Stochastic Risk Analysis and Management

Stochastic Models in Survival Analysis and Reliability Set

coordinated by
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Volume 2

Stochastic Risk Analysis and Management

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ISTE

WILEY

First published 2017 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

Library of Congress Control Number: 2016961651

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-78630-008-9

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Mathematical Bases

1.1. Introduction to stochastic risk analysis

1.1.1. *About the subject*

The concept of risk is diverse enough and is used in many areas of human activity. The object of interest in this book is the theory of collective risk. Swedish mathematicians Cramér and Lundberg established stochastic models of insurance based on this theory.

Stochastic risk analysis is a rather broad name for this volume. We will consider mathematical problems concerning the Cramér-Lundberg insurance model and some of its generalizations. The feature of this model is a random process, representing the dynamics of the capital of a company. These dynamics consists of alternations of slow accumulation (that may be not monotonous, but continuous) and fast waste with the characteristic of negative jumps.

All mathematical studies on the given subject continue to be relevant nowadays thanks to the absence of a compact analytical description of such a process. The stochastic analysis of risks which is the subject of interest has special aspects. For a long time, the most interesting problem within the framework of the considered model was ruin, which is understood as the capital of a company reaching a certain low level. Such problems are usually more difficult than those of the value of process at fixed times.

Stochastic Risk Analysis and Management, First Edition. Boris Harlamov.

1.1.2. *About the ruin model*

Let us consider the dynamics of the capital of an insurance company. It is supposed that the company serves several clients, which bring in insurance premiums, i.e. regular payments, filling up the cash desk of the insurance company. Insurance premiums are intended to compensate company losses resulting from single payments of great sums on claims of clients at unexpected incident times (the so-called insured events). They also compensate expenditures on maintenance, which are required for the normal operation of a company. The insurance company's activity is characterized by a random process which, as a rule, is not stationary. The company begins business with some initial capital. The majority of such undertakings come to ruin and only a few of them prosper. Usually they are the richest from the very beginning. Such statistical regularities can already be found in elementary mathematical models of dynamics of insurance capital.

The elementary mathematical model of dynamics of capital, the Cramér-Lundberg model, is constructed as follows. It uses a random process R_t $(t \geq 0)$

$$Rt = u + p\,t - \sum_{n=1}^{Nt} U_n, \qquad [1.1]$$

where $u \geq 0$ is the initial capital of the company, $p > 0$ is the growth rate of an insurance premium and $p\,t$ is the insurance premium at time t. $(U_n)_{n=1}^{\infty}$ is a sequence of suit sizes which the insurance company must pay immediately. It is a sequence of independent and identically distributed (i.i.d.) positive random variables. We will denote a cumulative distribution function of U_1 (i.e. of all remaining) as $B(x) \equiv P(U_1 \leq x)$ $(x \geq 0)$. The function (N_t) $(t \geq 0)$ is a homogeneous Poisson process, independent of the sequence of suit sizes, having time moments of discontinuity at points $(\sigma_n)_{n=1}^{\infty}$. Here, $0 \equiv \sigma_0 < \sigma_1 < \sigma_2 < \ldots$; values $T_n = \sigma_n - \sigma_{n-1}$ $(n \geq 1)$ are i.i.d. random variables with a common exponential distribution with a certain parameter $\beta > 0$.

Figure 1.1 shows the characteristics of the trajectories of the process.

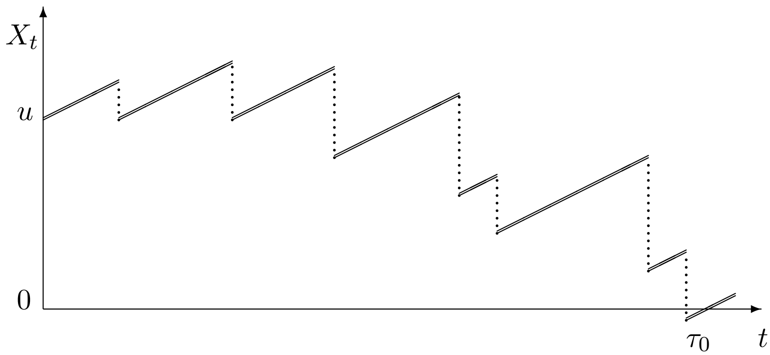

Figure 1.1. *Dynamics of capital*

This is a homogeneous process with independent increments (hence, it is a homogeneous Markov process). Furthermore, we will assume that process trajectories are continuous from the right at any point of discontinuity.

Let τ_0 be a moment of ruin of the company. This means that at this moment, the company reaches into the negative half-plane for the first time (see Figure 1.1). If this event does not occur, this moment is set as equal to infinity.

The first non-trivial mathematical results in risk theory were connected with the function:

$$\psi(u) = P_u(\tau_0 < \infty) \quad (u \geq 0),$$

i.e. a probability of ruin on an infinite interval for a process with the initial value u. Interest is also represented by the function $\psi(u,t) = P_u(\tau_0 \leq t)$. It is called the ruin function on “finite horizon”.

Nowadays many interesting outcomes have been reported for the Cramér-Lundberg model and its generalizations. In this volume, the basic results of such models are presented. In addition, we consider its generalizations, such as insurance premium inflow and distribution of suit sizes.

This is concentrated on the mathematical aspects of a problem. Full proofs (within reason) of all formulas, and volume theorems of the basic course are presented. They are based on the results of probability theory which are assumed to be known. Some of the information on probability theory is shortly presented at the start. In the last chapter some management problems in insurance business are considered.

1.2. Basic methods

1.2.1. *Some concepts of probability theory*

1.2.1.1. *Random variables*

The basis of construction of probability models is an abstract probability space $(\Omega, \mathcal{F}, P)$, where Ω is a set of elementary events; $\mathcal{F}$ is a sigma-algebra of subsets of the set Ω, representing the set of those random events, for which it makes sense to define the probability within the given problem; P is a probability measure on set Ω, i.e. non-negative denumerably additive function on $\mathcal{F}$. For any event $A \in \mathcal{F}$, the probability, $P(A)$, satisfies the condition $0 \leq P(A) \leq 1$. For any sequence of non-overlapping sets $(A_n)_1^\infty$ $(A_n \in \mathcal{F})$ the following equality holds:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n),$$

and $P(\Omega) = 1$. Random events A_1 and A_2 are called independent if $P(A_1, A_2) \equiv P(A_1 \cap A_2) = P(A_1)P(A_2)$. This definition is generalized on any final number of events. Events of infinite system of random events are called mutually independent if any of its final subsystem consists of independent events.

A random variable is a measurable function $\xi(\omega)$ $(\omega \in \Omega)$ with real values. It means that for any real x, the set $\{\omega : \xi(\omega) \leq x\}$ is a random event and hence, probability of it exists, designated as $F_\xi(x)$. Thus, the cumulative distribution function, F_ξ, is defined as follows :

$$F_\xi(x) = P(\xi \leq x) \quad (-\infty < x < \infty).$$

It is obvious that this function does not decrease when x increases. In this volume, we will deal with absolutely continuous distributions and discreet distributions (sometimes with their mixtures).

For an absolutely continuous distribution, there exists its distribution density $f_\xi(x) = dF_\xi(x)/dx$ for all $x \in (-\infty, \infty)$ such that

$$\int_{-\infty}^{\infty} f_\xi(x)\, dx = 1.$$

For discreet distributions, there exists a sequence of points (atoms) $(x_n)_1^\infty$ for which non-negative probabilities $p(x_n) = P(\xi = x_n)$ are defined as:

$$\sum_{n=1}^{\infty} p(x_n) = 1.$$

The random variable is called integer if it has a discreet distribution with atoms in the integer points of a numerical axis, denoted by $\mathbb{Z}$.

If $\mathbb{R}$ is the set of all real numbers, φ is a measurable function on $\mathbb{R}$, and ξ is a random variable, then superposition $\psi(\omega) \equiv \varphi(\xi(\omega))$ $(\omega \in \Omega)$ is a random variable too. Various compositions of random variables are possible, which are also random variables. Two random variables ξ_1 and ξ_2 are called independent, if for any x_1 and x_2 events $\{\xi_1 \leq x_1\}$ and $\{\xi_2 \leq x_2\}$ are independent.

Expectation (average) $E\xi$ of a random variable ξ is the integral of this function on Ω with respect to the probability measure P, i.e.:

$$E\xi = \int_\Omega \xi(\omega)\, P(d\omega) \equiv \int \xi\, dP$$

(an integral of Lebesgue). By a cumulative distribution function, this integral can be noted as an integral of Stieltjes:

$$E\xi = \int_{-\infty}^{\infty} x\, dF_\xi(x),$$

and for a random variable ξ with absolute continuous distribution, it can be represented as integral of Riemann:

$$E\xi = \int_{-\infty}^{\infty} x f_\xi(x)\, dx.$$

For a random variable ξ with a discreet distribution, it is possible to write an integral in the form of the sum:

$$E\xi = \sum_{n=1}^{\infty} x_n p(x_n).$$

When evaluating an expectation, it is necessary to be careful in case the integral from the module of this random variable is equal to infinity. Sometimes it useful to distinguish three cases: an integral equal to plus infinity, an integral equal to minus infinity and an integral does not exist.

Let us note that it is possible to consider separately a cumulative distribution function out of connection with random variables generating them and probability spaces. However, for any non-decreasing, continuous from the right, function F such that $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to \infty$ (the cumulative distribution function of any random variable possesses these properties), it is possible to construct a probability space and with random variable on this space, which has F as its cumulative distribution function on this probability space. Therefore, speaking about a cumulative distribution function, we will always mean some random variable within this distribution. It allows us to use equivalent expressions such as "distribution moment", "moment of a random variable", "generating function of a distribution" and "generating function of a random variable".

The following definitions are frequently used in probability theory. The moment of nth order of a random variable ξ is an integral $E\xi^n$ (if it exists). The central moment of nth order of a random variable ξ is an integral $E(\xi - E\xi)^n$ (if it exists). The variance (dispersion) $D\xi$ of a random variable ξ is its central moment of second order.

The generating function of a random variable is the integral $E\exp(\alpha\xi)$, considered as a function of α. Interest represents those generating functions which are finite for all α in the neighborhood of zero. In this case, there is one-to-one correspondence between the set of distributions and the set of generating functions. This function has received the name because of its property "to make" the moments under the formula:

$$E\xi^n = \left. \frac{d^n E\exp(\alpha\xi)}{d\alpha^n} \right|_{\alpha=0}.$$

A random n-dimensional vector is the ordered set of n random variables $\xi = (\xi_1, \ldots, \xi_n)$. Distribution of this random vector (joint distribution of its random coordinates) is a probability measure on space $\mathbb{R}^n$, defined by n-dimensional cumulative distribution function:

$$F_\xi(x_1, \ldots, x_n) = P(\xi_1 \le x_1, \ldots, \xi_n \le x_n) \quad (x_i \in \mathbb{R}, i = 1, \ldots, n).$$

As the generating function of a random vector is called function of n variables $E\exp(\alpha, \xi)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ $(\alpha_i \in \mathbb{R})$ and $(\alpha, \xi) = \sum_{i=1}^n \alpha_i \xi_i$. The mixed moment of order $m \ge 2$ of a random vector ξ is called $E(\xi_1^{m_1} \cdot \cdots \cdot \xi_n^{m_n})$, where $m_i \ge 0$, $\sum_{i=1}^n m_i = m$. Covariance of random variables ξ_1 and ξ_2 is called central joint moment of the second order:

$$\text{cov}(\xi_1, \xi_2) = E(\xi_1 - E\xi_1)(\xi_2 - E\xi_2).$$

1.2.1.2. *Random processes*

In classical probability theory, random process on an interval $T \subset \mathbb{R}$ is called a set of random variables $\xi = (\xi_t)_{t \in T}$, i.e. function of two arguments (t, ω) with values $\xi_t(\omega) \in \mathbb{R}$ $(t \in \mathbb{R}, \omega \in \Omega)$, satisfying measurability conditions. As random process, we can understand that an infinite-dimensional random vector, whose space is designated as $\mathbb{R}^T$, is a set of all functions on an interval T. Usually, it is assumed that a sigma-algebra of subsets of such set functions contains all so-called finite-dimensional cylindrical sets, i.e. sets of:

$$\{f \in \mathbb{R}^T : f_{t_1} \in A_1, \ldots, f_{t_n} \in A_n\} \quad (n \geq 1, t_i \in T, A_i \in \mathcal{B}(\mathbb{R})),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel sigma-algebra of the subsets of $\mathbb{R}$ (the sigma-algebra of subsets generated by all open intervals of a numerical straight line). For the problems connected with the first exit times, the minimal sigma-algebra $\widetilde{\mathcal{F}}$, containing all such cylindrical sets, is not sufficient. It is connected by that the set $\mathbb{R}^T$ "is too great". Functions belonging in this set are not connected by any relations considering an affinity of arguments t, such as a continuity or one-sided continuity.

For practical problems, it is preferable to use the other definition of the random process, namely not a set of random variables assuming the existence of the abstract probability spaces, but a random function as element of a certain set Ω, composed of all possible realizations within the given circle of problems. On this function space, a sigma-algebra of subsets and a probability measure on this sigma-algebra should be defined. For the majority of practical uses, it is enough to take as function space the set $\mathcal{D}$ of all functions $\xi : T \to \mathbb{R}$ continuous from the right and having a limit from the left at any point of an interval $T \subset \mathbb{R}$. The set $\mathcal{D}$ is a metric space with respect to the Skorokhod metric, which is a generalization of the uniform metric. A narrower set, that has numerous applications as a model of real processes, is the set $\mathcal{C}$ of all continuous functions on T with locally uniform metric. In some cases it is useful to consider other subsets of space $\mathcal{D}$, for example, all piece-wise constant function having a locally finite set of point of discontinuities. Sigma-algebra $\mathcal{F}$ of subsets of $\mathcal{D}$, generated by cylindrical sets with the one-dimensional foundation of an aspect $\{\xi \in \mathcal{D} : \xi(t) \in A\} \quad (t \in T, A \in \mathcal{B}(\mathbb{R}))$ that comprises all interesting subsets (events) connected with the moments of the first exit from open intervals belonging to a range of values of process.

Random process is determined if some probability measure on corresponding sigma-algebra of subsets of set of its trajectories is determined. In classical theory of random processes, a probability measure on $\widetilde{\mathcal{F}}$ is determined if and only if there exists a consistent system of finite-dimensional distributions determined on cylindrical sets with finite-dimensional foundations [KOL 36]. To represent a measure on the sigma-algebra $\mathcal{F}$, Kolmogorov's conditions for the coordination of distributions on the finite-dimensional cylindrical sets are not enough. In this case, some additional conditions are required. They, as a rule, are concerned with two-dimensional distributions $P(\xi(t_1) \in A_1, \xi(t_2) \in A_2)$ as $|t_1 - t_2| \to 0$. In problems of risk theory where, basically, Markov processes are used, these additional conditions are easily checked.

1.2.1.3. *Shift operator*

We will assume further that $T = [0, \infty) \equiv \mathbb{R}_+$. First, we define on set $\mathcal{D}$ an operator X_t "value of process in a point t": $X_t(\xi) \equiv \xi(t)$. We also use other labels for this operator, containing the information on concrete process, for example, R_t, N_t and S_t. They are operators with meaning: values of concrete processes at a point t. By an operator X_t we will represent the set $\{\xi \in \mathcal{D} : \xi(t_1) \in A_1, \dots, \xi(t_n) \in A_n\}$ as $\{X_{t_1} \in A_1, \dots, X_{t_n} \in A_n\}$. Thus, finite-dimensional distribution is possible to note as probability $P(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n)$. This rule of denotation when the argument in the subset exposition is omitted is also spread on other operators defined on $\mathcal{D}$.

A shift operator θ_t maps $\mathcal{D}$ on $\mathcal{D}$. It is possible to define function $\theta_t(\xi)$ $(t \geq 0)$ by its values at points $s \geq 0$. These values are defined as:

$$(\theta_t(\xi))(s) = \xi(t+s) \quad (t, s \geq 0).$$

Using an operator X_t this relation can be noted in an aspect $X_s(\theta_t(\xi)) = X_{t+s}(\xi)$ or, by lowering argument ξ, in an aspect $X_s(\theta_t) = X_{t+s}$. We also denote this relation (superposition) as $X_s \circ \theta_t = X_{t+s}$. Obviously, $\theta_s \circ \theta_t = \theta_{t+s}$.

An important place in the considered risk models is taken by the operator σ_Δ "the moment of the first exit from set Δ", defined as $\sigma_\Delta(\xi) = \inf\{t \geq 0 : \xi(t) \notin \Delta\}$, if the set in braces is not empty; otherwise, we suppose $\sigma_\Delta(\xi) = \infty$.

1.2.1.4. *Conditional probabilities and conditional averages*

From elementary probability theory, the concept of conditional probabilities $P(A|\,B)$ and a conditional average $E(f|\,B)$ concerning event B are well-known, where A and B are events, f is a random variable and $P(B) > 0$. The concept of conditional probability concerning a final partition of space on simple events $P(A\,|\,\mathcal{P})$ is not more complicated, where $\mathcal{P} = (B_1, \ldots, B_n)$ $(B_i \cap B_j = \emptyset$, $\bigcup_{k=1}^{n} B_k = \Omega)$ and $P(B_i) > 0$. In this case, the conditional probability can be understood as function on partition elements: on a partition element B_i, its value is $P(A|\,B_i)$. This function accepts n values. However, in this case, there is a problem as to how to calculate conditional probabilities with respect to some association of elements of the partition. It means to receive a function with a finite (no more 2^n) number of values, measurable with respect to the algebra of subsets generated by this finite partition. In this way, we can attempt to apply an infinite partition in the right part of the conditional probability. Obviously, this generalization is not possible for non-denumerable partition, for example, set of pre-images of function X_t, i.e. $(X_t^{-1}(x))_{x\in\mathbb{R}}$. In this case, conditional probability is accepted to define a function on $\mathbb{R}$ with special properties, contained in the considered example with a final partition. That is, the conditional probability $P(A\,|\,X_t)$ is defined as a function of $\xi \in \mathcal{D}$, measurable with respect to sigma-algebra, generated by all events $\{X_t < x\}$ (we denote such a sigma-algebra as $\sigma(X_t)$), which for any $B \in \mathcal{B}(\mathbb{R})$ satisfies the required conditions:

$$P(A, X_t \in B) = \int_{X_t \in B} P(A|\,X_t)(\xi)\,dP \equiv E(P(A|\,X_t); X_t \in B).$$

This integral can be rewritten in other form, while using representation of conditional probability in an aspect:

$$P(A|\,X_t) = g_A \circ X_t,$$

where g_A is a measurable function on $\mathbb{R}$, defined uniquely according to the known theorem from a course on probability theory [NEV 64]. Then, using a change of variables $x = X_t(\xi)$, we obtain the following representation:

$$P(A, X_t \in B) = \int_B g_A(x)\, p_t(dx),$$

where $p_t(S) = P(X_t \in S)$ $(S \in \mathcal{B}(\mathbb{R}))$. The value of function $g_A(x)$ can be designated as $P(A|\, X_t = x)$. This intuitively clear expression cannot be understood literally in the spirit of elementary probability theory. In certain cases, it can be justified as a limit of conditional probabilities, where the right side of conditional probability is changed with the condition that X_t belongs to a small neighborhood of a point x. Usually, function $g_A(x)$ may be identified using the value of function $P_x(A)$, where $A \to P_x(A)$ is a measure on $\mathcal{F}$ for each $x \in \mathbb{R}$ and $x \to P_x(A)$ is a $\mathcal{B}(\mathbb{R})$-measurable function for each $A \in \mathcal{F}$. Hence,

$$g_A \circ X_t = P_{X_t}(A).$$

1.2.1.5. *Filtration*

To define the Markov process, it is necessary to define the concepts of "past" and "future" of the process, which means to define conditional probability and average "future" relative to "past". For this purpose, together with a sigma-algebra $\mathcal{F}$, the ordered increasing family of sigma-algebras $(\mathcal{F}_t)$ $(t \geq 0)$ is considered. This family is called filtration if $\lim_{t\to\infty} \mathcal{F}_t \equiv \bigcup_{t=0}^{\infty} \mathcal{F}_t$. For example, such a family consists of sigma-algebras $\mathcal{F}_t$. The latter is generated by all one-dimensional cylindrical sets $\{X_s < x\}$, where $s \leq t$ and $x \in \mathbb{R}$. It is designated as $\sigma(X_s : s \leq t)$), which is called natural filtration. The sigma-algebra $\mathcal{F}_t$ contains all measurable events reflecting the past of the process until the moment t. In relation to it, any value X_{t+s} $(s > 0)$ is reasonably called "future".

Another feature of the considered example is a conditional probability (average) with respect to sigma-algebra $\mathcal{F}_t$. Under conditional probability $P(A \mid \mathcal{F}_t)$, it is understood that for such $\mathcal{F}_t$-measurable function (random variable) on $\mathcal{D}$, for any $B \in \mathcal{F}_t$ the equality is fulfilled:

$$P(A, B) = \int_B P(A| \mathcal{F}_t)(\xi)\, dP \equiv E(P(A| \mathcal{F}_t); B).$$

Conditional average $E(f \mid \mathcal{F}_t)$ is similarly defined. For any random variable f, the random variable $E(f \mid \mathcal{F}_t)$ is $\mathcal{F}_t$-measurable function on $\mathcal{D}$, for any $B \in \mathcal{F}_t$ the equality is fulfilled:

$$E(f; B) = \int_B E(f| \mathcal{F}_t)(\xi)\, dP \equiv E(E(f| \mathcal{F}_t); B).$$

Let us note that the second definition is more general than the conditional probability of event A because it can be presented as a conditional average from an indicator function of the set A. Let us note also that $\Omega \in \mathcal{F}_t$ for any t, and consequently

$$Ef = E(f;\, \Omega) = E(E(f \mid \mathcal{F}_t); \Omega) = E(E(f \mid \mathcal{F}_t)).$$

Existence and uniqueness (within set of a measure 0) of the conditional average is justified by the Radon-Nikodym theorem, which is one of the key theorems of the theory of measure [KOL 72].

1.2.1.6. *Martingale*

Random process (X_t) $(t \geq 0)$, defined on a measurable space $(\mathcal{D}, \mathcal{F})$, supplied with filtration $(\mathcal{F}_t)$ $(\mathcal{F}_t \subset \mathcal{F})$, is called martingale, if at any t value of process X_t measurable with respect to $\mathcal{F}_t$, such that $E|X_t| < \infty$ and at any $s, t \geq 0$ it is fulfilled $E(X_{t+s} \mid \mathcal{F}_t) = X_t$ P a.s. If for any $s, t \geq 0$ $E(X_{t+s}| \mathcal{F}_t) \geq X_t$ P-a. s, then the process $X(t)$ is called sub-martingale. Thus the martingale is a partial case of a sub-martingale. However, the martingale, unlike a sub-martingale, supposes many-dimensional generalizations. Some proofs of risk theory are based on the properties of martingales (sub-martingales).

Further, we will use the generalization of the sigma-algebra $\mathcal{F}_t$ with a random t of special aspect, which depends on the filtration $(\mathcal{F}_t)$. We consider a random variable $\tau : \mathcal{D} \to \widetilde{\mathbb{R}}_+$ such that for any $t \geq 0$, the event $\{\tau \leq t\}$ belongs to $\mathcal{F}_t$. It is the Markov time. In this definition, $\widetilde{\mathbb{R}}_+$ denotes the enlarged positive half-line where the point "infinity" is supplemented. Therefore, we can admit infinity meanings for a Markov time. Let τ be a Markov time. Then, we define a sigma-algebra:

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : (\forall\, t > 0)\, A \cap \{\tau \leq t\} \in \mathcal{F}_t\}.$$

Intuitively, $\mathcal{F}_\tau$ is a sigma-algebra of all events before the moment τ. Further, we will use the following properties of martingales (sub-martingales).

THEOREM 1.1.– *(theorem of Doob about Markov times)* Let process (X_t) be a sub-martingale and τ_1, τ_2 be Markov times, for which $E|X_{\tau_i}| < \infty$ $(i = 1, 2)$. Then, on set $\{\tau_1 \leq \tau_2 < \infty\}$

$$E(X_{\tau_2}|\,\mathcal{F}_{\tau_1}) \geq X_{\tau_1} \quad P\text{- a. s..}$$

PROOF.– (see, for example, [LIP 86]).

Using evident property: if (X_t) is a martingale then $(-X_t)$ is a martingale too, we receive a consequence: if (X_t) is a martingale, then on set $\{\tau_1 \leq \tau_2 < \infty\}$:

$$E(X_{\tau_2}|\,\mathcal{F}_{\tau_1}) = X_{\tau_1} \quad P\text{-a. s.,}$$

and for any finite Markov time $EX_\tau = EX_0$.

One of the most important properties of a martingale is the convergence of a martingale when its argument t tends to a limit. It is one of few processes for which such limit exists with probability 1.

THEOREM 1.2.– *(theorem of Doob about convergence of martingales).* Let a process $(X_t, \mathcal{F}_t)$ $(t \in [0.\infty))$ be a sub-martingale, for which $\sup_{t\geq 0} E|X_t| < \infty$. Then, $E|X_\infty| < \infty$ and with probability 1 there exists a limit:

$$\lim_{t\to\infty} X_t = X_\infty.$$

PROOF.– (see, for example, [LIP 86]).

It is clear that a martingale with the above properties satisfies the assertion of this theorem.

1.2.2. *Markov processes*

1.2.2.1. *Definition of Markov process*

Markov processes are defined in terms of the conditional probabilities (averages) considered above. The random process defined on measurable space $(\mathcal{D}, \mathcal{F})$, is called Markov, if for any $t \geq 0$, $n \geq 1$, $s_i \geq 0$, $A_i \in \mathcal{B}(\mathbb{R})$ $(i = 1, \ldots, n)$ and $B \in \mathcal{F}_t$ is fulfilled

$$P\left(\bigcap_{k=1}^{n} X_{t+s_k} \in A_k,\, B\right) = E\left(P\left(\bigcap_{k=1}^{n} X_{t+s_k} \in A_k \,\middle|\, X_t\right); B\right). \qquad [1.2]$$

Using definition of conditional probability, it follows:

$$P\left(\bigcap_{k=1}^{n} X_{t+s_k} \in A_k,\, B\right) = E\left(P\left(\bigcap_{k=1}^{n} X_{t+s_k} \in A_k \,\middle|\, \mathcal{F}_t\right); B\right).$$

Because $\sigma(X_t) \subset \mathcal{F}_t$ and B is an arbitrary set in $\mathcal{F}_t$, it follows that for any $t \geq 0$, $n \geq 1$, $s_i \geq 0$, $A_i \in \mathcal{B}(\mathbb{R})$ $(i = 1, \ldots, n)$ is fulfilled

$$P\left(\bigcap_{k=1}^{n} (X_{t+s_k} \in A_k) \,\middle|\, \mathcal{F}_t\right) = P\left(\bigcap_{k=1}^{n} (X_{t+s_k} \in A_k) \,\middle|\, X_t\right)$$

P-a.s. (almost sure, i.e. the set where these functions differ as P-measures zero).

A well-known Markov property: the conditional distribution of “future” at the fixed “past” depends only on the “present”.

Let us note that the shift operator θ_t, defined on set of trajectories, defines an inverse operator θ_t^{-1}, defined on set of all subsets of $\mathcal{D}$. Thus,

$$\{X_s \circ \theta_t \in A\} = \{\xi \in \mathcal{D} :\ X_s(\theta_t(\xi)) \in A\} =$$
$$= \{\xi \in \mathcal{D} :\ \theta_t(\xi) \in X_s^{-1}(A)\} = \{\xi \in \mathcal{D} :\ \xi \in \theta_t^{-1} X_s^{-1}(A)\} =$$
$$= \theta_t^{-1} X_s^{-1}(A) = \theta_t^{-1}\{\xi \in \mathcal{D} :\ X_s(\xi) \in A\} = \theta_t^{-1}\{X_s \in A\}.$$

From here,

$$\{X_{t+s_1} \in A_1, \ldots, X_{t+s_n} \in A_n\} = \bigcap_{k=1}^{n} \{X_{t+s_k} \in A_k\} =$$
$$= \bigcap_{k=1}^{n} \{X_{s_k} \circ \theta_t \in A_k\} = \bigcap_{k=1}^{n} \theta_t^{-1}\{X_{s_k} \in A_k\} =$$
$$= \theta_t^{-1} \bigcap_{k=1}^{n} \{X_{s_k} \in A_k\} = \theta_t^{-1} S,$$

where $S = \{X_{s_1} \in A_1, \ldots, X_{s_n} \in A_n\}$ is a cylindrical set with finite-dimensional foundation. From the well-known theorem of extension of measure from algebra on a sigma-algebra generated by it (see [DYN 63]), a Markov behavior condition [1.2] can be rewritten in the following aspect:

$$P(\theta_t^{-1} S,\, B) = E(P(\theta_t^{-1} S \,|\, X_t);\, B) \qquad [1.3]$$

for any set $S \in \mathcal{F}$, whence the relation for conditional probabilities follows.

In terms of averages, the condition of a Markov behavior of process looks as follows:

$$E(f(X_{t+s_1}, \ldots, X_{t+s_n});\, B) = E(E(f(X_{t+s_1}, \ldots, X_{t+s_n}) |\, X_t);\, B).$$

Using a shift operator, it is possible to note that for any measurable function f it holds:

$$f(X_{t+s_1}, \ldots, X_{t+s_n}) = f(X_{s_1} \circ \theta_t \ldots, X_{s_n} \circ \theta_t) = f(X_{s_1}, \ldots, X_{s_n}) \circ \theta_t.$$

From here, under the extension theorem, the Markov behavior condition can be rewritten in the following aspect:

$$E(g \circ \theta_t;\, B) = E(E(g \circ \theta_t |\, X_t);\, B), \qquad [1.4]$$

where g is arbitrary $\mathcal{F}$-measurable function on $\mathcal{D}$, whence the relation for conditional averages follows. Let us note that the condition [1.3] can be considered as a special case of conditions [1.4] where $f = I_S$. In this case, the following equality holds

$$E(I_S \circ \theta_t | \cdot) = P(\theta_t^{-1} S | \cdot).$$

1.2.2.2. *Temporally homogeneous Markov process*

A temporally homogeneous Markov process is usually defined in terms of transition functions.

A Markov transition function is called as a function $\widetilde{P}_{s,t}(S \,|\, x)$, where $0 \leq t < s$ and

1) $S \to \widetilde{P}_{s,t}(S \,|\, x)$ is a probability measure on $\mathcal{B}(\mathbb{R})$ for each $s,\, t$ and x;

2) $x \to \widetilde{P}_{s,t}(S \,|\, x)$ is $\mathcal{B}(\mathbb{R})$-measurable function for each $s,\, t$ and S;

3) if $0 \leq t < s < u$, then

$$\widetilde{P}u, t(S \,|\, x) = \int_{-\infty}^{\infty} \widetilde{P}s, t(dy \,|\, x)\, \widetilde{P}u, s(S \,|\, y) \qquad [1.5]$$

for all x and S.

Relationship [1.5] is called the Chapman - Kolmogorov equation.

A Markov transition function $\widetilde{P}_{s,t}(S \,|\, x)$ is said to be temporally homogeneous provided there exists a function $P_t(S \,|\, x)$ $(t > 0,\, x \in \mathbb{R},\, S \in \mathcal{B}(\mathbb{R}))$ such that $\widetilde{P}_{s,t}(S \,|\, x) = P_{s-t}(S \,|\, x)$. For this case, equation [1.5] becomes:

$$P_{s+t}(S \mid x) = \int_{-\infty}^{\infty} P_s(dy \mid x)\, P_t(S \mid y) \tag{1.6}$$

We define the distribution of a temporally homogeneous Markov process to within the initial distribution as a consistent measurable family of measures (P_x) on $\mathcal{F}$, where $P_x(X(0) = x) = 1$, and for any $x \in \mathbb{R}$, $t > 0$, $B \in \mathcal{F}_t$ and $S \in \mathcal{F}$ the following holds:

$$P_x(\theta_t^{-1}(S); B) = E_x(P_{X_t}(S); B), \tag{1.7}$$

and for any measurable function f:

$$E_x(f \circ \theta_t\,; B) = E_x(E_{X_t}(f); B). \tag{1.8}$$

Finite-dimensional distributions of a temporally homogeneous Markov process is constructed from the temporally homogeneous transition functions according to the formula:

$$P_x(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) =$$

$$= \int_{A_1} p_{t_1}(dx_1 |\, x) \int_{A_2} p_{t_2 - t_1}(dx_2 |\, x_1) \times \cdots \cdot \times \int_{A_n} p_{t_n - t_{n-1}}(dx_n |\, x_{n-1})$$

where $p_t(dx_1 |\, x_0)$ is a transition kernel.

However, *a priori* a set of transition functions submitting to coordination condition [1.6] do not necessarily define the probability measure on set of functions with given properties. In a class of Poisson processes, to verify the existence of a process with piece-wise constant trajectories requires a special proof.

1.2.3. *Poisson process*

1.2.3.1. *Poisson distribution*

The Poisson distribution is a discreet probability distribution on set of non-negative integers $\mathbb{Z}_+$ with values

$$p_n = \frac{\mu^n}{n!} e^{-\mu} \qquad (n = 0, 1, 2, \dots),$$

where $\mu > 0$ is the distribution parameter. Let us denote a class of Poisson distribution with a parameter μ as Pois(μ). Thus, from $\xi \in$ Pois(μ) we understand that ξ has Poisson distribution with parameter μ.

It is known that expectation, variance and the third central moment of a Poisson distribution have the same meaning as the parameter of this distribution, i.e.:

$$E\xi = D\xi = E(\xi - E\xi)^3 = \mu.$$

A mode of the Poisson distribution is n_{mod} such that $p_{n_{mod}} \geq p_n$ for each $n \in \mathbb{Z}_+$. This integer is determined by relations

$$p_{n+1}/p_n = \frac{\mu^{n+1}}{(n+1)!} \bigg/ \frac{\mu^n}{n!} = \frac{\mu}{n+1}.$$

1) If μ is an integer $n_1 + 1$, then $p_{n_1+1} = p_{n_1}$; for $n < n_1$, we have $p_{n+1}/p_n = \mu/(n+1) > \mu/(n_1+1) = 1$; this implies that in this case p_n increases; analogously for $n > n_1 + 1$, p_n decreases. Hence, there are two modes: $n_{mod}^{(1)} = n_1$ and $n_{mod}^{(2)} = n_1 + 1$.

2) Let μ be not an integer and $n_1 < \mu < n_1 + 1$; let us assume that $p_{n_1+1} \geq p_{n_1}$; this means that

$$\frac{\mu^{n_1+1}}{(n_1+1)!} \geq \frac{\mu^{n_1}}{n_1!};$$

which implies that $\mu \geq n_1 + 1$; from this contradiction, it follows that $p_{n_1+1} < p_{n_1}$; hence, $n_{mod} = n_1$ is a unique mode of this Poisson distribution.

Generating the function of a Poisson distribution (or corresponding random variable $\xi \in \text{Pois}(\mu)$) is a function of $\alpha \in \mathbb{R}$:

$$Ee^{\alpha\xi} = \sum_{n=0}^{\infty} e^{\alpha n} \frac{\mu^n}{n!} e^{-\mu} = \sum_{n=0}^{\infty} \frac{(\mu e^{\alpha})^n}{n!} e^{-\mu} = \exp\left(-\mu(1-e^{\alpha})\right). \tag{1.9}$$

Let ξ_1 and ξ_2 be independent Poisson variables with parameters μ_1 and μ_2, respectively. Then, the sum of these variables is a Poisson random variable with parameter $\mu_1 + \mu_2$. This can be proved easily by means of a generating function. Using independence, we have:

$$\begin{aligned} E\exp(\alpha(\xi_1+\xi_2)) &= E\exp(\alpha\xi_1)\, E\exp(\alpha\xi_2) \\ &= \exp\left(-(\mu_1+\mu_2)(1-e^{\alpha})\right). \end{aligned}$$

This corresponds to the distribution $\text{Pois}(\mu_1 + \mu_2)$ as the equality is fair at any $\alpha \in \mathbb{R}$.

1.2.3.2. *Poisson process*

A non-decreasing integer random process $(N(t))$ $(t \geq 0)$ with values from set $\mathbb{Z}_+$ is said to be a temporally homogeneous Poisson process if $N(0) = 0$ and if its increments on non-overlapping intervals are independent and have Poisson distributions. That is, there exists such a positive β, called the intensity of process, that $N(t) - N(s) \in$ Pois $(\beta\,(t-s))$ $(0 \leq s < t)$. For $N(t)$, we will also use a label N_t. This process has step-wise trajectories with unit jumps. By the additional definition, such a trajectory is right continuous at point of any jump.

The sequence of the moments of jumps of the process (σ_n) $(n \geq 1)$ completely characterizes a Poisson process. This sequence is called a point-wise Poisson process. Let us designate $T_n = \sigma_n - \sigma_{n-1}$ $(n \geq 1$, $\sigma_0 = 0)$, where (T_n) is a sequence of independent and identically distributed (i.i.d.) random variables with common exponential distribution $P(T_1 > t) = e^{-\beta t}$. Using a shift operator on set of sample trajectories of a Poisson process, it is possible to note that

$$\sigma_{n+1} = \sigma_n + \sigma_1 \circ \theta_{\sigma_n}.$$

A generalization of the above process, the so-called inhomogeneous Poisson process $(N(t))$, is characterized by means of a non-constant function of intensity $\beta(t) \geq 0$, setting Poisson distributions as independent increments, for $0 \leq s < t$

$$P(N(t) - N(s) = n) = \frac{\mu(s,t)^n}{n!} e^{-\mu(s,t)} \qquad (n \in \mathbb{Z}_+),$$

where

$$\mu(s,t) = \int_s^t \beta(u)\, du.$$

Gaps between unit jumps in this process are not independent and are not identically distributed.

1.2.3.3. *Stochastic continuity*

A random process $X(t)$ is called stochastically continuous from the right at a point t_0, if for any positive ϵ and $t > t_0$

$$\lim_{t \to t_0} P(|X(t) - X(t_0)| > \epsilon) = 0.$$

Similarly, a stochastic continuity from the left is defined. Bilateral stochastic continuity is both from the left and right. For a Poisson process with locally limited intensity $\beta(t)$, the stochastic continuity from the right at a point t_0 follows from inequality:

$$\mathbf{P}(X(t) - X(t_0) > \epsilon) \leq \mathbf{P}(X(t) - X(t_0) > 0) =$$

$$= 1 - \exp\left(-\int_{t_0}^t \lambda(s)\, ds\right) = \int_{t_0}^t \lambda(s)\, ds + o(t - t_0).$$

Stochastic continuity from the left is proved analogously.

1.2.3.4. *Composite Poisson process*

A random process (X_t) $(t \geq 0)$ is called a temporally homogeneous composite Poisson process if it is defined by means of temporally homogeneous Poisson process $(N(t))$ $(t \geq 0)$ and a sequence (U_n) $(n \geq 1)$ of i.i.d. random variables, where $(N(t))$ and (U_n) are independent. By definition:

$$X_t = \sum_{n=1}^{N_t} U_n \qquad (t \geq 0).$$

Let us designate $B(x) = P(U_1 \leq x)$ as the cumulative distribution function of U_1, $\mu_B = EU_1$ the expectation of U_1 and $\mu_B^{(n)} = EU_1^n$ the nth moment of U_1 $(n \geq 1)$. The sequence of jump times, (σ_n) $(n \geq 1)$, of the composite Poisson process coincides with sequence of jumps of the original Poisson process and hence it is possible to note that:

$$X_t = \sum_{n=1}^{\infty} U_n \, I_{\{\sigma_n \leq t\}}.$$

From here the formula for an average follows:

$$EX_t = \mu_B \sum_{n=1}^{\infty} P(\sigma_n \leq t) = \mu_B \, EN_t = \mu_B \, \beta t.$$

In the next chapters, this process will be considered in more detail.

1.2.4. *Gamma process*

A gamma function is called an integral depending on parameter $m > 0$,

$$\Gamma(m) = \int_0^{\infty} x^{m-1} e^{-x} \, dx.$$

For a case where $m = 1, 2, 3, \ldots$ we have $\Gamma(m) = (m-1)!$.

The non-negative random variable X has a gamma distribution if its distribution density resembles

$$f_X(x) = \frac{\delta}{\Gamma(\gamma)}(x\delta)^{\gamma-1}e^{-x\delta} \quad (x > 0),$$

where δ is a scale parameter and γ is a form parameter of the distribution. We designate such a class of random variables as Gam(γ, δ).

At $\gamma = 1$, the gamma distribution coincides with a exponential distribution with parameter δ. For the integer $\gamma = n$, where $n \geq 2$, the gamma distribution is called the Erlang distribution. It is an n-fold convolution of exponential distributions.

Let us obtain a Laplace transformation of a gamma distribution density:

$$\int_0^\infty e^{-\lambda x}\frac{\delta}{\Gamma(\gamma)}(x\delta)^{\gamma-1}e^{-x\delta}\,dx =$$

$$= \left(\frac{\delta}{\lambda+\delta}\right)^\gamma \int_0^\infty e^{-(\lambda+\delta)x}\frac{(\lambda+\delta)}{\Gamma(\gamma)}(x(\lambda+\delta))^{\gamma-1}\,dx = \left(\frac{\delta}{\lambda+\delta}\right)^\gamma.$$

From here it follows that the sum of two independent random variables $X_1 \in$ Gam (γ_1, δ) and $X_2 \in$ Gam (γ_2, δ) is a random variable from a class Gam $(\gamma_1 + \gamma_2, \delta)$.

A process $X(t)$ $(t \geq 0)$, possessing the following properties:

1) $X(0) = 0$, process trajectories do not decrease and are continuous from the right;

2) the process has independent increments;

3) at any $s \geq 0$ and $t > s$ the increment $X(t) - X(s)$ belongs to class Gam $(\gamma(t-s), \delta)$;

is called a gamma process with parameters γ and δ.

Let us prove that a homogeneous gamma process is stochastically continuous in any point of the area of the representation. Designate $\gamma_1 = \gamma\,(t{-}t_0)$. We have at $t > t_0$ and $0 < \gamma_1 < 1$ (without loss of generality):

$$P(|X(t){-}X(t_0)| > \epsilon) = P(X(t){-}X(t_0) > \epsilon) = P(X(t{-}t_0) > \epsilon) =$$

$$= \int_\epsilon^\infty \frac{\delta}{\Gamma(\gamma_1)}\,(\delta\,x)^{\gamma_1-1}\,e^{-\delta\,x}\,dx = \frac{1}{\Gamma(\gamma_1)}\int_{\epsilon\delta}^\infty y^{\gamma_1-1}\,e^{-y}\,dy \le$$

$$\le \frac{1}{\Gamma(\gamma_1)}\,(\epsilon\delta)^{\gamma_1-1}\int_{\epsilon\delta} e^{-y}\,dy \le \frac{1}{\Gamma(\gamma_1)}\,(\epsilon\delta)^{\gamma_1-1}.$$

From our definition of the function Γ follows that $\Gamma(z) \to \infty$ at $z \downarrow 0$. Hence, the inequality right member aspires to zero. The stochastic continuity of the gamma process is proved. □

Finite-dimensional distributions of a gamma process possess good analytical properties. The practical application of the gamma process model is hindered a little by the property of its sample trajectories because these trajectories may have ruptures such as in short intervals. It is possible to tell that trajectories "consist only of positive jumps". The inverse gamma process is more convenient for physical interpretation.

1.2.5. *Inverse gamma process*

The process $X(t)$ $(t \ge 0)$ is known as the inverse gamma process if it possesses the following properties:

1) $X(0) = 0$, process trajectories do not decrease and are continuous with probability 1;

2) process $X(t)$ possesses a Markov property with respect to the time of the first exit from any interval $[0, u)$;

3) the inverse process for the process $X(t)$, i.e. the process:

$$Y(u) \equiv \inf\{t:\ X(t) \ge u\} \qquad (u > 0),$$

is a gamma process with some parameters γ and δ.

Sample trajectories of this process are exotic enough. Almost everywhere (concerning a Lebesgue measure) for domain of definition, these trajectories are constant (they have zero derivatives). The increase of these trajectories on an interval $[0, \infty)$ is ensured with the presence of a non-enumerable set of points of growth (like in a Cantor curve), filling any interval from the moment of first reaching level u_1 (the random moment $Y(u_1)$) until moment $Y(u_2)$, where $0 \leq u_1 < u_2$. Intervals of constancy of a sample trajectory of $X(t)$ correspond to jumps of the trajectory $Y(u)$. It is important to note that the beginning time of each interval of constancy of the process $X(t)$ is not a Markov time. Thus, it is an example of a real stopping time, which is not a Markov time.

In modern terminology, processes such as the inverse gamma process are known as continuous semi-Markov processes (see [HAR 07]).

1.2.6. *Renewal process*

Renewal theory is commonly used in risk theory. For example, a renewal process can serve as a model when entering sequences of claim times into an insurance company (instead of using a Poisson process). Renewal equations arise during the analysis of probability of ruin. The asymptotics of a solution of such an equation allows to express probability of ruin in case of a high initial capital.

1.2.6.1. *Renewal process*

A simple temporally homogeneous renewal process is said to be a non-decreasing integer random process $(N(t))$ $(t \geq 0)$. It is assumed that $N(0) = 0$, where a process has jumps of unit magnitude, and where distances in time (T_n) $(n \geq 1)$ between the neighboring jumps (the renew times) are i.i.d. positive random variables, and at any jump time a sample trajectory of the process is continuous from the right. Such a process is determined by a distribution function of T_1, such as , $F(x) = P(T_1 \leq x)$ $(x \geq 0)$. Magnitude T_k is interpreted as distance in time between the $(k-1)$th and kth process jumps, thus $\sigma_n = \sum_{k=1}^{n} T_k$ is a time of the nth renew.

A temporally homogeneous Poisson process is a partial case of renewal process. In the Poisson case, $F(x) \equiv P(T_1 \leq x) = 1 - e^{-\beta x}$ $(x \geq 0)$ for some $\beta > 0$. Also for a Poisson process, we will sometimes use the notation N_t instead of $N(t)$.

1.2.6.2. *Renewal equation*

Outcomes of renewal theory are used in risk theory mainly in connection with a solution of so-called renew equations. First, we consider the so-called renewal function.

Renewal function $H(t)$ $(t \geq 0)$ is expressed as:

$$H(t) \equiv 1 + EN_t = 1 + \sum_{n=1}^{\infty} P(\sigma_n \leq t) = \sum_{n=0}^{\infty} F^{(n)}(x),$$

where $F^{(n)}$ is n-fold convolution of distribution functions F:

$$F^{(n)}(x) = \int_0^x F^{(n-1)}(x-y)\, dF(y) \quad (n \geq 1).$$

$F^{(0)}(x) = I_{[0,\infty)}(x)$, "zero convolution". It corresponds to the sum n of i.i.d. random variables; we will also use the notation H_t in addition to $H(t)$. Using a permutability of summation with convolution operation, we obtain the equation:

$$H(t) = 1 + \int_0^t H(t-x)\, dF(x).$$

For the given cumulative distribution function F on interval $[0, \infty)$, and a known function $y(t)$ $(t \geq 0)$, the equation

$$Z(t) = y(t) + \int_0^t Z(t-x)\, dF(x) \qquad [1.10]$$

is a renewal equation concerning unknown function $Z(t)$. The solution of the renewal equation always exists and is unique:

$$Z(t) = \int_0^t y(t-x)\, dH(x).$$

It is easy to prove this by substituting the right-hand-side of equation [1.10], where the function expressed, by the whole equation (i.e. iterating the equation).

Analytical expression for function H_t is known only in exceptional cases. For example, if F is an exponential distribution function with parameter β, then $H_t = 1 + \beta t$, in the a case of a Poisson process. The basic outcome of the theory is connected with an asymptotics of the renewal function and a limit of a solution of the equation [1.10].

THEOREM 1.3.– *Elementary renewal theorem*

$$\frac{H_t}{t} \to \frac{1}{ET_1} \quad (t \to \infty).$$

THEOREM 1.4.– *Blackwell theorem*

$$H_{t+s} - H_t \to \frac{s}{ET_1} \quad (t \to \infty).$$

THEOREM 1.5.– *Smith theorem*

For any function $y(t)$ immediately integrable by Riemann:

$$\int_0^t y(t-x)\, dH(x) \to \frac{1}{ET_1} \int_0^\infty y(t)\, dt \quad (t \to \infty).$$

PROOF.– (for all the three theorems, see Feller [FEL 66]). An immediately integrable function by Riemann on an interval (a, b) is called a function f for which there exist identical limits

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} m_k,$$

$$m_k = \min\left\{ f(x) : \frac{b-a}{n}(k-1) \le x \le \frac{b-a}{n}k \right\},$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} M_k,$$

$$M_k = \max\left\{ f(x) : \frac{b-a}{n}(k-1) \le x \le \frac{b-a}{n}k \right\}.$$

A function for which this condition is fulfilled for all its restrictions on final intervals is an immediately integrable function by Riemann on an infinite interval. An example of such function is any monotone function integrable by Riemann.

1.2.6.3. *Direct and inverse renewal times*

In risk theory, properties of the so-called direct and inverse renewal times are used. They are as follows:

$$\zeta(t) \equiv \sigma_{N_t+1} - t, \quad \eta(t) \equiv t - \sigma_{N_t}.$$

We have:

$$P(\eta(t) > x, \zeta(t) > y) =$$
$$= \sum_{n=1}^{\infty} \int_0^t P(t - \sigma_{n-1} > x, \sigma_n - t > y, \sigma_1 \in ds) =$$
$$= I_{t>x} P(\sigma_1 - t > y) + \sum_{n=2}^{\infty} \int_0^t P(t - \sigma_{n-1} > x, \sigma_n - t > y, \sigma_1 \in ds).$$

Using representation $\sigma_k = \sigma_1 + \sigma_{k-1} \circ \theta_{\sigma_1}$ $(k \geq 2)$ and property of a renewal process concerning the time σ_1, we know that this expression is:

$$I_{t>x} P(\sigma_1 - t > y) +$$
$$+ \sum_{n=2}^{\infty} \int_0^t P(t - s - \sigma_{n-2} \circ \theta_{\sigma_1} > x, s + \sigma_{n-1} \circ \theta_{\sigma_1} - t > y, \sigma_1 \in ds) =$$
$$= I_{t>x} P(\sigma_1 - t > y) +$$
$$+ \sum_{n=2}^{\infty} \int_0^t P(t - s - \sigma_{n-2} > x, s + \sigma_{n-1} - t > y) P(\sigma_1 \in ds).$$

Designating $P(\eta(t) > x, \zeta(t) > y) = Z(t)$, $I_{t>x} P(\sigma_1 - t > y) = y(t)$, we come to the equation [1.10] for which limit of a solution as $t \to \infty$ is:

$$\frac{1}{\mu} \int_0^{\infty} I_{t>x} P(T_1 - t > y)\, dt = \frac{1}{\mu} \int_x^{\infty} P(T_1 > y + t)\, dt = \frac{1}{\mu} \int_{x+y}^{\infty} \overline{F}(t)\, dt,$$

where $\mu = ET_1$ and $\overline{F}(t) = 1 - F(t)$. From here, both variables $\eta(t)$ and $\zeta(t)$ have the same limit distribution:

$$\widetilde{F}(x) = \frac{1}{\mu} \int_0^x \overline{F}(t)\, dt.$$

In risk theory, the following properties of the variable $\zeta(t)$ are useful.

THEOREM 1.6.– *Property of direct renewal time*

For renewal process, the following limits are true:

a) $\dfrac{\zeta(t)}{t} \xrightarrow{\text{a. s.}} 0 \quad (t \to \infty);$

b) $\dfrac{E\zeta(t)}{t} \to 0 \quad (t \to \infty).$

PROOF.– (see, for example, Asmussen [ASM 00]).

2

Cramér-Lundberg Model

2.1. Infinite horizon

2.1.1. *Initial probability space*

The natural initial space of elementary Ω events in the Cramér-Lundberg model is defined as a set of all sequences of view $\omega = (t_n, x_n)$, where $0 \le t_1 \le t_2 \le \dots$ and $t_n \to \infty$ as $n \to \infty$, and real x_n $(n = 1, 2, \dots)$. That is, in this case an initial probability measure where P is defined as a distribution of a random sequence of pairs (σ_n, U_n), for which (σ_n) is a sequence of jump points of a Poisson process (N_t) with intensity $\beta > 0$. The point σ_n is interpreted as a moment of the nth claim arrival in an insurance business. (U_n) is a sequence of claim sizes; the sequence (U_n) constitutes an i.i.d. sequence of non-negative random variables with a common distribution function $B(x)$. The claim size sequence (U_n) and the claim arrival sequence (σ_n) are assumed to be mutually independent. The random variables σ_n, U_n are considered as functions of $\omega \in \Omega$, and events connected with these random variables are measured by using the measure P. In particular,

$$B(x) = P(U_1 \le x) \quad (x \ge 0,\ B(0) = 0),$$

$$P(T_n > t) = e^{-\beta t},$$

where $T_n = \sigma_n - \sigma_{n-1}$ $(n \ge 1, \sigma_0 = 0)$.

Stochastic Risk Analysis and Management, First Edition. Boris Harlamov.

2.1.2. *Dynamics of a homogeneous insurance company portfolio*

On the basis of these elements, a part-wise linear random process (R_t) $(t \geq 0)$ is defined as follows:

$$R_t = u + p\,t - \sum_{k=1}^{Nt} U_k \quad (t \geq 0). \qquad [2.1]$$

This determines a reserve capital of the insurance company, where u is an initial capital of the company and p is a premium rate. Moreover, the sequence (σ_n) is a point Poisson process with some intensity $\beta > 0$ and (N_t) is the corresponding proper Poisson process. Thus, the sum $A_t \equiv \sum_{k=1}^{N_t} U_k$ is the corresponding composite Poisson process. The process R_t is a homogeneous process with independent increments. It means that this is a Markov process that is homogeneous in time and space. An analysis of this process composes the main content of investigation of the Cramér-Lundberg model. In this course, we will consider some generalizations of this model as well. In particular, it will be a model with the premium depending on the current capital of a company– a Markov process homogeneous in time but not in space.

With every initial capital u, we connect a probability measure P_u on a set $\mathcal{D}_0$ of part-wise linearly increasing trajectories ξ with no positive jumps and continuous from the right at points of discontinuity. Relations between measures P and P_u can be described as follows: denote Ω the set of all sequences (t_n, x_n) with some natural distance (metric) on this set, which generates a sigma-algebra $\mathcal{F}$. Let X_u be a map $\Omega \to \mathcal{D}_0$ such that:

$$\xi \equiv X_u((t_n, x_n)_1^\infty) = \left(u + pt - \sum_{n=1}^{\infty} x_n I_{t_n \leq t} \right)_{t \geq 0},$$

where I_S is an indicator of the subset $S \subset \Omega$. Thus, we have:

$$\{\xi \in A\} = \{X_u(\omega) \in A\} \equiv X_u^{-1} A$$

as a measurable subset of Ω, where A is a measurable subset of the set $\mathcal{D}_0$. From here,

$$P_u(A) \equiv P\{\xi \in A\} = P(X_u(\omega) \in A) \equiv (P \circ X_u^{-1})(A).$$

It means that this expression defines a measure $P_u \equiv P \circ X_u^{-1}$. It is the so-called induced probability measure corresponding to the original measure P on the set of sequences and the map X_u. In addition, for any measurable A the function $P_u(A)$ is measurable as a function of u.

Evidently, a shift operator θ_t $(t \geq 0)$ maps the set $\mathcal{D}_0$ on itself. Consider on this set, a consistent family of measures (P_u) $(u \geq 0)$ of a temporally homogeneous Markov process. This means that for any $A \in \mathcal{F}$ and $B \in \mathcal{F}_t$,

$$P_u(\theta_t^{-1}A, B) = E_u(P_{R_t}(A);\, B).$$

By using properties of conditional probabilities, it can be shown that this equality is equivalent to a P_u-a.s. equality of two random variables:

$$P_u(\theta_t^{-1}A|\,\mathcal{F}_t) = P_{R_t}(A),$$

where $\mathcal{F}_t$ is a sigma-algebra generated by all R_s $(s \leq t)$. Besides, the Cramér-Lundberg process is homogeneous in space. This means that for any $u_1, u_2 \geq 0$, $0 \leq t_1 < t_2$ and $x \in \mathbb{R}$:

$$P_{u_1}(R_{t_2} - R_{t_1} < x) = P_{u_2}(R_{t_2} - R_{t_1} < x).$$

It follows that all properties of the process depending on increments relative to the initial state can be evaluated by using the measure P_0 (of the process with zero initial state). A common rule of varying an integrand expression for a process that is homogeneous in space reduces to the following varying of operators R_t and σ_Δ:

$$E_x(f(R_t,\, \sigma_{(a,b)})) = E_0(f(R_t + x,\, \sigma_{(a-x,b-x)})),$$

where f is an arbitrary measurable function. The same is true for the equality relative to a function f with any finite number of similar arguments. In particular, the measure P_0 determines a distribution of the process (S_t), where

$$S_t = \sum_{n=1}^{N_t} U_n - pt,$$

which is called dynamics of losses and is equal to the increment with the opposite sign $S_t = R_0 - R_t$ (Figure 2.1).

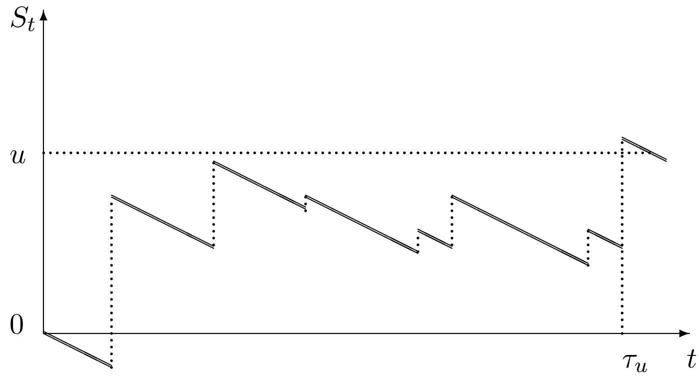

Figure 2.1. *Loss process in Cramér-Lundberg model*

REMARK–. Along with the measure P on the set of sequences, the family of measures (P_u) on the set of trajectories is defined. Applicability of this family of measures has certain advantages while analyzing homogeneous Markov processes. However, such a two-fold interpretation implies some questions. The first is where to use a measure on Ω, and $\mathcal{D}_0$. For example, we have $P_u(R_t \in S) = P(u + pt - A_t \in S)$. It would be a mistake to write $P(R_t \in S)$ in the second case, because the sense that applies to the denotation R_t is not clear: is it only the meaning of a trajectory at the point t or is it a the denotation of a function of ω (a sequence of view [2.1])? Such a two-fold meaning is admissible if an event is exclusive of an initial capital u. For example,

$$P_u(S_t \in S) = P_0(-R_t \in S) = P\left(\sum_{n=1}^{N_t} U_n - pt \in S\right).$$

In this case, it is not a mistake to write $P(S_t \in S)$ while understanding S_t as a function of ω.

2.1.3. *Ruin time*

It can be shown [DYN 63] that the process (R_t) is a strong Markov process (exhibiting the Markov property with respect to any Markov time). In particular, it possesses the Markov property with respect to the first exit time from an open interval. Besides, this process has the Markov property with respect to the first exit time from an interval closed from the left, because an exit from such an interval is possible only with a jump, and a meaning of the process at the jump time does not belong to this interval. A ruin time

$$\tau_0(\xi) \equiv \sigma_{[0,\infty)}(\xi) = \inf\{t \geq 0 : \xi(t) \notin [0, \infty)\}$$

is such a time. At the ruin time, the meaning of the process passes with a jump to the negative part of the real line. The distribution of this time essentially depends on the initial capital of the company.

Let us denote

$$\psi(u) = P_u(\tau_0 < \infty), \quad \psi(u, T) = P_u(\tau_0 \leq T).$$

Obtaining an explicit meaning of these probabilities is possible only in special cases. The first non-trivial results were obtained from asymptotics of the function $\psi(u)$ as $u \to \infty$.

2.1.4. *Parameters of the gain process*

Let us denote

$$\mu_B^{(n)} = EU_1^n \quad (n \geq 1), \quad \mu_B = EU_1, \quad \varrho = \beta\,\mu_B.$$

While deriving of the following formulas, a property of conditional expectations $Ef = EE(f|\,g)$ is used, where f and g are two random variables. By denoting $A_t = \sum_{k=1}^{N_t} U_k$, we have:

$$\begin{aligned} E_u R_t &= u + pt - EE(A_t|\,N_t) = \\ &= u + pt - EN_t\,EU_1 = u + pt - \beta t \mu_B = u + (p - \varrho)t, \quad [2.2] \end{aligned}$$

$$\begin{aligned}
D_u R_t \equiv E_u(R_t - E_u R_t)^2 &= DA_t = EA_t^2 - (EA_t)^2 = \\
&= EE(A_t^2|\, N_t) - (EA_t)^2 = \\
&= EE(A_t^2|\, N_t) - E(E(A_t|\, N_t))^2 + E(E(A_t|\, N_t))^2 \\
&\quad - (EE(A_t|\, N_t))^2 = ED_{N_t} A_t + DE(A_t|\, N_t) \\
&= E(N_t\, DU_1) + D(N_t\, EU_1) = \\
&= DU_1 EN_t + (EU_1)^2 DN_t = \mu_B^{(2)}\, \beta t.
\end{aligned} \qquad [2.3]$$

From the definition of the Poisson process, it follows that for $0 < z < 1$:

$$Ez^{N_t} = \sum_{n=0}^{\infty} z^n P(N_t = n) = e^{-\beta t(1-z)}.$$

For Poisson distributions and some others, this series converges into a finite limit when $0 < z < 1 + \epsilon$, for some $\epsilon > 0$. Furthermore,

$$\begin{aligned}
Ez^{pt-A_t} &= z^{pt} \sum_{n=0}^{\infty} E \exp\left(\sum_{k=1}^{n} U_k\right) \frac{(\beta t)^n}{n!}\, e^{-\beta t} = \\
&= z^{pt} \sum_{n=0}^{\infty} (E \exp(-U_1))^n \frac{(\beta t)^n}{n!}\, e^{-\beta t} \\
&= z^{pt} \exp(\beta t(E \exp(-U_1) - 1)).
\end{aligned}$$

Denote $z = e^{-\alpha}$. Thus, we obtain a useful formula

$$Ee^{-\alpha(Rt)} = e^{-\alpha u} Ee^{-\alpha(pt-At)} = e^{-\alpha u + t\, \kappa(\alpha)}, \qquad [2.4]$$

where

$$\kappa(\alpha) = \beta(\widehat{B}(\alpha) - 1) - p\, \alpha, \qquad [2.5]$$

where $\widehat{B}(\alpha) \equiv \int_0^{\infty} e^{\alpha\, x}\, dB(x) < \infty$ for some positive α. As it will be shown later, the function $\kappa(\alpha)$ plays a key role in the Cramér-Lundberg theory of ruin.

2.1.5. *Safety loading*

Let us consider a coefficient $\eta = p/\varrho - 1$ which is called safety loading. The asymptotics of the ruin probability depends on a sign of η.

THEOREM 2.1.– *(on the safety loading)* If $u \geq 0$ as $t \to \infty$ then:

a) $\dfrac{R_t}{t} \to p - \varrho \; P_u$-a. s.;

b) if $\eta > 0$, then $R_t \to \infty \quad P_u$-a. s.;

c) if $\eta < 0$, then $R_t \to -\infty \quad P_u$-a. s.;

d) if $\eta = 0$, then $\liminf_{t\to\infty} R_t = -\infty$, $\limsup_{t\to\infty} R_t = \infty$ P_u-a.s.

PROOF.– a) Due to the Markov property of exponential distribution, the process R_t is a process with independent increments; it implies that the strong law of large numbers can be applied to this process: P_u-a.s.

$$\frac{R_t}{t} \to \lim \frac{ER_t}{t} = p - \varrho;$$

b) and c) immediately follow from a); the assertion d) follows from the theory of a random walk with zero expectation [SPI 76]. □

THEOREM 2.2.– *(central limit theorem)* For any $u \geq 0$, as $t \to \infty$:

$$\frac{R_t - t(p-\varrho)}{\sigma\sqrt{t}} \xrightarrow{distr} Z, \quad P_u\text{-a. s.}$$

where $\sigma^2 = \beta\mu_B^{(2)}$ and Z is a standard normal random variable.

PROOF.– Because R_t is a process with independent increments, its meaning at times $t = n\,h$ $(h > 0,\ n = 1, 2, \dots)$ can be represented as u plus summation of i.i.d. random variables to which the central limit theorem is applicable as $n \to \infty$:

$$\frac{R_{nh} - nh(p-\varrho)}{\sqrt{nh\beta\mu_B^{(2)}}} \xrightarrow{distr} Z \quad (n \to \infty).$$

The general case reduces to application by substituting the following estimate in this formula:

$$R_{(n+1)h} - ph \le R_t \le R_{nh} + ph$$

for $nh \le t \le (n+1)h$. □

2.1.6. *Pollaczek-Khinchin formula*

2.1.6.1. *Ladder process*

According to the definition of the ruin time $\tau_0 \equiv \sigma_{[0,\infty)}$ and the ruin probability $\psi(u) = P_u(\tau_0 < \infty)$, we represent this probability in terms of the so-called ladder processes:

$$M(t) = \sup_{0\le s\le t} (R_0 - R_t)$$

(this process has to be non-decreasing by tradition). It is a part-wise constant process. Its jump times can be only positive and simultaneous with some jumps of the original Poisson process. Let α_1 be the first jump time of the process. In this case, $R_{\alpha_1} < R_0$ and $R_s \ge R_0$ for all $s > \alpha_1$. The time α_1 is a Markov time with respect to the natural filtration $(\mathcal{F}_t)_{t\ge 0}$. Due to that process, R_t possesses the Markov property relative to this time. According to the definition, $\alpha_1(\xi) = \inf\{t \ge 0 : \xi(t) < \xi(0)\}$. Consequently, it can be represented as $\alpha_1 = \sigma_{[R_0,\infty)}$. If the nth jump time of the process $M(t)$ is determined and finite, then $\alpha_{n+1} = \alpha_n \dot{+} \alpha_1 \equiv \alpha_n + \alpha_1 \circ \theta_{\alpha_n}$ $(n \ge 1)$. If $\eta > 0$, the ladder $M(t)$ has only a finite number of steps. It means a finite number of such n, for which $\alpha_n < \infty$, and sequently $M(\infty) < \infty$. In this case, $\psi(u) = P_u(M(\infty) > u)$. Homogeneity in space of the process (R_t) implies that $\psi(u) = P_0(M(\infty) > u)$.

Let us consider distribution of $M(\infty)$. By using the strong Markov property of the process R_t and setting $\alpha_0 = 0$, we have for $x > 0$:

$$P_0(M(\infty) \le x) = \sum_{n=0}^{\infty} P_0(\alpha_n < \infty,\ -R_{\alpha_n} \le x,\ \alpha_{n+1} = \infty) =$$

$$= P_0(\alpha_1 = \infty) + \sum_{n=1}^{\infty} P_0(\alpha_n < \infty,\, R_{\alpha_n} \geq -x,\, \alpha_1 \circ \theta_{\alpha_n} = \infty) =$$

$$= P_0(\alpha_1 = \infty) + \sum_{n=1}^{\infty} E_0(P_{R_{\alpha_n}}(\alpha_1 = \infty);\, \alpha_n < \infty,\, R_{\alpha_n} \geq -x) =$$

$$= P_0(\alpha_1 = \infty) + \sum_{n=1}^{\infty} P_0(\alpha_n < \infty,\, R_{\alpha_n} \geq -x) P_0(\alpha_1 = \infty).$$

We will substitute $M_n = R_0 - R_{\alpha_n}$. Then, by using the formula:

$$R_{\tau_1 \dot{+} \tau_2} = R_{\tau_2} \circ \theta_{\tau_1}$$

which is natural generalization of the formula $R_{t+s} = R_s \circ \theta_t$ [HAR 07], we have for $n \geq 1$:

$$P_0(\alpha_n < \infty,\, M_n \leq x) = P_0(\alpha_n < \infty,\, R_{\alpha_n} \geq -x) =$$
$$= P_0(\alpha_1 < \infty, \alpha_{n-1} \circ \theta_{\alpha_1} < \infty, R_{\alpha_{n-1}} \circ \theta_{\alpha_1} \geq -x) =$$
$$= E_0(P_{R_{\alpha_1}}(R_{\alpha_{n-1}} \geq -x, \alpha_{n-1} < \infty);\, \alpha_1 < \infty) =$$
$$= \int_{-\infty}^{0} P_y(R_{\alpha_{n-1}} \geq -x, \alpha_{n-1} < \infty) P_0(R_{\alpha_1} \in dy,\, \alpha_1 < \infty) =$$
$$= \int_{-\infty}^{0} P_y(R_{\alpha_{n-1}} - R_0 \geq -x - y, \alpha_{n-1} < \infty) P_0(R_{\alpha_1} \in dy,\, \alpha_1 < \infty) =$$
$$= \int_{-\infty}^{0} P_0(M_{n-1} \leq x + y, \alpha_{n-1} < \infty) P_0(-M_1 \in dy,\, \alpha_1 < \infty) =$$
$$= \int_{0}^{\infty} P_0(M_{n-1} \leq x - y, \alpha_{n-1} < \infty) P_0(M_1 \in dy,\, \alpha_1 < \infty).$$

By substituting $G_+(y) = P_0(M_1 \leq y,\, \alpha_1 < \infty)$ and defining a convolution in standard manner, we obtain:

$$P_0(\alpha_n < \infty,\, R_{\alpha_n} \geq -x) = G_+^{(n)}(x) = \int_0^x G_+^{(n-1)}(x-y)\, dG_+(y),$$

and derive:

$$P_0(M(\infty) \le x) = (1 - G_+(\infty)) \sum_{n=0}^{\infty} G_+^{(n)}(x),$$

where $G_+^{(0)} = I_{[0,\infty)}$. From here,

$$P_0(M(\infty) < \infty) = (1 - G_+(\infty)) \sum_{n=0}^{\infty} G_+^{(n)}(\infty).$$

For $\eta > 0$, this probability equals to 1, which implies the Pollaczek-Khinchin formula:

$$\begin{aligned}\psi(u) = P_0(M(\infty) > u) = 1 - P_0(M(\infty) \le u) = \\ = (1 - G_+(\infty)) \sum_{n=1}^{\infty} (G_+^{(n)}(\infty) - G_+^{(n)}(x)) = \\ = (1 - G + (\infty)) \sum_{n=1}^{\infty} \overline{G_+^{(n)}}(x)),\end{aligned} \quad [2.6]$$

where $\overline{G_+^{(n)}}(x)) = G_+^{(n)}(\infty) - G_+^{(n)}(x)$.

2.1.7. *Sub-probability distribution* G_+

By substituting $\Delta_{n,k}N = N_{k/n} - N_{(k-1)/n}$, $\Delta_{n,k}R = R_{k/n} - R_{(k-1)/n}$, we have for $x > 0$:

$$P_0(M_1 > x,\, \alpha_1 < \infty) = P_0(R_{\alpha_1} < -x,\, \alpha_1 < \infty) =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \int_0^{\infty} P_0(R_s \ge 0\, (\forall s < (k-1)/n),\, R_{(k-1)/n} \in dy,$$

$$\Delta_{n,k}N \ge 1,\, \Delta_{n,k}R < -(y + x)) =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \int_0^{\infty} P_0(R_s \ge 0\, (\forall s < (k-1)/n),\, R_{(k-1)/n} \in dy) \times$$

$$\times P_0(\Delta_{n,1}N \ge 1,\, \Delta_{n,1}R < -(x + y)) =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \int_0^{\infty} P_0(R_s \geq 0\,(\forall s < (k-1)/n),\, R_{(k-1)/n} \in dy) \times$$

$$\times\, (\beta\overline{B}(x+y)/n + o(1/n)) =$$

$$= \int_0^{\infty} \int_0^{\infty} P_0(R_s \geq 0\,(\forall s < t),\, R_t \in dy)\beta\overline{B}(x+y)\,dt =$$

$$= \int_0^{\infty} \beta\overline{B}(x+y)\,Q(dy),$$

where $Q(A) = \int_0^{\infty} P_0(R_s \geq 0\,(\forall s < t),\, R_t \in A)\,dt \quad (A \subset [0,\infty))$.

Let us consider the measure $Q(A)$. Denote $R_s^* = R_t - R_{t-s}$ $(0 \leq s \leq t)$. We have $R_0^* = 0$ and $R_t^* = R_t$ on the set $\{R_0 = 0\}$. Besides, due to this transformation of the process on interval $[0, t]$ it corresponds to the inversion of time and does not change the distribution of the composite Poisson process on this interval. Thus, we have:

$$P_0(R_s \geq 0\,(\forall s < t),\, R_t \in A) = P_0(R_s^* \geq 0\,(\forall s < t),\, R_t^* \in A) =$$

$$= P_0(R_t - R_{t-s} \geq 0\,(\forall s < t),\, R_t \in A) =$$

$$= P_0(R_t - R_s \geq 0\,(\forall s < t),\, R_t \in A) =$$

$$= P_0(R_t \geq R_s\,(\forall s < t),\, R_t \in A) = E_0 I_Y,$$

where $Y \equiv \{R_t \geq R_s\,(\forall s < t),\, R_t^* \in A\}$. It follows that:

$$Q(A) = E_0 \int_0^{\infty} I_{\{R_t \geq R_s\,(\forall s<t),\, R_t \in A\}}\,dt = |A|/p,$$

where $|A|$ is Lebesgue measure of A. The latter equality can be understood from Figure 2.2.

While restoring the distribution G_+,we have:

$$P_0(M_1 > x,\, \alpha_1 < \infty) = \frac{\beta}{p} \int_0^{\infty} \overline{B}(x+y)\,dy = \frac{\beta}{p} \int_x^{\infty} \overline{B}(y)\,dy.$$

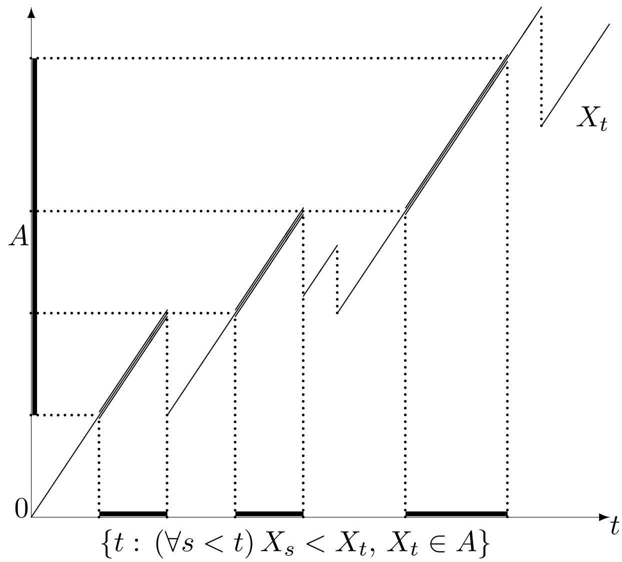

Figure 2.2. *Construction of measure* $Q(A)$

From here,

$$P_0(\alpha_1 < \infty) = \frac{\beta}{p} \int_0^\infty \overline{B}(y)\, dy = \frac{\beta \mu_B}{p},$$

and by using denotation $\varrho = \beta\, \mu_B$, we obtain

$$G_+(x) = \frac{\varrho}{p\, \mu_B} \int_0^x \overline{B}(y)\, dy. \qquad [2.7]$$

Let us denote $B_0(x) = \mu_B^{-1} \int_0^x \overline{B}(y)\, dy$ (probability distribution) and then, the Pollaczek-Khinchin formula obtains its final view

$$\psi(u) = \left(1 - \frac{\varrho}{p}\right) \sum_{n=1}^{\infty} \left(\frac{\varrho}{p}\right)^n \overline{B_0^{*n}}(u). \qquad [2.8]$$

2.1.8. *Consequences from the Pollaczek-Khinchin formula*

2.1.8.1. *Zero initial capital*

In terms of the Cramér-Lundberg model, it is not necessary to be ruined if the initial capital is equal to zero. In this case, if $\eta > 0$, then

$$\psi(0) = (1 - \varrho/p) \sum_{k=1}^{\infty} (\varrho/p)^n = \varrho/p.$$

2.1.8.2. *Exponential distribution B*

Let $B(x) = 1 - e^{-\delta x}$. Then, $B_0(x) = B(x)$. It is well known that $B^{(n)}(x)$ represents a distribution function of gamma-distribution with form parameter n and scale parameter δ (another name is the Erlang distribution of order n). If $b_n(x)$ is a density of this distribution, then

$$b_n(x) = b^{(n)}(x) = \frac{\delta}{\Gamma(n)} (\delta x)^{n-1} e^{-\delta x},$$

where $\Gamma(n) = (n - 1)!$. By using permutability of summing and integration, we obtain:

$$\begin{aligned}\psi(u) &= (1 - \varrho/p) \sum_{n=1}^{\infty} (\varrho/p)^n \int_u^{\infty} \delta e^{-x\delta} \frac{(x\delta)^{n-1}}{(n-1)!} \, dx = \\ &= \frac{\varrho\delta}{p} \left(1 - \frac{\varrho}{p}\right) \int_x^{\infty} e^{-x\delta} \sum_{n=0}^{\infty} \left(\frac{\varrho}{p}\right)^n \frac{(x\delta)^n}{n!} \, dx = \\ &= \frac{\varrho\delta}{p} \left(1 - \frac{\varrho}{p}\right) \int_x^{\infty} e^{-x\delta(1-\varrho/p)} \, dx = \frac{\varrho}{p} \, e^{-x\delta(1-\varrho/p)}.\end{aligned}$$

2.1.8.3. *Incomplete renewal equation*

Denoting $M \equiv M(\infty)$ and $M_n \equiv M(\alpha_n)$, from equality,

$$R_{\alpha_n} = R_{\alpha_1 \dot{+} \alpha_{n-1}} = R_{\alpha_{n-1}} \circ \theta_{\alpha_1}$$

we have for $u \geq 0$:

$$\psi(u) = P_0(M > u) = P_0(M > u, \, \alpha_1 < \infty) =$$

$$= P_0(M_1 > u,\, \alpha_1 < \infty) + P_0(M > u,\, M_1 \le u,\, \alpha_1 < \infty) =$$

$$= P_0(M_1 > u,\, \alpha_1 < \infty)+$$

$$+\sum_{n=1}^{\infty} P_0(M_{n-1} \le u,\, M_n > u,\, \alpha_n < \infty,\, \alpha_{n+1} = \infty) =$$

$$= \overline{G}_+(u) + \sum_{n=1}^{\infty} \int_{-u}^{0} P_0(R_{\alpha_1} \in dx,\, M_{n-1} \le u,\, M_n > u,\, \alpha_n$$

$$< \infty,\, \alpha_{n+1} = \infty) =$$

$$= \overline{G}_+(u) + \sum_{n=2}^{\infty} \int_{-u}^{0} P_0(R_{\alpha_1} \in dx,\, \alpha_1 < \infty)\times$$

$$\times P_x(M_{n-2} \le u,\, M_{n-1} > u,\, \alpha_{n-1} < \infty,\, \alpha_n = \infty) =$$

$$= \overline{G}_+(u) + \sum_{n=1}^{\infty} \int_{0}^{u} \times$$

$$\times P_x(M_{n-1} \le u,\, M_n > u,\, \alpha_n < \infty,\, \alpha_{n+1} = \infty)\, dG_+(x) =$$

$$= \overline{G}_+(u) + \sum_{n=1}^{\infty} \int_{0}^{u} \times$$

$$\times P_0(M_{n-1} \le u-x,\, M_n > u-x,\, \alpha_n < \infty,\, \alpha_{n+1} = \infty)\, dG_+(x) =$$

$$= \overline{G}_+(u) + \int_{0}^{u} \psi(u-x)\, dG_+(x).$$

Thus, an integral equation relative to ψ is derived as follows:

$$\psi(u) = \overline{G}_+(u) + \int_{0}^{u} \psi(u-x)\, dG_+(x). \qquad [2.9]$$

Such an equation is called to be an incomplete renewal equation. Its difference from a proper renewal equation is only the property $G_+(\infty) < 1$, i.e. this distribution function is not probabilistic.

By denoting $\varphi(u) = 1 - \psi(u)$ and because of $G_+(u) + \overline{G}_+(u) = P_0(\alpha_1 < \infty)$, we derive an integral equation relative to probability for absence of ruin:

$$\varphi(u) = P_0(\alpha_1 = \infty) + \int_0^u \varphi(u - x)\, dG_+(x).$$

2.1.8.4. *Probabilistic case*

Given the safety loading $\eta < 0$, the ruin probability $\psi(u)$ equals to 1 for any $u \geq 0$. It implies that the ladder process $M(t)$ tends to infinity and distribution G_+ is probabilistic. Such a process determines two renewal processes. The first one is the process along the X-axis (abscissa) with points $\alpha_1, \alpha_2, \ldots$. The second one is that along the Y-ax (ordinate) with points $M_1, M_2, \ldots$. Let us consider the second renewal process with respect to measure P_0 by denoting it as $N_x^{(M)}$. We have $M_1 = -R_{\alpha_1} \overset{distr}{=} -R_{\alpha_1 - 0} + U_1$ as the magnitude of the jump over the cross at the time of the first exit from $[0, \infty)$. The second time of the vertical renewal process is the meaning of the process $-R_t$ at the time of its first exit from interval $(-\infty, M_1]$, and so on. Due to homogeneity in the space of the process R_t, the differences $(M_k - M_{k-1})$ $(k \geq 1, M_0 = 0)$ are independent and identically distributed with common distribution function G_+. If we consider the horizontal level $u > 0$, we can see that, for the process $N_x^{(M)}$, it is a distance from u up to the nearest after its jump, i.e. it is a direct renewal time. For the process $-R_t$, it is a magnitude of the jump over the cross line u. If u is the initial capital, then ζ_u is a value of debt at the ruin time of the insurance company. From the renewal theory, it follows that for $\eta < 0$ and $u \to \infty$

$$\frac{\zeta_u}{u} \overset{\text{a. s.}}{\to} 0, \qquad \frac{E_0 \zeta_u}{u} \to 0. \qquad [2.10]$$

2.1.9. *Adjustment coefficient of Lundberg*

2.1.9.1. *Adjustment coefficient*

Let us consider a generating function of the process R_t:

$$Ee^{\alpha R_t},$$

represented by formula [2.12]. The linear dependence of $\log Ee^{\alpha R_t}$ on t shows that R_t is a process with independent increments. Let us denote:

$$\widehat{B}(\alpha) := \int_0^\infty e^{\alpha t}\, dB(t),$$

if this integral exists, and

$$\kappa(\alpha) := \beta(\widehat{B}(\alpha) - 1) - p\,\alpha.$$

A special interest challenges the first positive root γ of this function, if this root exists. The first who noted its interesting properties was Lundberg.

Evidently, that $\kappa(0) = 0$. A derivative of this function at zero depends on the adjustment coefficient:

$$\kappa'(0) = \frac{\partial}{\partial\alpha}(\beta(\widehat{B}(\alpha) - 1) - \alpha p)\Big|_{\alpha=0} =$$
$$= \beta \int_0^\infty x\, e^{\alpha x}\, dB(x) - p\Big|_{\alpha=0} = \varrho - p = -\eta\varrho.$$

Consider this function under a positive safety loading. That is, the ruin probability does not reduce to the trivial meaning 1. Thus, in this case, the derivative of the function κ at zero is negative. Noting that the second derivative of this function is positive on its domain of definition, we see that a positive root of this function exists at least for the case $\kappa(\alpha) \to \infty$ as $\alpha \to \infty$ (Figure 2.3). We will call this root as adjustment coefficient of Lundberg.

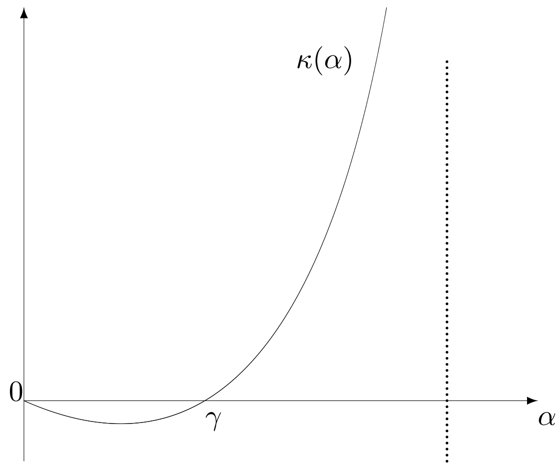

Figure 2.3. *Adjustment coefficient of Lundberg*

2.1.10. *Lundberg inequality*

The next theorem was the first non-trivial result in the theory of risk.

THEOREM 2.3.– *(Lundberg inequality).* If $\eta > 0$ and a positive root γ of the function $\kappa(\alpha)$ exists, then

$$\psi(u) \leq e^{-\gamma u}. \tag{2.11}$$

PROOF.– For given $u > 0$, let us denote:

$$\psi(u) = \lim_{n\to\infty} \psi_n(u),$$

where $\psi_n(u) = P_u(\tau_0 \leq \sigma_n)$. Prove that $\psi_n(u) \leq e^{-\gamma u}$ for any $n \geq 1$ and $u > 0$. We will prove it by induction. This inequality holds for $n = 0$. Let it hold for some $n \geq 0$. We have:

$$\psi_{n+1}(u) = P_u(R_{\sigma_1} < 0) + P_u(R_{\sigma_1} \geq u,\, \tau_0 \leq \sigma_{n+1}) =$$
$$= P_0(R_{\sigma_1} < -u) + P_0(R_{\sigma_1} \geq -u,\, \tau_{-u} \leq \sigma_{n+1}).$$

Denoting $F(x) = P_0(R_{\sigma_1} \le x)$ and noting that on the set $\{\sigma_1 < \tau_{-u}\}$ the representation $\tau_{-u} = \sigma_1 \dot{+} \tau_{-u} \equiv \sigma_1 + \tau_{-u} \circ \theta_{\sigma_1}$ is fulfilled, and also $\sigma_{n+1} = \sigma_1 \dot{+} \sigma_n$, we obtain:

$$
\begin{aligned}
\psi_{n+1}(u) &= F(-u) + E_0(P_{R_{\sigma_1}}(\tau_{-u} \le \sigma_n);\, R_{\sigma_1} \ge -u) = \\
&= F(-u) + \int_{-u}^{\infty} P_x(\tau_{-u} \le \sigma_n)\, dF(x) = \\
&= F(-u) + \int_{-u}^{\infty} P_0(\tau_{-u-x} \le \sigma_n)\, dF(x) = \\
&= F(-u) + \int_{-u}^{\infty} \psi_n(u+x)\, dF(x).
\end{aligned}
$$

By inductive proposition it is fair that:

$$
\begin{aligned}
\psi_{n+1}(u) &\le F(-u) + \int_{-u}^{\infty} e^{-\gamma(u+x)}\, dF(x) \le \\
&\le \int_{-\infty}^{-u} e^{-\gamma(u+x)}\, dF(x) + \int_{-u}^{\infty} e^{-\gamma(u+x)}\, dF(x) = \\
&= e^{-\gamma u} \int_{-\infty}^{\infty} e^{-\gamma x}\, dF(x) = e^{-\gamma u} E_0 e^{-\gamma R_{\sigma_1}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
E_0 e^{-\gamma R_{\sigma_1}} &= E_0 \exp(-\gamma(p\sigma_1 - U_1)) \\
&= E \exp(-\gamma(p\sigma_1))\, E \exp(\gamma U_1) = \\
&= \frac{\beta}{\beta + \gamma p} \widehat{B}(\gamma) = 1.
\end{aligned}
$$

The latter inequality follows from the definition of γ as a positive root of equation $\kappa(\alpha) = 0$, where $\kappa(\alpha) = \beta(\widehat{B}(\alpha) - 1) - \alpha p$. □

2.1.11. *Cramér asymptotics*

The next theorem is called to be the best theorem of the risk theory.

THEOREM 2.4.– *(theorem of Cramér).* For $\eta > 0$, there exists a limit:

$$\lim_{u\to\infty} \psi(u)e^{\gamma u} = C,$$

where

$$C = \frac{p-\varrho}{\beta\gamma \int_0^\infty x \, \exp(\gamma x)\, \overline{B}(x)\, dx}.$$

PROOF.– Formerly in [2.9] it was proved that:

$$\psi(u) = \overline{G_+}(u) + \int_0^u \psi(u-x)\, dG_+(x).$$

Denote $Z(x) = \psi(x)e^{\gamma x}$ $(x \geq 0)$. We have:

$$Z(u) = \overline{G_+}(u)e^{\gamma u} + \int_0^u Z(u-x)\, e^{\gamma x} dG_+(x).$$

Consider a function $F(y) = \int_0^y e^{\gamma x} dG_+(x)$. Evidently, this function does not decrease with respect to y. Besides, from formula [2.7], it follows that:

$$F(\infty) = \int_0^\infty e^{\gamma x} dG_+(x) = \int_0^\infty e^{\gamma x} \frac{\varrho}{p\,\mu_B} \overline{B}(x)\, dx =$$
$$= \frac{\varrho}{p\,\mu_B}\left(\frac{e^{\gamma x}}{\gamma}\overline{B}(x)\Big|_0^\infty + \int_0^\infty \frac{e^{\gamma x}}{\gamma}\, dB(x)\right).$$

Noting that:

$$e^{\gamma x}\overline{B}(x) \leq \int_x^\infty e^{\gamma s}\, dB(s) \to 0 \quad (x \to \infty),$$

we obtain:

$$F(\infty) = \frac{\varrho}{p\,\mu_B}\left(-\frac{1}{\gamma} + \frac{1}{\gamma}\,\widehat{B}(\gamma)\right) = \frac{\beta}{p\,\gamma}(\widehat{B}(\gamma) - 1) = 1.$$

Hence, the function $Z(x)$ satisfies a renewal equation (see section 1) with a known function $y(x) = \overline{G_+}(x)e^{\gamma x}$ and a distribution function $F(x)$ on half-line. From the theorem of Smith, it follows that:

$$Z(u) \to C \equiv \frac{1}{\mu_F} \int_0^\infty y(x)\, dx.$$

According to [2.7], we have:

$$\mu_F = \frac{\beta}{p} \int_0^\infty x\, e^{\gamma x} \overline{B}(x)\, dx.$$

We also have:

$$\begin{aligned}\int_0^\infty y(x)\, dx &= \int_0^\infty e^{\gamma x} \frac{\varrho}{p\mu_B} \int_x^\infty \overline{B}(s)\, ds = \\ &= \frac{\beta}{p} \int_0^\infty \overline{B}(s) \int_0^s e^{\gamma x}\, dx\, ds \\ &= \frac{\beta}{p} \int_0^\infty \overline{B}(s) \frac{1}{\gamma}(e^{\gamma s} - 1)\, ds = \\ &= \frac{\beta}{\gamma p} \left(\frac{1}{\gamma}(\widehat{B}(\gamma) - 1) - \mu_B \right) = \frac{1}{\gamma p}(p - \varrho),\end{aligned}$$

what implies C. Further simplification of this equation is possible only in exclusive cases. □

2.1.11.1. *Exponential B*

Let $\overline{B}(x) = e^{-\delta x}$. Then,

$$\int_0^\infty x e^{\gamma x} \overline{B}(x)\, dx = \int_0^\infty x e^{-(\delta-\gamma)x}\, dx = \frac{1}{(\delta - \gamma)^2}.$$

From here,

$$C = \frac{(p - \varrho)(\delta - \gamma)^2}{\beta\gamma}.$$

Let us find the meaning of γ for this case. We have $\mu_B = 1/\delta$, and also

$$\widehat{B}(\alpha) = \int_0^\infty e^{\alpha x}\delta e^{-\delta x}\,dx = \frac{\delta}{\delta-\alpha},$$

$$\kappa(\alpha) \equiv \beta(\delta/(\delta-\alpha)-1) - p\,\alpha = 0,\ \alpha > 0 \Rightarrow \alpha \equiv \gamma = \delta - \beta/p,$$

$$C = \frac{(p-\beta/\delta)(\delta-(\delta-\beta/p))^2}{\beta(\delta-\beta/p)} = \frac{\beta}{\delta p} = \frac{\varrho}{p}.$$

2.2. Finite horizon

In this subsection, the main object of investigation will be the random value τ_0 on a set $\{\tau_0 < \infty\}$. We will use a technical device, which essentially simplifies proofs of some useful theorems. This device is called to be the change of measure of a process and is a generalization for the change of measure of a random value.

2.2.1. *Change of measure*

Let us consider a random value X with a distribution function F. Assume that there exists a generating function of this function depending on α for any α in the region of the zero-point. Denote $\widetilde{\kappa}(\alpha) = \ln Ee^{\alpha X}$. Then, for any $\lambda > 0$, the distribution

$$dF_\lambda(x) \equiv e^{\lambda x - \widetilde{\kappa}(\lambda)}dF(x)$$

is a distribution function of some random value R_λ. If the generating function of this distribution exists for given α, then

$$\widetilde{\kappa}_\lambda(\alpha) \equiv \ln Ee^{\alpha R_\lambda} =$$

$$= \ln \int_{-\infty}^{\infty} e^{\alpha x}e^{\lambda x - \widetilde{\kappa}(\lambda)}\,dF(x) = \widetilde{\kappa}(\alpha+\lambda) - \widetilde{\kappa}(\lambda). \qquad [2.12]$$

A useful generalization of such a method for random processes is shown to be as follows: let $P^{(T)}$ and $P^{(L,T)}$ be projections of measures

P and $P^{(L)}$ (for some random processes) on interval $[0, T]$. This means the latter measures are considered only on sigma-algebra $\mathcal{F}_T$. Let us assume that the measure $P^{(L,T)}$ is absolutely continuous with respect to $P^{(T)}$ and $L(T)$ is the derivative of Radon-Nikodym denoted as:

$$L(T) = \frac{dP^{(L,T)}}{dP^{(T)}}.$$

Thus, for any $A \in \mathcal{F}_T$

$$E^L(A) = E(L(T);\ A) \qquad [2.13]$$

From this assumption, it follows that $L(T)$ is a martingale with respect to measure P. It means that:

$$E(L(t);\ A) = E(E(L(t+s)\,|\,\mathcal{F}_t);\ A)$$

for all $t \geq 0$ and $A \in \mathcal{F}_t$.

In fact, the main property of projections is:

$$P^{(L,t)}(A) = P^{(L,t+s)}(A) = P^{(L)}(A) \quad (A \in \mathcal{F}_t,\ t, s \geq 0).$$

From the definition of the Radon-Nikodym derivative, it follows that:

$$P^{(L,t)}(A) = E(L(t); A), \quad P^{(L,t+s)}(A) = E(L(t+s); A).$$

From the definition of conditional expectation, it follows that:

$$E(L(t+s); A) = E(E(L(t+s)\,|\,\mathcal{F}_t); A).$$

Noting that $D_0 \in \mathcal{F}_t$ (D_0 is the set of all trajectories), we obtain $EL(t) = 1$ for any $t \geq 0$.

Besides, we can prove that there exists a random process with measure $P^{(L)}$. The proof reduces to showing that Kolmogorov conditions hold in this case which is equivalent to equality:

$$P^{L,t}(A) = P^{L,t+s}(A)$$

for any $A \in \mathcal{F}_t,\ s,\, t \geq 0$. It follows from equalities:

$$P^{L,t+s}(A) = E(L(t+s); A) = E(P(L(t+s) \,|\, \mathcal{F}_t); A)$$
$$= E(L(t); A) = P^{L,t}(A).$$

From the general theory of martingales, it follows that $EL_\tau = 1$ for any Markov time τ and

$$E(L_T |\, \mathcal{F}_\tau) = L_\tau$$

on the set $\{\tau \leq T\}$. Further, we will use the next useful result.

THEOREM 2.5.– *(direct and inverse transformations).* If there exists a martingale (L_t) and measures P and $P^{(L)}$ with property [2.13], then for any Markov time τ and a set $G \in \mathcal{F}_\tau$ such that $G \subset \{\tau < \infty\}$ it holds the following expressions:

$$P^{(L)}(G) = E(L_\tau;\, G), \qquad [2.14]$$

$$P(G) = E^{(L)}\left(\frac{1}{L_\tau};\, G\right). \qquad [2.15]$$

PROOF.– Let $G \subset \{\tau < T\}$ for some finite T. By using property $G \in \mathcal{F}_\tau$ and the martingale property with respect to $\tau < T$, we obtain:

$$E^{(L)}\left(\frac{1}{L_\tau};\, G\right) = E\left(\frac{L_T}{L_\tau};\, G\right) =$$

$$= E\left(E(L_T |\, \mathcal{F}_\tau)\,\frac{1}{L_\tau};\, G\right) = P(G).$$

In the general case, we apply the last equality to set $G \cap \{\tau \leq T\}$. Thus,

$$E^{(L)}\left(\frac{1}{L_\tau};\, G \cap \{\tau \leq T\}\right) = P(G \cap \{\tau \leq T\}).$$

Noting that both sides of this equality increase monotonically with respect to $T \to \infty$, we obtain the equality for limit expressions. □

It is interesting to consider the transformation of measure P_u of the process (R_t) with the help of the random function $\Lambda_z(t) = e^{zS_t - t\kappa(z)}$, where

$$S_t = \sum_{k=1}^{N_t} U_k - p\,t, \qquad \kappa(z) = \beta(\widehat{B}(z) - 1) - p\,z \quad (z \geq 0).$$

With respect to the original measure P (without lower index) on the set of sequences (t_n, x_n), determining the composite Poisson process, this random function is a martingale with expectation 1. In fact,

$$E(e^{zS_t - t\kappa(z)}) = e^{-t\kappa(z)} E(e^{zS_t}) = e^{-t\kappa(z)} e^{t\kappa(z)} = 1,$$

and by using a property of conditional expectation, we obtain:

$$E(e^{zS_{t+s} - (t+s)\kappa(z)} |\, \mathcal{F}_t) = e^{zS_t - t\kappa(z)} E(e^{z(S_{t+s} - S_t) - s\kappa(z)} |\, \mathcal{F}_t) =$$

$$= e^{zS_t - t\kappa(z)} E(e^{zS_s \circ \theta_t - s\kappa(z)} |\, \mathcal{F}_t) =$$

$$= e^{zS_t - t\kappa(z)} E(e^{zS_s - s\kappa(z)}) = e^{zS_t - t\kappa(z)}.$$

Let us call the transformation defined by this martingale as the standard change of measure. Denote $P_u^{(z)}$ as a standard change of measure P_u with the help of this martingale

THEOREM 2.6.– *(transformed risk process).* A Cramér-Lundberg process of risk with parameters

1) $\beta_z = \beta \widehat{B}(z)$ (intensity of the Poisson process),

2) $dB_z(x) = \dfrac{e^{x\kappa(z)}}{\widehat{B}(z)} dB(x)$ (common function of the distribution of a claim size) and

3) $p_z = p$ (premium rate size)

has a distribution, which is equal to that of the transformed process with a standard change of measures $(P_u^{(z)})$ $(u \geq 0)$.

PROOF.– In order to prove it, we need to construct the family of measures $(\widetilde{P}_u^{(z)})$ $(u \geq 0)$ of the corresponding process of risk and to find (if it exists), for any $t > 0$, a Radon-Nikodym derivative with respect to the t-projection of the original measure.

Let us consider the Cramér-Lundberg process of risk with parameters β_z, B_z, p_z. Denote $(\widetilde{P}_u^{(z)})$ $(u \geq 0)$ as a family of measures of this process. A generating function of the random value S_t for this process is equal to:

$$\widetilde{E}_u^{(z)} e^{\alpha S_t} = e^{t\widetilde{\kappa}_z(\alpha)},$$

where $\widetilde{\kappa}_z(\alpha) = \beta_z(\widehat{B}_z(\alpha) - 1) - p\,\alpha$. We have:

$$\begin{aligned}\widetilde{\kappa}_z(\alpha) &= \beta(\widehat{B}(\alpha + z) - \widehat{B}(z)) - p\,\alpha = \\ &= \beta(\widehat{B}(\alpha + z) - 1) - p\,(\alpha + z) - \beta(\widehat{B}(z) - 1) + p\,z \\ &= \kappa(\alpha + z) - \kappa(z).\end{aligned}$$

On the other hand, as it was shown earlier [2.12], the same meaning has the corresponding function $\kappa_z(\alpha)$ of the last process, for which the distribution of S_t is transformed by the above method:

$$dF_{S_t}^{(z)}(x) = e^{xz - t\kappa(z)} dF_{S_t}(x).$$

Consequently, the $\widetilde{P}_u^{(z)}$-distribution of this random value has view $dF_{S_t}^{(z)}(x)$. Further, an assertion of the theorem is equivalent to equality:

$$\widetilde{E}_u^{(z)}(Z) = E_u(e^{z\,S_t - t\,\kappa(z)} Z)$$

for any $\mathcal{F}_t$-measurable random value Z.

From the general theory of measures, it is well-known that this equality is sufficient to prove for the $\mathcal{F}_t^{(n)}$-measurable random value Z, where $\mathcal{F}_t^{(n)}$ is a sigma-algebra, generated by random values $R_{kt/n}$ $(k = 1, \ldots, n)$, or (it is the same) by increments of the process $(R_s)_0^t$ at points kt/n. A general view of such a function is:

$$Z = g(S_{t/n}, S_{2t/n} - S_{t/n}, \ldots, S_t - S_{(n-1)t/n}).$$

Due to independence and identical distribution of increments of (S_t), we have:

$$\begin{aligned}\widetilde{E}_u^{(z)}(Z) &= \int_{\mathbb{R}^n} g(x_1, x_2, \ldots, x_n) \times \\ &\times dF^{(z)}_{S_{t/n}}(x_1)\, dF^{(z)}_{S_{2t/n}-S_{t/n}}(x_2) \ldots dF^{(z)}_{S_t - S_{(n-1)t/n}}(x_n) = \\ &= \int_{\mathbb{R}^n} g(x_1, x_2, \ldots, x_n)\, e^{z\,(x_1+x_2+\cdots+x_n) - t\,\kappa(z)} \times \\ &\times dF_{S_{t/n}}(x_1)\, dF_{S_{2t/n}-S_{t/n}}(x_2) \ldots dF_{S_t - S_{(n-1)t/n}}(x_n) = \\ &= E_u(e^{zS_t + t\,\kappa(z)} Z).\end{aligned}$$

□

From the theorem 2.5, it follows that for any τ and $G \subset \{\tau < \infty\}$:

$$P_u(G) = E_u^{(z)}(e^{-zS_\tau + \tau\kappa(z)};\, G).$$

The same equality holds in terms of expectations

$$E_u(f;\, \tau < \infty) = E_u^{(z)}(f\, e^{-zS\tau + \tau\kappa(z)};\, \tau < \infty). \qquad [2.16]$$

2.2.2. *Theorem of Gerber*

The following theorem can be considered as the Lundberg inequality for a finite interval. It is a very good example of using the change of measure.

THEOREM 2.7.– *(theorem of Gerber).* For the Cramér-Lundberg process under positive safety loading and adjustment coefficient γ, we have:

$$\psi(u, yu) \le e^{-\gamma_y u}, \quad y < \frac{1}{\kappa'(\gamma)}, \qquad [2.17]$$

$$\psi(u) - \psi(u, yu) \le e^{-\gamma_y u}, \quad y > \frac{1}{\kappa'(\gamma)}, \qquad [2.18]$$

where $\kappa(\alpha) \equiv \beta(\widehat{B}(\alpha)-1)-p\,\alpha\ (\alpha > 0)$; γ_y is an intersecting point of the abscissa axis and a tangent to the curve $\kappa(\alpha)$ at the point α_y, which is disposed to the right of a point of minimum of the curve $\kappa(\alpha)$; $1/y$ is angular coefficient of the tangent (Figure 2.4).

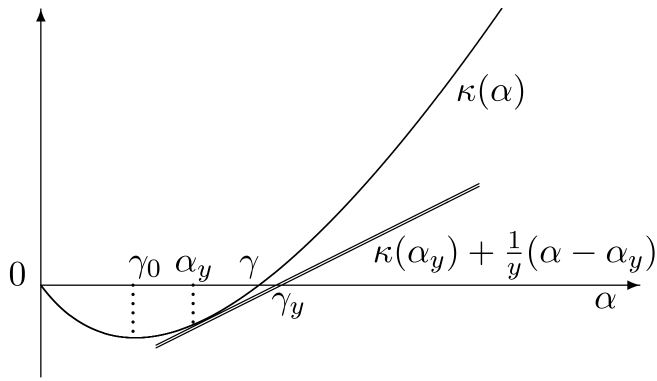

Figure 2.4. *For Gerber theorem*

PROOF.– Note that at the ruin time $\tau(-u)$, the random value $S_{\tau(-u)}$ can be represented in view of $u+\zeta_{-u}$, where $\zeta_{-u} \equiv \zeta(-u) > 0$ is a value of the jump over the cross line $(-u)$. Consider the first case $1/y > \kappa'(\gamma)$. We have $\kappa(\alpha_y) > 0$, and consequently by using the change of measure with parameter α_y, we obtain:

$$\begin{aligned}\psi(u,\, yu) &= E_0(\tau(-u) \le yu) = \\ &= e^{-\alpha_y u} E_0^{(\alpha_y)}\left(e^{-\alpha_y \zeta(-u)+\tau(-u)\kappa(\alpha_y)};\ \tau(-u) \le yu\right) \le \\ &\le e^{-\alpha_y u} E_0^{(\alpha_y)}\left(e^{\tau(-u)\kappa(\alpha_y)};\ \tau(-u) \le yu\right) \\ &\le e^{-\alpha_y u + yu\kappa(\alpha_y)} = e^{-\gamma_y u},\end{aligned}$$

because γ_y is a root of the tangent equation $\kappa(\alpha_y) + (\alpha - \alpha_y)/y = 0$.

Consider the second case $0 < 1/y < \kappa'(\gamma)$. In this case, $\kappa(\alpha_y)$ is less than zero. We have:

$$\psi(u) - \psi(u,\, yu) = E_0(yu \le \tau(-u) < \infty) =$$

$$= e^{-\alpha_y u} E_0^{(\alpha_y)} \left(e^{-\alpha_y \zeta(-u) + \tau(-u)\kappa(\alpha_y)};\; yu \le \tau(-u) < \infty \right) \le$$

$$\le e^{-\alpha_y u} E_0^{(\alpha_y)} \left(e^{\tau(-u)\kappa(\alpha_y)};\; yu \le \tau(-u) < \infty \right) \le$$

$$\le e^{-\alpha_y u + yu\kappa(\alpha_y)} = e^{-\gamma_y u}. \qquad \square$$

2.2.3. *Change of measure with parameter gamma*

Especially convenient to use the change of measure with $z = \gamma$ (coefficient of Lundberg). In this case,

$$E_u(f;\, \tau < \infty) = E_u^{(\gamma)}(f\, e^{-\gamma S_\tau};\, \tau < \infty). \qquad [2.19]$$

Let us use this property for the short proof of Lundberg inequality.

THEOREM 2.8.– *(another proof of Lundberg inequality).* For the Cramér-Lundberg process of risk with parameter $\eta > 0$ and coefficient of Lundberg γ, it is fair that:

$$\psi(u) \le e^{-\gamma u}.$$

PROOF.–

$$\psi(u) = P_u(\tau_0 < \infty) = P_0(\tau_{-u} < \infty) = E_0^{(\gamma)}(e^{-\gamma S_{\tau(-u)}};\, \tau_{-u} < \infty).$$

From here,

$$\psi(u) = e^{-\gamma u} E_0^{(\gamma)}(e^{-\gamma \zeta(-u)};\, \tau_{-u} < \infty) \le e^{-\gamma u}. \qquad \square$$

Another important property of this transformation consists in the difference of safety loadings signs between the original and transformed processes. In fact,

$$\eta_\gamma = \frac{p - \beta_\gamma \mu_{B_\gamma}}{\beta_\gamma \mu_{B_\gamma}}.$$

We have:

$$\beta_\gamma \mu_{B_\gamma} = \beta \int_0^\infty x e^{\gamma x}\, dB(x) = \beta \widehat{B}'(\gamma).$$

On the other hand, $\kappa'(\gamma) = \beta \widehat{B}'(\gamma) - p > 0$. Hence, $\eta_\gamma < 0$.

2.2.4. *Exponential distribution of claim size*

Assume that for the original process of the risk $B(x) = 1 - e^{-\delta x}$. Then, $\widehat{B}(\alpha) = \frac{\delta}{\delta-\alpha}$ $(\alpha < \delta)$. From here, $\kappa(\alpha) = \beta\left(\frac{\delta}{\delta-\alpha} - 1\right) - p\,\alpha$, and the coefficient of Lundberg is equal to

$$\gamma = \delta - \frac{\beta}{p}. \qquad [2.20]$$

Further $\varrho \equiv \beta\, \mu_B = \beta/\delta$, and the safety loading is equal to:

$$\eta = \frac{p\,\delta - \beta}{\beta}. \qquad [2.21]$$

For the transformed process with measures $P_u^{(z)}$ $(0 < z < \delta)$, we have:

$$\beta_z = \beta \widehat{B}(z) = \frac{\beta\delta}{\delta - z} > \beta,$$

$$\widehat{B}_z(\alpha) = \frac{\widehat{B}(\alpha + z)}{\widehat{B}(z)} = \frac{\delta - z}{\delta - \alpha - z},$$

which represents the generating function of an exponential distribution with parameter $\delta - z$. Hence, $\mu_{B_z} = 1/(\delta - z)$, and also:

$$\varrho_z \equiv \beta_z \mu_{B_z} = \frac{\beta\delta}{(\delta - z)^2} > \varrho,$$

$$\eta_z \equiv p/\varrho_z - 1 = \frac{p(\delta - z)^2}{\beta\delta} - 1.$$

This safety loading of the transformed process can have any sign at the expense of p.

For a process with measures $P_u^{(\gamma)}$, we have $\beta_\gamma = p\,\delta$, $\widehat{B}_\gamma(\alpha) = \frac{\beta}{\beta - p\,\alpha}$, which represents a generating function of the exponential distribution with parameter β/p. Hence, $\mu_{B_\gamma} = p/\beta$, and also:

$$\varrho_\gamma = p^2\delta/\beta, \quad \eta_\gamma = \frac{\beta - p\,\delta}{p\,\delta}.$$

Thus, we can see again that the safety loading of a transformed process with parameter γ has the opposite sign, comparative with that of the original one [2.21].

An exponential distribution of jump sizes of the process (R_t) has yet another important consequence. The value of the jump over the cross line zero $\zeta_0 = u - R_{\tau_0}$ has the same distribution as that of U_1. In fact, taking into account the independence of components, we have:

$$P_u(\zeta_0 > x,\, \tau_0 < z) = \sum_{n=1}^{\infty} P_u(\zeta_0 > x,\, \tau_0 = \sigma_n,\, \sigma_n < z) =$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} P_u(U_n - y > x,\, R_{\sigma_n - 0} \in dy,\, \tau_0 > \sigma_{n-1},\, \sigma_n < z) =$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\delta(x+y)} P_u(R_{\sigma_n - 0} \in dy,\, \tau_0 > \sigma_{n-1},\, \sigma_n < z) =$$

$$= e^{-\delta x} \sum_{n=1}^{\infty} \int_0^{\infty} P_u(U_n > y,\, R_{\sigma_n - 0} \in dy,\, \tau_0 > \sigma_{n-1},\, \sigma_n < z) =$$

$$= e^{-\delta x} \sum_{n=1}^{\infty} P_u(\tau_0 = \sigma_n,\, \sigma_n < z) = e^{-\delta x} P_u(\tau_0 < z).$$

Hence, for any $z > 0$ (including $z = \infty$):

$$P_u(\zeta_0 > x |\, \tau_0 < z) = e^{-\delta x}, \qquad [2.22]$$

which implies also that ζ_0 and τ_0 are independent with respect to measure P_u on a set $\tau_0 < \infty$.

THEOREM 2.9.– *(conditional distribution of a ruin time).* For the Cramér-Lundberg process of risk with a claim distribution $B(x) = 1 - e^{-\delta x}$ and safety loading $\eta > 0$, the following expression is fair that:

$$E_u(\tau_0 | \, \tau_0 < \infty) = \frac{u\,\beta + p}{p^2\,\delta - p\,\beta}\,. \qquad [2.23]$$

PROOF.– According to formula [2.16], and due to the negativeness of η_γ (from which follows ruin with probability 1), we have:

$$E_u(\tau_0;\, \tau_0 < \infty) = E_u^{(\gamma)}(\tau_0 e^{-\gamma S_{\tau_0}};\, \tau_0 < \infty) = E_u^{(\gamma)}(\tau_0 e^{-\gamma S_{\tau_0}}) =$$

$$= e^{-\gamma u} E_u^{(\gamma)}(\tau_0 e^{-\gamma\zeta_0}) = e^{-\gamma u} E_u^{(\gamma)} e^{-\gamma\zeta_0} E_u^{(\gamma)}(\tau_0) =$$

$$= E_u^{(\gamma)} e^{-\gamma S_{\tau_0}} E_u^{(\gamma)}(\tau_0) = E_u^{(\gamma)}(e^{-\gamma S_{\tau_0}};\, \tau_0 < \infty) E_u^{(\gamma)}(\tau_0) =$$

$$= E_u(\tau_0 < \infty) E_u^{(\gamma)}(\tau_0),$$

hence, $E_u(\tau_0 | \, \tau_0 < \infty) = E_u^{(\gamma)}(\tau_0)$. On the other hand, the process R_t with respect to the family $(P_u^{(\gamma)})$ is a homogenous process with independent increments for which the identity of Wald is fair:

$$E_u^{(\gamma)} S_{\tau_0} = E_u^{(\gamma)} S_1\, E_u^{(\gamma)} \tau_0 = (\varrho_\gamma - p) E_u^{(\gamma)} \tau_0.$$

Consequently,

$$E_u^{(\gamma)} \tau_0 = \frac{E_u^{(\gamma)} S_{\tau_0}}{\varrho_\gamma - p} = \frac{u + \mu_{B_\gamma}}{\varrho_\gamma - p} = \frac{u + p/\beta}{p^2\delta/\beta - p} = \frac{u\,\beta + p}{p^2\delta - p\,\beta}. \qquad \square$$

REMARK.– The identity of Wald for a discrete Markov time and an homogeneous process with independent increments (Y_t) can be proven as follows: let $\tau = Nh$, where N is an integer-valued Markov time: $\{N = n\} \in \mathcal{F}_{nh}$, and $h > 0$ is a given value. In this case,

$$\begin{aligned} EY_\tau = EY_{Nh} &= \sum_{n=1}^{\infty} E\left(\sum_{k=1}^{n} \Delta_k;\, N = n\right) \\ &= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} E(\Delta_k;\, N = n) = \\ &= \sum_{k=1}^{\infty} E(\Delta_k;\, N \geq k), \end{aligned}$$

where $\Delta_k = Y_{kh} - Y_{(k-1)h}$. Noting that $\{N \geq k\} = \{N > k-1\} \in \mathcal{F}_{(k-1)h}$, and by using independence Δ_k from the past, we obtain:

$$EY_\tau = E(\Delta_k)EN = hEY_1EN = EY_1E\tau.$$

In a general case, Wald's identity can be proven with the help of passage to a limit as $h \to 0$. □

THEOREM 2.10.– *(Laplace transformation of ruin time)* For the Cramér-Lundberg process of risk with a claim distribution $B(x) = 1 - e^{-\delta x}$ and with safety loading $\eta > 0$, the next representation of the Laplace image of the ruin time is true

$$E_u e^{-\lambda\tau 0} = e^{-z(\lambda)u}\frac{\delta - z(\lambda)}{\delta} \quad (\lambda \geq \lambda_0), \qquad [2.24]$$

where

$$\lambda_0 = \min\{k(\alpha) :\, 0 < \alpha < \delta\},$$

$$z(\lambda) = \frac{1}{2p}\left(p\,\delta - \beta - \lambda + \sqrt{(p\,\delta - \beta - \lambda)^2 + 4p\delta\lambda}\right),$$

PROOF.– According to formula [2.16], for any positive z:

$$E_u e^{-\lambda\tau_0} = E_u(e^{-\lambda\tau_0};\, \tau_0 < \infty) = E_u(e^{-\lambda\tau_0 - zS_{\tau_0} + \tau_0\kappa(z)};\, \tau_0 < \infty).$$

Choose z in such a way that $\kappa(z) = \lambda$ for all λ from the minimal meaning λ_0 up to infinity. Thus, z is a positive root of the equation:

$$\frac{\beta z}{\delta - z} - p\,z = \lambda,$$

By solving this square equation, we obtain $z(\lambda)$, appearing in assertion of the theorem. Hence,

$$\begin{aligned} E_u e^{-\lambda\tau_0} &= E_u^{(z(\lambda))}(e^{-z(\lambda)S_{\tau_0}}) = \\ &= E_u^{(z(\lambda))}(e^{-z(\lambda)(u+\zeta_0)}) = e^{-z(\lambda)u} E_u^{(z(\lambda))}(e^{-z(\lambda)U_1}) = \\ &= e^{-z(\lambda)u} \int_0^\infty e^{-z(\lambda)x}(\delta - z(\lambda))\, e^{-(\delta - z(\lambda))x}\, dx \\ &= e^{-z(\lambda)u} \frac{\delta - z(\lambda)}{\delta}. \end{aligned}$$

□

2.2.4.1. *Arbitrary distribution of claim size*

For the Cramér-Lundberg process of risk with claim size distribution B of the general view and with a finite first moment, we will consider some results for limit behavior of ruin probability $\psi(u, T)$ under consistent tending u and T to infinity.

THEOREM 2.11.– *(law of large numbers).* For the Cramér-Lundberg process with safety loading $\eta < 0$, the following limits hold:

a) $\tau_{-u}/u \to m \quad P_0 - a.s.$;

b) $E_u \tau_0 / u \to m$;

c) $\dfrac{\tau_{-u} - mu}{\sqrt{u}} \overset{distr}{\longrightarrow} \sigma Z$

as $u \to \infty$, where $m = 1/(\varrho - p)$; $\sigma^2 = \beta\, \mu_B^{(2)} m^3$; Z is a standard normal random value.

PROOF.– Due to condition $\eta < 0$, the ruin has probability 1. Besides, due to the homogeneity of the process in space and in time for any

$M > 0$, we have $P_u(\tau_0 < M) = P_0(\tau_{-u} < M) \to 0\ (u \to \infty)$ tending to infinity in probability. Because τ_{-u} is a non-decreasing random value, convergence in probability implies convergence with probability 1.

a) According to theorem 2.1, and due to P_0-a.s. convergence of τ_{-u} to infinity, we have:

$$m = \lim_{t\to\infty} (\text{a. s.}) \frac{t}{S_t} = \lim_{u\to\infty} (\text{a. s.}) \frac{\tau_{-u}}{S_{\tau_{-u}}} = \lim_{u\to\infty} (\text{a. s.}) \frac{\tau_{-u}}{u + \zeta_{-u}},$$

where ζ_{-u} is a value of the jump over the cross line $-u$ at the first crossing time of this level. Because $\zeta/u \overset{\text{a.s.}}{\to} 0$ by formula [2.10], the assertion is proved.

b) By using Wald's identity, we have:

$$u + E_0\zeta_{-u} = E_0 S_{\tau_{-u}} = E_0\tau_{-u} \cdot E_0 S_1 = (\varrho - p) E_0 \tau_{-u}.$$

According to property [2.10], it follows that $E_0\zeta_{-u}/u \to 0\ (u \to \infty)$. From here, the second assertion follows:

c) by using theorem 2.2:

$$\frac{S_t - t(\varrho - p)}{\sqrt{t}} \overset{\text{distr}}{\longrightarrow} \sigma Z \quad (t \to \infty),$$

where $\sigma^2 = \beta\, \mu_B^{(2)}$ and Z is a standard normal random value. Evidently, in this relation, argument t can be replaced on a random argument tending to infinity (see Remark). From here,

$$\frac{u + \zeta_{-u} - \tau_{-u}/m}{\sqrt{\tau_{-u}}} \overset{distr}{\longrightarrow} \sigma Z \quad (u \to \infty).$$

By multiplying the last expression by m and changing the sign, we obtain:

$$\frac{\tau_{-u} - mu}{\sqrt{\tau_{-u}}} \overset{distr}{\longrightarrow} m\,\sigma Z,$$

By multiplying it by $\sqrt{m}$, we obtain:

$$\frac{\tau_{-u} - mu}{\sqrt{u}} \overset{distr}{\longrightarrow} m^{3/2}\sigma Z.$$

□

REMARK.– Replacing t by τ_{-u} is possible due to the following reason: denote $\Omega_1 = \{f(t) \to c\ (t \to \infty)\}$ and $\Omega_2 = \{g(u) \to \infty\ (u \to \infty)\}$. If $P_0(\Omega_1) = P_0(\Omega_2) = 1$, then $P_0(\Omega_1 \cap \Omega_2) = 1$. But $\Omega_1 \cap \Omega_2 \subset \{f(g(u)) \to c\ (u \to \infty)\}$. Hence, $P_0(f(g(u)) \overset{u\to\infty}{\longrightarrow} c) = 1$.

THEOREM 2.12.– *(conditional law of large numbers).* For the Cramér-Lundberg process with parameter $\eta > 0$, when $u \to \infty$, the fraction τ_{-u}/u tends to $m_\gamma \equiv 1/(\varrho_\gamma - p)$ in conditional probability with respect to condition $\{\tau_{-u} < \infty\}$, i.e. for any $\epsilon > 0$:

$$P_0\left(\left|\frac{\tau_{-u}}{u} - m_\gamma\right| > \epsilon \middle|\ \tau_{-u} < \infty\right) \to 0.$$

Besides,

$$\frac{\psi(u, \alpha u)}{\psi(u)} \to \begin{cases} 0 & \alpha < m_\gamma, \\ 1 & \alpha > m_\gamma. \end{cases}$$

PROOF.– By using change of measure, we have:

$$P_0\left(\left|\frac{\tau_{-u}}{u} - m_\gamma\right| > \epsilon \middle|\ \tau_{-u} < \infty\right) =$$

$$= \frac{e^{-\gamma u}}{\psi(u)} E_0^{(\gamma)}\left(e^{-\gamma\zeta_{-u}};\ \left|\frac{\tau_{-u}}{u} - m_\gamma\right| > \epsilon\right) \leq$$

$$\leq \frac{e^{-\gamma u}}{\psi(u)} P_0^{(\gamma)}\left(\left|\frac{\tau_{-u}}{u} - m_\gamma\right| > \epsilon\right) \to 0,$$

because the first fraction tends to C^{-1} (C is Cramér's limit) and the second multiplier tends to zero, according to theorem 2.11(a). Besides,

$$\frac{\psi(u, \alpha u)}{\psi(u)} = P_0\left(\frac{\tau_{-u}}{u} < \alpha \middle|\ \tau_{-u} < \infty\right),$$

which in the first case is equal to:

$$P_0\left(m_\gamma - \frac{\tau_{-u}}{u} > m_\gamma - \alpha \middle|\ \tau_{-u} < \infty\right) \to 0,$$

in the second case:

$$1 - P_0\left(\frac{\tau_{-u}}{u} - m_\gamma \geq \alpha - m_\gamma \middle|\ \tau_{-u} < \infty\right) \to 1. \qquad \square$$

2.2.5. *Normal approximation*

Special properties of value m_γ were already remarked above. Now, we consider further specifications of these properties.

2.2.5.1. *Lemma of Stamm and theorem of Segerdal*

The theorem of Segerdal can be called as a generalization of theorem of Cramér for the ruin on a finite interval.

THEOREM 2.13.– *(theorem of Segerdal).* For any Cramér-Lundberg process with $\eta > 0$, if Lundberg root γ exists, then it is true

$$e^{\gamma u}\psi(u,\, um_\gamma + y\sigma_\gamma\sqrt{u}) \to C\Phi(y) \quad (u \to \infty), \qquad [2.25]$$

where $\sigma_\gamma^2 = \beta_\gamma E^{(\gamma)}U_1^2 m_\gamma^3$ and C is the limit of Cramér; Φ is the standard normal distribution function.

LEMMA 2.1.– *(lemma of Stamm).* If $\eta < 0$, then ζ_{-u} and τ_{-u} are asymptotically independent. It means that, for all continuous and bounded function f determined on $[0,\infty)$, and continuously differentiable and bounded functions g with bounded derivatives, determined on $(-\infty,\infty)$, there are limits:

$$Ef(\zeta_{-\infty}) = \lim_{u\to\infty} Ef(\zeta_{-u}),$$

$$E\left(f(\zeta_{-u})\, g\left(\frac{\tau_{-u} - mu}{\sigma\sqrt{u}}\right)\right) \to Ef(\zeta_{-\infty})Eg(Z),$$

where $\sigma^2 = \beta\mu_B^{(2)} m^3$, $m = (\varrho - p)^{-1}$; Z is a standard normal random variable.

PROOF OF LEMMA.– Existence of the first limit follows from the renewal theory (see [2.10] and above). Denote $u' = u - u^{1/4}$ and $Z_u = (\tau_{-u} - mu)/(\sigma\sqrt{u})$. Let us estimate this difference (in what follows, we will write $\tau_{-u} \equiv \tau(-u)$, $\zeta_{-u} \equiv \zeta(-u)$). By using the strong Markov property, and homogeneity in space and theorem 2.11(b), we obtain:

$$E_0(\tau(-u) - \tau(-u')) = E_0(\tau(-u) \circ \theta_{\tau(-u')};\ \tau(-u) > \tau(-u')) =$$
$$= E_0(E_{R_{\tau(-u')}}\tau(-u);\ R_{\tau(-u')} \geq -u) =$$
$$= \int_{-u}^{-u'} E_0(E_0\tau(-u-x);\ R_{\tau(-u')} \in dx)$$
$$\leq E_0\tau(-u^{1/4}) = O(u^{1/4}).$$

Furthermore,

$$g(Z_u) = g(Z_{u'}) + g'(c)(Z_u - Z_{u'}) =$$

$$= g(Z_{u'}) + g'(c)\left(\frac{\tau(-u)-\tau(-u')}{\sigma\sqrt{u}} + \frac{m(u'-u)}{\sigma\sqrt{u}} + \left(\frac{\sqrt{u'}}{\sqrt{u}} - 1\right)Z_{u'}\right),$$

where c is a point posed between $Z(u)$ and $Z(u')$. From here,

$$|E_0 f(\zeta(-u))(g(Z_u) - g(Z_{u'})| \leq$$
$$\leq \max_{x,c}(|f(x)| \cdot |g'(c)|)\left(\frac{E_0(\tau(-u)-\tau(-u'))}{\sigma\sqrt{u}} + \frac{mu^{-1/4}}{\sigma} + \right.$$
$$\left. + \left(1 - \sqrt{\frac{u'}{u}}\right) E_0|Z_{u'}|\right) \to 0 \quad (u \to \infty),$$

the latter summand tends to zero due to convergence $E_0|Z_u|$ to $E_0 Z$ and boundedness of the latter member. Now, we consider this expression with change Z_u by $Z_{u'}$. By applying formulas $\tau(-u) = \tau(-u') \dot{+} \tau(-u)$, and $R_{\tau(-u') \dot{+} \tau(-u)} = R_{\tau(-u)} \circ \theta_{\tau(-u')}$, and by using the Markov property, we obtain:

$$E_0(f(\zeta(-u))g(Z_{u'})) = E_0(f(-u - R_{\tau(-u)})g(Z_{u'})) =$$
$$= E_0(f(-u - R_{\tau(-u)}) \circ \theta_{\tau(-u')} \cdot g(Z_{u'})) =$$
$$= E_0(E_{R_{\tau(-u')}} f(-u - R_{\tau(-u)}) \cdot g(Z_{u'})) =$$

$$= \int_{-\infty}^{-u'} E_0(E_0 f(-u - (R_{\tau(-u-x)} + x)) \cdot g(Z_{u'});\ R_{\tau(-u')} \in dx) =$$

$$= \int_{-\infty}^{-u'} E_0 f(\zeta(-u - x)) \cdot E_0(g(Z_{u'});\ R_{\tau(-u')} \in dx).$$

Let us split this interval into two parts:

$$(-\infty,\ -u') = (-\infty, -u + u^{1/4}/2) \cup [-u + u^{1/4}/2,\ -u').$$

On the first interval, $R_{\tau(-u')} < -u + u^{1/4}/2$. It means that $\zeta(-u') \equiv -u' - R_{\tau(-u')} > -u' + u - u^{1/4}/2 = u^{1/4}/2$. The measure of this event tends to zero as $u \to \infty$ according to the limit theorem of the renewal theory for a direct renewal time. For additional events with its measure tending to 1, the argument $-(u + x)$ of the function ζ is posed between $-u + u' = -u^{1/4}$ and $-u - (-u + u^{1/4}/2) = -u^{1/4}/2$. Hence, it tends to infinity as $u \to \infty$. Consequently, the first integrand tends to a limit, which was denoted as $E_0\zeta(-\infty)$. The second integrand tends to $E_0 g(Z)$ by theorem 2.11(c). □

PROOF OF THEOREM.– We have:

$$e^{\gamma u}\psi(u,\ m_\gamma u + y\sigma_\gamma\sqrt{u}) = e^{\gamma u} P_0(\tau(-u) \le m_\gamma u + y\sigma_\gamma\sqrt{u}) =$$

$$= e^{\gamma u} P_0\left(\frac{\tau(-u) - m_\gamma u}{\sigma_\gamma\sqrt{u}} \le y\right) = e^{\gamma u} P_0(Z_u \le y),$$

where $Z_u = (\tau(-u) - m_\gamma u)/(\sigma_\gamma\sqrt{u})$. The latter probability can be written as expectation $E_0(I_{(-\infty,u]}(Z_u))$. By using the change of measure, we obtain:

$$e^{\gamma u}\psi(u,\ m_\gamma u + y\sigma_\gamma\sqrt{u}) = E_0^{(\gamma)}(e^{-\gamma\zeta(-u)} I_{(-\infty,u]}(Z_u)).$$

This expression initates the expression considered in *Lemma* of Stamm. We have $\eta_\gamma < 0$. A decreasing exponent plays a role of f. As a function g, we would take $I_{(-\infty,u]}$, but the latter function is not continuous. Due to this, we will consider a sequence of admissible functions converging (non-uniformly) to the indicator. Define:

$$g_\epsilon(x) = \int_x^\infty \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{(y-s)^2}{2\epsilon}\right) ds \equiv \overline{\Phi}\left(\frac{x-y}{\sqrt{\epsilon}}\right).$$

For $\epsilon \to 0$, this function converges to the indicator of interval $(-\infty, y]$ everywhere except the point y. For any $\epsilon > 0$, this function is bounded, continuous and has bounded and continuous derivatives. Given $\epsilon > 0$, the convergence to limit as $u \to \infty$ follows from *Lemma* of Stamm. We have:

$$E_0^{(\gamma)} e^{-\gamma\zeta(-u)} g_\epsilon(Z_u) \to E_0^{(\gamma)} e^{-\gamma\zeta(-\infty)}\, E_0 g_\epsilon(Z).$$

According to the theorem of Cramér in this expression:

$$E_0^{(\gamma)} e^{-\gamma\zeta(-\infty)} = \lim_{u\to\infty} E_0^{(\gamma)} e^{-\gamma\zeta(-u)} = \lim_{u\to\infty} e^{\gamma u} E_0(\tau(-u) < \infty) = C.$$

From the proof of *Lemma* of Stamm, it follows that a set on which $E_0^{(\gamma)} e^{-\gamma\zeta(-u)}$ converges to a limit does not depend on choice of g. Hence, it is sufficient to estimate the convergence of $E_0^{(\gamma)} g_\epsilon(Z_u)$. By denoting distribution functions of Z_u and Z as F_{Z_u} and F_Z correspondingly, we have:

$$\begin{aligned} E_0^{(\gamma)} g_\epsilon(Z_u) &= \int_{-\infty}^{\infty} \overline{\Phi}_{y,\epsilon}(x)\, dF_{Z_u}(x) = \\ &= \int_{-\infty}^{\infty} \int_x^\infty \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{(s-y)^2}{2\epsilon}\right) ds\, dF_{Z_u}(x) = \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{(s-y)^2}{2\epsilon}\right) \int_{-\infty}^{s} dF_{Z_u}(x)\, ds = \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{(s-y)^2}{2\epsilon}\right) F_{Z_u}(s)\, ds. \end{aligned}$$

The similar formula can be applied to for Z. Thus, we have:

$$|E_0^{(\gamma)} g_\epsilon(Z_u) - E_0^{(\gamma)} g_\epsilon(Z)| \leq$$
$$\leq \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{(s-y)^2}{2\epsilon}\right) |F_{Z_u}(s) - F_Z(s)|\, ds \leq$$
$$\leq \sup_s |F_{Z_u}(s) - F_Z(s)|,$$

which implies that, for $u \to \infty$, the value $E_0^{(\gamma)} e^{-\gamma\zeta(-u)} g_\epsilon(Z_u)$ converges to its limit uniformly on $\epsilon > 0$. Remark that when $\epsilon \to 0$, the following is true:

$$E_0^{(\gamma)} g_\epsilon(Z) \to E_0^{(\gamma)} I_{(-\infty,\, y]}(Z) = P(Z \leq y) = \Phi(y).$$

However, the passage to this limit from the original expression have to be justified, because we came to it by the passage firstly when $u \to \infty$, and secondly when $\epsilon \to 0$. But, we must go firstly when $\epsilon \to 0$, and secondly when $u \to \infty$. A uniform convergence for $u \to \infty$ saves us. In this case,

$$|e^{\gamma u}\psi(u,\, m_\gamma u + y\sigma_\gamma\sqrt{u}) - C\Phi(y)| \leq$$
$$\leq |e^{\gamma u}\psi(u,\, m_\gamma u + y\sigma_\gamma\sqrt{u}) - e^{\gamma u} E_0 g_\epsilon(Z_u)| +$$
$$+ |e^{\gamma u} E_0 g_\epsilon(Z_u) - C\, E_0^{(\gamma)} g_\epsilon(Z)| + |C\, E_0^{(\gamma)} g_\epsilon(Z) - C\Phi(y)|.$$

For any $\delta > 0$, there is u_0, that for all $u > u_0$ and $\epsilon > 0$, the second member is less than $\delta/3$. For any $\delta > 0$ and $u > 0$, there is ϵ_0, that for all $\epsilon < \epsilon_0$, both the first member and the third one are less than $\delta/3$. Thus, the convergence is proved. □

2.2.6. *Diffusion approximation*

For Cramér-Lundberg processes, there is no compact formula for the first exit time from an interval. Thus, it is constructed more complexly than, for example, that of the homogeneous Wiener process for which such a formula exists. On the other hand, the Cramér-Lundberg process as a homogeneous process with independent increments for some parameters is a good approximation for the Wiener process. It is reasonable to suppose that distributions of the first exit times of Cramér-Lundberg processes are similar to those of the Wiener process.

Consider this nearness.

THEOREM 2.14.– *(theorem of Donsker)* Let $(X_n)_0^\infty$ be a simple random walk (its sequence of increments $(X_n - X_{n-1})$ is a sequence of i.i.d. random variables); $\mu = E(X_n - X_{n-1})$, $\sigma^2 = D(X_n - X_{n-1})$. Then,

$$\left(\frac{1}{\sigma\sqrt{c}}(X_{[tc]} - t\,c\,\mu)\right)_{t\geq 0} \overset{distr}{\to} (W_t)_{t\geq 0} \quad (c \to \infty), \qquad [2.26]$$

where (W_t) is a standard Wiener process.

PROOF.– (see in Billingsley [BIL 70]). The first application of the Donsker's theorem relates to a family of Cramér-Lundberg processes with safety loadings near zero.

THEOREM 2.15.– *(process with drift)* Let $R_t^{(p)}$ be a Cramér-Lundberg process with a premium rate p and positive η. And then, given $R_0^{(p)} = 0$

$$\left(\frac{p-\varrho}{\sigma^2} R^{(p)}_{t\sigma^2/(p-\varrho)^2}\right)_{t\geq 0} \overset{distr}{\to} (W_t + t)_{t\geq 0} \quad (p \downarrow \varrho), \qquad [2.27]$$

where $p - \varrho = E_0 R_1^{(p)}$, $\varrho = \beta\,\mu_B$, $\sigma^2 = D_0 R_1^{(p)} = \beta\,\mu_B^{(2)}$.

PROOF.– We have $E_0 R_t^{(p)} = (p-\varrho)t$. Denote $X_t = R_t^{(p)}/(p-\varrho)$. According to Donsker's theorem applied to the process with independent increments (X_t),

$$\left(\frac{X_{tc} - tc}{\sqrt{cD_0 X_1}}\right) \to (W_t) \quad (c \to \infty)$$

in distribution. By substituting $c = \sqrt{cD_0X_1}$ (i.e. $c = D_0X_1$), we note that $c = \sigma^2/(p-\varrho)^2 \to \infty$ as $p \to \varrho$. Hence,

$$\left(\frac{X_{tc} - tc}{\sqrt{cD_0X_1}}\right) = \left(\frac{X_{tc}}{\sqrt{cD_0X_1}} - t\right) =$$

$$= \left(\frac{p-\varrho}{\sigma^2} R^{(p)}_{t\sigma^2/(p-\varrho)^2} - t\right) \to (W_t) \quad (p \downarrow \varrho),$$

what is equivalent to the assertion of the theorem. □

2.2.7. *The first exit time for the Wiener process*

Taking into account the nearness of the Cramér-Lundberg process with Wiener process, we would await that their hitting time of a given level be near.

Let us consider the distribution of the first hitting of a level $b > 0$ for a standard Wiener process. For given $t > 0$,

$$P_0^{(W)}(W_t > b) = \int_0^t P_0^{(W)}(\sigma_{(-\infty,b)} \in ds) P_b^{(W)}(W_{t-s} > b) =$$
$$= \frac{1}{2} P_0^{(W)}(\sigma_{(-\infty,b)} \le t).$$

Hence,

$$P_0^{(W)}(\sigma_{(-\infty,b)} \le t) = 2\int_b^\infty \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx.$$

Differentiating with respect to t, we obtain a distribution density for this time:

$$g_b(t) = 2\int_b^\infty \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\right) dx =$$
$$= \frac{1}{\sqrt{2\pi}} \int_b^\infty \left(-t^{-3/2} e^{-x^2/2t} + x^2 t^{-5/2} e^{-x^2/2t}\right) dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_b^\infty \left(-t^{-3/2} e^{-x^2/2t} - x t^{-3/2} \left(e^{-x^2/2t} \right)' \right) dx =$$

$$= -\frac{1}{\sqrt{2\pi}} \int_b^\infty t^{-3/2} e^{-x^2/2t} dx - \frac{1}{\sqrt{2\pi}} \left(x t^{-3/2} e^{-x^2/2t} \Big|_b^\infty - \right.$$

$$\left. - \int_b^\infty t^{-3/2} e^{-x^2/2t} dx \right) = \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t}.$$

Thus, the distribution density of the first hitting time of level $b > 0$ for a standard Wiener process is expressed by the formula:

$$gb(t) = \frac{b}{\sqrt{2\pi t^3}} e^{-b2/2t}. \qquad [2.28]$$

For a Wiener process with a constant multiplier of view $W_t^{(0,\sigma)} \equiv \sigma W_t$ $(t \geq 0)$, the corresponding formula follows by using similarity principle

$$\sigma_{(ac,ab)}(a\zeta) = \sigma_{(c,b)}(\zeta) \quad (a > 0,\, c < b,\, \zeta \in \mathcal{D}), \qquad [2.29]$$

from which it follows:

$$P_0^{(W^{(0,\sigma)})}(\sigma_{(-\infty,b)} \leq t) = P_0^{(W)}(\sigma_{(-\infty,b)}(\sigma\zeta) \leq t) =$$

$$= P_0^{(W)}(\sigma_{(-\infty,b/\sigma)}(\zeta) \leq t)$$

and density of this distribution:

$$g_b^{(0,\sigma)}(t) = g_{b/\sigma}(t) = \frac{b}{\sqrt{2\pi\sigma^2 t^3}} e^{-b^2/2\sigma^2 t}.$$

Let us consider the Wiener process with constant drift, i.e. a process $W_t^{(a)} \equiv W_t^{(a,1)} = W_t + at$ $(a \neq 0)$. The distribution density for the meaning of the process at time t is equal to:

$$f_t^{(a)}(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-at)^2/2t}.$$

We will investigate the first hitting time problem for this process.

For this aim, we will use a change of measure with the help of a martingale. We need to evaluate a derivative of Radon-Nikodym for the distribution of the process with drift with respect to the process without drift. Consider projections of this measures on sigma-algebra $\mathcal{F}_T$. According to well-known results of the measure theory [BIL 70], this derivative is equal to:

$$\frac{dP_{0,T}^{(W^{(a)})}}{dP_{0,T}^{(W)}} = \lim_{n\to\infty} \frac{f_{t_1,\ldots,t_n}^{(a)}(X_{t_1},\ldots,X_{t_n})}{f_{t_1,\ldots,t_n}(X_{t_1},\ldots,X_{t_n})} =$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n} \frac{f_{t_k-t_{k-1}}^{(a)}(X_{t_k}-X_{t_{k-1}})}{f_{t_k-t_{k-1}}(X_{t_k}-X_{t_{k-1}})} =$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n} \exp\left(-\frac{(X_{t_k}-X_{t_{k-1}}-a(t_k-t_{k-1}))^2}{2(t_k-t_{k-1})} + \frac{(X_{t_k}-X_{t_{k-1}})^2}{2(t_k-t_{k-1})}\right) =$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n} \exp\left(a(X_{t_k}-X_{t_{k-1}}) - \frac{a^2}{2}(t_k-t_{k-1})\right) = e^{aX_T - a^2T/2},$$

where $t_k = Tk/n$. Form here, for $S \in \mathcal{F}_T$, a following probability occurs:

$$P_0^{(W^{(a)})}(S) = P_0^{(W^{(a)},T)}(S) = E_0^{(W,T)}(e^{aX_T-a^2T/2};\, S) =$$
$$= E_0^{(W)}(e^{aX_T-a^2T/2};\, S).$$

Let us prove that $M_t \equiv e^{aX_t-a^2t/2}$ is a martingale. For this, it is sufficient to control the martingale equality $E(M_{t+s}|\,\mathcal{F}_t) = M_t$. By using the homogeneous Markov property of the Wiener process, we have for $S \in \mathcal{F}_t$:

$$E_0^{(W)}(e^{aX_{t+s}-a^2(t+s)/2};\, S) = e^{-a^2(t+s)/2} E_0^{(W)}(e^{aX_s\circ\theta_t};\, S) =$$

$$= e^{-a^2(t+s)/2} E_0^{(W)}(E_{X_t}^{(W)} e^{aX_s};\, S) =$$

$$= e^{-a^2(t+s)/2} \int_{-\infty}^{\infty} E_0^{(W)}(E_x^{(W)} e^{aX_s};\, S,\, X_t \in dx) =$$

$$= e^{-a^2(t+s)/2} \int_{-\infty}^{\infty} E_0^{(W)} (E_0^{(W)} e^{a(X_s+x)};\ S,\ X_t \in dx) =$$

$$= e^{-a^2(t+s)/2} E_0^{(W)} (e^{aX_t} E_0^{(W)} e^{aX_s};\ S).$$

In the latter expression,

$$E_0^{(W)} e^{aX_s} = \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{ax - x^2/2s}\, dx = e^{a^2 s/2}.$$

Hence, we obtain:

$$E_0^{(W)} (e^{aX_{t+s} - a^2(t+s)/2};\ S) = E_0^{(W)} (e^{aX_t - a^2 t/2};\ S),$$

Because S is arbitrary, the martingale equality is proved.

Now, we can apply theorem 2.5, and obtain for a Markov time $\tau \equiv \sigma_{(-\infty,b)}$:

$$P_0^{(W^{(a)})}(\tau < t) = E_0^{(W)} (e^{aX_\tau - a^2\tau/2};\ \tau < t)$$
$$= E_0^{(W)} (e^{ab - a^2\tau/2};\ \tau < t) =$$
$$= e^{ab} \int_0^t e^{-a^2 s/2} \frac{b}{\sqrt{2\pi s^3}}\, e^{-b^2/2s}\, ds.$$

Thus, the distribution density of the first exit time for the Wiener process with drift is equal to:

$$g_b^{(a)}(t) = e^{ab} e^{-a^2 t/2} \frac{b}{\sqrt{2\pi t^3}}\, e^{-b^2/2t} = \frac{b}{\sqrt{2\pi t^3}}\, e^{-(b-at)^2/2t}.$$

For fullness of picture, we consider also the Wiener process with constant drift and linear dispersion (such that $DW(t) = \sigma^2 t$). The distribution of such a process equals to the distribution of linear transformation of the standard Wiener process $W_t^{(a,\sigma)} = \sigma W_t + at$ $(t \geq 0)$. Because

$$W_t^{(a,\sigma)} = \sigma(W_t + ta/\sigma) = \sigma W_t^{(a/\sigma,1)} \equiv \sigma W_t^{(a/\sigma)},$$

we can use the similarity principle [2.29]:

$$P_0^{(W^{(a,\sigma)})}(\sigma_{(-\infty,b)} \le t) = P_0^{(W^{(a/\sigma)})}(\sigma_{(-\infty,b)}(\sigma\zeta) \le t) =$$
$$= P_0^{W^{(a/\sigma)}}(\sigma_{(-\infty,b/\sigma)}(\zeta) \le t),$$

hence,

$$g_b^{(a,\sigma)}(t) = g_{b/\sigma}^{(a/\sigma)}(t) = \frac{b}{\sqrt{2\pi\sigma^2 t^3}} e^{-(b-at)2/2\sigma 2t}. \qquad [2.30]$$

2.2.7.1. *Estimation of ruin probability*

In order to use the last result for estimation of ruin probability, we need in analytical form of distribution function of the first hitting time for a process $W^{(a)}$. For given level b, this function has view:

$$\text{IG}\,(t|\,a,b) \equiv P_0^{(W^{(a)})}(\sigma_{(-\infty,b)} \le t) =$$
$$= 1 - \Phi\left(\frac{b-at}{\sqrt{t}}\right) + e^{2ab}\Phi\left(\frac{-b-at}{\sqrt{t}}\right) \quad (t > 0), \qquad [2.31]$$

where Φ is the standard normal distribution function.

In fact, IG $(0|\,a,b) = 0$, and for a positive IG $(\infty|\,a,b) = 1$, and also:

$$\frac{\partial}{\partial t}\text{IG}\,(t|\,a,b) = \frac{1}{2}\,\Phi'\left(\frac{b-at}{\sqrt{t}}\right)(bt^{-3/2} + at^{-1/2}) -$$
$$- \frac{1}{2}e^{2ab}\Phi'\left(\frac{-b-at}{\sqrt{t}}\right)(-bt^{-3/2} + at^{-1/2}) =$$
$$= \frac{1}{2\sqrt{2\pi}}\exp(-(b-at)^2/2t)(bt^{-3/2} + at^{-1/2}) -$$
$$- \frac{1}{2\sqrt{2\pi}}\exp(-(b+at)^2/2t + 2ab)(-bt^{-3/2} + at^{-1/2}) =$$
$$= \frac{1}{2\sqrt{2\pi}}\exp(-(b-at)^2/2t)2bt^{-3/2} = \frac{b}{\sqrt{2\pi}}e^{-(b-at)^2/2t}.$$

Distribution IG $(t|\, a, b)$ is called to be an inverse Gaussian distribution.

Note that, for $a < 0$, the function IG $(t|\, a, b)$ is a defective distribution, because IG $(\infty|\, a, b) = e^{2ab} < 1$.

In the following theorem, we represent asymptotics of ruin probability, when the premium rate tends to ϱ.

THEOREM 2.16.– *(diffusion asymptotics).* For a family of Cramér-Lundberg processes with parameter $p \downarrow \varrho$, it is true that:

$$\psi^{(p)}\left(\frac{u\sigma^2}{p-\varrho}, \frac{T\sigma^2}{(p-\varrho)^2}\right) \to \text{IG}\,(T\,|-1, u),$$

where $\sigma^2 = \beta\mu_B^{(2)}$.

PROOF.– We have:

$$\psi^{(p)}\left(\frac{u\sigma^2}{p-\varrho}, \frac{T\sigma^2}{(p-\varrho)^2}\right) = P_0^{(p)}\left(\tau\left(-\frac{u\sigma^2}{p-\varrho}\right) < \frac{T\sigma^2}{(p-\varrho)^2}\right) =$$

$$= P_0^{(p)}\left(\inf\{R_t : 0 < t < T\sigma^2/(p-\varrho)^2\} < -u\right) =$$

$$= P_0^{(p)}\left(\inf\left\{\frac{p-\varrho}{\sigma^2}R_t : 0 < t < T\sigma^2/(p-\varrho)^2\right\} < -u\right) =$$

$$= P_0^{(p)}\left(\inf\left\{\frac{p-\varrho}{\sigma^2}R_{s\sigma^2/(p-\varrho)^2} : 0 < s < T\right\} < -u\right).$$

According to theorem 2.15, the latter expression has a limit, which is obtained by the change argument of operator inf by using the expression $W_s + s$; it means that this limit is equal to:

$$P_0^{(W^{(1)})}(\sigma_{(-u,\infty)} < T) = P_0^{(W^{(-1)})}(\sigma_{(-\infty,u)} < T) = \text{IG}\,(T\,|-1, u).$$

□

From the last theorem, a practical estimate for ruin probability is as follows:

$$\psi(u,T) \approx \text{IG}\,(T(p-\varrho)^2/\sigma^2 \,|\, -1,\, u(p-\varrho)/\sigma^2), \qquad [2.32]$$

and also

$$\psi(u) \approx \text{IG}\,(\infty \,|\, -1,\, u(p-\varrho)/\sigma^2) = e^{-2u(p-\varrho)/\sigma 2}. \qquad [2.33]$$

3

Models With the Premium Dependent on the Capital

3.1. Definitions and examples

In this chapter, a generalization of the Cramér–Lundberg model in the case of a non-constant premium rate, namely if the premium rate depends on the current capital of the insurance company, is considered:

$$R_t = R_0 + \int_0^t p(R_s)\, ds - \sum_{k=1}^{N_t} U_k, \qquad [3.1]$$

where N_t is the Poisson process for considering the number of claims (requirements, suits) that have arrived up to moment t; (U_n) is independent of the (N_t) sequence of i.i.d. positive random variables (claim sizes) executed by the insurance company; $p(x)$ $(x \in \mathbb{R})$ is a measurable positive function setting dependence of the premium rate on the current capital of the company. Such a process is defined as a temporally homogeneous Markov process, which, however, does not possess a homogeneity in space. This will be determined by a consistent set of probability measures (P_x), where $P_x(R_0 = x) = 1$. These probability measures are set on the initial space of elementary events. Such an elementary event is a sequence of pairs $(t_n, x_n)_0^\infty$, where $t_0 = 0 < t_1 < t_2 < \ldots t_n \to \infty$, and $x_n > 0$. Process trajectories are piecewise continuously increasing curves with negative

Stochastic Risk Analysis and Management, First Edition. Boris Harlamov.

jumps. As before, the probability of ruin of the insurance company is the main interest. These are functions $\psi(u) = P_u(\tau_0 < \infty)$ and $\psi(u, T) = P_u(\tau_0 < T)$, where $\tau_0(\xi) \equiv \sigma_{[0,\infty)}(\xi) = \inf\{t : \xi(t) < 0\}$ $(\xi \in \mathcal{D})$, and $T \geq 0$.

EXAMPLE 3.1.– Let us assume that the company changes the premium rate at the moment its capital reaches the level $v > 0$. Let us denote:

$$p(x) = \begin{cases} p_1 & x \leq v, \\ p_2 & x > v, \end{cases},$$

where $p_1 > p_2$. Such a policy can be called a competition reason, as it gives them advantages over other companies in a competitive struggle. Another possible interpretation is a payment of dividends to members of the company when the capital of the company goes beyond a certain level. This example will be discussed further in more detail.

EXAMPLE 3.2.– (Constant percent). In addition to the basic source of income as premium rates, paid by clients, the company uses a constant income without risk from the location of the capital of the company in bank at a percentage $\epsilon > 0$. In this case, it is possible to suppose

$$p(x) = \epsilon x.$$

This example will also be considered further.

3.1.1. *General properties*

3.1.1.1. *Comparison theorems*

Let us assume further that function $p(x)$ is bounded and

$$\int_0^\infty p(x)\, dx = \infty,$$

provided that process R_t with probability 1 goes beyond any limited interval, and with positive probability through any of the two boundaries of an interval.

Let us compare $\psi(u)$ and $\psi(v)$ at $u > v$. We have

$$\psi(v) = P_v(\tau_0 < \infty) = P_v(R_{\sigma_{[0,u)}} < 0) + P_v(R_{\sigma_{[0,u)}} = u,\ \tau_0 < \infty) =$$
$$= P_v(R_{\sigma_{[0,u)}} < 0) + P_v(R_{\sigma_{[0,u)}} = u,\ \tau_0 \circ \theta_{\sigma_{[0,u)}} < \infty) =$$
$$= P_v(R_{\sigma_{[0,u)}} < 0) + \psi(u) P_u(R_{\sigma_{[0,u)}} = u) =$$
$$= 1 - P_v(R_{\sigma_{[0,u)}} = u) + \psi(u) P_v(R_{\sigma_{[0,u)}} = u),$$

Thus, considering inequality $0 < P_v(R_{\sigma_{[0,u)}} = u) < 1$, we obtain an alternative:

or $(\forall u > v \geq 0) \quad 0 < \psi(u) < \psi(v) < 1$,

or $(\forall u \geq 0) \quad \psi(u) = 1$.

THEOREM 3.1.– *(comparison theorem).* Under the set conditions it holds:

a) if $p(x) \leq \beta\mu_B$ for all large enough x, then $\psi(u) \equiv 1$;

b) if there exists $\epsilon > 0$, for which $p(x) \geq \beta\mu_B + \epsilon$ for all large enough x, then $\psi(u) < 1$ and $P_u(R_t \to \infty\ (t \to \infty)) > 0$.

PROOF.–

a) For all $x \geq u$ and $p(x) \leq \varrho \equiv \beta\mu_B$, we will denote

$$\widetilde{R}_t = u + \varrho t - \sum_{k=1}^{N_t} U_k.$$

It is obvious that $\widetilde{R}_t > R_t$ at all $0 < t \leq \tau_u$ and, hence, $\widetilde{\tau}_u > \tau_u$ for the corresponding moments of the first exit from $[u, \infty)$. Let us denote $(\widetilde{P}_x)$ as a set of measures of the Cramer–Lundberg process with a constant premium rate equal to ϱ. Then,

$$P_u(\tau_u < \infty) = P_u(\inf\{R_t : t > 0\} < u)$$
$$\geq P_u(\inf\{\widetilde{R}_t : t > 0\} < u) =$$

$$= \widetilde{P}_u(\inf\{\widetilde{R}_t : t > 0\} < u)$$
$$= \widetilde{P}_0(\inf\{R_t : t > 0\} < 0) = \widetilde{\psi}(0 = 1.$$

We have furthermore

$$\psi(u) = P_u(\tau_0 < \infty) = P_u(R_{\tau_u} < 0, \tau_u < \infty)$$
$$+ P_u(R_{\tau_u} \in [0, u), \tau_0 < \infty).$$

By denoting the first term of the sum $r(u)$ and using the identity $\tau_0 = \tau_u + \tau_0 \circ \theta_{\tau_u}$, we have:

$$\psi(u) = r(u) + \int_0^u P_u(R_{\tau_u} \in dx, \tau_u < \infty, \tau_0 \circ \theta_{\tau_u} < \infty) =$$
$$= r(u) + \int_0^u P_x(\tau_0 < \infty) P_u(R_{\tau_u} \in dx, \tau_u < \infty).$$

We note that for $x \in [0, u)$:

$$P_x(\tau_0 < \infty) = P_x(R_{\sigma_{[0,u)}} < 0) + \psi(u) P_x(R_{\sigma_{[0,u)}} = u).$$

From here, $\psi(u) = r(u) +$

$$+ \int_0^u (P_x(R_{\sigma_{[0,u)}} < 0) + \psi(u) P_x(R_{\sigma_{[0,u)}} = u)) P_u(R_{\tau_u} \in dx, \tau_u < \infty).$$

We note that $P_u(\tau_u < \infty)$ is equal to:

$$r(u) + \int_0^u (P_x(R_{\sigma_{[0,u)}} < 0) + P_x(R_{\sigma_{[0,u)}} = u)) P_u(R_{\tau_u} \in dx, \tau_u < \infty).$$

As $P_u(\tau_u < \infty) = 1$, we obtain a relation:

$$1 - \psi(u) = (1 - \psi(u)) p_1,$$

where

$$p_1 = \int_0^u P_x(R_{\sigma_{[0,u)}} = u) P_u(R_{\tau_u} \in dx, \tau_u < \infty).$$

This integral is less than 1 (at least because $P_x(R_{\sigma_{[0,u)}} = u) < 1$). From here, we obtain $\psi(u) = 1$ and by the property proved earlier, this equality is true for all $u \geq 0$.

b) Using the fact that $R_t \geq \widetilde{R}_t \equiv u + (\varrho + \epsilon)t - \sum_{k=1}^{N_t} U_k$ up to the moment $\widetilde{\tau}_u$, we obtain $\tau_u \geq \widetilde{\tau}_u$ and

$$P_u(\tau_u < \infty) \leq P_u(\widetilde{\tau}_u < \infty) < 1.$$

$$P_u(\tau_0 < \infty) = P_u(\tau_u < \infty,\ \tau_0 \circ \theta_{\tau_u} < \infty) \leq P_u(\tau_u < \infty) < 1.$$

Let $P_u(\tau_u < \infty) = 1 - \epsilon_1$, where $\epsilon_1 > 0$. Using relations from point a), we obtain:

$$(1 - \psi(u))(1 - p_1) = \epsilon_1 > 0.$$

As $(1 - p_1) > 0$, it is true that $\psi(u) < 1$, and on the property proved earlier, this inequality is true for all $u \geq 0$. Furthermore, for any $x \geq 0$ and homogeneous process with a positive factor of safety, we have:

$$\widetilde{P}_x(R_t \to \infty) = \widetilde{P}_x(R_t/t \to p - \varrho,\ \tau_0 = \infty) = \widetilde{P}_x(\tau_0 = \infty) > 0.$$

From here, the latter statement for all initial points $y \geq u$ is justified. For $x < u$, we have:

$$P_x(R_t \to \infty) \geq P_x(R_t \to \infty,\ R_{\sigma_{[0,u)}} = u) =$$

$$= P_x(R_{\sigma_{[0,u)}+t} \to \infty,\ R_{\sigma_{[0,u)}} = u) =$$

$$= E_x(R_t \circ \theta_{\sigma_{[0,u)}} \to \infty;\ R_{\sigma_{[0,u)}} = u) =$$

$$= P_u(R_t \to \infty) P_x(R_{\sigma_{[0,u)}} = u) > 0. \qquad \square$$

3.1.2. *Accumulation process*

In the book [ASM 00], its author presents a method of how to use the duality between an accumulation process and the corresponding risk process in order to evaluate ruin probability $\psi(u)$. The accumulation process is constructed by the same probability measure on the space of all sequences of pairs (t_n, x_n) as the risk process. The probability measure P on this space is set by the Poisson condition of the sequence (t_n) with the known parameter β, and by the i.i.d. condition of the sequence (x_n) with the known common distribution function B. These sequences are assumed to be mutually independent. Keeping the previous labels, we will define the accumulation process as a non-negative process (V_t), defined by a stochastic integral equation:

$$V_t = V_0 - \int_0^t p^+(V_s)\, ds + \sum_{k=1}^{N_t} U_k \quad (t \geq 0),$$

where

$$p^+(x) = \begin{cases} 0 & x = 0, \\ p(x) & x > 0. \end{cases}$$

According to the definition, this process is a temporally homogeneous Markov process. It begins with zero and remains equal to zero until the moment of the first "inflow". At this moment, the stock increases instantaneously and at once begins "to be spent". If the interval of time to the following "inflow" is large enough, such that the accumulated stock is completely "spent", the process keeps a zero value to the following "inflow". The "inflow" moments represent a Poisson point process.

In Figure 3.1, an accumulation process is represented in the form of a bold dotted non-increasing line on a segment $[0, T]$. Asmussen proposed to connect some risk processes (R_t^*) with this accumulation process. He considered a risk process in the opposite time direction beginning from the point T, where this process has a value u. In Figure 3.1, we have two such processes beginning from points u_1 and u_2 respectively.

It is obvious that the graph of the risk process in the opposite time direction beginning from the point T, where it has a value $u_1 < V_T$, is situated below the graph of the accumulation process and reaches an area of negative value at some point $T - \tau_0$. This point of ruin takes place necessarily because the process V_t begins from the interval of zero values.

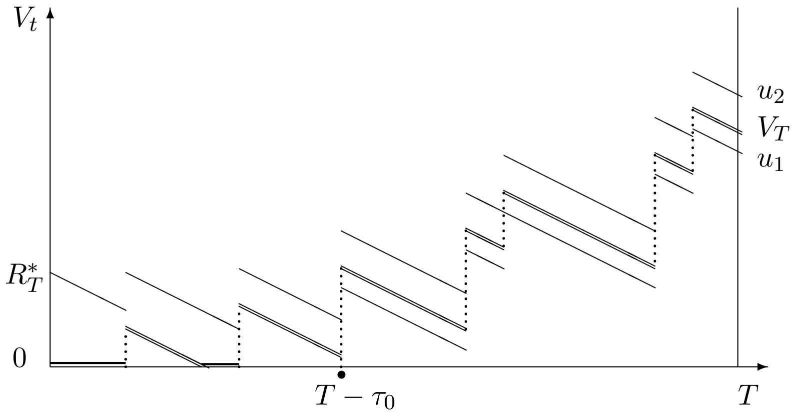

Figure 3.1. *Accumulation process*

Moreover, it is obvious that the graph of the process in the opposite time direction beginning from the point T, where it has a value $u_2 \geq V_T$, is not anywhere below the graph of accumulation process. Hence, it does not reach an area of negative values on an interval $[0, T]$.

Clearly, the distributions of the process of risk in direct and opposite time with respect to the measure P coincide. Thus, the formula is true:

$$P_u(\tau_0 < T) = P_0(V_T > u). \qquad [3.2]$$

Evaluation of the distribution of V_T is not easier than evaluation of the distribution of τ_0. The sense of this formula is uncovered when $T \to \infty$. At the suppositions made, there is a function G of limiting distribution of the random variable V_T. It is much easier to discover than to prelimit. In a limit, we have $\psi(u) = \overline{G}(u)$. Obviously, function G has some positive jump γ_0 in zero. Furthermore, it will be established that the distribution of positive values of an argument is set by some density g. Our next problem is to discover an integral equation concerning these parameters of the risk process.

Limiting distribution is a one-dimensional stationary distribution of a homogeneous Markov process. According to the Kolmogorov–Chapman equation, parameters of limiting distribution satisfy the equation:

$$\int_y^\infty g(x)\,dx = \gamma_0 P_h([y,\infty)\,|\,0) + \int_{0+}^\infty P_h([y,\infty)\,|\,x)g(x)\,dx.$$

It is possible to present a transition function of this process in an aspect:

$$p_h([y,\infty)\,|\,0) = \beta h\overline{B}(y) + o(h),$$
$$p_h([y,\infty)\,|\,x) = (1-\beta h)I_{[y,\infty)}(x - p(x)h) + \beta h\overline{B}(y-x) + o(h).$$

From here, we obtain the right member of an equation:

$$\gamma_0\beta h\overline{B}(y) + \int_{x-p(x)h\geq y}(1-\beta h)g(x)\,dx + \beta h\int_0^\infty \overline{B}(y-x)g(x)\,dx + o(h).$$

Supposing $\overline{B}(y-x) = 1$ at $x \geq y$, we obtain the equation:

$$(1\beta h)\int_y^\infty g(x)\,dx = \gamma_0\beta h\overline{B}(y) + (1-\beta h)\int_{x-p(x)h\geq y} g(x)\,dx + $$
$$+\ \beta h\int_0^y \overline{B}(y-x)g(x)\,dx + o(h)$$

or

$$(1-\beta h)\int_{y<x<y+p(x)h} g(x)\,dx$$
$$= \gamma_0\beta h\overline{B}(y) + \beta h\int_0^y \overline{B}(y-x)g(x)\,dx + o(h).$$

Let us consider an integral in the left part of the equation. Let functions $p(x)$ and $g(x)$ be continuous in some neighborhood of a point y. Then $p(x) = p(y) + \epsilon$, where $\epsilon \to 0$ at $x \to y$, and, hence, at $h \to 0$ thanks to the boundedness of function $p(x)$. From here, the integration area in the first integral has an order $p(y)h + o(h)$. Hence,

$$\frac{1}{h} \int_{y<x<y+p(x)h} g(x)\,dx \to p(y)g(y) \quad (h \to 0),$$

Thus, we obtain the equation:

$$p(y)g(y) = \gamma_0 \beta \overline{B}(y) + \beta \int_0^y \overline{B}(y-x)g(x)\,dx. \qquad [3.3]$$

With respect to $g(x)$, this equation is called an integral equation of Volterrá. Its analytical solution is known only in exceptional cases.

EXAMPLE 3.3.– Let $\overline{B}(x) = e^{-\delta x}$. We have:

$$g(y)p(y) = \gamma_0 \beta e^{-\delta y} + \beta \int_0^y e^{-\delta(y-x)} g(x)\,dx.$$

Multiplying both parts of equality on $e^{\delta y}$, we obtain:

$$g(y)e^{\delta y} = \frac{\gamma_0 \beta}{p(y)} + \frac{\beta}{p(y)} \int_0^y g(x)e^{\delta x}\,dx.$$

By denoting

$$f(y) = \gamma_0 + \int_0^y g(x)\,e^{\delta x}\,dx,$$

we see that, from an initial integral equation, the differential equation follows:

$$f'(y) = \frac{\beta}{p(y)} f(y), \quad f(0) = \gamma_0,$$

whose solution is:

$$f(y) = \gamma_0 \exp\left(\int_0^y \frac{\beta}{p(x)}\,dx \right),$$

and, hence,

$$g(y) = \frac{\beta\gamma_0}{p(y)} \exp\left(\int_0^y \frac{\beta}{p(x)}\,dx - \delta y\right).$$

The probability γ_0 is found from a normalization condition:

$$\gamma_0 + \int_0^\infty g(y)\,dy = 1.$$

From this, it follows that:

$$\gamma_0^{-1} = 1 + \int_0^\infty \frac{\beta}{p(y)} \exp\left(\int_0^y \frac{\beta}{p(x)}\,dx - \delta y\right)\,dy. \qquad \square$$

Further simplifications are connected with an integration possibility at a given function $p(x)$.

Let us note that $\omega(x) = \int_0^x 1/p(s)\,ds$ is possible to interpret as time during which the capital will grow on value x, if there are no single payments at the request of clients.

3.1.3. *Two levels*

Let us assume that the premium rate can accept two values:

$$p(x) = \begin{cases} p_1 & x \le v, \\ p_2 & x > v. \end{cases}$$

We will denote (P_x) as the set of measures of such a process. In addition, two processes homogeneous in space $(P_x^{(1)})$ and $(P_x^{(2)})$ with respective premium rates p_1 and p_2 will be considered. The probabilities of ruin connected with these measures will be denoted as $\psi(u)$, $\psi_1(u)$, $\psi_2(u)$. The following theorem expresses $\psi(u)$ through $(P_x^{(1)})$ and $(P_x^{(2)})$.

THEOREM 3.2.– *(theorem of two levels).* Let $0 \le x < v \le y$. Then,

$$\psi(x) = \psi(v) - q(x)(1 - \psi(v)),$$

$$\psi(y) = p(y) + \psi(v)(\psi_2(y - v) - r(y)),$$

$$\psi(v) = \frac{r(v)}{1 - \psi_2(0) + p(y)},$$

where

$$q(x) = \frac{\psi_1(x) - \psi_1(v)}{1 - q(x)},$$

$$r(y) = P_y^{(2)}(\tau_v < \infty,\ R_{\tau_v} < 0) + \int_0^v (1 - q(x))\, P_y^{(2)}(\tau_v < \infty,\ R_{\tau_v} \in dx);$$

Thus, $p(y)$ is a probability that the moment of reaching a negative half-plane will be before the moment of intersection of level v from below.

PROOF.– We have:

$$\psi_1(x) = P_x^{(1)}(R_{\sigma_{[0,v)}} < 0) + P_x^{(1)}(R_{\sigma_{[0,v)}} = v,\ \tau_0 \circ \theta_{\sigma_{[0,v)}} < \infty) = 1 - P_x^{(1)}(R_{\sigma_{[0,v)}} = v) + \psi_1(v) P_x^{(1)}(R_{\sigma_{[0,v)}} = v).$$

Let us denote $P_x^{(1)}(R_{\sigma_{[0,v)}} = v) = q(x)$ that corresponds to the definition of this value in the theorem condition. As measures P_x and $P_x^{(1)}$ coincide up to the moment of the first exit from set $[0, v)$ (i.e. on a sigma-algebra $\mathcal{F}_{\sigma_{[0,v)}}$),

$$\psi(x) = 1 - P_x(R_{\sigma_{[0,v)}} = v) + \psi(v) P_x(R_{\sigma_{[0,v)}} = v) = 1 - P_x^{(1)}(R_{\sigma_{[0,v)}} = v) + \psi(v) P_x^{(1)}(R_{\sigma_{[0,v)}} = v) = 1 - q(x) + \psi(v) q(x).$$

Furthermore,

$$\psi(y) = P_y(\tau_v < \infty,\, R_{\tau_v} < 0)$$
$$+ P_y(\tau_v < \infty,\, R_{\tau_v} \in [0,v),\, R_{\tau_v \dot{+} \sigma_{[0,v)}} < 0) +$$
$$+ P_y(\tau_v < \infty,\, R_{\tau_v} \in [0,v),\, R_{\tau_v \dot{+} \sigma_{[0,v)}}$$
$$= v,\, \tau_0 \circ \theta_{\tau_v \dot{+} \sigma_{[0,v)}} < \infty).$$

By denoting $p(y)$ for the sum of first two items, we have:

$$\psi(y) = p(y) + \psi(v) P_y(\tau_v < \infty,\, R_{\tau_v} \in [0,v),\, R_{\tau_v \dot{+} \sigma_{[0,v)}} = v) =$$
$$= p(y) + \psi(v)(P_y(\tau_v < \infty,\, R_{\tau_v} \in [0,v)) -$$
$$- P_y(\tau_v < \infty,\, R_{\tau_v} \in [0,v),\, R_{\tau_v \dot{+} \sigma_{[0,v)}} < 0)) =$$
$$= p(y) + \psi(v)(P_y(\tau_v < \infty) - P_y(\tau_v < \infty,\, R_{\tau_v} < 0)$$
$$- P_y(\tau_v < \infty,\, R_{\tau_v} \in [0,v),\, R_{\tau_v \dot{+} \sigma_{[0,v)}} < 0)) =$$
$$= p(y) + \psi(v)(P_y(\tau_v < \infty) - p(y)).$$

We note that the measure P_y coincides with a measure $P_y^{(2)}$ up to the moment of the first exit from set $[v,\infty)$, from which it follows that $P_y(\tau_v < \infty) = P_y^{(2)}(\tau_v < \infty)$, and because of homogeneity in space of the second process, the latter expression is equal to $P_{y-v}^{(2)}(\tau_0 < \infty) = \psi_2(y-v)$. The value of probability $\psi(v)$ turns out to be an equation solution in which the second formula is reduced at $y = v$. Using representation $R_{\tau_v \dot{+} \sigma_{[0,v)}} = R_{\sigma_{[0,v)}} \circ \theta_{\tau_v}$ and the Markov property of process, we obtain the second member in representation of $p(y)$:

$$P_y(\tau_v < \infty,\, R_{\tau_v} \in [0,v),\, R_{\tau_v \dot{+} \sigma_{[0,v)}} < 0) =$$
$$= E_y(P_{R_{\tau_u}}(R_{\sigma_{[0,v)}} < 0);\, \tau_v < \infty,\, R_{\tau_v} \in [0,v)) =$$
$$= \int_0^v P_x^{(1)}(R_{\sigma_{[0,v)}} < 0) P_y^{(2)}(\tau_v < \infty,\, R_{\tau_v} \in dx) =$$

$$= \int_0^v (1 - q(x))\, P_y^{(2)}(\tau_v < \infty,\, R_{\tau_v} \in dx). \qquad \square$$

EXAMPLE 3.4.– Let $\overline{B}(x) = e^{-\delta x}$. Then,

$$\psi_1(u) = \frac{\beta}{p_1\delta} e^{-\gamma_1 u}, \quad \psi_2(u) = \frac{\beta}{p_2\delta} e^{-\gamma_2 u},$$

where $\gamma_i = \delta - \beta/p_i$ $(i = 1, 2)$. From here, we deduce the formula for $q(x)$ at $0 \le x < v$. Furthermore, at $y \ge v$, thanks to the special property of the exponential distribution with independent time of ruin and value of the capital (debt) at the moment of ruin, we have:

$$P_y^{(2)}(\tau_v < \infty,\, R_{\tau_v} < s) = P_{y-v}^{(2)}(\tau_0 < \infty,\, R_{\tau_0} + v < s) =$$
$$= P_{y-v}^{(2)}(\tau_0 < \infty) P(-U_1 < s - v) = \psi_2(y - v) e^{-\delta(v-s)}.$$

From here,

$$r(y) = \psi_2(y - v) \left(e^{-\delta v} + \int_0^v (1 - q(x)) \delta e^{-\delta(v-x)}\, dx \right) =$$
$$= \psi_2(y - v) \left(1 - \int_0^v q(x) \delta e^{-\delta(v-x)}\, dx \right) =$$
$$= \frac{\psi_2(y - v)}{1 - \beta e^{-\gamma_1 v}/(p_1\delta)} \left(1 - \frac{\beta e^{-\gamma_1 v}}{p_1\delta} \right.$$
$$\left. - \int_0^v \left(1 - \frac{\beta e^{-\gamma_1 x}}{p_1\delta} \right) \delta e^{-\delta(v-x)}\, dx \right) =$$
$$= \frac{\psi_2(y - v)}{1 - \beta e^{-\gamma_1 v}/(p_1\delta)} e^{-\gamma_1 v} \left(1 - \frac{\beta}{p_1\delta} \right) =$$
$$= \frac{\beta}{p_2\delta} e^{-\gamma_2(y-v) - \gamma_1 v} \frac{1 - \beta/(p_1\delta)}{1 - \beta e^{-\gamma_1 v}/(p_1\delta)}. \qquad \square$$

3.1.4. *Interest rate*

Let us consider a model with the premium rate defined by formula $p(x) = p + \epsilon x$ (example 2). In this case, the current capital satisfies a stochastic integral equation:

$$R_t = R_0 + pt + \epsilon \int_0^t R_s \, ds - \sum_{k=1}^{N_t} U_k,$$

which can be considered as the determinate integral equation for each sample sequence (t_n, x_n) and an initial point u. The random series of pairs is denoted as (σ_n, U_n). We will further use the same labels also for sample sequences, stipulating the sense of a label every time it can lead to a misunderstanding. Therefore, at points $t = \sigma_k$, the process R_t has negative jumps $R_{\sigma_k} - R_{\sigma_k - 0} = U_k$, and on each interval (σ_k, σ_{k+1}), the sample function R_t satisfies a differential equation:

$$f' = p + \epsilon f, \quad f(\sigma_k) = R_{\sigma_k},$$

and, hence, it is given by:

$$R_t = R_{\sigma_k} e^{\epsilon(t-\sigma_k)} + \frac{p}{\epsilon}\left(e^{\epsilon(t-\sigma_k)} - 1\right).$$

EXAMPLE 3.5.– Let $\overline{B}(x) = e^{-\delta x}$ and $p(x) = p + \epsilon x$ $(x \geq 0,\ p > 0)$. According to example 3:

$$\begin{aligned} g(x) &= \frac{\gamma_0 \beta}{p + \epsilon x} \exp\left(\int_0^x \frac{\beta}{p + \epsilon s} \, ds - \delta x\right) = \\ &= \frac{\gamma_0 \beta}{p + \epsilon x} \exp\left(\frac{\beta}{\epsilon} \log(p + \epsilon x) - \frac{\beta}{\epsilon} \log p - \delta x\right) \\ &= \frac{\gamma_0 \beta}{p^{\beta/\epsilon}} (p + \epsilon x)^{\beta/\epsilon - 1} e^{-\delta x}. \end{aligned}$$

From here,

$$\psi(u) = \int_u^\infty g(x) \, dx = \frac{\gamma_0 \beta}{p^{\beta/\epsilon}} \int_u^\infty (p + \epsilon x)^{\beta/\epsilon - 1} e^{-\delta x} \, dx.$$

By substituting $s = (p + \epsilon x)\delta/\epsilon$, we have:

$$\psi(u) = \frac{\gamma_0 \beta}{\epsilon} \left(\frac{\epsilon}{\delta p}\right)^{\beta/\epsilon} e^{p\delta/\epsilon} \Gamma\left(\frac{\delta(p + \epsilon u)}{\epsilon}, \frac{\beta}{\epsilon}\right),$$

where $\Gamma(x, a) = \displaystyle\int_x^\infty t^{a-1} e^{-t}\, dt \quad (x > 0,\, a > 0)$ is an incomplete gamma function. The parameter γ_0 is determined from expression:

$$\frac{1}{\gamma_0} = 1 + \frac{\beta}{p^{\beta/\epsilon}} \int_0^\infty (p + \epsilon x)^{\beta/\epsilon - 1} e^{-\delta x}\, dx \equiv$$

$$\equiv 1 + \frac{\beta}{\epsilon} \left(\frac{\epsilon}{\delta p}\right)^{\beta/\epsilon} e^{p\delta/\epsilon} \Gamma\left(\frac{\delta p}{\epsilon}, \frac{\beta}{\epsilon}\right),$$

form which follows

$$\psi(u) = \frac{\Gamma(\delta(p/\epsilon + u),\, \beta/\epsilon)}{(\delta p/\epsilon)^{\beta/\epsilon} e^{-p\delta/\epsilon} \epsilon/\beta + \Gamma(\delta p/\epsilon,\, \beta/\epsilon)}.$$

□

3.1.5. *Shift on space*

It is interesting to compare the sample trajectory beginning from value $u > 0$, with the selective trajectory that begins with zero. We will prove the formula:

$$R_t^{(u)} = e^{\epsilon t} u + R_t^{(0)}. \qquad [3.4]$$

Let us denote $\widetilde{R}_t^{(u)}$ as the expression on the right-hand side. Both processes begin with the value u at $t = 0$. Both have negative jumps of the same magnitude in the same places. In this case:

$$\widetilde{R}_t^{(u)} = u\left(\int_0^t \epsilon e^{\epsilon s}\, ds + 1\right) + pt + \epsilon \int_0^t R_s^{(0)}\, ds - A_t =$$

$$= u + pt + \epsilon \int_0^t (R_s^{(0)} + e^{\epsilon s} u)\, ds - A_t$$

$$= u + pt + \epsilon \int_0^t \widetilde{R}_s^{(0)}\, ds - A_t,$$

where $A_t = \sum_{k=1}^{N_t} U_k$. Thus, $\widetilde{R}_t^{(u)}$ satisfies the same integral equation, as $R_t^{(u)}$. Hence, they are equal to each other. □

NOTE.– Furthermore, we denote $R_t^{(u)}$ $(u \geq 0)$ as the function set on space Ω of all admissible sequences of pairs (t_n, x_n) on which the measure P is set; similarly, let us denote R_t as the sample function from space $\mathcal{D}$ on which the family of measures P_u $(u \geq 0)$ is set. The relation of measures follows from the obvious identity:

$$P_u(R_t \geq x) = P(R_t^{(u)} \geq x).$$

From formula [3.4], it follows that $E_u f(R_t) = E_0 f(R_t + ue^{\epsilon t})$ $(\epsilon \geq 0)$. In particular, at $\epsilon = 0$, we come to a rule of replacement of argument in a set of measures of space homogeneous processes. The moment of the first exit from the given interval is transformed by replacement of a measure for a process with the interest rate in a more difficult way. That is, the interval with constant boundaries will be transformed to an interval with variable boundaries to which corresponds the first exit. In the given course, we will not introduce special labels for such Markov moments.

3.1.6. *Discounted process*

Let us consider a process with an exponential multiplier:

$$Z_t = e^{-\epsilon t} R_t^{(0)}, \qquad [3.5]$$

that is usually interpreted as a discounting initial cost (according to the previous note, Z_t is understood as a function on Ω).

Let $\widetilde{R}_t = \widetilde{R}_0 + pt - A_t$ $(t \geq 0)$ be the Cramér–Lundberg process corresponding to process R_t. We will prove the formula:

$$Z_t = \int_0^t e^{-\epsilon s}\, d\widetilde{R}_s^{(0)}, \qquad [3.6]$$

that represents a stochastic integral concerning a process with locally limited variations, such as $\widetilde{R}_t$ (note that in formula [3.6], the increment

$d\widetilde{R}_s^{(0)}$ can be substituted by $d\widetilde{R}_s^{(u)}$). Let us denote $\widetilde{Z}_t$ as a right member of this equality. At $t = 0$, both parts of equality are equal to zero. In a point σ_k, increments of both processes are equal to $-U_k$. At a point $t \in (\sigma_k, \sigma_{k+1})$, we have:

$$Z_t' = -\epsilon e^{-\epsilon t} R_t^{(0)} + e^{-\epsilon t}(p + \epsilon R_t^{(0)}) = e^{-\epsilon t} p,$$

$$\widetilde{Z}_t' = \left(\int_0^t e^{-\epsilon s}\, d(ps)\right)' = e^{-\epsilon t} p,$$

from which the demanded equality for each sample trajectory (i.e. for any sequence (σ_n, U_n)) follows. □

On the basis of the proved formula, it is obvious to derive the formulas expressing the probability of ruin in terms of the associated Cramér–Lundberg process. We denote:

$$Z = \int_0^\infty e^{-\epsilon s}\, d\widetilde{R}_s^{(0)},$$

if this integral exists in the sense of convergence P-a.s. Let $H(z) = P(Z \le z)$ be a cumulative distribution function of Z and $\widehat{H}(\alpha) = Ee^{\alpha Z}$.

THEOREM 3.3.– *(convergence of the discounted process).* For a considered process of risk:

$$Z_t \overset{\text{a.s.}}{\to} Z,$$

$$\widehat{H}(\alpha) = \exp\left(\int_0^\infty \kappa(-\alpha e^{-\epsilon t})\, dt\right) \exp\left(\int_0^\alpha \frac{1}{\epsilon y} \kappa(-y)\, dy\right),$$

where $\kappa(\alpha) = \beta(\widehat{B}(\alpha) - 1) - p\,\alpha$.

PROOF.– Denote $M_t = A_t - \varrho t$ ($\varrho = \beta\mu_B$). It is obvious that (M_t) is a martingale concerning natural filtration. From the theory of martingales, it follows that the stochastic integral of this martingale is a martingale too. For example:

$$X_t = \int_0^t e^{-\epsilon s}\, dM_s.$$

In this case:

$$EX_t = 0, \quad DX_t = \int_0^t e^{-2\epsilon s} \beta \mu_B^{(2)} \, ds = \frac{\beta \mu_B^{(2)}}{2\epsilon} \, (1 e^{-2\epsilon t}).$$

Under the well-known theorem on the convergence of quadratically integrable martingales [LIP 86], [HAR 06], we have:

$$Z_t = \int_0^t e^{-\epsilon s} \, d\widetilde{R}_s^{(0)} =$$

$$= \int_0^t e^{-\epsilon s} \left((p - \varrho) ds - dM_s\right) \xrightarrow{\text{a.s}} \int_0^\infty e^{-\epsilon s} \left((p - \varrho) ds - dM_s\right) =$$

$$= \int_0^\infty e^{-\epsilon s} \, d\widetilde{R}_s^{(0)} = Z.$$

On the other hand, using the independence of increments, we have:

$$E e^{\alpha Z} = E \exp\left(\alpha \int_0^\infty e^{-\epsilon t} \, d\widetilde{R}_t^{(0)}\right) =$$

$$= \lim_{h \to 0} E \exp\left(\alpha \sum_{n=1}^\infty e^{-\epsilon n h} (\widetilde{R}_{nh}^{(0)} - \widetilde{R}_{(n-1)h}^{(0)})\right) =$$

$$= \lim_{h \to 0} \prod_{n=1}^\infty E_0 \exp(\alpha e^{-\epsilon n h} \widetilde{R}_h^{(0)}) = \lim_{h \to 0} \prod_{n=1}^\infty \exp(h \kappa(-\alpha e^{-\epsilon n h})) =$$

$$= \lim_{h \to 0} \exp\left(\sum_{n=1}^\infty h \kappa(-\alpha e^{-\epsilon n h})\right) = \exp\left(\int_0^\infty \kappa(-\alpha e^{-\epsilon t}) \, dt\right).$$

By applying replacement $y = \alpha e^{-\epsilon t}$, we obtain the second representation of $\widehat{H}(\alpha)$. □

In the following theorem, the probability of ruin is presented through a conditional average.

THEOREM 3.4.– *(representation of probability of ruin).* For a considered process:

$$\psi(u) = \frac{H(-u)}{E_u(H(-R_{\tau_0}) \mid \tau_0 < \infty)}.$$

PROOF.– From formulas [3.5] and [3.6], it follows that $Z_t + u = e^{-\epsilon t} R_t^{(u)}$. This formula remains true at replacement t on any finite function $\tau(\omega)$ $(\omega \in \Omega)$. From here,

$$Z + u = (Z_\tau + u) + (Z - Z_\tau) = e^{-\epsilon\tau} R_\tau^{(u)} + \int_\tau^\infty e^{-\epsilon t}\, d\widetilde{R}_t^{(0)} =$$

$$= e^{-\epsilon\tau} \left(R_\tau^{(u)} + \int_\tau^\infty e^{-\epsilon(t-\tau)}\, d\widetilde{R}_t^{(0)} \right) =$$

$$= e^{-\epsilon\tau} \left(R_\tau^{(u)} + \int_\tau^\infty e^{-\epsilon(t-\tau)}\, d\widetilde{R}_t^{(u)} \right).$$

It is possible to write the latter expression as a function of a set of the trajectories which go out at point u, and for which it makes sense to consider $\tau \equiv \tau_0$. Therefore,

$$H(-u) = P(Z + u < 0)$$

$$= P_u \left(e^{-\epsilon\tau_0} \left(R_{\tau_0} + \int_{\tau_0}^\infty e^{-\epsilon(t-\tau_0)}\, d\widetilde{R}_t \right) < 0 \right) =$$

$$= P_u \left(R_{\tau_0} + \int_{\tau_0}^\infty e^{-\epsilon(t-\tau_0)}\, d\widetilde{R}_t < 0 \right).$$

Let us note that Z depends only on the increments of process R_t and consequently has the same value at different initial points of a trajectory. From here,

$$P_u(\tau_0 = \infty,\, u + Z < 0) = P_u((\forall t)\, R_t \geq 0,\, u + Z < 0) =$$

$$= P((\forall t)\, R_t^{(u)} \geq 0,\, u + Z < 0)$$

$$= P((\forall t)\, e^{\epsilon t} u + R_t^{(0)} \geq 0,\, u + Z < 0) =$$

$$= P((\forall t)\, u + Z_t \geq 0,\, u + Z < 0) = 0.$$

Hence,

$$H(-u) = P_u\left(R_{\tau_0} + \int_{\tau_0}^{\infty} e^{-\epsilon(t-\tau_0)}\, d\widetilde{R}_t < 0,\, \tau_0 < \infty\right).$$

Changing the variables $t_1 = t - \tau_0$ and using representation $R_{\tau+t_1} = R_{t_1} \circ \theta_{\tau_0}$, in this expression, it is possible to present an integral in an aspect:

$$\int_0^{\infty} e^{-\epsilon t_1}\, d\widetilde{R}_{t_1} \circ \theta_{\tau_0} = Z \circ \theta_{\tau_0}.$$

Thus, using the Markov property, we obtain:

$$H(-u) = \int_{-\infty}^{0} P_x(x + Z < 0) P_u(R_{\tau_0} \in dx,\, \tau_0 < \infty),$$

and as $P_x(x + Z < 0) = P(x + Z < 0) \equiv H(-x)$,

$$H(-u) = \int_{-\infty}^{0} H(-x) P_u(R_{\tau_0} \in dx,\, \tau_0 < \infty) =$$

$$= E_u(H(-R_{\tau_0}),\, \tau_0 < \infty) = \psi(u) E_u(H(-R_{\tau_0}) \,|\, \tau_0 < \infty).$$

□

EXAMPLE 3.6.– Let $\overline{B}(x) = e^{-\delta x}$ and $p(x) = p + \epsilon x$ $(x \geq 0,\, p > 0)$. We use theorem 3.4 for the determination of the probability of ruin. We have:

$$\kappa(c) = \beta c/(\delta - c) - pc,$$

$$\widehat{H}(\alpha) = \exp\left(\int_0^{\alpha} \frac{1}{\epsilon x} \kappa(-x)\, dx\right)$$

$$= \exp\left(\int_0^{\alpha} \left(-\frac{\beta/\epsilon}{\delta + x} + \frac{p}{\epsilon}\right) dx\right) =$$

$$= \exp\left(\frac{p\alpha}{\epsilon} - \frac{\beta}{\epsilon}\log(\delta + \alpha) + \frac{\beta}{\epsilon}\log\delta\right) = e^{p\alpha/\epsilon}\left(\frac{\delta}{\delta + \alpha}\right)^{\beta/\epsilon}.$$

From the latter expression, it follows that Z is distributed like $p/\epsilon - X$, where the random variable X has a gamma distribution with parameters β/ϵ (forms) and δ (scale), whence, in particular, it follows that $Z \le p/\epsilon$. Hence,

$$H(-u) = P(p/\epsilon - X \le -u) = P(X \ge p/\epsilon + u) =$$
$$= \int_{p/\epsilon+u}^{\infty} \frac{\delta^{\beta/\epsilon}}{\Gamma(\beta/\epsilon)} x^{\beta/\epsilon-1} e^{-\delta x}\, dx = \frac{\Gamma(\delta(p/\epsilon+u), \beta/\epsilon)}{\Gamma(\beta/\epsilon)}.$$

Using independence R_{τ_0} and τ_0 and that $-R_{\tau_0}$ and U_1 have the same exponential distribution, we obtain:

$$E_u(H(-R_{\tau_0}) \,|\, \tau_0 < \infty) = E(H(U_1)) = \int_0^\infty P(Z \le x)\delta e^{-\delta x}\, dx =$$
$$= \int_0^\infty P(X \ge p/\epsilon - x)\delta e^{-\delta x}\, dx =$$
$$= -e^{-\delta x} P(X \ge p/\epsilon - x)\,|_0^\infty + \int_0^\infty e^{-\delta x}(P(X \ge p/\epsilon - x))'_x\, dx.$$

The derivative of the integrant is equal to zero at $x \ge p/\epsilon$. By denoting f as the density of a gamma distribution, the estimated magnitude becomes equal to:

$$P(X \ge p/\epsilon) + \int_0^{p/\epsilon} e^{-\delta x} f(p/\epsilon - x)\, dx =$$
$$= \frac{\Gamma(\delta p/\epsilon,\, \beta/\epsilon)}{\Gamma(\beta/\epsilon)} + \int_0^{p/\epsilon} e^{-\delta x} \frac{\delta^{\beta/\epsilon}}{\Gamma(\beta/\epsilon)} (p/\epsilon - x)^{\beta/\epsilon-1} e^{-\delta(p/\epsilon - x)}\, dx =$$
$$= \frac{\Gamma(\delta p/\epsilon,\, \beta/\epsilon)}{\Gamma(\beta/\epsilon)} + \frac{\delta^{\beta/\epsilon}}{\Gamma(\beta/\epsilon)} e^{-\delta p/\epsilon} \int_0^{p/\epsilon} (p/\epsilon - x)^{\beta/\epsilon-1}\, dx =$$
$$= \frac{\Gamma(\delta p/\epsilon,\, \beta/\epsilon)}{\Gamma(\beta/\epsilon)} + \frac{\delta^{\beta/\epsilon}}{\Gamma(\beta/\epsilon)} e^{-\delta p/\epsilon} \int_0^{p/\epsilon} x^{\beta/\epsilon-1}\, dx =$$
$$= \frac{\Gamma(\delta p/\epsilon,\, \beta/\epsilon)}{\Gamma(\beta/\epsilon)} + \frac{\delta^{\beta/\epsilon}}{\Gamma(\beta/\epsilon)} e^{-\delta p/\epsilon} \frac{(p/\epsilon)^{\beta/\epsilon}}{\beta/\epsilon}.$$

As a result, we obtain:

$$\psi(u) = \frac{\Gamma(\delta(p/\epsilon + u), \beta/\epsilon)}{\Gamma(\delta p/\epsilon,\ \beta/\epsilon) + e^{-\delta p/\epsilon}(p\delta/\epsilon)^{\beta/\epsilon}\epsilon/\beta}$$

which coincides with the expression found in example 5. □

3.1.7. *Local factor of Lundberg*

We consider the local factor of Lundberg as a positive solution of an equation:

$$\beta(\widehat{B}(\alpha) - 1) - p(x)\alpha = 0.$$

Let us denote:

$$J(u) = \int_0^u \gamma(x)\, dx,$$

where $\gamma(x)$ is a local factor of Lundberg at a preset value $p(x)$ $(x \geq 0)$. Let us consider some problems of the risk theory with the premium rate depending on the current capital in which function $J(u)$ is used for an estimation of the probability of ruin.

EXAMPLE 3.7.– Again, we will consider $\overline{B}(x) = e^{-\delta x}$. In this case, $\gamma(x) = \delta - \beta/p(x)$ and $J(u) = \delta u - \beta\omega(u)$, where $\omega(u) = \int_0^u p(x)^{-1}dx$. We will assume that $J(\infty) = \infty$. Then, from the outcomes of example 3.3, it follows that:

$$\frac{1}{\gamma_0}\int_u^\infty g(x)\, dx = \int_u^\infty \frac{\beta}{p(x)} \exp\left(\beta \int_0^x p(s)^{-1}ds - \delta x\right) dx =$$

$$= \int_u^\infty \left(\exp\left(\beta\int_0^x p(s)^{-1}ds\right)\right)' e^{-\delta x}dx =$$

$$= \exp\left(\beta \int_0^x p(s)^{-1}ds - \delta x\right)\Bigg|_u^\infty + \delta\int_u^\infty \exp\left(\beta\int_0^x p(s)^{-1}ds - \delta x\right) dx =$$

$$= -\exp\left(\beta\int_0^u p(s)^{-1}ds - \delta x\right) + \delta\int_u^\infty \exp\left(\beta\int_0^x p(s)^{-1}ds - \delta x\right) dx =$$
$$= -e^{-J(u)} + \delta\int_u^\infty e^{-J(x)}dx,$$

and also

$$\frac{1}{\gamma_0} = 1 + \int_0^\infty \frac{\beta}{p(x)} \exp\left(\beta\int_0^x p(s)^{-1}ds - \delta x\right) dx = \delta\int_0^\infty e^{-J(x)}dx.$$

Hence,

$$\psi(u) = \frac{-e^{-J(u)} + \delta\int_u^\infty e^{-J(x)}dx}{\delta\int_0^\infty e^{-J(x)}dx} =$$
$$= e^{-J(u)}\frac{-1 + \delta\int_0^\infty \exp\left(-\int_0^x \gamma(u+s)\,ds\right)\,dx}{\delta\int_0^\infty e^{-J(x)}dx}. \qquad \square$$

3.1.7.1. *Generalization of Lundberg's inequality and Cramér's theorem*

THEOREM 3.5.– *(generalization of Lundberg's inequality).* Let us assume that $p(x)$ is a non-decreasing positive function of x. Then,

$$\psi(u) \leq e^{-J(u)}. \qquad [3.7]$$

PROOF.– Denoting $\gamma_1(u)$ as a positive solution of the next equation with respect to α:

$$E\exp(-\alpha(R_{\sigma_1}^{(u)} - u)) = 1 \quad (\alpha > 0).$$

LEMMA 3.1.– *(comparison of local factors).* If $p(x)$ is a non-decreasing function of x, then:

a) $\gamma(x)$ and $\gamma_1(x)$ are non-decreasing functions of x;

b) $\gamma(x) \leq \gamma_1(x)$.

PROOF OF LEMMA.–

a) Function $\gamma(x)$ does not decrease because values of this function are abscissas of cross points of the non-decreasing convex (downwards) function $\beta(\widehat{B}(\alpha) - 1)$ and a ray $p(x)\alpha$, going out of one point. For analysis of function $\gamma_1(x)$, we will note, at first, that, up to the moment of the first jump (a point σ_1), it is true that $R_t^{(u)} \leq R_t^{(v)}$ as $0 \leq u < v$, which follows from a solution uniqueness theorem for the differential equation $f' = p(f)$. Furthermore, both functions $f_u(t) = R_t^{(u)} - u$ and $f_v(t) = R_t^{(v)} - v$ at $t = 0$ are equal to zero, and, at any $t > 0$ on the given segment, $f'_u(t) \leq f'_v(t)$. From here, $f_u(t) \leq f_v(t)$ on this segment. Hence,

$$E\exp(-\alpha(R_{\sigma_1}^{(u)} - u)) \geq E\exp(-\alpha(R_{\sigma_1}^{(v)} - v)) \quad (\alpha \leq \gamma_1(u)),$$

from which it follows that $\gamma_1(v) \geq \gamma_1(u)$ because the second derivative with respect to α is positive.

b) As $p(x)$ does not decrease, $R_t^{(u)} - u \geq p(u)t$ $(0 \leq t < \sigma_1)$. This is enough to compare initial points and derivatives of these functions. From here,

$$\begin{aligned} 1 &= E\exp(-\gamma_1(u)(R_{\sigma_1}^{(u)} - u)) \\ &= E\exp(-\gamma_1(u)(R_{\sigma_1-0}^{(u)} - U_1 - u)) \leq \\ &\leq E\exp(-\gamma_1(u)(p(u)\sigma_1 - U_1)) = \frac{\beta}{\beta + \gamma_1(u)p(u)}\widehat{B}(\gamma_1(u)), \end{aligned}$$

that is:

$$\beta(\widehat{B}(\gamma_1(u)) - 1) - \gamma_1(u)p(u) \geq 0,$$

from which it follows that $\gamma_1(u) \geq \gamma(u)$. □

Prolongation of the proof of the theorem. Let

$$\psi_n(u) = P_u(\tau_0 \leq \sigma_n).$$

Let us prove on an induction that for any $n \geq 0$:

$$\psi_n(u) \leq \exp(-\int_0^u \gamma_1(x)\, dx).$$

It is true for $n = 0$. Let it be true for n. We denote:

$$F_u(x) = P(u - R_{\sigma_1}^{(u)} \leq x).$$

We have:

$$\begin{aligned}
\psi_{n+1}(u) &= P_u(R_{\sigma_1} < 0) + P_u(R_{\sigma_1} \geq 0,\, \tau_0 \leq \sigma_{n+1}) = \\
&= P_u(R_{\sigma_1} - u < -u) + P_u(R_{\sigma_1} - u \geq -u,\, \tau_0 \circ \theta_{\sigma_1} \leq \sigma_n \circ \theta_{\sigma_1}) = \\
&= P_u(u - R_{\sigma_1} > u) + E_u(P_{U-(u-R_{\sigma_1})}(\tau_0 \leq \sigma_n);\, u - R_{\sigma_1} \leq u) = \\
&= \overline{F}_u(u) + \int_{-\infty}^{u} \psi_n(u - x)\, dF_u(x) \leq \\
&\leq \int_u^{\infty} dF_u(x) + \int_{-\infty}^{u} e^{-\int_0^{u-x} \gamma_1(s)\, ds}\, dF_u(x) = \\
&= e^{-\int_0^u \gamma_1(s)\, ds} \left(\int_u^{\infty} e^{\int_0^u \gamma_1(s)\, ds} dF_u(x) + \int_{-\infty}^{u} e^{\int_{u-x}^{u} \gamma_1(s)\, ds}\, dF_u(x) \right).
\end{aligned}$$

We note that in the first integral:

$$\int_0^u \gamma_1(s)\, ds \leq \gamma_1(u)u \leq \gamma_1(u)x,$$

and in the second integral:

$$\int_{u-x}^{u} \gamma_1(s)\, ds \leq \gamma_1(u).$$

From here,

$$\begin{aligned}
\psi_{n+1}(u) &\leq e^{-\int_0^u \gamma_1(s)\, ds} \int_{-\infty}^{\infty} e^{\gamma_1(u)x}\, dF_u(x) = \\
&= e^{-\int_0^u \gamma_1(s)\, ds} E e^{\gamma_1(u)(u - R_{\sigma_1}^{(u)})} = e^{-\int_0^u \gamma_1(s)\, ds},
\end{aligned}$$

and, as $(\forall x)\ \gamma(x) \leq \gamma_1(x)$, inequality [3.7] is proved. □

The following theorem shows that the boundary of probability of ruin obtained from the previous theorem, can be reached by a process with a non-increasing premium rate with both sufficiently big frequency streams of requirements and sufficiently small values of requirements themselves. Let $(P_x^{(\epsilon)})$ be a set of measures of risk process with parameters $\beta_\epsilon \equiv \beta/\epsilon$ (intensity of a Poisson stream of requirements), $B_\epsilon(x) \equiv B(x/\epsilon)$ (a cumulative distribution function of a required value) and $p(x)$ be a non-decreasing function which is a premium rate of the risk process, dependent on the current capital. Other functions of the process with a given set of measures will also be denoted with the index ϵ.

THEOREM 3.6.– *(ruin at the big frequency of requirements).* The probability of ruin of a process with measures $(P_x^{(\epsilon)})$ satisfies a condition:

$$\lim_{\epsilon\downarrow 0} \epsilon \log \psi_\epsilon(u) = -J(u).$$

PROOF.– We have $\widehat{B}_\epsilon(\alpha) = \widehat{B}(\alpha\epsilon)$. Furthermore,

$$\begin{aligned} 0 &= \beta_\epsilon(\widehat{B}_\epsilon(\gamma_\epsilon(x)) - 1) - \gamma_\epsilon(x)p(x) \\ &= \beta(\widehat{B}(\epsilon\gamma_\epsilon(x)) - 1) - \epsilon\gamma_\epsilon(x)p(x) = \\ &= \beta(\widehat{B}(\gamma(x)) - 1) - \gamma(x)p(x). \end{aligned}$$

From here, $\epsilon\gamma_\epsilon(x) = \gamma(x)$ and $J_\epsilon(x) = J(x)/\epsilon$. Based on the previous theorem, it follows that:

$$\psi_\epsilon(u) \le e^{-J(x)/\epsilon},$$

Thus,

$$\liminf_{\epsilon\downarrow 0}(-\epsilon \log \psi_\epsilon(u)) \ge J(u).$$

It remains to prove that the upper bound of the considered function is no longer $J(u)$. We have:

$$\begin{aligned} \psi(u) = P_u(\tau_0 < \infty) &\ge P_u(R_{\sigma_{[u-r,u+r)}} < u-r,\, \tau_0 < \infty) \ge \\ &\ge \int_{-\infty}^{u-r} \psi(x) P_u(R_{\sigma_{[u-r,u+r)}} \in dx) \\ &\ge \psi(u-r)\, P_u(R_{\sigma_{[u-r,u+r)}} < u-r). \end{aligned}$$

From here, at $r = u/n$ $(n \geq 2)$ and $u_k = ku/n$, we obtain:

$$\psi(u) \geq \prod_{k=1}^{n} P_{u_k}(R_{\sigma_{[u_k-r,u_k+r)}} < u_k - r).$$

Furthermore, having designated $m_k = p(u_k + r)$ and $\widetilde{R}_{t,k}^{(u_k)}$ a trajectory of the Cramér–Lundberg process with an initial point u_k and parameters β, B, m_k (premium rate), we have $\widetilde{R}_{t,k}^{(u_k)} \geq R_t^{(u_k)}$, at $t \leq \sigma_{[u_k-r,u_k+r)}$ concerning a measure P, from which it follows that:

$$P_{u_k}(R_{\sigma_{[u_k-r,u_k+r)}} < u_k - r) = P(R^{(u_k)}_{\sigma_{[u_k-r,u_k+r)}} < u_k - r) \geq$$

$$\geq P(\widetilde{R}^{(u_k)}_{\sigma_{[u_k-r,u_k+r)},k} < u_k - r) = \widetilde{P}_{u_k}^{(k)}(R_{\sigma_{[u_k-r,u_k+r)}} < u_k - r),$$

where $(\widetilde{P}_x^{(k)})$ is a set of measures of the Cramér–Lundberg process with parameters β, B, m_k. Hence, using a homogeneity in the space of Cramér–Lundberg processes, we obtain:

$$\psi(u) \geq \prod_{k=1}^{n} \widetilde{P}_{u_k}^{(k)}(R_{\sigma_{[u_k-r,u_k+r)}} < u_k - r) = \prod_{k=1}^{n} \widetilde{P}_r^{(k)}(R_{\sigma_{[-0,2r)}} < 0).$$

Having designated $(\widetilde{P}_x^{(k,\epsilon)})$ as a set of measures of a Cramér–Lundberg process with parameters β_ϵ, B_ϵ, m_k, we obtain a similar formula:

$$\psi_\epsilon(u) \geq \prod_{k=1}^{n} \widetilde{P}_r^{(k,\epsilon)}(R_{\sigma_{[0,2r)}} < 0).$$

On the other hand, having designated $\widetilde{\psi}_{k,\epsilon}(u)$ as the corresponding probability of ruin, we have:

$$\widetilde{\psi}_{k,\epsilon}(r) = \widetilde{P}_r^{(k,\epsilon)}(\tau_0 < \infty) =$$

$$= \widetilde{P}_r^{(k,\epsilon)}(R_{\sigma_{[0,2r)}} < 0) + \widetilde{P}_r^{(k,\epsilon)}(R_{\sigma_{[0,2r)}} = 2r,\ \tau_0 < \infty) =$$

$$= \widetilde{P}_r^{(k,\epsilon)}(R_{\sigma_{[0,2r)}} < 0) + \widetilde{\psi}_{k,\epsilon}(2r)\widetilde{P}_r^{(k,\epsilon)}(R_{\sigma_{[0,2r)}} = 2r) \leq$$

$$\leq \widetilde{P}_r^{(k,\epsilon)}(R_{\sigma_{[0,2r)}} < 0) + \widetilde{\psi}_{k,\epsilon}(2r).$$

Thus,

$$\widetilde{P}_r^{(k,\epsilon)}(R_{\sigma_{[0,2r)}} < 0) \geq \widetilde{\psi}_{k,\epsilon}(r) - \widetilde{\psi}_{k,\epsilon}(2r).$$

It should be noted that replacement of the measure in the initial probabilistic space which determines the passage from parameters β, B to parameters β_ϵ, B_ϵ, corresponds to the time change $t \mapsto t/\epsilon$ in the Poisson process and scale change $U_k \mapsto U_k\epsilon$ in the sequence of independent variables, and consequently:

$$\begin{aligned}\widetilde{\psi}_{k,\epsilon}(u) &= \widetilde{P}_u^{(k,\epsilon)}((\exists t \geq 0)\, R_t < 0) = \\ &= P^{(\epsilon)}\left((\exists t \geq 0)\, u + m_k t - \sum_{i=1}^{N_t} U_i < 0\right) = \\ &= P\left((\exists t \geq 0)\, u + (m_k\epsilon)(t/\epsilon) - \sum_{i=1}^{N(t/\epsilon)} \epsilon U_i < 0\right) = \\ &= P\left((\exists t \geq 0)\, u + m_k\epsilon t - \sum_{i=1}^{N_t} \epsilon U_i < 0\right) = \\ &= P\left((\exists t \geq 0)\, u/\epsilon + m_k t - \sum_{i=1}^{N_t} U_i < 0\right) = \widetilde{\psi}_k(u/\epsilon).\end{aligned}$$

According to Cramér's theorem:

$$\widetilde{\psi}_k(u/\epsilon) \sim C_k e^{-\gamma_k u/\epsilon} \quad (\epsilon \to 0),$$

where γ_k corresponds to m_k, which is equal to $\gamma(u_k + r)$. From here,

$$\widetilde{\psi}_{k,\epsilon}(r) - \widetilde{\psi}_{k,\epsilon}(2r) \sim C_k e^{-\gamma_k r/\epsilon} \quad (\epsilon \to 0),$$

and, hence,

$$\begin{aligned}&-\log \psi_\epsilon(u) \leq -\sum_{k=1}^{n} \log(\widetilde{\psi}_{k,\epsilon}(r) - \widetilde{\psi}_{k,\epsilon}(2r)) \sim \\ &\sim \sum_{k=1}^{n} \gamma_k\, r/\epsilon - \sum_{k=1}^{n} \log C_k.\end{aligned}$$

From here, it follows that:

$$\limsup_{\epsilon \downarrow 0}(-\epsilon \log \psi_\epsilon(u)) \leq \sum_{k=1}^{n} \gamma((k+1)u/n)\, u/n.$$

Directing n to infinity and using a monotonicity of $\gamma(x)$, we obtain:

$$\limsup_{\epsilon \downarrow 0}(-\epsilon \log \psi_\epsilon(u)) \leq \int_0^u \gamma(x)\, dx \equiv J(u). \qquad \square$$

4

Heavy Tails

4.1. Problem of heavy tails

Among the general stream of requirements on compensation of damage resulting from an insured event, there exist rare, but very big requirements that result from force-major circumstances, such as earthquakes, tsunamis, hurricanes (typhoons, cyclones, storms), flooding, tornadoes, fires, acts of terrorism, revolts, wars and technical catastrophes. These cases sharply change the statistical picture made under simple requirements. Such modifications call into question the basic premise of the theory of Cramér-Lundberg; that the magnitude of the requirement as the random variable belongs to the so-called class of Cramér (the supposition about generating function existence). Therefore, there is a necessity to develop a section of the risk theory without this supposition. The risk theory in the presence of heavy tails of requirement magnitude distributions is such a theory.

4.1.1. *Tail of distribution*

A tail of distribution $P(X \in \, dx)$ of a positive random variable X was fixed to the function $\overline{F}(x) = P(X > x)$. The tail is called heavy if $Ee^{\alpha X} = \infty$ for any $\alpha > 0$. An example of a distribution with a heavy tail is the log-normal distribution, which is the distribution of the random variable $X = e^Z$, where Z is a standard normal variable. Other popular distributions with heavy tails are Pareto, Weibool and

Stochastic Risk Analysis and Management, First Edition. Boris Harlamov.

log-gamma distributions, each of which will be dealt with explicitly in frames of two basic classes of distributions with heavy tails: subexponential and regularly varying.

Log-normal distribution:

Let $Y = e^X$, where $X \in N(a, \sigma^2)$. We have:

$$F_Y(x) \equiv P(Y \leq x) = P(X \leq \log x),$$

$$f_Y(x) = F'_Y(x) = f_X(\log x)/x$$

$$= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-(\log x - a)^2/2\sigma^2),$$

and for any positive α, it holds that:

$$Ee^{\alpha Y} = \int_0^\infty \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(\alpha y - \frac{(\log y - a)^2}{2\sigma^2}\right) dy \geq$$

$$\geq \int_0^\infty \frac{1}{y\sigma\sqrt{2\pi}} \exp(\alpha y)\, dy = \infty.$$

Log-gamma distribution:

It is a distribution of the random variable $Y = e^X$, where $X \in \text{Gam}(a, \delta)$ (a gamma distribution with the form parameter a and the scale parameter δ). The density of this distribution is given by:

$$f_X(x) = \frac{\delta}{\Gamma(a)}(ax)^{a-1}e^{-\delta x} \quad (a,\, \delta > 0).$$

Thus,

$$Ee^{\alpha Y} = \int_0^\infty e^{\alpha y} \frac{\delta}{\Gamma(a)}(ay)^{a-1}e^{-\delta y}\, dy = \infty$$

at any $\alpha > 0$.

4.1.2. *Subexponential distribution*

4.1.2.1. *Maximum and the sum*

Let positive random variables X_1, X_2, ..., X_n be independent and have cumulative distribution functions $B_1, \dots, B_n$ respectively. Then,

$$P(\max\{X_1, \dots X_n\} > x) = P\left(\bigcup_{k=1}^{n} \{X_k > x\}\right) =$$

$$= \sum_{k=1}^{n} \overline{B}_k(x) \prod_{i=1}^{k-1} B_i(x) = \sum_{k=1}^{n} \overline{B}_k(x)(1 + o(1)) \quad (x \to \infty).$$

In addition,

$$P(\max\{X_1, \dots X_n\} > x) \leq P\left(\sum_{k=1}^{n} X_k > x\right) \equiv \overline{B_1 * \dots * B_n}(x).$$

From here, we have:

$$\liminf_{x\to\infty} \frac{\overline{B_1 * \dots * B_n}(x)}{\sum_{k=1}^{n} \overline{B}_k(x)} \geq 1. \qquad [4.1]$$

Let all B_i be identical and equal to B. Then,

$$P(\max\{X_1, \dots X_n\} > x) = n\overline{B}(x)(1 + o(1)) \quad (x \to \infty).$$

$$P(\max\{X_1, \dots X_n\} > x) \leq \overline{B^{(n)}}(x),$$

where $B^{(n)}$ is the n-fold convolution of a cumulative distribution function B. From here, we have:

$$\lim_{x\to\infty} \frac{P(\max\{X_1, \dots X_n\} > x)}{\overline{B}(x)} = n,$$

$$\liminf_{x\to\infty} \frac{\overline{B^{(n)}}(x)}{\overline{B}(x)} \geq n.$$

DEFINITION 4.1.– The distribution P of a positive random variable X is called a subexponential distribution if for any $n \geq 2$,

$$\frac{\overline{B^{(n)}}(x)}{\overline{B}(x)} \to n \quad (x \to \infty).$$

Furthermore, the adjective "subexponential" will also be used in relation to the corresponding cumulative distribution function and to the random variable. We will denote $\mathcal{S}$ as the class of subexponential distributions on $(0, \infty)$.

DEFINITION 4.2.– The distribution tail $\overline{B}$ is said to be regularly varying (at infinity) with an index $\alpha > 0$ if

$$\overline{B}(x) = \frac{L(x)}{x^{\alpha}} \quad (x \to \infty),$$

where L is a slowly varying function (by Karamata), i.e. for any $t > 0$,

$$\frac{L(xt)}{L(x)} \to 1 \quad (x \to \infty).$$

Let us denote RV_{α} as the class of all regularly varying functions with an index α; thus, RV_0 is the class of all the functions slowly varying at infinity.

An example of a slowly varying function is any function having a non-zero limit at infinity, and also $\log x$, any finite degree of a logarithm and any finite iteration of a logarithm (i.e. $\log^{(n)}(x)$, where $\log^{(1)}(x) = \log(x)$, $\log^{(n+1)}(x) = \log(\log^{(n)}(x))$ for $n \geq 1$ and at x for which this operation is determined).

EXAMPLE 4.1.– Pareto distribution

It is a distribution with a tail $\overline{B}(x) = (c/(c+x))^{a}$ $(c > 0,\ a > 0)$. A tail of this distribution is a regularly varying function with an index a and with a slowly varying function $L(x) = (cx/(c+x))^{a} \to c^{a}$.

LEMMA 4.1.– If $\overline{B}_1, \overline{B}_2 \in RV_\alpha$, then $\overline{B_1 * B_2} \in RV_\alpha$ and

$$\frac{\overline{B_1 * B_2}(x)}{\overline{B}_1(x) + \overline{B}_2(x)} \to 1 \quad (x \to \infty),$$

where $B_1 * B_2$ is a convolution of these functions.

PROOF.– Let X_1 and X_2 be the positive random variables with regularly varying tails $\overline{B}_1$, $\overline{B}_2$ with the same index α. If $\delta \in (0, 1/2)$ and $X_1 + X_2 > x$, then either the two magnitudes are greater than δx or at least one of the magnitudes is greater than $(1-\delta)x$ (Figure 4.1). It means that:

$$P(X_1+X_2 > x) \leq \overline{B}_1((1-\delta)x)+\overline{B}_2((1-\delta)x)+\overline{B}_1(\delta x)\overline{B}_1(\delta x) =$$
$$= (L_1((1-\delta)x) + L_2((1-\delta)x))(1+o(1))/((1-\delta)x)^\alpha \sim$$
$$\sim (L_1(x) + L_2(x))(1+o(1))/((1-\delta)x)^\alpha \quad (x \to \infty).$$

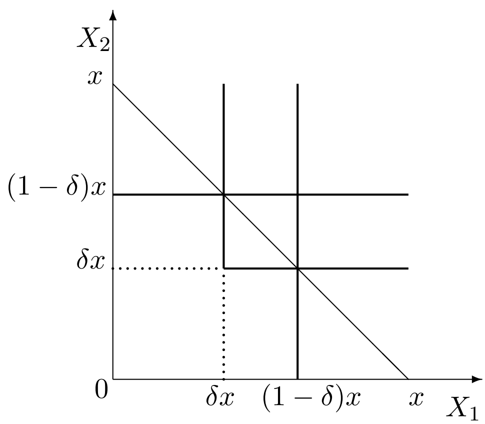

Figure 4.1. *Graphical representation of theorem 4.1*

From here and from an inequality (4.1), we obtain a comparison:

$$1 \leq \liminf_{x\to\infty} \frac{P(X_1 + X_2 > x)}{\overline{B}_1(x) + \overline{B}_2(x)} \leq \limsup_{x\to\infty} \frac{P(X_1 + X_2 > x)}{\overline{B}_1(x) + \overline{B}_2(x)} \leq (1-\delta)^{-\alpha}.$$

Supposing $\delta \to 0$, we obtain:

$$\lim_{x\to\infty} \frac{P(X_1 + X_2 > x)}{\overline{B}_1(x) + \overline{B}_2(x)} = 1.$$

It is obvious that the sum of slowly varying functions is a slowly varying function too. From this, we can see that $\overline{B_1 * B_2}$ is a regularly varying function with an index α, and

$$\overline{B_1 * B_2}(x) \sim \overline{B}_1(x) + \overline{B}_2(x) \quad (x \to \infty).$$ □

THEOREM 4.1.– *(on a regularly varying tail).* If the cumulative distribution function B has a regularly varying tail with an index $\alpha > 0$, then $B \in \mathcal{S}$.

PROOF.– Using lemma 4.1 and a mathematical induction, we obtain for any $n \geq 2$,

$$\overline{B^{(n)}}(x) \sim n\overline{B}(x) \quad (x \to \infty),$$

thus it follows that $B \in \mathcal{S}$. □

In the following theorem, the tail asymptotics are estimated.

THEOREM 4.2.– *(difference tail).*

If $\overline{B^{(2)}}(x)/\overline{B}(x) \to 2$, then for any $y_0 > 0$,

$$\frac{\overline{B}(x-y)}{\overline{B}(x)} \to 1 \quad (x \to \infty)$$

uniformly on $y \in [0, y_0]$.

PROOF.– It is given that:

$$\frac{\overline{B^{(2)}}(x)}{\overline{B}(x)} = 1 + \frac{B(x) - B^{(2)}(x)}{\overline{B}(x)} = 1 + \overline{B}(x)^{-1} \int_0^x (1 - B(x-s))\, dB(s) =$$

$$= 1 + \overline{B}(x)^{-1} \int_0^y \overline{B}(x-s)\,dB(s) + \overline{B}(x)^{-1} \int_y^x \overline{B}(x-s)\,dB(s) \geq$$

$$\geq 1 + B(y) + \frac{\overline{B}(x-y)}{\overline{B}(x)}(B(x) - B(y)).$$

At a large enough value of x, $B(x) - B(y) > 0$. From here, we have:

$$1 \leq \frac{\overline{B}(x-y)}{\overline{B}(x)} \leq \left(\frac{\overline{B^{(2)}}(x)}{\overline{B}(x)} - 1 - B(y) \right) (B(x) - B(y))^{-1}.$$

The right-hand side member of this inequality tends to 1 as $x \to \infty$. From this, it follows that there is a limit to this ratio which is equal to 1. In addition, for any $y \in (0, y_0)$,

$$\overline{B}(x-y)/\overline{B}(x) - 1 \leq \overline{B}(x-y_0)/\overline{B}(x) - 1,$$

a uniform convergence at y results from this interval. □

THEOREM 4.3.– *(convolution tail).* If $\overline{B^{(2)}}(x)/\overline{B}(x) \to 2$ as $x \to \infty$, then $B \in \mathcal{S}$.

PROOF.– Let $\overline{B^{(n-1)}}(x)/\overline{B}(x) \to n-1$ as $x \to \infty$ for some $n > 2$. Then,

$$\frac{\overline{B^{(n)}}(x)}{\overline{B}(x)} = 1 + \int_0^x \frac{\overline{B^{(n-1)}}(x-s)}{\overline{B}(x)}\,dB(s) =$$

$$= 1 + \int_0^{x-y} \frac{\overline{B^{(n-1)}}(x-s)}{\overline{B}(x-s)} \frac{\overline{B}(x-s)}{\overline{B}(x)}\,dB(s) +$$

$$+ \int_{x-y}^x \frac{\overline{B^{(n-1)}}(x-s)}{\overline{B}(x-s)} \frac{\overline{B}(x-s)}{\overline{B}(x)}\,dB(s).$$

Let us denote $J_1(x,y)$, $J_2(x,y)$ as the first and second integrals respectively. We have in the first integral $y \leq x - s \leq x$ and, consequently,

$$\inf_{s>y} \frac{\overline{B^{(n-1)}}(s)}{\overline{B}(s)} \int_0^{x-y} \frac{\overline{B}(x-s)}{\overline{B}(x)}\,dB(s) \leq J_1(x,y) \leq$$

$$\leq \sup_{s>y} \frac{\overline{B^{(n-1)}}(s)}{\overline{B}(s)} \int_0^{x-y} \frac{\overline{B}(x-s)}{\overline{B}(x)} \, dB(s).$$

Thus,

$$\int_0^{x-y} \frac{\overline{B}(x-s)}{\overline{B}(x)} \, dB(s) = \frac{B(x) - B^{(2)}(x)}{\overline{B}(x)} - \int_{x-y}^{x} \frac{\overline{B}(x-s)}{\overline{B}(x)} \, dB(s) \to 1,$$

as according to theorem 4.2 at any $y > 0$,

$$\int_{x-y}^{x} \frac{\overline{B}(x-s)}{\overline{B}(x)} \, dB(s) \leq \frac{(B(x) - B(x-y))\,\overline{B}(x-y)}{\overline{B}(x)} \to 0 \quad (x \to \infty).$$

From here, it follows that:

$$\inf_{s>y} \frac{\overline{B^{(n-1)}}(s)}{\overline{B}(s)} \leq \liminf_{x\to\infty} J_1(x,y) \leq \limsup_{x\to\infty} J_1(x,y) \leq \sup_{s>y} \frac{\overline{B^{(n-1)}}(s)}{\overline{B}(s)}.$$

Also, the second integral at any $y > 0$ is given by:

$$J_2(x,y) \leq \sup_{s\geq 0} \frac{\overline{B^{(n-1)}}(s)}{\overline{B}(s)} \int_{x-y}^{x} \frac{\overline{B}(x-s)}{\overline{B}(x)} \, dB(s) \to 0 \quad (x \to \infty).$$

If we choose a high enough value of y, we are convinced that the limit at $x \to \infty$ of the ratio $\overline{B^{(n)}}(x)/\overline{B}(x)$ differs from n on any $\epsilon > 0$. □

THEOREM 4.4.– *(subexponential decrease of a tail).*

If $B \in \mathcal{S}$, then $e^{\epsilon x}\overline{B}(x) \to \infty$ $(x \to \infty)$ and $\widehat{B}(\epsilon) = \infty$ at any $\epsilon > 0$, i.e. the distribution has the heavy tail.

PROOF.– From the theorem 4.2, it follows that for any $\delta \in (0, \epsilon)$, there exists n_0 such that at all $n \geq n_0$, it is true that $\overline{B}(n)/\overline{B}(n+1) \leq e^{\delta}$. From here, it follows that $\overline{B}(n+1) \geq e^{-\delta}\overline{B}(n)$ and for any $m \geq 1$,

$$\overline{B}(n+m) \geq e^{-\delta m}\overline{B}(n).$$

From this, it follows that for some $c > 0$, $\overline{B}(x) \geq ce^{-\delta x}$, and, hence, $e^{\epsilon x}\overline{B}(x) \to \infty$ $(x \to \infty)$. From here, we have:

$$\int_0^\infty e^{\epsilon s}\, dB(s) \geq \int_x^\infty e^{\epsilon s}\, dB(s) \geq e^{\epsilon x}\overline{B}(x) \to \infty \quad (x \to \infty).$$

□

The following theorem characterizes a class of subexponential distributions by properties of "hazard rate", i.e. a conditional distribution density function of the claim value $\lambda(x) = b(x)/\overline{B}(x)$ concerning a condition that the magnitude of the claim exceeds the given level (here $b(x) = B'(x)$).

THEOREM 4.5.– *(theorem of Pitman).* Let B have a derivative b, and suppose that the hazard rate $\lambda(x)$ strictly decreases as $x \to \infty$ since some $x_0 \geq 0$ and

$$\int_0^\infty e^{x\lambda(x)} b(x)\, dx < \infty.$$

Then $B \in \mathcal{S}$.

PROOF.– According to theorem 4.3, it is enough to prove that the ratio $\overline{B^{(2)}}(x)/\overline{B}(x)$ tends to 2 as $x \to \infty$. Without loss of generality, it is possible to consider that $\lambda(x)$ does not increase from zero. We have:

$$\frac{\overline{B^{(2)}}(x)}{\overline{B}(x)} - 1 = \int_0^x \frac{\overline{B}(x-s)}{\overline{B}(x)}\, dB(s) =$$
$$= \int_0^x e^{\Lambda(x)-\Lambda(x-s)-\Lambda(s)} \lambda(s)\, ds,$$

where $\Lambda(x) = \int_0^x \lambda(s)\, ds$. From here, we see that the integral is equal to:

$$\int_0^{x/2} e^{\Lambda(x)-\Lambda(x-s)-\Lambda(s)}\, \lambda(s)\, ds + \int_{x/2}^x e^{\Lambda(x)-\Lambda(x-s)-\Lambda(s)} \lambda(s)\, ds =$$

$$= \int_0^{x/2} e^{\Lambda(x)-\Lambda(x-s)-\Lambda(s)} \lambda(s)\, ds + \int_0^{x/2} e^{\Lambda(x)-\Lambda(x-z)-\Lambda(z)} \lambda(x-z)\, dz,$$

where the replacement $x-s=z$ is used. At $s<x/2$ in the first integral,

$$\Lambda(x)-\Lambda(x-s)=\int_{x-s}^{x}\lambda(y)\,dy\le\lambda(x-s)\cdot s\le\lambda(s)\cdot s.$$

From here, it follows that:

$$\int_0^{x/2}e^{\Lambda(x)-\Lambda(x-s)-\Lambda(s)}\,\lambda(s)\,ds\le\int_0^{\infty}e^{\lambda(s)s}\,b(s)\,ds<\infty.$$

Hence, in the first integral it is possible to pass to a limit of the integrand as $x\to\infty$. We obtain:

$$\int_0^{\infty}I_{[0,x/2)}(s)e^{\Lambda(x)-\Lambda(x-s)-\Lambda(s)}\lambda(s)\,ds\to 1,$$

because for any $s>0$, the limit of the integrand is equal to $b(s)$. At $z<x/2$ in the second integral, $\lambda(x-z)\le\lambda(z)$ and $\Lambda(x)-\Lambda(x-z)\le\lambda(x-z)\cdot z\le\lambda(z)\cdot z$. From here, it follows that:

$$\int_0^{x/2}e^{\Lambda(x)-\Lambda(x-z)-\Lambda(z)}\lambda(x-z)\,dz\le\int_0^{\infty}e^{\lambda(x)x}b(x)\,dx<\infty.$$

Passing to a limit of the integrand as $x\to\infty$, we see that the second integral tends to zero. From here, it follows that $\overline{B^{(2)}}(x)/\overline{B}(x)\to 2$. □

EXAMPLE 4.2.– Weibool distribution

Let us consider the Weibool distribution. It is a distribution with a tail:

$$\overline{B}(x)=e^{-\alpha x^{\beta}},$$

where α, $\beta>0$. For this distribution, the hazard rate is equal to:

$$\lambda(x)\equiv b(x)/\overline{B}(x)=\alpha\beta x^{\beta-1},$$

where $b(x)=-(\overline{B}(x))'$. We have:

$$\int_0^{\infty}e^{x\lambda(x)}b(x)\,dx=\int_0^{\infty}e^{\alpha\beta x^{\beta}}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx=$$

$$= \int_0^\infty e^{\alpha(\beta-1)x^\beta} \alpha\beta x^{\beta-1}\,dx,$$

which is equal to a finite value at $\beta < 1$. In this case, according to the theorem of Pitman, this distribution is subexponential and, hence, it has a heavy tail according to theorem 4.4.

4.1.3. *Cramér-Lundberg process*

4.1.3.1. *Theory of the process*

LEMMA 4.2.– *(geometrical decrease of the ratio).* If $B \in \mathcal{S}$, then for any $\epsilon > 0$, there exists $K \equiv K_\epsilon$ such that for any $n \geq 2$ and $x > 0$,

$$\frac{\overline{B^{(n)}}(x)}{\overline{B}(x)} \leq K(1+\epsilon)^n.$$

PROOF.– For any given $\epsilon > 0$, there exists $T > 0$ such that at any $x \geq T$,

$$\frac{B(x) - B^{(2)}(x)}{\overline{B}(x)} < 1 + \epsilon.$$

From here, it follows that:

$$\alpha_n \equiv \sup_{x\geq 0} \frac{\overline{B^{(n)}}(x)}{\overline{B}(x)} = \sup_{x\geq 0}\left(1 + \int_0^x \frac{\overline{B^{(n-1)}}(x-s)}{\overline{B}(x)}\,dB(s)\right) \leq$$

$$\leq 1 + \sup_{x<T} \int_0^x \frac{\overline{B^{(n-1)}}(x-s)}{\overline{B}(x)}\,dB(s) +$$

$$+ \sup_{x\geq T} \int_0^x \frac{\overline{B^{(n-1)}}(x-s)}{\overline{B}(x-s)}\,\frac{\overline{B}(x-s)}{\overline{B}(x)}\,dB(s) \leq$$

$$\leq 1 + (\overline{B}(T))^{-1} + \alpha_{n-1} \sup_{x\geq T} \int_0^x \frac{\overline{B}(x-s)}{\overline{B}(x)}\,dB(s) \leq$$

$$\leq 1 + (\overline{B}(T))^{-1} + \alpha_{n-1}(1+\epsilon) \leq$$

$$\leq (1 + (\overline{B}(T))^{-1}) \sum_{k=0}^{n-2} (1+\epsilon)^k + \alpha_1(1+\epsilon)^{n-1} \leq K(1+\epsilon)^n,$$

where $\alpha_1 = 1$ and $K = (1 + (\overline{B}(T))^{-1})/\epsilon$. □

LEMMA 4.3.– *(convolution of a random number of distributions).* Let $Y_1, Y_2, \ldots$ be the i.i.d. random variables with the common cumulative distribution function $G \in \mathcal{S}$, and let Z be a non-negative integer random variable independent of (Y_n) such that $Es^Z < \infty$ for some $s > 1$. Then,

$$\frac{P(Y_1 + \cdots + Y_Z > x)}{\overline{G}(x)} \to EZ \quad (x \to \infty).$$

PROOF.– According to lemma 4.2 for any $s > 1$, there exists K such that:

$$\frac{\overline{G^{(n)}}(x)}{\overline{G}(x)} \leq Ks^n$$

for any $x > 0$. From here, it follows that:

$$\frac{P(Y_1 + \cdots + Y_Z > x)}{\overline{G}(x)} = \sum_{n=0}^{\infty} P(Z = n) \frac{\overline{G^{(n)}}(x)}{\overline{G}(x)} \leq$$

$$\leq K \sum_{n=0}^{\infty} P(Z = n) s^n \equiv KEs^Z < \infty.$$

Hence, it is possible to pass to a limit of the summand:

$$\frac{P(Y_1 + \cdots + Y_Z > x)}{\overline{G}(x)} \to \sum_{n=0}^{\infty} P(Z = n) \cdot n = EZ.$$

□

In the following theorem, notations and some outcomes presented in Chapter 2 are used. First of all, it is a formula of Pollaczek-Khinchin:

$$\psi(u) = (1 - \varrho/p) \sum_{k=1}^{\infty} (\varrho/p)^k \overline{B_0^{(k)}}(u),$$

where $\varrho = \beta\mu_B$, $B_0(x) = \mu_B^{-1} \int_0^x \overline{B}(s)\, ds$.

THEOREM 4.6.– *(probability of ruin).*

If $B_0 \in \mathcal{S}$, then

$$\psi(u) \sim \frac{\varrho}{p-\varrho}\overline{B_0}(u) \quad (u \to \infty).$$

PROOF.– Derivation of the Pollaczek-Khinchin formula is based on the representation of the ladder height process in an aspect:

$$M = Y_1 + \cdots + Y_Z$$

where Z is the random number of steps of a ladder. It has a geometrical distribution with parameter $\varrho\,/p$, so:

$$P(Z = k) = (\varrho\,/p)^k(1 - \varrho\,/p) \quad (k \geq 0).$$

The conditional cumulative distribution function of the kth step, if it exists, is found to be B_0. By lemma 4.3, it follows that:

$$\psi(u) = P(Y_1 + \cdots + Y_Z > u) \sim \overline{B_0}(u)EZ \quad (u \to \infty).$$

Thus,

$$EZ = (1-\varrho\,/p)\sum_{k=1}^{\infty} k(\varrho\,/p)^k = (1-\varrho\,/p)\varrho\,/p\left(\sum_{k=0}^{\infty} x^k\right)'\Bigg|_{x=\varrho\,/p} =$$

$$= \frac{(1-\varrho\,/p)\,\varrho\,/p}{(1-\varrho\,/p)^2} = \frac{\varrho}{p-\varrho}.$$

□

The following theorem shows that the tail $\overline{B}_0$ is somewhat "heavier" than the tail $\overline{B}(x)$.

THEOREM 4.7.– *(comparison of tails).*

If $B \in \mathcal{S}$, $\overline{B}_0(x)/\overline{B}(x) \to \infty$ as $x \to \infty$.

PROOF.– According to theorem 4.2, we have at any $a > 0$:

$$\frac{\overline{B}_0(x)}{\overline{B}(x)} \geq \int_x^{x+a} \frac{\overline{B}(s)}{\mu_B \overline{B}(x)}\, ds \geq \frac{\overline{B}(x+a)\, a}{\mu_B \overline{B}(x)} \to \frac{a}{\mu_B} \quad (x \to \infty).$$

From here, it follows that:

$$\liminf_{x\to\infty} \frac{\overline{B}_0(x)}{\overline{B}(x)} \geq \frac{a}{\mu_B} \quad (x \to \infty).$$

However, a can be more then M for any $M > 0$. □

4.1.4. *Examples*

EXAMPLE 4.3.– Let us consider the Weibool distribution with the tail $\overline{B}(x) = e^{-\alpha x^\beta}$, where $\alpha > 0$ and $0 < \beta < 1$. Using the replacement $\alpha s^\beta = y$, we have:

$$\mu_B \overline{B}_0(x) = \int_x^\infty e^{-\alpha\, s^\beta}\, ds = \alpha^{-1/\beta}\beta^{-1} \int_{\alpha x^\beta}^\infty y^{1/\beta-1} e^{-y}\, ds =$$

$$= \alpha^{-1/\beta}\beta^{-1}\Gamma(\alpha x^\beta, 1/\beta).$$

From here, it follows that $\mu_B = \alpha^{-1/\beta}\beta^{-1}\Gamma(1/\beta)$ and $\overline{B}_0(x) = \Gamma(\alpha x^\beta, 1/\beta)/\Gamma(1/\beta)$. For the application of the theorem of Pitman, the equivalence is used:

$$\Gamma(x, a) \sim \gamma(x, a), \qquad [4.2]$$

where $\gamma(x, a) = x^{a-1}e^{-x}$ and $\Gamma(x, a)$ is a corresponding incomplete gamma function. In fact, by using the rule of L'Hospital, we have:

$$\frac{\Gamma(x,a)}{\gamma(x,a)} \sim \frac{\Gamma(x,a)'}{\gamma(x,a)'} = \frac{-\gamma(x,a)}{-\gamma(x,a) + (a-1)x^{a-2}e^{-x}} =$$

$$= \frac{1}{1-(a-1)/x} \to 1 \quad (x \to \infty).$$

Thus, we have at $\beta < 1$,

$$\lambda_0(x) \equiv \frac{-\overline{B}_0(x)'}{\overline{B}_0(x)} = \frac{\gamma(\alpha x^\beta, 1/\beta)\cdot \alpha\beta x^{\beta-1}}{\Gamma(\alpha x^\beta, 1/\beta)} \sim \alpha\beta x^{\beta-1} \quad (x \to \infty).$$

Thus,

$$\int_0^\infty e^{\lambda_0(x)x} \frac{\gamma(\alpha x^\beta, 1/\beta) \cdot \alpha\beta x^{\beta-1}}{\Gamma(1/\beta)}\, dx =$$

$$= \int_0^M e^{\lambda_0(x)x} \frac{\gamma(\alpha x^\beta, 1/\beta) \cdot \alpha\beta x^{\beta-1}}{\Gamma(1/\beta)}\, dx +$$

$$+ \int_M^\infty e^{\alpha\beta x^\beta} \frac{(\alpha x^\beta)^{(1/\beta-1)} e^{-\alpha x^\beta} \cdot \alpha\beta x^{\beta-1}}{\Gamma(1/\beta)}\, dx + o(M) =$$

$$= \int_0^M e^{\lambda_0(x)x} \frac{\gamma(\alpha x^\beta, 1/\beta) \cdot \alpha\beta x^{\beta-1}}{\Gamma(1/\beta)}\, dx +$$

$$+ \int_M^\infty e^{\alpha(\beta-1)x} \frac{\alpha^{1/\beta}\beta}{\Gamma(1/\beta)} dx + o(M) < \infty,$$

where $o(M)/M \to 0$ as $M \to \infty$. According to the theorem of Pitman, it follows that $B_0 \in \mathcal{S}$ and theorem 4.7 is applicable to the Weibool distribution. □

EXAMPLE 4.4.– Let us consider a class of regularly varying functions with an index α. If $f \in RM_\alpha$, then at any $t > 0$,

$$\frac{f(tx)}{f(x)} \to t^{-\alpha} \quad (x \to \infty).$$

On the other hand, if this condition is fulfilled, then $f \in RV_\alpha$. The corresponding slowly varying function has a kind of $f(x)x^\alpha$. Now let us consider a cumulative distribution function B with a tail belonging to a class RV_α. Let $\overline{B}(x) = L(x)/x^\alpha$, where $L(x)$ is a slowly varying function. We have:

$$\overline{B}_0(xt) = \frac{1}{\mu_B} \int_{xt}^\infty L(s)/s^\alpha\, ds = t^{-\alpha+1} \frac{1}{\mu_B} \int_x^\infty L(st)/s^\alpha\, ds,$$

thus, for any $\epsilon > 0$, there exists x_0 such that at any $x > x_0$,

$$(1-\epsilon)t^{-\alpha+1}\overline{B}_0(x) \le \overline{B}_0(xt) \le (1+\epsilon)t^{-\alpha+1}\overline{B}_0(x).$$

From here, it follows that $\overline{B}_0(x) \in RV_{\alpha-1}$, and if $\alpha > 1$, then $B_0 \in \mathcal{S}$. In the latter case, theorem 4.7 is applicable to this distribution. Hence, this theorem is applicable to the Pareto distribution with an index $\alpha > 1$. □

EXAMPLE 4.5.– Let us consider a log-gamma distribution. Let $Y = e^X$, where $X \in$ Gam(a, δ) (a gamma distribution with the form parameter a and the scale parameter δ). Using equivalence (4.2), we have:

$$\overline{F}_Y(x) \equiv P(Y > x) = P(X > \ln x)$$
$$= \frac{\Gamma(\delta \ln x,\, a)}{\Gamma(a)} \sim \frac{\gamma(\delta \ln x,\, a)}{\Gamma(a)} =$$
$$= \frac{(\delta \ln x)^{a-1} e^{-\delta \ln x}}{\Gamma(a)} = \frac{L(x)}{x^\delta} \quad (x \to \infty),$$

where $L(x) = (\delta \ln x)^{a-1}/\,\Gamma(a)$ is a slowly varying function. Thus, $\overline{F}_Y(x)$ is a regularly varying function with an index δ, and if $B = F_Y$, then $B_0 \in RV_{\delta-1}$ (see example 4.4). Thus, according to theorem 4.1, if $\delta > 1$, then $B_0 \in \mathcal{S}$, and theorem 4.7 is applicable to the log-gamma distribution. □

EXAMPLE 4.6.– Let us consider the log-normal distribution. We have $Y = e^{a+\sigma X}$ $(\sigma > 0)$, where X is a standard normal random variable. Then,

$$\overline{B}(x) \equiv P(Y > x) = \overline{\Phi}((\ln x - a)/\sigma).$$

By denoting φ as the density of the standard normal distribution, we have:

$$\mu_B \overline{B}_0(x) = \int_x^\infty \overline{\Phi}((\ln s - a)/\sigma)\, ds = \int_x^\infty \int_{(\ln s - a)/\sigma}^\infty \varphi(y)\, dy\, ds =$$

$$= \int_{(\ln x-a)/\sigma}^{\infty} \varphi(y) \int_{x}^{e^{y\,\sigma+a}} ds\,dy = \int_{(\ln x-a)/\sigma}^{\infty} \varphi(y)(e^{y\,\sigma+a} - x)\,dy =$$

$$= \int_{(\ln x-a)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2+y\,\sigma+a}\,dy - x\overline{\Phi}((\ln x - a)/\sigma) =$$

$$= \int_{(\ln x-a)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{a+\sigma^2/2} e^{-(y-\sigma)^2/2\sigma^2}\,dy - x\overline{\Phi}((\ln x - a)/\sigma) =$$

$$= e^{a+\sigma^2/2}\overline{\Phi}((\ln x - a)/\sigma - \sigma) - x\overline{\Phi}((\ln x - a)/\sigma).$$

From here, in particular, it follows that $\mu_B = e^{a+\sigma^2/2}$, and also

$$\lambda_0(x) = \frac{\overline{\Phi}((\ln x - a)/\sigma)}{e^{a+\sigma^2/2}\overline{\Phi}((\ln x - a)/\sigma - \sigma) - x\overline{\Phi}((\ln x - a)/\sigma)}.$$

For the determination of asymptotics of the hazard rate at infinity, we use the well-known equivalence:

$$\overline{\Phi}(x) \sim \varphi(x)/x \quad (x \to \infty), \tag{4.3}$$

(Mill's ratio). In fact, by using the rule of L'Hospital, we have:

$$\frac{\overline{\Phi}(x)}{\varphi(x)/x} \sim \frac{-\varphi(x)}{-\varphi(x) - \varphi(x)/x^2} = \frac{1}{1+1/x^2} \to 1 \quad (x \to \infty).$$

Thus,

$$e^{a+\sigma^2/2}\,\overline{\Phi}((\ln x - a)/\sigma - \sigma) \sim$$

$$\sim e^{a+\sigma^2/2} \frac{1}{\sqrt{2\pi}}\, e^{-((\ln x-a)/\sigma-\sigma)^2/2}/((\ln x - a)/\sigma - \sigma) =$$

$$= \frac{x}{\sqrt{2\pi}}\, e^{-((\ln x-a)/\sigma)^2/2}/((\ln x - a)/\sigma - \sigma) =$$

$$= x\varphi((\ln x - a)/\sigma)/((\ln x - a)/\sigma - \sigma) \sim$$

$$\sim x\overline{\Phi}((\ln x - a)/\sigma)\frac{(\ln x - a)/\sigma}{(\ln x - a)/\sigma - \sigma} \quad (x \to \infty).$$

Substituting this value in the formula for $\lambda(x)$, we obtain:

$$\lambda(x) \sim \frac{\ln x - a - \sigma^2}{x\sigma^2} \sim \frac{\ln x}{x\sigma^2} \quad (x \to \infty).$$

Let us verify the convergence of an integral of Pitman. It is enough to estimate a tail of this integral for a large value of M when it is possible to substitute the integrands by their asymptotic values:

$$\begin{aligned} &\mu_B^{-1} \int_M^\infty e^{\lambda(x)x} \overline{\Phi}((\ln x - a)/\sigma)\, dx \\ &\quad \sim \mu_B \int_M^\infty e^{\ln x/\sigma^2} \frac{\varphi((\ln x - a)/\sigma)}{(\ln x - a)/\sigma}\, dx \le \\ &\le C_1 \int_M^\infty e^{-(\ln x - a - 1)^2/2\sigma^2}\, dx < \infty, \end{aligned}$$

where C_1 is a positive number. Hence, according to the theorem of Pitman, $B_0 \in \mathcal{S}$, and theorem 4.7 is applicable to the log-normal distribution. □

4.2. Integro-differential equation

This section of the theory does not provide any restrictions on a distribution tail. However, the basic preconditions of the Cramér-Lundberg theory remain. Let us consider a process with the premium rate depending on the current capital. Let β be the intensity of the Poisson process of the moments of suit inflow, and (U_n) be the independent sequence of i.i.d. random variables with the common cumulative distribution function B. They represent the values of suits. Let $p(x)$ be a premium rate, $\psi(u) = 1 - \varphi(u)$, $\varrho = \beta\mu_B$, $\mu_B = EU_1$.

THEOREM 4.8.– *(integro-differential equation).* If $p(x) > \varrho$ at all $x \ge 0$ and the function p is continuous, then

$$\varphi(u) = \frac{\beta}{p(u)} \left(\varphi(u) - \int_0^u \varphi'(u - x)\, dB(x) \right), \qquad [4.4]$$

and if $(\forall x \geq 0)\ p'(x) \geq \beta$, then

$$\psi(u) \leq \beta \int_u^{\infty} \frac{\overline{B}(x)\, dx}{p(x) - \varrho} \quad (p(u) - \varrho \geq 0).$$

PROOF.– It follows that at $u \geq 0$,

$$\begin{aligned}
&\psi(u) = P_u(R_{\sigma_{[u,u+r)}} < 0) \\
&+ \int_0^u P_u(R_{\sigma_{[u,u+r)}} \in dx,\ \tau_0 \circ \theta_{\sigma_{[u,u+r)}} < \infty) + \\
&+\ P_u(R_{\sigma_{[u,u+r)}} = u + r,\ \tau_0 \circ \theta_{\sigma_{[u,u+r)}} < \infty)) = \\
&= P_u(R_{\sigma_{[u,u+r)}} < 0) + \int_0^u \psi(x) P_u(R_{\sigma_{[u,u+r)}} \in dx) + \\
&+\ \psi(u+r) P_u(R_{\sigma_{[u,u+r)}} = u + r) = \beta \frac{r}{p(u)} \overline{B}(u) + \\
&+\ \beta \frac{r}{p(u)} \int_0^u \psi(u - x)\, dB(x) + \left(1 - \beta \frac{r}{p(u)}\right) \psi(u + r) + o(r).
\end{aligned}$$

From here, it follows that the continuous and differentiable solution satisfies the equation:

$$-\psi'(u) = \frac{\beta}{p(u)} \left(\overline{B}(u) - \psi(u) + \int_0^u \psi(u - x)\, dB(x)\right),$$

which can be rewritten in an aspect:

$$\varphi'(u) = \frac{\beta}{p(u)} \left(\varphi(u) - \int_0^u \varphi(u - x)\, dB(x)\right).$$

Let us estimate the sign of a flexion function (second derivative). We have:

$$\varphi''(u) = -\frac{p'(u)}{p(u)} \varphi'(u) + \frac{\beta}{p(u)} \left(\varphi'(u) - \varphi(0) b(u) - \int_0^u \varphi'(u - x)\, dB(x)\right).$$

The condition $p' \geq \beta$ guarantees the negativity of the flexion on all semi-axes. Rewriting formula (4.4) in an aspect, we have:

$$\varphi'(u) = \frac{\beta}{p(u)} \left(\varphi(u) \overline{B}(u) + \int_0^u (\varphi(u) - \varphi(u - x))\, dB(x)\right).$$

Under the condition of the theorem, the integrand difference of the integral on the right-hand side of this equation is not less than $\varphi'(u)x$, which leads to a differential inequality:

$$\varphi'(u)\left(1-\frac{\beta}{p(u)}\int_0^u x\,dB(x)\right)\geq\frac{\beta}{p(u)}\varphi(u)\overline{B}(u).$$

Supposing $\varphi(\infty)=1$, we obtain a solution of this inequality:

$$\varphi(u)\geq\exp\left(-\int_u^\infty\frac{\beta\overline{B}(x)}{p(x)-\beta\int_0^x s\,dB(s)}\,dx\right),$$

thus it follows that:

$$\psi(u)\leq 1-\exp\left(-\int_u^\infty\frac{\beta\overline{B}(x)}{p(x)-\beta\int_0^x s\,dB(s)}\,dx\right)\leq$$

$$\leq\int_u^\infty\frac{\beta\overline{B}(x)}{p(x)-\beta\int_0^x s\,dB(s)}\,dx\leq\int_u^\infty\frac{\beta\overline{B}(x)}{p(x)-\beta\mu_B}\,dx.\qquad\square$$

Let us note that for the evaluation of an inequality from formula (4.4), a convexity upwards of the function $\varphi(u)$ on ray $u\geq 0$ is used. The method of the proof and an asymptotic inequality remain true if the convexity upwards begins from some $u_0>0$, i.e. for any $u\geq u_0$. The condition $p'\geq\beta$ is too burdensome. Thanks to it, the unique positive item in the sum of negative magnitudes is cleared. The account of these magnitudes depends on the factor:

$$f(u)\equiv\varphi'(u)-\varphi(0)b(u)-\int_0^u\varphi'(u-x)\,dB(x),$$

which in the given course is not analyzed.

At a constant premium rate $p(x)\equiv p$, it is possible to use a Laplace transformation for the solution of equation (4.4). Using the formula,

$$\int_0^\infty e^{-\lambda u}\varphi'(u)\,du=-\varphi(0)+\lambda\widehat{\varphi}(\lambda)\quad(\lambda>0),$$

where $\widehat{\varphi}(\lambda)$ is a Laplace image of a function φ, we have:

$$-\varphi(0) + \lambda\widehat{\varphi}(\lambda) = \frac{\beta}{p}(\widehat{\varphi}(\lambda) - \widehat{\varphi}(\lambda)\widehat{B}(-\lambda)),$$

thus,

$$\widehat{\varphi}(\lambda) = \frac{\varphi(0)}{\lambda - (\beta/p)(1 - \widehat{B}(-\lambda))}.$$

Let us obtain the value $\varphi(0)$. Integrating equation (4.4) on an interval $(0, u)$, we obtain:

$$\begin{aligned}
\varphi(u) - \varphi(0) &= \frac{\beta}{p}\int_0^u \varphi(x)\,dx + \frac{\beta}{p}\int_0^u \int_0^x \varphi(x-s)\,d\overline{B}(s)\,dx = \\
&= \frac{\beta}{p}\int_0^u \varphi(x)\,dx + \\
&+ \frac{\beta}{p}\int_0^u \left(\varphi(0)\overline{B}(x) - \varphi(x) + \int_0^x \varphi'(x-s)\overline{B}(s)\,ds\right)\,dx = \\
&= \frac{\beta\varphi(0)}{p}\int_0^u \overline{B}(x)\,dx + \frac{\beta}{p}\int_0^u \left(\int_s^u \varphi'(x-s)\,dx\right)\overline{B}(s)\,ds = \\
&= \frac{\beta\varphi(0)}{p}\int_0^u \overline{B}(x)\,dx + \frac{\beta}{p}\int_0^u (\varphi(u-s) - \varphi(0))\,\overline{B}(s)\,ds.
\end{aligned}$$

Passing to a limit as $u \to \infty$ and using the convergence of the integrand as a limit, we obtain the equation:

$$1 - \varphi(0) = \frac{\beta\varphi(0)}{p}\mu_B + \frac{\beta}{p}(1 - \varphi(0))\mu_B,$$

from which

$$\varphi(0) = 1 - \frac{\varrho}{p}.$$

Hence,

$$\widehat{\varphi}(\lambda) = \frac{p - \varrho}{\lambda p - \beta(1 - \widehat{B}(-\lambda))}. \qquad [4.5]$$

Let us note that to derive this formula, the membership condition B to Cramér's class is not used. □

EXAMPLE 4.7.– Let $\overline{B}(x) = e^{-\delta x}$. Thus, $\widehat{B}(-\lambda) = \delta/(\delta + \lambda)$, and also

$$\widehat{\varphi}(\lambda) = \frac{p - \varrho}{\lambda(p - \beta/(\delta + \lambda))} = 1/\lambda - \frac{\beta/\delta}{p(\delta + \lambda) - \beta} =$$
$$= 1/\lambda - \left(\frac{\beta/\delta}{p\delta - \beta}\right) \frac{\delta - \beta/p}{\lambda + \delta - \beta/p}.$$

The first member is the Laplace image of unit, and the second member is the Laplace image of an exponential function. That is,

$$\varphi(u) = 1 - \frac{\beta/\delta}{p\delta - \beta} \exp(-(\delta - \beta/p)u) = 1 - \frac{\varrho}{\delta(p - \varrho)} \exp(-\delta(1 - \varrho/p)u),$$

where $\beta/\delta = \varrho$. □

5

Some Problems of Control

5.1. Estimation of probability of ruin on a finite interval

For practical calculations, the probability of ruin on a finite interval can be easily obtained by a Monte-Carlo imitation method that simulates the dynamics of the company capital (R_t) by using pseudo-random numbers in a computer. The algorithm of such an estimation is transparent enough.

At first, in the program, the minor cycle "$while$" should be provided for deriving two sums:

1) The sum of independent pseudo-random variables Δ_k $(k \geq 1)$ distributed exponentially (with parameter β) that corresponds to observation time intervals of the pointwise Poisson process (σ_k) (where $\Delta_k = \sigma_k - \sigma_{k-1}$ and $\sigma_0 = 0$);

2) The sum of independent pseudo-random variables $c_k = p\,\Delta_k - U_k$, where (U_k) is a sequence of independent positive pseudo-random numbers distributed according to the common cumulative distribution function $B(x)$. The number c_k is an income (which can be negative) over one exponential time interval.

The minor cycle comes to the end when the sum of incomes $S_n = u + \sum_{k=1}^{n} c_k$ becomes negative for the first time (the first reason for

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stopping; it is an event A_1). We denote n_0 as the number of the event A_1 step and $\tau_0 = \sum_{k=1}^{n_0} \Delta_k \equiv \sigma_{n_0}$ (it is an instant of ruin). In the program, the possible variant of "cycling" (when at any n magnitude, S_n is non-negative) should be provided. This event has positive probability when $p > \beta\mu$, where $\mu = EU_1$. In order to prevent cycling, it is necessary to provide some computer "actual infinity", i.e. some positive numbers T_∞. The minor cycle comes to the end when the condition $\sigma_k > T_\infty$ is satisfied for the first time (the second reason for stopping).

Secondly, in the program, the major cycle "$for\,to$" should be provided. It consists of the independent and identically distributed minor cycles with their finishing sums $(\tau_0^{(k)})$ (k is an order number of the minor cycle). The major cycle proceeds until some determinate number N. For any $x < T_\infty$, the random variable $N_1(x) = \sum_{k=1}^{N} I_{(A_1,\, \tau_0^{(k)} < x)}$ (where I_A is an indicator of the set A) is calculated. The statistic $N_1(x)/N$ represents a consistent (pseudo-consistent) estimate of the probability $P(\tau_0 < x)$. Other statistics can be constructed analogously.

The estimation of these probabilities becomes particularly relevant for a problem concerning the optimal credit amount which the insurance company intends to take in the beginning of its business.

5.2. Probability of the credit contract realization

When starting a new business, the company is interested in its reliability in the long run. The company hopes that bank credit can help them. Let the magnitude of the company's initial resources be close to zero, but the company has a perspective business plan, and it goes to the bank for credit.

The credit contract provides three cores: (1) the credit sum, u, (2) the return time of the credit, T, and (3) the share in relation to the credit sum (usually in percentage expression) which the company has to pay over the credit sum, γ. A company that has satisfied a condition of the credit contract can expect with a new credit contract with the same or with a different bank, as it is a “good” client. Therefore, the company is interested in the greatest probability of realization of the credit contract.

Thus, dynamics of the company capital that uses the credit is a random process $R(t)$ $(t \geq 0)$, with a measure P_u, and the probability of realization of the credit contract is equal to $P_u(\tau_0 > T,\, R(T) > u + \gamma u)$, where τ_0 is a time of the first exit from the interval $[0, \infty)$. We will consider a situation when the parameters T and γ are defined by the bank, and the client can choose the magnitude u at his own discretion. A problem for the management of the company is choosing the optimum magnitude of the credit. In this case, the optimum means to define such credit u such that it ensures a maximum value of the function $r(u)$, where

$$r(u) \equiv P_u(\tau_0 > T,\, R(T) > u + \gamma u) \quad (u \geq 0).$$

We will assume that the process $(R(t))$ is homogeneous in space (as considered in Chapter 2). For such a process, the following representation holds:

$$\begin{aligned} r(u) &= P_0(\tau_{-u} > T,\, R(T) + u > u + \gamma u) = \\ &= P_0(\tau_{-u} > T,\, R(T) > \gamma u), \end{aligned}$$

where τ_{-u} is the moment of the first exit from the interval $[-u,\, \infty)$.

We know that usually it is impossible to count on a continuity of a sample trajectory of the process (see, for example, the Cramér-Lundberg process). Therefore, we will demand only the condition of a stochastic continuity. Moreover, under this supposition (taking into account some additional conditions that are fulfilled for the Cramér-Lundberg process), the function $r(x)$ is continuous on its domain. For such functions, the aforementioned problem about a maxima point makes sense. In our case, the maximum point x_{max} of the function $r(x)$ exists and belongs to an interval $[0, \infty)$. In fact, it is evident that as $u \to \infty$,

$$r(u) \leq P_0(R(T) > \gamma u) \to 0.$$

On the other hand, $r(u) \leq 1$ and is non-negative at any $u \geq 0$. Thus, the possibility that $x_{max} = 0$ is not excluded, i.e. it is not reasonable to take the credit. In certain cases, it is possible to prove that $x_{max} > 0$ and to estimate its value with a computer, using a mathematical software package.

5.2.1. *Dynamics of the diffusion-type capital*

We will use the results of Chapter 2 relating to the diffusion-type approximation (see theorem 2.15, Chapter 2) in order to show that a diffusion representation is reasonable.

THEOREM 5.1.– *(consequence of the Donsker theorem).* Let $(X(t))$ be a process with independent increments for which the asymptotics are true:

$$(\forall t > 0) \quad \alpha X(t/\alpha^2) \stackrel{distr}{\longrightarrow} W(t) + t \quad (\alpha \to 0).$$

where $(W(t))$ is a standard Wiener process. Then, for any $\sigma > 0$ and $A \neq 0$, it is true that:

$$(\forall t > 0) \quad \sigma\, \alpha_1 X(t/\alpha_1^2) \stackrel{distr}{\longrightarrow} \sigma W(t) + At \quad (\alpha \to 0),$$

where $\alpha_1 = \alpha\, \sigma / A$.

PROOF.– According to the condition, it follows that:

$$\sigma\, \alpha_1 X(t/\alpha_1^2) = \frac{\sigma^2}{A}\, (\alpha X(t_1/\alpha^2)),$$

where $t_1 = t\, A^2/\sigma^2$. Thus,

$$\sigma\, \alpha_1 X(t/\alpha_1^2) \stackrel{distr}{\longrightarrow} \frac{\sigma^2}{A}\, (W(t_1) + t_1) = \sigma \left(\frac{\sigma}{A}\, W\left(t\, \frac{A^2}{\sigma^2} \right) \right) + A\, t.$$

From here, using the auto-model property of a standard Wiener process, it follows that:

$$(\forall t > 0), (\forall b \neq 0) \quad b\, W(t/b^2) \stackrel{distr}{=} W(t),$$

we thus obtain the assertion of the theorem. □

From this outcome, we can see which transformations of the Cramér-Lundberg process lead to a process with a distribution arbitrarily close to a distribution of the Wiener process with constant values of drift and local variance. It has been noted above that for the Cramér-Lundberg classical process with a positive factor of safety, the probability of ruin on an infinite interval is less than 1 even in the case when the company's initial capital is equal to zero. In the case where the dynamics of the capital is considered as a homogeneous diffusion process $X(t)$, a zero initial capital guarantees ruin of the company with probability 1 on any interval of positive length. That is, at any $\epsilon > 0$ with probability 1, there exists a point $t_1 \in (0, \epsilon)$, such that $X(t_1) < 0$.

The aforementioned problem about an optimum choice of the magnitude of the credit is considered in the case of a homogeneous diffusion process expression in analytical terms. That is, for the process $W(t) + A\,t$, it is true that:

$$\begin{aligned} r(u) &= P_0(X(T) > \gamma u) - P_0(\tau_{-u} < T, X(T) > \gamma u) = \\ &= \overline{\Phi}\left(\frac{\gamma u - AT}{\sqrt{T}}\right) - \\ &- \int_0^T \frac{u}{\sqrt{2\pi t^3}} \exp\left(-\frac{(u+At)^2}{2t}\right) \overline{\Phi}\left(\frac{u + \gamma u - A\,(T-t)}{\sqrt{T-t}}\right) dt, \end{aligned}$$

where for any x and y

$$\overline{\Phi}(x) \equiv \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right), \qquad \mathrm{erfc}(y) \equiv \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-s^2}\, ds.$$

We do not know an analytical representation for the maximum point of the function $r(u)$. For its numerical evaluation, we used a software package MatLab13. In the examples considered, it is accepted that $\gamma = 10\%$, $T = 2$, $A = 1$ and $A = 3$, where u is expressed in number of million units, T in years and A in (million units)/year.

A graphical representation of the function $r(u)$ for two values of the parameter A (on interval $(0,\ 0.1)$) is shown in Figure 5.1. The whole graph at another scale along the time axis $(u \in (0,\ 100)\,)$ is shown in Figure 5.2.

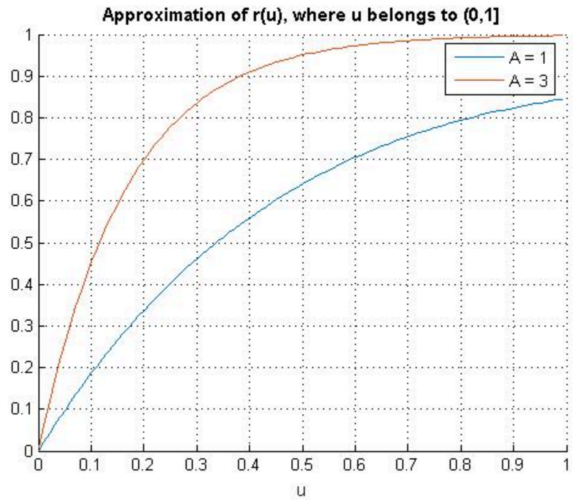

Figure 5.1. *Part of the graph close to zero time*

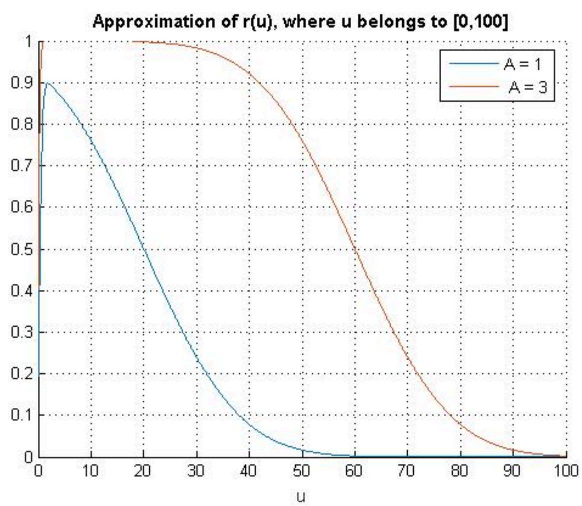

Figure 5.2. *Full graph of the probability* $r(u)$

The maximum value of the function $r(u)$ for $A = 1$ is found to be at $u = 0.98$ and is equal to $0.89650\ldots$ This maximum value for $A = 3$ is found to be at point $u = 2.04$ and is equal to $0.99997\ldots$

5.3. Choosing the moment at which insurance begins

5.3.1. *Model of voluntary individual insurance*

5.3.1.1. *About the insurance contract*

The problem concerning the time at which insurance begins is considered. It is assumed that information about the degree of deterioration is accessible for the insurant. Insurance considers the presence of two subjects: insured (the client named further by the insurant) and insuring (insurer). Thus, it is supposed that the danger of the insurance event (refusal) is proportional to the degree of deterioration and is known to the insurant. The insurant endeavors to reduce the risk connected with the loss of some value. He is ready to pay to the insurer the defined sum until the moment of the occurrence of an event leading to the loss of value. He hopes that in the case of an occurrence of an event, the insurer would compensate his loss completely or partially. The insurer agrees to pay the insurance compensation in the hope that the insurance event will not happen. The typical insurance contract is made up to a certain term, for example for a year. The cost of the insured value and a share of this cost are fixed, determining an insurance payment (the insurance tariff). In a discrete model of insurance, the insurant pays to the insurer an insurance payment (premium) at the beginning of each time unit from the insurance contract term. He renews the contract after the outflow of a certain term of insurance if there is no insurance event. In a continuous model of insurance, the insurant continuously pays to the insurer a premium rate until the moment of the insurance event. The insurer pays insurance compensation of the fixed size if the insurance event happens after the contract conclusion and before a certain term is finished.

5.3.1.2. *Continuous model*

The following model of voluntary insurance without a fixed termination date of insurance is considered. An insurance event (refusal) occurs at a random instant ζ. This moment may depend on different exterior and interior conditions. The insurant concludes the insurance contract at the moment τ ($\tau < \zeta$). From the moment τ and until the moment ζ, the client pays an insurance payment (insurance premium) with intensity V (payment in unit of time). At the moment ζ, the insurant receives insurance compensation W from the insurer. Expenditures of the insurant at this moment are equal to:

$$X_\tau = V\,(\zeta - \tau) + (H - W)\,I_{\{\zeta > \tau\}},$$

where H is the real damage evoked by an insurance event; $x_+ \equiv \max\{0, x\}$ and I_A is the indicator of event A. Average losses of the insurant are given by:

$$E\,X_\tau = V\,E(\zeta - \tau;\,\zeta > \tau) + (H - W)\,P(\zeta > \tau).$$

Our problem consists of justifying a choice of the moment of conclusion of the insurance contract τ from the insurant's point of view.

5.3.1.3. *The model improvement*

In our model, it is supposed that the danger of occurrence of an insurance event (hazard rate) is proportional to the degree of deterioration of some responsible detail, serving as the diagnostic parameter of the product. Such proportionality represents the sense of the so-called model of Cox in the reliability theory (see, for example, [KOV 07]). So, let us suppose that the insurant observes a continuous non-decreasing random process $\xi(t)$, using the hazard rate of the product. Thus, the following reliability function is known to the insurant:

$$P(\zeta > t) = \exp\left(-\int_0^t \xi(s)\,ds\right).$$

The time of the beginning of insurance τ, under our supposition, is a Markov time (concerning a natural filtration) for the random process ξ. Thus, $\tau = \tau(\xi)$, and the conditional probability that the potential insurant will be insured before the insurance event is equal to:

$$P(\zeta > \tau \,|\, \xi) = \exp\left(-\int_0^{\tau(\xi)} \xi(s)\, ds\right),$$

and the conditional expectation of the duration of phase insurance is equal to:

$$E(\zeta - \tau;\, \zeta > \tau \,|\, \xi) = \int_{\tau(\xi)}^{\infty} (t - \tau(\xi))\xi(t) \exp\left(-\int_0^t \xi(s)\, ds\right)\, dt.$$

Integrating piecemeal and assuming that:

$$x \exp\left(-\int_0^x \xi(s)\, ds\right) \to 0,$$

as $x \to \infty$, we obtain:

$$E(\zeta - \tau;\, \zeta > \tau \,|\, \xi) = \int_{\tau(\xi)}^{\infty} \exp\left(-\int_0^x \xi(s)\, ds\right)\, dx.$$

Unconditional probability and expectation are found as a result of the integration of these expressions on some measure Q, i.e. the distribution of the process ξ:

$$P(\zeta > \tau) = E^{(Q)}(P(\zeta > \tau |\, \xi)),$$

$$E(\zeta - \tau; \zeta > \tau) = E^{(Q)}(E(\zeta - \tau; \zeta > \tau |\, \xi)).$$

Hereinafter, we apply a notation:

$$E^{(Q)}(f) \equiv \int_{\mathcal{D}} f(\xi)\, Q(d\xi),$$

where f is an integrable function and $\mathcal{D}$ is a set of possible realizations of the process ξ (usually, it is the Skorokhod space).

5.3.1.4. *Strategy of the insurant*

Taking into account the distribution of ξ, we obtain the average losses of the insurant as:

$$E\,X_\tau = E^{(Q)}\left(\int_\tau^\infty \exp\left(-\int_0^t \xi(s)\,ds\right)\,(V + (H - W)\,\xi(t))\,dt\right).$$

This magnitude is positive if $H \ge W$ (real loss more than insurance compensation). In this case, the potential insurant does not always take the profit. We will consider a situation where because of the insufficient knowledge of the insurer who concludes the contract, the actual inequality $H < W$ is fulfilled, and the potential insurant knows about it. In this case, the problem of the insurant consists of choosing τ (the moment of conclusion of the insurance contract) in order to minimize the magnitude $E\,X_\tau$.

The necessary condition for a minimum of a functional EX_τ at a point τ is a realization of two conditions:

$$\liminf_{h\to 0} \tfrac{1}{h}(EX_{\tau+h\eta} - EX_\tau) \ge 0, \qquad [5.1]$$

$$\liminf_{h\to 0} \tfrac{1}{h}(EX_{\tau-h\eta} - EX_\tau) \ge 0, \qquad [5.2]$$

for any measurable, bounded, non-negative function $\eta(\xi)$. In this case,

$$\begin{aligned}
&\frac{1}{h}(EX_{\tau+h\eta} - EX_\tau) = \\
&= \frac{1}{h}E^{(Q)}\left(\int_{\tau+h\eta}^\infty \exp\left(-\int_0^t \xi(s)\,ds\right)\,(V + \xi(t)\,(H - W))\,DT -\right.\\
&\left.-\int_\tau^\infty \exp\left(-\int_0^t \xi(s)\,ds\right)\,(V + \xi(t)\,(H - W))\,dt\right) = \\
&= \frac{1}{h}E^{(Q)}\left(-\int_\tau^{\tau+h\eta} \exp\left(-\int_0^t \xi(s)\,ds\right)\,(V + \xi(t)\,(H - W))\,dt\right) \to \\
&\to E^{(Q)}\left(-\eta\,\exp\left(-\int_0^\tau \xi(s)\,ds\right)\,(V + \xi(\tau)\,(H - W))\right).
\end{aligned}$$

Thanks to the arbitrariness of η from condition [5.1], it follows that $V + \xi(\tau)\,(H - W) \leq 0 \quad Q$ - almost sure. Similarly, from [5.2], it follows that $V + \xi(\tau)\,(H - W) \geq 0 \quad Q$ - almost sure. Hence,

$$V + \xi(\tau)\,(H - W) = 0$$

Q - almost sure. Under the condition $W > H$, it means that for Q - almost all functions ξ the equality is fulfilled $\xi(\tau) = V/(W - H)$. For a stochastically continuous, non-decreasing process ξ in a class of Markov times, τ is only the moment where level $b = V/(W - H)$ is first reached by this process.

Let us denote $\tau_b(\xi) \equiv \sigma_{(-\infty,b)}(\xi)$ (the moment of the first exit from an interval $(-\infty, b)$).

Let us note that for a stochastically continuous non-decreasing process for almost all trajectories ξ, the moment τ_b (where $b > 0$) does not belong to an interval of constancy of this trajectory. Thus, for any $\epsilon > 0$, a strict inequality $\xi(\tau_b + \epsilon) > b$ is fulfilled Q - almost sure.

It is easy to verify that $E\,X_{\tau_b} < 0$ at $b = V/(W - H)$. It means that such an insurance strategy is profitable to the insurant (negative loss is a positive profit).

Usually, a determination of an explicit analytical expression $E\,X_{\tau_b}$ as a function of b is impossible, or represents a difficult analytical problem. We will consider how this dependence is affected if Q is the distribution of the continuous semi-Markov process with non-decreasing trajectories.

5.3.2. *Non-decreasing continuous semi-Markov process*

The random process X_t $(t \geq 0)$ is called a continuous semi-Markov process if it possesses the Markov property concerning the moment of the first exit from any open set [HAR 07]. Such a process is determined by the consistent family of measures (Q_x), where $(x \geq 0)$ is an initial point of a trajectory. This set of measures is called the distribution of process within its initial point. This process is not obliged to be

Markov with respect to a non-random (fixed) instant. However, any strictly Markov process will be at the same time a continuous semi-Markov one. Among the continuous semi-Markov processes, one-dimensional monotone processes are most simply arranged. An example of such a process is an inverse gamma process, as mentioned in Chapter 1. Generally, a non-decreasing semi-Markov process represents a converted process with independent strictly positive increments (not necessarily a gamma process). It means that the function τ_y $(y \geq 0)$ defined earlier represents a proper (not converted) process with independent positive increments. Thus, an argument y (the reached level) plays the role of time. The process (τ_y) is convenient for setting using semi-Markov transition functions of the process X_t. We will consider the process τ_y as a temporally homogeneous process. For such a process, it is a true Levy-Khinchin expansion: for any $\lambda \geq 0$, and $y > 0$

$$E_0 \exp(-\lambda\, \tau_y) = \exp(-y\, \beta(\lambda)), \qquad [5.3]$$

where

$$\beta(\lambda) \equiv \lambda\, m + \int_{0+}^{\infty} (1 - e^{-\lambda u})\, n(du),$$

$m \geq 0$ is a non-negative quantity (drift parameter), and $n(du)$ is the so-called Levy-Khinchin measure, such that

$$\int_{0+}^{\infty} \min\{1,\, u\}\, n(du) < \infty$$

(see, for example, [SKO 64]).

THEOREM 5.2.– Let τ_y be a homogeneous strictly increasing process with independent increments with a parameter m, and a measure $n(du)$ for which the function $\beta(\lambda)$ is continuous. Then,

$$P(\zeta > \tau_y) = \exp\left(-\int_0^y \beta(x)\, dx\right), \qquad [5.4]$$

$$E(\zeta - \tau_y;\, \zeta > \tau_y) = \int_y^{\infty} \exp\left(-\int_0^x \beta(u)\, du\right) \frac{\beta(x)}{x}\, dx. \qquad [5.5]$$

PROOF.– 1) Let $0 = y_0 < y_1 < \cdots < y_n = b$ (for simplification of notations, we sometimes use a label $y(k) \equiv y_k$). From identity (true for non-decreasing processes), we have:

$$\tau_{y_k} = \tau_{y_{k-1}} + \tau_{\Delta_k} \circ \theta_{\tau_{y(k-1)}}$$

(where θ_t is a shift operator on $\mathcal{D}$ and $\Delta_k \equiv y_k - y_{k-1}$). From a condition of the Markov behavior of a process with independent increments and from formula [5.3], it follows that:

$$E_0^{(Q)} \exp\left(-\int_0^{\tau_b} \xi(t)\,dt\right) =$$

$$= E_0^{(Q)} \exp\left(-\sum_{k=1}^{n} \int_{\tau_{y_{k-1}}}^{\tau_{y_k}} \xi(t)\,dt\right) =$$

$$= E_0^{(Q)} \prod_{k=1}^{n} \exp\left(-\int_{\tau_{y_{k-1}}}^{\tau_{y_k}} \xi(t)\,dt\right) =$$

$$= \prod_{k=1}^{n} E_0^{(Q)} \exp\left(-\int_{\tau_{y_{k-1}}}^{\tau_{y_k}} \xi(t)\,dt\right) \leq$$

$$\leq \prod_{k=1}^{n} E_0^{(Q)} \exp\left(-\xi(\tau_{y_{k-1}})(\tau_{y_k} - \tau_{y_{k-1}})\right) =$$

$$= \prod_{k=1}^{n} E_0^{(Q)} \exp\left(-y_{k-1}\left(\tau_{y_k} - \tau_{y_{k-1}}\right)\right) =$$

$$= \prod_{k=1}^{n} E_0^{(Q)} \exp\left(-y_{k-1}\ \tau_{\Delta_k} \circ \theta_{\tau_{y(k-1)}}\right) =$$

$$= \prod_{k=1}^{n} E_{y_{k-1}}^{(Q)} \exp\left(-y_{k-1}\ \tau_{\Delta_k}\right).$$

From the homogeneity in space of a process and from the Levy-Khinchin formula, the latter expression is equal to:

$$\prod_{k=1}^{n} E_0^{(Q)} \exp\left(-y_{k-1}\ \tau_{\Delta_k}\right) = \prod_{k=1}^{n} \exp\left(-\Delta_k\, \beta(y_{k-1})\right) =$$

$$= \exp\left(-\sum_{k=1}^{n} \Delta_k\, \beta(y_{k-1})\right) \to \exp\left(-\int_0^b \beta(y)\, dy\right)$$

as a fineness of a partition tends to zero.

Similarly, we obtain:

$$E_0^{(Q)} \exp\left(-\int_0^{\tau_b} \xi(t)\, dt\right) \geq$$

$$\geq \exp\left(-\int_0^b \beta(y)\, dy\right).$$

2) It is further given by:

$$E_0^{(Q)} \left(\int_{\tau_y}^{\infty} \exp\left(-\int_0^t \xi(s)\, ds\right)\, dt\right) \leq$$

$$\leq E_0^{(Q)} \left(\int_{\tau_y}^{\infty} \exp\left(-\int_{\tau_y}^t \xi(s)\, ds\right)\, dt\right) \leq$$

$$\leq E_0^{(Q)} \left(\int_{\tau_y}^{\infty} \exp\left(-y\,(t-\tau_y)\right)\, dt\right) = \frac{1}{y} \to 0 \quad (y \to \infty).$$

From here, it follows that:

$$E_0^{(Q)} \left(\int_{\tau_b}^{\infty} \exp\left(-\int_0^t \xi(s)\, ds\right)\, dt\right) =$$

$$= E_0^{(Q)} \left(\int_{\tau_b}^{\tau_y} \exp\left(-\int_0^t \xi(s)\, ds\right)\, dt\right) + O(1/y) =$$

$$= \sum_{k=1}^{n} E_0^{(Q)} \left(\int_{\tau_{y_{k-1}}}^{\tau_{y_k}} \exp\left(-\int_0^t \xi(s)\, ds\right)\, dt\right) + O(1/y) =$$

$$= \sum_{k=1}^{n} E_0^{(Q)} \exp\left(- \int_0^{\tau_{y_{k-1}}} \xi(s)\, ds\right) \times$$

$$\times \left(\int_{\tau_{y_{k-1}}}^{\tau_{y_k}} \exp\left(- \int_{\tau_{y_{k-1}}}^{t} \xi(s)\, ds\right)\, dt \right) + O(1/y) =$$

$$= \sum_{k=1}^{n} E_0^{(Q)} \exp\left(- \int_0^{\tau_{y_{k-1}}} \xi(s)\, ds\right) \times$$

$$\times\, E_0^{(Q)} \left(\int_{\tau_{y_{k-1}}}^{\tau_{y_k}} \exp\left(- \int_{\tau_{y_{k-1}}}^{t} \xi(s)\, ds\right)\, dt \right) + O(1/y) \leq$$

$$\leq \sum_{k=1}^{n} E_0^{(Q)} \exp\left(- \int_0^{\tau_{y_{k-1}}} \xi(s)\, ds\right) \times$$

$$\times\, E_0^{(Q)} \left(\int_{\tau_{y_{k-1}}}^{\tau_{y_k}} \exp\left(-y_{k-1}(t - \tau_{y_{k-1}})\right)\, dt \right) + O(1/y) =$$

$$= \sum_{k=1}^{n} E_0^{(Q)} \exp\left(- \int_0^{\tau_{y_{k-1}}} \xi(s)\, ds\right) \times$$

$$\times\, E_0^{(Q)} \frac{1}{y_{k-1}} (1 \exp(-y_{k-1}(\tau_{y_k} - \tau_{y_{k-1}})) + O(1/y) =$$

$$= \sum_{k=1}^{n} E_0^{(Q)} \exp\left(- \int_0^{\tau_{y_{k-1}}} \xi(s)\, ds\right) E_0^{(Q)} \frac{1}{y_{k-1}} \times$$

$$\times\, (1 \exp(-y_{k-1}(\tau_{\Delta_k} \circ \theta_{\tau_{y(k-1)}}))) + O(1/y) =$$

$$= \sum_{k=1}^{n} E_0^{(Q)} \exp\left(- \int_0^{\tau_{y_{k-1}}} \xi(s)\, ds\right) \frac{1}{y_{k-1}} \times$$

$$\times \left(1 - E_0^{(Q)} \exp(-y_{k-1}(\tau_{\Delta_k}))\right) + O(1/y) =$$

$$= \sum_{k=1}^{n} E_0^{(Q)} \exp\left(- \int_0^{\tau_{y_{k-1}}} \xi(s)\, ds\right) \frac{1}{y_{k-1}} \times$$

$$\times\, (1 - \exp(-\Delta_k\, \beta(y_{k-1}))) + O(1/y).$$

Considering only the first and second terms of the Taylor expansion of exponential members, and supposing that the partition fineness tends to zero, and also formula [5.4], we obtain a limit of the previous sum:

$$\int_b^y \exp\left(-\int_0^x \beta(t)\,dt\right)\frac{\beta(x)}{x}\,dx + O(1/y).$$

Supposing $y \to \infty$, we obtain formula [5.5]. □

Varying the diagnostic parameter under the law of the inverse process with independent positive increments seems quite justified. For example, for abrasive wear, the sense of this supposition is that times of deterioration, not intersected portions of a material, represent independent random variables. This supposition proves to be true for statistical data such as deterioration of automobile tires or contact brushes in electric motors.

5.3.2.1. *Examples*

Examples of monotone continuous semi-Markov processes can be found in [HAR 07].

1) One such process is an inverse gamma process, which is a homogeneous monotone semi-Markov process with independent positive increments of a random function τ_x $(x > 0)$, distributed according to a density:

$$f_{\tau_x}(t) = \frac{\delta}{\Gamma(x\gamma)}(\delta t)^{x\gamma-1}e^{-\delta t} \quad (t > 0),$$

where $\Gamma(x)$ is a gamma function, $\gamma > 0$ is the form parameter and $\delta > 0$ is the scale parameter.

Gamma distribution application in the reliability theory is justified in a number of works (see, for example, [GRA 66]).

The indicator of Levy-Khinchin's exponential representation of this process is of the form:

$$\beta(\lambda) = \int_0^\infty (1 - e^{-\lambda u})\frac{\gamma e^{-\delta\, u}}{u}\,du = \gamma \ln\frac{\delta+\lambda}{\delta}.$$

(see [HAR 07], p. 333). It can easily be proved by Taylor expansion of both members of this equality with respect to λ. The probability that the potential insurant will be insured before the insurance event [5.4] is equal to:

$$P(\zeta > \tau_b) = \exp\left(-\gamma(\delta + b)\ln\frac{\delta + b}{\delta} + \gamma\, b\right).$$

Using this formula, it is possible to obtain numerically on a computer the conditional expectation time of occurrence of the insurance event after the conclusion of the insurance contract (using formula [5.5]). This integral cannot be considered in a general view for the inverse gamma process.

2) The following example is associated with the so-called homogeneous process of Gut and Ahlberg [GUT 91], where the process of this aspect has been used for the summation of a random number of random summands. This process has been applied as a model of chromatography separations (see, for example, [HAR 07], p. 325). Trajectories of this process are continuous, do not decrease, and consist of independent intervals of linearly increasing movement (with exponentially distributed lengths), and independent intervals of constancy between increasing intervals with exponentially distributed lengths. For this process, the indicator of the Levy-Khinchin representation is equal to (within the magnitudes of three non-negative parameters):

$$\beta(\lambda) = m\,\lambda + v\int_{0+}^{\infty}(1 - e^{-\lambda\, u})\, e^{-k\, u}\, du = m\,\lambda + \frac{v\,\lambda}{k(k+\lambda)}.$$

It is not necessary to take an inverse Laplace transformation for the process of Gut and Ahlberg. The evaluation of numerical values of the probability that the potential insurant will conclude the insurance contract before the insurance event [5.4], and the conditional expectation time of the occurrence of the insurance event after the moment of an inference of the insurance contract [5.5] is possible, using any popular package of mathematical programs.

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