω where the economy ω is varied. In this chapter, the parameter space is $\Omega = X^m$, with $X = \mathbb{R}^\ell_{++}$. Other parameter spaces can and have been considered. The most extreme case occurs for a parameter space Ω reduced to just one point, i.e., $\Omega = \{\omega\}$. It follows from the Debreu-Mantel-Sonnenschein (DMS) theorem that the equilibrium price set $W(\omega)$, the analogue of the equilibrium manifold for that parameter space, can be any compact subset of the price set S. It is therefore all the more remarkable that it suffices to enlarge the parameter space to $\Omega = X^m = (\mathbb{R}^\ell_{++})^m$ to inject enough regularity to make the equilibrium manifold E a *smooth submanifold* of $S \times \Omega$ with its structure of linear fibers arranged through the no-trade equilibria.

The discontinuity property of equilibrium price selections contradicts an assumption that underlies, implicitly at least, many policy-oriented fields as, for example, international trade theory or public economics. Such is the case of the standard argument about the gains from trade used to justify international trade. That justification, which is fine as long as equilibrium is unique, does not resist confrontation with the chaotic world of multiple equilibria and discontinuous selections of price equilibria. The multiplicity of equilibria is a consequence of large volumes of trade. The development of international trade therefore inexorably puts the world economy into multiple equilibria territory. The consequence is that we should expect to observe from time to time sudden changes of market prices.

4.10 Notes and Comments

The poor understanding that the economic profession had of the equilibrium manifold in 1972 is illustrated by the picture of a set with kinks and multiple connected components in K. Hildenbrand (1972). An almost identical picture appears in W. Hildenbrand (1974).

The properties of pathconnectedness, simple connectedness, contractibility, homeomorphism, and diffeomorphism to a Euclidean space were proved for the case of variable total resources, the case considered in this chapter, by Balasko (1973; 1975a; 1975b). The diffeomorphism is proved as a consequence of the homeomorphism property with the help of a rather deep theorem of Stallings (1962) that says that smooth manifolds of dimension $\neq 4$ that are homeomorphic to Euclidean spaces are also diffeomorphic to these spaces. This excludes the case of two goods and two consumers, a case that can be handled directly by elementary techniques because the fibers are then line segments. A direct proof of the diffeomorphism is given by Schecter (1979).

Crucial to all these results are (1) the identification of the set of the no-trade equilibria, and (2) the partition of the equilibrium manifold in linear fibers "glued" together by the no-trade equilibria. Coordinate systems that reflect the partition of the equilibrium manifold into fibers were introduced by Balasko (1975a; 1975b).

The expression of the Jacobian matrix of aggregate excess demand as a function of the coordinates (proposition 4.5.6) appears in Balasko (1975a). Applications to the regularity of the no-trade equilibria (proposition 4.5.9) to the regularity of Pareto optima (proposition 4.6.2) and to the degree of the natural projection (proposition 4.6.3) are due to Balasko (1975a). The oddness of the number of equilibria of a regular economy (proposition 4.6.4) is due to Dierker (1972), who proves it with the help of the Poincaré-Hopf index theorem applied to the vector field defined on the price set by Walras tatonnement.

The surjectivity of the projection map stated in proposition 4.6.5 and its equivalence to the existence of equilibrium for all economies is due to Balasko (1975a). This approach has been applied quite effectively in other setups: by Smale (1974a) for nonconvex economies, and by Duffie and Shafer (1985) in the general equilibrium model with incomplete financial assets.

The proof that every no-trade equilibrium is tatonnement-stable (first part of proposition 4.8.3) and that the equilibrium is unique at Pareto optima (proposition 4.6.1) are due to Arrow and Hurwicz (1958). The pathconnectedness of the set of tatonnement-stable extended equilibria (third part of proposition 4.8.3) is due to Balasko (1978a).

The property that the set of Pareto optima is contained in one and only one connected component of the set of regular economies $\mathcal{R}$ (proposition 4.6.6), with the consequence that equilibrium is unique for all economies in that component (proposition 4.6.7), is due to Balasko (1975a). This component is also shown by Balasko (1978c; 1979a; 1995) to contain all the regular economies that have a unique equilibrium in the case of two consumers. In the general case of an arbitrary number of consumers and goods, sufficient conditions for this property to be satisfied are given in Balasko (1988).

The genericity of regular equilibria (proposition 4.7.1) and its application to a proof of Debreu's theorem on the genericity of regular economies (proposition 2.7.1), which avoids Sard's theorem, are due to Balasko (1992). This approach to proving the genericity of some form of regularity is particularly useful when Sard's theorem cannot be invoked. An example of this is Balasko's (1994) proof of the genericity of regular sequential equilibria in the intertemporal Arrow-Debreu model.

Formula (12.1) in Hicks's *Value and Capital* (1946, 31) is similar to proposition 4.5.6, expressing the Jacobian determinant of aggregate excess demand as a function of the coordinates of the equilibrium manifold. Hicks interprets the formula as the sum of a "substitution effect" and an "income effect." Hicks mentions that the income effect may be sufficiently large to offset the substitution effect, making the equilibrium unstable (in Hicks's sense). Nevertheless, he apparently never says that this loss of stability may be triggered by sufficiently large trade vectors. He also never says that no-trade equilibria are stable (in his sense). It can only be conjectured that he would have reformulated the conflicting effects of the substitution and income effects as a function of the trade vector had he had at his disposal the concept of the equilibrium manifold.

At least three properties of the Arrow-Debreu model related to the equilibrium manifold and the natural projection were discovered during the first two phases of general equilibrium theory.

Walras's (1874) theorem of "répartitions équivalentes" (lecture 14) appears to be the first mention in the literature of the existence of linear fibers in the equilibrium manifold. It is so poorly understood that it disappears from all treatments of the general equilibrium model that are posterior to Walras's book, starting with Pareto's *Cours d'Economie Politique* (1896).

The two results of Arrow and Hurwicz (1958)—that equilibrium is unique and tatonnement-stable (theorem 1, 532) when there is no trade—are not even mentioned in the two major references on uniqueness and stability of equilibrium, namely, Negishi (1962) and Arrow and Hahn (1971, chs. 9 and 11). It is hard to imagine that these properties were dropped from these otherwise excellent surveys simply because of ignorance. It is more likely that the idea of no-trade equilibria was considered at the time to be too unrealistic to be worthy of consideration.

Hicks's formula and the three results of Arrow, Hurwicz, and Walras are like some isolated pieces in a jigsaw puzzle where the global picture is missing. They acquire their real significance and importance only with the global picture provided by the equilibrium manifold and the natural projection.

5 The Equilibrium Equation and Its Geometric Interpretation

5.1 Introduction

The equation defined by the equality of aggregate supply and demand is the centerpiece of the Arrow-Debreu model. Most equation systems in mathematics, however, can take different, if equivalent, forms. Some of them lend themselves more easily than others to the study of specific questions.

The potential of the method of equivalent equation systems is already apparent in elementary mathematics. Consider, for example, the general quadratic equation $ax^2 + bx + c = 0$, where, to fix ideas, a, b, and c are real numbers. By writing

$$ax^2 + bx + c = a(x + b/2a)^2 - (b^2 - 4ac)/4a = 0,$$

the general quadratic equation is seen to be equivalent to the equation in which the coefficient of the term of the first degree is equal to zero.

It follows from that equivalence that $x \in \mathbb{R}$ is a solution of the general quadratic equation if it satisfies the equality

$$(x + b/2a)^2 = (b^2 - 4ac)/4a^2.$$

The scope of the method of equivalent equation systems is not limited to high school algebra. Many questions of classical mechanics are more easily handled in their Hamiltonian and Lagrangian reformulations than in Newton's original formulation.

This chapter shows that in economics the consideration of equivalent equation systems can be a powerful tool. The equilibrium equation is converted into an equivalent form that lends itself well to the study of questions dealing with the number and the selection of equilibria. This approach is useful for a number of other issues as well.

The main interest of the reformulation is the equivalence between the equilibrium equation and the intersection of two sets. These two sets are smooth submanifolds of some suitable Euclidean space. Intersection of smooth manifolds is one of the subjects of differential topology. Of particular interest in that case is the concept of transverse submanifolds and the related important and powerful transversality theorems. Milnor (1997) and Guillemin and Pollack (1974) provide a very accessible introduction to transversality theory. More advanced readers will find additional material in Abraham and Robin (1967), Golubitsky and Guillemin (1973), and Hirsch (1976).

In this chapter, I assume only the most elementary facts about smooth manifolds and explicitly state all the properties of transversality theory that are used. The hope is that readers will develop some intuition for the geometric approach even without having prior knowledge of transversality theory.

5.2 Geometric Formulation of the Equilibrium Equation

5.2.1 Generalized Price-Income Space

Denote by $B = S \times \mathbb{R}^m_{++}$ the set of prices and incomes. Then denote by $H = (\mathbb{R}^{\ell-1} \times \{1\}) \times \mathbb{R}^m$ the affine space generated by $B = S \times \mathbb{R}^m_{++}$. The set H is the *generalized price-income space*. Its dimension is equal to $m + \ell - 1$.

The difference between B and H is that the elements in H are not restricted to being strictly positive. If some coordinates of $b = (p, w_1, \dots, w_m) \in H$ happen to be negative, we speak of *generalized prices* or *wealths.*

5.2.2 Constant versus Variable Total Resources

In the previous chapters, the parameter space is the set $\Omega = X^m$ and the total resource vector $r = \omega_1 + \omega_2 + \cdots + \omega_m$ is variable. From an economic point of view, it is much more difficult to vary the total resource vector $r \in X$ than the distribution of these resources within the economy. There would be no economics of development, for example, if it were that easy to increase the output of an economy. This observation shows the importance of being able to apply the parametric approach with the restriction that total resources are held constant. This leads to a much smaller parameter space than $\Omega = X^m$, and with such a smaller parameter space, it can be expected that the properties of the equilibrium manifold and the natural projection can become more complex and far more difficult to establish.

It is therefore a remarkable by-product of the geometric approach that, by lending itself rather easily to the case of fixed total resources, it allows extending to the case of fixed total resources the properties established in previous chapters for the case of variable total resources. But the full set of assumptions about utility functions (see chapter 1) is then required.

Parameter Space

In this chapter, at variance with the previous chapters, the vector of total resources $r = \omega_1 + \omega_2 + \cdots + \omega_m \in X$ is held constant. The notation $\Omega = X^m$ is reserved for the set of strictly positive endowments with variable total resources. The notation $\Omega(r)$ denotes the set of strictly positive endowment vectors compatible with the total resources $r \in X$:

$$\Omega(r) = \{\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in X^m \mid \omega_1 + \omega_2 + \omega_m = r\}.$$

An extended endowment vector $\omega = (\omega_1, \ldots, \omega_m)$ is simply one where there is no sign restriction on the components of ω. The extended parameter space $\widetilde{\Omega(r)}$ consists of the extended endowment vectors $\omega = (\omega_1, \ldots, \omega_m)$ such that $\omega_1 + \cdots + \omega_m = r$.

Pareto Optima

The set $P(r)$ is defined as the set of Pareto optima compatible with the total resources $r \in X$:

$$P(r) = P \cap \Omega(r),$$

where P is the subset of Ω consisting of the Pareto optima.

5.2.3 Equivalent Equation System

Given the assumption that total resources are constant and equal to the vector $r \in X$, the equilibrium equation

$$f_1(p, p \cdot \omega_1) + f_2(p, p \cdot \omega_2) + \cdots + f_m(p, p \cdot \omega_m) = \omega_1 + \omega_2 + \cdots + \omega_m \tag{5.1}$$

is equivalent to

$$f_1(p, p \cdot \omega_1) + f_2(p, p \cdot \omega_2) + \cdots + f_m(p, p \cdot \omega_m) = r. \tag{5.2}$$

Let $w_1, w_2, \ldots, w_m$ denote the wealths of consumers $1, 2, \ldots, m$. It follows from the definition of wealth that

$$\begin{cases} w_1 = p \cdot \omega_1; \\ w_2 = p \cdot \omega_2; \\ \vdots \\ w_m = p \cdot \omega_m. \end{cases} \tag{5.3}$$

With the new variables $w_1, w_2, \ldots, w_m$, equation (5.2) becomes

$$f_1(p, w_1) + f_2(p, w_2) + \cdots + f_m(p, w_m) = r. \tag{5.4}$$

Let now $p \in S$ be a solution of equation (5.1). Then $b = (p, w_1, w_2, \ldots, w_m) \in B$, where $w_1, w_2, \ldots, w_m$ are determined by equation (5.3), is a solution of the equation system defined by (5.3) and (5.4).

Conversely, if $b = (p, w_1, \ldots, w_m) \in B$ is a solution of the equation system defined by (5.3) and (5.4), then the price component $p \in S$ of $b = (p, w_1, \ldots, w_m) \in B$ is a solution of equation (5.1) for the endowment vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$. The price vector $p \in S$ is then an equilibrium price vector associated with the economy $\omega \in \Omega(r)$. For a given $\omega \in \Omega(r)$, equation (5.1) is therefore equivalent to the equation system defined by (5.3) and (5.4).

5.3 Geometric Interpretation

Let us now interpret geometrically the equation system defined by (5.3) and (5.4). For this, we need to define three subsets of the *generalized price-income space* $H = (\mathbb{R}^{\ell-1} \times \{1\}) \times \mathbb{R}^m$. Each one of these subsets corresponds to a different economic concept.

5.3.1 Resource-Compatible Price-Income Space

The *resource-compatible price-income space* $H(r)$ consists of the (generalized) price-income vectors $b = (p, w_1, \ldots, w_m) \in H$ such that the equality

$$w_1 + w_2 + \cdots + w_m = p \cdot r \tag{5.5}$$

is satisfied. With $b = (p, w_1, \ldots, w_m) \in B$, (i.e., all components of b are > 0), the inner product $p \cdot r$ is the value of the total resources $r \in X$ for the price system $p \in S$. Equality (5.5) means that total wealth $p \cdot r$ is distributed among all the agents of the economy. We define the resource-compatible price-income space $H(r)$ as the subset of H defined by equation (5.5). This actually defines a hyperplane of H. Its dimension is equal to $\ell + m - 2$.

We also define the subset $H_{++}(r)$ of $H(r)$ that consists of the vectors $b = (p, w_1, \dots, w_m)$ with strictly positive coordinates.

5.3.2 Budget Manifold

The *budget manifold* $A(\omega)$ consists of the (generalized) price-income vectors $(p, w_1, \dots, w_m) \in H$ that satisfy the equations $w_1 = p \cdot \omega_1$, $w_2 = p \cdot \omega_2, \dots, w_m = p \cdot \omega_m$.

The (generalized) wealth w_1 (resp. $w_2, \dots, w_m$) is an affine function of the (generalized) price vector $p \in \mathbb{R}^{\ell-1} \times \{1\}$, i.e., a polynomial function of degree 1 of the coordinates $p_1, p_2, \dots, p_{\ell-1}$ of p. A truly linear function has no constant term different from zero. Here, the price normalization implies that the wealths $w_1, w_2, \dots, w_m$ are affine functions of the coordinates of p with constant terms equal to $\omega_1^\ell, \omega_2^\ell, \dots, \omega_m^\ell$, respectively. These terms are different from zero in general. The budget manifold $A(\omega)$ is an affine subspace of H of dimension $\ell - 1$.

Let us denote by $\mathcal{A}$ the set of dimension $\ell - 1$ affine subspaces of $H(r)$. The budget manifold $A(\omega)$ is an element of the set $\mathcal{A}$.

In the same way as the endowment vector $\omega = (\omega_1, \dots, \omega_m)$ is an "economy," the budget manifold $A \in \mathcal{A}$ is an "economy." Note, however, that not all budget manifolds $A \in \mathcal{A}$ are of the form $A(\omega)$ for some $\omega \in \Omega$.

5.3.3 Section Manifold

The *section manifold* $B(r)$ consists of the (strictly positive) price-income vectors $(p, w_1, \dots, w_m) \in B$ such that the demand $f_1(p, w_1) + \cdots + f_m(p, w_m)$ is equal to the vector of total resources $r \in X$. The section manifold is the subset of $B = S \times \mathbb{R}^m_{++}$ defined by the equation

$$f_1(p, w_1) + f_2(p, w_2) + \cdots + f_m(p, w_m) = r. \tag{5.6}$$

At this stage, the section manifold $B(r)$ is nothing more than a subset of B. The property that it is actually a smooth submanifold of B and several other important properties are proved in the following sections.

5.3.4 Geometric Interpretation of the Equilibrium Equation

Let $(p, \omega) \in S \times \Omega$. Denote by $\varphi(p, \omega)$ the price-income vector

$$\varphi(p, \omega) = (p, p \cdot \omega_1, \dots, p \cdot \omega_m) \in S \times \mathbb{R}^m.$$

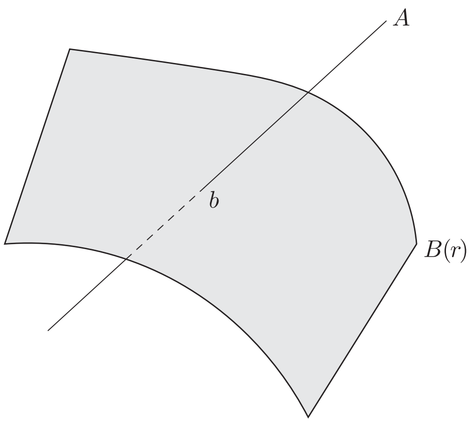

Figure 5.1
Intersection $A(\omega) \cap B(r)$

Proposition 5.3.1 The pair $(p, \omega) \in S \times \Omega(r)$ is an *equilibrium* if and only if the price-income vector $b = \varphi(p, \omega)$ belongs to the section manifold $B(r)$. The price-income vector $b = (p, w_1, \ldots, w_m) = \varphi(p, \omega) \in B$ is then a solution of the equation system defined by (5.3) and (5.4).

The proof is obvious.

The study of the equilibrium price vectors $p \in S$ associated with the endowment vector $\omega \in \Omega(r)$ is therefore equivalent to the study of the intersection points of the section manifold $B(r)$ with the budget manifold $A(\omega)$ (figure 5.1).

5.3.5 Section and Budget Manifolds as Subsets of $H(r)$

The section manifold $B(r)$ and the budget manifold $A(\omega)$ have been defined as subsets of H. In fact, they are contained in an even smaller set, which is not without its importance when dealing with the intersection of these two sets.

Proposition 5.3.2 The section manifold $B(r)$ and the budget manifold $A(\omega)$ are both contained in $H(r)$.

Proof of inclusion $B(r) \subset H(r)$ Let $b = (p, w_1, \ldots, w_m)$ belong to the section manifold $B(r)$. The equality $f_1(p, w_1) + \cdots + f_m(p, w_m) = r$ is satisfied by the definition of $B(r)$. The inner product of both sides of this equality by the price vector $p \in S$ yields the equality

$$p \cdot (f_1(p, w_1) + \cdots + f_m(p, w_m)) = p \cdot r;$$

hence

$$p \cdot f_1(p, w_1) + \cdots + p \cdot f_m(p, w_m) = p \cdot r,$$

which by Walras' law is applied to each term of the sum, yields

$$w_1 + \cdots + w_m = p \cdot r.$$

The element b therefore belongs to $H(r)$. This proves the inclusion $B(r) \subset H(r)$.

Proof of inclusion $\boldsymbol{A(\omega) \subset H(r)}$ Let now $b = (p, w_1, \dots, w_m)$ belong to the budget manifold $A(\omega)$. Adding the equalities $w_1 = p \cdot \omega_1$, $w_2 = p \cdot \omega_2, \dots,$ $w_m = p \cdot \omega_m$ yields the equality $w_1 + \cdots + w_m = p \cdot (\omega_1 + \cdots + \omega_m) = p \cdot r$, which proves that b belongs to $H(r)$, hence the inclusion $A(\omega) \subset H(r)$. ■

5.3.6 Geometric Equilibrium

Let $\mathcal{A}$ denote the set of dimension $\ell - 1$ affine subspaces of $H(r)$. Propositions 5.3.1 and 5.3.2 lead us to define a *geometric equilibrium* as a pair $(b, A) \in B(r) \times \mathcal{A}$ such that the point $b \in B(r)$ belongs to budget manifold $A \in \mathcal{A}$. In this definition, the affine space $A \in \mathcal{A}$ does not have to be of the form $A = A(\omega)$ for some $\omega \in \Omega(r)$.

The geometric equilibria associated with the budget manifold $A \in \mathcal{A}$ are then in bijection with the intersection points of A with $B(r)$.

5.3.7 Parametric Approach in the Geometric Setup

We can now formulate the parametric approach as the study of the intersection of the section manifold $B(r)$ with the affine subspaces A (the budget manifold) when the latter are varied in suitably defined subsets of $\mathcal{A}$, the set of dimension $\ell - 1$ affine subspaces of $H(r)$.

The Set $\boldsymbol{\mathcal{A}_{++}}$

Since not all affine subspaces $A \in \mathcal{A}$ are of the form $A = A(\omega)$, we define the set $\mathcal{A}_{++}$ as consisting of the elements $A \in \mathcal{A}$ for which there exists an endowment vector $\omega \in \Omega(r)$ such that $A = A(\omega)$. Such ω is unique when it exists.

The Sets $\boldsymbol{\mathcal{A}(b)}$ *and* $\boldsymbol{\mathcal{A}_{++}(b)}$

For several questions, it is convenient to consider other subsets of $\mathcal{A}$. For a given element $b \in H(r)$, let us define the subsets $\mathcal{A}(b)$ and $\mathcal{A}_{++}(b)$ of $\mathcal{A}$ and $\mathcal{A}_{++}$, respectively, comprising the $\ell - 1$ dimensional affine subspaces A of $H(r)$ that contain the point $b \in H(r)$ in addition to belonging to $\mathcal{A}$ and $\mathcal{A}_{++}$, respectively.

By taking the point b as an origin for the affine space $H(r)$, whose dimension is equal to $\ell + m - 2$, the set $\mathcal{A}(b)$ can be identified with the set of dimension $\ell - 1$ vector subspaces of $\mathbb{R}^{\ell+m-2}$. The latter set is known as a Grassmann manifold and denoted by $G_{\ell+m-2,\ell-1}$. Grassmann manifolds are generalizations of projective spaces. Here only elementary properties of these manifolds are needed.

Only the budget space A is varied in the parametric approach. This implies that all the nonlinearities of the equilibrium equation are encapsulated in the structure of the section manifold $B(r)$ and therefore do not depend on the variable parameter. This certainly contributes to the efficiency of the geometric formulation.

The properties of the section manifold $B(r)$ play a crucial role in the study of the intersection $A \cap B(r)$ when the affine space A is varied in $\mathcal{A}$ or in some of its subsets.

Exercises

5.1. Consider the map $F : B \to X$ where $F(p, w_1, \ldots, w_m) = f_1(p, w_1) + \cdots + f_m(p, w_m)$ from B into X.

a. Show that the Jacobian matrix of F at $b \in B$ can be written as the $\ell \times (\ell - 1 + m)$ matrix

$$DF(b) = \left(\sum \frac{\partial f_i}{\partial p_1} \cdots \sum \frac{\partial f_i}{\partial p_{\ell-1}} \, \frac{\partial f_1}{\partial w_1} \cdots \frac{\partial f_m}{\partial w_m} \right).$$

b. Show that matrix $DF(b)$ has the same rank as the matrix

$$\begin{bmatrix} & & & \dfrac{\partial f_1^1}{\partial w_1} & \cdots & \dfrac{\partial f_m^1}{\partial w_m} \\ & J & & \vdots & & \vdots \\ & & & \dfrac{\partial f_1^{\ell-1}}{\partial w_1} & \cdots & \dfrac{\partial f_m^{\ell-1}}{\partial w_m} \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix},$$

where J is the $\ell - 1 \times \ell - 1$ sum of the individual Slutsky matrices at $b = (p, w_1, \ldots, w_m)$.

c. By definition, $b \in B$ is a critical point for the map F if the linear map defined by the Jacobian matrix $DF(b)$ is not onto. Show that the map $F : B \to X$ has no critical point.
d. Apply the regular value theorem to the map $F : B \to X$ to show that the section manifold $B(r) = F^{-1}(r)$ is a smooth submanifold of B for any $r \in X$.

5.2. In $\mathbb{R}^n$, let $G_{n,k}$ denote the Grassmann manifold consisting of the dimension k vector subspaces of the vector space $\mathbb{R}^n$. Show that $G_{n,k}$ is pathconnected. (*Hint:* Let A and A' be two-dimensional k vector subspaces. Find a continuous path linking A to A'.)

5.4 Structure of the Section Manifold *B*(*r*)

The set of Pareto optima $P(r)$ associated with the total resources $r \in X$ is a smooth submanifold of $\Omega(r)$ diffeomorphic to $\mathbb{R}^{m-1}$, by proposition A.6.2 in the appendix. The structure of the section manifold $B(r)$ is related to the structure of the set of Pareto optima $P(r)$, as follows from proposition 5.4.1.

Proposition 5.4.1 The section manifold $B(r)$ is a smooth submanifold of $B = S \times \mathbb{R}^m_{++}$ diffeomorphic to the set of Pareto optima $P(r)$, hence to $\mathbb{R}^{m-1}$.

Proof The map $\pi \circ f : B \to P$ is such that

$$\pi \circ f(p, w_1, w_2, \ldots, w_m) = (f_1(p, w_1), f_2(p, w_2), \ldots, f_m(p, w_m)).$$

This map is a diffeomorphism between B and P, the set of Pareto optima for variable total resources, by corollary 3.3.2 in chapter 3.

The section manifold $B(r)$ is then the preimage by $\pi \circ f$ of the set of Pareto optima $P(r)$ associated with the total resources $r \in X$. The proposition then follows from proposition A.6.2 in the appendix. ∎

5.5 Another Useful Diffeomorphism Involving the Section Manifold

This section relies heavily on developments on the structure of the set of Pareto optima presented in the appendix. The proofs of this section can be skipped in a first reading.

The following proposition uses the set $\mathcal{U}_{m-1}$ that consists of the pairs $(r, u) \in X \times \mathbb{R}^{m-1}$ with the property that there exists $x = (x_1, x_2, \ldots, x_{m-1}) \in X^{m-1}$ such that $x_1 + x_2 + \cdots + x_{m-1} < r$ and $u_1(x_1) = u_1$, $u_2(x_2)$

$= u_2, \ldots, u_{m-1}(x_{m-1}) = u_{m-1}$. The set $\mathcal{U}_{m-1}$ can be interpreted as the set of feasible utility levels for the first $m-1$ consumers with incomplete utilization of total resources. (See appendix sections A.7 and A.8.)

Proposition 5.5.1 The map $\theta : B \to X \times \mathbb{R}^{m-1}$ defined by

$$\theta(p, w_1, \ldots, w_m) = \left(\sum_i f_i(p, w_i), u_1(f_1(p, w_1)), \ldots, u_{m-1}(f_{m-1}(p, w_{m-1})) \right)$$

is a diffeomorphism between B and $\mathcal{U}_{m-1}$.

Proof Note that if $m = 1$, the set $\mathcal{U}_0$ is equal to X, the set B is the Cartesian product $S \times \mathbb{R}_{++}$, and the map θ becomes the demand map $f_1 : S \times \mathbb{R}_{++} \to X$, which is a diffeomorphism (see exercise 1.10 in chapter 1). Therefore, proposition 5.5.1 is an extension to the case of several consumers of the diffeomorphism property of individual demand functions.

The strategy of proof is to show that θ is a smooth embedding from B into $X \times \mathbb{R}^{m-1}$, whose image is $\mathcal{U}_{m-1}$.

First, it follows from the formula defining θ that this map is smooth.

Let us prove the inclusion $\theta(B) \subset \mathcal{U}_{m-1}$. Let

$$(r, u_1, \ldots, u_{m-1}) = \theta(p, w_1, \ldots, w_m).$$

Let $x_i = f_i(p, w_i)$ for $i = 1, 2, \ldots, m$. The equality $u_i(x_i) = u_i$ for $1 \leq i \leq m-1$ combined with $x_1 + x_2 + \cdots + x_{m-1} < x_1 + x_2 + \cdots + x_{m-1} + x_m = r$ implies that $(r, u_1, \ldots, u_{m-1})$ belongs to $\mathcal{U}_{m-1}$.

Let now $(r, u_1, u_2, \ldots, u_{m-1}) \in \mathcal{U}_{m-1}$. Define

$$x' = (x'_1, x'_2, \ldots, x'_m) = R_{m-1}(r, u_1, u_2, \ldots, u_{m-1})$$

where the map $R_{m-1} : \mathcal{U}_{m-1} \to X^m$ is defined in the appendix, section A.3, p. 204.

The Pareto optimum x' is supported by some price vector $p' \in S$. Therefore, $x'_i = f_i(p', p' \cdot x'_i)$ for $i = 1, 2, \ldots, m$. Let $w'_i = p' \cdot x'_i$. Then it follows from the definition of θ that $\theta(p, w_1, w_2, \ldots, w_m)$ is equal to

$$(x'_1 + x'_2 + \cdots + x'_m, u_1(x'_1), \ldots, u_{m-1}(x'_{m-1}))$$

i.e., to $(r, u_1, u_2, \ldots, u_{m-1})$. This proves the inclusion $\mathcal{U}_{m-1} \subset \theta(B)$.

Second, the map $\alpha : X \times \mathcal{U}_{m-1}$ into B, defined by the composition of the map $R_{m-1} : X \times \mathcal{U} \to P$ with the map $P \to B$ that associates with every Pareto optimum $x = (x_1, x_2, \ldots, x_m)$ the element

$$(D_n u_1(x_i), D_n u_1(x_1) \cdot x_1, D_n u_1(x_1) \cdot x_2, \ldots, D_n u_1(x_1) \cdot x_m)$$

of B, is smooth as the composition of two smooth maps.

Let us show that the composition $\alpha \circ \theta : B \to B$ is the identity map of B. Let $b = (p, w_1, \ldots, w_m) \in B$. Then

$$x = (x_1, x_2, \ldots, x_m) = (f_1(p, w_1), f_2(p, w_2), \ldots, f_m(p, w_m))$$

is a Pareto optimum associated with the total resources $r = f_1(p, w_1) + f_2(p, w_2) + \cdots + f_m(p, w_m)$. This Pareto optimum gives the utility levels $u_1(x_1), u_2(x_2), \ldots, u_{m-1}(x_{m-1})$ to the first $m-1$ consumers. Therefore, the Pareto optimum $x = (x_1, \ldots, x_m)$ is such that

$$x = R_{m-1}(r, u_1(x_1), u_2(x_2), \ldots, u_{m-1}(x_{m-1})).$$

It also follows from $x_1 = f_1(p, w_1)$ that the gradient vector $Du_1(x_1)$ is collinear with p. This implies that the composition $\alpha \circ \theta$ is the identity map of B.

The combination of the preceding properties implies that the map θ is an embedding and therefore a diffeomorphism between its domain and its image. ■

5.6 Indirect Utility Functions

Now that readers have a better understanding of the structure of the section manifold $B(r)$, the question is whether the relation with the set of Pareto optima has an even deeper significance than the diffeomorphism properties. The answer comes from the concept of *indirect utility functions*.

By definition, the *indirect utility* $\hat{u}_i(p, w_i)$ for consumer i of the price-income pair $(p, w_i) \in S \times \mathbb{R}_{++}$ is the utility of the commodity bundle $f_i(p, w_i)$ that is demanded by consumer i as the result of maximizing the (direct) utility $u_i(x_i)$ subject to the budget constraint $p \cdot x_i = w_i$:

$$\hat{u}_i(p, w_i) = u_i(f_i(p, w_i)). \tag{5.7}$$

Associated with the indirect utility function $\hat{u}_i : S \times \mathbb{R}_{++} \to \mathbb{R}$ are the *indirect indifference surfaces* (or curves, in the case of two goods) defined by equation $\hat{u}_i(p, w_i) = K$, where K varies in $\mathbb{R}$.

After considering the example of indirect utility functions associated with log-linear direct utility functions, I prove a few properties of the indirect utility functions associated with the general direct utility functions (see chapter I).

5.6.1 An Example: Indirect Utility Functions Associated with Log-Linear Direct Utility Functions

To fix ideas, let $u_i(x_i) = \sum_j a_i^j \ln x_i^j$ be a log-linear utility function, with $a_i^j > 0$ and $\sum_j a_i^j = 1$. Its associated demand function is $f_i^j(p, w_i) = a_i^j \dfrac{w_i}{p_j}$, with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, \ell$.

The indirect utility function $\hat{u}_i(p, w_i)$ is then equal to

$$\hat{u}_i(p, w_i) = \sum_j a_i^j \ln a_i^j + \ln w_i - \sum_j a_i^j \ln p_j.$$

The equation of the indirect indifference (hyper)surface takes the form $w_i = K \prod_j p_j^{a_i^j}$, where K is any value > 0.

In the simple case $\ell = 2$, and $a_i^1 = a_i^2 = 1/2$, the indifference (hyper)-surfaces become curves, and in the plane with coordinates (p_1, w_i), these curves are actually parabola defined by equation $w_i = K\sqrt{p_1}$, with $K > 0$.

5.6.2 Smoothness of Indirect Utility Functions

In the general case, which is considered from now on, it is evident from the expression of the indirect utility function, equation (5.7), that the indirect utility function $\hat{u}_i : (p, w_i) \to \hat{u}_i(p, w_i)$ is the composition of two smooth functions, the demand function $(p, w_i) \to f_i(p, w_i)$ and the direct utility function $x_i \to u_i(x_i)$, and therefore is smooth.

5.6.3 Roy's Identities

Proposition 5.6.1 Roy's identities consist of the following equalities:

$$\frac{\partial \hat{u}_i}{\partial p_j}(p, w_i) = -v_i(p, w_i) f_i^j(p, w_i), \qquad j = 1, 2, \ldots, \ell - 1,$$

$$\frac{\partial \hat{u}_i}{\partial w_i}(p, w_i) = v_i(p, w_i),$$

where $v_i(p, w_i)$ is defined by the equality

$$Du_i(f_i(p, w_i)) = v_i(p, w_i) p. \tag{5.8}$$

Proof In order to prove these identities, let us take the derivatives of

$$\hat{u}_i(p, w_i) = u_i(f_i(p, w_i))$$

with respect to p_j and w_i and apply the chain rule. This yields

$$\frac{\partial \hat{u}_i}{\partial p_j}(p, w_i) = \sum_{1 \leq k \leq \ell} \frac{\partial u_i}{\partial x^k} \frac{\partial f_i^k(p, w_i)}{\partial p_j}, \tag{5.9}$$

$$\frac{\partial \hat{u}_i}{\partial w_i}(p, w_i) = \sum_{1 \leq k \leq \ell} \frac{\partial u_i}{\partial x^k} \frac{\partial f_i^k(p, w_i)}{\partial w_i}. \tag{5.10}$$

Taking the derivative of Walras' law, $p \cdot f_i(p, w_i) = w_i$, with respect to p_j and w_i yields

$$p \cdot \frac{\partial f_i(p, w_i)}{\partial p_j} = -f_i^j(p, w_i), \tag{5.11}$$

$$p \cdot \frac{\partial f_i(p, w_i)}{\partial w_i} = 1. \tag{5.12}$$

It then suffices to substitute equality (5.8) into (5.9) and (5.10) and to apply (5.11) and (5.12). ■

5.6.4 Monotonicity with respect to Income and Prices

It follows from the definition of $v_i(p, w_i)$ in proposition 5.6.1 that this term is necessarily > 0 because the gradient vector $Du_i(x_i)$ belongs to the strictly positive orthant X. This therefore implies the strict inequality

$$\frac{\partial \hat{u}_i}{\partial w_i}(p, w_i) > 0.$$

The same proof, combined with the observation that the commodity bundle $f_i(p, w_i)$ belongs to the strictly positive orthant X, shows us that the derivative of the indirect utility $\hat{u}_i$ with respect to the price p_j of any commodity j that is different from the numeraire (i.e., $j \neq \ell$) is < 0:

$$\frac{\partial \hat{u}_i}{\partial p_j}(p, w_i) < 0.$$

5.6.5 Strict Quasi-Convexity

By definition, strict quasi-convexity of the indirect utility function $\hat{u}_i(p, w_i)$ means that the set

$$C = \{(p, w_i) \in S \times \mathbb{R}_{++} \mid \hat{u}_i(p, w_i) \leq K)\}$$

is strictly convex if nonempty. The boundary ∂C of the set C is the indirect indifference hypersurface defined by equation $\hat{u}_i(p, w_i) = K$. This hypersurface ∂C is a smooth submanifold of $S \times \mathbb{R}_{++}$.

In order to prove that the set C is strictly convex, it suffices to show that, for any $(p^*, w_i^*) \in \partial C$, the set C minus the point (p^*, w_i^*) is contained in only one of the two open half-spaces defined by the tangent hyperplane $H_{(p^*, w_i^*)}$ at $(p^*, w_i^*) \in \partial C$.

The equation of the tangent hyperplane $H_{(p^*, w_i^*)}$ follows from equating to zero the first-order derivative of the equation of ∂C, namely, $\hat{u}_i(p, w_i) = \hat{u}_i(p^*, w_i^*)$. This yields

$$\sum_{j=1}^{\ell-1}(p_j - p_j^*)\frac{\partial \hat{u}_i}{\partial p_j}(p^*, w_i^*) + (w_i - w_i^*)\frac{\partial \hat{u}_i}{\partial w_i}(p^*, w_i^*) = 0.$$

It follows from proposition 5.6.1 that this equation can be rewritten as

$$\sum_{j=1}^{\ell-1}(p_j - p_j^*)f_i^j(p^*, w_i^*) - (w_i - w_i^*) = 0.$$

Combined with

$$w_i^* = \sum_{j=1}^{\ell-1} p_j^* f_i^j(p^*, w_i^*) + f_i^\ell(p^*, w_i^*),$$

it yields for the tangent hyperplane $H_{(p^*, w_i^*)}$ the following simple equation:

$$w_i = p \cdot f_i(p^*, w_i^*).$$

The set C is defined by the inequality

$$\hat{u}_i(p, w_i) \leq \hat{u}_i(p^*, w_i^*).$$

Let $x_i = f_i(p, w_i)$ and $x_i^* = f_i(p^*, w_i^*)$. It follows from the strict quasi-concavity of the utility function u_i that the inequality $u_i(x_i) \leq u_i(x_i^*)$ implies that x_i^* belongs to the strictly convex set $\{x_i' \in X \mid u_i(x_i) \leq u_i(x_i')\}$ of commodity bundles preferred to x_i.

Because of its strict convexity, this set minus the point x_i is contained in one of the two open half-spaces defined by the tangent hyperplane at x_i to the indifference hypersurface through x_i, namely, the set

$\{x_i' \in X \mid u_i(x_i') = u_i(x_i)\}$. The tangent hyperplane at $x_i = f_i(p, w_i)$ to the indifference hypersurface through $x_i = f_i(p, w_i)$ is perpendicular to the price vector $p \in S$. Its equation is therefore $p \cdot x_i' = p \cdot x_i = w_i$. The strict inequality $p \cdot x_i < p \cdot x_i^*$ then follows from the inequality $u_i(x_i) \leq u_i(x_i^*)$ combined with the inequality $x_i \neq x_i^*$.

This proves that every $(p, w_i) \in C$ different from (p^*, w_i^*) satisfies the strict inequality

$$p \cdot f_i(p^*, w_i^*) - w_i > 0, \tag{5.13}$$

a strict inequality that implies that the set C minus the point (p^*, w_i^*) is contained in one of the two half-spaces determined by the hyperplane $H_{(p^*, w_i^*)}$.

5.6.6 Smooth Strict Quasi-Convexity

By definition, the indirect utility function $\hat{u}_i(p, w_i)$ is *smoothly strictly quasi-convex* if, in addition to being strictly quasi-convex, the restriction of the quadratic form defined by the Hessian matrix $D^2\hat{u}_i\,(p^*, w_i^*)$ of the indirect utility function $\hat{u}_i$ at (p^*, w_i^*) to the hyperplane $H_{(p^*, w_i^*)}$ (the tangent hyperplane to the indirect indifference hypersurface ∂C at (p^*, w_i^*)) is positive definite.

The quadratic form defined by the Hessian matrix of a real-valued function at a given point coincides with the second-degree polynomial of the Taylor expansion of the function at that point. It follows from the uniqueness of the Taylor expansion that the restriction to a hyperplane of the quadratic form defined by the Hessian matrix coincides with the second-order polynomial of the Taylor expansion of the restriction of the function to that hyperplane. In the case of the tangent hyperplane $H_{(p^*, w_i^*)}$ to the hypersurface ∂C at (p^*, w_i^*) with equation $w_i = p \cdot f_i(p^*, w_i^*)$, the restriction of the indirect utility function $\hat{u}_i$ to $H_{(p^*, w_i^*)}$ is the map $p \to \hat{u}_i(p, p \cdot x_i^*)$, where $x_i^* = f_i(p^*, w_i^*)$. Let us show that the Hessian matrix of this map $p \to \hat{u}_i(p, p \cdot x_i^*)$ at p^* is positive definite.

Let us start by computing its first-order derivatives. We get

$$\begin{aligned}\frac{d\hat{u}_i}{dp_j}(p, p \cdot x_i^*) &= \frac{\partial \hat{u}_i}{\partial p_j}(p, p \cdot x_i^*) + \frac{\partial \hat{u}_i}{\partial w_i}(p, p \cdot x_i^*)x_i^{j*} \\ &= v_i(p, p \cdot x_i^*)(x_i^{j*} - f_i^j(p, p \cdot x_i^*)).\end{aligned}$$

Taking the derivative of this expression with respect to p_k yields

$$\frac{\partial^2 \hat{u}_i}{\partial p_j \partial p_k}(p, p\cdot x_i^*) = \frac{dv_i(p, p\cdot x_i^*)}{dp_k}(x_i^{j*} - f_i^j(p, p\cdot x_i^*))$$

$$- v_i(p, p\cdot x_i^*)\,\frac{df_i^j}{dp_k}(p, p\cdot x_i^*),$$

where

$$\frac{df_i^j}{dp_k}(p, p\cdot x_i^*) = \frac{\partial f_i^j}{\partial p_k}(p, p\cdot x_i^*) + \frac{\partial f_i^j}{\partial w_i}(p, p\cdot x_i^*) f_i^k(p, p\cdot x_i^*).$$

The Hessian matrix of the map $p \to \hat{u}_i(p, p\cdot x_i^*)$ at p^* is therefore equal to

$$-v_i(p^*, p^*\cdot x_i^*) M_i(p^*, p^*\cdot x_i^*), \tag{5.14}$$

where $M_i(p^*,\ p^*\cdot x_i^*)$ is the $\ell - 1 \times \ell - 1$ Slutsky matrix of the direct utility function u_i at $x_i^* = f_i(p^*, w_i^*)$. It follows from the assumptions regarding the direct utility function u_i that the Slutsky matrix $M_i(p^*, p\cdot x_i^*)$ is negative definite. It then follows from (5.14) that the restriction of the quadratic form defined by the Hessian matrix $D^2\hat{u}_i(p^*, w_i^*)$ to the tangent hyperplane $H_{(p^*, w_i^*)}$ is positive definite.

5.7 Section Manifold and Indirect Utilities: Special Case $(\ell, m) = (2, 2)$

The special case of two goods, two consumers, and fixed total resources has the very nice geometrical illustration of the well-known Edgeworth box. In this illustration, the set of Pareto optima is known as the contract curve. An economy is defined by the point of the box that represents the endowment vector. The equilibrium allocations associated with a given economy are the intersection points (different from the endowment point) of the curves representing consumers' demands when prices are varied.

A question is whether there exists in that setup a geometrical illustration of the section manifold.

Let us start by defining indirect Pareto "optima" for the special case of two goods and two consumers. The more general case of an arbitrary number of goods and consumers is addressed in section 5.8.

5.7.1 Consumer 1's Indirect Indifference Curves in the Plane (p_1, w_1)

Consider the indifference curve of consumer 1 for the indirect utility function $\hat{u}_1(p, w_1) = u_1(f_1(p, w_1))$ through the point (p^*, w_1^*). That curve has a tangent at the point of coordinates (p_1^*, w_1^*), whose equation is

$$(p_1 - p_1^*)\frac{\partial \hat{u}_1}{\partial p_1}(p^*, w_1^*) + (w_1 - w_1^*)\frac{\partial \hat{u}_1}{\partial w_1}(p^*, w_1^*) = 0.$$

It follows from Roy's identities of proposition 5.6.1 that this equation can be rewritten as

$$-v_1(p^*, w_1^*) f_1^1(p^*, w_1^*)(p_1 - p_1^*) + v_1(p^*, w_1^*)(w_1 - w_1^*) = 0.$$

After simplification by $v_1(p^*, w_1^*) \neq 0$, it becomes

$$w_1 = w_1^* + f_1^1(p^*, w_1^*)(p_1 - p_1^*).$$

Combined with $w_1^* = p_1 f_1^1(p^*, w_1^*) + f_1^2(p^*, w_1^*)$, we get the following simple equation for the tangent line:

$$w_1 = f_1^1(p^*, w_1^*) p_1 + f_1^2(p^*, w_1^*).$$

The slope of this line is equal to $f_1^1(p^*, w_1^*)$, and its ordinate at the origin is equal to $f_1^2(p^*, w_1^*)$ (figure 5.2).

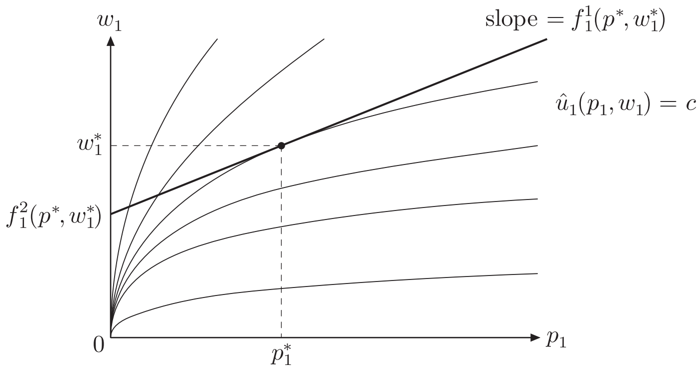

Figure 5.2
Indirect indifference curves and a tangent line

5.7.2 Consumer 2's Indirect Indifference Curves in the Plane (p_1, w_2)

The same computations give us in the coordinate system (p_1, w_2) the equation

$$w_2 = f_2^1(p^*, w_2^*)p_1 + f_2^2(p^*, w_2^*) \tag{5.15}$$

for the tangent line to the indirect indifference curve of consumer 2 through the point of coordinates (p_1^*, w_2^*).

5.7.3 Consumer 2's Indirect Indifference Curves in $H_{++}(r)$ with Coordinates (p_1, w_1)

Recall that the plane $H_{++}(r)$ is the set of price-income vectors $b = (p_1, 1, w_1, w_2)$ that satisfy the relation $w_1 + w_2 = p_1 r^1 + r^2$ $(p_1, w_1, w_2 > 0)$.

We can therefore use the pair (p_1, w_1) as coordinates for the point $b = (p_1, 1, w_1, w_2) \in H_{++}(r)$. The equation of the tangent line to the indirect indifference curve of consumer 2 in this coordinate system follows from substituting $w_2 = p_1 r^1 + r^2 - w_1$ into (5.15). This yields

$$p_1 r^1 + r^2 - w_1 = f_2^1(p^*, w_2^*)p_1 + f_2^2(p^*, w_2^*),$$

which can be rewritten under the simpler

$$w_1 = (r^1 - f_2^1(p^*, w_2^*))p_1 + r^2 - f_2^2(p^*, w_2^*).$$

This is the equation of a line with slope $r^1 - f_2^1(p^*, w_2^*)$ and ordinate at the origin $r^2 - f_2^2(p^*, w_2^*)$.

5.7.4 Geometric Reformulation of the Equilibrium Condition

The tangent lines to the indirect indifference curves of the two consumers through the point $b^* = (p^*, w_1^*, w_2^*)$ in $H_{++}(r)$ coincide if and only if the indifference curves are tangent to each other. This property is similar to the tangency condition of the two consumers' indifference curves that defines a Pareto optimum in the Edgeworth box. This tangency property is a first-order condition, i.e., a condition that involves only the first-order derivatives of the utility functions. The two tangent lines are identical if and only if the equalities

$$f_1^1(p^*, w_1^*) = r^1 - f_2^1(p^*, w_2^*) \quad \text{and} \quad f_1^2(p^*, w_1^*) = r^2 - f_2^2(p^*, w_2^*)$$

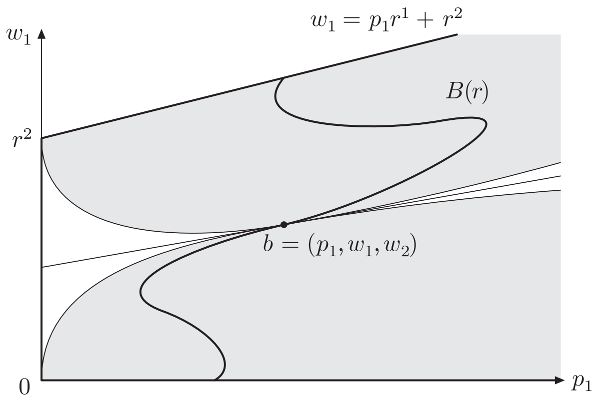

Figure 5.3
The tangency condition at $B(r)$

are satisfied. These two equalities are equivalent to the (vector) equality

$$f_1(p^*, w_1^*) + f_2(p^*, w_2^*) = r,$$

which is equivalent to $b^* = (p^*, w_1^*, w_2^*)$ belonging to the section manifold $B(r)$ (figure 5.3).

5.7.5 Indirect Contract Curve in Price-Income Space

The section manifold $B(r)$ is therefore the set of points where the two consumers′ indirect indifference curves are tangent to each other (figure 5.4). This is the same condition as for the contract curve in the Edgeworth box.

5.8 Section Manifold and Indirect Utilities: General Case

The goal is now to extend to an arbitrary number of goods and consumers the interpretation of the section manifold $B(r)$ as the set of common tangent points to the consumers′ indirect indifference curves in $H_{++}(r)$.

5.8.1 Pareto Minima for Indirect Utility Functions

The indirect utility function $\hat{u}_i(p, w_i)$ of consumer i depends on the price vector $p \in S$ and income w_i. Since we are interested in the price-income vectors $b = (p, w_1, \dots, w_i, \dots, w_m) \in S \times \mathbb{R}^m_{++}$, we use the nota-

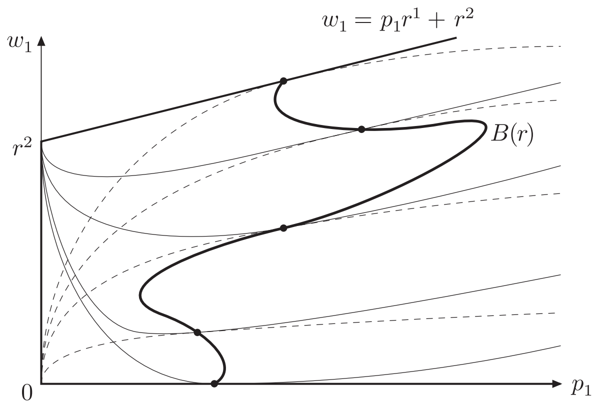

Figure 5.4
Parameterization of $B(r)$ by indirect utility levels

tion $\hat{u}_i(b)$, defined by $\hat{u}_i(b) = \hat{u}_i(p, w_i)$. Indirect indifference hypersurfaces contained in $H_{++}(r)$ are then associated with these indirect utility functions.

Let us formulate a concept similar to Pareto optimality for the m indirect utility functions $\hat{u}_i : H_{++}(r) \to \mathbb{R}$.

The price-income vector $b = (p, w_1, \ldots, w_m) \in H_{++}(r)$ is a *Pareto minimum* for the map $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_m) : H_{++}(r) \to \mathbb{R}^m$ if there is no $b' = (p', w'_1, \ldots, w'_m) \in H_{++}(r)$ such that the inequalities

$$\hat{u}_i(b') \leq u_i(b)$$

are satisfied for all i, one of them at least being strict.

The two concepts of Pareto minimum and Pareto optimum are obviously closely related. Strictly speaking, there is actually no need to introduce a new concept because a Pareto minimum for the map $\hat{u} : H_{++}(r) \to \mathbb{R}^m$ is a Pareto optimum (or maximum) for the map $-\hat{u}$. However, it is probably more intuitive to deal with the concept of a Pareto minimum for indirect utility functions than with a Pareto maximum for the *opposite* of indirect utility functions.

5.8.2 Pareto Minima for Indirect Utility Functions and Their Equivalence with Price-Income Equilibria

Let us formulate the first-order conditions associated with a Pareto minimum for indirect utility functions. Then apply this characterization to prove that every Pareto minimum belongs to the section manifold $B(r)$. Finally, the converse is also proved true, namely, every price-income equilibrium, i.e., every element of the section manifold $B(r)$, is also a Pareto minimum.

First-Order Conditions for the Pareto Minima for Indirect Utility Functions

It is straightforward that, just as in the case of the Pareto optima for direct utility functions, the price-income vector $b = (p, w_1, \ldots, w_m) \in H_{++}(r)$ is a Pareto minimum if it is a solution for the problem of finding $b' = (p', w'_1, \ldots, w'_m) \in H_{++}(r)$ that minimizes

$$\hat{u}_1(b')$$

subject to the constraints

$$\hat{u}_2(b') \leq \hat{u}_2(b);$$

$$\vdots$$

$$\hat{u}_m(b') \leq \hat{u}_m(b).$$

Recall that the element $b' = (p', w'_1, \ldots, w'_m)$ satisfies the equality $w'_1 + \cdots + w'_m = p' \cdot r$, as follows from the equation of the hyperplane defined by $H_{++}(r)$.

Positivity constraints are implicit. They are clearly not binding because of the assumptions on the direct or indirect utility functions. They can therefore be neglected when writing down the first-order conditions associated with this constrained minimization problem.

It follows from the monotonicity of the indirect utility functions $\hat{u}_1, \ldots, \hat{u}_m$ (see section 5.6.4) that the utility constraints are necessarily binding, which enables us to substitute equality constraints for the inequality constraints.

Associated with this constrained minimization problem with equality constraints is its Lagrangean function

$$\begin{aligned}\mathcal{L}(p, w_1, \ldots, w_m, \lambda_2, \ldots, \lambda_m, \mu) = \hat{u}_1(p, w_1) + \lambda_2 \hat{u}_2(p, w_2) + \cdots \\ + \lambda_m \hat{u}_m(p, w_m) \\ + \mu(w_1 + \cdots + w_m - p \cdot r). \qquad (5.16)\end{aligned}$$

The first-order necessary conditions consist of the first derivatives of the Lagrangean

$$\mathcal{L}(p_1, p_2, \ldots, p_{\ell-1}, w_1, \ldots, w_m, \lambda_2, \ldots, \lambda_m, \mu)$$

with respect to $p_1, \ldots, p_{\ell-1}$, $w_1, \ldots, w_m$ equated to zero, equalities to which are added the equalities induced by the binding constraints.

This proves the following lemma:

Lemma 5.8.1 The first-order necessary conditions for

$$b = (p, w_1, \ldots, w_m) \in H_{++}(r)$$

to be a Pareto minimum for the indirect utility functions take the form

$$\frac{\partial \mathcal{L}}{\partial p_1} = \frac{\partial \hat{u}_1}{\partial p_1} + \lambda_2 \frac{\partial \hat{u}_2}{\partial p_1} + \cdots + \lambda_m \frac{\partial \hat{u}_m}{\partial p_1} - \mu r^1 = 0;$$

$$\frac{\partial \mathcal{L}}{\partial p_2} = \frac{\partial \hat{u}_1}{\partial p_2} + \lambda_2 \frac{\partial \hat{u}_2}{\partial p_2} + \cdots + \lambda_m \frac{\partial \hat{u}_m}{\partial p_2} - \mu r^2 = 0;$$

$$\vdots$$

$$\frac{\partial \mathcal{L}}{\partial p_{\ell-1}} = \frac{\partial \hat{u}_1}{\partial p_{\ell-1}} + \lambda_2 \frac{\partial \hat{u}_2}{\partial p_{\ell-1}} + \cdots + \lambda_m \frac{\partial \hat{u}_m}{\partial p_{\ell-1}} - \mu r^{\ell-1} = 0;$$

$$\frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \hat{u}_1}{\partial w_1} + \mu = 0;$$

$$\vdots$$

$$\frac{\partial \mathcal{L}}{\partial w_m} = \frac{\partial \hat{u}_m}{\partial w_m} + \mu = 0,$$

where $(\lambda_2, \ldots, \lambda_m, \mu) \in \mathbb{R}^m$.

Equivalence between First-Order Conditions and an Equilibrium Condition

Roy's identities of proposition 5.6.1 applied to the preceding first-order conditions yield

$$-v_1 f_1^1(p, w_1) - v_2 \lambda_2 f_2^1(p, w_2) - \cdots - v_m \lambda_m f_m^1(p, w_m) - \mu r^1 = 0;$$

$$-v_1 f_1^2(p, w_1) - v_2 \lambda_2 f_2^2(p, w_2) - \cdots - v_m \lambda_m f_m^2(p, w_m) - \mu r^2 = 0;$$

$$\vdots$$

$$-v_1 f_1^{\ell-1}(p, w_1) - v_2 \lambda_2 f_2^{\ell-1}(p, w_2) - \cdots - v_m \lambda_m f_m^{\ell-1}(p, w_m) - \mu r^{\ell-1} = 0;$$

$$v_1 + \mu = 0;$$

$$v_2\lambda_2 + \mu = 0;$$

$$\vdots$$

$$v_m\lambda_m + \mu = 0,$$

with $v_1, \ldots, v_m$ defined by

$$Du_i(f_i(p, w_i)) = v_i(p, w_i)p \quad \text{for } i = 1, 2, \ldots, m.$$

We have seen that v_1, for example, is different from zero. These equalities imply

$$v_1 = \lambda_2 v_2 = \cdots = \lambda_m v_m = -\mu \neq 0.$$

Substituting $-\mu$ for $v_1, \lambda_2 v_2, \ldots, \lambda_m v_m$ in the first $\ell - 1$ equations and dividing them by μ yields

$$f_1^1(p, w_1) + f_2^1(p, w_2) + \cdots + f_m^1(p, w_m) - r^1 = 0;$$

$$f_1^2(p, w_1) + f_2^2(p, w_2) + \cdots + f_m^2(p, w_m) - r^2 = 0;$$

$$\vdots$$

$$f_1^{\ell-1}(p, w_1) + f_2^{\ell-1}(p, w_2) + \cdots + f_m^{\ell-1}(p, w_m) - r^{\ell-1} = 0.$$

Combined with the equality $w_1 + w_2 + \cdots + w_m = p \cdot r$, these $\ell - 1$ equalities imply the equality

$$f_1^{\ell}(p, w_1) + f_2^{\ell}(p, w_2) + \cdots + f_m^{\ell}(p, w_m) - r^{\ell} = 0.$$

We have therefore proved the following property.

Proposition 5.8.2 Every Pareto minimum $b \in H_{++}(r)$ for the indirect utility functions $\hat{u}_1, \ldots, \hat{u}_m$ is a price-income equilibrium, i.e., the equality

$$f_1(p, w_1) + \cdots + f_m(p, w_m) = r$$

is satisfied.

Section Manifold as Set of Pareto Minima

Let us now prove the converse property, namely, that every element of $B(r)$ is a Pareto minimum for the map $\hat{u} : H_{++}(r) \to \mathbb{R}^m$. Here is a proof by contradiction. Assume that the element $b \in B(r)$ is not a Pareto minimum. Then there exists some Pareto minimum $b' \in H_{++}(r)$ that satisfies the inequalities $\hat{u}_i(b') \leq \hat{u}_i(b)$ for all i, one of these inequalities at least being strict.

These inequalities take the form $u_i(f_i(p', w_i')) \leq u_i(f_i(p, w_i))$ for all i, one inequality at least being strict. Since the price vector p' supports the allocation $f_i(p', w_i')$, the inequality translates into the inequality $p' \cdot f_i(p', w_i') \leq p' \cdot f_i(p, w_i)$, the inequality being strict if the inequality in utilities is itself strict. Using Walras' law, these inequalities become $w_i' \leq p' \cdot f_i(p, w_i)$ for i varying from 1 to m, with at least one inequality being strict. Adding up these inequalities for all i's yields the strict inequality

$$\sum_i w_i' < p' \cdot \sum_i f_i(p, w_i).$$

It follows from the assumption that $b = (p, w_1, \ldots, w_m)$ belongs to $B(r)$ that we have $\sum_i f_i(p, w_i) = r$, which therefore implies the strict inequality

$$\sum_i w_i' < p' \cdot r,$$

a contradiction with the assumption that $b' = (p', w_1', \ldots, w_m')$ belongs to $H_{++}(r)$. This proves that the converse of proposition 5.8.2 is also true. In other words, the following proposition is satisfied.

Proposition 5.8.3 The section manifold $B(r)$ coincides with the set of Pareto minima in $H_{++}(r)$ for the indirect utility functions $\hat{u}_i$ with $i = 1, 2, \ldots, m$.

5.9 Geometric Equilibrium Manifold

The next two sections introduce the concepts that underlie the geometric approach to the equilibrium equation.

5.9.1 Definition

The *geometric equilibrium manifold* $\mathcal{E}$ associated with the parameter space $\mathcal{A}$ is the subset of $B(r) \times \mathcal{A}$ consisting of the pairs (b, A), where b belongs to $B(r)$ and the $\ell - 1$ dimensional affine subspace A of $H(r)$ contains the point b.

5.9.2 Structure

The collection $\mathcal{A}(b)$ of affine subspaces $A \in \mathcal{A}$ that contain the point b can be identified with the Grassmann manifold $G_{m+\ell-2,\ell-1}$, which consists, by definition, of the vector subspaces of $\mathbb{R}^{m+\ell-2}$ having dimension $\ell - 1$. Therefore, the geometric equilibrium manifold $\mathcal{E}$ can be identified with the Cartesian product $B(r) \times G_{m+\ell-2,\ell-1}$. This result shows that the structure of the section manifold $B(r)$ is a key component of the structure of the geometric equilibrium manifold $\mathcal{E}$.

5.9.3 No-trade Equilibria in the Geometric Reformulation

Notice that the set $\mathcal{A}(b)$ of affine subspaces $A \in \mathcal{A}$ containing the point $b \in H(r)$ is the geometric equivalent of the fiber associated with the price-income vector $b = (p, w_1, \ldots, w_m) \in B(r)$. The no-trade equilibrium of that fiber corresponds to the affine space $A(f_1(p, w_1), \ldots, f_m(p, w_m))$. It is left as an exercise to check that this affine space is tangent at the point $b = (p, w_1, \ldots, w_m)$ to all the indirect indifference (hyper)surfaces of the various consumers in $H_{++}(r)$ through b.

Parenthetically, having a smaller parameter space like $\mathcal{A}_{++}$ does not change the general line of the geometric approach even if some technical difficulties are encountered in the proofs. For example, in the case of the parameter space $\mathcal{A}_{++}$ (which corresponds to economies characterized by strictly positive endowment vectors $\omega = (\omega_1, \ldots, \omega_m) \in \Omega(r)$), the fiber of the geometric equilibrium manifold becomes the smooth manifold $\mathcal{A}_{++}(b)$ made of elements of $\mathcal{A}_{++}$ that contain the element $b \in B(r)$. It then suffices to observe that the smooth manifold $\mathcal{A}_{++}(b)$ is diffeomorphic to $\mathbb{R}^{(\ell-1)(m-1)}$ to get the global structure of the equilibrium manifold for this parameter space.

5.9.4 Pathconnectedness

Back to the parameter space $\mathcal{A}$. A lot is known about the structure of Grassmann manifolds and their global properties. For example, Grassmann manifolds are pathconnected. Combined with the pathconnectedness of the section manifold $B(r)$, this implies readily the pathconnectedness of the geometric equilibrium manifold $\mathcal{E}$.

5.10 Application of the Geometric Approach to Genericity

5.10.1 Regular Geometric Equilibria

Let $(b, A) \in \mathcal{E}$ be a geometric equilibrium. By definition, this geometric equilibrium is *regular* if the submanifolds A and $B(r)$ of $H(r)$ are *transverse* at $b \in A \cap B(r)$. Since these two submanifolds have dimension $\ell - 1$ and $m - 1$, respectively, in the dimension $\ell + m - 2$ space $H(r)$, transversality means here that the intersection of the affine space A with the tangent affine space $T_b(B(r))$ to $B(r)$ at b consists only of the point b. In other words, the geometric equilibrium $(b, A) \in \mathcal{E}$ is *regular* if $T_b(B(r)) \cap A = \{b\}$. We denote by $\mathcal{RE}$ the set of *regular* geometric equilibria. It is a subset of $\mathcal{E}$. Section 5.10.4 presents a proof showing that the set of regular equilibria $\mathcal{RE}$ is an open and dense subset of the geometric equilibrium manifold $\mathcal{E}$. (The density property can even be strengthened into a property of full measure once a suitable measure has been defined for the set $\mathcal{A}$ consisting of the dimension $\ell - 1$ affine subspaces of $H(r)$ and then for the geometric equilibrium manifold $\mathcal{E}$.)

It also follows from the implicit function theorem that the correspondence that associates with $A' \in \mathcal{A}$ the intersection $A' \cap B(r)$ has a selection map that is locally defined, i.e., defined on some neighborhood of A in $\mathcal{A}$ and that this map $A' \to b' \in B(r) \cap A'$ is smooth.

5.10.2 Critical Geometric Equilibria

If the two submanifolds A and $B(r)$ are not transverse, the geometric equilibrium (b, A) is said to be *critical*. The dimension of the intersection $T_b(B(r)) \cap A$ is then at least equal to 1. In other words, the affine space A contains at least one line that is tangent to the section manifold $B(r)$. Denote by $\mathcal{C}$ the set of critical geometric equilibria. Roughly speaking, the latter case corresponds to the intersection point $b \in B(r) \cap A$ having an order of multiplicity at least equal to 2 (figure 5.5).

It follows readily from the definitions that the equilibrium (p, ω) is critical in the sense of the earlier chapters (i.e., $p \in S$ is a solution of the equation $\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i$ with an order of multiplicity at least equal to 2 for given $\omega \in \Omega(r)$) if and only if the price-income vector $(p, p \cdot \omega_1, \ldots, p \cdot \omega_m)$ is a critical geometric equilibrium.

5.10.3 Classification of Critical Geometric Equilibria

The geometric approach gives a very simple classification of the critical geometric equilibria $(b, A) \in \mathcal{CE}$ by the dimension of the intersection $T_b(B(r)) \cap A$. The geometric equilibrium $(b, A) \in \mathcal{CE}$ is *critical of type k*, where k is an integer ≥ 1, if dim $T_b(B(r)) \cap A = k$.

The dimension k of the intersection $T_b(B(r)) \cap A$ is a measure of how "bad" the singularity or criticality of the geometric equilibrium (b, A) is. The most common type of singularity corresponds to type 1, i.e., to an intersection $A \cap B(r)$ of dimension 1.

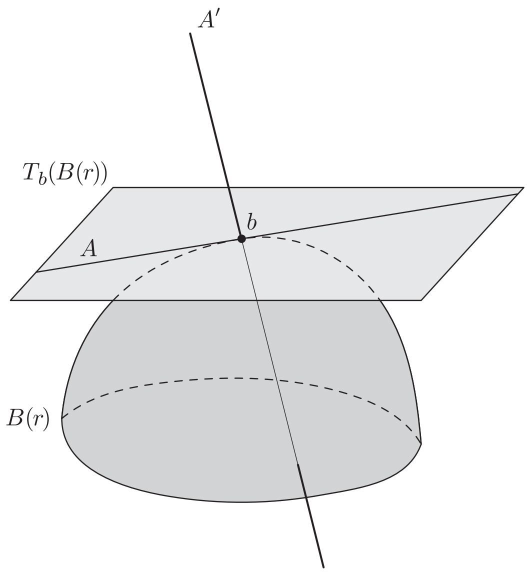

Figure 5.5
(b, A) regular equilibrium, (b, A') critical equilibrium

In terms of the equilibrium (p, ω), where $p \in S$ is a solution of the equation $\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i$ for given ω, criticality of type 1 means that the solution $p \in S$ is only a double root of the equilibrium equation. In other words, this solution has an order of multiplicity equal to 2. Roughly speaking, dimension $k \geq 2$ corresponds to solutions with orders of multiplicity at least equal to 2.

Let us denote by $\mathcal{C}_k\mathcal{E}$ the set of critical geometric equilibria $(b, A) \in \mathcal{E}$ of type k with $k \geq 1$. The set of critical geometric equilibria $\mathcal{CE}$ is partitioned into the subsets $\mathcal{C}_k\mathcal{E}$ for $k \geq 1$: $\mathcal{CE} = \bigcup_{k \geq 1} \mathcal{C}_k\mathcal{E}$. The structure of the set of critical geometric equilibria $\mathcal{CE}$ is discussed further in section 5.11.3.

5.10.4 Regular and Singular Geometric Economies

By definition, the *economy* $A \in \mathcal{A}$ is *regular* if it is transverse to the section manifold $B(r)$. In practice, this means that A is transverse to the section manifold $B(r)$ at all intersection points. (Note that the affine subspace A is also transverse to $B(r)$ if the intersection $A \cap B(r)$ is empty.)

By definition, the economy $A \in \mathcal{A}$ is *singular* if it is not transverse to the section manifold $B(r)$.

It follows readily from Thom's transversality theorems that the set of economies (i.e., the set of dimension $\ell - 1$ affine subspaces of $H(r)$) that are transverse to the section manifold $B(r)$ is open and dense in $\mathcal{A}$. (An alternative proof of this property that does not require Thom's transversality theorems is given in section 5.11.3.)

5.11 Global Properties of the Geometric Equilibria

Let us investigate the properties that deal with the pathconnectedness of the set of economies with multiple equilibria, the characterization of the set of regular economies with a unique equilibrium in the case of two consumers, and the structure of the set of critical geometric equilibria.

5.11.1 Economies with Multiple Equilibria

Let $b \in B(r)$ be fixed. Recall that $\mathcal{A}(b)$ denotes the set of dimension $\ell - 1$ affine subspaces of $H(r)$ through b. Let $\mathcal{M}(b)$ denote the set of affine subspaces $A \in \mathcal{A}(b)$ that either intersect $B(r)$ at another point $b' \neq b$ or are tangent to $B(r)$ at b. The former case corresponds to having at least two different geometric equilibria associated with A, whereas the latter case is simply the "infinitesimal version" when two equilibria merge.

Let $\mathcal{M}$ be the set of affine subspaces $A \in \mathcal{A}$ that have at least two intersection points with $B(r)$ or that are not transverse to $B(r)$. (Note that these two conditions are not exclusive: it may very well be that A intersects $B(r)$ at different points and, in addition, is tangent to $B(r)$ at one of these intersection points.) Similarly, define $\mathcal{U}$ as the set of (regular) economies that are transverse to $B(r)$ and with a unique intersection point with $B(r)$.

Pathconnectedness of $\mathcal{M}(b)$

Proposition 5.11.1 The set $\mathcal{M}(b)$ of economies with multiple equilibria through the price-income equilibrium b is pathconnected.

Proof of Case $\ell = 2$ Let us first prove the pathconnectedness of $\mathcal{M}(b)$ in the case $\ell = 2$. The affine subspace A is a line. The elements of $\mathcal{M}(b)$ are therefore the lines passing through the point $b \in B(r)$ that are either tangent to $B(r)$ at b or that intersect $B(r)$ at another point b'. This leads us to consider the manifold that is obtained by "blowing-up" $B(r)$ at point b. This manifold consists of the points $b' \neq b$ in $B(r)$ and of the

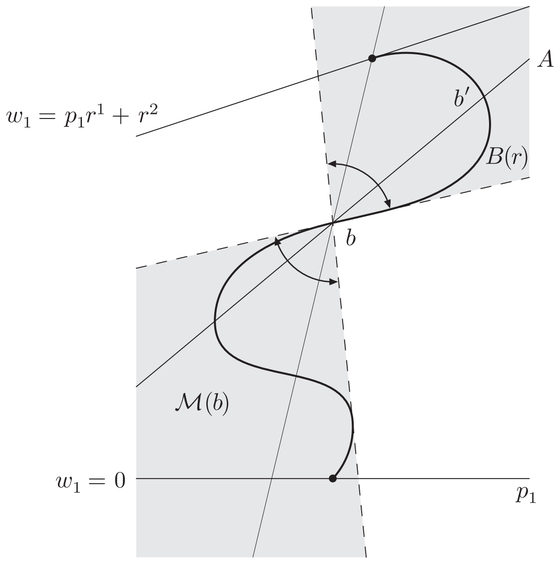

Figure 5.6
Set $\mathcal{M}(b)$

tangents to $B(r)$ at b. This manifold, denoted by $\widehat{B(r)}_b$, is known to be pathconnected, a property that can be checked rather easily given the construction of the blown-up manifold $\widehat{B(r)}_b$. The map $b' \in \widehat{B(r)}_b \to bb'$ defines a continuous map from the pathconnected manifold $\widehat{B(r)}_b$ into the set of lines of $H(r)$. The image of this map is the set $\mathcal{M}(b)$ (figure 5.6). This set is therefore pathconnected as the image of a pathconnected set by a continuous map.

Proof of Case $\ell \geq 3$ The preceding proof is now extended to the case $\ell \geq 3$. We have to accommodate the fact that two distinct points do not now determine a unique element $A \in \mathcal{A}(b)$. Let us consider the lines D in the space $H(r)$. The line D is a *multiple direction* for the affine subspace $A \in \mathcal{A}$ if the following two properties are satisfied: (1) $b \in D$ and $D \subset A$, and (2) the line D is tangent to $B(r)$ at b or D intersects $B(r)$ at some other point $b' \neq b$.

It follows from the definition of a multiple direction that $A \in \mathcal{A}(b)$ belongs to $\mathcal{M}(b)$ if and only if it admits a multiple direction.

Let now A' and A'' belong to $\mathcal{M}(b)$ and show that a continuous path can be found that links A' to A'' in the set $\mathcal{M}(b)$. The following two cases are distinguished.

The spaces A' and A'' have a multiple direction in common. Let D be this multiple direction. The projection parallel to the line D defines a bijection, actually a homeomorphism, between the set of dimension $\ell - 1$ linear subspaces of $H(r)$ that contain D and the set of $\ell - 2$ vector subspaces of $\mathbb{R}^{\ell+m-3}$, i.e., the Grassmann manifold $G_{\ell+m-3,\ell-2}$. The Grassmann manifold is pathconnected; there exists therefore a continuous path in $G_{\ell+m-3,\ell-2}$ defined by the continuous map $t \in [0,1] \rightarrow A(t)$ that links A' to A'' and such that $A(t)$ contains the line D. The line D is therefore a multiple direction of $A(t)$, from which follows that $A(t)$ does belong to $\mathcal{M}(b)$.

The spaces A' and A'' have no common multiple direction. Let D' (resp. D'') be a multiple direction associated with A' (resp. A''). Because $\{b\} \in D' \cap D''$, the lines D' and D'' generate a plane; because $\ell - 1 \geq 2$, there exists a dimension $\ell - 1$ linear space A of $H(r)$ that contains the plane generated by the lines D' and D'' and therefore the lines D' and D'' themselves. It is clear that D' and D'' are multiple directions for the space A, which implies that A belongs to $\mathcal{M}(b)$. The pairs (A, A') and (A, A'') both have a common multiple direction, so the preceding case can be applied to build a continuous path in $\mathcal{M}(b)$ that goes from A' to A and then from A to A''. ■

The results just obtained can be given the following simple and intuitive illustration. The line bb' for b and b' in $B(r)$ can be interpreted as the direction through which the point b' of $B(r)$ is viewed from the point $b \in B(r)$. Then—at least in the case of $\ell = 2$—the set $\mathcal{M}(b)$ corresponds to the "view" one gets of the (hyper)surface $B(r)$ from the point $b \in B(r)$.

Pathconnectedness of $\mathcal{M}$

Proposition 5.11.2 The set $\mathcal{M}$ of economies with multiple equilibria is pathconnected.

Proof Let A and A' be elements of $\mathcal{M}$. By definition there exist $b \in B(r) \cap A$ and $b' \in B(r) \cap A'$. If $b = b'$, then A and A' belong to $\mathcal{M}(b)$, which is pathconnected and can therefore be linked by a continuous path in that space. If $b \neq b'$, the line bb' is a multiple direction of any $A'' \in \mathcal{A}$ that contains b and b', so A'' belongs to $\mathcal{M}(b)$ and to $\mathcal{M}(b')$. Let A'' be such subspace. Then, we can find a continuous path linking A to A'' (in $\mathcal{M}(b)$) and another one linking A'' to A' (in $\mathcal{M}(b')$). ■

5.11.2 Set of Regular Economies with a Unique Equilibrium in the Case $m = 2$

Let $\mathcal{U}$ denote the set of regular economies with a unique equilibrium and $\mathcal{U}(b)$ the subset of $\mathcal{U}$ consisting of the affine spaces A that contain the price-income equilibrium $b \in B(r)$. For $A \in \mathcal{U}(b)$, the affine space A is transverse to the section manifold $B(r)$ at the point b and the intersection $A \cap B(r)$ is reduced to the point b.

The complement of $\mathcal{U}(b)$ in $\mathcal{A}(b)$ is the set $\mathcal{M}(b)$.

Characterization of $\mathcal{U}(b)$

Proposition 5.11.3 For $m = 2$, the set $\mathcal{U}(b)$ is the connected component of the set of regular economies in $\mathcal{A}(b)$ that contain the affine space $A(f(b))$.

Proof Let $b = (p, w_1, \ldots, w_m) \in B(r)$. Then,

$$f(b) = (p, f_1(p, w_1), \ldots, f_m(p, w_m)),$$

and $f(b)$ is a no-trade equilibrium. It is known that the equilibrium $f(b)$ is regular and that $p \in S$ is the unique equilibrium price vector associated with the endowment vector $\omega = (f_1(p, w_1), \ldots, f_m(p, w_m))$. This implies that the affine space $A(f(b))$ belongs to $\mathcal{U}(b)$.

The proposition therefore follows from the proof of the pathconnectedness of $\mathcal{U}(b)$.

For $m = 2$, the set $B(r)$ is actually a smooth pathconnected curve. The elements of $\mathcal{A}$ are hyperplanes of $H(r)$, an affine space of dimension $2 + \ell - 2 = \ell$. Let A' and A'' be two distinct hyperplanes of $H(r)$ passing through the point b, transverse to the curve $B(r)$ and intersecting $B(r)$ only at the point b. Let F be the dimension $\ell - 2$ space that is the intersection of A' and A''. The ambiant space $H(r)$ is decomposed by the hyperplanes A' and A'' into four quadrants. The tangent to the curve $B(r)$ at b being transverse to A' and to A'' is therefore contained in two opposite quadrants. We denote these quadrants by I and III. It follows from this definition of quadrants I and III that the curve $B(r)$ also has points in quadrants I and III.

The set $B(r) \backslash \{b\}$ is decomposed into two connected components. Each one of these components is contained in one of the quadrants defined by A' and A'' because otherwise A' and A'' would intersect $B(r)$ at another point in addition to b. One component contains a point of $B(r)$ that belongs to quadrant I. Call this component I$'$. This component is necessarily contained in quadrant I. The second component of $B(r)\setminus\{b\}$ necessarily contains an element of quadrant III. This implies that it is fully contained in quadrant III. Denote this component by III$'$. The curve $B(r)$ is the union of the components I$'$ and III$'$ and of the point $\{b\}$. Therefore, there is no element of $B(r)$ in the opposite quadrants II and IV.

Therefore, any hyperplane containing the subspace $F = A' \cap A''$ and contained in quadrants II and IV belongs to $\mathcal{U}(b)$. Because the set of hyperplanes contained in two opposite quadrants is pathconnected, there exists at least one continuous path in $\mathcal{U}(b)$ that links A' to A''. ∎

Pathconnectedness of $\mathcal{U}$

Proposition 5.11.4 The set $\mathcal{U}$ of economies with a unique equilibrium is in the case $m = 2$ the connected component of the set of regular economies that contains the affine spaces associated with the Pareto optima.

Proof The set $\mathcal{U}$ is the union of connected components of the set of regular economies such that, for each economy A in any such component, the intersection $A \cap B(r)$ is reduced to a point.

Let us show that $\mathcal{U}$ is made of only one connected component of the set of regular economies and that this component contains the affine subspaces associated with the Pareto optima.

Let $b = (p, w_1, w_2, \dots, w_m) \in B(r)$. Consider

$$A_b = A(f_1(p, w_1), f_2(p, w_2), \dots, f_m(p, w_m)),$$

the affine space that corresponds to the endowment vector

$$\omega = (f_1(p, w_1), \dots, f_m(p, w_m)),$$

which is a Pareto optimum. We have seen that A_b belongs to $\mathcal{U}(b)$ (and therefore to $\mathcal{U}$).

Let $A' \in \mathcal{U}(b')$ and $A'' \in \mathcal{U}(b'')$. By proposition 5.11.3, there exists a continuous path in $\mathcal{U}(b')$ that links A' to $A_{b'}$. Similarly, A'' is linked to $A_{b''}$ by a continuous path in $\mathcal{U}(b'')$. Because $B(r)$ is pathconnected, we can find a continuous path $t \in [0, 1] \to b(t) \in B(r)$ that links b' and b''. The continuous path $t \to A_{b(t)}$ in $\mathcal{U}$ then links $A_{b'}$ to $A_{b''}$. ∎

5.11.3 Structure of Set of Critical Equilibria

The next proposition clarifies the structure of the strata $\mathcal{C}_i\mathcal{E}$ of the set of critical equilibria $\mathcal{CE}$. Though a really rigorous proof would require technical notions involving vector and Grassmann bundles, the intuition provided by the geometric approach allows a highly intuitive presentation that is sufficient for the present purpose.

Proposition 5.11.5 The set $\mathcal{C}_k\mathcal{E}$ of critical equilibria of type k is a smooth submanifold of the set of equilibria $\mathcal{CE}$ diffeomorphic to $B(r) \times G_{m+\ell-2-k,\ell-1-k} \times G_{m-1,k}$.

Proof The stratified structure of the set $\mathcal{C}_k\mathcal{E}$ can be generated as follows. Pick some $b \in B(r)$. In the tangent space $T_b(B(r))$, consider the collection of dimension k linear subspaces of $T_b(B(r))$ through the point b. For a fixed $b \in B(r)$, this set can be identified with the Grassmann manifold $G_{m-1,k}$. Let F be one of these linear subspaces. Now consider the collection of dimension $\ell - 1$ linear subspaces of $H(r)$ that contains the linear subspace F. The latter space (for fixed $b \in B(r)$ and F) can be identified with the Grassmann manifold $G_{m+\ell-2-k,\ell-1-k}$. It then suffices to vary $b \in B(r)$ and F in $T_b(B(r))$ to get all the possible cases. With some mathematical experience, it is straightforward to show that this set has the structure of a smooth manifold and is a trivial "fiber bundle" over $B(r)$, the "typical fiber" being diffeomorphic to the Cartesian product of Grassmann manifolds $G_{m+\ell-2-k,\ell-1-k} \times G_{m-1,k}$. This proof can easily be made rigorous. ■

Emptiness of the Set of Critical Equilibria of Type k with k ≥ inf(m, ℓ)
As a special case of the preceding proposition—here we do not even need the advanced theory of Grassmann manifolds—we have that $\mathcal{C}_k\mathcal{E}$ is empty for $k \geq \inf(\ell, m)$ because the dimension of the intersection of $T_b(B(r)) \cap A$ cannot exceed the smaller of the dimensions of $T_b(B(r))$ and A, i.e., $\inf(m-1, \ell-1)$.

Measure Zero of the Set of Critical Geometric Equilibria
The dimension of the manifold $\mathcal{C}_k\mathcal{E}$ is strictly less than the dimension $(m-1)\ell$ of the geometric equilibrium manifold $\mathcal{E}$. Therefore, the Lebesgue measure of the smooth submanifold $\mathcal{C}_k\mathcal{E}$ in $\mathcal{E}$ is zero. A finite union of closed sets of measure zero is closed and still has measure zero. This proves that the set of critical geometric equilibria $\mathcal{CE}$ has measure zero in the geometric equilibrium manifold $\mathcal{E}$.

This readily implies that the set of singular economies, the image of the set of critical equilibria $(b, A) \in \mathcal{CE}$ by the projection $(b, A) \to A$ has also measure zero. Its complement, the set of regular economies, has full measure.

5.12 Number and Determinateness of Equilibria

This section addresses the issue of the relation between the number of equilibria and their determination. Let us denote by $W(A)$ the intersection $W(A) = B(r) \cap A$ so that W can be viewed as the equilibrium correspondence that associates with every economy $A \in \mathcal{A}$ the corresponding equilibria.

Let $N(A)$ be the number (finite or infinite) of elements of the set $W(A)$. This is the number of equilibria associated with the economy $A \in \mathcal{A}$.

Proposition 5.12.1 The function $N : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ determines the equilibrium correspondence W.

Thus, knowing the number of solutions for all possible values of the parameter $A \in \mathcal{A}$ is equivalent to knowing the precise value of these solutions. This peculiarity of the equilibrium equation is a far-reaching consequence of the assumptions that underlie the Arrow-Debreu model. This proves, if it is still necessary at this stage, that this model does indeed have a lot of structure.

The intuition behind proposition 5.12.1 is that the section manifold $B(r)$ is determined by its tangent spaces. These tangent spaces are determined by the singular economies $A \in \mathcal{A}$. The singular economies A are those at which the function N is not locally constant. Despite the simplicity of this observation, putting this into a rigorous proof is not totally straightforward.

The easiest way to prove proposition 5.12.1 is to use the Radon transform, which amounts to integrating the function $N(A)$ over the space $\mathcal{A}$. Such a proof is slightly beyond the scope of this book. For suitable mathematical references, see, for example, Helgason (1980). For an alternative proof of a closely related property in the case where affine subspaces A perpendicular to the price set S are excluded (those spaces correspond to endowments located at "infinity" in the parameter space $\widetilde{\Omega(r)}$), readers can consult Balasko (1988, theorem 7.3.9).

Exercises

5.3. Let $\mathcal{A}_{\geq n}$ be the subset of $\mathcal{A}$ that has at least n intersection points with the section manifold $B(r)$. Show that $\mathcal{A}_{\geq n}$ has a nonempty interior for $n \leq \ell$. (*Hint:* Pick n points of $B(r)$ and $\ell - n$ other points in $H(r)$.)

5.4. Let $\mathcal{A}_{\geq n}$ be defined as in exercise 5.3. Prove that for $2 \leq n \leq \ell$, the set $\mathcal{A}_{\geq n}$ is pathconnected.

5.13 Conclusion

The geometric approach of this chapter is more powerful than the direct approach of the previous chapters for issues involving the determinateness or the number of equilibria. Besides the technical simplification brought by intersections of sets instead of envelopes of linear manifolds, the geometric approach is efficient because in the definitions of the section and budget manifolds $B(r)$ and $A(\omega)$, respectively, preferences, individual resources, and total resources play well-differentiated roles. The section manifold $B(r)$ is defined by individual preferences and total resources, whereas the budget manifold $A(\omega)$ depends only on the individual endowments $\omega = (\omega_i)$.

5.14 Notes and Comments

The formulation of the equilibrium equation as the intersection of the section and budget manifolds is due to Balasko (1979a).

The observation that extending the properties of the equilibrium manifold and the natural projection to the case of fixed total resources crucially depends on the structure of the set of Pareto optima for fixed total resources given the linear structure of the fibers of the equilibrium manifold is due to Balasko (1979a). These properties include in particular the pathconnectedness of the equilibrium manifold and the openness and full measure of the set of regular economies.

The property that the set of Pareto optima for fixed total resources is homeomorphic to a Euclidean space is due to Bewley (1969; 1972). The diffeomorphism is proved by Balasko (1979b; 1988) for the case of consumption sets equal to the full commodity space. The proof for consumption sets limited to the strictly positive orthant given here is new.

The interpretation of the section manifold as the set of Pareto minima for the indirect utility functions is due to Balasko (1979b).

Indirect utility functions were first considered by Hotelling (1938) and Roy (1942). This chapter is in some sense an extension of the general equilibrium setup of an arbitrary number of consumers of the Hotelling and Roy approach, the latter being limited to a single consumer. Indirect utility functions for an arbitrary number of consumers are also used by Luenberger (1994) in issues of welfare economics.

The application of the geometric approach to issues related to the number of equilibria is due to Balasko (1979a; 1979b; 1988). Ghiglino and Tvede (1997) give a better generic lower bound on the number of equilibria than the one of exercise 5.3 for consumption sets equal to the full commodity space.

6 Economies with Price-Dependent Utility Functions

6.1 Introduction

So far, utility has depended only the consumer's own consumption. This representation of consumers' preferences ignores the many forms of externalities associated with consumption that exist in the real world. The full-scale analysis of general equilibrium models with general forms of consumption externalities would require developments that go far beyond the scope of this book. Nevertheless, there is one kind of consumption externality that can be handled by a simple adaptation of the techniques of the previous chapters. This externality occurs when the consumer's utility function depends on the price system in addition to the consumer's own consumption. This form of externality is indirect because it is through the value taken by the equilibrium price system that a consumer's utility depends on the other consumers' consumptions.

The goal of this chapter is to extend the properties of comparative statics associated with the structure of the equilibrium manifold to the setup of price-dependent utility functions. I show that pathconnectedness and homeomorphism/diffeomorphism with a Euclidean space of the equilibrium manifold, and the smoothness and properness of the natural projection, extend readily to the setup of price-dependent utility functions provided that total resources can vary. The fact that these properties do not hold true when total resources are held constant highlights a deep difference between the two models.

The main difference in the technical assumptions of this chapter compared to those of the previous chapters is in having non-normalized prices. The cost of this minor complication is offset by the simplicity it brings in disentangling the two different roles played by the price system when utility functions are price-dependent, one as an argument of utility functions, the other as defining the consumer's budget set.

6.2 Consumer's Theory with Price-Dependent Utility Functions

6.2.1 Price-Dependent Utility Functions

Preferences are represented by utility functions of the form $u_i(p, x_i)$, where the commodity bundle x_i belongs to the strictly positive orthant $X = \mathbb{R}^{\ell}_{++}$. At variance with the approach of the previous chapters, the price vector $p = (p_1, p_2, \ldots, p_\ell)$ belongs to the strictly positive orthant X and is not normalized. Since prices are not normalized, the utility function $u_i(p, x_i)$ is assumed to be homogeneous of degree 0 with respect to the price vector p. In other words, utility depends on relative prices and not on some absolute measure of the price level.

When dealing with homogeneous functions of degree 0, it is mathematically convenient to consider the set $\mathbb{P}_{\ell-1}$ of lines of $\mathbb{R}^{\ell}$ passing through the origin, a set known as the projective space of dimension $\ell - 1$. Since prices are strictly positive, this leads to consideration of the subset $\mathbb{P}(X)$ of lines defined by strictly positive vectors. Denote by $\overline{\mathbb{P}(X)}$ its closure in $\mathbb{P}_{\ell-1}$. Also needed is the set $\mathbb{P}(X \times \mathbb{R}_{++})$, which consists of the directions defined by the vectors in $X \times \mathbb{R}_{++}$, i.e., the strictly positive vectors of $\mathbb{R}^{\ell} \times \mathbb{R}$. The same notation is used for the line belonging to $\mathbb{P}(X)$ (resp. $\overline{\mathbb{P}(X)}$, $\mathbb{P}(X \times \mathbb{R}_{++})$), defined by the vector $p \in X$ (resp. $p \in \overline{X}$, $(p, w_i) \in X \times \mathbb{R}_{++}$) and the vector p (resp. p and (p, w_i)) itself.

The notation $u_i(p, x_i)$ therefore represents either a homogeneous function of degree 0 with respect to p if (p, x_i) represents an element of $X \times X$, or simply a function of (p, x_i) if (p, x_i) is considered an element of $\mathbb{P}(X) \times X$. No serious confusion can result from this simplification.

Exercise

6.1. Let $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$ be two points of $\mathbb{R}^{n+1} \backslash \{0\}$. Define the relation $x \sim y$ by the condition: there exists $\lambda \neq 0$ such that $x = \lambda y$.

a. Prove that $\sim$ is an equivalence relation on $\mathbb{R}^{n+1} \backslash \{0\}$. Let $\bar{x}$ denote the equivalence class of $x \in \mathbb{R}^{n+1} \backslash \{0\}$. The quotient space $\mathbb{R}^{n+1} \backslash \{0\} / \sim$ is the set of equivalence classes $\bar{x}$ for $x \in R^{n+1} \backslash \{0\}$. Let $p : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{P}_n$ denote the quotient map $x \to \bar{x}$.

b. Show that the quotient space $\mathbb{R}^{n+1} \backslash \{0\} / \sim$ is the projective space $\mathbb{P}_n$.

c. The coordinates $(x_1, x_2, \ldots, x_{n+1})$ of $x \in \bar{x}$ are known as a set of homogeneous coordinates of the element $\bar{x} \in \mathbb{P}_n$. Show that every function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ that is homogeneous of degree 0 defines a function $\bar{f} : \mathbb{P}_n \to \mathbb{R}$ such that $f = \bar{f} \circ p$.
d. The projective space $\mathbb{P}_n$ is equipped with the quotient topology, i.e., the strongest topology for which the quotient map $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}_n$ is continuous. Prove that if the map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous, the associated function $\bar{f} : \mathbb{P}_n \to \mathbb{R}$ is also continuous.
e. Show that the projective space $\mathbb{P}_n$ equipped with the quotient topology is compact.

6.2.2 Properties of Price-Dependent Utility Functions

The following assumptions extend to price-dependent utility functions the assumptions of continuity, differentiability, monotonicity, and quasi-concavity made in chapter 1, section 1.2.2, for non-price-dependent utility functions:

i. The price-dependent utility function $u_i : \overline{\mathbb{P}(X)} \times X \to \mathbb{R}$ is continuous, surjective, and differentiable up to any order in the interior $\mathbb{P}(X) \times X$.

ii. *Smooth monotonicity*. The partial utility function $u_i(p, .) : X \to \mathbb{R}$ is differentiable for all $p \in \overline{\mathbb{P}(X)}$, and every partial derivative with respect to $x^j > 0$.

iii. *Smooth quasi-concavity*. The restriction of the quadratic form defined by the Hessian matrix of second-order derivatives with respect to the coordinates of x_i to the tangent hyperplane at $x_i \in X$ to the indifference surface through x_i is negative definite for any $p \in \overline{\mathbb{P}(X)}$ and $x_i \in X$.

iv. *Necessity of every commodity*. For any $u_i \in \mathbb{R}$, the indifference surface $\{x_i \in X \mid u_i(p, x_i) = u_i\}$ is closed in $\mathbb{R}^\ell$ for all $p \in \overline{\mathbb{P}(X)}$.

The continuity property of the utility function on the boundary $(\overline{\mathbb{P}(X)} \setminus \mathbb{P}(X)) \times X$ expresses the continuity of the price effect on preferences even when some relative prices tend to zero or infinity. Properties (ii)–(iv) are the direct extensions to the setup of price-dependent preferences of the equivalent properties when preferences do not depend on prices.

Exercise

6.2. The study of temporary financial equilibria (see chapter 9) requires consideration of individual consumption sets that are different from the strictly positive orthant X. More precisely, let X_i be the subset of the commodity space $\mathbb{R}^\ell$ defined by linear equalities of the form $p(1) \cdot x_i \geq w_i(1)$, $p(2) \cdot x_i \geq w_i(2), \ldots,$ $p(k) \cdot x_i \geq w_i(k)$, where $p(1), p(2), \ldots,$ $p(k)$ belong to X. The vector $y \in \mathbb{R}^\ell$ is a *direction of recession* if, starting at any $x \in X_i$, the point $x + \alpha y$ belongs to X_i for any $\alpha > 0$. The *recession cone* of the convex set X_i is the set of all directions of recession.

a. Show that all the sets X_i have the same recession cone P.

b. Show that, for every i, there exists $x_i \in \mathbb{R}^\ell$ such that the set X_i is contained in $x_i + P$.

An example of a price-dependent utility function is the utility function

$$u_i(p, x_i) = \sum_{j=1}^{\ell} a_i^j(p) \ln x_i^j, \tag{6.1}$$

where it is customary to add the restrictions $a_i^j(p) > 0$ and $\sum_{j=1}^{\ell} a_i^j(p) = 1$. This function depends on prices through its coefficients $a_i^j(p)$, which are homogeneous functions of degree 0 of $p \in X$.

6.2.3 Extended Demand Function

Consumer's Maximization Problem

For given $p \in \mathbb{P}(X)$, consumer i maximizes the utility $u_i(p, x_i)$ subject to the budget constraint $p \cdot x_i = w_i$. In this maximization problem, the price vector $p \in \mathbb{P}(X)$ plays two different roles: one as argument of the utility function, another in defining the budget constraint.

Disentangling the Two Roles of the Price System

These two different roles are disentangled by considering the more general problem of maximizing $u_i(q, x_i)$ for a given $q \in \overline{\mathbb{P}(X)}$ subject to the budget constraint $p \cdot x_i = w_i$ for $p \in \mathbb{P}(X)$, the price vectors p and q being not necessarily equal. This maximization problem is purely mathematical or abstract in the sense that it has no direct economic interpretation.

Extended Demand Function

With the price vector $q \in \overline{\mathbb{P}(X)}$ being kept fixed, the preceding maximization problem is identical to the consumer's maximization problem for non-price-dependent utility functions. Under our assumptions, this problem has a unique solution denoted by $\varphi_i(q, p, w_i)$ for any $(p, w_i) \in X \times \mathbb{R}_{++}$ and $q \in \overline{\mathbb{P}(X)}$.

The solution of this maximization problem defines the *extended demand function* $\varphi_i(q, p, w_i)$ of consumer *i*, a function that is homogeneous of degree 0 with respect to $(p, w_i) \in X \times \mathbb{R}_{++}$. The same notation is used for the map

$$\varphi_i : \overline{\mathbb{P}(X)} \times \mathbb{P}(X \times \mathbb{R}_{++}) \to X,$$

which represents consumer *i*'s extended demand given $q \in \overline{\mathbb{P}(X)}$ and $(p, w_i) \in \mathbb{P}(X \times \mathbb{R}_{++})$.

Continuity and Smoothness

One way of proving the smoothness of individual demand functions when utility does not depend on prices is to apply the implicit function theorem to the first-order conditions of the consumer's maximization problem (see exercise 1.6 in chapter 1). This technique can be extended to the case of the extended individual demand function $\varphi_i(q, (p, w_i))$ by using the parametric version of the implicit function theorem, where continuity and differentiability follow from the continuity and differentiability of the utility function $u_i(q, x_i)$ on $\overline{\mathbb{P}(X)} \times X$ (resp. $\mathbb{P}(X) \times X$) (Balasko 1997).

Diffeomorphism

The following property extends to the price-dependent setup an important property of the individual demand functions for non-price-dependent preferences.

Proposition 6.2.1 The partial extended demand function

$$\varphi_i(q, ., .) : \mathbb{P}(X \times \mathbb{R}_{++}) \to X$$

is a diffeomorphism for any $q \in \overline{\mathbb{P}(X)}$ and satisfies Walras' law $p \cdot \varphi(q, p, w_i) = w_i$ for $(p, w_i) \in \mathbb{P}(X \times \mathbb{R}_{++})$.

Proof Let $D_x u_i(q, x_i)$ denote the gradient vector of the partial utility function $u_i(q, .)$ at $x_i \in X$. Then $D_x u_i(q, x_i) \in X$.

For $q \in \overline{\mathbb{P}(X)}$, and $x_i \in X$, define

$$g_i(q, x_i) = (D_x u_i(q, x_i), D_x u_i(q, x_i) \cdot x_i),$$

where the notation $(D_x u_i(q, x_i), D_x u_i(q, x_i) \cdot x_i)$ is used to represent this element of $X \times \mathbb{R}_{++}$ and also the corresponding element in $\mathbb{P}(X \times \mathbb{R}_{++})$.

It follows from this formula that the partial map $g_i(q, .) : X \to \mathbb{P}(X \times \mathbb{R}_{++})$ is smooth (i.e., differentiable up to any order) because $u_i(q, .)$ is. In addition, it follows from the necessity and sufficiency of the first-order conditions for the maximization of $u_i(q, x_i)$ subject to the constraint $p \cdot x_i = w_i$ that the map $g_i(q, .) : X \to \mathbb{P}(X \times \mathbb{R}_{++})$ is the inverse of the map $\varphi_i(q, ., .) : \mathbb{P}(X \times \mathbb{R}_{++}) \to X$.

Walras' law means that the budget constraint is binding, which follows from the monotonicity of the utility function.

Exercise

6.3. Let X_i be a consumption set as in exercise 6.2. Let Y denote the interior of the dual of the recession cone P of X_i. Let $\mathbb{P}(Y)$ denote the set of lines defined by the elements of Y. Consider now the following extension of the assumptions regarding price-dependent utility functions:

- The price-dependent utility function is a continuous map $u_i : \overline{\mathbb{P}(Y)} \times X_i \to \mathbb{R}$ that is onto and differentiable up to any order in the interior $\mathbb{P}(Y) \times X_i$.
- *Smooth monotonicity* of the (partial) function $u_i(p, .)$. The partial utility function $u_i(p, .) : X_i \to \mathbb{R}$ is differentiable for all $p \in \overline{\mathbb{P}(Y)}$, and every partial derivative with respect to $x_i^j > 0$.
- *Smooth quasi-concavity* of the (partial) function $u_i(p, .)$. (The restriction of the quadratic form defined by $D^2 u_i(p, .)$ to the tangent hyperplane at $x_i^* \in X$ to the hypersurface

$$\{x_i \in X_i \mid u_i(p, x_i) = u_i(p, x_i^*)\}$$

is negative definite for any $p \in \overline{\mathbb{P}(Y)}$ and $x_i^* \in X_i$.

- *Necessity of every commodity*. For any $u_i \in \mathbb{R}$, the indifference hypersurface $\{x_i \in X_i \mid u_i(p, x_i) = u_i\}$ is closed in $\mathbb{R}^\ell$ for all $p \in \overline{\mathbb{P}(Y)}$.

a. Show that for every $p \in Y$, there exists $w_i^* \in \mathbb{R}$ such that $p \cdot x_i \geq w_i^*$ for $x_i \in X_i$.

b. Show that for every $q \in \overline{\mathbb{P}(Y)}$, $p \in Y$, and $w_i > w_i^*$, there exists a unique element $x_i \in X_i$ that maximizes $u_i(q, p, x_i)$ subject to the constraint $p \cdot x_i = w_i$.

c. Define the extended demand function by $\varphi(q, (p, w_i)) = x_i$. Show that the extended demand function is smooth.

6.2.4 Strict Demand Function

The *strict demand function* $f_i(p, w_i)$ is simply the solution of the problem of maximizing the utility $u_i(p, x_i)$ subject to the constraint $p \cdot x_i = w_i$ for $(p, w_i) \in X \times \mathbb{R}_{++}$.

We have

$$f_i(p, w_i) = \varphi_i(p, (p, w_i)). \tag{6.2}$$

The strict demand function is homogeneous of degree 0. As before, we use the same notation $f_i(\mathbb{P}(X \times \mathbb{R}_{++})) \to X$ for the demand viewed as a function of elements of the projective space.

It follows readily from formula (6.2) combined with the differentiability of the composition of differentiable functions that the (homogeneous of degree 0) strict demand function $f_i : X \times \mathbb{R}_{++} \to \mathbb{R}^\ell$ is smooth. It also satisfies Walras' law, namely, the identity $p \cdot f_i(p, w_i) = w_i$.

Following are examples of demand functions associated with price-dependent log-linear utility functions.

Consider again the price-dependent log-linear utility function

$$u_i(q, x_i) = \sum_{j=1}^{\ell} a_i^j(q) \ln x_i^j.$$

The extended demand function is then equal to

$$\varphi_i(q, p, w_i) = \left(\frac{a_i^1(q)}{p_1}, \frac{a_i^2(q)}{p_2}, \dots, \frac{a_i^\ell(q)}{p_\ell} \right) w_i, \tag{6.3}$$

for $q \in \overline{\mathbb{P}(X)}$, and $(p, w_i) \in X \times \mathbb{R}_{++}$.

The strict demand function is equal to

$$f_i(p, w_i) = \left(\frac{a_i^1(p)}{p_1}, \frac{a_i^2(p)}{p_2}, \dots, \frac{a_i^\ell(p)}{p_\ell} \right) w_i. \tag{6.4}$$

Note that the diffeomorphism property of the individual demand function, the symmetry and negative definiteness of the Slutsky matrix—all properties satisfied in the case of non-price-dependent preferences (see, e.g., exercises 1.10, 1.11, 1.14, 1.16, and 1.17 in chapter 1)—are not satisfied by the strict demand functions resulting from the maximization of price-dependent preferences.

6.2.5 Equilibrium

The definition of an equilibrium extends readily to the price-dependent setup. The pair $(p, \omega) \in \mathbb{P}(X) \times \Omega$ is an equilibrium if the equilibrium equation

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i \tag{6.5}$$

is satisfied.

6.3 Equilibrium Manifold

The *equilibrium manifold* E is the subset of $\mathbb{P}(X) \times \Omega$ consisting of the pairs (p, ω) that satisfy equation (6.5).

As in the case of price-independent preferences, it is not obvious from this definition that the equilibrium manifold E is actually a smooth submanifold of $\mathbb{P}(X) \times \Omega$ nor even a smooth manifold in itself.

The main results of this section consist in the extension to the setup of price-dependent utility functions of the local and global properties of the equilibrium manifold proved for the setup of non-price-dependent utility functions.

6.3.1 Local Structure: Smooth Manifold

Proposition 6.3.1 The equilibrium manifold E is a smooth submanifold of $\mathbb{P}(X) \times \Omega$.

Proof The proof of proposition 2.4.1 in chapter 2 for the case of total variable resources does not require more than the differentiability of the demand functions $f_i(p, w_i)$ combined with Walras' law. These properties are also satisfied by the strict demand functions defined in the previous section. This proves readily the smooth submanifold structure of the equilibrium manifold E for the case of price-dependent utility functions and variable total resources. ■

6.3.2 Global Structure: From Pathconnectedness to Diffeomorphism with a Euclidean Space

Here, too, all the global properties proved for the equilibrium manifold in chapter 4 remain true in the setup of price-dependent utility functions and variable total resources. In particular, the equilibrium manifold E is diffeomorphic to $\mathbb{R}^{\ell m}$.

It is also possible to consider in that setup the *extended equilibria* where the parameter ω is not restricted to consist of only strictly positive endowment vectors. Again, the *extended equilibrium manifold* $\tilde{E}$ is diffeomorphic to $(\mathbb{R}^\ell)^m$.

6.4 Natural Projection

The natural projection $\pi : E \to \Omega$ is now the restriction to the equilibrium manifold E of the projection map $(p, \omega) \to \omega$ from $\mathbb{P}(X) \times \Omega$ into Ω. The following are two crucial properties for the natural projection.

Proposition 6.4.1 The natural projection $\pi : \mathbb{P}(X) \times \Omega$ is smooth and proper.

6.4.1 Proof of Smoothness

The natural projection $\pi : E \to \Omega$ is smooth as the composition of two smooth maps, the embedding map $E \to \mathbb{P}(X) \times \Omega$, which is smooth because E is a smooth submanifold of $\mathbb{P}(X) \times \Omega$, and the projection map $\mathbb{P}(X) \times \Omega \to \Omega$, whose smoothness follows readily from its linearity. This is exactly the same proof as for proposition 2.5.1 in chapter 2 for the case of non-price-dependent preferences.

6.4.2 Proof of Properness

Let us now show that the natural projection $\pi : E \to \Omega$ is a proper map, i.e., the preimage $\pi^{-1}(K)$ of every compact subset K of Ω is a compact subset of the equilibrium manifold E.

The map $(\omega_1, \omega_2, \dots, \omega_m) \to \sum_i \omega_i$ is continuous. The image of the compact set K by this map is compact and therefore bounded from above by some element $r^* \in X$.

It follows from the equilibrium equation (6.5) that the sum $\sum_i f_i(p, p \cdot \omega_i)$ is bounded from above by $r^* \in X$. Each term $f_i(p, p \cdot \omega_i)$ also belongs to X, so this implies that for any $(p, \omega) \in \pi^{-1}(K)$ the demand $f_i(p, p \cdot \omega_i)$ is bounded from above by r^*.

Pick arbitrarily some consumer i. Then $\omega_i \in X$ belongs to the image K_i of the compact set K by the projection $(\omega_1, \omega_2, \dots, \omega_i, \dots, \omega_m) \to \omega_i$. The set K_i is compact as the image of a compact set by a continuous map. The compact set K_i, being a subset of the strictly positive orthant X, is bounded away from zero: there exists $x_i^* \in X$ such that $x_i^* \leq \omega_i$ for $\omega \in K$.

Let us now show that the demand $f_i(p, p \cdot \omega_i)$ is also bounded away from zero for all $(p, \omega) \in \pi^{-1}(K)$. For a given $q \in \overline{\mathbb{P}(X)}$, the utility $u_i(q, f_i(p, p \cdot \omega_i)) \geq u_i^*(x_i^*) = \inf_{q \in \overline{\mathbb{P}(X)}} u_i(q, x_i^*)$. The set

$$\{x_i \in X \mid u_i(q, x_i) \geq u_i(x_i^*),\ x_i \leq r^* \in X\}$$

is then bounded away from zero by some $\varepsilon(q) > 0$ that depends continuously on $q \in \overline{\mathbb{P}(X)}$. It follows from the compactness of $\overline{\mathbb{P}(X)}$ that $\varepsilon = \inf_{q \in \overline{\mathbb{P}(X)}} \varepsilon(q) > 0$.

It has therefore been proved that for any $(p, \omega) \in \pi^{-1}(K)$, the demand $f_i(p, p \cdot \omega_i)$ belongs to some compact set H_i of X. Let us prove that this implies that p then belongs to a compact subset of $\mathbb{P}(X)$.

Consider the map $(q, x_i) \to D_x u_i(q, x_i)$ from $\overline{\mathbb{P}(X)} \times X$ into X. This map is continuous. Therefore, the image of the compact set $\overline{\mathbb{P}(X)} \times H_i$ by this map is a compact subset L of $\mathbb{P}(X)$. It suffices to observe that for $f_i(p, p \cdot \omega_i)$ belonging to the compact set H_i, p necessarily belongs to the compact subset L.

It is also obvious that, in addition to being contained in $L \times K$, the set $\pi^{-1}(K)$ is closed. This implies that $\pi^{-1}(K)$ is compact.

6.4.3 Applications to Comparative Statics and Number of Equilibria

Proposition 6.4.1 implies the important property that the natural projection is a finite ramified covering of the parameter space Ω. This implies in particular the following proposition.

Proposition 6.4.2 There exists a closed set of measure zero Σ in Ω such that, for any endowment vector $\omega \notin \Sigma$, the set $\pi^{-1}(\omega)$ of equilibria is finite, the equilibria being locally unique.

The finite ramified covering property also implies that, at the regular value $\omega \in \mathcal{R}$, there exists an open neighborhood U of ω such that $\pi^{-1}(U)$ consists of a finite number of disjoint (open) sets such that the restriction of π to each of these subsets is a diffeomorphism with the open set U. The equilibrium price vectors $p \in \mathbb{P}(X)$ associated with $\omega \in U$ are smooth functions of the parameter $\omega \in U$.

In addition, the same kind of upper bound on the number of equilibria as the one given by proposition 2.8.1 in chapter 2 exists for economies with price-dependent utility functions.

It is also true that the modulo 2 degree of the natural projection is equal to 1, just as in the non-price-dependent setup.

Proposition 6.4.3 The modulo 2 degree of the natural projection $\pi : E \to \Omega$ is equal to 1.

Proof The proof of this property cannot follow the approach of proposition 4.6.3 in chapter 4 because there is no equivalent here of Pareto efficiency or of the regularity property of equilibrium allocations.

A solution is to construct a *proper homotopy*, i.e., a continuous path going from one map to the other in the set of proper maps, here from the natural projection with non-price-dependent utility functions to the natural projection for price-dependent utility functions. In the process, the domain of the natural projection, namely, the equilibrium manifold, also changes. This requires using the diffeomorphism or at least the homeomorphism of the equilibrium manifold with $\mathbb{R}^{\ell m}$ to work with a domain that does not change.

Let us use the homeomorphism $\theta_E : E \to B \times B(0; r^*)$, defined by equation (4.3) in chapter 4, between the equilibrium manifold E and the Cartesian product $B \times B(0; r)$, where $B(0; r)$ is the open ball of $\mathbb{R}^{(\ell-1)(m-1)}$ centered at zero with radius $r > 0$. The composition $\pi \circ (\theta_E)^{-1} : B \times B(0; r) \to \Omega$ that associates with $(b, \rho) \in B \times B(0; r)$ the endowment vector $\omega = (\omega_1, \ldots, \omega_m)$ corresponds to the natural projection π, but its domain $B \times B(0; r)$ is now independent of the demand functions $f_1, f_2, \ldots, f_m$.

Let $(f_1, f_2, \ldots, f_m)$ and $(f'_1, f'_2, \ldots, f'_m)$ be two m-tuples of individual demand functions. They satisfy Walras' law, and the associated natural projection maps are proper. In addition, assume that the m individual demand functions $f_1, f_2, \ldots, f_m$ are associated with non-price-dependent utility functions.

The natural projection maps now correspond to the maps denoted by $\omega(b, \rho)$ and $\omega'(b, \rho)$ from $B \times B(0; r)$ into Ω. These maps are proper.

Let $f_i^t = (1-t)f_i + tf'_i$ for $i = 1, 2, \ldots, m$ with $t \in [0, 1]$. Define the map $\omega^t : B \times B(0; r)$ by the formula

$$\omega^t(b, \rho) = (1-t)\omega(b, \rho) + t\omega'(b, \rho).$$

We have $\omega^0(b, \rho) = \omega(b, \rho)$ and $\omega^1(b, \rho) = \omega'(b, \rho)$.

Let us show that the map $\omega^t : B \times B(0; r) \to \Omega$ is proper for any $t \in [0, 1]$. We can assume $t \in (0, 1)$ because ω^0 and ω^1 are already proper. Let K be a compact subset of Ω. The preimage $(\omega^t)^{-1}(K)$ consists of the price-income vectors $b = (p, w_1, w_2, \ldots, w_m) \in B$ and $\rho \in B(0; r)$ such that $\omega^t(b, \rho) \in K$.

The element $\omega^t(b,\rho) = (1-t)\omega(b,\rho) + t\omega'(b,\rho)$ belongs to K. This implies that the sum

$$(1-t)\left(\sum_i f_i(p,w_i)\right) + t\left(\sum_i f_1'(p,w_i)\right)$$

is bounded from above by some vector r^*.

Each demand vector belongs to X and has therefore strictly positive components and $1-t>0$, so each term $f_i(p,w_i)$ is bounded from above.

Pick arbitrarily consumer i. Compactness of K implies that the component ω_i is bounded away from zero: there exists a vector $x_i^* \in X$ such that $x_i^* \leq \omega_i$. Then, the non-price-dependent utility $u_i(f_i(p, p\cdot\omega_i)) \geq u_i(x_i^*)$ and is therefore bounded from below by $u_i(x_i^*)$.

Boundedness from above combined with the utility $u_i(f_i(p,w_i))$ being bounded from below implies that (p,w_i) belongs to a compact subset of $\mathbb{P}(X \times \mathbb{R}_{++})$. (Use the analogue for non-normalized prices and incomes of lemma 2.5.3.) This proves that the projection on the price set $\mathbb{P}(X)$ of the preimage $(\omega^t)^{-1}(K)$ is a closed subset of a compact set and is therefore compact. Then, for $i=1,2,\ldots,m$, we have $(p,w_i) = (p, p\cdot\omega_i)$, which belongs to some compact subset of $\mathbb{P}(X\times\mathbb{R}_{++})$ because ω_i belongs to a compact set.

The proof ends with the observation that for a given $b \in B$, the set of $\rho \in B(0;r)$ such that $\theta_E(b,\rho) = (p,\omega)$ is such that ω belongs to the compact subset K of Ω is also compact. ■

Corollary 6.4.4 Equilibrium exists for any $\omega \in \Omega$.

The degree property of the natural projection implies that $\pi : E \to \Omega$ is onto. In other words, equilibrium exists when preferences are price-dependent.

6.5 Conclusion

In this chapter the consumption set is the strictly positive orthant $X = \mathbb{R}^\ell_{++}$. Some exercises here are devoted to extending the properties of chapter 6 to general consumption sets of the kind required for applying the price-dependent model to the temporary equilibrium model (see chapter 9).

These general consumption sets satisfy the following properties: (1) consumption sets are polyhedra, i.e., defined by linear inequalities; (2)

all consumption sets have the same recession cone P; (3) for every consumer i, there exists a commodity bundle $x_i \in \mathbb{R}^\ell$ (its coordinates can be < 0) such that the set $x_i + P$ contains the consumption set X_i. Prices are then restricted to belong to the interior of the dual cone associated with the recession cone P. The common recession cone P of all the consumption sets becomes the analogue of the strictly positive orthant X for the model considered in this chapter.

6.6 Notes and Comments

Price-dependent utility functions (or preferences) were considered the main explanation for conspicuous consumption by Veblen (1899). Later, Scitovsky (1944) and Pollak (1977) showed that inferring the quality of a product by its price is equivalent to having price-dependent preferences. Shafer (1974) observed that consumer's maximization behavior subject to a budget constraint is in the case of nontransitive preferences formally equivalent to the maximization of some price-dependent function. A first study of the Slutsky matrix associated with a price-dependent utility function is due to Kalman (1968).

Existence of equilibrium in the Arrow-Debreu model with price-dependent preferences was proved by Arrow and Hahn (1971) under fairly general assumptions. The properties of the equilibrium manifold and natural projection for price-dependent preferences are due to Balasko (2003a).

A possible direction for future research would be to assume that preferences depend not only on the relative but also on the absolute level of commodity prices q. The utility function $u_i(q, x_i)$ is therefore not homogeneous of degree 0 with respect to q. The properties of the equilibrium model with such price-dependent preferences will then be somewhat different from those studied in this chapter. The main reason is that the equilibrium equation has in that case ℓ real unknowns, with only $\ell - 1$ equations independent. Unsurprisingly, this creates indeterminacy for the solutions. Nevertheless, the equilibrium set remains a smooth submanifold, diffeomorphic to $\mathbb{R}^{\ell m+1}$, and the projection map π is still a proper map. The study of the indeterminacy of the equilibrium solution could then follow the same lines as for the model with purely financial instruments studied by Balasko and Cass (1989).

7 Out-of-Equilibrium Price Dynamics

7.1 Introduction

Any treatment of the properties of the general equilibrium model would be incomplete without some mention of how equilibrium is attained and sustained. At the base of the general equilibrium model is the idea that an excess of total supply over total demand, or of total demand over total supply, triggers the forces of competition into driving the price system toward a state where aggregate demand and supply eventually become equal.

The study of these out-of-equilibrium price dynamics is very difficult, starting with the dynamics formulation itself. One of the nicest features of the theory of general equilibrium is that it avoids getting into the details of these dynamics by directly addressing the properties of their stationary points, the competitive equilibria. This is the right strategy to follow if only equilibrium solutions are considered to be of interest and if there are no issues related to the selection and stability properties of these equilibria.

Nevertheless, the selection of equilibria and their stability properties are issues sufficiently important not to be neglected. In addition, the behavior of competitive markets when there is no equality between total supply and total demand is of independent interest.

Walras (1874) observed that in organized markets like the stock exchange, an auctioneer determines the prices that equate aggregate supply and demand by increasing (resp. reducing) the prices of the commodities that are in excess supply (resp. demand) until equilibrium is reached. He made the bold assumption that in markets without auctioneers—markets where commodity prices are set by the economic agents themselves, whether buyers or sellers—the forces of competition operate along the same lines as the auctioneers of the organized markets. This behavior of competitive markets when prices are not at equilibrium had already been described by Adam Smith in 1776.

This description of price behavior out of equilibrium is purely qualitative. It does not indicate by how much the market or the auctioneer revises the price of a commodity for which there is no equality of supply and demand. In other words, there is no information on the speed of adjustment. This question was solved somewhat arbitrarily by Samuelson (1941) when he defined *Walras tatonnement* by the differential equation

$$\begin{cases} \dot{\bar{p}}(t) = \bar{z}(p(t), \omega)), \\ p(0) = p^*. \end{cases}$$

Let $\bar{a} = (a^1, a^2, \ldots, a^{\ell-1})$ denote a row vector in $\mathbb{R}^{\ell-1}_{++}$. Define the product, represented by the symbol $\square$, by the formula $\bar{a} \,\square\, \bar{z} = (a^1 z^1, a^2 z^2, \ldots, a^{\ell-1} z^{\ell-1})$. The qualitative behavior postulated by Walras is satisfied by all the dynamic systems with the differential equation

$$\begin{cases} \dot{\bar{p}}(t) = \bar{a} \,\square\, \bar{z}(p(t), \omega)), \\ p(0) = p^*. \end{cases}$$

Walras tatonnement corresponds to the special case $\bar{a} = (1, 1, \ldots, 1)$. Varying $\bar{a}$ amounts to changing the speed of adjustment.

The problem is that different speeds of adjustment yield different dynamic systems, in the sense that their trajectories are different even if the fixed or stationary points of these dynamic systems are all the same. The property for an equilibrium of being (locally) stable therefore depends on the adjustment speeds. The qualitative behavior postulated by Walras leaves too much arbitrariness for a meaningful definition of stability. More information is required about the determination of the speeds at which markets adjust when they are not at equilibrium.

7.2 Structure of the Exchange Process

The starting point is to impose more structure on the exchange process than there is in the Arrow-Debreu model. The formulation of the price adjustment process gains in realism if goods are exchanged for money, and money for goods, a role of money highlighted in Shapley and Shubik's (1977) trading post model. Trading post models have been quite successful in monetary economics. For example, Starr (2003) and Starr and Stinchcombe (1999) use them to analyze the emergence of commodity money in an exchange setup. A variant where "specialized shops" are substituted for the trading posts is used by Howitt (2005) to endogenize fiat money.

Here, the trading posts supply the additional structure needed to deal with the out-of-equilibrium price dynamics. There is one trading post for every commodity, and trade takes place only in the trading posts. Goods are traded for "money." The buying (resp. selling) operations done by the trading posts are synchronized all over the economy. The selling operations occur only after the completion of the buying operations.

7.3 Scenario: Endogenous Money Creation

A scenario consists of a sequence of rounds. A round starts with bid prices that are exogenously given. It ends with the determination of the selling prices by the trading posts. Money is created and used in the middle of every round, but there is no money at the beginning or end of each round. However, because of the existence of money in the middle of each round, *prices are not normalized.* There is no numeraire in this model. Consequently, the price vector $p = (p_1, p_2, \dots, p_{\ell-1}, p_\ell) \in X$ is not normalized.

7.3.1 Description of a Round

Phase 1

A round starts with a bid price vector $p \in X$ that is exogenously given. The trading posts buy consumers' resources at the bid price vector $p \in X$. They buy all the resources that are available in the economy. For that purpose, they write IOUs in exchange for the goods delivered to them.

The quantity of IOUs received by consumer i from selling the quantity ω_i^j of commodity j is equal to $p_j \omega_i^j$. Consumer i's wealth is the sum of the IOUs received from the various trading posts and is equal to $p \cdot \omega_i$.

The quantity of IOUs issued by trading post j is $m_j = p_j r^j$.

Phase 2

Consumers know the price system $p \in X$ and their own wealth. In an Arrow-Debreu world, they maximize utility subject to budget constraints. In this operation, they assume that the price vector $p' \in X$ at which the goods are to be sold by the trading posts will coincide with the bid price vector $p \in X$ of phase 1.

Consumer i's demand for commodity j is then equal to $f_i^j(p, p \cdot \omega_i)$. Therefore, consumer i brings to trading post j the quantity $p_j f_i^j(p, p \cdot \omega_i)$ in cash, and cash consists of the IOUs created by the various trading posts.

Trading post j ends up facing a total quantity of IOUs equal to $m_j = \sum_i p_j f_i^j(p, p \cdot \omega_i)$ that is offered in exchange for commodity j.

Phase 3

In order to satisfy all individual demands, trading post j sets the selling price p_j' of commodity j at a level such that the value $p_j' r^j$ of commodity j held by the trading post is equal to the demand made up of IOUs, demand equal to m_j. This implies the equality $p_j' r^j = m_j$.

The selling price vector $p' = (p_1', p_2', \ldots, p_\ell') \in X$ is by definition the *round's output.*

7.3.2 Recontracting, or the Trade Feasibility Condition

If the price vector p', the round's output, is different from the round's input p, *no trade actually takes place.* A new round then starts with the vector $p' \in X$, the selling price vector of the previous round, as the new bid price vector. This assumption is similar to the recontracting assumption used in Walras tatonnement.

If the price vectors p and p' are equal, the equality $\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i$ is satisfied. Then, the price vector $p \in X$ is actually an equilibrium price vector, and trade can take place.

7.4 Alternative Scenario: Fiat Money, Bid and Selling Prices Endogenously Determined

This second scenario may sound more interesting because it starts from a given stock of fiat money. Money is held by the trading posts between each round. As a consequence, the trading posts do not issue IOUs. Another consequence is that the trading posts cannot take the bid prices as exogenously given because they may not have enough money to buy all the goods they are being offered at the begining of a round.

7.4.1 Description of a Round

A round still comprises three phases. But the round's input and output are now different from those in the previous scenario. A round's input now consists in the distribution $m = (m_j) \in \mathbb{R}^\ell_{++}$ of money holdings between the various trading posts. The three phases now take the following forms:

Phase 1

The trading post j, with j varying from 1 to ℓ, determines the bid price p_j for commodity j by the condition that the trading post can buy with its money holdings m_j the quantity r^j of commodity j brought by all the consumers in the economy. This implies that the buying price p_j must satisfy the equality $p_j r^j = m_j$.

Phase 2

Same as in the scenario with IOUs.

Phase 3

The trading post j determines the selling price p'_j, as in the version with IOUs, by the condition

$$p'_j r^j = p_j \left(\sum_i f_i^j(p, p \cdot \omega_i) \right).$$

Trading post j's money holdings at the end of phase 3 are $m'_j = p'_j r^j$. The round's output is the distribution of the trading posts' money holdings $m' = (m'_j) \in \mathbb{R}^\ell_{++}$.

7.4.2 Trade Feasibility Condition

The trade feasibility condition is again the equality $p = p'$. This condition is equivalent to equality $m = m'$. If the trade feasibility condition is not satisfied, in which case $m' \neq m$, another round starts with a new input that is the distribution $m' = (m'_j)$.

7.5 *D*-Dynamics (Discrete Time)

Note that recontracting, whether in the first or second scenario, leads to an infinite sequence of rounds if $p \neq p'$. These iterations define a discrete-time dynamic system on the set of prices, a system of which the fixed prices are the equilibrium price vectors.

Using the $\square$ notation for coordinatewise multiplication, let us define the map $\mathfrak{a} : X \to X$ by the formula

$$\mathfrak{a}(p) = r^{-1} \square \left(\sum_i f_i(p, p \cdot \omega_i) \right) \square p.$$

The bid $p \in X$ and selling prices $p' \in X$ in a given round, whether in the first or second scenario, are related by the formula

$$p' = \mathfrak{a}(p).$$

The equilibrium price vectors of the Arrow-Debreu exchange economy defined by the endowment vector $\omega = (\omega_i)$ are the fixed points of the map $\mathfrak{a} : X \to X$.

Map $\mathfrak{a}$ being homogeneous of degree 1, then λp is a fixed point of $\mathfrak{a}$ for any $\lambda > 0$ if p is also a fixed point.

The D-dynamics is the discrete dynamic system on X defined by the map $\mathfrak{a}$ and its iterates, the maps $\mathfrak{a}^t$, with $t \in \mathbb{Z}$, where

$$\mathfrak{a}^t = \begin{cases} \underbrace{\mathfrak{a} \circ \mathfrak{a} \circ \cdots \circ \mathfrak{a}}_{t \text{ times}} & \text{for } t \geq 1, \\ \text{identity map} & t = 0, \\ \underbrace{\mathfrak{a}^{-1} \circ \mathfrak{a}^{-1} \circ \cdots \circ \mathfrak{a}^{-1}}_{-t \text{ times}} & \text{for } t \leq -1. \end{cases}$$

See Hirsch and Smale (1974, ch. 13, sec. 2).

7.6 *C*-Dynamics (Continuous Time)

The C-dynamics is the limit of the discrete-time D-dynamics defined by the map $\mathfrak{a} : X \to X$ when the duration of a round tends to zero.

The C-dynamics is then defined by the differential equation

$$\begin{cases} \dot{p}(t) = r^{-1} \square [(\sum_i f_i(p(t), p(t) \cdot \omega_i)) - r] \square p(t), & t \geq 0, \\ p(0) = p^*. \end{cases}$$

7.6.1 Value Normalization of the Price Vector

Let us denote by $S_w = \{p \in X \mid p \cdot r = w\}$ the set of price vectors that give the value w to the total resources $r \in X$.

It follows from the equality $\dot{p}(t) \cdot r = p \cdot z(p(t), \omega) = 0$ that the inner product $p(t) \cdot r$ is constant and necessarily equal to w for $p^* \in S_w$.

Therefore, the C-dynamics define a dynamic system on the set of value-normalized price vectors S_w. The zeros or equilibria of the C-dynamics are the equilibrium price vectors of the exchange economy defined by the endowment vector $\omega = (\omega_i)$.

7.6.2 *C*-Stability

The (value-normalized) equilibrium price vector $p \in S_w$ of the economy $\omega = (\omega_i)$ is C-stable if it is a locally asymptotically stable fixed point of the C-dynamics. We denote by $E(\mathfrak{C})$ the subset of the equilibrium manifold E consisting of the set of C-stable equilibria.

Let $(p, \omega) \in E$ be an equilibrium, with $p \in S_w$. The linearized dynamic system at p of the C-dynamics is the linear differential equation

$$\begin{cases} \dot{p}(t) = T(p,\omega)(p(t) - p), & t \geq 0, \\ p(0) = p^*, \end{cases}$$

where $T(p, \omega)$ is the Jacobian matrix of the map

$$p \to r^{-1} \square \left[\left(\sum_i f_i(p, p \cdot \omega_i) \right) - r \right] \square\, p.$$

Let us introduce matrix $J(p, \omega)$ as the $\ell \times \ell$ Jacobian matrix of the aggregate excess demand map

$$p \in X = \mathbb{R}^{\ell}_{++} \to z(p,\omega) = \left(\sum_i f_i(p, p \cdot \omega_i) \right) - r \in \mathbb{R}^{\ell}.$$

Note that prices are not normalized by the numeraire convention in this chapter. The Jacobian matrix of the map

$$(\bar{p}, 1) \in S \to \bar{z}(p,\omega) \in \mathbb{R}^{\ell-1}$$

that has been considered in the previous chapters becomes the $\ell - 1 \times \ell - 1$ matrix obtained from $J(p, \omega)$ by deleting the ℓ-th row and column, a matrix denoted in this chapter by $J_{\ell\ell}(p, \omega)$. Note that $\det J(p, \omega)$ is equal to zero. The equilibrium (p, ω) is regular if $\det J_{\ell\ell}(p, \omega)$ is different from zero. This is equivalent to $\operatorname{rank} J(p, \omega) = \ell - 1$.

Let Π denote the diagonal matrix whose j-th diagonal coefficient is p_j / r^j, with $j = 1, \ldots, \ell$. It follows readily from the definitions that we have

$$T(p,\omega) = J(p,\omega)\Pi.$$

The C-stability properties of the equilibrium (p, ω) in E can be derived from those of the linearized differential equation in the following case:

Definition 7.6.1 The equilibrium (p, ω) is *C-hyperbolic* if it is a regular equilibrium and if all the nonzero eigenvalues of the matrix $T(p, \omega)$ have nonzero real parts.

A straightforward adaptation of the proof of proposition 4.7.1 in chapter 4 shows us that the complement of the subset of the equilibrium manifold consisting of the C-hyperbolic equilibria is closed with measure zero.

The C-hyperbolic equilibrium (p, ω) is C-stable if and only if the nonzero eigenvalues of the matrix $T(p, \omega)$ all have strictly negative real parts. This algebraic characterization of the C-hyperbolic $\mathfrak{C}$-equilibria is particularly helpful.

7.6.3 Nonequivalence of *C*-Dynamics with Walras Tatonnement

Provided that we limit ourselves to equilibria that are both tatonnement-hyperbolic and C-hyperbolic—the sets of tatonnement-hyperbolic equilibria and of C-hyperbolic equilibria are both open dense subsets of the equilibrium or extended equilibrium manifolds and, as such, their intersection is open dense—C-stability boils down to the matrix $T(p, \omega)$ having $\ell - 1$ eigenvalues with strictly negative real parts, and tatonnement stability amounts to the matrix $J(p, \omega)$ having all its eigenvalues with strictly negative real parts. These matrices, although related to each other, have different characteristic polynomials. For suitable choices of the price vector $p \in S_w$ and the total resource vector $r \in \mathbb{R}^{\ell}$, some stable (resp. unstable) matrices $J_{\ell\ell}$ correspond to unstable (resp. stable) matrices $T(p, \omega)$. Therefore, C-stability and tatonnement stability are different concepts.

7.7 Classes of *C*-Stable Equilibria

The goal of this section is to identify several classes of C-stable equilibria or, alternatively, several remarkable subsets of the set $E(\mathfrak{C})$ of C-stable equilibria.

7.7.1 Inclusion $T \subset E(\mathfrak{C})$

This inclusion follows from proposition 7.7.1.

Proposition 7.7.1 Every no-trade equilibrium (p, ω) is C-hyperbolic and C-stable.

Proof The no-trade equilibrium (p, ω) is regular. Therefore, the matrix $T(p, \omega)$ has $\ell - 1$ nonzero eigenvalues (counted with their order of multiplicity). The idea of the proof is to show that these $\ell - 1$ nonzero eigenvalues are < 0.

Step 1. The eigenvalues of $T(p, \omega)$ are real Recall that Π denotes the $\ell \times \ell$ diagonal matrix whose coefficient of row j is equal to p_j/r^j. Recall that $T = J\Pi$.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of T. We have $\det(T - \lambda I) = \det(J\Pi - \lambda I) = 0$. This is equivalent to $\det(J - \lambda\Pi^{-1}) = 0$.

There exists a column vector $X \in \mathbb{C}^\ell$, with $X \neq 0$, such that

$$(J - \lambda\Pi^{-1})X = 0. \tag{7.1}$$

The left matrix multiplication of equality (7.1) by the row matrix $\overline{X}^T$, where $\overline{X}$ denotes the complex conjugate of X, yields

$$\overline{X}^T J X = \lambda \overline{X}^T \Pi^{-1} X. \tag{7.2}$$

The complex conjugate of the transposed equality (7.1) yields the equality $\overline{X}^T(J - \bar{\lambda}\Pi^{-1}) = 0$ because J is real and symmetric at a no-trade equilibrium. The right multiplication of this equality by the column matrix X yields

$$\overline{X}^T(J - \bar{\lambda}\Pi^{-1})X = 0;$$

hence

$$\overline{X}^T J X = \bar{\lambda} \overline{X}^T \Pi^{-1} X. \tag{7.3}$$

It follows from $X \neq 0$ that $\overline{X}^T \Pi^{-1} X \neq 0$. The combination of equalities (7.2) and (7.3) then implies the equality $\lambda = \bar{\lambda}$, which proves that the eigenvalue λ is real.

Step 2. The nonzero eigenvalues of $T(p, \omega)$ are < 0 Let now $\lambda \neq 0$ be an eigenvalue of $T(p, \omega) = J(p, \omega)\Pi$. There exists a column matrix $X \neq 0$ (with real coefficients) such that

$$JX - \lambda\Pi^{-1}X = 0.$$

Left matrix multiplication by the row matrix X^T yields the equality

$$X^T J X - \lambda X^T \Pi^{-1} X = 0;$$

hence

$$\lambda X^T \Pi^{-1} X = X^T J X.$$

The matrix J is negative semidefinite at the no-trade equilibrium; hence $X^T J X \leq 0$, from which follows $\lambda X^T \Pi^{-1} X \leq 0$. Then $X^T \Pi^{-1} X > 0$, and $\lambda \leq 0$, and therefore $\lambda < 0$ because we assumed $\lambda \neq 0$. This readily implies that the no-trade equilibrium (p, ω) is C-hyperbolic because no eigenvalue of $T(p, \omega)$ that is not equal to zero has a real part equal to zero. In addition, the no-trade equilibrium (p, ω) is C-stable because the real parts of the nonzero eigenvalues of M are all < 0. ∎

Remark As for all properties of the no-trade equilibria, the importance of proposition 7.7.1 lies in the implication that C-stability holds true if the intensity of trade as measured by the vector of net exchanges $(f_i(p, p \cdot \omega_i) - \omega_i)$ is not too large.

The study of the bifurcation occurring when the competitive equilibrium loses its C-stability because of, for example, a relative increase of net trade is an open problem and should be the subject for further research.

7.7.2 Inclusion $E(\mathfrak{S}) \subset E(\mathfrak{C})$

The equilibrium (p, ω) satisfies the gross substitutability property (the $\mathfrak{S}(p, \omega)$ property) whenever the aggregate demand for any commodity increases with an increase of the price of any other commodity (see section 4.8.2).

Proposition 7.7.2 Inclusion $E(\mathfrak{S}) \subset E(\mathfrak{C})$.

Proof The off-diagonal coefficients of matrix $T(p, \omega)$ are > 0, whereas its diagonal coefficients are < 0. Multiply row j of matrix $T(p, \omega) - \lambda I$ by r^j for j varying from 1 to ℓ. This yields matrix $L(p, \omega) - \lambda K$, where the coefficient of the jth row of the $\ell \times \ell$ diagonal matrix K is equal to r^j. Let b_{jk} denote the coefficient of row j and column k of matrix $L(p, \omega)$. Recall that $\sum_j b_{jk} = 0$ for every k.

Let $\lambda \neq 0$ be some complex number with a real part ≥ 0. Let us show that matrix $L(p, \omega) - \lambda K$ is invertible.

It follows from $b_{kk} < 0$ that $|b_{kk}| = -b_{kk} = \sum_{j \neq k} b_{jk}$. In addition, the real part of λ being ≥ 0, this implies the strict inequality $|b_{kk} - \lambda r^k| > |b_{kk}|$. Therefore,

$$|b_{kk} - \lambda r^k| > \sum_{j \neq k} b_{jk} = \sum_{j \neq k} |b_{jk}|.$$

Matrix $L - \lambda K$ is therefore strictly diagonal-dominant and, by the Levy-Desplanques theorem (Horn and Johnson 1985, corollary 5.6.17), invertible. Every nonzero eigenvalue of $T(p, \omega)$ has therefore a strictly negative real part. This proves that every equilibrium belonging to $E(\mathfrak{S})$ is C-hyperbolic and C-stable, and hence belongs to $E(\mathfrak{C})$. ■

7.8 Pathconnectedness of the Set of Extended *C*-Stable Equilibria

Let $\tilde{E}(\mathfrak{C})$ denote the set of C-stable extended equilibria. Let $\tilde{E}_h(\mathfrak{C})$ denote the set of C-hyperbolic C-stable equilibria. The main result of this section is the following.

Proposition 7.8.1 The set $\tilde{E}(\mathfrak{C})$ of C-stable extended equilibria is pathconnected.

Proof It follows from the mathematical definition of pathconnectedness that two C-stable extended equilibria can always be linked by a continuous path of C-stable extended equilibria.

The line of reasoning is similar to the one used to prove proposition 4.8.3 in chapter 4, the pathconnectedness of the set of tatonnement-stable extended equilibria. The goal is to show that the following properties are satisfied in every fiber: (1) the set of equilibria (p, ω) whose matrix $T(p, \omega)$ has $\ell - 1$ eigenvalues with strictly negative real parts is pathconnected and contains the no-trade equilibrium of the fiber; and (2) the closure of that set contains the set of C-stable equilibria belonging to the fiber.

The crux of the proof is to reduce the study of sets of matrices that have one eigenvalue equal to zero, their other eigenvalues having strictly negative or negative real parts, to sets of matrices with nonzero eigenvalues and strictly negative or negative real parts.

Lemma 7.8.2 The intersection $\widetilde{E(\mathfrak{C})} \cap \widetilde{F(b)}$ contains the no-trade equilibrium $f(b)$ and is pathconnected for every $b \in B$.

Step 1. Parameterization of the extended fiber $\widetilde{F(b)}$ by the $\ell \times m$ real matrices Y Let Y be the $\ell \times m$ matrix

$$Y = \begin{bmatrix} y_1^1 & y_1^2 & \cdots & y_1^\ell \\ y_2^1 & y_2^2 & \cdots & y_2^\ell \\ \vdots & \vdots & \vdots & \vdots \\ y_m^1 & y_m^2 & \cdots & y_m^\ell \end{bmatrix}.$$

Denote the rows of Y by the vectors $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m$ of $\mathbb{R}^\ell$, and write

$$Y = \begin{bmatrix} \vec{y}_1 \\ \vec{y}_2 \\ \vdots \\ \vec{y}_m \end{bmatrix}.$$

It follows from $\vec{y}_i = \omega_i - f_i(p, w_i) \in \mathbb{R}^\ell$, i varying from 1 to m, that we can associate with the equilibrium $(p, \omega) \in \widetilde{F(b)}$ the $\ell \times m$ matrix $Y(p, \omega)$ defined by the row vectors $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m$. Conversely, every matrix Y such that $p \cdot \vec{y}_1 = p \cdot \vec{y}_2 = \cdots = p \cdot \vec{y}_m = 0$ and $\vec{y}_1 + \vec{y}_2 + \cdots + \vec{y}_m = 0$ corresponds to a uniquely defined equilibrium $(p, \omega) \in \widetilde{F(b)}$. Note that matrix $Y = 0$ corresponds to the no-trade equilibrium $f(b)$ of the fiber $\widetilde{F(b)}$.

Denote by $\mathfrak{Y}(b)$ the subset of the set of $\ell \times m$ matrices Y such that

$$p \cdot \vec{y}_1 = p \cdot \vec{y}_2 = \cdots = p \cdot \vec{y}_m = 0$$

$$\vec{y}_1 + \vec{y}_2 + \cdots + \vec{y}_m = 0.$$

Step 2. Formula $T(p, \omega) = T(f(b)) + FY(p, \omega)$ Let $(p, \omega) = f(b) = (p, f_1(p, w_1), \ldots, f_m(p, w_m))$ be a no-trade equilibrium. A straightforward computation of the derivatives of aggregate demand with respect to prices yields the formula

$$T(p, \omega) = T(f(b)) + FY(p, \omega), \tag{7.4}$$

where the $\ell \times m$ matrix F is equal to

$$F = \begin{bmatrix} \dfrac{\partial f_1^1}{\partial w_1} & \dfrac{\partial f_2^1}{\partial w_2} & \cdots & \dfrac{\partial f_m^1}{\partial w_m} \\ \dfrac{\partial f_1^2}{\partial w_1} & \dfrac{\partial f_2^2}{\partial w_2} & \cdots & \dfrac{\partial f_m^2}{\partial w_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_1^\ell}{\partial w_1} & \dfrac{\partial f_2^\ell}{\partial w_2} & \cdots & \dfrac{\partial f_m^\ell}{\partial w_m} \end{bmatrix}.$$

Let $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_\ell$ denote the column vectors of matrix $T(f(b))$. Let also $\frac{\partial \vec{f}_1}{\partial w_1}, \frac{\partial \vec{f}_2}{\partial w_2}, \ldots, \frac{\partial \vec{f}_m}{\partial w_m}$ denote the column vectors of matrix F. Then

$$r \cdot \frac{\partial \vec{f}_1}{\partial w_1} = r \cdot \frac{\partial \vec{f}_2}{\partial w_2} = \cdots = r \cdot \frac{\partial \vec{f}_m}{\partial w_m} = 1. \tag{7.5}$$

The column vectors of matrix $T(p, \omega)$ associated with $Y(p, \omega)$ are then the ℓ vectors

$$\vec{t}_1 + \frac{\partial \vec{f}_1}{\partial w_1} y_1^1 + \cdots + \frac{\partial \vec{f}_m}{\partial w_m} y_m^1, \ldots, \vec{t}_\ell + \frac{\partial \vec{f}_1}{\partial w_1} y_1^\ell + \cdots + \frac{\partial \vec{f}_m}{\partial w_m} y_m^\ell. \tag{7.6}$$

The eigenvalues of matrix $T(p, \omega)$ are therefore the eigenvalues of the ℓ column vectors making up matrix T. We now compute these eigenvalues in a new base.

Step 3. Reduction to dimension $\ell - 1$ and to matrices with no zero eigenvalue The matrices $T = T(f(b)) + FY$, where Y belongs to $\mathfrak{Y}(b)$, have the eigenvalue $\lambda = 0$. The goal is to get rid of the eigenvalue $\lambda = 0$ by reducing the dimension to $\ell - 1$.

The no-trade equilibrium $(p, \omega) = f(b)$ is regular. One sees readily that this is equivalent here to the eigenvalue $\lambda = 0$ of $T(f(b))$ having an order of multiplicity equal to 1. This implies that the rank of $T(f(b))$ is equal to $\ell - 1$. Let $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_\ell$ be the column vectors of matrix $T(f(b))$. A straightforward computation shows that they satisfy the condition $r \cdot \vec{t}_1 = r \cdot \vec{t}_2 = \cdots = r \cdot \vec{t}_\ell = 0$. These vectors belong to the hyperplane $H(r)$ of $\mathbb{R}^\ell$ that is perpendicular to r.

The rank condition implies that the $\ell - 1$ vectors $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{\ell-1}$, for example, are linearly independent. These vectors are perpendicular to the vector r, so they define with the vector r a base of $\mathbb{R}^\ell$. (The vectors $\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{\ell-1}$ define a base of the hyperplane $H(r)$.)

The ℓ column vectors of T whose expression is given in (7.6) define in the base $(\vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{\ell-1}, r)$ a matrix N that has the same eigenvalues as matrix T. (Matrix T defines for the canonical base a linear map from $\mathbb{R}^\ell$ into itself, a map whose matrix in the base $(r, \vec{t}_1, \vec{t}_2, \ldots, \vec{t}_{\ell-1})$ is N.)

It follows from (7.5) combined with (7.4) that the coefficients of the last row of N are all equal to zero. Therefore, the eigenvalues of T consist of $\lambda = 0$ and the eigenvalues of the collection of $\ell - 1$ vectors that belong to the hyperplane $H(r)$:

$$\vec{t}_1 + \frac{\partial \bar{f}_1}{\partial w_1} y_1^1 + \cdots + \frac{\partial \bar{f}_m}{\partial w_m} y_m^1, \ldots, \vec{t}_{\ell-1} + \frac{\partial \bar{f}_1}{\partial w_1} y_1^{\ell-1} + \cdots + \frac{\partial \bar{f}_m}{\partial w_m} y_m^{\ell-1}, \tag{7.7}$$

where

$$\frac{\partial \bar{f}_1}{\partial w_1}, \frac{\partial \bar{f}_2}{\partial w_2}, \ldots, \frac{\partial \bar{f}_m}{\partial w_m}$$

are the orthogonal projections of the vectors

$$\frac{\partial \vec{f}_1}{\partial w_1}, \frac{\partial \vec{f}_2}{\partial w_2}, \ldots, \frac{\partial \vec{f}_m}{\partial w_m}$$

into the hyperplane $H(r)$. (Recall that the vector r is orthogonal to $H(r)$.)

It follows from $y_m = -(y_1 + y_2 + \cdots + y_{m-1})$ that the $\ell - 1$ column vectors (7.7) define a matrix N such that (7.7) can be interpreted as the matrix equality

$$N = N_0 + G\bar{Y}, \tag{7.8}$$

where $\bar{Y}$ is the $(m-1) \times (\ell - 1)$ matrix obtained from matrix Y by deleting the last row and column. Note that $\bar{Y}$ can be any $(m-1) \times (\ell - 1)$ real matrix. The matrix N_0 is symmetric and negative definite.

Step 4. Application of Balasko (1978a, theorems 2 and 4) The set of matrices $\bar{Y}$ such that the eigenvalues of N have strictly negative real parts contains the matrix $\bar{Y} = 0$ and is therefore nonempty. It then suffices to apply theorems 2 and 4 of Balasko (1978a). ■

Remark It is an open problem whether the pathconnectedness property remains true when the parameter space is limited to the set Ω, i.e., when endowments are strictly positive.

7.9 Conclusion

The out-of-equilibrium D- and C-dynamics share with Walras tatonnement the qualitative property that excess demand (resp. supply) drives prices higher (resp. lower). For Walras, this property is an axiom. Here, the property follows from the structure of the exchange process. But the fact that there is a commodity numeraire in Walras tatonnement while there is none in the D- and C-dynamics makes the D- and C-dynamics very different from Walras tatonnement. For example, a competitive equilibrium may be tatonnement-stable and C-unstable, or C-stable and tatonnement-unstable. Despite these differences, the properties of C-stability that have been proved look very similar to those of tatonnement stability in the sense that no trade or gross substitutability implies C-stability. Note also the pathconnectedness of the set $\widetilde{E(\mathfrak{C})}$ of C-stable extended equilibria.

Another important assumption here is the trade feasibility condition, namely, that bid and selling prices have to be equal for trade to take place. Relaxing this assumption gives rise to expectation formation and learning in a totally new setup. Much research remains to be done in that direction.

7.10 Notes and Comments

Much of the material of this chapter is taken from Balasko (2007).

8 Economic Fluctuations and the Arrow-Debreu Model

8.1 Introduction

The main goal of this chapter is to explore whether the Arrow-Debreu model can shed light on the long-standing problem of economic fluctuations. This subject is sufficiently important not to neglect any kind of explanation.

A first difficulty is that the general version of the Arrow-Debreu model has no time component. Its structure is characterized by the following properties: (1) goods are just enumerated; (2) the utility functions that represent individual preferences are monotone and strictly quasi-concave; and (3) consumers maximize their utility functions subject to unique budget constraints. The other properties of utility functions are of lesser importance because their role is merely to simplify the mathematics. The fact that goods are just enumerated implies that there is no notion of distance or, alternatively, of proximity between goods. The commodity space is simply some Euclidean space where each commodity defines an element of the base of that vector space. The physical and economic properties of the goods are encapsulated in the properties of the consumers' utility functions.

To apply the Arrow-Debreu model to issues of economic dynamics, the structure of the model needs to be enriched by the explicit introduction of time. Time is introduced into the model by differentiating goods by their date of delivery. Two commodity bundles that differ by their delivery dates should nevertheless make about the same contributions to consumers' utility if the dates of delivery are very close, a property that does not follow from continuity, monotonicity, and quasi-concavity.

Some new assumptions have to be made to reflect the dependence of utility on the date of delivery. A solution is to define the utility of a commodity bundle as the sum of the discounted "utilities" of the goods delivered at the different dates that make up the commodity bundle. In this chapter the utility that depends only on the physical characteristics of the goods (not on the time of delivery) is called a *short-run utility function*. A discount factor enables the consumer to compare the short-run utilities of goods delivered at different dates and to aggregate them into an expression giving the long-run utility.

The *long-run utility function* is the sum of discounted short-run utilities. It plays the same role in the intertemporal Arrow-Debreu model as the standard utility function in the timeless version of the same model.

The impact of the definition of dated goods is not limited to utility functions. The assumption of consumers facing a unique budget constraint becomes much harder to justify once goods are dated. The hypothesis of a unique budget constraint is natural for lack of a better alternative in the standard timeless version of the Arrow-Debreu model. This assumption becomes much more heavily loaded when it is formulated for an environment of dated goods. The uniqueness of the budget constraint means that the typical consumer can freely transfer wealth between time periods. In the real world, this cannot be done without durable goods and money and, more generally, financial assets like stocks and bonds. Not all consumers have sufficient access to financial markets for the assumption of a unique budget constraint for each consumer to be fully realistic. Therefore, even if the Arrow-Debreu model does not feature money or financial assets, the latter are not very far away once goods are dated and not all consumers have equal access to markets for these assets. The implication is that some consumers at least may face multiple budget constraints instead of a unique one.

To sum up, the introduction of time into the Arrow-Debreu model leads to a much richer structure where some consumers can face more than one budget constraint and where the utility functions belong to a much smaller class than the monotone and strictly quasi-concave functions.

This chapter is devoted to a study of this specialized version of the Arrow-Debreu model. In the process, I exploit properties of the general version of the Arrow-Debreu model considered in the previous chapters. In addition to offering a representation of market economies, the Arrow-Debreu model reveals itself to be a powerful (mostly mathematical) tool for the study of equilibrium equation systems rendered more complex by the introduction of time and uncertainty.

8.2 Intertemporal Arrow-Debreu Model

8.2.1 Goods and Prices

In the intertemporal version of the Arrow-Debreu model, goods are indexed by their date of delivery. There are $T+1$ time periods, starting from period 0. The number T is finite. The physical goods are the same from one time period to the other. Denote by ℓ the number of physical goods in each time period, by $x(t)$ a commodity bundle made of goods delivered at date t, and by $x = (x(t))$ the commodity bundle defined by varying the date of delivery of those goods t from 0 to T.

Let H denote the set of consumers, with $h \in H$ an arbitrary consumer, and by $x_h = (x_h(t))$ and $\omega_h = (\omega_h(t))$ consumer h's consumption and endowment bundles. As in the previous chapters, consumption is assumed to only be strictly positive. Let $x = (x_h)_{h \in H}$ and $\omega = (\omega_h)_{h \in H}$ denote the consumption and endowment vectors of the economy.

Let $p(t) \in X$ denote the price vector of the (physical) goods to be delivered in period t. Prices are all strictly positive. The physical commodity 1 that is delivered in period 0 plays the role of the numeraire, i.e., $p_1(0) = 1$. Denote by $p = (p(t))$ the price vector of all the goods in the economy and by $\mathbb{S}$ the set of numeraire normalized price vectors.

The *value* of the commodity bundle x_h given the price vector $p \in \mathbb{S}$ is by definition the inner product

$$p \cdot x_h = \sum_{0 \leq t \leq T} p(t) \cdot x_h(t).$$

8.2.2 Short-Run Utility Functions

The *short-run utility function* of consumer h is a map $v_h : X \to \mathbb{R}$ (where X is the strictly positive orthant $X = \mathbb{R}^{\ell}_{++}$) that (1) is surjective and smooth, with (2) first-order derivatives > 0, and (3) Hessian matrix (second-order derivatives) negative definite, and (4) every indifference (hyper)surface closed in $\mathbb{R}^{\ell}$.

The short-run utility function v_h can be viewed as the utility function of some *fictitious* consumer whose consumption space is X in the commodity space $\mathbb{R}^{\ell}$. The properties of the pure exchange economy defined by these fictitious consumers will actually play a crucial role in the study of the intertemporal model. Note that the short-run utility function is strictly concave instead of just strictly quasi-concave, as in the previous chapters.

Denote by $\bar{f}_h : \mathbb{R}^{\ell}_{++} \times \mathbb{R}_{++} \to X$ the demand function of this fictitious consumer. (The price vector is not normalized.) This demand function is also called the *short-run demand function* of consumer h.

8.2.3 Long-Run Utility Functions

Consumers' preferences are defined by their *long-run utility functions*.

Consumer h's *long-run utility function* $u_h : X^{T+1} \to \mathbb{R}$ is defined by

$$u_h(x_h) = \sum_{0 \leq t \leq T} \delta_h^t v_h(x_h(t)),$$

where the discount factor $0 < \delta_h < 1$ defines the *impatience level* of consumer h.

The long-run utility function u_h then satisfies (1) surjectivity and smoothness, (2) first-order derivatives > 0; (3) Hessian matrix (second-order derivatives) negative definite; and (4) indifference (hyper)-surfaces closed in $(\mathbb{R}^{\ell})^{T+1}$.

The long-run utility functions are strictly concave as a sum of strictly concave functions. This strict concavity enables parameterization of the set of Pareto optima by consumers' welfare weights (see proposition A.9.4 in the appendix).

8.2.4 Endowments

Each consumer h is endowed in every time period with some quantities of the goods delivered in that period. Let $\omega_h(t) \in \mathbb{R}^{\ell}$ denote the vector of those goods for period t. Note that I do not impose that all components $\omega_h(t)$ are strictly positive. Some coordinates may be equal to zero or even negative. The endowment vector of consumer h is the vector $\omega_h = (\omega_h(t)) \in (\mathbb{R}^{\ell})^{T+1}$.

The *endowment vector* of the economy is the vector $\omega = (\omega_h)_{h \in H}$ whose components are the endowment vectors of all the consumers in the economy. With utility functions kept fixed, the endowment vector $\omega = (\omega_h)$ is the only variable parameter.

Let us denote by $r = (r(t))$ the vector of total resources, the sum of individual resources: $r = \sum_{h \in H} \omega_h$.

8.2.5 Budget Constraints

The consumers' budget constraints result from the quantity of wealth that can be transferred from one time period to another. One extreme case corresponds to the total inability of transferring wealth. The opposite case corresponds to the lack of any restriction on intertemporal transfers. These two kinds of budget constraints define the following two categories of consumers: a consumer of type I faces a unique lifetime budget constraint, and a consumer of type J faces one budget constraint per time period.

Type I	**Type J**
$\sum_{t=0}^{T} p(t) \cdot (x_i(t) - \omega_i(t)) = 0$	$p(t) \cdot (x_j(t) - \omega_j(t)) = 0$ $t = 0, 1, \ldots, T.$

The terms *unrestricted* and *restricted* identify the consumers of type I and J, respectively. Also denote by the letter I the subset of the set of consumers H that consists of type I (or unrestricted) consumers and by the letter J the subset consisting of type J (or restricted) consumers. The set H of all consumers is therefore partioned into the two subsets I and J. The subscripts i and j refer to type I and J consumers, respectively.

Given the price vector $p = (p(t))$ and the endowment vector $\omega_h = (\omega_h(t))$, $w_h(t) = p(t) \cdot \omega_h(t)$ is defined as the value of consumer h's endowments in period t. The *total wealth* w_h of consumer h is the sum $w_h = \sum_{t=0}^{T} w_h(t)$.

8.2.6 Consumer's Long-Term Demand Function

Consumer h, whether restricted or unrestricted, maximizes the long-run utility function $u_h(x_h)$. The difference between restricted and unrestricted consumers is only in the nature of the constraints. The *unrestricted consumer* is no different from the consumer considered in the previous chapters. The maximization of the long-run utility function of the unrestricted consumer $i \in I$ defines a demand denoted by $\varphi_i(p, \omega_i)$, a demand that depends on the price vector p and the endowment ω_i. This new function $\varphi_i(p, \omega_i)$ should not be confused with the demand function $f_i(p, w_i)$, whose arguments are the price vector p and income w_i. In fact, $\varphi_i(p, \omega_i) = f_i(p, p \cdot \omega_i)$. The reason for introducing this new function and notation is that the demand function $f_i(p, w_i)$ has no equivalent in the case of the restricted consumers $j \in J$.

Denote by $\varphi_h(p, \omega_h)$ the long-run demand of consumer h, unrestricted or restricted, given the price vector p and the endowment vector $\omega_h \in (\mathbb{R}^{\ell})^{T+1}$.

8.2.7 Long-Run Economy $\mathcal{E}(I, J)[r](\omega)$

The *long-run economy* $\mathcal{E}(I, J)[r](\omega)$ consists of a set of consumers H that is the (disjoint) union $I \cup J$. Consumer h's preferences and endowments, for $h \in H$, are defined by the long-run utility function u_h and the vector $\omega_h = (\omega_h(t))$, respectively. The partition of the set H of consumers into the two subsets I and J is exogenously given. (The endogenization of this partition into restricted and unrestricted consumers is discussed in the following sections.)

The only variable parameter is the endowment vector $\omega = (\omega_h)_{h \in H}$. Total resources are kept fixed and equal to the vector $r \in X^{T+1}$.

8.2.8 Long-Run Equilibria

The price vector $p = (p(t)) \in \mathbb{S}$ is an *equilibrium price vector* of the long-run economy $\mathcal{E}(I,J)[r](\omega)$ if the *individual demand vector* $\varphi_h(p, \omega_h)$ is defined for every consumer $h \in H$ and if there is equality of aggregate supply and demand:

$$\sum_{h \in H} \varphi_h(p, \omega_h) = r. \tag{8.1}$$

An *equilibrium* is therefore a pair $(p, \omega) \in \mathbb{S} \times ((\mathbb{R}^{\ell})^{T+1})^m$ that satisfies equation (8.1). The *equilibrium manifold* $\mathbb{E}$ is the subset of $\mathbb{S} \times ((\mathbb{R}^{\ell})^{T+1})^m$ defined by equation (8.1).

The *equilibrium allocation* $x = (x_h)_{h \in H}$ associated with the equilibrium $(p, \omega) \in \mathbb{E}$ is such that $x_h = \varphi_h(p, \omega_h)$ for all $h \in H = I \cup J$.

The equilibrium price vector $p = (p(t))$ and the equilibrium allocation $x = (x(t))$ of the long-run economy $\mathcal{E}(I,J)[r](\omega)$ are called a *long-run equilibrium price vector* and a *long-run equilibrium allocation*, respectively. Similarly, a Pareto optimum for the long-run economy $\mathcal{E}(I,J)[r](\omega)$ is termed a *long-run Pareto optimum*. Denote by $\mathfrak{P}(r)$ the set of long-run Pareto optima. This set is a smooth manifold that is diffeomorphic to $\mathbb{R}^{m-1}$. Of particular interest is its parameterization by welfare weights. Unless the contrary is specified, any m-tuple of welfare weights $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is normalized by the convention $\lambda_1 = 1$. Incidentally, the set of long-run Pareto optima $\mathfrak{P}(r)$ does not depend on the partition of the consumer set H into the two subsets I and J of unrestricted and restricted consumers.

8.2.9 Stationary Long-Run Economies

Now that the intertemporal dimension has been added to the Arrow-Debreu model, I address the issue of the fluctuations of equilibrium allocations and, more particularly, whether the market mechanism may add some form of volatility to the economy.

With fundamentals of the economy that vary through time, it is clear that equilibrium allocations are also going to vary. Comparing the level of variability of equilibrium allocations with fundamentals that are themselves variable is not easy in its most general form. Since the objective is limited to whether the market mechanism adds some level of volatility, I start with stationary fundamentals, in which case market-induced volatility becomes equivalent to the existence of nonstationary equilibrium allocations. The following sections therefore consider stationary or quasi-stationary fundamentals.

Stationary Preferences

Preferences are represented by long-run utility functions that are the sum of discounted short-run utilities. In this model, we consider consumers born at date 0. But a definition of preferences that are stationary or time invariant requires us to consider the preferences of consumers born at various dates. If the horizon T were infinite, these discounted utility functions define for a consumer born at date t the same preferences as for a consumer born at date 0, commodity bundles being shifted by t time periods. If the horizon T is finite but large, this invariance property becomes only approximately true.

Stationary Endowments

From now on, endowments are stationary. For every consumer $h \in H$, there exists a vector $\bar{\omega}_h \in \mathbb{R}^\ell$ that represents consumer h's constant endowments in every time period:

$$\omega_h = (\bar{\omega}_h, \bar{\omega}_h, \ldots, \bar{\omega}_h) \in (\mathbb{R}^\ell)^{T+1}.$$

Nonstationarity of Equilibrium Allocations

Let $x = (x(t))$ be a long-run equilibrium allocation of the stationary long-run economy $\mathcal{E}(I, J)[r](\omega)$. Is the component $x(t)$ stationary? And if it is not stationary, how does it depend on time? What can be said of the asymptotic behavior of $x(t)$ in such a case? By asymptotic behavior is meant when time t tends to infinity. The meaning of such an expression has to be explained given the fact that the number $T+1$ of time periods is finite, and t is necessarily smaller than T, an apparent contradiction with t tending to infinity. The expression "t tends to infinity" is to be taken here as meaning that t can take arbitrarily large values. This is made compatible with the finite number $T+1$ of time periods because the expression of the period t component $x(t)$ does not depend on the value T of the horizon, provided that T is larger than t. In issues dealing with equilibrium allocations at a given date t, it therefore suffices to take T finite but arbitrarily large, larger than t.

8.2.10 Short-Run Economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$

The long-run economy having stationary endowments, define the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$ by its ℓ (physical) goods and m consumers indexed by $h \in H$, with consumer h's utility function the short-run utility function $v_h : X \to \mathbb{R}$ and consumer h's endowment the vector $\bar{\omega}_h \in \mathbb{R}^\ell$. Denote by $\overline{P(\bar{r})}$ the set of Pareto optima of the short-run economy. A Pareto optimum of the short-run economy is called a short-run Pareto optimum.

The short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$ can be thought of as some kind of cross-section of the long-run economy $\mathcal{E}(I, J)[r](\omega)$. The short-run economy does not depend on the partition between restricted and unrestricted consumers.

8.3 Long-Run Equilibria: Fully Unrestricted Case

8.3.1 Set Δ of Most Patient Consumers

In this section, all consumers are unrestricted. These are standard Arrow-Debreu economies where no consumer faces restrictions in transferring wealth between time periods. Nevertheless, these unrestricted consumers may have different time discount factors. This is reflected in the following definition of the "most patient" consumers.

The set of unrestricted consumers $I = H$ is divided into two subsets. The first subset Δ consists of the "most patient" consumers, i.e., the consumers with the highest factor δ_h. There is no loss in generality in assuming that consumer 1 belongs to that subset, i.e., $1 \in \Delta$. Therefore $\delta_h = \delta_1$ for every consumer $h \in \Delta$. The second set of consumers consists of all the other consumers: $\delta_h < \delta_1$ for $h \notin \Delta$.

8.3.2 Long-Run Equilibria for $\Delta \neq H$

Proposition 8.3.1 The limit when t tends to ∞ of the period t component $x(t) = (x_h(t))$ of the equilibrium allocation $x = (x_h)$ of the long-run economy $\mathcal{E}(H, \emptyset)[r](\omega)$ is $x^* = (x_h^*)$, where $x_h^* = 0$ for $h \notin \Delta$ and $(x_h^*)_{h \in \Delta}$ is a Pareto optimum of the short-run economy consisting the most patient consumers, i.e., $h \in \Delta$ with total resources $\bar{r} \in X$.

Proof The long-run economy $\mathcal{E}(H, \emptyset)[r](\omega)$ satisfies the first welfare theorem: every (long-run) equilibrium allocation $x = (x(t))$ of that economy is a long-run Pareto optimum. The properties to be proved for the (long-run) equilibrium allocations of the long-run economy $\mathcal{E}(H, \emptyset)[r](\omega)$ are in fact properties of all long-run Pareto-optimal allocations $x = (x(t)) \in \mathfrak{P}(r)$.

It follows from the parameterization of the set of (long-run) Pareto optima $\mathfrak{P}(r)$ by the welfare weights that, if $x = (x(t)) \in \mathfrak{P}(r)$ is a long-run Pareto optimum, there exists an m-tuple $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m_{++}$ such that $x = (x_1, x_2, \dots, x_m)$ maximizes the collective utility function

$$\lambda_1 u_1(x_1) + \lambda_2 u_2(x_2) + \cdots + \lambda_m u_m(x_m)$$

subject to the total resource constraint

$$x_1 + x_2 + \cdots + x_m = r.$$

The collective utility function can be written as the sum

$$\sum_{0 \le t \le T} [\lambda_1 \delta_1^t v_1(x_1(t)) + \lambda_2 \delta_2^t v_2(x_2(t)) + \cdots + \lambda_m \delta_m^t v_m(x_m(t))],$$

and the resource constraint takes the form

$$x_1(t) + x_2(t) + \cdots + x_m(t) = \bar{r} \qquad t = 0, 1, \dots, T.$$

This maximization problem is therefore equivalent to $T+1$ separate maximization problems, one for each time period, where the problem for period t takes the form

$$\text{maximize} \quad \lambda_1 \delta_1^t v_1(x_1(t)) + \lambda_2 \delta_2^t v_2(x_2(t)) + \cdots + \lambda_m \delta_m^t v_m(x_m(t))$$

subject to the total resource constraint

$$x_1(t) + x_2(t) + \cdots + x_m(t) = \bar{r}.$$

The solution $x(t)$ of the maximization problem corresponding to period t is therefore the Pareto optimum of the short-run economy $\overline{\mathcal{E}[\bar{r}](\overline{\omega})}$ that is associated with the m-tuple of welfare weights $(\lambda_1 \delta_1^t, \lambda_2 \delta_2^t, \dots, \lambda_m \delta_m^t)$. Let us normalize this m-tuple by the convention that the welfare weight of the first consumer is equal to 1. The result therefore is that the component $x(t) \in \overline{P(\bar{r})}$ is parameterized by the normalized m-tuple $(1, \mu_2(t), \dots, \mu_h(t), \dots, \mu_m(t))$, where

$$\mu_h(t) = \left(\frac{\delta_h}{\delta_1}\right)^t \lambda_h$$

for $h \in H$ and $t = 0, 1, \ldots, T$.

The rest of the proof consists in exploiting this expression.

Incidentally, the normalized m-tuple $(1, \mu_2(t), \ldots, \mu_m(t))$ does not depend on the horizon T. The latter can therefore be taken arbitrarily large, which justifies taking the limit of t up to infinity.

8.3.3 Asymptotic Equilibrium Consumption

For all consumers except the most patient ones, i.e., $h \notin \Delta$, we have $\delta_h < \delta_1$, so that $\mu_h(t) = \left(\frac{\delta_h}{\delta_1}\right)^t \lambda_h$ tends to zero as t tends to infinity.

The limit of the preceding maximization problem is the problem of maximizing

$$\sum_{h \in \Delta} \lambda_h v_h(x_h^*)$$

subject to the constraint

$$\sum_h x_h^* = \bar{r}.$$

The solution $x_\Delta^* = (x_h^*)_{h \in \Delta}$ is the Pareto optimum of the short-run economy consisting of the most patient consumers $h \in \Delta$ defined by their short-run utilities $v_h : X \to \mathbb{R}$ and total resources $\bar{r}$ associated with the weights $(\lambda_1, \ldots, \lambda_h, \ldots)$ with $h \in \Delta$. For $h \notin \Delta$, define $x_h^* = 0 \in \mathbb{R}^\ell$, and let $x^* = (x_h^*)$ with $h \in H$.

A simple continuity argument applied to the first maximization problem and to its limit shows that $x(t)$, solution of the first maximization problem, tends to x^*, solution of the limit maximization problem. ∎

Economically speaking, it is as if all the consumers except the most patient ones are "eliminated" from the economy after some time. Elimination means here that their consumption tends to zero, even if the flow of their resources remains constant through time.

Note that the components $x(t)$ of the equilibrium allocation x are not stationary. Nevertheless, they converge to some limit when time tends to infinity, which is some approximation to being stationary. In other words, the equilibrium allocation $x = (x(t))$ is asymptotically stationary.

8.3.4 Long-Run Equilibria for $H = \Delta$

Because the equilibrium consumption of the less than patient consumers tends to zero, now assume that all consumers have the same level of impatience, i.e., $\Delta = H$.

Proposition 8.3.2 For $\Delta = H$, the period t component $x(t)$ of the long-run equilibrium allocation $x = (x(t))$ is stationary and equal to $\bar{x}$, where $\bar{x}$ is an equilibrium allocation of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$ composed of the h consumers with short-run utility functions $v_h : X \to \mathbb{R}$ and endowments $\bar{\omega}_h \in \mathbb{R}^{\ell}$.

Proof Let $x = (x(t)) \in \mathfrak{P}(r)$ be the (long-run) Pareto optimum parameterized by the normalized m-tuple of welfare weights $(\lambda_1, \lambda_2, \dots, \lambda_m)$. The component $x(t)$ is therefore the Pareto optimum of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$ parameterized by the m-tuple of welfare weights $\delta^t(\lambda_1, \lambda_2, \dots, \lambda_m)$. These (non-normalized) m-tuples differ for different time periods t only by the factor δ^t. Therefore, the component $x(t)$ is equal to the vector $\bar{x} = (\bar{x}_h)_{h \in H} \in \overline{P(\bar{r})}$ that is parameterized by the m-tuple of weights $(\lambda_1, \lambda_2, \dots, \lambda_m)$. The component $x(t)$ does not depend on time t. ∎

Stationarity strengthens the property regarding the asymptotic behavior of the consumption of the most patient consumers proved in proposition 8.3.1.

8.3.5 Long-Run Price Vectors

Supporting Price Vectors of Long-Run Pareto Optima

Let $p = (p(t))$ be the supporting price vector (normalized by the numeraire convention $p_1(0) = 1$) of the long-run Pareto optimum $x = (x_h)_{h \in H}$. This price vector p is collinear with the gradient vector of the utility function u_h at $x_h = (x_h(t))$ for any arbitrarily chosen consumer $h \in H$. It follows from the expression of the utility function $u_h(x_h)$ that there exists a real number $\kappa > 0$ such that $p = \kappa D u_h(x_h)$. (The role of κ is to produce a normalized price vector.) The period t component therefore satisfies

$$p(t) = \kappa \delta^t D v_h(\bar{x}_h) \tag{8.2}$$

for $t = 0, 1, \dots, T$, and $x_h = (\bar{x}_h)$ is stationary because x is a long-run Pareto optimum.

It then follows from (8.2) that $p(t) = \delta^t \bar{p}$ for $t = 0, 1, \ldots, T$, where $\bar{p} = p(0) \in \mathbb{R}^{\ell}_{++}$ is a numeraire normalized price vector of the short-run economy with ℓ goods. It also follows from (8.2) that, for $t = 0$, the first-order conditions are satisfied at $\bar{x}$ by $\bar{p} = p(0)$. This implies that the price vector $\bar{p} = p(0)$ does support the (short-run) Pareto optimum $\bar{x} = (\bar{x}_h)_{h \in H}$.

Application to Equilibrium Prices
Recall $H = \Delta = I$ and $J = \emptyset$.

Proposition 8.3.3 The long-run equilibrium price vector $p = (p(t))$ of the long-run economy $\mathcal{E}(\Delta, \emptyset)[r](\omega)$ is of the form $p(t) = \delta^t \bar{p}$, where $\bar{p}$ is an equilibrium price vector of the short-run economy $\overline{\mathcal{E}(\Delta)[\bar{r}](\varpi)}$.

Proof Let now $p = (p(t))$ be a long-run equilibrium price vector and $x = (x(t))$ be the corresponding equilibrium allocation for the long-run economy $\mathcal{E}(H, \emptyset)[r](\omega)$. Because the allocation $x = (x_h)_{h \in H}$ is a Pareto optimum, the equilibrium price vector $p = (p(t))$ supports the Pareto optimum $x = (x_h)_{h \in H}$. This proves that $p(t) = \delta^t \bar{p}$ for $t = 0, 1, \ldots, T$, where $\bar{p}$ supports the short-run Pareto optimum $\bar{x} = (\bar{x}_h)_{h \in H}$. To conclude that $\bar{p}$ is an equilibrium price vector and $\bar{x}$ the corresponding equilibrium allocation of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\varpi)}$, it is only required to show that the budget equality $\bar{p} \cdot (\bar{x}_h - \varpi_h) = 0$ is satisfied for every $h \in H$.

Let us write the budget equality satisfied by the long-run equilibrium price vector $p = (\delta^t \bar{p})$ and allocation $x = (\bar{x})$ for every consumer $h \in H$. This yields the equality

$$p \cdot (x_h - \omega_h) = (1 + \delta + \cdots + \delta^T) \bar{p} \cdot (\bar{x}_h - \varpi_h) = 0,$$

from which follows the equality

$$\bar{p} \cdot (\bar{x}_h - \varpi_h) = 0 \quad \text{for every } h \in H.$$

Conversely, let $\bar{p}$ be a short-run equilibrium price vector of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\varpi)}$, and let $\bar{x} = (\bar{x}_h)_{h \in H}$ denote the corresponding equilibrium allocation. Define $p(t) = \delta^t \bar{p}$ and $x(t) = \bar{x}$ for $t = 0, 1, \ldots, T$. Then it is straightforward to check that the price vector $p = (p(t))$ supports the allocation $x = (x(t))$. In addition, the long-run budget constraint is satisfied for every consumer $h \in H$. This implies that $p = (p(t))$ is an equilibrium price vector of the long-run economy $\mathcal{E}(H, \emptyset)[r](\omega)$, with $x = (x(t))$ as the corresponding equilibrium allocation. ∎

Absence of Wealth Transfer at Equilibrium

The following corollary is not surprising. Some of its economic real-world implications have certainly been overlooked up to now.

Corollary 8.3.4 Let $x = (x(t))$ be a long-run equilibrium allocation. Then $p(t) \cdot (x_h(t) - \omega_h(t)) = 0$ for every consumer $h \in H$.

In other words, there is at equilibrium no transfer of wealth from one time period to another. And this property is satisfied despite the fact that consumers face no restrictions on making such transfers.

In the real world, intertemporal transfers are done through, for example, banks. If there is no transfer, the necessity of banks becomes questionable. Then why not close the banks? But then the budget set of every consumer becomes much smaller because of the liquidity constraints resulting from bank closures. The proper model when there are no banks corresponds to the case $(I, J) = (\emptyset, H)$, a model that, quite surprisingly, may feature non-Pareto-optimal equilibria. Therefore, a service like the one offered by the banking system may contribute to efficiency in ways that are not necessarily properly measured by the intensity of trade or by intertemporal wealth transfers.

8.4 Long-Run Equilibria: Fully Restricted Case

The case where $I = \emptyset$ and $J = H$ turns out to be mathematically very simple. Let $j \in J$ be some restricted consumer. The demand $\varphi_j(p, \omega)$ maximizes

$$u_j(x_j) = \sum_{t=0}^{T} \delta^t v_j(x_j(t))$$

subject to the constraints

$$p(t) \cdot (x_j(t) - \omega_j(t)) = 0, \qquad t = 0, 1, \ldots, T.$$

This problem decomposes into $T + 1$ maximization problems. For the time period t, the maximization problem takes the form

$$\text{maximize} \quad \delta^t v_j(x_j(t))$$

subject to the (unique) constraint

$$p(t) \cdot (x_j(t) - \omega_j(t)) = 0.$$

The solution $x_j(t)$ can be interpreted as the demand of consumer j given the (non-normalized) price vector $p(t)$ in the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$. Using the demand function $\bar{f}_j : \mathbb{R}^{\ell}_{++} \times \mathbb{R}_{++} \to X$ of this consumer j, we have $x_j(t) = \bar{f}_j(p(t), p(t) \cdot \omega_j(t))$.

The period t component of the equilibrium equation $\sum_{j \in J} x_j = \sum_{j \in J} \omega_j = r$ can be written as

$$\sum_{j \in J} \bar{f}_j(p(t), p(t) \cdot \bar{\omega}_j) = \bar{r}.$$

This equation implies that $p(t)$ is a non-normalized equilibrium price vector of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$. Conversely, it is obvious that if the price vectors $p(t)$ are (non-normalized) equilibrium price vectors (except for $p(0)$, which is necessarily normalized by the numeraire convention $p_1(0) = 1$), then the price vector $p = (p(t))$ is an equilibrium price vector of the economy $\mathcal{E}(\emptyset, H)[r](\omega)$.

8.4.1 Characterization of the Long-Run Equilibrium Allocations

A first consequence is that equilibrium prices are not determined in period $t \geq 1$. There is a continuum of equilibrium price vectors. Despite this indeterminacy of the equilibrium price vectors, equilibrium allocations turn out to be determined and locally unique (at least generically) because of the homogeneity of degree 0 of consumers' demand functions in the short-run economy.

Let us denote by $\bar{\mathcal{A}}(H)[\bar{r}](\bar{\omega})$ and $\mathcal{A}(\emptyset, H)[r](\omega)$ the set of equilibrium allocations of the short-run and long-run economies $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$ and $\mathcal{E}(\emptyset, H)[r](\omega)$, respectively.

Proposition 8.4.1 Long-run and short-run equilibrium allocations are related by the formula

$$\mathcal{A}(\emptyset, H)[r](\omega) = (\overline{\mathcal{A}(H)[\bar{r}](\bar{\omega})})^{T+1}.$$

The proof is obvious.

8.4.2 Nonstationary Long-Run Equilibrium Allocations

It follows from proposition 8.4.1 that a necessary and sufficient condition for the existence of *nonstationary long-run equilibrium allocations* is that the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$ has more than one equilibrium allocation. More precisely, if n is the number of equilibrium allocations of the short-run economy, then the long-run economy $\mathcal{E}(\emptyset, H)[r](\omega)$ features n^{T+1} long-run equilibrium allocations, out of which only n are stationary and $n(n^T - 1)$ are nonstationary. Except for the case $n = 2$ and $T = 1$, (actually, the case $n = 2$ is not generic because $n = 3$ is the minimum number of multiple equilibria for generic economies), there are more long-run nonstationary equilibria than stationary ones.

At this point, it may be worth recalling that we know from the previous chapters that the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\varpi)}$ has multiple equilibrium allocations if the endowment parameter ϖ is sufficiently far away from the set of short-run Pareto optima $\overline{P(\bar{r})}$.

8.5 Long-Run Equilibria: General Case

The two polar cases where all consumers are unrestricted (i.e., $H = I$ and $J = \emptyset$) or restricted (i.e., $H = J$ and $I = \emptyset$) highlight the role played by restrictions in wealth transfers between time periods in generating nonstationary equilibrium allocations in stationary long-run economies.

We now explore the nature of the equilibria of the long-run stationary economy $\mathcal{E}(I, J)[r](\omega)$, with the subsets I and J assumed to be nonempty. There is no loss of generality in assuming that consumer 1 is unrestricted, i.e., $1 \in I$. The analysis is also simplified by assuming $H = \Delta$: $\delta_h = \delta$ for all $h \in H$.

8.5.1 Stationary Long-Run Equilibria

First it is shown that the stationary long-run equilibria correspond to equilibria of the short-run economy independently of the decomposition of H into the two subsets I and J.

Proposition 8.5.1 The stationary allocation $x = (x(t))$, with $x(t) = \bar{x}$, is an equilibrium allocation of the long-run economy $\mathcal{E}(I, J)[r](\omega)$ if and only if $\bar{x}$ is an equilibrium allocation of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\varpi)}$.

Proof The proof is a slight variation of the one given for the case $H = J$, $I = \emptyset$. Let $x = (x(t)) = (\bar{x})$ be the stationary allocation associated with the long-run equilibrium price vector $p = (p(t))$. Since consumer 1 is unrestricted, $x_1 = (x_1(t))$ maximizes $u_1(x_1)$ subject to the unique budget constraint $p \cdot (x_1 - \omega_1) = 0$.

The (necessary) first-order conditions for this maximization problem take the form $p(t) = \kappa \delta_1^t D v_1(x_1(t))$ for $t = 0, 1, \ldots, T$, with $\kappa > 0$. It follows from the assumption of stationarity that $x_1(t) = \bar{x}_1$ for every t, from which follows $p(t) = \delta_1^t p(0)$.

The budget equation

$$p \cdot (x_i - \omega_i) = 0$$

of the unrestricted consumer $i \in I$ then becomes

$$p \cdot (x_i - \omega_i) = (1 + \delta_1 + \cdots + \delta_1^T) p(0) \cdot (\bar{x}_i - \overline{\omega}_i),$$

from which follows

$$p(0) \cdot (\bar{x}_i - \overline{\omega}_i) = 0, \qquad i \in I.$$

In addition, the (sufficient) first-order conditions for the maximization of $v_i(x_i(t))$ subject to the constraint $p(0) \cdot (x_i(t) - \omega_i(t))$ are satisfied at $\bar{x}_i$ for every $i \in I$.

This proves that the budget constraint is satisfied at the price vector $\bar{p} = p(0)$ by the allocation $\bar{x} = (\bar{x}_h)_{h \in H}$ for all consumers $h \in H$. This implies that $\bar{x} = (\bar{x}_h)_{h \in H}$ is an equilibrium allocation of the short-run economy $\bar{\mathcal{E}}(H)[\bar{r}](\overline{\omega})$. The converse is obvious. ∎

8.5.2 Nonstationary Long-Run Equilibria

The goal is now to investigate the existence of nonstationary long-run equilibrium allocations. The key to their existence is the failure of the first welfare theorem to hold true in the long-run stationary economy $\mathcal{E}(I, J)[r](\omega)$. Had the first welfare theorem been satisfied, every long-run equilibrium allocation would have been a long-run Pareto optimum, which is stationary.

In fact, this observation does not convey very good news about the mathematical difficulty of this problem. It is indeed much easier to study the properties of solutions of optimization problems than of general equation systems. This is the explanation for the wide success of variational methods (i.e., the identification with maximization problems) in mathematics in general and in numerical analysis in particular. In the Arrow-Debreu model, the power of the variational methods enabled by the first welfare theorem is illustrated by Negishi's approach to the existence of equilibrium. This chapter also contains a few examples of the power of the variational approach in the study of equilibrium solutions of models like the Arrow-Debreu.

The strategy in this section is to restore some version of the first welfare theorem for the stationary long-run economies $\mathcal{E}(I, J)[r](\omega)$. This leads to defining another concept of Pareto optimality for long-run economies. The interest of this concept is mostly mathematical because any really convincing economic interpretation is lacking.

8.5.3 Deconstructing Restricted Consumers

Every restricted consumer $j \in J$ is deconstructed into $T+1$ new consumers denoted by j_t, with t varying from 0 to $T+1$. The idea is that the deconstructed consumer j_t "lives" only in period t: endowments and consumption bundles are different from $0 \in \mathbb{R}^\ell$ only in that period. Endowments and consumption bundles are formally extended to the $T+1$ periods by being equal to $0 \in \mathbb{R}^\ell$ in the T time periods that are different from t.

The endowment vector ω_{j_t} of the deconstructed consumer j_t is therefore the vector of $(\mathbb{R}^\ell)^{T+1}$ whose components are equal to $\overline{\omega}_j$ in period t and zero in all other periods.

If $x_{j_t} = (x_{j_t}(t'))_{0 \leq t' \leq T}$ denotes the consumption bundle of the deconstructed consumer j_t, then $x_{j_t}(t') = 0$ for $t' \neq t$ while $x_{j_t}(t) \in X = \mathbb{R}^\ell_{++}$.

The utility function u_{j_t} of consumer j_t is simply

$$u_{j_t}(x_{j_t}) = v_j(x_{j_t}(t)),$$

where x_{j_t} is the consumption bundle of the deconstructed consumer j_t.

Consumer j_t's consumption bundle maximizes the utility $u_{j_t}(x_{j_t})$ subject to the budget constraint $p \cdot (x_{j_t} - \omega_{j_t}) = 0$. The nonzero components of x_{j_t} and ω_{j_t} are such that $x_{j_t}(t)$ maximizes $v_j(x_{j_t}(t))$ subject to the budget constraint $p(t) \cdot (x_{j_t}(t) - \overline{\omega}_j) = 0$.

The demand $x_{j_t}(t)$ of consumer j_t can thus be expressed in terms of the *short-run demand function* $\bar{f}_j(\bar{p}, w_j)$ of consumer j:

$$x_{j_t}(t) = \bar{f}_j(p(t), p(t) \cdot \overline{\omega}_j).$$

The deconstructed consumers j_t differ from the consumers of the standard versions of the Arrow-Debreu model by their consumption set, which has a much smaller dimension than the commodity space. But this difference has only minor implications, especially when it comes to satisfying the first welfare theorem for an economy that includes these deconstructed consumers.

8.5.4 The Deconstructed Economy $\mathcal{D}(I, J)[r](\omega)$

The deconstructed economy $\mathcal{D}(I, J)[r](\omega)$ is defined as follows:

Commodity space. $(\mathbb{R}^\ell)^{T+1}$;

Consumers of type I, for $i \in I$. Consumption set X^{T+1}; endowments $\omega_i = (\overline{\omega}_i, \overline{\omega}_i, \ldots, \overline{\omega}_i)$; utility function $u_i : X^{T+1} \to \mathbb{R}$;

Consumers of type J, for j_t such that $j \in J$ and $t = 0, 1, \ldots, T$. Consumption set $\{0\} \times \{0\} \times \cdots \times \{0\} \times X \times \{0\} \times \cdots \times \{0\}$; endowments $\omega_{j_t} =$

$(0, 0, \ldots, 0, \overline{\omega}_j, 0, \ldots, 0)$; utility function $u_{j_t} : \{0\} \times \{0\} \times \cdots \times \{0\} \times X \times \{0\} \times \cdots \times \{0\} \to \mathbb{R}$.

Consumers of type I and J maximize the utility of their consumption subject to a unique budget constraint. Let $x_i = (x_i(t))$ and $x_{j_t} = (x_{j_t}(t'))_{t'=0,1,\ldots,T}$ be the demand of consumer i and j_t, respectively, for the price vector $p = (p(t))$. That price vector is an *equilibrium price vector* of the *deconstructed economy* $\mathcal{D}(I, J)[r](\omega)$ if there is equality of aggregate supply and demand:

$$\sum_{i \in I} x_i + \sum_{(j,t) \in J \times \{0,1,\ldots,T\}} x_{j_t} = \sum_{i \in I} \omega_i + \sum_{(j,t) \in J \times \{0,1,\ldots,T\}} \omega_{j_t}.$$

8.5.5 Long-Run and Deconstructed Economies

Proposition 8.5.2 The price vector $p = (p(t))$ is an equilibrium price vector of the long-run economy $\mathcal{E}(I, J)[r](\omega)$ if and only if it is an equilibrium price vector of the deconstructed economy $\mathcal{D}(I, J)[r](\omega)$.

Proof It suffices to observe that the demand of the restricted consumer $j \in J$ in the long-run economy maximizes $u_j(x_j)$ subject to the $T + 1$ budget constraints $p(t) \cdot (x_j(t) - \omega_j(t)) = 0$. Because of the separability of the utility $u_j(x_j)$ with respect to its various time period components, this maximization problem is equivalent to solving $T + 1$ maximization problems, where each problem consists in the maximization of $v_j(x_j(t))$ subject to the unique budget constraint $p(t) \cdot (x_j(t) - \omega_j(t))$, with t varying from 0 to T. The commodity bundle $x_j(t)$ coincides with the period t component of the demand x_{j_t} of consumer j_t given the endowment $\omega_{j_t} = (0, \ldots, 0, \omega_j(t), 0, \ldots, 0)$. It then suffices to observe that the equilibrium conditions take the same form for the long-run and the deconstructed economies, which ends the proof of their equivalence. ∎

Remark Comparing prices between the long-run and the deconstructed economies is easy. But allocations between these two economies must also be compared. And there the notation may become a little bit heavy. The allocation $(x_i, x_{j_t})_{(i,j,t) \in I \times J \times \{0,1,\ldots,T\}}$ of the deconstructed economy corresponds to the allocation $(x_i, x_j)_{(i,j) \in I \times J}$ of the long-run economy, and conversely. Often, these two allocations can be "identified." In case of doubt, (x_i, x_j) is the *trace* in the long-run economy or the *long-run trace* of the allocation $(x_i, x_{j_t})_{(i,j,t) \in I \times J \times \{0,1,\ldots,T\}}$ of the deconstructed economy.

8.5.6 First Welfare Theorem for Deconstructed Economies

What makes the deconstructed economy interesting is that it differs very little from a standard Arrow-Debreu economy. The uniqueness of the consumer's budget constraint being crucial to the validity of the first welfare theorem, it can be expected that some version of that theorem is going to hold true for deconstructed economies.

Pareto Optima of Deconstructed Economies

The allocation $x = (x_i, x_{j_t})_{(i,j,t)\in I\times J\times\{0,1,\dots,T\}}$ is a *Pareto optimum for the deconstructed economy* $\mathcal{D}(I,J)[r](\omega)$ if there is no $x' = (x'_i, x'_{j_t})$ such that the inequalities $u_i(x_i) \le u_i(x'_i)$ and $u_{j_t}(x_{j_t}) \le u_{j_t}(x'_{j_t})$ are all satisfied, one of them being strict, and the total resource constraint being also satisfied, i.e.,

$$\sum_{i\in I} x_i + \sum_{(j,t)\in J\times\{0,1,\dots,T\}} x_{j_t} = \sum_{i\in I} x_i + \sum_{(j,t)\in J\times\{0,1,\dots,T\}} x'_{j_t} = r.$$

Denote by $DP(I,J)(r)$ the set of Pareto optima of the deconstructed economy $\mathcal{D}(I,J)[r](\omega)$. A *long-run Pareto optimum* is obviously a Pareto optimum for the deconstructed economy but *the converse is not true* in general. We therefore have the strict inclusion $\mathfrak{P}(r) \subset DP(I,J)(r)$.

First Welfare Theorem for Deconstructed Economies

Proposition 8.5.3 Every equilibrium allocation of the deconstructed economy $\mathcal{D}(I,J)[r](\omega)$ is a Pareto optimum.

The following proof is just an adaptation to the setup of deconstructed consumers of the standard proof of the first welfare theorem for the Arrow-Debreu model.

Proof Let $x = ((x_i),(x_{j_t}))_{(i,j,t)\in I\times J\times\{0,1,\dots,T\}}$ be the allocation associated with some equilibrium price vector $p = (p(t))$ for the deconstructed economy.

Let us argue by contradiction. Assume that there exists an allocation $x' = ((x'_i),(x'_{j_t}))_{(i,j,t)\in I\times J\times\{0,1,\dots,T\}}$ that is Pareto-superior to x. The inequalities $u_i(x_i) \le u_i(x'_i)$ are satisfied for all $i \in I$, and $v_j(x_j(t)) \le v_j(x'_j(t))$ for all j in J and $t = 0,1,\dots,T$, one inequality at least being strict. It follows from the definition of the x_i (resp. $x_j(t)$) as maximizing $u_i(x_i)$ (resp. $v_j(x_j(t))$) subject to the constraint $p\cdot(x_i - \omega_i) = 0$ (resp. $p(t)\cdot(x_j(t) - \omega_j(t)) = 0$) that the inequalities $p\cdot(x'_i - \omega_i) \ge 0$ (resp. $p(t)\cdot(x'_j(t) - \omega_j(t)) \ge 0$) are satisfied and that one inequality at least is strict. Adding up all these inequalities yields the strict inequality

$$p \cdot \left(\sum_{i \in I} (x_i - \omega_i) + \sum_{j \in J} (x_j - \omega_j) \right) > 0,$$

which contradicts the equality

$$\sum_{i \in I} (x_i - \omega_i) + \sum_{j \in J} (x_j - \omega_j) = 0,$$

which is satisfied by any equilibrium allocation. ■

Parameterization of DP(I, J)(r) by Welfare Weights

In addition to the first welfare theorem, the properties regarding the parameterization of the set of Pareto optima by the welfare weights associated with the consumers making up the economy remain true. We therefore have the following.

Proposition 8.5.4 The set of Pareto optima $DP(I, J)(r)$ of the deconstructed economy is parameterized by the weights (λ_i) for $i \in I$ and λ_{j_t} for $(j, t) \in J \times \{0, 1, \ldots, T\}$.

(It is often convenient to normalize these weights. Since $1 \in I$, we use the convention $\lambda_1 = 1$.)

It follows from this parameterization that the set $DP(I, J)(r)$ of Pareto optima of the deconstructed economy $\mathcal{D}(I, J)[r](\omega)$ is diffeomorphic to $\mathbb{R}^{\#I+(T+1)\#J-1}$.

8.5.7 Period *t* Component of a Pareto Optimum of the Deconstructed Economy

Proposition 8.5.5 The allocation $x = (x(t))$ is the long-run trace of a Pareto optimum of the deconstructed economy $\mathcal{D}(I, J)[r](\omega)$ if and only if the period t component $x(t)$ is a Pareto optimum of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\overline{\omega})}$ and the (normalized) weights associated with $x(t)$ for the unrestricted consumers $i \in I$ are equal for all t.

Proof This proposition is also an obvious consequence of the characterization of the Pareto optima of the deconstructed economy $\mathcal{D}(I, J)[r](\omega)$ as solutions of the problem of maximizing

$$u_1(x_1) + \cdots + \lambda_i u_i(x_i) + \cdots + \lambda_{j_t} v_j(x_{j_t}) + \cdots$$

subject to the resource constraints

$$\sum_{i \in I} x_i + \sum_{j \in J} x_j = r.$$

This problem is decomposable into the $T+1$ smaller problems

$$\text{maximize} \quad v_1(x_1(t)) + \cdots + \lambda_i v_i(x_i(t)) + \cdots + \lambda_{j_t} v_j(x_{j_t}) + \cdots,$$

where t is now fixed (with $0 \le t \le T$) subject to the resource constraints

$$\sum_{i \in I} x_i(t) + \sum_{j \in J} x_j(t) = r(t) = \bar{r}.$$

The solutions of these $T+1$ optimization problems are the Pareto optima of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\overline{\omega})}$ associated with the welfare weights $(1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_{j_t}, \ldots)$. ■

8.5.8 Long-Run Pareto Optima in the Set of Pareto Optima of the Deconstructed Economy

Proposition 8.5.6 The long-run trace $(x_i, x_j)_{I \times J}$ of the Pareto optimum $(x_i, x_{j_t})_{(i,j,t) \in I \times J \times \{0,1,\ldots,T\}}$ of the deconstructed economy is a long-run Pareto optimum if and only if the weights λ_{j_t} associated with the deconstructed consumers j_t, with $j \in J$ and $t = 0, 1, \ldots, T$, satisfy the equalities $\lambda_{j_0} = \lambda_{j_1} = \cdots = \lambda_{j_t} = \cdots = \lambda_{j_T}$ for every $j \in J$.

Proof This property follows readily from the stationarity of the long-run Pareto optimum $(x_i, x_j)_{I \times J}$ combined with proposition 8.5.5. ■

If we "identify" allocations of the long-run and deconstructed economies, we can say that the set of long-run Pareto optima is a subset of the set of deconstructed Pareto optima. It is in fact the subset consisting of the *stationary deconstructed Pareto optima.*

The set of long-run Pareto optima $\mathfrak{P}(r)$ associated with given total resources $r \in X$ has dimension $(\#I) + (\#J) - 1$. The set of disconstructed Pareto optima $DP(I, J)(r)$ for the same $r \in X$ has dimension $(\#I) + (T+1)(\#J) - 1$. The difference between these two numbers gives an idea of the degree of fluctuations that can be achieved by the deconstructed Pareto optima.

Exercise

8.1. Prove proposition 8.5.6 without the assumption $\Delta = H$, i.e., all δ_h's are equal. (*Hint:* Use directly the maximization problem instead of the stationarity of the allocation.)

8.5.9 Limited Version of the Second Welfare Theorem for Deconstructed Economies

The k-Regularity of Long-Run Allocations

Let $x = (x(t))$ be an allocation in the long-run economy $\mathcal{E}(I, J)[r](\omega)$ whose period t components $x(t)$ are all *short-run Pareto optima,* i.e., $x(t) \in \overline{P[\bar{r}]}$. Let $\bar{p}(t)$ denote the normalized price vector that supports the short-run Pareto optimum $x(t) \in \overline{P[\bar{r}]}$. Then the long-run allocation $x = (x(t))$ is *k-regular* if the sequence $x(t)$ takes k different values and if the linear equation system

$$\bar{p}(t) \cdot \bar{\omega}_h = \bar{p}(t) \cdot x_h(t) \quad \text{for } t = 0, \ldots, T \text{ and } h \in H,$$

where $\bar{\omega}_h = (\omega_n^1, \ldots, \omega_n^\ell)$ is the unknown, has a solution. Note that only k of these equations are really different.

Proposition 8.5.7 The long-run allocation $x = (x(t))$ is k-regular if and only if there exists an economy $\bar{\omega} \in \overline{\Omega(\bar{r})}$ such that the components $x(t)$ for $t = 0, 1, \ldots, T$ are equilibrium allocations of the short-run economy $\overline{\mathcal{E}(H)[\bar{r}](\bar{\omega})}$.

The proof is obvious.

Let us now define the set $\mathcal{R}A_k(I, J)[r]$ as consisting of the k-regular Pareto optima of the deconstructed economy $\mathcal{D}(I, J)[r](\omega)$. Note that $\mathcal{R}A_1(I, J)[r]$ consists of the Pareto optima (x_i, x_{j_t}) of the deconstructed economy that have a long-run stationary trace (x_i, x_j). Therefore, $\mathcal{R}A_1(I, J)[r]$ can be identified with the set of long-run Pareto optima $\mathfrak{P}(r)$ through the "identification" of the allocations of the long-run and deconstructed economies.

Proposition 8.5.8 The set $\mathcal{R}A_k(I, J)[r]$ is nonempty for an open and dense set of utility functions for $2 \le k \le \ell$.

Proof The idea of the proof is very simple, details are a bit technical. Let us use the geometric approach of chapter 5. The allocation $x(t)$ and its supporting price vector $\bar{p}(t)$ define a price-income equilibrium $b(t) = (\bar{p}(t), (w_h(t) = \bar{p}(t) \cdot x_h(t))_{h \in H})$. These k price-income equilibria $b(t)$ belong to the section manifold $B(r)$. Because k is $\le \ell$, they also belong to at least one affine subspace A of $H(r)$ with dimension $\ell - 1$. If the affine space A corresponds to an economy $\bar{\omega}$, i.e., takes the form $A = A(\bar{\omega})$, the proof is over. The last remaining step is therefore to show that the k different points of $B(r)$ can determine an affine space of the form $A(\bar{\omega})$. It is easy to see that this property is satisfied if the price system $(p(0), p(1), \dots, p(t), \dots, p(T))$ has rank k. The latter property is true for a generic set of utility functions. This follows from the property of the section manifold $B(r)$ of being also the set of Pareto minima for the indirect utility functions, without forgetting the complication that the welfare weights of the unconstrained consumers also must be equal. Details are left to the reader. ∎

Second Welfare Theorem for the Elements of $\mathcal{R}A_k(I,J)[r]$ with $k \leq \ell$
The analogue of the second theorem of welfare economics for the setup of deconstructed economies is as follows.

Proposition 8.5.9 The long-run trace of every k-regular Pareto optimum $\tilde{x} = (x_i, x_{j_t}) \in \mathcal{R}A_k(I,J)[r]$ with $k \leq \ell$ of the deconstructed economy is the equilibrium allocation of some long-run stationary economy.

Proof It follows from the definition of k-regularity that the (normalized) price vectors $\bar{p}(t)$ that support the short-run Pareto optimum $x(t) \in \bar{P}[\bar{r}]$ define for t varying from 0 to T a system of $T+1$ vectors that has rank k.

Let us compute the coordinates of $\bar{\omega}_i$ and $\bar{\omega}_j$ of the unrestricted and restricted consumers $i \in I$ and $j \in J$, respectively, that are compatible with the k-regular extended Pareto optimum $\tilde{x} = (x_i, x_{j_t})$. In other words, let us determine the fiber consisting of the stationary endowments that are compatible with the given Pareto optimum. As in chapter 4, for the fiber of the equilibrium manifold of the standard Arrow-Debreu model, the fiber here also satisfies a set of linear equations whose unknowns are the coordinates of the endowment vector $\omega = ((\omega_i)_{i\in I}, (\omega_j)_{j\in J})$.

Determination of $\bar{\omega}_j$ for $j \in J$ The linear equation system that determines the endowment vector $\bar{\omega}_j$, with $j \in J$, involves ℓ unknowns and $T+1$ linear equations:

$$p(0) \cdot \bar{\omega}_j = p(0) \cdot x_j(0)$$

$$p(1) \cdot \bar{\omega}_j = p(1) \cdot x_j(1)$$

$$\vdots$$

$$p(T) \cdot \bar{\omega}_j = p(T) \cdot x_j(T).$$

It follows from k-regularity that only $k \leq \ell$ values of $x(t)$ are different. This implies that out of these $T+1$ linear equations, only k are really different and have to be solved to determine the ℓ unknowns $\bar{\omega}_j^1, \bar{\omega}_j^2, \dots, \bar{\omega}_j^\ell$. In addition, these k linear equations are independent, i.e., have full rank, which implies that this linear equation system always has a solution. The solution set for $\bar{\omega}_j$ is therefore a linear manifold parameterized by $\ell - k - 1$ parameters.

Determination of $\bar{\omega}_i$ for $i \in I \backslash \{1\}$ The stationary period t component $\bar{\omega}_i$ of the endowment vector of consumer i, with $i \in I$ but $i \neq 1$, has to satisfy the budget equality:

$$\left(\sum_{0 \leq t \leq T} p(t)\right) \cdot \bar{\omega}_i = \sum_{0 \leq t \leq T} p(t) \cdot x_i(t) \quad \text{for } i \in I. \tag{8.3}$$

Solutions to this linear equation with ℓ unknowns $\bar{\omega}_i^1, \bar{\omega}_i^2, \dots, \bar{\omega}_i^\ell$ always exist. They depend on $\ell - 1$ parameters.

Determination of $\bar{\omega}_1$ Once $\bar{\omega}_j$ and $\bar{\omega}_i$ for $j \in J$ and $i \in I \backslash \{1\}$ have been determined, the stationary period t component $\bar{\omega}_1$ for consumer 1 follows from the total resource constraint:

$$\bar{\omega}_1 = \bar{r} - \sum_{j \in J} \bar{\omega}_j - \sum_{i \in I \backslash \{1\}} \bar{\omega}_i.$$

The set of stationary endowments $\bar{\omega} = ((\bar{\omega}_i)_{i \in I}, (\bar{\omega}_j)_{j \in J})$ compatible with the equilibrium allocation $x = (x(t))$ (i.e., the associated fiber) is therefore a linear manifold of dimension $(\#J)(\ell - k - 1) + ((\#I) - 1)(\ell - 1)$. ∎

Remark. Budget constraint of consumer 1 Let us check that $\bar{\omega}_1 \in \mathbb{R}^\ell$ satisfies the budget constraint for consumer 1. Multiplication of the two sides of equality (8.3) by the price vector $p(t)$ yields

$$p(t) \cdot \bar{\omega}_1 = p(t) \cdot \bar{r} - \sum_{j \in J} p(t) \cdot \bar{\omega}_j - \sum_{i \in I \backslash \{1\}} p(t) \cdot \bar{\omega}_i.$$

Let us add up all these equalities side by side for t varying from $t = 0$ to $t = T$:

$$p \cdot \omega_1 = p \cdot r - \sum_{j \in J} \left(\sum_{t=0}^{T} p(t) \cdot x_j(t) \right) - \sum_{i \in I \backslash \{1\}} p \cdot x_i,$$

from which follows

$$p \cdot \omega_1 = p(\cdot r - \sum_{j \in J} x_j - \sum_{i \in I \setminus \{1\}} x_i) = p \cdot x_1.$$

Remark. Importance of the fibers The preceding proof of existence exploits the linearity of the budget and total resource constraints, in other words, the linearity of the equations of the fibers.

Generic Existence of Nonstationary Equilibrium Allocations

Proposition 8.5.10 Generically on preferences, stationary long-run economies have nonstationary equilibrium allocations whenever the set of constrained consumers J is nonempty.

The proposition follows readily from the combination of propositions 8.5.9 and 8.5.8.

8.5.10 Intensity of Trade and Nonstationarity of the Long-Run Equilibrium Allocation

The existence of nonstationary equilibrium allocations in stationary long-run economies raises many new questions. For example, can anything be said about the size of the set of economies that feature nonstationary equilibrium allocations? Or, since stationary solutions exist for any decomposition of the consumer set H into restricted and unrestricted consumers, how can an economy go from a stationary, long-run, Pareto-optimal equilibrium to a nonstationary, non-long-run, Pareto-optimal one?

Here, I give a very limited answer to the first question, which highlights, as in previous chapters, the importance of the amount of trade at equilibrium.

Proposition 8.5.11 There is an open neighborhood U of the set of short-run Pareto optima $\overline{P(\bar{r})}$ such that, for $\bar{\omega} \in U$, the long-run economy $\mathcal{E}(I, J)[r](\omega)$ has a unique, stationary equilibrium for any decomposition of the consumer set H into the sets I and J of unrestricted and restricted consumers.

Proof Here I just give an idea of the proof. For any given partition (I, J) of H, one can study solutions of the equilibrium equation of the long-run economy $\mathcal{E}(I, J)[r](\omega)$ along the lines of chapters 2–4, with the equilibrium manifold, the natural projection, and the remarkable role of the no-trade equilibria and the Pareto optima. The same lines of reasoning imply that for $\bar{\omega} \in \overline{P(\bar{r})}$, the equilibrium is unique and regular, the corresponding equilibrium allocation is equal to $\bar{\omega}$. As in the standard version of the Arrow-Debreu model, equilibrium is then unique for $\bar{\omega}$ in a neighborhood of $\overline{P(\bar{r})}$ and therefore necessarily stationary. There is no room for nonstationary equilibrium allocations for such $\bar{\omega}$'s.

With all possible partitions of H into the subsets I and J, it suffices to consider the intersection of the open neighborhoods defined for each pair (I, J) to prove the proposition. ∎

8.6 Conclusion

This chapter showed that a highly stationary version of the intertemporal Arrow-Debreu model can feature nonstationary equilibrium allocations provided some consumers (at least one) are prevented from participating in financial markets. The phenomenon of nonstationarity also requires that the intensity of trade at equilibrium goes beyond some threshold value. These results throw new light on the question of business fluctuations, even if many questions of major economic interest remain unanswered.

The structure and properties of the standard version of the Arrow-Debreu model have played a major role in our study of properties of long-run economies with restricted market participation. The interest of the standard Arrow-Debreu model is therefore not limited to a simple representation of market economies. The properties of this model make it a powerful tool for the study of more elaborate economic models.

The proof of the existence of nonstationary equilibrium allocations in the intertemporal Arrow-Debreu model with restricted market participation is at best a first step in a general equilibrium approach to the theory of business cycles and fluctuations. This result raises more questions than it solves. Presumably the most important question is why and how an economy would migrate from a stationary equilibrium solution to one of the nonstationary solutions whose existence was proved in this chapter. This question is a special case of the problem of equilibrium selection when there are multiple equilibria, a problem still open in the case of the standard Arrow-Debreu model. It would be particularly interesting to see whether the structure of the intertemporal Arrow-Debreu model with restricted market participation is sufficiently rich to bring the problem of equilibrium selection closer to a solution that would make economic sense.

8.7 Notes and Comments

The content of this chapter is new in printed form. Some of its results were presented at Malinvaud's seminar in Paris in 1987, at the London School of Economics in 1989, and in a few other places during the 1990s.

The formulation of the impact of monetary policy by way of liquidity constraints in a general equilibrium model is due to Tobin and Dolde (1971). They used explicit computations to analyze the properties of equilibrium allocations associated with given liquidity constraints. Their approach was generalized by Bewley (1986), Scheinkman, and Weiss (1986), and Woodford (1989) to models whose main features are the combination of an infinite horizon and the existence of two types of agents, only one type of agent being able to transfer wealth over time. The main results obtained in these papers consist mostly of examples of economies with nonstationary equilibria. If the consideration of an infinite horizon in such dynamic issues has a long tradition that goes back to growth theory, the infinite horizon makes the analysis of the properties of equilibria as a function of initial endowments and of the restrictions in market participation a task that is almost impossible to complete in sufficiently general models. Restricting the horizon to a finite number of time periods, as in this chapter, makes these issues more amenable to analysis.

The intertemporal deterministic model with restricted market participation considered in this chapter is, for the case of two periods, formally identical to the extension in Balasko, Cass, and Shell (1995) of the sunspot model of Cass and Shell (1983). Existence of nonstationary solutions is then equivalent to the existence of sunspot equilibria. For a general formulation of an equivariant Arrow-Debreu model that generalizes the sunspot model, see Balasko (1990). The specificity of the intertemporal model compared to the sunspot model is in its number of time periods, which is large, certainly much larger than two.

9 The Temporary Equilibrium Model

9.1 Introduction

A basic tenet of the multiperiod intertemporal Arrow-Debreu model is that all goods are traded at the same time and at the beginning of the first period, often known as period 0, at the latest. This simplifying assumption is extremely convenient in bringing consistency to the Arrow-Debreu model. It is so extreme, however, that it amply justifies the exploration of the consequences of restricting actual trade to some subset of all the goods available in the economy. In practice, the subset consists of the goods delivered in period 0. In other words, the assumption is that only spot markets are open in period 0. There are no forward markets for the goods to be delivered in period 1. This assumption raises the question of how economic agents can transfer wealth between time periods and states of nature, because the goods considered in the Arrow-Debreu model are perishable. It is therefore necessary to extend the Arrow-Debreu model by introducing suitably defined financial assets. This leads to a new model that features goods and assets. Assets are traded in period 0 as well as the goods delivered in period 0.

In the multiperiod intertemporal Arrow-Debreu model, consumers' choices are determined by budget constraints and prices. In the absence of forward markets, the goods delivered in period 1 and later have no prices. Therefore, consumers cannot base their intertemporal consumption decisions on the publicly known prices that are determined by the equality of supply and demand in actual markets. The best these consumers can do is to substitute for nonexisting market prices their own forecasts of the values these prices will take in due time. These forecasts suffice, however, to determine individual demand and supply, and therefore total demand and supply, of the goods delivered in period 0. Temporary equilibrium is nothing more than an equilibrium price vector and an equilibrium allocation of the goods and assets delivered in period 0 given consumers' forecasts of the prices of goods delivered in future periods.

Several versions of the temporary equilibrium model exist in the literature. They are all closely related. The one considered in this chapter is characterized by the fact that the values of future prices are included in the states of nature. One nice feature of this approach is that the temporary equilibrium model then turns out to possess a reduced form that is mathematically very close to the Arrow-Debreu model. The main differences between the reduced form and the Arrow-Debreu model are (1) the consumption sets contain the positive orthant and are usually larger, and (2) preferences are price-dependent. We have already seen the impact of the latter assumption on the properties of the Arrow-Debreu model in chapter 6. The more general consumption sets turn out not to alter significantly the properties of the model. Thus, the basic properties of temporary equilibria like existence and generic determinateness can be derived from the properties of the Arrow-Debreu model with price-dependent preferences. In addition, the global properties of the equilibrium manifold are also satisfied in this model.

9.2 General Two-Period Model with Financial Assets

The perfect foresight Arrow-Debreu model with a finite number of time periods (see chapter 8) is extended into a two-period model with uncertainty, uncertainty being concentrated in the second time period and all goods are traded at the beginning of period 0 whether delivered in period 0 or in period 1. This model differs on two accounts from the standard two-period Arrow-Debreu model with uncertainty: (1) consumers face budget constraints for every time period and every state of nature; and (2) the model features suitably defined assets in addition to physical goods. These assets enable consumers to transfer wealth between time periods and states of nature. Assets are not arguments of the consumers' utility functions.

9.2.1 Goods and Assets

Physical Goods

There are ℓ and k physical goods delivered in period 0 and 1, respectively. Period 1 can be viewed as a way of representing all future time periods. There is therefore no particular reason for the number of goods in periods 0 and 1 to be equal. In general, we have $\ell \neq k$. All physical goods are traded at the beginning of period 0. They are *perishable*. Therefore, the goods delivered in period 0 simply disappear at the end of period 0.

Uncertainty in Period 1 and Contingent Goods

Uncertainty is represented by way of a finite set of states of nature $\mathfrak{S}$, with $S = \#\mathfrak{S}$ denoting the number of states of nature. The delivery of the k physical goods available in period 1 is made contingent to the realization of the states of nature $s \in \mathfrak{S}$. Therefore, there are kS contingent goods.

Prices of Goods

Denote by $p(0) \in \mathbb{R}^{\ell}_{++}$ the price vector of the goods delivered in period 0 and by $p^{[1]} = (p(1), \ldots, p(S))$ the price vector of the kS contingent goods delivered in period 1. The interpretation of the model being easier if these prices are not normalized, no physical commodity is chosen as numeraire.

Assets and Their Prices

There is a finite number N of assets. Assets are traded at the beginning of period 0 at the same time as the physical goods. Asset j, with j varying from 1 to N, guarantees the owner of one unit of the asset that the quantity $\rho_j(s)$ of wealth will be delivered to the owner of the asset if state of nature s occurs in period 1. This quantity is a real number. For the sake of simplicity, it is assumed that the payoff $\rho_j(s)$ is a nonnegative number for every state of nature and is strictly positive for at least one state of nature: $\rho_j(s) \geq 0$ for every $s \in \mathfrak{S}$, and > 0 for some $s \in \mathfrak{S}$. The actual significance of the payoff $\rho_j(s)$ becomes clearer when consumers' budget constraints are considered. Denote by $q_j > 0$ the price of asset j, with $1 \leq j \leq N$, and $q = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N_{++}$ the price vector for the N assets.

More on Prices

In this model, which is *not* the temporary equilibrium model yet, all goods and assets are traded, and trade takes place at the beginning of period 0. It is assumed that the first asset has many features of money. This justifies normalizing the price vector of the goods and assets $(p(0), p^{[1]}, q)$ by the convention $q_1 = 1$.

Asset Payoff Matrix
The payoffs $\rho^j(s)$ of all N assets for the state of nature $s \in \mathfrak{S}$ define the row matrix

$$\rho(s) = (\rho^1(s), \rho^2(s), \ldots, \rho^N(s)) \in \mathbb{R}^N_+.$$

The *asset payoff matrix* for the N assets is defined as the $S \times N$ matrix R with row s equal to $\rho(s)$ for $1 \leq s \leq S$.

Money as an Asset
In this setup money becomes the asset that has in period 0 a price equal to 1 and payoffs equal to 1 in every state of nature $s \in \mathfrak{S}$. Assume from now on that asset 1 is money: $q_1 = 1$, and $\rho^1(s) = 1$ for $s \in \mathfrak{S}$.

Having an asset that is money simplifies the interpretation of the model. The payoffs of the other assets can then be interpreted in terms of money. Incidentally, the properties of the model regarding the existence and structure of its equilibria do not depend on whether the specific asset that is identified with money exists or not.

9.2.2 Consumers: Budget Constraints and Preferences

Assume a finite number m of consumers. What follows applies to consumer i, with $1 \leq i \leq m$.

Consumption Space
Consumer i's consumption space is the strictly positive orthant $\mathfrak{X} = \mathbb{R}^{\ell+kS}_{++}$, i.e., only strictly positive quantities of physical goods are consumed.

Asset Portfolio
The asset portfolio held by consumer i at the end of period 0 is a vector $b_i \in \mathbb{R}^N$. The consumer still holds this porfolio at the beginning of period 1, which enables him to transfer wealth directly between time 0 and time 1 and indirectly between different states of nature by exploiting the differences in payoffs of the various assets. At variance with the consumption of physical goods, no sign or quantity constraints are imposed on the assets owned by consumer i.

Budget Constraint for Period 0
In period 0, consumer i buys physical goods to be consumed in period 0 and assets to transfer wealth between period 0 and the various states of nature of period 1.

Consumer i's endowments in period 0 consist of the commodity bundle $\omega_i(0) \in \mathbb{R}^\ell$ and of the portfolio of assets $\beta_i = (\beta_i^1, \ldots, \beta_i^j, \ldots, \beta_i^N) \in \mathbb{R}^N$. At this stage, no sign restrictions are imposed on endowments. Consumer i's bundle of goods consumed in period 0 and the portfolio of assets held at the end of period 0, assets to be transferred to period 1, are denoted by $x_i(0) \in \mathbb{R}^\ell_{++}$ and $b_i = (b_i^1, \ldots, b_i^j, \ldots, b_i^N) \in \mathbb{R}^N$, respectively.

The budget constraint that must be satisfied by consumption $x_i(0)$ and the asset porfolio b_i in period 0 is therefore equal to

$$p(0) \cdot x_i(0) + qb_i(0) \leq p(0) \cdot \omega_i(0) + q\beta_i.$$

Budget Constraint for State of Nature $s \in \mathfrak{S}$

If state of nature $s \in \mathfrak{S}$ occurs in period 1, consumer i must have enough resources to pay for the consumption of the commodity bundle $x_i(s) \in \mathbb{R}^k_{++}$. Consumer i's resources consist of the commodity bundle $\omega_i(s)$ and of the portfolio of assets $b_i = (b_i^1, \ldots, b_i^j, \ldots, b_i^N)$ that has been transferred from period 0 to period 1. In practice, a unit of asset j yields wealth $\rho_j(s)$ in state $s \in \mathfrak{S}$. Therefore, the porfolio of assets $b_i \in \mathbb{R}^N$ contributes to the wealth of consumer i in state of nature s by the quantity

$$b_i \cdot \rho(s) = b_i^1 \rho^1(s) + \cdots + b_i^j \rho^j(s) + \cdots + b_i^N \rho^N(s).$$

Consumer i's consumption for state of nature $s \in \mathfrak{S}$ must therefore satisfy the constraint

$$p(s) \cdot x_i(s) \leq p(s) \cdot \omega_i(s) + b_i \cdot \rho(s).$$

Utility Functions

Let Σ denote the unit simplex of $\mathbb{R}^S$,

$$\Sigma = \left\{ (\pi(s)) \in \mathbb{R}^S \;\middle|\; \sum_{s=1}^{S} \pi(s) = 1, \qquad \pi(s) \geq 0, \qquad s = 1, \ldots, S \right\}.$$

A point in the simplex Σ corresponds to a probability distribution on the set of states of nature $\mathfrak{S}$. Note that the probability distribution π can be purely subjective.

Consumer i's preferences are represented by a utility function $u_i : \mathfrak{X} \times \Sigma \to \mathbb{R}$, which is surjective and smooth and such that the map $u_i(\cdot, \pi)$ is (1) smoothly increasing, (2) smoothly strictly quasi-concave, and (3) with indifference surfaces closed in $\mathbb{R}^{\ell+kS}$ for any $\pi \in \Sigma$. In addition, the utility function u_i depends smoothly on the probability distribution $\pi \in \Sigma$.

9.2.3 Consumer's Maximization Problem

S* + 1 *Budget Constraints

Consumer i maximizes utility $u_i(x_i(0), x_i^{[1]})$ subject to the identified $S+1$ budget constraints, namely, the one for period 0 and the S budget constraints for each one of the S states of nature in period 1. Given the monotonicity property of the utility function u_i, the budget constraints are necessarily binding so equalities can be substituted for inequalities. In addition, the notation is simplified with the help of the $\square$ operation, which is defined by

$$p^{[1]} \square (x_i^{[1]} - \omega_i^{[1]}) = \begin{pmatrix} p(1) \cdot (x_i(1) - \omega_i(1)) \\ \vdots \\ p(s) \cdot (x_i(s) - \omega_i(s)) \\ \vdots \\ p(S) \cdot (x_i(S) - \omega_i(S)) \end{pmatrix}.$$

A more compact notation for consumer i's $S+1$ budget equations is then

$$p(0) \cdot (x_i(0) - \omega_i(0)) = -q(b_i - \beta_i), \tag{9.1}$$

$$p^{[1]} \square (x_i^{[1]} - \omega_i^{[1]}) = Rb_i, \tag{9.2}$$

where the matrix equality (9.2) corresponds to S real-valued equalities.

Utility Maximization

Given the price vectors $p = (p(0), p^{[1]}) \in \mathbb{R}_{++}^{\ell+kS}$ for the physical goods and $q \in \mathbb{R}_{++}^{N}$ for the assets, and given the probability distribution $\pi \in \Sigma$, consumer i solves the following maximization problem:

Problem 9.2.1 Utility maximization. Find $(x_i, b_i) \in \mathfrak{X} \times \mathbb{R}^N$ that maximizes $u_i(x_i, \pi)$ subject to the budget constraints (9.1) and (9.2).

The existence of solutions to problem 9.2.1 depends on the value of the payoff matrix R and of the asset price vector q.

Assuming there is a solution for just one consumer implies that there exists a solution for every consumer. For such price vectors $(p(0), p^{[1]},$

q), a consumer i's demand is a vector $(x_i(0), x_i^{[1]}, b_i)$. An equilibrium price vector $(p(0), p^{[1]}, q)$ is by definition such that the equation system describing the equality of supply and demand of goods and assets

$$\sum_i x_i(0) = \sum_i \omega_i(0),$$

$$\sum_i b_i = \sum_i \beta_i,$$

$$\sum_i x_i^{[1]} = \sum_i \omega_i^{[1]},$$

is satisfied.

This model has been extensively studied during the past two decades in the case $N \leq S$ and rank $R = N$ and $\beta_i = 0$ for $i = 1$ to m. Its properties are by now well understood. Because this model is used here only as a step in exploring the selected version of the temporary equilibrium model, I do not pursue here a description of the model's properties. Readers may refer to the existing literature, for example, Magill and Shaffer (1991) and Magill and Quinzii (1995).

9.3 The Temporary Equilibrium Model

The temporary equilibrium model differs from the previous two-period model with uncertainty by the absence of markets at the beginning of period 0 for the goods delivered in period 1. The only markets that are open in period 0 consist of (1) the asset or financial markets, and (2) the spot markets for the goods delivered in period 0. This implies that there do not exist market prices for the physical goods delivered in period 1.

The markets open in period 0 do not determine the price vector $p^{[1]}$ of the goods delivered in period 1. If consumers still want to maximize their utility over the two time periods, they have to substitute for the nonexistent market prices $p^{[1]}$ their own forecasts of these prices.

9.3.1 Forecasts and the Consumer's Utility Maximization Problem

In the consumer's utility maximization problem considered when markets for the goods delivered in period 1 are open, the price vector $p^{[1]}$ is known to every consumer. The individual budget constraints associated with these prices are such that the utility maximization problems of the various consumers are sufficiently similar for all of them to have a solution if only one of them has a solution. This property simplifies the study of that model.

In the temporary equilibrium model, where there is no market in period 0 for the goods delivered in period 1, forecasts may vary from one consumer to another. There is no reason for every consumer to come up with the same forecast. Then, the consumer's utility maximization problem loses the remarkable property of having a solution for all consumers once it has a solution for one consumer. This difficulty is not insuperable and has been tackled in the literature at the cost of additional assumptions regarding the forecast functions, for example. Here, I circumvent this difficulty by exploiting the freedom to select the set of states of nature $\mathfrak{S}$ that represents uncertainty. Assume that the set of states of nature $\mathfrak{S}$ is such that the price vector $p(s)$, the payoff vector $\rho(s)$, and the endowment vector $\omega_i(s)$ of consumer i, with i varying from 1 to m, are determined by the state of nature $s \in \mathfrak{S}$.

9.3.2 Forecasts as Probability Distributions on the Set of States of Nature

The likelihood of observing the price vector $p(s)$, the payoff vector $\rho(s)$, and the endowment vector $(\omega_i(s))$ is simply expressed by the probability $\pi(s)$ of state $s \in \mathfrak{S}$.

The two assumptions of the incomplete asset market literature, namely, that the number S of states of nature is larger than the number of assets N, and that the rank of the asset payoff matrix R is equal to N and therefore maximal, become natural consequences of having endowments, payoffs, and future prices part of the set of states of nature $\mathfrak{S}$.

The central assumption of all temporary equilibrium models is that every consumer's forecasts are based on the market prices observed in period 0, namely, the price vector $(p(0), q) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{N}_{++}$. Here this takes the form that the probability distribution on the set of states of nature $\mathfrak{S}$ that expresses consumer i's forecasts is some function

$$\varphi_i : \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{N}_{++} \to \Sigma,$$

with i varying from 1 to m.

9.3.3 Consumer's Maximization Problem in the Temporary Equilibrium Setup

An important consequence of having endowments, payoffs, and future prices as part of the set of states of nature $\mathfrak{S}$ is that the $S + 1$ budget constraints that consumer i is facing in maximizing utility are now independent of the forecasts. Consumer i's maximization problem in this temporary equilibrium model is now formally identical to problem 9.2.1, where the price vector $(p(0), p^{[1]}, q)$ is taken as given. Note that in the current setup, the price vector $p^{[1]}$ represents a very large collection of prices of the physical goods delivered in period 1, a collection indexed by the states of nature $s \in \mathfrak{S}$. In particular and at variance with the previous two-period model, the component $p^{[1]}$ does not realize any kind of equality between the aggregate demand and supply of the goods delivered in period 1 because there are no markets open for the goods to be delivered in period 1.

Existence of a solution to the consumer's maximization problem depends only on the structure of the matrix $\begin{bmatrix} -q \\ R \end{bmatrix}$, not on the precise specification of the utility function $u_i(\cdot, \pi)$ nor on the form of the forecast function $\varphi_i(p(0), q)$.

Every consumer's utility maximization problem has a solution once a solution exists for one consumer. Consumer i's demand in period 0 given the period 0 price vector $(p(0), q)$ is the component $(x_i(0), b_i)$ of the solution $(x_i, b_i) = ((x_i(0), x_i^{[1]}), b_i)$ of problem 9.2.1 for the probability distribution $\pi = \varphi_i(p(0), q)$.

9.3.4 Temporary Equilibrium

It follows from the nature of the set of states of nature $\mathfrak{S}$ that an economy is characterized by its endowment in physical goods to be delivered in period 0, namely, the vector $\omega_i(0) \in \mathbb{R}^\ell$ and the asset portfolio $\beta_i \in \mathbb{R}^N$, with i varying from 1 to m.

A *temporary equilibrium* of the economy defined by the endowment vector $(\omega_i(0), \beta_i)$, the utility functions $u_i(x_i, \pi)$, and the forecast functions $\varphi_i : \mathbb{R}^\ell_{++} \times \mathbb{R}^N_{++} \to \Sigma$, for $i = 1, \ldots, m$, is a pair $(p(0), q) \in \mathbb{R}^\ell_{++} \times \mathbb{R}^N_{++}$ such that equalities

$$\sum_i x_i(0) = \sum_i \omega_i(0),$$

$$\sum_i b_i = \sum \beta_i,$$

are satisfied.

9.4 Properties of Temporary Equilibria

The goal is now to study the properties of the temporary equilibria. I show that such equilibria exist under sufficiently general assumptions for this equilibrium concept to be relevant. In addition and at variance with the two-period model with incomplete asset markets used as an introduction to the temporary equilibrium model, equilibria are determined, in other words, locally isolated in case there is a multiplicity of them. This striking difference between the two models highlights the important role played by consumers' forecasts in the determination of equilibria in these models.

9.4.1 Consumer *i*'s Demand in Period 0

Here, we express the period 0 component $(x_i(0), b_i)$ of consumer i's demand $((x_i(0), b_i), x_i^{[1]})$ as a function of the period 0 prices $(p(0), q)$. This follows from a reinterpretation of the utility maximization problem 9.2.1 as a two-step maximization problem, a standard approach in dynamic programming. This leads to consideration of the following problem:

Problem 9.4.1 Partial utility maximization. Given $(x_i(0), b_i)$ and π, find $x_i^{[1]} \in \mathbb{R}^{\ell S}_{++}$ that maximizes

$$u_i((x_i(0), x_i^{[1]}), \pi)$$

subject to the constraint

$$p^{[1]} \,\square\, (x_i^{[1]} - \omega_i^{[1]}) = Rb_i.$$

We define the *value function* of problem 9.4.1 by

$$v_i(x_i(0), b_i, \pi) = u_i((x_i(0), x_i^{[1]}), \pi),$$

where $x_i^{[1]}$ is a solution of the problem.

Set of Acceptable Portfolios

Problem 9.4.1 has a solution, which is then unique, if and only if the vector inequality

$$p^{[1]} \,\square\, \omega_i^{[1]} + R\, b_i > 0 \qquad (9.3)$$

is satisfied. The portfolios $b_i \in \mathbb{R}^N$ that satisfy these inequalities are called *acceptable* for consumer i.

The set

$$B_i = \{b_i \in \mathbb{R}^N \mid p^{[1]} \Box \omega_i^{[1]} + R\, b_i > 0\}$$

of *acceptable portfolios* is independent of the endowment vector $\omega_i(0)$ in period 0. This set is comparable to the consumption set in the standard Arrow-Debreu model. In fact, the Cartesian product $\mathbb{R}^{\ell}_{++} \times B_i$ is identified with the consumption space of a (fictitious) consumer having a suitably defined utility function for the bundle $(x_i(0), b_i)$. The properties of the set of *acceptable portfolios* B_i play an important role in the sequel.

The set B_i is a convex polyhedron whose facets are perpendicular to the vectors $\rho_i(s)$ for $s \in \mathfrak{S}$. The origin 0 belongs to B_i. In addition, the inclusion $b_i + \mathbb{R}^N_+ \subset B_i$ is satisfied for any $b_i \in B_i$.

The recession cone of the convex polyhedron B_i is defined as follows. Pick some element $b_i \in B_i$. The recession cone consists of the vectors $d \in \mathbb{R}^N$ such that $b_i + td \in B_i$ for $t \geq 0$. The recession cone is easily shown to be independent of the choice of the element $b_i \in B_i$. In addition, the facets of the convex polyhedron B_i are perpendicular to the vectors $\rho_i(s)$, for $s \in \mathfrak{S}$. Their directions do not depend on consumer i. This implies that the recession cone Γ of the set of *acceptable portfolios* B_i is the same for all consumers $i = 1, \ldots, m$.

To sum up, the sets B_i of *acceptable portfolios* for consumer i, with i varying from 1 to m, are convex polyhedra that may be different but they all have the same recession cone Γ.

Exercises

9.1. Show that with $N = 2$, $S = 2$, $\rho_i(1) = (1, 1)$, and $\rho_1(2) = (1, 3)$, the other parameters being arbitrarily chosen, the set of acceptable portfolios B_i is not bounded from below.

9.2. Show that with $N = 2$, $S = 3$, $\rho_i(1) = (1, 1)$, $\rho_1(2) = (1, 3)$, and $\rho_i(3) = (1, 2)$, and with $\omega_i(1) = (5, 5)$, $\omega_i(2) = (5, 5)$, $\omega_i(3) = (1, 1)$, the set of acceptable portfolios B_i is not a cone.

Value Function of Problem 9.4.1

It follows from the definition of an *acceptable portfolio* that the value function $v_i(x_i(0), b_i, \pi)$ is defined on the set $\mathbb{R}^{\ell}_{++} \times B_i \times \Sigma$.

This value function behaves very much like the utility functions considered in the Arrow-Debreu model in the sense that $v_i : \mathbb{R}^{\ell}_{++} \times B_i \times \Sigma \to \mathbb{R}$ is smooth and the (partial) value function $v_i(.,.,\pi)$ is smooth, smoothly monotonic, and smoothly strictly quasi-concave on $\mathbb{R}^{\ell}_{++} \times B_i$, with level sets closed in $\mathbb{R}^{\ell} \times \mathbb{R}^N$. The proofs of these properties present no major difficulties. They are developed in the following exercise.

Exercise

9.3. Let $w_i^{[1]} = (w_i(1), \ldots, w_i(S)) \in \mathbb{R}^S$ be a distribution of wealth across the S states of nature. The map $A : \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{kS}_{++} \to \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{S}_{++}$ is defined by the formula $A(x_i(0), x_i^{[1]}) = (x_i(0), w_i^{[1]})$, where $w_i(s) = p(s) \cdot x_i(s)$ for $s \in \mathfrak{S}$. For $x_i(0) \in \mathbb{R}^{\ell}_{++}$ and $w_i^{[1]} \in \mathbb{R}^{S}_{++}$ given, maximize $u_i(x_i(0), x_i^{[1]}, \pi)$ subject to the constraint $A(x_i(0), x_i^{[1]}) = (x_i(0), w_i^{[1]})$.
a. Show that this problem has a unique solution for any $(x_i(0), w_i^{[1]}) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{S}_{++}$ and that this solution defines a smooth function of $(x_i(0), w_i^{[1]})$.
b. Let $U_i(x_i(0), w_i^{[1]}, \pi) = u_i(x_i(0), x_i^{[1]}, \pi)$ be the associated value function. Show that for given $\pi \in \Sigma$, this function of $(x_i(0), w_i^{[1]})$ is smooth, smoothly monotonic, and smoothly quasi-concave, with level sets $\{(x_i(0), w_i^{[1]}) \in \mathbb{R}^{\ell+S}_{++} \mid U_i(x_i(0), w_i^{[1]}, \pi) = u_i\}$ closed in $\mathbb{R}^{\ell+S}$ for all u_i.
c. Prove the equality $v_i(x_i(0), b_i, \pi) = U_i(x_i(0), w_i^{[1]}, \pi)$ where $w_i^{[1]} = (w_i(s))$, with $s = 1, \ldots, S$, is such that $w_i(s) = p(s) \cdot \omega_i(s) + \rho(s) \cdot b_i$.
d. Apply the equality $v_i(x_i(0), b_i, \pi) = u_i((x_i(0), x_i^{[1]}), b_i, \pi)$ to prove the smoothness of v_i. (For the smooth dependence on π, apply the parametric version of the implicit function theorem to the first order-conditions.)
e. Apply the results of the previous parts to show that $v_i(., \pi)$ is smoothly monotonic, and strictly quasi-concave, with level sets closed in $\mathbb{R}^{\ell} \times \mathbb{R}^N$.
f. Prove the strict smooth quasi-concavity of $v_i(., \pi)$. (This part is more technical than just proving strict quasi-concavity. It requires exploiting the fact that the Gaussian curvature of the level sets is different from zero everywhere.)

9.4.2 Reformulation of Consumer *i*'s Period 0 Utility Maximization Problem

The solution to problem 9.4.1 determines the value function $v_i(x_i(0), b_i, \pi)$ for $x_i(0) \in \mathbb{R}^{\ell}_{++}$, $b_i \in B_i$, and $\pi \in \Sigma$. Given that value function $v_i : \mathbb{R}^{\ell}_{++} \times B_i \times \Sigma \to \mathbb{R}$, let us now consider the following maximization problem.

Problem 9.4.2 Given π, find $(x_i(0), b_i) \in \mathbb{R}^{\ell}_{++} \times B_i$ that maximizes

$$v_i(x_i(0), b_i, \pi)$$

subject to the constraint

$$p(0) \cdot (x_i(0) - \omega_i(0)) + q \cdot (b_i - \beta_i) = 0. \tag{9.4}$$

Problem 9.4.2 is very similar to the utility maximization problem under a single budget constraint of standard Arrow-Debreu model. The roles of the utility function, commodity bundle, price vector, and budget constraint are played respectively by v_i, $(x_i(0), b_i)$, $(p(0), q)$, and equality (9.4). Consumer i's problem 9.2.1 is now decomposed into the following two-stage maximization problem.

Theorem 9.4.1 The element $((x_i(0), b_i), x_i^{[1]}) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}^N \times \mathbb{R}^{\ell S}_{++}$ is a solution of problem 9.2.1 if and only if $(x_i(0), b_i)$ is a solution of problem 9.4.2. Conversely, let $(x_i(0), b_i)$ be a solution of problem 9.4.2; then $((x_i(0), b_i), x_i^{[1]})$ is a solution of problem 9.2.1 for $x_i^{[1]}$ solution of problem 9.4.1 given $(x_i(0), b_i)$.

Proof The condition is necessary. Let us assume that

$$((x_i(0), b_i), x_i^{[1]}) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}^N \times \mathbb{R}^{\ell S}_{++}$$

is the unique solution of problem 9.2.1. Then it follows from the constraints (9.1) and (9.2) of problem 9.2.1 that b_i necessarily belongs to B_i. It suffices to keep the period 1 component $x_i^{[1]}$ of the (unique) solution to problem 9.2.1 constant to observe that the period 0 component $(x_i(0), b_i)$ maximizes $v_i((x_i(0), x_i^{[1]}), \pi)$ given the period 0 constraint (9.4), which amounts to $(x_i(0), b_i) \in \mathbb{R}^{\ell}_{++} \times B_i$ being a solution to problem 9.4.2. By keeping the period 0 component $(x_i(0), b_i)$ of the solution to problem 9.2.1 constant, we see that the period 1 component $x_i^{[1]}$ maximizes $u_i(x_i(0), x_i^{[1]}, \pi)$ and therefore solves problem 9.4.1 subject to the constraint $p^{[1]} \,\square\, (x_i^{[1]} - \omega_i^{[1]}) = Rb_i$.

The condition is sufficient. Let $(x_i(0), b_i) \in \mathbb{R}^{\ell}_{++} \times B_i$ be a solution of problem 9.4.2 and $x_i^{[1]}$ a solution of problem 9.4.1 associated with the right-hand constraints $p^{[1]} \,\square\, (x_i^{[1]} - \omega_i^{[1]}) = Rb_i$. If problem 9.2.1 has a solution, it follows from the necessary conditions that the period 0 component of that solution is indeed the solution to problem 9.4.2 and therefore coincides with $(x_i(0), b_i)$. The same line of reasoning based on the uniqueness of the solution to problem 9.4.1 implies that the period 1 component of the solution of problem 9.2.1 is equal to $x_i^{[1]}$. ∎

9.4.3 Reduction to an Arrow-Debreu Economy with Price-Dependent Preferences

Theorem 9.4.2 The temporary equilibrium model is equivalent to an Arrow-Debreu pure exchange economy with $\ell + N$ goods, the set of prices being the set $\mathbb{R}^{\ell}_{++} \times \mathbb{R}^{N}_{++}$, consumer i's consumption set, endowments and (price-dependent) utility function being the set $\mathbb{R}^{\ell}_{++} \times B_i$, the vector $(\omega_i(0), \beta_i) \in \mathbb{R}^{\ell}_{++} \times B_i$, and the function $v_i((x_i(0), b_i), \varphi_i(p(0), q))$, respectively.

This follows readily from theorem 9.4.1 combined with the definition of a temporary equilibrium given in section 9.3.4.

9.4.4 Properties of Temporary Equilibria

Properties of temporary equilibria are therefore those of the equilibria of an Arrow-Debreu economy with price-dependent preferences, the model considered in chapter 6. There is a difference, however. In that model, the consumption sets are the strictly positive orthant. Here the consumption sets are more general. They satisfy the following properties: (1) the recession cone Γ associated with the consumption space $X_i \subset \mathbb{R}^{\ell+N}$ of consumer i is the same for all consumers; (2) for every i, there exists an element $a_i \in \mathbb{R}^{\ell+N}$ such that the inclusion $X_i \subset a_i + \Gamma$ is satisfied; and (3) indifference sets are closed in the commodity space $\mathbb{R}^{\ell+N}$. The price set then consists of the interior of the dual of the recession cone Γ. The recession cone Γ plays the same role as the positive orthant in models where the consumption sets are the strictly positive orthant.

The following properties are satisfied by the temporary equilibrium model.

Proposition 9.4.3 For exogenously given forecast functions of future prices and payoffs and for endowment vectors $(\omega_i(0), \beta_i) \in X \times B_i$ for $i = 1, 2, \dots, m$, temporary equilibria always exist.

Another important result is the following version of Debreu's theorem on the generic determinateness of the temporary equilibrium solution.

Proposition 9.4.4 For exogenously given preferences and forecast functions, there exists a generic set of endowments $(\omega_i(0), \beta_i)_i$ such that the associated temporary equilibria are locally isolated.

It also follows from theorem 9.4.2 that the equilibrium manifold of the temporary equilibrium model can be identified with the equilibrium manifold of an Arrow-Debreu pure exchange economy with $\ell + N$ goods and parameter space $((X \times B_1) \times \cdots \times (X \times B_m))$, utilities being price dependent. It then follows from section 6.3.2 that the equilibrium manifold is diffeomorphic to the Euclidean space $\mathbb{R}^{(\ell+N)m}$.

9.5 Conclusion

Temporary equilibria are locally isolated even when asset markets are incomplete and payoffs made in money. This property holds true for individual forecasts of future prices and payoffs exogenously given. Any change in the forecast functions induces a change in the corresponding temporary equilibrium allocations. This parameterization of temporary equilibria by the forecast functions reconcile the results of this chapter with those on incomplete asset markets with purely financial assets. See, for example, Balasko and Cass (1989). In view of the results of this chapter, the indeterminacy property of general equilibrium incomplete asset models can be interpreted as equivalent to the existence of an infinite number of individual forecast functions that are compatible with the equilibria of those models.

9.6 Notes and Comments

The first versions of the temporary equilibrium model were those of Lindahl (1939) and Hicks (1946) with subsequent contributions by Grandmont, Green, and Stigum focusing on the stochastic aspects of the model (see Grandmont 1988).

The original two-period model with financial assets is due to Arrow (1953; 1964). The study of the structure of the solutions of this model when asset markets are incomplete was initiated by Cass (1984) and Werner (1985). For a survey of that literature, see, for example, Magill and Shaffer (1991).

The extension of the equilibrium manifold and natural projection approach to the temporary equilibrium model is due to Balasko (2003b).

Appendix: The Set of Pareto Optima and Its Parameterizations

This appendix describes the structure of the set of Pareto optima when total resources are fixed. The first part is devoted to parameterization by utility levels, the second to parameterization by welfare weights. These properties are all well known. Nevertheless, no exhaustive rigorous exposition that covers exactly the assumptions made in this book, in particular those regarding the consumption sets, seems to exist in the literature. This appendix is motivated by the close relations that exist between section manifolds and sets of Pareto optima. It deals also with the properties of sets of feasible utility levels, which are of independent interest.

A.1 Sets of Feasible Utility Levels

Let $P(r) = P \cap \Omega(r)$ denote the set of Pareto optima compatible with the total resource vector $r \in X$.

In order to understand the properties of the set of Pareto optima $P(r)$, it is necessary to consider the utility levels that can be reached with a given vector of total resources $r \in X$. I use induction on the number of consumers at times because the definitions that follow are relative to subsets of the m consumers that make up the economy. The usual treatments consider all consumers at once.

A.1.1 Definitions

The *set of feasible utility levels for the first k consumers with incomplete resource utilization* $\mathcal{U}_k(r)$, with $1 \le k \le m$, is the subset of $\mathbb{R}^k$ such that

$$\mathcal{U}_k(r) = \{(u_1(x_1), \ldots, u_k(x_k)) \in \mathbb{R}^k \mid x_1, \ldots, x_k \in X, x_1 + \cdots + x_k < r\}.$$

The term *incomplete resource utilization* refers to the fact that the resource inequality $x_1 + x_2 + \cdots + x_k < r$ is strict coordinatewise.

The set $\mathcal{U}_k(r)$ is therefore the image of the set

$$\Omega_k(< r) = \{(x_1, x_2, \ldots, x_k) \in X^k \mid x_1 + x_2 + \cdots + x_k < r\}$$

by the map

$$(u_1, u_2, \ldots, u_k) : (x_1, x_2, \ldots, x_k) \to (u_1(x_1), u_2(x_2), \ldots, u_k(x_k))$$

from X^k into $\mathbb{R}^k$.

Similarly, the *set of feasible utility levels for the first k consumers with full utilization of resources* is the image of the set

$$\Omega_k(r) = \{(x_1, x_2, \ldots, x_k) \in X^k \mid x_1 + x_2 + \cdots + x_k = r\}$$

by the map

$$(x_1, x_2, \ldots, x_k) \to (u_1(x_1), u_2(x_2), \ldots, u_k(x_k)).$$

The definition of the set of Pareto optima $P(r)$ is extended to define $P_k(r)$ as the set of Pareto optima associated with the total resources $r \in X$ for the first k consumers. This implies that $P(r) = P_m(r)$.

The set of feasible utility levels for the first k consumers with *full utilization of resources* is the closure of $\mathcal{U}_k(r)$, the set of feasible utility levels with *incomplete utilization of resources*. Before proving this property, I look more closely at some properties of the set $\mathcal{U}_k(r)$.

A.1.2 Free Disposal of Feasible Utility Levels

The idea of the free disposability of utility levels takes the following form.

Proposition A.1.1 For $u = (u_1, u_2, \ldots, u_k) \in \mathcal{U}_k$, the inclusion $u - \mathbb{R}^k_+ \subset \mathcal{U}_k(r)$ is satisfied.

Proof Let $(x_1, x_2, \ldots, x_k) \in X^k$ be such that $u_1(x_1) = u_1$, $u_2(x_2) = u_2, \ldots, u_k(x_k) = u_k$, and $x_1 + x_2 + \cdots + x_k < r$. Let $u' = (u'_1, u'_2, \ldots, u'_k) \leq u = (u_1, u_2, \ldots, u_k)$ with at least one strict inequality. For every i between 1 and k, let $x'_i \in X$ be the unique point collinear with x_i and such that $u_i(x'_i) = u'_i \leq u_i(x_i) = u_i$. By construction, $x'_i \leq x_i$ for every i, one inequality at least being strict. Therefore, the sum $x'_1 + x'_2 + \cdots + x'_k$ is strictly less than $x_1 + x_2 + \cdots + x_k$ and therefore less than r. ■

A.1.3 Openness of the Set of Feasible Utility Levels with Incomplete Resource Utilization

Proposition A.1.2 The set $\mathcal{U}_k(r)$ is nonempty and open in $\mathbb{R}^k$.

Proof The set $\mathcal{U}_k(r)$ is nonempty: take, for example, $x_1 = x_2 = \cdots = x_k = \dfrac{1}{k+1} r$. Then $(u_1(x_1), u_2(x_2), \ldots, u_k(x_k))$ obviously belongs to $\mathcal{U}_k(r)$.

Let now $u = (u_1, u_2, \ldots, u_k) \in \mathcal{U}_k(r)$. To prove openness, it suffices to show that there exists an open cube centered at $u \in \mathbb{R}^k$ such that, if u' belongs to that cube, then u' belongs to $\mathcal{U}_k(r)$.

By the definition of $\mathcal{U}_k(r)$, there exists $x = (x_1, x_2, \ldots, x_k) \in X^k$ with $u_1(x_1) = u_1, u_2(x_2) = u_2, \ldots, u_k(x_k) = u_k$ and $x_1 + x_2 + \cdots + x_k = y < r$. Define $\varepsilon = \frac{1}{2k}(r - y)$ and let $x_1' = x_1 + \varepsilon, x_2' = x_2 + \varepsilon, \ldots, x_k' = x_k + \varepsilon$. By the definition of ε, we have $x_1' + x_2' + \cdots + x_k' = y + \frac{1}{2}(r - y) < y + (r - y) = r$. Then $u' = (u_1', u_2', \ldots, u_k') = (u_1(x_1), u_2(x_2), \ldots, u_m(x_m))$ belongs to $\mathcal{U}_k(r)$. The inequality $u < u'$ implies that $u = (u_1, u_2, \ldots, u_k)$ belongs to the set $u' - \mathbb{R}^k_{++}$, which is open and contained in $\mathcal{U}_k(r)$. The set $u' - \mathbb{R}^k_{++}$ is an open neighborhood of u and therefore contains an open cube centered at u. This ends the proof of the openness of $\mathcal{U}_k(r)$.

■

A.1.4 Closedness of the Set of Feasible Utility Levels with Full Resource Utilization

Proposition A.1.3 The set of feasible utility levels for the first k consumers *with full resource utilization* is the closure in $\mathbb{R}^k$ of the set $\mathcal{U}_k(r)$ of feasible utility levels for the first k consumers *with incomplete resource utilization.*

Proof Let $u = (u_1, u_2, \ldots, u_k)$ such that there exists $x = (x_1, x_2, \ldots, x_k) \in X^k$ with $u_1(x_1) = u_1, u_2(x_2) = u_2, \ldots, u_k(x_k) = u_k$ and $x_1 + x_2 + \cdots + x_k = r \in X$. Let $x^{(n)} = \left(1 - \frac{1}{n}\right)x$. Define $u^{(n)} = (u_1(x_1^{(n)}), u_2(x_2^{(n)}), \ldots, u_k(x_k^{(n)}))$. Then $u^{(n)} \in \mathcal{U}_k(r)$ and $\lim_{n\to\infty} u^{(n)} = u$. ■

A.1.5 Characterization of the Boundary $\partial\mathcal{U}_k(r)$

Recall that $\mathbb{R}^k_+ = \{(v_1, v_2, \ldots, v_k) \in \mathbb{R}^k \mid v_1 \geq 0, v_2 \geq 0, \ldots, v_k \geq 0\}$.

Proposition A.1.4 The k-tuple $u = (u_1, u_2, \ldots, u_k) \in \overline{\mathcal{U}_k(r)}$ belongs to the boundary $\partial\mathcal{U}_k(r)$ if and only if

$$(u + \mathbb{R}^k_+) \cap \overline{\mathcal{U}_k(r)} = \{u\}.$$

Proof The set $\mathcal{U}_k(r)$ being open, its boundary is the set

$\partial\mathcal{U}_k(r) = \overline{\mathcal{U}_k(r)} \backslash \mathcal{U}_k(r).$

The k-tuple $u = (u_1, u_2, \ldots, u_k) \in \overline{\mathcal{U}_k(r)}$ therefore belongs to the boundary $\partial\mathcal{U}_k(r)$ if and only if u does not belong to the open set $\mathcal{U}_k(r)$. It follows from the formula $(u + \mathbb{R}^k_+) \cap \overline{\mathcal{U}_k(r)} = \{u\}$ that u belongs to $\overline{\mathcal{U}_k(r)}$. Let us show that this formula is equivalent to u not in $\mathcal{U}_k(r)$. If u belongs to $\mathcal{U}_k(r)$, which is open, there is an open neighborhood of u that is contained in $\mathcal{U}_k(r)$. The intersection of this neighborhood with the set $u + \mathbb{R}^k_+$ then necessarily contains other elements than u. ■

A.1.6 The Boundary $\partial\mathcal{U}_k(r)$ and the Pareto Optima

A necessary condition for $x = (x_1, x_2, \ldots, x_k) \in X^k$ to be such that its image $u(x) = (u_1(x_1), u_2(x_2), \ldots, u_k(x_k))$ belongs to the boundary $\partial\mathcal{U}_k(r)$ is that the sum $x_1 + x_2 + \cdots + x_k$ be equal to r. This condition is obviously not sufficient for $u(x) = (u_1(x_1), u_2(x_2), \ldots, u_k(x_k))$ to belong to $\partial\mathcal{U}_k(r)$. A first result in this direction is the following.

Lemma A.1.5 The image of the set of Pareto optima $P_k(r)$ by the map $u : (x_1, x_2, \ldots, x_k) \to (u_1(x_1), u_2(x_2), \ldots, u_k(x_k))$ from X^k to $\mathbb{R}^k$ is contained in the boundary $\partial\mathcal{U}_k(r)$.

Proof Let $x = (x_1, x_2, \ldots, x_k) \in P_k(r)$ and assume $u(x) \in \mathcal{U}_k(r)$. There exists therefore some $y = (y_1, y_2, \ldots, y_k) \in X^k$ such that $u(y) = u(x)$ and $y_1 + y_2 + \cdots + y_k < r$. Let $\varepsilon = \frac{1}{k}(r - (y_1 + y_2 + \cdots + y_k))$ and define $x' = (x'_1, x'_2, \ldots, x'_k)$ by $x'_1 = y_1 + \varepsilon$, $x'_2 = y_2 + \varepsilon, \ldots,$ $x'_k = y_k + \varepsilon$. It follows from $u_1(x_1) < u_1(x'_1), u_2(x_2) < u_2(x'_2), \ldots, u_k(x_k) < u_k(x'_k)$ that x' is Pareto-superior to x, hence a contradiction. This proves the inclusion $u(P_k(r)) \subset \partial\mathcal{U}_k(r)$. ■

To prove the equality $u(P_k(r)) = \partial\mathcal{U}_k(r)$, it is necessary to have some information on how to obtain the Pareto optima associated with given utility levels.

A.2 Classical Optimization Problem for Pareto Optima

By definition, problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$ is to find $x \in X^{k+1}$ that maximizes $u_{k+1}(x_{k+1})$ subject to the constraints

$$u_1(x_1) \geq u_1,$$

$$u_2(x_2) \geq u_2,$$

$$\vdots$$

$$u_k(x_k) \geq u_k,$$

$$x_1 + x_2 + \cdots + x_k + x_{k+1} \leq r.$$

Proposition A.2.1 Maximization problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$ has a solution if and only if $(u_1, u_2, \ldots, u_k)$ belongs to $\mathcal{U}_k(r)$. The solution is then unique and defines a function $R_k(r, u_1, u_2, \ldots, u_k)$ from $\mathcal{U}_k(r)$ into X^{k+1}.

A.2.1 Necessity

Let us show that if problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$ has a solution, then $u = (u_1, u_2, \ldots, u_k)$ belongs to $\mathcal{U}_k(r)$. It follows from monotonicity of utility that the constraints of problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$ are necessarily binding at a solution. Therefore, if

$$x = (x_1, x_2, \ldots, x_k, x_{k+1}) \in X^{k+1}$$

denotes such a solution, we have $u_1(x_1) = u_1$, $u_2(x_2) = u_2, \ldots,$ $u_k(x_k) = u_k$ and $x_1 + x_2 + \cdots + x_k + x_{k+1} = r$. The last constraint combined with $x_{k+1} \in X$ (i.e., all coordinates of x_{k+1} are strictly positive) implies the strict inequality $x_1 + x_2 + \cdots + x_k < r$, which proves that $u = (u_1, u_2, \ldots, u_k)$ belongs to $\mathcal{U}_k(r)$.

A.2.2 Sufficiency

Let $u = (u_1, u_2, \ldots, u_k) \in \mathcal{U}_k(r)$. There exists $x \in X^k$ such that $u_1(x_1) = u_1$, $u_2(x_2) = u_2, \ldots,$ $u_k(x_k) = u_k$ and $x_1 + x_2 + \cdots + x_k < r$. Define $x_{k+1} = r - (x_1 + x_2 + \cdots + x_k) \in X$.

Let C denote the subset of X^{k+1} defined by the inequalities $u_1(x'_1) \geq u_1(x_1) = u_1$, $u_2(x'_2) \geq u_2(x_2) = u_2, \ldots,$ $u_k(x'_k) \geq u_k(x_k) = u_k$, $x'_1 + x'_2 + \cdots + x'_k + x'_{k+1} \leq r$ and $u_{k+1}(x'_{k+1}) \geq u_{k+1}(x_{k+1})$. A solution of problem $\mathcal{P}_{k+1}(r, u_1, \ldots, u_k)$ is a solution of the new problem that consists in maximizing $u_{k+1}(x'_{k+1})$ but now subject to the constraint

$$x' = (x'_1, x'_2, \ldots, x'_k, x'_{k+1}) \in C$$

and conversely. The new problem turns out to be easier to handle because of its constraint set C, which is compact, as is now shown.

It follows from $x'_1 + x'_2 + \cdots + x'_{k+1} \leq r$ that the inequality $x'_i \leq r$ is satisfied for $1 \leq i \leq k+1$. Combined with $u_i(x'_i) \geq u_i(x_i)$, this implies that x'_i belongs for each i, with $1 \leq i \leq k+1$, to the set

$$L_i = \{x'_i \in X \,|\, u_i(x'_i) \geq u_i(x_i), x'_i \leq r\},$$

which is compact by lemma 2.5.3. Therefore, the constraint set C is closed as a subset defined by equalities and large inequalities in the compact set that is the Cartesian product of compact sets $L_1 \times L_2 \times \cdots \times L_k \times L_{k+1}$. The set C is therefore compact.

The continuous function $u_{k+1}(x_{k+1})$ reaches its maximum on the compact set C. This proves the existence of a solution to the problem of maximizing $u_{k+1}(x'_{k+1})$ subject to the constraint $x' \in C$ and therefore to problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$ for $(u_1, u_2, \ldots, u_k) \in \mathcal{U}_k(r)$. ■

A.2.3 Uniqueness

Let us show that the solution is unique. Assume the contrary. Let x and x' be two solutions, with $x \neq x'$. Recall that at those solutions the constraints are binding. Therefore, $u_i(x_i) = u_i(x'_i) = u_i$ for $i = 1, 2, \ldots, k$ and $x_1 + \cdots + x_{k+1} = x'_1 + \cdots + x'_{k+1} = r$.

The element $x'' = (x + x')/2$ satisfies $x''_1 + \cdots + x''_{k+1} = r$ and $u_i(x''_i) \geq u_i$ for $i = 1, 2, \ldots, k$, the inequality being strict for $x_i \neq x'_i$ by the strict quasi-concavity of the utility functions u_i. If $x_i = x'_i$ for $i = 1, 2, \ldots, k$, then necessarily x_{k+1} and x'_{k+1} are equal to $r - (x_1 + \cdots + x_k)$ and $r - (x'_1 + \cdots + x'_k)$, respectively, and therefore they are equal, which contradicts the assumption. Therefore, assume without loss of generality that $x_1 \neq x'_1$. Then $u_1(x''_1) > u_1(x_1) = u_1(x'_1) = u_1$. Let $x'''_1 \in X$ collinear with x''_1 and such that $u_1(x'''_1) = u_1$. We have $x'''_1 < x''_1$. Let $h = x''_1 - x'''_1 \in X$ (i.e., all coordinates of h are > 0). Define $y \in X^{k+1}$ by $y_1 = x'''_1, y_2 = x''_2, \ldots, y_k = x''_k$ and $y_{k+1} = x''_{k+1} + h$. Then y satisfies all the constraints of problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$ and $u_{k+1}(y_{k+1}) > u_{k+1}(x''_{k+1}) \geq u_{k+1}(x_{k+1})$. The last inequality again contradicts the definition of x as a solution of problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$, which ends the proof of the uniqueness of the solution.

A.3 The Map $R_k(r, .) : \mathcal{U}_k(r) \to X^{k+1}$

Let $R_k(r, u_1, u_2, \ldots, u_k)$ denote the solution of $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$. The solution being unique, this defines a map $R_k(r, .) : \mathcal{U}_k(r) \to X^{k+1}$.

Proposition A.3.1 The map $R_k(r, .) : \mathcal{U}_k(r) \to X^{k+1}$ is smooth.

A.3.1 First-Order Conditions

Substitute for problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$ its first-order conditions. It is left for the reader to check the well-known property that these conditions are necessary and sufficient. Therefore, the vector $x = (x_1, x_2, \ldots, x_k, x_{k+1})$ is a solution to problem $\mathcal{P}_{k+1}(r, u_1, u_2, \ldots, u_k)$ if and only if $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{R}^k$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{R}^\ell$ exist such that (x, μ, ν) solves

$$\mu_1 Du_1(x_1) = \nu,$$

$$\mu_2 Du_2(x_2) = \nu,$$

$$\vdots$$

$$\mu_k Du_k(x_k) = \nu,$$

$$Du_{k+1}(x_{k+1}) = \nu,$$

$$r - (x_1 + x_2 + \cdots + x_k + x_{k+1}) = 0,$$

$$u_1(x_1) - u_1 = 0,$$

$$u_2(x_2) - u_2 = 0,$$

$$\vdots$$

$$u_k(x_k) - u_k = 0.$$

Elimination of $\nu \in \mathbb{R}^k$ yields the equivalent equation system

$$\mu_1 Du_1(x_1) - Du_{k+1}(x_{k+1}) = 0,$$

$$\mu_2 Du_2(x_2) - Du_{k+1}(x_{k+1}) = 0,$$

$$\vdots$$

$$\mu_k Du_k(x_k) - Du_{k+1}(x_{k+1}) = 0,$$

$$r - (x_1 + x_2 + \cdots + x_k + x_{k+1}) = 0,$$

$$u_1(x_1) - u_1 = 0,$$

$$u_2(x_2) - u_2 = 0,$$

$$\vdots$$

$$u_k(x_k) - u_k = 0.$$

Note that it follows from this equation system that $\mu = (\mu_1, \dots, \mu_k) > 0$. In addition, the gradient vectors $Du_1(x_1), Du_2(x_2), \dots, Du_{k+1}(x_{k+1})$ are collinear.

A.3.2 Nonvanishing of the Associated Jacobian Determinant

The solution $(x, \mu) \in X^{k+1} \times \mathbb{R}^k$ is a smooth function of the parameters of this equation system, namely, $r \in X$ and $u = (u_1, u_2, \dots, u_k) \in \mathbb{R}^k$ if we can apply the implicit function theorem. Here, this amounts to the Jacobian matrix of that equation system with respect to the unknowns (x, μ) in $X^{k+1} \times \mathbb{R}^k$ being invertible.

This Jacobian matrix takes the following bloc form:

$$A = \begin{bmatrix} \mu_1 D^2 u_1 & 0 & \cdots & 0 & -D^2 u_{k+1} & Du_1 & 0 & \cdots & 0 \\ 0 & \mu_2 D^2 u_2 & \cdots & 0 & -D^2 u_{k+1} & 0 & Du_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \mu_k D^2 u_k & -D^2 u_{k+1} & 0 & 0 & \cdots & Du_k \\ -I & -I & \cdots & -I & -I & 0 & 0 & \cdots & 0 \\ Du_1^T & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & Du_2^T & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & Du_k^T & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let us show that $\det A \neq 0$. Assume the contrary. There exists a row vector $Z^T = (X_1^T, X_2^T, \dots, X_{k+1}^T, y_1, y_2, \dots, y_k)$ (where $X_1^T, X_2^T, \dots, X_{k+1}^T$ are $1 \times \ell$ row matrices) such that $AZ = 0$ with $Z \neq 0$. Spelling out this condition yields

$$\mu_1 D^2 u_1 X_1 - D^2 u_{k+1} X_{k+1} + y_1 Du_1 = 0,$$

$$\mu_2 D^2 u_2 X_2 - D^2 u_{k+1} X_{k+1} + y_2 Du_2 = 0,$$

$$\vdots$$

$$\mu_k D^2 u_k X_k - D^2 u_{k+1} X_{k+1} + y_k Du_k = 0,$$

$$X_1 + X_2 + \cdots + X_k + X_{k+1} = 0,$$

$$Du_1^T X_1 = Du_2^T X_2 = \cdots = Du_k^T X_k = 0.$$

This equation system yields the following:

$$\mu_1 X_1^T D^2 u_1 X_1 - X_1^T D^2 u_{k+1} X_{k+1} = 0,$$

$$\mu_2 X_2^T D^2 u_2 X_2 - X_2^T D^2 u_{k+1} X_{k+1} = 0,$$

$$\vdots$$

$$\mu_k X_k^T D^2 u_k X_k - X_k^T D^2 u_{k+1} X_{k+1} = 0.$$

Adding up all these equalities and substituting

$$X_{k+1} = -(X_1 + X_2 + \cdots + X_k)$$

yields

$$\mu_1 X_1^T D^2 u_1 X_1 + \mu_2 X_2^T D^2 u_2 X_2 + \cdots + \mu_k X_k^T D^2 u_k X_k$$
$$+ X_{k+1}^T D^2 u_{k+1} X_{k+1} = 0.$$

The quadratic forms defined by the Hessian matrics $D^2 u_1, \ldots,$ $D^2 u_{k+1}$ when restricted to the set of row matrices X^T such that

$$X^T D u_1 = X^T D u_2 = \cdots = X^T D u_{k+1} = 0$$

are all negative definite. The variables $\mu_1, \ldots, \mu_k$ are all > 0; therefore these quadratic forms add up to zero. Each term of this sum is equal to zero:

$$X_1^T D^2 u_1 X_1 = X_2^T D^2 u_2 X_2 = \cdots = X_{k+1}^T D^2 u_{k+1} X_{k+1} = 0,$$

which implies because of the negative definiteness $X_1 = X_2 = \cdots = X_{k+1} = 0$. This readily implies $y_1 = y_2 = \cdots = y_k = 0$. Therefore, matrix Z such that $AZ = 0$ must be equal to zero, a contradiction.

This ends the proof that the map $R_k(r, .) : \mathcal{U}_k(r) \to X^{k+1}$ is smooth.

A.4 Set of Pareto Optima and Feasible Utility Levels

Proposition A.4.1 The set $P_{k+1}(r)$ of Pareto optima for the first $k+1$ consumers is diffeomorphic to $\mathcal{U}_k(r)$ by the map that associates the Pareto optimum for the first $k+1$ consumers $R_k(r, u_1, u_2, \ldots, u_k)$ with $(u_1, u_2, \ldots, u_k) \in \mathcal{U}_k(r)$.

This proposition is equivalent to the statement that the map $R_k(r, .) : \mathcal{U}_k(r) \to X^{k+1}$ is an embedding and that its image is the set

$P_{k+1}(r)$ of Pareto optima for the first $k+1$ consumers. The second part of this statement has already been shown to be true.

Proof To prove the first part of the statement, compose the map $R_k(r,.)$ with the map $P_{k+1}(r) \to \mathbb{R}^k$ defined by $(x_1,\ldots,x_k) \to (u_1(x_1),\ldots, u_k(x_k))$. This gives the identity map of $\mathcal{U}_k(r)$. It then suffices to apply lemma 3.2.2 in chapter 3 to prove that the map $R_k(r,.)$ is an embedding from $\mathcal{U}_k(r)$ into $\mathbb{R}^{k+1}$ whose image is the set $P_{k+1}(r)$ of Pareto optima for the first $k+1$ consumers. This proves that $P_{k+1}(r)$ and $\mathcal{U}_k(r)$ are diffeomorphic. ∎

A.5 Structure of $\mathcal{U}_k(r)$

A.5.1 Value Function of Problem $\mathcal{P}_{k+1}(r,u_1,u_2,\ldots,u_k)$

Let $R_k^j(r,u_1,u_2,\ldots,u_k) \in X$ denote the component for consumer $j \leq k+1$ of the solution $R_k(r,u_1,u_2,\ldots,u_k)$ of problem $\mathcal{P}_{k+1}(r,u_1,u_2,\ldots, u_k)$.

Define $\rho_k(r,u_1,u_2,\ldots,u_k) = u_{k+1}(R_k^{k+1}(r,u_1,u_2,\ldots,u_k))$, the *value function* associated with problem $\mathcal{P}_{k+1}(r,u_1,u_2,\ldots,u_k)$.

Proposition A.5.1 The map $\rho_k(r,.) : \mathcal{U}_k(r) \to \mathbb{R}$ is smooth.

The value function $\rho_k(r,.)$ is the composition of two smooth maps and is therefore smooth.

A.5.2 Relation between $\mathcal{U}_{k+1}(r)$ and $\mathcal{U}_k(r)$

The following propositions are useful when analyzing the structure of the sets of feasible utility levels. Their proofs follow readily from the properties of the maps ρ_k and R_k.

Proposition A.5.2 The set $\mathcal{U}_{k+1}(r)$ consists of the elements $(u_1,\ldots,u_k, u_{k+1}) \in \mathbb{R}^{k+1}$ such that

$$(u_1,\ldots,u_k) \in \mathcal{U}_k(r),$$

$$u_{k+1} < \rho_k(r,u_1,\ldots,u_k).$$

This follows readily from the definition of the set $\mathcal{U}_{k+1}(r)$.

Proposition A.5.3 The open set $\mathcal{U}_{k+1}(r)$ is diffeomorphic to $\mathcal{U}_k(r) \times (0,+\infty)$.

Let

$$\kappa(u_1,\ldots,u_k,u_{k+1}) = (u_1,\ldots,u_k,\rho_k(r,u_1,\ldots,u_k) - u_{k+1}).$$

This defines a map $\kappa : \mathcal{U}_{k+1}(r) \to \mathcal{U}_k(r) \times (0, +\infty)$. This map is obviously a bijection. It is also smooth. It therefore suffices to check that it is locally a diffeomorphism, in other words, that its Jacobian matrix is everywhere invertible. This matrix is equal to

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ * & * & \cdots & * & -1 \end{bmatrix}$$

and is invertible.

Proposition A.5.4 The boundary $\partial\mathcal{U}_{k+1}(r)$ is the graph of the map $\rho_k(r,.) : \mathcal{U}_k(r) \to \mathbb{R}$.

This follows readily from the definitions.

Corollary A.5.5 The boundary $\partial\mathcal{U}_{k+1}(r)$ is a smooth submanifold of $\mathbb{R}^{k+1}$ diffeomorphic to $\mathcal{U}_k(r)$.

This follows from the identification of the boundary $\partial\mathcal{U}_{k+1}(r)$ with the graph of a smooth map, namely, the map $\rho_k(r,.) : \mathcal{U}_k(r) \to \mathbb{R}$. Such graph is a smooth submanifold of $\mathbb{R}^k \times \mathbb{R}$ diffeomorphic to the domain of the map $\rho_k(r,.)$, namely, the set $\mathcal{U}_k(r)$.

A.5.3 Diffeomorphism of $\mathcal{U}_k(r)$ with $\mathbb{R}^k$

Proposition A.5.6 The open set $\mathcal{U}_k(r)$ is diffeomorphic to $\mathbb{R}^k$ for $1 \leq k \leq m$.

Proof We proceed by induction on k. For $k = 1$, then obviously $\mathcal{U}_1(r)$ is the interval $(-\infty, u_1(r))$, which is diffeomorphic to $\mathbb{R}$.

It now follows from proposition A.5.3 that $\mathcal{U}_k(r)$ is diffeomorphic to $\mathcal{U}_{k-1}(r) \times (0, +\infty)$. It then follows from the induction assumption that $\mathcal{U}_{k-1}(r)$ is diffeomorphic to $\mathbb{R}^{k-1}$ and $\mathcal{U}_k(r)$ to $\mathbb{R}^{k-1} \times (0, +\infty)$ and consequently to $\mathbb{R}^k$. ∎

A.6 Application to the Set of Pareto Optima *P*(*r*)

These properties follow from the previous developments applied to $P(r) = P_m(r)$.

Proposition A.6.1 The set of Pareto optima $P(r)$ is the image of the smooth map $R_{m-1}(r,.) : \mathcal{U}_{m-1} \to X^m$.

Proposition A.6.2 The set of Pareto optima $P(r)$ is a smooth submanifold of $\Omega(r)$ diffeomorphic to $\mathbb{R}^{m-1}$.

Corollary A.6.3 The Pareto optimum $x = (x_1, x_2, \ldots, x_m) \in P(r)$ is parameterized by the utility levels $u_1(x_1) = u_1, u_2(x_2) = u_2, \ldots, u_{m-1}(x_{m-1}) = u_{m-1}$ of the first $m-1$ consumers.

This is just another way of saying that the map $R_{m-1}(r,.) : \mathcal{U}_{m-1}(r) \to \Omega(r)$ defines a diffeomorphism between $\mathcal{U}_{m-1}(r)$ and $P(r)$.

A.7 Making Total Resources Variable: The Set $\mathcal{U}_k$

For some questions it is also useful to consider the set $\mathcal{U}_k$, which is the graph of the correspondence from X into $\mathbb{R}^k$ that associates with $r \in X$ the set $\mathcal{U}_k(r)$. In other words, the set $\mathcal{U}_k$ is the subset of $X \times \mathbb{R}^k$ consisting of the pairs (r, u) with $u = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k$ such that $u \in \mathcal{U}_k(r)$.

Proposition A.7.1 The set $\mathcal{U}_k$ is a smooth manifold diffeomorphic to $X \times \mathbb{R}^k$.

Proof It suffices to adapt the proof of proposition A.5.6 to the fact that $r \in X$ is now variable. Consider therefore the map

$$(r, u_1, \ldots, u_k) \to (r, u_1, \ldots, u_{k-1}, \rho_k(r, u_1, \ldots, u_{k-1}) - u_k)$$

from $X \times \mathcal{U}_k$ into $X \times \mathcal{U}_{k-1} \times (0, +\infty)$. It follows from the preceding developments that this map is a diffeomorphism. An induction argument then implies that $X \times \mathcal{U}_k$ is diffeomorphic with $X \times (0, +\infty)^k$ and therefore with $X \times \mathbb{R}^k$. ∎

A.8 Set of Pareto Optima $P(r)$ and Boundary of the Set $\mathcal{U}_m(r)$

Another interesting feature of the set of feasible utility levels $\mathcal{U}_m(r)$ is the existence of a relation between its boundary $\partial\mathcal{U}_m(r)$ and the set of Pareto optima $P(r)$, more specifically, as follows:

Proposition A.8.1 Let $u : (x_1, \ldots, x_m) \to (u_1(x_1), \ldots, u_m(x_m))$ from $\Omega(r)$ into $\mathbb{R}^m$. Its image is the closure $\overline{\mathcal{U}_m(r)}$. The restriction of map u to $P(r)$ is a diffeomorphism with $\partial\mathcal{U}_m(r)$, the boundary of $\mathcal{U}_m(r)$.

See corollary A.5.5.

A.9 Parameterization by Welfare Weights

A.9.1 Strict Concavity of Utility Functions

From now on in this appendix, assume that all utility functions are *strictly smoothly concave,* i.e., that the Hessian matrices of their second-order partial derivatives are all negative definite. This assumption is important because the sum of concave functions is concave, which is not the case for quasi-concave functions.

Restricted Lagrangean

The restricted Lagrangean is by definition the expression

$$\mathcal{L}_k(x, \lambda_1, \lambda_2, \dots, \lambda_k) = \lambda_1 u_1(x_1) + \lambda_2 u_2(x_2) + \cdots + \lambda_k u_k(x_k) + u_{k+1}(x_{k+1}).$$

The following proposition is obvious.

Proposition A.9.1 For $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k_{++}$ given, the restricted Lagrangean defines a smoothly strictly concave function $\mathcal{L}_k(., \lambda_1, \lambda_2, \dots, \lambda_k)$ on X^{k+1}.

Maximization of the Restricted Lagrangean under a Resource Constraint

Problem $\mathcal{Q}_{k+1}(r, \lambda_1, \dots, \lambda_k)$ is the maximization of $\mathcal{L}_k(x, \lambda_1, \dots, \lambda_k)$ for $(x, \lambda) \in X^{k+1} \times \mathbb{R}^k_{++}$ subject to the resource constraint $x_1 + x_2 + \cdots + x_k + x_{k+1} \leq r$.

Proposition A.9.2 Problem $\mathcal{Q}_{k+1}(r, \lambda_1, \dots, \lambda_k)$ has a unique solution that is a smooth function of $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k_{++}$.

It is left to readers to check the well-known part that the first-order conditions are necessary and sufficient here. The end of the proof consists in observing that these first-order conditions coincide with those of section A.3.1, the first-order conditions for problem $\mathcal{P}_{k+1}(r, u_1, u_2, \dots, u_k)$.

Corollary A.9.3 The solution $T_{k+1}(r, \lambda_1, \dots, \lambda_k)$ of the maximization problem $\mathcal{Q}_{k+1}(r, \lambda_1, \dots, \lambda_k)$ is a smooth function $T_{k+1}(r, .) : \mathbb{R}^k_{++} \to P_k(r)$.

Parameterization by Welfare Weights of P(r)

Proposition A.9.4 follows from the application of the previous results to the special case $k = m - 1$.

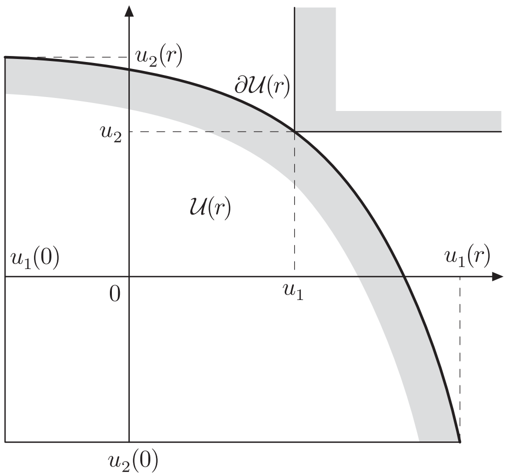

Figure A.1
Feasible utility levels with strictly concave utility functions

Proposition A.9.4 The set of Pareto optima $P(r)$ can be parameterized by the welfare weights $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ and $\lambda_m = 1$ for the m consumers, with $\lambda_i > 0$ for all consumers. The parameterization is smooth and defines a diffeomorphism between $\mathbb{R}^{m-1}_{++}$ and the set of Pareto optima $P(r)$.

A.9.2 Convexity of the Set of Feasible Utility Levels

Proposition A.9.5 The set of feasible utility levels $\overline{\mathcal{U}_k(r)}$ with full resource utilization is strictly convex. (See figure A.1.)

Proof Let $u = (u_1(x_1), u_2(x_2), \ldots, u_k(x_k))$ and $u' = (u_1(x'_1), u_2(x'_2), \ldots, u_k(x'_k))$ with $x_1 + x_2 + \cdots + x_k$ and $x'_1 + x'_2 + \cdots + x'_k \leq r \in X$ and $u \neq u'$.

Let $x'' = (x + x')/2$. We then have $x''_1 + x''_2 + \cdots + x''_k \leq r$. The inequality

$$u''_i = u_i((x_i + x'_i)/2) \geq (u_i + u'_i)/2$$

for every i follows from the concavity of the utility function u_i, which proves the inequality $(u + u')/2 \leq u''$. In addition, this inequality is strict for at least one consumer because $u \neq u'$, i.e., $u'' \in (u + u')/2 + \mathbb{R}^k_+$.

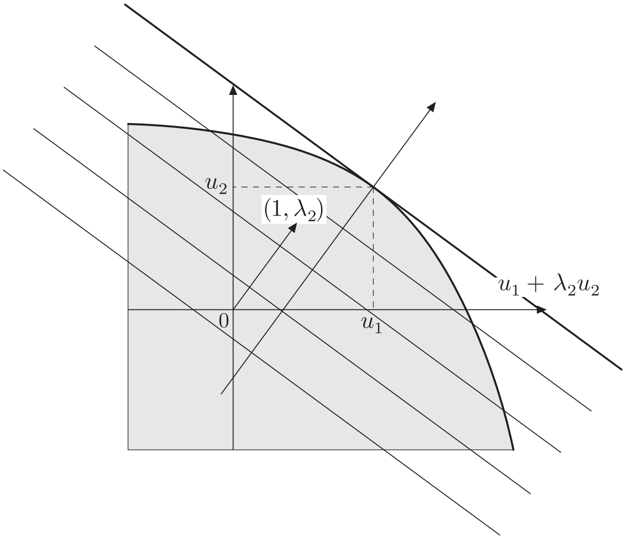

Figure A.2
Maximization of $\lambda_1 u_1 + u_2$ subject to the constraint $(u_1, u_2) \in \overline{\mathcal{U}_2(r)}$

It follows from $u'' \in \overline{\mathcal{U}_k(r)}$ combined with proposition A.1.4 that $(u + u')/2$ belongs to $\mathcal{U}_k(r)$, the interior of $\overline{\mathcal{U}_k(r)}$. ∎

Exercises

A.1. Show that a strictly increasing map $u_i : \mathbb{R}_{++} \to \mathbb{R}$ is quasi-concave. Give examples of strictly quasi-concave functions that are not strictly concave.

A.2. Apply exercise A.1 to give an example of an economy with a set $\mathcal{U}_k(r)$ for $k = 2$ that is not strictly convex. (*Hint:* Give the example for an economy with only one commodity.)

A.3. Define $\mathcal{L}(\lambda, u) = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_{k-1} u_{k-1} + u_k$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \in \mathbb{R}^{k-1}_{++}$ and $u = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k$. Show that the problem of maximizing $\mathcal{L}(\lambda, u)$ for $\lambda \in \mathbb{R}^{k-1}_{++}$ subject to the constraint $u \in \overline{\mathcal{U}_{k-1}(r)}$ has a unique solution and that this solution depends smoothly on $\lambda \in \mathbb{R}^{k-1}_{++}$. (See figure A.2.)

A.4. Show that the value function $\rho_k(r, .) : \mathcal{U}_{k-1}(r) \to \mathbb{R}$ has a negative definite Hessian matrix.

A.10 Notes and Comments

The parameterization of the boundary of the set of feasible utility levels by the utility levels of all but one consumers (corollary A.6.3) was used by Samuelson (1947, 244). Bewley (1969) was the first to describe the global topological structure (without differentiability assumptions) of the set of Pareto optima (for fixed total resources) by showing that the boundary of the set of feasible utility levels is homeomorphic to the unit simplex, a property he used in his existence proof for the infinite horizon model. See also Arrow and Hahn (1971, ch. 5, sec. 2, lemma 3), which reproduces what is essentially Bewley's existence proof but adapted to the finite general equilibrium model. Smale (1974c) proved that the set of Pareto optima is a smooth manifold. The relation between the set of Pareto optima and the section manifold in the set of price-income equilibria is due to Balasko (1978a).

The parameterization of the set of Pareto optima by the welfare weights is a classic of the literature on welfare economics. The restricted Lagrangean belongs to the larger category of collective utility functions considered by Samuelson (1947). The homeomorphism between the set of Pareto optima and the set of welfare weights can easily be derived from the discussion in Varian (1978). Nevertheless, a rigorous proof that the homeomorphism is actually a diffeomorphism requires the application of the implicit function theorem to the first-order conditions, which is almost never done.

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