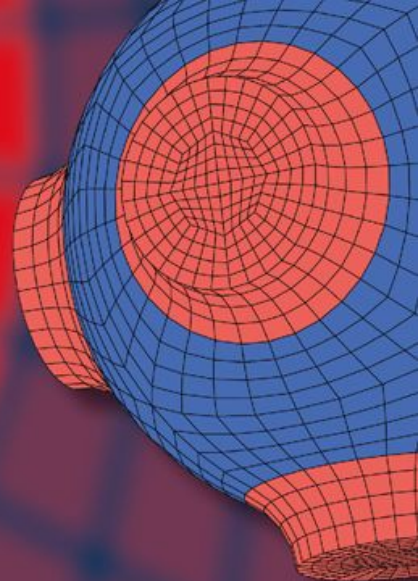


Advanced Structured Materials

Oxana Sadovskaya
Vladimir Sadovskii



Mathematical Modeling in Mechanics of Granular Materials

Advanced Structured Materials

Volume 21

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Foreword

The new monograph “Mathematical Modeling in Mechanics of Granular Materials” written by Oxana & Vladimir Sadovskii is based on a previous Russian version published in 2008. The Russian version was significantly revised and extended. The References were updated with respect to the readers not being familiar with the Russian language. Instead of eight chapters of the Russian original version there are now ten chapters—a new chapter devoted to continua with independent rotational degrees of freedom is added.

Looking on the basics of this book it is obvious that the starting point is the method of rheological models. In Continuum Mechanics one can split the approaches in material modeling into three different directions:

- the deductive approach (top-down modeling), which starts with some general mathematical structures restricted by the constitutive axioms and after that special cases will be deduced,
- the inductive approach (bottom-up modeling), which starts with special cases that are generalized step by step to derive more complex models, and
- last but not least the method of rheological modeling lying in-between the first and the second approaches.

The last approach is related to a pure phenomenological modeling without taking into account the microstructural behavior. On the other hand, this approach is an engineering method in material modeling since the parameter identification is very simple and can be computer-assisted performed.

Since the new monograph is based on the method of rheological models the question arises why we need a new book on rheological models. In this field there exist a lot of outstanding monographs, among them being:

- Deformation, Strain and Flow: an Elementary Introduction to Rheology, written by Markus Reiner and published by H. K. Lewis (London, 1960) and which was translated later into German and Russian,

- Vibrations of Elasto-plastic Bodies, written by Vladimir A. Pal'mov and published by Springer (Berlin, 1998), which is based on the original Russian edition from 1976,
- Materialtheorie—Mathematische Beschreibung des phänomenologischen thermomechanischen Verhaltens (Theory of Materials—Mathematical Description of the Phenomenological Thermo-mechanical Behavior), written by Arnold Krawietz and published by Springer (Berlin et al., 1986),
- Phänomenologische Rheologie—eine Einführung (Phenomenological Rheology—an Introduction), written by Hanswalter Giesekus and published by Springer (Berlin et al., 1994),
- Continuum Mechanics and Theory of Materials, written by Peter Haupt and published by Springer (Berlin et al., 2002, 2nd edition).

The new monograph is an excellent addition to the existing literature since the following items are new and have not been discussed in the previous books:

- a new rheological model (the rigid contact model) is introduced,
- the application fields of rheological models are extended to granular materials,
- a consequent and new mathematical description, necessary for the new element, is given and used also for the plastic rheological model, and
- several new examples are introduced, solved, and discussed.

It is desirable that this monograph will be accepted by the scientific community as well as the other monographs in this field.

Magdeburg, Germany, January 2012

Holm Altenbach

Preface

This monograph contains original results in the field of mathematical and numerical modeling of mechanical behavior of granular materials and materials with different strengths. Zones of the strains localization are defined by means of proposed models. The processes of propagation of elastic and elastic-plastic waves in loosened materials are analyzed. Mixed type models, describing the flow of granular materials in the presence of quasi-static deformation zones, are constructed. Numerical realizations of mechanics models of granular materials on multiprocessor computer systems are considered.

The book is intended for scientific researchers, university lecturers, post-graduates, and senior students, who specialize in the field of the mechanics of deformable bodies, mathematical modeling, and adjacent fields of applied mathematics and scientific computing.

This monograph is a revised and supplemented edition of the book “Mathematical Modeling in the Problems of Mechanics of Granular Materials”, published by “Fizmatlit” (Moscow) in 2008 in Russian. Compared with the Russian edition, its content is expanded by a new [Chap. 10](#), devoted to mathematical modeling of dynamic deformations of structurally inhomogeneous media, taking into account the rotational degrees of freedom of the particles. Besides, in [Chap. 7](#) the [Sect. 7.4](#), containing new results on the analysis of wave motions in layered media with viscoelastic interlayers, is added, and [Chap. 9](#), [Sect. 9.8](#) is added with the results of solving the problem of radial expansion of spherical and cylindrical layers of a granular material under finite strains.

The results presented in the monograph were used when reading special courses in the Siberian Federal University. The work was performed at the Institute of Computational Modeling of the Siberian Branch of Russian Academy of Sciences. It was partially supported by the Russian Foundation for Basic Research (grants no. 04-01-00267, 07-01-07008, 08-01-00148, 11-01-00053), the Krasnoyarsk Regional Science Foundation (grant no. 14F45), the Complex Fundamental Research Program no. 17 “Parallel Computations on Multiprocessor Computer Systems” of the Presidium of RAS, the Program no. 14 “Fundamental Problems of Informavtics and Informational Technologies” of the Presidium of RAS, the

Program no. 2 “Intelligent Information Technologies, Mathematical Modeling, System Analysis and Automation” of the Presidium of RAS, the Interdisciplinary Integration Project no. 40 of the Siberian Branch of RAS, the grant no. MK–982.2004.1 of the President of Russian Federation, and the grant of the Russian Science Support Foundation.

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It should be noted that significant improvements in the presentation of the material in comparison with the Russian edition was achieved through the attentive participation of the scientific editor of the monograph—Prof. Holm Altenbach, who has made many invaluable comments on the content.

Last but not least the authors wish to express special thanks, for supporting this project, to Dr. Christoph Baumann as a responsible person from Springer Publishers Group.

Krasnoyarsk, Russia, January 2012

Oxana Sadovskaya
Vladimir Sadovskii

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Chapter 1

Introduction

The theory of granular materials is among the most interesting and intensively developing fields of mechanics because the area of its application is very wide. It involves problems of mechanics of geomaterials (soils and rocks) related to the estimation of strength and stability of mine openings, bases and slopes when performing designed construction engineering work, problems of transportation of granular materials of minerals industry and agriculture production, problems of design of storage bunkers and grain tanks, problems of design of chemical machines with a boiling granular layer, problems of modeling of avalanching, etc.

In spite of the fact that the foundations of the theory have been laid even at the beginning of the development of continuum mechanics in the classical works by Coulomb and Reynolds, by now the theory is still far from completeness. The situation differs essentially from that in the elasticity theory, hydrodynamics, and gas dynamics where the constitutive equations have been formulated conclusively almost two centuries ago, and is similar to that in the plasticity theory where, with a number of particular models being available, the problem on an adequate description of kinematics of irreversible deformation for an arbitrary value of strains is not still conclusively solved [17, 18, 23–26].

The main difficulties are caused by significant difference in behavior of granular materials in tension and compression experiments. Such a behavior is also named strength-different effect and (this must be noted separately) is one type of the material behavior which cannot be modeled by the so-called unique stress-strain curve [1, 31]. Essentially all of known natural and artificial materials possess this property of heteroresistance (heteromodular) to some extent. For some of them, differences in modulus of elasticity, yield point, or creep diagram obtained with tension and compression are small to an extent that they should be neglected. However, in the studies of alternating-sign strains in granular materials, these differences may not be neglected. For example, when compressing, an ideal medium whose particles freely come in contact with each other behaves as if it is an elastic or elastic-plastic body depending on the stress level and does not offer resistance to tension. In cohesive media (soils and rocks) admissible tensile stresses are substantially smaller than

compressive ones and do not exceed a critical value defined by cohesion of particles. For a comparatively wide class of rocks, the ratio between ultimate tensile and compressive strengths varies in the range from 8 to 10, but for some types it reaches 50 and higher values [4]. In addition, mechanical properties of granular materials, as a rule, depend on a number of side factors such as inhomogeneity in size of particles and in composition, anisotropy, fissuring, moisture, etc. This results in low accuracy of experimental measurements of phenomenological parameters of models.

At the present time, two classes of mathematical models corresponding to two different conditions of deformation of a granular material (quasistatic conditions and fast motion ones) have been formed [9]. The first class describes behaviour of a closely packed medium at compression load on the basis of the theory of plastic flow with the Coulomb–Mohr or von Mises–Schleicher failure¹ condition. In the space of stress tensors conical domains of admissible stresses rather than cylindrical ones, as with the perfect plasticity theory, satisfy these conditions. In the second class a loosened medium modeled as an ensemble of a large number of particles in the context of the kinetic gas theory is considered.

To study quasi-static conditions of deformation, the stress theory in statically determinate problems which is applied in soil mechanics is developed [29]. The case of plane strain is best studied by Sokolovskii [33], and the axially symmetric case—by Ishlinskii [14]. Velocity fields in these problems are defined according to the associated flow rule considered by Drucker and Prager [7]. Mróz and Szymanski [22] showed that the special non-associated rule provides more accurate results in the problem on penetration of a rigid stamp into sand. A common disadvantage of these approaches lies in the fact that, when unloading, in the kinematic laws of the plastic flow theory a strain rate tensor is assumed to be zero, hence, deformation of a material is possible only as stresses achieve a limiting surface. From this it follows, for example, that a loosened granular material whose stressed state corresponds to a vertex of the Coulomb–Mohr or von Mises–Schleicher cone can not be compressed by hydrostatic pressure since to any state of hydrostatic compression there corresponds an interior point on the axis of the cone. This is in contradiction with a qualitative pattern. Kinematic laws turn out to be applicable in practice only in the case of monotone loading. Constitutive equations of the hypoplasticity in application to soil mechanics have a similar disadvantage [6, 12, 30, 34] because tension and compression states in them differ from one another in sign of instantaneous strain rate rather than in sign of total strain.

The equations of uniaxial dynamic deformation of a granular material with elastic particles, correct from the mechanical point of view, being a limiting case of the equations of heteromolecular elastic medium [2, 20] were studied by Maslov and Mosolov [19]. It is shown that along with velocity discontinuities (shock waves) they also describe displacement discontinuities. Maslov et al. [21] applied these equations to analysis of the “dry boiling” process, i.e. spontaneous appearance and collapse of voids in a granular material. Phenomenological models of a spatial stress-strain state

¹ The term failure is used in the generalized sense that means failure occurs if the material starts to yield, to damage, to break (fracture), etc.

of a cohesive soil for finite strains were proposed by Grigoryan [11] and Nikolaevskii [28]. The works [5, 8, 13, 15, 16] are devoted to generalization of fundamentals of the plasticity theory for description of dynamics and statics of granular materials.

Bagnold [3] stated experimentally that appearance of relatively small nonzero tangential stresses in a loosened granular material with an intensive shear flow is caused by two factors: particle collision provided that rarefaction of a medium is low, and impulse interchange between different layers due to displacement of particles in the case of higher degree of rarefaction. A spatial model of fast motions was proposed by Savage [32] who compared the solution of the problem on channel flow with experimental results, in particular, with those of Bagnold. Goodman and Cowin [10] developed a model for the analysis of gravity flows of a granular material. Nedderman and Tüzün [27] constructed a simple kinematic model which allows one to simulate an experimental pattern of steady-state outflow from funnel-shaped bunkers.

In this monograph a radically new approach, where constitutive relationships of heteromaterials are constructed with the help of rheological schemes including a special element called rigid contact, is worked out. By the combination of this element with traditional ones (elastic spring, viscous damper, and plastic hinge), special mathematical models of mechanics of granular materials taking into account features of the deformation process are obtained. The static and kinematic theorems of the limit equilibrium theory are extended to the case of heteromaterials. On the basis of the finite element method, computational algorithms are developed. Using them, the numerical analysis of strain localization zones in samples with cuts is performed.

In the framework of the small strains theory, the propagation of compression shock waves (signotons) in a pre-loosened granular material possessing of elastic and plastic properties is analyzed. Exact solutions of the one-dimensional problems with plane waves are obtained. Several problems related to the numerical implementation of the proposed models on supercomputers with parallel architecture are considered. Parallel program systems for the computation of dynamic problems in two-dimensional and three-dimensional formulations on multiprocessor computer systems of the MVS series intended for the application to problems of geophysics (seismicity) are worked out.

A model of mixed type taking into account stagnation regions of quasi-static deformation in a moving flow of a loosened granular material is constructed. In the context of this model, an exact solution describing the Couette stationary rotational flow between coaxial cylinders is obtained. Nonstationary avalanche-like motion of a granular material along an inclined plane is described. An exact solution of the problem on stationary motion of a layer caused by horizontal displacement of a heavy plate along its surface is constructed.

In what follows, we use the notations:

- the numeration of formulas, theorems and figures is given as (i.j), where i is the number of the chapter and j is the number inside the chapter;
- the vectors are denoted by bold italic font like $\mathbf{x}$, $\mathbf{u}$, $\mathbf{v}$;

- the second- and higher tensors are denoted by bold italic font like $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$;
- the vectors and matrices in the vector-matrix notation are denoted as
 - the vectors by bold font like $\mathbf{U}$, $\mathbf{V}$;
 - the matrices by bold font like $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$;
- the maps of vector spaces onto vector sets are denoted by bold font like π , Π ;
- the spaces and special sets of vectors and functions are denoted by bold italic font like $\mathbf{C}$, $\mathbf{F}$, $\mathbf{K}$;
- the spaces with blackboard bold font like $\mathbb{R}^3$, $\mathbb{R}^m$.

We also employ the Einstein summation convention with respect to repeated indices and use the $\text{\LaTeX}$'s notations like $\Re z$ and $\Im z$ for real and imaginary parts of a complex number $z = x_1 + \imath x_2$.

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Chapter 2

Rheological Schemes

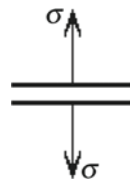
Abstract The traditional rheological method is supplemented by a new element—rigid contact, which serves to take into account different resistance of a material to tension and compression. A rigid contact describes mechanical properties of an ideal granular material involving rigid particles for an uniaxial stress state. Combining it with elastic, plastic, and viscous elements, one can construct rheological models of different complexity.

2.1 Granular Material With Rigid Particles

The method of rheological models is the basis of the phenomenological approach to the description of a stress-strain state of media with complex mechanical properties, [18, 22, 30]. Ignoring the physical nature of deformation, this method enables one to construct mathematical models which describe quantitative characteristics with a satisfactory accuracy (from the point of view of engineering applications) and are of a good mathematical structure. As a rule, for the models obtained with the help of the rheological method, solvability of main boundary-value problems can be analyzed and efficient algorithms for numerical implementation can be easily constructed. At the same time, with the use of conventional rheological elements (a spring simulating elastic properties of a material, a viscous damper, and a plastic hinge) only, it is impossible to construct a rheological scheme for a medium with different resistance to tension and compression or for a medium with different ultimate strengths under tension and compression.

To make it possible, we supplement the method by a new, fourth element, namely, a rigid contact, [26–28]. It is represented schematically as two plates being in contact (Fig. 2.1). A granular material with rigid particles, i.e. a system of absolutely rigid balls being in contact with each other, is an ideal material whose behavior at a uniaxial stress-strain state corresponds to this element. With tension of a system, balls roll about and stress turns out to be zero. Following previous tension, compression goes

Fig. 2.1 Rigid contact element



on with zero stresses until the balls touch each other and the system in fact returns to its original position. Compressive strains are impermissible and compressive stresses can be arbitrary with strain being equal to zero.

With the conventional notations, we represent the constitutive relationships of a rigid contact as the system

$$\sigma \leq 0, \quad \varepsilon \geq 0, \quad \sigma \varepsilon = 0. \quad (2.1)$$

The inequalities involved in this system exclude arising tensile stresses and compressive strains in a granular material with rigid particles. From the equation (so-called complementing condition) it follows that one of the quantities being considered (stress or strain) must be zero.

It should be noted that the constitutive relationships (2.1) are incorrect in the mechanical sense because in the general case they do not enable one to determine uniquely acting stress from given strain and, conversely, to determine strain from given stress. However, as will be shown further, this incorrectness can be easily eliminated by adding regularizing elements to the rheological scheme.

Similar systems of inequalities with complementing conditions arise, for example, in mathematical economics when solving problems of multiple objective optimization (see, [8, 23]). It is known that such a system can be reduced to two variational inequalities equivalent to one another (arbitrary varying quantities are marked by tilde):

$$\sigma (\tilde{\varepsilon} - \varepsilon) \leq 0, \quad \varepsilon, \tilde{\varepsilon} \geq 0; \quad (\tilde{\sigma} - \sigma) \varepsilon \leq 0, \quad \sigma, \tilde{\sigma} \leq 0. \quad (2.2)$$

Indeed, let the system (2.1) be valid for σ and ε . Then either $\sigma = 0$ and $\varepsilon \geq 0$, or $\sigma < 0$ and $\varepsilon = 0$. In either case both inequalities (2.2) hold since, on the one hand, $\sigma \tilde{\varepsilon} \leq 0$ and, on the other hand, $\tilde{\sigma} \varepsilon \leq 0$. Now assume that on the contrary σ and ε satisfy the first inequality of (2.2). Then either $\varepsilon = 0$ and the relationships (2.1) are evident, or $\varepsilon > 0$ and from the fact that strain variation may be positive ($\tilde{\varepsilon} > \varepsilon$) as well as negative ($\varepsilon > \tilde{\varepsilon} \geq 0$) it follows that σ equals zero. In this case the relationships (2.1) are also evident. If σ and ε satisfy the second inequality of (2.2) rather than the first one, then the system (2.1) is valid for them. This is proved in a similar way.

The advantage of the formulation of constitutive relationships of a rigid contact in terms of variational inequalities over the equivalent formulation (2.1) lies in the fact that these inequalities admit a generalization to the case of a spatial stress-strain state of a medium. This generalization is given in Chap. 4. It is performed with the help

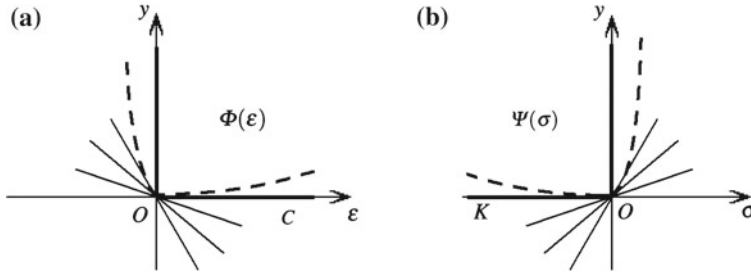


Fig. 2.2 Stress potential **a** and strain potential **b**

of tensor representations by introducing cones of admissible strains and stresses. In the uniaxial state considered now these cones are equal to $C = \{\varepsilon \geq 0\}$ and $K = \{\sigma \leq 0\}$, respectively. To state the potential nature of the relationships, we represent (2.2) in the following form:

$$\sigma \in \partial\Phi(\varepsilon), \quad \varepsilon \in \partial\Psi(\sigma). \quad (2.3)$$

Here Φ and Ψ are the stress and strain potentials, the symbol ∂ denotes subdifferential.

Contrary to the classical models of mechanics of deformable media, in this case the potentials are not differentiable and even continuous. They are defined in terms of the indicator functions of the cones C and K :

$$\Phi(\varepsilon) = \begin{cases} 0, & \text{if } \varepsilon \in C, \\ +\infty, & \text{if } \varepsilon \notin C, \end{cases} \quad \Psi(\sigma) = \begin{cases} 0, & \text{if } \sigma \in K, \\ +\infty, & \text{if } \sigma \notin K, \end{cases}$$

for which the conventional notations $\delta_C(\varepsilon)$ and $\delta_K(\sigma)$ are used further. The graph of the former function is formed by two positive semi-axes on the ε y plane and the graph of the latter one by negative and positive semi-axes on the σ y plane (Fig. 2.2). Both of them can be obtained by passage to the limit with the help of sequences of continuously differentiable functions whose graphs are shown as dashed lines. Smoothed functions can be considered as potentials of special nonlinearly elastic media with different strength properties to tension and compression. For such media the nonlinear Hooke law is valid: stresses are expressed in terms of derivatives with respect to strains and vice versa. In the limit the derivatives, with which the angular coefficients of tangents to graphs of smooth potentials are identified, are transformed to subdifferentials of the indicator functions. For the interior points of the cones C and K they tend to zero and for the boundary points ($\varepsilon = 0$ and $\sigma = 0$, respectively) they may take any limit position shown in Fig. 2.2 as a fan of straight lines.

A rigorous mathematical definition of subdifferential of a convex function and some its properties required for the study of models of spatial deformation of a granular material are given in Chap. 3. Here, basing on the intuitive notion described above, we only state that subdifferential of a function at a given point is the set formed

by angular coefficients of all straight lines, “tangent” to the graph of the function at this point and lying below the graph. Thus, if $\varepsilon \in C$ and $\sigma \in K$ then

$$\partial \delta_C(\varepsilon) = \{\tilde{\sigma} \mid \tilde{\sigma}(\tilde{\varepsilon} - \varepsilon) \leq 0 \quad \forall \tilde{\varepsilon} \geq 0\},$$

$$\partial \delta_K(\sigma) = \{\tilde{\varepsilon} \mid (\tilde{\sigma} - \sigma)\tilde{\varepsilon} \leq 0 \quad \forall \tilde{\sigma} \leq 0\},$$

and going from Eq. (2.2) to (2.3) is a trivial change of notations for a more illustrative geometric interpretation. We also note that it makes no sense to look for a form of phenomenological constitutive relationships for an ideal granular material with rigid particles which is more simple than (2.3) since the notions and notations being used describe a threshold nature of deformation of a material with extreme precision. Besides, they are a simple generalization of the constitutive equations of the nonlinear elasticity theory to the case of non-differentiable potentials.

2.2 Elastic-Visco-Plastic Materials

A known way of regularization of incorrect mechanical model is in going to a more complex model describing adequately special features of deformation of a material which are not taken into account. As a version of complication, we consider the model of an ideal granular material with elastic particles whose rheological scheme is given in Fig. 2.3a. According to this scheme, strain is equal to the sum of an elastic component $\varepsilon^e = a \sigma$ (computed by the Hooke law), where $a > 0$ is the modulus of elastic compliance of a spring, and strain $\varepsilon^c = \varepsilon - \varepsilon^e$ of a rigid contact. If $\sigma < 0$ then $\varepsilon^c = 0$ and $\varepsilon = a \sigma < 0$, i.e. elastic compression takes place. If $\sigma = 0$ then $\varepsilon^e = 0$ and $\varepsilon \geq 0$, i.e. the loosening of a material is observed. In the general case the real stress is determined in terms of the strain by the formula

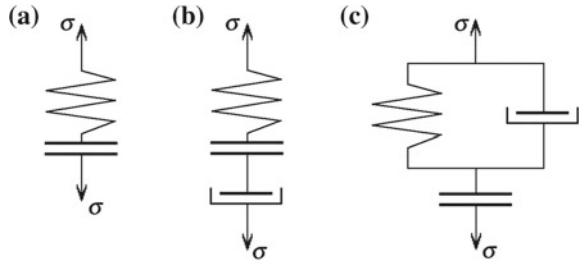
$$\sigma = \frac{\varepsilon - |\varepsilon|}{2a}. \quad (2.4)$$

On the contrary, generally speaking, the strain is not uniquely determined in terms of given stress. Thus, the model of an elastic granular material is as much incorrect as the model of an elastic-plastic material with hardening being not taken into account, [9]. The constitutive relationships can be represented in the potential form (2.3) with potentials

$$\Phi(\varepsilon) = \begin{cases} \varepsilon^2/(2a), & \text{if } \varepsilon < 0, \\ 0, & \text{if } \varepsilon \geq 0, \end{cases} \quad \Psi(\sigma) = \frac{a\sigma^2}{2} + \delta_K(\sigma).$$

The former potential is a differentiable function and the latter one takes infinite values exterior to the cone K . This expression for the stress potential is obtained as a solution of the differential equation $\partial \Phi / \partial \varepsilon = \sigma$ with the right-hand side (2.4), and

Fig. 2.3 Rheological schemes: **a** elastic granular material, **b** viscoelastic material (Maxwell model), **c** viscoelastic material (Kelvin–Voigt model)



for the strain potential it is obtained as a consequence of an additive representation in the form of the sum of potentials of an elastic spring and a rigid contact.

The rheological schemes shown in Figs. 2.3b,c correspond to granular materials which show viscoelastic properties in the compression process. In both cases, ideal (cohesionless) materials are considered. The scheme in Fig. 2.3b describes compression with the help of the Maxwell model and the scheme in Fig. 2.3c with the help of the Kelvin–Voigt model. For the former scheme from Eq. (2.4), taking into account the Newton law $\sigma = \eta \dot{\varepsilon}^v$, we have

$$2a\eta\dot{\varepsilon}^v = \varepsilon - \varepsilon^v - |\varepsilon - \varepsilon^v| \leq 0. \quad (2.5)$$

Here η is the viscosity coefficient and $\dot{\varepsilon}^v$ is the rate of the viscous strain. If the time-dependence of stress $\sigma(t) \leq 0$ is known, then the viscous strain component is determined by integration of the equation corresponding to the Newton law. To determine total deformation, Eq. (2.5) whose solution is, in general, ambiguous is used. When, on the contrary, the dependence $\varepsilon(t)$ is given, then, integrating the differential Eq. (2.5), we can determine the dependence $\varepsilon^v(t)$ and, hence, $\sigma(t)$.

The solution of the differential equation is conveniently interpreted geometrically on the $\varepsilon - \varepsilon^v$ plane. For $\varepsilon \geq \varepsilon^v$ the rate of viscous strain equals zero and for $\varepsilon < \varepsilon^v$ the equation $a\eta\dot{\varepsilon}^v = \varepsilon - \varepsilon^v$ holds. Hence,

$$\varepsilon^v = \varepsilon_0^v \exp\left(-\frac{t-t_0}{a\eta}\right) + \frac{1}{a\eta} \int_{t_0}^t \varepsilon(t_1) \exp\left(-\frac{t-t_1}{a\eta}\right) dt_1,$$

where ε_0^v and t_0 are constants. In Fig. 2.4 the typical deformation curve is shown. The ray OP_0 corresponds to tension of a material for $\varepsilon_0^v = 0$ and the curve OP_1P_2 depending on $\varepsilon(t)$ corresponds to compression. At the point P_1 the strain rate changes its sign from negative to positive. At the point P_2 an irreversibly compressed material transforms to a loosened state. In the case of slow (quasistatic) compression, the curve OP_1P_2 tends to the rectilinear segment OP_2 of the ray $\varepsilon = \varepsilon^v \leq 0$ shown as a dashed line. When repeating a deformation cycle, a similar curve issues out of the point P_2 rather than of O .

Fig. 2.4 Deformation curve
(Maxwell model)

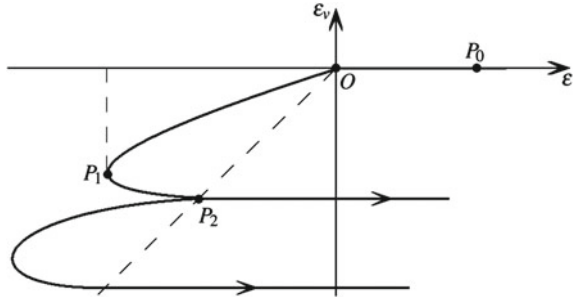
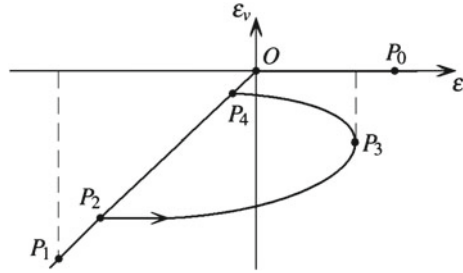


Fig. 2.5 Deformation curve
(Kelvin–Voigt model)



For the latter scheme, stress consists of two components (elastic and viscous) $\sigma = \sigma^e + \sigma^v$ and strains of viscous and elastic elements coincide. Thus,

$$\varepsilon = \varepsilon^e + \varepsilon^v, \quad \sigma = \frac{\varepsilon^v}{a} + \eta \dot{\varepsilon}^v. \quad (2.6)$$

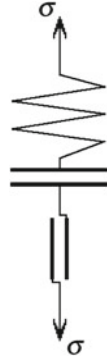
For $\varepsilon^e > 0$, when a material is loosened, stress equals zero, hence,

$$\varepsilon^v = \varepsilon_0^v \exp\left(-\frac{t - t_0}{a \eta}\right). \quad (2.7)$$

For $\varepsilon^e = 0$, when a material is in a compact state, stress $\sigma \leq 0$ is calculated from given strain by Eq. (2.6) for $\varepsilon^v = \varepsilon$. The typical deformation curve for given dependence $\varepsilon(t)$ is shown in Fig. 2.5. Tension is described by the ray OP_0 and compression by the rectilinear segment OP_1 . At the point P_1 the strain rate $\dot{\varepsilon}$ changes sign. In the segment P_1P_2 unloading is performed for $\varepsilon^e = 0$ and $\sigma < 0$. The viscoelastic component of strain relaxes. Stress turns out to be equal to zero at some point P_2 and the further process is consistent with Eq. (2.7). The curve $P_2P_3P_4$ is associated with this equation. At the point P_4 a cycle of repeated deformation starts.

Total strain is uniquely determined from a given dependence $\sigma(t) \leq 0$ only in a viscoelastic compression state for $\varepsilon^e = 0$,

Fig. 2.6 Rheological scheme with plastic element



$$\varepsilon^v = \varepsilon_0^v \exp\left(-\frac{t-t_0}{a\eta}\right) + \frac{1}{\eta} \int_{t_0}^t \sigma(t_1) \exp\left(-\frac{t-t_1}{a\eta}\right) dt_1.$$

In a tension state the model remains incorrect due to ambiguity of strain of a contact.

The rheological scheme of an ideal elastic-plastic granular material is shown in Fig. 2.6. With tension or compression under the action of stress whose absolute value does not exceed the yield point σ_s of a plastic hinge, such a material behaves according to the scheme shown in Fig. 2.3a. As the yield point is achieved, with compression the material passes to a plastic flow state. In this state, the strain rate $\dot{\varepsilon}$ can take an arbitrary negative value. If, following a plastic flow state, stress decreases (unloading occurs) but remains compressing, then the strain rate is expressed in terms of the stress rate by the linear Hooke law. Stresses exceeding σ_s are impermissible.

Total strain involves three components associated with three elements of the scheme: $\varepsilon = \varepsilon^e + \varepsilon^c + \varepsilon^p$. Due to (2.4)

$$2a\sigma = \varepsilon - \varepsilon^p - |\varepsilon - \varepsilon^p| \leq 0.$$

Taking into account the sign of stress, we write the constitutive relationships of a plastic hinge as a system of inequalities with the complementing condition

$$\dot{\varepsilon}^p \leq 0, \quad \sigma \geq -\sigma_s, \quad (\sigma + \sigma_s) \dot{\varepsilon}^p = 0.$$

Similarly to (2.1), this system can be transformed to equivalent variational inequalities or reduced to the potential form. To this end, we first consider the potential representation of the Newton law for a viscous flow:

$$\sigma = \frac{\partial D(\dot{\varepsilon}^v)}{\partial \dot{\varepsilon}^v}, \quad \dot{\varepsilon}^v = \frac{\partial H(\sigma)}{\partial \sigma}.$$

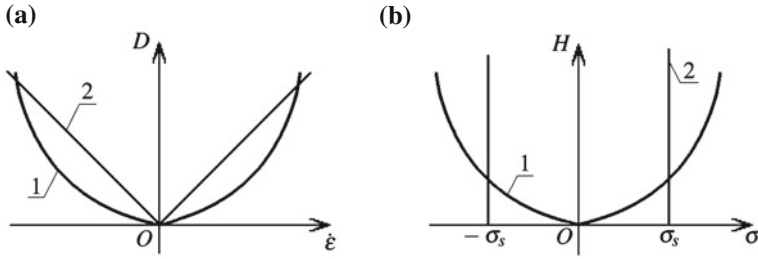


Fig. 2.7 Dissipative potentials of stresses **a** and strain rates **b**

If the coefficient of viscosity is constant, the dissipative potentials $D = \eta (\dot{\varepsilon}^v)^2/2$ and $H = \sigma^2/(2\eta)$ are quadratic functions (curves 1 in Fig. 2.7). Deforming the graphs with preservation of convexity, we can obtain potentials for a material with a variable viscosity coefficient depending on achieved stress or instant strain rate. To retain consistency of potentials, the graphs D and H should be deformed so that these functions are expressed in terms of one another with the help of the Legendre tangent transform

$$H(\sigma) = \sigma \dot{\varepsilon}^v - D(\dot{\varepsilon}^v).$$

Convexity is required for a viscosity coefficient to be positive. The limit version of convex curves (the piecewise linear curves 2) corresponds to the plastic state of a material. The existence of corner points on graphs of plastic dissipative potentials leads to the necessity of using subdifferential which generalizes the notion of derivative. The constitutive relationships $\sigma \in \partial D(\dot{\varepsilon}^p)$ and $\dot{\varepsilon}^p \in \partial H(\sigma)$ in terms of subdifferentials result in two inequalities

$$\sigma (\tilde{\varepsilon} - \dot{\varepsilon}^p) \leq D(\tilde{\varepsilon}) - D(\dot{\varepsilon}^p) \quad \forall \tilde{\varepsilon},$$

$$(\tilde{\sigma} - \sigma) \dot{\varepsilon}^p \leq 0, \quad |\sigma| \leq \sigma_s, \quad |\tilde{\sigma}| \leq \sigma_s.$$

Their equivalence can be proved on the basis of the results given in the next chapter.

For an elastic-plastic granular material (Fig. 2.6), this leads to the variational inequality

$$(\tilde{\sigma} - \sigma)(a \dot{\sigma} - \dot{\varepsilon}) \geq 0, \quad |\sigma| \leq \sigma_s, \quad |\tilde{\sigma}| \leq \sigma_s, \quad (2.8)$$

which provides an exact description of rheology of a plastic element. Consider the $\sigma - \varepsilon$ diagrams of the uniaxial deformation for such a material (Fig. 2.8) constructed with the help of (2.8). The $\sigma - \varepsilon$ diagram shows the active loading as a three-segment broken line whose segments correspond to the loosening of a material (the segment OP_0) and to the elastic and plastic compression (OP_1 and P_1P_2 , respectively). The unloading following the plastic flow of a material is described as the rectilinear segment P_2P_3 which is parallel to the original elastic segment of the diagram.

Fig. 2.8 Diagram of uniaxial tension–compression

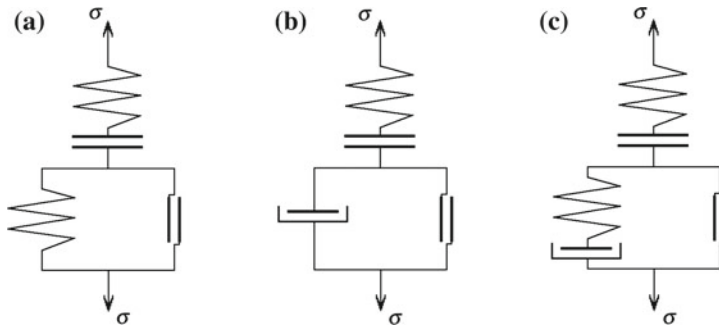
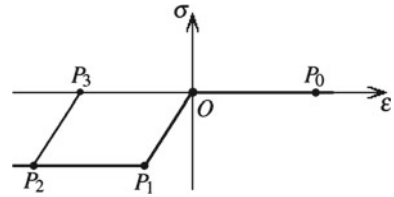


Fig. 2.9 Complex rheological schemes: **a** elastic-plastic granular material, **b** elastic-visco-plastic granular material (Schwedoff–Bingham model), **c** regularized variant of previous scheme

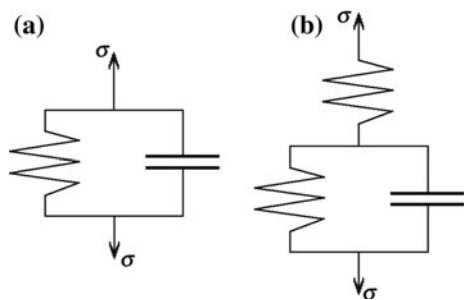
Combining elastic, plastic and viscous elements with a rigid contact, we can construct constitutive relationships for granular materials of more complex rheology. Examples of more complex schemes are given in Fig. 2.9. The scheme in Fig. 2.9a describes a granular material whose deformation with compressive stresses is defined by the theory of elastic-plastic flow with linear hardening. The schemes in Figs. 2.9b,c correspond to the theory of viscoplastic Schwedoff–Bingham flow.

In conclusion, it should be noted that a rigid contact, using in the given approach to take into account different compression and tension strength properties of the granular material and being in fact a nonlinearly elastic element, describes a thermodynamically reversible process. Irreversible deformation of a material which results in dissipation of mechanical energy is taken into account only when viscous or plastic elements are involved into the rheological scheme.

2.3 Cohesive Granular Materials

Further development of the model of a granular material leading to constitutive relationships correct in the mechanical sense consists in the phenomenological description of connections between particles. To this end, in parallel with a rigid contact, an elastic, viscous, or plastic element is involved into a scheme depending on properties of the binder. The simplest rheological scheme taking into account elastic connections between absolutely rigid particles is given in Fig. 2.10a. Figure 2.10b

Fig. 2.10 Elastic connections: **a** elastic material with rigid particles, **b** heteromodular elastic material



corresponds to a model of a heteromodular elastic material whose elastic properties with tension are characterized by two series-connected springs and with compression by only one of these springs. In this case the constitutive equations

$$\varepsilon = \begin{cases} (a + b) \sigma, & \text{if } \sigma \geq 0, \\ a \sigma, & \text{if } \sigma < 0, \end{cases}$$

(a and b are the moduli of elastic compliance) describe a one-to-one dependence between stress and strain.

Viscous properties of a binder are taken into account in the rheological schemes in Fig. 2.11. The scheme given in Fig. 2.11a serves to describe a cohesive granular material with absolutely rigid particles. In the scheme shown in Fig. 2.11b particles with compression are deformed according to the elastic law. More complicated rheology can be taken into account with the help of the models considered in the above section. According to the second scheme, for $\varepsilon^c = \varepsilon - a \sigma > 0$ the strain of the material obeys the Maxwell model. If the dependence $\sigma(t)$ is given, the unknown time-dependence of strain is uniquely determined by integrating the equation

$$\dot{\varepsilon} = a \dot{\sigma} + \frac{\sigma}{\eta}, \quad (2.9)$$

whose solution describes a real process provided that $\varepsilon \geq a \sigma$. With violating this condition, strain is determined from the Hooke law as $\varepsilon = a \sigma$. If the dependence $\varepsilon(t)$ is given, then the function $\sigma(t)$ describing the stress state of a material with the same condition is determined by integrating Eq. (2.9). Otherwise real stress is determined from the Hooke law. Thus, the model is correct for an arbitrary program of deformation or loading.

In Fig. 2.12a the rheological scheme of a material involving rigid particles with plastic connections is given. Deformation of such a material is possible provided that the absolute value of stress is equal to the yield point of a plastic hinge. Compression is admissible only after previous tension. Any deformation is thermodynamically irreversible. The rheological scheme given in Fig. 2.12b takes into account, along with plastic properties of a binder, its elastic properties and elastic properties of

Fig. 2.11 Viscous connections: **a** cohesive material with rigid particles, **b** viscoelastic granular material

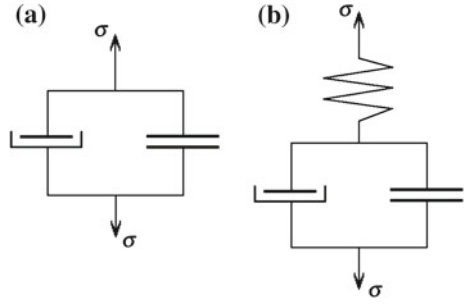
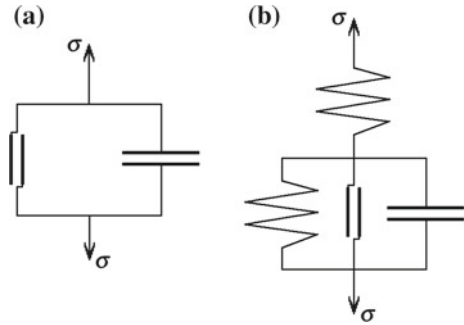


Fig. 2.12 Plastic connections: **a** plastic material with rigid particles, **b** elastic-plastic granular material



particles. This model is quite correct since the corresponding diagram of uniaxial tension–compression (Fig. 2.13) is strictly monotone on the active loading segments OP_0 and OP_1P_2 as well as on the unloading segment $P_2P_3P_4$. On the segments OP_0 and OP_1 elastic deformation of a material is observed. The segment P_1P_2 of the diagram describes the process of plastic tension of natural (no hardening) material. In this case

$$\varepsilon = a\sigma + b(\sigma - \sigma_s).$$

On the elastic unloading segment P_2P_3 strain of the upper spring is equal to $a\sigma$ and the strain of the system of parallel elements is constant:

$$\varepsilon^e = \varepsilon^p = \text{const}, \quad \sigma^e = \frac{\varepsilon^p}{b}, \quad \sigma^p = \sigma - \sigma^e.$$

At the point P_3 stress of a plastic hinge achieves the yield point ($-\sigma_s$) with compression and a material is transformed into a state of plastic flow (the segment P_3P_4) described by the equation

$$\varepsilon = a\sigma + b(\sigma + \sigma_s).$$

At the point P_4 symmetric to P_1 contact is closed up, i.e. its strain $\varepsilon^c = b(\sigma + \sigma_s)$ turned out to be zero. Thus, in the framework of the model with the rheological scheme given in Fig. 2.12b, with the cyclic loading, the translational strain hardening of the

Fig. 2.13 Diagram of uniaxial tension–compression

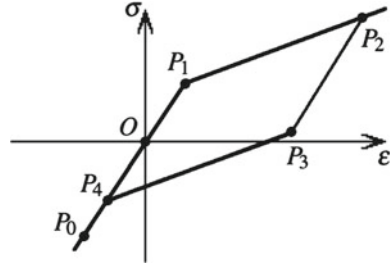
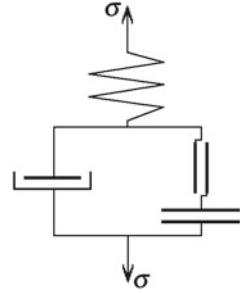


Fig. 2.14 Scheme involving four different elements



material is observed. However, as the cycle is completed, the yield surface takes its original position.

More complex rheological properties of particles and the binder are taken into account in the scheme involving four elements of different types shown in Fig. 2.14. This is probably the only version of the configuration of four elements which results in a model correct in the mechanical sense. Judging by this scheme, in the tension state, where $\varepsilon^c = \varepsilon^v - \varepsilon^p > 0$, a plastic hinge has no effect, hence, behavior of a material is described by the Maxwell model of a viscoelastic medium. In the compression state, where a contact is closed up, a material behaves as the Schwedoff–Bingham elastic-visco-plastic medium. The equations

$$\sigma = \sigma^e = \sigma^v, \quad \varepsilon = \varepsilon^e + \varepsilon^v, \quad \varepsilon^e = a \sigma, \quad \eta \dot{\varepsilon}^v = \sigma$$

form a total system for determining strain from given stress or stress from given strain for $\varepsilon^v > \varepsilon^p$. Hence it follows that strain is determined in terms of stress by integrating the differential equation (2.9) with respect to ε and stress is determined in terms of strain with the help of the same equation with respect to σ . In this case the general solution is given by the integral

$$\sigma = s(t) \equiv \sigma_0 \exp\left(-\frac{t-t_0}{a\eta}\right) + \frac{1}{a} \int_{t_0}^t \exp\left(-\frac{t-t_1}{a\eta}\right) d\varepsilon(t_1), \quad (2.10)$$

where the integration constant σ_0 is determined from the condition for continuity of stress with change of mode, t_0 is the instant of going to a given mode. Thus, all unknown functions turn out to be uniquely determined.

With compression, different variants may take place. If compression follows previous tension of a material and, hence, $\varepsilon^c > 0$, then the process is described by Eqs. (2.9), (2.10) up to the instant at which a contact is closed up. For $\varepsilon^c = 0$ the equations

$$\sigma = \sigma^e = \sigma^v + \sigma^c, \quad \sigma^c = \sigma^p, \quad \varepsilon = \varepsilon^e + \varepsilon^v, \quad \varepsilon^v = \varepsilon^p$$

are valid. For $0 > \sigma > -\sigma_s$ a plastic element blocks deformation of a viscous damper and viscous stress σ^v turns out to be zero. In this state a material is deformed according to the Hooke law $\varepsilon = a\sigma$. Finally, for $\sigma \leq -\sigma_s$ the equality $\sigma^p = -\sigma_s$ holds, hence,

$$\dot{\varepsilon} = a\dot{\sigma} + \frac{\sigma + \sigma_s}{\eta}, \quad \sigma = s(t) - \sigma_s. \quad (2.11)$$

All unknown functions are also uniquely determined.

If a loading or deformation program involves alternating tension and compression segments then the unknown time-dependences of strain and stress, respectively, can be obtained in a closed form with the help of Eqs. (2.9), (2.10), and (2.11). Only a choice of an appropriate mode presents difficulties. To give a rigorous mathematical formulation of the problem in the general case (for arbitrary loading and deformation programs) which allows one to solve the problem of choice implicitly, we supplement the equations with universal constitutive relationships for a rigid contact and a plastic hinge in the form of variational inequalities (2.2) and (2.8):

$$\sigma^c (\tilde{\varepsilon} - \varepsilon^c) \leq 0, \quad \varepsilon^c \geq 0, \quad \tilde{\varepsilon} \geq 0,$$

$$(\tilde{\sigma} - \sigma^p) \dot{\varepsilon}^p \leq 0, \quad |\sigma^p| \leq \sigma_s, \quad |\tilde{\sigma}| \leq \sigma_s,$$

which involve arbitrary admissible variations of stress and strain. Besides, we formulate initial conditions for viscous and plastic elements for which the constitutive relationships are of the differential form: $\varepsilon^v(0) = \varepsilon^p(0) = 0$.

2.4 Computer Modeling

The construction and study of constitutive relationships for materials whose rheological schemes involve a reasonably large number of elements are rather tedious. In this section, a way of the solution of this problem with the use of general-purpose computational algorithms implemented in the form of a computer system with elements of visual design, [31], is proposed.

In the general case the analysis of rheological properties of materials with uniaxial deformation is reduced to two problems considered above. In the first problem a

loading program (the time-dependence of stress σ) is known and the time-dependence of strain $\varepsilon(t)$ is unknown. In the second one, conversely, a deformation program is given and the time-dependence of stress is to be determined. In the case of constant tensile or compressive stress, the solution of the first problem enables one to construct the creep diagrams of a material. The second problem provides curves of the stress relaxation in the case of constant strain.

If the scheme of a material being studied involves nonlinear elements (plastic hinges or rigid contacts) then in a natural way the question of correctness of a rheological model arises. A model is assumed to be correct if both problems are uniquely solvable and stable. For example, the model of ideal plasticity whose rheological scheme involves a single element, namely, a plastic hinge, is among incorrect models. For given stress being equal to the yield point, strain can not be uniquely determined in the framework of this model. The model of an ideal granular material with absolutely rigid particles, whose rheological scheme is represented by a rigid contact (Fig. 2.1), is another example of an incorrect model. In this case, for $\sigma = 0$ strain is not uniquely determined, besides, stress is not uniquely determined for $\varepsilon = 0$.

Among correct models, further we consider only the models which enable us to determine stresses and strains of all elements of a rheological scheme. Most likely it is difficult to formulate in the general case the conditions under which a model has this property. Because of this, further this question is related to correctness of a computational algorithm being applied. An example of a rheological scheme involving four base elements of different types which is correct in this sense is given in Fig. 2.14 of the previous section.

In the general case a rheological scheme involving n elements is subdivided into m levels depending on the position of connective elements. Each level is characterized by strain ε_i , $i = 1, \dots, m$. Elements are numbered in a strictly specified order: first elastic elements, next viscous ones, then rigid contacts, and finally plastic hinges. To each of them, there corresponds stress σ_j , $j = 1, \dots, n$. Let $\mathbf{U}$ be a vector of dimension $N = m + n + 1$ such that these $m + n$ quantities and one more quantity (the unknown value of total strain or of resulting stress, according to the type of a problem) are its components. A rheological scheme of the general form leads to a system involving algebraic equations (equilibrium conditions and constitutive equations for elastic elements)

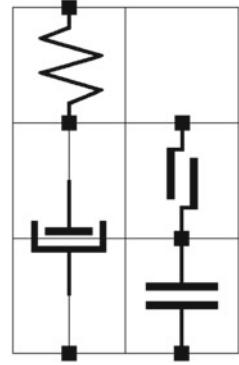
$$\sum_{j=1}^N a_{ij} U_j = f_i(t), \quad i = 1, \dots, N_1, \quad (2.12)$$

ordinary differential equations specifying viscous elements

$$\sum_{j=1}^N a_{ij} \dot{U}_j = U_i, \quad i = N_1 + 1, \dots, N_2, \quad (2.13)$$

and variational inequalities for rigid contacts

Fig. 2.15 Graphical representation of a scheme



$$(\tilde{V}_i - V_i) U_i \leq 0, \quad V_i \equiv \sum_{j=1}^N a_{ij} U_j \geq 0, \quad \tilde{V}_i \geq 0, \quad i = N_2 + 1, \dots, N_3, \quad (2.14)$$

and for plastic hinges

$$(\tilde{U}_i - U_i) \sum_{j=1}^N a_{ij} \dot{U}_j \leq 0, \quad |U_i| \leq U_i^*, \quad |\tilde{U}_i| \leq U_i^*, \quad i = N_3 + 1, \dots, N, \quad (2.15)$$

with the initial conditions

$$\sum_{j=1}^N a_{ij} U_j(0) = 0, \quad i = N_1 + 1, \dots, N_2, N_3 + 1, \dots, N.$$

The coefficients a_{ij} involved in the system (2.12)–(2.15) may be equal to 0, ± 1 or take the values of the moduli of elasticity and the viscosity coefficients.

For example, the rheological scheme shown in Fig. 2.14 has three levels ($m = 3$, $n = 4$) whose boundaries pass through the nodes of connections (see Fig. 2.15). Strains of elements are determined by the formulae $\varepsilon^e = \varepsilon_1$, $\varepsilon^v = \varepsilon_2 + \varepsilon_3$, $\varepsilon^p = \varepsilon_2$, $\varepsilon^c = \varepsilon_3$. The elements are numbered in the following order: elastic (e), viscous (v), a rigid contact (c), and a plastic hinge (p). For this scheme $N_1 = 5$, $N_2 = 6$, $N_3 = 7$, $N = 8$. In the first problem a vector of unknown functions is represented in the form $\mathbf{U} = (\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$. In the second problem, in place of total strain $\varepsilon = \varepsilon^e + \varepsilon^v$, stress $\sigma = \sigma^e = \sigma_1$ is repeated in this vector. Rectangular matrix $\mathbf{A} \sim a_{ij}$ and vector $\mathbf{F} \sim f_i$ can be composed of the coefficients of the equations and inequalities. For the first problem

$$A = \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & \eta & \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ \sigma(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For the second problem

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & \eta & \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} \varepsilon(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is evident that a rheological scheme of any level of complexity can be described with the help of the system (2.12)–(2.15). Approximation of the derivatives involved in this system leads to the equations and inequalities

$$\sum_{j=1}^N a_{ij} U_j^{k+1} = f_i^{k+1}, \quad \sum_{j=1}^N a_{ij} (U_j^{k+1} - U_j^k) = \Delta t U_i^{k+1}, \quad (2.16)$$

$$(\tilde{V}_i - V_i^{k+1}) U_i^{k+1} \leq 0, \quad V_i^{k+1} = \sum_{j=1}^N a_{ij} U_j^{k+1} \geq 0, \quad \tilde{V}_i \geq 0, \quad (2.17)$$

$$(\tilde{U}_i - U_i^{k+1}) \sum_{j=1}^N a_{ij} (U_j^{k+1} - U_j^k) \leq 0, \quad |U_i^{k+1}| \leq U_i^*, \quad |\tilde{U}_i| \leq U_i^*. \quad (2.18)$$

The limits of variation of subscript i which are given in the corresponding formulae (2.12)–(2.15) are omitted here for brevity.

Repeating the reasoning, given in Sect. 2.1 when justifying the formulation of constitutive relationships for a rigid contact in the form of the variational inequalities (2.2), it is easy to show that the inequality (2.17) is equivalent to the alternative: either $V_i^{k+1} = 0$ and at the same time $U_i^{k+1} \leq 0$ or $V_i^{k+1} > 0$ and $U_i^{k+1} = 0$. In a similar way, the inequality (2.18) is reduced to the choice of one of three alternatives:

$$(i) \quad |U_i^{k+1}| < U_i^*, \quad \sum_{j=1}^N a_{ij}(U_j^{k+1} - U_j^k) = 0,$$

$$(ii) \quad U_i^{k+1} = U_i^*, \quad \sum_{j=1}^N a_{ij}(U_j^{k+1} - U_j^k) \geq 0,$$

$$(iii) \quad U_i^{k+1} = -U_i^*, \quad \sum_{j=1}^N a_{ij}(U_j^{k+1} - U_j^k) \leq 0.$$

For the first variant, stress of a plastic hinge is lower than the limit level, hence, the plastic strain rate equals zero. For the second variant, on tension the stress coincides with the yield point, hence, the strain rate is non-negative. For the third variant, on compression the stress achieves the yield point, therefore the strain rate is less than or equal to zero. Thus, the system (2.16)–(2.18) can be solved numerically with the help of an search algorithm among a finite number of admissible variants. At each step of this algorithm, a linear system involving Eqs. (2.16) and equations corresponding to the variational inequalities (2.17), (2.18) is solved. Going to the next step is performed only if an obtained solution does not satisfy some restriction (inequality). In this case the corresponding equation is replaced with the alternative one. If all restrictions are satisfied, then the process of search is finished with going to the next time level.

To accelerate calculation, the multiple solution of the system (2.16) may be eliminated. To this end, all components of the vector $\mathbf{U}$ except stresses of plastic hinges are determined from Eqs. (2.16) and the equations for V_i^{k+1} involved in (2.17). Stresses of plastic hinges are assumed to be arbitrary. Strains of rigid contacts remain undetermined as well. More exactly, a basis of the space of solutions of the system of linear algebraic Eqs. (2.16), (2.17) is constructed. The dimension of this space must be equal to the number of rigid contacts and plastic hinges. This requirement is among the conditions of correctness of a rheological scheme. In practice this condition is easily verified. If the rank of a matrix consisting of the coefficients of the system is less than N_3 then the scheme is inappropriate.

Then the problem is reduced to the solution of the variational inequalities (2.17), (2.18) for stresses in plastic hinges and strains of rigid contacts with the help of the search algorithm described above. At each step of the algorithm the equations sets of dimension $N - N_2$ are solved. The requirement of existence and uniqueness of a solution of the variational inequalities as well as the condition of convergence of the algorithm impose additional restrictions on correctness of a scheme.

In the general case the system of linear algebraic equations, which follows from (2.16), (2.17), at the $(k + 1)$ st time step has the form

$$\begin{aligned} \sum_{j=1}^{N_3} a_{ij} U_j^{k+1} &= f_i^{k+1} - \sum_{j=N_3+1}^N a_{ij} U_j^{k+1}, \quad i = 1, \dots, N_1, \\ \sum_{j=1}^{N_3} \bar{a}_{ij} U_j^{k+1} &= \sum_{j=1}^N a_{ij} U_j^k - \sum_{j=N_3+1}^N a_{ij} U_j^{k+1}, \quad i = N_1 + 1, \dots, N_2, \\ \sum_{j=1}^{N_3} a_{ij} U_j^{k+1} &= V_i^{k+1} - \sum_{j=N_3+1}^N a_{ij} U_j^{k+1}, \quad i = N_2 + 1, \dots, N_3, \end{aligned}$$

where $\bar{a}_{ij} = a_{ij} - \Delta t \delta_{ij}$ (δ_{ij} is the Kronecker symbol). We assume that the determinant of the square matrix, composed of the coefficients in the left-hand side of this system, differs from zero. Thus, from the system we can express

$$U_i^{k+1} = \sum_{j=N_2+1}^N b_{ij} V_j^{k+1} + g_i^{k+1}, \quad i = 1, \dots, N_3. \quad (2.19)$$

Here b_{ij} are the coefficients calculated from the coefficients a_{ij} and the inverse matrix of the system, g_j^{k+1} are the quantities depending on f_j^{k+1} and U_j^k , $V_j^{k+1} = U_j^{k+1}$ for $j = N_3 + 1, \dots, N$.

Substituting Eqs. (2.19) into the inequalities (2.17), (2.18), we obtain a variational inequality in the matrix form with simple constraints (individual for each component of the unknown vector)

$$(\tilde{\mathbf{V}} - \mathbf{V})(\mathbf{C} \mathbf{V} - \mathbf{Y}) \geq 0, \quad \mathbf{V}^- \leq \mathbf{V} \leq \mathbf{V}^+, \quad \mathbf{V}^- \leq \tilde{\mathbf{V}} \leq \mathbf{V}^+, \quad (2.20)$$

where $\mathbf{C}$ and $\mathbf{Y}$ are a square matrix and a vector, respectively, (rows and columns are numbered beginning with $N_2 + 1$ rather than with 1) composed of the coefficients $c_{ij} = -b_{ij}$, $y_i^{k+1} = g_i^{k+1}$ for $i = N_2 + 1, \dots, N_3$ and

$$c_{ij} = \begin{cases} -\sum_{l=1}^{N_3} a_{il} b_{lj}, & \text{if } j = N_2 + 1, \dots, N_3, \\ -a_{ij} - \sum_{l=1}^{N_3} a_{il} b_{lj}, & \text{if } j = N_3 + 1, \dots, N, \end{cases}$$

$$y_i^{k+1} = \sum_{j=1}^{N_3} a_{ij} g_j^{k+1} - \sum_{j=1}^N a_{ij} U_j^k$$

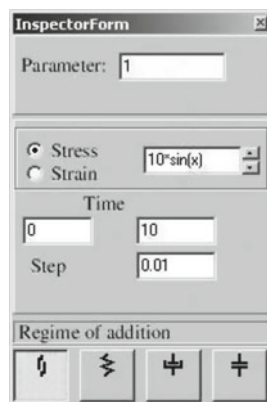
for $i = N_3 + 1, \dots, N$. The components of the vector $\mathbf{V}^+$ corresponding to rigid contacts are equal to $+\infty$, for plastic elements $V_i^+ = U_i^*$. The vector $\mathbf{V}^-$ is composed of zeroes and negative quantities $-U_i^*$, respectively.

If the matrix $\mathbf{C}$ is positive definite then by the existence and uniqueness theorem (its proof is given in Sect. 3.2 of the next chapter) the variational inequality (2.20) has a unique solution. Numerical experiments with different rheological schemes show that the search process also converges. At the same time, for a wide class of schemes, in particular, for the scheme involving four rheological elements of different types considered above, the matrix $\mathbf{C}$ is nonnegative definite rather than positive definite. In this case the algorithm sometimes leads to infinitely repeating cycles. This situation can be improved by adding a small regularizing parameter to the diagonal elements which corresponds to involving a system of elastic elements of small rigidity in a rheological scheme in parallel with plastic hinges and rigid contacts. There is no assurance that a sequence of successive solutions converges as a regularization parameter tends to zero, however, it is sufficient to make an a posteriori convergence test by monotone decreasing the value of this parameter in computations.

This algorithm is implemented in the general form in the Delphy 5 object programming environment. The values of phenomenological parameters for elastic, viscous, and plastic elements are input variables for the computer system worked out. A scheme to be studied is constructed by tools of visual design with the use of graphic primitives. An example of the assignment of a concrete scheme involving four elements of different types is shown in Fig. 2.15. At the output the system enables one to obtain graphs of the strains and stresses variation in elements of a scheme depending on time as well as graphs of total strain $\varepsilon(t)$ and resulting stress $\sigma(t)$. Testing the algorithm was performed on the solutions obtained by the formulae (2.10), (2.11) and showed that the computational error of the algorithm corresponds to the first order of approximation of the implicit scheme.

The work technique with the system is as follows. To develop a new rheological scheme, one should choose an option or to press a key on the toolbar. In so doing, the scheme editor, i.e. a program dealing with a set of tools and objects with the help of which an arbitrary scheme is created, is started. Rheological elements are successively marked on a workspace with simultaneously specifying the parameters. A workspace is a space with a grid marked on it intended for the exact positioning of elements. For convenience, the so-called object inspector being a set of elements which can be placed in the workspace is located at the top right (Fig. 2.16). The object inspector has several editing fields which serve for the change of parameters and elements of a scheme. Thus, parameters of elements remain available for editing after they are introduced into a rheological scheme. It is sufficient to click the required elements and to change the corresponding values in the object inspector.

When placed in the workspace, an element can be stretched or compressed according to topology of a scheme. An element introduced mistakenly can be deleted or moved to other part of the workspace by the choice of corresponding option of the contextual menu which is defined for each element. The operation with elements can be also performed by “hot keys”.

Fig. 2.16 Object inspector

The results of the assignment of a rheological scheme are saved in a file of special format formed by the system. Moreover, the system by itself keeps track of changes in a scheme and, if any, on exit a dialog window on its save is displayed. An alternative way of the assignment of a scheme, namely, loading from an existing file, is provided. When loading, a file name is displayed and file format is tested for compliance with the system.

Once the system has been defined, it should be pointed out which of two problems is to be solved (the problem on determining strain from given stress or, conversely, stress from given strain). The possibility to use time-dependent functions is realized with the help of a syntax analyzer of formulae. A syntax analyzer is a special function subprogram exported from a dynamic library involved in the project. A line with a formula and a list of values of the variables involved in the formula are transferred to the subprogram as parameters and at the output the calculated value of the function is obtained. The formula may involve the signs of mathematical operations (addition, subtraction, multiplication, division, raising to a power, extraction of a root) as well as all elementary functions.

Further computational procedures implementing this algorithm are started. In these procedures, a basis of the space of the solutions of the system (2.16) is constructed with the Gauss method with the choice of principal element. This method is also applied in the solution of the systems of equations which arise when implementing the variational inequality (2.20). Once computations have been performed, the system provides a possibility to output data in the form of graphs on a display or to an output file (a text file with separators) which can be used for analysis in other graphic editors.

The results of calculations obtained with the help of the computer system for the scheme involving four rheological elements (Fig. 2.14) are shown in Figs. 2.17–2.23. They represented in the form of diagrams of variation of strain with time for the cyclic loading with a constant and linearly increasing stress amplitude. On all graphs the curves 1 correspond to the time-dependence of stress $\sigma(t)$, the curves 2 to total strain $\varepsilon(t)$, and the curves 3 to strain $\varepsilon^c(t)$ of a rigid contact. The solution is given in the

Fig. 2.17 Loading for $\sigma(t) = -0.01 \cos t$

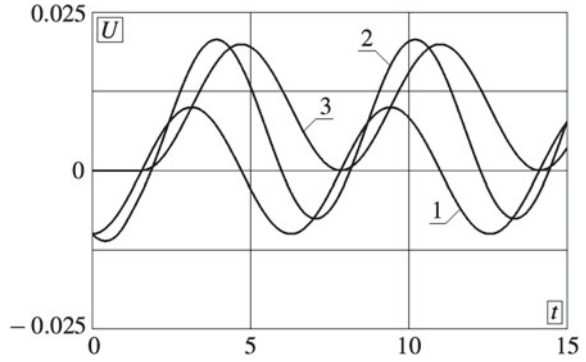


Fig. 2.18 Loading for $\sigma(t) = -0.001 t \cos t$

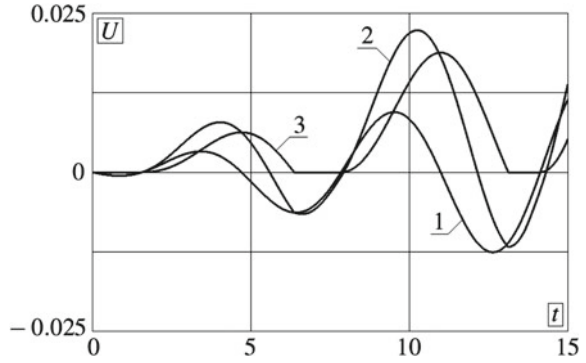
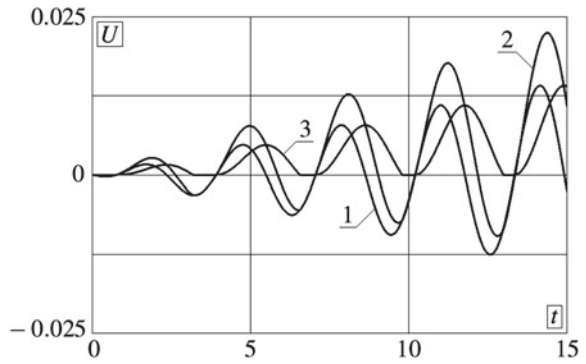


Fig. 2.19 Loading for $\sigma(t) = -0.001 t \cos 2t$



dimensionless variables ($\tau = a \eta$ is the characteristic time of relaxation):

$$\bar{t} = \frac{t}{\tau}, \quad \bar{\sigma} = a \sigma, \quad \bar{\varepsilon} = \varepsilon, \quad \bar{a} = 1, \quad \bar{\sigma}_s = a \sigma_s, \quad \bar{\eta} = \frac{a \eta}{\tau} = 1.$$

Fig. 2.20 Loading for
 $\sigma(t) = -0.001 t \cos 3 t$

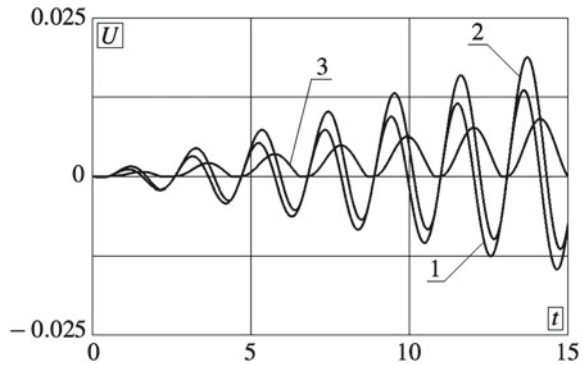
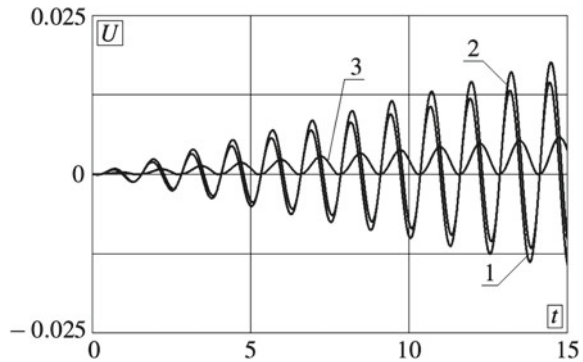


Fig. 2.21 Loading for
 $\sigma(t) = -0.001 t \cos 5 t$



The dimensionless yield point $\bar{\sigma}_s$ (further a bar over dimensionless quantities is omitted) is equal to 0.005.

Analysis shows that for a constant amplitude a rigid contact is in a closed state only during the initial time interval within the first period of loading. Later on a medium adapts itself to the periodic load and never achieves a compaction mode (Fig. 2.17). In the case of an increasing amplitude, the interval of a closed state of a contact is periodically repeated (Figs. 2.18 and 2.19).

With increasing frequency (see Figs. 2.20 and 2.21), the curve 2 approaches to the curve 1. These curves can coincide exactly only in the case of a rheological scheme involving a single elastic element, hence, the influence of viscosity, plasticity, and heterostrength in comparison with elastic properties of a material becomes insignificant with increasing a loading frequency.

The graphs which describe variations of stresses and strains for the same rheological scheme with the same parameters for given total strain $\varepsilon(t)$ varying by a periodic law are given in Figs. 2.22 and 2.23. As before, the curves 1 correspond to time-dependence of stress and the curves 2—of strain. From the graphs of strain of a rigid contact (curves 3) it follows that a medium does not adapt itself to periodic de-

Fig. 2.22 Deformation for $\varepsilon(t) = -0.015 \cos t$

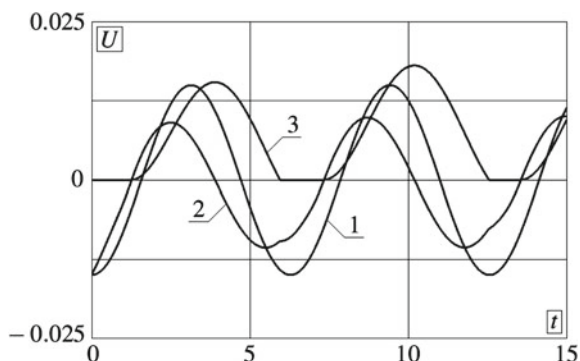
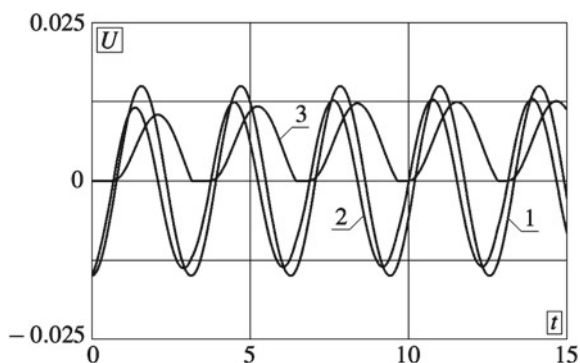


Fig. 2.23 Deformation for $\varepsilon(t) = -0.015 \cos 2t$



formation with a constant amplitude. At each cycle a compaction state of a medium where a contact is closed up is changed by a loosening state.

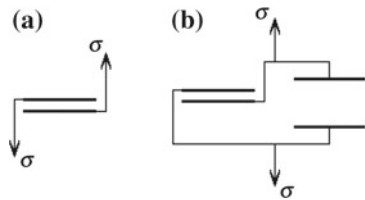
A comparison of Figs. 2.22 and 2.23 shows that with increasing a frequency the stress and strain curves approach to each other. Thus, in the case of a high-frequency deformation viscoplastic properties have a weak effect on a stress state of a material. An elastic element of a rheological scheme for which dimensionless dependencies $\sigma(t)$ and $\varepsilon(t)$ coincide is of first importance.

Other examples of the studies of rheological schemes for materials with different tensile and compressive strengths with the help of the computer system presented above are given in the following sections.

2.5 Fiber Composite Model

At the present level of development of a production technology of artificial materials, including those of engineering plastics, high-polymeric materials, and composites of various structures, rheology being a classical field of mechanics goes from the

Fig. 2.24 Rheological scheme of a thread: **a** rigid contact of opposite polarity, **b** rigid contact with initial strain



solution of the direct problem of description of mechanical properties of existing materials to the study of the inverse problem of production of materials with preassigned properties. This approach requires development of new theoretical methods and improvement of known ones as well as software for mathematical modeling of the behaviour of continuous media which, at first glance, possess exotic properties. It seems likely that materials which are compression compliant more than tension compliant belong to this class.

A flexible non-stretchable thread (membrane) is the simplest example of a heteroresistant mechanical system without the compression strength and the tensile stress deformation. The stretched state is a natural state of a thread for zero strain and stress. Positive strain is impermissible, i.e. $\varepsilon \leq 0$, and negative stress is also impermissible, so $\sigma \geq 0$. If $\varepsilon < 0$ then $\sigma = 0$, and if $\sigma > 0$ then $\varepsilon = 0$. Thus, the constitutive relationships for a thread for uniaxial tension–compression coincide with the relationships (2.1) for an ideal granular medium accurate within the change of signs of the inequalities. The corresponding rheological scheme (Fig. 2.24a) is a rigid contact of opposite polarity.

When modeling finite strains of a thread, it is necessary to take into account restrictions from below related to the fact that, when changing positions of the ends in the compression process, a thread is stretched again. So, a more detailed rheological scheme must involve a rigid contact with a given value of initial strain (Fig. 2.24b). But this purpose is not pursued here.

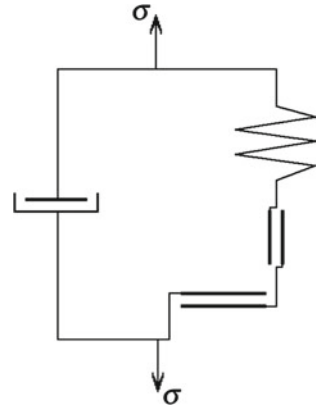
A rheological scheme which describes the compression–tension process for a unidirectional fiber composite consisting of elastic-plastic fibres in a viscous binder is shown in Fig. 2.25. According to this scheme, with compression a sequential chain consisting of elastic and plastic elements is broken and only a viscous damper is deformed. By the Newton law in this case $\sigma = \eta \dot{\varepsilon}$. With tension, matched deformation of all elements takes place except for a rigid contact being in the closed state. For $\sigma \leq \sigma_s$ a plastic hinge also is not deformed, hence,

$$\sigma = \sigma^e + \sigma^v = \frac{\varepsilon}{a} + \eta \dot{\varepsilon}.$$

For $\sigma > \sigma_s$ stress in a chain equals σ_s , therefore

$$\sigma = \sigma_s + \sigma^v = \sigma_s + \eta \dot{\varepsilon}, \quad \varepsilon^e = a \sigma_s, \quad \varepsilon^p = \varepsilon - \varepsilon^e.$$

Fig. 2.25 Rheological scheme of a composite

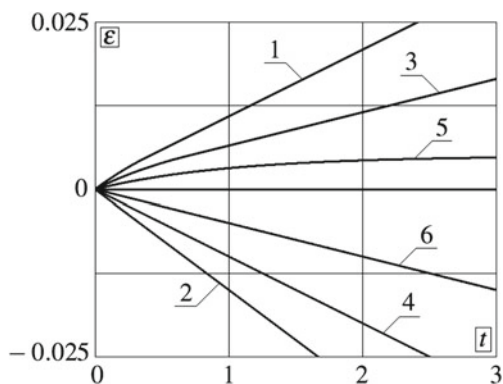
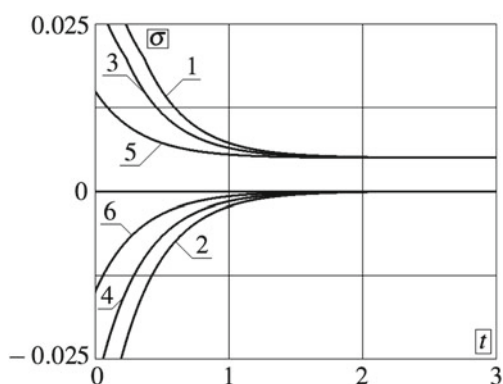
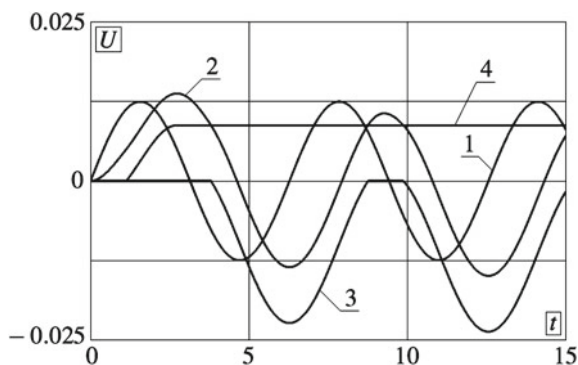


Stresses and strains of all elements are uniquely determined in any version of loading considered here. Thus, this system is correct.

Results of computations for this scheme are given in Figs. 2.26–2.29, [25]. A series of creep diagrams for a constant stress level $\sigma = \pm 0.015, \pm 0.01$, and ± 0.005 (the curves 1, ..., 6, respectively) is presented in Fig. 2.26. For compressive stresses the diagrams are linear, they describe the strain of a viscous element. For tensile stresses the curves are of nonlinear nature, besides, on the initial segment where their convexity is more pronounced elastic deformation takes place and with arising plastic deformation the curves become more flat. The stress relaxation diagrams for strain $\varepsilon(t) = \varepsilon_0(1 - e^{-3t})$ which exponentially tends to a constant value ε_0 are shown in Fig. 2.27. Here the curves 1, ..., 6 correspond to $\varepsilon_0 = \pm 0.015, \pm 0.01$, and ± 0.005 , respectively. The salient points on the upper graphs correspond to going from an elastic stage of the fibres tension to a plastic one. Observe that with compression stresses are rapidly reduced to zero whereas with tension they relax to a constant value equal to the yield point of a plastic hinge. It turns out that for a moderate level of tensile strain ε_0 , when a hinge remains in the rigid state, dimensionless stress $\sigma_0 = \varepsilon_0$ of an elastic spring is the limit value in the relaxation process.

Results of calculation for the uniaxial cyclic loading and the cyclic deformation of a composite are shown in Figs. 2.28 and 2.29. The curves 1 correspond to the dependence $\sigma(t)$, the curves 2 characterize the dependence $\varepsilon(t)$, the curves 3 describe the variation of strain of a rigid contact with time, and the curves 4 describe the variation of strain of a plastic element. Analysis shows that plastic strain takes place only in the first cycle of loading or deformation of a material. After irreversible elongation the fibres remain elastic in all subsequent cycles.

Consider a more complicated rheological scheme of a fiber composite which is heteroresistant with respect to tension and compression (Fig. 2.30). A coupled chain of elastic and plastic elements in this scheme serves to model a double system of reinforcing fibres differing in elastic and plastic properties which show themselves only with tension. As in the previous scheme, compression of a composite is de-

Fig. 2.26 Creep diagrams**Fig. 2.27** Stress relaxation curves**Fig. 2.28** Loading for $\sigma(t) = 0.0125 \sin t$ 

scribed by a model of a viscous medium. With tension, one of three alternatives is implemented. If strain does not exceed the critical value equal to $\min \{a \sigma_s, a' \sigma'_s\}$ where a , a' and σ_s , σ'_s are the moduli of elastic compliance and the yield points of elements, respectively, then visco-elastic deformation of a material takes place according to the Kelvin–Voigt theory:

Fig. 2.29 Deformation for $\varepsilon(t) = 0.0125 \sin t$

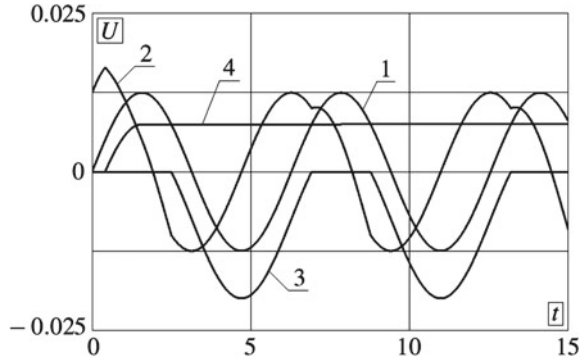
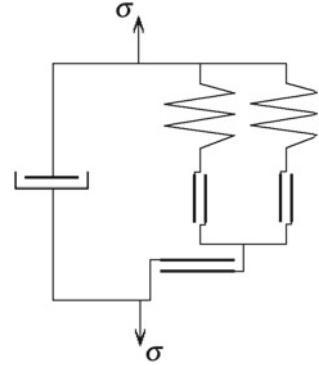


Fig. 2.30 A double system of fibres



$$\sigma = \left(\frac{1}{a} + \frac{1}{a'} \right) \varepsilon + \eta \dot{\varepsilon}.$$

Exceeding a critical level results in the fact that in the chain with a lesser value of the product $a \sigma_s$ a plastic hinge is broken. In this case

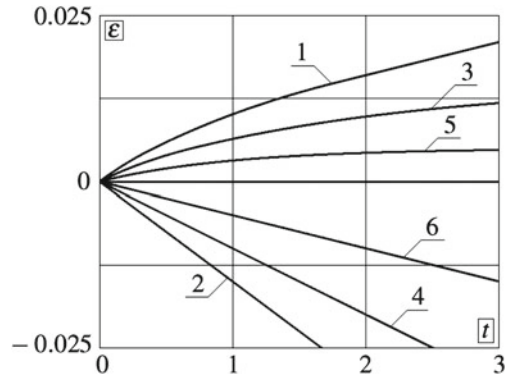
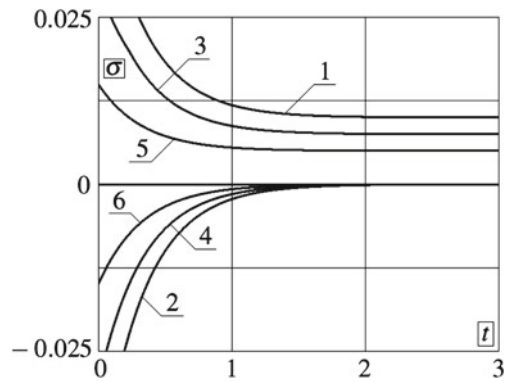
$$\sigma = \sigma_s + \frac{\varepsilon}{a'} + \eta \dot{\varepsilon}, \quad a \sigma_s < \varepsilon \leq a' \sigma'_s.$$

Finally, if strain is higher than $a' \sigma'_s$ then both hinges are broken. In this case the constitutive equation has the form:

$$\sigma = \sigma_s + \sigma'_s + \eta \dot{\varepsilon}.$$

Thus, the active loading takes place for $\dot{\varepsilon} \geq 0$. When unloading, for $\dot{\varepsilon} < 0$, both hinges are simultaneously blocked. The increments of total stress and strain satisfy the equation

$$d\sigma = \left(\frac{1}{a} + \frac{1}{a'} \right) d\varepsilon + \eta d\dot{\varepsilon},$$

Fig. 2.31 Creep diagrams**Fig. 2.32** Stress relaxation curves

from which it follows that

$$\sigma = \sigma_s + \sigma'_s + \left(\frac{1}{a} + \frac{1}{a'} \right) (\varepsilon - \varepsilon_{\max}) + \eta \dot{\varepsilon},$$

where $\varepsilon_{\max}$ is the maximal tensile strain achieved at the loading stage. A rigid contact is broken if stress $\sigma^c = \sigma - \eta \dot{\varepsilon}$ calculated by this formula becomes zero in the unloading process. Further deformation for $\dot{\varepsilon} < 0$ is described by the Newton equation.

One can make sure that in each of these variants stresses and strains of all elements of the rheological scheme are uniquely determined, so this scheme is correct. In Figs. 2.31 and 2.32, the diagrams of creep and stress relaxation similar to those in Figs. 2.26 and 2.27 are presented. The results are obtained with the help of the computer system for the following values of the dimensionless parameters of the model: $a = a' = 2$, $\sigma_s = 0.0025$, $\sigma'_s = 0.0075$. Comparison of the diagrams shows that under tension of a material with a double reinforcing system the rigidity essentially increases. Besides, the level of residual stresses after relaxation increases.

Including additional elements in the rheological schemes considered above enables one to take into account elastic and plastic properties of a composite binder as well as viscous properties of reinforcing fibers.

2.6 Porous Materials

A number of books (see bibliography of [3–5, 10, 15, 20]) is devoted to the mathematical modeling of behavior of porous materials under the action of static and dynamic loads. However, up to now there has been no unique theory. Main difficulties here are related to the fact that porous materials also have the heteroresistance property. Under the action of compressive stresses up to the instant of collapse of pores these materials turn out to be more compliant than with subsequent compression. The unloading process for a compressed porous medium may be reversible or irreversible. In the first case a pore space restores completely for zero stress and in the second case in the limits of the “loading–unloading” cycle pores vary in size, [19, 29]. Collapse of pores can be modeled as a result of loss of stability of a porous skeleton, [6, 11].

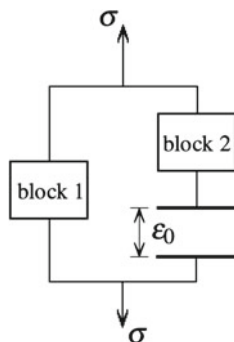
New application of mechanics of porous materials is porous metals. These artificial materials can be widely used in engineering because of their low density and good damping properties, [16]. The ability of porous metals effectively absorb mechanical energy on the stage of plastic deformation opens up prospects for their use in the manufacture of bumper cars and elements of the car body, the so-called crushed zones. They also can be used in gearboxes and actuators as destructible fuses which dissipate the energy of dynamic impact preventing the destruction of all mechanical system.

Deformation properties of porous metals are significantly different in tension and in compression. In tension there are the stage of elastic deformation of skeleton and the stage of plastic flow up to the destruction. In compression there are the stages of elastic and plastic deformation of skeleton until the collapse of pores and the subsequent stage of elastic or elastic-plastic deformation of a solid material without pores. At small sizes of pores the collapse can occur on the elastic stage with the appearance of plasticity only under a sufficiently high level of load.

At present the technologies of production of metal foams based on aluminum, copper, nickel, tin, zinc and other metals are developed. According to the information published in Internet, A. Rabiney from the University of North Carolina (USA) in 2010 created a technology for production of the most durable foam in the world. High strength of this material is achieved by ensuring that the surface of a thin-wall skeleton in the foam practically has no dislocations, i.e. defects that are initiators of the destruction.

Extensive experimental researches of the mechanical properties of such materials were carried out. The diagrams of uniaxial tension and uniaxial compression on the example of aluminum foam and porous copper were obtained, [1, 2]. The problems of durability and cyclic fatigue of porous metals are considered in [17] etc.

Fig. 2.33 General scheme of porous material



Theoretical questions of the constructing constitutive equations and of the analysis on this basis the spatial stress-strain state of structural elements of metal foams, according to the available publications, have not been studied almost. At the level of physical and mechanical representations, the deformation of metal foam is rather complex process. Under compression it leads to the elastic-plastic loss of stability of metal skeleton at high porosity and to the stable mechanism of collapse of pores at low porosity. The collapse is accompanied by a contact interaction of skeleton walls which is difficult for modeling at the discrete level. Besides, it is necessary to consider the presence of compressed gas in closed pores. It is rather difficult to describe the process of shear when, according to the experiments, the volume of a material changes. Even more difficult to construct a universal model of the spatial stress-strain state of a material under complex loading. The performance of adequate computations based on discrete models of the metal foam as a structurally inhomogeneous material is only possible with the use of multiprocessor systems with high performance and large amount of random-access memory.

A simple and effective solution to these problems gives the rheological approach, in the framework of which one can describe the main qualitative and quantitative effects such as a significant difference of diagrams of uniaxial deformation before and after the collapse of pores and a significant dissipation of energy at the stage of plastic flow of a material.

A rheological scheme of the general form for a porous material is shown in Fig. 2.33. Here block 1 consisting of elements placed parallel to a prestretched rigid contact describes mechanical properties of a skeleton with open pores. Initial strain ε_0 of a rigid contact is defined by a specific volume of pores. Block 2 which consists of elements placed in series with a rigid contact describes the hardening of a compressed skeleton. In a more general case, one more block of rheological elements modeling strain of a medium which does not depend on a pores state can be added to this scheme in series, [24].

Replacing blocks 1 and 2 with elastic springs, we obtain the simplest model of the ideal elastic porous medium whose rheological scheme is given in Fig. 2.34a. According to this model, with tension of a material and with compression to the critical value $\varepsilon = -\varepsilon_0$ of strain the equation $\sigma = \varepsilon/a$ holds, with compression

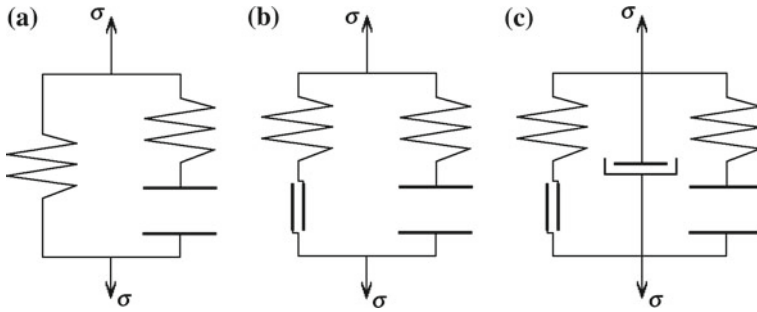
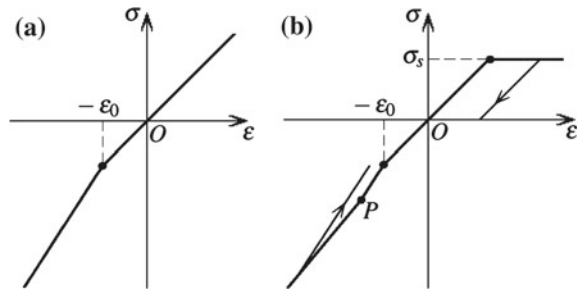


Fig. 2.34 Rheological schemes of porous materials: **a** elastic material, **b** elastic-plastic material, **c** elastic-visco-plastic material

Fig. 2.35 Diagrams of uniaxial tension–compression: **a** elastic porous material, **b** elastic-plastic porous material in the case of elastic collapse of pores



over the critical value ($\varepsilon < -\varepsilon_0$) the equation $d\sigma = (1/a + 1/b)d\varepsilon$ holds (a and b are moduli of elastic compliance of springs). The deformation process for such a medium is thermodynamically reversible. A diagram of uniaxial tension–compression is shown in Fig. 2.35a as a two-segment broken line.

A rheological scheme of an elastic-plastic porous material, where the compression process after collapse of pores is described with the model of linear hardening, is presented in Fig. 2.34b. If stress in modulus does not exceed the yield point σ_s of a plastic hinge, then deformation of a material exactly corresponds to the elastic model. It is impossible to apply tensile stress higher than σ_s since for $\sigma = \sigma_s$ flow of a material due to increase of size of pores is observed. With compression the effect of plastic hardening takes place.

Typical diagrams of the active loading and unloading are shown in Figs. 2.35b and 2.36a. Figure 2.35b corresponds to the case where the phenomenological parameters satisfy the condition $\varepsilon_0 < a\sigma_s$. This condition holds for weakly porous materials with small value of ε_0 and means that the process of collapse of pores takes place in the range of elastic strain. According to this scheme, at the point P of the transition of a medium to the plastic state

$$\sigma = \frac{\varepsilon}{a} + \frac{\varepsilon + \varepsilon_0}{b}, \quad \frac{\varepsilon}{a} = -\sigma_s.$$

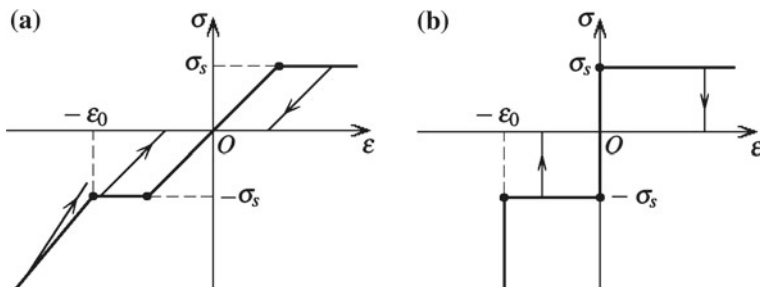


Fig. 2.36 Diagrams of uniaxial tension-compression in the case of plastic collapse of pores: **a** elastic-plastic material, **b** rigid-plastic material

Hence, at this point $\sigma = -\sigma_s(1 + a/b) + \varepsilon_0/b$. The effect of the plastic hardening of a compressed skeleton is described by the linear equation

$$\sigma = -\sigma_s + \frac{\varepsilon + \varepsilon_0}{b}.$$

The case of $\varepsilon_0 > a\sigma_s$, where compression of pores is accompanied by plastic dissipation of energy, is considered in Fig. 2.36a. It should be noted that the tangent of the angle of slope of the segment, which corresponds to the unloading of a plastically compressed material, equals $1/a$ until the instant of collapse of pores and after this instant it increases to $1/a + 1/b$.

Using the rheological scheme given in Fig. 2.34b, with a tending a to zero, we can obtain a model in which strain of pores is completely irreversible and strain of a compressed skeleton follows the law of linear hardening. With b tending to zero, a model of an elastic-plastic porous medium with an absolutely rigid skeleton is obtained. If $a \rightarrow 0$ and $b \rightarrow 0$ simultaneously, then a model of a rigid-plastic porous medium with a rigid skeleton, whose diagram of uniaxial deformation is given in Fig. 2.36b, can be obtained.

The diagrams show that the constitutive relationships of an elastic-plastic porous medium are as much incorrect in the mechanical sense as ones of the classical theory of ideal plasticity. To obtain an unambiguous description of deformation of a material for a given loading program, we add a regularizing viscous element to the scheme (see Fig. 2.34c). The model corresponding to this scheme takes into account viscous properties of the skeleton. With η tending to zero, it is transformed into a model of an elastic-plastic porous medium.

Graphs of the changing of resulting stress (the curve 1) and stress in a rigid contact (the curve 2) for cyclic deformation of a medium with zero viscosity are presented in Fig. 2.37. The results are obtained for $a = b = 1$, $\sigma_s = 0.005$, and $\varepsilon_0 = 0.0025$. This choice of the parameters corresponds to the case of elastic collapse of pores. However, according to computations elastic collapse takes place only at the first cycle. At the second and subsequent cycles, specific horizontal portions of compressive stresses corresponding to the value of the yield point of a plastic hinge

Fig. 2.37 Deformation for $\varepsilon(t) = -0.015 \sin t$ ($\varepsilon_0 = 0.0025$)

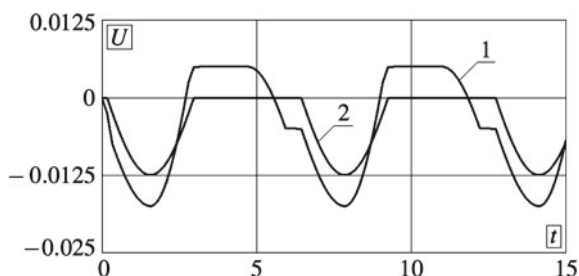


Fig. 2.38 Deformation for $\varepsilon(t) = -0.015 \sin t$ ($\varepsilon_0 = 0.0075$)

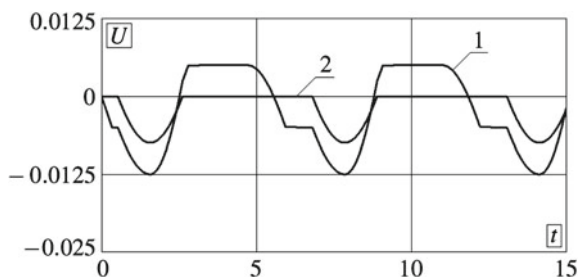
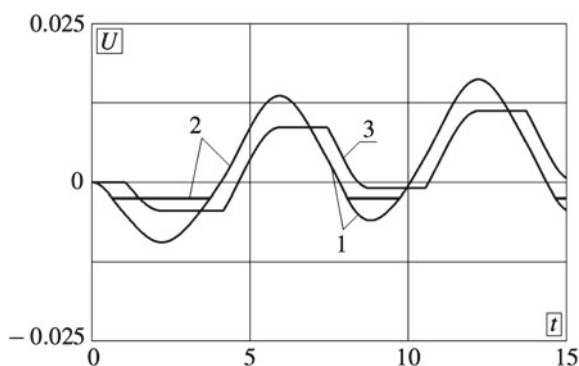


Fig. 2.39 Loading for $\sigma(t) = -0.015 \sin t$ ($\varepsilon_0 = 0.0025$)



appear on the curve 1 as a result of preliminary irreversible strain of a material. According to the curve 2, in this case a rigid contact is in the broken state since its stress turns out to be zero. Thus, even at the second cycle compression of pores is accompanied by plastic strain of a material.

Similar graphs for $\varepsilon_0 = 0.0075$ in the case of plastic collapse at the first cycle of the deformation program are shown in Fig. 2.38. Comparison shows that as the porosity increases, the level of compressive stresses in a medium considerably decreases.

Graphs of the changing of characteristics of the strained state for a viscoplastic porous medium with the dimensionless viscosity coefficient $\eta = 1$ for the cyclic loading are shown in Figs. 2.39 and 2.40. The curves 1 describe total strain $\varepsilon(t)$, the curves 2 describe strain of a rigid contact, and the curves 3 describe strain of a plastic

Fig. 2.40 Loading for $\sigma(t) = -0.015 \sin t$ ($\varepsilon_0 = 0.0075$)

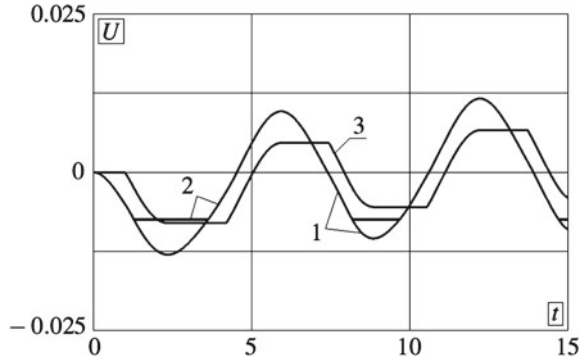


Fig. 2.41 Loading for $\sigma(t) = -0.015 \sin 2t$ ($\varepsilon_0 = 0.0025$)

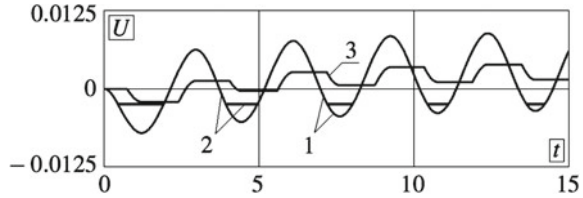
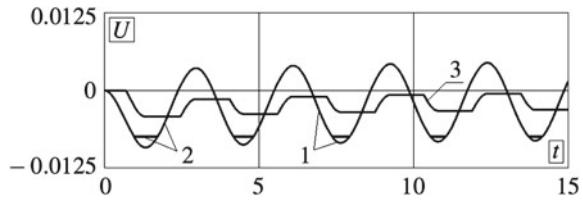


Fig. 2.42 Loading for $\sigma(t) = -0.015 \sin 2t$ ($\varepsilon_0 = 0.0075$)



element. Fig. 2.39 corresponds to the case of $\varepsilon_0 < a \sigma_s$ and Fig. 2.40 corresponds to the case of $\varepsilon_0 > a \sigma_s$. In the first case plastic strain of a medium is observed after the instant of collapse of pores and in the second case until this instant. However, as before, this holds true only for the first cycle of loading, at all subsequent cycles the compression of open pores is accompanied by plastic dissipation of energy. From the presented graphs it follows that strain of a rigid contact differs from total strain only when a contact is closed. In addition, the interval of the closed state decreases depending on the number of a cycle and for a sufficiently large value of time corresponding graphs completely coincide. Hence, for the multiple cyclic loading a material gradually loses the heteroresistance property and collapse of pores comes to a stop. Analysing the curves, we can notice that in a viscous medium the creep is developed with time, i.e. the maximal value of strain increases during one cycle.

Similar graphs for the cyclic loading with doubled frequency (in comparison with the graphs shown in Figs. 2.39 and 2.40) are represented in Figs. 2.41 and 2.42.

To conclude, we notice that a large series of methodical calculations performed with the help of the computer system described above shows that as frequency of the

cyclic loading increases, the level of strain of a medium considerably decreases, a given stress amplitude turns out to be insufficient to provide collapse of pores, plastic strain takes a negative value and remains constant during all loading process, and the creep property completely disappears. In this case the behaviour of a porous material is adequately described by the elasticity theory with initial (plastic) strain taken into account.

2.7 Rheologically Complex Materials

In the previous sections, constitutive relationships of uniaxial deformation of materials with different strengths were constructed only on the basis of the method of rheological schemes where four types of elements associated with elementary properties are used. If in a tension or compression state a material shows complicated set of properties, then one has to use a large number of rheological elements to describe it. This results in a multiparametric model which may be difficult of access for practical applications. An alternative approach which is well-developed in the application to standard viscoelastic materials with symmetric properties with respect to tension and compression, [12, 14, 21], is as follows. Constitutive equations are postulated in the special form for which, generally speaking, there exists no appropriate rheological scheme. Generalizing this approach to materials with different strengths we notice that, in combination with a system of rigid contacts, in a nonstandard rheological scheme blocks, which have no traditional rheological schemes, may be used along with blocks involving traditional elements. For example, in a general scheme of a porous material shown in Fig. 2.33, constitutive equations of the special form mentioned above may correspond to the blocks 1 and 2. Similar conditional schemes can be constructed for the description of composite and granular materials. However, this way of the construction of constitutive relationships has some restrictions. One of the most important restrictions is related to the requirement of dissipativity of the derivable model.

For a model with the standard rheological scheme involving an arbitrary number of elements of four types the dissipativity property holds automatically. In the classical viscoelasticity theory this statement is assumed to be obvious since by the physical sense dissipation of energy in a system is formed from dissipations on its elements. We prove it more rigorously basing on the formal representation of rheological schemes given in Sect. 2.4.

In the general case the sum of stress powers of elements of a rheological scheme for corresponding strains is expressed by the next formula:

$$\dot{W} = \sum_{j=1}^n \sigma_j \dot{\varepsilon}^j, \quad \varepsilon^j = \sum_{k=b_j}^{e_j} \varepsilon_k.$$

Here, when calculating strain of the j -th element, the summation is performed over the layers of a scheme numbered from b_j where the element begins to e_j where it ends. Collecting similar terms, we can rewrite the expression for $\dot{W}$ in the form of a linear combination of strain rates of layers

$$\dot{W} = \sum_{k=1}^m \bar{\sigma}_k \dot{\varepsilon}_k$$

with the coefficients $\bar{\sigma}_k$ equal to the sum of stresses of those elements whose ends pass through the given layer. In view of the equilibrium equation, at the upper boundary of the first layer $\bar{\sigma}_1 = \sigma(t)$. To show that the remaining coefficients are equal to $\sigma(t)$ as well, we break in mind the ends of elements intersecting the interfaces between layers and add paired systems of stresses σ_j and $-\sigma_j$, which are equivalent to zero, to the corresponding boundaries. From the equilibrium equations written for interfaces we successively obtain

$$\bar{\sigma}_1 = \bar{\sigma}_2, \quad \bar{\sigma}_2 = \bar{\sigma}_3, \quad \dots, \quad \bar{\sigma}_{m-1} = \bar{\sigma}_m.$$

Finally, we have

$$\dot{W} = \sigma \sum_{k=1}^m \dot{\varepsilon}_k = \sigma \dot{\varepsilon}.$$

Among rheological elements of four types, two elements (an elastic spring and a rigid contact) represent conservative systems. For them

$$\sigma_j \dot{\varepsilon}^j = \dot{\Phi}_j(\varepsilon^j),$$

where Φ_j are the corresponding stress potentials. The remaining two elements (a viscous damper and a plastic hinge) are dissipative because for them

$$\sigma_j \dot{\varepsilon}^j = D_j(\dot{\varepsilon}^j) \geq 0.$$

Thus, the dissipativity property of a model consists in the following equality:

$$\sigma \dot{\varepsilon} = \dot{\Phi} + D, \tag{2.21}$$

where the functions Φ and $D \geq 0$ are the sums of conservative and dissipative potentials of the elements of a scheme.

Now consider the case of a nonstandard rheological scheme. Repeating the above considerations, we can show that a model of uniaxial tension–compression is dissipative if and only if all blocks involved in the scheme are dissipative. Keeping in mind applications to granular materials, for description of which it makes sense to use mathematical models of the hereditary elasticity theory as individual blocks in combination with standard rheological elements, we present an independent deriva-

tion of the dissipation condition for such block in the Breuer and Onat form, [7, 21].

Assume that the constitutive equation of a model is written in the convolution form

$$\sigma(t) = (\varphi * \varepsilon)(t) \equiv \int_{-\infty}^t \varphi(t - t_1) \varepsilon(t_1) dt_1. \quad (2.22)$$

Here the relaxation kernel $\varphi(t)$ is an arbitrary function constructed from experimental data. We can extend this function putting it equal to zero for negative values of t , since (2.22) involves only the values corresponding to $t \geq 0$. Then Eq. (2.22) is reduced to the form

$$\sigma(t) = \int_{-\infty}^{\infty} \varphi(t - t_1) \varepsilon(t_1) dt_1 = \int_{-\infty}^{\infty} \varphi(t_2) \varepsilon(t - t_2) dt_2 = (\varepsilon * \varphi)(t).$$

It is known that the Fourier transform

$$\widehat{\sigma}(\omega) = \int_{-\infty}^{\infty} \sigma(t) e^{-i\omega t} dt$$

of the convolution of two functions, one of which vanishes for $t < 0$, is the product of the Fourier transform. Therefore the equality

$$\widehat{\sigma}(\omega) = \widehat{\varphi}(\omega) \widehat{\varepsilon}(\omega)$$

is valid. Hence, Eq. (2.22) can be represented in the inverted form

$$\varepsilon(t) = (\psi * \sigma)(t), \quad (2.23)$$

where $\psi(t)$ is the creep kernel being a function which is put to be equal to zero for negative t and for which $\widehat{\psi}(\omega) = 1/\widehat{\varphi}(\omega)$.

For the Kelvin–Voigt classical model $\widehat{\varphi}(\omega) = 1/a + i\omega\eta$. For the Maxwell model $\widehat{\varphi}(\omega) = 1/(a + i\omega\eta)$. For the parallel connection of two blocks of elastic and viscous elements, the Fourier transform of the relaxation kernel is equal to the sum of the Fourier transforms of the kernels of the blocks. For the series connection it is equal to the reciprocal value of the sum of the reciprocal values. Hence, in the case of a viscoelastic scheme of the general form the quantity $\widehat{\varphi}(\omega)$ is a fractionally rational complex-valued function $P(i\omega)/Q(i\omega)$ for which the difference of degrees of the polynomials $P(z)$ and $Q(z)$ does not exceed one.

If in the constitutive Eq. (2.23) $\varepsilon(t)$ is a periodic function with period T then the function $\sigma(t)$ is also periodic with the same period. According to the second principle of thermodynamics, the energy produced by stress per one deformation

period (cycle) can not be negative, i.e.

$$W = \int_0^T \sigma(t) d\varepsilon(t) \geq 0. \quad (2.24)$$

If this property is violated, then a mathematical model with the constitutive Eqs. (2.22), (2.23) is incorrect in the physical sense, since in its framework there exists “perpetuum mobile”. Expand an arbitrary periodic function into the Fourier series

$$\varepsilon(t) = \frac{c_0}{2} + \sum_{k=1}^{\infty} (c_k \cos \omega_k t + s_k \sin \omega_k t),$$

where $\omega_k = 2\pi k/T$. For an individual term of the type $\varepsilon_k(t) = \sin \omega_k t$ the inequality (2.24) takes the following form:

$$W_k = \omega_k \int_0^T \sigma_k(t) \cos \omega_k t dt = \omega_k \Re \int_0^T \sigma_k(t) e^{i\omega_k t} dt \geq 0.$$

Here the fact that the value of stress is real is taken into account. Due to Eq. (2.22)

$$\begin{aligned} \sigma_k(t) &= \int_{-\infty}^{\infty} \varphi(t-t_1) \sin \omega_k t_1 dt_1 = \int_{-\infty}^{\infty} \varphi(t-t_1) \frac{e^{i\omega_k t_1} - e^{-i\omega_k t_1}}{2i} dt_1 \\ &= \frac{e^{i\omega_k t}}{2i} \int_{-\infty}^{\infty} \varphi(t-t_1) e^{-i\omega_k(t-t_1)} dt_1 - \frac{e^{-i\omega_k t}}{2i} \int_{-\infty}^{\infty} \varphi(t-t_1) e^{i\omega_k(t-t_1)} dt_1 \\ &= \frac{e^{i\omega_k t}}{2i} \widehat{\varphi}(\omega_k) - \frac{e^{-i\omega_k t}}{2i} \widehat{\varphi}(-\omega_k). \end{aligned}$$

The complex-conjugate value is calculated by the formula

$$\overline{\sigma_k(t)} = -\frac{e^{-i\omega_k t}}{2i} \overline{\widehat{\varphi}(\omega_k)} + \frac{e^{i\omega_k t}}{2i} \overline{\widehat{\varphi}(-\omega_k)}.$$

Stress turns out to be real, i.e. $\sigma_k(t) = \overline{\sigma_k(t)}$, provided that

$$\overline{\widehat{\varphi}(\omega)} = \widehat{\varphi}(-\omega). \quad (2.25)$$

In particular, from this condition it follows that if $\widehat{\varphi}(\omega)$ is a fractionally rational function then $P(z)$ and $Q(z)$ must be polynomials with real coefficients. The immediate calculation of the integral

$$\int_0^T \sigma_k(t) e^{\imath \omega_k t} dt = \frac{\imath T}{2} \widehat{\varphi}(-\omega_k)$$

enables one to determine

$$W_k = \frac{\omega_k T}{2} \Im \widehat{\varphi}(\omega_k) = \pi k \Im \widehat{\varphi}(\omega_k)$$

taking into account (2.25). For a term of the type $\varepsilon_k(t) = \cos \omega_k t$ stress is determined by the formula

$$\sigma_k(t) = \frac{e^{\imath \omega_k t}}{2} \widehat{\varphi}(\omega_k) + \frac{e^{-\imath \omega_k t}}{2} \widehat{\varphi}(-\omega_k).$$

It also is real provided that the condition (2.25) holds. The value of the energy dissipation per one deformation cycle is calculated as the integral

$$W_k = -\omega_k \int_0^T \sigma_k(t) \sin \omega_k t dt = \omega_k \Im \int_0^T \sigma_k(t) e^{-\imath \omega_k t} dt.$$

Considering that $\int_0^T \sigma_k(t) e^{-\imath \omega_k t} dt = \frac{\imath T}{2} \widehat{\varphi}(\omega_k)$, we arrive at the previous expression for W_k .

Thus, the condition of non-negativity of internal dissipation of energy for individual harmonics of the Fourier series is reduced to Eq. (2.25) and the inequality

$$\Im \widehat{\varphi}(\omega) \geq 0. \quad (2.26)$$

To write it for an arbitrary deformation program, we represent the Fourier series in the complex form

$$\varepsilon(t) = \sum_{k=-\infty}^{\infty} a_k e^{\imath \omega_k t}, \quad a_k = \frac{1}{T} \int_0^T \varepsilon(t) e^{-\imath \omega_k t} dt.$$

Here $a_0 = c_0/2$ and $a_k = (c_k - \imath s_k)/2$, $a_{-k} = (c_k + \imath s_k)/2$ for $k = 1, 2, \dots$. Because of (2.22), stress is expanded into the series

$$\sigma(t) = \sum_{l=-\infty}^{\infty} a_l \int_{-\infty}^{\infty} \varphi(t - t_1) e^{\omega_l t_1} dt_1 = \sum_{l=-\infty}^{\infty} a_l \widehat{\varphi}(\omega_l) e^{\imath \omega_l t}.$$

Taking into account orthogonality of the system of the functions $e^{i\omega_l t}$ on the segment $[0, T]$, the inequality (2.24) can be written in the form:

$$\begin{aligned} 0 \leq W &= i \sum_{k,l=-\infty}^{\infty} a_k a_l \omega_k \widehat{\varphi}(\omega_l) \int_0^T e^{i(\omega_k + \omega_l)t} dt = i T \sum_{k=-\infty}^{\infty} a_k a_{-k} \omega_k \widehat{\varphi}(-\omega_k) \\ &= \frac{i T}{4} \sum_{k=1}^{\infty} (c_k^2 + s_k^2) \omega_k (\widehat{\varphi}(-\omega_k) - \widehat{\varphi}(\omega_k)). \end{aligned}$$

It holds automatically provided that the conditions (2.25) and (2.26) are valid since

$$i(\widehat{\varphi}(-\omega_k) - \widehat{\varphi}(\omega_k)) = i(\overline{\widehat{\varphi}(\omega_k)} - \widehat{\varphi}(\omega_k)) = 2 \Im \widehat{\varphi}(\omega_k)$$

and, hence, $W = \sum_{k=1}^{\infty} (c_k^2 + s_k^2) W_k$.

Considering that the period T is arbitrary, we can make the following conclusion: a model of a linear hereditary medium with a relaxation kernel $\varphi(t)$ is dissipative if and only if the Eq. (2.25) and the inequality (2.26) are valid for any $\omega > 0$. In terms of a creep kernel this system of conditions is written as

$$\overline{\widehat{\psi}(\omega)} = \widehat{\psi}(-\omega), \quad \Im \widehat{\psi}(\omega) \leq 0.$$

In an equivalent form the inequality (2.26) was first obtained by Breuer and Onat from other considerations related to the fact that work of stresses on strains, which are identically equal to zero for $t \leq 0$, is positive. For viscoelastic materials whose constitutive equations are constructed with the help of standard rheological schemes this inequality holds automatically since integration of Eq. (2.21) within the limits of one cycle yields

$$\int_0^T \sigma(t) d\varepsilon(t) = \int_0^T D(t) dt \geq 0.$$

The inequality of internal dissipation holds for a number of known models as well, in particular, for models with fractional exponential kernels, [21]. It gives nontrivial restrictions on the form of constitutive equations of the differential type. As an example we consider the Hohenemser–Prager equation, [22]:

$$a_0 \sigma + a_1 \dot{\sigma} = b_0 \varepsilon + b_1 \dot{\varepsilon}$$

with constant phenomenological coefficients a_k and b_k . In this case the Fourier transform of the relaxation kernel is as follows

$$\widehat{\varphi}(\omega) = \frac{b_0 + \iota \omega b_1}{a_0 + \iota \omega a_1} = \frac{(b_0 + \iota \omega b_1)(a_0 - \iota \omega a_1)}{a_0^2 + \omega^2 a_1^2},$$

and the inequality (2.26) results in the condition $a_0 b_1 \geq a_1 b_0$.

For the description of rheological properties of materials the Cole–Cole model, [13], with coefficients E_0 , E_∞ , η , and $\alpha \geq 0$, for which

$$\widehat{\varphi}(\omega) = E_\infty - \frac{E_\infty - E_0}{1 + (\iota \omega / \omega_\eta)^\alpha}, \quad \omega_\eta = \frac{E_\infty - E_0}{\eta},$$

is often used. This is a version of a model with a fractional exponential relaxation kernel. For it Eq. (2.25) holds only if all coefficients are real. From (2.25) for $\omega \rightarrow 0$ it follows that $\bar{E}_0 = E_0$. Similarly for $\omega \rightarrow \infty$ we have $\bar{E}_\infty = E_\infty$. If $\Im \omega_\eta \neq 0$ or $\Im \alpha \neq 0$ then the equation is not valid for intermediate values of ω .

To satisfy the inequality (2.26), the difference $E_\infty - E_0$ must be non-negative and the coefficient α may not exceed 2. Indeed, since

$$\iota^\alpha = e^{\iota \pi \alpha / 2} = \cos \frac{\pi \alpha}{2} + \iota \sin \frac{\pi \alpha}{2},$$

the quantity

$$\Im \widehat{\varphi}(\omega) = \frac{(E_\infty - E_0)(\omega / \omega_\eta)^\alpha}{1 + (\omega / \omega_\eta)^{2\alpha}} \Im \iota^\alpha$$

is non-negative that provides dissipation of a model.

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Chapter 3

Mathematical Apparatus

Abstract Basic notions of convex analysis required for the generalization of constitutive relationships of uniaxial deformation of granular materials to the spatial case are considered. Proofs of some theorems from subdifferential calculus and duality theory which are used below in the study of models of the spatial stress-strain state are presented.

3.1 Convex Sets and Convex Functions

Let $\mathbb{R}^m$ be a finite-dimensional arithmetic space. Further it is assumed to be a space of strain or stress tensors. Its dimension m depends on symmetry of a problem. A set F is said to be convex in $\mathbb{R}^m$ if, along with any two points $u, \tilde{u} \in F$, it involves the segment joining these points:

$$u^\lambda = u + \lambda(\tilde{u} - u) = \lambda \tilde{u} + (1 - \lambda)u \in F, \quad 0 \leq \lambda \leq 1.$$

A real-valued function $f = f(u)$ mapping from $\mathbb{R}^m$ into $\mathbb{R}$ is said to be convex if its epigraph

$$\text{epi } f = \{(u, z) \mid z \geq f(u)\}$$

is a convex set in $\mathbb{R}^{m+1}$.

Notice that the definition of convexity of a function does not exclude the case that an epigraph is an empty set ($f \equiv +\infty$) or that it coincides with the whole space ($f \equiv -\infty$). The indicator function of a convex set F :

$$\delta_F(u) = \begin{cases} 0, & \text{if } u \in F, \\ +\infty, & \text{if } u \notin F, \end{cases}$$

is an important example of a convex function taking infinite values. A convex cylinder with the base $\mathbf{F}$ is the epigraph of such a function.

We restrict our consideration to the class of convex functions such that $\text{epi } f \neq \emptyset$ and $f(\mathbf{u}) > -\infty$. On the basis of the above definition one can prove that the domain of finite values

$$\text{dom } f = \{\mathbf{u} \in \mathbb{R}^m \mid f(\mathbf{u}) < +\infty\}$$

of a convex function is convex. Indeed, if $f(\tilde{\mathbf{u}}), f(\mathbf{u}) \leq z < +\infty$ then $(\tilde{\mathbf{u}}, z) \in \text{epi } f$ and $(\mathbf{u}, z) \in \text{epi } f$. Hence,

$$(\mathbf{u}, z)^\lambda \equiv \lambda (\tilde{\mathbf{u}}, z) + (1 - \lambda)(\mathbf{u}, z) = (\mathbf{u}^\lambda, z) \in \text{epi } f,$$

i.e. $f(\mathbf{u}^\lambda) \leq z < +\infty$.

An equivalent definition can be given: a function $f(\mathbf{u})$ taking finite values on a convex set $\mathbf{F} \subset \mathbb{R}^m$ is said to be convex if the inequality

$$f(\mathbf{u}^\lambda) \leq \lambda f(\tilde{\mathbf{u}}) + (1 - \lambda) f(\mathbf{u}), \quad 0 \leq \lambda \leq 1, \quad \tilde{\mathbf{u}}, \mathbf{u} \in \mathbf{F}, \quad (3.1)$$

is valid.

A convex function turns out to be continuous at any point $\tilde{\mathbf{u}}$ in a neighborhood of which it is bounded.

It is sufficient to prove this statement assuming that $\tilde{\mathbf{u}} = \mathbf{0}$, $f(\mathbf{0}) = 0$ and that in the unit ball $|\mathbf{u}| \leq 1$ the function is bounded by a unit: $f(\mathbf{u}) \leq 1$. We can arrive at this case due to a translation and extension transformations of the range of definition and the range of values of a function.

For any $\varepsilon > 0$ we assume that $\delta > 0$ is equal to the minimum of ε and unit. Let $|\mathbf{u}| \leq \delta$, then due to the inequality (3.1) and the obvious identity

$$\mathbf{u} = \delta \frac{\mathbf{u}}{\delta} + (1 - \delta) \cdot \mathbf{0}$$

the condition $f(\mathbf{u}) \leq \delta \cdot f(\mathbf{u}/\delta) \leq \varepsilon$ holds. In a similar way, due to (3.1) and the identity

$$\mathbf{0} = \frac{\mathbf{u}}{1 + \varepsilon} + \frac{\varepsilon}{1 + \varepsilon} \cdot \left(-\frac{\mathbf{u}}{\varepsilon}\right)$$

the condition $f(\mathbf{u}) \geq -\varepsilon f(-\mathbf{u}/\varepsilon) \geq -\varepsilon$ takes place. Thus, $|f(\mathbf{u})| \leq \varepsilon$ which proves that $f(\mathbf{u})$ is continuous at zero.

We cite two known criteria of convexity of a function, [1, 13, 19].

The first criterion. A differentiable function $f(\mathbf{u})$ is convex if and only if for any two vectors $\tilde{\mathbf{u}}$ and $\mathbf{u} \in \mathbf{F}$ the inequality

$$f(\tilde{\mathbf{u}}) - f(\mathbf{u}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} \quad (3.2)$$

is valid, where $\partial f(\mathbf{u})/\partial \mathbf{u}$ is the gradient vector of $f(\mathbf{u})$.

A proof of this criterion follows immediately from the definition of convexity in the form of the inequality (3.1). Because of (3.1),

$$f(\tilde{\mathbf{u}}) - f(\mathbf{u}) \geq \frac{f(\mathbf{u}^\lambda) - f(\mathbf{u})}{\lambda}.$$

Passage to the limit for $\lambda \rightarrow 0$ yields that

$$(\tilde{\mathbf{u}} - \mathbf{u}) \geq \frac{d f(\lambda \tilde{\mathbf{u}} + (1 - \lambda) \mathbf{u})}{d\lambda} \Big|_{\lambda=0} = (\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}}.$$

Conversely, if the inequality (3.2) holds for any $\tilde{\mathbf{u}}$ and $\mathbf{u} \in \mathbf{F}$ then

$$f(\tilde{\mathbf{u}}) - f(\mathbf{u}^\lambda) \geq (\tilde{\mathbf{u}} - \mathbf{u}^\lambda) \frac{\partial f(\mathbf{u}^\lambda)}{\partial \mathbf{u}} = (1 - \lambda)(\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u}^\lambda)}{\partial \mathbf{u}},$$

$$f(\mathbf{u}) - f(\mathbf{u}^\lambda) \geq (\mathbf{u} - \mathbf{u}^\lambda) \frac{\partial f(\mathbf{u}^\lambda)}{\partial \mathbf{u}} = -\lambda (\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u}^\lambda)}{\partial \mathbf{u}}.$$

Multiplying the first inequality by λ and the second one by $1 - \lambda$ and summing up the left-hand and right-hand sides, respectively, we obtain the inequality (3.1) which implies convexity of $f(\mathbf{u})$. $\square$

From this criterion it follows that if a convex function is differentiable then its gradient is a monotone mapping:

$$(\tilde{\mathbf{u}} - \mathbf{u}) \left(\frac{\partial f(\tilde{\mathbf{u}})}{\partial \mathbf{u}} - \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} \right) \geq 0 \quad \forall \mathbf{u}, \tilde{\mathbf{u}}.$$

This inequality (the monotonicity condition for the mapping $\partial f(\mathbf{u})/\partial \mathbf{u}$) results from the term-by-term summing up of the inequality (3.2) and the similar inequality obtained by interchanging vectors $\tilde{\mathbf{u}}$ and $\mathbf{u}$.

As a simple example of application of the first criterion let us prove a convexity of the function $f(\mathbf{u}) = \sqrt{\mathbf{u} \mathbf{A} \mathbf{u}}$ where $\mathbf{A}$ is a symmetrical non-negatively definite $(m \times m)$ -dimensional matrix. In this case by means of the formula for gradient $f(\mathbf{u})$

$$\frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} = \frac{\mathbf{A} \mathbf{u}}{\sqrt{\mathbf{u} \mathbf{A} \mathbf{u}}}$$

the inequality (3.2) after some elementary transformations reduces to the Cauchy–Bunyakovskii inequality

$$\sqrt{\tilde{\mathbf{u}} \mathbf{A} \tilde{\mathbf{u}}} \sqrt{\mathbf{u} \mathbf{A} \mathbf{u}} \geq \tilde{\mathbf{u}} \mathbf{A} \mathbf{u},$$

which is valid due to the sign-definiteness of the matrix.

The second criterion. A twice continuously differentiable function $f(\mathbf{u})$ is convex if and only if the matrix of the second-order partial derivatives (the Hesse matrix)

$$\frac{\partial^2 f(\mathbf{u})}{\partial \mathbf{u}^2} = \begin{pmatrix} \frac{\partial^2 f(\mathbf{u})}{\partial u_1^2} & \frac{\partial^2 f(\mathbf{u})}{\partial u_1 \partial u_2} & \dots & \frac{\partial^2 f(\mathbf{u})}{\partial u_1 \partial u_m} \\ \dots & \dots & \dots & \dots \\ \frac{\partial^2 f(\mathbf{u})}{\partial u_m \partial u_1} & \frac{\partial^2 f(\mathbf{u})}{\partial u_m \partial u_2} & \dots & \frac{\partial^2 f(\mathbf{u})}{\partial u_m^2} \end{pmatrix}$$

is non-negative definite at each point of the set $\mathbf{F}$.

A proof of this statement is based on the first convexity criterion. Let $f(\mathbf{u})$ be convex. Then the inequality (3.2) holds. Using the Taylor expansion of its left-hand side in the neighborhood of the point $\mathbf{u}$ with the estimate of a remainder term in the Peano form, we obtain the inequality

$$\frac{1}{2}(\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial^2 f(\mathbf{u})}{\partial \mathbf{u}^2} (\tilde{\mathbf{u}} - \mathbf{u}) + o(|\tilde{\mathbf{u}} - \mathbf{u}|^2) \geq 0,$$

which leads to the condition of non-negative definiteness of $\partial^2 f / \partial \mathbf{u}^2$ at the point $\mathbf{u} \in \mathbf{F}$ since $\tilde{\mathbf{u}}$ is arbitrary. A proof of the converse statement is based on the Taylor expansion of the function $f(\tilde{\mathbf{u}})$ with a remainder term in the Lagrange form:

$$f(\tilde{\mathbf{u}}) = f(\mathbf{u}) + (\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} + \frac{1}{2} (\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial^2 f(\mathbf{u}^\lambda)}{\partial \mathbf{u}^2} (\tilde{\mathbf{u}} - \mathbf{u}),$$

where $\lambda \in (0, 1)$. Since the Hesse matrix is non-negative definite, the third term in the right-hand side of the expansion is non-negative. Thus, the inequality (3.2) is valid which implies convexity of the function $f(\mathbf{u})$. $\square$

Convex function $f(\mathbf{u})$ is called to be strongly convex if, instead of (3.1), it satisfies the stronger inequality

$$f(\mathbf{u}^\lambda) \leq \lambda f(\tilde{\mathbf{u}}) + (1 - \lambda) f(\mathbf{u}) - \lambda(1 - \lambda) \frac{a}{2} |\tilde{\mathbf{u}} - \mathbf{u}|^2$$

with some small positive constant $a > 0$. Taking into account obvious identity

$$|\mathbf{u}^\lambda|^2 = \lambda |\tilde{\mathbf{u}}|^2 + (1 - \lambda) |\mathbf{u}|^2 - \lambda(1 - \lambda) |\tilde{\mathbf{u}}|^2,$$

one can show that $f(\mathbf{u})$ is strongly convex if and only if for some $a > 0$ the function $f_a(\mathbf{u}) = f(\mathbf{u}) - a |\mathbf{u}|^2 / 2$ is convex. Thus, if $f(\mathbf{u})$ is twice continuously differentiable function, then for its strong convexity is necessary and sufficient that the matrix $\partial^2 f(\mathbf{u}) / \partial \mathbf{u}^2$ is strictly positive definite. This condition provides a uniformly positive

definiteness of the Hesse matrix for the function $f(\mathbf{u})$ —an important property which guarantees the thermodynamic correctness of the mechanical models in many cases.

With the help of convex functions one can define convex sets. First we prove the following simple statement. If each function $f_l(\mathbf{u})$, where $l = 1, \dots, n$, is convex then the set

$$\mathbf{F} = \{\mathbf{u} \in \mathbb{R}^m \mid f_l(\mathbf{u}) \leq 0, l = 1, \dots, n\}$$

is convex as well. To prove it, we consider two elements $\mathbf{u}$ and $\tilde{\mathbf{u}} \in \mathbf{F}$. Since $f_l(\mathbf{u}) \leq 0$ and $f_l(\tilde{\mathbf{u}}) \leq 0$, we have $f_l(\mathbf{u}^\lambda) \leq \lambda f_l(\tilde{\mathbf{u}}) + (1 - \lambda) f_l(\mathbf{u}) \leq 0$, i.e. $\mathbf{u}^\lambda \in \mathbf{F}$. Hence, $\mathbf{F}$ is convex.

It turns out that any convex set can be described with the help of some convex function. The Minkowski function yields one way of such a description. Assume that $\mathbf{F}$ is a convex closed set in the space $\mathbb{R}^m(\mathbf{u})$ for which the point $\mathbf{0}$ (the origin of coordinates) is an interior point (with at least one interior point of the set, this requirement can be always satisfied due to translation of a coordinate system). The function

$$f_0(\mathbf{u}) = \min \left\{ r > 0 \mid \frac{\mathbf{u}}{r} \in \mathbf{F} \right\}$$

is called the Minkowski function of the set $\mathbf{F}$. It is easy to verify that $f_0(\mathbf{u})$ is convex, positive homogeneous: $f_0(\lambda \mathbf{u}) = \lambda f_0(\mathbf{u})$ ($\lambda > 0$), and that for any vector $\mathbf{u} \in \mathbf{F}$ the inequality $f_0(\mathbf{u}) \leq 1$ holds, and for $\mathbf{u} \notin \mathbf{F}$ the inequality $f_0(\mathbf{u}) > 1$ holds, i.e. with the help of the Minkowski function the set $\mathbf{F}$ is parametrized in the form

$$\mathbf{F} = \{\mathbf{u} \in \mathbb{R}^m \mid f_0(\mathbf{u}) \leq 1\}. \quad (3.3)$$

Indeed, since by the definition of the Minkowski function the inclusions $\tilde{\mathbf{u}}/f_0(\tilde{\mathbf{u}})$, $\mathbf{u}/f_0(\mathbf{u}) \in \mathbf{F}$ take place, due to convexity of the set $\mathbf{F}$ for any $\lambda \in [0, 1]$ we have

$$\begin{aligned} \frac{\mathbf{u}^\lambda}{\lambda f_0(\tilde{\mathbf{u}}) + (1 - \lambda) f_0(\mathbf{u})} &= \alpha \frac{\tilde{\mathbf{u}}}{f_0(\tilde{\mathbf{u}})} + (1 - \alpha) \frac{\mathbf{u}}{f_0(\mathbf{u})} \in \mathbf{F}, \\ \alpha &= \frac{\lambda f_0(\tilde{\mathbf{u}})}{\lambda f_0(\tilde{\mathbf{u}}) + (1 - \lambda) f_0(\mathbf{u})}, \quad 0 \leq \alpha \leq 1. \end{aligned}$$

Besides, by this definition $f_0(\mathbf{u}^\lambda)$ is the smallest of positive numbers $r > 0$ for which $\mathbf{u}^\lambda/r \in \mathbf{F}$. Putting $r = \lambda f_0(\tilde{\mathbf{u}}) + (1 - \lambda) f_0(\mathbf{u})$, from this condition we obtain

$$f_0(\mathbf{u}^\lambda) \leq \lambda f_0(\tilde{\mathbf{u}}) + (1 - \lambda) f_0(\mathbf{u}).$$

Convexity of the Minkowski function is proved, homogeneity follows from the chain of the equalities

$$f_0(\lambda \mathbf{u}) \equiv \min \left\{ r > 0 \mid \frac{\lambda \mathbf{u}}{r} \in \mathbf{F} \right\} = \lambda \min \left\{ \frac{r}{\lambda} > 0 \mid \frac{\mathbf{u}}{r/\lambda} \in \mathbf{F} \right\} = \lambda f_0(\mathbf{u}).$$

The property (3.3) is also proved taking into account convexity of $\mathbf{F}$. Let $f_0(\mathbf{u}) \leq 1$. Then since $\mathbf{u}/f_0(\mathbf{u}) \in \mathbf{F}$ we have

$$\mathbf{u} = f_0(\mathbf{u}) \frac{\mathbf{u}}{f_0(\mathbf{u})} + (1 - f_0(\mathbf{u})) \mathbf{0} \in \mathbf{F}.$$

If $\mathbf{u} \in \mathbf{F}$ then according to the definition $f_0(\mathbf{u}) \leq 1$.

The converse statement is valid. If $f(\mathbf{u}) \geq 0$ is an arbitrary convex positive homogeneous function defined on the whole space, then the set defined by the formula (3.3) is convex and its Minkowski function coincides with the function $f(\mathbf{u})$. Indeed, consider an arbitrary number $r > 0$. If $\mathbf{u}/r \in \mathbf{F}$ then because of homogeneity of the function f we have $f(\mathbf{u})/r = f(\mathbf{u}/r) \leq 1$, hence, $f(\mathbf{u}) \leq r$. Besides,

$$f\left(\frac{\mathbf{u}}{f(\mathbf{u})}\right) = \frac{f(\mathbf{u})}{f(\mathbf{u})} = 1,$$

hence, $\mathbf{u}/f(\mathbf{u}) \in \mathbf{F}$. Thus,

$$f(\mathbf{u}) = \min \left\{ r > 0 \mid \frac{\mathbf{u}}{r} \in \mathbf{F} \right\}.$$

Convexity of $\mathbf{F}$ follows from the statement proved above considering convexity of the function $f(\mathbf{u}) - 1$.

Notice that the graph of the Minkowski function $f_0(\mathbf{u})$ is a conic hypersurface in the space $\mathbb{R}^{m+1}$ formed by the rays issuing out of the point $\mathbf{0}$. The level surface corresponding to the value 1 coincides with the boundary of the set $\mathbf{F}$.

If the Minkowski function is continuously differentiable in a neighbourhood of a point $\mathbf{u} \neq \mathbf{0}$ (the case of a regular point) then the Euler theorem is valid for it:

$$\mathbf{u} \frac{\partial f_0(\mathbf{u})}{\partial \mathbf{u}} = f_0(\mathbf{u}).$$

A proof of this theorem is based on the differentiation of identity $f_0(\lambda \mathbf{u}) = \lambda f_0(\mathbf{u})$ with respect to λ for $\lambda = 1$.

As a rule, in the irregular case a set $\mathbf{F}$ is represented in the form of the intersection of a finite number of sets $\mathbf{F}_1, \dots, \mathbf{F}_n$ which have continuously differentiable Minkowski function $f_1(\mathbf{u}), \dots, f_n(\mathbf{u})$. Besides, for the Minkowski function of the set $\mathbf{F}$ the identity

$$f_0(\mathbf{u}) = \min_{l=1, \dots, n} f_l(\mathbf{u}) \tag{3.4}$$

holds, and for each $f_l(\mathbf{u})$ the Euler theorem is valid.

Convex sets and convex functions defined on a space of stress tensors are applied in models of elastic-plastic media when constructing yield surfaces. We present a proof of a statement which is of crucial importance in the studies of models of this type. This statement ensures convexity of a set of admissible stress tensors provided

that a yield function, which parameterizes it, is convex and symmetric with respect to principal stresses, [24]. More exactly, assume that $f(\sigma_1, \sigma_2, \sigma_3)$ is a convex twice continuously differentiable function which depends on principal stresses, in addition, $f(\sigma_1, \sigma_2, \sigma_3) = f(\sigma_2, \sigma_1, \sigma_3) = f(\sigma_1, \sigma_3, \sigma_2)$. Then the generated yield function $f(\boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ is a symmetric stress tensor with components σ_{ij} ($i, j = 1, 2, 3$) in an arbitrary fixed coordinate system, is convex as well.

To prove this, we consider three linearly independent combinations of principal stresses which are previously numbered in decreasing order ($\sigma_1 \geq \sigma_2 \geq \sigma_3$):

$$\tau_1 = \sigma_1, \quad \tau_2 = \sigma_1 + \sigma_2, \quad \tau_3 = \sigma_1 + \sigma_2 + \sigma_3,$$

and show that each of them is a convex function with respect to σ_{ij} . Convexity of $\tau_1(\boldsymbol{\sigma})$ is proved with the help of the known representation for the maximal eigenvalue of a symmetric matrix (the rule of summing over repetitive indices is assumed):

$$\tau_1 = \max_{|\mathbf{v}|=1} \sigma_{ij} v_i v_j.$$

Addressing immediately of equivalent definition (3.1) we have

$$\begin{aligned} \tau_1(\boldsymbol{\sigma}^\lambda) &= \max_{|\mathbf{v}|=1} \sigma_{ij}^\lambda v_i v_j = \max_{|\mathbf{v}|=1} (\lambda \tilde{\sigma}_{ij} + (1 - \lambda) \sigma_{ij}) v_i v_j \\ &\leq \lambda \max_{|\mathbf{v}|=1} \tilde{\sigma}_{ij} v_i v_j + (1 - \lambda) \max_{|\mathbf{v}|=1} \sigma_{ij} v_i v_j = \lambda \tau_1(\tilde{\boldsymbol{\sigma}}) + (1 - \lambda) \tau_1(\boldsymbol{\sigma}). \end{aligned}$$

Convexity of $\tau_3(\boldsymbol{\sigma})$ is obvious since $\tau_3 = \sigma_{11} + \sigma_{22} + \sigma_{33}$ is a linear function of a stress tensor. Finally, the combination τ_2 can be represented as

$$\tau_2 = \tau_3 - \sigma_3 = \tau_3 - \min_{|\mathbf{v}|=1} \sigma_{ij} v_i v_j = \tau_3 + \max_{|\mathbf{v}|=1} \{-\sigma_{ij} v_i v_j\},$$

and its convexity follows from convexity of both terms.

Further we establish the inequalities

$$\frac{\partial f}{\partial \sigma_1} \geq \frac{\partial f}{\partial \sigma_2} \geq \frac{\partial f}{\partial \sigma_3}, \quad (3.5)$$

which follow from convexity and symmetry of $f(\sigma_1, \sigma_2, \sigma_3)$. Putting

$$f(\lambda) = f(\lambda \sigma_1 + (1 - \lambda) \sigma_2, \lambda \sigma_2 + (1 - \lambda) \sigma_1, \sigma_3),$$

we can show that

$$f(\lambda) \leq \lambda f(\sigma_1, \sigma_2, \sigma_3) + (1 - \lambda) f(\sigma_2, \sigma_1, \sigma_3) = f(1),$$

i.e. the function $f(\lambda)$ defined here does not decrease in a neighbourhood of the point $\lambda = 1$. Thus,

$$\frac{d f(1)}{d \lambda} \geq 0 : (\sigma_1 - \sigma_2) \frac{\partial f}{\partial \sigma_1} + (\sigma_2 - \sigma_1) \frac{\partial f}{\partial \sigma_2} \geq 0,$$

whence it follows that $\partial f / \partial \sigma_1 \geq \partial f / \partial \sigma_2$ provided that $\sigma_1 > \sigma_2$. If σ_1 and σ_2 are equal then the derivatives $\partial f / \partial \sigma_1$ and $\partial f / \partial \sigma_2$ turn out to be equal due to symmetry of the function $f(\sigma_1, \sigma_2, \sigma_3)$. The inequality $\partial f / \partial \sigma_2 \geq \partial f / \partial \sigma_3$ is proved in a similar way.

Considering that $\sigma_1 = \tau_1$, $\sigma_2 = \tau_2 - \tau_1$, $\sigma_3 = \tau_3 - \tau_2$, from (3.5) we obtain the inequalities

$$\frac{\partial f}{\partial \tau_1} = \frac{\partial f}{\partial \sigma_1} - \frac{\partial f}{\partial \sigma_2} \geq 0, \quad \frac{\partial f}{\partial \tau_2} = \frac{\partial f}{\partial \sigma_2} - \frac{\partial f}{\partial \sigma_3} \geq 0,$$

which are used when verifying convexity of $f(\sigma)$. Applying the second criterion of convexity, we define

$$\frac{\partial^2 f}{\partial \sigma_{ij} \partial \sigma_{kl}} = \sum_{p,q=1}^3 \frac{\partial \tau_q}{\partial \sigma_{kl}} \frac{\partial^2 f}{\partial \tau_p \partial \tau_q} \frac{\partial \tau_p}{\partial \sigma_{ij}} + \sum_{p=1}^2 \frac{\partial f}{\partial \tau_p} \frac{\partial^2 \tau_p}{\partial \sigma_{ij} \partial \sigma_{kl}}.$$

Here the term corresponding to $p = 3$ is omitted since τ_3 is a linear function of σ_{ij} and, hence, $\frac{\partial^2 \tau_3}{\partial \sigma_{ij} \partial \sigma_{kl}} = 0$. Nonnegative definiteness of matrices composed of the terms of the first sum in the right-hand side of this inequality is proved immediately: if α_{ij} is an arbitrary tensor then

$$\alpha_{kl} \frac{\partial \tau_q}{\partial \sigma_{kl}} \frac{\partial^2 f}{\partial \tau_p \partial \tau_q} \frac{\partial \tau_p}{\partial \sigma_{ij}} \alpha_{ij} = \alpha_q \frac{\partial^2 f}{\partial \tau_p \partial \tau_q} \alpha_p,$$

where $\alpha_p = \alpha_{ij} \partial \tau_p / \partial \sigma_{ij}$. This expression is non-negative due to the second criterion of convexity written with respect to the function $f = f(\tau_1, \tau_2, \tau_3)$ whose convexity can be proved on the basis of the inequality (3.1) taking into account that τ_q is linearly dependent of σ_p . Nonnegative definiteness of the matrices corresponding to the terms of the second sum follows from convexity of $\tau_1(\sigma)$ and $\tau_2(\sigma)$, and also from the inequalities (3.5). Thus, the statement is completely proved. $\square$

In fact, the converse statement is true: if the isotropic function is convex with respect to the components of a symmetric tensor in an arbitrary coordinate system, then it is convex in the space of principal values of this tensor. In addition, it is symmetric, since the isotropy is allowed an arbitrary enumeration of principal values. Thus, under the supposed conditions the convexity is necessary and sufficient.

Taking into account that by the invariance of the Euclidean norm of tensor

$$f(\sigma) - \frac{a}{2} \sigma_{ij} \sigma_{ij} = f(\sigma_1, \sigma_2, \sigma_3) - \frac{a}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2),$$

it is easy to prove the following.

In order to function $f(\sigma_1, \sigma_2, \sigma_3)$ be strongly convex as a function of the components of symmetric tensor in an arbitrary Cartesian coordinate system, it is necessary and sufficient that it is strongly convex as a function of three variables—principal values of tensor, and symmetric with respect to these variables.

A more general statement, useful in constructing of convex potentials for isotropic couple-stress elastic media, is valid. In such media the nonsymmetric second-rank tensors are used under the formulation of constitutive relationships. This statement is as follows, [21].

Proposition 3.1 *Assume that the isotropic twice continuously differentiable scalar function $\Phi(\mathbf{A})$ depends only on four arguments – the three principal values of the symmetric part $\mathbf{A}^s = (\mathbf{A} + \mathbf{A}^*)/2$ of tensor $\mathbf{A}$ (the asterisk denotes transposition) and the modulus of vector associated with the antisymmetric part $\mathbf{A}^a = (\mathbf{A} - \mathbf{A}^*)/2$. Then this function is convex relative to the components of nonsymmetric tensor $\mathbf{A}$, written in an arbitrary Cartesian coordinate system, if and only if it is symmetric relative to the principal values of $\mathbf{A}^s$, i.e.*

$$\Phi(\Lambda_1^s, \Lambda_2^s, \Lambda_3^s, |\mathbf{A}^a|) = \Phi(\Lambda_2^s, \Lambda_1^s, \Lambda_3^s, |\mathbf{A}^a|) = \Phi(\Lambda_1^s, \Lambda_3^s, \Lambda_2^s, |\mathbf{A}^a|),$$

non-decreasing for $|\mathbf{A}^a|$ and convex in all arguments.

Proof Indeed, let Φ be a convex function relative to the components $\Lambda_{ij} \neq \Lambda_{ji}$ of tensor $\mathbf{A}$ in an arbitrary fixed Cartesian coordinate system, depending only on specified four arguments—the invariants of tensor $\mathbf{A}$. In the system of principal axes of the symmetric part of this tensor, which can be obtained by a special rotation of original coordinate system, the tensors $\mathbf{A}^s$ and $\mathbf{A}^a$ take the following form

$$\mathbf{A}^s = \begin{pmatrix} \Lambda_1^s & 0 & 0 \\ 0 & \Lambda_2^s & 0 \\ 0 & 0 & \Lambda_3^s \end{pmatrix}, \quad \mathbf{A}^a = \begin{pmatrix} 0 & -\Lambda_3^a & \Lambda_2^a \\ \Lambda_3^a & 0 & -\Lambda_1^a \\ -\Lambda_2^a & \Lambda_1^a & 0 \end{pmatrix}.$$

At first it is necessary to show that the function $\Phi(\Lambda_1^s, \Lambda_2^s, \Lambda_3^s, |\mathbf{A}^a|)$ is convex, non-decreasing for $|\mathbf{A}^a| = \sqrt{(\Lambda_1^a)^2 + (\Lambda_2^a)^2 + (\Lambda_3^a)^2}$ and symmetric relative to Λ_k^s .

The symmetry of this function follows from its isotropy (independence from the rotation of coordinate system), the convexity follows from the convexity of $\Phi(\mathbf{A})$ on the set of tensors of a particular form

$$\mathbf{A} = \begin{pmatrix} \Lambda_1^s - \Lambda_3^a & 0 \\ \Lambda_3^a & \Lambda_2^s & 0 \\ 0 & 0 & \Lambda_3^s \end{pmatrix}, \quad \Lambda_3^a \geq 0.$$

The fact, that the function Φ is not decreasing in the last argument, is proved by contradiction. Assume that in a neighborhood of a point $\Lambda_1^s, \Lambda_2^s, \Lambda_3^s, |\mathbf{A}^a| = \chi$ this function is strictly decreasing. Then $\partial\Phi(\Lambda_1^s, \Lambda_2^s, \Lambda_3^s, \chi)/\partial|\mathbf{A}^a| < 0$. Consider the

following pair of tensors with the same systems of invariants

$$\mathbf{A} = \begin{pmatrix} \Lambda_1^s - \chi & 0 \\ \chi & \Lambda_2^s & 0 \\ 0 & 0 & \Lambda_3^s \end{pmatrix}, \quad \tilde{\mathbf{A}} = \begin{pmatrix} \Lambda_1^s & -\chi \cos \varphi & \chi \sin \varphi \\ \chi \cos \varphi & \Lambda_2^s & 0 \\ -\chi \sin \varphi & 0 & \Lambda_3^s \end{pmatrix}$$

(φ is an arbitrary angle), on where Φ takes the same value. For this pair

$$\tilde{\mathbf{A}} - \mathbf{A} = \begin{pmatrix} 0 & \chi(1 - \cos \varphi) & \chi \sin \varphi \\ -\chi(1 - \cos \varphi) & 0 & 0 \\ -\chi \sin \varphi & 0 & 0 \end{pmatrix}.$$

In the general case

$$4|\mathbf{A}^a|^2 = (\Lambda_{32} - \Lambda_{23})^2 + (\Lambda_{13} - \Lambda_{31})^2 + (\Lambda_{21} - \Lambda_{12})^2,$$

therefore

$$\frac{\partial |\mathbf{A}^a|}{\partial \mathbf{A}} = \frac{1}{4|\mathbf{A}^a|} \begin{pmatrix} 0 & \Lambda_{12} - \Lambda_{21} & \Lambda_{13} - \Lambda_{31} \\ \Lambda_{21} - \Lambda_{12} & 0 & \Lambda_{23} - \Lambda_{32} \\ \Lambda_{31} - \Lambda_{13} & \Lambda_{32} - \Lambda_{23} & 0 \end{pmatrix} = \frac{1}{2\chi} \begin{pmatrix} 0 & -\chi & 0 \\ \chi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Based on the second criterion of convexity, we obtain a contradiction

$$\begin{aligned} 0 &\geq \sum_{i,j=1}^3 (\tilde{\Lambda}_{ij} - \Lambda_{ij}) \frac{\partial \Phi}{\partial \Lambda_{ij}} = \frac{\partial \Phi}{\partial |\mathbf{A}^a|} \sum_{i,j=1}^3 (\tilde{\Lambda}_{ij} - \Lambda_{ij}) \frac{\partial |\mathbf{A}^a|}{\partial \Lambda_{ij}} \\ &= -\frac{\partial \Phi}{\partial |\mathbf{A}^a|} \chi (1 - \cos \varphi) > 0, \end{aligned}$$

which completes the proof of the first part of the statement.

To prove the converse proposition about convexity of the function Φ , depending on four invariants, relative to the components of tensor in an arbitrary coordinate system after renumbering the principal values of the symmetric part Λ_k^s in decreasing order, let us consider the convex invariants

$$\mu_1 = \Lambda_1^s, \quad \mu_2 = \Lambda_1^s + \Lambda_2^s, \quad \mu_3 = \Lambda_1^s + \Lambda_2^s + \Lambda_3^s.$$

The following representation

$$\mu_1(\mathbf{A}) = \max_{|\mathbf{v}|=1} \Lambda_{ij}^s v_i v_j, \quad \mu_2(\mathbf{A}) = \mu_3(\mathbf{A}) + \max_{|\mathbf{v}|=1} \{-\Lambda_{ij}^s v_i v_j\},$$

$$\mu_3(\mathbf{A}) = \Lambda_{11} + \Lambda_{22} + \Lambda_{33}$$

is valid. In addition, the invariant $\mu_4(\mathbf{A}) = |\mathbf{A}^a|$ is convex itself on Λ_{ij} , because it is a special case of the function $\sqrt{\mathbf{u} \mathbf{A} \mathbf{u}}$ with non-negative definite matrix $\mathbf{A}$. Convexity of this function is proved as an example at the beginning of this section.

Repeating a fragment of the above proof for the case of symmetric tensor, we can establish that

$$\frac{\partial \Phi}{\partial \Lambda_1^s} \geq \frac{\partial \Phi}{\partial \Lambda_2^s} \geq \frac{\partial \Phi}{\partial \Lambda_3^s},$$

therefore

$$\frac{\partial \Phi}{\partial \mu_1} = \frac{\partial \Phi}{\partial \Lambda_1^s} - \frac{\partial \Phi}{\partial \Lambda_2^s} \geq 0, \quad \frac{\partial \Phi}{\partial \mu_2} = \frac{\partial \Phi}{\partial \Lambda_2^s} - \frac{\partial \Phi}{\partial \Lambda_3^s} \geq 0.$$

The function $\Phi(\Lambda_1^s, \Lambda_2^s, \Lambda_3^s, |\mathbf{A}^a|)$ assumed to be convex. Taking into account the linearity of the equations connecting its arguments with the invariants μ_k , directly by definition one can set the convexity of the function $\Phi(\mu_1, \mu_2, \mu_3, \mu_4)$:

$$\begin{aligned} \Phi(\mu_1^\lambda, \mu_2^\lambda, \mu_3^\lambda, \mu_4^\lambda) &= \Phi((\Lambda_1^s)^\lambda, (\Lambda_2^s)^\lambda, (\Lambda_3^s)^\lambda, |\mathbf{A}^a|^\lambda) \\ &\leq \lambda \Phi(\tilde{\Lambda}_1^s, \tilde{\Lambda}_2^s, \tilde{\Lambda}_3^s, |\tilde{\mathbf{A}}^a|) + (1 - \lambda) \Phi(\Lambda_1^s, \Lambda_2^s, \Lambda_3^s, |\mathbf{A}^a|) \\ &= \lambda \Phi(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3, \tilde{\mu}_4) + (1 - \lambda) \Phi(\mu_1, \mu_2, \mu_3, \mu_4). \end{aligned}$$

By the second criterion of convexity, the corresponding Hesse matrix is non-negative definite. Then, repeating the proof of statement for a symmetric tensor, we obtain

$$\frac{\partial^2 \Phi}{\partial \Lambda_{ij} \partial \Lambda_{kl}} = \sum_{p,q=1}^4 \frac{\partial \mu_q}{\partial \Lambda_{kl}} \frac{\partial^2 \Phi}{\partial \mu_p \partial \mu_q} \frac{\partial \mu_p}{\partial \Lambda_{ij}} + \sum_{p \neq 3} \frac{\partial \Phi}{\partial \mu_p} \frac{\partial^2 \mu_p}{\partial \Lambda_{ij} \partial \Lambda_{kl}}.$$

Taking into account the above inequalities, we can prove that this matrix is non-negative definite, hence the function $\Phi(\mathbf{A})$ is convex. $\square$

The next variant of this statement is valid. If the isotropic twice differentiable function $\Phi(\mathbf{A})$ depends on three principal values of the symmetric part $\mathbf{A}^s$ of $\mathbf{A}$ and the modulus of vector associated with the antisymmetric part $\mathbf{A}^a$, then this function is strongly convex with respect to the components of nonsymmetric tensor $\mathbf{A}$ in an arbitrary Cartesian coordinate system, if and only if it is symmetric onto the principal values of $\mathbf{A}^s$, it satisfies the condition $\partial \Phi / \partial |\mathbf{A}^a| \geq 2a |\mathbf{A}^a|$ for some $a > 0$ and it is a strongly convex function of its arguments.

The proof of strong convexity under these conditions can be obtained by passing from the function $\Phi(\mathbf{A})$ to the function

$$\Phi(\mathbf{A}) - \frac{a}{2} \Lambda_{ij} \Lambda_{ij} = \Phi(\Lambda_1^s, \Lambda_2^s, \Lambda_3^s, |\mathbf{A}^a|) - \frac{a}{2} ((\Lambda_1^s)^2 + (\Lambda_2^s)^2 + (\Lambda_3^s)^2 + 2|\mathbf{A}^a|^2).$$

In practice the proved statement is applicable only in the theory of small strains of a couple-stress medium. In the general case, a nonsymmetric tensor has six functionally

independent invariants. Among them are invariants Λ_k^s and Λ_k^a . A simple example of an isotropic function, which can not be expressed only by means of four invariant Λ_k^s and $|\mathbf{A}^a|$, is the determinant of the tensor $\mathbf{A}$ describing the change in volume of a particle under finite strains:

$$\det \mathbf{A} = \Lambda_1^s \Lambda_2^s \Lambda_3^s + \Lambda_1^s (\Lambda_1^a)^2 + \Lambda_2^s (\Lambda_2^a)^2 + \Lambda_3^s (\Lambda_3^a)^2.$$

Fixing the principal values Λ_k^s , it is easy to find two systems of invariants Λ_k^a and $\tilde{\Lambda}_k^a$ with identical modules $|\mathbf{A}^a|$ and $|\tilde{\mathbf{A}}^a|$ but with different determinants.

Let us construct a more meaningful system of convex invariants, by means of which the determinant and other characteristics of the strain state of a medium are calculated unambiguously. Let $\mathbf{o}$ and $\mathbf{V}$ be orthogonal and symmetric tensors involved in the polar decomposition:

$$\mathbf{A} = \mathbf{o} \cdot \mathbf{V}, \quad \mathbf{o} \cdot \mathbf{o}^* = \mathbf{o}^* \cdot \mathbf{o} = \delta, \quad \mathbf{V} = \mathbf{V}^*, \quad \det \mathbf{V} \geq 0,$$

where δ is the unit tensor. Symmetric tensor $\mathbf{V}$ has three functionally independent invariants—the principal values $V_1 \geq V_2 \geq V_3$, describing the deformation of a material. The largest of them is nonnegative by virtue of the condition $\det \mathbf{V} \geq 0$. The number of invariants for the orthogonal tensor $\mathbf{o}$ is also three, but only one of them is of particular interest—the rotation angle of a particle.

Now we show that the following combinations of invariants are convex:

$$\mu_1 = V_1, \quad \mu_2 = \sqrt{V_1^2 + V_2^2}, \quad \mu_3 = \sqrt{V_1^2 + V_2^2 + V_3^2}. \quad (3.6)$$

Indeed, since $\mathbf{A}^* \cdot \mathbf{A} = \mathbf{V}^2$, then

$$\mu_1 = \max_{|\mathbf{v}|=1} \sqrt{\Lambda_{ki} \Lambda_{kj} v_i v_j}.$$

Included here radicand is a non-negative quadratic form with respect to the components of $\mathbf{A}$, so the function $\sqrt{\Lambda_{ki} \Lambda_{kj} v_i v_j}$ is convex and, as consequence, the function μ_1 is convex too. Based on the formula $\mu_3 = \sqrt{\Lambda_{ij} \Lambda_{ij}}$, one can prove the convexity of μ_3 . Finally, μ_2 can be written as follows:

$$\begin{aligned} \mu_2 &= \sqrt{\mu_3^2 - V_3^2} = \sqrt{\Lambda_{ij} \Lambda_{ij} - \min_{|\mathbf{v}|=1} \Lambda_{ki} \Lambda_{kj} v_i v_j} \\ &= \max_{|\mathbf{v}|=1} \sqrt{\Lambda_{ij} \Lambda_{ij} - \Lambda_{ki} \Lambda_{kj} v_i v_j}. \end{aligned} \quad (3.7)$$

From the chain of equations and inequalities

$$\Lambda_{ki} \Lambda_{kj} v_i v_j \leq \max_{|\mathbf{v}|=1} \Lambda_{ki} \Lambda_{kj} v_i v_j = V_1^2 \leq V_1^2 + V_2^2 + V_3^2 = \Lambda_{ij} \Lambda_{ij}$$

it follows that the radicand in the right-hand side of (3.7) is also a non-negative quadratic form with respect to Λ_{ij} . Consequently, the function μ_2 is convex.

In addition to the system (3.6) we can take $\mu_4 = |\mathbf{A}^a|$ as a convex invariant characterizing the rotation. This invariant is identically zero in the absence of rotation of a particle when $\mathbf{o} = \delta$, and is nonzero when $\mathbf{o} \neq \delta$.

The following statement is valid. If the function $\Phi(\mu_1, \mu_2, \mu_3, \mu_4)$ is convex and its first derivatives $\partial\Phi/\partial\mu_k$ are nonnegative for all arguments, then it is convex with respect to the components of $\mathbf{A}$. If, moreover, $\Phi(\mu_1, \mu_2, \mu_3, \mu_4) - \varepsilon\mu_3^2$ is a convex function and the inequality $\partial\Phi/\partial\mu_3 \geq 2\varepsilon\mu_3$ is performed for some sufficiently small $\varepsilon > 0$, then Φ is strongly convex with respect to the components of $\mathbf{A}$.

Notice that in a similar way one can write four convex invariants of a nonsymmetric tensor, based on the left polar decomposition $\mathbf{A} = \mathbf{W} \cdot \mathbf{o}$ as a product of symmetric and orthogonal tensors, and that the question of constructing complete system of six functionally independent convex invariants remains open.

3.2 Discrete Variational Inequalities

We consider the minimization problem for a differentiable convex function $f(\mathbf{u})$ on a convex set $\mathbf{F} \subset \mathbb{R}^m$. It turns out that a point $\mathbf{u} \in \mathbf{F}$ is a minimum point if and only if for any vector $\tilde{\mathbf{u}} \in \mathbf{F}$ the inequality

$$(\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} \geq 0 \quad (3.8)$$

holds. Thus, it is a necessary and sufficient minimum condition. A more general case, where a function involves a nondifferentiable term, will be considered in the next section. A proof of the necessary condition follows immediately from the minimum condition according to which for any positive $\lambda \leq 1$

$$\frac{f(\mathbf{u}^\lambda) - f(\mathbf{u})}{\lambda} \geq 0.$$

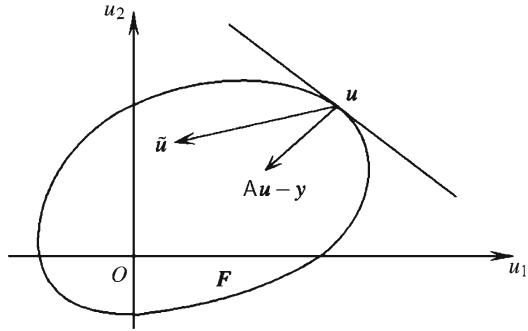
Passing to the limit for $\lambda \rightarrow 0$ and considering that the left-hand side tends to the derivative $df(\mathbf{u}^\lambda)/d\lambda$ at the point $\lambda = 0$, we arrive at the inequality (3.8). The sufficient condition follows immediately from (3.8) in view of the first convexity criterion:

$$f(\tilde{\mathbf{u}}) - f(\mathbf{u}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} \geq 0.$$

Now we consider a similar problem. Let $\mathbf{A}$ be an $(m \times m)$ -matrix with entries a_{ij} , $\mathbf{y}$ be an m -dimensional column-vector, and $\mathbf{F}$ be a given subset of $\mathbb{R}^m$. The problem is to determine a vector $\mathbf{u}$ satisfying the condition $\mathbf{u} \in \mathbf{F}$ and the inequality

$$(\tilde{\mathbf{u}} - \mathbf{u})(\mathbf{A}\mathbf{u} - \mathbf{y}) \geq 0 \quad \forall \tilde{\mathbf{u}} \in \mathbf{F}. \quad (3.9)$$

Fig. 3.1 Geometric interpretation



The scalar product of the vector of an arbitrary admissible variation $\tilde{u} - u$ of a solution and the vector $Au - y$ involved in the inequality in the case of $m = 2, 3$ is expressed as the product of the modules of these vectors by cosine of the angle between them. Therefore the problem (3.9) has the following geometric interpretation (see Fig. 3.1): find a point $u \in F$ such that the angle between these vectors is acute for any choice of $\tilde{u} \in F$.

If a solution u to be found is an interior point of the set F then the system of equations $Au = y$ is valid. If u is a boundary point then the vector $Au - y$ is directed toward the boundary along an inner normal. In other words, if the vector $\tilde{u}$ being a solution of the system of linear algebraic equations $A\tilde{u} = y$ belongs to the set F then $u = \tilde{u}$. If $\tilde{u} \notin F$ then the vector $A(u - \tilde{u})$ is perpendicular to the boundary and is directed inward F . In the case that a matrix A is symmetric and satisfies a condition of nonnegative definiteness and a set F is convex, the problem (3.9) is equivalent to the problem on minimum of a convex quadratic function

$$f(u) = \frac{1}{2} u A u - u y, \quad \frac{\partial f}{\partial u} = A u - y.$$

In the case of a nonsymmetric matrix the relation to the minimization problem is lost. A version of a variational inequality with a nonsymmetric matrix was obtained in Sect. 2.4 when developing a computational algorithm for a rheological scheme of the general form that involves rigid contacts and plastic elements. In this section the aim is to formulate conditions under which the problem is correct, i.e. it has a unique solution that continuously depends on input data.

The proofs of following statements in more general case of variational inequalities in infinite-dimensional space can be found, for example, in [5, 8, 9, 11, 13, 14].

Proposition 3.2 *If a matrix A involved in the variational inequality is positive definite then there can not exist more than one solution of the problem.*

Proof The statement is proved by contradiction. Let u and $u' \in F$ be two solution. Putting $\tilde{u} = u'$ in the inequality (3.9) and $\tilde{u} = u$ in the similar inequality for u' , we obtain

$$(u' - u)(Au - y) \geq 0, \quad (u - u')(Au' - y) \geq 0.$$

Summing up, we have

$$(\mathbf{u}' - \mathbf{u})\mathbf{A}(\mathbf{u}' - \mathbf{u}) \leq 0.$$

Since the matrix is positive definite, the quadratic form $\tilde{\mathbf{u}}\mathbf{A}\tilde{\mathbf{u}}$ is strictly positive for any vector $\tilde{\mathbf{u}} \neq \mathbf{0}$. Hence, $\mathbf{u}' - \mathbf{u} = \mathbf{0}$ and the solutions coincide. $\square$

Theorem 3.1 *If a matrix $\mathbf{A}$ is symmetric and positive definite and a set $\mathbf{F}$ is convex and closed, then the problem (3.9) has a unique solution.*

To prove the theorem, it is sufficient to establish existence and uniqueness of a minimum point for a quadratic function $f(\mathbf{u})$. Let $\mathbf{u}^0$ be a fixed element of $\mathbf{F}$, and $\mathbf{F}_0 = \{\tilde{\mathbf{u}} \in \mathbf{F} \mid f(\tilde{\mathbf{u}}) \leq f(\mathbf{u}^0)\}$. The set $\mathbf{F}_0$ is obtained by eliminating the points, at which the values of the function $f(\mathbf{u})$ are “reasonably large”, therefore it is clear that the greatest lower bounds of this function on $\mathbf{F}$ and on $\mathbf{F}_0$ coincide. The set $\mathbf{F}_0$ is turned out to be closed and bounded, hence, it is compact in $\mathbb{R}^m$.

Proof of closeness. Since $f(\mathbf{u})$ is continuous for any sequence $\mathbf{u}^n \in \mathbf{F}_0$ that converges to the vector $\mathbf{u} \in \mathbb{R}^m$, the following inequality holds:

$$f(\mathbf{u}) = \lim_{n \rightarrow \infty} f(\mathbf{u}^n) \leq f(\mathbf{u}^0).$$

Besides, since $\mathbf{F}$ is closed we have $\mathbf{u} \in \mathbf{F}$. Thus, $\mathbf{u} \in \mathbf{F}_0$, i.e. any limit point of $\mathbf{F}_0$ belongs to this set.

Boundedness is proved by contradiction. Assume that there exists an unbounded sequence of vectors $\mathbf{u}^n \in \mathbf{F}_0$ such that $|\mathbf{u}^n| \rightarrow +\infty$. The matrix $\mathbf{A}$ is positive definite, hence, for any vector $\mathbf{u}$

$$\mathbf{u}\mathbf{A}\mathbf{u} \geq a|\mathbf{u}|^2, \quad a = \inf_{|\mathbf{u}|=1} \mathbf{u}\mathbf{A}\mathbf{u} > 0.$$

Applying this property and the Cauchy–Bunyakovskii inequality for an scalar product $(\mathbf{u}|\mathbf{y}) \leq |\mathbf{u}||\mathbf{y}|$, we can estimate $f(\mathbf{u}^n)$:

$$f(\mathbf{u}^n) \geq \frac{a|\mathbf{u}^n|^2}{2} - |\mathbf{u}^n||\mathbf{y}| \rightarrow +\infty.$$

This estimate is in contradiction with the condition $f(\mathbf{u}^n) \leq f(\mathbf{u}^0)$.

Now the existence of a minimum point (a solution of the variational inequality) follows from the Weierstrass theorem which says that a continuous function on a compact set reaches its lower bound, [15]. The uniqueness follows from the statement proved above. $\square$

According to this theorem, in the case of a convex and closed set $\mathbf{F}$ for any vector $\tilde{\mathbf{u}} \in \mathbb{R}^m$ there exists a unique solution of the problem

$$(\tilde{\mathbf{u}} - \mathbf{u})(\mathbf{u} - \tilde{\mathbf{u}}) \geq 0, \quad \mathbf{u}, \tilde{\mathbf{u}} \in \mathbf{F}, \quad (3.10)$$

which is a minimum point for the quadratic function $f(\mathbf{u}) = \mathbf{u}^2/2 - \mathbf{u}\bar{\mathbf{u}}$, as well as for the distance between points $\mathbf{u}$ and $\bar{\mathbf{u}}$ being equal to $|\mathbf{u} - \bar{\mathbf{u}}| = \sqrt{2f(\mathbf{u}) + \bar{\mathbf{u}}^2}$.

A mapping $\pi : \mathbb{R}^m \rightarrow \mathbf{F}$ which associates with a vector $\bar{\mathbf{u}}$ the vector $\mathbf{u} = \pi(\bar{\mathbf{u}}) \in \mathbf{F}$ defines a projection operator onto the set $\mathbf{F}$. We establish an important property of a projection operator onto a convex set, namely, it is a non-expanding mapping, i.e. the distance between projections of two vectors does not exceed the distance between the vectors.

Let $\bar{\mathbf{u}}$ and $\bar{\mathbf{u}}'$ be arbitrary vectors. Taking $\tilde{\mathbf{u}} = \pi(\bar{\mathbf{u}}') \in \mathbf{F}$ in the inequality (3.10) and $\tilde{\mathbf{u}} = \pi(\bar{\mathbf{u}}) \in \mathbf{F}$ in the similar inequality for $\mathbf{u}' = \pi(\bar{\mathbf{u}}')$ we obtain

$$(\mathbf{u}' - \mathbf{u})(\mathbf{u} - \bar{\mathbf{u}}) \geq 0 \quad \text{and} \quad (\mathbf{u} - \mathbf{u}')(\mathbf{u}' - \bar{\mathbf{u}}') \geq 0.$$

Summing up and rearranging the terms, we get

$$|\mathbf{u}' - \mathbf{u}|^2 \leq (\mathbf{u}' - \mathbf{u})(\bar{\mathbf{u}}' - \bar{\mathbf{u}}).$$

By the Cauchy–Bunyakovskii inequality: $(\mathbf{u}' - \mathbf{u})(\bar{\mathbf{u}}' - \bar{\mathbf{u}}) \leq |\mathbf{u}' - \mathbf{u}| |\bar{\mathbf{u}}' - \bar{\mathbf{u}}|$. Hence,

$$|\mathbf{u}' - \mathbf{u}| \leq |\bar{\mathbf{u}}' - \bar{\mathbf{u}}|.$$

The projection operator with respect to the norm $|\mathbf{u}|_{\mathbf{A}} = \sqrt{\mathbf{u}\mathbf{A}\mathbf{u}}$ associated with a symmetric positive definite matrix $\mathbf{A}$ is a generalization of $\pi(\bar{\mathbf{u}})$. This operator establishes a correspondence between a vector $\bar{\mathbf{u}} \in \mathbb{R}^m$ and the point $\mathbf{u} = \pi_{\mathbf{A}}(\bar{\mathbf{u}})$ of the set $\mathbf{F}$ which is nearest with respect to this norm:

$$|\mathbf{u} - \bar{\mathbf{u}}|_{\mathbf{A}} \leq |\tilde{\mathbf{u}} - \bar{\mathbf{u}}|_{\mathbf{A}}, \quad \tilde{\mathbf{u}} \in \mathbf{F},$$

and which is also a minimum point on $\mathbf{F}$ for the quadratic function

$$f(\mathbf{u}) = \frac{1}{2} \mathbf{u}\mathbf{A}\mathbf{u} - \mathbf{u}\mathbf{A}\bar{\mathbf{u}} = \frac{1}{2} (|\mathbf{u} - \bar{\mathbf{u}}|_{\mathbf{A}}^2 - |\bar{\mathbf{u}}|_{\mathbf{A}}^2)$$

and a solution of the variational inequality

$$(\tilde{\mathbf{u}} - \mathbf{u})\mathbf{A}(\mathbf{u} - \bar{\mathbf{u}}) \geq 0, \quad \mathbf{u}, \tilde{\mathbf{u}} \in \mathbf{F}. \quad (3.11)$$

The associated norm is a Euclidean one. The scalar product for it is defined by the bilinear form $\mathbf{u}\mathbf{A}\tilde{\mathbf{u}}$ which satisfies the Cauchy–Bunyakovskii inequality:

$$\mathbf{u}\mathbf{A}\tilde{\mathbf{u}} \leq |\mathbf{u}|_{\mathbf{A}} |\tilde{\mathbf{u}}|_{\mathbf{A}}.$$

Therefore, repeating the proof of the statement that $\pi(\bar{\mathbf{u}})$ is a non-expanding mapping, we can establish a similar property for the operator $\pi_{\mathbf{A}}(\bar{\mathbf{u}})$:

$$|\pi_{\mathbf{A}}(\bar{\mathbf{u}}') - \pi_{\mathbf{A}}(\bar{\mathbf{u}})|_{\mathbf{A}} \leq |\bar{\mathbf{u}}' - \bar{\mathbf{u}}|_{\mathbf{A}}.$$

Consider the function $f_\pi(\bar{\mathbf{u}}) = |\bar{\mathbf{u}} - \bar{\mathbf{u}}^\pi|_A^2$ where for brevity we use the notation $\bar{\mathbf{u}}^\pi = \pi_A(\bar{\mathbf{u}})$. We prove that in the general case this function is convex and differentiable and its derivative is calculated by the formula

$$\frac{\partial f_\pi(\bar{\mathbf{u}})}{\partial \bar{\mathbf{u}}} = 2 \mathbf{A}(\bar{\mathbf{u}} - \bar{\mathbf{u}}^\pi). \quad (3.12)$$

To this end we form the identity

$$f_\pi(\tilde{\mathbf{u}}) - f_\pi(\bar{\mathbf{u}}) = 2(\tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}})\mathbf{A}(\tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}}^\pi - \tilde{\mathbf{u}} + \bar{\mathbf{u}}) + |\tilde{\mathbf{u}}^\pi - \tilde{\mathbf{u}} - \bar{\mathbf{u}}^\pi + \bar{\mathbf{u}}|_A^2,$$

which can be easily verified. The second term in the right-hand side of this identity is estimated by the triangle inequality

$$|\tilde{\mathbf{u}}^\pi - \tilde{\mathbf{u}} - \bar{\mathbf{u}}^\pi + \bar{\mathbf{u}}|_A \leq |\tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}}^\pi|_A + |\tilde{\mathbf{u}} - \bar{\mathbf{u}}|_A \leq 2|\tilde{\mathbf{u}} - \bar{\mathbf{u}}|_A$$

as a quantity of order $|\tilde{\mathbf{u}} - \bar{\mathbf{u}}|_A^2$. The first term is equal to

$$2(\tilde{\mathbf{u}} - \bar{\mathbf{u}})\mathbf{A}(\bar{\mathbf{u}} - \bar{\mathbf{u}}^\pi) + 2(\tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}}^\pi)\mathbf{A}(\bar{\mathbf{u}}^\pi - \bar{\mathbf{u}}).$$

The last term of this expression can be estimated in the following way. On the one hand, because of the inequality (3.11) it is nonnegative. On the other hand, it is equal to the sum

$$2(\tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}}^\pi)\mathbf{A}(\tilde{\mathbf{u}}^\pi - \tilde{\mathbf{u}}) + 2(\tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}}^\pi)\mathbf{A}(\bar{\mathbf{u}}^\pi - \tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}} + \tilde{\mathbf{u}}),$$

where the first term is less than or equal to zero since $\tilde{\mathbf{u}}^\pi$ is a projection of the vector $\tilde{\mathbf{u}}$ and the second term does not exceed the quantity

$$2|\tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}}^\pi|_A \left(|\tilde{\mathbf{u}}^\pi - \bar{\mathbf{u}}^\pi|_A + |\tilde{\mathbf{u}} - \bar{\mathbf{u}}|_A \right) \leq 4|\tilde{\mathbf{u}} - \bar{\mathbf{u}}|_A^2.$$

Taking into account the above estimates, we can obtain the following equality:

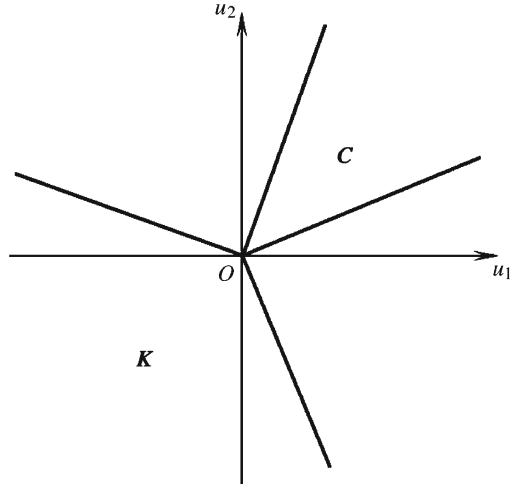
$$f_\pi(\tilde{\mathbf{u}}) - f_\pi(\bar{\mathbf{u}}) = 2(\tilde{\mathbf{u}} - \bar{\mathbf{u}})\mathbf{A}(\bar{\mathbf{u}} - \bar{\mathbf{u}}^\pi) + O(|\tilde{\mathbf{u}} - \bar{\mathbf{u}}|_A^2),$$

which proves the formula (3.12). Convexity of the function $f_\pi(\bar{\mathbf{u}})$ follows from the inequality

$$f_\pi(\tilde{\mathbf{u}}) - f_\pi(\bar{\mathbf{u}}) \geq (\tilde{\mathbf{u}} - \bar{\mathbf{u}}) \frac{\partial f_\pi(\bar{\mathbf{u}})}{\partial \bar{\mathbf{u}}},$$

which is actually established in the previous proof.

The projection operator has some special properties if the set of admissible variations is a convex and closed cone $\mathbf{K}$ with the vertex at zero, i.e. if along with any its element $\mathbf{u} \in \mathbf{K}$ a ray $\lambda \mathbf{u}$ ($\lambda \geq 0$) which issues out of the origin also belongs to $\mathbf{K}$.

Fig. 3.2 Dual cones

In the case of a cone the variational inequality (3.11), which can be thought of as defining a projection, is equivalent to the system

$$u A(u - \bar{u}) = 0, \quad \bar{u} A(u - \bar{u}) \geq 0, \quad u, \bar{u} \in K. \quad (3.13)$$

The equation involved in this system can be obtained from (3.11) if we first put $\bar{u} = 0 \in K$ and then put $\bar{u} = 2u \in K$. The inequality in (3.13) follows from (3.11) taking into account this equation. Conversely, the inequality (3.11) obviously follows from the system (3.13).

Consider a cone dual to K , i.e. a set of the form

$$C = \{ y \in \mathbb{R}^m \mid u y \leq 0 \quad \forall u \in K \}$$

consisting of vectors which form obtuse angles with vectors of K (Fig. 3.2). The fact that it is a cone with the vertex at zero is verified immediately by the definition: $\lambda y \in C$ for any nonnegative λ provided that $y \in C$. Convexity has place as well since if $u y \leq 0$ and $u \tilde{y} \leq 0$ then $u y^\lambda \leq 0$ for any $\lambda \in [0, 1]$. The cone

$$K = \{ u \in \mathbb{R}^m \mid u y \leq 0 \quad \forall y \in C \}$$

turns out to be dual to the dual cone C . In the general case this property will be established in Sect. 3.3 as a simple consequence of involution of the the Young–Fenchel transform. Here, assuming it to be valid, we prove that an arbitrary vector $\bar{u}$ of the space $\mathbb{R}^m$ can be uniquely represented as a sum of projections onto mutually dual cones:

$$\bar{u} = \pi_A(\bar{u}) + A^{-1} \Pi_{A^{-1}}(A \bar{u}), \quad (3.14)$$

where $\Pi_{A^{-1}}(\mathbf{y})$ is a projection operator onto the cone $\mathbf{C}$ with respect to the norm $|\mathbf{y}|_{A^{-1}} = \sqrt{\mathbf{y}^T \mathbf{A}^{-1} \mathbf{y}}$. Notice that in the special case that $\mathbf{K}$ is a subspace and, thus, $\mathbf{C}$ is its orthogonal complement the formula (3.14) gives a decomposition of $\mathbb{R}^m$ into a direct sum of orthogonal subspaces.

We introduce the notation $\mathbf{y} = \mathbf{A}(\bar{\mathbf{u}} - \pi_{\mathbf{A}}(\bar{\mathbf{u}}))$. Then in view of the system (3.13) we have

$$\pi_{\mathbf{A}}(\bar{\mathbf{u}}) \mathbf{y} = 0, \quad \tilde{\mathbf{u}}^T \mathbf{y} \leq 0 \quad \forall \tilde{\mathbf{u}} \in \mathbf{K}.$$

By the definition of a dual cone it follows that the vector $\mathbf{y}$ belongs to $\mathbf{C}$ and since $\pi_{\mathbf{A}}(\bar{\mathbf{u}}) = \bar{\mathbf{u}} - \mathbf{A}^{-1} \mathbf{y}$ we have

$$\mathbf{y}^T (\bar{\mathbf{u}} - \mathbf{A}^{-1} \mathbf{y}) = \mathbf{y}^T \mathbf{A}^{-1} (\mathbf{A} \bar{\mathbf{u}} - \mathbf{y}) = 0.$$

In addition, $\pi_{\mathbf{A}}(\bar{\mathbf{u}}) \in \mathbf{K}$ and $\mathbf{K}$ is a cone dual to $\mathbf{C}$, hence,

$$\tilde{\mathbf{y}}^T \mathbf{A}^{-1} (\mathbf{A} \bar{\mathbf{u}} - \mathbf{y}) \leq 0 \quad \forall \tilde{\mathbf{y}} \in \mathbf{C}.$$

The equation obtained above and the last inequality taken together form the system (3.13) up to replacing in it the vector $\mathbf{u}$ by $\mathbf{y}$, the vector $\bar{\mathbf{u}}$ by $\mathbf{A} \bar{\mathbf{u}}$, and the matrix $\mathbf{A}$ by the inverse matrix. Hence, by the definition of the projection $\mathbf{y} = \Pi_{A^{-1}}(\mathbf{A} \bar{\mathbf{u}})$ or, considering the notation for $\mathbf{y}$,

$$\mathbf{A}(\bar{\mathbf{u}} - \pi_{\mathbf{A}}(\bar{\mathbf{u}})) = \Pi_{A^{-1}}(\mathbf{A} \bar{\mathbf{u}}).$$

Thus, the equality (3.14) is valid.

Notice that in the case of a projection onto a cone the function

$$f_{\pi}(\bar{\mathbf{u}}) = |\bar{\mathbf{u}} - \pi_{\mathbf{A}}(\bar{\mathbf{u}})|_{\mathbf{A}}^2,$$

whose differentiability is established above, is transformed to the form

$$f_{\pi}(\bar{\mathbf{u}}) = (\bar{\mathbf{u}} + \pi_{\mathbf{A}}(\bar{\mathbf{u}}))^T \mathbf{A}(\bar{\mathbf{u}} - \pi_{\mathbf{A}}(\bar{\mathbf{u}})) - 2 \pi_{\mathbf{A}}(\bar{\mathbf{u}})^T \mathbf{A}(\bar{\mathbf{u}} - \pi_{\mathbf{A}}(\bar{\mathbf{u}})) = |\bar{\mathbf{u}}|_{\mathbf{A}}^2 - |\pi_{\mathbf{A}}(\bar{\mathbf{u}})|_{\mathbf{A}}^2$$

and, hence, by the formula (3.12) for the derivative

$$\frac{\partial |\pi_{\mathbf{A}}(\bar{\mathbf{u}})|_{\mathbf{A}}^2}{\partial \bar{\mathbf{u}}} = 2 \mathbf{A} \pi_{\mathbf{A}}(\bar{\mathbf{u}}). \quad (3.15)$$

Taking into account (3.14), it follows that

$$|\bar{\mathbf{u}}|_{\mathbf{A}}^2 = |\pi_{\mathbf{A}}(\bar{\mathbf{u}})|_{\mathbf{A}}^2 + |\Pi_{A^{-1}}(\mathbf{A} \bar{\mathbf{u}})|_{A^{-1}}^2. \quad (3.16)$$

Further we study a more general problem which is in the solution of the variational inequality

$$(\tilde{u} - u)(A(u) - y) \geq 0, \quad u, \tilde{u} \in F, \quad (3.17)$$

for a nonlinear mapping $A : F \rightarrow \mathbb{R}^m$. If, in particular, this mapping is linear and the corresponding matrix is symmetric, then the problem (3.17) is reduced to (3.9). However, further no symmetry is assumed.

For the problem (3.17) to have a unique solution, the mapping is required to be strongly monotone in the following sense:

$$(u' - u)(A(u') - A(u)) \geq a|u' - u|^2, \quad a > 0,$$

for any u and $u' \in F$.

It is obvious that the notion of strong monotonicity generalizes the condition of positive definiteness of a matrix since, applying it to a linear mapping $A(u) = Au$, we can establish that $uAu \geq a|u|^2$. To verify positive definiteness of a nonsymmetric matrix, one should decompose A into a sum of symmetric and antisymmetric parts:

$$A^s = \frac{1}{2}(A + A^*), \quad A^a = \frac{1}{2}(A - A^*).$$

For an antisymmetric matrix the quadratic form identically equals zero. The quadratic forms for A and A^s coincide, hence, the problem on positive definiteness of the matrix A is solved with the help of the Sylvester criterion applied to the matrix A^s .

Assume that the mapping A satisfies the Lipschitz condition: for any vectors u and $u' \in F$ the inequality $|A(u') - A(u)| \leq L|u' - u|$ holds with a constant L which does not depend on u and u' .

It is known that any linear mapping $A(u) = Au$ in a finite-dimensional space satisfies the Lipschitz condition with $L = |A|$. By the definition of a matrix norm

$$|A| = \max_{|\tilde{u}|=1} |A\tilde{u}| = \max_{\tilde{u} \neq 0} \frac{|A\tilde{u}|}{|\tilde{u}|} \geq \frac{|A(u' - u)|}{|u' - u|}, \quad u' \neq u,$$

hence $|Au' - Au| \leq |A||u' - u|$.

Theorem 3.2 *If a set $F \subset \mathbb{R}^m$ is convex and closed and a mapping $A(u)$ is strongly monotone and satisfies the Lipschitz condition, then the variational inequality (3.17) has a unique solution for any vector $y \in \mathbb{R}^m$.*

To prove the theorem, we rewrite (3.17) in an equivalent form

$$(\tilde{u} - u)(u - (u - \tau A(u) + \tau y)) \geq 0, \quad u, \tilde{u} \in F,$$

assuming that τ is a positive number. By the definition of a projection operator this means that

$$u = Q(u) \equiv \pi(u - \tau A(u) + \tau y).$$

Thus, the problem is reduced to looking for a fixed point of the mapping $Q(u)$. We show that for some $\tau > 0$ this mapping is contractive, i.e. it satisfies the Lipschitz condition with a constant $q < 1$. Then the statement about existence and uniqueness of a solution of the variational inequality follows from the principle of contractive mappings (the Banach principle) considering that the remaining conditions in the formulation of this principle are easily verified, namely, a set F is a complete metric space as a closed subset of $\mathbb{R}^m$, the mapping $Q(u)$ maps F into F .

Let $u, u' \in F$. Since a projection operator is a non-expanding mapping, the following chain of relationships is valid:

$$\begin{aligned} |Q(u') - Q(u)|^2 &\leq |u' - \tau A(u') - u + \tau A(u)|^2 = |u' - u|^2 \\ &- 2\tau(u' - u)(A(u') - A(u)) + \tau^2|A(u') - A(u)|^2 \leq (1 - 2a\tau + L^2\tau^2)|u' - u|^2, \end{aligned}$$

where the last inequality is obtained with the use of the Lipschitz condition and strong monotonicity of $A(u)$.

The expression $q^2(\tau) = 1 - 2a\tau + L^2\tau^2$ is a quadratic trinomial relative to τ , which decreases in a neighbourhood of the point $\tau = 0$:

$$q^2(0) = 1, \quad \frac{dq^2(0)}{d\tau} = -2a < 0.$$

Therefore in the interval $0 < \tau < 2a/L^2$ the inequality $q < 1$ holds. For any value of τ from this interval the mapping $Q(u)$ is contraction. The theorem is completely proved. $\square$

The problem on solvability for a more general case where a mapping $A(u)$ does not satisfy the monotonicity condition remains unsolved. We show that a solution of the variational inequality (3.17) exists provided that the mapping is continuous and a convex and closed set F is bounded. To this end, we use the Brouwer theorem about the existence of a fixed point whose complete proof can be found, for example, in [12].

Theorem 3.3 (Brouwer) *If a mapping $Q(u)$ continuously maps a closed ball $S_r = \{u \in \mathbb{R}^m \mid |u| \leq r\}$ of radius r into itself, then there exists a fixed point for it.*

This theorem remains valid for a mapping $Q(u)$ that maps a bounded closed convex set F into itself. Indeed, if F is bounded then it can be completely inscribed into a closed ball of some radius. A mapping $Q'(u) = Q(\pi(u))$, where π is an operator of projection on F , is continuous as a superposition of continuous and non-expanding (and, hence, continuous) mappings and maps S_r into $F \subset S_r$, therefore, by the Brouwer theorem, there exists a point u such that $u = Q'(u)$. Moreover, $u = Q'(u) \in F$, hence, $Q'(u) = Q(u)$. Existence of a fixed point is proved.

Repeating the stage of going from the variational inequality (3.17) to the problem of determining a fixed point (the value of the parameter $\tau > 0$ can be taken arbitrary), on the basis of the Brouwer theorem we obtain a proof of the following statement.

Let a set F be convex and closed in $\mathbb{R}^m$ and a mapping $A(u)$ be continuous. Then for any $r > 0$ there exists a solution of the auxiliary problem

$$(\tilde{u}_r - u_r)(A(u_r) - y) \geq 0, \quad u_r, \tilde{u}_r \in F_r,$$

obtained from (3.17) by replacement of the set F by $F_r = F \cap S_r$.

The following two theorems in infinite-dimensional variant were formulated and proved in [14].

Theorem 3.4 *Under the assumptions of the above statement, the problem (3.17) has a solution if and only if for some r there exists a solution u_r satisfying the condition $|u_r| < r$.*

Proof To prove the necessary condition, we assume that a solution u of the problem exists under the constraint $\tilde{u} \in F$. Because of $F_r \subset F$, for any $r > |u|$ it can be taken as a solution u_r under the constraint $\tilde{u}_r \in F_r$.

To prove the sufficient condition, let us assume that for some r the inequality $|u_r| < r$ holds. Since this inequality is strict, for any vector $\tilde{u} \in F$ there exists some $\varepsilon > 0$ such that $\tilde{u}_r = u_r + \varepsilon(\tilde{u} - u_r) \in F_r$. Then from the variational inequality of the auxiliary problem, on dividing by ε , we obtain that u_r satisfies (3.17) as well. $\square$

On the basis of the theorem proved we can formulate sufficient criteria for existence of a solution. One of these criteria is related to the notion of a *coercivity* of mapping, [14]. A mapping $A : F \rightarrow \mathbb{R}^m$ is said to be *coercive* if for some $\bar{u} \in F$

$$\frac{(u - \bar{u})(A(u) - A(\bar{u}))}{|u - \bar{u}|} \rightarrow +\infty \quad \text{for } |u| \rightarrow \infty, \quad u \in F.$$

It is easy to show that any strongly monotone mapping is coercive. However, the class of coercive mappings is more wide, it involves, for example, all mappings in the case of a bounded set F .

Theorem 3.5 *If a set $F \subset \mathbb{R}^m$ is convex and closed and a mapping $A(u)$ is continuous and coercive on F , then there exists at least one solution of the variational inequality (3.17).*

Proof By the definition of a coercivity, for any $a > |A(\bar{u}) - y|$ there exists $r > |\bar{u}|$ such that

$$\frac{(u - \bar{u})(A(u) - A(\bar{u}))}{|u - \bar{u}|} \geq a$$

for $|u| \geq r$, $u \in F$. This leads to the chain of inequalities

$$\begin{aligned} (u - \bar{u})(A(u) - y) &\geq a|u - \bar{u}| + (u - \bar{u})(A(\bar{u}) - y) \\ &\geq a|u - \bar{u}| - |u - \bar{u}||A(\bar{u}) - y| \geq (|u| - |\bar{u}|)(a - |A(\bar{u}) - y|) \geq 0, \end{aligned}$$

obtained by rearrangement of expressions with the use of the Cauchy–Bunyakovskii inequality and the triangle inequality in the form $|\mathbf{u} - \bar{\mathbf{u}}| \geq |\mathbf{u}| - |\bar{\mathbf{u}}|$.

The final inequality is valid for any $\mathbf{u} \in \mathbf{F}$, for which $|\mathbf{u}| \geq r$. A solution $\mathbf{u}_r \in \mathbf{F}_r$ of the auxiliary problem satisfies the opposite inequality

$$(\mathbf{u}_r - \bar{\mathbf{u}})(\mathbf{A}(\mathbf{u}_r) - \mathbf{y}) = -(\bar{\mathbf{u}} - \mathbf{u}_r)(\mathbf{A}(\mathbf{u}_r) - \mathbf{y}) \leq 0.$$

Thus, $|\mathbf{u}_r| < r$ and, hence, $\mathbf{u}_r$ is a solution of the problem (3.17). The theorem is proved. $\square$

To conclude, we notice that, generally speaking, this problem does not provide uniqueness of a solution of the variational inequality. As a sufficient condition for uniqueness one may use the condition of strict monotonicity of a mapping:

$$(\mathbf{u}' - \mathbf{u})(\mathbf{A}(\mathbf{u}') - \mathbf{A}(\mathbf{u})) > 0, \quad \text{if } \mathbf{u}' \neq \mathbf{u}.$$

A proof of uniqueness of a solution in this case exactly repeats that of the similar statement about uniqueness of a solution of the variational inequality with a positive definite matrix.

3.3 Subdifferential Calculus

In this section, the finite-dimensional version of the separability theorem, which is of first importance in convex analysis, is essentially used. This theorem has an illustrative geometric interpretation: if some point of a finite-dimensional space (this point can be assumed to be $\mathbf{0}$) does not belong to a convex set, then there exists a hyperplane separating it from this set.

Theorem 3.6 *Let $\mathbf{F} \neq \emptyset$ be a convex set in $\mathbb{R}^m$ and $\mathbf{0} \notin \mathbf{F}$. Then there exists a vector $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{u} \mathbf{y} \geq 0$ for any $\mathbf{u} \in \mathbf{F}$. If a set $\mathbf{F}$ is closed then $\mathbf{u} \mathbf{y} > 0$.*

Proof First we assume that $\mathbf{0}$ does not belong to $\bar{\mathbf{F}}$ being the closure of set $\mathbf{F}$. In this case, assuming that $\mathbf{y}$ is equal to the projection of zero onto $\bar{\mathbf{F}}$ and using the definition of a projection, we have

$$(\bar{\mathbf{u}} - \mathbf{y}) \mathbf{y} \geq 0, \quad \bar{\mathbf{u}} \in \bar{\mathbf{F}},$$

which results in the required inequality.

Now we assume that $\mathbf{0}$ is a boundary point of $\mathbf{F}$. In this case, there exists a sequence $\mathbf{u}^n \notin \bar{\mathbf{F}}$ that converges to $\mathbf{0}$. Consider a sequence of unit vectors (π is a projection operator onto $\bar{\mathbf{F}}$)

$$\mathbf{y}^n = \frac{\pi(\mathbf{u}^n) - \mathbf{u}^n}{|\pi(\mathbf{u}^n) - \mathbf{u}^n|}, \quad |\mathbf{y}^n| = 1.$$

Each of these vectors satisfies the inequality

$$(\tilde{u} - \pi(u^n))y^n = \frac{(\tilde{u} - \pi(u^n))(\pi(u^n) - u^n)}{|\pi(u^n) - u^n|} \geq 0 \quad \forall \tilde{u} \in \bar{F}.$$

Since a unit sphere in $\mathbb{R}^m$ is compact, we can select a convergent subsequence of y^n . For simplicity of notations, we assume that the sequence y^n itself is convergent and its limit is a vector y . Passing to the limit as $n \rightarrow \infty$ in the last inequality and taking into account that the sequence $\pi(u^n)$ of points converges to zero, we arrive at the statement of the theorem. $\square$

Corollary 3.1 *Any two nonempty and nonintersecting convex sets F and G can be separated by a hyperplane in the following sense: there exist a vector $y \in \mathbb{R}^m$ and a number d such that $u y \geq d$ for any $u \in F$ and $u' y \leq d$ for any $u' \in G$.*

Proof To prove the corollary, we consider a set being the difference of F and G :

$$F - G = \{u - u' \mid u \in F, u' \in G\}.$$

We can show that this set is convex. Indeed, if $\tilde{u} - \tilde{u}', u - u' \in F - G$ then because of convexity of F and G and the obvious equality $(u - u')^\lambda = u^\lambda - u'^\lambda$ we have the inclusion $(u - u')^\lambda \in F - G$. Besides, $0 \notin F - G$ since the condition $0 \in F - G$ means that $0 = u - u'$ ($u \in F, u' \in G$), i.e. that $u = u' \in F \cap G$. This is in contradiction with the condition that the sets do not intersect.

By the separability theorem, there exists a vector y such that $(u - u') y \geq 0$ for any $u \in F, u' \in G$. Hence,

$$u y \geq \inf_{\tilde{u} \in F} \{\tilde{u} y\} \geq \sup_{\tilde{u}' \in G} \{\tilde{u}' y\} \geq u' y.$$

Taking d as any number from the segment whose left endpoint is infimum and right endpoint is supremum, from the chain of inequalities we get

$$u y \geq d, \quad u' y \leq d.$$

$\square$

Notice that this corollary provides a more general statement than the separability theorem. This theorem follows from this statement provided that a set G consists of a single point 0 .

Consider a common property of convex sets which will be used further in the finite-dimensional version as well as in the infinite-dimensional one.

Proposition 3.3 *The interior $\text{int } F$ of a convex set F is a convex set. If it is not empty, then its closure coincides with that of F : $\overline{\text{int } F} = \bar{F}$.*

Proof First we prove the following auxiliary statement. Assume that $\mathbf{u} \in \text{int } \mathbf{F}$, $\tilde{\mathbf{u}} \in \mathbf{F}$ and $0 < \lambda \leq 1$, then $\mathbf{u}^\lambda \in \text{int } \mathbf{F}$. Indeed, along with the point $\mathbf{u}$, some its ε -neighbourhood belongs to the set $\mathbf{F}$. If $|\mathbf{u}' - \mathbf{u}^\lambda| \leq \lambda \varepsilon$ then it is obvious that the point $\mathbf{u} + (\mathbf{u}' - \mathbf{u}^\lambda)/\lambda$ belongs to this neighbourhood. Hence,

$$\mathbf{u}' \equiv \lambda \left(\mathbf{u} + \frac{\mathbf{u}' - \mathbf{u}^\lambda}{\lambda} \right) + (1 - \lambda) \tilde{\mathbf{u}} \in \mathbf{F}.$$

It follows from this that the $\lambda \varepsilon$ -neighbourhood of the point $\mathbf{u}^\lambda$ entirely belongs to $\mathbf{F}$, i.e. $\mathbf{u}^\lambda \in \text{int } \mathbf{F}$.

Convexity of $\text{int } \mathbf{F}$ is proved in a similar way. Any two points $\mathbf{u}$ and $\tilde{\mathbf{u}} \in \text{int } \mathbf{F}$ belong to the set $\mathbf{F}$ together with some ε -neighbourhoods. Therefore, if $|\mathbf{u}' - \mathbf{u}^\lambda| \leq \varepsilon$ then the point

$$\mathbf{u}' \equiv \lambda (\mathbf{u} + \mathbf{u}' - \mathbf{u}^\lambda) + (1 - \lambda) (\tilde{\mathbf{u}} + \mathbf{u}' - \mathbf{u}^\lambda)$$

belongs to $\mathbf{F}$ since it is a convex combination of the points $\mathbf{u} + \mathbf{u}' - \mathbf{u}^\lambda$ and $\tilde{\mathbf{u}} + \mathbf{u}' - \mathbf{u}^\lambda$ of the given neighbourhoods. Thus, the ε -neighbourhood of $\mathbf{u}^\lambda$ belongs to $\mathbf{F}$, i.e. $\mathbf{u}^\lambda$ is an interior point of $\mathbf{F}$.

From the above auxiliary statement it follows that if $\text{int } \mathbf{F} \neq \emptyset$ then in any neighborhood of an arbitrary point $\tilde{\mathbf{u}} \in \mathbf{F}$ there exists a point $\mathbf{u}^\lambda \in \text{int } \mathbf{F}$ for sufficiently small $\lambda > 0$. Hence, $\mathbf{F} \subset \overline{\text{int } \mathbf{F}}$ and $\bar{\mathbf{F}} \subset \overline{\text{int } \mathbf{F}}$. The opposite inclusion is always valid since $\mathbf{F} \supset \text{int } \mathbf{F}$. $\square$

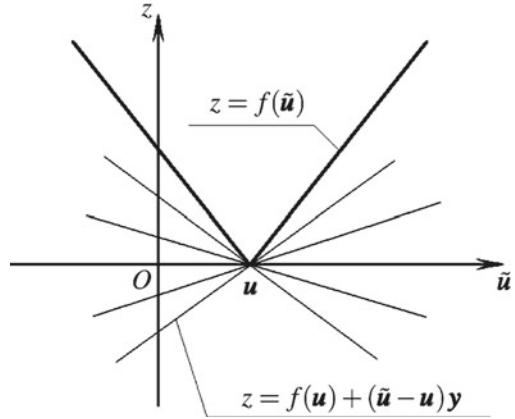
The set of vectors

$$\partial f(\mathbf{u}) = \{ \mathbf{y} \in \mathbb{R}^m \mid f(\tilde{\mathbf{u}}) - f(\mathbf{u}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \mathbf{y} \quad \forall \tilde{\mathbf{u}} \in \mathbb{R}^m \}$$

is called a *subdifferential of a convex function* $f(\mathbf{u})$ at the point $\mathbf{u}$. An element of this set is called a *subgradient*. In the geometric sense a subdifferential is a set of “angular coefficients” of linear functions $z = f(\mathbf{u}) + (\tilde{\mathbf{u}} - \mathbf{u}) \mathbf{y}$ of all kinds which take the value $f(\mathbf{u})$ for $\tilde{\mathbf{u}} = \mathbf{u}$ and whose graphs lie below the graph of $f(\tilde{\mathbf{u}})$ (Fig. 3.3). From the definition it immediately follows that a subdifferential is convex set in $\mathbb{R}^m$. Indeed, if $\mathbf{y}$ and $\tilde{\mathbf{y}} \in \partial f(\mathbf{u})$ then for any $\tilde{\mathbf{u}} \in \mathbb{R}^m$ and $\lambda \in [0, 1]$ the inequality

$$f(\tilde{\mathbf{u}}) - f(\mathbf{u}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \mathbf{y}^\lambda$$

is valid. This inequality is obtained by multiplication of the similar inequality for $\tilde{\mathbf{y}}$ by λ , the inequality for $\mathbf{y}$ by $1 - \lambda$, and by subsequent summation. Thus, the vector $\mathbf{y}^\lambda$ belongs to $\partial f(\mathbf{u})$. If a function is differentiable at a point $\mathbf{u}$ then its subdifferential at this point consists of the single element $\mathbf{y} = \partial f(\mathbf{u})/\partial \mathbf{u}$. The fact that the vector $\mathbf{y}$ belongs to the subdifferential follows from the first criterion of convexity of a function. Uniqueness is proved by contradiction. Assume, that there exists another vector $\mathbf{y}' \in \partial f(\mathbf{u})$ such that $\mathbf{y}' \neq \mathbf{y}$, then

Fig. 3.3 Subdifferential

$$\begin{aligned}
 -\mathbf{v} \mathbf{y} &\leq f(\mathbf{u} - \mathbf{v}) - f(\mathbf{u}) = -\mathbf{v} \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} + o(|\mathbf{v}|), \\
 \mathbf{v} \mathbf{y}' &\leq f(\mathbf{u} + \mathbf{v}) - f(\mathbf{u}) = \mathbf{v} \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} + o(|\mathbf{v}|).
 \end{aligned}$$

Summation of these inequalities yields

$$\mathbf{v}(\mathbf{y}' - \mathbf{y}) \leq o(|\mathbf{v}|) \quad \forall \mathbf{v} \in \mathbb{R}^m.$$

Substituting $\mathbf{v} = \lambda \mathbf{v}'$ ($\lambda > 0$), dividing by λ , and passing to the limit as $\lambda \rightarrow 0$, we obtain from this inequality that $\mathbf{v}'(\mathbf{y}' - \mathbf{y}) \leq 0$ for any vector $\mathbf{v}' \in \mathbb{R}^m$, which is possible if and only if $\mathbf{y}' - \mathbf{y} = \mathbf{0}$.

As an useful example we consider the function $\omega(\mathbf{u}) = |\mathbf{u}^0|$, where $\mathbf{u}^0$ is the projection of vector $\mathbf{u}$ onto the subspace $\mathbf{U} \subset \mathbb{R}^m$, and show that

$$\partial\omega(\mathbf{0}) = \{ \mathbf{y}^0 \in \mathbf{U} \mid |\mathbf{y}^0| \leq 1 \}. \quad (3.18)$$

Convexity of the function $\omega(\mathbf{u})$ follows from the norm axioms. We denote the orthogonal complement to $\mathbf{U}$ via $\mathbf{U}^\perp$. It is evident that $\mathbf{u}' = \mathbf{u} - \mathbf{u}^0 \in \mathbf{U}^\perp$. Let $\mathbf{y}^0$ be an arbitrary element of the set in the right-hand side of (3.18). By the definition of a subdifferential

$$\partial\omega(\mathbf{0}) = \{ \mathbf{y} \in \mathbb{R}^m \mid \mathbf{u} \mathbf{y} \leq \omega(\mathbf{u}) \quad \forall \mathbf{u} \in \mathbb{R}^m \}.$$

According to the Cauchy–Bunyakovskii inequality for any $\mathbf{u} \in \mathbb{R}^m$

$$\mathbf{u} \mathbf{y}^0 = (\mathbf{u}^0 + \mathbf{u}') \mathbf{y}^0 = \mathbf{u}^0 \mathbf{y}^0 \leq |\mathbf{u}^0| |\mathbf{y}^0| \leq \omega(\mathbf{u}),$$

i.e. $\partial\omega(\mathbf{0})$ contains the set indicated in (3.18). To complete a proof of the formula (3.18) we assume that the vector $\mathbf{y}$ exists which belongs to $\partial\omega(\mathbf{0})$ and not belongs

to the right-hand side of (3.18). Then either $\mathbf{y} \notin U$ or $\mathbf{y} \in U$ but $|\mathbf{y}| > 1$. In the first case $\mathbf{y} = \mathbf{y}^0 + \mathbf{y}'$ where $\mathbf{y}' \in U^\perp$ and $\mathbf{y}' \neq \mathbf{0}$. Hence, for $\mathbf{u} = \mathbf{y}'$

$$\mathbf{u} \mathbf{y} = \mathbf{u} (\mathbf{y}^0 + \mathbf{y}') = \mathbf{u} \mathbf{y}' = |\mathbf{y}'|^2 > 0 = \omega(\mathbf{u}),$$

which is in contradiction with the definition of a subdifferential. In the second case for $\mathbf{u} = \mathbf{y} \in U$

$$\mathbf{u} \mathbf{y} = |\mathbf{u}^0| |\mathbf{y}| > |\mathbf{u}^0| = \omega(\mathbf{u}),$$

which is in contradiction with this definition as well.

The subdifferential of a sum of two functions is calculated by the standard rule (see, for example, [1]).

Theorem 3.7 (Moreau–Rockafellar) *Assume that $f(\mathbf{u})$ and $g(\mathbf{u})$ are convex functions, besides, one function is continuous at some point $\bar{\mathbf{u}} \in \mathbb{R}^m$ and another one takes a finite value at this point. Then*

$$\partial(f + g)(\mathbf{u}) = \partial f(\mathbf{u}) + \partial g(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^m.$$

Proof Let $\mathbf{y} \in \partial f(\mathbf{u})$, $\mathbf{y}' \in \partial g(\mathbf{u})$. Then by the definition of a subdifferential for any $\tilde{\mathbf{u}} \in \mathbb{R}^m$ the following two inequalities are valid:

$$f(\tilde{\mathbf{u}}) - f(\mathbf{u}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \mathbf{y}, \quad g(\tilde{\mathbf{u}}) - g(\mathbf{u}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \mathbf{y}'.$$

Summing up them, we obtain the inequality which means that $\mathbf{y} + \mathbf{y}' \in \partial(f + g)(\mathbf{u})$. Thus, the inclusion $\partial(f + g)(\mathbf{u}) \supset \partial f(\mathbf{u}) + \partial g(\mathbf{u})$ is valid. To prove the opposite inclusion, we introduce two auxiliary convex functions:

$$p(\mathbf{v}) = f(\mathbf{u} + \mathbf{v}) - f(\mathbf{u}) - \mathbf{v} \mathbf{y} \quad \text{and} \quad q(\mathbf{v}) = g(\mathbf{u} + \mathbf{v}) - g(\mathbf{u}).$$

For these functions the condition $\mathbf{0} \in \partial(p + q)(\mathbf{0})$, i.e. the inequality

$$p(\mathbf{v}) + q(\mathbf{v}) \geq p(\mathbf{0}) + q(\mathbf{0}) = 0, \quad \mathbf{v} \in \mathbb{R}^m,$$

exactly coincides with the condition $\mathbf{y} \in \partial(f + g)(\mathbf{u})$. In another form this condition can be represented as

$$\min_{\mathbf{v} \in \mathbb{R}^m} \{p(\mathbf{v}) + q(\mathbf{v})\} = 0.$$

Assuming for definiteness that at the point $\bar{\mathbf{v}} = \bar{\mathbf{u}} - \mathbf{u}$ the function $p(\mathbf{v})$ is continuous and $q(\mathbf{v})$ is finite, we construct two convex sets in $\mathbb{R}^{m+1}$:

$$\mathbf{F} = \text{int epi } p, \quad \mathbf{G} = \{(\mathbf{v}, z) \mid -z \geq q(\mathbf{v})\},$$

where int denotes the set of interior points.

The set G is non-empty since $q(\bar{\mathbf{v}}) < +\infty$ and therefore $(\bar{\mathbf{v}}, -q(\bar{\mathbf{v}})) \in G$, and the set F is open and nonempty because the function $p(\mathbf{v})$ continuous at the point $\bar{\mathbf{v}}$ is bounded in a neighborhood of this point. In addition, $F \cap G = \emptyset$. Indeed, by contradiction assume that $(\mathbf{v}, z) \in F \cap G$, then $p(\mathbf{v}) < z \leq -q(\mathbf{v})$. Then we arrive at the contradiction:

$$0 = \min_{\tilde{\mathbf{v}} \in \mathbb{R}^m} (p + q)(\tilde{\mathbf{v}}) \leq p(\mathbf{v}) + q(\mathbf{v}) < 0.$$

By the separability theorem there exists a vector $(y, a) \in \mathbb{R}^{m+1}$ for which the following inequality holds:

$$\inf_{(\mathbf{v}, z) \in F} \{\mathbf{v} y + z a\} \geq \sup_{(\mathbf{v}', z') \in G} \{\mathbf{v}' y + z' a\}.$$

It is easy to see that $a \geq 0$, otherwise supremum in the right-hand side of this inequality is equal to $+\infty$ since at least for $\mathbf{v}' = \bar{\mathbf{v}}$ we can tend z' to $-\infty$ without violating the condition $-z' \geq q(\bar{\mathbf{v}})$. Now assume that $a = 0$. From the above inequality, taking into account that a lower bound of a linear function can not be achieved at an interior point, we have the condition

$$\bar{\mathbf{v}} y > \inf_{(\mathbf{v}, z) \in F} \{\mathbf{v} y\} \geq \sup_{(\mathbf{v}', z') \in G} \{\mathbf{v}' y\},$$

which is in contradiction with the definition of supremum. Hence, $a > 0$.

The set F is convex, open, and nonempty, hence, by the statement proved at the beginning of this section its closure coincides with that of $\text{epi } p$. Therefore the obtained inequality holds not only for points of F but also for any point of epigraph of p . Putting $y' = y/a$, $z = p(\mathbf{v})$ and $z' = -q(\mathbf{v})$, we have

$$\inf_{\mathbf{v} \in \mathbb{R}^m} \{\mathbf{v} y' + p(\mathbf{v})\} \geq \sup_{\mathbf{v}' \in \mathbb{R}^m} \{\mathbf{v}' y' - q(\mathbf{v}')\}. \quad (3.19)$$

Since both functions in braces vanish for $\mathbf{v} = \mathbf{v}' = \mathbf{0}$, this gives two inequalities

$$\inf_{\mathbf{v} \in \mathbb{R}^m} \{\mathbf{v} y' + p(\mathbf{v})\} \geq 0 \quad \text{and} \quad 0 \geq \sup_{\mathbf{v}' \in \mathbb{R}^m} \{\mathbf{v}' y' - q(\mathbf{v}')\}.$$

The former inequality follows from the fact that supremum in the right-hand side of (3.19) is nonnegative and the latter one follows from the fact that infimum in the left-hand side is non-positive. These inequalities can be written in an equivalent form

$$p(\mathbf{v}) - p(\mathbf{0}) \geq -\mathbf{v} y' \quad \text{and} \quad q(\mathbf{v}') - q(\mathbf{0}) \geq \mathbf{v}' y',$$

or in the inclusion form $-y' \in \partial p(\mathbf{0})$ and $y' \in \partial q(\mathbf{0})$. Thus, $\mathbf{0} \in \partial p(\mathbf{0}) + \partial q(\mathbf{0})$, i.e. $y \in \partial f(\mathbf{u}) + \partial g(\mathbf{u})$. The theorem is completely proved. $\square$

On the basis of this theorem we can prove that if a convex function $f(\mathbf{u})$ is continuous at a point $\bar{\mathbf{u}}$ then $\partial f(\bar{\mathbf{u}}) \neq \emptyset$.

Indeed, let $\delta_{\bar{\mathbf{u}}}(\mathbf{u})$ be the indicator function of the set consisting of the single point $\bar{\mathbf{u}}$. Since both functions satisfy the conditions of the Moreau–Rockafellar theorem and

$$f(\mathbf{u}) + \delta_{\bar{\mathbf{u}}}(\mathbf{u}) = \begin{cases} f(\bar{\mathbf{u}}), & \text{if } \mathbf{u} = \bar{\mathbf{u}}, \\ +\infty, & \text{if } \mathbf{u} \neq \bar{\mathbf{u}}, \end{cases}$$

we have the equality

$$\mathbb{R}^m = \partial(f + \delta_{\bar{\mathbf{u}}})(\bar{\mathbf{u}}) = \partial f(\bar{\mathbf{u}}) + \partial \delta_{\bar{\mathbf{u}}}(\bar{\mathbf{u}}) = \partial f(\bar{\mathbf{u}}) + \mathbb{R}^m,$$

which means that $\partial f(\bar{\mathbf{u}}) \neq \emptyset$. The statement that the subdifferential of a function, taking finite value at one point only, is equal to $\mathbb{R}^m$ at this point, which is used in the last chain, is verified immediately with the help of the definition of a subdifferential.

We give one more corollary of the Moreau–Rockafellar theorem. Assume that $\mathbf{u}$ is a minimum point for the sum $f(\mathbf{u}) + j(\mathbf{u})$ of two convex functions, besides, the first function is differentiable in the domain $\text{dom } f$ and the second one takes a finite value at least at one point of this domain. According to the definition of a subdifferential, the minimum condition is equivalent to the inclusion $\mathbf{0} \in \partial(f + j)(\mathbf{u}) = \partial f(\mathbf{u}) + \partial j(\mathbf{u})$. The set $\partial f(\mathbf{u})$ consists of the single element (the gradient of the function), hence, $\mathbf{0} = \partial f(\mathbf{u})/\partial \mathbf{u} + \mathbf{y}$ where $\mathbf{y} \in \partial j(\mathbf{u})$, or

$$(\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} + j(\tilde{\mathbf{u}}) - j(\mathbf{u}) \geq 0 \quad \forall \tilde{\mathbf{u}}. \quad (3.20)$$

The reverse considerations show that any solution of (3.20) is a minimum point of the function $f(\mathbf{u}) + j(\mathbf{u})$. Thus, the obtained variational inequality is a necessary and sufficient minimum condition for a convex function involving a non-differentiable term.

In the case that $j(\mathbf{u}) = \delta_F(\mathbf{u})$ the condition (3.20) is transformed into the condition (3.8) obtained earlier. With $j(\mathbf{u}) + \delta_F(\mathbf{u})$ in place of $j(\mathbf{u})$, it is transformed into a necessary and sufficient condition for a more general problem on minimization of a convex function, involving a non-differentiable term, on a convex set F :

$$(\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial f(\mathbf{u})}{\partial \mathbf{u}} + j(\tilde{\mathbf{u}}) - j(\mathbf{u}) \geq 0, \quad \mathbf{u}, \tilde{\mathbf{u}} \in F.$$

Consider main properties of the Young–Fenchel transform, [7], which consists in calculation of a dual function by the formula

$$g(\mathbf{y}) = \sup_{\mathbf{u} \in \mathbb{R}^m} \{\mathbf{u} \mathbf{y} - f(\mathbf{u})\}.$$

It can be shown that the function $g(\mathbf{y})$ is convex. Supremum of a sum does not exceed the sum of suprema, therefore for any $\lambda \in [0, 1]$

$$\begin{aligned}
g(\mathbf{y}^\lambda) &= \sup_{\mathbf{u} \in \mathbb{R}^m} \{ \lambda (\mathbf{u} \tilde{\mathbf{y}} - f(\mathbf{u})) + (1 - \lambda)(\mathbf{u} \mathbf{y} - f(\mathbf{u})) \} \\
&\leq \lambda \sup_{\mathbf{u} \in \mathbb{R}^m} \{ \mathbf{u} \tilde{\mathbf{y}} - f(\mathbf{u}) \} + (1 - \lambda) \sup_{\mathbf{u} \in \mathbb{R}^m} \{ \mathbf{u} \mathbf{y} - f(\mathbf{u}) \} \\
&= \lambda g(\tilde{\mathbf{y}}) + (1 - \lambda) g(\mathbf{y}).
\end{aligned}$$

In the case of a differentiable function, this transformation coincides with the Legendre transform. The case that the function $f(\mathbf{u})$ is non-convex is of prime interest in applications to some problems of mechanics and physics, [2, 3]. However, further in the modeling of granular materials the Young–Fenchel transform is applied to convex potentials only, therefore the function $f(\mathbf{u})$ is assumed to be convex as well.

Theorem 3.8 *If a subdifferential $\partial f(\mathbf{u})$ of a convex function is a nonempty set (for example, in the case that a function is continuous at a point $\mathbf{u}$ or is bounded in a neighborhood of this point) and $\mathbf{y} \in \partial f(\mathbf{u})$ then $g(\mathbf{y}) + f(\mathbf{u}) = \mathbf{u} \mathbf{y}$. In addition, the Young–Fenchel transform is involutive, i.e. the transformation of the conjugate function at the point $\mathbf{u}$ turns out to be equal to $f(\mathbf{u})$.*

Proof By the definition of a subgradient

$$f(\tilde{\mathbf{u}}) - f(\mathbf{u}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \mathbf{y},$$

i.e. $\tilde{\mathbf{u}} \mathbf{y} - f(\tilde{\mathbf{u}}) \leq \mathbf{u} \mathbf{y} - f(\mathbf{u})$, hence,

$$g(\mathbf{y}) = \max_{\tilde{\mathbf{u}} \in \mathbb{R}^m} \{ \tilde{\mathbf{u}} \mathbf{y} - f(\tilde{\mathbf{u}}) \} = \mathbf{u} \mathbf{y} - f(\mathbf{u}).$$

The equality $g(\mathbf{y}) + f(\mathbf{u}) = \mathbf{u} \mathbf{y}$ is proved. To prove that the transformation is involutive, we consider the obvious inequality

$$\sup_{\tilde{\mathbf{y}} \in \mathbb{R}^m} \{ \mathbf{u} \tilde{\mathbf{y}} - g(\tilde{\mathbf{y}}) \} \geq \mathbf{u} \mathbf{y} - g(\mathbf{y}) = f(\mathbf{u}).$$

On the other hand, $g(\tilde{\mathbf{y}}) \geq \mathbf{u} \tilde{\mathbf{y}} - f(\mathbf{u})$, therefore

$$\sup_{\tilde{\mathbf{y}} \in \mathbb{R}^m} \{ \mathbf{u} \tilde{\mathbf{y}} - g(\tilde{\mathbf{y}}) \} \leq \sup_{\tilde{\mathbf{y}} \in \mathbb{R}^m} \{ \mathbf{u} \tilde{\mathbf{y}} - (\mathbf{u} \tilde{\mathbf{y}} - f(\mathbf{u})) \} = f(\mathbf{u}).$$

□

The inclusion $\mathbf{u} \in \partial g(\mathbf{y})$ is a consequence of the theorem. It follows from the fact that

$$g(\mathbf{y}) + f(\mathbf{u}) = \mathbf{u} \mathbf{y} \geq g(\mathbf{y}) + \mathbf{u} \tilde{\mathbf{y}} - g(\tilde{\mathbf{y}}).$$

As an example, we determine the Young–Fenchel transform of the indicator function $\delta_F(\mathbf{u})$ of a convex closed set F :

$$D_F(y) = \sup_{u \in \mathbb{R}^m} \{u y - \delta_F(u)\} = \sup_{u \in F} \{u y\}.$$

This function is positive homogeneous since for $\lambda \geq 0$

$$D_F(\lambda y) = \sup_{u \in F} \{\lambda u y\} = \lambda \sup_{u \in F} \{u y\} = \lambda D_F(y).$$

It is equal to infinity if the set F is unbounded in the direction of the vector y , i.e. if there exists a sequence $u^n \in F$ such that $u^n y \rightarrow +\infty$ as $n \rightarrow \infty$.

For any $u \in F$ the subdifferential of the indicator function is a nonempty set since, obviously, $0 \in \partial \delta_F(u)$. Therefore the Young–Fenchel transform is involutive everywhere on F . We show that for $\bar{u} \notin F$ the Young–Fenchel transform is involutive as well:

$$\sup_{y \in \mathbb{R}^m} \{\bar{u} y - D_F(y)\} = \delta_F(\bar{u}) = +\infty.$$

Indeed, assuming that supremum is finite, replacing the vector y with $\lambda \tilde{y}$ ($\lambda > 0$), and taking into account that $D_F(y)$ is positive homogeneous, we can bring it to the form

$$\lambda \sup_{\tilde{y} \in \mathbb{R}^m} \{\bar{u} \tilde{y} - D_F(\tilde{y})\}.$$

It follows that it is equal to zero and for any $\tilde{y} \in \mathbb{R}^m$

$$\bar{u} \tilde{y} \leq D_F(\tilde{y}).$$

The quantity $D_F(0)$ is equal to zero, therefore this inequality means exactly that $\bar{u} \in \partial D_F(0) \neq \emptyset$. According to the above theorem $D_F(0) + \delta_F(\bar{u}) = 0$, i.e. $\bar{u} \in F$. This contradiction proves the statement.

The following property is that the Young–Fenchel transform of the indicator function of a convex closed cone K with the vertex at the origin is the indicator function of the dual cone:

$$D_K(y) = \delta_C(y), \quad C = \{y \mid \tilde{u} y \leq 0 \quad \forall \tilde{u} \in K\}.$$

This is proved with the use of the equality

$$D_K(y) = \sup_{u \in K} \{u y\} = \lambda \sup_{\tilde{u} \in K} \{\tilde{u} y\},$$

which is obtained by replacing u with $\lambda \tilde{u}$ ($\lambda \geq 0$). Hence, $D_K(y) = 0$ if $\tilde{u} y \leq 0$ for any vector $\tilde{u} \in K$, and $D_K(y) = +\infty$ if $\tilde{u} y > 0$ for some vector.

The Young–Fenchel transform of $\delta_C(y)$ coincides with $\delta_K(u)$ since it is involutive. Thus, the cone dual to C is K . This property was used in the previous section when

decomposing an arbitrary vector into the sum of projections onto mutually dual cones.

The notion of subdifferential enables one to relate the problem (3.17) of the solution of a discrete variational inequality of the general form to some equation or, more exactly, to an inclusion for a multivalued operator. This is based on an equivalent form of the inequality

$$(\tilde{\mathbf{u}} - \mathbf{u})(\mathbf{A}(\mathbf{u}) - \mathbf{y}) + \delta_{\mathbf{F}}(\tilde{\mathbf{u}}) - \delta_{\mathbf{F}}(\mathbf{u}) \geq 0, \quad \mathbf{u}, \tilde{\mathbf{u}} \in \mathbb{R}^m. \quad (3.21)$$

If $\mathbf{u} \in \mathbf{F}$ is a solution of (3.17) then this inequality holds for any $\tilde{\mathbf{u}} \in \mathbb{R}^m$ since in the case of $\tilde{\mathbf{u}} \in \mathbf{F}$ it coincides with (3.17) and in the case of $\tilde{\mathbf{u}} \notin \mathbf{F}$ it is transformed to the obvious inequality $+\infty \geq 0$. On the other hand, if $\mathbf{u}$ is a solution of (3.20) then, firstly, from this inequality it follows that $\delta_{\mathbf{F}}(\mathbf{u}) < +\infty$, hence, $\mathbf{u} \in \mathbf{F}$, secondly, the inequality (3.17), which follows from (3.21) as a special case for $\tilde{\mathbf{u}} \in \mathbf{F}$, is valid. Thus, the solutions of these problems coincide.

Taking into account the definition of a subdifferential, we can represent the last problem in the inclusion form

$$\mathbf{y} \in \mathbf{A}(\mathbf{u}) + \partial \delta_{\mathbf{F}}(\mathbf{u}). \quad (3.22)$$

Such a form of the variational inequality (3.17) enables one to generalize some known methods for the solution of nonlinear equations, [9, 20].

To conclude, we consider a more general problem

$$\mathbf{y} \in \mathbf{A}(\mathbf{u}) + \partial j(\mathbf{u}),$$

where $j(\mathbf{u})$ is an arbitrary and, generally speaking, nondifferentiable convex function. Actually, this problem is reduced to the solution of the variational inequality

$$(\tilde{\mathbf{u}} - \mathbf{u})(\mathbf{A}(\mathbf{u}) - \mathbf{y}) + j(\tilde{\mathbf{u}}) - j(\mathbf{u}) \geq 0, \quad \mathbf{u}, \tilde{\mathbf{u}} \in \mathbb{R}^m. \quad (3.23)$$

In the case that $j(\mathbf{u})$ is differentiable, this is a system of nonlinear equations $\mathbf{A}(\mathbf{u}) + \partial j(\mathbf{u})/\partial \mathbf{u} = \mathbf{y}$. In the case of $\mathbf{A}(\mathbf{u}) = \mathbf{y} = \mathbf{0}$ the inequality (3.23) defines an unconditional minimum point of a function $j(\mathbf{u})$. With some additional assumptions of $j(\mathbf{u})$, we can prove that the problem (3.23) has a unique solution.

Assume that the set $\mathbf{F} = \text{dom } j$ of finite values of a function $j(\mathbf{u})$ is nonempty and closed and the function itself is convex, continuous on $\mathbf{F}$, and satisfies the condition

$$\frac{j(\mathbf{u})}{|\mathbf{u}|^2} \geq -\frac{1}{2} + \varepsilon \quad \text{for } |\mathbf{u}| \rightarrow +\infty$$

($\varepsilon > 0$), which bounds decreasing at infinity. Consider the inequality of a special form

$$(\tilde{\mathbf{u}} - \mathbf{u})(\mathbf{u} - \tilde{\mathbf{u}}) + j(\tilde{\mathbf{u}}) - j(\mathbf{u}) \geq 0, \quad \mathbf{u}, \tilde{\mathbf{u}} \in \mathbb{R}^m. \quad (3.24)$$

This inequality is a necessary and sufficient minimum condition on the set $\mathbf{F}$ for the function

$$f(\mathbf{u}) = \frac{|\mathbf{u}|^2}{2} - \mathbf{u} \bar{\mathbf{u}} + j(\mathbf{u}).$$

Eliminating from $\mathbf{F}$ “superfluous” elements at which minimum can not be obviously achieved we arrive at $\mathbf{F}_0 = \{\mathbf{u} \in \mathbf{F} \mid f(\mathbf{u}) \leq f(\mathbf{u}^0)\}$ for an arbitrarily taken $\mathbf{u}^0 \in \mathbf{F}$. This set is closed and bounded since the existence of a sequence $\mathbf{u}^n \in \mathbf{F}_0$, such that $|\mathbf{u}^n| \rightarrow +\infty$, means that

$$\begin{aligned} f(\mathbf{u}^0) \geq f(\mathbf{u}^n) &\geq \frac{|\mathbf{u}^n|^2}{2} - |\mathbf{u}^n| |\bar{\mathbf{u}}| + j(\mathbf{u}^n) \\ &= |\mathbf{u}^n|^2 \left(\frac{1}{2} - \frac{|\bar{\mathbf{u}}|}{|\mathbf{u}^n|} + \frac{j(\mathbf{u}^n)}{|\mathbf{u}^n|^2} \right) \geq |\mathbf{u}^n|^2 \left(\varepsilon - \frac{|\bar{\mathbf{u}}|}{|\mathbf{u}^n|} \right) \rightarrow +\infty. \end{aligned}$$

By the Weierstrass theorem a minimum point exists.

Now we prove uniqueness. Let $\mathbf{u}$ be a solution of (3.24) and $\mathbf{u}'$ be a solution of the similar problem for a vector $\bar{\mathbf{u}}' \in \mathbb{R}^m$. Then two inequalities

$$(\mathbf{u}' - \mathbf{u})(\mathbf{u} - \bar{\mathbf{u}}) + j(\mathbf{u}') - j(\mathbf{u}) \geq 0,$$

$$(\mathbf{u} - \mathbf{u}')(\mathbf{u}' - \bar{\mathbf{u}}') + j(\mathbf{u}) - j(\mathbf{u}') \geq 0$$

are valid and summation of them yields

$$|\mathbf{u}' - \mathbf{u}|^2 \leq (\mathbf{u}' - \mathbf{u})(\bar{\mathbf{u}}' - \bar{\mathbf{u}}) \leq |\mathbf{u}' - \mathbf{u}| |\bar{\mathbf{u}}' - \bar{\mathbf{u}}|.$$

Thus, $|\mathbf{u}' - \mathbf{u}| \leq |\bar{\mathbf{u}}' - \bar{\mathbf{u}}|$. This immediately results in uniqueness of the solution for $\bar{\mathbf{u}}' = \bar{\mathbf{u}}$. In addition, it follows from this that the mapping $\Pi : \mathbb{R}^m \rightarrow \mathbf{F}$, which associates with a vector $\bar{\mathbf{u}}$ a solution $\mathbf{u} = \Pi(\bar{\mathbf{u}})$ of the variational inequality (3.24), (generalized projection operator) is non-expanding.

Theorem 3.9 Assume that a convex function $j(\mathbf{u})$ is continuous on a closed domain $\mathbf{F} = \text{dom } j$, the ratio $j(\mathbf{u})/|\mathbf{u}|^2$ is bounded below on $\mathbf{F}$ as $|\mathbf{u}| \rightarrow \infty$, and a mapping $\mathbf{A}(\mathbf{u})$ is strongly monotone and satisfies the Lipschitz condition. Then the variational inequality (3.23) has a unique solution.

Proof We rearrange the inequality (3.23) to the next form:

$$(\bar{\mathbf{u}} - \mathbf{u}) \left(\mathbf{u} - (\mathbf{u} - \tau \mathbf{A}(\mathbf{u}) + \tau \mathbf{y}) \right) + \tau j(\bar{\mathbf{u}}) - \tau j(\mathbf{u}) \geq 0.$$

For sufficiently small $\tau > 0$ the function $\tau j(\mathbf{u})$ satisfies the condition

$$\frac{\tau j(\mathbf{u})}{|\mathbf{u}|^2} \geq -\frac{1}{2} + \varepsilon \quad \text{for } |\mathbf{u}| \rightarrow +\infty,$$

hence, the problem (3.23) is equivalent to looking for a fixed point of the mapping $Q(\mathbf{u}) = \Pi(\mathbf{u} - \tau \mathbf{A}(\mathbf{u}) + \tau \mathbf{y})$. The further considerations coincide with the proof of the existence and uniqueness theorem in the case of a strongly monotone mapping with the only difference that a projection operator $\pi(\mathbf{u})$ is replaced everywhere with a generalized projection operator $\Pi(\mathbf{u})$. $\square$

3.4 Kuhn–Tucker’s Theorem

The problem of convex programming is to determine a minimum point of a convex function $f(\mathbf{u})$ with constraints $f_l(\mathbf{u}) \leq 0$, $l = 1, \dots, n$, where $f_l(\mathbf{u})$ is a system of continuous convex functions. A statement about existence of the Lagrange multipliers in the problem of convex programming which will be used further is formulated as follows [13].

Theorem 3.10 (Kuhn–Tucker) *Let $\mathbf{u}$ be a minimum point of a function $f(\mathbf{u})$ with constraints in the form of inequalities. Then there exists a set of nonnegative numbers (Lagrange multipliers) $\lambda_0 \geq 0$, $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$, satisfying the complementary conditions*

$$\lambda_l f_l(\mathbf{u}) = 0, \quad l = 1, \dots, n,$$

for which the Lagrange function

$$L(\mathbf{u}, \boldsymbol{\lambda}) = \lambda_0 f(\mathbf{u}) + \sum_{l=1}^n \lambda_l f_l(\mathbf{u})$$

takes at this point a minimal value $L(\mathbf{u}, \boldsymbol{\lambda}) \leq L(\tilde{\mathbf{u}}, \boldsymbol{\lambda})$ for any vector $\tilde{\mathbf{u}} \in \mathbb{R}^m$.

Conversely, assume that some pair $\mathbf{u}, \boldsymbol{\lambda}$ of vectors satisfies the constraints $f_l(\mathbf{u}) \leq 0$, the nonnegativity and complementing conditions, and the minimum condition for the Lagrange function, besides, $\lambda_0 \neq 0$. Then $\mathbf{u}$ is a solution of the problem of convex programming.

For the strict inequality $\lambda_0 > 0$ to be valid it is sufficient that the Slater condition holds, i.e. that there exists a vector $\bar{\mathbf{u}} \in \mathbb{R}^m$ such that $f_l(\bar{\mathbf{u}}) < 0$, $l = 1, \dots, n$. For definiteness sake, in this case we can put $\lambda_0 = 1$.

Under the Slater condition a pair $\mathbf{u}, \boldsymbol{\lambda}$ satisfies all above conditions if and only if it is a saddle point of the Lagrange function, i.e.

$$L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) \geq L(\mathbf{u}, \boldsymbol{\lambda}) \geq L(\mathbf{u}, \tilde{\boldsymbol{\lambda}}), \quad \tilde{\lambda}_0, \lambda_0 = 1, \quad \tilde{\lambda}_l, \lambda_l \geq 0.$$

Proof We assume that $f(\mathbf{u}) = 0$. This can be achieved by subtracting a constant. Assume that $\mathbf{M}$ is a set of all vectors $\boldsymbol{\mu} \in \mathbb{R}^{n+1}$ such that for each of them there exists an appropriate point $\mathbf{u}' \in \mathbb{R}^m$ such that $f(\mathbf{u}') < \mu_0$ and $f_l(\mathbf{u}') \leq \mu_l$, $l = 1, \dots, n$. It can be directly verified that this set is convex and that any vector $\boldsymbol{\mu}$ with positive

coordinates belongs to $\mathbf{M}$ since in this case a minimum point $\mathbf{u}$ can be taken as $\mathbf{u}'$. Thus, $\mathbf{M} \neq \emptyset$. In addition, $\mathbf{0} \notin \mathbf{M}$. Otherwise there would exist a vector $\mathbf{u}'$ such that $f_l(\mathbf{u}') \leq 0$ and $f(\mathbf{u}') < 0 = f(\mathbf{u})$. This is in contradiction with the minimum condition.

By the separability theorem there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^{n+1}$ such that for any $\boldsymbol{\mu} \in \mathbf{M}$

$$\lambda_0 \mu_0 + \sum_{l=1}^n \lambda_l \mu_l \geq 0.$$

The vector of the Lagrange multipliers is determined. Before verification of all conditions in the formulation of the theorem, we notice that this vector is determined to within an arbitrary positive factor. For $\lambda_0 \neq 0$ this arbitrariness can be eliminated by dividing all components by λ_0 .

The vector

$$\boldsymbol{\mu} = (\varepsilon, \dots, \varepsilon, 1, \varepsilon, \dots, \varepsilon)$$

for $\varepsilon > 0$ obviously belongs to $\mathbf{M}$ (for this vector $\mathbf{u}$ plays the role of $\mathbf{u}'$ again). Hence,

$$\varepsilon \sum_{l \neq k} \lambda_l + \lambda_k \geq 0.$$

For $\varepsilon \rightarrow 0$ this results in the nonnegativity condition: $\lambda_k \geq 0$. The complementing conditions are proved by contradiction. Assume that $f_k(\mathbf{u}) < 0$ and $\lambda_k > 0$ for some k . Since $\boldsymbol{\mu} = (\varepsilon, \dots, \varepsilon, f_k(\mathbf{u}), \varepsilon, \dots, \varepsilon) \in \mathbf{M}$, we have the inequality

$$\varepsilon \sum_{l \neq k} \lambda_l + \lambda_k f_k(\mathbf{u}) \geq 0,$$

which is in contradiction with the assumption for $\varepsilon \rightarrow 0$.

Assuming that $\tilde{\mathbf{u}}$ is an arbitrary point of $\mathbb{R}^m$, we consider the vector

$$\boldsymbol{\mu} = (f(\tilde{\mathbf{u}}) + \varepsilon, f_1(\tilde{\mathbf{u}}), \dots, f_n(\tilde{\mathbf{u}})) \in \mathbf{M}.$$

This vector satisfies the inequality

$$\lambda_0(f(\tilde{\mathbf{u}}) + \varepsilon) + \sum_{l=1}^n \lambda_l f_l(\tilde{\mathbf{u}}) = L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) + \lambda_0 \varepsilon \geq 0 = L(\mathbf{u}, \boldsymbol{\lambda}),$$

which results in the minimum conditions for the Lagrange function for $\varepsilon \rightarrow 0$.

The converse statement of the theorem for $\lambda_0 \neq 0$ follows from the chain of inequalities which are valid provided that $\tilde{\mathbf{u}}$ satisfies the constraints $f_l(\tilde{\mathbf{u}}) \leq 0$:

$$\lambda_0 f(\tilde{\mathbf{u}}) \geq \lambda_0 f(\tilde{\mathbf{u}}) + \sum_{l=1}^n \lambda_l f_l(\tilde{\mathbf{u}}) = L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) \geq L(\mathbf{u}, \boldsymbol{\lambda}) = \lambda_0 f(\mathbf{u}).$$

The Slater condition eliminates the case of $\lambda_0 = 0$ since in this case

$$L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) = \sum_{l=1}^n \lambda_l f_l(\tilde{\mathbf{u}}) < 0 = L(\mathbf{u}, \boldsymbol{\lambda}),$$

which is in contradiction with the minimum condition for the Lagrange function.

We prove the last statement concerning a saddle point. Let $\mathbf{u}$ be a solution of the problem of convex programming. It has already been proved that under the Slater condition there exists a vector $\boldsymbol{\lambda}$ ($\lambda_0 = 1$) of the Lagrange multipliers for which

$$L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) \geq L(\mathbf{u}, \boldsymbol{\lambda}) = f(\mathbf{u}) \geq f(\mathbf{u}) + \sum_{l=1}^n \tilde{\lambda}_l f_l(\mathbf{u}) = L(\mathbf{u}, \tilde{\boldsymbol{\lambda}})$$

($\tilde{\mathbf{u}} \in \mathbb{R}^m$, $\tilde{\boldsymbol{\lambda}} \geq 0$). Thus, the pair $(\mathbf{u}, \boldsymbol{\lambda})$ is a saddle point of the Lagrange function.

Conversely, assume that $(\mathbf{u}, \boldsymbol{\lambda})$ is an arbitrary saddle point, besides, $\lambda_0 = 1$ and $\lambda_l \geq 0$. Then for any nonnegative numbers $\tilde{\lambda}_l$ ($\tilde{\lambda}_0 = 1$) we have the inequality $L(\mathbf{u}, \boldsymbol{\lambda}) \geq L(\mathbf{u}, \tilde{\boldsymbol{\lambda}})$ which can be rearranged to give the variational inequality

$$-\sum_{l=1}^n (\tilde{\lambda}_l - \lambda_l) f_l(\mathbf{u}) \geq 0, \quad \tilde{\lambda}_l, \lambda_l \geq 0. \quad (3.25)$$

Putting $\tilde{\lambda}_l = \lambda_l$ for all numbers l , except k , and varying $\tilde{\lambda}_k$, we can show that (3.25) admits an equivalent formulation in the form of a system of inequalities with complementing conditions:

$$\lambda_k \geq 0, \quad -f_k(\mathbf{u}) \geq 0, \quad \lambda_k f_k(\mathbf{u}) = 0.$$

Besides, the vectors $\mathbf{u}, \boldsymbol{\lambda}$ satisfy the minimum condition for the Lagrange function. Hence, for this saddle point all conditions providing the converse statement of the theorem are valid. By this statement, the point $\mathbf{u}$ is a solution of the problem of convex programming. The theorem is completely proved. $\square$

With the help of the Kuhn–Tucker theorem, we calculate the subdifferential of the indicator function of the set

$$\mathbf{F} = \{\mathbf{u} \mid f_l(\mathbf{u}) \leq 0, \quad l = 1, \dots, n\},$$

assuming that the Slater condition being the condition of existence of an interior point of $\mathbf{F}$ is valid. If $\mathbf{u} \in \mathbf{F}$ and $\mathbf{y} \in \partial \delta_{\mathbf{F}}(\mathbf{u})$ then for any vector $\tilde{\mathbf{u}} \in \mathbb{R}^m$

$$\delta_F(\tilde{\mathbf{u}}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \mathbf{y}, \quad \text{i.e.} \quad -\mathbf{u} \mathbf{y} = \min_{\tilde{\mathbf{u}} \in F} \{-\tilde{\mathbf{u}} \mathbf{y}\}.$$

The Lagrange function associated with the obtained problem of minimization of the linear function $\tilde{f} = -\tilde{\mathbf{u}} \mathbf{y}$ has the form

$$L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) = -\tilde{\mathbf{u}} \mathbf{y} + \sum_{l=1}^n \lambda_l f_l(\tilde{\mathbf{u}}).$$

The minimum point satisfies the system of equations

$$\frac{\partial L}{\partial \mathbf{u}} = 0 \Rightarrow \mathbf{y} = \sum_{l=1}^n \lambda_l \frac{\partial f_l}{\partial \mathbf{u}}, \quad (3.26)$$

which is the necessary and sufficient minimum condition for the function $L(\mathbf{u}, \boldsymbol{\lambda})$ since due to the first convexity criterion we have the inequality

$$L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) - L(\mathbf{u}, \boldsymbol{\lambda}) \geq (\tilde{\mathbf{u}} - \mathbf{u}) \frac{\partial L}{\partial \mathbf{u}} = 0.$$

By the converse statement of the Kuhn–Tucker theorem, the solution of the system (3.26) under the conditions $\lambda_l \geq 0$, $f_l(\mathbf{u}) \leq 0$ and $\lambda_l f_l(\mathbf{u}) = 0$, $l = 1, \dots, n$, yields a minimum point for the linear function $\tilde{f} = -\tilde{\mathbf{u}} \mathbf{y}$ on the set F . Thus, the expression (3.26) in combination with the system of conditions describes an arbitrary element of the set $\partial \delta_F(\mathbf{u})$.

Of course, a subgradient has such a form only in the case of differentiable functions $f_l(\mathbf{u})$ which parametrize the set F . If these functions are not of this kind then it is necessary to generalize the system taking into account their subdifferentiability. In the general case, the minimum principle for the Lagrange function, written in the form of the inequality

$$-(\tilde{\mathbf{u}} - \mathbf{u}) \mathbf{y} + \sum_{l=1}^n \lambda_l (f_l(\tilde{\mathbf{u}}) - f_l(\mathbf{u})) \geq 0,$$

results in a subdifferential form of the system (3.26):

$$\mathbf{y} \in \partial \sum_{l=1}^n \lambda_l f_l(\mathbf{u}).$$

By the Moreau–Rockafellar theorem, the subdifferential of the sum of continuous convex functions is equal to the sum of the subdifferentials, hence, the generalization of the system (3.26) takes the following form:

$$\mathbf{y} \in \sum_{l=1}^n \lambda_l \partial f_l(\mathbf{u}), \quad (3.27)$$

where $\mathbf{u} \in \mathbf{F}$ as before, and coefficients λ_l are nonnegative and satisfy the complementing conditions.

Taking into account the representation of the subgradient of the indicator function, we present equivalent formulations of the variational inequality (3.17):

$$\mathbf{A}(\mathbf{u}) + \sum_{l=1}^n \lambda_l \frac{\partial f_l(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{y}$$

in the case of differentiable functions $f_l(\mathbf{u})$ and

$$\mathbf{y} \in \mathbf{A}(\mathbf{u}) + \sum_{l=1}^n \lambda_l \partial f_l(\mathbf{u})$$

in a more general case. These formulations are applied in the development of numerical methods for the solution of variational inequalities, [9, 11]. In fact, they reduce a problem with constraints of a relatively general form to a problem with simple constraints on the Lagrange multipliers in the form $\lambda_l \geq 0$.

When constructing dual cones in spatial models of mechanics of granular materials in the next chapter, one more well-known statement is used (see [6, 13]).

Theorem 3.11 (Farkas) *Let $\mathbf{y}, \mathbf{y}_l, l = 1, \dots, n$, be a given system of m -dimensional vectors. The inequality $\mathbf{u} \mathbf{y} \leq 0$ holds for any $\mathbf{u} \in \mathbf{K} = \{\tilde{\mathbf{u}} \in \mathbb{R}^m \mid \tilde{\mathbf{u}} \mathbf{y}_l \leq 0, l = 1, \dots, n\}$ if and only if there exists a set of nonnegative numbers λ_l such that*

$$\mathbf{y} = \sum_{l=1}^n \lambda_l \mathbf{y}_l. \quad (3.28)$$

Or, in equivalent terms, a vector $\mathbf{y}$ belongs to the cone

$$\mathbf{C} = \{\tilde{\mathbf{y}} \in \mathbb{R}^m \mid \mathbf{u} \tilde{\mathbf{y}} \leq 0 \quad \forall \mathbf{u} \in \mathbf{K}\},$$

dual to $\mathbf{K}$, if and only if it is a linear combination of vectors $\mathbf{y}_l$ with nonnegative coefficients.

Under the fulfillment of the Slater condition (the condition of existence of an interior point of the cone $\mathbf{K}$) the formula (3.28) follows immediately from (3.26) since the inequality $\mathbf{u} \mathbf{y} \leq 0$ can be represented in the form

$$\mathbf{u} \mathbf{y} \leq \delta_{\mathbf{K}}(\mathbf{u}) \quad \forall \mathbf{u} \in \mathbb{R}^m,$$

which means that $\mathbf{y} \in \partial\delta_K(\mathbf{0})$. In addition, the system of functions $f_l(\mathbf{u}) = \mathbf{u} \mathbf{y}_l$ which parametrize the cone $\mathbf{K}$ is continuously differentiable which is assumed when deriving the formula (3.26).

However, the interior of $\mathbf{K}$ may be empty. In the general case the proof of sufficiency is trivial. If the formula (3.28) is valid then multiplying both its sides scalar by $\mathbf{u} \in \mathbf{K}$ we obtain that $\mathbf{y} \in \mathbf{C}$. To prove necessity, we consider the set of vectors

$$\bar{\mathbf{C}} = \left\{ \tilde{\mathbf{y}} = \sum_{l=1}^n \tilde{\lambda}_l \mathbf{y}_l \mid \tilde{\lambda}_l \geq 0 \right\}$$

being a closed cone with the vertex at the origin. Contrary, assume that the inequality $\mathbf{u} \mathbf{y} \leq 0$ holds for any $\mathbf{u} \in \mathbf{K}$ but $\mathbf{y} \notin \bar{\mathbf{C}}$. Then by the separability theorem there exists a vector $\bar{\mathbf{u}}$ such that

$$\bar{\mathbf{u}} \mathbf{y} > \bar{\mathbf{u}} \tilde{\mathbf{y}} \quad \forall \tilde{\mathbf{y}} \in \bar{\mathbf{C}}. \quad (3.29)$$

Since $\bar{\mathbf{C}}$ is a cone, we can replace $\tilde{\mathbf{y}}$ by $\lambda \tilde{\mathbf{y}}$. Then, assuming that λ tends to infinity, we get the inequality $\bar{\mathbf{u}} \tilde{\mathbf{y}} \leq 0$ from which it follows that $\bar{\mathbf{u}} \mathbf{y}_l \leq 0$ for $l = 1, \dots, n$. On the other hand, the inequality $\bar{\mathbf{u}} \mathbf{y} > 0$ follows from (3.29) for $\tilde{\mathbf{y}} = \mathbf{0}$. This is in contradiction with the assumption. $\square$

Now, bearing in mind the results of the following chapters, we consider sufficient conditions which provide interchanging infimum and supremum when calculating minimax of the Lagrange function. These results can be found in [5, 19].

Let $L(\mathbf{u}, \boldsymbol{\lambda})$ be a function taking finite values for $\mathbf{u} \in U \subset \mathbb{R}^m$ and $\boldsymbol{\lambda} \in \mathbf{A} \subset \mathbb{R}^n$. Since infimum is a lower bound, we have

$$\inf_{\tilde{\mathbf{u}} \in U} L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) \leq L(\mathbf{u}, \boldsymbol{\lambda}) \quad \forall \boldsymbol{\lambda} \in \mathbf{A}.$$

Hence,

$$\sup_{\boldsymbol{\lambda} \in \mathbf{A}} \inf_{\tilde{\mathbf{u}} \in U} L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) \leq \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\mathbf{u}, \boldsymbol{\lambda}) \quad \forall \mathbf{u} \in U.$$

Infimum is the greatest lower bound, therefore

$$\sup_{\boldsymbol{\lambda} \in \mathbf{A}} \inf_{\tilde{\mathbf{u}} \in U} L(\tilde{\mathbf{u}}, \boldsymbol{\lambda}) \leq \inf_{\mathbf{u} \in U} \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\mathbf{u}, \boldsymbol{\lambda}). \quad (3.30)$$

This inequality holds in the most general case. It is required to define conditions under which the opposite inequality holds.

The following saddle point criterion related to this problem is known.

Criterion. A point $(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}}) \in U \times \mathbf{A}$ is a saddle point, i.e.

$$L(\bar{\mathbf{u}}, \boldsymbol{\lambda}) \leq L(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}}) \leq L(\mathbf{u}, \bar{\boldsymbol{\lambda}}),$$

if and only if

$$L(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}}) = \max_{\boldsymbol{\lambda} \in \mathbf{A}} \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \boldsymbol{\lambda}) = \min_{\mathbf{u} \in \mathbf{U}} \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\mathbf{u}, \boldsymbol{\lambda}). \quad (3.31)$$

Proof If $(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}})$ is a saddle point then

$$\sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\bar{\mathbf{u}}, \boldsymbol{\lambda}) = L(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}}) = \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \bar{\boldsymbol{\lambda}}).$$

Taking into account that

$$\inf_{\mathbf{u} \in \mathbf{U}} \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\mathbf{u}, \boldsymbol{\lambda}) \leq \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\bar{\mathbf{u}}, \boldsymbol{\lambda}) \quad \text{and} \quad \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \bar{\boldsymbol{\lambda}}) \leq \sup_{\boldsymbol{\lambda} \in \mathbf{A}} \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \boldsymbol{\lambda}),$$

we can obtain the inequality

$$\inf_{\mathbf{u} \in \mathbf{U}} \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\mathbf{u}, \boldsymbol{\lambda}) \leq \sup_{\boldsymbol{\lambda} \in \mathbf{A}} \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \boldsymbol{\lambda})$$

opposite to (3.30). Therefore this inequality in fact is an equality. We can make the same conclusion for two previous inequalities. Thus, the relationships (3.31) take place.

Conversely, on the one hand, from (3.31) it follows that

$$\inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \bar{\boldsymbol{\lambda}}) \leq L(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}}) \leq \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\bar{\mathbf{u}}, \boldsymbol{\lambda}),$$

and from the other hand, maximum in the left-hand side of (3.31) is achieved at the point $\bar{\boldsymbol{\lambda}}$ and minimum in the right-hand side is achieved at $\bar{\mathbf{u}}$, hence, inequalities are transformed into equalities. Thus, $(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}})$ is a saddle point of the function $L(\mathbf{u}, \boldsymbol{\lambda})$. $\square$

An obvious consequence of this criterion is interchanging inf and sup with at least one saddle point of the Lagrange function. To conclude, we present several almost obvious existence theorems for a saddle point.

Assume that $\mathbf{U}$ and $\mathbf{A}$ are convex sets, $L(\mathbf{u}, \boldsymbol{\lambda})$ is a differentiable convex function in the variable $\mathbf{u}$ for any fixed $\boldsymbol{\lambda} \in \mathbf{A}$ and a differentiable concave function in $\boldsymbol{\lambda}$ for any $\mathbf{u} \in \mathbf{U}$. A point $(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}}) \in \mathbf{U} \times \mathbf{A}$ is a saddle point if and only if the following two variational inequalities related to each other are valid:

$$(\mathbf{u} - \bar{\mathbf{u}}) \frac{\partial L(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}})}{\partial \mathbf{u}} \geq 0 \quad \forall \mathbf{u} \in \mathbf{U}, \quad -(\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}}) \frac{\partial L(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}})}{\partial \boldsymbol{\lambda}} \geq 0 \quad \forall \boldsymbol{\lambda} \in \mathbf{A}. \quad (3.32)$$

This statement does not require a detailed proof since due to the criterion (3.8) the former inequality defines a minimum point for the convex function $L(\mathbf{u}, \bar{\boldsymbol{\lambda}})$ and the latter one defines a minimum point for the convex function $-L(\bar{\mathbf{u}}, \boldsymbol{\lambda})$.

Theorem 3.12 Assume that $\mathbf{U}$ and $\mathbf{A}$ are convex closed subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and the mapping

$$(\mathbf{u}, \boldsymbol{\lambda}) \mapsto \left(\frac{\partial L(\mathbf{u}, \boldsymbol{\lambda})}{\partial \mathbf{u}}, -\frac{\partial L(\mathbf{u}, \boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} \right) \quad (3.33)$$

from $\mathbf{U} \times \mathbf{A}$ into $\mathbb{R}^{m+n}$ is strongly monotone and satisfies the Lipschitz condition. Then the Lagrange function L has a unique saddle point.

A theorem follows immediately from the theorem of existence and uniqueness of a solution of the variational inequality (3.17) for a strictly monotone mapping.

Theorem 3.13 *Let $\mathbf{U}$ and $\mathbf{A}$ be convex, closed and bounded sets and the mapping (3.33) be continuous on $\mathbf{U} \times \mathbf{A}$. Then there exists a saddle point but, generally speaking, it is non unique (in this case inf and sup can be interchanged as well).*

A proof follows from the Brouwer theorem which should be applied to the problem on a fixed point

$$\bar{\mathbf{u}} = \pi \left(\bar{\mathbf{u}} - \tau \frac{\partial L(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}})}{\partial \mathbf{u}} \right), \quad \bar{\boldsymbol{\lambda}} = \Pi \left(\bar{\boldsymbol{\lambda}} + \tau \frac{\partial L(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}})}{\partial \boldsymbol{\lambda}} \right)$$

being equivalent to the inequalities (3.32). Here π and Π are projections onto the sets $\mathbf{U}$ and $\mathbf{A}$ with respect to the standard Euclidean norms, and τ is an arbitrary positive parameter.

This theorem remains valid in a more general case of unbounded sets under the next additional conditions: if $\mathbf{U}$ is unbounded then

$$L(\mathbf{u}, \boldsymbol{\lambda}^0) \rightarrow +\infty \quad \text{as} \quad |\mathbf{u}| \rightarrow \infty$$

for some $\boldsymbol{\lambda}^0 \in \mathbf{A}$, and if $\mathbf{A}$ is unbounded then

$$L(\mathbf{u}^0, \boldsymbol{\lambda}) \rightarrow -\infty \quad \text{as} \quad \|\boldsymbol{\lambda}\| \rightarrow \infty$$

for some $\mathbf{u}^0 \in \mathbf{U}$.

To prove the theorem in this case, we consider bounded sets $\mathbf{U}_r$ and $\mathbf{A}_r$ being intersections of $\mathbf{U}$ and $\mathbf{A}$ with balls of radius r in the corresponding spaces. For these sets there exists a saddle point $(\mathbf{u}_r, \boldsymbol{\lambda}_r)$ such that

$$L(\mathbf{u}_r, \boldsymbol{\lambda}) \leq L(\mathbf{u}_r, \boldsymbol{\lambda}_r) \leq L(\mathbf{u}, \boldsymbol{\lambda}_r) \quad \forall (\mathbf{u}, \boldsymbol{\lambda}) \in \mathbf{U}_r \times \mathbf{A}_r. \quad (3.34)$$

Assume that the value of r is large to an extent that $\mathbf{u}^0 \in \mathbf{U}_r$ and $\boldsymbol{\lambda}^0 \in \mathbf{A}_r$. Then because of (3.34)

$$L(\mathbf{u}_r, \boldsymbol{\lambda}^0) \leq L(\mathbf{u}_r, \boldsymbol{\lambda}_r) \leq L(\mathbf{u}^0, \boldsymbol{\lambda}_r).$$

Of course, both inequalities hold provided that both sets $\mathbf{U}$ and $\mathbf{A}$ are unbounded. If $\mathbf{U}$ is bounded then only the left-hand inequality holds, if $\mathbf{A}$ is bounded then only the right-hand inequality holds.

By contradiction we can show that $|\mathbf{u}_r| \leq C_1$ and $|\boldsymbol{\lambda}_r| \leq C_1$ where C_1 is a constant independent of r . Indeed, for $|\mathbf{u}_r| \rightarrow \infty$ and for $|\boldsymbol{\lambda}_r| \rightarrow \infty$ the inequalities result in $L(\mathbf{u}_r, \boldsymbol{\lambda}_r) \leq -\infty$ and $+\infty \leq L(\mathbf{u}_r, \boldsymbol{\lambda}_r)$, respectively. Both inequalities make no sense.

In a finite-dimensional spaces a bounded sets are compact, hence, there exist convergent sequences $\mathbf{u}_{r_k} \rightarrow \bar{\mathbf{u}}$ and $\boldsymbol{\lambda}_{r_k} \rightarrow \bar{\boldsymbol{\lambda}}$ as $r_k \rightarrow \infty$. Passage to the limit in the inequalities (3.34) for $r = r_k$ shows that $(\bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}})$ is a required saddle point.

We formulate one more statement about interchanging infimum and supremum for an unbounded set $\mathbf{A}$ in the case that a saddle point may not exist.

Theorem 3.14 *Assume that a continuously differentiable Lagrange function is convex in $\mathbf{u}$ and concave in $\boldsymbol{\lambda}$, sets $\mathbf{U}$ and $\mathbf{A}$ are convex and closed, besides, if $\mathbf{U}$ is unbounded then there exists a vector $\boldsymbol{\lambda}^0 \in \mathbf{A}$ such that $L(\mathbf{u}, \boldsymbol{\lambda}^0) \rightarrow +\infty$ as $|\mathbf{u}| \rightarrow \infty$. Then*

$$\sup_{\boldsymbol{\lambda} \in \mathbf{A}} \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \boldsymbol{\lambda}) = \inf_{\mathbf{u} \in \mathbf{U}} \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\mathbf{u}, \boldsymbol{\lambda}).$$

Proof We go from $L(\mathbf{u}, \boldsymbol{\lambda})$ to the regularized function $L(\mathbf{u}, \boldsymbol{\lambda}) - \varepsilon |\boldsymbol{\lambda}|^2$ with a small parameter $\varepsilon > 0$. Fix an arbitrary vector $\mathbf{u}^0 \in \mathbf{U}$. Due to the convexity criterion (3.2) for $-L(\mathbf{u}^0, \boldsymbol{\lambda})$

$$L(\mathbf{u}^0, \boldsymbol{\lambda}) - \varepsilon |\boldsymbol{\lambda}|^2 \leq L(\mathbf{u}^0, \boldsymbol{\lambda}^0) + (\boldsymbol{\lambda} - \boldsymbol{\lambda}^0) \frac{\partial L(\mathbf{u}^0, \boldsymbol{\lambda}^0)}{\partial \boldsymbol{\lambda}} - \varepsilon |\boldsymbol{\lambda}|^2 \rightarrow -\infty$$

as $|\boldsymbol{\lambda}| \rightarrow \infty$ (if $\mathbf{U}$ is bounded then we can take an arbitrary element of $\mathbf{A}$ as $\boldsymbol{\lambda}^0$). This function satisfies all conditions of the above theorem, hence, it has a saddle point:

$$L(\mathbf{u}_\varepsilon, \boldsymbol{\lambda}) - \varepsilon |\boldsymbol{\lambda}|^2 \leq L(\mathbf{u}_\varepsilon, \boldsymbol{\lambda}_\varepsilon) - \varepsilon |\boldsymbol{\lambda}_\varepsilon|^2 \leq L(\mathbf{u}, \boldsymbol{\lambda}_\varepsilon) - \varepsilon |\boldsymbol{\lambda}_\varepsilon|^2. \quad (3.35)$$

Therefore $L(\mathbf{u}_\varepsilon, \boldsymbol{\lambda}_\varepsilon) \leq L(\mathbf{u}, \boldsymbol{\lambda}_\varepsilon)$, so,

$$L(\mathbf{u}_\varepsilon, \boldsymbol{\lambda}_\varepsilon) = \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \boldsymbol{\lambda}_\varepsilon) \leq \sup_{\boldsymbol{\lambda} \in \mathbf{A}} \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \boldsymbol{\lambda}).$$

From (3.35) for $\mathbf{u} = \mathbf{u}^0$ it follows that $\boldsymbol{\lambda}_\varepsilon$ is bounded by a constant, independent of ε , otherwise $L(\mathbf{u}_\varepsilon, \boldsymbol{\lambda}^0) \rightarrow -\infty$ as $\varepsilon \rightarrow 0$ which is in contradiction with assumption of the theorem. Selecting a convergent sequence $\boldsymbol{\lambda}_{\varepsilon_k} \rightarrow \bar{\boldsymbol{\lambda}}$ ($\varepsilon_k \rightarrow 0$) and passing to the limit in (3.35), we obtain

$$\inf_{\mathbf{u} \in \mathbf{U}} \sup_{\boldsymbol{\lambda} \in \mathbf{A}} L(\mathbf{u}, \boldsymbol{\lambda}) \leq \sup_{\boldsymbol{\lambda} \in \mathbf{A}} \inf_{\mathbf{u} \in \mathbf{U}} L(\mathbf{u}, \boldsymbol{\lambda}).$$

Since the opposite inequality holds, the theorem is completely proved. $\square$

More general versions of the minimax theorem for a non-differentiable Lagrange functions in an infinite-dimensional spaces can be found, for example, in [5].

3.5 Duality Theory

The duality principle plays an important role in the study of solvability of boundary-value problems in elasticity and plasticity theory as well as in mechanics of granular materials, [5, 23]. We present one of a number of proofs of this principle, [10]. The proof is essentially based on a fundamental statement of functional analysis, namely, the Hahn–Banach theorem about extension of a linear functional (see [12, 15]). A geometric consequence of this theorem being the infinite-dimensional version of the separability theorem is as follows.

Theorem 3.15 *Let F be a convex subset of a normalized space U with nonempty interior $\text{int } F \neq \emptyset$. Assume that $\bar{u} \notin F$ or, in a weak form, $\bar{u} \notin \text{int } F$. Then there exists a linear continuous functional $y \neq 0$ being an element of the conjugate space Y which satisfies the condition*

$$(u, y) \leq (\bar{u}, y) \quad \forall u \in F. \quad (3.36)$$

Proof Compared to the separability theorem in a finite-dimensional space, in this case the essential assumption of existence of an interior point of F is made. Without loss of generality, we can suppose that this point is 0 . Consider the Minkowski function of the set F :

$$f_0(u) = \inf \left\{ r > 0 \mid \frac{u}{r} \in F \right\}.$$

This function is nonnegative, convex, and positive homogeneous. At each point of F its value does not exceed one and at each point, which does not belong to F , its value is greater than or equal to one. In particular, $f_0(\bar{u}) \geq 1$.

The set $U_1 = \{\lambda \bar{u} \mid \lambda \in \mathbb{R}\}$ of vectors is a linear one-dimensional subspace of U . The formula $(\lambda \bar{u}, y) = \lambda$ defines a linear functional on U_1 which satisfies the inequality

$$(u, y) \leq f_0(u) \quad \forall u \in U_1, \quad (3.37)$$

since $(\lambda \bar{u}, y) = \lambda \leq \lambda f_0(\bar{u}) = f_0(\lambda \bar{u})$. By the Hahn–Banach theorem, this functional can be extended to the whole space U with the inequality (3.37) remaining valid.

In a neighborhood of zero, which belongs to the set F , the Minkowski function is bounded, therefore the functional is bounded as well and, hence, is continuous. Thus, $y \in Y$,

$$(u, y) \leq f_0(u) \leq 1 = (\bar{u}, y) \quad \forall u \in F,$$

which proves the theorem. $\square$

If the condition $\bar{u} \notin F$ holds in a weak form $\bar{u} \notin \text{int } F$, then we have to apply the statement proved immediately above to the open convex set $\text{int } F$. The inequality (3.36) turns out to be valid at interior points of F , but by the statement proved in Sect. 3.3 any boundary point $u \in F$ is a limit of a convergent sequence of interior

points, hence, passing to the limit in (3.36) for elements of a sequence, we can prove this inequality everywhere in F . In the case that $\bar{u}$ is a boundary point, the corresponding functional y is said to be support. Considering the set Y_F of all support functionals of a convex closed set F , we can represent it in the form

$$F = \{u \in U \mid (u, y) \leq 1 \quad \forall y \in Y_F\}. \quad (3.38)$$

Let a convex function $f(u)$, which is not identically equal to $-\infty$ on U , and a linear normalized subspace $U_0 \subset U$ be given. The main problem is to determine $\inf \{f(u) \mid u \in U_0\}$.

Denote the Young–Fenchel transform for $f(u)$ by $g(y)$ and the subspace of conjugate space, orthogonal to the subspace U_0 , by Y_0 :

$$g(y) = \sup_{u \in U} \{(u, y) - f(u)\}, \quad Y_0 = \{y \in Y \mid (u, y) = 0 \quad \forall u \in U_0\}.$$

The problem of determining $\sup \{-g(y) \mid y \in Y_0\}$ is called the dual problem.

Theorem 3.16 (Duality) *Assume that U is a linear normalized space and $f(u)$ is a convex function on U which is continuous at some point $\bar{u} \in U_0$. Then*

$$\inf_{u \in U_0} f(u) = \sup_{y \in Y_0} \{-g(y)\} = - \inf_{y \in Y_0} g(y). \quad (3.39)$$

In addition, assume that $f(u)$ is bounded below on U_0 . Then supremum in the dual problem is achieved on some element of Y_0 .

Proof On the basis of the definition of the Young–Fenchel transform, we can obtain the inequality $(u, y) - g(y) \leq f(u)$ which is valid for any vectors $u \in U$ and $y \in Y$. If $u \in U_0$ and $y \in Y_0$ then $-g(y) \leq f(u)$. Hence,

$$\sup_{y \in Y_0} \{-g(y)\} \leq \inf_{u \in U_0} f(u). \quad (3.40)$$

It remains to prove the opposite inequality.

If $\inf$ in the right-hand side of (3.40) is equal to $-\infty$, then $\sup$ in the left-hand side is equal to $-\infty$ as well. In this case the statement of the theorem remains valid. Now consider the case that the function $f(u)$ is bounded below on U_0 :

$$-\infty < m = \inf_{u \in U_0} f(u) \leq f(\bar{u}) < +\infty.$$

Construct a special set in the space $U \times \mathbb{R}$:

$$G = \{(v, z) \mid \exists u_0 \in U_0 : z > f(u_0 + v)\}.$$

It can be shown that this set is convex. Indeed, if $(\mathbf{v}, z)$ and $(\tilde{\mathbf{v}}, \tilde{z}) \in \mathbf{G}$, i.e. if $f(\mathbf{u}_0 + \mathbf{v}) < z$ and $f(\tilde{\mathbf{u}}_0 + \tilde{\mathbf{v}}) < \tilde{z}$ for some $\mathbf{u}_0$ and $\tilde{\mathbf{u}}_0 \in U_0$, then due to convexity of $f(\mathbf{u})$

$$f(\mathbf{u}_0^\lambda + \mathbf{v}^\lambda) \leq \lambda f(\tilde{\mathbf{u}}_0 + \tilde{\mathbf{v}}) + (1 - \lambda) f(\mathbf{u}_0 + \mathbf{v}) < z^\lambda,$$

hence, $(\mathbf{v}, z)^\lambda = (\mathbf{v}^\lambda, z^\lambda) \in \mathbf{G}$ for any $\lambda \in [0, 1]$.

Let $\bar{z} > f(\bar{\mathbf{u}})$ and $\varepsilon > 0$ such that $\bar{z} - 2\varepsilon > f(\bar{\mathbf{u}})$. The function $f(\mathbf{u})$ is continuous at the point $\bar{\mathbf{u}}$, hence, for given ε there exists $\delta > 0$ such that

$$|f(\bar{\mathbf{u}} + \mathbf{v}) - f(\bar{\mathbf{u}})| \leq \varepsilon \quad \forall \mathbf{v} \in U : \|\mathbf{v}\| \leq \delta.$$

Taking into account the inequality $f(\bar{\mathbf{u}} + \mathbf{v}) \leq \varepsilon + f(\bar{\mathbf{u}}) < \bar{z} - \varepsilon$, we obtain that the point $(\mathbf{0}, \bar{z})$ belongs to the set $\mathbf{G}$ (with the element $\bar{\mathbf{u}} \in U_0$ being taken as $\mathbf{u}_0$) together with its neighborhood consisting of the points $(\mathbf{v}, z)$ for which $\|\mathbf{v}\| \leq \delta$, $|z - \bar{z}| \leq \varepsilon$. Thus, $(\mathbf{0}, \bar{z})$ is an interior point of $\mathbf{G}$.

In addition, $(\mathbf{0}, m) \notin \mathbf{G}$. This is the case, since otherwise there would exist an element $\mathbf{u}_0 \in U_0$ such that

$$f(\mathbf{u}_0) < m = \inf \{f(\mathbf{u}) \mid \mathbf{u} \in U_0\},$$

which is in contradiction with the definition of infimum.

One more property of the set $\mathbf{G}$, which is used in the proof of the theorem, is that for any $z > m$ and $\mathbf{v} \in U_0$ the pair $(\mathbf{v}, z)$ is an element of $\mathbf{G}$. This also follows from the definition of infimum according to which there exists an element $\mathbf{u}' \in U_0$, generally speaking, depending on z , such that $m \leq f(\mathbf{u}') < z$. Putting in this case $\mathbf{u}_0 = \mathbf{u}' - \mathbf{v} \in U_0$, we obtain the inequality $f(\mathbf{u}_0 + \mathbf{v}) < z$ which proves this property.

The set $\mathbf{G}$ and the point $(\mathbf{0}, m)$ satisfy the conditions of the separability theorem, hence, there exists an element $(\mathbf{y}, a) \in Y \times \mathbb{R}$ of the conjugate space which is different from zero such that

$$(\mathbf{v}, \mathbf{y}) + a z \leq a m \quad \forall (\mathbf{v}, z) \in \mathbf{G}. \quad (3.41)$$

Taking here an arbitrary vector of U_0 as $\mathbf{v}$, for fixed $z > m$ we have the inequality

$$(\mathbf{v}, \mathbf{y}) \leq a(m - z).$$

Since U_0 is a subspace, in this inequality we can replace $\mathbf{v}$ by $\pm \lambda \mathbf{v}$. When passing to the limit as $\lambda \rightarrow \infty$, this results in the equation $(\mathbf{v}, \mathbf{y}) = 0 \quad \forall \mathbf{v} \in U_0$ which means that $\mathbf{y} \in Y_0$.

By the proved above property $(\mathbf{0}, z) \in \mathbf{G}$ for $z > m$, therefore due to (3.41) we have $a(z - m) \leq 0$, i.e. $a \leq 0$. Assume that $a = 0$. Then from (3.41) it follows that $(\mathbf{v}, \mathbf{y}) \leq 0$ for any $(\mathbf{v}, z) \in \mathbf{G}$. It was proved that the point $(\mathbf{0}, z)$ is an interior point

of $\mathbf{G}$ for $z > f(\bar{\mathbf{u}})$. Hence, the inequality $(\mathbf{v}, \mathbf{y}) \leq 0$ holds in some neighbourhood of zero in the space $\mathbf{U}$. Replacing $\mathbf{v}$ by $-\mathbf{v}$, we can see that in fact in this neighbourhood the equation $(\mathbf{v}, \mathbf{y}) = 0$ holds. Then $\mathbf{y} \equiv \mathbf{0}$. The functional $(\mathbf{y}, a)$ turns out to be equal to zero, which is in contradiction with the separability theorem. Thus, $a < 0$.

Dividing both sides of the inequality (3.41) by a , we arrive at

$$(\mathbf{v}, \bar{\mathbf{y}}) + z \geq m, \quad \bar{\mathbf{y}} = \frac{\mathbf{y}}{a}.$$

The last inequality can be written for $(\mathbf{u}, f(\mathbf{u}) + \varepsilon) \in \mathbf{G}$. Thereafter, assuming that $\varepsilon \rightarrow 0$, we obtain $(\mathbf{u}, \bar{\mathbf{y}}) + f(\mathbf{u}) \geq m$. Thus,

$$g(-\bar{\mathbf{y}}) = \sup_{\mathbf{u} \in \mathbf{U}_0} \{-(\mathbf{u}, \bar{\mathbf{y}}) - f(\mathbf{u})\} = - \inf_{\mathbf{u} \in \mathbf{U}_0} \{(\mathbf{u}, \bar{\mathbf{y}}) + f(\mathbf{u})\} \leq -m,$$

or, in another form,

$$-g(-\bar{\mathbf{y}}) \geq m = \inf_{\mathbf{u} \in \mathbf{U}_0} f(\mathbf{u}).$$

Since a value of function at a point does not exceed supremum of all values, we have the inequality opposite to (3.40). Besides, in the case of a finite value of m supremum is achieved on the element $-\bar{\mathbf{y}} \in \mathbf{Y}_0$. The theorem is completely proved. $\square$

Let $\mathbf{U}$ be a reflexive Banach space. Reflexivity means that a unit ball in $\mathbf{U}$ is weakly compact, [5, 12], i.e. out of any infinite sequence of elements such that $\|\mathbf{u}_n\| \leq 1$ we can select a subsequence $\mathbf{u}_{n_k}$ which converges to an element $\mathbf{u} \in \mathbf{U}$ in the weak sense:

$$(\mathbf{u}_{n_k}, \mathbf{y}) \rightarrow (\mathbf{u}, \mathbf{y}) \quad \forall \mathbf{y} \in \mathbf{Y}.$$

Basing on the representation (3.38), which follows from the separability theorem, we can prove that any convex closed set is weakly closed. In the case of a unit ball, this ensures that the limit of a weakly convergent sequence belongs to the unit ball as well ($\|\mathbf{u}\| \leq 1$).

A function $f(\mathbf{u})$ is said to be lower semicontinuous at a point $\mathbf{u} \in \mathbf{F} \subset \mathbf{U}$ if for any sequence $\mathbf{u}_n \in \mathbf{F}$, that converges to $\mathbf{u}$ with respect to a norm, the inequality

$$\liminf_{n \rightarrow \infty} f(\mathbf{u}_n) \geq f(\mathbf{u})$$

is valid. It is obvious that a function continuous on $\mathbf{F}$ is lower semicontinuous. It is easy to show that a function is lower semicontinuous on the whole space $\mathbf{U}$ if and only if its epigraph $\text{epi } f$ is a closed set. Thus, a convex lower semicontinuous function is weakly lower semicontinuous (its epigraph is a weakly closed set).

In the following chapters we will use one more general statement, namely, the theorem about existence and uniqueness of a minimum point for a strictly convex function on a convex set of an infinite-dimensional vector space, [16, 17].

Theorem 3.17 *Let a function $f(\mathbf{u})$ be convex and weakly lower semi-continuous at the points of a convex closed set $\mathbf{F}$ of a reflexive Banach space $\mathbf{U}$. If $f(\mathbf{u})$ is coercive, i.e. $f(\mathbf{u}_n) \rightarrow +\infty$ for any sequence $\mathbf{u}_n \in \mathbf{F}$ such that $\|\mathbf{u}_n\| \rightarrow \infty$, then there exists an element $\mathbf{u} \in \mathbf{F}$:*

$$f(\mathbf{u}) = \min_{\tilde{\mathbf{u}} \in \mathbf{F}} f(\tilde{\mathbf{u}}).$$

If $f(\mathbf{u})$ is strictly convex, i.e.

$$f(\mathbf{u}^\lambda) < \lambda f(\tilde{\mathbf{u}}) + (1 - \lambda) f(\mathbf{u}), \quad \mathbf{u} \neq \tilde{\mathbf{u}} \in \mathbf{F}, \quad \lambda \in (0, 1),$$

then a solution of the minimization problem is unique. If at each point $\mathbf{u} \in \mathbf{F}$ a function $f(\mathbf{u})$ has a weak derivative, i.e. a continuous linear functional $f'(\mathbf{u}) \in \mathbf{Y}$:

$$(\tilde{\mathbf{u}}, f'(\mathbf{u})) = \lim_{\varepsilon \rightarrow 0} \frac{f(\mathbf{u} + \varepsilon \tilde{\mathbf{u}}) - f(\mathbf{u})}{\varepsilon},$$

then the necessary and sufficient minimum condition has the form

$$(\tilde{\mathbf{u}} - \mathbf{u}, f'(\mathbf{u})) \geq 0, \quad \mathbf{u}, \tilde{\mathbf{u}} \in \mathbf{F}. \quad (3.42)$$

In a more general case that a function to be minimized involves a non-differentiable term, i.e. a function $j(\mathbf{u})$ which is continuous at some point of $\mathbf{U}$, this inequality is replaced with the following one:

$$(\tilde{\mathbf{u}} - \mathbf{u}, f'(\mathbf{u})) + j(\tilde{\mathbf{u}}) - j(\mathbf{u}) \geq 0, \quad \mathbf{u}, \tilde{\mathbf{u}} \in \mathbf{F}.$$

If a set $\mathbf{F}$ coincides with the whole space or it has an interior point, then the requirement of continuity at a point can be weakened, namely, a function $j(\mathbf{u})$ must take a finite value at least at one point.

Proof Assume that $\mathbf{u}_n$ is a minimizing sequence:

$$f(\mathbf{u}_n) \rightarrow m = \inf \{ f(\tilde{\mathbf{u}}) \mid \tilde{\mathbf{u}} \in \mathbf{F} \},$$

which can be constructed, for example, with the help of the procedure of the choice of elements $\mathbf{u}_n \in \mathbf{F}$ according to the inequality $f(\mathbf{u}_n) \leq m + 1/n$. If a set $\mathbf{F}$ is bounded, then this sequence is bounded as well. In the case that $\mathbf{F}$ is unbounded, the fact that the sequence is bounded follows from coercitivity of $f(\mathbf{u})$ since for $\|\mathbf{u}_n\| \rightarrow \infty$ we would have $f(\mathbf{u}_n) \rightarrow m = +\infty$ which is in contradiction with the definition of infimum. So, existence of the solution is proved.

In a reflexive Banach space any bounded set is weakly compact, hence, we can select a weakly convergent subsequence out of $\mathbf{u}_n$. For simplicity of notations we suppose that the sequence $\mathbf{u}_n$ converges itself. By the conditions of the theorem, the set $\mathbf{F}$ is convex and closed, hence, it is weakly closed. Therefore the limit point $\mathbf{u}$

belongs to $\mathbf{F}$. Besides, $f(\mathbf{u})$ is weakly lower semicontinuous, so

$$f(\mathbf{u}) \leq \liminf_{n \rightarrow \infty} f(\mathbf{u}_n) = m.$$

In fact this inequality is an equality since m is infimum. From this equality it follows that $\mathbf{u}$ is the required minimum point.

Uniqueness of the minimum point is proved by contradiction. Assume that $\mathbf{u}$ and $\mathbf{u}' \in \mathbf{F}$: $f(\mathbf{u}) = f(\mathbf{u}') = m \equiv \min \{f(\tilde{\mathbf{u}}) \mid \tilde{\mathbf{u}} \in \mathbf{F}\}$. Then because of strict convexity for $\lambda \in (0, 1)$ we have

$$f(\lambda \mathbf{u}' + (1 - \lambda) \mathbf{u}) < \lambda f(\mathbf{u}') + (1 - \lambda) f(\mathbf{u}) = m,$$

which is in contradiction with the minimum condition.

A proof of equivalence of the minimization problem and the variational inequality (3.42) up to insignificant details coincides with that for the finite-dimensional version of this statement presented in the beginning of Sect. 3.2. A more general case, where the function involves a nondifferentiable term, can be studied with the help of the infinite-dimensional version of the Moreau–Rockafellar theorem (see, for example, [1]). $\square$

To conclude, we cite Korn's inequality. Different versions of its proof can be found in [4, 8, 18, 23].

Assume that $\Omega \subset \mathbb{R}^m$ ($m = 2, 3$) is a bounded open domain with a regular boundary (a star domain, [18], a domain with a Lipschitz boundary, [4], a domain with a boundary satisfying the cone condition, [23], or the strong cone condition, [8]) and on this domain a vector field of displacements $\mathbf{u}(\mathbf{x})$, which belongs to the Sobolev space $\mathbf{H}^1(\Omega)$ of generalized functions integrable over Ω together with first-order distributional derivatives, is given. Then there exists a positive constant a_0 , which depends only on the domain Ω , such that

$$\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}) d\Omega + \int_{\Omega} \mathbf{u} \cdot \mathbf{u} d\Omega \geq a_0 \|\mathbf{u}\|_1^2. \quad (3.43)$$

Here $\boldsymbol{\varepsilon}(\mathbf{u})$ is a small strains tensor calculated by the formula

$$2 \boldsymbol{\varepsilon}(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^*,$$

$\nabla \mathbf{u}$ is a strain gradients tensor, asterisk means transposition, point and colon mean scalar product and double tensors convolution. The norm on the space $\mathbf{H}^1(\Omega)$ is

$$\|\mathbf{u}\|_1 = \sqrt{\int_{\Omega} (\nabla \mathbf{u} : (\nabla \mathbf{u})^* + \mathbf{u} \cdot \mathbf{u}) d\Omega}.$$

Korn's inequality enables one to introduce an equivalent norm on $\mathbf{H}^1(\Omega)$, namely, the square root of the left-hand side of (3.43). This result is quite nontrivial and hard to prove since the left-hand side involves certain combinations of partial derivatives of a vector function $\mathbf{u}(\mathbf{x})$ whereas the right-hand side involves all derivatives.

Assume that $\mathbf{u}(\mathbf{x}) = \mathbf{0}$ on a part $\Gamma_u \subset \Gamma$ of the boundary of a domain Ω , which has a strictly positive surface measure. Then there exists a constant $a_1 > 0$ such that

$$I(\mathbf{u}) \equiv \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}) d\Omega \geq a_1 \|\mathbf{u}\|_1^2. \quad (3.44)$$

This statement can be proved on the basis of Korn's inequality by contradiction. According to the opposite statement, there exists a sequence of functions $\mathbf{u}_n \in \mathbf{H}^1(\Omega)$ satisfying a homogeneous boundary condition on Γ_u such that $\|\mathbf{u}_n\|_1 = 1$ and $I(\mathbf{u}_n) \rightarrow 0$. The space $\mathbf{H}^1(\Omega)$ is a Hilbert space and, hence, is reflexive, therefore a unit ball in it is weakly compact. Because of this, we can select a weakly convergent subsequence of $\mathbf{u}_n$. Denote it by $\mathbf{u}_n$ as well.

The functional $I(\mathbf{u})$ is convex, since it is the integral of the convex function $f(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}$, and continuous on the whole space $\mathbf{H}^1(\Omega)$. Hence, it is weakly lower semicontinuous, thus

$$\liminf_{n \rightarrow \infty} I(\mathbf{u}_n) \geq I(\mathbf{u}) = 0.$$

In the general case the equation $I(\mathbf{u}) = 0$, i.e. $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}$, has nontrivial solutions of the form $\mathbf{u} = \mathbf{w} + \boldsymbol{\omega} \cdot \mathbf{x}$ ($\mathbf{w}$ is an arbitrary constant vector and $\boldsymbol{\omega} = -\boldsymbol{\omega}^*$ is an antisymmetric tensor) which describe displacement of a domain Ω as of a rigid unit. The set of these solutions is a subspace of $\mathbf{H}^1(\Omega)$ which is further denoted by $\mathbf{R}_0$. Taking into account the boundary condition on Γ_u , we have $\mathbf{w} = \boldsymbol{\omega} = \mathbf{0}$. Hence, $\mathbf{u} \equiv \mathbf{0}$.

By the Sobolev embedding theorem, [22], a unit ball of $\mathbf{H}^1(\Omega)$ is a compact set in $L_2(\Omega)$, hence, the sequence $\mathbf{u}_n$ can be assumed to converge to zero with respect to the norm of this space:

$$\|\mathbf{u}\|_0 = \sqrt{\int_{\Omega} \mathbf{u} \cdot \mathbf{u} d\Omega}.$$

Passing to the limit in inequality (3.43) written for the elements of this sequence

$$I(\mathbf{u}_n) + \|\mathbf{u}_n\|_0^2 \geq a_0 \|\mathbf{u}_n\|_1^2 = a_0,$$

we arrive at contradiction with the condition $a_0 > 0$ which proves (3.44).

The inequality (3.44) means that $\sqrt{I(\mathbf{u})}$ is a norm equivalent to $\|\mathbf{u}\|_1$. Let $\Gamma_u = \emptyset$. In this case $\sqrt{I(\mathbf{u})}$ is only a seminorm on $\mathbf{H}^1(\Omega)$ and a norm on the factor space $\mathbf{H}^1(\Omega)/\mathbf{R}_0$. Equivalence classes (functions $\mathbf{u}$ and $\mathbf{u}'$ belongs to the same class U_0 provided that $\mathbf{u}' - \mathbf{u} \in \mathbf{R}_0$) are elements of the factor space. One more norm on the factor space induced by the norm of the main space is defined by the equality

$\|U_0\| = \inf_{u \in U_0} \|u\|_1$. We prove equivalence of the norms:

$$I(U_0) \geq a_2 \|U_0\|^2, \quad a_2 > 0, \quad (3.45)$$

where $I(U_0)$ is defined as $I(u)$ for any element $u \in U_0$.

To this end, we replace $\|u\|_1$ by the equivalent norm whose square is equal to $I(u) + \|u\|_0^2$. Then the problem is reduced to a proof of the inequality

$$I(u) \geq a_3 \left(I(u) + \inf_{\tilde{u} \in R_0} \|u - \tilde{u}\|_0^2 \right), \quad a_3 > 0, \quad (3.46)$$

since it results in (3.45) due to Korn's inequality.

In fact, the expression for $I(u)$ involves only strains which vanish on vectors of the space R_0 , therefore both sides of the inequality (3.46) do not depend on the choice of a particular element u out of the corresponding equivalence class U_0 . We specify the element $u \in U_0$ on which infimum in (3.46) is achieved. It satisfies the condition

$$\|u\|_0^2 \leq \|u - w - \omega \cdot x\|_0^2 \quad \forall w, \omega = -\omega^*.$$

Rearranging the squared norm, we obtain

$$2(u, w + \omega \cdot x)_0 \leq \|w + \omega \cdot x\|_0^2,$$

where $(u, v)_0$ is the scalar product in $L_2(\Omega)$. Replacing w and ω by $\pm \lambda w$ and $\pm \lambda \omega$, respectively, and passing to the limit as $\lambda \rightarrow 0$, we get

$$(u, w + \omega \cdot x) \equiv \int_{\Omega} u \cdot (w + \omega \cdot x) d\Omega = 0.$$

Since the product of an antisymmetric tensor and a vector in Cartesian coordinate system

$$\omega \cdot x = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \omega \times x$$

is equal to the vector product of the vector $\omega = (\omega_1, \omega_2, \omega_3)$ and x , using the formula $u \cdot (\omega \times x) = \omega \cdot (x \times u)$ for a mixed product and taking into account that vectors w and ω are arbitrary, we obtain the following conditions for u :

$$\int_{\Omega} u d\Omega = \int_{\Omega} x \times u d\Omega = 0. \quad (3.47)$$

By the definition of a projection $\pi_0(\mathbf{u})$ onto the subspace $\mathbf{R}_0$

$$\inf_{\tilde{\mathbf{u}} \in \mathbf{R}_0} \|\mathbf{u} - \tilde{\mathbf{u}}\|_0 = \|\mathbf{u} - \pi_0(\mathbf{u})\|_0,$$

hence, to prove (3.46) it is sufficient to prove the inequality

$$I(\mathbf{u}) \geq a_4 \|\mathbf{u} - \pi_0(\mathbf{u})\|_0^2, \quad a_4 > 0.$$

We assume the opposite, i.e. that there exists a sequence of functions $\mathbf{u}_n \in \mathbf{H}^1(\Omega)$ such that $I(\mathbf{u}_n) / \|\mathbf{u}_n - \pi_0(\mathbf{u}_n)\|_0^2$ tends to zero. Taking into account the homogeneity of an orthoprojector onto a subspace: $\pi_0(\lambda \mathbf{u}) = \lambda \pi_0(\mathbf{u})$, we suppose that $\|\mathbf{u}_n - \pi_0(\mathbf{u}_n)\|_0 = 1$ since this case can be obtained by passing to the sequence $\mathbf{u}_n / \|\mathbf{u}_n - \pi_0(\mathbf{u}_n)\|_0$.

The sequence of functions $\mathbf{v}_n = \mathbf{u}_n - \pi_0(\mathbf{u}_n)$ satisfies two conditions

$$I(\mathbf{v}_n) = I(\mathbf{u}_n) \rightarrow 0, \quad \|\mathbf{v}_n\|_0 = 1,$$

hence, it is bounded with respect to the norm $\sqrt{I(\mathbf{u}) + \|\mathbf{u}\|_0^2}$ and to the norm of the space $\mathbf{H}^1(\Omega)$. A bounded closed set in $\mathbf{H}^1(\Omega)$ is weakly compact, hence, the subsequence $\mathbf{v}_n$ weakly converges to some element $\mathbf{v}$. Since the functional $I(\mathbf{u})$ is weakly lower semicontinuous, we have

$$I(\mathbf{v}) \leq \liminf_{n \rightarrow \infty} I(\mathbf{v}_n) = 0.$$

Therefore $I(\mathbf{v}) = 0$ and $\mathbf{v} \in \mathbf{R}_0$. On the other hand, $\mathbf{v}_n = \mathbf{u}_n - \pi_0(\mathbf{u}_n)$ are elements of the orthogonal complement $\mathbf{R}_0^\perp$ of the subspace $\mathbf{R}_0$ with respect to the whole space. The subspace $\mathbf{R}_0^\perp$ is closed and weakly closed, hence, $\mathbf{v} \in \mathbf{R}_0^\perp$. Thus, $\mathbf{v} \equiv \mathbf{0}$. By the embedding theorem $\mathbf{v}_n \rightarrow \mathbf{0}$ in the space $\mathbf{L}_2(\Omega)$ which is in contradiction with the condition $\|\mathbf{v}_n\|_0 = 1$. The inequalities (3.45) and (3.46) are both proved.

From the inequality (3.46) it follows that on the class of functions $\mathbf{u} \in \mathbf{H}^1(\Omega)$, satisfying the conditions (3.47), the inequality (3.44) of norm equivalence holds.

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Chapter 4

Spatial Constitutive Relationships

Abstract The constitutive relationships for a granular material involving absolutely rigid and elastic particles as well as the constitutive equations for a heteromodular elastic material are generalized to the case of a spatial stress-strain state under small strains. Versions of constraints on admissible stress tensors for an isotropic material, which are defined with the help of the Coulomb–Mohr and von Mises–Schleicher cones, are considered. Dual cones of admissible strain tensors are constructed. The projection operators, which are used further in algorithms for numerical realization of spatial models, are presented.

4.1 Granular Material With Elastic Properties

In this chapter, when constructing constitutive relationships for spatial deformation of a material being heteroresistant to tension and compression, we use the symmetric stress tensor σ and the symmetric strain tensor ε which can be considered as elements of two different six-dimensional spaces. The double convolution operation $\sigma : \varepsilon$ defines the duality relation between these spaces. The stress tensor space becomes conjugate to the strain tensors space and conversely. Let $\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(6)}$ and $\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(6)}$ be reciprocal bases such that $\sigma^{(i)} : \varepsilon^{(j)} = \delta_{ij}$, then

$$\sigma = \sigma_i \sigma^{(i)}, \quad \varepsilon = \varepsilon_j \varepsilon^{(j)}, \quad \sigma : \varepsilon = \sigma_i \varepsilon_i$$

Here, as before, summation over repeating indices $i, j = 1, 2, \dots, 6$ is assumed. To every tensor there corresponds a vector which belongs to $\mathbb{R}^6$. The tensors $\mathbf{a}, \hat{\mathbf{a}} = \mathbf{a}^{-1}$ of rank four of elastic coefficients (of moduli of elastic compliance and moduli of elasticity)

$$\varepsilon = \mathbf{a} : \sigma, \quad \sigma = \hat{\mathbf{a}} : \varepsilon$$

are linear operators on the considered spaces, hence, as any linear operator, in these bases they are defined by square matrices with coefficients a_{ij} and $\hat{a}_{ij}$. The corresponding images are calculated as matrix–vector products.

Consider a stress-strain state of an ideal granular material, consisting of absolutely rigid particles, whose rheological scheme of uniaxial deformation involves a single element, namely, a rigid contact. In the spatial case, the variety of versions of a deformation process of such a material, which qualitatively differ from each other, obviously is not limited to tension and compression. In this process size, shape, packing density of particles, non-homogeneity in size and shape, moisture, etc. are of considerable importance. Assume that a granular material, whose behavior is described in the context of mechanics of isotropic continua, is a set of fine-dispersed particles of spherical shape of equal radius being before deformation in the natural state of dense packing where effective density $\rho = m N$ (m is the mass of a particle, N is the number of particles in the volume unit) of a material with a porous space taken into account is maximal. Deformation for this state can be accompanied only by loosening, therefore the actual strain tensor, which is equal to zero in the natural state, may not be arbitrary. It satisfies the constraint $\boldsymbol{\varepsilon} \in \mathcal{C}$ which excludes, for example, uniaxial compression. The specific form of the set $\mathcal{C}$ of admissible strain tensors depends on main properties of a material listed above.

The interior points of $\mathcal{C}$ correspond to a loose material whose particles do not come into contact with each other. In this state the actual stress tensor is equal to zero. Neglecting sizes of particles, we assume that if $\boldsymbol{\varepsilon}$ is an interior point of the set $\mathcal{C}$ then $\lambda \boldsymbol{\varepsilon}$ is an interior point of $\mathcal{C}$ as well for any positive λ , i.e. that both tensors $\boldsymbol{\varepsilon}$ and $\lambda \boldsymbol{\varepsilon}$ simultaneously describe the loose state of a material. In this case $\mathcal{C}$ is a cone with the vertex at zero. In addition, provided that a superposition (sum) of the loosening strains leads to the loosening of a material as well, the cone $\mathcal{C}$ is assumed to be convex.

On the phenomenological level, the boundary points of $\mathcal{C}$ define the process of coordinated displacement of the particles of a granular material being in contact with each other. When moving in such a way, stresses arise in a material. Strain tensors, which do not belong to $\mathcal{C}$, are excluded since interpenetration of rigid particles in each other is impossible. For an isotropic material, constraints are formulated in terms of invariants of the tensor $\boldsymbol{\varepsilon}$, hence, in the three-dimensional space of principal strains $\varepsilon_1, \varepsilon_2, \varepsilon_3$ the cone $\mathcal{C}$ is shown as a convex cone symmetric with respect to change of principal axes. With the help of the statement about convexity of a function depending on invariants of symmetric tensor, presented in Sect. 3.1, we can prove that in this case $\mathcal{C}$ is a convex cone in the six-dimensional tensor space.

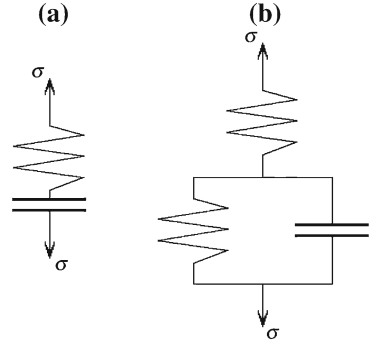
The constitutive relationships for a material with absolutely rigid particles can be obtained by formal generalization of the inclusion (2.3):

$$\boldsymbol{\sigma} \in \partial \Phi(\boldsymbol{\varepsilon}), \quad (4.1)$$

where $\Phi(\boldsymbol{\varepsilon}) = \delta_{\mathcal{C}}(\boldsymbol{\varepsilon})$ (stress potential) is the indicator function of the cone $\mathcal{C}$. Besides, it is implicitly assumed that the deformation process for such a material is a thermodynamically reversible process where the energy dissipation is entirely absent and it differs from nonlinearly elastic deformation by a special form of potential only.

The strain potential $\Psi(\boldsymbol{\sigma})$ is determined as the Young–Fenchel transform for $\Phi(\boldsymbol{\varepsilon})$. According to the results presented in Chap. 3, it is the indicator function $\delta_{\mathcal{K}}(\boldsymbol{\sigma})$ of

Fig. 4.1 Rheological schemes: **a** elastic granular material, **b** heteromodular elastic material



the dual cone

$$\mathbf{K} = \{\boldsymbol{\sigma} \mid \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \leq 0 \quad \forall \boldsymbol{\varepsilon} \in \mathbf{C}\}.$$

By the corollary of the theorem 3.8 about involutivity of the Young–Fenchel transform, we have the inclusion

$$\boldsymbol{\varepsilon} \in \partial\Psi(\boldsymbol{\sigma}). \quad (4.2)$$

Inside $\mathbf{K}$ the indicator function is differentiable and its derivative is equal to zero, hence, $\boldsymbol{\varepsilon} = \mathbf{0}$. On the boundary the subdifferential involves nonzero subgradients $\boldsymbol{\varepsilon} \neq \mathbf{0}$. Thus, the cone $\mathbf{K}$ is the set of admissible stress tensors for a granular material involving rigid particles whose interior points describe the state of a material as that of a solid unit, and the boundary points correspond to admissible strain states.

In the case of elastic granular material, whose rheological scheme is shown in Fig. 4.1a, the strain tensor is decomposed into a sum of two terms which correspond to an elastic spring and a rigid contact. Hence, the strain potential is also decomposed into the sum

$$\Psi(\boldsymbol{\sigma}) = \frac{1}{2} \boldsymbol{\sigma} : \mathbf{a} : \boldsymbol{\sigma} + \delta_{\mathbf{K}}(\boldsymbol{\sigma}),$$

where $\mathbf{a}$ is the tensor of moduli of elastic compliance of a spring. By the definition of a subdifferential, the inclusion (4.2) is equivalent to the variational inequality

$$(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}) : (\mathbf{a} : \boldsymbol{\sigma} - \boldsymbol{\varepsilon}) \geq 0, \quad \boldsymbol{\sigma}, \tilde{\boldsymbol{\sigma}} \in \mathbf{K}, \quad (4.3)$$

which is studied in Sect. 3.2 in a more general form. As usually, we assume that the tensor $\mathbf{a}$ is symmetric and positive definite (its matrix is symmetric and positive definite). Then the stress tensor $\boldsymbol{\sigma} = \pi_{\mathbf{a}}(s)$, which is equal to the projection of the conditional stress tensor s , determined from the linear Hooke law $\mathbf{a} : s = \boldsymbol{\varepsilon}$, onto the cone $\mathbf{K}$ with respect to the norm $|\boldsymbol{\sigma}|_{\mathbf{a}} = \sqrt{\boldsymbol{\sigma} : \mathbf{a} : \boldsymbol{\sigma}}$, is a solution of the variational inequality (4.3).

It turns out that in the context of a model, which takes into account elastic properties of particles, the actual stress tensor is subject to the constraint $\sigma \in K$ as well, whereas the strain tensor can be arbitrary. If σ belongs to the interior of K , then the process of deformation of a material is described in the framework of the elasticity theory. When σ is on the boundary, strain caused by the slip of particles is added to elastic one. The vertex of the cone K (the point $\sigma = 0$) corresponds to the loosened state where particles do not come into contact with each other.

Assume that the cone K can be represented in the form $f_l(\sigma) \leq 0$, $l = 1, \dots, n$, where f_l is a system of convex and differentiable functions. Then by the Kuhn–Tucker theorem the problem of determining a projection (the problem of minimization of the distance $|\sigma - s|_a$ on the cone K) is equivalent to finding a saddle point of the Lagrange function:

$$L(\sigma, \lambda) = \frac{1}{2} |\sigma - s|_a^2 + \sum_{l=1}^n \lambda_l f_l(\sigma), \quad \lambda_l \geq 0.$$

In this case the following equations hold:

$$\lambda_l f_l(\sigma) = 0, \quad a : (\sigma - s) + \sum_{l=1}^n \lambda_l \frac{\partial f_l}{\partial \sigma} = 0. \quad (4.4)$$

Constitutive equations of this type were first used by Haar and von Kármán, [3]. In an explicit form they were obtained in the more recent work of Hencky, [4], where the deformation plasticity theory of metals has its origin. Koiter has shown, [5], that the deformation theory in this form is valid only provided that in the zone of plastic strain stresses do not vary with time. This condition allows one to integrate the associated plastic flow rule which is formulated in fact not in terms of the irreversible strain tensor $\epsilon^p = \epsilon - a : \sigma$ as in (4.4) but in terms of the rate of change of this tensor. In the general case, the replacement of the associated plastic flow rule by the finite relationship (4.4) leads to a qualitatively wrong result, namely, when unloading, instead of the strain rate tensor $\dot{\epsilon}^p$, the strain tensor itself turns out to be equal to zero.

In fact, the Haar–von Kármán variational principle, as well as the Hencky constitutive equations obtained basing on it, is intended to describe the behavior of a granular material with elastic particles for which the condition of going a material from the state of limit deformation to the compacted elastic state, replacing the unloading condition in the plasticity theory, is formulated in terms of strains rather than strain rates. Moreover, the use of the associated plastic flow rule for determining a field of velocities in a granular material (see, for example, [1, 6]) should be considered as incorrect provided that this rule does not allow integration with respect to time. The corresponding loading program at least may not involve the unloading stage where a material goes from the limit state to the dense one but, according to the associated flow rule, inelastic strains remain constant.

The immediate calculation of the dual stress potential for a granular material with elastic properties results in the following chain of equalities:

$$\begin{aligned} 2\Phi(\boldsymbol{\varepsilon}) &= \sup_{\boldsymbol{\sigma} \in K} \{2\boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma} : \mathbf{a} : \boldsymbol{\sigma}\} = \boldsymbol{\varepsilon} : \mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \inf_{\boldsymbol{\sigma} \in K} |\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}|_{\mathbf{a}}^2 \\ &= |\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - |\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \pi_{\mathbf{a}}(\mathbf{a}^{-1} : \boldsymbol{\varepsilon})|_{\mathbf{a}}^2 = |\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - |\Pi_{\mathbf{a}^{-1}}(\boldsymbol{\varepsilon})|_{\mathbf{a}^{-1}}^2. \end{aligned}$$

Here $\mathbf{a}^{-1}$ is the inverse tensor of the elasticity moduli, $\Pi_{\mathbf{a}^{-1}}$ is the projection operator onto the cone $\mathbf{C}$ with respect to the norm associated with the tensor $\mathbf{a}^{-1}$. This chain is obtained with the help of the obvious formula of separating perfect square

$$2\boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma} : \mathbf{a} : \boldsymbol{\sigma} = \boldsymbol{\varepsilon} : \mathbf{a}^{-1} : \boldsymbol{\varepsilon} - |\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}|_{\mathbf{a}}^2$$

and of Eq. (3.14) which relates operators of projection onto dual cones:

$$\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \pi_{\mathbf{a}}(\mathbf{a}^{-1} : \boldsymbol{\varepsilon}) = \mathbf{a}^{-1} : \Pi_{\mathbf{a}^{-1}}(\boldsymbol{\varepsilon}).$$

Analyzing the obtained expression for Φ , we can see that $\Phi(\boldsymbol{\varepsilon}) = 0$ for any $\boldsymbol{\varepsilon} \in \mathbf{C}$: any such a tensor describes the loosened state of a material for the stress tensor equal to zero. From the results of Sect. 3.2 it follows that $\Phi(\boldsymbol{\varepsilon})$ is a convex and differentiable function, besides, according to (3.14) and (3.15)

$$\boldsymbol{\sigma} = \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}} = \mathbf{a}^{-1} : (\boldsymbol{\varepsilon} - \Pi_{\mathbf{a}^{-1}}(\boldsymbol{\varepsilon})) \equiv \pi_{\mathbf{a}}(\boldsymbol{\varepsilon}).$$

An equivalent approach to the description of an ideal granular material with elastic particles is to define a differentiable convex stress potential, which is equal to zero on a convex cone $\mathbf{C}$ in the strain tensor space and is strictly positive exterior to $\mathbf{C}$. According to the physically nonlinear Hooke law $\boldsymbol{\sigma} = \partial \Phi / \partial \boldsymbol{\varepsilon}$, stresses corresponding to strains of this cone are equal to zero as well. The dual strain potential, which is determined with the help of the Young–Fenchel transform, turns out to be infinite exterior to the dual cone $\mathbf{K}$. The degenerate potential of a heteromodular elastic medium by Myasnikov and Oleinikov, [7–9], is an example of the stress potential which is equal to zero on a convex cone.

As noticed in Chap. 2, the constitutive relationships of an ideal granular material are incorrect, namely, for the stress tensor equal to zero, various strained states of a material are possible. We consider a model of the heteromodular elasticity theory, whose rheological scheme is given in Fig. 4.1b, as a regularizing one. Let $\boldsymbol{\sigma}^c$ and $\boldsymbol{\varepsilon}^c$ be the stress and strain tensors in a rigid contact, respectively, $\mathbf{b}$ be the tensor of moduli of elastic compliance of an elastic element parallel to the contact. Then, according to this scheme,

$$\boldsymbol{\varepsilon}^c = \mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}^c), \quad (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^c) : \boldsymbol{\varepsilon}^c \leq 0, \quad \boldsymbol{\sigma}^c, \tilde{\boldsymbol{\sigma}} \in \mathbf{K}.$$

Hence, $\sigma^c = \pi_b(\sigma)$ where π_b is the operator of projection onto the cone K with respect to the norm $|\sigma|_b$. The constitutive equations for this model take the following form:

$$\varepsilon = a : \sigma + b : (\sigma - \pi_b(\sigma)). \quad (4.5)$$

Due to the formula (3.12) for differentiation, the strain potential is given by

$$\Psi(\sigma) = \frac{1}{2} \left(|\sigma|_a^2 + |\sigma - \pi_b(\sigma)|_b^2 \right).$$

Taking into account (3.14) and (3.16), we can obtain it in the next equivalent forms:

$$\begin{aligned} 2\Psi(\sigma) &= |\sigma|_a^2 + |\Pi_{b^{-1}}(b : \sigma)|_{b^{-1}}^2 = |\sigma|_a^2 + |\sigma|_b^2 - |\pi_b(\sigma)|_b^2 \\ &= |\sigma|_a^2 + |\sigma|_b^2 - |b : \sigma - \Pi_{b^{-1}}(b : \sigma)|_{b^{-1}}^2 \\ &= |\sigma|_a^2 + |\sigma|_b^2 - \inf_{\tilde{\varepsilon} \in C} |b : \sigma - \tilde{\varepsilon}|_{b^{-1}}^2. \end{aligned}$$

Unlike the strain potential for an ideal granular material, the obtained potential is a differentiable function. It simulates the behavior of a material whose particles are related to each other by compliant elastic connections which vanish as $b \rightarrow \infty$. For such a material, the stress tensor as well as the strain tensor is not subject to any constraint. The dual potential is determined as the Young–Fenchel transform for $\Psi(\sigma)$:

$$2\Phi(\varepsilon) = 2 \sup_{\sigma} \{ \sigma : \varepsilon - \Psi(\sigma) \} = \sup_{\sigma} \inf_{\tilde{\varepsilon} \in C} L_{\varepsilon}(\sigma, \tilde{\varepsilon}).$$

Here L_{ε} is the Lagrange function which depends on the tensor ε as on a parameter:

$$L_{\varepsilon}(\sigma, \tilde{\varepsilon}) = 2\sigma : \varepsilon - |\sigma|_a^2 - |\sigma|_b^2 + |b : \sigma - \tilde{\varepsilon}|_{b^{-1}}^2 = 2\sigma : (\varepsilon - \tilde{\varepsilon}) - |\sigma|_a^2 + |\tilde{\varepsilon}|_{b^{-1}}^2.$$

For the fixed argument σ the function L_{ε} is convex and for fixed $\tilde{\varepsilon}$ it is concave. For the tensors σ^0 and $\tilde{\varepsilon}^0 \in C$ which can be taken arbitrarily the following conditions are fulfilled:

$$L_{\varepsilon}(\sigma^0, \tilde{\varepsilon}) \rightarrow +\infty \text{ as } |\tilde{\varepsilon}| \rightarrow \infty, \quad L_{\varepsilon}(\sigma, \tilde{\varepsilon}^0) \rightarrow -\infty \text{ as } |\sigma| \rightarrow \infty.$$

Besides, the mapping

$$(\sigma, \tilde{\varepsilon}) \mapsto \left(-\frac{\partial L_{\varepsilon}(\sigma, \tilde{\varepsilon})}{\partial \sigma} = 2(a : \sigma + \tilde{\varepsilon} - \varepsilon), \frac{\partial L_{\varepsilon}(\sigma, \tilde{\varepsilon})}{\partial \tilde{\varepsilon}} = 2(b^{-1} : \tilde{\varepsilon} - \sigma) \right)$$

is continuous. Hence, as established in Sect. 3.4, there exists a saddle point $(\bar{\sigma}, \bar{\varepsilon}) \in \mathbb{R}^6 \times C$:

$$L_{\varepsilon}(\sigma, \bar{\varepsilon}) \leq L_{\varepsilon}(\bar{\sigma}, \bar{\varepsilon}) \leq L_{\varepsilon}(\bar{\sigma}, \tilde{\varepsilon}) \quad \forall \sigma, \forall \tilde{\varepsilon} \in C.$$

The existence of a saddle point is the sufficient condition for interchanging supremum and infimum, hence,

$$2\Phi(\boldsymbol{\varepsilon}) = \inf_{\tilde{\boldsymbol{\varepsilon}} \in C} \sup_{\boldsymbol{\sigma}} L_{\boldsymbol{\varepsilon}}(\boldsymbol{\sigma}, \tilde{\boldsymbol{\varepsilon}}).$$

Supremum is achieved on the tensor $\tilde{\boldsymbol{\sigma}} = \boldsymbol{a}^{-1} : (\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}})$ since $\partial L_{\boldsymbol{\varepsilon}}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\varepsilon}})/\partial \boldsymbol{\sigma} = \mathbf{0}$. Substituting $\tilde{\boldsymbol{\sigma}}$, we rearrange the expression for $L_{\boldsymbol{\varepsilon}}(\boldsymbol{\sigma}, \tilde{\boldsymbol{\varepsilon}})$ to the form

$$\begin{aligned} L_{\boldsymbol{\varepsilon}}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\varepsilon}}) &= |\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}|_{\boldsymbol{a}^{-1}}^2 + |\tilde{\boldsymbol{\varepsilon}}|_{\boldsymbol{b}^{-1}}^2 = |\boldsymbol{\varepsilon}|_{\boldsymbol{a}^{-1}}^2 + \tilde{\boldsymbol{\varepsilon}} : \boldsymbol{c}^{-1} : \tilde{\boldsymbol{\varepsilon}} - 2\boldsymbol{\varepsilon} : \boldsymbol{a}^{-1} : \tilde{\boldsymbol{\varepsilon}} \\ &= |\boldsymbol{\varepsilon}|_{\boldsymbol{a}^{-1}}^2 - |\boldsymbol{c} : \boldsymbol{a}^{-1} : \boldsymbol{\varepsilon}|_{\boldsymbol{c}^{-1}}^2 + |\boldsymbol{c} : \boldsymbol{a}^{-1} : \boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}|_{\boldsymbol{c}^{-1}}^2. \end{aligned}$$

Here $\boldsymbol{c}^{-1} = \boldsymbol{a}^{-1} + \boldsymbol{b}^{-1}$ is a symmetric and positive definite tensor. When rearranging the above expression and separating perfect square with respect to the tensor $\tilde{\boldsymbol{\varepsilon}}$, the formula

$$(\boldsymbol{c} : \boldsymbol{a}^{-1} : \boldsymbol{\varepsilon}) : \boldsymbol{c}^{-1} : \tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} : (\boldsymbol{c} : \boldsymbol{a}^{-1})^* : \boldsymbol{c}^{-1} : \tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} : \boldsymbol{a}^{-1} : \tilde{\boldsymbol{\varepsilon}},$$

where asterisk means transposition, is essentially used. This formula becomes obvious when going to matrix and vector terms.

Infimum is calculated taking into account the definition of a projection and the equality (3.16):

$$\begin{aligned} 2\Phi(\boldsymbol{\varepsilon}) &= |\boldsymbol{\varepsilon}|_{\boldsymbol{a}^{-1}}^2 - |\boldsymbol{c} : \boldsymbol{a}^{-1} : \boldsymbol{\varepsilon}|_{\boldsymbol{c}^{-1}}^2 + |\boldsymbol{c} : \boldsymbol{a}^{-1} : \boldsymbol{\varepsilon} - \Pi_{\boldsymbol{c}^{-1}}(\boldsymbol{c} : \boldsymbol{a}^{-1} : \boldsymbol{\varepsilon})|_{\boldsymbol{c}^{-1}}^2 \\ &= |\boldsymbol{\varepsilon}|_{\boldsymbol{a}^{-1}}^2 - |\Pi_{\boldsymbol{c}^{-1}}(\boldsymbol{c} : \boldsymbol{a}^{-1} : \boldsymbol{\varepsilon})|_{\boldsymbol{c}^{-1}}^2. \end{aligned}$$

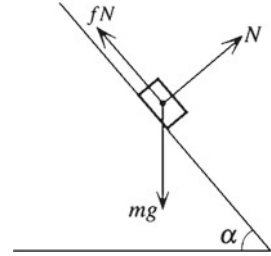
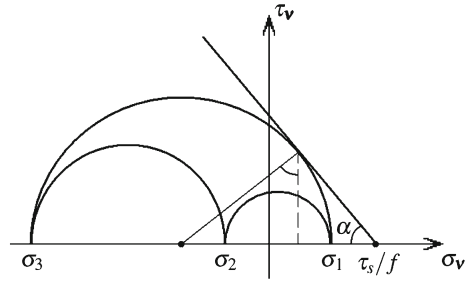
Using the obtained expression for Φ , due to (3.15) we can establish the dependence inverse with respect to (4.5):

$$\boldsymbol{\sigma} = \boldsymbol{a}^{-1} : \left(\boldsymbol{\varepsilon} - \Pi_{\boldsymbol{c}^{-1}}(\boldsymbol{c} : \boldsymbol{a}^{-1} : \boldsymbol{\varepsilon}) \right). \quad (4.6)$$

Thus, the regularized constitutive equations are correct in the mechanical sense. They allow to determine the strain tensor of a material from the given stress tensor and, conversely, the stress tensor from the given strain tensor.

4.2 Coulomb–Mohr Cone

Consider specific versions of cones of admissible stresses and strains. In the theory of soils, the stress cone constructed by the Coulomb–Mohr strength condition is known (see, [13]). According to this condition, the stress state is admissible at a point of a material provided that on any area element passing through this point the tangential stress τ_v does not exceed the limit friction stress: $\tau_v \leq -f\sigma_v + \tau_s$. Here σ_v is the

Fig. 4.2 Limiting equilibrium**Fig. 4.3** Mohr diagram

normal stress, f and τ_s are the coefficients of internal friction and adhesion. When on a slip area elements the equals sign is achieved, the limit state is considered.

Instead of the friction coefficient f , we can introduce the angle α of internal friction which is equal to the slope angle of a plane surface of a granular material such that a particle of mass m lying on it is in the limiting equilibrium state (see Fig. 4.2) under the action of three forces: its own weight mg , the reaction force N , and the friction force fN . By the equilibrium conditions $N = mg \cos \alpha$ and $fN = mg \sin \alpha$. Hence, $\tan \alpha = f$.

On the Mohr diagram, to the limit state of a granular material there corresponds tangency of the larger circle and the limiting straight line $\tau_v = -f\sigma_v + \tau_s$ (Fig. 4.3). Tangent of the slope angle of the limiting straight line is equal to f , hence, this angle coincides with the angle α of internal friction. If principal stresses are renumbered in decreasing order, then the radius of the larger Mohr circle is equal to $(\sigma_1 - \sigma_3)/2$. From geometric considerations

$$\begin{aligned} \tau_v &= \frac{\sigma_1 - \sigma_3}{2} \cos \alpha, & \sigma_1 - \sigma_v &= \frac{\sigma_1 - \sigma_3}{2} (1 - \sin \alpha), \\ \sigma_v - \sigma_3 &= \frac{\sigma_1 - \sigma_3}{2} (1 + \sin \alpha). \end{aligned} \quad (4.7)$$

Besides,

$$\sin \alpha = \frac{(\sigma_1 - \sigma_3)/2}{(\sigma_1 - \sigma_3)/2 - \sigma_1 + \tau_s/f}.$$

Thus, the limiting condition can be written in the form

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3) \sin \alpha = 2 \tau_s \cos \alpha. \quad (4.8)$$

By the Cauchy formula for stresses we have

$$\sigma_{\mathbf{v}} = \sigma_1 v_1^2 + \sigma_2 v_2^2 + \sigma_3 v_3^2, \quad \tau_{\mathbf{v}}^2 = \sigma_1^2 v_1^2 + \sigma_2^2 v_2^2 + \sigma_3^2 v_3^2 - \sigma_{\mathbf{v}}^2,$$

hence, for determining the orientation of slip area elements, defined by the normal vector $\mathbf{v}$: $v_1^2 + v_2^2 + v_3^2 = 1$, we get

$$v_i^2 = \frac{(\sigma_{\mathbf{v}} - \sigma_j)(\sigma_{\mathbf{v}} - \sigma_l) + \tau_{\mathbf{v}}^2}{(\sigma_i - \sigma_j)(\sigma_i - \sigma_l)},$$

where the subscripts i, j and l form an even permutation. With the use of (4.7) we can show that $v_2 = 0$. Hence,

$$\sigma_{\mathbf{v}} = \sigma_1 v_1^2 + \sigma_3 v_3^2, \quad v_3^2 = 1 - v_1^2, \quad v_1^2 = \frac{\sigma_{\mathbf{v}} - \sigma_3}{\sigma_1 - \sigma_3} = \frac{1 + \sin \alpha}{2}.$$

The angle β between the normal vector and the axis x_1 is determined from the equation $\sin \alpha = 2 \cos^2 \beta - 1 = \cos 2\beta$ and is equal to

$$\beta = \pm \left(\frac{\pi}{4} - \frac{\alpha}{2} \right).$$

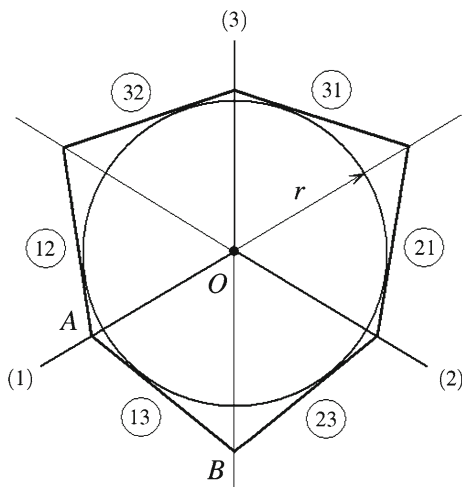
For $\alpha \rightarrow 0$ the Coulomb–Mohr condition is transformed into the Tresca–Saint-Venant plasticity condition for which slip area elements are oriented at an angle $\beta = \pm \pi/4$. For $\alpha \rightarrow \pi/2$ the strength condition for a granular material with maximal angle of internal friction is obtained. In this case $\beta \rightarrow 0$, hence, slip area elements coincide with the coordinate surfaces x_2, x_3 .

In the space of principal stresses the condition (4.8) defines a conical surface with the vertex at the point $\sigma_1 = \sigma_2 = \sigma_3 = \tau_s/f$. This surface can be obtained by shear of the limiting surface for an ideal granular material with the vertex at the origin along the principal octahedral direction for the vector of length $\sqrt{3} \tau_s/f$. In the case that principal stresses are numbered arbitrarily, the admissible stress cone for an ideal granular material is defined in the form

$$\mathbf{K} = \{ \boldsymbol{\sigma} \mid \sigma_i - \sigma_j + (\sigma_i + \sigma_j) \sin \alpha \leq 0, \quad i \neq j \}.$$

The section of the cone with the plane $\sigma_1 + \sigma_2 + \sigma_3 = -3p$ ($p > 0$ is a given hydrostatic pressure) parallel to the deviatoric plane is an irregular hexagon (Fig. 4.4). The side AB is described by the equation $\sigma_1 (1 + \sin \alpha) = \sigma_3 (1 - \sin \alpha)$. In view of symmetry, at the vertex A the equality $\sigma_2 = \sigma_3$ holds, hence,

Fig. 4.4 Section of the Coulomb–Mohr cone



$$\sigma_{1,3} = -3p \frac{1 \mp \sin \alpha}{3 + \sin \alpha}.$$

At the vertex B

$$\sigma_1 = \sigma_2, \quad \sigma_{1,3} = -3p \frac{1 \mp \sin \alpha}{3 - \sin \alpha}.$$

The considered plane lies at the distance of $\sqrt{3}p$ from the origin of the system of principal axes, hence, the squared distances from the point O to the vertices are calculated by the formula $R^2 = |\sigma|^2 - 3p^2$:

$$OA = 2\sqrt{6}p \frac{\sin \alpha}{3 + \sin \alpha}, \quad OB = 2\sqrt{6}p \frac{\sin \alpha}{3 - \sin \alpha}.$$

The length of AB is determined by the cosine rule

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos \frac{\pi}{3},$$

hence,

$$AB = 6\sqrt{2}p \frac{\sqrt{3 + \sin^2 \alpha}}{9 - \sin^2 \alpha} \sin \alpha.$$

The radius of the circumscribed circle is equal to OB . The radius of the inscribed circle is determined due to the obvious equation $AB \cdot r = OA \cdot OB \sin(\pi/3)$ for doubled area of the triangle OAB :

$$r = \sqrt{6} p \frac{\sin \alpha}{\sqrt{3 + \sin^2 \alpha}}.$$

In the absence of internal friction ($\alpha = 0$) the cone degenerates into a ray $\sigma_1 = \sigma_2 = \sigma_3 \leq 0$. In such a medium tangential stresses are impossible. This is a hydrostatic medium wherein pressure is nonnegative. For infinite friction ($\alpha = \pi/2$) the cone coincides with the negative octant of the principal stress space. In this case on elementary area elements positive normal stresses are impermissible and tangential stresses can be arbitrary.

Due to the Farkas theorem from Sect. 3.4, the cone $\mathbf{C}$ dual to the Coulomb–Mohr cone is a convex shell of the system of normal vectors to faces of the cone $\mathbf{K}$:

$$\mathbf{C} = \left\{ \boldsymbol{\varepsilon} = \sum_{i \neq j} \lambda_{ij} \mathbf{f}^{ij} \mid \lambda_{ij} \geq 0 \right\}.$$

The coordinates $\mathbf{f}^{ij}$ of the vectors for all six faces, numbered by corresponding pairs of superscripts in Fig. 4.4, are as follows:

$$\begin{aligned} \mathbf{f}^{23} &= (0, \sin \alpha + 1, \sin \alpha - 1), & \mathbf{f}^{32} &= (0, \sin \alpha - 1, \sin \alpha + 1), \\ \mathbf{f}^{31} &= (\sin \alpha - 1, 0, \sin \alpha + 1), & \mathbf{f}^{13} &= (\sin \alpha + 1, 0, \sin \alpha - 1), \\ \mathbf{f}^{12} &= (\sin \alpha + 1, \sin \alpha - 1, 0), & \mathbf{f}^{21} &= (\sin \alpha - 1, \sin \alpha + 1, 0). \end{aligned}$$

The convex shell is a hexahedral cone with the vertex at zero. Normals to its faces are defined by pair vector products of the vectors given above. The direct calculation of the vector products via formal determinants yields

$$\begin{aligned} \mathbf{f}^{13} \times \mathbf{f}^{12} &= \begin{vmatrix} (1) & (2) & (3) \\ \sin \alpha + 1 & 0 & \sin \alpha - 1 \\ \sin \alpha + 1 & \sin \alpha - 1 & 0 \end{vmatrix} = \left(-(\sin \alpha - 1)^2, -\cos^2 \alpha, -\cos^2 \alpha \right), \\ \mathbf{f}^{31} \times \mathbf{f}^{12} &= \begin{vmatrix} (1) & (2) & (3) \\ \sin \alpha - 1 & 0 & \sin \alpha + 1 \\ \sin \alpha - 1 & \sin \alpha + 1 & 0 \end{vmatrix} = \left(-(\sin \alpha + 1)^2, -\cos^2 \alpha, -\cos^2 \alpha \right). \end{aligned}$$

The former vector is the normal vector to the face spanned by $\mathbf{f}^{13}$ and $\mathbf{f}^{12}$, and the latter one is the normal vector to the face spanned by $\mathbf{f}^{31}$ and $\mathbf{f}^{12}$. The normal vectors to the remaining four faces are determined in a similar way. As a result, the cone $\mathbf{C}$ takes the form

$$\mathbf{C} = \left\{ \boldsymbol{\varepsilon} \mid \varepsilon_i (\sin \alpha \pm 1)^2 + (\varepsilon_j + \varepsilon_l) \cos^2 \alpha \geq 0, \quad i \neq j, j \neq l, l \neq i \right\}.$$

Assume that in the neighborhood of the state of hydrostatic compression a granular material defined by the Coulomb–Mohr cone of admissible stresses behaves like an isotropic elastic solid with the bulk modulus k and the shear modulus μ . In such

a material, stresses for given strains are calculated with the help of Eqs. (4.4). If the strain tensor is given in an arbitrary coordinate system, then at the first stage of calculations the system of principal axes is constructed. In the general case this problem is solved, for example, with the help of the Jacobi rotation method. Once the process has been completed and principal strains have been determined, the rotation of axes to the original coordinate system is performed.

If strains are such that $\boldsymbol{\varepsilon} \in \mathbf{C}$, then, as established above, $\sigma_1 = \sigma_2 = \sigma_3 = 0$. For $\boldsymbol{\varepsilon} \notin \mathbf{C}$ unknown principal stresses are determined as projections of the conditional stress vector

$$s_i = \left(k - \frac{2\mu}{3}\right) \theta + 2\mu \varepsilon_i, \quad \theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

depending on the location of this vector relative to the cone $\mathbf{K}$. In particular, for $\mathbf{s} \in \mathbf{K}$ $\sigma_i = s_i$. If conditional stresses are previously renumbered in decreasing order ($s_1 \geq s_2 \geq s_3$), then three more alternatives are possible. In the first case the point $\boldsymbol{\sigma}$ belongs to the face of the conical surface marked by 1 and 3 in Fig. 4.4. The solution of the system (4.4) has the following form:

$$\begin{aligned} \sigma_{1,3} &= s_{1,3} - \lambda_1 \left(\left(k + \frac{\mu}{3}\right) \sin \alpha \pm \mu \right), \quad \sigma_2 = s_2 - \lambda_1 \left(k - \frac{2\mu}{3} \right) \sin \alpha, \\ \lambda_1 &= \frac{s_1 (1 + \sin \alpha) - s_3 (1 - \sin \alpha)}{2 (k + \mu/3) \sin^2 \alpha + 2\mu} \geq 0. \end{aligned} \quad (4.9)$$

This alternative takes place for

$$\mu (1 + \sin \alpha) \lambda_1 \leq s_1 - s_2, \quad \mu (1 - \sin \alpha) \lambda_1 \leq s_2 - s_3.$$

In the remaining cases the point to be found lies on one of two edges of the cone. In the second case the solution of the system (4.4) after elimination of one of the Lagrange multipliers has the next form:

$$\begin{aligned} \sigma_1 &= s_1 - \lambda_2 \left(\left(k + \frac{\mu}{3}\right) \sin \alpha + \mu \right), \quad \sigma_2 = \sigma_3 = \sigma_1 \frac{1 + \sin \alpha}{1 - \sin \alpha}, \\ \lambda_2 &= \frac{2s_1(1 + \sin \alpha) - (s_2 + s_3)(1 - \sin \alpha)}{(4k + \mu/3) \sin^2 \alpha + 2\mu \sin \alpha + 3\mu}, \quad \lambda_2 \geq \frac{s_2 - s_3}{\mu (1 - \sin \alpha)}. \end{aligned} \quad (4.10)$$

In the third case

$$\begin{aligned} \sigma_1 &= \sigma_2 = \sigma_3 \frac{1 - \sin \alpha}{1 + \sin \alpha}, \quad \sigma_3 = s_3 - \lambda_3 \left(\left(k + \frac{\mu}{3}\right) \sin \alpha - \mu \right), \\ \lambda_3 &= \frac{(s_1 + s_2)(1 + \sin \alpha) - 2s_3(1 - \sin \alpha)}{(4k + \mu/3) \sin^2 \alpha - 2\mu \sin \alpha + 3\mu}, \quad \lambda_3 \geq \frac{s_1 - s_2}{\mu (1 + \sin \alpha)}. \end{aligned} \quad (4.11)$$

The limiting case of these formulae for $\alpha = \pi/2$ is applied in the numerical solution of problems of dynamics of an ideal elastic granular material in [12]. For numerical computations the fact that Eqs. (4.9)–(4.11) define a projection is of considerable importance. Any operator of projection onto a convex set is a non-expanding mapping, therefore recalculation of stresses σ via s by these formulae does not lead to increase of round-off errors. The calculation process turns out to be stable.

4.3 Von Mises–Schleicher Cone

To describe approximately admissible stresses in an ideal granular material, one can use the circular cone related to the von Mises–Schleicher strength condition:

$$K = \{\sigma \mid \tau(\sigma) \leq \mathfrak{x} p(\sigma)\}.$$

Here $\tau(\sigma) = \sqrt{\sigma' : \sigma' / 2}$ is the intensity of tangential stresses, prime means deviator of a tensor: $\sigma' = \sigma + p(\sigma) \delta$, $p(\sigma) = -\sigma : \delta / 3$ is the hydrostatic pressure, δ is the metric tensor, $\mathfrak{x}$ is the parameter of internal friction. In the space of principal stresses

$$\tau(\sigma) = \sqrt{\frac{1}{6} \sum_{i>j} (\sigma_i - \sigma_j)^2}, \quad p(\sigma) = -\frac{\sigma_1 + \sigma_2 + \sigma_3}{3},$$

the equation of conical surface is written in the form

$$\sqrt{(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2} = -\sqrt{\frac{2}{3}} \mathfrak{x} (\sigma_1 + \sigma_2 + \sigma_3).$$

The cone axis coincides with the principal octahedral axis. Cosine of the angle φ at the cone vertex can be calculated via the scalar product of the unit vector $(\sigma_1, \sigma_2, \sigma_3)$ belonging to the surface and the octahedral direction vector $-(1, 1, 1)/\sqrt{3}$:

$$\cos \varphi = -\frac{1}{\sqrt{3}} (\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{\sqrt{2\mathfrak{x}}} \sqrt{(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2}.$$

We square the first equation of this chain and, taking into account the equality $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1$, get

$$\sigma_2 \sigma_3 + \sigma_3 \sigma_1 + \sigma_1 \sigma_2 = \frac{1}{2} (3 \cos^2 \varphi - 1).$$

In a similar way from the second equation we obtain

$$\mathfrak{x}^2 \cos^2 \varphi = 1 - (\sigma_2 \sigma_3 + \sigma_3 \sigma_1 + \sigma_1 \sigma_2) = \frac{3}{2} \sin^2 \varphi.$$

Hence,

$$\tan \varphi = \sqrt{\frac{2}{3}} \mathfrak{x}. \quad (4.12)$$

The dual cone $\mathcal{C}$ is circular as well and its axis coincides with the principal octahedral axis. The angle at the vertex is equal to $\pi/2 - \varphi$. Taking the equation of a dual cone in the following general form

$$\mathcal{C} = \{\boldsymbol{\varepsilon} \mid \gamma(\boldsymbol{\varepsilon}) \leq \kappa \theta(\boldsymbol{\varepsilon})\},$$

where $\gamma = \sqrt{2 \boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}'}$ is the shear intensity and $\theta(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} : \boldsymbol{\delta}$ is the volume strain, we determine the parameter κ . Since

$$\gamma(\boldsymbol{\varepsilon}) = \sqrt{\frac{2}{3} \sum_{i>j} (\varepsilon_i - \varepsilon_j)^2}, \quad \theta(\boldsymbol{\varepsilon}) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

in the system of principal axes of the strain tensor the equation of conical surface for $\mathcal{C}$ is reduced to the form

$$\sqrt{(\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 + (\varepsilon_1 - \varepsilon_2)^2} = \sqrt{\frac{3}{2}} \kappa (\varepsilon_1 + \varepsilon_2 + \varepsilon_3).$$

Repeating the calculations for determining the angle, we finally arrive at

$$\cot \varphi = \tan\left(\frac{\pi}{2} - \varphi\right) = \sqrt{\frac{3}{2}} \kappa,$$

hence, due to (4.12) $\kappa = 1/\mathfrak{x}$.

Notice that in the model of a granular material with absolutely rigid particles, where the stress potential is equal to $\delta_{\mathcal{C}}(\boldsymbol{\varepsilon})$, the shear strain may not exceed the quantity proportional to the volume strain. This model describes the dilatancy effect which was established experimentally by Reynolds, [11]: with shear the volume of a dense packed material increases. If the volume is bounded by external conditions, then shear of a material involving rigid particles is impossible. With elasticity of particles taken into account, the quantity of shear in a bounded volume is of order of the ratio of tangential stresses and modulus μ . However, in the free state an elastic material dilates as well. This will be shown in Chap. 9.

In the case of the von Mises–Schleicher circular cone, the formulae for calculation of stresses from given strains in a material involving elastic particles are considerably simplified in comparison with (4.9)–(4.11). They take the tensor form and going to principal axes is not required. As before, the conditional stress tensor is determined

according to the linear Hooke law:

$$p(s) = -k \theta(\epsilon), \quad s' = 2 \mu \epsilon'. \quad (4.13)$$

If $\epsilon \in C$, then stresses turn out to be zero. This takes place under the condition $\mu p(s) + \alpha k \tau(s) \leq 0$ which follows from the inequality $\alpha \gamma(\epsilon) \leq \theta(\epsilon)$ in view of (4.13). If $s \in K$, then $\sigma = s$. If $s \notin K$ and the above condition is not valid, then the stress tensor is determined as a projection of s onto a conical surface. The conical surface is defined with the help of the differentiable convex function

$$f(\sigma) = \tau(\sigma) - \alpha p(\sigma), \quad \frac{\partial f}{\partial \sigma} = \frac{\sigma'}{2 \tau(\sigma)} + \frac{\alpha}{3} \delta,$$

hence, the system of Eqs. (4.4) characterizing the projection has the form

$$p(\sigma) - p(s) = \alpha k \lambda, \quad \left(1 + \frac{\mu \lambda}{\tau(\sigma)}\right) \sigma' = s'.$$

Calculating the convolution $s' : s'$, from the last tensor equality we get

$$\tau(\sigma) + \mu \lambda = \tau(s), \quad \lambda = \frac{1}{\mu} (\tau(s) - \alpha p(\sigma)).$$

Thus,

$$p(\sigma) = \frac{\mu p(s) + \alpha k \tau(s)}{\mu + \alpha^2 k}, \quad \sigma' = \alpha p(\sigma) \frac{s'}{\tau(s)}. \quad (4.14)$$

The essential distinction of the von Mises–Schleicher cone from the Coulomb–Mohr cone is that for sufficiently large values of α on some area elements, passing through a given point of a material, tensile (positive) normal stresses are admissible. Such a minimal value of α corresponds to tangency of the conical surface and the principal coordinate planes of the negative octant. Cosine of the angle φ for which tangency takes place is equal to the scalar product of unit vectors $-(1, 1, 1)/\sqrt{3}$ and $-(1, 1, 0)/\sqrt{2}$ directed along the cone axis and along the tangency line. Hence,

$$\cos \varphi = \sqrt{\frac{2}{3}}, \quad \tan \varphi = \frac{1}{\sqrt{2}}, \quad \alpha = \frac{\sqrt{3}}{2}.$$

In fact, tensile stresses can arise in a granular material involving particles of relatively large size and specific form, for example, in masonry. In this case equilibrium of solid mass, being in the gravity field, with formation of overhanging free surfaces is possible. In such a material the angle of internal friction exceeds $\pi/2$. In the limiting equilibrium condition in the von Mises–Schleicher form the parameter α exceeds the value of $\sqrt{3}/2$ and the Coulomb–Mohr condition makes no sense since $f = \tan \alpha < 0$.

To obtain a more complete notion about the range of variation of the internal friction parameter, we consider the simplest cases of uniaxial and biaxial tension of a granular material with rigid particles. In the first case

$$\varepsilon_1 > 0, \quad \varepsilon_2 = \varepsilon_3 = 0, \quad \gamma(\boldsymbol{\varepsilon}) = \frac{2}{\sqrt{3}} \varepsilon_1, \quad \theta(\boldsymbol{\varepsilon}) = \varepsilon_1.$$

The condition of admissibility of the strain $\boldsymbol{\varepsilon} \in \mathbf{C}$: $\gamma(\boldsymbol{\varepsilon}) \leq \theta(\boldsymbol{\varepsilon})$ after cancelation by ε_1 is reduced to the inequality $\gamma \leq \sqrt{3}/2$. For $\gamma > \sqrt{3}/2$ uniaxial tension (loosening) of a material is impossible, because the degree of freedom of the volume variation turns out to be deficient for shear and the dilatancy process is blocked similarly to that in a material of bounded volume.

In the case of tension along two principal axes

$$\varepsilon_1 = \varepsilon_2 > 0, \quad \varepsilon_3 = 0, \quad \gamma(\boldsymbol{\varepsilon}) = \frac{2}{\sqrt{3}} \varepsilon_1, \quad \theta(\boldsymbol{\varepsilon}) = 2 \varepsilon_1.$$

The condition of admissibility of the strained state of a material takes the form $\gamma \leq \sqrt{3}$. Thus, for $\gamma > \sqrt{3}$ biaxial tension of a material is impossible.

For $\gamma = \sqrt{3}$ the model of an ideal granular material with rigid particles results in a relatively simple and unconventional description of kinematics of the plane strain state. In this state

$$\gamma(\boldsymbol{\varepsilon}) = 2 \sqrt{\frac{1}{3} (\varepsilon_{11}^2 - \varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}^2) + \varepsilon_{12}^2}, \quad \theta(\boldsymbol{\varepsilon}) = \varepsilon_{11} + \varepsilon_{22}.$$

The condition $\boldsymbol{\varepsilon} \in \mathbf{C}$ is reduced to two inequalities

$$(\varepsilon_{11} - \varepsilon_{22})^2 + 4 \varepsilon_{12}^2 \leq 0, \quad \varepsilon_{11} + \varepsilon_{22} \geq 0,$$

from which it follows that

$$\varepsilon_{11} = \varepsilon_{22} \geq 0, \quad \varepsilon_{12} = 0. \quad (4.15)$$

The absence of shears in the $x_1 x_2$ plane means the conservation of angles when mapping the initial unstrained configuration into the actual one, and the equality of axial strains means the conservation of scales. Therefore the transformation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{u}$, where $\mathbf{u}$ is the displacement vector, is conformal. This can be justified in another way. The relationships (4.15) in terms of the small displacement vector lead to the Cauchy–Riemann equations

$$u_{1,1} = u_{2,2}, \quad u_{1,2} = -u_{2,1}.$$

Here the subscripts after comma mean derivatives with respect to spatial coordinates. With the additional condition $u_{1,1} \geq 0$. Hence, the function $w(z) = u_1 + i u_2$

of the complex variable $z = x_1 + i x_2$ (i is the imaginary unit) is analytic in the whole domain occupied by a material, besides, $\Re w'(z) \geq 0$.

Consider examples of the displacement fields defined with the help of the simplest analytic functions of complex variable and construct a family of the current-flow lines for them. The system of equations

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = d\lambda$$

of these lines is reduced to the differential equation $dz/d\lambda = w(z)$ and is integrated after separation of variables:

$$\int \frac{dz}{w(z)} = \lambda. \quad (4.16)$$

In the case of $w(z) = z$ the condition $\Re w'(z) \geq 0$ holds automatically on the whole plane. The corresponding displacement field describes biaxial uniform tension of a material. The rays issuing out of the point $z = 0$: $z = C_1 e^{\lambda}$, where C_1 is an arbitrary complex constant of integration, serve as the current-flow lines.

For the analytic function $w(z) = z^2$ the condition $\Re w'(z) = 2x_1 \geq 0$ holds everywhere in the right-hand half-plane. The displacement vector is

$$u_1 = x_1^2 - x_2^2, \quad u_2 = 2x_1 x_2.$$

The family of current-flow lines is defined by the equation

$$z = \frac{1}{C_2 - \lambda},$$

hence,

$$x_1 = \frac{A_2 - \lambda}{(A_2 - \lambda)^2 + B_2^2}, \quad x_2 = -\frac{B_2}{(A_2 - \lambda)^2 + B_2^2},$$

where A_2 and B_2 are real and imaginary parts, respectively, of the complex constant C_2 . This family consists of semicircles, whose centers lie on the imaginary axis, which tangent the real axis at the point $z = 0$ (see Fig. 4.5). The arrows point to the direction of displacement of particles of a granular material.

As an example we also consider the analytic function $w(z) = \ln z$. For it the condition $\Re w'(z) = \Re \bar{z}/|z|^2 \geq 0$ of the volume tension of a material (bar means complex conjugation) is valid everywhere in the right-hand half-plane with the point $z = 0$ deleted. The displacement vector is defined by the formulae

$$u_1 = \ln \sqrt{x_1^2 + x_2^2}, \quad u_2 = \arctan \frac{x_2}{x_1}.$$

The integral in (4.16) is calculated via a special function (integral logarithm), [2, 10]:

Fig. 4.5 Current-flow lines
for $w(z) = z^2$

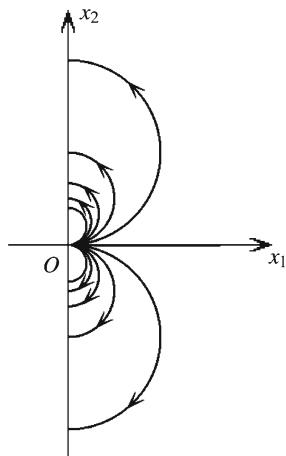
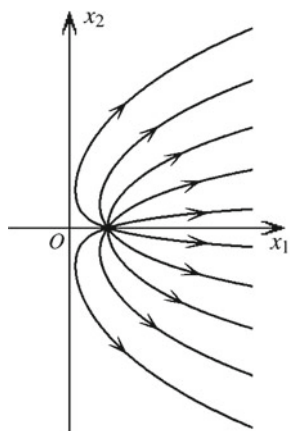


Fig. 4.6 Current-flow lines
for $w(z) = \ln z$



$$\int_0^z \frac{dp}{\ln p} = \text{li}(z).$$

The family of current-flow lines is defined by the equation $\text{li}(z) = C_3 + \lambda$. The point $z = 1$ at which displacements are equal to zero is a degenerate current-flow line. The remaining lines issue out of this point and go to infinity as λ tends to infinity (Fig. 4.6).

As $\varkappa \rightarrow \infty$, the von Mises–Schleicher cone goes to the half-space lying below the deviatoric plane. The dual cone $\mathcal{C}$ is transformed into the ray $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \geq 0$. If proper strain of particles is not taken into account, then the corresponding model describes the behavior of an exotic material wherein only the volume tension without variation of shears is possible.

In this case the vector of small displacements with respect to an arbitrary Cartesian coordinate system is determined from the system of equations

$$u_{i,j} + u_{j,i} = \frac{2}{3} \theta \delta_{ij}. \quad (4.17)$$

Differentiating the equation $u_{1,2} + u_{2,1} = 0$, involved in the system (4.17), with respect to x_1 and then with respect to x_2 , we obtain $\theta_{,22} + \theta_{,11} = 0$. In a similar way $\theta_{,33} + \theta_{,11} = 0$ and $\theta_{,33} + \theta_{,22} = 0$. Hence, $\theta_{,11} = \theta_{,22} = \theta_{,33} = 0$. Thus, the volume strain linearly depends on each of the spatial variables:

$$\theta = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_2 x_3 + c_5 x_1 x_3 + c_6 x_1 x_2 + c_7 x_1 x_2 x_3.$$

Due to linearity, a solution of the system (4.17) for the right-hand side of this form is equal to the sum of the general solution of the homogeneous system and a linear combination of particular solutions for $\theta = 1$, $\theta = x_1$, $\theta = x_1 x_3$, $\theta = x_1 x_2 x_3$ and for right-hand sides obtained by the change of subscripts. The general solution

$$\begin{aligned} u_1 &= w_1 - \omega_3 x_2 + \omega_2 x_3, \\ u_2 &= w_2 + \omega_3 x_1 - \omega_1 x_3, \\ u_3 &= w_3 - \omega_2 x_1 + \omega_1 x_2 \end{aligned}$$

of the homogeneous system (superposition of plane-parallel displacement and infinitesimal rotation) corresponds to the motion of a material as of a rigid body. The particular solution corresponding to the right-hand side $\theta = 1$ is the solution of the system of equations

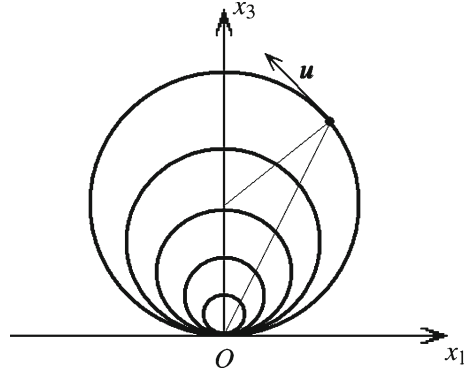
$$u_{1,1} = \frac{1}{3}, \quad u_{1,22} = u_{1,33} = 0$$

for u_1 and of similar systems for u_2 and u_3 . It can be taken in the form $u_j = x_j/3$ which describes the uniform volume expansion of a material with respect to a fixed origin. The particular solution for $\theta = x_1$ is determined in an explicit form

$$u_1 = \frac{1}{6}(x_1^2 - x_2^2 - x_3^2), \quad u_2 = \frac{1}{3} x_1 x_2, \quad u_3 = \frac{1}{3} x_1 x_3 \quad (4.18)$$

by integration of the system of equations

$$\begin{aligned} u_{1,1} &= \frac{x_1}{3}, \quad u_{1,22} = -\frac{1}{3}, \quad u_{1,33} = -\frac{1}{3}, \\ u_{2,11} &= 0, \quad u_{2,2} = \frac{x_1}{3}, \quad u_{2,33} = 0, \\ u_{3,11} &= 0, \quad u_{3,22} = 0, \quad u_{3,3} = \frac{x_1}{3}, \end{aligned}$$

Fig. 4.7 Current-flow lines

which follows from (4.17). It can be shown that in the case of $\theta = x_1 x_2$ and $\theta = x_1 x_2 x_3$ the system (4.17) is inconsistent.

The displacement field (4.18) corresponds to a specific deformation of a material such that the shape of elements remains unchanged and the volume linearly depends on the x_1 space coordinate. Integrating the differential equations

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3},$$

we construct spatial current-flow lines which point to the direction of the displacement vector at each their point. First from the equation

$$\frac{dx_2}{x_2} = \frac{dx_3}{x_3}$$

we get the equality $x_2 = C_4 x_3$, then from

$$x_3 dx_1^2 = (x_1^2 - (1 + C_4^2) x_3^2) dx_3$$

we find that $x_1^2 = 2 C_5 x_3 - (1 + C_4^2) x_3^2$ (C_4 and C_5 are arbitrary constants). For $C_4 = 0$ obtained equation gives a family of current-flow lines in the $x_1 x_3$ plane. This family is formed by the system of circles $x_1^2 + (x_3 - C_5)^2 = C_5^2$, which are tangent to the axis x_1 at the origin. Displacement along the circles occurs in a positive counterclockwise direction as shown in Fig. 4.7. The direct calculation by the formulae (4.18) shows that modulus of the vector of displacement of a point is proportional to the squared distance from this point to the origin. In other planes passing through the x_1 axis the same qualitative pattern is observed.

In fact, this displacement field takes place only in the domain $x_1 \geq 0$ since θ may not be negative. The general solution of (4.17) is an admissible combination of displacement of a material as of a rigid body, uniform tension–compression, and three displacements of the form (4.18), two of which are obtained from (4.18) by

cyclic change of subscripts. This solution may be used when testing computational algorithms and in program debugging. The question of its practical application is open since it seems likely that there does not exist in nature a granular material wherein under mechanical actions the volume strain takes place but the shear is impossible.

Emphasizing the distinction between the von Mises–Schleicher and Coulomb–Mohr cones, we notice that the problem of practical determination of the internal friction parameter for a granular material is not trivial. One way of the solution is to relate this parameter and the angle or coefficient of internal friction. For example, we can take it in such a way that the corresponding cone is inscribed into the Coulomb–Mohr cone. For this choice $\tan \varphi = r/(\sqrt{3} p)$ and we obtain

$$\mathfrak{x} = \frac{\sqrt{3} \sin \alpha}{\sqrt{3} + \sin^2 \alpha}. \quad (4.19)$$

The value of $\mathfrak{x}$ varies in the range from 0 to $\sqrt{3}/2$. In the case of a circumscribed cone

$$\mathfrak{x} = \frac{2\sqrt{3} \sin \alpha}{3 - \sin \alpha}.$$

The range of variation is $0 \leq \mathfrak{x} \leq \sqrt{3}$. In this case, positive normal stresses are admissible in the model of a material with relatively large internal friction.

One more way of the choice of $\mathfrak{x}$ is to require that limit tangential stresses of pure shear for given hydrostatic pressure coincide for the von Mises–Schleicher and Coulomb–Mohr cones. For the stress state of this type $\sigma_1 = \tau - p$, $\sigma_2 = -p$, $\sigma_3 = -\tau - p$ where τ is the value of tangential stress. The substitution into the Coulomb–Mohr condition yields $\mathfrak{x} = \tau/p = \sin \alpha$, $0 \leq \mathfrak{x} \leq 1$. The alternative way is based on the solution of the problem on limiting equilibrium of a slope, i.e. of an infinite solid mass of a granular material bounded by a plane free surface at an angle α to the horizon. In the subsequent chapters it will be shown that this approach enables one to justify the formula (4.19) for the case of the inscribed cone.

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Chapter 5

Limiting Equilibrium of a Material With Load Dependent Strength Properties

Abstract Solvability of static boundary-value problems within the framework of a model describing small strains in a material with load dependent strength properties, for example at tension and compression, is studied. A generalization of the static and kinematic theorems of the theory of limiting equilibrium is given. As an example of the application of the kinematic theorem, an upper estimate of the limit load and of the angle of departure of linear zone of the strain localization for the problem on discontinuity of a notched sample under the action of pressure on the edges of a notch is found. It is shown that logarithmic spirals serve as localization lines. With the help of two-sided estimates, an expression for the angle of natural slope of an ideal granular material is obtained. For the numerical solution of boundary-value problems, an iterative algorithm based on the finite-element approximation of a model is worked out. Results of computations which confirm the obtained estimating solutions are presented.

5.1 Model of a Material With Load Dependent Strength Properties

To describe the stress-strain state of a material having different ultimate strengths under tension and compression, we use a model of a granular material with plastic connections whose rheological scheme is the parallel coupling of two elements: a plastic hinge and a rigid contact (see Fig. 2.12a). Under the action of compressive stresses or tensile stresses which are less than the cohesion coefficient σ_s (the yield point of a plastic element) such a material is not deformed. Achieving σ_s corresponds to the limiting equilibrium where strain can be an arbitrary positive value. Stresses beyond this limit are impossible. Thus, defining displacement fields in the limiting state we can study kinematic mechanisms of fracture of a material with different strengths within the framework of the model under consideration. However, such mechanisms may differ essentially from those in mechanics of brittle fracture,

[4, 11, 17], since they correspond to fracture of other type, namely, to plastic flow of a material.

For the monotone loading without unloading, the constitutive relationships of uniaxial deformation are reduced to the system

$$\sigma \leq \sigma_s, \quad \varepsilon \geq 0, \quad (\sigma - \sigma_s) \varepsilon = 0. \quad (5.1)$$

Repeating the considerations from Sect. 2.1, we can prove that this system is equivalent to the variational inequalities

$$\begin{aligned} (\sigma - \sigma_s)(\tilde{\varepsilon} - \varepsilon) &\leq 0, & \varepsilon, \tilde{\varepsilon} &\geq 0, \\ (\tilde{\sigma} - \sigma)\varepsilon &\leq 0, & \sigma, \tilde{\sigma} &\leq \sigma_s, \end{aligned}$$

($\tilde{\varepsilon}$ and $\tilde{\sigma}$ are varying quantities). Indeed, if $\varepsilon > 0$ then from the first variational inequality due to the fact that the strain variation $\tilde{\varepsilon} - \varepsilon$ may be positive as well as negative it follows that $\sigma = \sigma_s$. Then all relationships of the system (5.1) (two inequalities and the complementing condition) are valid. If $\varepsilon = 0$ then $\tilde{\varepsilon} - \varepsilon \geq 0$. The product of two factors is negative only when the factors are of opposite sign, hence, $\sigma \leq \sigma_s$. The constitutive relationships are valid again. The third case of $\varepsilon < 0$ which excludes the two cases considered above does not take place. Thus, stress σ and strain ε satisfying the first variational inequality are a solution of the system (5.1). To prove the converse statement, we multiply the left-hand side of the inequality $\tilde{\sigma} - \sigma_s \leq 0$ from (5.1) by a non-negative quantity $\tilde{\varepsilon}$ and subtract the complementing condition from the result obtained. We arrive at the first variational inequality. The equivalence is completely proved. For the second variational inequality a proof is performed in a similar way accurate within the replacement of strain by stress and conversely.

Using the definition of a subdifferential, we can show that each of these inequalities admits the potential representation

$$\sigma \in \partial\Phi(\varepsilon), \quad \varepsilon \in \partial\Psi(\sigma), \quad (5.2)$$

besides, in this model the stress potential and the strain potential are equal to $\Phi(\varepsilon) = \sigma_s \varepsilon + \delta_C(\varepsilon)$ and $\Psi(\sigma) = \delta_K(\sigma - \sigma_s)$, respectively, and the strain and stress cones dual to one another are defined as before:

$$C = \{\varepsilon \geq 0\}, \quad K = \{\sigma \leq 0\}.$$

Generalization of the constitutive relationships to the case of the spatial state is constructed on the basis of the inclusions (5.2). To this end, the symmetric cohesion tensor σ_0 and a convex closed cone C with the vertex at zero in the six-dimensional strain tensor space or a similar cone K in the stress tensor space are introduced. If one of these cones is known, then another one is determined as dual to it. The cone K and the tensor σ_0 must satisfy the condition $-\sigma_0 \in K$ which means that the

natural unstressed state of a material is admissible. The potentials $\Phi(\boldsymbol{\varepsilon})$ and $\Psi(\boldsymbol{\sigma})$ are automatically obtained on the basis of the above expressions with the replacement of scalar quantities by tensor ones and of the product by the tensor convolution. These potentials are dual convex functions, i.e. they are expressed in terms of one another with the help of the Young–Fenchel transform:

$$\begin{aligned}\Phi(\boldsymbol{\varepsilon}) &= \sup_{\boldsymbol{\sigma}} \{ \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \Psi(\boldsymbol{\sigma}) \} = \sup_{\boldsymbol{\sigma} - \boldsymbol{\sigma}_0 \in \mathbf{K}} \{ \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \} = \boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon} + \delta_{\mathbf{C}}(\boldsymbol{\varepsilon}), \\ \Psi(\boldsymbol{\sigma}) &= \sup_{\boldsymbol{\varepsilon}} \{ \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \Phi(\boldsymbol{\varepsilon}) \} = \sup_{\boldsymbol{\varepsilon} \in \mathbf{C}} \{ (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) : \boldsymbol{\varepsilon} \} = \delta_{\mathbf{K}}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0).\end{aligned}$$

It turns out that the spatial constitutive relationships with potentials of this kind can be used in calculation of the limit loads in problems of statics of a material with different strengths, i.e. of the loads starting with which strains are developed in a material while for the lower loads strains are absent. It is obvious that the constitutive relationships (5.2) are incorrect, i.e. they do not allow one to determine uniquely the strain state of a material from given stresses and the stress state from given strains. The model of a cohesive granular material involving elastic particles whose rheological scheme is given in Fig. 2.12b serves as regularization.

Let $\mathbf{a}$ and $\mathbf{b}$ be symmetric positive definite tensors of rank four involving moduli of elastic compliance of regularizing elements. In view of the rheological scheme, a material with elastic properties is characterized by the system

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_a + \boldsymbol{\varepsilon}_b, \quad \boldsymbol{\varepsilon}_a = \mathbf{a} : \boldsymbol{\sigma}, \quad (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0 - \mathbf{b}^{-1} : \boldsymbol{\varepsilon}_b) : (\tilde{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_b) \leq 0, \quad \boldsymbol{\varepsilon}_b, \tilde{\boldsymbol{\varepsilon}} \in \mathbf{C}.$$

From this system it follows that

$$\boldsymbol{\varepsilon} = \mathbf{a} : \boldsymbol{\sigma} + \Pi_{\mathbf{b}^{-1}}(\mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)). \quad (5.3)$$

Using (5.3), with the help of Eq. (3.12) for differentiation and Eq. (3.16) we can obtain the strain potential

$$\begin{aligned}\Psi(\boldsymbol{\sigma}) &= \frac{1}{2} \left(|\boldsymbol{\sigma}|_a^2 + \left| \Pi_{\mathbf{b}^{-1}}(\mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)) \right|_{\mathbf{b}^{-1}}^2 \right) \\ &= \frac{1}{2} \left(|\boldsymbol{\sigma}|_a^2 + |\boldsymbol{\sigma} - \boldsymbol{\sigma}_0|_b^2 - |\pi_{\mathbf{b}}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)|_b^2 \right).\end{aligned} \quad (5.4)$$

Due to (3.16) the norms of a tensor and of its projection onto a cone coincide only for elements of the cone. Hence, the potential (5.4) is equal to the strain energy of an elastic element with the compliance tensor $\mathbf{a}$ if and only if $\boldsymbol{\sigma} - \boldsymbol{\sigma}_0 \in \mathbf{K}$. It tends to

$$\Psi_a(\boldsymbol{\sigma}) = \frac{|\boldsymbol{\sigma}|_a^2}{2} + \delta_{\mathbf{K}}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)$$

as $\mathbf{b} \rightarrow \infty$, and to

$$\psi_{\mathbf{b}}(\boldsymbol{\sigma}) = \frac{1}{2} \left| \Pi_{\mathbf{b}^{-1}}(\mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)) \right|_{\mathbf{b}^{-1}}^2$$

as $\mathbf{a} \rightarrow \mathbf{0}$.

Since for the projection onto a convex cone with the vertex at zero the next chain of equalities is valid:

$$\begin{aligned} & \left| \mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) \right|_{\mathbf{b}^{-1}}^2 - \left| \Pi_{\mathbf{b}^{-1}}(\mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)) \right|_{\mathbf{b}^{-1}}^2 \\ &= \left| \mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) - \Pi_{\mathbf{b}^{-1}}(\mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)) \right|_{\mathbf{b}^{-1}}^2 \\ &= \inf_{\tilde{\boldsymbol{\varepsilon}} \in \mathbf{C}} \left| \mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) - \tilde{\boldsymbol{\varepsilon}} \right|_{\mathbf{b}^{-1}}^2 = \inf_{\tilde{\boldsymbol{\varepsilon}} \in \mathbf{C}} \left| \boldsymbol{\sigma} - \boldsymbol{\sigma}_0 - \mathbf{b}^{-1} : \tilde{\boldsymbol{\varepsilon}} \right|_{\mathbf{b}}^2, \end{aligned}$$

then the dual stress potential (the Young–Fenchel transform of the function $\Psi(\boldsymbol{\sigma})$) is equal to

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2} \sup_{\boldsymbol{\sigma}} \inf_{\tilde{\boldsymbol{\varepsilon}} \in \mathbf{C}} \left\{ 2 \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - |\boldsymbol{\sigma}|_{\mathbf{a}}^2 - |\boldsymbol{\sigma} - \boldsymbol{\sigma}_0|_{\mathbf{b}}^2 + |\boldsymbol{\sigma} - \boldsymbol{\sigma}_0 - \mathbf{b}^{-1} : \tilde{\boldsymbol{\varepsilon}}|_{\mathbf{b}}^2 \right\}.$$

By considerations similar to those in Sect. 4.1, we can prove that sup and inf can be interchanged here. After interchanging the least upper bound is calculated. For fixed $\tilde{\boldsymbol{\varepsilon}}$ it is achieved on the tensor $\boldsymbol{\sigma} = \mathbf{a}^{-1} : (\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}})$, hence,

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2} \inf_{\tilde{\boldsymbol{\varepsilon}} \in \mathbf{C}} \left\{ |\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}|_{\mathbf{a}^{-1}}^2 + |\tilde{\boldsymbol{\varepsilon}}|_{\mathbf{b}^{-1}}^2 + 2 \boldsymbol{\sigma}_0 : \tilde{\boldsymbol{\varepsilon}} \right\}.$$

Separating out the perfect square with respect to the tensor $\tilde{\boldsymbol{\varepsilon}}$ results in the following equality:

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2} \left(|\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - |\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0|_{\mathbf{c}}^2 + \inf_{\tilde{\boldsymbol{\varepsilon}} \in \mathbf{C}} |\mathbf{c} : (\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0) - \tilde{\boldsymbol{\varepsilon}}|_{\mathbf{c}^{-1}}^2 \right),$$

where $\mathbf{c}^{-1} = \mathbf{a}^{-1} + \mathbf{b}^{-1}$. Thus,

$$\begin{aligned} \Phi(\boldsymbol{\varepsilon}) &= \frac{1}{2} \left(|\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - \left| \Pi_{\mathbf{c}^{-1}}(\mathbf{c} : (\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0)) \right|_{\mathbf{c}^{-1}}^2 \right) \\ &= \frac{1}{2} \left(|\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - |\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0|_{\mathbf{c}}^2 + |\pi_{\mathbf{c}}(\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0)|_{\mathbf{c}}^2 \right). \end{aligned} \quad (5.5)$$

In view of (5.5) the stress potential is a differentiable function. The direct calculation of derivatives by the formula (3.12) gives

$$\mathbf{a} : \boldsymbol{\sigma} = \boldsymbol{\varepsilon} - \Pi_{\mathbf{c}^{-1}}(\mathbf{c} : (\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0)). \quad (5.6)$$

Passing to the limit as $\mathbf{b} \rightarrow \infty$ ($\mathbf{c} \rightarrow \mathbf{a}$) we can obtain the stress potential, dual to $\Psi_{\mathbf{a}}(\boldsymbol{\sigma})$, for a material whose rheological scheme involves a single elastic element:

$$\Phi_{\mathbf{a}} = \frac{1}{2} \left(|\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - |\Pi_{\mathbf{a}^{-1}}(\boldsymbol{\varepsilon} - \mathbf{a} : \boldsymbol{\sigma}_0)|_{\mathbf{a}^{-1}}^2 \right).$$

In a similar way, as $\mathbf{a} \rightarrow \mathbf{0}$ we determine the potential dual to $\Psi_{\mathbf{b}}(\boldsymbol{\sigma})$:

$$\Phi_{\mathbf{b}} = \frac{|\boldsymbol{\varepsilon}|_{\mathbf{b}^{-1}}^2}{2} + \boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon} + \delta_C(\boldsymbol{\varepsilon}).$$

The constitutive equations in the form (5.3) or (5.6) with the equilibrium conditions and the geometrical relationships

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}, \quad 2\boldsymbol{\varepsilon}(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^* \quad (5.7)$$

($\mathbf{f}$ is the vector of the volume forces) form a closed mathematical model that describes equilibrium of a material with the potentials $\Phi(\boldsymbol{\varepsilon})$ and $\Psi(\boldsymbol{\sigma})$ for small strains.

Let Ω be a space or plane domain, occupied by a material, with the boundary Γ consisting of two nonintersecting parts Γ_u and Γ_σ . On the first part displacements are absent and on the second part the distributed load is given ($\mathbf{v}$ is a normal vector):

$$\begin{cases} \mathbf{u} = \mathbf{0} & \text{on } \Gamma_u, \\ \boldsymbol{\sigma} \cdot \mathbf{v} = \mathbf{q} & \text{on } \Gamma_\sigma. \end{cases} \quad (5.8)$$

The problem is to determine the displacement vector field $\mathbf{u}(\mathbf{x})$ and the stress tensor field $\boldsymbol{\sigma}(\mathbf{x})$ satisfying the differential equations (5.6), (5.7) and the boundary conditions (5.8).

Assume that Ω and Γ_σ are such that Korn's inequality (see Sect. 3.5) is valid, for example, Ω is a bounded domain satisfying the cone condition and Γ_σ is an open set in Γ . For simplicity we take notations of the same type for corresponding spaces of scalar, vector, and tensor functions. Assume that $\boldsymbol{\sigma}_0 \in L_2(\Omega)$, $\nabla \cdot \boldsymbol{\sigma}_0$, $\mathbf{f} \in L_2(\Omega)$, $\mathbf{q} \in L_2(\Gamma_\sigma)$, $\mathbf{a}$, $\mathbf{b} \in L_\infty(\Omega)$, besides, the tensor functions $\mathbf{a}$ and $\mathbf{b}$ are positive definite uniformly in Ω , i.e. there exist constants $a_0 > 0$ and $b_0 > 0$ such that $|\boldsymbol{\sigma}|_{\mathbf{a}}^2 \geq a_0 \boldsymbol{\sigma} : \boldsymbol{\sigma}$ and $|\boldsymbol{\sigma}|_{\mathbf{b}}^2 \geq b_0 \boldsymbol{\sigma} : \boldsymbol{\sigma}$ almost everywhere in Ω for any $\boldsymbol{\sigma}$.

In this case the tensor function $\mathbf{a}^{-1} - \mathbf{a}^{-1} : \mathbf{c} : \mathbf{a}^{-1} \in L_\infty(\Omega)$ is uniformly positive definite. This is easily proved on the basis of the formula

$$\mathbf{a}^{-1} - \mathbf{a}^{-1} : \mathbf{c} : \mathbf{a}^{-1} = (\mathbf{a} + \mathbf{b})^{-1},$$

which follows from the chain of equalities

$$\begin{aligned} \mathbf{a} - \mathbf{c} = \mathbf{c} : (\mathbf{c}^{-1} - \mathbf{a}^{-1}) : \mathbf{a} = (\mathbf{a}^{-1} + \mathbf{b}^{-1})^{-1} : \mathbf{b}^{-1} : \mathbf{a} = \mathbf{a} : (\mathbf{a}^{-1} : (\mathbf{a}^{-1} \\ + \mathbf{b}^{-1})^{-1} : \mathbf{b}^{-1}) : \mathbf{a} = \mathbf{a} : (\mathbf{b} : (\mathbf{a}^{-1} + \mathbf{b}^{-1}) : \mathbf{a})^{-1} : \mathbf{a} = \mathbf{a} : (\mathbf{a} + \mathbf{b})^{-1} : \mathbf{a}, \end{aligned}$$

being easily verified. With the above assumptions for $\mathbf{a}$ and $\mathbf{b}$ the tensor $\mathbf{a} + \mathbf{b}$ as well as the inverse tensor $(\mathbf{a} + \mathbf{b})^{-1}$ is positive definite. Therefore there exists a constant $\lambda_0 > 0$ such that

$$\boldsymbol{\varepsilon} : (\mathbf{a} + \mathbf{b})^{-1} : \boldsymbol{\varepsilon} \geq \lambda_0 |\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2$$

almost everywhere in Ω for any fixed tensor $\boldsymbol{\varepsilon}$.

The last inequality enables one to obtain the required estimates for the potentials $\Phi(\boldsymbol{\varepsilon})$ and $\Psi(\boldsymbol{\sigma})$. Due to (5.5)

$$2\Phi \geq |\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - |\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0|_{\mathbf{c}}^2 = |\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - |\mathbf{a}^{-1} : \boldsymbol{\varepsilon}|_{\mathbf{c}}^2 + 2\boldsymbol{\varepsilon} : \mathbf{a}^{-1} : \mathbf{c} : \boldsymbol{\sigma}_0 - |\boldsymbol{\sigma}_0|_{\mathbf{c}}^2,$$

hence, applying the obvious inequality (a consequence of non-negativity of the squared norm of the difference of tensors $\sqrt{\lambda_1} \boldsymbol{\varepsilon}$ and $\mathbf{c} : \boldsymbol{\sigma}_0 / \sqrt{\lambda_1}$)

$$2\boldsymbol{\varepsilon} : \mathbf{a}^{-1} : \mathbf{c} : \boldsymbol{\sigma}_0 \geq -\lambda_1 |\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - \frac{|\mathbf{c} : \boldsymbol{\sigma}_0|_{\mathbf{a}^{-1}}^2}{\lambda_1}, \quad \lambda_1 > 0,$$

we can show that

$$|\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 \geq 2\Phi(\boldsymbol{\varepsilon}) \geq (\lambda_0 - \lambda_1) |\boldsymbol{\varepsilon}|_{\mathbf{a}^{-1}}^2 - \frac{1}{\lambda_1} |\mathbf{c} : \boldsymbol{\sigma}_0|_{\mathbf{a}^{-1}}^2 - |\boldsymbol{\sigma}_0|_{\mathbf{c}}^2. \quad (5.9)$$

For $\Psi(\boldsymbol{\sigma})$ from (5.4) it follows that

$$|\boldsymbol{\sigma}|_{\mathbf{a}}^2 \leq 2\Psi(\boldsymbol{\sigma}) \leq |\boldsymbol{\sigma}|_{\mathbf{a}}^2 + |\boldsymbol{\sigma} - \boldsymbol{\sigma}_0|_{\mathbf{b}}^2. \quad (5.10)$$

Generally speaking, in an inhomogeneous material not only tensors of the elasticity coefficients but also the cones $\mathbf{C}$ and $\mathbf{K}$ depend on a point $\mathbf{x} \in \Omega$. Further we assume that in this case the functions

$$\mathbf{x} \mapsto \Phi(\mathbf{x}, \boldsymbol{\varepsilon}), \quad \mathbf{x} \mapsto \Psi(\mathbf{x}, \boldsymbol{\sigma})$$

are measurable in Ω for any $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$. Taking into account the estimates (5.9) and (5.10), we can show that this condition is sufficient for the potentials to be integrable in the sense of Lebesgue provided that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma} \in \mathbf{L}_2(\Omega)$. In addition, with this condition the constitutive equations (5.3) and (5.6) define a one-to-one mapping of the space $\mathbf{L}_2(\Omega)$ onto $\mathbf{L}_2(\Omega)$ since the inequalities

$$\begin{aligned}
\|\boldsymbol{\varepsilon}\|_0 &\leq \|\mathbf{a} : \boldsymbol{\sigma}\|_0 + \|\Pi_{\mathbf{b}^{-1}}(\mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0))\|_0 \leq \|\mathbf{a} : \boldsymbol{\sigma}\|_0 \\
&+ C_1 \|\Pi_{\mathbf{b}^{-1}}(\mathbf{b} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0))\|_{\mathbf{b}^{-1}} \leq \|\mathbf{a} : \boldsymbol{\sigma}\|_0 + C_1 \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_0\|_{\mathbf{b}}, \\
\|\mathbf{a} : \boldsymbol{\sigma}\|_0 &\leq \|\boldsymbol{\varepsilon}\|_0 + \|\Pi_{\mathbf{c}^{-1}}(\mathbf{c} : (\mathbf{a}^{-1}\boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0))\|_0 \leq \|\boldsymbol{\varepsilon}\|_0 + C_2 \|\mathbf{a}^{-1} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}_0\|_{\mathbf{c}}
\end{aligned}$$

are valid with constants depending on the tensors $\mathbf{b}$ and $\mathbf{c}$ only.

The assumptions formulated above enables one to prove solvability of the problem which leads to two independent variational principles (see, for example, [1, 2, 24, 26]). The displacement field to be found minimizes the integral

$$I(\mathbf{u}) = \int_{\Omega} (\Phi(\boldsymbol{\varepsilon}(\mathbf{u})) - \mathbf{f} \cdot \mathbf{u}) d\Omega - \int_{\Gamma_{\sigma}} \mathbf{q} \cdot \mathbf{u} d\Gamma \quad (5.11)$$

on the linear space $\mathbf{U}$ of generalized functions $\mathbf{u} \in \mathbf{H}^1(\Omega)$ satisfying the boundary condition (5.8) on the part Γ_u of the boundary. This follows from the chain of relationships

$$\begin{aligned}
0 &= - \int_{\Omega} (\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) d\Omega = \int_{\Omega} \boldsymbol{\sigma} : (\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) - \boldsymbol{\varepsilon}(\mathbf{u})) d\Omega \\
&- \int_{\Gamma} \mathbf{v} \cdot \boldsymbol{\sigma} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) d\Gamma - \int_{\Omega} \mathbf{f} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) d\Omega \leq I(\tilde{\mathbf{u}}) - I(\mathbf{u}),
\end{aligned}$$

obtained with the help of the Green formula considering the boundary conditions (5.8) and the definition of a subdifferential according to which

$$\boldsymbol{\sigma} : (\tilde{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}) \leq \Phi(\tilde{\boldsymbol{\varepsilon}}) - \Phi(\boldsymbol{\varepsilon}).$$

The stress field minimizes the integral

$$J(\boldsymbol{\sigma}) = \int_{\Omega} \Psi(\boldsymbol{\sigma}) d\Omega \quad (5.12)$$

on the affine space Σ of the tensor functions $\boldsymbol{\sigma} \in \mathbf{L}_2(\Omega)$, for which the equilibrium equations (5.7) and the boundary conditions (5.8) on Γ_{σ} are fulfilled in the generalized sense (in the form of the integral virtual work principle), i.e.

$$\int_{\Omega} \boldsymbol{\sigma} : \nabla \tilde{\mathbf{u}} d\Omega = \int_{\Gamma_{\sigma}} \mathbf{q} \cdot \tilde{\mathbf{u}} d\Gamma + \int_{\Omega} \mathbf{f} \cdot \tilde{\mathbf{u}} d\Omega \quad \forall \tilde{\mathbf{u}} \in \mathbf{U}. \quad (5.13)$$

This statement follows from the relationships

$$0 = \int_{\Omega} (\tilde{\sigma} - \sigma) \cdot (\varepsilon - \varepsilon(u)) d\Omega = \int_{\Omega} (\tilde{\sigma} - \sigma) : \varepsilon d\Omega - \int_{\Omega} (\tilde{\sigma} - \sigma) : \nabla u d\Omega \leq J(\tilde{\sigma}) - J(\sigma),$$

since in view of (5.2)

$$(\tilde{\sigma} - \sigma) : \varepsilon \leq \Psi(\tilde{\sigma}) - \Psi(\sigma).$$

Both integrals are strictly convex continuous and, hence, weakly lower semicontinuous functionals on corresponding spaces. The estimates (5.9) and (5.10) provide coercitivity of these functionals. The subspaces $\mathbf{U}$ and Σ are closed. On the basis of the theorem whose proof is given in Sect. 3.5 we can conclude that solutions exist and are unique.

It turns out that the minimization problems (5.11) and (5.12) are dual to one another. This can be proved on the basis of the general duality principles presented in [6, 24]. To use a more simple version of the duality theorem from Sect. 3.5, we reformulate the main problem in the following way.

Let $\bar{\sigma} \in \Sigma$ be an arbitrary stress field satisfying the equilibrium equations in Ω and boundary conditions on Γ_σ . Taking into account Eq. (5.13) and the obvious equality

$$\bar{\sigma} : \varepsilon(u) = \bar{\sigma} : \nabla u,$$

which follows from symmetry of the strain tensor, we can represent the expression (5.11) in the form

$$I^*(\varepsilon(u)) = \int_{\Omega} (\Phi(\varepsilon(u)) - \bar{\sigma} : \varepsilon(u)) d\Omega.$$

In this form it depends on $\varepsilon(u)$ only and defines a convex functional on the linear subspace

$$\mathbf{E} = \{\varepsilon = \varepsilon(u) \mid u \in \mathbf{U}\}$$

of the space of symmetric strain tensors of $L_2(\Omega)$. The dual space to $L_2(\Omega)$ coincides with $L_2(\Omega)$. Its linear subspace

$$\Sigma_0 = \Sigma - \Sigma = \{\sigma - \bar{\sigma} \mid \sigma \in \Sigma\}$$

is orthogonal to $\mathbf{E}$ since due to (5.13) for any $\Delta\sigma \in \Sigma_0$ and $\varepsilon \in \mathbf{E}$

$$\int_{\Omega} \Delta\sigma : \varepsilon(u) d\Omega = 0.$$

From the inequality

$$\int_{\Omega} \Delta \sigma : \varepsilon \, d\Omega - I^*(\varepsilon) \equiv \int_{\Omega} ((\bar{\sigma} + \Delta \sigma) : \varepsilon - \Phi(\varepsilon)) \, d\Omega \leq \int_{\Omega} \Psi(\bar{\sigma} + \Delta \sigma) \, d\Omega,$$

which is easily verified and becomes an equality for $\varepsilon = \partial \Psi(\bar{\sigma} + \Delta \sigma) / \partial \sigma$, we can show that the Young–Fenchel transform for $I^*(\varepsilon)$, which is equal to

$$\max_{\varepsilon \in L_2(\Omega)} \left\{ \int_{\Omega} \Delta \sigma : \varepsilon \, d\Omega - I^*(\varepsilon) \right\},$$

coincides with $J(\bar{\sigma} + \Delta \sigma)$. Thus, the dual problem is in minimization of the functional $J(\bar{\sigma} + \Delta \sigma)$ on the linear subspace $\Delta \sigma \in \Sigma_0$ or, what is the same, of the functional $J(\sigma)$ on the affine subspace Σ .

In its turn, with the help of the inequality

$$\begin{aligned} \int_{\Omega} \Delta \sigma : \varepsilon \, d\Omega - J(\bar{\sigma} + \Delta \sigma) &\equiv \int_{\Omega} (\sigma : \varepsilon - \Psi(\sigma)) \, d\Omega - \int_{\Omega} \bar{\sigma} : \varepsilon \, d\Omega \\ &\leq \int_{\Omega} (\Phi(\varepsilon) - \bar{\sigma} : \varepsilon) \, d\Omega, \quad \sigma = \bar{\sigma} + \Delta \sigma, \end{aligned}$$

which becomes an equality for $\Delta \sigma = \partial \Phi(\varepsilon) / \partial \varepsilon - \bar{\sigma}$, we can prove that the Young–Fenchel transform of $J(\bar{\sigma} + \Delta \sigma)$ is equal to $I^*(\varepsilon)$. Hence, in the subsequent considerations we can interchange the main problem and the dual one.

According to the duality theorem, the next equality is valid:

$$\min_{u \in U} I(u) = - \min_{\sigma \in \Sigma} J(\sigma).$$

In the case of $\Gamma_u = \emptyset$, where static boundary conditions are given on the whole boundary, which is of practical importance, a displacement field certainly is not unique. It is defined accurate within rigid displacements

$$u(x) = w + \omega \cdot x,$$

where w is an arbitrary vector, ω is an antisymmetric tensor of rank two. The space $H^1(\Omega)$ is decomposed into the direct sum of the subspace R_0 of rigid displacements and its orthogonal complement

$$R_0^\perp(\Omega) = \left\{ u \in H^1(\Omega) \mid \int_{\Omega} u \, d\Omega = \int_{\Omega} x \times u \, d\Omega = 0 \right\}.$$

If Ω is a Lipschitz domain then Korn's inequality is valid. Taking into account the estimate (5.9) for $\lambda_1 < \lambda_0$, this means that the functional $I(\mathbf{u})$ is coercitive on the given subspace and, as a consequence, this ensures existence of a solution provided that the resultant vector and the resultant moment are equal to zero:

$$\int_{\Omega} \mathbf{f} d\Omega + \int_{\Gamma} \mathbf{q} d\Gamma = \int_{\Omega} \mathbf{x} \times \mathbf{f} d\Omega + \int_{\Gamma} \mathbf{x} \times \mathbf{q} d\Gamma = 0. \quad (5.14)$$

The statement that solutions exist and are unique remains valid when on Γ_u instead of (5.8) inhomogeneous boundary conditions in terms of displacements are given:

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \Gamma_u, \quad (5.15)$$

moreover, $\mathbf{u}_0 \in \mathbf{H}^{1/2}(\Gamma_u)$. Then we can turn back to homogeneous boundary conditions with the help of the standard change of the unknown function going from displacements to auxiliary functions obtained by subtraction of an arbitrary field $\bar{\mathbf{u}} \in \mathbf{H}^1(\Omega)$ satisfying (5.15) from displacements.

Solvability of boundary-value problems in the context of limiting models turns out to be a more complicated problem. The problem in terms of stresses resulting in minimization of the quadratic functional

$$J_a(\sigma) = \frac{1}{2} \int_{\Omega} |\sigma|_a^2 d\Omega$$

on the convex closed set

$$\Sigma_K = \{ \sigma \in \Sigma \mid \sigma - \sigma_0 \in K \text{ almost everywhere in } \Omega \}$$

is well-posed for $b \rightarrow \infty$. If the set Σ_K is nonempty, then a minimum point exists and is unique. A proof of existence of a displacement field requires special constructions since the functional $I_a(\mathbf{u})$, defined by the formula (5.11) in terms of $\Phi_a(\varepsilon)$, is not coercitive in $\mathbf{H}^1(\Omega)$. Generally speaking, the uniqueness theorem is not valid.

For $a \rightarrow 0$ the problem in terms of displacements is well-posed. It is reduced to determining a minimum point of the quadratic functional

$$I_b(\mathbf{u}) = \int_{\Omega} \left(\frac{1}{2} |\varepsilon(\mathbf{u})|_{b^{-1}}^2 + \sigma_0 : \varepsilon(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u} \right) d\Omega - \int_{\Gamma_\sigma} \mathbf{q} \cdot \mathbf{u} d\Gamma \quad (5.16)$$

on the convex closed cone

$$U_C = \{ \mathbf{u} \in U \mid \varepsilon(\mathbf{u}) \in C \text{ almost everywhere in } \Omega \}.$$

If $\Gamma_u = \emptyset$, then it is necessary to consider the cone U_C as a subset of $R_0^\perp(\Omega)$ with the additional requirement that the conditions (5.14) are valid. A solution of the minimization problem exists and is unique (accurate within rigid displacements for $\Gamma_u = \emptyset$). The construction of stresses is related to minimization of the noncoercitive functional $J_b(\sigma)$, obtained from (5.12) by means of replacement of $\Psi(\sigma)$ by $\Psi_b(\sigma)$, on σ . In the case of $\Sigma_K \neq \emptyset$ where, as will be shown further, a material is in the absolutely rigid state, any tensor field $\sigma \in \Sigma_K$ satisfies the equation $J_b(\sigma) = 0$ and is a minimum point to be found. Thus, a stress field certainly is not unique. In the general case a proof of existence of a solution of the problem in terms of stresses is of no practical interest in view of obvious incorrectness of the model from a mechanical standpoint.

5.2 Static and Kinematic Theorems

According to the model of a material with different strengths, described by the potentials $\Phi_b(\epsilon)$ and $\Psi_b(\sigma)$, in the equilibrium state a domain Ω occupied by a material is subdivided into two parts: a rigid zone where a material is not deformed and a zone of nonzero strain. An applied external load (f, q) is said to be safe if there is no a strain zone at all. In this case the displacement vector or (provided that on the whole boundary of a domain boundary conditions in terms of stresses are given) its projection onto $R_0^\perp(\Omega)$ is equal to zero almost everywhere in Ω .

Let Σ_K be a nonempty set. Then by the Green formula for any $\tilde{\sigma} \in \Sigma_K$ and $\tilde{u} \in U_C$

$$\int_{\Omega} (\nabla \cdot \tilde{\sigma} + f) \cdot \tilde{u} \, d\Omega = \int_{\Omega} (f \cdot \tilde{u} - \tilde{\sigma} : \epsilon(\tilde{u})) \, d\Omega + \int_{\Gamma_\sigma} q \cdot \tilde{u} \, d\Gamma = 0.$$

Consequently, since

$$(\tilde{\sigma} - \sigma_0) : \epsilon(\tilde{u}) \leq 0,$$

we have

$$\int_{\Omega} (f \cdot \tilde{u} - \sigma_0 : \epsilon(\tilde{u})) \, d\Omega + \int_{\Gamma_\sigma} q \cdot \tilde{u} \, d\Gamma \leq 0. \quad (5.17)$$

The actual displacement field is determined as a solution of the problem (5.16) of minimization of the functional $I_b(u)$ on the cone U_C . The point $\mathbf{0} \in U_C$ is the vertex of this cone, hence,

$$I_b(u) = \min_{\tilde{u} \in U_C} I_b(\tilde{u}) = \min_{\tilde{u} \in U_C} \min_{\lambda \geq 0} I_b(\lambda \tilde{u}).$$

The direct calculation of the minimum of the expression

$$I_b(\lambda \tilde{\mathbf{u}}) = \int_{\Omega} \left(\frac{\lambda^2}{2} |\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})|_{b^{-1}}^2 + \lambda \boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) - \lambda \mathbf{f} \cdot \tilde{\mathbf{u}} \right) d\Omega - \lambda \int_{\Gamma_\sigma} \mathbf{q} \cdot \tilde{\mathbf{u}} d\Gamma$$

with respect to $\lambda \geq 0$ gives

$$\lambda = \frac{\left(\int_{\Omega} (\mathbf{f} \cdot \tilde{\mathbf{u}} - \boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}})) d\Omega + \int_{\Gamma_\sigma} \mathbf{q} \cdot \tilde{\mathbf{u}} d\Gamma \right)_+}{\int_{\Omega} |\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})|_{b^{-1}}^2 d\Omega},$$

where the subscript “+” means the positive part of the expression: $(z)_+ = (z + |z|)/2$. After elimination of λ the minimization problem can be reduced to determining the maximum:

$$I_b(\mathbf{u}) = - \max_{\mathbf{0} \neq \tilde{\mathbf{u}} \in U_C} \frac{\left(\int_{\Omega} (\mathbf{f} \cdot \tilde{\mathbf{u}} - \boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}})) d\Omega + \int_{\Gamma_\sigma} \mathbf{q} \cdot \tilde{\mathbf{u}} d\Gamma \right)_+^2}{2 \int_{\Omega} |\boldsymbol{\varepsilon}(\tilde{\mathbf{u}})|_{b^{-1}}^2 d\Omega}.$$

According to this formula, the value of the functional $I_b(\mathbf{u})$ is nonzero (strictly negative) only if the condition (5.17) is violated for some element $\tilde{\mathbf{u}} \in U_C$. Under this condition for all such elements $I_b(\mathbf{u}) = 0$. In this case the unique solution of the minimization problem is identically equal to zero. Thus, if $\Sigma_K \neq \emptyset$ then the load $(\mathbf{f}, \mathbf{q})$ is safe.

Using the duality theorem, we can prove the converse statement, [20], in the following weakened form: if an actual load is safe, then there exists a sequence of tensor functions $\boldsymbol{\sigma}_n \in \Sigma$ such that the sequence $\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0$ tends to the cone $\mathbf{K}$ as $n \rightarrow \infty$. This means that almost everywhere in Ω the sequence of projections onto the dual cone $\mathbf{C}$ formed by the second terms of the decompositions (3.14):

$$\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0 = \pi_b(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0) + \mathbf{b}^{-1} : \Pi_{b^{-1}}(\mathbf{b} : (\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_0)) \quad (5.18)$$

tends to zero. If $\boldsymbol{\sigma}_n$ is a convergent sequence in $L_2(\Omega)$, then its limit belongs to the closed set Σ_K and, hence, this set is nonempty. However, generally speaking, the limit does not have to exist, but even in this case existence of such a sequence provides the condition (5.17).

Indeed, the problem of determining the greatest lower bound of the functional

$$J_b(\sigma) = \frac{1}{2} \int_{\Omega} \left| \Pi_{b^{-1}}(b : (\sigma - \sigma_0)) \right|_{b^{-1}}^2 d\Omega$$

on the affine subspace Σ due to changing the argument is reduced to the similar problem for the functional $J(\bar{\sigma} + \Delta\sigma)$ on the linear subspace $\Sigma_0 = \Sigma - \Sigma$ in $L_2(\Omega)$. The considered functional is bounded below on σ_0 and continuous, for example, at the point $\mathbf{0} \in \Sigma_0$ since using properties of the operator of projection onto a cone we can obtain the estimates

$$\begin{aligned} 2 |J_b(\sigma) - J_b(\bar{\sigma})| &\leq \int_{\Omega} \left| \left| \Pi_{b^{-1}}(b : (\sigma - \sigma_0)) \right|_{b^{-1}}^2 - \left| \Pi_{b^{-1}}(b : (\bar{\sigma} - \sigma_0)) \right|_{b^{-1}}^2 \right| d\Omega \\ &= \int_{\Omega} \left| \left(\Pi_{b^{-1}}(b : (\sigma - \sigma_0)) - \Pi_{b^{-1}}(b : (\bar{\sigma} - \sigma_0)) \right) : b^{-1} : \left(\Pi_{b^{-1}}(b : (\sigma - \sigma_0)) \right. \right. \\ &\quad \left. \left. + \Pi_{b^{-1}}(b : (\bar{\sigma} - \sigma_0)) \right) \right| d\Omega \leq \int_{\Omega} \left| \Pi_{b^{-1}}(b : (\sigma - \sigma_0)) - \Pi_{b^{-1}}(b : (\bar{\sigma} - \sigma_0)) \right|_{b^{-1}} \\ &\quad \times \left(\left| \Pi_{b^{-1}}(b : (\sigma - \sigma_0)) \right|_{b^{-1}} + \left| \Pi_{b^{-1}}(b : (\bar{\sigma} - \sigma_0)) \right|_{b^{-1}} \right) d\Omega \\ &\leq \int_{\Omega} |\sigma - \bar{\sigma}|_b (|\sigma - \sigma_0|_b + |\bar{\sigma} - \sigma_0|_b) d\Omega, \end{aligned}$$

in which the Cauchy–Bunyakovskii inequality for the Hilbert norm with subscript b^{-1} as well as the triangle inequality for this norm is used.

Applying the Cauchy–Bunyakovskii inequality for $L_2(\Omega)$ -norm, from these estimates we get

$$|J_b(\sigma) - J_b(\bar{\sigma})| \leq C_3 \|\sigma - \bar{\sigma}\|_0,$$

where a constant C_3 depends on the norm $\|\sigma\|_0$.

The stress potential $\Phi_b(\varepsilon)$ and the strain potential $\Psi_b(\sigma)$ are dual convex functions. Besides, $\Psi_b(\sigma) \leq \Psi(\sigma)$, hence,

$$\begin{aligned} \int_{\Omega} (\Delta\sigma : \varepsilon - \Psi(\bar{\sigma} + \Delta\sigma)) d\Omega &\leq \int_{\Omega} (\Delta\sigma : \varepsilon - \Psi_b(\bar{\sigma} + \Delta\sigma)) d\Omega \\ &= \int_{\Omega} (\sigma : \varepsilon - \Psi_b(\sigma)) d\Omega - \int_{\Omega} \bar{\sigma} : \varepsilon d\Omega \\ &\leq \int_{\Omega} (\Phi_b(\varepsilon) - \bar{\sigma} : \varepsilon) d\Omega. \end{aligned}$$

Therefore, passing to sup for all $\Delta\sigma \in L_2(\Omega)$, we can show that the functional

$$I_b^*(\varepsilon) = \sup_{\Delta\sigma \in L_2(\Omega)} \left\{ \int_{\Omega} \Delta\sigma : \varepsilon \, d\Omega - J_b(\bar{\sigma} + \Delta\sigma) \right\}$$

dual to $J_b(\bar{\sigma} + \Delta\sigma)$ for $\varepsilon = \varepsilon(u) \in E$ satisfies the system of inequalities

$$I(u) \leq I_b^*(\varepsilon(u)) \leq I_b(u).$$

Since the value of $I(u)$ in the left-hand side of this system tends to $I_b(u)$ as $a \rightarrow 0$, we have $I_b^*(\varepsilon(u)) \equiv I_b(u)$.

Thus, all assumptions of the duality theorem are fulfilled and according to its statement we have the equality

$$\inf_{\tilde{\Sigma} \in \Sigma} J_b(\tilde{\sigma}) = -I_b(u), \quad (5.19)$$

where the right-hand side is equal to zero due to the safety of load and, generally speaking, the greatest lower bound does not have to be achieved on some element of Σ .

Denote a minimizing sequence by $\sigma_n \in \Sigma$. Since $J_b(\sigma_n) \rightarrow 0$, then the expression $\Pi_{b-1}(b : (\sigma_n - \sigma_0))$ under the integral sign of the functional on the elements of this sequence tends to zero with respect to the norm of the space $L_2(\Omega)$. It is known that out of a sequence convergent in $L_2(\Omega)$ one can separate a subsequence which converges almost everywhere, [12]. It is obvious that this subsequence satisfies the required property, i.e. for it $\sigma_n - \sigma_0$ tends to the cone K . The limit of σ_n (provided that it exists) belongs to the set Σ_K since, firstly, the set Σ is closed and, secondly, passage to the limit in (5.18) for $n \rightarrow \infty$ yields $\sigma - \sigma_0 = \pi_b(\sigma - \sigma_0) \in K$.

On the other hand, if a sequence $\sigma_n \in \Sigma$, for which $\sigma_n - \sigma_0$ tends to the cone K , exists then the condition (5.17) holds. A proof of this statement is trivial since if a sequence of elements of $L_2(\Omega)$ converges almost everywhere then it converges with respect to the norm of $L_2(\Omega)$. Therefore the numerical sequence $J_b(\sigma_n)$ tends to zero and is minimizing, moreover, the greatest lower bound in (5.19) is equal to zero. Hence, $I_b(u) = 0$ and $u \equiv 0$.

Thus, a load (f, q) is safe if and only if two equivalent conditions (the condition $\Sigma_K \neq \emptyset$ in the weak form described above and the condition (5.17)) are valid. In essence, these conditions are the content of two theorems about limiting equilibrium, namely, the static and kinematic theorems. Using the terminology of the plasticity theory, [3, 10, 18], we represent their formulations for the case of the proportional loading where the vector of surface forces varies proportional to some scalar parameter m which plays the role of time:

$$q(x) = m q^0(x), \quad x \in \Gamma_\sigma,$$

and the vector of mass forces does not depend on this parameter. The loads turn out to be safe, i.e. there are no displacements in the body of a material with different strengths provided that the parameter does not exceed the limit value $\bar{m}$. Exceeding this value results in appearance of a domain of nonzero strain. As a rule, in applications the limit value is an unknown to be determined. Further, when considering examples, we will use the asterisk subscript for notation of safe loads (lower estimates $m_* \leq \bar{m}$) and the asterisk superscript for notation of loads greater than the limit one (upper estimates $\bar{m} \leq m^*$).

Theorem 5.1 (about lower estimate). *The value of the loading parameter does not exceed the limit one ($m_* \leq \bar{m}$) if there exists at least one statically admissible stress field ($\Sigma_K \neq \emptyset$) corresponding to this parameter or if there exists a sequence of fields $\sigma_n \in \Sigma$ that converges almost everywhere and such that $\sigma_n - \sigma_0$ tends to the cone K .*

Theorem 5.2 (about upper estimate). *The limit value $\bar{m}$ does not exceed the coefficient*

$$m^* = \frac{\int_{\Omega} (\sigma_0 : \varepsilon(u) - f \cdot u) d\Omega}{\left(\int_{\Gamma_\sigma} q^0 \cdot u d\Gamma \right)_+},$$

which is called a kinematic coefficient and is calculated from an arbitrary kinematically admissible displacement field $u \in U_C$.

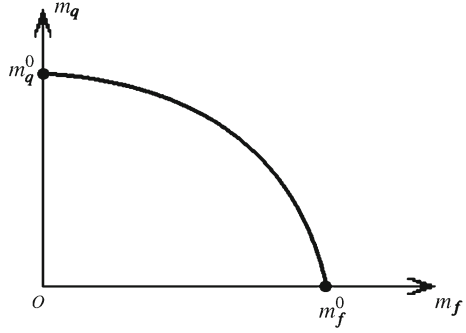
It should be noticed that, contrary to the plasticity theory, in this case the consideration of the class of the proportional loadings is of methodological interest only since the constitutive equations of the model do not depend on time.

In the Cartesian product of the spaces $L_2(\Omega)$ and $L_2(\Gamma_\sigma)$, safe loads form a convex closed set S . Convexity is verified basing on the definition: if (f, q) and $(\tilde{f}, \tilde{q})$ are elements of S then for them the condition (5.17) holds. It is obvious that this condition holds for a convex combination $\lambda(\tilde{f}, \tilde{q}) + (1 - \lambda)(f, q)$ of loads for a parameter $\lambda \in [0, 1]$ as well. The set S is closed since a solution of the problem of minimization of the functional $I_b(u)$ continuously depends on f and q . This problem is equivalent to the variational inequality ($u, \tilde{u} \in U_C$)

$$\begin{aligned} \int_{\Omega} \left\{ (b^{-1} : \varepsilon(u) + \sigma_0) : (\varepsilon(\tilde{u}) - \varepsilon(u)) - f \cdot (\tilde{u} - u) \right\} d\Omega \\ - \int_{\Gamma_\sigma} q \cdot (\tilde{u} - u) d\Gamma \geq 0. \end{aligned} \quad (5.20)$$

Taking here a solution corresponding to the external load $(\tilde{f}, \tilde{q})$ as an arbitrary varying function $\tilde{u}$ and summing up (5.20) with the similar variational inequality, which defines the solution $\tilde{u}$ and where u is taken as a varying function, we can obtain

Fig. 5.1 Dependence m_q versus m_f



$$\int_{\Omega} |\varepsilon(\tilde{u}) - \varepsilon(u)|_{b^{-1}}^2 d\Omega \leq \int_{\Omega} (\tilde{f} - f) \cdot (\tilde{u} - u) d\Omega + \int_{\Gamma_{\sigma}} (\tilde{q} - q) \cdot (\tilde{u} - u) d\Gamma.$$

Applying Korn's inequality to the left-hand side of this inequality and normative inequalities to the right-hand side we get the estimate

$$a_1 \|\tilde{u} - u\|_1 \leq \|\tilde{f} - f\|_0 + \|\tilde{q} - q\|_{H^{-1/2}(\Gamma_{\sigma})}, \quad a_1 > 0, \quad (5.21)$$

which provides continuous dependence of a solution in the space $H^1(\Omega)$. If a sequence (f_n, q_n) converging to (f, q) is formed by safe loads, then the corresponding displacement fields u_n are identically equal to zero. Then in view of the estimate (5.21) the limit of the sequence u_n is equal to zero as well, i.e. the load (f, q) is safe. We proved that S is closed.

The boundary points of the set S correspond to limit loads. For any load (f, q) that is not necessarily safe we can define the safety factors, i.e. the nonnegative numbers m_f and m_q for which the load $(m_f f, m_q q)$ is the limit one. If the factor m_f is given, besides, $(m_f f, 0) \in S$, then in view of the criterion (5.17)

$$m_q(m_f) = \inf_{\tilde{u} \in U_C} \frac{\int_{\Omega} (\sigma_0 : \varepsilon(\tilde{u}) - m_f f \cdot \tilde{u}) d\Omega}{\left(\int_{\Gamma_{\sigma}} q \cdot \tilde{u} d\Gamma \right)_+}. \quad (5.22)$$

The function $m_q(m_f)$ is concave (Fig. 5.1) since the greatest lower bound of a sum is greater than or equal to the sum of the greatest lower bounds:

$$\begin{aligned}
m_q(\lambda \tilde{m}_f + (1 - \lambda) m_f) &= \inf_{\tilde{\mathbf{u}} \in U_C} \left\{ \left(\lambda \int_{\Omega} (\boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) - \tilde{m}_f \mathbf{f} \cdot \tilde{\mathbf{u}}) d\Omega \right. \right. \\
&\quad \left. \left. + (1 - \lambda) \int_{\Omega} (\boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) - m_f \mathbf{f} \cdot \tilde{\mathbf{u}}) d\Omega \right) / \left(\int_{\Gamma_\sigma} \mathbf{q} \cdot \tilde{\mathbf{u}} d\Gamma \right)_+ \right\} \\
&\geq \lambda m_q(\tilde{m}_f) + (1 - \lambda) m_q(m_f).
\end{aligned}$$

The characteristic points at which the graph of this function intersects the coordinate axes on the $m_f m_q$ plane can be determined with the help of (5.22):

$$m_q^0 = m_q(0), \quad m_f^0 = \inf_{\tilde{\mathbf{u}} \in U_C} \frac{\int_{\Omega} \boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) d\Omega}{\left(\int_{\Omega} \mathbf{f} \cdot \tilde{\mathbf{u}} d\Omega \right)_+}.$$

The last expression can be obtained by taking $m_q = 0$ in (5.22). It gives the inequality

$$\int_{\Omega} (\boldsymbol{\sigma}_0 : \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) - m_f^0 \mathbf{f} \cdot \tilde{\mathbf{u}}) d\Omega \geq 0$$

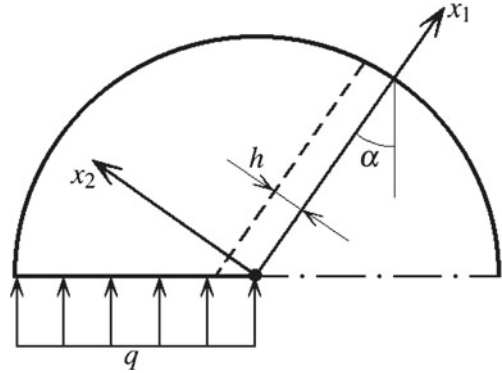
for each $\tilde{\mathbf{u}} \in U_C$. Hence, m_f^0 is the lower bound of given ratio. The fact that it is the greatest lower bound can be proved by the consideration of minimizing sequence for (5.22).

For a model of a viscous rigid-plastic material a similar dependence between safety factors was first studied in [19].

5.3 Examples of Estimates

The obtained formulae provide a simple way to estimate the safety factors of load. As an example we consider the plane strain state of a homogeneous cylindrical sample of radius R with a radial notch on the edges of which pressure $q > 0$, caused by the temperature expansion of a thin metal plate engaged in the notch, acts. Properties of a material with different strengths are described on the basis of the von Mises–Schleicher condition. According to this condition, in the limiting state $\tau(\boldsymbol{\sigma}) = \tau_s + \alpha p(\boldsymbol{\sigma})$, where τ_s is the cohesion coefficient. To go to the notations used in this chapter, we put

$$\boldsymbol{\sigma}_0 = \frac{\tau_s}{\alpha} \boldsymbol{\delta}, \quad \mathbf{K} = \left\{ \boldsymbol{\sigma} \mid \tau(\boldsymbol{\sigma}) \leq \alpha p(\boldsymbol{\sigma}) \right\}, \quad \mathbf{C} = \left\{ \boldsymbol{\varepsilon} \mid \gamma(\boldsymbol{\varepsilon}) \leq \frac{\theta(\boldsymbol{\varepsilon})}{\alpha} \right\}.$$

Fig. 5.2 Loading scheme

Let $\mathbf{u}(\mathbf{x})$ be an admissible displacement field that describes the strain localization for a simple shear with dilatancy in a narrow linear zone of thickness h inclined at an angle α to the line of action of pressure (see Fig. 5.2). In the Cartesian coordinates related to this zone

$$u_1 = \gamma_0 x_2, \quad u_2 = \varepsilon_0 x_2, \quad 0 \leq x_2 \leq h. \quad (5.23)$$

In the remaining part exterior to the localization zone, displacements are constant and are determined from continuity. For this field

$$\gamma(\boldsymbol{\varepsilon}) = \sqrt{\frac{4}{3} \varepsilon_0^2 + \Gamma_0^2}, \quad \theta(\boldsymbol{\varepsilon}) = \varepsilon_0,$$

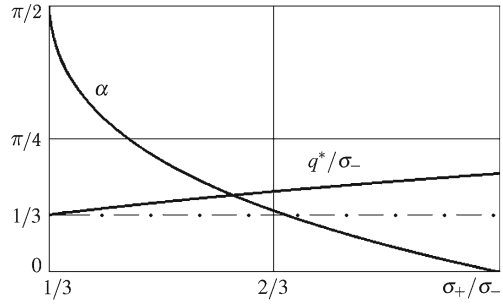
hence, the admissibility condition for a displacement field $\mathbf{u} \in U_C$ is written as the inequality $\gamma_0 \leq v \varepsilon_0$ with the parameter $v = \sqrt{1/\varkappa^2 - 4/3}$ and makes sense only in the case of $\varkappa \leq \sqrt{3}/2$. Limiting pressure is estimated by the formula

$$q^* = \frac{\tau_s}{\varkappa} \inf_{\mathbf{u} \in U_C} \frac{\int_{\Gamma} \mathbf{u} \cdot \mathbf{v} d\Gamma}{\left(\int_{\Gamma_\sigma} \mathbf{u} \cdot \mathbf{v} d\Gamma \right)_+},$$

obtained from (5.22) with the help of the Green formula. Putting $h \rightarrow 0$ we can obtain an upper estimate of the form

$$q^* = \frac{\tau_s}{\varkappa} \frac{\varepsilon_0}{\gamma_0 \cos \alpha + \varepsilon_0 \sin \alpha}.$$

The parameters involved here should be taken from the minimum condition for q^* . Hence,

Fig. 5.3 Solution with a linear zone

$$q^* = \frac{\tau_s}{\sqrt{1 - \varkappa^2/3}}, \quad \alpha = \arctan \frac{1}{\nu}. \quad (5.24)$$

This is the best upper estimate for pressure and the most probable angle of departure of the strain localization zone. The angle α turns out to be equal to the angle of internal friction of a material with different strengths. This will be proved below in this section when considering the next example.

According to (5.24), in the limit as $\varkappa \rightarrow 0$ the direction of the linear zone is perpendicular to the notch. As $\varkappa \rightarrow \sqrt{3}/2$, the zone is turned and becomes a continuation of the notch. In the case of $\varkappa > \sqrt{3}/2$ uniaxial tension of a material is impossible, hence, the localization of this type does not take place.

For most of natural and artificial materials with different strengths, the ultimate tension strength σ_+ is less than the ultimate compression strength σ_- . Some of them, for instance, graphitized carbons, differ in that the ratio σ_+/σ_- varies practically in the whole range from 0 to 1 depending on the kind of a material, [7, 16]. By the von Mises–Schleicher condition, ultimate strengths for the uniaxial stress state are equal to

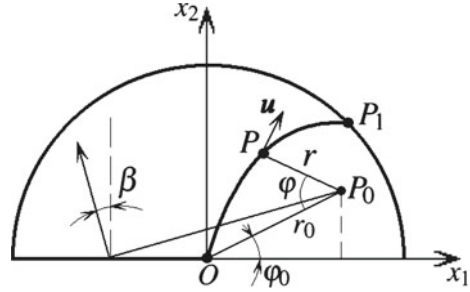
$$\sigma_{\pm} = \frac{3\tau_s}{\sqrt{3} \pm \varkappa}.$$

The internal friction parameter is expressed in terms of the ratio of limits by the formula

$$0 < \varkappa = \sqrt{3} \frac{1 - \sigma_+/\sigma_-}{1 + \sigma_+/\sigma_-} < \sqrt{3}.$$

The maximal value $\varkappa = \sqrt{3}/2$ is achieved for $\sigma_- = 3\sigma_+$. If the ultimate strengths differ from one another more than by a factor of three, we fail to obtain an estimate with the help of the displacement field (5.23). The dimensionless dependences of the angle α and the quantity q^*/σ_- on the ratio σ_+/σ_- obtained by the formulae (5.24) are presented in Fig. 5.3.

The upper estimate (5.24) knowingly is not exact. It can be improved by introducing a curvilinear zone of the strain localization. Drucker and Prager, [5], basing on the associated plastic flow rule, proposed that only logarithmic spirals can serve

Fig. 5.4 Localization line

as localization lines for the plane strain state of a cohesive granular material. This is related to the fact that for a logarithmic spiral the angle between the vector of a point and the direction of a tangent remains constant along the whole line. Hence, in a narrow localization zone the ratio of the normal and tangential components of the velocity vector (the vector of small displacements) turns out to be constant which is exactly in agreement with a deformation kinematics.

We present a more rigorous justification of this hypothesis, namely, we prove that, among all possible localization lines, a logarithmic spiral provides the least upper estimate.

Assume that P_0 is an instantaneous center of rotations in a plane absolutely rigid motion executed by the part of a cylindrical sample which is adjacent to the notch and is separated from the remaining fixed part by the strain localization zone (see Fig. 5.4). Denote the polar coordinates of a point P_0 by r_0 and φ_0 . Consider the auxiliary coordinates r, φ with the origin at the point P_0 . The equation of the localization line can be written as $r = r(\varphi)$. Tangential and normal displacements in a layer of thickness h near this line vary linearly in thickness, hence, strains are determined in terms of displacements on the upper boundary of the layer by the formulae

$$\varepsilon_{\tau\tau} = O(1), \quad \varepsilon_{vv} = \frac{u_v}{h} + O(1), \quad 2\varepsilon_{v\tau} = \frac{u_\tau}{h} + O(1).$$

Invariants of the strain tensor are as follows:

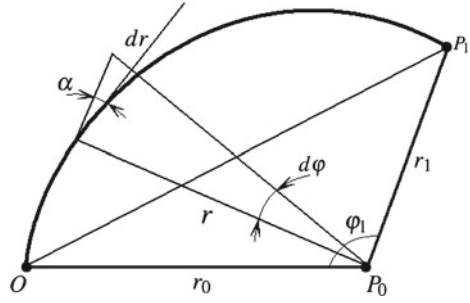
$$\theta(\varepsilon) = \frac{u_v}{h} + O(1), \quad \gamma(\varepsilon) = \sqrt{\frac{4}{3} \left(\frac{u_v}{h} \right)^2 + \left(\frac{u_\tau}{h} \right)^2} + O(1).$$

The condition $\mathbf{u} \in U_C$ holds provided that $\gamma(\varepsilon) \leq \theta(\varepsilon)$, i.e. that $u_\tau \leq \nu u_v + O(h)$. Putting $h \rightarrow 0$, we obtain

$$u_\tau \leq \nu u_v. \quad (5.25)$$

In rotational motion about the center P_0 , the vector of small displacements is perpendicular to the radius-vector of the point and its modulus is calculated via the angle Ψ of rotation as the product $r \Psi$. Therefore

$$u_\tau^2 + u_v^2 = r^2 \Psi^2, \quad u_\tau = r \Psi \cos \alpha, \quad u_v = r \Psi \sin \alpha,$$

Fig. 5.5 Geometrical constructions

where α is the angle between the vector $\mathbf{u}$ and the tangent which depends on the position of the point on the curve.

In view of (5.25), $\tan \alpha \geq 1/\nu$. From the geometrical illustration shown in Fig. 5.5 it follows that

$$\tan \alpha = -\frac{dr}{r d\varphi}.$$

By integration we can show that only the localization lines such that

$$-\int_0^\varphi \frac{dr}{r} \geq \frac{\varphi}{\nu} : \quad r \leq r_0 e^{-\varphi/\nu}$$

are admissible.

The upper estimate of limiting pressure is reduced to the form

$$q^* = \frac{\tau_s}{\alpha} \frac{\int_0^{P_1} u_\nu ds}{\int_{-R}^0 u_2 dx_1}. \quad (5.26)$$

Here P_1 is the point at which the localization line reaches the boundary of a sample, ds is an element of the arc which, according to the formulae

$$ds^2 = dr^2 + r^2 d\varphi^2 = dr^2(1 + \cot^2 \alpha),$$

is equal to $ds = -dr/\sin \alpha$. The numerator of the fraction in (5.26) is determined in an explicit form for an arbitrary localization line:

$$\int_O^{P_1} u_v ds = -\Psi \int_{r_0}^{r_1} r dr = \Psi \frac{r_0^2 - r_1^2}{2}.$$

It depends only on the position of the point at which the line reaches the boundary of a sample. The vertical displacement u_2 on the notch is determined from Fig. 5.4:

$$u_2 = r \Psi \cos \beta, \quad \cos \beta = \frac{1}{r} (r_0 \cos \varphi_0 - x_1).$$

After calculation of the integral the denominator of the fraction turns out to be equal to

$$\int_{-R}^0 u_2 dx_1 = \left(\frac{R^2}{2} + r_0 R \cos \varphi_0 \right) \Psi.$$

Thus, in the case of the curvilinear localization zone

$$q^* = \frac{\tau_s}{\alpha} \frac{r_0^2 - r_1^2}{R^2 + 2 r_0 R \cos \varphi_0}.$$

Taking into account the condition of admissibility of a localization line, in this expression we choose free parameters providing minimum of q^* in the following way:

$$\varphi_0 = 0, \quad r_1 = r_0 e^{-\varphi_1/\nu}.$$

Hence, a logarithmic spiral really has the extremal property, while any other localization line that reaches the boundary of a sample at a fixed point provides the same value of q^* . The polar angle of this point can be determined from Fig. 5.5 by the cosine rule for the triangle OP_0P_1 , according to which

$$R^2 = r_0^2 + r_1^2 - 2 r_0 r_1 \cos \varphi_1,$$

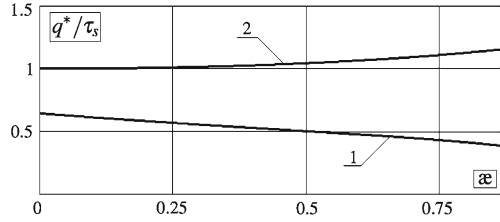
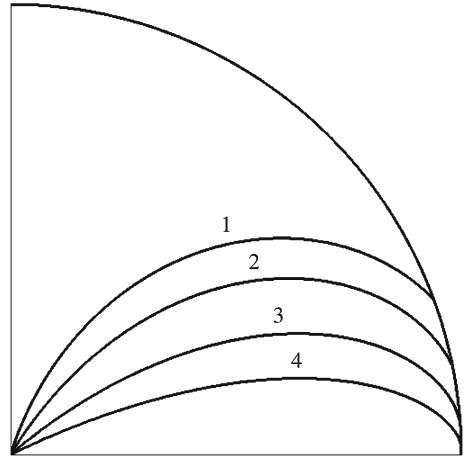
as a solution of the equation

$$\zeta(\varphi_1) \equiv \sqrt{1 + e^{-2\varphi_1/\nu} - 2 e^{-\varphi_1/\nu} \cos \varphi_1} = \frac{R}{r_0}.$$

The distance to the instantaneous center r_0 of rotations is a free parameter in the formula for upper estimate as well, hence, the problem is reduced to finding the minimum of the function

$$q^* = \frac{\tau_s}{\alpha} \frac{1 - e^{-2\varphi_1/\nu}}{\zeta^2 + 2\zeta}. \quad (5.27)$$

Applying the asymptotic expansion

Fig. 5.6 Upper estimates**Fig. 5.7** Logarithmic spirals

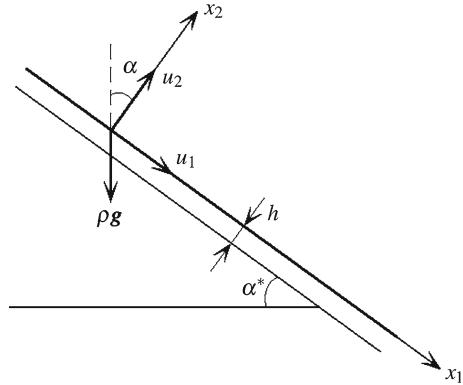
$$\zeta^2 \approx 1 + \left(1 - \frac{2\varphi_1}{\nu} + \frac{2\varphi_1^2}{\nu^2}\right) - 2\left(1 - \frac{\varphi_1}{\nu} + \frac{\varphi_1^2}{2\nu}\right)\left(1 - \frac{\varphi_1^2}{2}\right) \approx \left(1 + \frac{1}{\nu^2}\right)\varphi_1^2,$$

we can show that for small φ_1 (5.27) coincides with (5.24).

Results of the numerical solution of the minimization problem for $0 < x < \sqrt{3}/2$ are presented by the curve 1 in Fig. 5.6. For comparison, the curve 2 corresponding to the estimate (5.24) is shown. Observe that there is a considerable difference between these estimates and it increases as x increases. Additional analysis shows that this difference is due to the consideration of non-homogeneity of the displacement field in a curvilinear zone of the strain localization. In particular, to the parameter $x = \sqrt{3}/2$ in both cases there corresponds a linear localization zone which continues the notch. However, the difference is that the estimate (5.24) is obtained with the help of a displacement field homogeneous in length and for the estimate (5.27) a displacement field with the instantaneous center of rotations at the point $(R, 0)$ is used.

In the case when $x \rightarrow 0$ ($\nu \rightarrow \infty$), minimum of the estimate (5.27) achieves at $\zeta \rightarrow 0$ ($\varphi_1 \rightarrow \pi/2$, $r_0 \rightarrow \infty$). As in the solution with a linear zone, the line of localization coincides with the radius which is perpendicular to the notch.

A number of logarithmic spirals for different values of the internal friction parameter x in the range of its variation is shown in Fig. 5.7 (the curves 1, 2, 3, and 4 correspond to $x = 0.3, 0.5, 0.7$, and 0.8 , respectively).

Fig. 5.8 Natural slope

Unfortunately, one fails to obtain a lower estimate close to (5.27) by construction of a statically admissible stress field. We consider a more simple example where upper and lower estimates turn out to be the same. Assume that a body of an ideal granular material of infinite length with the cohesion coefficient $\tau_s = 0$, bounded by a free plane surface inclined to the horizon, is in the equilibrium state under the action of its own weight. Basing on general statements being proved we determine the angle α of natural slope at which the stress state of a body is the limit one.

We construct the upper estimate with the help of the linear displacement field of the form (5.23). To this end, we take the coordinate system related to the plane of strain localization (Fig. 5.8) which is set at the angle α^* . The state of the body is not safe, i.e. the slope angle is greater than the limit one provided that for this field the following inequality opposite to (5.17) is valid:

$$\int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, d\Omega + \int_{\Gamma_{\sigma}} \mathbf{q} \cdot \mathbf{u} \, d\Gamma > 0.$$

The surface integral involved in this inequality vanishes and only the integral over the localization domain (a strip of thickness h) caused by gravity $\mathbf{f} = \rho \mathbf{g}$ (ρ is the density, $\mathbf{g}$ is the gravitational acceleration vector) remains. In the strict sense, we have to go to a finite domain of integration taking a part of the strip since in the proof of the theorems a domain Ω is assumed to be bounded. After the calculation of the integral, in the limit as $h \rightarrow 0$ we arrive at a condition of positivity of the scalar product of vectors $\mathbf{f}$ and $\mathbf{u}$:

$$u_1 \sin \alpha^* - u_2 \cos \alpha^* > 0 \quad \Rightarrow \quad \tan \alpha^* > \frac{\varepsilon_0}{\gamma_0} \geq \frac{1}{\nu}.$$

For the lower estimate we direct the x_1 axis along the free surface and the x_2 axis along an external normal. We construct a statically admissible stress field σ satisfying the equilibrium equations

$$d\sigma_{12} = -\rho g \sin \alpha_* dx_2, \quad d\sigma_{22} = \rho g \cos \alpha_* dx_2$$

and the boundary conditions $\sigma_{12} = \sigma_{22} = 0$ on the free surface $x_2 = 0$:

$$\sigma_{12} = -\rho g x_2 \sin \alpha_*, \quad \sigma_{22} = \rho g x_2 \cos \alpha_*.$$

Putting $\sigma_{11} = \sigma_{33} = \lambda \sigma_{22}$, where λ is a constant value, we rewrite the condition $\tau(\sigma) = \mathfrak{x} p(\sigma)$ under which $\Sigma_K \neq \emptyset$ in the form of equality:

$$\sqrt{\frac{(\lambda - 1)^2}{3} + \tan^2 \alpha_*} = (2\lambda + 1) \frac{\mathfrak{x}}{3}.$$

This equality enables one to determine the angle α_* depending on λ . It makes sense only for $\lambda \geq -1/2$ where hydrostatic pressure is positive. When squared it gives

$$3 \tan^2 \alpha_* = (2\lambda + 1)^2 \frac{\mathfrak{x}^2}{3} - (\lambda - 1)^2. \quad (5.28)$$

The right-hand side of (5.28) is a quadratic trinomial relative to λ . For $\mathfrak{x} < \sqrt{3}/2$ its graph is a parabola whose branches are directed downwards as shown in Fig. 5.9a. Independently of the value of $\mathfrak{x}$, there are two intersection points of the graph and the abscissa axis:

$$\lambda_+ = \frac{\sqrt{3} + \mathfrak{x}}{\sqrt{3} - 2\mathfrak{x}}, \quad \lambda_- = \frac{\sqrt{3} - \mathfrak{x}}{\sqrt{3} + 2\mathfrak{x}}.$$

At the maximum point

$$\lambda = \frac{\lambda_+ + \lambda_-}{2} = \frac{3 + 2\mathfrak{x}^2}{3 - 4\mathfrak{x}^2},$$

and according to (5.28)

$$\tan \alpha_* = \frac{1}{\sqrt{1/\mathfrak{x}^2 - 4/3}} \equiv \frac{1}{v}. \quad (5.29)$$

This formula gives a lower estimate for the angle α . Observe that the lower estimate coincides with the least upper estimate. Hence, the angle α of the natural slope is calculated by the formula (5.29). The left-hand side of this formula is equal to the internal friction coefficient of a granular material. Solving (5.29) relative to $\mathfrak{x}$, we can obtain the formula (4.19) for the case where the von Mises–Schleicher cone is inscribed into the Coulomb–Mohr cone.

It is interesting to notice that in the problem on a slope equilibrium the accurate upper estimate is obtained with the help of a linear rather than logarithmic zone of a strain localization.

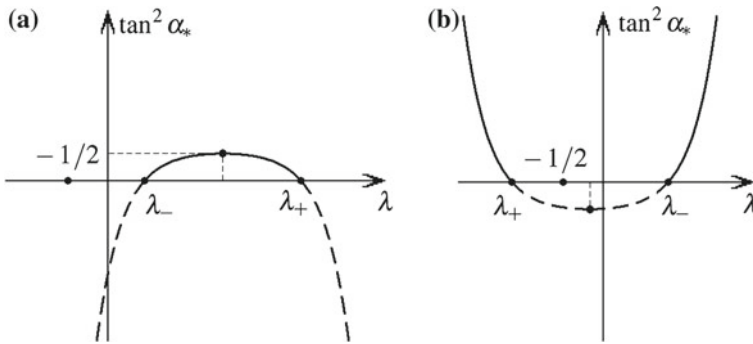


Fig. 5.9 Dependence $\tan^2 \alpha_*$ versus λ : **a** $a < \sqrt{3}/2$, **b** $a > \sqrt{3}/2$

Now let $a > \sqrt{3}/2$. In this case the branches of the parabola are directed upwards (Fig. 5.9b). As before, there are two points of intersection with the abscissa axis. The right point λ_- belongs to the range of definition $\lambda \geq -1/2$ and the left point λ_+ lies outside of it. The tolerance range of $\tan^2 \alpha_*$ coincides with the positive semiaxis, hence, for any angle α_* , taking the corresponding value of λ , we can construct a statically admissible stress field, i.e. for any α_* the stress state of a slope is safe.

5.4 Computational Algorithm

From the above examples we see that it is easy to determine analytically upper estimates for the limit load in the problems of statics of a material with different strengths. In the general case the problem of the construction of statically admissible stress fields for the lower estimate turns out to be somewhat more complicated than in the perfect plasticity theory. Some known ways to solve it with the help of the numerical method of characteristics will be considered in the next section. Here a computational algorithm, in which the theorems about an estimate of the limit load are not used, is proposed, [13, 14]. This algorithm, based on a finite-element approximation of the regularized model, reduces the problem of determining a displacement field in a material with different strengths to the solution of a number of static problems of the linear elasticity theory with initial stresses. Using it, we can determine the limit loads only approximately, analyzing the obtained numerical solution, as the loads such that exceeding them results in intensive deformation of a material.

The idea of the algorithm is to replace the constitutive equations (5.6) by the iterative formulae ($n = 1, 2, 3, \dots$)

$$\begin{aligned} \varepsilon(u^n) &= a : (\sigma^n + \Delta \sigma^{n-1}), \\ \Delta \sigma^{n-1} &= a^{-1} : \Pi_{c^{-1}} \left(c : (a^{-1} : \varepsilon(u^{n-1}) - \sigma_0) \right). \end{aligned} \quad (5.30)$$

At the first step the field $\Delta \sigma^0$ of initial stresses is assumed to be identically equal to zero, and the elastic problem for a non-stressed material with the tensor $\mathbf{a}$ of compliance moduli is solved. At the next steps initial stresses are calculated from the strain field obtained from the previous solution. The iteration steps proceed until the norm of the difference of two approximate solutions at neighboring steps becomes less than a given accuracy.

At the n th step of the algorithm the problem is reduced to the minimization of the integral functional

$$I^n(\mathbf{u}) = \int_{\Omega} \left(\frac{|\boldsymbol{\varepsilon}(\mathbf{u})|_{\mathbf{a}^{-1}}^2}{2} - \Delta \boldsymbol{\sigma}^{n-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u} \right) d\Omega - \int_{\Gamma_\sigma} \mathbf{q} \cdot \mathbf{u} d\Gamma$$

or to the solution of the variational equation

$$\int_{\Omega} \left((\mathbf{a}^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}^n) - \Delta \boldsymbol{\sigma}^{n-1}) : (\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}) - \boldsymbol{\varepsilon}(\mathbf{u}^n)) - \mathbf{f} \cdot (\tilde{\mathbf{u}} - \mathbf{u}^n) \right) d\Omega = \int_{\Gamma_\sigma} \mathbf{q} \cdot (\tilde{\mathbf{u}} - \mathbf{u}^n) d\Gamma, \quad \tilde{\mathbf{u}}, \mathbf{u}^n \in U, \quad (5.31)$$

for which the standard technique of the finite-element method, [8, 22], can be applied. As usual, the space U of displacements is approximated by a finite-dimensional subspace U_h (h is the discretization parameter) spanned by a given system of basic functions of U . As a result, a finite-dimensional problem of quadratic programming, which leads to a large-scale system of linear algebraic equations, is obtained. In the computations described below, the standard piecewise-linear splines defined on a non-regular triangular grid were taken as basic functions and the system of linear equations was solved by the conjugate gradient method. Convenience of this method is that it provides an easy implementation of the programming technology such that only nonzero coefficients of the matrix of a system and corresponding indices (numbers of rows and columns) are constantly saved in memory of a computer. Basing on this information, a subprogram for calculation of the product of the matrix and a given arbitrary vector which is used many times in the conjugate gradient method can be easily written. The concrete calculations show that in fact the matrix of the system is very sparse, more than 98 % of its coefficients are zeroes. Thus, elimination of operations with zeroes provides high performance of the computational process with small volume of RAM.

This matrix is formed in the compact form once in the calculation process at the first iteration step and is not recalculated at the subsequent iteration steps. The algorithm of the construction of the matrix is extremely simple. In a loop in elements of a grid domain, to any (nonzero) coefficient, which already exists and is an element of a one-dimensional array, corresponding coefficients of a local stiffness matrix are added. If by the time of adding coefficients with the required numbers of rows and columns do not exist yet, then coefficients of a local matrix subsequently take new places in the array of nonzero elements and corresponding numbers take

places in the array of indices. An insignificant technical problem, related to the fact that the dimension of these arrays is not known beforehand and is determined after completing an iteration loop, can be easily solved provided that programming is performed in the Fortran language with the use of the technology of embedding of all arrays of the same type being used into one-dimensional vector-containers of “infinitely” large dimensionality. This technology will be written in detail in Sect. 8.2, where algorithms of parallel computations for the analysis of dynamic problems are represented.

The exact solution of the problem, minimizing the functional $I(\mathbf{u})$ on the space $\mathbf{U}$, obviously satisfies the variational equation (5.31) with replacement of stress $\Delta \sigma^{n-1}$ by stress

$$\Delta \sigma = \mathbf{a}^{-1} : \Pi_{c^{-1}}(\mathbf{c} : (\mathbf{a}^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - \sigma_0)). \quad (5.32)$$

Substituting the exact solution $\mathbf{u}$ into Eq. (5.31) as the varying element $\tilde{\mathbf{u}}$ and the approximate solution $\mathbf{u}^n$ into the similar equation for $\Delta \sigma$ and summing up the results we obtain

$$\int_{\Omega} \left(|\boldsymbol{\varepsilon}(\mathbf{u}^n - \mathbf{u})|_{\mathbf{a}^{-1}}^2 - (\Delta \sigma^{n-1} - \Delta \sigma) : \boldsymbol{\varepsilon}(\mathbf{u}^n - \mathbf{u}) \right) d\Omega = 0.$$

In terms of the scalar product

$$(\tilde{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}) = \int_{\Omega} \tilde{\boldsymbol{\varepsilon}} : \mathbf{a}^{-1} : \boldsymbol{\varepsilon} d\Omega$$

and the corresponding Hilbert norm $\|\boldsymbol{\varepsilon}\|_0 = \sqrt{(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon})}$ the last equation can be represented in the form

$$\|\boldsymbol{\varepsilon}(\mathbf{u}^n - \mathbf{u})\|_0 = (\mathbf{a} : (\Delta \sigma^{n-1} - \Delta \sigma), \boldsymbol{\varepsilon}(\mathbf{u}^n - \mathbf{u})),$$

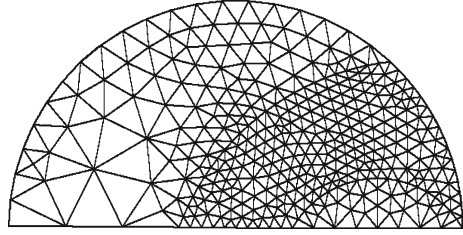
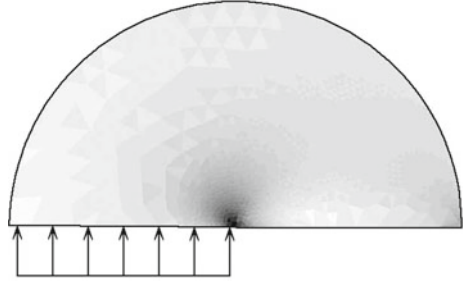
hence, by the Cauchy–Bunyakovskii inequality we have

$$\|\boldsymbol{\varepsilon}(\mathbf{u}^n - \mathbf{u})\|_0 \leq \|\mathbf{a} : (\Delta \sigma^{n-1} - \Delta \sigma)\|_0. \quad (5.33)$$

For simplicity we put $\mathbf{a} = \varsigma \mathbf{b}$, $\mathbf{c} = (1 + \varsigma)^{-1} \mathbf{a}$, where σ is a small dimensionless parameter.

Due to Korn’s inequality, on the space $\mathbf{U}$ the left-hand side of (5.33) defines the norm $\|\mathbf{u}^n - \mathbf{u}\|_1$ equivalent to the $\mathbf{H}^1(\Omega)$ -norm. The right-hand side is estimated with the help of formulae (5.30) and (5.32), taking into account the fact that a projection operator is a non-expanding mapping:

$$\begin{aligned} (1 + \varsigma) \|\mathbf{a} : (\Delta \sigma^{n-1} - \Delta \sigma)\|_0 &= \|\Pi_{\mathbf{a}^{-1}}(\boldsymbol{\varepsilon}(\mathbf{u}^{n-1}) - \mathbf{a} : \sigma_0) - \Pi_{\mathbf{a}^{-1}}(\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{a} : \sigma_0)\|_0 \\ &\leq \|\boldsymbol{\varepsilon}(\mathbf{u}^{n-1} - \mathbf{u})\|_0 = \|\mathbf{u}^{n-1} - \mathbf{u}\|_1. \end{aligned}$$

Fig. 5.10 Finite-element grid**Fig. 5.11** Shear intensity: an elastic material

Thus, we have the estimate

$$\|u^n - u\|_1 \leq \frac{1}{1 + \varsigma} \|u^{n-1} - u\|_1 \leq \frac{1}{(1 + \varsigma)^n} \|u^0 - u\|_1,$$

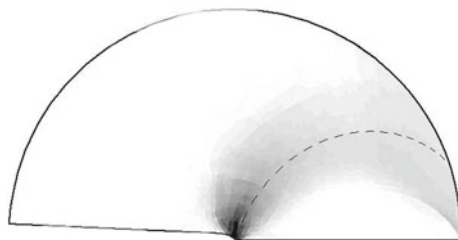
which provides convergence of a sequence of approximate solutions to an exact one with the rate of geometric progression, whose ratio is equal to $1/(1 + \varsigma) < 1$.

Repeating the above considerations for the finite-dimensional space U_h instead of U , we can prove convergence of a sequence of solutions of discrete problems of the finite-element method, which is constructed in the numerical implementation of the problem.

We present results of the numerical analysis of some problems with the help of the iterative algorithm described above. In Fig. 5.10 a finite-element grid with refinement in the neighbourhood of the assumed strain localization zone (of a logarithmic spiral), which is used in computations for a cylindrical sample with a notch, is shown. This grid involves 366 nodes and 673 elements. To improve accuracy, the results presented below are obtained on a finer grid of a similar structure consisting of 1369 nodes and 2634 elements. A comparison of numerical results on different grids shows qualitative and quantitative agreement between them. In computations the values of mechanical parameters corresponding to graphitized carbons used in aluminium industry as lining materials were taken: $k = 8$ and $\mu = 4.8$ GPa, $\tau_s = 14$ MPa.

In Fig. 5.11 the shear intensity distribution in a computational domain, obtained in the framework of the linear elasticity theory, is given. The loading scheme is shown as well. The part of the domain, where the shear intensity exceeds 1.5 % that is half its maximal value on a fine grid, is shown by black. In the remaining part the intensity

Fig. 5.12 Shear intensity:
a material with different
strengths ($\alpha = 0.3$)



level below this value is shown by shades of gray. In Fig. 5.12 the shear intensity field for a material with different strengths with the internal friction parameter $\alpha = 0.3$ is presented. The corresponding strain localization line is shown by a dashed line. In computations the regularization parameter ζ was taken equal to 0.001. It has been found that the rate of convergence of the iterative process, as well as the limit value of the load such that a nonlinear increase of displacements of the loaded boundary of a sample is observed, essentially depends on this parameter. The value of the limit load q turns out to be about 10 MPa. In Fig. 5.12 the part of the domain where the shear intensity exceeds 35 % is shown by black. To obtain a solution with relative error of 0.5 % requires about hundred iteration steps of the method of successive approximations.

Notice that the calculated value of limit load is slightly greater than the upper estimate obtained in the previous section. Analysis shows that this is primarily related to the fact that in the numerical solution of the problem fixed parameters of elasticity of a material are used. With the assumption that they tend to infinity, this value decreases but convergence of the iterative method becomes poor. Thus, the strain localization in a reasonable number of steps being expected does not take place. The second cause is that in computations the parameter ζ differs from zero and there is no point in its further decrease since this results in slower convergence of the algorithm. Finally, the size of grid elements in the neighbourhood of the vertex of the notch is the third cause which decreases the limit load and has influence on the structure of a solution. However, nonuniform grid refinement in subdomains results in the ill-conditioned matrix of a system, round-off errors increase, and a solution becomes unsatisfactory, even though arithmetical calculations are performed with double precision.

In Figs. 5.13 and 5.14 the results of computations for $\alpha = 0.7$ and $\alpha = 1.3$ with the same elasticity moduli and coefficient of cohesion of a material with different strengths are presented. In Fig. 5.13 the localization line for $\alpha = 0.7$ is shown by a dashed line. Shades of gray correspond to the values of the shear intensity from zero to 56 %. Figure 5.14 refers to a solution of the problem for $\alpha = 1.3$. This value is in the range from $\sqrt{3}/2$ to $\sqrt{3}$, hence, in this case there does not exist a solution with a strain zone localized along the line. The boundary of a domain shown by black corresponds to the shear intensity level of 32 % which is half its maximal value at the vertex of the notch.

Fig. 5.13 Shear intensity:
a material with different
strengths ($\alpha = 0.7$)



Fig. 5.14 Shear intensity:
a material with different
strengths ($\alpha = 1.3$)

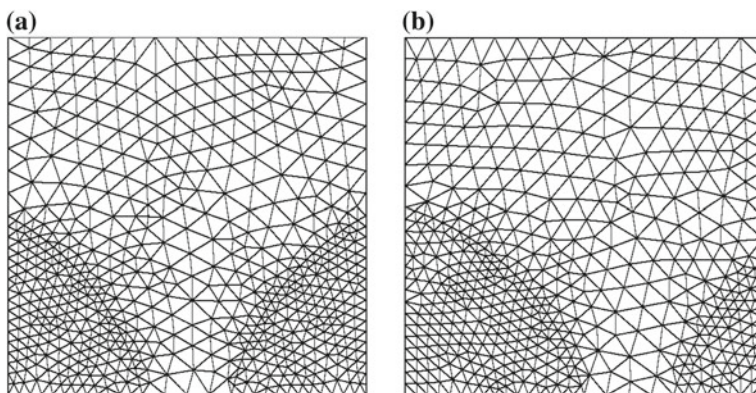
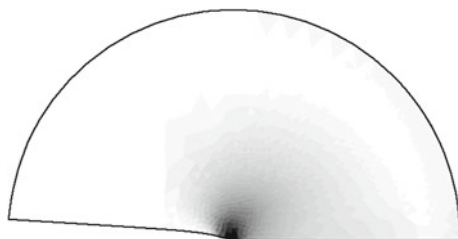


Fig. 5.15 Finite-element grids: **a** central crack, **b** displaced crack

The results of similar computations for $\alpha > \sqrt{3}$ are close to the elastic solution shown in Fig. 5.11. This is well substantiated since for plane deformation of a material with the internal friction parameter greater than $\sqrt{3}$ a rigid contact in the rheological scheme of a material is blocked and the property to have different strengths is not exhibited.

In Figs. 5.16 and 5.17 the results of numerical solution of the problem on tension of a rectangular sample with a horizontal notch (crack), being a classical problem of the theory of strength, are illustrated. The solution is obtained for half the sample, the underside of the boundary of computational domain is a symmetry line. A finite-element grid for the problem with a central crack whose length is equal to half the width of the sample is shown in Fig. 5.15a. It consists of 642 nodes and 1187

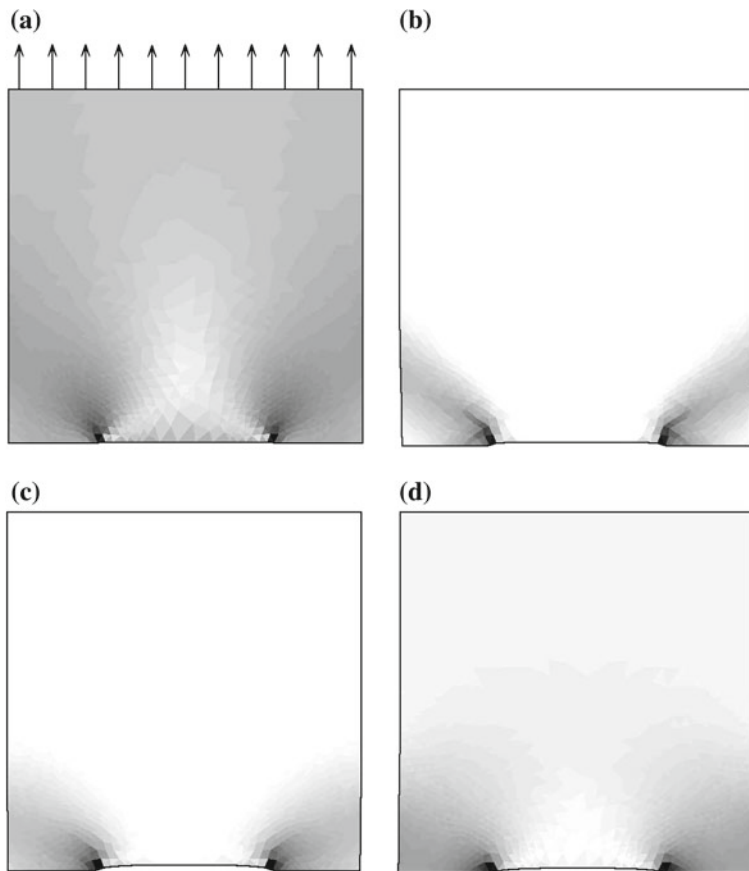


Fig. 5.16 Tension of a sample with a crack. Shear intensity: **a** an elastic material, **b** a material with different strengths ($\alpha = 0.3$), **c** a material with different strengths ($\alpha = 0.7$), **d** a material with different strengths ($\alpha = 1.3$)

elements. More precise computations were performed on a finer grid consisting of 1329 nodes and 2522 elements. A grid for the problem with a displaced crack of the same length, consisting of 557 nodes and 1022 elements, is presented in Fig. 5.15b. Its finer version consists of 1090 nodes and 2052 elements which are refined in the same subdomains near the vertices of the crack.

In Fig. 5.16a the elastic solution is presented. The maximal value of the shear intensity on the fine grid is 0.75 %. All the range of the intensity variation from the maximal value (black) to zero (white) is shown by shades of gray. In the computations the given external load $q = 15$ MPa approximately corresponds to the level of the limit load for a material with different strengths. In Fig. 5.16b the intensity distribution in a material for $\alpha = 0.3$ is shown. The regularization parameter is $\sigma = 0.001$. The maximum of the shear intensity is 30 %. Figure 5.16c,d illustrate the solutions

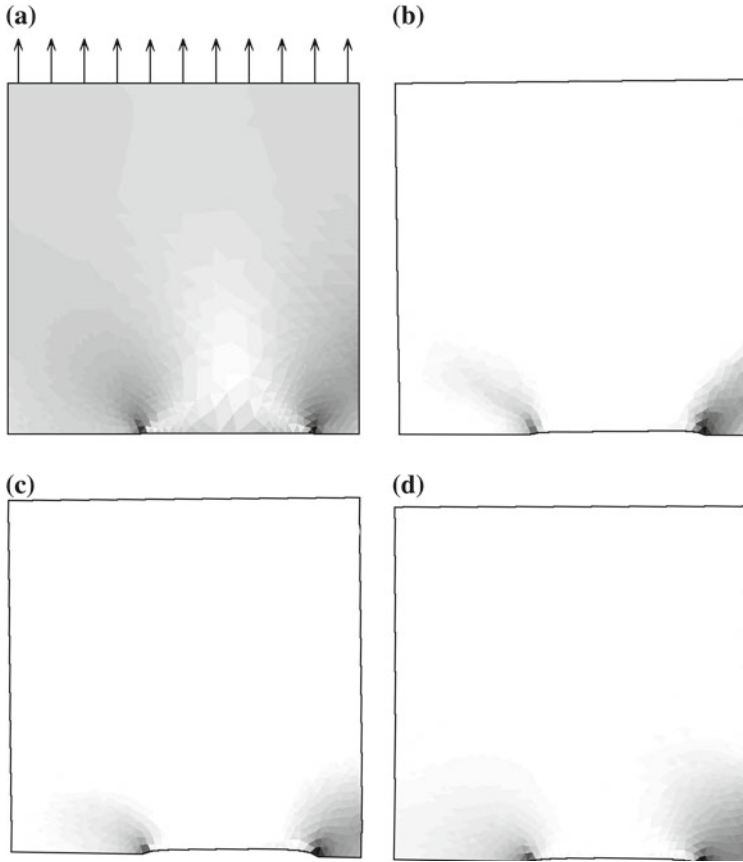


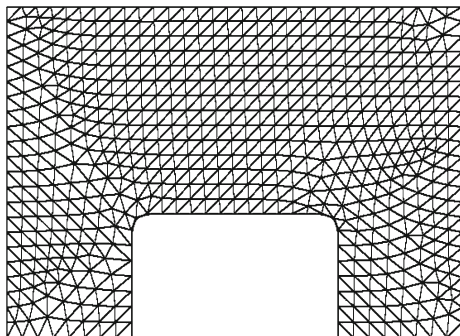
Fig. 5.17 Tension of a sample with an asymmetric crack. Shear intensity: **a** an elastic material, **b** a material with different strengths ($\alpha = 0.3$), **c** a material with different strengths ($\alpha = 0.7$), **d** a material with different strengths ($\alpha = 1.3$)

for $\alpha = 0.7$ and $\alpha = 1.3$, respectively. The maximal values of the shear intensity for these solutions are 36 and 20 %, respectively.

In Fig. 5.17 similar results for the problem with a crack displaced relative to the middle of a sample are presented. The calculated value of the limit load is $q = 14$ MPa. The maximum of the shear intensity in the elastic sample (Fig. 5.17a) is 1 %, for $\alpha = 0.3$ (Fig. 5.17b) it is 36 %, for $\alpha = 0.7$ (Fig. 5.17c) it is 59 %, and for $\alpha = 1.3$ (Fig. 5.17d) it is 29 %. Analyzing the obtained solutions, we can observe that the strain localization for tension of a sample with a crack, as in the previous problem, takes place for relatively small values of α .

Now we consider the problem on thermal expansion of a bottom block being a lining element of an aluminium electrolytic cell, [9]. The lining prevents aluminium from contact with a metal bath and current-carrying elements of an electrolytic cell in

Fig. 5.18 Cross-section of a block



the production process. In general terms, an individual block is a graphitized carbon parallelepiped with a groove (cut) into which a steel bloom, serving as a cathode, is placed. The cross-section of a block and the coarser finite-element grid consisting of 623 nodes and 1123 elements are shown in Fig. 5.18.

With bottom burning conditions after complete overhaul of an electrolytic cell, the block–bloom system heats to 700–900°C. The thermal expansion coefficient of a steel considerably exceeds that of graphitized carbon, hence, from a bloom along vertical boundaries of a cut normal pressure acts which may result in the cracking of a block. In practice, it is important to know the direction of probable propagation of a main crack. If in the operating process of an electrolytic cell a crack has achieved an upper boundary of a block, which is overlain by liquid aluminium, then aluminium comes through to the surface of a bloom and as a result of a chemical reaction quality of metal is lost. At the same time, cracks propagating in the direction of back and lateral surfaces of a block are not so much dangerous.

In Fig. 5.19a the calculated levels of the shear intensity for an elastic material with elasticity moduli of graphitized carbon are presented. The values of the levels vary from zero (white) to the maximum of intensity equal to 1.5 % (black). The computations were performed on a grid consisting of 1937 nodes and 3654 elements. Given pressure of a bloom to a block (15 MPa) approximately corresponds to the limiting pressure for a material with different strengths. The results of computations for $\alpha = 0.3$ and $\zeta = 0.01$ are shown in Fig. 5.19b. In this case the maximal shear intensity is 12 %. It turns out that in this problem decreasing the regularization parameter ζ results in an extremely low rate of convergence of the iterative process. In Figs. 5.19c,d the internal friction parameters are $\alpha = 0.7$ and $\alpha = 1.3$, respectively, for the maximal values of the shear intensity of 14 and 17 %, respectively. Analysis shows that for relatively small α the safe strain localization in narrow strips, which reach the lateral surfaces of a block, takes place. At the same time, for large α the localization domain extends up to the surface of contact with liquid metal. This means that a life cycle of an electrolytic cell between complete overhauls can be extended due to the use of lining materials with a small internal friction coefficient, i.e. such that their ultimate strengths under compression and tension differ only slightly.

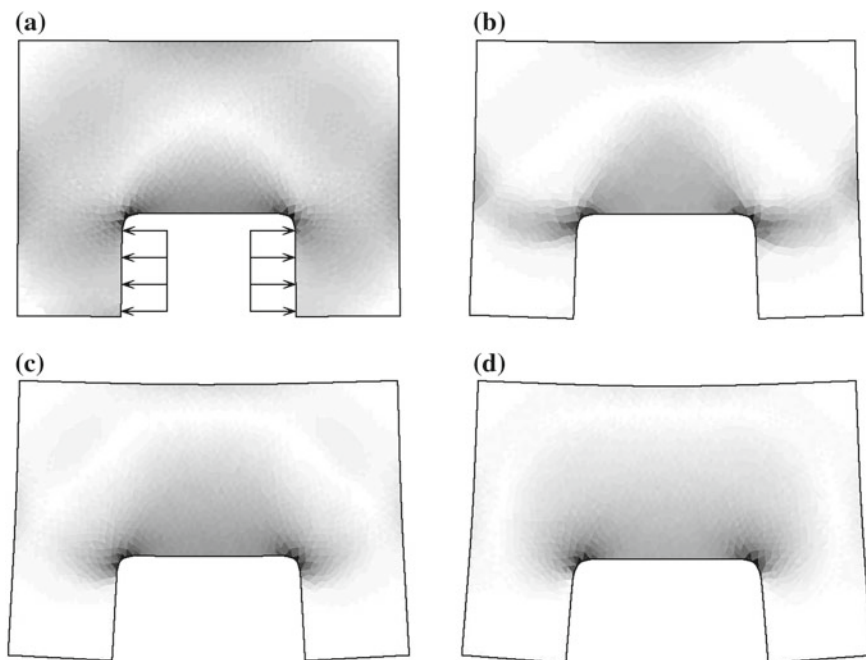


Fig. 5.19 Shear intensity in a block of an electrolytic cell: **a** an elastic material, **b** a material with different strengths ($\alpha = 0.3$), **c** a material with different strengths ($\alpha = 0.7$), **d** a material with different strengths ($\alpha = 1.3$)

The actual sizes of the cross-section of blocks used at the Krasnoyarsk aluminium plant were taken as geometric parameters of the problem: width is 0.55 m, height is 0.4 m, width of a cut for a bloom is 0.25 m, depth of a cut is 0.15 m. A large number of computations for different values of the parameters shows that the direction of a crack, arising in a block due to thermal expansion of a bloom, essentially depends on geometry of cross-section, on quality of sealing of interblock joints, and on the ratio of the ultimate strengths of graphitized carbon.

5.5 Plane Strain State

The computational algorithm being worked out enables one to determine indirectly the strain localization zones with the help of a regularized model where these zones are represented as fuzzy domains of relatively large values of the strain intensity. We describe another approach based on the numerical solution of equations of a model of an ideal material with different strengths and with degenerated moduli of elastic compliance $a = 0$ and $b = \infty$. We consider the plane strain state. In this case

$$\begin{aligned} 2 \tau^2(\boldsymbol{\sigma}) &= (\sigma_{11} + p(\boldsymbol{\sigma}))^2 + (\sigma_{22} + p(\boldsymbol{\sigma}))^2 + (\sigma_{33} + p(\boldsymbol{\sigma}))^2 + 2 \sigma_{12}^2, \\ 3 p(\boldsymbol{\sigma}) &= -\sigma_{11} - \sigma_{22} - \sigma_{33}. \end{aligned}$$

The constitutive relationship (4.2) in the form of the variational inequality

$$(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}) : \boldsymbol{\varepsilon} \leq 0 \quad (-\boldsymbol{\sigma} : \boldsymbol{\varepsilon} \leq -\tilde{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon})$$

with an arbitrary admissible variation of the stress tensor, which satisfies the constraint $\tau(\tilde{\boldsymbol{\sigma}}) \leq \tau_s + \mathfrak{x} p(\tilde{\boldsymbol{\sigma}})$, leads to the problem of finding the unconditional minimum of the convex Lagrangian

$$L(\tilde{\boldsymbol{\sigma}}, \lambda) = -\tilde{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} + \lambda (\tau(\tilde{\boldsymbol{\sigma}}) - \mathfrak{x} p(\tilde{\boldsymbol{\sigma}}))$$

in $\tilde{\boldsymbol{\sigma}}$. The necessary and sufficient minimum condition in differential form

$$\frac{\varepsilon_{jj}}{\lambda} = \frac{\sigma_{jj} + p(\boldsymbol{\sigma})}{2 \tau(\boldsymbol{\sigma})} + \frac{\mathfrak{x}}{3}, \quad \frac{\varepsilon_{12}}{\lambda} = \frac{\sigma_{12}}{2 \tau(\boldsymbol{\sigma})}, \quad \lambda \geq 0, \quad j = 1, 2, 3,$$

constitutes the deformation law associated with the von Mises–Schleicher strength condition.

We obtain equations, describing the stress-strain state of a material, only in the limiting equilibrium zone where $\tau(\boldsymbol{\sigma}) = \tau_s + \mathfrak{x} p(\boldsymbol{\sigma})$. In the rigid zone where strains vanish, within the framework of the model under consideration stresses, generally speaking, may not be determined. For plane strain $\varepsilon_{33} \equiv 0$, therefore

$$3(\sigma_{33} + p(\boldsymbol{\sigma})) = -2 \mathfrak{x} (\tau_s + \mathfrak{x} p(\boldsymbol{\sigma})).$$

Hence,

$$\begin{aligned} \left(1 - \frac{\mathfrak{x}^2}{3}\right) \sigma_{33} &= \left(1 + \frac{2 \mathfrak{x}^2}{3}\right) \frac{\sigma_{11} + \sigma_{22}}{2} - \mathfrak{x} \tau_s, \\ \left(1 - \frac{\mathfrak{x}^2}{3}\right) p(\boldsymbol{\sigma}) &= \frac{\mathfrak{x} \tau_s}{3} - \frac{\sigma_{11} + \sigma_{22}}{2}, \\ \tau^2(\boldsymbol{\sigma}) &= \frac{1}{4} (\sigma_{22} - \sigma_{11})^2 + \sigma_{12}^2 + \frac{\mathfrak{x}^2/3}{(1 - \mathfrak{x}^2/3)^2} \left(\tau_s - \frac{\sigma_{11} + \sigma_{22}}{2}\right). \end{aligned}$$

After transformation, the limiting equilibrium condition is reduced to the equation

$$\sqrt{\frac{(\sigma_{22} - \sigma_{11})^2}{4} + \sigma_{12}^2} + \kappa s = 0, \quad \kappa = \frac{\mathfrak{x}}{\sqrt{1 - \mathfrak{x}^2/3}}, \quad s = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\tau_s}{\mathfrak{x}}. \quad (5.34)$$

Notice that (5.34) makes sense only if $\mathfrak{x} < \sqrt{3}$. For greater $\mathfrak{x}$ only rigid motions of a material with zero strains are possible in the plane strain state. Comparing

Eq. (5.34) and the equation of the circle of radius $-\kappa s \geq 0$, we introduce a polar angle 2φ and arrive at the formulae

$$\begin{aligned}\sigma_{11} &= \frac{\tau_s}{\mathfrak{a}} + (1 + \kappa \cos 2\varphi) s, \\ \sigma_{22} &= \frac{\tau_s}{\mathfrak{a}} + (1 - \kappa \cos 2\varphi) s, \\ \sigma_{12} &= \kappa s \sin 2\varphi.\end{aligned}\tag{5.35}$$

If the internal friction coefficient is in the range of $\mathfrak{a} \leq \sqrt{3}/2$, then it is related to the angle α of internal friction by Eq. (4.19) from which it follows that $\kappa = \sin \alpha \leq 1$. In this case the formulae (5.35) coincide with similar formulae of the theory of soils where the Coulomb–Mohr strength condition is applied, [23]. The corresponding equations for stresses are well-known. They can be obtained by substitution of (5.35) in the differential equations of equilibrium

$$\sigma_{11,1} + \sigma_{12,2} + f_1 = 0, \quad \sigma_{12,1} + \sigma_{22,2} + f_2 = 0$$

(f_i are the projections of vector of the volume forces). This substitution results in two first-order equations in two unknown functions s and φ . For an arbitrary value of the parameter $\kappa > 0$ the equations for stresses have the form

$$\begin{aligned}(1 + \kappa \cos 2\varphi) s_{,1} - 2\kappa s \sin 2\varphi \varphi_{,1} + \kappa \sin 2\varphi s_{,2} \\ + 2\kappa s \cos 2\varphi \varphi_{,2} + f_1 &= 0, \\ \kappa \sin 2\varphi s_{,1} + 2\kappa s \cos 2\varphi \varphi_{,1} + (1 - \kappa \cos 2\varphi) s_{,2} \\ + 2\kappa s \sin 2\varphi \varphi_{,2} + f_2 &= 0.\end{aligned}\tag{5.36}$$

To determine the type of the system (5.36), we write it in the matrix form

$$\mathbf{A}_\sigma \frac{\partial}{\partial x_1} \begin{pmatrix} s \\ \varphi \end{pmatrix} + \mathbf{B}_\sigma \frac{\partial}{\partial x_2} \begin{pmatrix} s \\ \varphi \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbf{0},$$

where $\mathbf{A}_\sigma$ and $\mathbf{B}_\sigma$ are the square matrices:

$$\mathbf{A}_\sigma = \begin{pmatrix} 1 + \kappa \cos 2\varphi & -2\kappa s \sin 2\varphi \\ \kappa \sin 2\varphi & 2\kappa s \cos 2\varphi \end{pmatrix}, \quad \mathbf{B}_\sigma = \begin{pmatrix} \kappa \sin 2\varphi & 2\kappa s \cos 2\varphi \\ 1 - \kappa \cos 2\varphi & 2\kappa s \sin 2\varphi \end{pmatrix}.$$

By simplification the characteristic equation $\det(\mathbf{B}_\sigma - c \mathbf{A}_\sigma) = 0$ of the system is reduced to the quadratic equation

$$c^2(\kappa + \cos 2\varphi) - 2c \sin 2\varphi + \kappa - \cos 2\varphi = 0,$$

which has the roots

$$c^{\pm} = \frac{\sin 2\varphi \pm \sqrt{1 - \kappa^2}}{\cos 2\varphi + \kappa}.$$

Hence, the system is of hyperbolic type for $\kappa \leq 1$ and of elliptic type for $\kappa > 1$.

In the case of hyperbolic system, recalculation of eigenvalues in terms of the angle of internal friction yields

$$c^{\pm} = \tan(\varphi \pm \beta),$$

where $\beta = \pi/4 - \alpha/2$ is the angle between a normal to the slip area element and the first principal direction of the stress tensor, which was introduced in Sect. 4.2. This can be easily checked, verifying the simple trigonometric identities

$$\frac{\sin 2\varphi \pm \cos \alpha}{\cos 2\varphi + \sin \alpha} = \tan(\varphi \pm \beta).$$

Left eigenvectors (rows satisfying the system of the linear algebraic equations $\mathbf{Y}\mathbf{B}_{\sigma} = c \mathbf{Y}\mathbf{A}_{\sigma}$ for $c = c^{\pm}$) can be taken in the following form:

$$\mathbf{Y}^{\pm} = \frac{(\cos(2\varphi \pm \alpha), \sin(2\varphi \pm \alpha) \pm 1)}{\cos 2\varphi + \sin \alpha}.$$

For this choice

$$\mathbf{Y}^{\pm}\mathbf{A}_{\sigma} = (\cos \alpha, \pm 2s \sin \alpha).$$

Multiplying the matrix system from the left by eigenvectors in turn, we transform (5.36) to two equations on characteristics which can be obtained from the general equation

$$\mathbf{Y}\mathbf{A}_{\sigma} \frac{d}{dx_1} \begin{pmatrix} s \\ \varphi \end{pmatrix} + \mathbf{Y} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbf{0}, \quad \frac{d}{dx_1} = \frac{\partial}{\partial x_1} + c \frac{\partial}{\partial x_2}.$$

In the expanded form they are as follows:

$$ds \pm 2s \tan \alpha d\varphi + f^{\pm} dx_1 = 0, \quad dx_2 = \tan(\varphi \pm \beta) dx_1, \quad (5.37)$$

where

$$f^{\pm} = \frac{f_1 \cos(2\varphi \pm \alpha) + f_2 (\sin(2\varphi \pm \alpha) \pm 1)}{(\cos 2\varphi + \sin \alpha) \cos \alpha}.$$

Equations (5.37) are a generalization of the Hencky equations in the plasticity theory, [3]. In some special problems they can be easily integrated with the help of the numerical method of characteristics, [23]. However, before going to examples, we obtain an analogue of the Geiringer equations for determining a displacement field in a material with different strengths.

We represent the constitutive equations, associated with the condition (5.34), in the form

$$\frac{\varepsilon_{11,22}}{\lambda} = \pm \frac{\sigma_{22} - \sigma_{11}}{4\kappa q} + \frac{\kappa}{2}, \quad \frac{\varepsilon_{12}}{\lambda} = -\frac{\sigma_{12}}{2\kappa q}, \quad \lambda \geq 0,$$

where λ , generally speaking, differs from the Lagrange multiplier introduced at the beginning of this section. Taking into account (5.35), we obtain

$$\varepsilon_{11} + \varepsilon_{22} = \lambda \kappa, \quad \varepsilon_{22} - \varepsilon_{11} = \lambda \cos 2\varphi, \quad 2\varepsilon_{12} = -\lambda \sin 2\varphi.$$

The invariants $\theta(\boldsymbol{\varepsilon}) = \lambda \kappa$ and

$$\gamma^2(\boldsymbol{\varepsilon}) = \frac{2}{3} (\varepsilon_{11}^2 + \varepsilon_{22}^2 + (\varepsilon_{22} - \varepsilon_{11})^2) + 4\varepsilon_{12}^2 = \lambda^2 \left(1 + \frac{\kappa^2}{3}\right)$$

of the strain tensor are related between themselves by means of the equation $\theta(\boldsymbol{\varepsilon}) = \kappa \gamma(\boldsymbol{\varepsilon})$. The displacement field is determined from the system

$$\frac{u_{1,1} + u_{2,2}}{\kappa} = \frac{u_{2,2} - u_{1,1}}{\cos 2\varphi} = -\frac{u_{1,2} + u_{2,1}}{\sin 2\varphi} \geq 0. \quad (5.38)$$

Two independent equations of this system can be written in the matrix form

$$\mathbf{A}_u \frac{\partial}{\partial x_1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \mathbf{B}_u \frac{\partial}{\partial x_2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{0}$$

with the matrices

$$\mathbf{A}_u = \begin{pmatrix} \cos 2\varphi + \kappa & 0 \\ \sin 2\varphi & \kappa \end{pmatrix}, \quad \mathbf{B}_u = \begin{pmatrix} 0 & \cos 2\varphi - \kappa \\ \kappa & \sin 2\varphi \end{pmatrix}.$$

The characteristic equation $\det(\mathbf{B}_u - c \mathbf{A}_u) = 0$ coincides with the quadratic equation obtained above. Hence, the eigenvalues are equal to $c^\pm$ and the system is of hyperbolic type for $\kappa = \sin \alpha \leq 1$ and of elliptic type for $\kappa > 1$. For the hyperbolic system the left eigenvectors are equal to

$$\mathbf{Y}^\pm = \frac{(-\sin 2\varphi \cos 2\varphi \pm \sin \alpha \cos \alpha, \cos^2 2\varphi - \sin^2 \alpha)}{\cos 2\varphi + \sin \alpha}.$$

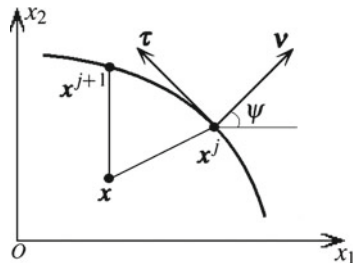
For them

$$\mathbf{Y}^\pm \mathbf{A}_u = \sin \alpha (-\sin 2\varphi \pm \cos \alpha, \cos 2\varphi - \sin \alpha).$$

With the help of the identities

$$\frac{\cos 2\varphi - \sin \alpha}{-\sin 2\varphi \pm \cos \alpha} = \tan(\varphi \pm \beta),$$

Fig. 5.20 System of nodes for method of characteristics



which are easily verified, the equations on characteristics, obtained by multiplying the system from the left by the eigenvectors, are reduced to the form

$$du_1 + \tan(\varphi \pm \beta) du_2 = 0, \quad dx_2 = \tan(\varphi \pm \beta) dx_1. \quad (5.39)$$

For the numerical solution of the system (5.37), (5.39) we use the direct characteristic method, [15], which enables one to construct the stress and displacement fields in a neighbourhood of an arbitrary curve Γ_0 on which the boundary conditions are given in terms of stresses and displacements simultaneously. As in the ideal plasticity theory, the necessity to redefine the boundary conditions is a consequence of the fact that the model is degenerate.

Going to the coordinate system connected with vectors of normal and tangent to the curve (see Fig. 5.20), we reformulate the boundary conditions, given in terms of stresses, in terms of the unknown functions q and φ with the help of the formulae (5.35):

$$\frac{\tau_s}{\alpha} + (1 + \kappa \cos 2(\varphi - \Psi)) s = q_v, \quad \kappa s \sin 2(\varphi - \Psi) = q_\tau,$$

where Ψ is the angle between the normal and the x_1 axis, q_v and q_τ are projections of the vector of external load. Hence,

$$\frac{\sin 2(\varphi - \Psi)}{1 + \kappa \cos 2(\varphi - \Psi)} = \frac{q_\tau}{q_v - \tau_s/\alpha}, \quad s = \frac{q_v - \tau_s/\alpha}{1 + \kappa \cos 2(\varphi - \Psi)}. \quad (5.40)$$

In the general case, to determine the boundary values of the function φ , it is required to solve a transcendental equation. To this end, for example, Newton's method can be used. The equation becomes trivial provided that the load acts along a normal to the boundary ($q_\tau = 0$). Then $\varphi = \Psi$ and $\varphi = \Psi \pm \pi/2$ are the solutions of (5.40).

According to the method of characteristics, on the curve Γ_0 a system of nodes, numbered in the direction of positive path tracing, is taken. Characteristics issued out of each pair of neighbouring nodes are constructed. Coordinates of a point of intersection of characteristics belonging to different families are determined from the difference equations

$$x_2 - x_2^j = (x_1 - x_1^j) \tan(\varphi^j - \beta), \quad x_2 - x_2^{j+1} = (x_1 - x_1^{j+1}) \tan(\varphi^{j+1} + \beta).$$

Equations (5.37), (5.39) are approximated in the following way:

$$\begin{aligned} \ln \frac{s}{s^j} - 2(\varphi - \varphi^j) \tan \alpha + \frac{f_j^-(x_1 - x_1^j)}{s^j} &= 0, \\ \ln \frac{s}{s^{j+1}} + 2(\varphi - \varphi^{j+1}) \tan \alpha + \frac{f_{j+1}^+(x_1 - x_1^{j+1})}{s^{j+1}} &= 0, \\ u_1 - u_1^j + (u_2 - u_2^j) \tan(\varphi^j - \beta) &= 0, \\ u_1 - u_1^{j+1} + (u_2 - u_2^{j+1}) \tan(\varphi^{j+1} + \beta) &= 0. \end{aligned}$$

Thus, at a new node lying at the intersection of characteristics:

$$\begin{aligned} x_1 &= \frac{x_1^{j+1} \tan(\varphi^{j+1} + \beta) - x_1^j \tan(\varphi^j - \beta) - x_2^{j+1} + x_2^j}{\tan(\varphi^{j+1} + \beta) - \tan(\varphi^j - \beta)}, \\ x_2 &= \frac{x_2^{j+1} \cot(\varphi^{j+1} + \beta) - x_2^j \cot(\varphi^j - \beta) - x_1^{j+1} + x_1^j}{\cot(\varphi^{j+1} + \beta) - \cot(\varphi^j - \beta)}, \\ \varphi &= \frac{\varphi^{j+1} + \varphi^j}{2} + \frac{1}{4 \tan \alpha} \left(\ln \frac{s^{j+1}}{s^j} - \frac{f_{j+1}^+(x_1 - x_1^{j+1})}{s^{j+1}} + \frac{f_j^-(x_1 - x_1^j)}{s^j} \right), \\ s &= \sqrt{s^{j+1} s^j} \exp \left((\varphi^{j+1} - \varphi^j) \tan \alpha - \frac{f_{j+1}^+(x_1 - x_1^{j+1})}{2 s^{j+1}} - \frac{f_j^-(x_1 - x_1^j)}{2 s^j} \right), \\ u_1 &= \frac{u_1^{j+1} \cot(\varphi^{j+1} + \beta) - u_1^j \cot(\varphi^j - \beta) + u_2^{j+1} - u_2^j}{\cot(\varphi^{j+1} + \beta) - \cot(\varphi^j - \beta)}, \\ u_2 &= \frac{u_2^{j+1} \tan(\varphi^{j+1} + \beta) - u_2^j \tan(\varphi^j - \beta) + u_1^{j+1} - u_1^j}{\tan(\varphi^{j+1} + \beta) - \tan(\varphi^j - \beta)}. \end{aligned} \quad (5.41)$$

Once a solution at all such nodes has been calculated by the formulae (5.41), the process is recursively repeated.

The algorithm and the program were tested by comparison of the results of computations and known approximate solutions for centered simple waves, [23]. The method turns out to be of high accuracy. However, it should be noticed that it may not serve as an universal tool in the development of computer technologies for calculation of the stress-strain state of a material with different strengths. It can be applied only to a rather special class of problems. The simplest examples of such problems are represented in Figs. 5.21 and 5.22. The fields of characteristics (Figs. 5.21a and 5.22a) and the vector fields of displacements (Figs. 5.21b and 5.22b) for the problem about action of distributed pressure on the surface of a half-space filled by a material with different strengths (a soil) are shown. Normal pressure is given in the form of a smooth cubic spline varying in the range from q_0 to q_1 on half the boundary (on the left) and taking a constant value q_1 on another half (on the right). The following values of the parameters are taken: $q_0 = 0$, $q_1 = 5 \tau_s$ (Fig. 5.21) and $q_0 = 5 \tau_s$, $q_1 = 0$ (Fig. 5.22). The displacement vector on the boundary is given in a similar way. Here and in the subsequent computations the internal friction parameter is $\alpha = 0.5$. The solutions are obtained for the boundary value $\varphi = -\pi/2$. For $\varphi = 0$ we have

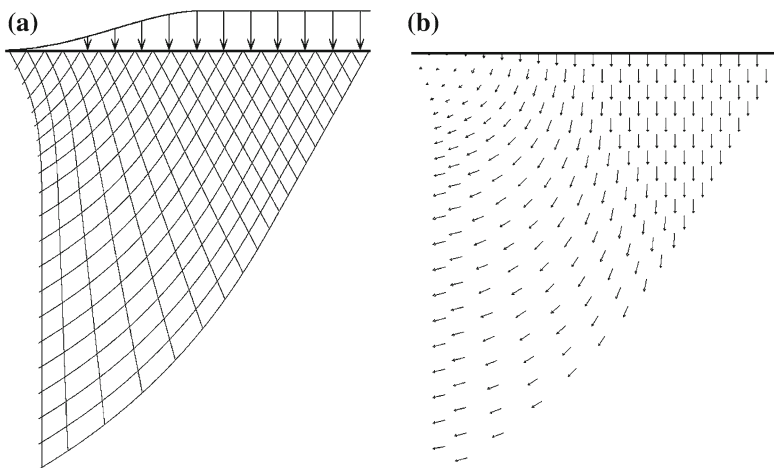


Fig. 5.21 Half-space under action of pressure: **a** field of characteristics, **b** vector field of displacements

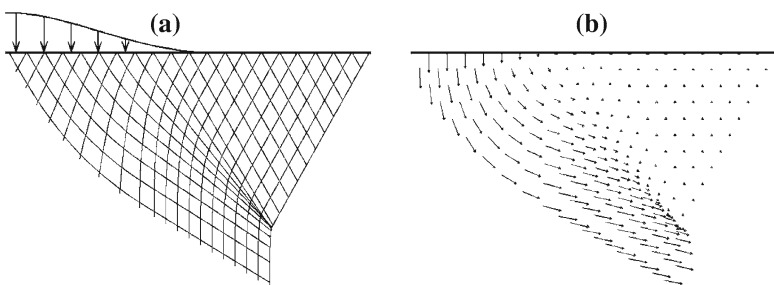


Fig. 5.22 Gradient catastrophe: **a** field of characteristics, **b** vector field of displacements

other solutions. They differ from the presented ones in slope of characteristics and, as a consequence, in size of characteristic zones. The inequality (5.38) is used as a criterion of solution selection.

Analysis shows that, even in the case of smooth variation of external load, characteristics of one family may intersect each other (see Fig. 5.22). This results in a gradient catastrophe, i.e. in destruction of a solution accompanied by the stress discontinuities. Though the required relationships on a discontinuity line can be easily obtained from the equilibrium equations, [23], to use them when separating out discontinuities on a nonregular grid for the direct method of characteristics is a cumbersome problem. The need for pasting together the solutions corresponding to different variants of the choice of the function φ on the boundary introduces additional difficulties.

In Fig. 5.23 similar results of numerical solution of the problem on thermal expansion of a block of an aluminium electrolytic cell with the assumption, that everywhere

Fig. 5.23 Kinematic mechanism of destruction of a block:
a field of characteristics,
b vector field of displacements

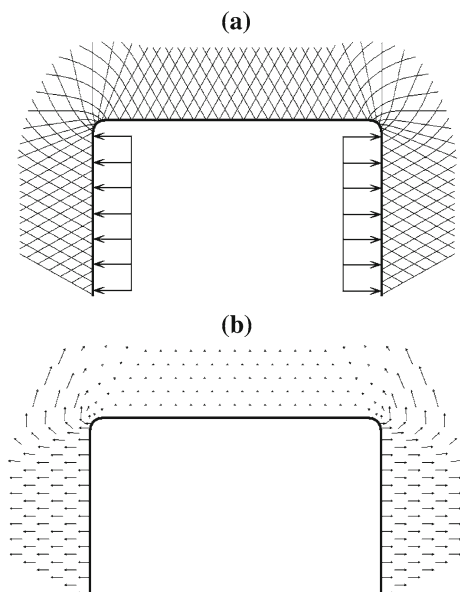
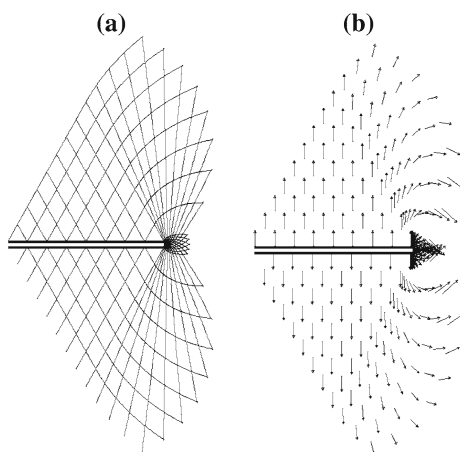


Fig. 5.24 Limit state of a sample at vertex of a notch:
a field of characteristics,
b vector field of displacements



on the surface of a cut for a bloom the limiting stress state is achieved, are presented. The solution illustrates the kinematically admissible mechanism of destruction of a block near edges of a cut which may differ from the actual mechanism since the presence of domains with the subcritical stress level is not taken into account.

The results of computations of the limiting state of a cylindrical sample with a notch are presented in Fig. 5.24. Pressure acting on the edges of the notch is taken according to the curve 1 in Fig. 5.6.

To conclude, we notice that the numerical integration of the quasi-linear systems (5.36) and (5.38) for $\kappa > 1$ (when they are elliptic) is a far more complicated problem than in the hyperbolic case considered here. The exception is the case of $\kappa = \sqrt{3}$, where $\kappa \rightarrow \infty$. As noticed in Sect. 4.3, in this case the system of kinematic equations is transformed to the system of the Cauchy–Riemann equations from which it follows that the function $w(z) = u_1 + \iota u_2$ of the complex variable $z = x_1 + \iota x_2$ is analytic. Hence, the displacement field can be constructed by selection of an appropriate analytic function.

The von Mises–Schleicher condition (5.34) of the limiting equilibrium of a granular material takes the form

$$s = 0 \Rightarrow \sigma_{11} + \sigma_{22} = \frac{2\tau_s}{\sqrt{3}}.$$

If in the problem under consideration the vector of volume forces is equal to zero, then the equilibrium equations are also reduced to the Cauchy–Riemann equations for the analytic function

$$\zeta(z) = \sigma_{11} - \frac{\tau_s}{\sqrt{3}} - \iota \sigma_{12}.$$

Thus, by selection of the analytic function $\zeta(z)$ we can obtain the stress fields

$$\sigma_{11} = \frac{\tau_s}{\sqrt{3}} + \Re \zeta(z), \quad \sigma_{22} = \frac{\tau_s}{\sqrt{3}} - \Re \zeta(z), \quad \sigma_{12} = -\Im \zeta(z),$$

which satisfy some boundary conditions on the boundary of the body of a granular material. In this case for calculation of the tangential stress intensity we can apply the following formula:

$$\sqrt{\frac{(\sigma_{22} - \sigma_{11})^2}{4} + \sigma_{12}^2} = |\zeta(z)|.$$

When the volume forces are taken into account, we have first to construct a particular solution $\sigma_{11}^0, \sigma_{22}^0, \sigma_{12}^0$ of the equilibrium equations which satisfies the condition $\sigma_{22}^0 = 2\tau_s/\sqrt{3} - \sigma_{11}^0$, then the problem is reduced to the selection of the analytic function

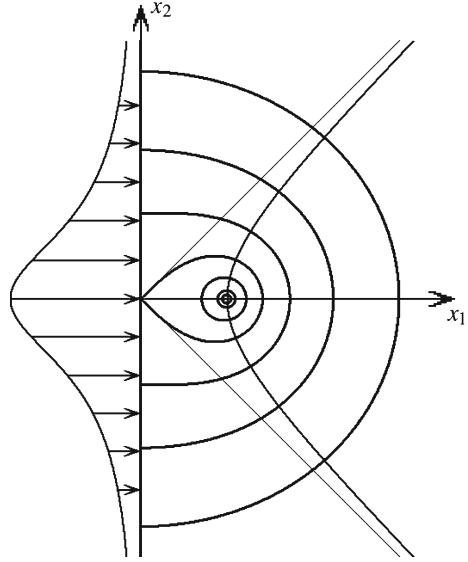
$$\zeta(z) = \sigma_{11} - \sigma_{11}^0 - \iota (\sigma_{12} - \sigma_{12}^0),$$

with the help of which the actual stress field is determined by the formulae

$$\sigma_{11} = \sigma_{11}^0 + \Re \zeta(z), \quad \sigma_{22} = \sigma_{22}^0 - \Re \zeta(z), \quad \sigma_{12} = \sigma_{12}^0 - \Im \zeta(z).$$

Applying this approach, we can, for example, determine the stress state of the body of a weightless material bounded by the plane $x_1 = 0$ and being under the action of normal pressure $\sigma_{11} = -q(x_2)$ where the function $q(x_2)$ is analytic. In

Fig. 5.25 Level curves of the intensity of tangential stresses



this case $\zeta(z) = -q(-iz) - \tau_s/\sqrt{3}$. If, in particular, a granular material is ideal, i.e. $\tau_s = 0$, and $q(x_2) = 1/(1 + x_2^2)$, then stresses are constructed with the help of the function $\zeta(z) = 1/(z^2 - 1)$ which is analytic in the right half-plane everywhere except the point $z = 1$. The direct calculation of the real and imaginary parts of this function yields

$$\sigma_{22} = -\sigma_{11} = \frac{1 - x_1^2 + x_2^2}{(1 - x_1^2 + x_2^2)^2 + 4x_1^2 x_2^2}, \quad \sigma_{12} = \frac{2x_1 x_2}{(1 - x_1^2 + x_2^2)^2 + 4x_1^2 x_2^2}.$$

The level curves of the intensity of tangential stresses are described by the equation $|1 - z^2| = 1/C_4$ which is reduced to the biquadratic equation in the x_2 coordinate:

$$(1 - x_1^2)^2 + 2(1 + x_1^2)x_2^2 + x_2^4 = \frac{1}{C_4^2}.$$

Hence, $x_2 = \pm \sqrt{\sqrt{4x_1^2 + 1/C_4^2} - x_1^2} - 1$. The system of such curves for the level constant C_4 varying in the range from 0.125 to 10 is shown in Fig. 5.25. For $C_4 < 1$ the level curves are open-circuit curves, for $C_4 > 1$ they are closed about the point $z = 1$ being the stress concentrator. In Fig. 5.25 the parabola $x_1 = \sqrt{1 + x_2^2}$ with two asymptotes is shown. When going across this parabola, both normal stresses σ_{11} and σ_{22} change sign. Stresses tend to zero at infinity.

Calculation of the principal vector of external load gives a finite value equal to

$$\int_{-\infty}^{\infty} q(x_2) dx_2 = \arctan x_2 \Big|_{-\infty}^{+\infty} = \pi.$$

With the help of the transformation $\hat{x}_{1,2} = \lambda x_{1,2}$ of contraction along the axes we can obtain a solution of the problem about action of a concentrated force on the boundary of a half-plane. To this end, we have to go to the solution $\hat{\sigma}(\hat{x}) = \sigma(x)/\lambda$ which corresponds to the load $\hat{q}(\hat{x}_2) = q(x_2)/\lambda$ and assume that the parameter λ tends to zero. From the chain of relationships

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} \hat{q}(\hat{x}_2) \xi(\hat{x}_2) d\hat{x}_2 - \pi \xi(0) \right| &= \left| \left(\int_{-\infty}^{-1/\sqrt{\lambda}} + \int_{-1/\sqrt{\lambda}}^{1/\sqrt{\lambda}} + \int_{1/\sqrt{\lambda}}^{+\infty} \right) \frac{(\xi(\lambda x_2) - \xi(0)) dx_2}{1 + x_2^2} \right| \\ &\leq \left(M \int_{-\infty}^{-1/\sqrt{\lambda}} + \max_{|x_2| \leq 1/\sqrt{\lambda}} |\xi(\lambda x_2) - \xi(0)| \int_{-\infty}^{+\infty} + M \int_{1/\sqrt{\lambda}}^{+\infty} \right) \frac{dx_2}{1 + x_2^2} \\ &\leq M \left(\pi - 2 \arctan \frac{1}{\sqrt{\lambda}} \right) + \pi \max_{|\hat{x}_2| \leq \sqrt{\lambda}} |\xi(\hat{x}_2) - \xi(0)| \rightarrow 0, \end{aligned}$$

where $\xi(\hat{x}_2)$ is an arbitrary function of the space $S(\mathbb{R})$ of trial functions, [21, 25], and $M = \sup\{|\xi(x_2) - \xi(0)|, x_2 \in \mathbb{R}\}$, it follows that the sequence $\hat{q}(\hat{x}_2)$ converges to $\pi \delta(x_2)$ in the space $S'(\mathbb{R})$ of generalized temperate functions. We can show that passing to the limit in λ we obtain the stress field concentrated at the point $z = 0$.

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Chapter 6

Elastic–Plastic Waves in a Loosened Material

Abstract A priori estimates for solutions in characteristic cones, which provide assurance that a boundary-value problem with initial data and dissipative boundary conditions is well-posed in the framework of a model describing dynamic deformation of an elastic–plastic granular material, are obtained. Shock adiabatic curves for plane longitudinal compression waves, propagating in an unbounded body, are constructed for various combinations of mechanical parameters of a material. Computational algorithm for the analysis of propagation of shock waves of small amplitude in a granular material, based on the method of splitting with respect to physical processes and with respect to spatial variables, is proposed. The results of two-dimensional computations of interaction of signotons in an inhomogeneous loosened material accompanied by a transverse cumulative ejection as well as the results of modeling of the “dry boiling” process (formation of continuity jumps in a material under the action of periodic load and their collapse) are presented.

6.1 Model of an Elastic–Plastic Granular Material

The rheological scheme of an ideal elastic–plastic granular material being a series connection of an elastic spring, a plastic hinge, and a rigid contact is shown in Fig. 2.6. The tensor of small strains for such material is decomposed into the sum of elastic, granular, and plastic components: $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^c + \boldsymbol{\varepsilon}^p$. For the sum of elastic and granular components of tensor the strain potential has the form

$$\Psi(\boldsymbol{\sigma}) = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{a} : \boldsymbol{\sigma} + \delta_K(\boldsymbol{\sigma})$$

(see Sect. 4.1). If the stress tensor $\boldsymbol{\sigma}$ lies inside K , then strain obeys the linear law of the elasticity theory, and if $\boldsymbol{\sigma}$ belongs to the conical surface, then the transition (limit) state of a medium is achieved. The constitutive relationships (2.3) are reduced to the Haar–von Kármán inequality [15]:

$$(\tilde{\sigma} - \sigma) : (a : \sigma - \varepsilon^e - \varepsilon^c) \geq 0, \quad \sigma, \tilde{\sigma} \in \mathbf{K}. \quad (6.1)$$

For the plastic strain rate tensor we use the constitutive relationships of the flow theory:

$$\sigma \in \partial D(\dot{\varepsilon}^P). \quad (6.2)$$

Here D is the dissipative potential of stresses being a first-degree convex positive homogeneous function of the strain rates. Homogeneity of the potential

$$D(\lambda \dot{\varepsilon}^P) = \lambda D(\dot{\varepsilon}^P) \quad \forall \lambda \geq 0$$

is related to the fact that the plastic deformation process does not depend on time scale (on rate of loading or rate of deformation of a material). The point is that for a homogeneous convex function, even though it is nondifferentiable, the generalized Euler theorem is valid:

$$D(\dot{\varepsilon}^P) = \sigma : \dot{\varepsilon}^P.$$

This follows immediately from the definition of a subdifferential. On the one hand, the inequality

$$D(\tilde{\varepsilon}) - D(\dot{\varepsilon}^P) \geq \sigma : (\tilde{\varepsilon} - \dot{\varepsilon}^P)$$

for $\tilde{\varepsilon} = 2\dot{\varepsilon}^P$ gives $D(\dot{\varepsilon}^P) \geq \sigma : \dot{\varepsilon}^P$, on the other hand, this inequality with replacement of $\tilde{\varepsilon}$ by $\lambda \tilde{\varepsilon}$ is rearranged to the inequality

$$\lambda D(\tilde{\varepsilon}) - D(\dot{\varepsilon}^P) \geq \sigma : (\lambda \tilde{\varepsilon} - \dot{\varepsilon}^P),$$

from which for $\lambda \rightarrow 0$ we have $D(\dot{\varepsilon}^P) \leq \sigma : \dot{\varepsilon}^P$.

The effect of energy dissipation in an ideal plastic material is described by the equation (see, for example, [20, 21, 43])

$$T\dot{S} = \sigma : \dot{\varepsilon}^P - \frac{\nabla T \cdot \mathbf{h}}{T} \geq 0,$$

where T is the absolute temperature, S is the entropy of a particle, $\mathbf{h}$ is the vector of heat flow. The convolution $\sigma : \dot{\varepsilon}^P = D(\dot{\varepsilon}^P)$ defines internal dissipation, the term $-\nabla T \cdot \mathbf{h}/T$ is related to irreversible cross-flow of heat energy between particles. In the adiabatic case this equation takes the form

$$T\dot{S} = D(\dot{\varepsilon}^P),$$

and is an invariant with respect to the transformation of time stretching, i.e. it does not depend on scale. Notice that the complete model, taking into account heat conduction of a material, does not have this property.

In view of homogeneity of the potential $D(\dot{\varepsilon}^P)$, the dual dissipative strain rate potential, equal to the Young–Fenchel transform

$$H(\sigma) = \sup_{\lambda \dot{\epsilon}^p} \{ \sigma : (\lambda \dot{\epsilon}^p) - D(\lambda \dot{\epsilon}^p) \} = \sup_{\lambda \geq 0} \{ \lambda H(\sigma) \},$$

is the indicator function of the convex closed set

$$F = \{ \sigma \mid \sigma : \dot{\epsilon}^p \leq D(\dot{\epsilon}^p) \quad \forall \dot{\epsilon}^p \}.$$

The boundary of F in the stress space defines the yield surface of a material. If the set F is the cylinder with the principal octahedral direction in the system of principal axes of the stress tensor being its axis of hydrostatic stresses, then the volume strain of a material obeys the linearly elastic law. In the opposite case the model describes irreversible volumetric compaction of a material. The inclusion (6.2) in the equivalent form $\dot{\epsilon}^p \in \partial H(\sigma)$ is reduced to the von Mises inequality

$$(\tilde{\sigma} - \sigma) : \dot{\epsilon}^p \leq 0, \quad \sigma, \tilde{\sigma} \in F. \quad (6.3)$$

Due to (6.3) the tensor ϵ^p varies with time only in the case of irreversible deformation of a material where σ is a boundary point of F .

Notice that the need to apply subdifferential inclusions in formulation of constitutive relationships in the theory of rigid-plastic and visco-plastic materials, for which convex dissipative potentials are obviously nondifferentiable, was first suggested in [29, 30]. Problems of correctness of the use of nondifferentiable and nonconvex potentials in the analysis of deformation of nonlinearly elastic materials are discussed in detail in [32]. In [5] piecewise-linear potentials of the creep theory being nondifferentiable functions as well are considered.

The variational inequalities (6.1) and (6.3) together with the equations of motion and the kinematic equations

$$\rho \dot{\mathbf{v}} = \nabla \cdot \sigma + \rho \mathbf{g}, \quad 2\dot{\epsilon} = \nabla \mathbf{v} + (\nabla \mathbf{v})^* \quad (6.4)$$

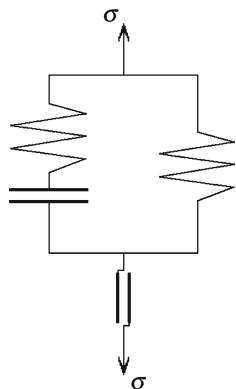
form a closed model of the dynamics of a granular material. Here $\mathbf{v} = \dot{\mathbf{u}}$ is the velocity vector, $\mathbf{g}$ is the vector of mass forces. The remaining notations were used in the previous chapters.

Let s be the tensor of conditional stresses expressed in terms of the tensor $\epsilon^e + \epsilon^c = \epsilon - \epsilon^p$ by the linear Hooke law $\mathbf{a} : s = \epsilon^e + \epsilon^c$, and s^π be the projection of s onto the cone K with respect to the norm $|s|_a$. Since the tensor $\mathbf{a}$ is symmetric and positive definite, we can show that $\sigma = s^\pi$ is the unique solution of the inequality (6.1).

The dependence of σ on s and, thus, on $\epsilon^e + \epsilon^c$ is not one-to-one which means that the model is not well-defined in the mechanical sense, i.e. generally speaking, we may not determine the strained state of a material by given stresses. We regularize the model (6.1), (6.3), and (6.4) taking

$$\sigma = \varsigma s + (1 - \varsigma) s^\pi, \quad 0 < \varsigma \leq 1. \quad (6.5)$$

Fig. 6.1 Rheological scheme of regularized model



This way of regularization corresponds to the replacement of the tensor $\mathbf{a}$ of moduli of elastic compliance by $\mathbf{a}/(1 - \varsigma)$ and simultaneously introducing elastic connections between particles in the loosened state which are defined by the tensor $\mathbf{a}/\varsigma$ of compliance moduli. For nonzero ς Eq. (6.5) describes the behaviour of an elastic material with different moduli, for $\varsigma \rightarrow 0$ it defines a solution of the variational inequality (6.1). For $\varsigma \rightarrow 1$ it results in the Hooke law. The rheological scheme of an elastic–plastic granular material, corresponding to the regularized model, is shown in Fig. 6.1.

We can prove that $\boldsymbol{\sigma}^\pi = \mathbf{s}^\pi$. Indeed, by the definition of a projection onto a cone (see Sect. 3.2) the tensor $\mathbf{s}^\pi$ is a solution of the variational inequality

$$(\tilde{\mathbf{s}} - \mathbf{s}^\pi) : \mathbf{a} : (\mathbf{s}^\pi - \mathbf{s}) \geq 0 \quad \forall \tilde{\mathbf{s}} \in \mathbf{K},$$

which is equivalent to the system

$$\mathbf{s}^\pi : \mathbf{a} : (\mathbf{s}^\pi - \mathbf{s}) = 0, \quad \tilde{\mathbf{s}} : \mathbf{a} : (\mathbf{s}^\pi - \mathbf{s}) \geq 0 \quad \forall \tilde{\mathbf{s}} \in \mathbf{K}.$$

Substituting $\boldsymbol{\sigma}$ instead of $\mathbf{s}$ into each of these relationships and taking into account (6.5), we obtain

$$\mathbf{s}^\pi : \mathbf{a} : (\mathbf{s}^\pi - \boldsymbol{\sigma}) = \varsigma \mathbf{s}^\pi : \mathbf{a} : (\mathbf{s}^\pi - \mathbf{s}) = 0,$$

$$\tilde{\mathbf{s}} : \mathbf{a} : (\mathbf{s}^\pi - \boldsymbol{\sigma}) = \varsigma \tilde{\mathbf{s}} : \mathbf{a} : (\mathbf{s}^\pi - \mathbf{s}) \geq 0.$$

Thus, $\boldsymbol{\sigma}^\pi = \mathbf{s}^\pi$. The dependence of $\boldsymbol{\sigma}$ on $\mathbf{s}$ becomes one-to-one:

$$\mathbf{s} = \frac{1}{\varsigma} \boldsymbol{\sigma} - \frac{1 - \varsigma}{\varsigma} \boldsymbol{\sigma}^\pi. \quad (6.6)$$

For simplicity, further we consider the case of an isotropic granular material whose tensor of elastic compliance in an arbitrary Cartesian coordinates x_1, x_2, x_3 is defined by two coefficients, namely, by the volume compression modulus k and the shear modulus μ :

$$a_{ijlh} = \frac{1}{2\mu} \delta_{il} \delta_{jh} + \frac{1}{3} \left(\frac{1}{3k} - \frac{1}{2\mu} \right) \delta_{ij} \delta_{lh}.$$

Admissible stresses are described with the help of the von Mises–Schleicher circular cone $\mathbf{K}$. The set $\mathbf{F}$ is approximated by the von Mises cylinder

$$\mathbf{F} = \{\boldsymbol{\sigma} \mid \tau(\boldsymbol{\sigma}) \leq \tau_s\},$$

where τ_s is the yield point of particles.

The procedure of calculation of a projection onto the von Mises–Schleicher cone required further is described in Sect. 4.3. It results in the following three alternatives. If $\tau(\mathbf{s}) \leq \kappa p(\mathbf{s})$ (i.e. $\mathbf{s} \in \mathbf{K}$), then $\mathbf{s}^\pi$ coincides with $\mathbf{s}$. If $\mu p(\mathbf{s}) + \kappa k \tau(\mathbf{s}) \leq 0$, then $\mathbf{s}^\pi = 0$ is the vertex of the cone. If $\tau(\mathbf{s}) > \kappa p(\mathbf{s})$ and $\mu p(\mathbf{s}) + \kappa k \tau(\mathbf{s}) > 0$, then the projection belongs to the conical surface and is calculated by the formulae (4.14). The formulae for the projection onto the von Mises cylinder, used in the numerical implementation of the model, are as follows [35]:

$$(\mathbf{s}^\pi)' = \kappa \mathbf{s}', \quad p(\mathbf{s}^\pi) = p(\mathbf{s}). \quad (6.7)$$

Here $\kappa = 1$ provided that $\mathbf{s} \in \mathbf{F}$: $\tau(\mathbf{s}) \leq \tau_s$ and $\kappa = \tau_s/\tau(\mathbf{s})$ in the opposite case.

Assume that $\mathbf{V}(t, \mathbf{x})$ and $\mathbf{U}(t, \mathbf{x})$ are m -dimensional vector-functions such that the first of them is composed of nonzero components of the velocity vector $\mathbf{v}$ and the actual stress tensor $\boldsymbol{\sigma}$ and the second of them involves the conditional stress tensor $\mathbf{s}$ instead of $\boldsymbol{\sigma}$. In terms of these functions the model (6.1), (6.3) and (6.4) is reduced to the variational inequality

$$(\tilde{\mathbf{V}} - \mathbf{V}) \left(\mathbf{A} \mathbf{U}_t - \sum_{i=1}^n \mathbf{B}^i \mathbf{V}_{x_i} - \mathbf{Q} \mathbf{V} - \mathbf{G} \right) \geq 0, \quad \mathbf{V}, \tilde{\mathbf{V}} \in \mathbf{F}, \quad (6.8)$$

where $\tilde{\mathbf{V}}$ is a varied vector, $\mathbf{A}$ and $\mathbf{B}^i$ are symmetrical $(m \times m)$ -dimensional matrix-coefficients (the matrix $\mathbf{A}$ is positive definite), $\mathbf{Q}$ and $\mathbf{G}$ are given matrix and vector, respectively, n is the spatial dimension of the problem, subscripts serve for notation of partial derivatives with respect to the corresponding coordinates. According to the relationships (6.5) and (6.6)

$$\mathbf{V} = \varsigma \mathbf{U} + (1 - \varsigma) \mathbf{U}^\pi, \quad \mathbf{U} = \frac{1}{\varsigma} \mathbf{V} - \frac{1 - \varsigma}{\varsigma} \mathbf{V}^\pi, \quad (6.9)$$

where superscript π means the projection of vectors onto the cone $\mathbf{K}$ imbedded into the space $\mathbb{R}^m$ with the energy norm $|\mathbf{U}|_{\mathbf{A}} = \sqrt{\mathbf{U} \mathbf{A} \mathbf{U}}$.

In spatial problems $n = 3$, $m = 9$. With plane or axial symmetry taken into account $n = 2$ and $m = 6$. The matrix $\mathbf{Q}$ is equal to zero in the Cartesian system and is expressed in terms of the Lamé coefficients in an arbitrary curvilinear coordinate system. The vector $\mathbf{G}$ is equal to zero as well provided that the mass forces are neglected.

In the study of the plane strained state of a granular material in the Cartesian coordinate system, $v_1, v_2, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}$ and $v_1, v_2, s_{11}, s_{22}, s_{33}, s_{12}$ are components of $\mathbf{V}$ and $\mathbf{U}$, respectively, and the matrices $\mathbf{A}, \mathbf{B}^1, \mathbf{B}^2$ and the vector $\mathbf{G}$ are as follows:

$$\mathbf{A} = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1111} & a_{1122} & a_{1133} & 2a_{1112} \\ 0 & 0 & a_{2211} & a_{2222} & a_{2233} & 2a_{2212} \\ 0 & 0 & a_{3311} & a_{3322} & a_{3333} & 2a_{3312} \\ 0 & 0 & 2a_{1211} & 2a_{1222} & 2a_{1233} & 2a_{1212} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \rho g_1 \\ \rho g_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{B}^1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If the cone $\mathbf{K}$ coincides with the space $\mathbb{R}^m$ or the regularization parameter ς defining the ratio of elasticity moduli of a material under tension and compression is equal to one, then the system (6.8), (6.9) describes the Prandtl–Reuss elastic–plastic material [4], whose diagram of deformation is symmetric with respect to tension and compression:

$$(\tilde{\mathbf{U}} - \mathbf{U}) \left(\mathbf{A} \mathbf{U}_t - \sum_{i=1}^n \mathbf{B}^i \mathbf{U}_{x_i} - \mathbf{Q} \mathbf{U} - \mathbf{G} \right) \geq 0, \quad \mathbf{U}, \tilde{\mathbf{U}} \in \mathbf{F}.$$

If $\mathbf{F} = \mathbb{R}^m$, then the variational inequality (6.8) is reduced to the system of equations

$$\mathbf{A} \mathbf{U}_t = \sum_{i=1}^n \mathbf{B}^i \mathbf{V}_{x_i} + \mathbf{Q} \mathbf{V} + \mathbf{G}, \quad (6.10)$$

which describes together with (6.9) the deformation of a heteromodal elastic material. If, in addition, $\mathbf{K} = \mathbb{R}^m$ or $\varsigma = 1$, then the last system is reduced to the system of linear equations of the dynamic elasticity theory written in the hyperbolic form in the sense of Friedrichs, [10, 12, 33].

6.2 A Priori Estimates of Solutions

When obtaining a priori estimates for the variational inequality (6.8), the Hamilton–Jacobi equation plays an important role. In essence, solutions of this equation enable one to construct the domains of uniqueness of a solution of the Cauchy problem for (6.8) in the form of truncated cones in the space of the $t, \mathbf{x}$ variables. We describe a method for integration of the Hamilton–Jacobi equation [12].

Let $c(t, \mathbf{x}, \mathbf{v})$ be the maximal of m roots $c_1(t, \mathbf{x}, \mathbf{v}), c_2(t, \mathbf{x}, \mathbf{v}), \dots, c_m(t, \mathbf{x}, \mathbf{v})$ of the characteristic equation

$$\det \left(c \mathbf{A} - \sum_{i=1}^n v_i \mathbf{B}^i \right) = 0.$$

The corresponding Hamilton–Jacobi equation has the next form:

$$z_t = c(t, \mathbf{x}, \nabla z). \quad (6.11)$$

Assume that $z(t, \mathbf{x})$ is a sufficiently smooth solution of (6.11).

Multiplication of the matrix under the determinant sign by an arbitrary nonzero constant does not change the characteristic equation, hence, each root is a first-degree homogeneous function relative to the vector $\mathbf{v}$. If, in addition, the constant is positive, then the order of roots does not change with multiplication. Thus, the function $c(t, \mathbf{x}, \mathbf{v})$ is positively homogeneous: $c(t, \mathbf{x}, \lambda \mathbf{v}) = \lambda c(t, \mathbf{x}, \mathbf{v})$ ($\lambda \geq 0$). For $\mathbf{v} \neq \mathbf{0}$ this function is differentiable with respect to $\mathbf{v}$. According to the Euler theorem for homogeneous functions

$$c = \sum_{i=1}^n v_i \frac{\partial c}{\partial v_i},$$

and Eq. (6.11) can be represented in the form

$$z_t = \sum_{i=1}^n z_{x_i} \frac{\partial c}{\partial v_i}.$$

In this form it can be integrated by the method of characteristics. Let $\mathbf{x} = \mathbf{x}(t)$ be the equation of bicharacteristic for (6.11). Then

$$\dot{x}_i = -\frac{\partial c}{\partial v_i}. \quad (6.12)$$

The corresponding relationship on the bicharacteristic $\dot{z} = 0$ means that along the line $\mathbf{x} = \mathbf{x}(t)$ the function $z(t, \mathbf{x})$ takes a constant value.

We redenote the components z_{x_i} of the gradient vector by v_i . Differentiation of Eq. (6.11) with respect to x_i enables one to obtain the following system of equations for v_i :

$$(v_i)_t = \sum_{j=1}^n (v_i)_{x_j} \frac{\partial c}{\partial v_j} + \frac{\partial c}{\partial x_i}.$$

Bicharacteristics of each of n equations of this system coincide with (6.12). The relationships on bicharacteristics have the form

$$\dot{v}_i = \frac{\partial c}{\partial x_i}. \quad (6.13)$$

Thus, to determine a solution, we have the canonical system of ordinary differential Eqs. (6.12), (6.13) written relative to x_i , v_i and the finite relationship $z = \text{const}$ for $\mathbf{x} = \mathbf{x}(t)$.

Assume that $z^0(\mathbf{x}) = z(t_0, \mathbf{x})$ is a function which defines the bounded closed domain

$$\Omega_0 = \{\mathbf{x} \in \mathbb{R}^n \mid z^0(\mathbf{x}) \leq 0\},$$

lying entirely in the intersection of definitional domain of the smooth solution of the variational inequality (6.8) and the hyperplane $t = t_0$. The process of construction of the function $z(t, \mathbf{x})$ in a space-time neighbourhood of the domain Ω_0 is reduced to the repeated solution of the Cauchy problem for the system (6.12), (6.13) with the initial data

$$x_i|_{t=t_0} = x_i^0, \quad v_i|_{t=t_0} = z_{x_i}^0(\mathbf{x}^0),$$

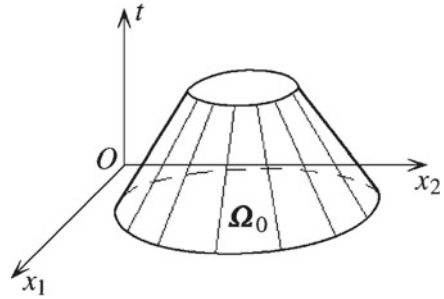
where $\mathbf{x}^0$ is an arbitrary point of Ω_0 . Besides, $z(t, \mathbf{x}(t)) = 0$ provided that $z^0(\mathbf{x}^0) = 0$.

As a result, we obtain a domain of type of a truncated cone in the direction of the t axis. The lateral surface of this domain, defined by the equation $z(t, \mathbf{x}) = 0$, is the hypersurface formed by the bicharacteristics intersecting the boundary of domain Ω_0 (Fig. 6.2). It is of fundamental importance that everywhere at the points of lateral surface the characteristic matrix is nonnegative definite, i.e.

$$\tilde{\mathbf{V}} \left(c \mathbf{A} - \sum_{i=1}^n v_i \mathbf{B}^i \right) \tilde{\mathbf{V}} \geq 0$$

for any vector $\tilde{\mathbf{V}}$. Indeed, there exists a transformation that reduces simultaneously the positive definite quadratic form of the matrix $\mathbf{A}$ to the sum of squares of corresponding components of the vector $\tilde{\mathbf{V}}$ with unit coefficients and the quadratic form of the matrix $\sum_{i=1}^n v_i \mathbf{B}^i$ to the sum of squares with the coefficients c_j . Hence, the numbers $c - c_j \geq 0$ are the coefficients of the quadratic form of the characteristic matrix.

Fig. 6.2 Bicharacteristics of the Hamilton–Jacobi equation



Often in the case that the matrices A and B^i depend on x we have to construct characteristic cones using the Hamilton–Jacobi inequality

$$z_t \geq c(t, x, \nabla z) \quad (6.14)$$

rather than the equation. A solution of this inequality can be obtained, for example, in the following way. Given some function $c_0(t, x, v)$ homogeneous with respect to the vector v and satisfying the condition $c_0(t, x, v) \geq c(t, x, v)$, we solve the Hamilton–Jacobi equation for c_0 . One of the ways to choose such a function is to calculate maximum of the greatest eigenvalue with respect to variables t and x .

Notice that if the function $z(t, x)$ satisfies the Hamilton–Jacobi inequality, then the matrix

$$z_t A - \sum_{i=1}^n z_{x_i} B^i$$

is nonnegative definite. This is so, because due to (6.14) the quadratic form of this matrix is bounded below by the quadratic form of the characteristic matrix which is reduced to the sum of squares with negative coefficients $c - c_j$.

As an example we consider the Hamilton–Jacobi equation for a homogeneous isotropic elastic material. The greatest eigenvalue $c(v)$ is equal to $c_p |v|$, where $c_p = \sqrt{(k + 4/3 \mu)/\rho}$ is the velocity of longitudinal elastic waves. The canonical system of Eqs. (6.12), (6.13) takes the form

$$\dot{x}_i = -c_p \frac{v_i}{|v|}, \quad \dot{v}_i = 0.$$

A solution of the system

$$x_i = x_i^0 - c_p v_i^0 \frac{t - t_0}{|v^0|}$$

defines bicharacteristics, i.e. straight lines in the space of variables t and x lying in hyperplanes, parallel to the t axis and passing through normals to the boundary of Ω_0 . In particular, if Ω_0 is a ball of radius R with its center at zero and $t_0 = 0$, then the solution of the Hamilton–Jacobi equation turns out to be equal to

$$z(t, \mathbf{x}) = |\mathbf{x}| - R + c_p t.$$

The domain Ω_0 is a circular cone whose lateral surface is defined by the equation $|\mathbf{x}| = R - c_p t$. Below, when obtaining estimates, we essentially use the fact that everywhere on this surface the quantity c is greater than zero.

We make an unessential assumption which considerably simplifies the further notations: we assume that the matrices $\mathbf{A}$ and $\mathbf{B}^i$ are constant. We prove the following auxiliary formulae

$$\mathbf{V} \mathbf{A} \mathbf{V}_t^\pi = \mathbf{V}^\pi \mathbf{A} \mathbf{V}_t = \mathbf{V}^\pi \mathbf{A} \mathbf{V}_t^\pi, \quad (6.15)$$

which involve the rate of change of the projection of the vector $\mathbf{V}$ onto the cone $\mathbf{K}$. By the definition of a projection we have

$$\mathbf{V}^\pi \mathbf{A} (\mathbf{V}^\pi - \mathbf{V}) = 0, \quad \tilde{\mathbf{V}} \mathbf{A} (\mathbf{V}^\pi - \mathbf{V}) \geq 0 \quad \forall \tilde{\mathbf{V}} \in \mathbf{K}.$$

Putting here $\tilde{\mathbf{V}} = \mathbf{V}^\pi(t \pm \Delta t)$, we arrive at two inequalities

$$(\mathbf{V}^\pi(t \pm \Delta t) - \mathbf{V}^\pi(t)) \mathbf{A} (\mathbf{V}^\pi(t) - \mathbf{V}(t)) \geq 0.$$

Dividing the first inequality by Δt and the second one by $-\Delta t$ and passing to the limit, we obtain

$$\mathbf{V}_t^\pi \mathbf{A} (\mathbf{V}^\pi - \mathbf{V}) = 0.$$

This is one of the formulae (6.15). Another formula is a consequence of the chain of equalities

$$\mathbf{V}_t^\pi \mathbf{A} \mathbf{V}^\pi = \mathbf{V}_t^\pi \mathbf{A} \mathbf{V} = (\mathbf{V}^\pi \mathbf{A} \mathbf{V})_t - \mathbf{V}^\pi \mathbf{A} \mathbf{V}_t = (\mathbf{V}^\pi \mathbf{A} \mathbf{V}^\pi)_t - \mathbf{V}^\pi \mathbf{A} \mathbf{V}_t,$$

which is obtained taking into account the fact that the matrix $\mathbf{A}$ is time-independent and the obvious identity

$$\mathbf{V}_t^\pi \mathbf{A} \mathbf{V}^\pi = \frac{1}{2} (\mathbf{V}^\pi \mathbf{A} \mathbf{V}^\pi)_t.$$

Assume that $\mathbf{W}$ is a truncated cone with bases Ω_0 and Ω_1 lying in the hyperplanes $t = t_0$ and $t = t_1$, and its lateral surface satisfies the Hamilton–Jacobi inequality. The unit vector of an outward normal to this surface in the space $\mathbb{R}^{n+1}$ is

$$\frac{(z_t, \nabla z)}{\sqrt{z_t^2 + |\nabla z|^2}}.$$

With the help of the known formula $c = z_t/|\nabla z|$ for the velocity of motion of a surface along a normal [41], we can represent this vector in the form $(c, \mathbf{v})/\sqrt{1 + c^2}$,

where $\nu = \nabla z / |\nabla z|$ is a unit normal vector in the space $\mathbb{R}^n$. In these notations, $c = 1$, $\nu = \mathbf{0}$ on the upper base of the truncated cone and $c = -1$, $\nu = \mathbf{0}$ on the lower base.

Take $\tilde{V} = \mathbf{0} \in F$ as a varying vector in (6.8). Since

$$V A U_t = \frac{1}{\varsigma} V A V_t - \frac{1-\varsigma}{\varsigma} V A V_t^\pi = \frac{1}{2} E(V)_t,$$

where $\varsigma E(V) = V A V - (1 - \varsigma) V^\pi A V^\pi$, we have the inequality

$$\frac{1}{2} E(V)_t - \frac{1}{2} \sum_{i=1}^n (V B^i V)_{x_i} - V Q V - V G \leq 0.$$

Integrating it over the cone W with the use of the Green formula, we can show that any sufficiently smooth solution of (6.8) satisfies the inequality

$$\int_S \left(c E(V) - \sum_{i=1}^n v_i V B^i V \right) \frac{dS}{\sqrt{1+c^2}} \leq 2 \iint_W (V Q V + V G) d\Omega dt \quad (6.16)$$

(S is the boundary of W). The norm of a projection of the vector onto a cone with the vertex at zero does not exceed the norm of the vector, hence, the function $E(V)$ satisfies the two-sided estimate

$$\frac{|V|_A^2}{\varsigma} \geq E(V) \geq |V|_A^2, \quad (6.17)$$

from which in the case of $c \geq 0$ we have

$$c E(V) - \sum_{i=1}^n v_i V B^i V \geq V \left(c A - \sum_{i=1}^n v_i B^i \right) V.$$

Taking into account that the characteristic matrix is nonnegative definite on the lateral surface, from the inequality (6.16) we obtain

$$|||V|||^2(t_1) \leq |||V|||^2(t_0) + 2 \iint_W (V Q V + V G) d\Omega dt,$$

where $|||V|||^2(t) = \int_{\Omega(t)} E(V) d\Omega$ is the integral of doubled elastic energy over the domain $\Omega(t)$ being the section of the cone W by the hyperplane $t = \text{const}$.

Since the matrix A is positive definite, there exist constants $C_1 > 0$ and $C_2 > 0$ independent of V such that

$$\mathbf{V} \mathbf{Q} \mathbf{V} \leq C_1 |\mathbf{V}|_{\mathbf{A}}^2 \leq C_1 E(\mathbf{V}), \quad \mathbf{V} \mathbf{G} \leq C_2 |\mathbf{V}|_{\mathbf{A}} |\mathbf{G}|_{\mathbf{A}} \leq C_2 \sqrt{E(\mathbf{V})} |\mathbf{G}|_{\mathbf{A}}.$$

This enables one, with the help of the Cauchy–Bunyakovskii inequality, to estimate the integrals

$$\begin{aligned} \iint_W \mathbf{V} \mathbf{Q} \mathbf{V} d\Omega dt &= \int_{t_0}^{t_1} \int_{\Omega(t)} \mathbf{V} \mathbf{Q} \mathbf{V} d\Omega dt \leq C_1 \int_{t_0}^{t_1} |||\mathbf{V}|||^2 dt, \\ \iint_W \mathbf{V} \mathbf{G} d\Omega dt &= \int_{t_0}^{t_1} \int_{\Omega(t)} \mathbf{V} \mathbf{G} d\Omega dt \leq C_2 \int_{t_0}^{t_1} \int_{\Omega(t)} \sqrt{E(\mathbf{V})} |\mathbf{G}|_{\mathbf{A}} d\Omega dt \\ &\leq C_2 \int_{t_0}^{t_1} \left(\int_{\Omega(t)} E(\mathbf{V}) d\Omega \int_{\Omega(t)} |\mathbf{G}|_{\mathbf{A}}^2 d\Omega \right)^{1/2} dt \\ &\equiv C_2 \int_{t_0}^{t_1} |||\mathbf{V}||| \|\mathbf{G}\| dt, \end{aligned}$$

where $\|\mathbf{G}\|$ is the weighted norm of $L_2(\Omega(t))$. Thus, we have the inequality

$$|||\mathbf{V}|||^2(t_1) \leq |||\mathbf{V}|||^2(t_0) + 2C_1 \int_{t_0}^{t_1} |||\mathbf{V}|||^2 dt + 2C_2 \int_{t_0}^{t_1} |||\mathbf{V}||| \|\mathbf{G}\| dt.$$

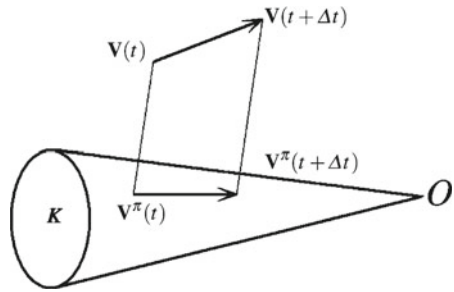
It is also valid in the case that the integration limits t_0 and t_1 are replaced by an arbitrary instants $t'_0 \in (t_0, t_1)$ and $t'_1 \in (t'_0, t_1)$, respectively. Assuming that t'_0 tends to t and t'_1 tends to t and dividing both sides of the inequality by the positive number $t'_1 - t'_0$, we get

$$\begin{aligned} \frac{d}{dt} |||\mathbf{V}|||^2 &\leq 2C_1 |||\mathbf{V}|||^2 + 2C_2 |||\mathbf{V}||| \|\mathbf{G}\|, \\ \exp(C_1 t) \frac{d}{dt} (|||\mathbf{V}||| \exp(-C_1 t)) &\leq C_2 \|\mathbf{G}\|. \end{aligned}$$

Hence, on integration with respect to time over the interval (t_0, t_1) , we arrive at the estimate

$$|||\mathbf{V}|||(t_1) \leq |||\mathbf{V}|||(t_0) \exp(C_1(t_1 - t_0)) + C_2 \int_{t_0}^{t_1} \|\mathbf{G}\| \exp(C_1(t_1 - t)) dt. \quad (6.18)$$

Fig. 6.3 Projection onto the supporting hyperplane



This estimate gives an idea of whether the Cauchy problem for the variational inequality (6.8) with the initial condition of the form

$$V|_{t=t_0} = V^0(x)$$

is well-posed. Due to (6.17) the expression $|||V|||(t)$ is equivalent to the norm of the space $L_2(\Omega(t))$. Hence, from (6.18) it follows that a solution of the Cauchy problem in the characteristic cone is bounded and the trivial solution for $V_0 \equiv \mathbf{0}$ is unique and continuously depends on initial data. Besides, according to (6.18), the norm of the solution on the upper base of the cone turns out to be independent of initial data outside the domain $\Omega(t_0)$ which indicates that the domain of dependence of the solution is finite.

We give the scheme of the construction of a more informative estimate of the norm of difference of two solutions of the variational inequality (6.8) in the framework of the following assumption which can be rigorously justified only with the use of subtle results of convex analysis. Assume that for an operator of projection onto the cone K there exists a symmetric matrix $\Pi(V)$, satisfying the Lipschitz condition

$$|\Pi(\tilde{V}) - \Pi(V)| \leq L |\tilde{V} - V|, \quad V, \tilde{V} \in \mathbb{R}^m,$$

such that

$$\Pi(V) V_t = A V_t^\pi. \quad (6.19)$$

It is obvious that if for $\Delta t \rightarrow 0$ the inclusion $V(t + \Delta t) \in K$ is valid, then $\Pi(V) = A$. According to Fig. 6.3, in the opposite case $\Pi(V)$ is the matrix of the linear operator being a projector onto the supporting hyperplane for the cone at the point V^π . The equation of the supporting hyperplane has the form

$$(\tilde{V} - V^\pi(t)) A N = 0, \quad N = \frac{(V(t) - V^\pi(t))}{|V(t) - V^\pi(t)|_A}$$

(N is a unit normal vector with respect to the inner product in $\mathbb{R}^m$ associated with the matrix A). A projection onto the hyperplane is defined by the approximation

$$\mathbf{V}^\pi(t + \Delta t) \approx \mathbf{V}(t + \Delta t) - (\mathbf{V}(t + \Delta t) - \mathbf{V}^\pi(t)) \mathbf{A} \mathbf{N} \cdot \mathbf{N},$$

which can be used for approximate calculation of a projection onto the cone $\mathbf{K}$ in a small neighbourhood of the point $\mathbf{V}^\pi(t)$. The vector $\mathbf{V}(t)$ satisfies the exact equation since its projections onto the cone and onto the hyperplane coincide. For the difference we get

$$\mathbf{V}^\pi(t + \Delta t) - \mathbf{V}^\pi(t) \approx \mathbf{V}(t + \Delta t) - \mathbf{V}(t) - (\mathbf{V}(t + \Delta t) - \mathbf{V}(t)) \mathbf{A} \mathbf{N} \cdot \mathbf{N}.$$

Hence, for $\Delta t \rightarrow 0$ we have

$$\Pi(\mathbf{V}) \mathbf{V}_t \equiv \mathbf{A} \mathbf{V}_t^\pi = \mathbf{A} \mathbf{V}_t - (\mathbf{V}_t \mathbf{A} \mathbf{N}) \cdot \mathbf{A} \mathbf{N}.$$

Symmetry of $\Pi(\mathbf{V})$ follows from the chain of equalities:

$$\bar{\mathbf{V}} \Pi(\mathbf{V}) \tilde{\mathbf{V}} = \bar{\mathbf{V}} \mathbf{A} \tilde{\mathbf{V}} - (\tilde{\mathbf{V}} \mathbf{A} \mathbf{N})(\bar{\mathbf{V}} \mathbf{A} \mathbf{N}) = \tilde{\mathbf{V}} \Pi(\mathbf{V}) \bar{\mathbf{V}}.$$

Applying (6.9) and (6.19), we can represent the product $\mathbf{A} \mathbf{U}_t$, involved in (6.8), in the form

$$\frac{1}{\varsigma} \mathbf{A} \mathbf{V}_t - \frac{1 - \varsigma}{\varsigma} \Pi(\mathbf{V}) \mathbf{V}_t = \mathbf{A} \mathbf{V}_t + \frac{1 - \varsigma}{\varsigma} (\mathbf{V}_t \mathbf{A} \mathbf{N}) \cdot \mathbf{A} \mathbf{N} \equiv \Lambda(\mathbf{V}) \mathbf{V}_t,$$

where $\Lambda(\mathbf{V})$ is a symmetric positive definite matrix and its quadratic form satisfies the obvious condition

$$\tilde{\mathbf{V}} \Lambda(\mathbf{V}) \tilde{\mathbf{V}} \geq \tilde{\mathbf{V}} \mathbf{A} \tilde{\mathbf{V}}.$$

Let $\mathbf{V}(t, \mathbf{x})$ be a solution of the system (6.8), (6.9) and $\mathbf{V}'(t, \mathbf{x})$ be a solution of the similar system where the vector-function $\mathbf{G}(t, \mathbf{x})$ is replaced by $\mathbf{G}'(t, \mathbf{x})$. Going to the sum of inequalities, where the first inequality is the result of substitution of the varied vector $\tilde{\mathbf{V}} = \mathbf{V}'(t, \mathbf{x})$ into (6.8) and the second inequality coincides with the first one to within interchanging solutions, we obtain

$$(\mathbf{V}' - \mathbf{V}) \left(\Lambda(\mathbf{V}') \mathbf{V}'_t - \Lambda(\mathbf{V}) \mathbf{V}_t - \sum_{i=1}^n \mathbf{B}^i (\mathbf{V}' - \mathbf{V})_{x_i} - \mathbf{Q}(\mathbf{V}' - \mathbf{V}) - \mathbf{G}' + \mathbf{G} \right) \leq 0.$$

To derive the estimate, we have to rearrange this inequality according to the formula

$$\begin{aligned} (\mathbf{V}' - \mathbf{V}) \left(\Lambda(\mathbf{V}') \mathbf{V}'_t - \Lambda(\mathbf{V}) \mathbf{V}_t \right) &= \frac{1}{2} \left((\mathbf{V}' - \mathbf{V}) \Lambda(\mathbf{V}) (\mathbf{V}' - \mathbf{V}) \right)_t \\ &\quad - \frac{1}{2} (\mathbf{V}' - \mathbf{V}) \Lambda(\mathbf{V})_t (\mathbf{V}' - \mathbf{V}) + (\mathbf{V}' - \mathbf{V}) \left(\Lambda(\mathbf{V}') - \Lambda(\mathbf{V}) \right) \mathbf{V}'_t \end{aligned}$$

and then to integrate it over the conical domain $\mathbf{W}$. Application of the Green formula to the divergent terms yields

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega_1} (\mathbf{V}' - \mathbf{V}) \Lambda(\mathbf{V}) (\mathbf{V}' - \mathbf{V}) d\Omega - \frac{1}{2} \int_{\Omega_0} (\mathbf{V}' - \mathbf{V}) \Lambda(\mathbf{V}) (\mathbf{V}' - \mathbf{V}) d\Omega \\
& + \frac{1}{2} \int_{S_c} (\mathbf{V}' - \mathbf{V}) \left(c \Lambda(\mathbf{V}) - \sum_{i=1}^n v_i \mathbf{B}^i \right) (\mathbf{V}' - \mathbf{V}) \frac{dS}{\sqrt{1+c^2}} \\
& \leq \iint_{\mathbf{W}} (\mathbf{V}' - \mathbf{V}) \left(\mathbf{Q} + \frac{1}{2} \Lambda(\mathbf{V})_t \right) (\mathbf{V}' - \mathbf{V}) d\Omega dt \\
& + \iint_{\mathbf{W}} (\mathbf{V}' - \mathbf{V}) \left(\mathbf{G}' - \mathbf{G} - (\Lambda(\mathbf{V}') - \Lambda(\mathbf{V})) \mathbf{V}'_t \right) d\Omega dt.
\end{aligned}$$

By the construction of the cone $\mathbf{W}$ the integral over the conical surface S_c is nonnegative. The matrix $\Lambda(\mathbf{V})$ is positive definite, hence, the terms in the right-hand side do not exceed the expression

$$C_3 \iint_{\mathbf{W}} (\mathbf{V}' - \mathbf{V}) \Lambda(\mathbf{V}) (\mathbf{V}' - \mathbf{V}) d\Omega dt + \iint_{\mathbf{W}} (\mathbf{V}' - \mathbf{V}) (\mathbf{G}' - \mathbf{G}) d\Omega dt$$

with a constant C_3 which depends on the Lipschitz constant L , on both solutions, and on their first-order derivatives with respect to time. Hence,

$$\begin{aligned}
||| \mathbf{V}' - \mathbf{V} |||^2(t_1) & \leq ||| \mathbf{V}' - \mathbf{V} |||^2(t_0) + 2 C_3 \int_{t_0}^{t_1} ||| \mathbf{V}' - \mathbf{V} |||^2 dt \\
& + 2 C_4 \int_{t_0}^{t_1} ||| \mathbf{V}' - \mathbf{V} ||| \| \mathbf{G}' - \mathbf{G} \| dt,
\end{aligned}$$

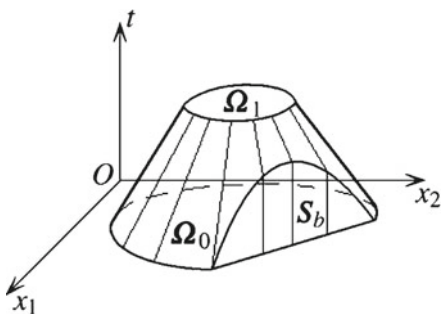
where $||| \mathbf{V}' - \mathbf{V} |||^2(t) = \int_{\Omega(t)} (\mathbf{V}' - \mathbf{V}) \Lambda(\mathbf{V}) (\mathbf{V}' - \mathbf{V}) d\Omega$ is the squared norm of

the difference of solutions. Repeating the derivation of the estimate (6.18), from this inequality we obtain

$$\begin{aligned}
||| \mathbf{V}' - \mathbf{V} |||(t_1) & \leq ||| \mathbf{V}' - \mathbf{V} |||(t_0) \exp(C_3(t_1 - t_0)) \\
& + C_4 \int_{t_0}^{t_1} \| \mathbf{G}' - \mathbf{G} \| \exp(C_3(t_1 - t)) dt. \quad (6.20)
\end{aligned}$$

This estimate provides local uniqueness of the solution of the Cauchy problem and its continuous dependence on initial data and the right-hand side since, according to (6.20), any perturbation of initial data and the vector-function $\mathbf{G}$ as small as is wished results in a small perturbation of a solution on the upper base of the truncated cone.

Fig. 6.4 The integration domain



In addition, from this estimate it follows that the domain of dependence of solutions of the variational inequality (6.8), which describes dynamics of a granular material, is not wider than that of the system of equations of the linear elasticity theory with the compliance moduli tensor α .

In a similar way we can obtain an integral estimate of the difference of solutions in a neighbourhood of a fixed hypersurface S_b with dissipative boundary conditions given on it. Fulfilment of these boundary conditions for any two vector-functions $\tilde{\mathbf{V}}$ and $\bar{\mathbf{V}}$ provides fulfilment of the inequality

$$(\tilde{\mathbf{V}} - \bar{\mathbf{V}}) \sum_{i=1}^n v_i \mathbf{B}^i (\tilde{\mathbf{V}} - \bar{\mathbf{V}}) \leq 0$$

at the points of S_b , where $\mathbf{v}$ is an outward normal vector to the hypersurface with respect to the domain of a solution of the problem. It is easy to show that in the framework of this model the boundary conditions satisfying the inequality

$$\mathbf{v} \cdot (\tilde{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}) \cdot (\tilde{\mathbf{v}} - \bar{\mathbf{v}}) \leq 0$$

are dissipative. Among them, in particular, are kinematic boundary conditions of the type $\mathbf{v} = \mathbf{w}(t, \mathbf{x})$ and the conditions $\boldsymbol{\sigma} \cdot \mathbf{v} = \mathbf{q}(t, \mathbf{x})$ in terms of stresses. In this case, for integration of the divergent inequality, the part of the truncated cone bearing on the hypersurface S_b (Fig. 6.4), whose conical surface satisfies the Hamilton-Jacobi inequality is used as the domain W . The arising integral over the lateral surface of W involves the term

$$-\int_{S_b} (\mathbf{V}' - \mathbf{V}) \sum_{i=1}^n v_i \mathbf{B}^i (\mathbf{V}' - \mathbf{V}) dS,$$

which is nonnegative due to dissipativity of boundary conditions. This enables one to ensure that the estimate (6.20) is valid. In this case from the estimate it follows that a boundary-value problem with dissipative boundary conditions is well-posed.

6.3 Shock-Capturing Method

In subsequent chapters, the shock-capturing method [35], which enables one to solve wave problems for continuous velocity and stress fields as well as those with discontinuity surfaces (shock waves), is applied to the numerical analysis of a model of a granular material. Although the accuracy of a numerical solution is not too high, this method has obvious advantages in comparison with the discontinuity separation methods, which are related to the fact that in models of the plasticity theory the relationships of strong discontinuity are of rather complicated form even for materials whose mechanical properties are symmetric with respect to tension and compression, [2, 3, 37]. For a granular material exact discontinuous solutions in a particular case of one-dimensional motions with plane longitudinal waves are considered in Sect. 6.4. It turns out that with different strengths of a material taken into account the situation is much more complicated, there arise shock waves of various configurations which depend on parameters of a material and on intensity of a shock wave.

An algorithm explicit in time for the numerical implementation of the variational inequality (6.8) is constructed with the help of the method of splitting with respect to physical processes in the following way. First the problem on deformation of a heteromodal elastic material is solved at each time level, then the obtained solution is corrected to take into account plastic properties. For $t = 0$ initial data $U(0, \mathbf{x}) = U_0(\mathbf{x})$ are given. The general form of boundary conditions is given below.

Assume that $\bar{U}$ is a vector-function which is defined in terms of an unknown solution of a boundary-value problem due to the system

$$A \bar{U}_t = \sum_{i=1}^n B^i V_{x_i} + Q V + G, \quad (6.21)$$

besides, at given fixed instant t_0 is satisfies the condition $\bar{U}(t_0, \mathbf{x}) = U(t_0, \mathbf{x})$. Then from (6.8) we get the inequality

$$(\tilde{V} - V) A (U_t - \bar{U}_t) \geq 0, \quad V, \tilde{V} \in F. \quad (6.22)$$

After approximation of the time derivative by a finite difference on the interval $(t_0, t_0 + \Delta t)$, at each mesh of a spatial grid we obtain a problem

$$(\tilde{V} - V) A (U - \bar{U}) \geq 0,$$

which is reduced to the form

$$(\tilde{V} - V) A (V - (1 - \varsigma) V^\pi - \varsigma \bar{U}) \geq 0$$

with the help of (6.9). Hence, by definition of a projection,

$$\mathbf{V} = \left((1 - \varsigma) \mathbf{V}^\pi + \varsigma \bar{\mathbf{U}} \right)^\Pi. \quad (6.23)$$

It turns out that for $\varsigma > 0$ the mapping $\mathbf{P}(\mathbf{V})$ defined by the right-hand side of (6.23) is contractive. Indeed, since an operator of projection onto a convex closed set is a non-expanding mapping, we have

$$|\mathbf{P}(\tilde{\mathbf{V}}) - \mathbf{P}(\mathbf{V})|_A \leq (1 - \varsigma) |\tilde{\mathbf{V}}^\pi - \mathbf{V}^\pi|_A \leq (1 - \varsigma) |\tilde{\mathbf{V}} - \mathbf{V}|_A.$$

Thus, the procedure of a solution correction is in determining a fixed point of the contractive mapping $\mathbf{P}(\mathbf{V})$ and can be implemented by the method of successive approximations. Taking into account Eqs. (4.14), (6.7) for projections onto the von Mises–Schleicher circular cone and the von Mises cylinder, we can propose a direct method for the solution of (6.23) consisting in searching a finite number of variants which are not presented because of cumbersome expressions.

Notice that for the case of the Prandtl–Reuss elastic–plastic material ($\mathbf{K} = \mathbb{R}^m$ or $\varsigma = 1$) this procedure coincides with the well-known Wilkins procedure for correction of stresses proposed as a stage of an algorithm for the numerical analysis of elastic–plastic flows [44]. There exist other variants of correction which have smaller scheme viscosity and, thus, are more accurate [1, 22, 37]. However, they are unsuitable for models where heteromodularity is taken into account. To generalize one of such variants to the case of a granular material, we approximate (6.22) in the following way:

$$(\tilde{\mathbf{V}} - \xi \mathbf{V} - (1 - \xi) \bar{\mathbf{V}}) \mathbf{A} (\mathbf{U} - \bar{\mathbf{U}}) \geq 0, \quad \xi \mathbf{V} + (1 - \xi) \bar{\mathbf{V}}, \quad \tilde{\mathbf{V}} \in \mathbf{F}. \quad (6.24)$$

Here ξ is a weighting coefficient, $\bar{\mathbf{V}}$ is an auxiliary vector related to $\bar{\mathbf{U}}$ by the formulae (6.9). This problem is equivalent to the problem on a fixed point

$$\mathbf{V} = \frac{1}{\xi} \left(\xi (1 - \varsigma) \mathbf{V}^\pi + \xi \varsigma \bar{\mathbf{U}} + (1 - \xi) \bar{\mathbf{V}} \right)^\Pi - \frac{1 - \xi}{\xi} \bar{\mathbf{V}}, \quad (6.25)$$

and the corresponding mapping is contractive for any positive ξ . Improving a solution with the help of the correction, consisting in calculation of a fixed point (6.25) by the method of successive approximations, was verified numerically when performed calculations for one-dimensional problems with shock waves, whose exact solutions are known, described below. It turns out that in fact improving takes place, besides the best value of the weighting coefficient ξ is close to 0.5 [35]. It is likely that one fails to construct a direct method for determining a fixed point for this variant of correction. However, computational work can be considerably reduced provided that particular cases are implemented separately (bypassing an iterative process): $\mathbf{V} \in \mathbf{K}$ where a corrected vector is determined by the formula

$$\mathbf{V} = \frac{1}{\xi} \left(\xi \bar{\mathbf{U}} + (1 - \xi) \bar{\mathbf{V}} \right)^\Pi - \frac{1 - \xi}{\xi} \bar{\mathbf{V}},$$

$V^\pi = \mathbf{0}$ where

$$V = \frac{1}{\xi} \left(\xi \bar{U} + (1 - \xi) \bar{V} \right)^\Pi - \frac{1 - \xi}{\xi} \bar{V},$$

and $\bar{V} \in F$ where $V = \bar{V}$. The above formulae for the vector V are easily derived taking into account a definition of projection onto a convex set. Thus, in the first case

$$\xi V + (1 - \xi) \bar{V} = \left(\xi \bar{U} + (1 - \xi) \bar{V} \right)^\Pi,$$

hence, we have the variational inequality

$$(\tilde{V} - \xi V - (1 - \xi) \bar{V}) A(V - \bar{U}) \geq 0,$$

which coincides with (6.24), since because of (6.9) for $V = V^\pi$ the vectors V and U are equal to one another. The second and third cases are considered in a similar way.

For the solution of the problem (6.21), (6.9) one of variants of the fractional step method [40, 47], namely, the method of two-cyclic splitting with respect to spatial variables in combination with a monotone ENO-scheme of the “predictor–corrector” type with linear reconstruction, is applied.

In the three-dimensional case, on the time interval $(t_0, t_0 + \Delta t)$ the splitting method [11, 26], involves seven stages: the solution of a one-dimensional problem in the x_1 direction on the interval $(t_0, t_0 + \Delta t/2)$, similar stages in the x_2 and x_3 directions, the solution of a system of linear ordinary differential equations with the matrix Q , the recalculation of a problem in the x_3 direction on the interval $(t_0 + \Delta t/2, t_0 + \Delta t)$, and the recalculation in the x_2 and x_1 directions. The application of the splitting procedure to the system of equations (6.21) results in the following one-dimensional systems:

$$\begin{aligned} A U_t^1 &= B^1 V_{x_1}^1 + G^1, & U^1(t_0, x) &= U(t_0, x), \\ A U_t^2 &= B^2 V_{x_2}^2 + G^2, & U^2(t_0, x) &= U^1(t_0 + \Delta t/2, x), \\ A U_t^3 &= B^3 V_{x_3}^3 + G^3, & U^3(t_0, x) &= U^2(t_0 + \Delta t/2, x), \\ A U_t^4 &= Q V^4, & U^4(t_0, x) &= U^3(t_0 + \Delta t/2, x), \\ A U_t^5 &= B^3 V_{x_3}^5 + G^3, & U^5(t_0 + \Delta t/2, x) &= U^4(t_0 + \Delta t, x), \\ A U_t^6 &= B^2 V_{x_2}^6 + G^2, & U^6(t_0 + \Delta t/2, x) &= U^5(t_0 + \Delta t, x), \\ A U_t^7 &= B^1 V_{x_1}^7 + G^1, & U^7(t_0 + \Delta t/2, x) &= U^6(t_0 + \Delta t, x). \end{aligned} \tag{6.26}$$

Here $V^i = \varsigma U^i + (1 - \varsigma) (U^i)^\pi$, $G^1 + G^2 + G^3 = G$. The desired value $U(t_0 + \Delta t, x)$ is equal to $U^7(t_0 + \Delta t, x)$.

The vectors G^i are taken on the basis of the principle of physical adequacy of one-dimensional models. For example, taking into account the gravity force in the x_1 direction, the vector G of mass forces is entirely transferred to the first and last stages. This leads to adequate description of static stresses in a body of a ponderable medium. In the calculation of a two-dimensional (plane or axially symmetric) problem,

the splitting method does not involve the third and fifth stages related to the x_3 direction. In the plane case $\mathbf{Q} = 0$ and, hence, the fourth stage is not involved as well.

It is known that the method of two-cyclic splitting under consideration is of second-order accuracy provided that at its stages second-order schemes are used. Besides, it ensures the stability of a numerical solution in spatial case provided that stability conditions for one-dimensional systems are fulfilled. To maintain second-order accuracy, at the fourth stage, when solving a system of ordinary differential equations in linear problems where the vectors $\mathbf{V}$ and $\mathbf{U}$ are equal to one another, the Crank–Nicholson implicit difference scheme [25]

$$\mathbf{A} \frac{\mathbf{U}^{k+1} - \mathbf{U}^k}{\Delta t} = \mathbf{Q} \frac{\mathbf{U}^{k+1} + \mathbf{U}^k}{2}$$

is applied (k is the number of time step). This scheme has good computational properties: it is conservative in the sense of consistency with the corresponding energy balance equation and, hence, it is stable independently of the value of a time step. For the nonlinear system

$$\mathbf{A} \mathbf{U}_t = \mathbf{Q} \mathbf{V}, \quad \mathbf{V} = \varsigma \mathbf{U} + (1 - \varsigma) \mathbf{U}^\pi \quad (6.27)$$

the Crank–Nicholson scheme is not conservative, therefore in this case the more general scheme

$$\mathbf{A} \frac{\mathbf{U}^{k+1} - \mathbf{U}^k}{\Delta t} = \mathbf{Q} \mathbf{V}^{k+1/2} \quad (6.28)$$

is used, where

$$\mathbf{V}^{k+1/2} = \varsigma \frac{\mathbf{U}^{k+1} + \mathbf{U}^k}{2} + (1 - \varsigma) \left(\zeta (\mathbf{U}^\pi)^{k+1} + (1 - \zeta) (\mathbf{U}^\pi)^k \right),$$

ζ is a free parameter which, generally speaking, is not equal to $1/2$ and in each mesh of a grid domain it is defined from considerations of conservatism.

The energy balance equation for the system (6.27) is obtained as a result of scalar multiplication of the system by the vector $\mathbf{V}$ taking into account the equality $\mathbf{U}^\pi \mathbf{A} \mathbf{U}_t = \mathbf{U}^\pi \mathbf{A} \mathbf{U}_t^\pi$ derived in Sect. 6.2:

$$E(\mathbf{U})_t = 2 \mathbf{V} \mathbf{Q} \mathbf{V}, \quad E(\mathbf{U}) = \varsigma \mathbf{U} \mathbf{A} \mathbf{U} + (1 - \varsigma) \mathbf{U}^\pi \mathbf{A} \mathbf{U}^\pi.$$

It is proved that the discrete analog of this equation

$$\frac{E(\mathbf{U}^{k+1}) - E(\mathbf{U}^k)}{2 \Delta t} = \mathbf{V}^{k+1/2} \mathbf{Q} \mathbf{V}^{k+1/2}$$

is fulfilled automatically provided that

$$\begin{aligned}
& \left(\zeta (U^\pi)^{k+1} + (1 - \zeta)(U^\pi)^k \right) \mathbf{A} \frac{U^{k+1} - U^k}{\Delta t} \\
&= \frac{(U^\pi)^{k+1} + (U^\pi)^k}{2} \mathbf{A} \frac{(U^\pi)^{k+1} - (U^\pi)^k}{\Delta t}, \tag{6.29}
\end{aligned}$$

since in this case the discrete energy balance equation can be obtained by means of multiplication of both sides of the system (6.28) by the vector $\mathbf{V}^{k+1/2}$.

By the definition of projection onto a cone, we have the relationships

$$(U^\pi)^{k+1} \mathbf{A} ((U^\pi)^k - U^k) \geq 0, \quad (U^\pi)^{k+1} \mathbf{A} ((U^\pi)^{k+1} - U^{k+1}) = 0.$$

Taking into account that the matrix $\mathbf{A}$ is symmetric and positive definite, subtraction of the above relationships gives

$$\begin{aligned}
& (U^\pi)^{k+1} \mathbf{A} (U^{k+1} - U^k) \geq (U^\pi)^{k+1} \mathbf{A} ((U^\pi)^{k+1} - (U^\pi)^k) \\
&= \frac{(U^\pi)^{k+1} + (U^\pi)^k}{2} \mathbf{A} ((U^\pi)^{k+1} - (U^\pi)^k) \\
&\quad + \frac{(U^\pi)^{k+1} - (U^\pi)^k}{2} \mathbf{A} ((U^\pi)^{k+1} - (U^\pi)^k) \\
&\geq \frac{(U^\pi)^{k+1} \mathbf{A} (U^\pi)^{k+1} - (U^\pi)^k \mathbf{A} (U^\pi)^k}{2}.
\end{aligned}$$

Besides, the inequality

$$(U^\pi)^k \mathbf{A} (U^{k+1} - U^k) \leq \frac{(U^\pi)^{k+1} \mathbf{A} (U^\pi)^{k+1} - (U^\pi)^k \mathbf{A} (U^\pi)^k}{2}$$

is valid which coincides with the previous one to within the replacement of U^{k+1} by U^k , and conversely. The obtained inequalities show that the function, linear relative to parameter ζ , with the help of which Eq. (6.29) is written in the form $f(\zeta) = 0$, changes its sign on the segment $[0, 1]$ and has a single root on this segment. The special case, that the coefficient for ζ is zero, is an exception. But in this case the scheme is conservative independently of the value of ζ .

Under numerical implementation of the nonlinear scheme (6.28) the equation for the vector U^{k+1} is reduced to the fixed point problem

$$\begin{aligned}
& \left(\mathbf{A} - \frac{\varsigma \Delta t}{2} \mathbf{Q} \right) U^{k+1} = \left(\mathbf{A} + \frac{\varsigma \Delta t}{2} \mathbf{Q} \right) U^k \\
& \quad + (1 - \varsigma) \Delta t \mathbf{Q} \left(\zeta (U^\pi)^{k+1} + (1 - \zeta)(U^\pi)^k \right),
\end{aligned}$$

whose solution is constructed by the method of successive approximations. At each step of this method, the value ζ is calculated from a previous approximation according to Eq. (6.29). It can be shown that the mapping, for which a fixed point

is determined, is contractive in the space with the norm $|\mathbf{U}|_A$ at least for small Δt which provides convergence of the method.

For the solution of each one-dimensional problem (6.26) an explicit monotone difference ENO-scheme is applied. This scheme is of second-order accuracy in domains of monotone change of a solution and it is stable under fulfillment of the Courant–Friedrichs–Levi condition. This scheme is a generalization of the Godunov scheme [13, 18], with piecewise-linear distributions of velocities and stresses over meshes. We consider a procedure of the construction of the scheme for a one-dimensional problem, using the system of equations

$$\mathbf{A} \mathbf{U}_t = \mathbf{B} \mathbf{V}_x + \mathbf{G}, \quad (6.30)$$

which is of a common form for six stages of the splitting, as an example. In the case of constant matrix-coefficients, at the “corrector” step the relationships

$$\mathbf{U}^{j+1/2} = \mathbf{U}_{j+1/2} + \frac{\Delta t}{2} \mathbf{A}^{-1} \left(\mathbf{B} \frac{\mathbf{V}_{j+1} - \mathbf{V}_j}{\Delta x_{j+1/2}} + \mathbf{G} \right)$$

are used. Here index $j + 1/2$ is related to the center of a mesh of a spatial difference grid, a superscript corresponds to an actual time level and a subscript corresponds to a previous level. The vector $\mathbf{V}^{j+1/2}$ is calculated from $\mathbf{U}^{j+1/2}$ by the formula (6.9). If the matrices are variable then the corresponding terms of the conservative approximation are taken as a difference derivative with respect to x .

To close the scheme, we have to complement it with the relationships of the “predictor” step which are obtained with the use of the idea of grid-characteristic methods [24]. Let $\mathbf{Y}_l$ and c_l be a complete system of left eigenvectors and eigenvalues of the matrix $\mathbf{B} \mathbf{A}^{-1}$, respectively:

$$\mathbf{Y}_l \mathbf{B} = c_l \mathbf{Y}_l \mathbf{A}, \quad \mathbf{Y}_l \mathbf{A} \mathbf{Y}_h = \delta_{lh}.$$

This system exists since the matrices $\mathbf{A}, \mathbf{B}$ are symmetric and, in addition, $\mathbf{A}$ is positive definite. In the case of an inhomogeneous material, the eigenvalues involved in this system are functions of x of constant signs. Multiplying the equality (6.30) by the vector $\mathbf{Y}_l$ from the left, we arrive at a system of differential equations. For a model of an elastic medium these equations are equations on characteristics:

$$\mathbf{Y}_l \mathbf{A} \mathbf{U}_t = c_l \mathbf{Y}_l \mathbf{A} \mathbf{V}_x + \mathbf{Y}_l \mathbf{G}, \quad l = 1, \dots, m.$$

After approximation we obtain

$$(I_l^{j+1/2})^\pm = I_{lj+1/2} \pm \alpha_{lj+1/2} \frac{\Delta x_{j+1/2}}{2} + (c_l \beta_l + \mathbf{Y}_l \mathbf{G})_{j+1/2} \frac{\Delta t}{4}, \quad (6.31)$$

where $\alpha_{lj+1/2}$ and $\beta_{lj+1/2}$ are derivatives of coefficients of the decomposition of the vector-functions $\mathbf{U}$ and $\mathbf{V}$ in the basis $\mathbf{Y}_l$:

$$I_l = (Y_l \mathbf{A})_{j+1/2} \mathbf{U}, \quad J_l = (Y_l \mathbf{A})_{j+1/2} \mathbf{V},$$

received with the help of the iterative procedure of limit reconstruction (see [19, 23]). Indices “−” and “+” in (6.31) mark the values of these coefficients on the left and on the right boundaries of a mesh, respectively. The procedure of limit reconstruction enables one to improve an accuracy of a numerical solution and consists in the construction of monotone piecewise-linear splines which approximate I_l and J_l with minimal discontinuities on boundaries of neighbouring meshes of a grid. First each of these coefficients is approximated with the help of piecewise-constant splines which take the values $I_{l\ j+1/2}$ and $J_{l\ j+1/2}$ on the mesh (x_j, x_{j+1}) . Then jumps are determined by the formulae

$$\Delta I_{l\ j}^0 = I_{l\ j+1/2} - I_{l\ j-1/2}, \quad \Delta J_{l\ j}^0 = J_{l\ j+1/2} - J_{l\ j-1/2}.$$

When solving boundary-value problems, jumps related to boundaries of a computational domain are assumed to be zero. After this the functions

$$I_l = I_{l\ j+1/2} + \alpha_{l\ j+1/2}^0 (x - x_{j+1/2}), \quad J_l = J_{l\ j+1/2} + \beta_{l\ j+1/2}^0 (x - x_{j+1/2}),$$

linear on a mesh, with angular coefficients

$$\alpha_{l\ j+1/2}^0 = \frac{1}{\Delta x_{j+1/2}} \operatorname{sgn} \Delta I_{l\ j}^0 \min \left\{ |\Delta I_{l\ j}^0|, |\Delta I_{l\ j+1}^0| \right\} \quad \text{for } \Delta I_{l\ j}^0 \Delta I_{l\ j+1}^0 > 0$$

$$\text{or } \alpha_{l\ j+1/2}^0 = 0 \quad \text{for } \Delta I_{l\ j}^0 \Delta I_{l\ j+1}^0 \leq 0,$$

$$\beta_{l\ j+1/2}^0 = \frac{1}{\Delta x_{j+1/2}} \operatorname{sgn} \Delta J_{l\ j}^0 \min \left\{ |\Delta J_{l\ j}^0|, |\Delta J_{l\ j+1}^0| \right\} \quad \text{for } \Delta J_{l\ j}^0 \Delta J_{l\ j+1}^0 > 0$$

$$\text{or } \beta_{l\ j+1/2}^0 = 0 \quad \text{for } \Delta J_{l\ j}^0 \Delta J_{l\ j+1}^0 \leq 0,$$

become approximations. Each of these functions halves a minimal jump of a corresponding piecewise-constant spline with respect to the nodes $x = x_j$ and $x = x_{j+1}$. As a result, piecewise-linear discontinuous splines being monotone in domains of monotone change of piecewise-constant splines are obtained. Then new jumps

$$\Delta I_{l\ j}^1 = \Delta I_{l\ j}^0 - \frac{1}{2} (\alpha_{l\ j+1/2}^0 \Delta x_{j+1/2} + \alpha_{l\ j-1/2}^0 \Delta x_{j-1/2}),$$

$$\Delta J_{l\ j}^1 = \Delta J_{l\ j}^0 - \frac{1}{2} (\beta_{l\ j+1/2}^0 \Delta x_{j+1/2} + \beta_{l\ j-1/2}^0 \Delta x_{j-1/2})$$

are calculated and the procedure of decreasing jumps is repeated with preservation of monotonicity. To this end, to the old values of angular coefficients $\alpha_{l\ j+1/2}^0$ and

$\beta_{lj+1/2}^0$ their increments $\alpha_{lj+1/2}^1$ and $\beta_{lj+1/2}^1$ calculated by the above formulae in terms of ΔI_{lj}^1 and ΔJ_{lj}^1 , respectively, are added. The process is repeated four or five times and then is terminated since, as shown by practical calculations, further iterations do not improve a solution.

It is known [23], that as a result of the application of the procedure of limit reconstruction a piecewise-linear discontinuous spline is obtained with the conservation of integral mean values over meshes. For such a spline the sum of moduli of jumps on boundaries of meshes is minimal. This procedure improves accuracy of a scheme up to the second order provided that a solution of each equation on characteristics is monotone. However, in the general case, such that there are points of local extremum, we obtain a hybrid scheme of first-order accuracy.

At internal nodes of a computational domain, at the “predictor” step the quantities U_j are determined by the averaging formula

$$U_j = \frac{1}{2} (U_j^- + U_j^+)$$

via the quantities $U_j^\pm$ relating to different sides of a boundary between meshes and satisfying the system of nonlinear algebraic equations

$$\begin{cases} (Y_l A)_{j+1/2} U_j^+ = I_{lj+1/2}^- & \text{for } c_l \geq 0, \\ (Y_l A)_{j-1/2} U_j^- = I_{lj-1/2}^+ & \text{for } c_l \leq 0, \\ D_l V_j^+ = D_l V_j^-. \end{cases}$$

In this system the equations involving vectors D_l represent the continuity conditions for a velocity vector and a stress vector in passing across a boundary and the number of equations, including these conditions, is equal to the number of unknowns. According to (6.9), the continuity conditions are rewritten in the following way

$$D_l(U_j^+ - U_j^-) = (1 - \varsigma^+) D_l(U_j^+ - U_j^{+\pi}) - (1 - \varsigma^-) D_l(U_j^- - U_j^{-\pi})$$

and are implemented numerically with the help of the method of successive approximations. Convergence of the method is due to the fact that the mapping $U - U^\pi$ is nonexpanding and the multipliers $1 - \varsigma^\pm$ are strictly less than one. The quantities V_j are recalculated via U_j according to (6.9).

We present an algorithm for calculation of the vector V on the right-hand boundary of the domain of solution of a problem (on the left-hand boundary this vector is calculated in a similar way). Assume that boundary conditions are formulated in terms of velocities or strains. Then they can be represented in the form

$$D_l U = q_l, \quad l = 1, \dots, m_+. \quad (6.32)$$

Here m_+ is the number of strictly positive values of c_l corresponding to “outgoing” characteristics, D_l and q_l are given vectors and coefficients. The $m - m_+$ missing

equations for negative and zero values of c_h ($h = m_+ + 1, \dots, m$) have the form $Y_h A U = r_h$ where r_h are the expressions in the right-hand side of the equality (6.31). The boundary conditions (6.32) are well-posed if m vectors D_l and $Y_h A$ form a linearly independent system. In this case the equations enable one to find all invariants I_l for $l = 1, \dots, m$. Hence, U is determined from the decomposition in the basis Y_l and then V is recalculated via U .

If boundary conditions are formulated in terms of stresses, then the vector U in the system (6.32) is replaced by V . Involving $m - m_+$ additional equations, we obtain a more complicated problem

$$D_l U = q_l + (1 - \varsigma) D_l (U - U^\pi), \quad Y_h A U = r_h, \quad (6.33)$$

which is solved by means of the method of successive approximations. The vector $\bar{U}$, corresponding to the boundary conditions (6.32), is taken as an initial approximation. In the case that $\bar{U} \in K$ it is obviously a required solution. To avoid superfluous iterations, the case of $U^\pi = 0$, where U is also determined in the explicit form as a solution of the system of linear algebraic equations following from (6.33), is considered separately.

By simple manipulations, the problem is reduced to finding a fixed point

$$U = \bar{U} + (1 - \varsigma) C Z(U),$$

where C is an $(m \times m)$ -matrix and $Z(U) = U - U^\pi$. Using properties of a projection onto a convex set, we can prove that the mapping Z is nonexpanding:

$$|Z(\tilde{U}) - Z(U)|_A \leq |\tilde{U} - U|_A.$$

Indeed, due to the variational inequality (3.11), defining a projection,

$$(\tilde{U}^\pi - U^\pi) A Z(U) \leq 0, \quad (U^\pi - \tilde{U}^\pi) A Z(\tilde{U}) \leq 0.$$

Hence, $-(\tilde{U}^\pi - U^\pi) A (Z(\tilde{U}) - Z(U)) \leq 0$, or, taking into account the notations for $Z(U)$,

$$|Z(\tilde{U}) - Z(U)|_A^2 \leq (\tilde{U} - U) A (Z(\tilde{U}) - Z(U)) \leq |\tilde{U} - U|_A |Z(\tilde{U}) - Z(U)|_A.$$

In view of the principle of contractive mappings, the method of successive approximations converges provided that the following inequality holds:

$$(1 - \varsigma) |C|_A < 1, \quad |C|_A = \max_{|U|_A=1} |C U|_A. \quad (6.34)$$

Here $|C|_A$ is the matrix norm associated with the energy norm of the vector space. According to the Lagrange principle, the maximization problem which must be solved when calculating this norm is reduced to finding the unconstrained maximum

of the quadratic function

$$|\mathbf{C}\mathbf{U}|_{\mathbf{A}}^2 - \lambda |\mathbf{U}|_{\mathbf{A}}^2 \equiv \mathbf{U} \mathbf{C}^* \mathbf{A} \mathbf{C} \mathbf{U} - \lambda \mathbf{U} \mathbf{A} \mathbf{U}.$$

At the maximum point we have the system of equations

$$\mathbf{C}^* \mathbf{A} \mathbf{C} \mathbf{U} = \lambda \mathbf{A} \mathbf{U},$$

besides, $\lambda = |\mathbf{C}|_{\mathbf{A}}^2$. Hence, the norm $|\mathbf{C}|_{\mathbf{A}}$ can be determined as the square root of the maximal eigenvalue of the matrix $\mathbf{C}^* \mathbf{A} \mathbf{C} \mathbf{A}^{-1}$. The inequality (6.34) is a sufficient condition for unique solvability of the system of equations (6.33) and provides correctness of the computational algorithm. The method converges for any $\varsigma \in (0, 1]$ provided that $|\mathbf{C}|_{\mathbf{A}} \leq 1$.

In fact, when implementing the algorithm, it is inefficient to calculate approximations to the vector $\mathbf{U}$ at each step of an iterative process. To reduce the number of arithmetic operations, one should use the boundary conditions (6.33) for finding the first m_+ invariants I_l corresponding to positive values of c_l . The remaining invariants are known: $I_h = r_h$. Then the formulae for recursive recalculation take the following form:

$$I'_l = \bar{I}_l + (1 - \varsigma) \left(I_l - I_l^\pi + \sum_{h=m_++1}^m p_{lh} (r_h - I_h^\pi) \right),$$

where p_{lh} are constant coefficients. Once iterations have been performed, $\mathbf{U}$ and $\mathbf{V}$ are determined. The energy norm of the vector $\mathbf{U}$ in the space of invariants coincides with the standard Euclidean norm, hence, the convergence condition (6.34) for a sequence of approximations can be formulated in terms of p_{lh} in a simpler form: the maximum eigenvalue of the matrix $\mathbf{P}^* \mathbf{P}$, where

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & \dots & 0 & p_{1m_++1} & \dots & p_{1m} \\ 0 & 1 & \dots & 0 & p_{2m_++1} & \dots & p_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & p_{m_+m_++1} & \dots & p_{m_+m} \end{pmatrix},$$

may not exceed one.

Along with the boundary conditions considered above, in practice conditions on artificial boundaries are often used. These conditions arise because of necessity to reduce a computational domain (so-called nonreflecting, radiating, or absorbing boundary conditions [17, 23], modeling unobstructed passage of waves). Formulation of such conditions is a complicated problem which can be considered to be solved to some extent only for the equations of linear elasticity and viscoelasticity theory [6, 8, 16]. Under numerical analysis of wave propagation in a unbounded body within the framework of the nonlinear model of mechanics of granular materials we used the simplest variant of nonreflecting conditions which is as follows.

Assume that at an instant t a wave moving in the direction of a vector $\mathbf{v}$ arrives at a point $\mathbf{x}$ of an artificial boundary. In concrete problems this vector can be determined, for example, approximating a wave by a plane spherical or cylindrical one propagating from a given source. We can also construct the vector $\mathbf{v}$ analyzing position of fronts on a wave field with the help of differential analyzers, based on calculation of derivatives of displacements, velocities, or stresses with respect to spatial variables, and then determining the inflection points [9, 42, 46]. This vector remains constant as far as a new wave arrives at the point and varies as the wave passes.

The vector $\mathbf{v}$ is a base vector of the local Cartesian coordinate system related to the front. The projections of the vector $\mathbf{v}$ form the first column of the transition matrix $\mathbf{D}$ from the original Cartesian coordinate system to the local one. The second and third columns corresponding to two remaining base vectors can be taken to within an arbitrary rotation about the vector $\mathbf{v}$. Assuming that a wave is plane in a neighbourhood of the point $\mathbf{x}$, we write artificial boundary conditions on the right-hand boundary in the form of equations for invariants which transfer perturbations in the direction opposite to that of $\mathbf{v}$:

$$\hat{I}_l \equiv \mathbf{Y}_l \mathbf{A} \hat{\mathbf{U}} = 0, \quad \text{if } c_l > 0.$$

Here $\hat{\mathbf{U}}$ is a vector of unknown functions, projections of the velocity vector $\hat{\mathbf{v}} = \mathbf{D}^* \mathbf{v}$ and the tensor $\hat{\mathbf{s}} = \mathbf{D}^* \mathbf{s} \mathbf{D}$ of conditional stresses with respect to the local coordinate system are its components. Such equations were used in the implementation of boundary conditions for one-dimensional problems in the splitting method together with equations which define the values of invariants I_h on the incoming characteristics calculated in the original coordinate system. The obtained system of linear algebraic equations is a particular case of the system (6.32), and it can be solved by means of direct methods which do not require iterations.

6.4 Plane Signotons

We construct exact solutions, describing shock waves in a loosened granular material, and apply them when testing the developed algorithm. With a more appropriate variant being not found, we a priori simplified the terminology introduced in [27] and used in some other works (see, for example, [7, 28]). Further all kinds of waves, such that when passing across them strain changes its sign, are called signotons.

Consider a plane longitudinal shock wave propagating in an infinite body of a granular material in the x_1 direction of a rectangular Cartesian coordinate system $O x_1 x_2 x_3$. If the wave amplitude is sufficiently small then on the front of a wave the linear equations of dynamic and kinematic compatibility [41]

$$\rho c [[v_1]] = -[[\sigma_1]], \quad c [[\varepsilon_1]] = -[[v_1]] \quad (6.35)$$

are fulfilled. Here c is the velocity of wave propagation, ρ is the density of a material, v_1 is the mass velocity of particles, $\sigma_1, \sigma_2 = \sigma_3$, and ε_1 are nonzero components of stress and strain tensors, square brackets mean a jump of a function at discontinuity.

To close the system (6.35), it is required to add an equation connecting σ_1 and ε_1 . Then a shock adiabat, i.e. a curve of admissible shock-wave transitions from the fixed state (ε_1^0, v_1^0) before the front of the wave to the state (ε_1, v_1) behind the front, is shown by a curve on the $\varepsilon_1 v_1$ plane. The equation of this curve takes the form

$$\rho(v_1 - v_1^0)^2 = (\sigma_1 - \sigma_1^0)(\varepsilon_1 - \varepsilon_1^0). \quad (6.36)$$

The equation $\sigma_1 = \sigma_1(\varepsilon_1)$ depends on the sign of strain and on the nature of a process (elastic, elastic–plastic). We derive this equation basing on the constitutive relationships (6.1), (6.3) [34, 36, 39].

The elastic state takes place in a neighborhood of compressive hydrostatic stresses provided that the von Mises–Schleicher strength condition

$$\tau(\sigma) \leq \alpha p(\sigma)$$

holds, where $\tau(\sigma) = |\sigma_1 - \sigma_3|/\sqrt{3}$, $p(\sigma) = -(\sigma_1 + 2\sigma_3)/3$. The equality sign in this condition corresponds to the limit equilibrium of a granular material. For a sufficiently high pressure, tangential stresses may achieve limit values, corresponding to the onset of a plastic process, without violating the von Mises–Schleicher condition. In this case the von Mises condition

$$\tau(\sigma) = \tau_s$$

holds. Dependence of the yield point on the value of pressure and on the degree of strengthening of particles due to plastic deformation is not taken into account in the model.

For an elastic–plastic process, the strain ε_1 is equal to the sum of elastic ε_1^e , granular ε_1^c , and plastic ε_1^p components, respectively. Conditional stresses s_1 and $s_2 = s_3$ are calculated with the help of the Hooke law:

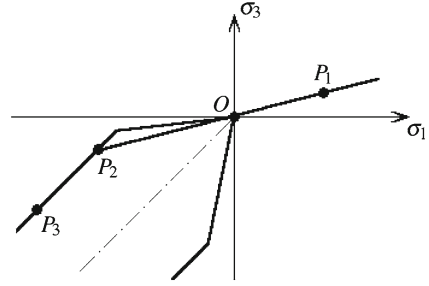
$$s_1 = \left(k + \frac{4\mu}{3}\right)(\varepsilon_1 - \varepsilon_1^p), \quad s_3 = \left(k - \frac{2\mu}{3}\right)(\varepsilon_1 - \varepsilon_1^p). \quad (6.37)$$

Actual stresses are determined as projections of conditional stresses onto the von Mises–Schleicher cone: $\sigma_1 = s_1^\pi, \sigma_3 = s_3^\pi$ with respect to the energy norm, the square of which is equal to $|\sigma|_a^2 = p^2(\sigma)/k + \tau^2(\sigma)/(2\mu)$. There are three alternatives (see Sect. 4.3). If $\tau(s) \leq \alpha p(s)$, i.e. conditional stresses lie in the cone, then $\sigma_1 = s_1, \sigma_3 = s_3$. If $\mu p(s) + \alpha k \tau(s) \leq 0$, then $\sigma_1 = \sigma_3 = 0$. In this case the vertex of the cone is the projection. If $\tau(s) > \alpha p(s)$ and $\mu p(s) + \alpha k \tau(s) > 0$, then the projection belongs to the conical surface

$$\frac{\sigma_1}{p(\sigma)} = \alpha \frac{s_1 + p(s)}{\tau(s)} - 1, \quad \frac{\sigma_3}{p(\sigma)} = \alpha \frac{s_3 + p(s)}{\tau(s)} - 1, \quad (6.38)$$

Fig. 6.5 Trajectories of stresses:

$$2\mu/(\sqrt{3}k) \leq \varkappa \leq \sqrt{3}/2$$



in addition, $p(\sigma) = (\mu p(s) + \varkappa k \tau(s))/(\mu + \varkappa^2 k)$. For plastic flow the von Mises plasticity condition and the elastic law of the volume change

$$\sigma_1 + 2\sigma_3 = 3k\varepsilon_1$$

are valid. Depending on sign of ε_1 , in the case of active loading we have stresses

$$\sigma_1 = k\varepsilon_1 + \frac{2\tau_s}{\sqrt{3}} \operatorname{sgn} \varepsilon_1, \quad \sigma_3 = k\varepsilon_1 - \frac{\tau_s}{\sqrt{3}} \operatorname{sgn} \varepsilon_1. \quad (6.39)$$

For unloading increments of stresses are calculated from increments of strain ε_1 by the Hooke linear law.

Considering ε_1 as a parameter, we construct trajectories of stresses $s(\varepsilon_1)$ and $\sigma(\varepsilon_1)$ on the $\sigma_1\sigma_3$ plane [34].

Let $2\mu/(\sqrt{3}k) \leq \varkappa \leq \sqrt{3}/2$. The position of the straight line P_1P_2 of elastic deformation with respect to the von Mises–Schleicher cone and the von Mises cylinder for this case is shown in Fig. 6.5. With tension, where the strain ε_1 is positive, conditional stresses s_1, s_3 are given by the formulae (6.37) with zero instead of ε_1^p and actual stresses σ_1, σ_3 (projections of conditional stresses onto the cone) are equal to zero. With compression in the range of $\varepsilon_1^2 \leq \varepsilon_1 \leq 0$, where $\varepsilon_1^2 = -\sqrt{3}\tau_s/(2\mu)$, a material is in the elastic state and

$$\sigma_1 = s_1 = \left(k + \frac{4\mu}{3}\right)\varepsilon_1, \quad \sigma_3 = s_3 = \left(k - \frac{2\mu}{3}\right)\varepsilon_1.$$

Further compression is accompanied by plastic strain with stresses $\sigma = s$ which are determined by the formulae (6.39). Thus, the two-element broken line $P_1P_2P_3$ is the trajectory of $s(\varepsilon_1)$ and its part OP_2P_3 is the trajectory of $\sigma(\varepsilon_1)$. Graph of the dependence of σ_1 on ε_1 is shown in Fig. 6.6.

Further we consider only waves moving in the positive direction of the x_1 axis, for which $c > 0$. Assuming that $v_1^0 = 0$ we construct shock adiabats of the compression waves ($\varepsilon_1 < \varepsilon_1^0$). If $\varepsilon_1^0 < \varepsilon_1^2$, i.e. before the front of a wave the material is compressed to the plastic state, then a plastic shock wave characteristic of a standard elastic–plastic material is realized there [31, 37]. Equation (6.36) of a shock adiabat has the

Fig. 6.6 Dependence of σ_1 versus ε_1 :
 $2\mu/(\sqrt{3}k) \leq \varepsilon \leq \sqrt{3}/2$

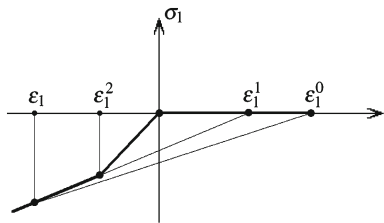


Fig. 6.7 Shock adiabat of a plastic wave: $\varepsilon_1^0 < \varepsilon_1^2$

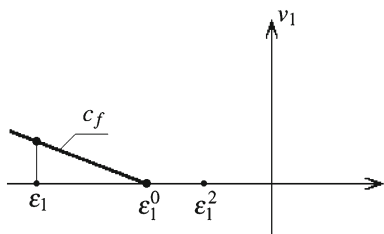
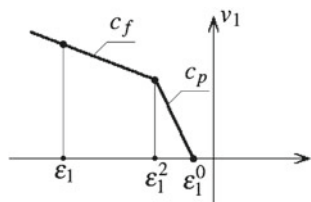


Fig. 6.8 Shock adiabats of elastic and plastic waves: $\varepsilon_1^2 \leq \varepsilon_1^0 \leq 0$



following form:

$$v_1 = -c_f(\varepsilon_1 - \varepsilon_1^0),$$

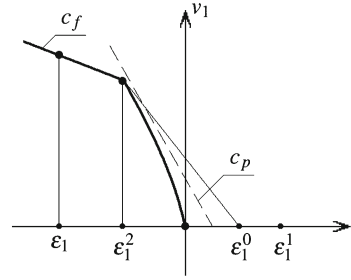
where $c_f = \sqrt{k/\rho}$ is the velocity of plastic shock waves. On the $\varepsilon_1 v_1$ plane a shock adiabat is shown as a ray (Fig. 6.7). If $\varepsilon_1^2 \leq \varepsilon_1^0 \leq 0$ then under intensive compression for $\varepsilon_1 < \varepsilon_1^2$ there arises two-wave configuration of discontinuities: an elastic precursor, which transfers a material from the state ε_1^0 to the limit elastic state ε_1^2 and whose velocity is equal to the velocity $c_p = \sqrt{(k + 4\mu/3)/\rho}$ of elastic longitudinal waves, and a plastic shock wave. The shock adiabat consists of two linear elements (Fig. 6.8). Under weak compression, where the value of ε_1 is in the range of elastic strain, a plastic wave does not arise.

If before the front a material is loosened ($\varepsilon_1^0 > 0$), then the equation of shock adiabat for the segment $P_1 P_2$ takes the form

$$\left(\frac{v_1}{c_p}\right)^2 = \varepsilon_1(\varepsilon_1 - \varepsilon_1^0). \quad (6.40)$$

This is the equation of adiabat of an elastic signoton, behind its front a material is in the compressed state below the yield point.

Fig. 6.9 Shock adiabats of an elastic signoton and a plastic wave: $0 < \varepsilon_1^0 \leq \varepsilon_1^1$



Under intensive compression in the case of $\varepsilon_1 < \varepsilon_1^2$, an elastic signoton, such that behind its front strain is equal to the limit value ε_1^2 , is adjacent to a plastic shock wave with velocity c_f (Fig. 6.9). There does not exist a shock wave from one loosened state to another loosened state ($\varepsilon_1 \geq 0$), since in this state particles of a material move independently of each other.

In fact, a configuration consisting of a signoton–precursor and a plastic wave is not always admissible. Velocity of an elastic signoton

$$c = c_p \sqrt{\frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_1^0}} \quad (6.41)$$

decreases as a material is loosened, it approaches zero as ε_1^0 tends to infinity, and for some critical loosening ε_1^1 it becomes equal to c_p . Exceeding ε_1^1 results in overturning of wave with formation of a solitary plastic signoton which describes transition from the state ε_1^0 immediately to the state ε_1 (Fig. 6.10). The equation of shock adiabat for such transitions has the next form:

$$\left(\frac{v_1}{c_f}\right)^2 = (\varepsilon_1 - \varepsilon_1^1)(\varepsilon_1 - \varepsilon_1^0), \quad \varepsilon_1^1 = \frac{2\tau_s}{\sqrt{3}k}. \quad (6.42)$$

The graphs of shock adiabats (6.40), (6.42) are branches of hyperbolas whose asymptotes are inclined at angles $\arctan c_p$ and $\arctan c_f$, respectively, relative to the abscissa axis. The asymptote of the former adiabat issues out of the point $\varepsilon_1^0/2$ (Fig. 6.9). The asymptote of the latter adiabat issues out of the middle point between ε_1^1 and ε_1^0 (Fig. 6.10). At the intersection of adiabats $\varepsilon_1 = \varepsilon_1^2$. From Eq. (6.35) it follows that velocity of each signoton to within a sign is equal to the angular coefficient of the secant which joins the center $(\varepsilon_1^0, 0)$ of the adiabat with a current point (ε_1, v_1) on the $\varepsilon_1 v_1$ plane. Thus, for a plastic signoton

$$c = c_f \sqrt{\frac{\varepsilon_1 - \varepsilon_1^1}{\varepsilon_1 - \varepsilon_1^0}}. \quad (6.43)$$

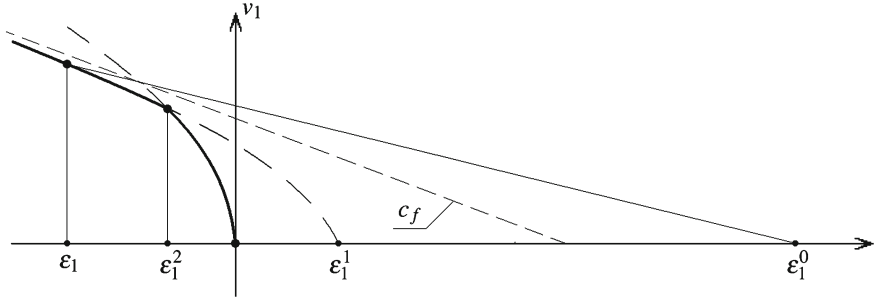
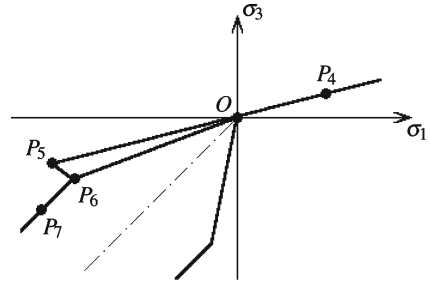


Fig. 6.10 Shock adiabats of elastic and plastic signotons: $\varepsilon_1^0 > \varepsilon_1^1$

Fig. 6.11 Trajectories of stresses: $\varkappa \leq \sqrt{3}/2$,
 $\varkappa < 2\mu/(\sqrt{3}k)$



Let $\varkappa < 2\mu/(\sqrt{3}k)$ and $\varkappa \leq \sqrt{3}/2$. In this case the relative position of the straight line (6.37), the von Mises–Schleicher cone, and the von Mises cylinder corresponds to Fig. 6.11. Elastic section of the trajectory of conditional stresses $s(\varepsilon_1)$, as before, is defined by Eqs. (6.37) for $\varepsilon_1 = \varepsilon_1^e$. Actual stresses are determined as the projections of conditional stresses onto the conical surface

$$\sigma_1 = \left(1 + \frac{2\varkappa}{\sqrt{3}}\right)\beta\varepsilon_1, \quad \sigma_3 = \left(1 - \frac{\varkappa}{\sqrt{3}}\right)\beta\varepsilon_1, \quad \beta = \frac{1 + 2\varkappa/\sqrt{3}}{1 + \varkappa^2 k/\mu} k.$$

The point P_5 , at which $\varepsilon_1^5 = -\tau_s/(\beta\varkappa)$, corresponds to transition to the plastic state. At this point a material no longer shows the shear strength, so additional compaction is required for recovery of bearing capacity. This compaction is observed on the segment P_5P_6 for constant stresses. The strain of compaction is given by

$$\Delta\varepsilon_1 = \varepsilon_1^5 - \varepsilon_1^6 = \frac{\tau_s}{k} \frac{2/\sqrt{3} - \varkappa k/\mu}{1 + 2\varkappa/\sqrt{3}}.$$

The parametric equations of the straight line P_5P_6 have the next form:

$$s_1 = \left(k - \frac{2\mu}{\sqrt{3}\varkappa}\right)\varepsilon_1 - \frac{2\tau_s}{\sqrt{3}}\left(1 + \frac{\mu}{\varkappa^2 k}\right),$$

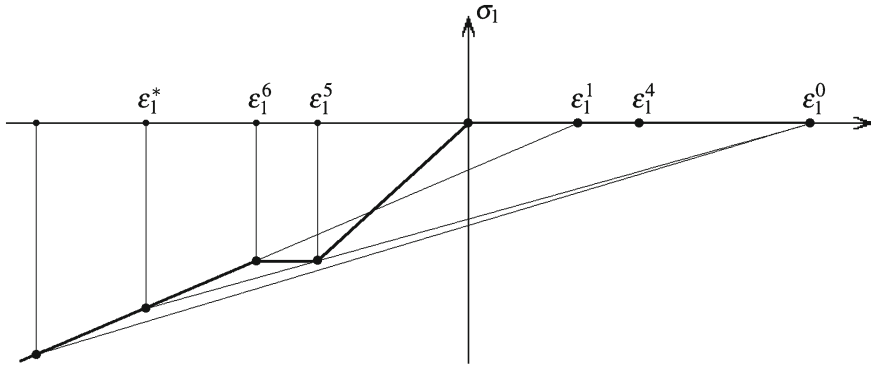
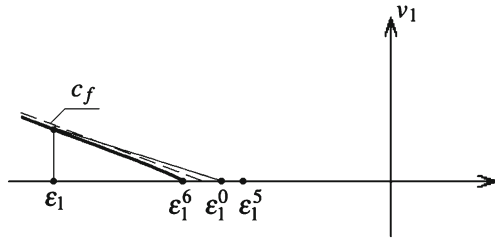


Fig. 6.12 Dependence of σ_1 versus ε_1 : $\varkappa \leq \sqrt{3}/2$, $\varkappa < 2\mu/(\sqrt{3}k)$

Fig. 6.13 Shock adiabat of a wave of plastic compaction:

$$\varepsilon_1^6 < \varepsilon_1^0 < \varepsilon_1^5$$



$$s_3 = \left(k + \frac{\mu}{\sqrt{3}\varkappa}\right)\varepsilon_1 + \frac{\tau_s}{\sqrt{3}}\left(1 + \frac{\mu}{\varkappa^2 k}\right).$$

As the segment P_5P_6 is passed, contrary to other parts of the trajectory, conditional stress s_1 increases, hence, a part of elastic strain is transformed into plastic one. At the point P_6 strain is $\varepsilon_1^6 = -\tau_s/(\varkappa k)$. To the interval $\varepsilon_1 < \varepsilon_1^6$ there corresponds a plastic process of compression (the ray P_6P_7). On this interval stresses $\sigma = s$ are expressed by the formulae (6.39). Thus, the broken line $P_4P_5P_6P_7$ is the trajectory of conditional stresses and the broken line OP_6P_7 is the trajectory of actual stresses. The graph of the function $\sigma_1(\varepsilon_1)$ is shown in Fig. 6.12.

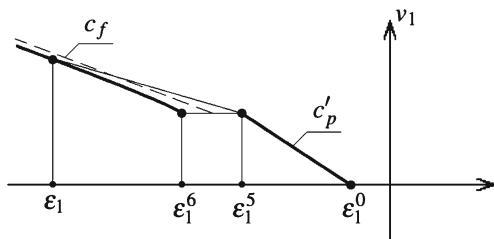
If at a compression wave $\varepsilon_1^0 \leq \varepsilon_1^6$, then this is a plastic shock wave corresponding to Fig. 6.7. If $\varepsilon_1^6 < \varepsilon_1^0 < \varepsilon_1^5$, then the shock adiabat is described by the equation

$$\left(\frac{v_1}{c_f}\right)^2 = (\varepsilon_1 - \varepsilon_1^6)(\varepsilon_1 - \varepsilon_1^0). \quad (6.44)$$

Such a wave arises only if $\varepsilon_1 < \varepsilon_1^6$ (see Fig. 6.13). This is a wave of plastic compaction of a material loosened at the instant, at which granules go to the plastic state. Its velocity is calculated by the formula similar to (6.43):

Fig. 6.14 Shock adiabats of an elastic wave and a plastic compaction wave:

$$\varepsilon_1^5 \leq \varepsilon_1^0 \leq 0$$



$$c = c_f \sqrt{\frac{\varepsilon_1 - \varepsilon_1^6}{\varepsilon_1 - \varepsilon_1^0}}. \quad (6.45)$$

If $\varepsilon_1^5 \leq \varepsilon_1^0 \leq 0$, then depending on the compression degree one or two waves propagate (Fig. 6.14). The former wave is an elastic precursor in a loosened material. It moves with velocity

$$c'_p = \sqrt{\frac{\beta(1 + 2\alpha/\sqrt{3})}{\rho}}.$$

For $\varepsilon_1 < \varepsilon_1^6$ the precursor is followed by a plastic compaction wave whose velocity is determined by the formula (6.45) with ε_1^5 instead of ε_1^0 . The branch of the shock adiabat of this wave is obtained by the parallel transition of the curve (6.44) along the v_1 axis in the value of v_1^6 . In the interval $\varepsilon_1^6 \leq \varepsilon_1 < \varepsilon_1^5$ bearing capacity of a material behind the front of an elastic precursor has no time to be recovered, hence, for this case there does not exist a system of waves.

If $\varepsilon_1^0 > 0$, then either a solitary elastic signoton described by the elastic adiabat (6.40) with c'_p instead of c_p or two-wave configuration of an elastic signoton-precursor and a plastic compaction wave is possible (Fig. 6.15). Such a pattern is observed only for sufficiently small loosening $\varepsilon_1^0 \leq \varepsilon_1^4$. Exceeding the critical value $\varepsilon_1^4 = \varepsilon_1^1 + \Delta\varepsilon_1$ results in the overturning of waves of strong compression, since velocity of a signoton-precursor becomes less than the maximal velocity of such waves equal to c_p .

The shock adiabat for $\varepsilon_1^0 > \varepsilon_1^4$ (Fig. 6.16) consists of three branches: the adiabat of elastic signotons, the adiabat of a plastic compaction wave, and the adiabat of plastic signotons (6.42). The actual system of waves for a given value of compressive strain ε_1 is easily determined from Fig. 6.12, taking into account that the value of ρc^2 is equal to the angular coefficient of a secant on the ε_1 σ_1 plane. According to Fig. 6.12, if $\varepsilon_1^5 \leq \varepsilon_1 < 0$, then a solitary elastic signoton is observed. If $\varepsilon_1^* < \varepsilon_1 < \varepsilon_1^6$, where

$$\varepsilon_1^* = \varepsilon_1^6 \frac{\varepsilon_1^0 + 2\alpha\varepsilon_1^5/\sqrt{3}}{\varepsilon_1^0 - \varepsilon_1^4}$$

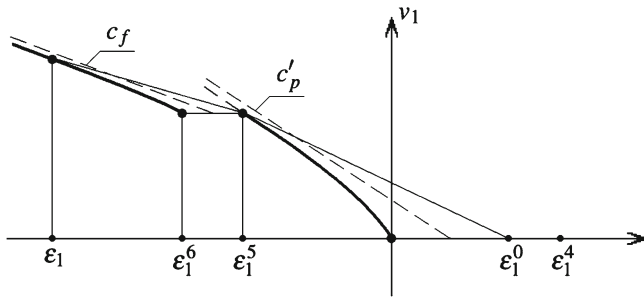


Fig. 6.15 Shock adiabats of an elastic signoton and a plastic compaction wave: $0 < \varepsilon_1^0 \leq \varepsilon_1^4$

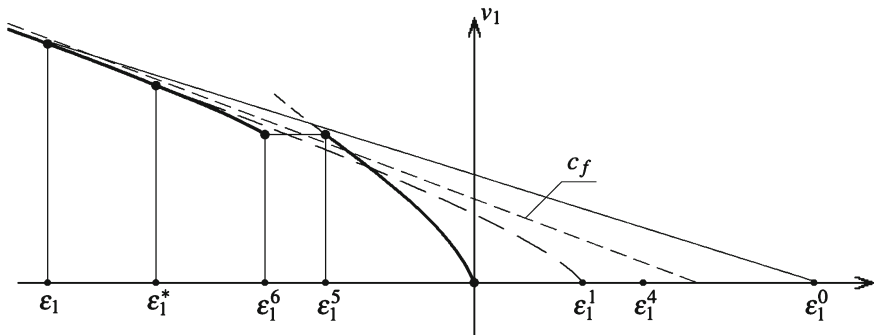


Fig. 6.16 Shock adiabats of an elastic signoton, a plastic compaction wave, and a plastic signoton: $\varepsilon_1^0 > \varepsilon_1^4$

is the strain at the point of intersection of the ray issued out of the initial state to the state P_5 and the graph of the function $\sigma_1(\varepsilon_1)$, then an elastic signoton is followed by a plastic compaction wave. For the strain $\varepsilon_1 = \varepsilon_1^*$ velocities of these waves become equal to one another. For $\varepsilon_1 < \varepsilon_1^*$ the shock-wave transition is described by a solitary plastic signoton.

Let $\varkappa > \sqrt{3}/2$. Then the von Mises–Schleicher cone intersects with the fourth quadrant of the $\sigma_1 \sigma_3$ plane. Trajectories of stresses $s(\varepsilon_1)$ and $\sigma(\varepsilon_1)$ for tensile strain vary significantly. It turns out that in this case a granular material resists to tension by going to the plastic state for some critical value of tensile strain. The trajectory of conditional stresses on the elastic segment OP_9 is defined by Eqs. (6.37). The equations of the trajectory of actual stresses on OP_8 have the form

$$\sigma_1 = \left(1 - \frac{2\varkappa}{\sqrt{3}}\right)\psi \varepsilon_1, \quad \sigma_3 = \left(1 + \frac{\varkappa}{\sqrt{3}}\right)\psi \varepsilon_1, \quad \psi = \frac{1 - 2\varkappa/\sqrt{3}}{1 + \varkappa^2 k/\mu} k. \quad (6.46)$$

The limit elastic strain is equal to $\varepsilon_1^9 = -\tau_s/(\psi \varkappa)$. On the plastic segment P_9P_{10} conditional stresses vary according to the equations

Fig. 6.17 Trajectories of stresses: $\mathfrak{x} > \sqrt{3}/2$,
 $\mathfrak{x} \geq 2\mu/(\sqrt{3}k)$

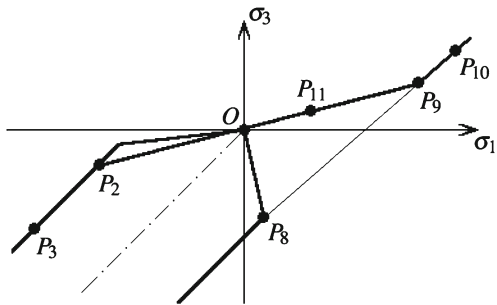
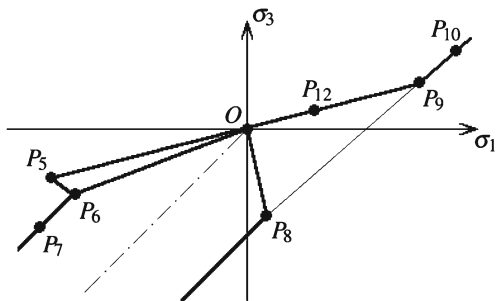


Fig. 6.18 Trajectories of stresses:
 $\sqrt{3}/2 < \mathfrak{x} < 2\mu/(\sqrt{3}k)$



$$s_1 = \left(k + \frac{2\mu}{\sqrt{3}\mathfrak{x}}\right)\varepsilon_1 + \frac{2\tau_s}{\sqrt{3}}\left(1 + \frac{\mu}{\mathfrak{x}^2k}\right),$$

$$s_3 = \left(k - \frac{\mu}{\sqrt{3}\mathfrak{x}}\right)\varepsilon_1 - \frac{\tau_s}{\sqrt{3}}\left(1 + \frac{\mu}{\mathfrak{x}^2k}\right),$$

actual stresses do not vary remaining at the point P_8 and are determined by the formulae (6.46) for $\varepsilon_1 = \varepsilon_1^9$.

In Figs. 6.17 and 6.18 the trajectory of $s(\varepsilon_1)$ for $\varepsilon_1 > 0$ is shown by the broken line OP_9P_{10} and the trajectory of $\sigma(\varepsilon_1)$ is shown by the segment OP_8 . Here two different variants of a trajectory of elastic compression with respect to the von Mises–Schleicher cone are presented. In the first case, where $\mathfrak{x} \geq 2\mu/(\sqrt{3}k)$, the continuation of the ray OP_9 lies inside the cone. In the second case, where the inequality has an opposite sign, there is no intersection. The graphs of the dependence $\sigma_1(\varepsilon_1)$ for both cases are given in Figs. 6.19 and 6.20.

Consider the configurations of shock-wave transitions in each case. If a material is in the compressed state ($\varepsilon_1^0 \leq 0$) before a front, then one of the systems of waves described above can be propagated through it. In the case of the weak initial loosening, an elastic precursor is replaced by an elastic signoton whose equation differs from (6.40) by stress $\sigma_1^0 > 0$ before a front. The qualitative wave pattern remains the same except that there arise two new waves of compression from a loosened state to a loosened one. They are an elastic wave propagating with the velocity

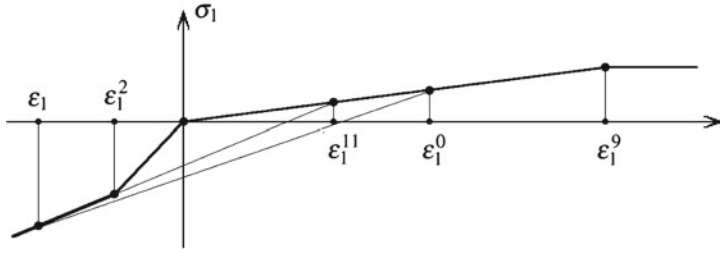


Fig. 6.19 Dependence of σ_1 versus ε_1 : $\varkappa > \sqrt{3}/2$, $\varkappa \geq 2\mu/(\sqrt{3}k)$

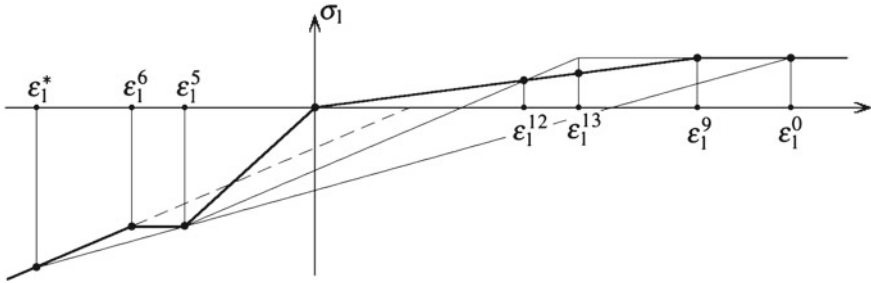


Fig. 6.20 Dependence of σ_1 versus ε_1 : $\sqrt{3}/2 < \varkappa < 2\mu/(\sqrt{3}k)$

$$c_p'' = \sqrt{\frac{\psi(1 - 2\varkappa/\sqrt{3})}{\rho}}$$

and a wave of the elastic unloading joining the state of plastic tension and the elastic state.

Omitting routine calculations, we give expressions for characteristic values of strain which enable one to construct easily discontinuous solutions for any degree of compression in a state behind a front from the graphs shown in Figs. 6.19 and 6.20.

In the first variant (Fig. 6.19) strain at the point P_{11} of intersection of the straight line of plastic deformation and the two-element broken line describing elastic-plastic tension is given by

$$\varepsilon_1^{11} = \begin{cases} \frac{\sqrt{3}\tau_s}{2\beta\varkappa} \frac{1 + 2\varkappa/\sqrt{3}}{\sqrt{3} - \varkappa(1 - 3k/(4\mu))}, & \text{if } \varkappa\left(1 - \frac{3k}{8\mu}\right) < \frac{3\sqrt{3}}{4}, \\ \frac{\tau_s}{k\varkappa} \left(\frac{4\varkappa}{\sqrt{3}} - 1\right), & \text{if } \varkappa\left(1 - \frac{3k}{8\mu}\right) \geq \frac{3\sqrt{3}}{4}. \end{cases}$$

Two cases arise here due to the fact that the required point of intersection can belong to the domain of elastic tension as well as to the domain of plastic one, where $\sigma_1 = \sigma_1^8$. Only for $\varepsilon_1^0 > \varepsilon_1^{11}$ a solitary plastic signoton can arise.

In the second variant (Fig. 6.20) two cases are distinguished as well. In the first case, where $\varepsilon_1^0 \leq \varepsilon_1^9$, compressive strain, such that velocity of a signoton–precursor becomes equal to velocity of a plastic compaction wave, is given by

$$\varepsilon_1^* = \varepsilon_1^5 \frac{(3 - 2\alpha/\sqrt{3})\varepsilon_1^0 + (1 + 2\alpha/\sqrt{3})\varepsilon_1^5}{2\omega(\varepsilon_1^0 - \varepsilon_1^{12})}, \quad \omega = 1 - \frac{\alpha}{\sqrt{3}} \left(1 - \frac{3k}{4\mu}\right),$$

where ε_1^{12} is the limit value of ε_1^0 as $\varepsilon_1^* \rightarrow \infty$:

$$\varepsilon_1^{12} = -\varepsilon_1^5 \frac{\sqrt{3} + \alpha(1 - 3k/(4\mu))}{\sqrt{3} - \alpha(1 - 3k/(4\mu))}.$$

A solitary plastic signoton is observed for $\varepsilon_1^0 \geq \varepsilon_1^{12}$ and $\varepsilon_1 \leq \varepsilon_1^*$. The condition $\varepsilon_1^{12} < \varepsilon_1^9$ leads to the inequality

$$\alpha^2 \left(1 - \frac{3k}{4\mu}\right) < \frac{3}{2}. \quad (6.47)$$

If this inequality is violated, then a plastic signoton for $\varepsilon_1^0 \leq \varepsilon_1^9$ does not arise for any value of ε_1 . This takes place, for example, where the compression modulus of a material is sufficiently small. In the second case ($\varepsilon_1^0 > \varepsilon_1^9$) we have

$$\varepsilon_1^* = \varepsilon_1^6 \frac{\varepsilon_1^0 - (1 - 4\alpha/\sqrt{3})\varepsilon_1^5}{\varepsilon_1^0 - \varepsilon_1^{13}}, \quad \varepsilon_1^{13} = -\frac{4\alpha\varepsilon_1^6}{\sqrt{3}} + \varepsilon_1^5.$$

The inequality (6.47) is violated if $\varepsilon_1^{13} \geq \varepsilon_1^9$. With this conditions of the appearance of a plastic signoton are $\varepsilon_1^0 \geq \varepsilon_1^{13}$, $\varepsilon_1 \leq \varepsilon_1^*$.

6.5 Cumulative Interaction of Signotons

We present results of computations for plane signotons on the basis of a spatially one-dimensional model. The numerical results were compared with the exact solutions from the previous section. In particular, velocities of propagation of elastic and plastic signotons defined by Eqs. (6.41) and (6.43) are in good correspondence in the case, where the coefficient α of internal friction lies in the range of $2\mu/(\sqrt{3}k) \leq \alpha \leq \sqrt{3}/2$. The strain profiles in the case of a two-wave configuration involving an elastic signoton–precursor and a plastic shock wave (for $0 < \varepsilon_1^0 \leq \varepsilon_1^1$) and in the case of a solitary plastic signoton ($\varepsilon_1^0 > \varepsilon_1^1$) are shown in Figs. 6.21 and 6.22, respectively. The exact solutions are shown by thin lines. The shock adiabats of compression waves corresponding to Figs. 6.21 and 6.22 are given in Figs. 6.9 and 6.10, respectively.

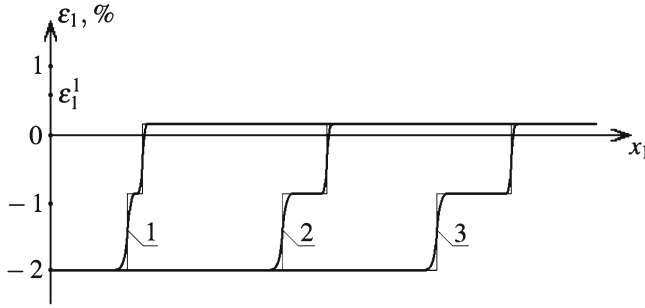


Fig. 6.21 Elastic signoton-precursor and a plastic shock wave

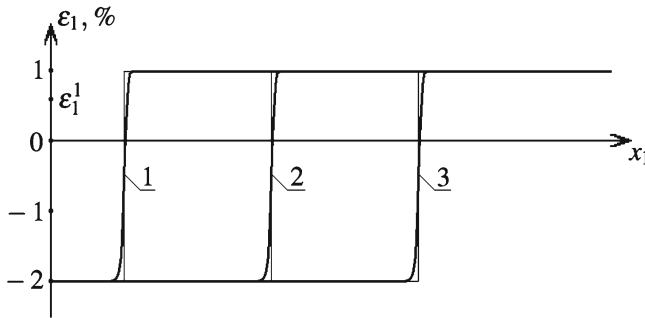
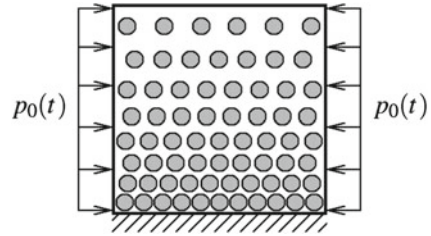
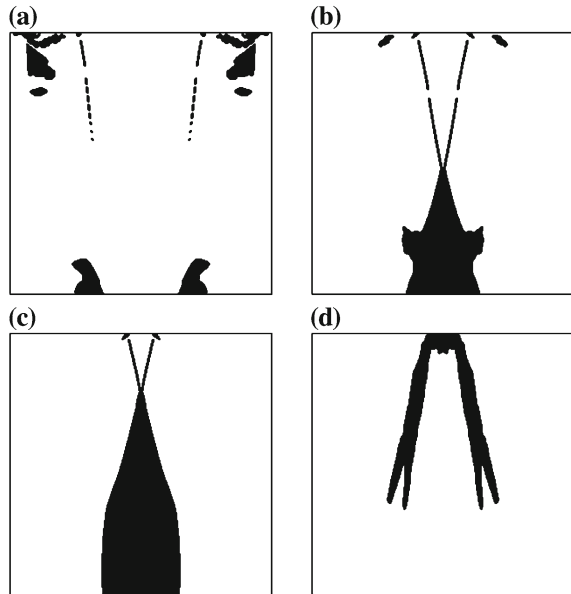


Fig. 6.22 Solitary plastic signoton

Computations were performed after the dimensionless of the model in the conventional form: $\tilde{x} = x/H$, $\tilde{t} = c_s t/H$, $\tilde{v} = v/c_s$, $\tilde{s} = s/\mu$, $\tilde{\sigma} = \sigma/\mu$ (H is the characteristic length scale, $c_s = \sqrt{\mu/\rho}$ is the velocity of transverse elastic waves), on an uniform grid consisting of 300 nodes for the following parameters: $k/\mu = 2$, $\tau_s/\mu = 0.01$, $\alpha = 0.7$. The graphs corresponding to instants 0.1, 0.3, and 0.5 of dimensionless time are marked by numbers 1, 2, and 3, respectively. The Courant parameter $c_p \Delta t/\Delta x$ is taken equal to 0.75. Notice that the limiting value of this parameter with respect to the stability condition is equal to 1 for a one-dimensional scheme and is equal to 2 in the two-dimensional case since the one-dimensional systems of Eqs. (6.26) are solved at splitting stages for the step $\Delta t/2$. Numerical experiments shows that an approximate solution weakly depends on the value of the Courant parameter which is characteristic for monotone schemes of more than first order of accuracy. In particular, moving fronts of elastic shock waves (of elastic precursor, of elastic signoton) are smeared over 2–3 meshes and their shape does not essentially vary with time. The width of a plastic shock waves considerably depends on the parameter ξ which is involved in the correction formulae (6.25) and for $\xi \rightarrow 0.5$ it is reduced to 3–4 meshes. At the same time, slight oscillations of a numerical solution within 1–3 % arise near fronts. They are reduced as ξ increases to

Fig. 6.23 Loosened material**Fig. 6.24** Configurations of plastic zones:**a** $t = 0.23 H/c_s$,**b** $t = 0.35 H/c_s$,**c** $t = 0.39 H/c_s$,**d** $t = 0.47 H/c_s$ 

0.75 but with smearing increasing to 7–8 meshes. In addition, the smoothing effect can be achieved by decreasing the Courant parameter.

We also present results of two-dimensional computations for cumulative interaction of signotons on a grid consisting of 300×300 meshes. The problem is as follows: a rectangular body of a granular material, initially loosened in the x_1 direction by the linear law $\varepsilon_{11}^0 = \varepsilon_0 x_2/H$, is symmetrically loaded by lateral Π -shaped shock impulses (see Fig. 6.23). On the lower side the condition of a rigid wall is imposed and the upper side is assumed to be free of stresses. On the left-hand and right-hand sides constant and uniformly distributed pressure is given, it acts some time and as time passes unloading occurs and lateral sides become free boundaries. Dimensionless pressure and time of action of impulses used in calculations are 0.027 and 0.2, respectively, $\varepsilon_0 = 0.012$. In Fig. 6.24 plastic zones in a body of a material are presented (shown by black), in Fig. 6.25 level curves of pressure are given (hatching becomes denser as pressure increases), and in Fig. 6.26 vector fields of velocities are represented at the next instants of dimensionless time: 0.23, 0.35, 0.39, and 0.47.

Fig. 6.25 Level curves of a pressure: **a** $t = 0.23 H/c_s$,
b $t = 0.35 H/c_s$,
c $t = 0.39 H/c_s$,
d $t = 0.47 H/c_s$

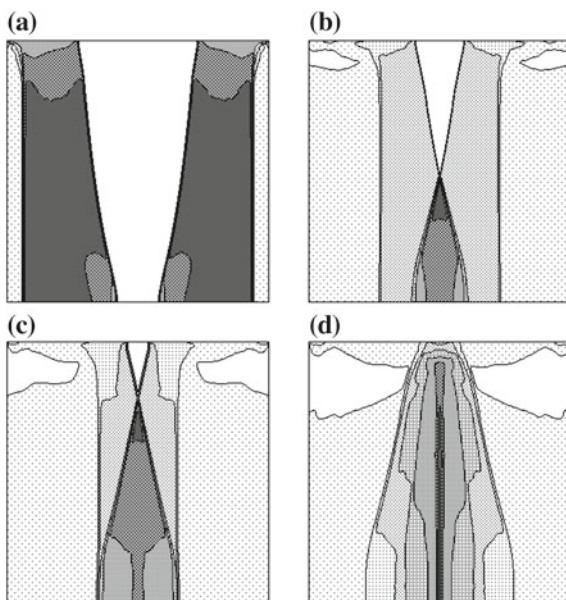
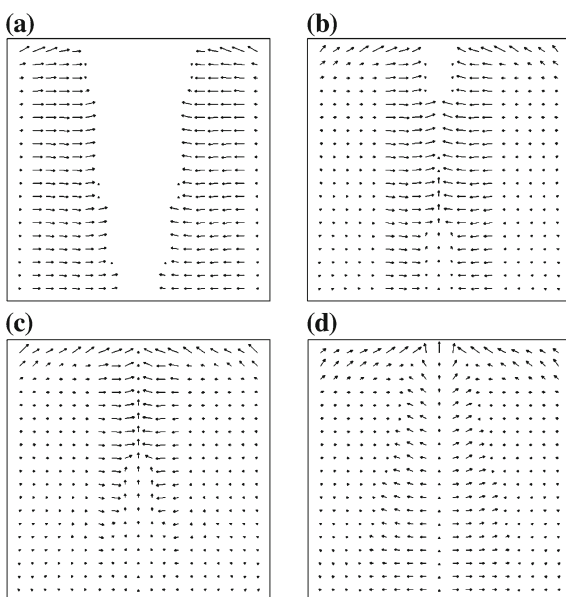


Fig. 6.26 Vector fields of velocities: **a** $t = 0.23 H/c_s$,
b $t = 0.35 H/c_s$,
c $t = 0.39 H/c_s$,
d $t = 0.47 H/c_s$



The qualitative pattern of the deformation process is as follows. Propagating in an inhomogeneous loosened material, plane fronts of shock waves (signotons) are gradually curved and slow down in a domain of strong loosening in comparison with

a denser domain. Unloading waves follows signotons over a compressed material where velocities of waves are constant, hence, their fronts remain almost plane up to the instant of meeting. This can be observed from the concentration of level curves of pressure in Fig. 6.25. At the point of meeting of signotons near the lower boundary, as a result of interaction of curved fronts, a cumulative splash arises which moves vertically upward with time (see Fig. 6.26).

It seems likely that the mechanism of cumulation is identical to that for oblique collision of plates [45]. When describing the yield of a cumulative jet on a free boundary, this model unsuitable. It is necessary to take into account finite strains of a material.

Judging by the results, the shock-capturing method provides satisfactory accuracy of a numerical solution of one- and two-dimensional problems of propagation of elastic–plastic waves of small amplitude in a granular material.

6.6 Periodic Disturbing Loads

Interaction of waves in a granular material may lead to spontaneous formation of the displacements discontinuities (voids bounded by free surfaces) inside a body. Results of computations for the one-dimensional problem on the interaction of two unloading waves propagating in a compressed material are shown in Fig. 6.27. For the taken values of the parameters, unloading waves are the elastic longitudinal waves (Fig. 6.27a) moving towards each other. At the point of their meeting the decay of discontinuity happens with formation of the displacements discontinuity, i.e. connected undeformed parts of a material fly away in opposite directions. With time, strain increases infinitely on the discontinuity (Fig. 6.27b).

To analyze a numerical solution, within the framework of the regularized model used in the computations we obtain the exact solution of the Riemann problem on decay of an arbitrary discontinuity without considering plasticity where a homogeneous state (v_1^-, ε_1^-) to the left of the plane of initial discontinuity and a state (v_1^+, ε_1^+) to the right lie in the domain of elastic strain of a material. This problem is solved in the class of piecewise-constant functions with the help of the strong discontinuity relationships (6.35):

$$\rho c(v_1 - v_1^0) + \sigma_1 - \sigma_1^0 = 0, \quad c(\varepsilon_1 - \varepsilon_1^0) + v_1 - v_1^0 = 0.$$

Here, as before, the superscript “0” marks quantities before the front of strong discontinuity and quantities without superscript are related to the state behind the front.

In the case of $\varepsilon_1^0 \leq 0$ admissible shock-wave transitions on the $\varepsilon_1 v_1$ plane are described by two two-element broken lines shown in Fig. 6.28. The elements of the decreasing broken line correspond to pair waves moving in the x_1 direction with velocities c_p and $\hat{c}_p = \sqrt{\zeta} c_p$ (ζ is the parameter of regularization) of longitudinal elastic waves in compressed and loosened materials, respectively. Clearly, the second wave occurs only if $\varepsilon_1 > 0$. The increasing broken line describes processes

Fig. 6.27 Interaction of unloading waves: **a** strain distribution before meeting of wave fronts, **b** strain distribution after meeting of wave fronts

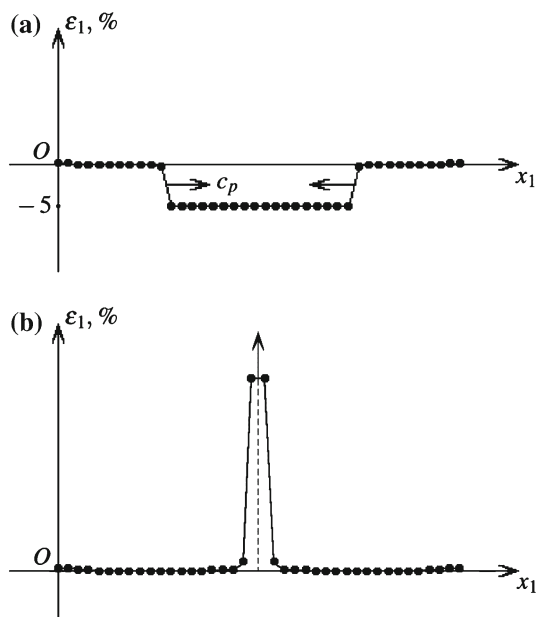
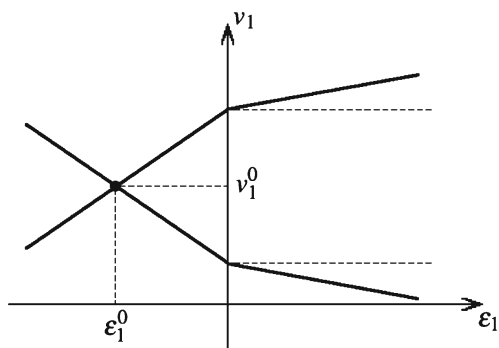


Fig. 6.28 Diagrams of shock-wave transitions: $\varepsilon_1^0 \leq 0$

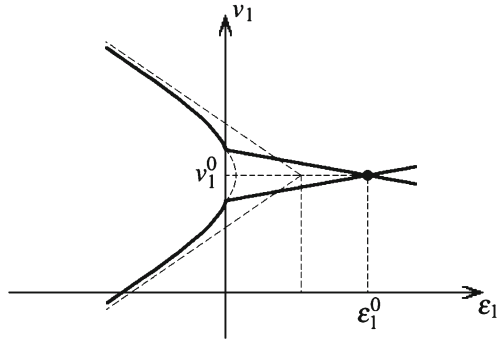


propagating in the opposite direction. The equations of the elements of the broken lines have the form

$$v_1 = v_1^0 \mp c_p (\varepsilon_1 - \varepsilon_1^0), \quad v_1 = v_1^0 \mp (\hat{c}_p \varepsilon_1 - c_p \varepsilon_1^0).$$

The similar shock-wave diagrams for $\varepsilon_1^0 > 0$ are represented in Fig. 6.29. These are two-element curves being a conjugation of rays corresponding to longitudinal waves in a stretched material and branches of the hyperbola

Fig. 6.29 Diagrams of shock-wave transitions:
 $\varepsilon_1^0 > 0$



$$\left(\frac{v_1 - v_1^0}{c_p} \right)^2 = (\varepsilon_1 - \varepsilon_1^0)(\varepsilon_1 - \varepsilon_1^0).$$

The equation of the hyperbola follows from (6.36) assuming that $\varepsilon_1 < 0$. The rays corresponding to longitudinal waves in the compression state are its asymptotes. In this case, independently of sign of strain ε_1 , a solitary shock wave occurs. Its velocity is equal to the angular coefficient of the secant drawn at the point (ε_1, v_1) of the curve from the center (ε_1^0, v_1^0) . Velocity and strain of an element after the discontinuity decay are determined by the point of intersection of the increasing diagram with its center at (ε_1^-, v_1^-) and the decreasing diagram with its center at (ε_1^+, v_1^+) .

Passage to the limit with respect to the regularization parameter ς as $\varsigma \rightarrow 0$ enables one to construct shock diagrams for an ideal granular material. All possible configurations of the discontinuity decay in such a material are represented in Figs. 6.30, 6.31, 6.32 and 6.33. In Fig. 6.30 a configuration where two longitudinal waves arise is represented schematically on the upper half-plane. The corresponding wave pattern is shown at the upper right of Fig. 6.30. In the case of the discontinuity decay, given on the lower half-plane, two longitudinal waves occur as well but there is no the point of intersection of shock diagrams. This is the limit case of the discontinuity decay with two pair longitudinal waves in the regularized model. It is interpreted as the displacements discontinuity in a granular material. In Fig. 6.31 a configuration with two shock waves is shown on the upper half-plane, and a configuration without shock waves and with the displacements discontinuity of a material is shown on the lower half-plane. In Figs. 6.32 and 6.33 two similar configurations are presented: a longitudinal wave with the shock wave and a longitudinal wave with the displacements discontinuity.

Thus, for one-dimensional motion with plane waves, the displacements discontinuities occur in four of the eight qualitatively different variants of decay of an arbitrary discontinuity. It is clear that for oblique interaction of waves the displacements discontinuities in the model may arise along arbitrary stationary surfaces. For example, on a surface a positive jump of the normal component of a velocity vector

Fig. 6.30 Diagrams of shock-wave transitions in an ideal material: $\varepsilon_1^\pm \leq 0$

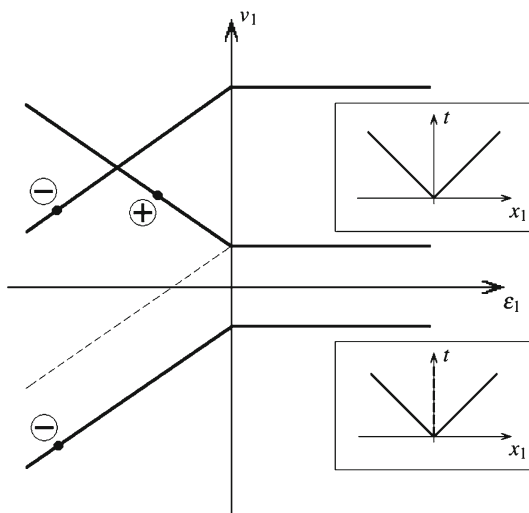
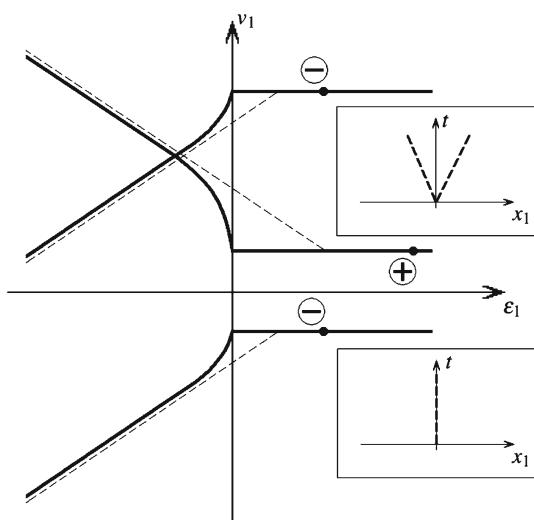


Fig. 6.31 Diagrams of shock-wave transitions in an ideal material: $\varepsilon_1^\pm > 0$



and an instantaneous stress-free state may occur due to the wave diffraction. The constitutive relationships in the form of the variational inequality (6.1) are close to the constitutive equations (6.5), hence, discontinuities can be considered as a result of the localization of tensile strains in narrow domains. It is natural from this point of view that in deformation process the displacements discontinuities in a material may collapse spontaneously.

Fig. 6.32 Diagrams of shock-wave transitions in an ideal material:
 $\varepsilon_1^- \leq 0, \varepsilon_1^+ > 0$

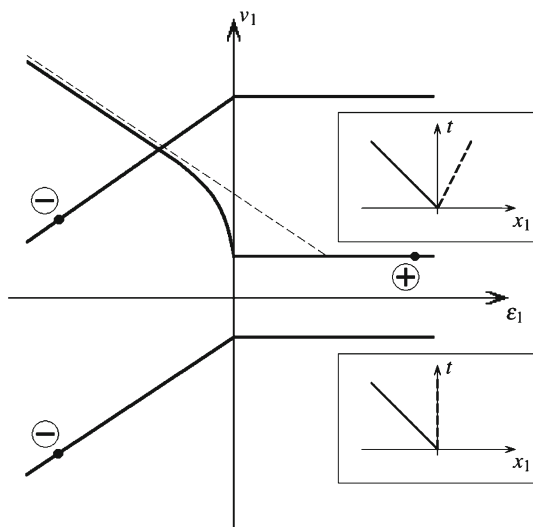
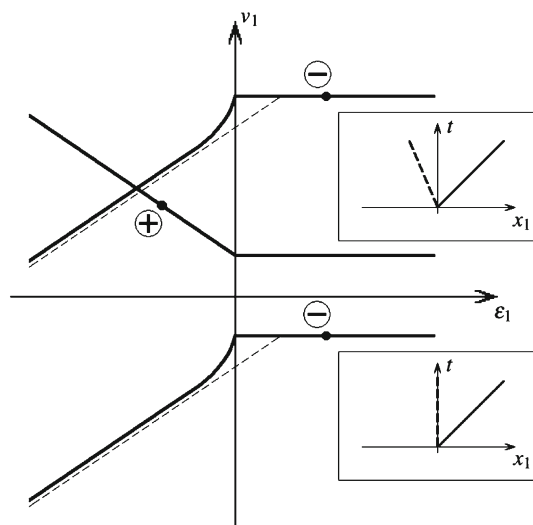


Fig. 6.33 Diagrams of shock-wave transitions in an ideal material:
 $\varepsilon_1^- > 0, \varepsilon_1^+ \leq 0$



In Fig. 6.34a a configuration of the displacements discontinuity in the problem of plane deformation of a rectangular body of a granular material is shown by level curves of the strain ε_{22} . The left boundary is a symmetry axis, on the right boundary the conditions of rigid wall are imposed, the upper boundary is free of stresses. Discontinuity is a result of the action of Π -shaped impulse of velocity on the lower boundary of a rectangle. The results are obtained on the basis of a variant of the model with the Coulomb–Mohr cone where the internal friction angle is $\pi/2$ [38].

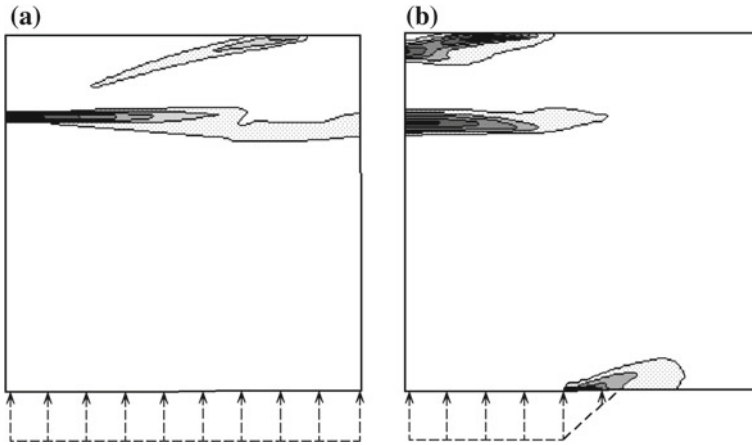


Fig. 6.34 Strain localization zones: **a** distributed load, **b** localized load

At the initial instant of time, in a neighborhood of the lower boundary of a body the loading wave arises. Away from the vertical boundaries it is a plane longitudinal wave. Near the boundaries a lateral unloading takes place due to which a solution of the problem is not one-dimensional. When meeting with the upper boundary, a loading wave transforms into a reflected wave which moves in the opposite direction. At the instant of a stress relief, an unloading wave moves from the lower boundary and its interaction with a reflected wave results in the discontinuity formation.

Similar results for the case of a spatial localized impulse of velocity are presented in Fig. 6.34b. The results show that the configurations of loading and unloading waves and the surfaces of the displacements discontinuity essentially depend on the degree of localization and on the shape of an impulse in time.

In Fig. 6.35 the results of numerical modeling of the “dry boiling” process described, for example, in [14, 28] are presented. This process takes under the action of the periodically repeating localized impulsive load in the gravity field, and it is a pattern of randomly arising displacements discontinuities (voids) in a material which collapse by turns with time. The results are obtained on a grid consisting of 100×100 nodes for three loading impulses and for the following values of dimensionless complexes consisting of mechanical and geometrical parameters:

$$\frac{c_p}{c_s} = 1.875, \quad \frac{v_0}{c_s} = 0.005, \quad \frac{g H}{c_s^2} = 0.005, \quad \frac{c_s t_0}{H} = 0.25,$$

where H is the height of a body, t_0 is the time of action of an impulse of velocity (loading half-period), v_0 is the given velocity of particles.

With some assumptions of the nature of loading, “dry boiling” can be described in the framework of a simplified model. Assume that the perturbation action is a

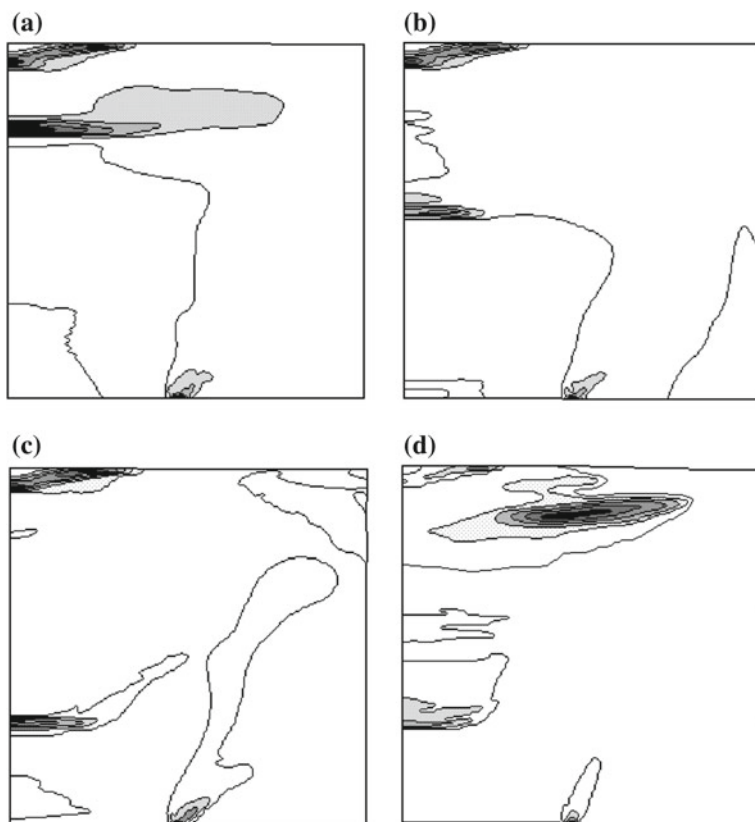
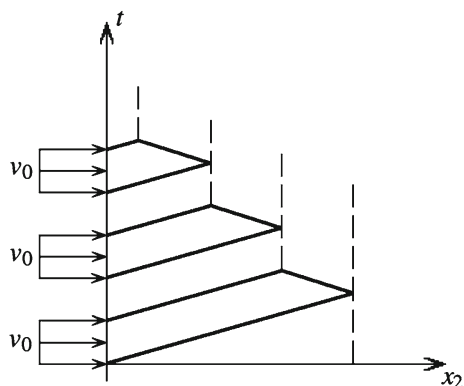


Fig. 6.35 Modeling of “dry boiling”: **a** $t = 0.7 H/c_s$, **b** $t = 1.07 H/c_s$, **c** $t = 1.44 H/c_s$, **d** $t = 1.81 H/c_s$

repeating sequence of Π -shaped impulses of velocity on the lower boundary of a body and that parameters of the problem satisfy the condition $c_p t_0 < 2H$, under which the first loading wave interacts with the unloading wave after one reflection from the upper free boundary.

If the weight of a material makes no visible influence, then the wave pattern of deformation corresponds to Fig. 6.36. At the point of meeting of waves the displacements discontinuity, shown as a dashed line, arises. Reflection of the next loading wave occurs not from the boundary of a body but from a free surface being formed. After interaction of the reflected wave and the unloading wave new displacements discontinuity arises. Thus, we have a system of discontinuities. According to the solution of the Riemann problem, relative velocity of banks of the displacements discontinuity at the instant of the discontinuity formation is equal to doubled velocity of action on the lower boundary of a body. The distance between the banks can be estimated by the formula $l(t) = 2v_0 t - g t^2/2$ of uniformly accelerated motion.

Fig. 6.36 Wave pattern

The condition of discontinuity formation at the point of the meeting of a reflected wave and an unloading wave has the form $v_0 \geq g t_0/2$. It provides the existence of free boundary in the time interval of reflection of the second loading wave. In the opposite case, with gravity taken into account, an elastic scheme of decay with two longitudinal waves takes place and the displacements discontinuity may arise only as one of waves moves towards a free surface. If this condition is satisfied, then the discontinuity does not collapse as long as the next loading wave arrives since $l(t_0) \geq g t_0^2/2 > 0$.

A large number of two-dimensional computations shows validity of the qualitative description of the process on the basis of the simplified one-dimensional model. A wave pattern significantly changes and becomes much more complicated when modeling the behaviour of a heavy material.

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Chapter 7

Contact Interaction of Layers

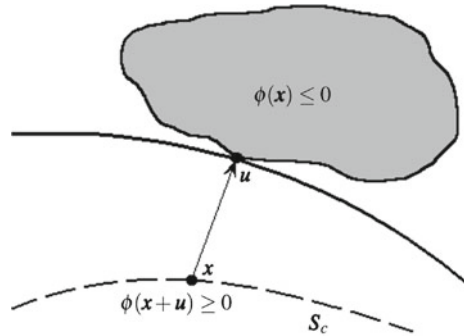
Abstract Algorithms for numerical implementation of conditions of dynamic contact interaction of deformable materials with a beforehand unknown zone of contact which varies in the process of motion are constructed. These algorithms take into account the influence of friction forces in a contact zone. On the basis of these algorithms a method for the numerical modeling of deformation of a body of a granular material in the presence of sliding surfaces is worked out. Results of testing an algorithm and results of the numerical solution of a problem for two layers of a medium consisting of an elastic-plastic material are presented. Computational algorithms are developed that simulate the dynamic interaction of elastic blocks through thin viscoelastic layers in structurally inhomogeneous media such as rocks.

7.1 Formulation of Contact Conditions

If at the initial stage of deformation a system of surfaces of localized strain is formed in a body of a granular material, then further they may serve as sliding surfaces, i.e. surfaces of discontinuity of the tangential component of a displacement vector. Depending on acting external loads, the normal component may be discontinuous as well. Thus, a gap of continuity is formed. This effect is observed, for example, when a reflected wave interacts with an unloading wave in an ideal granular material (see Sect. 6.6) or when a rarefaction wave is reflected from an interface with a rigid inclusion.

The modeling of motions of a granular material with sliding surfaces is a more simple problem than the modeling of developed flow in the general case since in this case there is no need to take into account finite strains of particles inside sliding layers. It is necessary only to describe adequately the behavior of particles on contact surfaces staying in the framework of the geometrically linear theory which takes into account finite displacements.

Fig. 7.1 Contact with a rigid inclusion



The problems of the construction of simplified models of mechanics of granular materials with sliding surfaces were considered in [22–26]. In this chapter we present a way of the modeling of contact interaction of layers with possible separation of contacting surfaces taken into account, keeping in mind the construction of universal computational technologies, [4, 5, 7, 27, 28]. This approach is based on the exact formulation of the conditions of contact of deformable bodies in the form of variational inequalities. This formulation is similar to the formulation arising in the Signorini contact problem, [32], for an elastic body and a smooth and absolutely rigid plane. A generalization of such approach to the case of contact of several deformable bodies with friction on contact surfaces taken into account is presented in [16–20].

For simplicity we first assume that one of layers is a fixed and absolutely rigid body (rigid inclusion) occupying a spacial domain $\phi(\mathbf{x}) \leq 0$ with a piecewise-smooth surface $\phi(\mathbf{x}) = 0$. We also assume that in an initial undeformed state of another (granular) layer we can separate the part of its boundary S_c such that its material points are in contact with the rigid body or are free of stresses at each following instant of time whereas on the remaining part of boundary the conditions of a rather general form are fulfilled but there is no contact with the body. The displacement vector $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ in the Lagrange coordinate system satisfies the geometric constraint $\phi(\mathbf{x} + \mathbf{u}) \geq 0$ on S_c (see Fig. 7.1). In Fig. 7.1 the position of the boundary of a layer at the initial instant of time is shown by a dashed line and the position of the boundary at the instant $t - \Delta t$, where Δt is a small time interval, is shown by a solid line. From the geometric constraint, taking into account the Taylor expansion

$$\mathbf{u} = \mathbf{u}|_{t-\Delta t} + \mathbf{v} \Delta t + O(\Delta t^2)$$

of the displacement vector, we obtain the approximate constraint on the velocity vector $\mathbf{v}$:

$$\mathbf{v} \cdot \nabla \phi(\mathbf{x} + \mathbf{u}|_{t-\Delta t}) \geq -\frac{\phi(\mathbf{x} + \mathbf{u}|_{t-\Delta t})}{\Delta t}. \quad (7.1)$$

This constraint is a condition of non-penetration of a deformable layer inside a body. It can be used in the numerical implementation of the contact conditions. Going to

the limit with respect to Δt as $\Delta t \rightarrow 0$, we obtain the exact constraint

$$\mathbf{v} \cdot \nabla \phi(\mathbf{x} + \mathbf{u}) \geq -\delta(\mathbf{x} + \mathbf{u}), \quad \delta(\mathbf{x}) = \begin{cases} 0, & \text{if } \phi(\mathbf{x}) = 0, \\ +\infty, & \text{if } \phi(\mathbf{x}) > 0. \end{cases} \quad (7.2)$$

Note that a normal vector to the surface of a deformable body in the contact zone $S_t = \{\mathbf{x} \in S_c \mid \phi(\mathbf{x} + \mathbf{u}) = 0\}$ is calculated by the formula

$$\mathbf{v} = -\frac{\nabla \phi(\mathbf{x} + \mathbf{u})}{|\nabla \phi(\mathbf{x} + \mathbf{u})|},$$

hence, in fact (7.1) and (7.2) involve only the normal component v_v of the velocity vector.

In the case of the full sliding, the boundary conditions of contact interaction are formulated as the equation

$$\sigma_v = \lambda \nabla \phi(\mathbf{x} + \mathbf{u}), \quad \lambda \geq 0. \quad (7.3)$$

Here $\sigma_v(t, \mathbf{x})$ is the stress vector acting on an area element of a deformed surface with normal $\mathbf{v}$, the scalar coefficient λ vanishes outside the contact zone S_t and is proportional to contact pressure in this zone. This equation means that, when sliding, there are no tangential friction stresses, i.e. the stress vector is directed along the normal.

Let $\delta \mathbf{v} = \tilde{\mathbf{v}} - \mathbf{v}$ denotes an arbitrary admissible variation of the velocity vector at a point of the boundary S_c for which the varying vector $\tilde{\mathbf{v}}$ satisfies the constraint (7.1). Multiplying both sides of Eq. (7.3) by the variation $\delta \mathbf{v}$, we obtain the identity

$$\sigma_v \cdot \delta \mathbf{v} = \frac{\lambda}{\Delta t} \delta \phi(\mathbf{x} + \mathbf{u}|_{t-\Delta t} + \mathbf{v} \Delta t), \quad (7.4)$$

since according to the variation rules

$$\delta \phi(\mathbf{x} + \mathbf{u}|_{t-\Delta t} + \mathbf{v} \Delta t) = \Delta t \delta \mathbf{v} \cdot \nabla \phi(\mathbf{x} + \mathbf{u}|_{t-\Delta t}).$$

Note that the right-hand side of identity (7.4) is non-negative in the case of contact, where

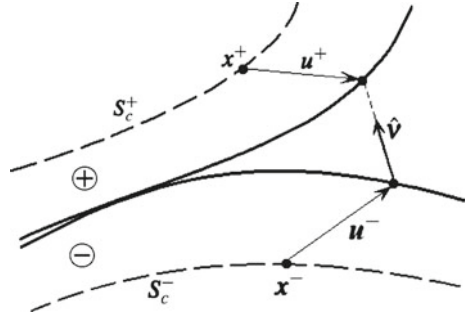
$$\phi(\mathbf{x} + \mathbf{u}|_{t-\Delta t} + \mathbf{v} \Delta t) = 0, \quad \phi(\mathbf{x} + \mathbf{u}|_{t-\Delta t} + \tilde{\mathbf{v}} \Delta t) \geq 0,$$

as well as in the absence of contact, where $\lambda = 0$. Thus, in the general case the inequality

$$\sigma_v \cdot (\tilde{\mathbf{v}} - \mathbf{v}) \geq 0 \quad (7.5)$$

holds which means that the virtual power $\tilde{\mathbf{v}} \cdot \sigma_v$ of surface stresses achieves its minimal value on the actual velocity vector $\tilde{\mathbf{v}} = \mathbf{v}$.

Fig. 7.2 Constraint in the contact zone of two deformable bodies



With friction taken into account, following [8, 18, 19], we can formulate the boundary contact conditions as

$$\sigma_v \cdot (\tilde{v} - v) + f |\sigma_{vv}| (|\tilde{v}_\tau| - |v_\tau|) \geq 0. \quad (7.6)$$

Here f is the coefficient of the sliding friction, the subscript τ denotes projections of vectors onto a tangential plane, σ_{vv} is the normal projection of σ_v , $\tilde{v} = \tilde{v}(t, x)$ is an arbitrary admissible variation of the velocity vector at a point of the boundary S_c which, along with the vector v , satisfies the constraint (7.2).

Generally speaking, the inequalities (7.5) and (7.6) are quasi-variational rather than variational ones since the constraints (7.1) and (7.2) depend on an unknown solution u . The latter inequality is a fundamental principle according to which virtual power of normal stress in a contact zone, equal to the difference between the power $\sigma_v \cdot \tilde{v}$ of surface stresses and the power $-f |\sigma_{vv}| |\tilde{v}_\tau|$ of friction forces, takes its minimal value on the actual velocity vector $\tilde{v} = v$. Below we prove that the inequality (7.6) is equivalent to the Amontons–Coulomb friction law.

The boundary contact conditions for two deformable bodies can be formulated in a similar way (Fig. 7.2). Assume that S_c^+ and S_c^- are the parts of boundaries of these bodies in the Lagrange variables which include the whole contact zones

$$S_t^\pm = \{x^\pm \in S_c^\pm \mid x^+ + u^+(t, x^+) = x^- + u^-(t, x^-)\}$$

at each fixed instant t . Here the superscripts $\pm$ denote quantities related to different layers. In Fig. 7.2 the position of the boundaries of layers at the initial instant of time is shown by dashed lines and that of at the instant $t - \Delta t$ is shown by solid lines. An approximate constraint on velocities of the points $x^+ \in S_c^+$ and $x^- \in S_c^-$ can be written in the following way:

$$(\tilde{v}^- - \tilde{v}^+) \cdot \hat{n} \leq \frac{1}{\Delta t} |x^+ + u^+_{|t-\Delta t} - x^- - u^-_{|t-\Delta t}|. \quad (7.7)$$

This constraint is a condition that does not allow the layers to penetrate into one another. The boundary points involved in (7.7) are assumed to be related by a one-to-one correspondence that depends on time as on a parameter. With this

correspondence, a point $\mathbf{x}^+$ of the contact zone S_t^+ is related to the point $\mathbf{x}^-$ of S_t^- determined by equality of positions at an actual instant of time:

$$\mathbf{x}^+ + \mathbf{u}^+(t, \mathbf{x}^+) = \mathbf{x}^- + \mathbf{u}^-(t, \mathbf{x}^-).$$

In the noncontact domains $S_c^\pm \setminus S_t^\pm$ the correspondence may be arbitrary. The unit vector $\hat{\mathbf{v}}$ indicating a local direction of approach of bodies is given by

$$\hat{\mathbf{v}} = \begin{cases} \mathbf{v}^- = -\mathbf{v}^+, & \text{if } \mathbf{x}^\pm \in S_{t-\Delta t}^\pm, \\ \frac{\mathbf{x}^+ + \mathbf{u}_{|t-\Delta t}^+ - \mathbf{x}^- - \mathbf{u}_{|t-\Delta t}^-}{|\mathbf{x}^+ + \mathbf{u}_{|t-\Delta t}^+ - \mathbf{x}^- - \mathbf{u}_{|t-\Delta t}^-|}, & \text{if } \mathbf{x}^\pm \notin S_{t-\Delta t}^\pm. \end{cases}$$

The right-hand side of (7.7), which is proportional to the distance between points at the instant $t - \Delta t$, has the sense of maximal possible velocity of approach. When solving contact problems, the choice of a one-to-one correspondence for the approximate constraint (7.7) is one of the stages of constructing a computational algorithm.

The exact constraint on velocity vectors is obtained from (7.7) by going to the limit with respect to Δt as $\Delta t \rightarrow 0$:

$$\tilde{\mathbf{v}}^- \cdot \mathbf{v}^- + \tilde{\mathbf{v}}^+ \cdot \mathbf{v}^+ \leq \begin{cases} 0, & \text{if } \mathbf{x}^\pm \in S_t^\pm, \\ +\infty, & \text{if } \mathbf{x}^\pm \notin S_t^\pm. \end{cases} \quad (7.8)$$

In a similar way to (7.6), the conditions of contact interaction can be formulated as follows:

$$\begin{aligned} & \sigma_v^- \cdot (\tilde{\mathbf{v}}^- - \mathbf{v}^-) + \sigma_v^+ \cdot (\tilde{\mathbf{v}}^+ - \mathbf{v}^+) + \frac{1}{2} f |\sigma_{vv}^+ + \sigma_{vv}^-| \\ & \times (|\tilde{\mathbf{v}}_\tau^+ - \tilde{\mathbf{v}}_\tau^-| - |\mathbf{v}_\tau^+ - \mathbf{v}_\tau^-|) \geq 0. \end{aligned} \quad (7.9)$$

The inequality (7.9), as well as (7.6), can be interpreted as the minimum principle for power of normal stress in a contact zone. The equivalence of this principle to the Amontons–Coulomb friction law for two deformable layers can be established similarly to the case of contact of a deformable layer with an absolutely rigid inclusion. To the approximate formulation of conditions of contact interaction of bodies there corresponds the quasi-variational inequality obtained from (7.9) by replacement of normal stresses with stresses $\pm \sigma_v^\pm \cdot \hat{\mathbf{v}}$ and of the difference of tangential components of velocity vectors with the projection of the vector $\mathbf{v}^+ - \mathbf{v}^-$ onto a plane orthogonal to $\hat{\mathbf{v}}$:

$$\sigma_v^- \cdot (\tilde{\mathbf{v}}^- - \mathbf{v}^-) + \sigma_v^+ \cdot (\tilde{\mathbf{v}}^+ - \mathbf{v}^+) + \frac{1}{2} f |(\sigma_v^+ - \sigma_v^-) \cdot \hat{\mathbf{v}}| (|\tilde{\mathbf{v}}_\tau^+ - \tilde{\mathbf{v}}_\tau^-| - |\mathbf{v}_\tau^+ - \mathbf{v}_\tau^-|) \geq 0.$$

In this case the arbitrary variable vectors $\tilde{\mathbf{v}}^\pm$ as well as the vectors $\mathbf{v}^\pm$ of actual velocities satisfy the constraint (7.7).

We can prove that the quasivariational inequality (7.6) with the constraint (7.1) is equivalent to the Amontons–Coulomb friction law being the system of the following

conditions:

$$\begin{cases} \sigma_{vv} \leq 0, & |\sigma_{v\tau}| \leq f |\sigma_{vv}|, \\ \sigma_{vv} = 0, & \sigma_{v\tau} = \mathbf{0}, \\ |\sigma_{v\tau}| = f |\sigma_{vv}| \text{ and } \frac{\sigma_{v\tau}}{|\sigma_{v\tau}|} = -\frac{\mathbf{v}_\tau}{|\mathbf{v}_\tau|}, \end{cases} \quad \begin{array}{l} \text{if } \mathbf{v} \cdot \mathbf{v} < h, \\ \text{if } \mathbf{v} \cdot \mathbf{v} = h \text{ and } \mathbf{v}_\tau \neq \mathbf{0}. \end{array} \quad (7.10)$$

Here $\mathbf{v} = -\frac{\nabla\phi(\mathbf{x}+\mathbf{u}|_{t-\Delta t})}{|\nabla\phi(\mathbf{x}+\mathbf{u}|_{t-\Delta t})|}$ and $h = \frac{\phi(\mathbf{x}+\mathbf{u}|_{t-\Delta t})}{|\nabla\phi(\mathbf{x}+\mathbf{u}|_{t-\Delta t})| \Delta t}$ are the outward normal vector to the surface of a deformable layer and the maximal admissible velocity of a point of contact surface, respectively. The condition $\mathbf{v} \cdot \mathbf{v} \leq h$ is consistent with (7.1). The first group of the formulae in (7.10) defines the constraint on stresses in a contact zone for any interaction mode, the second group corresponds to the conditions of a free surface in the absence of contact, and the third group is related to the contact with sliding. Contact interaction without sliding takes place in the case of $\mathbf{v} \cdot \mathbf{v} = h$ and $\mathbf{v}_\tau = \mathbf{0}$.

Assume that the inequality (7.6) holds. Then the minimum principle for virtual power mentioned above is valid. It is equivalent to the problem on the unconstrained minimum of the Lagrangian

$$L(\tilde{\mathbf{v}}, \lambda) = \sigma_v \cdot \tilde{\mathbf{v}} + f |\sigma_{vv}| |\tilde{\mathbf{v}}_\tau| + \lambda (\tilde{\mathbf{v}} \cdot \mathbf{v} - h),$$

where $\lambda \geq 0$ is the Lagrange multiplier corresponding to the constraint (7.1) and vanishing in the absence of contact. Taking into account that

$$\sigma_v \cdot \tilde{\mathbf{v}} = \sigma_{vv} \tilde{v}_v + \sigma_{v\tau} \cdot \tilde{\mathbf{v}}_\tau,$$

we obtain

$$L(\tilde{\mathbf{v}}, \lambda) = (\sigma_{vv} + \lambda) \tilde{v}_v + \sigma_{v\tau} \cdot \tilde{\mathbf{v}}_\tau + f |\sigma_{vv}| |\tilde{\mathbf{v}}_\tau| - \lambda h.$$

The function L involves the nondifferentiable term $|\tilde{\mathbf{v}}_\tau|$, hence, when analyzing the unconstrained minimum, it is necessary to consider two cases: $|\mathbf{v}_\tau| > 0$ and $\mathbf{v}_\tau = \mathbf{0}$.

First we assume that $|\mathbf{v}_\tau| > 0$, i.e. sliding takes place. Then the minimum conditions in the differential form are valid:

$$\frac{\partial L}{\partial v_v} = \sigma_{vv} + \lambda = 0, \quad \frac{\partial L}{\partial \mathbf{v}_\tau} = \sigma_{v\tau} + f |\sigma_{vv}| \frac{\mathbf{v}_\tau}{|\mathbf{v}_\tau|} = \mathbf{0}.$$

Thus, in the case of $\mathbf{v}_\tau \neq \mathbf{0}$ the inequality (7.6) results in the equations

$$\sigma_{vv} = -\lambda, \quad \sigma_{v\tau} = -\frac{f |\sigma_{vv}| \mathbf{v}_\tau}{|\mathbf{v}_\tau|}, \quad |\sigma_{v\tau}| = f |\sigma_{vv}|.$$

If $\lambda = 0$, then from these equations it follows that $\sigma_{vv} = 0$ and $\sigma_{v\tau} = \mathbf{0}$, i.e. in the noncontact zone the conditions of a part of a boundary being free of stresses

are valid. In the contact zone, where $\lambda > 0$, normal stress is compressive and the tangential stress vector has the opposite direction to the tangential velocity vector.

Now assume that $\mathbf{v}_\tau = \mathbf{0}$. In this case a Lagrangian derivative does not exist. Denote an arbitrary vector by $\tilde{\mathbf{v}}$. From the minimum condition $L(\tilde{\mathbf{v}}, \lambda) \geq L(\mathbf{v}, \lambda)$ we have

$$(\sigma_{vv} + \lambda) \tilde{v}_v + \sigma_{v\tau} \cdot \tilde{\mathbf{v}}_\tau + f |\sigma_{vv}| |\tilde{\mathbf{v}}_\tau| \geq (\sigma_{vv} + \lambda) v_v.$$

If we take $\tilde{v}_v = v_v$ and $\tilde{\mathbf{v}}_\tau = -\sigma_{v\tau}$, then from the last inequality we obtain

$$-|\sigma_{v\tau}|^2 + f |\sigma_{vv}| |\sigma_{v\tau}| \geq 0.$$

Hence, $|\sigma_{v\tau}| \leq f |\sigma_{vv}|$. For arbitrary $\tilde{v}_v$ and $\tilde{\mathbf{v}}_\tau = \mathbf{0}$ this inequality yields $\sigma_{vv} = -\lambda$. Thus, for $\mathbf{v}_\tau = \mathbf{0}$ all formulae of the system (7.10) related to this case are valid.

Now assume that the system (7.10) holds. To prove the fact that in this case the inequality (7.6) is valid, it is sufficient to establish that the Lagrange function $L(\tilde{\mathbf{v}}, \lambda)$ achieves its absolute minimum on the actual velocity vector $\mathbf{v}$. Let λ denotes the following quantity: $\lambda = -\sigma_{vv} \geq 0$. If there is no contact, then due to (7.10) $\sigma_{vv} = 0$, $\sigma_{v\tau} = \mathbf{0}$ and $\lambda = 0$. The minimum condition is valid since the Lagrange function is identically equal to zero. If contact takes place, then it is necessary to consider two cases: $\mathbf{v}_\tau = \mathbf{0}$ and $|\mathbf{v}_\tau| > 0$. In both cases the following formula is valid:

$$L(\tilde{\mathbf{v}}, \lambda) - L(\mathbf{v}, \lambda) = (\sigma_{vv} + \lambda)(\tilde{v}_v - v_v) + \sigma_{v\tau} \cdot (\tilde{\mathbf{v}}_\tau - \mathbf{v}_\tau) + f |\sigma_{vv}| (|\tilde{\mathbf{v}}_\tau| - |\mathbf{v}_\tau|).$$

In the first case, where $\mathbf{v}_\tau = \mathbf{0}$, taking into account the notation for λ we have

$$L(\tilde{\mathbf{v}}, \lambda) - L(\mathbf{v}, \lambda) = \sigma_{v\tau} \cdot \tilde{\mathbf{v}}_\tau + f |\sigma_{vv}| |\tilde{\mathbf{v}}_\tau| \geq -|\sigma_{v\tau}| |\tilde{\mathbf{v}}_\tau| + f |\sigma_{vv}| |\tilde{\mathbf{v}}_\tau|.$$

Hence, $L(\tilde{\mathbf{v}}, \lambda) - L(\mathbf{v}, \lambda) \geq 0$, i.e. the quantity $L(\mathbf{v}, \lambda)$ is minimal. In the second case, where $\mathbf{v}_\tau \neq \mathbf{0}$,

$$\begin{aligned} L(\tilde{\mathbf{v}}, \lambda) - L(\mathbf{v}, \lambda) &= -|\sigma_{v\tau}| (\tilde{\mathbf{v}}_\tau - \mathbf{v}_\tau) \cdot \frac{\mathbf{v}_\tau}{|\mathbf{v}_\tau|} + f |\sigma_{vv}| (|\tilde{\mathbf{v}}_\tau| - |\mathbf{v}_\tau|) \\ &= f |\sigma_{vv}| \left(-(\tilde{\mathbf{v}}_\tau - \mathbf{v}_\tau) \cdot \frac{\mathbf{v}_\tau}{|\mathbf{v}_\tau|} + |\tilde{\mathbf{v}}_\tau| - |\mathbf{v}_\tau| \right) \geq 0. \end{aligned}$$

The fact that the last expression in parentheses is nonnegative for an arbitrary vector $\tilde{\mathbf{v}}_\tau$ is proved basing on obvious property of convexity of the function $\omega(\mathbf{v}) = |\mathbf{v}_\tau|$. Indeed, due to the inequality

$$\omega(\tilde{\mathbf{v}}) - \omega(\mathbf{v}) \geq (\tilde{\mathbf{v}} - \mathbf{v}) \cdot \frac{\partial \omega(\mathbf{v})}{\partial \mathbf{v}},$$

which is valid for an arbitrary differential convex function, we have

$$|\tilde{\mathbf{v}}_\tau| - |\mathbf{v}_\tau| \geq (\tilde{\mathbf{v}}_\tau - \mathbf{v}_\tau) \cdot \frac{\mathbf{v}_\tau}{|\mathbf{v}_\tau|}.$$

Thus, the function L takes its minimum value provided that the relations (7.10) hold. Hence, by the Kuhn–Tucker theorem (see Sect. 3.4) the minimum principle (7.6) for virtual power under the constraint (7.1) is valid.

This proof of equivalence of the conditions (7.10) and the inequality (7.6) can be easily extended to the nonlinear friction law with the condition $|\sigma_{v\tau}| \leq \varphi(\sigma_{vv})$, where φ is a rather arbitrary nonnegative function satisfying the condition $\varphi(0) = 0$ and the Lipschitz condition

$$|\varphi(\tilde{\sigma}_{vv}) - \varphi(\sigma_{vv})| \leq f |\tilde{\sigma}_{vv} - \sigma_{vv}|$$

with a constant f . A generalization of the Amontons–Coulomb law:

$$\left\{ \begin{array}{ll} \sigma_{vv} \leq 0, & |\sigma_{v\tau}| \leq f |\sigma_{vv}|, \quad |\sigma_{v\tau}| \leq \tau_s, \\ \sigma_{vv} = 0, & \sigma_{v\tau} = \mathbf{0}, \\ |\sigma_{v\tau}| = f |\sigma_{vv}| \text{ or } |\sigma_{v\tau}| = \tau_s & \end{array} \right. \quad \text{if } \mathbf{v} \cdot \mathbf{v} < h, \quad (7.11)$$

$$\text{and } \frac{\sigma_{v\tau}}{|\sigma_{v\tau}|} = -\frac{\mathbf{v}_\tau}{|\mathbf{v}_\tau|}, \quad \text{if } \mathbf{v} \cdot \mathbf{v} = h \text{ and } \mathbf{v}_\tau \neq \mathbf{0},$$

where τ_s is the yields point of a material for shear, is an example of the nonlinear friction law. In this case the function φ has the form

$$\varphi(\sigma_{vv}) = \min \{f |\sigma_{vv}|, \tau_s\}.$$

When modeling contact of a deformable layer and a rigid body, instead of the quasi-variational inequality (7.6) one should take the similar inequality

$$\sigma_v \cdot (\tilde{\mathbf{v}} - \mathbf{v}) + \varphi(\sigma_{vv})(|\tilde{\mathbf{v}}_\tau| - |\mathbf{v}_\tau|) \geq 0. \quad (7.12)$$

Note that the friction law (7.10) can be applied only for the solution of contact problems in the framework of the elasticity theory. When solving elastic-plastic problems, it may lead to absurd results from the physical point of view. Since in the case of ideal plasticity the tangential stress does not exceed a yield point, in a sliding zone the normal stress turns out to be bounded: $|\sigma_{vv}| = |\sigma_{v\tau}|/f \leq \tau_s/f$. If contact pressure is higher than this limit, then sliding becomes impossible. Adhesion of contact surfaces is observed which has no reasonable explanation. In this case one should use the inequality (7.12).

Under the interaction of two elastic-plastic layers, to describe boundary conditions of contact, taking into account friction, instead of the inequality (7.9) we use the inequality

$$\begin{aligned} \sigma_v^- \cdot (\tilde{\mathbf{v}}^- - \mathbf{v}^-) + \sigma_v^+ \cdot (\tilde{\mathbf{v}}^+ - \mathbf{v}^+) + \varphi\left(\frac{\sigma_{v\hat{\mathbf{v}}}^+ - \sigma_{v\hat{\mathbf{v}}}^-}{2}\right) \\ \times \left(|\tilde{\mathbf{v}}_\tau^+ - \tilde{\mathbf{v}}_\tau^-| - |\mathbf{v}_\tau^+ - \mathbf{v}_\tau^-|\right) \geq 0, \end{aligned} \quad (7.13)$$

where $\varphi(\sigma_{vv}) = \min \{f |\sigma_{vv}|, \tau_s^-, \tau_s^+\}$. Taking into account obvious identity

$$\sigma_{\mathbf{v}} \cdot \tilde{\mathbf{v}} = (\sigma_{\mathbf{v}\hat{\mathbf{v}}} \hat{\mathbf{v}} + \sigma_{\mathbf{v}\hat{\mathbf{t}}} \hat{\mathbf{t}}) \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{v}}} \hat{\mathbf{v}} + \tilde{\mathbf{v}}_{\hat{\mathbf{t}}} \hat{\mathbf{t}}) = \sigma_{\mathbf{v}\hat{\mathbf{v}}} \tilde{\mathbf{v}}_{\hat{\mathbf{v}}} + \sigma_{\mathbf{v}\hat{\mathbf{t}}} \cdot \tilde{\mathbf{v}}_{\hat{\mathbf{t}}},$$

we write the corresponding Lagrange function

$$\begin{aligned} L(\tilde{\mathbf{v}}^-, \tilde{\mathbf{v}}^+, \lambda) = & \sigma_{\mathbf{v}\hat{\mathbf{v}}}^- \tilde{\mathbf{v}}_{\hat{\mathbf{v}}}^- + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- \cdot \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- + \sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ \tilde{\mathbf{v}}_{\hat{\mathbf{v}}}^+ + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ \cdot \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ \\ & + \varphi \left(\frac{\sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ - \sigma_{\mathbf{v}\hat{\mathbf{v}}}^-}{2} \right) |\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| + \lambda (\tilde{\mathbf{v}}_{\hat{\mathbf{v}}}^- - \tilde{\mathbf{v}}_{\hat{\mathbf{v}}}^+ - h). \end{aligned}$$

Here $\lambda \geq 0$ is the Lagrange multiplier corresponding to the constraint (7.7), and $h = |\mathbf{x}^+ + \mathbf{u}_{|t-\Delta t}^+ - \mathbf{x}^- - \mathbf{u}_{|t-\Delta t}^-| / \Delta t$. Further it is necessary to consider two cases: $|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-| > 0$ and $\mathbf{v}_{\hat{\mathbf{t}}}^- = \mathbf{v}_{\hat{\mathbf{t}}}^+$. In the first case the following minimum conditions for the Lagrangian in the differential form are fulfilled:

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{v}_{\hat{\mathbf{v}}}^-} &= \sigma_{\mathbf{v}\hat{\mathbf{v}}}^- + \lambda = 0, \quad \frac{\partial L}{\partial \mathbf{v}_{\hat{\mathbf{v}}}^+} = \sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ - \lambda = 0, \\ \frac{\partial L}{\partial \mathbf{v}_{\hat{\mathbf{t}}}^-} &= \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- - \varphi \left(\frac{\sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ - \sigma_{\mathbf{v}\hat{\mathbf{v}}}^-}{2} \right) \frac{\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-}{|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|} = 0, \\ \frac{\partial L}{\partial \mathbf{v}_{\hat{\mathbf{t}}}^+} &= \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ + \varphi \left(\frac{\sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ - \sigma_{\mathbf{v}\hat{\mathbf{v}}}^-}{2} \right) \frac{\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-}{|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|} = 0. \end{aligned}$$

Hence, for $\mathbf{v}_{\hat{\mathbf{t}}}^- \neq \mathbf{v}_{\hat{\mathbf{t}}}^+$ the inequality (7.13) results in the equations

$$\begin{aligned} \sigma_{\mathbf{v}\hat{\mathbf{v}}}^- &= -\lambda, \quad \sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ = \lambda, \\ \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- &= \varphi(\lambda) \frac{\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-}{|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|}, \quad \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ = -\varphi(\lambda) \frac{\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-}{|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|}, \\ |\sigma_{\mathbf{v}\hat{\mathbf{t}}}^-| &= |\sigma_{\mathbf{v}\hat{\mathbf{t}}}^+| = \varphi(\lambda), \quad \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ = \mathbf{0}. \end{aligned}$$

If $\lambda = 0$, then, taking into account that $\varphi(0) = 0$, from these equations we obtain the conditions of surfaces free of stresses:

$$\sigma_{\mathbf{v}\hat{\mathbf{v}}}^- = \sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ = 0, \quad \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- = \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ = \mathbf{0}.$$

In a contact zone $\lambda > 0$ normal stresses of interacting layers are equal in magnitude and opposite in sign, and vectors of tangential stresses and vectors of relative sliding velocities have opposite directions.

In the second case ($\mathbf{v}_{\hat{\mathbf{t}}}^- = \mathbf{v}_{\hat{\mathbf{t}}}^+ = \mathbf{v}_{\hat{\mathbf{t}}}$) there does not exist a derivative of the Lagrangian. Denote arbitrary vectors by $\tilde{\mathbf{v}}^-$ and $\tilde{\mathbf{v}}^+$. From the minimum condition $L(\tilde{\mathbf{v}}^-, \tilde{\mathbf{v}}^+, \lambda) \geq L(\mathbf{v}^-, \mathbf{v}^+, \lambda)$ we have

$$\begin{aligned}
& (\sigma_{\mathbf{v}\hat{\mathbf{p}}}^- + \lambda)(\tilde{\mathbf{v}}_{\hat{\mathbf{p}}}^- - \mathbf{v}_{\hat{\mathbf{p}}}^-) + (\sigma_{\mathbf{v}\hat{\mathbf{p}}}^+ - \lambda)(\tilde{\mathbf{v}}_{\hat{\mathbf{p}}}^+ - \mathbf{v}_{\hat{\mathbf{p}}}^+) \\
& + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- - \mathbf{v}_{\hat{\mathbf{t}}}^-) + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^+) + \varphi\left(\frac{\sigma_{\mathbf{v}\hat{\mathbf{p}}}^+ - \sigma_{\mathbf{v}\hat{\mathbf{p}}}^-}{2}\right)|\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| \geq 0. \quad (7.14)
\end{aligned}$$

Assume that in (7.14) $\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- = \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ = \mathbf{v}_{\hat{\mathbf{t}}}$, and $\tilde{\mathbf{v}}_{\hat{\mathbf{p}}}^-$ and $\tilde{\mathbf{v}}_{\hat{\mathbf{p}}}^+$ are arbitrary. Then

$$(\sigma_{\mathbf{v}\hat{\mathbf{p}}}^- + \lambda)(\tilde{\mathbf{v}}_{\hat{\mathbf{p}}}^- - \mathbf{v}_{\hat{\mathbf{p}}}^-) + (\sigma_{\mathbf{v}\hat{\mathbf{p}}}^+ - \lambda)(\tilde{\mathbf{v}}_{\hat{\mathbf{p}}}^+ - \mathbf{v}_{\hat{\mathbf{p}}}^+) \geq 0.$$

Hence, $\sigma_{\mathbf{v}\hat{\mathbf{p}}}^- = -\lambda$, $\sigma_{\mathbf{v}\hat{\mathbf{p}}}^+ = \lambda$. Now assume that $\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-$ and $\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+$ are arbitrary, and $\tilde{\mathbf{v}}_{\hat{\mathbf{p}}}^\pm = \mathbf{v}_{\hat{\mathbf{p}}}^\pm$. Then the inequality (7.14) is reduced to the form

$$\sigma_{\mathbf{v}\hat{\mathbf{t}}}^- \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- - \mathbf{v}_{\hat{\mathbf{t}}}^-) + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^+) + \varphi(\lambda)|\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| \geq 0.$$

If $\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- = \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ = \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}$, then $(\sigma_{\mathbf{v}\hat{\mathbf{t}}}^- + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+)(\tilde{\mathbf{v}}_{\hat{\mathbf{t}}} - \mathbf{v}_{\hat{\mathbf{t}}}) \geq 0$. Hence, $\sigma_{\mathbf{v}\hat{\mathbf{t}}}^- = -\sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ = \sigma_{\mathbf{v}\hat{\mathbf{t}}}$. Thus, for any $\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-$ and $\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+$

$$-\sigma_{\mathbf{v}\hat{\mathbf{t}}} \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-) + \varphi(\lambda)|\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| \geq 0.$$

From this inequality, taking $\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- = \sigma_{\mathbf{v}\hat{\mathbf{t}}}$, we obtain

$$-|\sigma_{\mathbf{v}\hat{\mathbf{t}}}|^2 + \varphi(\lambda)|\sigma_{\mathbf{v}\hat{\mathbf{t}}}| \geq 0.$$

Hence, $|\sigma_{\mathbf{v}\hat{\mathbf{t}}}| \leq \varphi(\lambda)$.

Thus, the quasi-variational inequality (7.13) with the constraint (7.7) results in the system of relationships of the friction law:

$$\begin{cases} \sigma_{\mathbf{v}\hat{\mathbf{p}}} \leq 0, & |\sigma_{\mathbf{v}\hat{\mathbf{t}}}| \leq \varphi(\sigma_{\mathbf{v}\hat{\mathbf{p}}}), \\ \sigma_{\mathbf{v}\hat{\mathbf{p}}} = 0, & \sigma_{\mathbf{v}\hat{\mathbf{t}}} = \mathbf{0}, & \text{if } \mathbf{v}_{\hat{\mathbf{p}}}^- - \mathbf{v}_{\hat{\mathbf{p}}}^+ < h, \\ |\sigma_{\mathbf{v}\hat{\mathbf{t}}}| = \varphi(\sigma_{\mathbf{v}\hat{\mathbf{p}}}) \text{ and} \\ \frac{\sigma_{\mathbf{v}\hat{\mathbf{t}}}}{|\sigma_{\mathbf{v}\hat{\mathbf{t}}}|} = \frac{\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-}{|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|}, & \text{if } \mathbf{v}_{\hat{\mathbf{p}}}^- - \mathbf{v}_{\hat{\mathbf{p}}}^+ = h \text{ and } \mathbf{v}_{\hat{\mathbf{t}}}^- \neq \mathbf{v}_{\hat{\mathbf{t}}}^+, \end{cases} \quad (7.15)$$

where $\sigma_{\mathbf{v}\hat{\mathbf{p}}} = \sigma_{\mathbf{v}\hat{\mathbf{p}}}^- = -\sigma_{\mathbf{v}\hat{\mathbf{p}}}^+$, $\sigma_{\mathbf{v}\hat{\mathbf{t}}} = \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- = -\sigma_{\mathbf{v}\hat{\mathbf{t}}}^+$.

Now we show that, conversely, the relationships (7.15) lead to the inequality (7.13). To do this, it is sufficient to show that for some $\lambda \geq 0$, which vanishes when (7.7) becomes a strict inequality, the following minimum condition holds:

$$L(\tilde{\mathbf{v}}^-, \tilde{\mathbf{v}}^+, \lambda) \geq L(\mathbf{v}^-, \mathbf{v}^+, \lambda).$$

Let $\lambda = -\sigma_{\mathbf{v}\hat{\mathbf{p}}}^- = \sigma_{\mathbf{v}\hat{\mathbf{p}}}^+ \geq 0$. According to the system (7.15), if there is no contact, then $\lambda = 0$,

$$\sigma_{\mathbf{v}\hat{\mathbf{v}}} = \sigma_{\mathbf{v}\hat{\mathbf{v}}}^- = -\sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ = 0, \quad \sigma_{\mathbf{v}\hat{\mathbf{t}}} = \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- = -\sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ = \mathbf{0}.$$

The minimum condition holds since the Lagrange function is identically equal to zero. If contact takes place, the following two cases are possible: $\mathbf{v}_{\hat{\mathbf{t}}}^- = \mathbf{v}_{\hat{\mathbf{t}}}^+$ and $\mathbf{v}_{\hat{\mathbf{t}}}^- \neq \mathbf{v}_{\hat{\mathbf{t}}}^+$. In both cases we have the equality

$$\begin{aligned} L(\tilde{\mathbf{v}}^-, \tilde{\mathbf{v}}^+, \lambda) - L(\mathbf{v}^-, \mathbf{v}^+, \lambda) &= (\sigma_{\mathbf{v}\hat{\mathbf{v}}}^- + \lambda)(\tilde{\mathbf{v}}_{\hat{\mathbf{v}}}^- - \mathbf{v}_{\hat{\mathbf{v}}}^-) + (\sigma_{\mathbf{v}\hat{\mathbf{v}}}^+ - \lambda)(\tilde{\mathbf{v}}_{\hat{\mathbf{v}}}^+ - \mathbf{v}_{\hat{\mathbf{v}}}^+) \\ &\quad + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- - \mathbf{v}_{\hat{\mathbf{t}}}^-) + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^+) \\ &\quad + \varphi(\lambda)(|\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| - |\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|). \end{aligned}$$

In the first case, where $\mathbf{v}_{\hat{\mathbf{t}}}^- = \mathbf{v}_{\hat{\mathbf{t}}}^+ = \mathbf{v}_{\hat{\mathbf{t}}}$, taking into account the notation for λ , we get

$$\begin{aligned} L(\tilde{\mathbf{v}}^-, \tilde{\mathbf{v}}^+, \lambda) - L(\mathbf{v}^-, \mathbf{v}^+, \lambda) &= \sigma_{\mathbf{v}\hat{\mathbf{t}}}^- \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- - \mathbf{v}_{\hat{\mathbf{t}}}) + \sigma_{\mathbf{v}\hat{\mathbf{t}}}^+ \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}) + \varphi(\lambda)|\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| \\ &= -\sigma_{\mathbf{v}\hat{\mathbf{t}}} \cdot (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-) + \varphi(\sigma_{\mathbf{v}\hat{\mathbf{v}}})|\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| \\ &\geq \left(\varphi(\sigma_{\mathbf{v}\hat{\mathbf{v}}}) - |\sigma_{\mathbf{v}\hat{\mathbf{t}}}| \right) |\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| \geq 0. \end{aligned}$$

Here the expression in parentheses is non-negative since (7.15) involves the condition $|\sigma_{\mathbf{v}\hat{\mathbf{t}}}| \leq \varphi(\sigma_{\mathbf{v}\hat{\mathbf{v}}})$. In the second case, where $\mathbf{v}_{\hat{\mathbf{t}}}^- \neq \mathbf{v}_{\hat{\mathbf{t}}}^+$,

$$\begin{aligned} L(\tilde{\mathbf{v}}^-, \tilde{\mathbf{v}}^+, \lambda) - L(\mathbf{v}^-, \mathbf{v}^+, \lambda) &= |\sigma_{\mathbf{v}\hat{\mathbf{t}}}|(\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- - \mathbf{v}_{\hat{\mathbf{t}}}^-) \cdot \frac{\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-}{|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|} \\ &\quad - |\sigma_{\mathbf{v}\hat{\mathbf{t}}}|(\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^+) \cdot \frac{\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-}{|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|} + \varphi(\sigma_{\mathbf{v}\hat{\mathbf{v}}})(|\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| - |\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|) \\ &= \varphi(\sigma_{\mathbf{v}\hat{\mathbf{v}}}) \left(-(\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- - \mathbf{v}_{\hat{\mathbf{t}}}^+ + \mathbf{v}_{\hat{\mathbf{t}}}^-) \cdot \frac{\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-}{|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|} + |\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| - |\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-| \right) \geq 0. \end{aligned}$$

The last inequality follows from the convexity criterion for the function $|\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|$, according to which

$$|\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^-| - |\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-| \geq (\tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^+ - \tilde{\mathbf{v}}_{\hat{\mathbf{t}}}^- - \mathbf{v}_{\hat{\mathbf{t}}}^+ + \mathbf{v}_{\hat{\mathbf{t}}}^-) \cdot \frac{\partial |\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-|}{\partial (\mathbf{v}_{\hat{\mathbf{t}}}^+ - \mathbf{v}_{\hat{\mathbf{t}}}^-)}.$$

By the Kuhn–Tucker theorem, the problem on the unconstrained minimum of the Lagrangian L is equivalent to the minimum principle (7.13) of virtual power of normal stress in a contact zone under the constraint (7.7). Thus, equivalence of

the quasi-variational inequality (7.13) with the constraint (7.7) and the generalized friction law (7.15) is proved.

7.2 Algorithm of Correction of Velocities

For convenience, the boundary conditions (7.6) of contact of a deformable layer with an absolutely rigid body and the conditions (7.9) of contact interaction of two deformable layers can be represented uniformly:

$$(\tilde{\mathbf{w}} - \mathbf{w})\mathbf{A}(\mathbf{w} - \bar{\mathbf{w}}) + f|\mathbf{b}(\mathbf{w} - \bar{\mathbf{w}})|(\omega(\tilde{\mathbf{w}}) - \omega(\mathbf{w})) \geq 0. \quad (7.16)$$

The approximate constraints (7.1) and (7.7) for velocities are reduced to the general form:

$$\tilde{\mathbf{w}} \nu \leq h, \mathbf{w} \nu \leq h. \quad (7.17)$$

Here the vector $\mathbf{w}$ is equal to the velocity vector ν in the case of contact of a layer and a rigid body or consists of nonzero components of the velocity vectors ν^- and ν^+ in the case of contact of two deformable layers, $\tilde{\mathbf{w}}$ is an arbitrary admissible variation of $\mathbf{w}$, $\mathbf{A}$ is a positive-definite square matrix which relates velocities and stresses, $\bar{\mathbf{w}}$ is the velocity vector corresponding to setting the conditions of a free surface in a contact zone, ν is the vector consisting of components of outward normals to contacting surfaces, $\mathbf{b} = \nu \mathbf{A}$. The expression $\mathbf{b}(\mathbf{w} - \bar{\mathbf{w}})$ coincides with normal stress in the problem on contact of a layer with a rigid inclusion and is equal to the half-sum of normal stresses in the case of two layers. As usual, the product in (7.17) is a product of matrix–row and matrix–column. In (7.16) $\omega(\mathbf{w})$ denotes the module of tangential component of a velocity vector or the modulus of the difference between the tangential components of velocities, depending on the context. This function can be represented in the form

$$\omega(\mathbf{w}) = \max_{\tilde{\mathbf{l}} \in \Lambda} \{\mathbf{w} \tilde{\mathbf{l}}\} = \mathbf{w} \mathbf{l},$$

where the bounded closed convex set Λ consists of vectors which are orthogonal to a normal vector and whose length does not exceed one:

$$\Lambda = \{\tilde{\mathbf{l}} | \tilde{\mathbf{l}} \nu = 0, \quad |\tilde{\mathbf{l}}| \leq 1\}.$$

The vectors $\mathbf{w}$ and $\tilde{\mathbf{w}}$ involved in the inequality (7.16) satisfy the one-sided constraint (7.17): they belong to the convex and closed set of admissible variations, i.e. to the half-space $\mathbf{W} = \{\mathbf{w} | \mathbf{w} \nu \leq h\}$. The scalar quantity h is defined by the shape of an inclusion or by the distance between corresponding points of contacting layers at an actual instant of time.

The constraint (7.17) is the condition of non-penetration of interacting bodies into one another. The matrix $\mathbf{A}$ in the inequality (7.16) consists of coefficients of linear equations in boundary meshes which are constructed by approximation of relationships on the bicharacteristics of the system of equations which describes the process of dynamic deformation of a material (see Sect. 6.3). The inequality (7.16) is equivalent to the Amontons–Coulomb friction law (7.10). In a more general case, where boundary conditions of contact are described by the quasi-variational inequality (7.12) or by the inequality (7.13), instead of (7.16) one should use the inequality

$$(\tilde{\mathbf{w}} - \mathbf{w})\mathbf{A}(\mathbf{w} - \bar{\mathbf{w}}) + \varphi(\mathbf{b}(\mathbf{w} - \bar{\mathbf{w}}))(\omega(\tilde{\mathbf{w}}) - \omega(\mathbf{w})) \geq 0, \quad (7.18)$$

which corresponds to the friction law (7.11) or (7.15).

For the numerical solution of the variational inequality (7.18) (and, in particular, for the solution of (7.16)) the algorithm of correction of velocities is proposed. In the boundary meshes, where contact may take place, a convergent iterative process is constructed. At each step of this process, projections of velocities and of auxiliary vectors, which define the direction of sliding, onto convex and closed sets of a special form are successively calculated. We describe this algorithm.

Specifying the vector $\mathbf{w} = \hat{\mathbf{w}}$ in the inequality (7.18), we fix the value of the function φ : $\hat{\varphi} = \varphi(\mathbf{b}(\hat{\mathbf{w}} - \bar{\mathbf{w}})) \geq 0$. We obtain the more simple inequality

$$(\tilde{\mathbf{w}} - \mathbf{w})\mathbf{A}(\mathbf{w} - \bar{\mathbf{w}}) + \hat{\varphi}(\omega(\tilde{\mathbf{w}}) - \omega(\mathbf{w})) \geq 0. \quad (7.19)$$

The matrix $\mathbf{A}$ is positive definite, the function $\omega(\mathbf{w})$ and the set $\mathbf{W}$ are convex, hence, this inequality has a unique solution. We can construct a mapping $\mathbf{Q} : \hat{\mathbf{w}} \mapsto \hat{\varphi} \mapsto \mathbf{w}$, such that the solution of (7.18) is its fixed point: $\mathbf{w} = \mathbf{Q}(\mathbf{w})$.

We prove that for a sufficiently small friction coefficient the mapping $\mathbf{Q}$ is contractive. Let $\mathbf{w} = \mathbf{Q}(\hat{\mathbf{w}})$ and $\mathbf{w}' = \mathbf{Q}(\hat{\mathbf{w}}')$. Put $\tilde{\mathbf{w}} = \mathbf{w}'$ in the inequality (7.19) and $\tilde{\mathbf{w}} = \mathbf{w}$ in the similar inequality for $\mathbf{w}'$ with the fixed value $\hat{\varphi}' = \varphi(\mathbf{b}(\hat{\mathbf{w}}' - \bar{\mathbf{w}}))$. Summing up the results, we obtain

$$(\mathbf{w}' - \mathbf{w})\mathbf{A}(\mathbf{w}' - \mathbf{w}) \leq (\hat{\varphi} - \hat{\varphi}')(\omega(\mathbf{w}') - \omega(\mathbf{w})).$$

Hence, taking into account that the matrix $\mathbf{A}$ is positive definite, due to the Cauchy–Bunyakovskii inequality we have

$$a_0|\mathbf{w}' - \mathbf{w}|^2 \leq |\hat{\varphi} - \hat{\varphi}'||\mathbf{w}' - \mathbf{w}|, \quad a_0 = \min_{\tilde{\mathbf{w}} \neq 0} \frac{\tilde{\mathbf{w}} \mathbf{A} \tilde{\mathbf{w}}}{|\tilde{\mathbf{w}}|^2}.$$

Using the Lipschitz condition, we rearrange and estimate the expression

$$\begin{aligned}
|\hat{\varphi} - \hat{\varphi}'| &= \left| \varphi(\mathbf{b}(\hat{\mathbf{w}} - \bar{\mathbf{w}})) - \varphi(\mathbf{b}(\hat{\mathbf{w}}' - \bar{\mathbf{w}})) \right| \\
&\leq f |\mathbf{b}(\hat{\mathbf{w}} - \bar{\mathbf{w}}) - \mathbf{b}(\hat{\mathbf{w}}' - \bar{\mathbf{w}})| \leq f |\mathbf{b}| |\hat{\mathbf{w}} - \hat{\mathbf{w}}'|.
\end{aligned}$$

Consequently,

$$a_0 |\mathbf{w}' - \mathbf{w}| \leq f |\mathbf{b}| |\hat{\mathbf{w}}' - \hat{\mathbf{w}}|.$$

Thus, for $f < a_0/|\mathbf{b}|$ the mapping $\mathbf{Q}$ is contractive. Because of the principle of contractive mappings, there exists a unique fixed point being a solution of (7.18). This solution can be determined by the method of successive approximations where at each step it is necessary to solve the inequality (7.19) with the coefficient $\hat{\varphi}$ calculated from the previous approximation. To construct the solution of (7.19), we consider the following auxiliary inequality:

$$(\tilde{\mathbf{w}} - \mathbf{w})(\mathbf{A}(\mathbf{w} - \bar{\mathbf{w}}) + \hat{\varphi} \mathbf{l}) \geq 0, \quad (7.20)$$

where $\tilde{\mathbf{w}} \in \mathbf{W}$ is an arbitrary vector. For given vector $\mathbf{l} \in \mathbf{A}$ there exists a unique solution $\mathbf{w} \in \mathbf{W}$. We show that for two solutions of (7.20), corresponding to vectors $\mathbf{l}$ and $\mathbf{l}'$, the estimate

$$|\mathbf{w}' - \mathbf{w}| \leq \frac{\hat{\varphi} |\mathbf{l}' - \mathbf{l}|}{a_0} \quad (7.21)$$

is valid. Assuming that $\tilde{\mathbf{w}} = \mathbf{w}'$ in the inequality (7.20) and $\tilde{\mathbf{w}} = \mathbf{w}$ in the similar inequality for $\mathbf{w}'$ and summing up these inequalities, after additional rearrangement we obtain

$$(\mathbf{w}' - \mathbf{w})\mathbf{A}(\mathbf{w}' - \mathbf{w}) \leq \hat{\varphi}(\mathbf{l} - \mathbf{l}')(\mathbf{w}' - \mathbf{w}).$$

With the help of the Cauchy–Bunyakovskii inequality and the fact that the matrix $\mathbf{A}$ is positive definite, this gives the estimate (7.21).

Now we construct a mapping $\mathbf{P} : \mathbf{l} \mapsto \mathbf{w} \mapsto \pi_{\mathbf{A}}(\mathbf{l} + \alpha \mathbf{w}) = \tilde{\mathbf{l}}$ with a constant $\alpha > 0$, where $\pi_{\mathbf{A}}$ is a projector onto the convex set $\mathbf{A}$ with respect to the Euclidean norm. This mapping is continuous since the vectors $\mathbf{w}$ and $\mathbf{w}'$, corresponding to $\mathbf{l}$ and $\mathbf{l}'$, satisfy the estimate (7.21) and the operator $\pi_{\mathbf{A}}$ satisfies the Lipschitz condition with the constant equal to one:

$$|\tilde{\mathbf{l}}' - \tilde{\mathbf{l}}| \leq |\mathbf{l}' + \alpha \mathbf{w}' - \mathbf{l} - \alpha \mathbf{w}| \leq |\mathbf{l}' - \mathbf{l}| + \alpha |\mathbf{w}' - \mathbf{w}| \leq \left(1 + \frac{\alpha \hat{\varphi}}{a_0}\right) |\mathbf{l}' - \mathbf{l}|.$$

The set $\mathbf{A}$ is convex and compact. Hence, the Brouwer theorem is valid, according to which $\mathbf{P}$ has a fixed point $\mathbf{l} \in \mathbf{A}$.

Thus, the variational inequality (7.20) has a solution which satisfies the condition $\mathbf{l} = \pi_{\mathbf{A}}(\mathbf{l} + \alpha \mathbf{w})$. We can show that this solution satisfies (7.19) independently of α . Indeed, for given vector $\mathbf{l}$

$$\begin{aligned}
& (\tilde{\mathbf{w}} - \mathbf{w})\mathbf{A}(\mathbf{w} - \bar{\mathbf{w}}) + \hat{\varphi}(\omega(\tilde{\mathbf{w}}) - \omega(\mathbf{w})) \\
& = (\tilde{\mathbf{w}} - \mathbf{w})(\mathbf{A}(\mathbf{w} - \bar{\mathbf{w}}) + \hat{\varphi}\mathbf{I}) + \hat{\varphi}(\omega(\tilde{\mathbf{w}}) - \tilde{\mathbf{w}}\mathbf{I}) - \hat{\varphi}(\omega(\mathbf{w}) - \mathbf{w}\mathbf{I}).
\end{aligned}$$

The first term in the right-hand side is nonnegative because the vector $\mathbf{w}$ is a solution of the inequality (7.20). Since

$$\omega(\tilde{\mathbf{w}}) = \max_{\tilde{\mathbf{I}} \in \Lambda} \{\tilde{\mathbf{w}}\tilde{\mathbf{I}}\},$$

we have $\omega(\tilde{\mathbf{w}}) - \tilde{\mathbf{w}}\mathbf{I} \geq 0$. Hence, the second term is nonnegative as well. By one of the properties of a projection operator, the equation $\mathbf{I} = \pi_{\Lambda}(\mathbf{I} + \alpha \mathbf{w})$ is equivalent to the inequality $(\tilde{\mathbf{I}} - \mathbf{I})(\mathbf{I} - \mathbf{I} - \alpha \mathbf{w}) \geq 0$, hence, $(\tilde{\mathbf{I}} - \mathbf{I})\mathbf{w} \leq 0$. Thus, $\mathbf{w}\tilde{\mathbf{I}} \leq \mathbf{w}\mathbf{I}$ for any $\tilde{\mathbf{I}} \in \Lambda$. Since $\omega(\mathbf{w})$ is maximum of the scalar product $\mathbf{w}\tilde{\mathbf{I}}$, the third term vanishes. Hence, the left-hand side of the equation is nonnegative, and $\mathbf{w}$ is a solution of the variational inequality (7.19).

When constructing a solution of (7.19), an algorithm of the Uzawa type is applied. At the n -th step of this algorithm the vector $\mathbf{w}^n$ is determined as a projection of the vector $\bar{\mathbf{w}}^n = \bar{\mathbf{w}} - \hat{\varphi}\mathbf{A}^{-1}\mathbf{I}^n$ onto the set $\mathbf{W}$: $\mathbf{w}^n = \pi_{\mathbf{W}}(\bar{\mathbf{w}}^n)$. In an explicit form

$$\mathbf{w}^n = \begin{cases} \bar{\mathbf{w}}^n, & \text{if } \bar{\mathbf{w}}^n \mathbf{v} \leq h, \\ \bar{\mathbf{w}}^n + \frac{h - \bar{\mathbf{w}}^n \mathbf{v}}{\mathbf{v} \mathbf{A}^{-1} \mathbf{v}} \mathbf{A}^{-1} \mathbf{v}, & \text{if } \bar{\mathbf{w}}^n \mathbf{v} > h. \end{cases}$$

Then the vector $\mathbf{I}$ is recalculated by the formula $\mathbf{I}^{n+1} = \pi_{\Lambda}(\bar{\mathbf{I}}^n)$, where $\bar{\mathbf{I}}^n = \mathbf{I}^n + \alpha \mathbf{w}^n$, or in an explicit form

$$\mathbf{I}^{n+1} = \begin{cases} \bar{\mathbf{I}}^n - \beta^n \mathbf{v}, & \text{if } |\bar{\mathbf{I}}^n - \beta^n \mathbf{v}| \leq 1, \\ \frac{\bar{\mathbf{I}}^n - \beta^n \mathbf{v}}{|\bar{\mathbf{I}}^n - \beta^n \mathbf{v}|}, & \text{if } |\bar{\mathbf{I}}^n - \beta^n \mathbf{v}| > 1, \end{cases} \quad \beta^n = \frac{\bar{\mathbf{I}}^n \mathbf{v}}{|\mathbf{v}|^2}.$$

An initial approximation $\mathbf{I}^0$ is arbitrary.

We prove that the sequence of vectors $\mathbf{w}^n$ converges to the vector $\mathbf{w}$ as $n \rightarrow \infty$ provided that $\alpha < 2a_0/\hat{\varphi}$. Take $\mathbf{w}$ as an admissible variation in the inequality (7.20) defining the solution $\mathbf{w}^n$ and take $\mathbf{w}^n$ as a variation in (7.19). Summing up the inequalities, we get

$$a_0|\mathbf{w}^n - \mathbf{w}|^2 \leq -\hat{\varphi}(\mathbf{w}^n - \mathbf{w})(\mathbf{I}^n - \mathbf{I}).$$

Hence, $(\mathbf{w}^n - \mathbf{w})(\mathbf{I}^n - \mathbf{I}) \leq -a_0|\mathbf{w}^n - \mathbf{w}|^2/\hat{\varphi}$. On the other hand, any projector onto a convex set is a non-expanding mapping, hence, $|\mathbf{I}^{n+1} - \mathbf{I}| \leq |\mathbf{I}^n - \mathbf{I} + \alpha(\mathbf{w}^n - \mathbf{w})|$. Thus,

$$\begin{aligned}
|\mathbf{I}^{n+1} - \mathbf{I}|^2 & \leq |\mathbf{I}^n - \mathbf{I}|^2 + 2\alpha(\mathbf{w}^n - \mathbf{w})(\mathbf{I}^n - \mathbf{I}) + \alpha^2|\mathbf{w}^n - \mathbf{w}|^2 \\
& \leq |\mathbf{I}^n - \mathbf{I}|^2 - \alpha\left(\frac{2a_0}{\hat{\varphi}} - \alpha\right)|\mathbf{w}^n - \mathbf{w}|^2.
\end{aligned}$$

The sequence of non-negative numbers $|\mathbf{l}^{n+1} - \mathbf{l}| < |\mathbf{l}^n - \mathbf{l}|$ decreases and converges to some limit. Besides,

$$\alpha \left(\frac{2a_0}{\hat{\varphi}} - \alpha \right) |\mathbf{w}^n - \mathbf{w}|^2 \leq |\mathbf{l}^n - \mathbf{l}|^2 - |\mathbf{l}^{n+1} - \mathbf{l}|^2 \rightarrow 0,$$

hence, $\mathbf{w}^n$ converges to $\mathbf{w}$.

In particular, if the matrix $\mathbf{A}$ is symmetrical, then the problem (7.19) is equivalent to the problem of minimization of the convex function

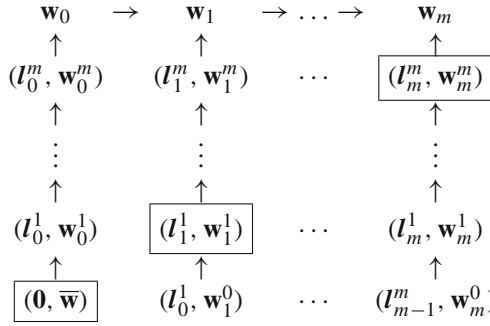
$$\psi(\mathbf{w}) = \max_{\tilde{\mathbf{l}} \in \mathbf{A}} \Psi(\mathbf{w}, \tilde{\mathbf{l}}), \quad \Psi(\mathbf{w}, \mathbf{l}) = \frac{1}{2} (\mathbf{w} - \bar{\mathbf{w}}) \mathbf{A} (\mathbf{w} - \bar{\mathbf{w}}) + \hat{\varphi} \mathbf{w} \mathbf{l},$$

on the set $\mathbf{W}$. The function $\Psi(\mathbf{w}, \mathbf{l})$ and the sets $\mathbf{W}$, $\mathbf{A}$ satisfy the conditions of the minimax theorem presented in Sect. 3.4, hence,

$$\min_{\tilde{\mathbf{w}} \in \mathbf{W}} \max_{\tilde{\mathbf{l}} \in \mathbf{A}} \Psi(\tilde{\mathbf{w}}, \tilde{\mathbf{l}}) = \max_{\tilde{\mathbf{l}} \in \mathbf{A}} \min_{\tilde{\mathbf{w}} \in \mathbf{W}} \Psi(\tilde{\mathbf{w}}, \tilde{\mathbf{l}}).$$

In this case the described algorithm is the Uzawa algorithm for the saddle point finding, [10].

It should be noted that recurrent calculation of iterations in the algorithm for numerical solution of the variational inequalities (7.16) and (7.18) is performed on the basis of the contractive mapping $\mathbf{Q}$ and two non-expanding operators $\pi_{\mathbf{W}}$ and $\pi_{\mathbf{A}}$. Such algorithm is stable with respect to round-off errors, i.e. going to the next iteration step does not lead to increasing errors. When implementing this algorithm, it is sufficient to restrict oneself to the construction of the so-called diagonal sequence, calculating at the m -th step of the method of successive approximations only m iteration steps of the Uzawa algorithm. We describe the scheme of choice of the diagonal sequence. Assume that $\mathbf{w}_m$ is a solution of the variational inequality (7.19) for $\hat{\varphi} = \varphi(\mathbf{b}(\mathbf{w}_{m-1} - \bar{\mathbf{w}}))$ obtained by the method of successive approximations with the initial approximation $\mathbf{w}_0 = \bar{\mathbf{w}}$, and $\mathbf{w}_m^n$ is a sequence of the Uzawa algorithm which converges to $\mathbf{w}_m$ as $n \rightarrow \infty$. In practice the vectors $\mathbf{w}_m^n$ are calculated only for $n = 0, 1, \dots, m$. As an initial approximation $\mathbf{l}_m^0$ for the sequence of auxiliary vectors $\mathbf{l}_m^n$, which is determined for $n = 1, \dots, m + 1$, we take the vector $\mathbf{l}_{m-1}^n$ obtained at the previous step of the algorithm. At each iteration step for $n < m$ the quantity $|\mathbf{w}_m^n - \mathbf{w}_m^{n-1}|$ is controlled. If it turns out to be less than given error ε , then the calculation of the vectors $\mathbf{w}_m^n$ is terminated and we go to the next $(m + 1)$ -st step. The scheme of choice of the diagonal sequence looks as follows:



The elements of the diagonal sequence are calculated until $|\mathbf{w}_m^m - \mathbf{w}_{m-1}^{m-1}| \leq \varepsilon$.

7.3 Results of Computations

To test the algorithm of correction of velocities, the quasistatic contact problem on indentation of a rigid stamp with a cylindrical contact surface into an elastic layer lying on a rigid plane was solved. Normal load is applied to a stamp and under its action a stamp is indented into a layer. The problem is solved in the two-dimensional formulation within the framework of the linear elasticity theory. The point of initial contact is taken as the origin of a rectangular coordinate system.

Computations were performed for a thick elastic layer, for a layer of medium thickness, and for a thin layer. A square difference grid consisting of 120×60 , 120×18 , or 120×6 nodes depending on thickness of a layer was used. Slow loading with the value of load increasing from zero to a given value P_0 was modeled. In Fig. 7.3 the shape of a contact surface in the process of deformation as well as the level curves of the normal stress σ_{22} are shown. Areas of maximal in magnitude compressive stresses are indicated by black and areas of maximal tensile stresses are indicated by white. In a thick layer (Fig. 7.3a) maximal compressive stresses are concentrated in the contact zone near the contact surface. In a thin layer (Fig. 7.3c) compressive stresses are immediately under the zone of contact with a stamp as well, but level lines are vertical. In a layer of medium thickness (Fig. 7.3b) maximal compressive stresses are concentrated near the upper boundary in the contact zone and the remaining level curves are almost vertical. In Fig. 7.4 graphs of pressure at the upper boundaries of layers in the area of contact with a stamp are presented. The curves 1 correspond to the numerical solution, the curves 2 correspond to the exact solution of the Hertz problem for a rigid cylinder and an elastic half-space, and the curves 3 correspond to the solution of the Hertz problem for a cylinder and a thin layer. Exact solutions are obtained by methods of the theory of functions of a complex variable (see, for example, [13]). For an elastic half-space and an absolutely rigid stamp in the form of a paraboloid of revolution

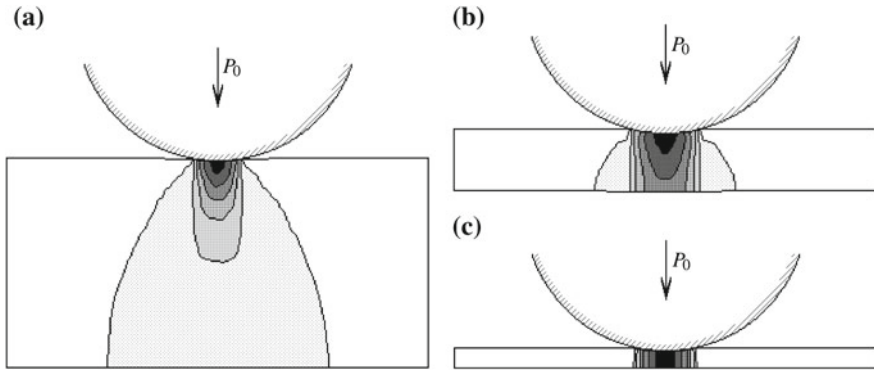


Fig. 7.3 Indentation of a stamp into an elastic layer (level curves of stress σ_{22}): **a** a thick layer, **b** a layer of medium thickness, **c** a thin layer

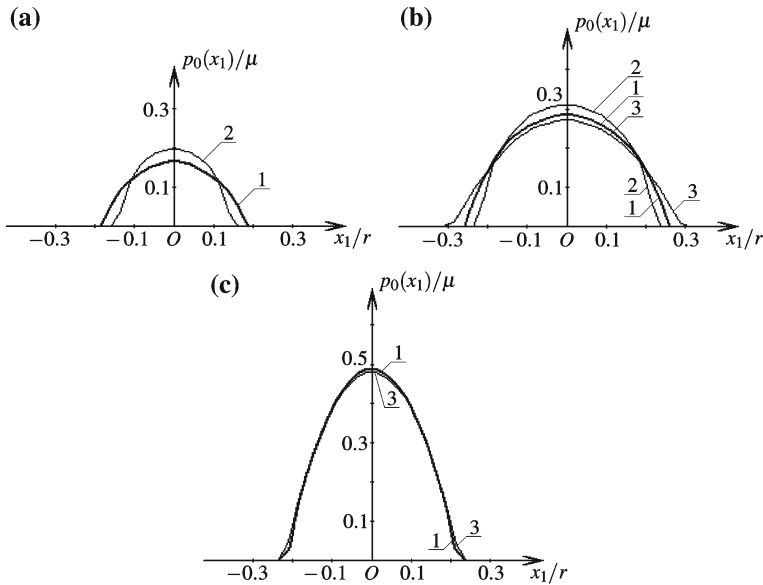
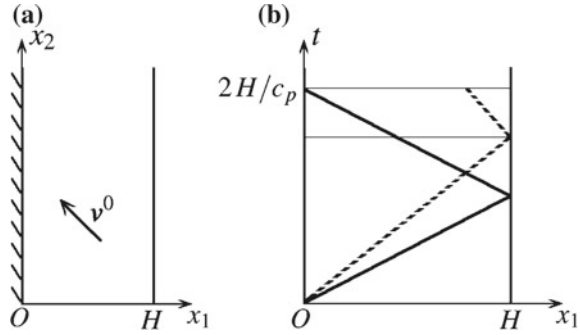


Fig. 7.4 Distribution of pressure $p_0(x_1)$ in a contact zone: **a** a thick layer, **b** a layer of medium thickness, **c** a thin layer

$$p_0(x_1) = \frac{2 P_0}{\pi d_c} \sqrt{1 - \frac{x_1^2}{d_c^2}}, \quad d_c = \sqrt{\frac{4 P_0 r}{\pi \mu_0}},$$

where d_c is the half-width of a contact zone, r is the radius of a cylindrical surface, $\mu_0 = 4 \mu (3k + \mu) / (3k + 4 \mu)$ is the modulus of plane strain. For a thin elastic layer

Fig. 7.5 Contact of an elastic layer and a rigid plane: **a** direction of the layer motion, **b** wave pattern



$$p_0(x_1) = \frac{d_c^2 \mu_0}{2rH} \left(1 - \frac{x_1^2}{d_c^2}\right), \quad d_c = \sqrt[3]{\frac{3P_0 r H}{2\mu_0}},$$

where H is the thickness of a layer, $H \ll d_c$. The numerical solution for a thick layer (the curve 1 in Fig. 7.4a) turns out to be close to the solution of the Hertz problem for a half-space (the curve 2). Some difference in half-width of a contact zone for analytical and numerical solutions with the same area under the curves of pressure $p_0(x_1)$ is observed. On the one hand, this can be explained by errors of the algorithm and the computational model being used where a layer is of finite size and a lateral surface is assumed to be free of stresses. On the other hand, it should be noticed that the size of a contact zone in the Hertz solution is determined approximately. A contact zone is assumed to be beforehand known and equal to the length of the segment obtained by intersection of a stamp and a undeformed half-space when a stamp is intended at given depth. The actual size of a zone turns out to be somewhat greater due to extrusion of an elastic material from under a stamp. The numerical solution for a layer of medium thickness (the curve 1 in Fig. 7.4b) lies between the solution of the Hertz problem for a half-space (the curve 2) and that for a thin layer (the curve 3). For a thin layer the numerical and analytical solutions (the curves 1 and 3 in Fig. 7.4c) almost coincide. The Hertz solution for an infinite half-space is inappropriate in this case.

In general, from the results of comparison we can make a conclusion that numerical solutions obtained on rather coarse grids are in good agreement with known analytical ones.

Further we consider the one-dimensional dynamic problem on interaction of an elastic layer and an absolutely rigid plane. An exact solution of this problem with friction on a plane taken into account can be constructed by the method of characteristics. This solution was used for testing the shock-capturing method described in the previous chapter and the algorithm implementing contact boundary conditions with friction forces taken into account which is proposed in Sect. 7.2. An elastic layer of thickness H falls onto a rigid plane with velocity v^0 (Fig. 7.5a). At the initial instant of time, a layer is free of stresses and velocities of particles are given: $v_1 = -v_v^0 < 0$, $v_2 = v_\tau^0 > 0$. At the instant of contact, longitudinal and transverse waves propagate

into the layer with velocities c_p and c_s , respectively. Their fronts are represented in Fig. 7.5b by solid and dashed lines. From the relationships on characteristics of the one-dimensional system of equations of the linear elasticity theory

$$\begin{cases} d(\rho c_p v_1 \pm \sigma_{11}) = 0, & \text{if } dx_1 = \mp c_p dt, \\ d(\rho c_s v_2 \pm \sigma_{12}) = 0, & \text{if } dx_1 = \mp c_s dt, \end{cases} \quad (7.22)$$

we obtain that behind the front of the longitudinal wave $v_1 = 0$,

$$\rho c_p v_1 + \sigma_{11} = -\rho c_p v_v^0 \Rightarrow \sigma_{11} = -\rho c_p v_v^0,$$

and behind the front of the reflected longitudinal wave $\sigma_{11} = 0$,

$$\rho c_p v_1 - \sigma_{11} = \rho c_p v_v^0 \Rightarrow v_1 = v_v^0.$$

By the instant $t_c = 2H/c_p$ (time of contact) all points of a layer move in the normal direction with velocity v_v^0 . This is natural since velocity of rebound coincides in magnitude with velocity of fall. The normal component of the vector of velocity averaged over thickness of a layer is calculated by the formula

$$v_1^{av} = \begin{cases} v_v^0 (c_p t/H - 1), & \text{if } 0 \leq t < t_c, \\ v_v^0, & \text{if } t \geq t_c. \end{cases}$$

Normal stress on the contact surface is given by

$$\sigma_{11} = \begin{cases} -\rho c_p v_v^0, & \text{if } 0 < t < t_c, \\ 0, & \text{if } t \geq t_c. \end{cases}$$

Transverse waves have influence on the tangential component of velocity after reflection. Two modes of interaction (slowing-down and sliding) are possible. In the slowing-down mode the state of rest is established behind the front of the transverse wave. This mode is implemented under the condition $|\sigma_{12}| \leq f|\sigma_{11}|$. In this case behind the front of the transverse loading wave $v_2 = 0$,

$$\rho c_s v_2 + \sigma_{12} = \rho c_s v_\tau^0 \Rightarrow \sigma_{12} = \rho c_s v_\tau^0,$$

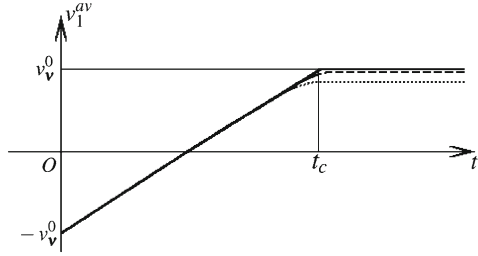
and behind the front of the reflected transverse wave $\sigma_{12} = 0$,

$$\rho c_s v_2 - \sigma_{12} = -\rho c_s v_\tau^0 \Rightarrow v_2 = -v_\tau^0.$$

Thus,

$$\sigma_{12} = \begin{cases} \rho c_s v_\tau^0, & \text{if } 0 < t < t_c, \\ 0, & \text{if } t \geq t_c. \end{cases}$$

Fig. 7.6 Normal component of velocity averaged over thickness of a layer



In addition, $v_{\tau}^0 \leq f c_p v_v^0 / c_s$. In the slowing-down mode the tangential component of the vector of average velocity is as follows:

$$v_2^{av} = \begin{cases} v_{\tau}^0 (1 - c_s t / H), & \text{if } 0 \leq t < t_c, \\ v_{\tau}^0 (1 - 2 c_s / c_p), & \text{if } t \geq t_c. \end{cases}$$

If $c_s < c_p/2$, i.e. the time when the transverse wave arrives at the free surface of a layer is greater than the time of contact, then v_2^{av} does not change its sign. If the transverse wave is reflected from the free surface before rebound ($c_s > c_p/2$), then the tangential component of average velocity changes its sign. In the case of the equality $c_s = c_p/2$, after rebound the velocity vector is perpendicular to the plane of contact. The sliding mode is implemented provided that $v_{\tau}^0 > f c_p v_v^0 / c_s$. In this case behind the front of the transverse wave $\sigma_{12} = f |\sigma_{11}| = \rho f c_p v_v^0$,

$$\rho c_s v_2 + \sigma_{12} = \rho c_s v_{\tau}^0 \Rightarrow v_2 = v_{\tau}^0 - \frac{f c_p v_v^0}{c_s},$$

and behind the front of the reflected wave $\sigma_{12} = 0$,

$$\rho c_s v_2 - \sigma_{12} = \rho c_s v_{\tau}^0 - 2 \rho f c_p v_v^0 \Rightarrow v_2 = v_{\tau}^0 - \frac{2 f c_p v_v^0}{c_s}.$$

Hence,

$$\sigma_{12} = \begin{cases} \rho f c_p v_v^0, & \text{if } 0 < t < t_c, \\ 0, & \text{if } t \geq t_c. \end{cases}$$

Then from the relationships (7.22) on characteristics we obtain

$$v_2^{av} = \begin{cases} v_{\tau}^0 - f v_v^0 c_p t / H, & \text{if } 0 \leq t < t_c, \\ v_{\tau}^0 - 2 f v_v^0, & \text{if } t \geq t_c. \end{cases}$$

In this mode the directions of the tangential component of velocity averaged over thickness of a layer before and after rebound coincide for $v_{\tau}^0 > 2 f v_v^0$ and are opposite to one another for $v_{\tau}^0 < 2 f v_v^0$. If $v_{\tau}^0 = 2 f v_v^0$, then after rebound the velocity vector is directed along the x_1 axis.

Fig. 7.7 Tangential component of average velocity for slowing-down

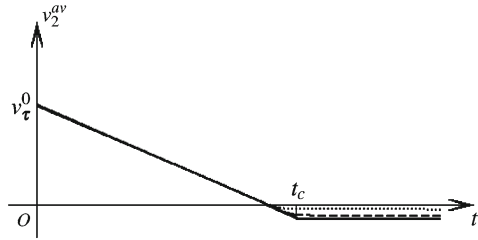
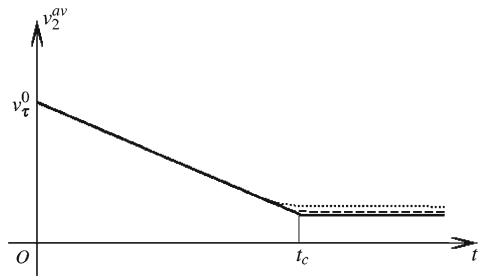


Fig. 7.8 Tangential component of average velocity for sliding



Numerical calculations were performed for different modes of interaction of an elastic layer and a rigid plane. The results of calculations by the Godunov scheme without reconstruction and by the scheme with reconstruction of a solution with the help of the algorithm of implementation of the contact conditions were compared with an exact solution. In Fig. 7.6 the dependence of the normal component of average velocity on time is shown. Here the Courant parameter is 0.5, a solid line corresponds to an exact solution, a dotted line corresponds to an approximate solution calculated by the Godunov scheme, and a dashed line corresponds to that obtained by the scheme of higher-order accuracy. In Fig. 7.7 the time-dependence of the tangential component of average velocity for the slowing-down mode ($c_s > c_p/2$) and in Fig. 7.8 that for the sliding mode ($v_{\tau}^0 > 2f v_v^0$) are shown.

The algorithms developed were applied when solving problems which are not immediately related to granular materials, namely, problems of contact deformation of laminated plates, of mechanical treatment of metals by cutting, of oblique collision of plates, [3, 4, 29]. In particular, with the help of the presented algorithm the problem on traveling load was solved. A thick layer, such that localized load p_0 is applied to its upper boundary, lying on a rigid plane, was considered. The remaining parts of the boundary of a layer are free of stresses. With time the area of application of load moves along the upper boundary of a layer from left to right with constant velocity v_c . Load is assumed to be caused by a plane absolutely rigid stamp with sliding friction forces on its surface.

For supersonic velocity $v_c > c_p$ for this problem in the two-dimensional elastic formulation numerical results were compared with an exact solution obtained with the help of the method of characteristics. A good correspondence of these results is observed. According to results of numerical solution of the elastic-plastic problem,

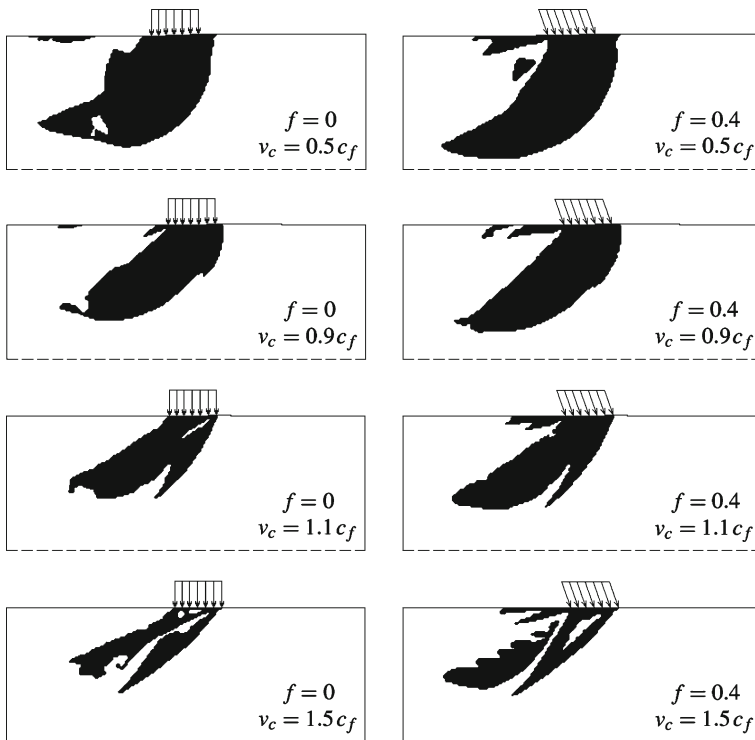


Fig. 7.9 Problem of traveling load

with time a plastic zone is established. The configuration of the zone depends on three factors: on the magnitude of load p_0 , on the velocity v_c and on the friction coefficient f . In Fig. 7.9 a plastic zone for moderate load $p_0 = 5 \tau_s$ for two values of the friction coefficient $f = 0$ and $f = 0.4$ is presented. A dashed line denotes the middle of the plate in thickness. In calculations a difference grid consisting of 200×150 nodes was used. If $v_c < c_f$, where c_f is the velocity of plastic shock waves, then a plastic zone is located not only under the area where load is applied but also ahead of it, besides, the less is velocity v_c , the longer is the zone. In addition, the size of a plastic zone increases with increasing friction. The graphs for $v_c > c_f$ show that with friction as well as without it a plastic zone is concentrated behind the front of load and does not run ahead. For any values of v_c the friction is a factor for arising plasticity near the upper boundary of the plate behind the area where load is applied. Friction also has influence on the shape of a plastic zone inside the plate. It should be noticed that with increasing intensity of load p_0 the size of a plastic zone increases as well but its shape remains almost the same.

Numerical results confirm a known hypothesis which explains the mechanism of waveformation of a connective joint in the process of the welding of metals by explosion, [15]. This hypothesis says that if velocity of a point of contact is less

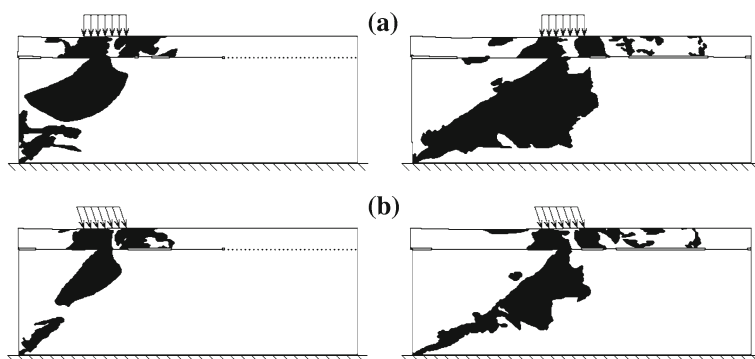


Fig. 7.10 Contact interaction of two layers ($v_c = 0.5 c_f$): **a** without friction, **b** for $f = 0.4$

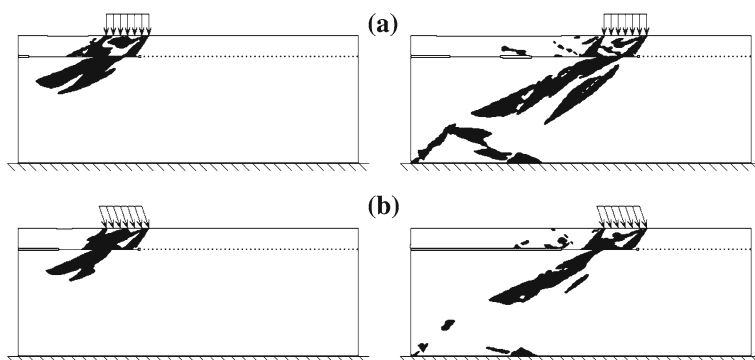


Fig. 7.11 Contact interaction of two layers ($v_c = 1.5 c_f$): **a** without friction, **b** for $f = 0.4$

than velocity of propagation of plastic waves, then a plasticity zone runs ahead and occupies some area ahead of this point. The results are obtained in the framework of the theory of elastic-plastic flow without consideration of different strengths of a material. If load travels along the surface of a layer of an (ideal granular) material with different strengths, then the displacement discontinuity is formed inside a layer as a result of interaction of a reflected wave and an unloading wave. This follows from analysis of a solution of the one-dimensional problem with plane waves on the impulsive loading of a layer (see Fig. 5.36). The depth at which discontinuity arises depends on the time of action of an impulse.

The mechanism of arising displacement discontinuity was analyzed with the help of a model of a quasilaminated package consisting of two layers of the same material separated by a plane contact surface. Variants of numerical computations are presented in Figs. 7.10 and 7.11 where plastic zones and zones of contact for $v_c = 0.5 c_f$ and $v_c = 1.5 c_f$ at different instants of time are shown. Adhesion zones are shown by a dotted line, sliding zones are shown by a thin solid line, and

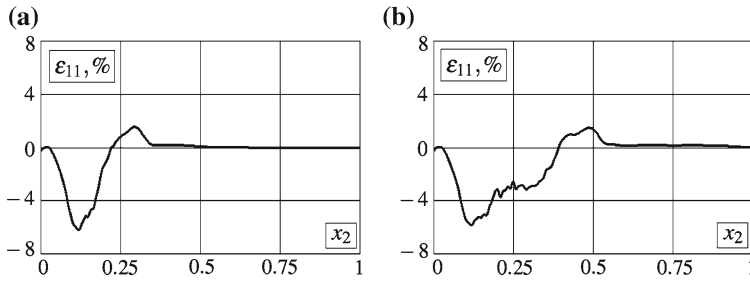


Fig. 7.12 Strain ε_{11} on the contact boundary of an upper layer: **a** $t = 0.625/c_f$, **b** $t = 1/c_f$

zones where contacting surfaces are separated from one another are shown by grey. The results are obtained considering friction as well as without considering it. They show that in both cases, as an impulse travels, a package delaminates due to arising normal tensile stresses near a contact surface, in addition, the degree of delamination essentially depends on velocity of displacement of an impulse on the boundary of a laminated package as well as on the friction coefficient.

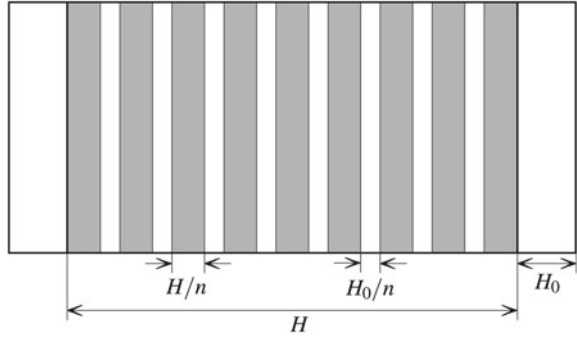
In Fig. 7.12 graphs of distribution of the transverse strain ε_{11} along the contact boundary of an upper layer for $v_c = 0.5 c_f$ are shown (at different instants of time; the length of package in horizontal direction is taken as unit). It turns out that for relatively small velocities of displacement of an impulse the typical peak of positive transverse strain arises in the area ahead of the impulse. On a contact surface a hump traveling with velocity v_c is formed. Its height practically does not vary with varying the friction coefficient. This hump contributes to delamination at the head of a laminated package. For high velocity v_c there are no peak and corresponding delamination.

It should be noted that with the use of the algorithms worked out for the numerical implementation of conditions of contact interaction of deformable bodies we can model the behaviour of granular materials for approximation of a block medium consisting of a considerably large number of uniform blocks.

7.4 Interaction of Blocks Through Viscoelastic Layers

In conclusion of this chapter we consider the simplified computational models of contact interaction of blocks through a thin viscoelastic layers to describe processes of the wave propagation in rocks, which are characterized by structurally inhomogeneous block-hierarchical structure. The block structure of rocks is observed at different levels of scale: from the size of crystal grains to large blocks of a rock body, separated by faults. The blocks are separated by layers with substantially compliant mechanical properties, [30].

Fig. 7.13 Laminated structure of a rock



Conditional scheme of the hierarchical structure of a rock is shown in Fig. 7.13. Ideally, it is a nested laminated structure with an invariant ratio of the characteristic scales of blocks and layers between them. Let us consider a single fragment of this structure in one-dimensional case as a system of n elastic blocks of the thickness H and elastic layers of the thickness H_0 .

Let ρ and ρ_0 be the densities, c and c_0 be the velocities of longitudinal waves, $a = 1/(\rho c^2)$ and $a_0 = 1/(\rho_0 c_0^2)$ be the elastic compliances of materials of blocks and layers, respectively. One-dimensional equations of the elasticity theory

$$\rho \frac{\partial v^k}{\partial t} = \frac{\partial \sigma^k}{\partial x}, \quad a \frac{\partial \sigma^k}{\partial t} = \frac{\partial v^k}{\partial x}, \quad (7.23)$$

are fulfilled inside of the layer with the number k . Here v^k is the longitudinal velocity in the direction of the x axis (x varies from 0 to H within each layer), σ^k is the normal stress. Behavior of a material of the layer can be described by equations

$$\rho_0 \frac{d}{dt} \frac{v^{k+1} + v^k}{2} = \frac{\sigma^{k+1} - \sigma^k}{H_0}, \quad a_0 \frac{d}{dt} \frac{\sigma^{k+1} + \sigma^k}{2} = \frac{v^{k+1} - v^k}{H_0}, \quad (7.24)$$

including the boundary values of velocities and stresses, the left for $(k+1)$ -th block and the right for k -th block, respectively. Such a system can be obtained by averaging of equations of an elastic medium approximated as a thin layer ($H_0 \ll H$). It takes into account the inertial properties of the layer. A simplified description, without these properties, follows from (7.24) in the limit when the specific gravity $\rho_0 H_0$ of a layer tends to zero. Then, obviously, the condition of continuity of stress at the interface between layers is valid. If the compliance $a_0 H_0$ of a layer tends to zero too, then the conditions of continuity of velocity follow from the system (7.24). Thus, we obtain the mathematical model of a homogeneous elastic medium, which does not describe the block structure of a material.

The system of Eqs. (7.23) and (7.24) together with initial data $v^k = \sigma^k = 0$ ($k = 1, \dots, n$) and boundary conditions $\sigma^1(0, t) = -p_0(t)$, $v^n(H, t) = 0$ ($p_0(t)$ is a given external force action) form a well-posed boundary-value problem. Its

correctness can be proved by the methods, outlined in [11], on the basis of integral estimates, following from the energy conservation law

$$\begin{aligned}
 & \frac{1}{2} \frac{\partial}{\partial t} \sum_{k=1}^n \int_0^H \left(\rho |v^k(x, t)|^2 + a |\sigma^k(x, t)|^2 \right) dx \\
 & + \frac{H_0}{2} \frac{d}{dt} \sum_{k=1}^{n-1} \left(\rho_0 \left| \frac{v^{k+1}(0, t) + v^k(H, t)}{2} \right|^2 + a_0 \left| \frac{\sigma^{k+1}(0, t) + \sigma^k(H, t)}{2} \right|^2 \right) \\
 & = \sigma^n(H, t) v^n(H, t) - \sigma^1(0, t) v^1(0, t).
 \end{aligned} \tag{7.25}$$

The conservation law (7.25) shows also on the thermodynamic self-consistency of the mathematical model being under consideration.

Numerical solution of the problem is constructed by means of the Godunov scheme on a uniform grid with maximum permissible value of the time step $\Delta t = \Delta x/c$ (according to the Courant–Friedrichs–Levi condition). In this case the scheme does not have an artificial energy dissipation. At smaller values of the time step a piecewise-linear ENO-reconstruction of the second order of accuracy is used, which was described in Sect. 6.3.

Matching conditions at the interfaces in the form of Eqs. (7.24) are calculated by the Godunov scheme as well. For this at each artificially introduced mesh of the length H_0 , simulating the layer, the scheme of discontinuity decay with the independent time step $\Delta t_0 = H_0/c_0 \ll \Delta t$, ultimate by the Courant–Friedrichs–Levi condition for a material of layers, is applied. Used for calculating a number of steps corresponds to the time step of basic scheme. Grid-characteristic interpretation of this method is schematically represented in Fig. 7.14. At the stage of solution of the system (7.24) with time step Δt_0 the equations of discontinuity decay at interfaces are used (predictor of the scheme in a layer):

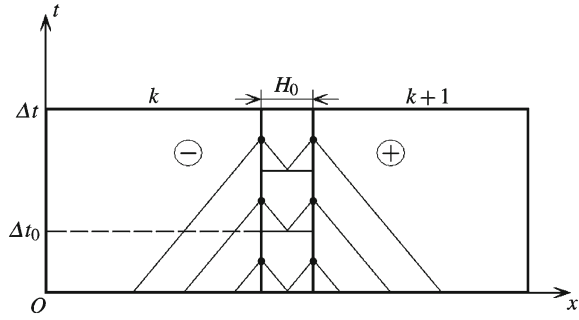
$$\begin{aligned}
 z_0 v_+ - \sigma_+ &= z_0 v - \sigma, & z v_+ + \sigma_+ &= z v^{k+1} + \sigma^{k+1}, \\
 z_0 v_- + \sigma_- &= z_0 v + \sigma, & z v_- - \sigma_- &= z v^k - \sigma^k.
 \end{aligned} \tag{7.26}$$

Here $z_0 = \rho_0 c_0$ and $z = \rho c$ are the acoustic impedances of materials, the quantities with superscripts relate to the boundary meshes of interacting blocks, the quantities marked by “+” and “-” relate to the right and left boundaries of a layer, respectively. Recalculation of the solution (corrector of the scheme) is carried out by formulae

$$\hat{v} = v + (\sigma_+ - \sigma_-) \frac{\Delta t_0}{\rho_0 H_0}, \quad \hat{\sigma} = \sigma + (v_+ - v_-) \frac{\Delta t_0}{a_0 H_0}, \tag{7.27}$$

where the values with a hat relate to a new time step. The predictor values of grid quantities of basic scheme with the Δt step are calculated by averaging the values, received at the boundaries of meshes in small steps. Corrector is fulfilled in the usual way on the basis of integral analogues of differential Eqs. (7.23).

Fig. 7.14 Grid-characteristic scheme



Substituting the exact solution of Eqs. (7.23) and (7.24) in this explicit difference scheme, after the expansion in the Taylor series one can show that this scheme has at least the first order of approximation with respect to the steps in time and in spatial variable. Besides, one can establish that it satisfies the difference analog of the energy conservation law (7.25).

Verification of the scheme was carried out on the exact solution of the problem on reflection and propagation of a monochromatic wave through the layer between two extended blocks occupying the left and right half-spaces. In this problem the unknowns are coefficients of the transmission $A(\omega)$ and of the reflection $B(\omega)$ of a wave, dependent on the frequency ω . Solution of Eqs. (7.23) for $x < 0$ is a superposition of incident and reflected waves:

$$v = f_0\left(t - \frac{x}{c}\right) + B f_0\left(t + \frac{x}{c}\right), \quad \sigma = -z f_0\left(t - \frac{x}{c}\right) + B z f_0\left(t + \frac{x}{c}\right).$$

For $x > 0$ it corresponds to the transmitted wave:

$$v = A f_0\left(t - \frac{x}{c}\right), \quad \sigma = -z A f_0\left(t - \frac{x}{c}\right).$$

At the interface $x = 0$ due to Eqs. (7.24):

$$\begin{aligned} \rho_0 H_0 (1 + A + B) \frac{df_0}{dt} &= 2z (1 - A - B) f_0, \\ a_0 H_0 z (1 + A - B) \frac{df_0}{dt} &= 2(1 - A + B) f_0, \end{aligned}$$

hence, $f_0 = C_1 e^{i\omega t}$, where C_1 is an arbitrary constant,

$$A = \frac{1 - \alpha \beta}{(1 + \alpha)(1 + \beta)}, \quad B = \frac{\beta - \alpha}{(1 + \alpha)(1 + \beta)}, \quad \alpha = \frac{i\omega H_0}{2c_0} \frac{z_0}{z}, \quad \beta = \frac{i\omega H_0}{2c_0} \frac{z}{z_0}.$$

These formulae, in particular, show that at the same impedances $z = z_0$ the equalities $|A| = 1$ and $B = 0$ are valid. In this case the incident wave passes through the layer

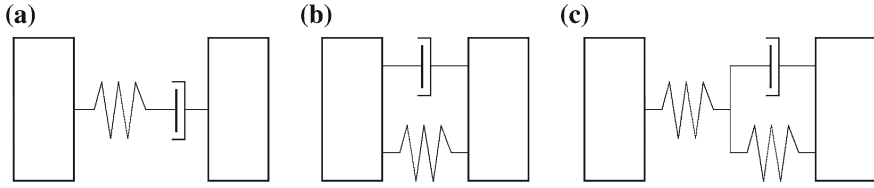


Fig. 7.15 Rheological schemes of interaction between blocks: **a** viscoelastic interaction (the Maxwell model), **b** viscoelastic interaction (the Kelvin–Voigt model), **c** complicated scheme of interaction

without hindrance and without reflection, that is the simplest test for the correctness of algorithms and programs.

Analysis of the experimental data shows that the layers in rocks behave inelastic even at very small amplitudes of the waves. More complex variants of models that take into account the natural dissipative processes in layers are presented by rheological schemes in Fig. 7.15.

Viscoelastic interaction based on the Maxwell model (Fig. 7.15a), according to which the strain of a layer is composed from elastic and viscous components, is described by (7.24) after replacement of the second equation by the next one:

$$a_0 \frac{d}{dt} \frac{\sigma^{k+1} + \sigma^k}{2} = \frac{v^{k+1} - v^k}{H_0} - \frac{\sigma^{k+1} + \sigma^k}{2\eta},$$

where η is a coefficient of viscosity. In this case instead of (7.27) the approximation of the equations of a viscoelastic medium by the Crank–Nicholson scheme is used on the corrector stage:

$$\rho_0 \frac{\hat{v} - v}{\Delta t_0} = \frac{\sigma_+ - \sigma_-}{H_0}, \quad a_0 \frac{\hat{\sigma} - \sigma}{\Delta t_0} = \frac{v_+ - v_-}{H_0} - \frac{\hat{\sigma} + \sigma}{2\eta},$$

which also implemented with an explicit algorithm.

In the Kelvin–Voigt model (Fig. 7.15b) the stress in a layer consists of elastic and viscous stresses. Constitutive equation becomes as follows:

$$a_0 \frac{d}{dt} \frac{\sigma^{k+1} + \sigma^k}{2} = a_0 \eta \frac{d}{dt} \frac{v^{k+1} - v^k}{H_0} + \frac{v^{k+1} - v^k}{H_0}.$$

Numerical implementation in individual meshes, corresponding to layers, is based on the difference scheme of the predictor–corrector type which approximates the equations of a viscoelastic material. Predictor stage is calculated by Eqs. (7.26) of elastic model. Corrector is carried out on the basis of the system

$$\rho_0 \frac{\hat{v} - v}{\Delta t_0} = \frac{\sigma_+ - \sigma_-}{H_0}, \quad a_0 \frac{\hat{s} - s}{\Delta t_0} = \frac{v_+ - v_-}{H_0}, \quad \sigma = s + \eta \frac{v_+ - v_-}{H_0}. \quad (7.28)$$

Here s is used to denote the elastic stress. According to (7.26) and (7.28) the difference between these values is equal to

$$v_+ - v_- = \frac{z(v^{k+1} - v^k) + \sigma^{k+1} - \sigma^k - 2s}{z_0 + z + 2\eta/H_0}.$$

Taking into account this formula, the last equation of (7.28) allows to determine the stress σ and by (7.26) to find

$$\begin{aligned} v_+ &= \frac{z v^{k+1} + z_0 v + \sigma^{k+1} - \sigma}{z_0 + z}, & \sigma_+ &= \frac{z_0 \sigma^{k+1} + z \sigma + z_0 z (v^{k+1} - v)}{z_0 + z}, \\ v_- &= \frac{z v^k + z_0 v + \sigma - \sigma^k}{z_0 + z}, & \sigma_- &= \frac{z_0 \sigma^k + z \sigma + z_0 z (v - v^k)}{z_0 + z}. \end{aligned}$$

The solution at the new time step is determined from the system (7.28).

In a more complicated model by Poynting and Thomson (Fig. 7.15c) the constitutive equation is integro-differential and has the form:

$$\begin{aligned} &(a_0 + a_1) \frac{\sigma^{k+1} + \sigma^k}{2} + a_0 a_1 \eta \frac{d}{dt} \frac{\sigma^{k+1} + \sigma^k}{2} \\ &= \int_0^t \frac{v^{k+1}(t_1) - v^k(t_1)}{H_0} dt_1 + a_1 \eta \frac{v^{k+1} - v^k}{H_0}, \end{aligned}$$

where a_1 is the modulus of elastic compliance of a spring parallel connected with a viscous damper. Numerical solution of the problem in meshes corresponding to the layers is carried out on the basis of the Crank–Nicholson implicit scheme:

$$\begin{aligned} \rho_0 \frac{\hat{v} - v}{\Delta t_0} &= \frac{\sigma_+ - \sigma_-}{H_0}, & \frac{\hat{\varepsilon} - \varepsilon}{\Delta t_0} &= \frac{v_+ - v_-}{H_0}, \\ (a_0 + a_1) \frac{\hat{\sigma} + \sigma}{2} + a_0 a_1 \eta \frac{\hat{\sigma} - \sigma}{\Delta t_0} &= \frac{\hat{\varepsilon} + \varepsilon}{2} + a_1 \eta \frac{v_+ - v_-}{H_0}, \end{aligned}$$

where the predictor values are determined from the system (7.26).

To verify algorithms and programs we use the exact solution of the problem on propagation and reflection of a monochromatic wave, which can be obtained from the above solution for the elastic layer with replacement of the modulus a_0 of elastic compliance in the expression for the coefficient $\beta = i \omega a_0 z H_0/2$ by the modulus $a_0 + 1/(i \omega \eta)$ in the Maxwell model, the modulus $a_0/(1 + i \omega a_0 \eta)$ in the Kelvin–Voigt model and the modulus $(a_0 + a_1 + i \omega a_0 a_1 \eta)/(1 + i \omega a_1 \eta)$ in the Poynting–Thomson model.

In Fig. 7.16 the graphs of dependence of the transmission coefficient A and of the reflection coefficient B on the dimensionless frequency $\bar{\omega} = \omega H_0/c_0$ are presented for a layer of elastic material (curves 1) and for layers of viscoelastic materials: the Maxwell material (curves 2), the Kelvin–Voigt material (curves 3), the Poynting–Thomson material (curves 4). The ratio of acoustic impedances $z_0/z =$

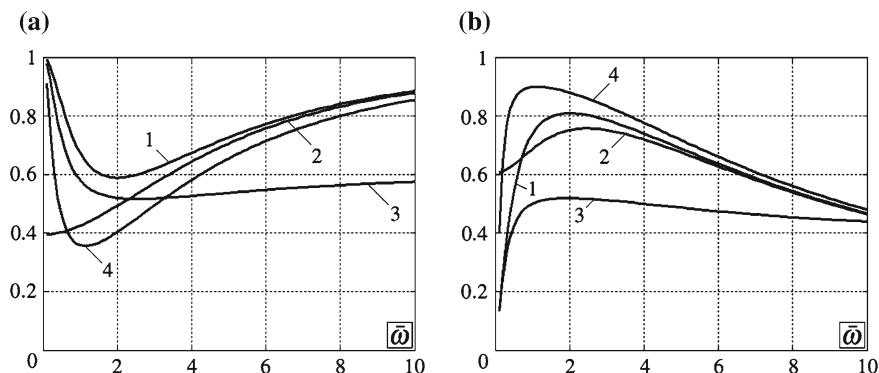


Fig. 7.16 Dependencies of the transmission coefficient **a** and the reflection coefficient **b** on a frequency: 1 a layer of elastic material, 2 a layer of the Maxwell viscoelastic material, 3 a layer of the Kelvin–Voigt material, 4 a layer of the Poynting–Thomson material

0.326 is given, based on the calculation of mechanical parameters for a rock with microfractures layers. The value of the dimensionless viscosity coefficient $\bar{\eta} = \eta / (z_0 H_0)$ for all models is equal to unity. The deviation of this coefficient from unity leads to the results which can be predict by means of rheological schemes of the models in Fig. 7.15. In the Maxwell model with a decrease in viscosity the transmission coefficient for low-frequency waves tends to zero, and the reflection coefficient tends to unity, that corresponds to the reflection from free surface. High-frequency waves are reflected from a layer, because the inertia forces are taken into account in the equations of motion of a layer. With an increase in viscosity the curves 2, corresponding to the solution in the framework of the Maxwell model, approach the curves 1 for an elastic layer. In the Kelvin–Voigt model the same effect is observed at an decrease in viscosity. If it increases then the coefficient of transmission tends to unity, and the coefficient of reflection tends to zero. The wave passes through the layer freely. But it is so only for the low-frequency waves, the high-frequency waves almost do not pass through the layer due to the dissipation of mechanical energy. In the Poynting–Thomson model the curves 4 come near the curves 1 with an increase in viscosity, and take the limit position, corresponding to the solution for an elastic layer with the compliance modulus $a_0 + a_1$, when viscosity decreases to zero.

Solution of the problem on passing and reflection of a wave has the independent interest, since it can be used to estimate the frequency range in which the layered structure of a medium has a significant influence on the overall wave pattern. Comparison of exact and numerical solutions for low-frequency waves, the length of which is greater than the thickness of a layer, shows a good correspondence of results.

Algorithms for computation of viscoelastic layers are implemented as subroutines, intended for numerical analysis of 2D and 3D problems of the dynamics of elastic-plastic and granular media, and are included in the parallel program systems described in Chap. 8. Program systems are oriented on multiprocessor computers of the cluster type. In these program systems the boundary-value problems,

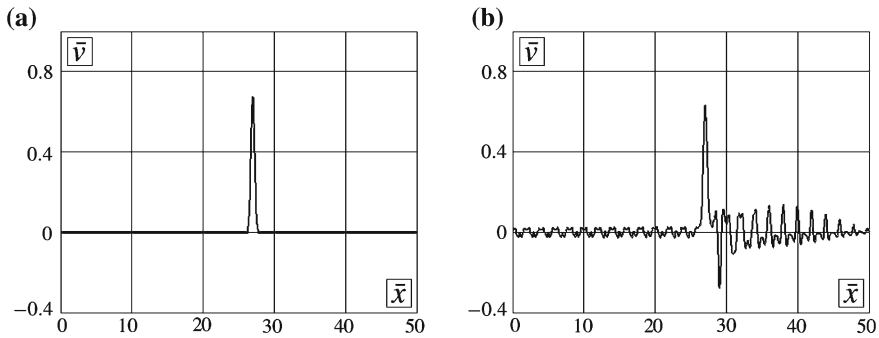


Fig. 7.17 Waves propagation in a laminated medium: **a** equal impedances of layers and blocks, **b** impedance of layers is 5 times less than impedance of blocks

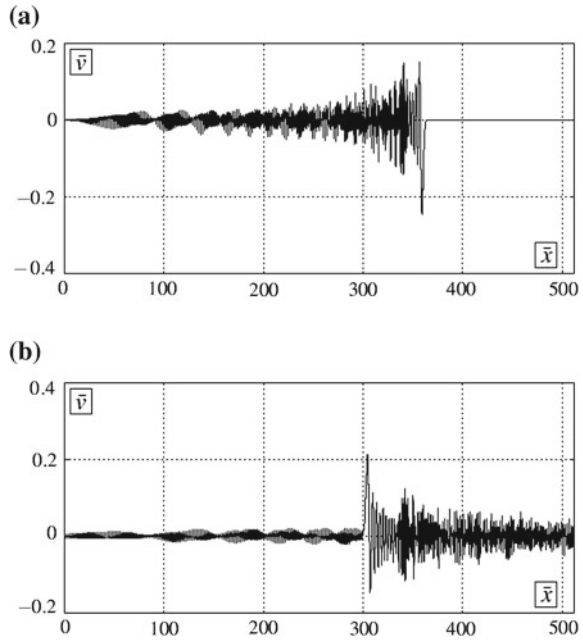
describing the propagation of waves of stresses and strains in the block body composed of heterogeneous materials, are solved in the plane and spatial statements.

Simulation of viscoelastic contact is realized on the basis of the system of Eqs.(7.24) and its generalizations, taking into account the viscosity of a material of layers. Such a systems are written separately for normal and tangential with respect to the interface stresses and velocities. They are implemented with independent time steps for longitudinal and transverse waves.

To demonstrate the availability of algorithms, the results of computations of plane longitudinal waves generated by short-time Π -shaped impulse of the stress of unit amplitude on the boundary of a laminated medium, consisting of 50 blocks with elastic layers, are presented in Fig. 7.17. In Fig. 7.17a, the acoustic impedance of the layers material coincides with the impedance of the blocks material, but the density is five times greater. In Fig. 7.17b, a material of layers has five times less acoustic impedance. The distribution of dimensionless velocity is shown. The absence of reflections from interfaces under the same impedances and weak change in form of the impulse after 1500 time steps, when the wave reflected from the right boundary passes about half of the thickness of a laminated body, indicates of sufficient accuracy of computations. The results presented in Fig. 7.17b, show a characteristic distinction of the wave pattern obtained with different impedances in blocks and layers. This difference is in the appearance of reflections and in a significant change of the shape of loading wave as it passing through the interfaces.

In Figs. 7.18 and 7.19 one can see the graphs of dimensionless velocity $\bar{v}$ on the spatial coordinate divided by the block thickness H for the problem on the action of Λ -shaped impulse of a pressure on boundary of a layered medium composed of 512 blocks of a rock with microfractures elastic layers, [33]. Computations were carried out after reduction of the system of equations to dimensionless variables with the following parameters: $\rho_0/\rho = 0.76$, $a_0/a = 7.17$, $H_0/H = 0.027$. A uniform finite-difference grid in blocks consists of 16 meshes. One mesh is used within each layer in accordance with the proposed method. Choice of the integer parameters in the form of powers of two is related to the specific features of programming and

Fig. 7.18 Velocity distribution behind the front of the incident wave **a** and the reflected wave **b**, caused in a layered medium by the influence of a short impulse



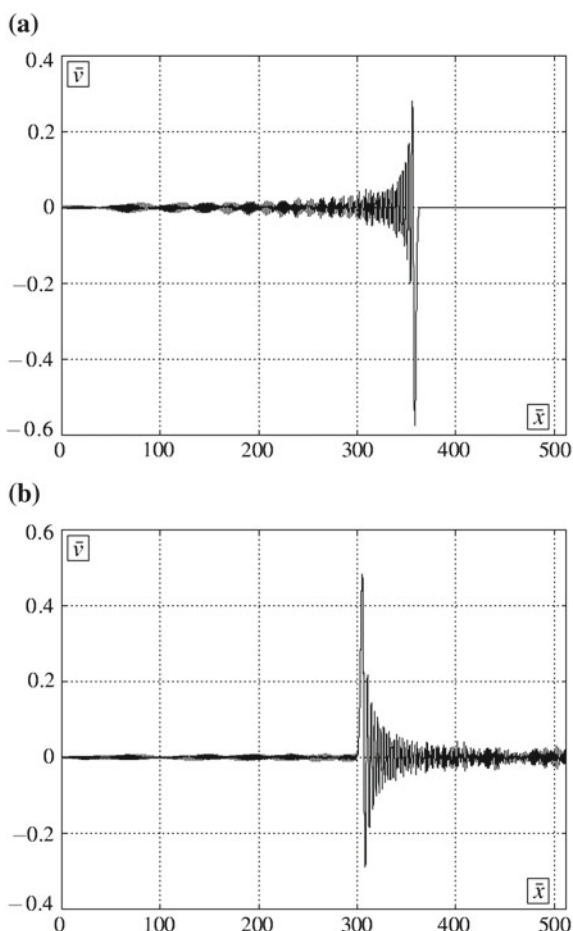
memory allocation in CUDA, [6, 9, 12, 14]. Computations were performed on the eight-core computer with graphics card Tesla C2050.

Figure 7.18 corresponds to the impulse duration which is equal to the time of passage of elastic wave through one block, Fig. 7.19 corresponds to the duration which is two and a half times greater. The impulse of unit amplitude acts on the left boundary of computational domain, the right boundary is fixed. In Figs. 7.18a and 7.19a the velocity profiles are shown at the moment when the incident wave goes about 370 blocks (6000-th time step of the basic scheme). In Figs. 7.18b and 7.19b the reflected wave goes in the opposite direction about 200 blocks (12000-th time step of the basic scheme).

These results demonstrate a qualitative difference of the wave pattern in layered media as compared with a homogeneous medium. At the initial stage this difference is the appearance of waves reflected from the layers—the characteristic oscillations behind the front of loading wave as it passes through the interface. With time, after multiple reflections behind the front of a head wave appears stationary wave pattern, the so-called pendulum wave, whose existence was predicted in [1, 2, 21, 31].

A comparison of Figs. 7.18 and 7.19 show that with an increase in impulse duration the amplitude of a head wave increases up to unity, and the amplitude of oscillations behind the front decreases and tends to zero. This is due to the fact that waves, which lengths are considerably greater than the thickness of the layer, are practically not reflected from the layers. Thus, it is possible to detect a weakened microstructure of layered or block medium only with the help of sufficiently short waves.

Fig. 7.19 Velocity distribution behind the front of the incident wave **a** and the reflected wave **b**, caused in a layered medium by the influence of a long impulse



Similar computations were performed in the process of testing algorithms for elastic layers, which impedance coincides with the impedance of blocks and which density is on the order less than the blocks density. In this case the wave, caused by impulsive action at the boundary, passes through the layers as in a homogeneous medium in the form of a solitary impulse, but its velocity is adjusted in a natural way, since the velocities of propagation of disturbances in blocks and in layers are different. The amplitude of a wave is practically not distorted over hundreds of thousands of time steps of basic scheme provided that Δt and Δt_0 correspond to the maximum permissible values by the Courant–Friedrichs–Levy condition. If they decrease then the wave amplitudes decay very quickly in a layered medium, since the scheme viscosity reveals itself.

A large series of numerical experiments showed that if the mechanical properties of materials and the thicknesses of blocks and layers do not allow to perform

a balanced computation of the problem with the maximum allowable time steps, then in order to achieve satisfactory accuracy it is necessary to select the limit time step in layers and to apply the scheme of high accuracy with the reconstruction of solution, making computations in blocks with a time step which is below the limit value.

In conclusion it should be noted that the choice of finite-difference scheme for numerical solution of the problems on propagation of high-frequency waves in layered media is extremely important and delicate question. In such problems are not applicable the methods based on non-monotone schemes of the Neumann–Richtmyer type (the scheme “cross”) on staggered grids, generating spurious oscillations at the wave fronts, and monotone schemes with large artificial viscosity in which the amplitude inadequately decays when the wave passes through a sufficiently large number of layers.

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Chapter 8

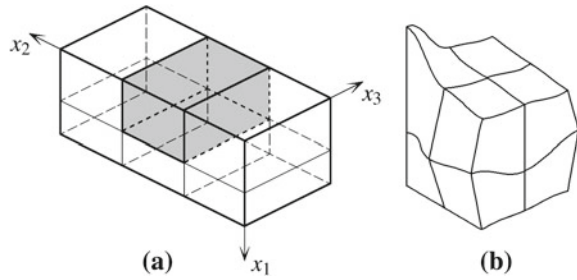
Results of High-Performance Computing

Abstract Algorithms for numerical implementation of the shock-capturing method for solving the problems of dynamics of a granular material are constructed. In these algorithms computations are parallelized at the stage of splitting a problem with respect to spatial variables. Different ways of distribution of a computational domain among parallel computational nodes are considered. It is shown that the minimal number of exchanges between nodes is achieved when a domain is decomposed into regular cubes. Numerical results for propagation of elastic–plastic waves in two-dimensional and three-dimensional formulations obtained with the help of multiprocessor computer systems of the MVS series are presented.

8.1 Generalization of the Method

The application of high-performance distributed computations provides wide possibilities for the mathematical modeling in problems of mechanics of granular materials. This is primarily related to spatial problems solved on fine grids. With the help of simple estimates we can show, for example, that for direct numerical solution of a three-dimensional problem on a modern single-processor personal computer the size of a grid is restricted to approximately $200 \times 200 \times 200$ meshes by the volume of random-access memory. Indeed, a modern Pentium-IV based on the Intel processor with 512 MB RAM provides approximately 256 MB for variables used in a program in the Fortran algorithmic language. When numerically solving a spatial problem, it is required to allocate in memory three-dimensional arrays for coordinates of a velocity vector and for components of a symmetric stress tensor (9 functions), as well as for coordinates of nodes of a grid (3 functions) if a solution is constructed in a curvilinear domain. Thus, about 24 MB are allocated for one array. In the case of the 8-byte representation of real numbers, required for arithmetic calculations with double precision, the size of an array turns out to be equal to $3 \times 10^6 \approx 144 \times 144 \times 144$.

Fig. 8.1 Examples of block media: **a** decomposition of a computational domain of parallelepiped form (homogeneous medium with a rigid inclusion) into 6 blocks, **b** decomposition of the computational domain with curvilinear boundaries into 8 blocks



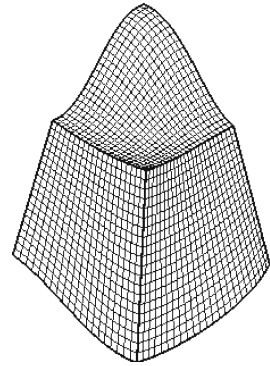
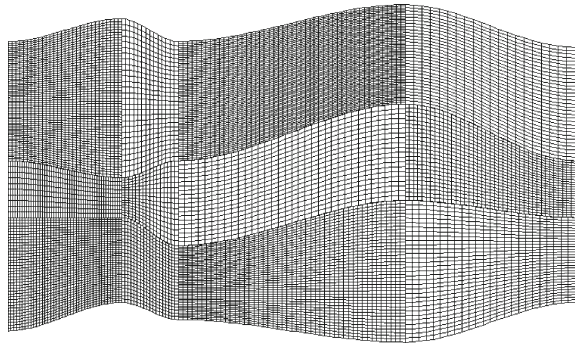
One can extend available memory space by doubling RAM, then the maximal dimension of a problem approaches the limit mentioned above.

An accuracy of numerical solution obtained on such a grid may be unsatisfactory especially when a computational domain has a complex structure (when it involves a large number of internal interfaces of materials with considerably different mechanical properties, rigid inclusions of small size, etc.). In addition, when implementing the iterative processes in each mesh of a grid domain, there arise natural restrictions for running time. The use of multiprocessor computer systems removes these restrictions. However, when going from a PC to a supercomputer with parallel architecture, there are some difficulties related, firstly, with the need to parallelize a computational algorithm and, secondly, with a fundamentally more complicated writing of a program code.

Focusing on applications to problems of geophysics (seismicity), we give a generalization of the shock-capturing method, based on the splitting of a mathematical model of dynamic deformation of a granular material with respect to physical processes and spatial variables, in order to describe non-stationary wave processes in a medium body consisting of an arbitrary number of curvilinear blocks of different structure, [8, 9]. We assume that the block structure of a body is regular in the sense that its separated blocks consisting of homogeneous materials can be numbered with three indices l_1, l_2, l_3 along the x_1, x_2, x_3 axes of a Cartesian coordinate system and these indices vary in the range from 1 to N_1, N_2 and N_3 , respectively.

This numbering can be introduced only if interfaces of blocks are consistent with each other. If there exist inconsistent interfaces, then it is necessary to extend these interfaces and to perform a fictitious regular decomposition of a medium body with a large number of blocks of the same material being involved. As an example, in Fig. 8.1a the scheme of fictitious decomposition of a body of a homogeneous medium with a rigid inclusion of parallelepiped form (it is marked by grey) into $2 \times 3 \times 1 = 6$ blocks is shown. An example of regular decomposition of a medium body into $2 \times 2 \times 2 = 8$ blocks bounded by curvilinear surfaces is presented in Fig. 8.1b.

Curvilinear grids in blocks are constructed with the help of an algebraic approach which consists in calculation of one-to-one mappings of a unit cube with a uniform grid in the space of parameters ξ_1, ξ_2, ξ_3 on a physical domain of blocks. This

Fig. 8.2 Difference grid**Fig. 8.3** Inconsistent grids in blocks

approach is taken mainly due to relative simplicity of implementation of algorithms for pasting together of solutions on interfaces since such mappings coincide on common boundaries. A function $x = x(\xi)$ that implements a mapping is constructed in the form of a multidimensional cubic spline

$$x = \sum_{i_1, i_2, i_3=0}^3 C_{i_1 i_2 i_3} \xi_1^{i_1} \xi_2^{i_2} \xi_3^{i_3},$$

whose vector coefficients $C_{i_1 i_2 i_3}$ are determined in the explicit form from the conjugation conditions at vertices of blocks (positions of vertices are given and directions of coordinate lines at vertices are determined from conditions of mutual orthogonality). An example of a grid constructed in such a way in an individual spatial block is shown in Fig. 8.2. An example of a grid in a plane cross-section of a body consisting of 12 curvilinear blocks is presented in Fig. 8.3.

In Sect. 6.1 it was shown that a model of an elastic–plastic material, differently resistant to tension and compression, admits a formulation in the form of the variational inequality (6.8) with the additional Eq. (6.9). After transformation of variables this model is reduced to the next system:

$$\begin{aligned}
 (\tilde{\mathbf{V}} - \mathbf{V}) \left(\mathbf{A} \mathbf{U}_t - \sum_{i=1}^n \bar{\mathbf{B}}^i \mathbf{V}_{\xi_i} - \mathbf{Q} \mathbf{V} - \mathbf{G} \right) &\geq 0, \quad \mathbf{V}, \tilde{\mathbf{V}} \in \mathbf{F}, \\
 \mathbf{U} &= \frac{1}{\varsigma} \mathbf{V} - \frac{1-\varsigma}{\varsigma} \mathbf{V}^\pi, \quad \bar{\mathbf{B}}^i = \sum_{l=1}^n \frac{\partial \xi_i}{\partial x_l} \mathbf{B}^l.
 \end{aligned} \tag{8.1}$$

In the spatial case the vectors $\mathbf{U}$ and $\mathbf{V}$ are given by

$$\begin{aligned}
 \mathbf{U} &= (v_1, v_2, v_3, s_{11}, s_{22}, s_{33}, s_{23}, s_{13}, s_{12}), \\
 \mathbf{V} &= (v_1, v_2, v_3, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}),
 \end{aligned}$$

and the matrices $\mathbf{A}$ and $\mathbf{B}^i$ are of the block form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_v & 0 & 0 \\ 0 & \mathbf{A}_\sigma & \mathbf{A}_{\sigma\tau} \\ 0 & \mathbf{A}_{\sigma\tau}^* & \mathbf{A}_\tau \end{pmatrix}, \quad \mathbf{B}^i = \begin{pmatrix} 0 & \mathbf{B}_\sigma^i & \mathbf{B}_\tau^i \\ \mathbf{B}_\sigma^i & 0 & 0 \\ \mathbf{B}_\tau^i & 0 & 0 \end{pmatrix},$$

where asterisk, as before, means transposition, $\mathbf{A}_v = \rho \delta$, δ is the (3×3) identity matrix,

$$\begin{aligned}
 \mathbf{A}_\sigma &= \begin{pmatrix} a_{1111} & a_{1122} & a_{1133} \\ a_{2211} & a_{2222} & a_{2233} \\ a_{3311} & a_{3322} & a_{3333} \end{pmatrix}, \quad \mathbf{A}_\tau = 2 \begin{pmatrix} a_{2323} & a_{2313} & a_{2312} \\ a_{1323} & a_{1313} & a_{1312} \\ a_{1223} & a_{1213} & a_{1212} \end{pmatrix}, \\
 \mathbf{A}_{\sigma\tau} &= 2 \begin{pmatrix} a_{1123} & a_{1113} & a_{1112} \\ a_{2223} & a_{2213} & a_{2212} \\ a_{3323} & a_{3313} & a_{3312} \end{pmatrix}, \quad \mathbf{B}_\sigma^i = \begin{pmatrix} \delta_{23}^i & 0 & 0 \\ 0 & \delta_{13}^i & 0 \\ 0 & 0 & \delta_{12}^i \end{pmatrix}, \\
 \mathbf{B}_\tau^i &= \begin{pmatrix} 0 & \delta_{12}^i & \delta_{13}^i \\ \delta_{12}^i & 0 & \delta_{23}^i \\ \delta_{13}^i & \delta_{23}^i & 0 \end{pmatrix}, \quad \delta_{jl}^i = \begin{cases} 1, & \text{if } i \neq j \text{ and } i \neq l, \\ 0, & \text{if } i = j \text{ or } i = l. \end{cases}
 \end{aligned}$$

In the case of an isotropic material the system of left eigenvectors $\mathbf{Z}_l = \mathbf{Y}_l \mathbf{A}$ of the matrix $\mathbf{A}^{-1}(v_1 \mathbf{B}^1 + v_2 \mathbf{B}^2 + v_3 \mathbf{B}^3)$, which is required for numerical implementation of one-dimensional schemes, consists of two vectors associated with longitudinal waves with velocities $\pm c_p$, four vectors for $\pm c_s$ (velocities of transverse waves are double eigenvalues) and three vectors for $c = 0$. Depending on the values of v_1 , v_2 , and v_3 , eigenvectors should be taken in different ways. If $v_1 \neq 0$, then the following system is linearly independent $\left(v = \sqrt{v_1^2 + v_2^2 + v_3^2} \right)$:

$$\begin{aligned}
 &(\pm \rho c_p v v_1, \pm \rho c_p v v_2, \pm \rho c_p v v_3, v_1^2, v_2^2, v_3^2, 2 v_2 v_3, 2 v_1 v_3, 2 v_1 v_2), \\
 &(\mp \rho c_s v v_2, \pm \rho c_s v v_1, 0, -v_1 v_2, v_1 v_2, 0, v_1 v_3, -v_2 v_3, v_1^2 - v_2^2),
 \end{aligned}$$

$$\begin{aligned}
&(\mp \rho c_s \nu v_3, 0, \pm \rho c_s \nu v_1, -\nu_1 v_3, 0, \nu_1 v_3, \nu_1 v_2, \nu_1^2 - \nu_3^2, -\nu_2 v_3), \\
&(0, 0, 0, a_1 \nu_2^2 + a_2 \nu_1^2, a_1 \nu_1^2 + a_2 \nu_2^2, a_2(\nu_1^2 + \nu_2^2), 0, 0, -\nu_1 v_2), \\
&(0, 0, 0, a_1 \nu_3^2 + a_2 \nu_1^2, a_2(\nu_1^2 + \nu_3^2), a_1 \nu_1^2 + a_2 \nu_3^2, 0, -\nu_1 v_3, 0), \\
&(0, 0, 0, 2 a_1 \nu_2 v_3, 2 a_2 \nu_2 v_3, 2 a_2 \nu_2 v_3, \nu_1^2, -\nu_1 v_2, -\nu_1 v_3),
\end{aligned}$$

where $a_1 = (k + \mu/3)/(3k)$, $a_2 = -(k - 2\mu/3)/(6k)$. If $\nu_2 \neq 0$, then the linearly independent system is the system of vectors, in which the first two vectors are the same and the remaining ones are as follows:

$$\begin{aligned}
&(\mp \rho c_s \nu v_2, \pm \rho c_s \nu v_1, 0, -\nu_1 v_2, \nu_1 v_2, 0, \nu_1 v_3, -\nu_2 v_3, \nu_1^2 - \nu_2^2), \\
&(0, \mp \rho c_s \nu v_3, \pm \rho c_s \nu v_2, 0, -\nu_2 v_3, \nu_2 v_3, \nu_2^2 - \nu_3^2, \nu_1 v_2, -\nu_1 v_3), \\
&(0, 0, 0, a_1 \nu_2^2 + a_2 \nu_1^2, a_1 \nu_1^2 + a_2 \nu_2^2, a_2(\nu_1^2 + \nu_2^2), 0, 0, -\nu_1 v_2), \\
&(0, 0, 0, a_2(\nu_2^2 + \nu_3^2), a_1 \nu_3^2 + a_2 \nu_2^2, a_1 \nu_2^2 + a_2 \nu_3^2, -\nu_2 v_3, 0, 0), \\
&(0, 0, 0, 2 a_2 \nu_1 v_3, 2 a_1 \nu_1 v_3, 2 a_2 \nu_1 v_3, -\nu_1 v_2, \nu_2^2, -\nu_2 v_3).
\end{aligned}$$

If $\nu_3 \neq 0$, then the last vectors must be replaced by the next ones:

$$\begin{aligned}
&(\mp \rho c_s \nu v_3, 0, \pm \rho c_s \nu v_1, -\nu_1 v_3, 0, \nu_1 v_3, \nu_1 v_2, \nu_1^2 - \nu_3^2, -\nu_2 v_3), \\
&(0, \mp \rho c_s \nu v_3, \pm \rho c_s \nu v_2, 0, -\nu_2 v_3, \nu_2 v_3, \nu_2^2 - \nu_3^2, \nu_1 v_2, -\nu_1 v_3), \\
&(0, 0, 0, a_1 \nu_3^2 + a_2 \nu_1^2, a_2(\nu_1^2 + \nu_3^2), a_1 \nu_1^2 + a_2 \nu_3^2, 0, -\nu_1 v_3, 0), \\
&(0, 0, 0, a_2(\nu_2^2 + \nu_3^2), a_1 \nu_3^2 + a_2 \nu_2^2, a_1 \nu_2^2 + a_2 \nu_3^2, -\nu_2 v_3, 0, 0), \\
&(0, 0, 0, 2 a_2 \nu_1 v_2, 2 a_2 \nu_1 v_2, 2 a_1 \nu_1 v_2, -\nu_1 v_3, -\nu_2 v_3, \nu_3^2).
\end{aligned}$$

When using rectangular computational grids formed by coordinate planes, in one-dimensional problems of the splitting method only one of the coefficients ν_1, ν_2, ν_3 is nonzero. For a curvilinear grid these coefficients, generally speaking, may take any values satisfying the condition $\nu > 0$. They are calculated in each mesh according to the formula for $\bar{\mathbf{B}}^i$ from the system (8.1).

The splitting method with respect to spatial variables is applied not in a physical space but in a parametric one. According to this method, at each time step the sequence of seven one-dimensional problems

$$\begin{aligned}
A U_t^1 &= \bar{B}^1 V_{\xi_1}^1 + G^1, & U^1(t_0, \xi) &= U(t_0, \xi), \\
A U_t^2 &= \bar{B}^2 V_{\xi_2}^2 + G^2, & U^2(t_0, \xi) &= U^1(t_0 + \Delta t/2, \xi), \\
A U_t^3 &= \bar{B}^3 V_{\xi_3}^3 + G^3, & U^3(t_0, \xi) &= U^2(t_0 + \Delta t/2, \xi), \\
A U_t^4 &= Q V^4, & U^4(t_0, \xi) &= U^3(t_0 + \Delta t/2, \xi), \\
A U_t^5 &= \bar{B}^3 V_{\xi_3}^5 + G^3, & U^5(t_0 + \Delta t/2, \xi) &= U^4(t_0 + \Delta t, \xi), \\
A U_t^6 &= \bar{B}^2 V_{\xi_2}^6 + G^2, & U^6(t_0 + \Delta t/2, \xi) &= U^5(t_0 + \Delta t, \xi), \\
A U_t^7 &= \bar{B}^1 V_{\xi_1}^7 + G^1, & U^7(t_0 + \Delta t/2, \xi) &= U^6(t_0 + \Delta t, \xi),
\end{aligned} \tag{8.2}$$

is successively solved. As a result, we obtain $U(t_0 + \Delta t, \xi) = U^7(t_0 + \Delta t, \xi)$.

When integrating one-dimensional systems of equations, at the “predictor” step the method of limiting reconstruction of a solution described in Sect. 6.3 is applied. In order to a scheme remains conservative on a curvilinear grid where the matrix-coefficients $\bar{B}^i$, generally speaking, depend on coordinates, the approximation of these systems at the “corrector” step is performed with the help of the integro-interpolation method. Integrating the divergent system of Eqs. (7.10) over a curvilinear mesh in a physical domain and then applying Green’s formula, we obtain the equality

$$A \bar{U}_t = \frac{1}{\Omega} \sum_{l=1}^6 \Gamma^l \left(v_1^l B^1 + v_2^l B^2 + v_3^l B^3 \right) V^l + Q \bar{V} + \bar{G}, \tag{8.3}$$

where v_i^l are the direction cosines of an outward normal, Ω is the volume of a mesh, a bar means the average integral value over a mesh, and the superscript l is related to the faces of a mesh, in particular, Γ^l is the area of a corresponding face. Then the sum in the right-hand side of (8.3) is decomposed into three pairs of terms for opposite faces and each of them corresponds to a special approximation of derivatives with respect to spatial variables in the one-dimensional systems (8.2). For example, the terms associated with the coordinate surfaces $\xi_2 \xi_3$ provide an approximation of the expression $\bar{B}^1 V_{\xi_1}$, the terms associated with the surfaces $\xi_1 \xi_3$ approximate $\bar{B}^2 V_{\xi_2}$, and the terms associated with $\xi_1 \xi_2$ approximate $\bar{B}^3 V_{\xi_3}$.

The conditions of conjugation (pasting together) of solutions on interfaces, being boundary conditions for the systems (8.2), are formulated as the continuity conditions for the velocity vector and the stress vector on area elements (faces of meshes) and are added with equations on incoming characteristics. In the numerical implementation of these conditions, a refined grid obtained by intersection of grids on sides of interface is considered. After the calculation of required values of a solution on faces of a fine grid, the backward going to the meshes of original grids is performed by the averaging method.

Boundary conditions for contact of blocks with possible separation of contacting surfaces in the deformation process are implemented in a more complicated way. In this case the computational algorithms, worked out in Chap. 7, are applied. The algorithm of calculation of the main types of boundary conditions (in terms of velocities, stresses, or strains), presented in Sect. 6.3, and the procedure of correction of

a solution, in order for plasticity to be taken into account, according to the splitting method with respect to physical processes are implemented in the same way as on a rectangular grid.

8.2 Distinctive Features of Parallel Realization

On the basis of the computational algorithm presented above, a complex of applied programs for the numerical solution of two-dimensional (plane, axially symmetric) and three-dimensional problems of the dynamics of granular materials on multi-processor computer systems is worked out, [7–11]. The programming is fulfilled by means of the Single Program–Multiple Data (SPMD) technology in Fortran-90 using the Message Passing Interface (MPI) library, [1–3, 13–17].

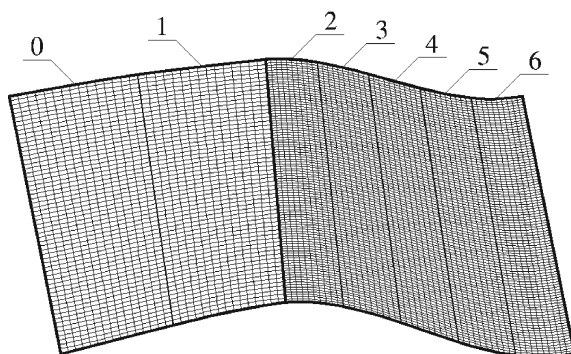
Program system allows to simulate the propagation of elastic–plastic waves produced by external mechanical effects in a granular medium body, aggregated of arbitrary number of heterogeneous blocks with curvilinear boundaries. It consists of a preprocessor program, a main program for computation of velocities and stresses, subroutines for implementation of boundary conditions and conditions of pasting together of solutions on inconsistent grids of neighboring blocks, and a postprocessor program. The universality of programs is achieved by a special packing of the variables used at each of computational nodes of a cluster into large one-dimensional arrays. Program system is equipped with subroutines which carry out 1D, 2D and 3D decomposition of a computational domain between computational nodes.

First we consider a plane problem with 1D decomposition of a computational domain between nodes of a cluster. Assume that blocks of a granular material body are arranged layer-by-layer in the horizontal and vertical directions and their boundaries are consistent. One may pass to a more general case of inconsistent boundaries by introducing the fictitious decomposition of a body discussed in the previous section.

Initial data of a problem required for computations are presented as text files organized similarly to relational tables. One of such files contains mechanical parameters of materials, in another file information on the block structure of a body (the number of horizontal layers, the total number of blocks in a layer, coordinates of vertices of blocks, identification numbers of materials, and spatial dimensions of grids) is stored. Computational grids, which, generally speaking, are not consistent on interblock boundaries, are constructed with the help of the Hermitian cubic splines in the simplest version. The pasting together is performed by a special procedure where a solution on a refined grid, obtained by the intersection of boundary meshes of neighboring blocks, is determined with the help of equations on characteristics and then is transferred on initial grids by the averaging method.

The preprocessor program packs initial data into two binary files of direct access: a file of reals where parameters of a material, a grid, and initial values of a solution are written block-by-block, and a file of integers which contains corresponding addresses (pointers), i.e. serial numbers of the first elements. Further real files of the same structure are created by the main program for conservation of resulting data in the

Fig. 8.4 Decomposition of a material body consisting of 2 blocks between 7 processes



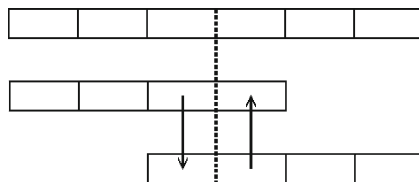
control points in the process of computations and for subsequent analysis of obtained results. The size of each file may considerably exceed the RAM memory space of an individual node of a cluster. On start of the program, each node reads all integer file and only the part of real file which is related to its process. Then the array of integers (the image of integer file) is reduced: grid parameters and pointers take individual values for this node. The balancing of computational load is achieved by means of uniform distribution of a grid domain between the cluster nodes. The simplest algorithm of decomposition is applied. If grid dimension in a block is greater than the average dimension per one node of a cluster, then this block is served by several nodes and, conversely, one and the same node serves several blocks following one after another if their total dimension does not exceed the average one.

Distribution of a domain between computational nodes is consistent with interfaces that essentially simplifies the algorithm of pasting together of solutions. An example of the construction of grids for a granular material body consisting of two blocks with decomposition of a problem between seven processes is shown in Fig. 8.4. The boundaries between blocks are shown by solid thick lines and interfaces between processes are shown by thinner lines.

The main program on each node of cluster makes a (on each time step). Data interchange between the processes is carried out at the level of coefficients of the solution decomposition on the basis from left eigenvectors at the predictor step of the solution of one-dimensional systems. Standard technology of the contour meshes

At each node of a cluster, the main program performs similar computations consisting of mutually coordinated step-by-step realization of the space-variable splitting method. The exception is represented by the processes which, in addition, perform the pasting together of solutions on inner boundaries. The pasting together conditions on horizontal boundaries are realized in the following way. Processes, which serve neighboring blocks (placed one under another), transfer required information to the left upper process which performs calculation of the whole boundary and sends results in the opposite direction. Horizontal boundaries are calculated at the first and fifth steps of the splitting method with respect to the spatial variables. This is performed almost simultaneously, insignificant delay may arise only because of

Fig. 8.5 Scheme of exchange between processes with contour meshes



the need for the executing process to transfer data for pasting together of solutions on the upper boundary of its block. However, all processes except left upper ones are in the wait state.

In the general case the pasting together on vertical boundaries at the second and fourth stages of splitting is implemented by pairs of processes. For reasons of minimization of the number of sends, both processes partially perform the same arithmetical operations. If a boundary lies inside a domain served by one process, then this pasting together is performed independently.

The solution of one-dimensional systems of equations in the vertical direction, parallel to the lines decomposing a domain between processes, is performed simultaneously as well. In the horizontal direction only coefficients of the decomposition of a solution with respect to the basis of left eigenvectors at the limit reconstruction stage are involved in the exchange between processes within one block. The standard technique of contour meshes is used, [3]. The exchange scheme corresponding to this technique is shown in Fig. 8.5. The arrays α_l and β_l , whose elements are shown by a set of meshes in the upper part of Fig. 8.5, turn out to be placed into different processes (a dotted line shows decomposition). The procedure of limit reconstruction performs iterative recalculation of values in meshes by the three-point stencil involving neighboring values. For convenience of recalculation in meshes adjoining with a dividing line, in the parallel program arrays with contour meshes are used and data exchange is performed with the help of the *MPI_SendRecv* function. To minimize running time, the volume of data block to be transferred (and, hence, the number of sends) is varied by the simultaneous solution of a certain number of one-dimensional systems.

The postprocessor program performs re-sampling down of files with results of computations in the control points, running meshes of a grid in each block with given step. Such re-sampling down is necessary because files may be of very large size and it requires considerable time to transport them through a network. Graphical output of results is carried out with the help of special programs for an usual personal computer.

Now let us consider a three-dimensional case. In Fig. 8.6 one can see an example of a finite-difference grid in 24-block domain with curvilinear boundaries. Examples of 1D, 2D and 3D decompositions of computational domain between processes for an individual spatial block in the form of a cube are represented in Fig. 8.7.

When solving spatial dynamic problems, the work technique with program system is as follows. First it is necessary to input initial data of a problem: a domain of solution consisting of three-dimensional curvilinear blocks, mechanical properties

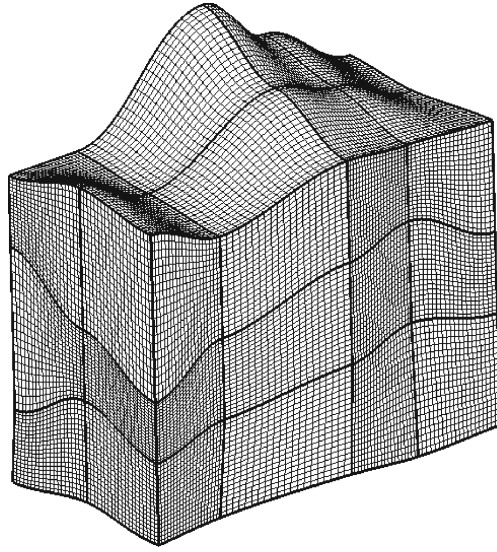


Fig. 8.6 Example of a grid in 24-block computational domain

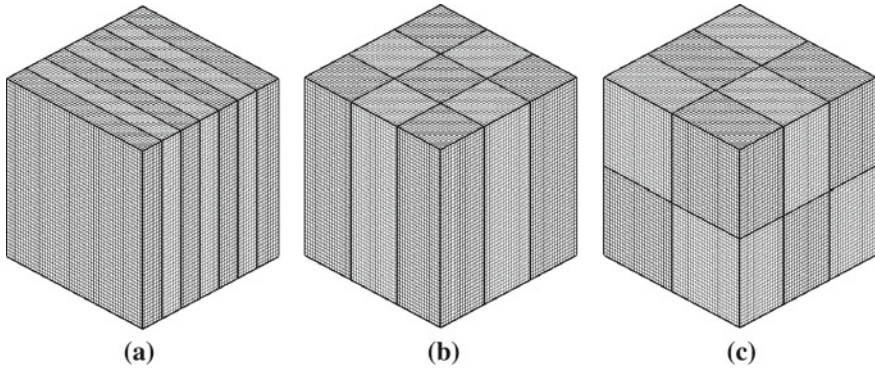


Fig. 8.7 Examples of decompositions of a block: **a** 1D decomposition between 7 processes, **b** 2D decomposition between 9 processes, **c** 3D decomposition between 12 processes

of materials in blocks, and loading conditions. Geometry of a block domain is defined by coordinates of block vertices in a Cartesian coordinate system. They are formed as a text data file. At the beginning of this file the number N_1 of layers with respect to x_1 , the number N_2 of strips in a layer with respect to x_2 , and the number N_3 of blocks in a strip with respect to x_3 are stored. Boundaries of blocks are assumed to be consistent. Thus, the number of blocks in a granular material body is equal to the product $N_1 N_2 N_3$.

In the second text file of initial data, phenomenological parameters of materials in a certain system of units (for instance, in SI) and dimensions of finite-difference grids

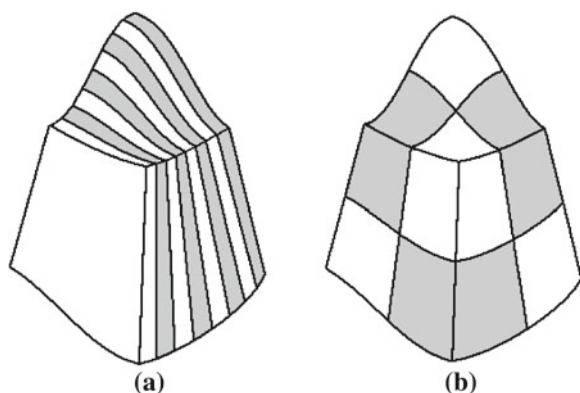


Fig. 8.8 Different ways of distribution of a curvilinear computational domain between 8 processes: **a** 1D decomposition, **b** 3D decomposition

in blocks with respect to three spatial coordinates are stored. Each material is defined by density, two elastic constants (velocities of longitudinal and transverse elastic waves), the yield point, and the internal friction coefficient. Some of materials may be ideal elastic or elastic–plastic. In this case the corresponding part of parameters is not given.

With the use of coordinates of vertices and parameters of materials, the special preprocessor program constructs independent (inconsistent) curvilinear difference grids in blocks and calculates the maximum permissible step in the sense of Courant–Friedrichs–Levi for integration of a system of equations on the basis of explicit difference schemes. In addition, the program distributes a computational domain between parallel computational nodes, specifies initial data (homogeneous in the case that the initial state of a body is natural), and packs coordinates of nodes of a grid and initial values of a solution into a binary data file which is a starting point for the main program.

Examples of 1D and 3D decompositions of a curvilinear block when parallelizing a problem into eight processes are shown in Fig. 8.8.

A comparison of different methods of distribution of a problem between nodes shows that 3D decomposition of a domain has the obvious advantages over 1D one. The number of surfaces through which data exchange is performed in the calculation process is minimal (see Fig. 8.8: three surfaces with near equal dimension of a grid as compared with seven ones). Data exchange operations require much more time in comparison with independent arithmetic operations, hence, a gain in program execution time is quite significant. However, the situation may change drastically when, for example, one dimension of a block considerably differs from two other ones. In this connection, the program system includes all variants of decompositions mentioned above, among them the variant with the minimal number of exchanges, which requires more time for the preprocessor program processing but minimizes execution time of a computational program due to exchanges [4, 5].

Without going into details of this variant (it is based on the algorithm of simple enumeration), we consider the problem on minimization of the number of exchanges from the general theoretical point of view. Assume that $k_1 \geq 1$, $k_2 \geq 1$, and $k_3 \geq 1$ are the quantity of segments into which an individual fixed block of a computational domain is subdivided with respect to three spatial directions. A block turns out to be subdivided by $k_i - 1$ surfaces along each direction. The number of exchanges through surfaces is proportional to the number of meshes. This number can be calculated by the formula

$$M = (k_1 - 1) n_2 n_3 + (k_2 - 1) n_1 n_3 + (k_3 - 1) n_1 n_2,$$

where n_1 , n_2 , and n_3 are dimensions of a spatial difference grid in a block. It is required, varying k_i , to find the least value of M for given number $n_p = k_1 k_2 k_3$ of processors.

This problem relates to integer programming, nevertheless, continuous methods of conditional optimization can be used to obtain its rough approximate solution. Let $L = M - \lambda k_1 k_2 k_3$ be the Lagrange function with an undetermined multiplier λ . From the minimum condition for L we have

$$n_2 n_3 = \lambda k_2 k_3, \quad n_1 n_3 = \lambda k_1 k_3, \quad n_1 n_2 = \lambda k_1 k_2.$$

Eliminating λ , we obtain

$$\frac{n_1}{k_1} = \frac{n_2}{k_2} = \frac{n_3}{k_3}.$$

Hence, the optimal way of decomposition of a grid of a block is close to that into equal cubes in the space of parameters ξ_1, ξ_2, ξ_3 .

We can follow this strategy without knowing an exact solution of the problem on minimization of the number of exchanges. We notice that the decomposition into cubes provides uniform load of executing processors at the stage of realization of a computational algorithm which results in decrease of downtime when exchanging data.

Conditions for pasting together of solution on inconsistent grids of interfaces are realized at each time step by the following scheme. Processes, which serve neighboring blocks, transfer required information to one of these processes with the least number. This process performs independent calculation of the whole boundary and transfers results in the opposite direction. This proceeds in all blocks of a body almost simultaneously, insignificant delay may arise only due to the need for an executing process to transfer data for pasting together of solutions on the opposite boundary of its block. But all processes except executing ones are in the wait state.

When calculating elastic waves in a homogeneous block, execution time of the program was measured for different number of processors of a computational system. A test example with a curvilinear grid consisting of $60 \times 60 \times 60$ meshes in the spatial domain shown in Fig. 8.2 was used. For each start, 10 time steps were calculated. Running time of the program was fixed with the help of the *MPI_Wtime*

Fig. 8.9 Dependence of running time on the number of processors

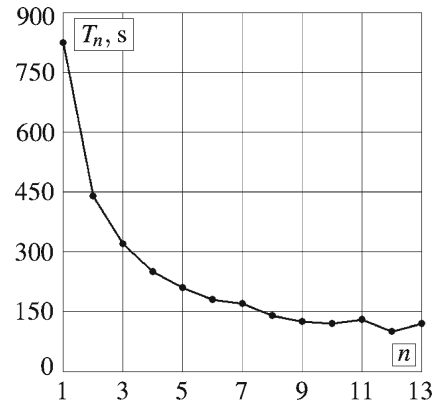
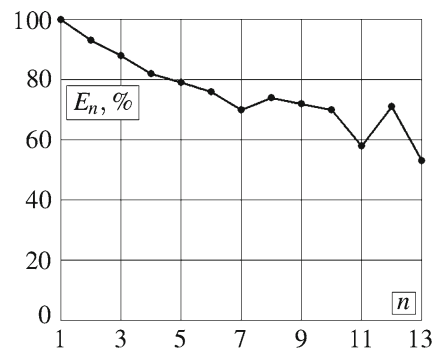


Fig. 8.10 Efficiency of parallelizing the algorithm



function called at the beginning and at the end of the program. This function returns astronomical time in seconds from a certain instant in the past. Efficiency of parallelizing the algorithm was calculated from obtained values of time. Efficiency of parallelizing into n processors is

$$E_n = \frac{T_1}{T_n n} \times 100\%,$$

where T_1 is the execution time of a program with one processor, T_n is the execution time of a program with n processors. Thus, E_n is the ratio of acceleration of execution of a program with n processors and the number n .

The graph of dependence of execution time of the program on the number of loaded processors is shown in Fig. 8.9. Acceleration of calculations with increasing n is observed, however, efficiency of parallelizing gradually decreases to 60% (see Fig. 8.10) due to large volume of data transferred at each step of the computational algorithm. This is typical for problems of moderate dimension. The number of nodes of a difference grid, processed with one processor, increases with refining subdivision of a block. At the same time, the number of arithmetic operations performed independently and the efficiency of parallelizing the problem increase as well. Thus, the

known thesis that the number of processors should be in reasonable correspondence with dimension of a problem is confirmed.

It should be noticed that, depending on the number of processors, different ways of decomposition of a domain were applied. In the case that n is a prime number (2, 3, 5, 7, 11, 13), 1D decomposition in the x_3 direction was used. In the remaining cases (4, 6, 8, 9, 10, 12) a domain was subdivided in two or three directions (see Fig. 8.7). As mentioned above, 3D decomposition considerably decreases the volume of transfers. This explains non-monotone variation of running time and efficiency of parallelizing the algorithm. For example, efficiency of parallelizing with 12 processors is significantly higher than with 11 or 13 ones which is due to three-dimensional decomposition of the domain and decreasing volume of transfers.

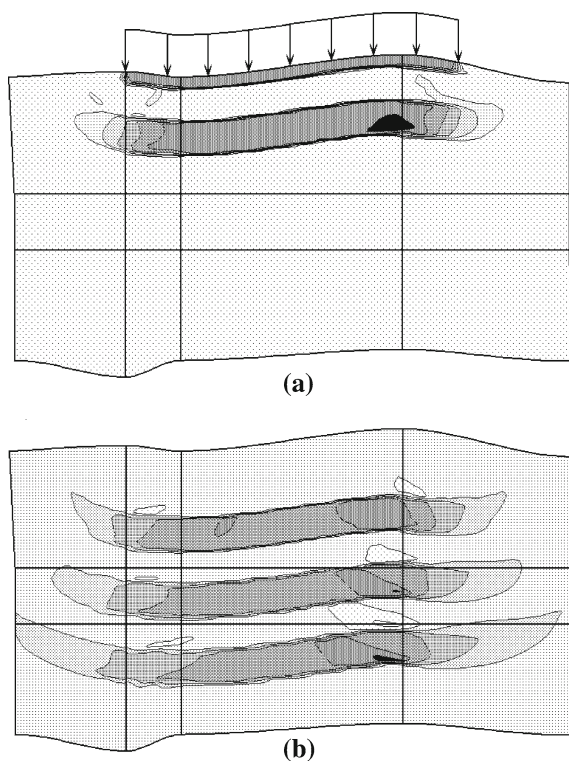
8.3 Results of Two-Dimensional Computations

For algorithm and program debugging, computations for a number of problems on propagation of stress waves in block materials were performed on multiprocessor computer systems of the MVS type of the Institute of Computational Modeling (ICM) of Siberian Branch of the Russian Academy of Sciences and the Joint Supercomputer Center (JSCC) of the Russian Academy of Sciences. Comparison of the results of computations with exact solutions and results obtained with a PC on the basis of successive programs show high efficiency of the parallel program system.

The sequential version of the method for the solution of one- and two-dimensional dynamic problems for a granular material with elastic and plastic properties was tested in [6]. Comparative computations with a cluster system gave the same results independently of the number of nodes being used. In additions, computations for the plane problem on interaction of compression waves (signotons) in an inhomogeneously loosened granular material with formation of transverse cumulative splash, considered in Sect. 6.5, were repeated. For this problem, running time for 600 time steps on a spatial grid consisting of 300×300 meshes with an Intel Pentium(R)-IV CPU 2.8 GHz PC is about 37 h. For comparison, with the MVS-1000M cluster of the Joint Supercomputer Center (based on Alpha 21264 processors) with ten nodes being loaded it takes 2 h 53 min (29 h of total time). With the use of twenty nodes it takes 1 h 38 min (about 33 h), and in the case of thirty nodes it takes 1 h 10 min (35 h). High efficiency of parallelizing in this problem is achieved due to the fact that iterative algorithms, applied for the numerical implementation of nonlinear constitutive relationships of a granular material, require considerable time of independent work of processors. However, excessive increasing the number of nodes loaded turns out to be unreasonable because of the increase of time costs for exchanges. The size of files where information on a solution at each time step is stored is 6.3 MB.

To control the work of the procedure of pasting together of solutions on boundaries of inconsistent grids of blocks, computations for the two-dimensional problem on propagation of elastic waves generated by the action of localized impulsive load on the upper boundary of a homogeneous body, which is fictitiously subdivided

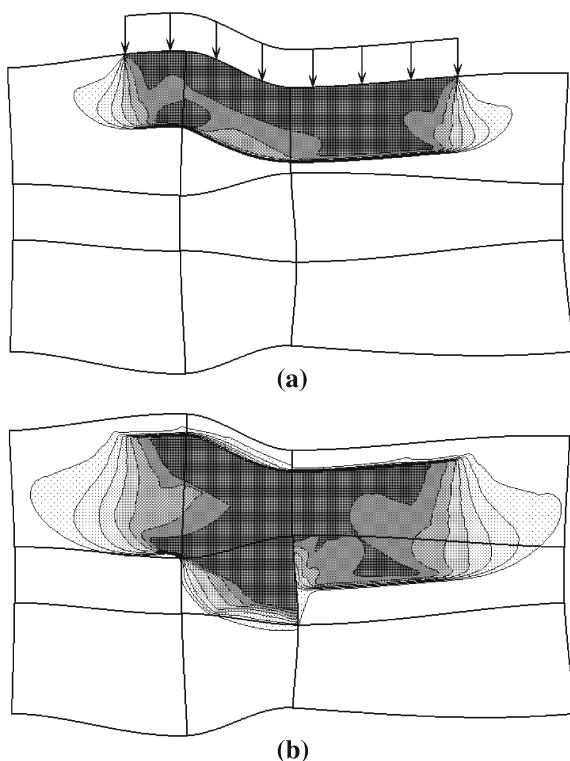
Fig. 8.11 Propagation of elastic waves of stresses: level curves of normal stress σ_{11} ; **a** $t = 25 \mu\text{s}$, **b** $t = 70 \mu\text{s}$



into 12 blocks, were performed. The lateral and lower boundaries are non-reflecting surfaces, through them waves freely go to infinity. As a hypothetical material in these numerical experiments was used iron with the next phenomenological parameters: the density is 7850 kg/m^3 , velocities of longitudinal and transverse waves are 6000 and 3210 m/s, respectively. Horizontal length of a computational domain is 1 m, the impulse duration is equal to $10 \mu\text{s}$.

In Fig. 8.11 level curves of the stress σ_{11} , corresponding to the vertical direction x_1 , are presented as three Π -shaped impulses travel. Crowding of level curves is observed on fronts of shock waves. Figure 8.11a relates to the time of action of the second impulse. One can see fronts of the loading wave, generated by the first impulse, the corresponding unloading wave, and the repeated loading wave. In Fig. 8.11b there are six typical fronts. These are loading and unloading waves generated by each of three impulses. Computations are performed with the MVS-1000M cluster on a very fine inhomogeneous grid whose meshes in neighboring blocks differ in size by a factor of two–three. The size of a file of reals is 46.5 MB. Taking into account distribution of memory between nodes, this quantity is not the limit for a multiprocessor system, however, it exceeds available RAM memory space of PC of average power. Running time for 1000 time steps with 43 nodes of a cluster being loaded is about 20 min

Fig. 8.12 Reflection of waves from a rigid inclusion: level curves of normal stress σ_{11} ; **a** $t = 40 \mu\text{s}$, **b** $t = 80 \mu\text{s}$



which is equivalent to 14 h of work of a successive program. The presented results show that there is no inadequate reflection or refraction of waves, when going through inner vertical and horizontal rectilinear boundaries, caused by errors of the pasting together of solutions on interblock boundaries.

Similar computations for the case of curvilinear boundaries, which subdivide fictitious blocks of the same material, give the same results. It turns out that the presence of interfaces has no considerable influence on a numerical solution.

Computations for the process of reflection of compression shock waves, generated by localized impulsive load, from a rigid inclusion in a granular material body, were performed on the MVS-1000M cluster of the Joint Supercomputer Center. A material of inclusion is iron, its parameters are given above. Phenomenological parameters of a granular material in a compact state: the density is 2500 kg/m^3 , velocities of longitudinal and transverse waves are 3000 and 1500 m/s, respectively. The impulse duration is $70 \mu\text{s}$. In Fig. 8.12 level curves of the stress σ_{11} in the plane problem are shown for two instants of time. A material body is subdivided into nine blocks. The block in the center is elastic and the remaining blocks consist of homogeneous compliant granular material (soil) with elastic and plastic properties. Boundary conditions are formulated in the same way as in the previous case. A single Π -shaped impulse of

external load is considered. Figure 8.12a corresponds to the instant of time, by which an incident shock wave has not achieved the surface of the inclusion yet. Up to this instant, it moves freely through a homogeneous material. Then interaction takes place and a reflected wave of small amplitude which moves in the opposite direction from the inclusion is generated (in Fig. 8.12b one of typical instants is presented). The front of the incident wave accelerates in the domain of inclusion since the velocity of longitudinal elastic waves here is higher than in a granular material. This results in generation of additional compression waves propagating to both directions (to the right and to the left) of the inclusion. Computations show that in a neighborhood of the points, at which load varies drastically, the strain concentration zones typical to a granular material arise with time. The size of the file, where information on a solution is stored, is 13.5 MB. Running time for 300 time steps with 34 nodes of a cluster is 45 min, whereas computations with a successive program for this problem take about 22 h.

8.4 Numerical Solution of Three-Dimensional Problems

With the help of the program system described above, computations of interaction of the compression shock waves (signotons) in an inhomogeneously loosened granular material in the three-dimensional formulations were performed on the MVS-1000M cluster of ICM SB RAS, [10–12]. It is shown that, propagating in a loose material, plane fronts of signotons (shock-wave transitions changing a sign of strain), generated by periodic impulsive load, are curved gradually and slow down in a domain of strong loosening in comparison with a denser domain. Signotons are followed by unloading waves propagating through a compressed material where velocities of waves are constant, hence, their fronts remain almost plane up to the instant when they meet. At the point of meeting of signotons, as a result of interaction of curved fronts, a cumulative splash (a characteristic zone of compressive stresses) arises. It travels with time in the vertical direction to the loosened domain.

Numerical solution of the problem of cumulative interaction of signotons in plane geometry is given in Sect. 6.5. Results of three-dimensional computations are shown in Fig. 8.13. A quarter of a rectangular block that is dissected by two planes of symmetry along the vertical axis is presented. The upper boundary of the block is free of stresses and the lower boundary is a non-reflecting interface. At the lateral faces uniformly distributed impulsive load equal to 750 MPa acts. The time of action of impulse is 25 ms. Initial strain of loosening a material along the horizontal axes is assumed to be a linear function with respect to the vertical coordinate that increases towards the upper boundary in the range from 0 to 7%.

Computations are performed for a compact ground. Velocities of longitudinal and transverse waves are 1500 and 160 m/s, respectively, the density is 1990 kg/m³, the yield point is 15 MPa, the angle of internal friction is 27°. In Fig. 8.13 level curves of normal stress σ_{11} at different instants of time ($t = 25, 32, 39, 42, 46$, and 56 ms) are shown. The values of levels vary in the range from –3600 to 100 MPa. Thus, in this

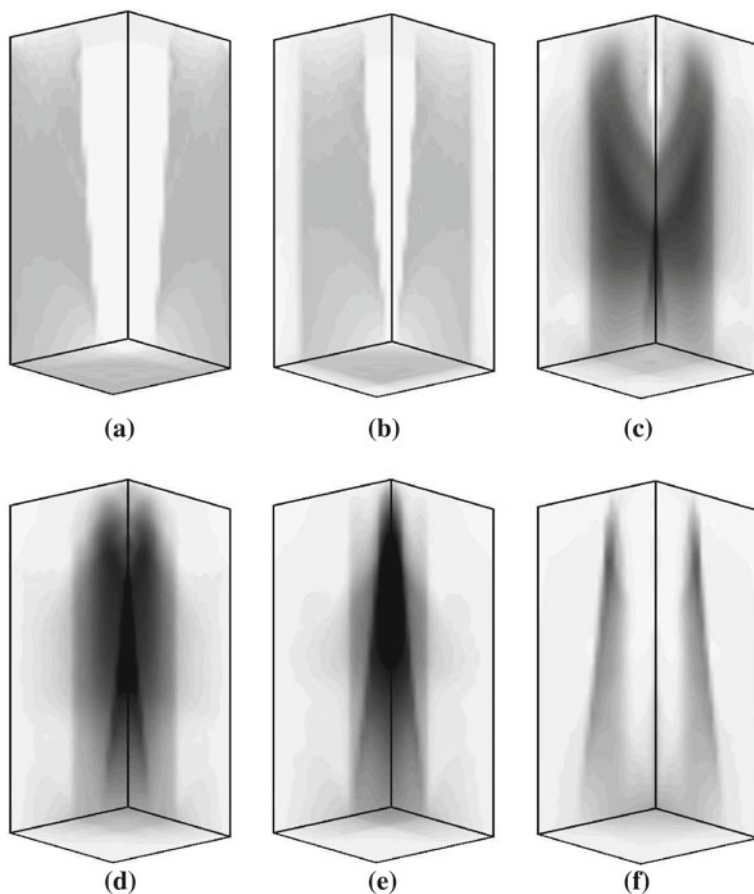
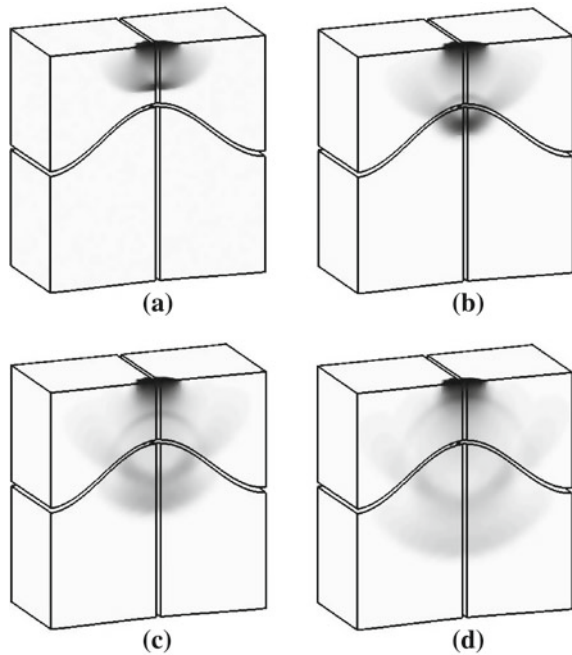


Fig. 8.13 Cumulative interaction of signotons: level curves of normal stress σ_{11} ; **a** $t = 25$ ms, **b** $t = 32$ ms, **c** $t = 39$ ms, **d** $t = 42$ ms, **e** $t = 46$ ms, **f** $t = 56$ ms

variant of the problem compressive stress in the cumulation zone is approximately five times higher than stress on the fronts of signotons before their interaction.

A difference grid consisting of $100 \times 100 \times 100$ meshes was used. Running time for 40 time steps (this is about 50 ms) on such a grid with 8 processors being loaded is approximately 4h. For comparison, running time for 20 time steps (this is about 50 ms as well) on a difference grid consisting of $50 \times 50 \times 50$ meshes is 15 min and running time for 60 time steps (50 ms) on a grid consisting of $150 \times 150 \times 150$ meshes is 20h. We can see that with k -times refining a spatial grid and decreasing time step by a factor of k as well, running time increases approximately by a factor of k^4 . This means that in this case volume of data transmissions between processors, increasing by a factor of k^3 , does not reduce excessively efficiency of a parallel

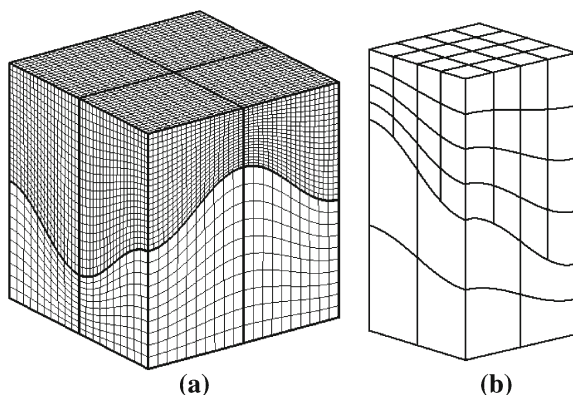
Fig. 8.14 Passage of waves through a convex interface: level curves of normal stress σ_{11} ; **a** $t = 12$ ms, **b** $t = 20$ ms, **c** $t = 25$ ms, **d** $t = 30$ ms



program. The index E_8 of parallelizing efficiency (see Sect. 8.2) on a grid consisting of $100 \times 100 \times 100$ meshes is about 90%.

In Fig. 8.14 results of the numerical analysis of stress waves generated by a periodic perturbation source on the surface of a body are presented. The upper boundary excepting a zone of application of impulsive load is assumed to be free of stresses. On the lower boundary and lateral boundaries non-reflecting conditions are imposed. They correspond to infinite length of a body and are formulated for one-dimensional systems of Eqs. (8.2). The body consists of two layers divided by a surface of the type of elliptic paraboloid of revolution which is convex in both horizontal directions. The upper layer is compact ground with the same mechanical parameters as in the previous problem. The lower layer is strong rock where velocities of elastic waves are 3500 and 1950 m/s, the density is 2620 kg/m^3 , the yield point is 150 MPa, and the angle of internal friction is 60° . On the upper boundary of a body (within the radial distance of 5 m from its center) the impulsive load of 250 MPa is applied. The time of action of impulse is 30 ms. We consider the case that the epicenter of load is over the upper point of the interface of layers. In computations the domain of solution is decomposed into eight curvilinear blocks. The plane vertical interface are introduced fictitiously in order to simplify analysis of results with the help of graphical programs. The interfaces are planes of symmetry of the problem. In Fig. 8.14 level curves of the stress σ_{11} at the instants $t = 12, 20, 25$, and 30 ms are shown. The values of levels vary in the range from -250 MPa to zero. For each block a difference grid consisting of $50 \times 50 \times 50$ meshes is used. Running time for 100 time steps (this

Fig. 8.15 Computational domain for two-layered medium with a curvilinear interface: **a** difference grids in layers, **b** a quarter of computational domain with decomposition between processes



is about 5 ms) on this grid with loading a program on 8 processors of a cluster is 3 h. Due to convexity of the interface, waves are focused in the lower layer. Conversely, reflected waves are defocused in the upper layer. Visible reflection of waves from the lateral faces of the body and from the lower boundary is not observed.

In Figs. 8.16 and 8.17 the results of numerical solution of Lamb's problem on the action of normal concentrated force on the boundary of a half-space in the three-dimensional formulation are presented. In contrast to the classical case, a half-space here is inhomogeneous. It consists of two elastic layers divided by a surface of the type of hyperbolic paraboloid whose convexity directions along two horizontal axes are opposite to one another. As before, on the lower and lateral faces of a computational domain the non-reflecting conditions are imposed. Hence, the continuation of the interface outside this domain should be imagined as a ruled surface with generating lines being perpendicular to lateral planes. Moduli of elasticity for the upper and lower layers are taken as for compact ground and strong rock, respectively. Different resistance of both materials to tension and compression are not taken into account in computations. In Fig. 8.15a a computational domain in the form of a cube with sides of 100 m decomposed into eight curvilinear blocks with a coarse difference grid constructed in each block independently is shown. In Fig. 8.15b a quarter of the domain (two blocks) is allocated taking into account the symmetry of the problem. It is distributed by the preprocessor program between 72 processes according to the principle of uniform load: $4^3 = 64$ processes in the upper block and $2^3 = 8$ processes in the lower one. In the lower block, filled with a strong rock, a step of grid is about twice more than in the upper one. In this case the maximum permissible time step, selected in accordance with the Courant–Friedrichs–Levi stability condition, turns out to be almost equal for both materials since velocity of longitudinal waves in a compact ground is twice less than in a strong rock.

When performing computations by this scheme on the MVS-15000 cluster of the Joint Supercomputer Center (Moscow), each of 72 nodes solved a part of the problem on a subgrid consisting of $50 \times 50 \times 50$ meshes. Running time is about 10 h. Resampling down of the obtained results was performed with respect to three spatial

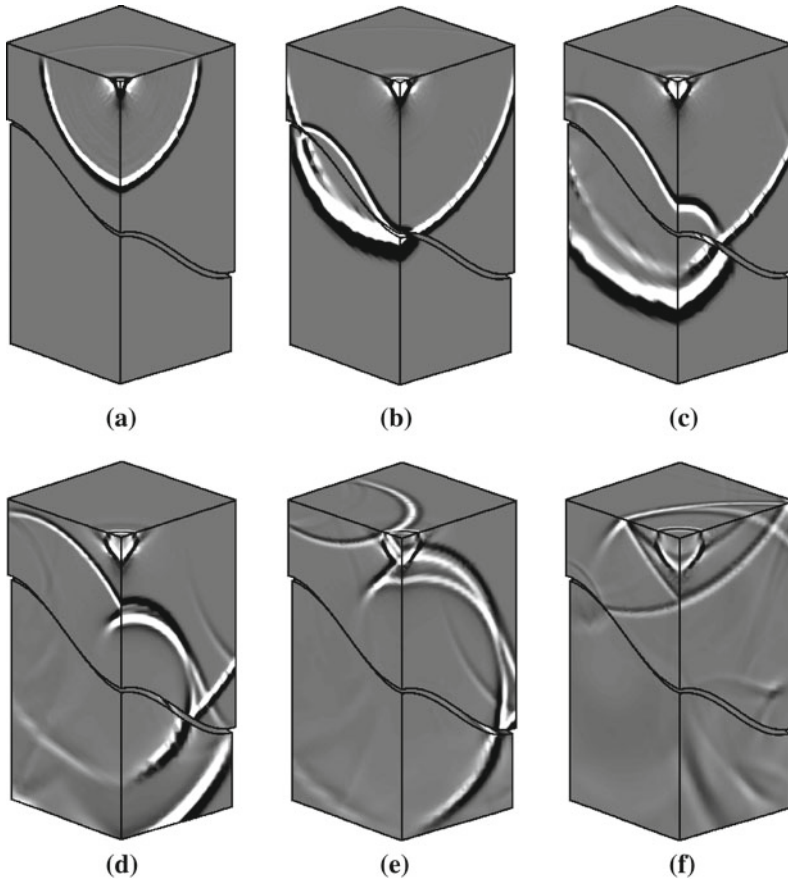


Fig. 8.16 Reflection of waves of the stress σ_{11} from an interface: **a** $t = 23$ ms, **b** $t = 34$ ms, **c** $t = 40$ ms, **d** $t = 49$ ms, **e** $t = 57$ ms, **f** $t = 71$ ms

coordinates in the ratio of 1:4. If graphs were constructed on the basis of numerical solution of the problem in full, wave fronts would be four times thinner, however, copying output files would require to increase the time of loading of a global network, which is rather long, by a factor of 64. Besides, for pictorial presentation, wave fronts were artificially thickened: only level surfaces lying in much narrower range than the range between minimal and maximal values were shown.

In Fig. 8.16 level surfaces of the normal stress σ_{11} at the instants $t = 23, 34, 40, 49, 57$, and 71 ms are represented. The position of fronts of incident and reflected longitudinal waves can be uniquely determined from these surfaces. Velocity of transverse waves in a compact ground is on the order of magnitude smaller than velocity of longitudinal waves, hence, a front of an incident transverse wave almost does not move towards the interface and stays in the neighborhood of the point of load application during the whole time interval being considered.

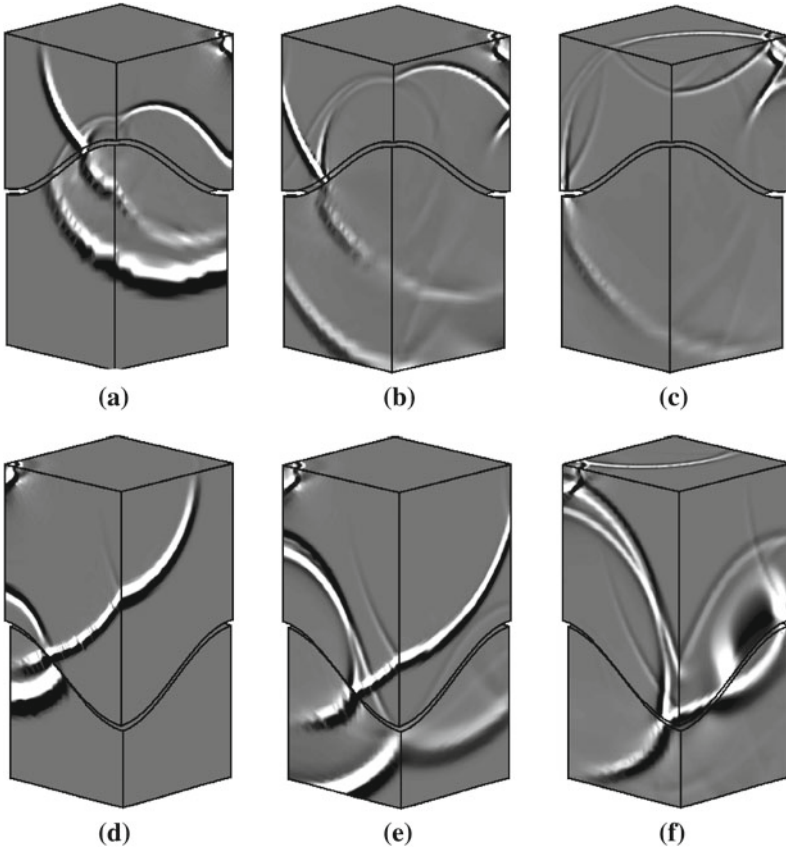
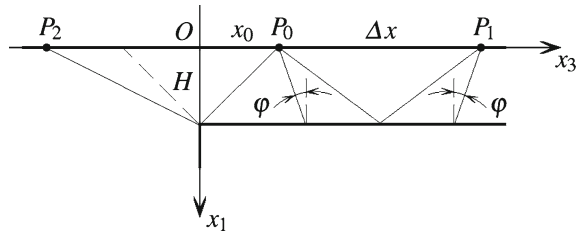


Fig. 8.17 Level surfaces of σ_{11} from the backside; *left side* of a body **a** $t = 40$ ms, **b** $t = 49$ ms, **c** $t = 57$ ms; *right side* of a body **d** $t = 40$ ms, **e** $t = 49$ ms, **f** $t = 57$ ms

In Fig. 8.17 level surfaces of the normal stress are shown as viewed from two backsides. Their spatial orientation can be determined from the position of an incident transverse wave being in the neighborhood of the point of load application. Fig. 8.17a–c (the left side of a body) and Fig. 8.17d–f (the right side of a body) relate to the instants $t = 40$, 49, and 57 ms, respectively.

According to computations, impulsive load concentrated in space and time generates in a material body two paired waves (adjacent to one another) rather than one, namely, a loading wave and an unloading wave. Stresses on these waves are equal in magnitude and opposite in sign. Fronts of loading waves are shown by black and fronts of unloading waves are shown by white (intermediate grey corresponds to the stress-free state of a material). After reflection, the sequence of waves remains the same. We notice that at the instant of reflection of an incident wave from an interface, in addition to a longitudinal wave, a transverse wave arises in strong rock. It travels downwards gradually passing through non-reflecting boundaries. We also

Fig. 8.18 System of reflected and refracted waves

notice that the wave diffraction may result in the formation of caustics, i.e. zones of increased stresses (see the last fragment in Fig. 8.17), which disappear in the process of unsteady motion.

Further we present the results of numerical solution of Lamb's problem for a medium with rigid inclusion in 3D formulation. In the upper layer of a double-layer elastic material of the thickness H and in the part $x_3 \leq 0$ of the lower layer, parameters of elasticity of a compact ground are given. In the remaining part of the lower layer, parameters of a strong rock (presented above) are given. In the direction of the horizontal axis x_2 the body of a material is of infinite length. Thickness of the lower layer is infinite as well.

The main system of waves, generated by an impulsive source at the point P_0 on the surface of the body, is shown in Fig. 8.18. It involves an incident longitudinal wave arriving at the instant $t = \Delta x/c_p$ at the point P_1 being at the distance Δx from the point P_0 , a reflected longitudinal wave arriving at the instant $t = \sqrt{4H^2 + \Delta x^2}/c_p$, and a refracted wave. A reflected wave is formed by two rays joining the points P_0 and P_1 with the reflection depth point lying on the interface midway between them. A refracted longitudinal wave is formed by a three-segment broken line. Its horizontal segment corresponds to a longitudinal wave moving along the interface with velocity c'_p in a strong rock. The slope angle φ of the first and third segments with respect to the vertical direction is determined by Snell's law:

$$\frac{c_p}{\sin \varphi} = \frac{c'_p}{\sin \varphi'}, \quad \varphi' = \frac{\pi}{2}.$$

The arrival time of wave at the point P_1 can be found by the formula

$$t = \frac{2H}{c_p \cos \varphi} + \frac{\Delta x - 2H \tan \varphi}{c'_p} \equiv \frac{2H \cos \varphi + \Delta x \sin \varphi}{c_p}.$$

At the instant of formation, a refracted wave is separated from the leading front of a reflected wave and then moves with velocity $c'_p > c_p$, overtaking the incident transverse wave at some instant of time. The distance $\Delta x = 2H \tan \varphi$ from the source to the point of formation is determined from the condition that reflected and refracted waves arrive at the same time, and the distance $\Delta x = 2H \cos \varphi / (1 - \sin \varphi)$

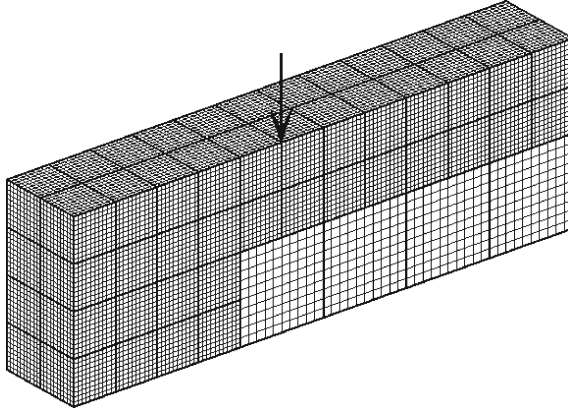


Fig. 8.19 Distribution of a computational domain between 68 processes

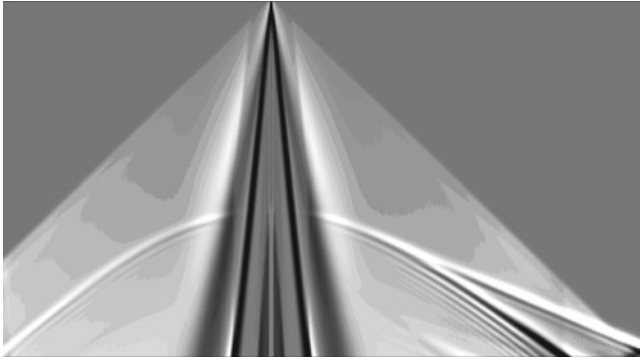


Fig. 8.20 Seismogram of u_1 for $x_0 = H/2$, $x_1 = 0$, $x_2 = H$, $x_3 \in (0, 6H)$

to the overtaking point is determined from the condition that incident longitudinal and refracted waves arrive at the same time.

To the left of impulsive source, a refracted wave does not occur provided that the distance x_0 between the point P_0 and the vertical boundary of a rigid inclusion is less than $2H \tan \varphi$. Then, depending on the position of the receiving point P_2 , along with an incident wave, a wave reflected from the interface ($\Delta x \leq 2x_0$) or a wave reflected from the angle of inclusion ($\Delta x > 2x_0$) travels to the left. The arrival time in the first case is determined from the formula for a reflected wave and in the second case

$$t = \frac{\sqrt{H^2 + x_0^2} + \sqrt{H^2 + (\Delta x - x_0)^2}}{c_p}.$$

We can observe characteristic features of a wave pattern, analyzing seismograms obtained with the help of the SeisView program system after the computations on 68

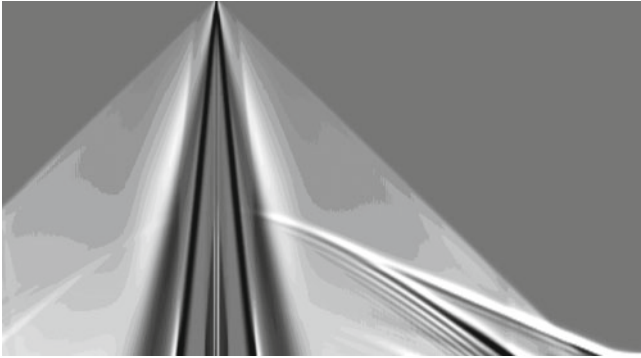


Fig. 8.21 Seismogram of u_1 for $x_0 = 0, x_1 = 0, x_2 = H, x_3 \in (0, 6H)$

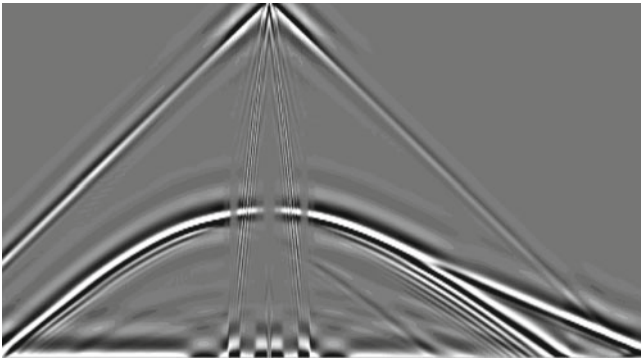


Fig. 8.22 Seismogram of u_1 for $x_0 = H/2$, result of filtration

nodes of the MVS-100k cluster of JSCC RAS. A coarsened grid with distribution of a domain between processes for such computations is shown in Fig. 8.19. In fact, the dimension of a grid processed by each processor is $50 \times 50 \times 50$ meshes. A concentrated normal load $\sigma_{11} = -\bar{p}_1 \delta(\mathbf{x}) \delta(t)$ acts instantaneously at one mesh of the upper boundary of the computational domain. On the frontal boundary of the domain the symmetry conditions are imposed, and on the remaining parts of the lateral surface and on the lower boundary the non-reflecting conditions are imposed. The upper boundary is a free surface. The point P_0 of the impulse action is shown by a vertical arrow.

Numerical results shown in Figs. 8.20 and 8.21 correspond to the cases of $x_0 = H/2$ and $x_0 = 0$, respectively. The time-distance curves of incident longitudinal and refracted waves are rectilinear segments. The time-distance curves of reflected waves are piecewise parabolic splines changing their curvature at the points of passage from reflection from an interface to that from an angle of inclusion (in the latter case, where a perturbation source is situated immediately over the edge of an inclusion, the point of the change of curvature is at the same place). Reflected

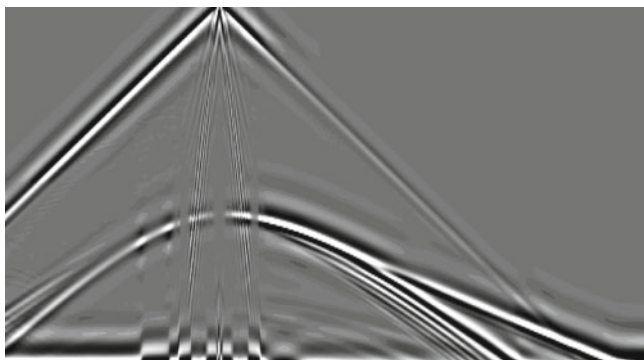


Fig. 8.23 Seismogram of u_1 for $x_0 = 0$, result of filtration

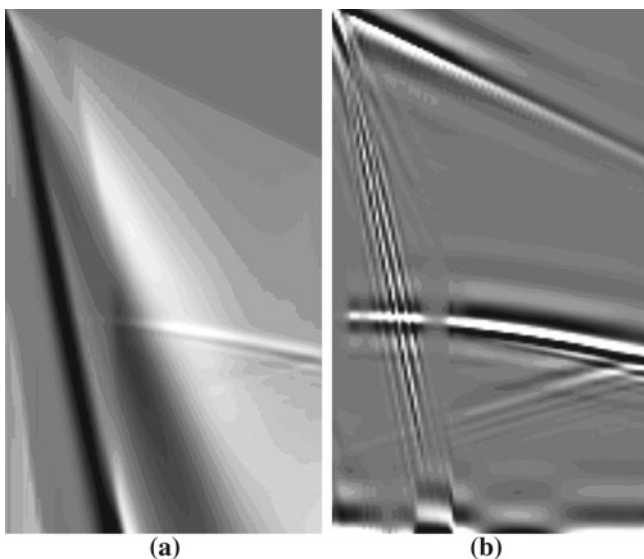
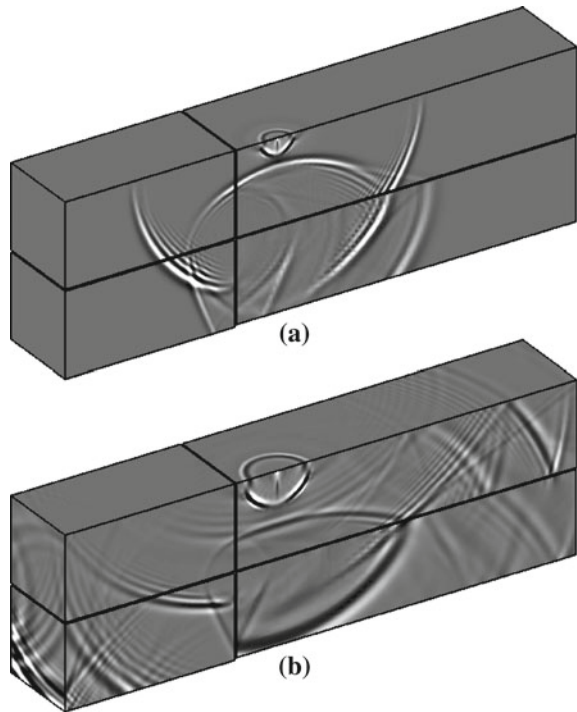


Fig. 8.24 Seismogram of u_1 for $x_0 = H/2$, $x_1 = 0$, $x_2 \in (0, H)$, $x_3 = 3.5H$: **a** seismogram, **b** result of filtration

and refracted waves are more clearly visible in Figs. 8.22 and 8.23 obtained by filtration with the help of SeisView. Coordinates of the points of intersection of wave fronts obtained numerically are in good quantitative agreement with the exact values calculated by the above formulae.

The seismograms given above describe the behavior of the vertical component u_1 of displacement depending on time in the case that receivers are on the upper boundary of a computational domain along the line passing parallel to the x_3 axis through the point of load application. In Fig. 8.24 seismogram of u_1 in the transverse

Fig. 8.25 Level surfaces of the stress σ_{11} for elastic medium with rigid inclusion ($x_0 = H/2$): **a** $t = 23$ ms, **b** $t = 46$ ms



direction (along the x_2 axis) for $x_0 = H/2$ and the result of its filtration are shown. For $x_0 = 0$ the seismogram in this direction looks similarly.

In Fig. 8.25 systems of waves propagating inside a body are shown at different instants of time. Symmetry of the problem is taken into account. Figure 8.25a relates to the instant of time when a reflected wave did not achieved yet the free surface of a body. Figure 8.25b relates to the instant of time when a refracted wave outruns an incident longitudinal wave.

To analyze the influence of transverse waves, it is required to perform computations on a longer time interval since, as mentioned above, in compact ground the velocity of transverse waves is lower than that of longitudinal waves almost by a factor of 10. Using fine computational grids, one can analyze qualitative variation of seismograms taking into account different resistance of a ground to compression and tension. However, to this end it is necessary to use special methods of the computer processing and visualization of results since, as shown by the numerical experiments performed, immediate analysis of seismograms with the help of SeisView (including the application of filtration algorithms) does not allow one to fix new waves characteristic for a granular material.

To conclude, we notice that the main purpose of the numerical experiments described in this chapter is in careful verification of adequacy of the program system. The necessity of this verification is aggravated by the fact that the debugging of

parallel programs is a more complicated and laborious process in comparison with usual sequential programming. Adequacy of the program system is confirmed by comparison of numerical results with exact solutions of one-dimensional problems, by verification of the symmetry of velocity and stress fields in a number of two-dimensional and three-dimensional problems, and by comparison of the qualitative behavior of solutions with results of analysis on the basis of methods of geometrical optics. On the whole, the results show that the multiprocessor computer systems have considerable advantages over the personal computers provided that, when loading a problem, the number of processors is taken in reasonable correspondence with the dimension of a grid being used. For a wide class of spatial problems, parallel techniques are probably the unique tool for analysis of solutions with acceptable accuracy.

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Chapter 9

Finite Strains of a Granular Material

Abstract A mathematical model of developed flow of a granular material is considered. On the phenomenological level, elastic properties characteristic for a compacted material and viscous properties appearing in loosening are taken into account. Exact solutions of problems on rotational and plane-parallel motion of a material with stagnant zones are constructed. Using them, influence of viscosity on a flow pattern is analyzed.

9.1 Dilatancy Effect

The effect of dilatancy established experimentally by Reynolds is in increasing a volume of a densely packed granular material with shear. As a rule, the dilatancy dependence between the shear intensity and the value of a volume strain is introduced into a model axiomatically, for example, as in [5, 6, 10]. We study the effect of dilatancy in the framework of the model of an elastic-plastic granular material constructed in Chap. 6 [25].

Let us consider the process of shear in the $x_1 x_2$ plane of a body of a material previously compressed in the x_3 axis by the pressure $\sigma_3 = -p_0$ ($p_0 > 0$) accompanied by strain in the axial direction. This process takes place, for example, in an experimental facility for the dilatancy measurement, described in monographs [22, 23], or in rotational motion between two coaxial cylinders of sufficiently large radius. According to the model notion of strain of a granular material possessing elastic and plastic properties, several qualitatively different variants of the process realized depending on the value of pressure p_0 and the relation between mechanical parameters of a material are possible. For high pressure, a material may pass from an original state, where it behaves as an ideal elastic body, to the plastic state without ability for dilatancy. For lower pressure, dilatancy may take place only in the limits of elastic deformation. A two-stage process, where the elastic stage is followed by

the stage of plastic dilatancy accompanied by irreversible deformation of a material, may be developed as well.

For simplicity we further assume that a material is incompressible. Principal strains are calculated from the shear angle χ and the axial strain Δ by the formulae

$$\varepsilon_{1,2} = \pm \frac{\chi}{2}, \quad \varepsilon_3 = \Delta.$$

The quantity Δ may be nonzero only from the instant when stresses achieve the von Mises–Schleicher conical surface. At the elastic stage of deformation, due to incompressibility of particles, the axial strain turns out to be zero. Indeed, from the Hooke law

$$\sigma_1 + \sigma_2 - p_0 = 3k\Delta, \quad \sigma_{1,2} + p_0 = \mu(\pm\chi - 2\Delta)$$

it follows that $\Delta \rightarrow 0$ if $k \rightarrow \infty$. In the shear plane principal stresses are

$$\sigma_{1,2} = -p_0 \pm \mu\chi.$$

If the limiting value $\chi_s = \tau_s/\mu$ of elastic shear determined by the von Mises plasticity condition

$$\tau(\sigma) = \tau_s \Rightarrow (\sigma_1 + p_0)^2 + (\sigma_2 + p_0)^2 + (\sigma_1 - \sigma_2)^2 = 6\tau_s^2 \quad (9.1)$$

is less than the limiting value $\chi_0 = \alpha p_0/\mu$ calculated by the von Mises–Schleicher condition $\tau(\sigma) = \alpha p_0$, then with increasing χ the elastic stage is followed by the stage of plastic deformation described by the associated flow rule

$$\frac{d(\varepsilon_1^p - \varepsilon_3^p)}{\sigma_1 + p_0} = \frac{d(\varepsilon_2^p - \varepsilon_3^p)}{\sigma_2 + p_0} > 0. \quad (9.2)$$

Substituting the plastic components of the strain tensor, i.e. differences between total and elastic components expressed by the Hooke law

$$\varepsilon_{1,2}^p - \varepsilon_3^p = \pm \frac{\chi}{2} - \frac{\sigma_{1,2} + p_0}{2\mu},$$

into (9.2) we obtain the ordinary differential equation

$$\frac{\mu d\chi - d\sigma_1}{\sigma_1 + p_0} = -\frac{\mu d\chi + d\sigma_2}{\sigma_2 + p_0}.$$

Integrating it together with (9.1), we can determine the dependence of stresses on shear angle for the plastic stage. Taking into account initial data (conditions of continuity of stresses when going from the elastic state), from this equation we obtain

$$\sigma_{1,2} = -p_0 \pm \tau_s, \quad \Delta = 0, \quad d\chi \geq 0. \quad (9.3)$$

Thus, if pressure is sufficiently large, i.e. if $p_0 \geq p_s = \tau_s/\mathfrak{x}$, then shear does not lead to variation of axial strain and dilatancy is not observed. In the opposite case, where $\chi_0 < \chi_s$, at the stage of elastic shear the process of loosening a material with $\Delta > 0$ starts. The corresponding point in the stress space slides over the von Mises–Schleicher conical surface. Principal conditional stresses in a material with the finite compression modulus k are

$$s_{1,2} = \left(k - \frac{2\mu}{3}\right)\Delta \pm \mu\chi, \quad s_3 = \left(k + \frac{4\mu}{3}\right)\Delta.$$

Actual stresses, obtained as projections of conditional stresses onto the von Mises–Schleicher cone by the formulae (4.14), for $k \rightarrow \infty$ tend to

$$\begin{aligned} \sigma_{1,2} &= \left(\pm \mathfrak{x} \left(\chi \mp \frac{2\Delta}{3}\right) - \sqrt{\chi^2 + \frac{4\Delta^2}{3}}\right) \left(\mathfrak{x} - \Delta / \sqrt{\chi^2 + \frac{4\Delta^2}{3}}\right) \frac{\mu}{\mathfrak{x}^2}, \\ \sigma_3 &= \left(\frac{4}{3} \mathfrak{x} \Delta - \sqrt{\chi^2 + \frac{4\Delta^2}{3}}\right) \left(\mathfrak{x} - \Delta / \sqrt{\chi^2 + \frac{4\Delta^2}{3}}\right) \frac{\mu}{\mathfrak{x}^2}. \end{aligned} \quad (9.4)$$

The equation $\sigma_3 = -p_0$ is reduced to a biquadratic equation relative to χ and its positive root

$$\begin{aligned} \chi^2 &= -\frac{4\Delta^2}{3} + \frac{1}{4\mathfrak{x}^2} \left(\left(\frac{4\mathfrak{x}^2}{3} + 1 \right) \Delta + \mathfrak{x} \chi_0 \right. \\ &\quad \left. + \sqrt{\left[\left(\frac{2\mathfrak{x}}{\sqrt{3}} - 1 \right)^2 \Delta + \mathfrak{x} \chi_0 \right] \left[\left(\frac{2\mathfrak{x}}{\sqrt{3}} + 1 \right)^2 \Delta + \mathfrak{x} \chi_0 \right]} \right)^2 \end{aligned} \quad (9.5)$$

is the equation of the dilatancy curve 1 in Fig. 9.1a, b. This curve intersects the χ axis at the point χ_0 and the Δ axis at the point $\Delta_0 = -\mathfrak{x} \chi_0 / (1 + 2\mathfrak{x}/\sqrt{3})^2$. To the elastic loosening there corresponds the curve piece AB which ends with the point B of transition of a material into the plastic state. The coordinates of the point B can be determined from the plasticity condition according to which

$$p(\sigma) = \frac{\tau_s}{\mathfrak{x}} \Rightarrow \sqrt{\chi^2 + \frac{4\Delta^2}{3}} = \chi_s + \frac{\Delta}{\mathfrak{x}}.$$

The equation of the dilatancy curve in the original form $\sigma_3 = -p_0$ is transformed into a quadratic equation for Δ . Solving it, we obtain

$$\Delta_* = \mathfrak{x} \chi_s \frac{\varsigma}{4\mathfrak{x}^2/3 - \varsigma}, \quad \chi_* = \frac{2\mathfrak{x}}{\sqrt{3}} \chi_s \frac{\sqrt{4\mathfrak{x}^2/3 - \varsigma^2}}{4\mathfrak{x}^2/3 - \varsigma},$$

where $\varsigma = 1 - \chi_0/\chi_s$.

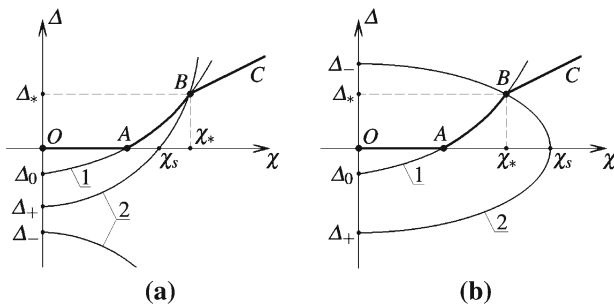


Fig. 9.1 Dilatancy dependence: **a** $\alpha < \sqrt{3}/2$, **b** $\alpha \geq \sqrt{3}/2$; 1—curve of elastic dilatancy, 2—plastic constraints

To the plasticity condition for $\alpha < \sqrt{3}/2$ there corresponds a pair of curves 2 on the $\chi\Delta$ plane shown in Fig. 9.1a and for $\alpha \geq \sqrt{3}/2$ there corresponds a single curve 2 in Fig. 9.1b. The points $\Delta_{\pm} = -\alpha \chi_s / (1 \pm 2\alpha/\sqrt{3})$ on the ordinate axis are the endpoints of these curves. The point χ_s is the point of intersection with the abscissa axis. For any α the inequality $\Delta_0 > \Delta_+$ holds, hence, the point $(0, \Delta_0)$ belongs to the admissible domain $p(\sigma) \leq \tau_s/\alpha$. Along with it, the curve piece OAB of elastic deformation lies entirely in this domain.

It turns out that for $\alpha < \sqrt{3}/2$ and $p_0 \leq p_* = \tau_s(1/\alpha - 4\alpha/3)$ the values of Δ_* and χ_* are infinite. In this case plasticity is not observed and dilatancy of a material is completely described by the curve 1. If stress p_0 lies in the interval (p_*, p_s) then, starting with the point B , the dilatancy curve changes. By the formulae (9.4), principal stresses for the point B in the shear plane are

$$\sigma_{1,2} = p_s \left(\pm \sqrt{\alpha^2 - \frac{3\tau_s^2}{4}} - \frac{\tau_s}{2} - 1 \right). \quad (9.6)$$

Further, with plastic deformation, stresses simultaneously satisfy the plasticity condition in the form of Eq. (9.1) and the von Mises–Schleicher condition which is equivalent to the equation $p(\sigma) = \tau_s/\alpha$. Hence, they remain constant and have the same form. To determine conditional stresses, we apply passage to the limit in constitutive relationships of the model of an elastically compressible material. If the value of modulus k is finite, then because of plastic incompressibility of a material the equation $p(s) = -k\Delta$ holds. By Eq. (4.14) for calculation of a projection we have

$$\tau(s) = \frac{\mu\Delta}{\alpha} + \frac{\mu + \alpha^2 k}{\alpha^2 k} \tau_s, \quad s_{1,2} - s_3 = \frac{\tau(s)}{\tau_s} (\sigma_{1,2} + p_0). \quad (9.7)$$

Taking into account the Hooke law for the elastic components of the strain tensor

$$\frac{s_{1,2} - s_3}{2\mu} = \pm \frac{\chi}{2} - \Delta - \varepsilon_{1,2}^p + \varepsilon_3^p,$$

we reduce the relationships (9.2) of the associated flow rule to the following form:

$$\frac{\mu d\chi - 2\mu d\Delta - d(s_1 - s_3)}{\sigma_1 + p_0} = -\frac{\mu d\chi + 2\mu d\Delta + d(s_2 - s_3)}{\sigma_2 + p_0} > 0.$$

Stresses σ_1 and σ_2 are constant, hence, due to (9.7),

$$d(s_{1,2} - s_3) = (\sigma_{1,2} + p_0) \frac{d\tau(s)}{\tau_s}.$$

This enables one to simplify the differential equation, which follows from (9.2), reduced it to the form

$$(\sigma_1 + \sigma_2 + 2p_0) d\chi + 2(\sigma_1 - \sigma_2) d\Delta = 0.$$

Hence, taking into account (9.6), we have

$$\frac{d\chi}{d\Delta} = \frac{4}{3\varsigma} \sqrt{\mathfrak{x}^2 - \frac{3\varsigma^2}{4}}. \quad (9.8)$$

Thus, at the point B the dilatancy curve gives place to the ray BC whose linear coefficient is defined by the right-hand side of (9.8). With the help of cumbersome calculations omitted here we can establish that for any $\mathfrak{x}$ the slope angle of this ray is less than that of the tangent line to the elastic curve at the point B , which corresponds to their mutual arrangement in Fig. 9.1.

We prove that the obtained solution satisfies the inequality (9.2) which provides nonnegativity of plastic dissipation of energy. To this end, using the formulae for derivatives obtained by differentiation of (9.7), we rewrite it in the form

$$\frac{d\chi - 2d\Delta}{\sigma_1 + p_0} > \frac{d\Delta}{\mathfrak{x}\tau_s}.$$

Then, taking into consideration the relation between the differentials $d\chi$ and $d\Delta$ and the expressions (9.6) for stresses, we obtain the inequality

$$-\sigma_1 - \sigma_2 - 2p_0 \equiv 3(p_s - p_0) < 4\mathfrak{x}\tau_s,$$

which is obviously valid since $p_0 > p_*$.

Assuming that μ tends to infinity, we obtain the solution which defines the effect of dilatancy in a rigid-plastic granular material. For $p_0 \geq p_s$ in such a material the stress-strain state (9.3) without dilatancy is realized. For $p_0 \leq p_*$ and $\mathfrak{x} < \sqrt{3}/2$ the loosening of a material is not accompanied by plastic strain and the dilatancy equation takes the form

$$\frac{\chi}{\Delta} = \sqrt{\frac{1}{\mathfrak{x}^2} - \frac{4}{3}}. \quad (9.9)$$

In this case stresses are determined by the formulae

$$\sigma_{1,2} = p_0 \frac{\pm \sqrt{1 - 4\alpha^2/3} - 2\alpha/3 - 1/\alpha}{1/\alpha - 4\alpha/3}. \quad (9.10)$$

For $p_* < p_0 < p_s$ the plastic deformation is observed. The stresses are given by Eq. (9.6) and the dilatancy dependence is defined by Eq. (9.8).

For a material with absolutely rigid particles the single linear dependence (9.9) remains as $\tau_s \rightarrow \infty$. In such a material the dilatancy is achieved only if $\alpha < \sqrt{3}/2$. For greater values of α the shear is impossible and the “jamming” effect is observed.

The process of plane shear of a granular material compressed along the axis can be also described completely on the basis of a more general model of a compressible material. However, the obtained pattern is complicated by the need to consider the additional case of $\alpha \leq 2\mu/(\sqrt{3}k)$ where the stress state of previous compression in the absence of shear belongs to the surface of the von Mises–Schleicher cone and, thus, a material initially is in the limiting state.

Attention should be given to obvious and quite regular disadvantage of the description of dilatancy in the theory of small strains. According to this theory, the volume of a material increases infinitely with increasing the angle of shear. We consider possible variants of generalization, for finite strains and rotations of particles to be taken into account, in the framework of the simplest model of a material with rigid particles [19]. For homogeneous shear and tension, the Eulerian coordinates of particles are defined in terms of the Lagrangian ones by the formulae

$$x_1 = \xi_1, \quad x_2 = \xi_2 + \chi \xi_1, \quad x_3 = z \xi_3. \quad (9.11)$$

Here χ is the tangent of shear angle in the $x_1 x_2$ plane and z is the coefficient of linear tension along the x_3 axis. The tensors of gradients are

$$\nabla_{\xi} \mathbf{x} = \begin{pmatrix} 1 & \chi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix}, \quad \nabla_x \xi = \begin{pmatrix} 1 & -\chi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{z} \end{pmatrix}.$$

In the geometrically nonlinear case, admissible states in the strain space are subject to some constraint as well. Assume that this constraint can be obtained from the inequality

$$\alpha \gamma(\boldsymbol{\varepsilon}) \leq \theta(\boldsymbol{\varepsilon})$$

by replacement of the tensor of small strains with the Almansi strain tensor

$$2\mathbf{e} = \boldsymbol{\delta} - \nabla_x \xi \cdot (\nabla_x \xi)^* = \begin{pmatrix} -\chi^2 & \chi & 0 \\ \chi & 0 & 0 \\ 0 & 0 & \frac{z^2 - 1}{z^2} \end{pmatrix}.$$

Then, taking into account the expressions

$$4e_{1,2} = -\chi^2 \pm \chi\sqrt{4 + \chi^2}, \quad 2e_3 = \frac{z^2 - 1}{z^2}$$

for principal strains, we can write the dilatancy equation in the form

$$\sqrt{\frac{1}{3} \left(2 \frac{z^2 - 1}{z^2} + \chi^2 \right)^2 + \chi^2(4 + \chi^2)} = \frac{1}{\mathfrak{a}} \left(\frac{z^2 - 1}{z^2} - \chi^2 \right). \quad (9.12)$$

On the χ z plane (Fig. 9.2) the monotone curve 1 with the vertical asymptote $\chi = \chi_\infty$ corresponds to this equation. For $\mathfrak{a} < \sqrt{3}/2$ the value of $\chi_\infty \leq 1$ is determined as a solution of the equation which follows from (9.12) as $z \rightarrow \infty$. The tangent to the curve at the point $\chi = 0$ is defined by the dilatancy Eq. (9.9) of the theory of small strains:

$$\chi = \nu(z - 1), \quad \text{where } \nu = \sqrt{\frac{1}{\mathfrak{a}^2} - \frac{4}{3}}, \quad \Delta = z - 1.$$

For $\mathfrak{a} \geq \sqrt{3}/2$ the dilatancy curve degenerates into a point. The “jamming” of particles is observed: in such a material the shear without the preliminary volume loosening is impossible. Disadvantage of the description of dilatancy with the help of the Almansi finite strain tensor is in the fact that in a dilatant material the angle of shear may not exceed the limiting value $\arctan \chi_\infty \leq \pi/4$. As it is approached, the volume of a material tends to infinity.

Assume that the constraint is formulated in terms of the Cauchy–Green strain tensor

$$2\boldsymbol{\varepsilon} = \nabla_{\boldsymbol{\xi}} \mathbf{x} \cdot (\nabla_{\boldsymbol{\xi}} \mathbf{x})^* - \boldsymbol{\delta} = \begin{pmatrix} \chi^2 & \chi & 0 \\ \chi & 0 & 0 \\ 0 & 0 & z^2 - 1 \end{pmatrix}.$$

For this tensor the principal strains have the form

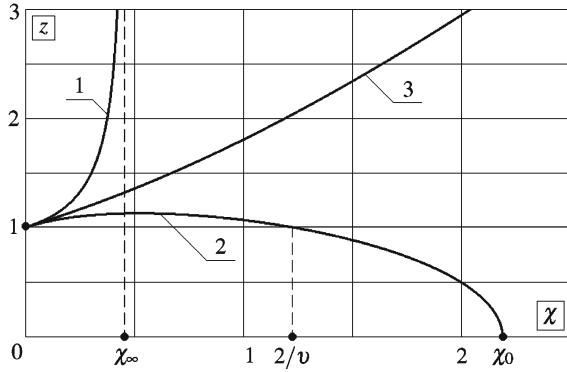
$$4\varepsilon_{1,2} = \chi^2 \pm \chi\sqrt{4 + \chi^2}, \quad 2\varepsilon_3 = z^2 - 1.$$

Then the dilatancy equation looks as follows:

$$\sqrt{\frac{1}{3} (2z^2 - 2 - \chi^2)^2 + \chi^2(4 + \chi^2)} = \frac{z^2 - 1 + \chi^2}{\mathfrak{a}}. \quad (9.13)$$

In Fig. 9.2 the non-monotone curve 2 corresponds to this equation. According to Eq. (9.13), the expansion of a material due to shear for $\chi = 2/\nu$ is followed by compression and the volume tends to zero as $\chi \rightarrow \chi_0 \geq 1$. The limiting value of χ_0 can be determined as a root of Eq. (9.13) for $z = 0$, however, the range of shear

Fig. 9.2 Dilatancy curves:
1 for the Almansi finite strain tensor, 2 for the Cauchy–Green strain tensor, 3 for the Hencky logarithmic strain tensor



angles where dilatancy is described satisfactory is bounded by a quantity which is considerably less than $\arctan \chi_0$.

To the formulation of the constraints we apply the Hencky logarithmic tensor of finite strains

$$\mathbf{h} = \frac{1}{2} \ln \mathbf{l},$$

where $\mathbf{l} = (\nabla_{\xi} \mathbf{x})^* \cdot \nabla_{\xi} \mathbf{x}$ is the left Cauchy–Green tensor. Some exclusive properties of the logarithmic tensor, established in [17, 32], are considered in the next section. In the state of joint tension and simple shear we have

$$\mathbf{l} = \begin{pmatrix} 1 & \chi & 0 \\ \chi & 1 + \chi^2 & 0 \\ 0 & 0 & z^2 \end{pmatrix}.$$

From the eigenvalues of this tensor we can find the principal logarithmic strains

$$h_j = \ln \sqrt{l_j}, \quad 2l_{1,2} = 2 + \chi^2 \pm \chi \sqrt{4 + \chi^2}, \quad l_3 = z^2.$$

With the help of the expressions

$$\theta(\mathbf{h}) = \ln z, \quad \gamma(\mathbf{h}) = \sqrt{\ln^2 l_1 + \frac{4}{3} \ln^2 z}$$

the dilatancy equation is written in the form $l_1 = z^v$ or in the form

$$\chi = z^{v/2} - z^{-v/2}, \quad (9.14)$$

solved for χ . In Fig. 9.2 the curve 3 corresponds to this equation.

From Eq. (9.14) it follows that for a constant coefficient v the volume of a material increases infinitely with increasing the shear angle. In fact, the internal friction

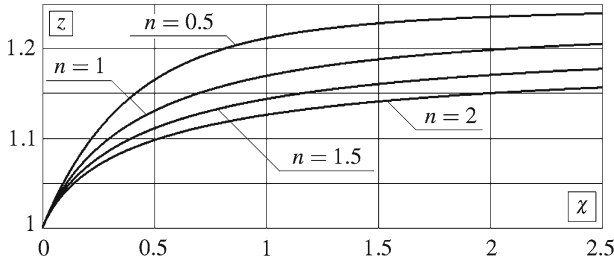


Fig. 9.3 Dilatancy curves for the dependence $\varkappa(\rho)$

parameter $\varkappa$ depends on density and tends to zero when some critical value ρ_* is achieved. The dilatancy curve approaches to the horizontal asymptote $z = z_*$. For actual granular materials the value of z_* is defined by size and shape of particles. Measuring it with an experimental facility [22, 24], it is easy to see that $\rho_* = \rho_0/z_*$ (ρ_0 is the initial density of a material).

When modeling fine-grained materials where dilatational change of density is moderate, we can use the power dependence

$$\varkappa = \begin{cases} \varkappa_0 \left(\frac{1 - \rho_*/\rho}{1 - \rho_*/\rho_0} \right)^n, & \text{if } \rho \geq \rho_*, \\ 0, & \text{if } \rho < \rho_*. \end{cases} \quad (9.15)$$

The dilatancy curves (9.14) with the dependence (9.15) for $\varkappa_0 = 0.5$, $z_* = 1.25$ and different values of n are shown in Fig. 9.3. Due to the choice of the coefficients $\varkappa_0$ and n we can achieve satisfactory approximation of experimental curves.

It should be noticed that taking into account the dependence of the coefficient $\varkappa$ on density does not eliminate the errors of the models with the use of the Almansi and Cauchy–Green finite strain tensors considered above which lead to rather hard restrictions on the value of shear.

9.2 Basic Properties of the Hencky Tensor

We consider the motion of a continuous medium in the following general form: $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$. The tensor of distortion

$$\mathbf{x}_{\boldsymbol{\xi}} = (\nabla_{\boldsymbol{\xi}} \mathbf{x})^* : d\mathbf{x} = \mathbf{x}_{\boldsymbol{\xi}} \cdot d\boldsymbol{\xi},$$

which defines the linear transformation of a particle from an initial configuration ($t = 0$) to an actual one corresponding to the instant of time t , can be represented as the polar decomposition (see, for example, [8, 9])

$$\mathbf{x}_{\boldsymbol{\xi}} = \mathbf{z}_- \cdot \mathbf{o} = \mathbf{o} \cdot \mathbf{z}_+, \quad \mathbf{o} \cdot \mathbf{o}^* = \mathbf{o}^* \cdot \mathbf{o} = \boldsymbol{\delta}, \quad \mathbf{z}_-^* = \mathbf{z}_-, \quad \mathbf{z}_+^* = \mathbf{z}_+.$$

Here $\mathbf{o}$ is the orthogonal rotation tensor, $\mathbf{z}_-$ and $\mathbf{z}_+$ are symmetric tensors describing strain of a particle. These tensors can be reduced to the Jordan diagonal form with relative elongations of linear elements along the principal diagonal. Under given tensor of the distortion, the squared $\mathbf{z}_-$ and $\mathbf{z}_+$, i.e. the so-called left and right Cauchy–Green tensors, are easily calculated:

$$\mathbf{l} = \mathbf{z}_-^2 = (\mathbf{z}_- \cdot \mathbf{o}) \cdot (\mathbf{o}^* \cdot \mathbf{z}_-) = \mathbf{x}_\xi \cdot \mathbf{x}_\xi^*, \quad \mathbf{d} = \mathbf{z}_+^2 = (\mathbf{z}_+ \cdot \mathbf{o}^*) \cdot (\mathbf{o} \cdot \mathbf{z}_+) = \mathbf{x}_\xi^* \cdot \mathbf{x}_\xi.$$

Let $\mathbf{q}$ be an arbitrary orthogonal tensor. The transformation $d\hat{\xi} = \mathbf{q} \cdot d\xi$ performs a rotation of initial configuration. According to the conventional terminology [15, 16], a tensor $\mathbf{z}$ is said to be indifferent if it does not vary with this rotation, i.e. if $\hat{\mathbf{z}} = \mathbf{z}$. In its turn, $\mathbf{z}$ is called an invariant tensor if it does not vary with an arbitrary rotation of actual configuration: $d\hat{\mathbf{x}} = \mathbf{q} \cdot d\mathbf{x}$.

We can show that $\mathbf{l}$ is an indifferent tensor and $\mathbf{d}$ is an invariant one. Indeed, in the case of the transformation of initial configuration $d\mathbf{x} = \mathbf{x}_\xi \cdot d\xi = \mathbf{x}_\xi \cdot \mathbf{q}^* \cdot d\hat{\xi}$, hence,

$$\hat{\mathbf{l}} = (\mathbf{x}_\xi \cdot \mathbf{q}^*) \cdot (\mathbf{q} \cdot \mathbf{x}_\xi^*) = \mathbf{l}.$$

Thus, $\mathbf{l}$ is indifferent. The fact that $\mathbf{d}$ is invariant is proved in a similar way. In the case of the transformation of actual configuration $d\hat{\mathbf{x}} = \mathbf{q} \cdot \mathbf{x}_\xi \cdot d\xi$, hence,

$$\hat{\mathbf{d}} = (\mathbf{x}_\xi^* \cdot \mathbf{q}^*) \cdot (\mathbf{q} \cdot \mathbf{x}_\xi) = \mathbf{d}.$$

Assume that $f_0(y)$ is an analytical function in the neighborhood of the point $y = 1$, which is expanded in a power series

$$f_0(y) = \sum_{j=0}^{\infty} \alpha_j (y - 1)^j$$

and satisfies the following conditions:

$$f_0(1) = 0, \quad \frac{df_0(1)}{dy} = \frac{1}{2}, \quad \frac{df_0(y)}{dy} > 0. \quad (9.16)$$

We introduce two classes of tensors of finite strains of a material, namely, the Eulerian tensor whose elements are primarily used in the Eulerian description of motion and the Lagrangian tensor which is more often used in the Lagrangian approach [32]. Tensors of the form

$$f_0(\mathbf{l}) = \sum_{j=0}^{\infty} \alpha_j (\mathbf{l} - \delta)^j$$

belong to the first class. They are obviously indifferent. Invariant tensors of the form $f_0(\mathbf{d})$ belong to the second class. The conditions (9.16) provides that in the

geometrically linear case all elements of both classes coincide with the Cauchy small strain tensor.

The function $2f_0(y) = 1 - 1/y$ corresponds to the Almansi strain tensor, which is a tensor of the Eulerian class. The Hencky logarithmic tensor

$$\mathbf{h} = \frac{1}{2}(\mathbf{I} - \boldsymbol{\delta}) - \frac{1}{4}(\mathbf{I} - \boldsymbol{\delta})^2 + \frac{1}{6}(\mathbf{I} - \boldsymbol{\delta})^3 - \frac{1}{8}(\mathbf{I} - \boldsymbol{\delta})^4 + \dots \quad (9.17)$$

is the Eulerian tensor as well. For it $2f_0(y) = \ln y$. The Cauchy–Green tensor, for which $2f_0(y) = y - 1$, is an example of the Lagrangian tensor.

There exist universal and sufficiently simple algorithms for calculation of the Hencky tensor which can be used in numerical computations as well as in analytical treatment. It is obvious that the expansion (9.17) is of little use from this viewpoint. If the kinematics of motion of a material in some coordinate system is given then, according to the algebraic method, it is required, firstly, to determine the left Cauchy–Green tensor at each point, secondly, to reduce it to the diagonal form by rotation of coordinate axes, thirdly, to calculate one-halves logarithms of the diagonal elements and, fourthly, to perform the transformation of the obtained diagonal matrix to the original coordinate system.

As a rule, this approach is laborious for analytical treatment since it requires the construction of a matrix of rotation transformation in an explicit form. In calculations we can use, for example, the Jacobi rotation method [3]. An alternative approach is related to the application of the Lagrange–Sylvester formula:

$$\mathbf{h} = a_1 \boldsymbol{\delta} + a_2 \mathbf{I} + a_3 \mathbf{I}^2. \quad (9.18)$$

Here a_1 , a_2 and a_3 are the scalar coefficients to be determined which depend on invariants of the tensor $\mathbf{I}$. The passage to the principal axes results in three scalar equations

$$h_j = a_1 + a_2 l_j + a_3 l_j^2. \quad (9.19)$$

From these equations we can see that these coefficients are the coefficients of the interpolation polynomial of degree two which takes the values $h_j = \ln \sqrt{l_j}$ at the nodes l_j . If some l_j coincide, then we have the problem on interpolation with multiple nodes. Its solution is the Hermite polynomial [3]. In the general case, when constructing the interpolation polynomial (in the Newton form), we use the table of divided differences

$$\begin{array}{c|ccc} l_1 & h_1 & & \\ & & h_{1;2} & \\ l_2 & h_2 & & h_{1;2;3} \\ & & h_{2;3} & \\ l_3 & h_3 & & \end{array}.$$

If all l_j are different, then the differences of the first and second order involved in the table are calculated as follows:

$$h_{i;j} = \frac{h_j - h_i}{l_j - l_i}, \quad h_{1;2;3} = \frac{h_{2;3} - h_{1;2}}{l_3 - l_1}.$$

In the case of multiple nodes they are determined by the formulae

$$h_{j;j} = \frac{1}{2} \frac{d \ln l_j}{d l_j} = \frac{1}{2 l_j}, \quad h_{j;j;j} = \frac{1}{2} \frac{d^2 \ln l_j}{d l_j^2} = -\frac{1}{4 l_j^2},$$

obtained from the previous ones with the help of the passage to the limit as $l_i \rightarrow l_j$.

The expression for the interpolation polynomial is constructed by the first elements of the columns of the table:

$$\mathbf{h} = h_1 \boldsymbol{\delta} + h_{1;2}(\mathbf{l} - l_1 \boldsymbol{\delta}) + h_{1;2;3}(\mathbf{l} - l_1 \boldsymbol{\delta}) \cdot (\mathbf{l} - l_2 \boldsymbol{\delta}). \quad (9.20)$$

To determine l_j , it is required to solve the characteristic equation for the tensor $\mathbf{l}$:

$$\mathbf{l}^3 - b_1 \mathbf{l}^2 + b_2 \mathbf{l} - b_3 = \mathbf{0}.$$

Its coefficients are

$$b_1 = \theta(\mathbf{l}), \quad b_2 = \frac{\theta^2(\mathbf{l}) - \theta(\mathbf{l}^2)}{2}, \quad b_3 = \frac{\theta^3(\mathbf{l})}{6} - \frac{\theta(\mathbf{l}) \theta(\mathbf{l}^2)}{2} + \frac{\theta(\mathbf{l}^3)}{3}.$$

By the Cardano formulae

$$l_j = \frac{1}{3} \left(b_1 + 2 \sqrt{b_1^2 - 3 b_2} \cos \left(\frac{\beta - 2 \pi j}{3} \right) \right), \quad \beta = \arccos \frac{2 b_1^3 - 9 b_1 b_2 + 27 b_3}{2 (b_1^2 - 3 b_2)^{3/2}}.$$

It turns out that the logarithmic tensor has an extremely important property: among the Eulerian tensors, it is the unique one which is related to the Cauchy strain rate tensor

$$\mathbf{e} = \frac{\nabla_x \mathbf{v} + (\nabla_x \mathbf{v})^*}{2}$$

by the operation of corotational differentiation. This means that there exists an orthogonal tensor $\mathbf{q}(t)$ such that

$$\mathbf{e} = \mathbf{q} \cdot \frac{d}{dt} (\mathbf{q}^* \cdot \mathbf{h} \cdot \mathbf{q}) \cdot \mathbf{q}^* \equiv \dot{\mathbf{h}} + \mathbf{h} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathbf{h}. \quad (9.21)$$

Here $\boldsymbol{\omega} = \dot{\mathbf{q}} \cdot \mathbf{q}^*$ (an antisymmetric tensor) is the spin of corotational derivative, a dot over a symbol denotes the total time derivative. Antisymmetry of $\boldsymbol{\omega}$ is proved by

differentiation of both sides of the equality $\mathbf{q} \cdot \mathbf{q}^* = \delta$. Thus, we have

$$\omega = \dot{\mathbf{q}} \cdot \mathbf{q}^* = -\mathbf{q} \cdot \dot{\mathbf{q}}^* = -(\dot{\mathbf{q}} \cdot \mathbf{q}^*)^* = -\omega^*.$$

We give proof of this property which seems to be more simple than that in the original paper [17] and in later papers [20, 32]. We consider only the case where l_j are different since in the case of multiple roots a proof is obtained by assuming that l_i tends to l_j .

We assume that an arbitrary strain tensor $\mathbf{h}$ of the Eulerian class is an isotropic function relative to a tensor $\mathbf{l}$ of an arbitrary form. Differentiating the equation in $\mathbf{l}$ and taking into account the formulae $(\nabla_{\xi} \mathbf{v})^* = (\nabla_x \mathbf{v})^* \cdot \mathbf{x}_{\xi}$ and $\nabla_{\xi} \mathbf{v} = \mathbf{x}_{\xi}^* \cdot \nabla_x \mathbf{v}$, we get

$$\dot{\mathbf{l}} = (\nabla_{\xi} \mathbf{v})^* \cdot \mathbf{x}_{\xi}^* + \mathbf{x}_{\xi} \cdot \nabla_{\xi} \mathbf{v} = (\nabla_x \mathbf{v})^* \cdot \mathbf{l} + \mathbf{l} \cdot \nabla_x \mathbf{v}.$$

Hence, in the coordinate system related to the principal axes of $\mathbf{l}$ the components of the tensor $\dot{\mathbf{l}}$ are

$$\dot{l}_{ij} = v_{i,j} l_j + l_i v_{j,i},$$

where $v_{i,j}$ are the components of $(\nabla_x \mathbf{v})^*$.

Denote an orthogonal matrix of transition to the principal axes of the tensor $\mathbf{l}$ from some fixed Cartesian coordinate system by $\mathbf{O}$ and the diagonal matrix corresponding to $\mathbf{l}$ by $\mathbf{l}^0$. It is known that the eigenvectors of the matrix of tensor in the original coordinate system are the columns of $\mathbf{O}$ and the eigenvalues are the diagonal elements of $\mathbf{l}^0$. The following equation, which implies similarity of the matrices, is valid:

$$\mathbf{l} = \mathbf{O} \mathbf{l}^0 \mathbf{O}^*.$$

Differentiating both its sides with respect to time, we obtain

$$\dot{\mathbf{l}} = \dot{\mathbf{O}} \mathbf{l}^0 \mathbf{O}^* + \mathbf{O} \dot{\mathbf{l}}^0 \mathbf{O}^* + \mathbf{O} \mathbf{l}^0 \dot{\mathbf{O}}^*.$$

Multiplying this equation by $\mathbf{O}^*$ from the left and by $\mathbf{O}$ from the right, we have

$$\mathbf{O}^* \dot{\mathbf{l}} \mathbf{O} = \mathbf{O}^* \dot{\mathbf{O}} \mathbf{l}^0 + \dot{\mathbf{l}}^0 + \mathbf{l}^0 \dot{\mathbf{O}}^* \mathbf{O}. \quad (9.22)$$

A matrix being the product of the antisymmetric matrix $\mathbf{O}^* \dot{\mathbf{O}} = -\dot{\mathbf{O}}^* \mathbf{O}$ by a diagonal one (from the left as well as from the right) involves zeroes on the principal diagonal, hence, from (9.22) it follows that the derivatives of the eigenvalues of $\dot{\mathbf{l}}_j$ are equal to the diagonal elements of the matrix $\dot{\mathbf{l}}$ in the principal axes of the tensor $\mathbf{l}$, i.e. that

$$\dot{l}_j = \dot{l}_{jj} = 2 v_{j,j} l_j. \quad (9.23)$$

Differentiating Eq. (9.19) with respect to time and taking into account that h_j depends on l_j only, we can obtain

$$\dot{a}_1 + \dot{a}_2 l_j + \dot{a}_3 l_j^2 = \left(\frac{dh_j}{dl_j} - a_2 - 2a_3 l_j \right) \dot{l}_j.$$

From (9.18) it follows that

$$\dot{\mathbf{h}} = \dot{a}_1 \boldsymbol{\delta} + \dot{a}_2 \mathbf{l} + \dot{a}_3 \mathbf{l}^2 + a_2 \dot{\mathbf{l}} + a_3 (\dot{\mathbf{l}} \cdot \mathbf{l} + \mathbf{l} \cdot \dot{\mathbf{l}}).$$

Passing to the system of principal axes of the tensor $\mathbf{l}$ and taking into account (9.23), we express the diagonal and off-diagonal components of the tensor $\dot{\mathbf{h}}$:

$$\begin{aligned} \dot{h}_{jj} &= \frac{dh_j}{dl_j} \dot{l}_j - (a_2 + 2a_3 l_j)(\dot{l}_j - 2v_{j,j} l_j) = \frac{dh_j}{dl_j} \dot{l}_j, \\ \dot{h}_{ij} &= (a_2 + a_3 (l_i + l_j))(v_{i,j} l_j + v_{j,i} l_i). \end{aligned}$$

Immediate calculation of the matrix for the tensor $\mathbf{h} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathbf{h}$ in this system yields the following result:

$$\mathbf{h} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathbf{h} = \begin{pmatrix} 0 & \omega_3(h_2 - h_1) & \omega_2(h_1 - h_3) \\ \omega_3(h_2 - h_1) & 0 & \omega_1(h_3 - h_2) \\ \omega_2(h_1 - h_3) & \omega_1(h_3 - h_2) & 0 \end{pmatrix},$$

where ω_j are the nonzero components of

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Substituting the obtained expressions into (9.21) and taking into account symmetry of the matrices, we arrive at three equations for the off-diagonal elements. From these equations the components of the spin tensor are uniquely determined:

$$\begin{aligned} \omega_1 &= \frac{(v_{2,3} + v_{3,2})/2 - (a_2 + a_3 (l_2 + l_3))(v_{2,3} l_3 + v_{3,2} l_2)}{h_3 - h_2}, \\ \omega_2 &= \frac{(v_{1,3} + v_{3,1})/2 - (a_2 + a_3 (l_1 + l_3))(v_{1,3} l_3 + v_{3,1} l_1)}{h_1 - h_3}, \\ \omega_3 &= \frac{(v_{1,2} + v_{2,1})/2 - (a_2 + a_3 (l_1 + l_2))(v_{1,2} l_2 + v_{2,1} l_1)}{h_2 - h_1}. \end{aligned}$$

The orthogonal tensor $\mathbf{q}$ can be restored from $\boldsymbol{\omega}$ as a solution of the system of ordinary differential equations

$$\dot{\mathbf{q}} = \mathbf{q} \cdot \boldsymbol{\omega}.$$

To the diagonal elements of the matrices there correspond the equations

$$\frac{dh_j}{dl_j} \dot{l}_j = v_{j,j} \quad (j = 1, 2, 3),$$

which are simplified due to (9.23): $2 dh_j/dl_j = 1/l_j$. Hence, $h_j = \ln \sqrt{l_j}$.

Proved property establishes the energy consistency of the Hencky logarithmic strain tensor and the Cauchy stress tensor. For a hyper-elastic material under adiabatic deformation there exists the stress potential, i.e. internal energy Φ which depends on the strain tensor and temperature. From the heat influx equation

$$\rho \dot{\Phi} = \sigma : \dot{e} = \sigma : \left(\mathbf{q} \cdot \frac{d(\mathbf{q}^* \cdot \mathbf{h} \cdot \mathbf{q})}{dt} \cdot \mathbf{q}^* \right) = (\mathbf{q}^* \cdot \sigma \cdot \mathbf{q}) : \frac{d(\mathbf{q}^* \cdot \mathbf{h} \cdot \mathbf{q})}{dt},$$

due to the fact that a potential does not depend on deformation history, we have the constitutive equation

$$\mathbf{q}^* \cdot \sigma \cdot \mathbf{q} = \rho \frac{\partial \Phi}{\partial (\mathbf{q}^* \cdot \mathbf{h} \cdot \mathbf{q})}.$$

It is written in terms of tensors related to actual rotated configuration. The rotation transformation is defined by the tensor $\mathbf{q}$. In the case of an isotropic material that does not change its mechanical properties with rotation, the constitutive equation of hyperelasticity has the following general form:

$$\sigma = \rho \frac{\partial \Phi}{\partial \mathbf{h}}. \quad (9.24)$$

In [1] it is established that the use of the two-constant quadratic potential of the linear elasticity theory written in terms of invariants of the Hencky logarithmic tensor enables one to describe the behavior of a wide class of materials in the range of moderate strains with reasonable accuracy. One more remarkable property of the Hencky tensor is that its additive decomposition

$$\mathbf{h} = \mathbf{h}' + \frac{\theta(\mathbf{h})}{3} \delta$$

into deviator and spherical part corresponds to the representation of the deformation process as a superposition of form change and volume strain of a material. Indeed, volume strain is described by the equation

$$\frac{\rho_0}{\rho} = \det \mathbf{x}_\xi = \det \mathbf{z}_- = \sqrt{\det \mathbf{l}} = \sqrt{l_1 l_2 l_3} = \exp(h_1 + h_2 + h_3)$$

and, thus, depends only on the quantity $\theta(\mathbf{h})$. The continuity equation takes the form

$$\theta(\mathbf{h}) = \ln \frac{\rho_0}{\rho}. \quad (9.25)$$

It is obvious that among all considered strain tensors only the logarithmic tensor satisfies this property.

9.3 Model of a Viscous Material with Rigid Particles

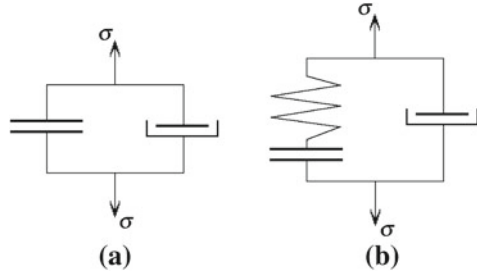
We construct the constitutive relationships of isothermal deformation of a granular material according to the rheological scheme shown in Fig. 9.4a [26]. For compressive stresses a rigid contact involved in the scheme is blocked and a material behaves as an absolutely rigid body. With tension, stress in a contact is zero, hence, the behavior of a material is described by a model of viscous fluid. Generally speaking, the stress field in an absolutely rigid body is not uniquely determined, hence, this model is ill-defined. One way of regularization is to add an elastic element that takes into account the compliance of particles (see Fig. 9.4b). According to the regularized scheme, the state of a material with compression is defined by the Kelvin–Voigt viscoelastic model. The viscous element in these schemes plays a regularizing role as well. Without it the model becomes ill-defined since the field of tensile strains is not uniquely defined.

The mechanism of arising viscous stresses in a loosened granular material that moves in a fluid flow was studied experimentally in the paper [2] giving rise to the theory of fast motions [11, 12, 28]. It turns out that in such a material considerable disperse pressure, caused by the contact interaction of rigid particles, and tangential pressure proportional to it are observed even in the case of weakly viscous fluid. Models of the quasi-static stress-strain state of a densely packed granular material for finite strains were presented in [13, 21]. Models based on rheological schemes of Fig. 9.4 can serve for the description of mixed flows where domains of fast motions are conjugated with stagnant zones of quasi-static deformation. Taking into account disperse nature of viscous stresses, we can also suppose that these models can describe a flow of moderately humid and dry media with acceptable accuracy. However, the problem on determining the phenomenological viscosity coefficient depending on the extent to which a material is loosened and on viscous properties of a fluid that may be in a pore space is rather complicated and is not considered here.

First we assume that the coefficient α is constant and as the next step we consider the case of variable α depending on density of a material. Taking into account energy consistency of the Hencky strain tensor and the Cauchy stress tensor [17], we represent the constitutive relationships for the scheme shown in Fig. 9.4a in the form

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}^v + \boldsymbol{\sigma}^c, \quad \boldsymbol{\sigma}^v = \left(m - \frac{2\eta}{3}\right) \theta(\mathbf{e}) \boldsymbol{\delta} + 2\eta \mathbf{e}, \\ \boldsymbol{\sigma}^c : (\tilde{\mathbf{h}} - \mathbf{h}) &\leq 0, \quad \mathbf{h}, \tilde{\mathbf{h}} \in \mathbf{C}. \end{aligned} \quad (9.26)$$

Fig. 9.4 Rheological schemes with a viscous element: **a** cohesive material with rigid particles, **b** elastic regularization



Here σ^v is the viscous stress tensor which is expressed in terms of the strain rate tensor by the Stokes law, m and η are the coefficients of volume and shear viscosity which depend, generally speaking, on density of a material. In the case of small strains for m , $\eta \rightarrow 0$, from (9.26) we can obtain relationships corresponding to the rheological scheme which consists of a single element, namely, of a rigid contact.

By Kuhn–Tucker’s theorem, the variational inequality (9.26) is reduced to the problem of maximization of the Lagrangian

$$\sigma^c : \tilde{h} - \lambda(\mathfrak{x}\gamma(\tilde{h}) - \theta(\tilde{h}))$$

with respect to $\tilde{h}$. Here the nonnegative Lagrange multiplier λ vanishes provided that $\mathfrak{x}\gamma(h) < \theta(h)$. In this case the viscous flow conditions for $\sigma = \sigma^v$ are achieved. In rigid zones where $h = 0$ the maximum condition

$$\sigma^c : \tilde{h} \leq 0 \quad \forall \tilde{h} \in C$$

means that $\sigma^c \in K$. Here the stress tensor and the multiplier λ remain undetermined. Finally, if $\mathfrak{x}\gamma(h) = \theta(h)$ and $h \neq 0$, then from the maximum conditions for the Lagrangian in the differential form we obtain

$$\frac{\sigma^c}{\lambda} = -\left(1 + \frac{2\mathfrak{x}^2}{3}\right)\delta + \frac{2\mathfrak{x}}{\gamma(h)}h. \quad (9.27)$$

Hence, $\tau(\sigma^c) = \mathfrak{x} p(\sigma^c)$, i.e. a material is in the limiting state and deviators of the tensors σ^c and h are proportional to one another. Indeed, equating traces of the left-hand and right-hand sides of the tensor equality (9.27), we can establish that $p(\sigma^c) = \lambda$. Then, subtracting the tensor δ from both sides of the equality, we obtain the proportionality condition for the deviators:

$$\frac{\sigma^c + p(\sigma^c)\delta}{\lambda} = \frac{2\mathfrak{x}}{\gamma(h)}h - \frac{2\mathfrak{x}^2}{3}\delta = \frac{2\mathfrak{x}}{\gamma(h)}\left(h - \frac{1}{3}\theta(h)\delta\right).$$

From this condition by calculation of a quadratic invariant we obtain $\tau(\sigma^c) = \mathfrak{x}\lambda$.

The variational inequality (9.26) is reduced to the equivalent potential form

$$\boldsymbol{\sigma}^c \in \rho \partial \delta_C(\mathbf{h}).$$

The potential form implies that in the state of quasi-static deformation, where viscous stresses are small, the constitutive relationships (9.26) describe a thermodynamically reversible process and relate to the nonlinear elasticity theory. To a certain extent, reversibility is contradictory to the notion of internal friction between particles. Hence, it seems likely that in a granular material tangential stresses should be related to the surface tension, electromagnetic forces etc. rather than to friction.

Dual strain potential of a granular material is calculated with the help of the Young–Fenchel transform:

$$\Psi(s) = \sup_{\tilde{\mathbf{h}}} \{s : \tilde{\mathbf{h}} - \Phi(\tilde{\mathbf{h}})\}.$$

The Young–Fenchel transform of the indicator function of a convex cone is the indicator function of the dual cone, hence, the inequality (9.26) is also reduced to the form $\mathbf{h} \in \partial \delta_K(\boldsymbol{\sigma}^c/\rho)$.

Let k and μ be the volume compression modulus and the shear modulus of a compacted material, respectively. Decomposition of the strain tensor for the combination of a rigid contact and an elastic element in the scheme shown in Fig. 9.4b is additive, hence, we can obtain the constitutive relationships of a granular material with elastic properties:

$$\mathbf{h} \in \partial \Psi\left(\frac{\boldsymbol{\sigma}^c}{\rho}\right), \quad \Psi(s) = \frac{1}{2} \rho_0 |s|^2 + \delta_K(s). \quad (9.28)$$

Here $|s| = \sqrt{p^2(s)/k + \tau^2(s)/\mu}$ is the energy norm of a tensor. Assuming that k and μ tend to infinity, from the relationships (9.28) we can obtain the constitutive relationships (9.26) for a material with rigid particles.

Physically linear Hooke's law

$$\rho_0 s = \left(k - \frac{2\mu}{3}\right) \theta(\mathbf{h}) \boldsymbol{\delta} + 2\mu \mathbf{h}$$

establishes a one-to-one correspondence between the spaces of tensors s and $\mathbf{h}$. The symmetric bilinear form

$$(\tilde{s}, s) = \frac{\tilde{s} : \mathbf{h}}{\rho_0} = \frac{s : \tilde{\mathbf{h}}}{\rho_0}$$

on the space s defines an inner product associated with the norm introduced above.

Since the Young–Fenchel transform is involutive, the stress potential is determined by the formula

$$\Phi(\mathbf{h}) = \sup_{\tilde{s}} \{\tilde{s} : \mathbf{h} - \Psi(\tilde{s})\}.$$

Thus,

$$\begin{aligned} \frac{\Phi(\mathbf{h})}{\rho_0} &= \sup_{\tilde{s} \in \mathbf{K}} \left\{ (\tilde{s}, s) - \frac{1}{2} |\tilde{s}|^2 \right\} = \frac{1}{2} \sup_{\tilde{s} \in \mathbf{K}} \{ |s|^2 - |s - \tilde{s}|^2 \} \\ &= \frac{1}{2} |s|^2 - \frac{1}{2} \inf_{\tilde{s} \in \mathbf{K}} |s - \tilde{s}|^2 = \frac{1}{2} |s|^2 - \frac{1}{2} |s - s^\pi|^2, \end{aligned}$$

where the superscript π means the projection of a tensor onto the cone $\mathbf{K}$. For the projection onto the cone the inner product $(s^\pi, s - s^\pi)$ is zero, hence,

$$\Phi(\mathbf{h}) = \frac{1}{2} \rho_0 |s^\pi|^2.$$

Besides, for the cone

$$|s - s^\pi|^2 = (s - s^\pi, s - s^\pi) = (s - s^\pi, s + s^\pi) = |s|^2 - |s^\pi|^2,$$

hence, using Eq. (3.12) for the derivative of the expression $|s - s^\pi|^2$, we can show that the potential is a continuously differentiable function and

$$\sigma^c = \rho \frac{\partial \Phi}{\partial \mathbf{h}} = \rho s^\pi. \quad (9.29)$$

The projection onto the von Mises–Schleicher cone is determined by (4.14). For $\tau(s) \leq \mathfrak{x} p(s)$, where $s \in \mathbf{K}$, the equality $s^\pi = s$ holds. In this case, a material is deformed by the Kelvin–Voigt viscoelasticity law. For $\mu p(s) + \mathfrak{x} k \tau(s) \leq 0$, where $\mathbf{h} \in \mathbf{C}$, the vertex $s^\pi = \mathbf{0}$ of the cone is the projection to be determined and viscous flow of a material is observed. For $\tau(s) > \mathfrak{x} p(s)$, $\mu p(s) + \mathfrak{x} k \tau(s) > 0$ the projection belongs to the conical surface and is calculated by the formulae describing the limiting deformation conditions

$$p(s^\pi) = \frac{\mu p(s) + \mathfrak{x} k \tau(s)}{\mu + \mathfrak{x}^2 k}, \quad \frac{s^\pi}{p(s^\pi)} = - \left(1 - \frac{\mathfrak{x} p(s)}{\tau(s)} \right) \delta + \frac{\mathfrak{x}}{\tau(s)} s.$$

The immediate calculation of the potential taking into account the definition of the energy norm and the formulae

$$p(s) = -\frac{k \theta(\mathbf{h})}{\rho_0}, \quad \tau(s) = \frac{\mu \gamma(\mathbf{h})}{\rho_0},$$

which follows from the Hooke law, yields

$$2\rho_0\Phi = \begin{cases} 0, & \text{if } \mathfrak{x}\gamma(\mathbf{h}) \leq \theta(\mathbf{h}), \\ k\theta^2(\mathbf{h}) + \mu\gamma^2(\mathbf{h}), & \text{if } \mu\gamma(\mathbf{h}) + \mathfrak{x}k\theta(\mathbf{h}) \leq 0, \\ \frac{\mu k (\mathfrak{x}\gamma(\mathbf{h}) - \theta(\mathbf{h}))^2}{\mu + \mathfrak{x}^2 k}, & \text{if } \mathfrak{x}\gamma(\mathbf{h}) > \theta(\mathbf{h}) \text{ and } \mu\gamma(\mathbf{h}) + \mathfrak{x}k\theta(\mathbf{h}) > 0. \end{cases}$$

As k and μ tend to infinity, the potential tends to the indicator function $\delta_C(\mathbf{h})$, hence, the tensor equation (9.29) is transformed into the variational inequality (9.26).

Now assume that the strain potential has the same form provided that the internal friction parameter $\mathfrak{x}$ depends on density of a material. With the help of the formula $\partial\rho/\partial\mathbf{h} = -\rho\boldsymbol{\delta}$, which follows from the continuity equation (9.25), we represent the relation (9.29) in a more general form

$$\begin{aligned} \boldsymbol{\sigma}^c &= \rho \left(\frac{\partial\Phi}{\partial\mathbf{h}} \Big|_{\mathfrak{x}} + \frac{\partial\Phi}{\partial\mathfrak{x}} \Big|_{\mathbf{h}} \frac{\partial\mathfrak{x}}{\partial\mathbf{h}} \right) = \rho s^\pi - \Delta p \boldsymbol{\delta}, \\ \Delta p &= \frac{\mu k}{(\mu + \mathfrak{x}^2 k)^2} (\mu\gamma(\mathbf{h}) + \mathfrak{x}k\theta(\mathbf{h}))_+ (\mathfrak{x}\gamma(\mathbf{h}) - \theta(\mathbf{h}))_+ \frac{\rho^2}{\rho_0} \frac{d\mathfrak{x}}{d\rho}. \end{aligned} \quad (9.30)$$

Here Δp is the correction pressure caused by ρ -dependence of $\mathfrak{x}$. Notice that the quantity Δp is nonzero only under the limiting conditions where, in accordance with the relation (9.30), the stress tensor $\boldsymbol{\sigma}^c + \Delta p \boldsymbol{\delta}$ belongs to the conical surface $\mathbf{K}$:

$$\frac{\tau(\boldsymbol{\sigma}^c)}{\mathfrak{x}} = p(\boldsymbol{\sigma}^c) - \Delta p. \quad (9.31)$$

We can show that the constitutive relationships, which take into account elastic and viscous properties of a granular material, are correct in the thermodynamic sense: since a material is isotropic, for them the heat influx equation

$$\begin{aligned} \boldsymbol{\sigma} : \mathbf{e} &= (\mathbf{q}^* \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^v) \cdot \mathbf{q}) : \frac{d}{dt} (\mathbf{q}^* \cdot \mathbf{h} \cdot \mathbf{q}) + \boldsymbol{\sigma}^v : \mathbf{e} \\ &= \rho \frac{\partial\Phi}{\partial(\mathbf{q}^* \cdot \mathbf{h} \cdot \mathbf{q})} : \frac{d}{dt} (\mathbf{q}^* \cdot \mathbf{h} \cdot \mathbf{q}) + \boldsymbol{\sigma}^v : \mathbf{e} = \rho \dot{\Phi}(\mathbf{h}) + m\theta^2(\mathbf{e}) + \eta\gamma^2(\mathbf{e}) \end{aligned}$$

is valid, where the term $\rho \dot{\Phi}(\mathbf{h})$ is the velocity of variation of potential energy of reversible deformation and the sum of the last two terms is the power of dissipative forces. Since $s^\pi \in \mathbf{K}$, for any $\tilde{\mathbf{h}} \in \mathbf{C}$ the inequality $s^\pi : \tilde{\mathbf{h}} \leq 0$ holds. In addition, according to the equation $(s^\pi, s - s^\pi) = 0$, we have

$$s^\pi : \mathbf{h} = \rho_0(s^\pi, s) = \rho_0 |s^\pi|^2 \geq 0.$$

Hence, due to (9.30)

$$\boldsymbol{\sigma}^c : (\tilde{\mathbf{h}} - \mathbf{h}) + \Delta p (\theta(\tilde{\mathbf{h}}) - \theta(\mathbf{h})) \leq 0 \quad \forall \tilde{\mathbf{h}} \in \mathbf{C}. \quad (9.32)$$

Assuming that k and μ tend to infinity, we obtain a generalization of the variational inequality (9.26), which serves to describe a material with rigid particles, to the case of the variable coefficient $\mathfrak{x}$. This generalization coincides with the inequality (9.32) to within the replacement of the correction pressure Δp by Δp_0 .

In the limiting state of a material, the tensor equality (9.30) is reduced to the proportionality condition for deviators of the tensors σ^c and $\mathbf{h}$ and to the equation for pressure

$$p(\sigma^c) - \Delta p = \frac{\rho}{\rho_0} \frac{\mu k}{\mu + \mathfrak{x}^2 k} (\mathfrak{x} \gamma(\mathbf{h}) - \theta(\mathbf{h})).$$

The expression for Δp can be rearranged to the following form:

$$\Delta p = \frac{p(\sigma^c) - \Delta p}{\mu + \mathfrak{x}^2 k} (\mu \gamma(\mathbf{h}) + \mathfrak{x} k \theta(\mathbf{h})) \rho \frac{d\mathfrak{x}}{d\rho}.$$

Hence, taking into account that $\mathfrak{x} \gamma(\mathbf{h}) \rightarrow \theta(\mathbf{h})$, for $k, \mu \rightarrow \infty$ we obtain

$$\Delta p_0 = \frac{f-1}{f} p(\sigma^c), \quad f = 1 + \frac{\rho}{\mathfrak{x}} \frac{d\mathfrak{x}}{d\rho} \ln \frac{\rho_0}{\rho},$$

and from the limiting state condition (9.31) it follows that

$$\tau(\sigma^c) = \frac{\mathfrak{x} p(\sigma^c)}{f}. \quad (9.33)$$

Applying the Kuhn–Tucker theorem to the variational inequality (9.32), we arrive at the equation which is obtained by the replacement of the tensor σ^c with the tensor $\sigma^c + \Delta p_0 \delta$ in the equality (9.27).

The closed model of dynamics of a granular material, well-defined in the thermodynamic sense, consists of the constitutive relationships (9.30) combined with the tensor equality in (9.26) for viscous stresses, the equations of motion, and the kinematic relationships connecting the strain tensor and the velocity vector ($\mathbf{g}$ is the vector of mass forces):

$$\rho \dot{\mathbf{v}} = \nabla \cdot \sigma + \rho \mathbf{g}, \quad \mathbf{h} = \frac{1}{2} \ln l, \quad l = (\nabla_{\xi} \mathbf{x})^* \cdot \nabla_{\xi} \mathbf{x}, \quad \dot{\mathbf{x}} = \mathbf{v}.$$

The proposed model is sufficiently simple for computer calculations. Besides, neglecting elasticity of particles, with its help we can analyze some exact solutions.

9.4 Shear Stresses

As a simple example, we determine the stress state of an ideal granular material with rigid particles, which is previously compressed in the x_3 direction by constant

pressure p_0 , for homogeneous shear in the $x_1 x_2$ plane. We assume that at the instant $t = 0$ a material is densely packed and, thus, initial strains of particles are zero. In this case the shear motion is accompanied by dilatancy described in Sect. 9.1. As the critical value of density, for which a material no longer increases in volume, is achieved, the passage to the viscous flow conditions according to the Newton law happens.

Deformation at the dilatancy stage is described by the kinematic equation (9.11). Due to (9.14)

$$l_1 = z^v, \quad l_2 = z^{-v}, \quad l_3 = z^2,$$

$$h_1 = -h_2 = \frac{v}{2} \ln z, \quad h_{1,2} = \frac{v}{z^v - z^{-v}} \ln z.$$

The tensors $\mathbf{l} - l_1 \delta$ and $\mathbf{l} - l_2 \delta$ involved in (9.20) are calculated by the formula

$$\mathbf{l} - l_{1,2} \delta = \begin{pmatrix} 1 - z^{\pm v} & z^{v/2} - z^{-v/2} & 0 \\ z^{v/2} - z^{-v/2} & z^{\mp v} - 1 & 0 \\ 0 & 0 & z^2 - z^{\pm v} \end{pmatrix}.$$

Their product is given by

$$(\mathbf{l} - l_1 \delta) \cdot (\mathbf{l} - l_2 \delta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (z^2 - z^v)(z^2 - z^{-v}) \end{pmatrix}.$$

The components of the Hencky logarithmic tensor in a Cartesian coordinate system are determined from the decomposition (9.18) and have the form

$$h_{11} = -h_{22} = -\frac{v}{2} \frac{z^{v/2} - z^{-v/2}}{z^{v/2} + z^{-v/2}} \ln z, \quad h_{12} = \frac{v \ln z}{z^{v/2} + z^{-v/2}}, \quad h_{33} = \ln z. \quad (9.34)$$

If the deformation program is defined by the equation $\chi = \chi(t)$, then the projections of the velocity vector and the nonzero components of the strain rate tensor can be calculated by differentiating Eqs. (9.11):

$$v_1 = 0, \quad v_2 = \dot{\chi} x_1, \quad v_3 = \frac{\dot{z} x_3}{z}, \quad 2e_{12} = \dot{\chi}, \quad e_{33} = \frac{\dot{z}}{z}.$$

The tensor of viscous stresses in a material is determined by the formulae

$$\sigma_{11}^v = \sigma_{22}^v = \left(m - \frac{2\eta}{3}\right) \frac{\dot{z}}{z}, \quad \sigma_{12}^v = \eta \dot{\chi}, \quad \sigma_{33}^v = \left(m + \frac{4\eta}{3}\right) \frac{\dot{z}}{z}. \quad (9.35)$$

Applying the expressions (9.34), the proportionality condition for deviators which follows from (9.27), the limiting surface equation (9.33), and the condition $\sigma_{33} = -p_0$, finally we arrive at

$$\begin{aligned} \frac{\sigma_{jj}^c}{p(\sigma^c)} &= \left((-1)^j v \frac{z^{v/2} - z^{-v/2}}{z^{v/2} + z^{-v/2}} - \frac{2}{3} \right) \frac{\mathfrak{x}^2}{f} - 1, \quad j = 1, 2, \\ \frac{\sigma_{12}^c}{p(\sigma^c)} &= \frac{2 v \mathfrak{x}^2}{(z^{v/2} + z^{-v/2}) f}, \quad p(\sigma^c) = \frac{p_0 + \sigma_{33}^v}{1 - 4 \mathfrak{x}^2 / (3 f)}. \end{aligned} \quad (9.36)$$

It turns out that as the angle increases, the deviator components of the tensor σ^c relax and in the limit the state of viscous shear is achieved. Normal stresses in a material tend to $-p_0$. Graphs of the χ -dependence of quasi-static stresses are presented in Fig. 9.5. They are constructed with the use of the expression (9.15) ($n = 0.5$) and the corresponding expression for f :

$$f = 1 + \frac{n \rho_*}{\rho - \rho_*} \ln \frac{\rho_0}{\rho}.$$

Judging by Fig. 9.5, tangential stress σ_{12}^c being zero before deformation varies step-wise for $\chi = 0$. This is related to the fact that in this model elastic properties of a material are not taken into account. In fact, the dilatancy process is preceded by the elastic stage of deformation without variation of volume where tangential stress increases monotonically. The elastic stage can be described in the framework of the small strain theory. This is confirmed by coinciding initial stresses calculated with the help of the formulae (9.36) for $z \rightarrow 1$:

$$\sigma_{11} = \sigma_{22} = -p_0 \left(1 + \frac{2}{v_0^2} \right), \quad \sigma_{12} = \frac{p_0}{v_0}, \quad (9.37)$$

with stresses of the limiting shear of an elastic granular material for small strains determined by Eq. (9.10).

Notice that the solution (9.36) can be applied to the analysis of stress state of a granular material for homogeneous shear with its weight taken into account. In this case from the equilibrium equation $\partial p_0 / \partial x_3 = -\rho g$, taking into account that $dx_3 = z d\xi_3$, $\rho z = \rho_0$ and that z does not depend on x_3 , we obtain

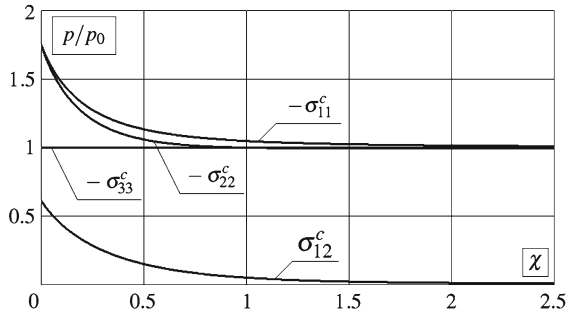
$$p_0 = \rho g (H_0 z - x_3),$$

where H_0 is the initial thickness of a layer. Thus, viscous stresses (9.35) are constant and quasi-static ones are linear in depth.

9.5 Couette Flow

Stagnant zones of a quasi-static deformation arise, for example, in rotational motion of a densely packed granular material in the space between two extended coaxial cylinders [26]. Assume that an inner cylinder of radius r_0 rotates about its axis with constant angular velocity ω_0 and a fixed outer cylinder is of infinite radius. If friction

Fig. 9.5 Relaxation curves of quasi-static stresses



of a material on walls is considerable, then at the initial stage in some domain near the inner cylinder the non-stationary shear motion inhomogeneous with respect to radius and accompanied by dilatancy is observed. Volumetric expansion results in displacement of particles in the direction of the axis and, as a consequence, in the increase of pressure in this domain. With time, stationary conditions are established. The part of a material being far from the rotating cylinder remains fixed and in the moving domain the density of a material achieves the critical value ρ_* .

The deformation process can be described in detail in the framework of the proposed mathematical model only with the use of numerical methods. Neglecting the own weight of a granular material and the influence of inertia forces, we construct the simplest stationary solution of the problem on slow motion which does not depend on the axial coordinate ζ . Assume that r_1 is the radius of an unknown boundary dividing the domain of viscous flow and the stagnant zone, p_0 is the value of initial pressure in the axial direction, M_0 is the rotational moment per a length unit of the inner cylinder. For $r_1 > r_0$ the flow occurs only if M_0 is higher than the limiting value $M_0^* = 2\pi r_0^2 p_0 / \nu_0$ for which the stress state (9.37) is achieved on the inner cylinder. In the strict sense, it is required to determine this state, corresponding to the start of the dilatancy process, for an infinitely thin layer of a granular material adjacent to the cylinder of radius r_0 . However, this problem again leads to Eqs. (9.37) to within the replacement of a Cartesian coordinate system by a cylindrical one $r \varphi \zeta$.

In the case of steady-state motion, the vector of velocity of particles and the velocity of shear are determined in terms of local angular velocity $\omega(r)$ by the formulae

$$v_r = v_\zeta = 0, \quad v_\varphi = \omega r, \quad 2e_{r\varphi} = \frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} = r \frac{\partial \omega}{\partial r}. \quad (9.38)$$

The remaining components of the strain rate tensor are zero. Nonzero components of the stress tensor satisfy the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \quad \frac{\partial \sigma_{r\varphi}}{\partial r} + 2 \frac{\sigma_{r\varphi}}{r} = 0, \quad \frac{\partial \sigma_\zeta}{\partial \zeta} = 0. \quad (9.39)$$

The parameter $\varkappa$ vanishes in the flow domain, hence, by the condition (9.33) the intensity of tangential stresses for the tensor σ^c is zero, i.e. this tensor is spherical. Due to the relationships (9.26) and (9.38), we have

$$\sigma_r = \sigma_\varphi = \sigma_\zeta = -p_1, \quad \sigma_{r\varphi} = \eta r \frac{\partial \omega}{\partial r}.$$

The first and the third equations of the system (9.39) hold automatically provided that $p_1 = \text{const}$. Integrating the second equation, taking into account the boundary conditions for ω under $r = r_0$ and $r = r_1$, we obtain

$$\sigma_{r\varphi} = -2\eta C_1 \frac{r_1^2}{r^2}, \quad \omega = C_1 \left(\frac{r_1^2}{r^2} - 1 \right), \quad C_1 = \frac{\omega_0 r_0^2}{r_1^2 - r_0^2}. \quad (9.40)$$

Assuming that on the interface a material is in the limiting state (9.37) for $\sigma_r = \sigma_{11}$, $\sigma_\varphi = \sigma_{22}$ and $\sigma_{r\varphi} = -\sigma_{12}$, from the conditions of continuity of stresses σ_r and $\sigma_{r\varphi}$ we can determine the position of the unknown boundary and the value of pressure:

$$r_1 = r_0 \sqrt{1 + 2\eta \omega_0 \frac{v_0}{p_0}}, \quad p_1 = p_0 \left(1 + \frac{2}{v_0^2} \right).$$

In the stagnant zone stresses are statically indeterminate. When calculating them, it is required to pass to the model, which takes into account elastic properties of a material, and then to assume that elasticity moduli k and μ tend to infinity. In this zone strains h_r and h_φ are equal and vanish, hence, by the Hooke law we have $\sigma_r = \sigma_\varphi$. Due to Eqs. (9.39) the stress σ_r is constant. Thus, in the stagnant zone $\sigma_r = \sigma_\varphi = -p_1$, $\sigma_\zeta = -p_0$, and $\sigma_{r\varphi}$ is defined by Eqs. (9.40). It can be proved that the obtained stress tensor is admissible, i.e. it belongs to the cone $\mathbf{K}$ for any $r > r_1$.

Since $M_0 = -2\pi r_0^2 \sigma_{r\varphi}(r_0)$, the formulae

$$\frac{1}{v_0} = \frac{M_0}{2\pi r_1^2 p_0}, \quad \eta = \frac{M_0}{4\pi \omega_0} \left(\frac{1}{r_0^2} - \frac{1}{r_1^2} \right) \quad (9.41)$$

are valid. They can serve in order to determine experimentally the internal friction parameter and the viscosity coefficient of a granular material from the measurement of angular velocity, rotational moment, and the width of a domain of viscous flow. A feature of this experiment is that pressure along the ζ axis in a flow domain depends on v_0 .

It should be noticed that if viscosity is small, then the width of a flow domain is proportional to η :

$$\frac{r_1 - r_0}{r_0} \approx \eta \omega_0 \frac{v_0}{p_0}.$$

As the viscosity coefficient tends to zero, the flow degenerates and the rotating cylinder slips as in the case of absolutely smooth surface. With decreasing pressure,

the stagnant zone decreases and vanishes as $p_0 \rightarrow 0$. The increase of pressure, as well as the decrease of viscosity of a material, has influence on the position of a boundary.

A solution describing stationary flow of a granular material between coaxial cylinders, where inertia of rotation (the right-hand side of the form $-\rho \omega^2 r$ in the first equation of the system (9.39)) is taken into account, can be constructed in a similar way. The distinction is in the expression for pressure in the domain $r_0 < r < r_1$ which is determined from this equation:

$$p = p_1 + \rho_* \int_{r_1}^r \omega^2(y) y dy = p_1 - \frac{\rho_* C_1^2}{2} \left(\frac{r_1^4}{r^2} + 4r_1^2 \ln \frac{r}{r_1} - r^2 \right). \quad (9.42)$$

For $p_0 \rightarrow 0$ ($r_1 \rightarrow \infty$) with the help of auxiliary limits

$$C_1 \rightarrow 0, \quad C_1 r_1^2 \rightarrow \omega_0 r_0^2, \quad C_1^2 r_1^2 \ln r_1 \rightarrow 0$$

from the relationships (9.40), (9.42) we can obtain a solution with infinite flow domain:

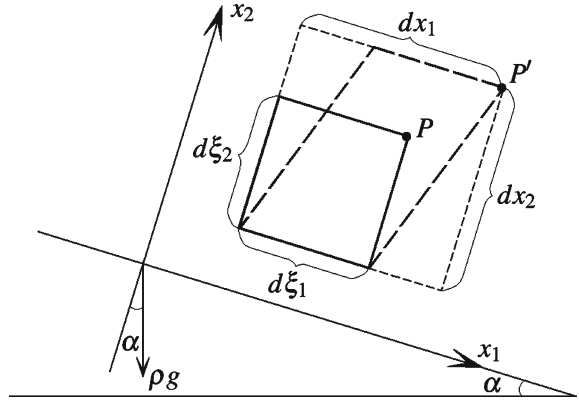
$$\sigma_{r\varphi} = -2\eta\omega_0 \frac{r_0^2}{r^2}, \quad \omega = \omega_0 \frac{r_0^2}{r^2}, \quad p = \rho_* \omega_0^2 \frac{r_0^4}{2r^2}. \quad (9.43)$$

Passage to the limit in (9.43) as $\eta \rightarrow 0$ shows that, independently of the value of angular velocity, non-viscous stationary flow is achieved as rotational moment $M_0 = 4\pi r_0^2 \eta \omega_0$ tends to zero, however, kinematic characteristics of the motion remain unchanged.

To conclude, we notice that for considerably high angular velocities the uniform rotational Couette motion becomes unstable [14, 29]. In this case the pattern of stationary flow varies along the axis, hence, the interface of the flow domain and the stagnation zone is no longer cylindrical. The rotational Couette flow as well as some other types of motions of a granular material was analyzed by experimental methods in [7, 12, 28, 31].

9.6 Motion Over an Inclined Plane

We study slow non-stationary motion of a layer of a granular material under the action of own weight over a rough inclined plane at an slope angle α to horizon in the plane formulation. It is known that such motion is realized only if the angle α is greater than the internal friction angle α_0 . If in the initial state of a layer strains are zero (particles are densely packed), then the initial stage of a motion is accompanied by dilatancy of a material with increasing the volume up to the critical value which corresponds to zero value of the internal friction parameter. Then viscous hydrodynamic flow is observed. Under the condition of adhesion of particles of a granular material to an

Fig. 9.6 Kinematics of gravitational motion

inclined surface, kinematics of motion is determined by the equations

$$dx_1 = d\xi_1 + z \chi d\xi_2, \quad dx_2 = z d\xi_2, \quad dx_3 = d\xi_3, \quad (9.44)$$

where the quantities $z(t, x_2)$ and $\chi(t, x_2)$ are defined above (see Fig. 9.6). The distortion tensor and the left Cauchy–Green tensor look as follows:

$$\nabla_{\xi} \mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ z\chi & z & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{l} = \begin{pmatrix} 1 + z^2\chi^2 & z^2\chi & 0 \\ z^2\chi & z^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The principal values of the tensor $\mathbf{l}$ are defined by the equalities

$$2l_{1,2} = 1 + z^2(1 + \chi^2) \pm \sqrt{(1 + z^2(1 + \chi^2))^2 - 4z^2}, \quad l_3 = 1.$$

Since $l_1 l_2 = z^2$, we have

$$\theta(\mathbf{h}) = h_1 + h_2 = \ln z, \quad \gamma(\mathbf{h}) = \sqrt{\ln^2\left(\frac{l_1}{z}\right) + \frac{1}{3} \ln^2 z}.$$

After obvious transformations, the dilatancy equation is written in the form

$$l_1 = z^{1+\kappa},$$

where $\kappa = \sqrt{v^2 + 1}$. Solving it for χ , we obtain

$$\chi = \sqrt{(z^{-1+\kappa} - 1)(1 - z^{-1-\kappa})}. \quad (9.45)$$

Due to (9.45), principal logarithmic strains are as follows:

$$h_{1,2} = \frac{1}{2} (1 \pm \kappa) \ln z, \quad h_3 = 0.$$

The tensors involved in the right-hand side of (9.20) turn out to be equal to

$$\begin{aligned} \mathbf{l} - l_{1,2} \boldsymbol{\delta} &= \begin{pmatrix} z^{1 \mp \kappa} - z^2 & z^2 \chi & 0 \\ z^2 \chi & z^2 - z^{1 \pm \kappa} & 0 \\ 0 & 0 & 1 - z^{1 \pm \kappa} \end{pmatrix}, \\ (\mathbf{l} - l_1 \boldsymbol{\delta}) \cdot (\mathbf{l} - l_2 \boldsymbol{\delta}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1 - z^{1+\kappa})(1 - z^{1-\kappa}) \end{pmatrix}. \end{aligned}$$

The table of divided differences has the form

$$\begin{array}{c|ccc} z^{1+\kappa} & \frac{1+\kappa}{2} \ln z & & \\ & \frac{\kappa \ln z}{z^{1+\kappa} - z^{1-\kappa}} & & \\ z^{1-\kappa} & \frac{1-\kappa}{2} \ln z & \left(\frac{\kappa}{z^{1+\kappa} - z^{1-\kappa}} - \frac{1}{2} \frac{1-\kappa}{z^{1-\kappa} - 1} \right) \frac{\ln z}{z^{1+\kappa} - 1} & \\ & \frac{1-\kappa}{2} \frac{\ln z}{z^{1-\kappa} - 1} & & \\ 1 & 0 & & \end{array}$$

Due to the equality (9.20), nonzero components of the logarithmic tensor in the $x_1 x_2$ coordinate system are as follows:

$$\begin{aligned} h_{11} &= \left(\frac{1+\kappa}{2} - \kappa \frac{z - z^{-\kappa}}{z^\kappa - z^{-\kappa}} \right) \ln z, \quad h_{22} = \left(\frac{1+\kappa}{2} - \kappa \frac{z^\kappa - z}{z^\kappa - z^{-\kappa}} \right) \ln z, \\ h_{12} &= \frac{\kappa \ln z}{z^\kappa - z^{-\kappa}} \sqrt{(z^\kappa - z)(z - z^{-\kappa})}. \end{aligned}$$

Strain rates in a layer are calculated according to the kinematic equation (9.44)

$$e_{22} = \frac{\dot{z}}{z}, \quad 2e_{12} = \frac{\dot{z}\chi}{z} + \dot{\chi}.$$

Viscous stresses are defined by the equalities

$$\sigma_{11}^v = \sigma_{33}^v = \left(m - \frac{2\eta}{3} \right) \frac{\dot{z}}{z}, \quad \sigma_{22}^v = \left(m + \frac{\eta}{3} \right) \frac{\dot{z}}{z}, \quad \sigma_{12}^v = \frac{\eta(z\dot{\chi})}{z}. \quad (9.46)$$

Quasi-static stresses are determined from the proportionality condition for the deviators of tensors σ^c and $\mathbf{h}$ and from Eq. (9.33):

$$\frac{\sigma_{jj}^c}{p(\sigma^c)} = \left(\frac{h_{jj}}{\theta(\mathbf{h})} - \frac{1}{3} \right) \frac{2\mathfrak{x}^2}{f} - 1, \quad j = 1, 2, 3, \quad \frac{\sigma_{12}^c}{p(\sigma^c)} = \frac{2\mathfrak{x}^2 h_{12}}{\theta(\mathbf{h}) f}. \quad (9.47)$$

Neglecting inertia forces in the case of slow motion of a material, from the equilibrium equations

$$\frac{\partial \sigma_{12}}{\partial x_2} + \rho g \sin \alpha = 0, \quad \frac{\partial \sigma_{22}}{\partial x_2} - \rho g \cos \alpha = 0$$

with the use of the formulae $dx_2 = z d\xi_2$ and $\rho z = \rho_0$ we obtain

$$\sigma_{12} = \sigma_{12}^v + \sigma_{12}^c = \rho_0 g (H_0 - \xi_2) \sin \alpha,$$

$$\sigma_{22} = \sigma_{22}^v + \sigma_{22}^c = -\rho_0 g (H_0 - \xi_2) \cos \alpha$$

(H_0 is the initial thickness of a layer). With the help of the relationships (9.46) and (9.47) we can derive the ordinary differential equation

$$\frac{\eta(z\dot{\chi}) - z\sigma_{12}}{(m + \eta/3)\dot{z} - z\sigma_{22}} = \frac{\kappa \sqrt{(z^\kappa - z)(z - z^{-\kappa})}}{((1 + 3\kappa)/6 - f/(2\mathfrak{x}^2))(z^\kappa - z^{-\kappa}) - \kappa(z^\kappa - z)}. \quad (9.48)$$

From the equality (9.45) it follows that

$$(z\dot{\chi}) = \frac{(\kappa + 1)(z^\kappa - z) + (\kappa - 1)(z - z^{-\kappa}) + (d\kappa/dz)(z^\kappa - z^{-\kappa})z \ln z}{2\sqrt{(z^\kappa - z)(z - z^{-\kappa})}} \dot{z},$$

where prime means a derivative with respect to z . Integrating Eq. (9.48) for the initial condition $z(0) = 1$, we can determine the dependence $z(t, \xi_2)$.

At the initial stage of deformation where the parameter $\mathfrak{x}$ is bounded away from zero by a positive constant, Eqs. (9.47) for stresses can be simplified with the use of the asymptotic decompositions

$$z = 1 + \Delta, \quad z^\kappa - z \approx (\kappa - 1)\Delta, \quad z - z^{-\kappa} \approx (\kappa + 1)\Delta \quad (\Delta \ll 1). \quad (9.49)$$

We have

$$\begin{aligned} \frac{\sigma_{11}^c}{p(\sigma^c)} &= \frac{\sigma_{33}^c}{p(\sigma^c)} = -1 - \frac{2\mathfrak{x}^2}{3f}, \\ \frac{\sigma_{22}^c}{p(\sigma^c)} &= -1 + \frac{4\mathfrak{x}^2}{3f}, \quad \frac{\sigma_{12}^c}{p(\sigma^c)} = \frac{\mathfrak{x}^2 v}{f}. \end{aligned} \quad (9.50)$$

Here, according to the notations introduced above, $v = \sqrt{1/\mathfrak{x}^2 - 4/3}$. Equation (9.48) is also reduced to a more simple form

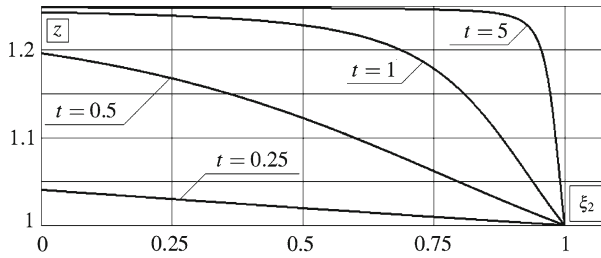


Fig. 9.7 Development of dilatancy in a moving layer

$$\frac{\eta v \dot{z} - \rho_0 g (H_0 - \xi_2) \sin \alpha}{(m + \eta/3) \dot{z} + \rho_0 g (H_0 - \xi_2) \cos \alpha} = \frac{v}{4/3 - f/\alpha^2}. \quad (9.51)$$

A material is in the state of limit equilibrium if $\dot{z} = 0$. In this state $\alpha = \alpha_0$ and $f = 1$, hence,

$$\alpha = \alpha_0, \quad \cot \alpha_0 = v_0, \quad \alpha_0 = \frac{\sqrt{3} \sin \alpha_0}{\sqrt{3} + \sin^2 \alpha_0}.$$

The formula, which relates the parameter α_0 and the internal friction angle α_0 , shows that for adequate description of the surface of a natural slope of a granular material in the framework of this model the von Mises–Schleicher cone must be inscribed into the Coulomb–Mohr cone [30]. This formula was obtained in Chap. 5 from other considerations, namely, with the help of the theorems about limit equilibrium of a granular material. Besides, from this formula it follows that the left-hand side of the first Eq. (9.41) is equal to the internal friction coefficient $\tan \alpha_0$.

A solution of Eq. (9.51) is correct only for $\alpha \geq \alpha_0$. For smaller angles, the equality $\alpha \gamma(\mathbf{h}) = \theta(\mathbf{h})$ loses its meaning, its right-hand side becomes negative since in this case $\dot{z} < 0$ and $z < 1$. For such angles a motion under the action of own weight is impossible.

In the process of deformation of a layer where α tends to zero, Eqs. (9.49) of decompositions fail, hence, Eq. (9.51) does not hold. After transition to the dimensionless quantities, the solution of Eq. (9.48) besides the slope angle and characteristics of the dilatancy curve depends only on the ratio m/η of the viscosity coefficients and on the dimensionless complex $R_0 = \rho_0 g H_0^2 / (\eta v_0)$ equal to the ratio of the Reynolds number to the squared Froude number (v_0 is the characteristic velocity of motion of particles in the x_2 direction). According to (9.48), near the free surface of a layer $\dot{z} \rightarrow 0$, hence, the dilatancy and the shear flow take place mainly in the lower part of a layer in the domain of increased pressure. We can observe the development of the process analyzing the change of the quantity z over the layer depth.

In Fig. 9.7 the curves obtained by the Runge–Kutta numerical method of fourth-order accuracy for $R_0 = 1$, $\alpha = 45^\circ$, $m = 0$, $\alpha_0 = 0.5$, $z_* = 1.25$, $n = 0.5$ at different instants of dimensionless time are shown. The derivative $d\kappa/dz$ was calculated with the use of the dependence (9.15):

$$\frac{d\kappa}{dz} = \frac{n}{\mathfrak{x}^2 \kappa (z_* - z)}.$$

Computations show that the qualitative pattern of a flow does not vary with increasing R_0 . As viscosity coefficients approach zero ($R_0 \rightarrow \infty$), Eq. (9.51) is reduced to the nonlinear algebraic equation

$$\tan \alpha = \frac{v}{f/\mathfrak{x}^2 - 4/3},$$

which has no solutions for $\alpha > \alpha_0$, since due to the inequalities $f > 1$ and $\mathfrak{x} < \mathfrak{x}_0$ the right-hand side of this equation does not exceed $1/\nu_0 = \tan \alpha_0$. Thus, the problem on slow motion of a non-viscous material without considering inertia forces is ill-posed. It is obvious that in this case a shear zone degenerates into a line and a layer slides over an inclined plane as a rigid body. Introducing inertia terms to describe the localization of a flow with increasing R_0 results in a system of nonlinear partial differential equations whose solution is beyond the scope of this chapter.

If viscosity of a material is nonzero, then $z \rightarrow z_*$ and $\mathfrak{x} \rightarrow 0$ in all flow domain when passing to steady-state conditions. Equation (9.48) is simplified ($H_* = H_0 z_*$ is the thickness of a dilatant layer):

$$\eta \dot{\chi} = \rho_* g (H_* - x_2) \sin \alpha.$$

It describes the flow of a viscous fluid with quadratic velocity profile

$$v_1 = \frac{\rho_* g}{2\eta} (2 H_* x_2 - x_2^2) \sin \alpha. \quad (9.52)$$

Passage to the limit in (9.52) as $\eta \rightarrow 0$ is incorrect since this solution is obtained without considering inertia forces.

Thus, the mathematical model proposed in Sect. 9.3 adequately describes the qualitative pattern of motion of a layer of a granular material over an inclined plane.

9.7 Plane-Parallel Motion

We consider stationary flow of a granular material occupying the half-space $x_2 \leq 0$ on the boundary of which a heavy rough plate is placed. This plate moves in the direction of the x_1 axis with constant velocity v_0 (Fig. 9.8). Let p_0 be the pressure caused by weight of a plate and q_0 be the tangential stress on a contact surface. Motion takes place only if q_0 is higher than the stress $q_0^* = p_0/\nu_0$ corresponding to the limit equilibrium state starting with which the dilatancy is observed in a thin surface layer of a material. Limit equilibrium stresses are determined from the relationships (9.50) for $\mathfrak{x} = \mathfrak{x}_0$ and $f = 1$. Taking into account the boundary condition $\sigma_{22} = -p_0$, we have

$$\sigma_{11} = \sigma_{33} = -p_0 \left(1 + \frac{2}{v_0^2} \right), \quad \sigma_{12} = \frac{p_0}{v_0}. \quad (9.53)$$

In the dilatancy process a domain of stationary flow is formed: a layer of finite depth H_* , on the lower boundary of which the stress state (9.53) to within the replacement of p_0 by pressure $p_1 = p_0 + \rho_* g H_*$ on this boundary is achieved from the side of the stagnant zone. In the flow domain $x_2 = 0$, hence, the stress tensor σ^c is spherical:

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p_0 + \rho_* g x_2.$$

The viscous stress tensor σ^v contains a unique nonzero component $\sigma_{12} = \eta \partial v_1 / \partial x_2$ which is constant with depth and equals q_0 . Thus, in the flow domain the velocity profile is linear in thickness:

$$v_1 = \frac{q_0 (x_2 + H_*)}{\eta}, \quad q_0 = \frac{\eta v_0}{H_*}. \quad (9.54)$$

Tangential stress varies continuously when passing across the interface, hence, the quadratic equation

$$p_0 H_* + \rho_* g H_*^2 = \eta v_0 v_0$$

holds. It has a positive root

$$H_* = \frac{\sqrt{p_0^2 + 4 \eta \rho_* g v_0 v_0} - p_0}{2 \rho_* g}. \quad (9.55)$$

In the stagnant zone the stress state of a material is determined with the use of the equilibrium equations and the von Mises–Schleicher condition. Taking into account the equality $\sigma_{11} = \sigma_{33}$, which follows from the constitutive relationships (9.27) for $h_{11} = h_{33} = 0$, we obtain

$$\sigma_{11} = \sigma_{33} = -p_2 \left(1 + \frac{2}{v_0^2} \right) + \frac{\sqrt{3(p_2^2 - p_1^2)}}{\varepsilon_0 v_0^2}, \quad \sigma_{22} = -p_2, \quad \sigma_{12} = \frac{p_1}{v_0},$$

where $p_2 = p_1 - \rho_0 g (x_2 + H_*)$.

To determine the internal friction and viscosity coefficients experimentally, it is convenient to use the equivalent form of the solution:

$$\frac{1}{v_0} = \frac{q_0}{p_0 + \rho_* g H_*}, \quad \eta = \frac{q_0 H_*}{v_0}.$$

It turns out that if a plate is weightless ($p_0 \rightarrow 0$), then due to own weight of a material the layer of viscous flow of thickness

Here u is the radial displacement of the point, σ_r , σ_φ and σ_ψ are the components of the stress tensor in a spherical coordinate system, p is the pressure at infinity, Q is the change in the cavity volume (mass flow), and μ is the shear modulus. The same distribution of displacements and stresses is formed in an expanding elastic layer with a given pressure applied to its outer boundary. The solution of the problem with cylindrical symmetry has the form

$$u = \frac{Q}{2\pi r}, \quad \sigma_r = -p - \frac{\mu Q}{2\pi r^2}, \quad \sigma_\varphi = -p + \frac{\mu Q}{2\pi r^2}, \quad \sigma_\psi = -p.$$

Deformation of a granular medium, in contrast to an elastic medium, involves a dilatational increase in volume owing to shears, and the deformation process becomes much more complicated, especially if the change in the cavity volume is rather large. In this case the dilatancy occurs only until the state of ultimate loosening of a material is reached (in this state a material becomes incompressible). Let us study the arising stress-strain state with both small and finite strains of a medium.

Let us assume that the inner surface of a spherical layer $r_0 < r < r_1$ of a densely packed granular medium with rigid particles is expanding, its radial displacement is u_0 , and a compressive stress equal to $-p_0$ is applied to the outer surface. For such a medium, the dilatancy equation has the form

$$\varkappa \gamma(\boldsymbol{\varepsilon}) = \theta(\boldsymbol{\varepsilon}), \quad (9.56)$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor. In a spherically symmetric case, the principal values of $\boldsymbol{\varepsilon}$ are $\varepsilon_r = du/dr$ and $\varepsilon_\varphi = \varepsilon_\psi = u/r$. Taking into account that $\varepsilon_\varphi > \varepsilon_r$ in the case of layer expansion, we obtain

$$\frac{2\varkappa}{\sqrt{3}} \left(\frac{u}{r} - \frac{du}{dr} \right) = \frac{du}{dr} + 2 \frac{u}{r}.$$

The solution of this equation satisfying the boundary condition at $r = r_0$ has the form

$$u = u_0 \left(\frac{r_0}{r} \right)^\vartheta, \quad \vartheta = 2 \frac{\sqrt{3} - \varkappa}{\sqrt{3} + 2\varkappa}.$$

If $\varkappa = 0$, the solution coincides with given above solution for an incompressible medium accurate within the recalculation of u_0 via Q . If $\varkappa < \sqrt{3}$, the displacement decreases with increasing r and tends to zero in an infinite layer. If $\varkappa > \sqrt{3}$, the displacement increases, whereas the strains

$$\varepsilon_r = -\vartheta \varepsilon_\varphi, \quad \varepsilon_\varphi = \varepsilon_\psi = \frac{u_0}{r_0} \left(\frac{r_0}{r} \right)^{\vartheta+1}$$

decrease independent of $\mathfrak{x}$ and remain small at sufficiently small values of u_0 . The condition $\varepsilon_r + 2\varepsilon_\varphi > 0$ is always satisfied; if this condition is violated, Eq. (9.56) makes no sense.

Note that another differential equation follows from (9.56) in the case of compression of a layer ($u_0 < 0$) with $\varepsilon_r > 0$ and $\varepsilon_\varphi < 0$:

$$\frac{2\mathfrak{x}}{\sqrt{3}} \left(\frac{du}{dr} - \frac{u}{r} \right) = \frac{du}{dr} + 2\frac{u}{r}.$$

The solution of this equation

$$u = u_0 \left(\frac{r_0}{r} \right)^\beta, \quad \beta = 2 \frac{\sqrt{3} + \mathfrak{x}}{\sqrt{3} - 2\mathfrak{x}}$$

makes sense only if $\mathfrak{x} < \sqrt{3}/2$, because the volume strain $\theta(\boldsymbol{\varepsilon}) = \varepsilon_r + 2\varepsilon_\varphi$ becomes negative at greater values of $\mathfrak{x}$. In this case no solution exists for (9.56): the effect of particle jamming is observed; a medium is not deformed and remains in a rigid state.

To find the stresses, we write the constitutive relationships for an ideal granular material in the form of a variational inequality

$$\sigma_i (\tilde{\varepsilon}_i - \varepsilon_i) \leq 0,$$

where $\tilde{\varepsilon}_i$ are the components of an arbitrary tensor $\tilde{\boldsymbol{\varepsilon}}$ satisfying the constraint $\mathfrak{x}\gamma(\tilde{\boldsymbol{\varepsilon}}) \leq \theta(\tilde{\boldsymbol{\varepsilon}})$. With the use of Kuhn–Tucker's theorem, this inequality is transformed to the equations

$$\frac{\sigma_i}{p} = \frac{4\mathfrak{x}}{3\gamma(\boldsymbol{\varepsilon})} \left(\varepsilon_i - \frac{\varepsilon_j + \varepsilon_k}{2} \right) - 1, \quad (9.57)$$

where $i \neq j \neq k$, the Lagrangian multiplier $p > 0$ is equal to hydrostatic pressure. In the case of spherical symmetry, Eqs. (9.57) taking into account (9.56) are transformed to

$$\sigma_r = - \left(\frac{2\mathfrak{x}}{\sqrt{3}} + 1 \right) p, \quad \sigma_\varphi = \sigma_\psi = \left(\frac{\mathfrak{x}}{\sqrt{3}} - 1 \right) p.$$

From the equilibrium equation

$$\frac{\sigma_r}{dr} + 2 \frac{\sigma_r - \sigma_\varphi}{r} = 0 \quad (9.58)$$

and the boundary condition $\sigma_r = -p_0$ on the outer surface, we find

$$p = \frac{\sqrt{3} p_0}{\sqrt{3} + 2\mathfrak{x}} \left(\frac{r_1}{r} \right)^{2-\vartheta}.$$

The stress on the inner surface of a layer is determined via the hydrostatic pressure calculated by this formula with $r = r_0$.

As the power index $2 - \vartheta = 6\mathfrak{x}/(\sqrt{3} + 2\mathfrak{x})$ is strictly positive, the received solution makes no sense for an infinite layer ($r_1 \rightarrow \infty$) if the stress p_0 differs from zero. In this case expansion of the spherical cavity is impossible, because it requires an infinite stress to act from the side of a cavity on a medium.

At finite strains, when the displacement u_0 is not small, the strain state of a granular medium is described by the Hencky logarithmic tensor [26] with non-zero components $h_r = \ln(dR/dr)$ and $h_\varphi = h_\psi = \ln(R/r)$, where $R = r + u$ is the Eulerian coordinate of the particle. The dilatancy equation derived from (9.56) by means of replacing the small strain tensor with the logarithmic tensor is transformed to

$$\frac{dR}{dr} = \left(\frac{r}{R}\right)^\vartheta. \quad (9.59)$$

In contrast to the case of small strains, the parameter ϑ calculated by the formula given above depends here on density:

$$\rho = \rho_0 e^{-\theta(\mathbf{h})}, \quad \theta(\mathbf{h}) = h_r + 2h_\varphi = \ln\left(\frac{R^2}{r^2} \frac{dR}{dr}\right).$$

For a medium with moderate dilatational expansion, such a dependence is approximately described by the expression (9.15). Thus, Eq. (9.59) is a differential equation, which is not resolved with respect to the derivative. Direct calculations, however, show that

$$\frac{d\vartheta}{d\mathfrak{x}} < 0, \quad \frac{d\mathfrak{x}}{d\rho} \geq 0, \quad \frac{d\rho}{dR'} < 0 \quad \left(R' = \frac{dR}{dr}\right).$$

For $R \geq r$, therefore, the condition

$$1 - \frac{d}{dR'} \left(\frac{r}{R}\right)^\vartheta = 1 + \left(\frac{r}{R}\right)^\vartheta \ln\left(\frac{r}{R}\right) \frac{d\vartheta}{d\mathfrak{x}} \frac{d\mathfrak{x}}{d\rho} \frac{d\rho}{dR'} \geq 1$$

is satisfied, which ensures the problem solvability on the basis of the theorem on an implicit function.

By virtue of Eq. (9.59), we have

$$\frac{\rho}{\rho_0} = \frac{r^2}{R^2} \frac{dr}{dR} = \left(\frac{r}{R}\right)^{2-\vartheta}.$$

Using this relation and assuming that $r = r_0$, we can express the radial displacement on the inner surface of the layer as a function of the medium density near this surface:

$$\frac{u_0}{r_0} = \left(\frac{\rho_0}{\rho}\right)^{1/(2-\vartheta)} - 1.$$

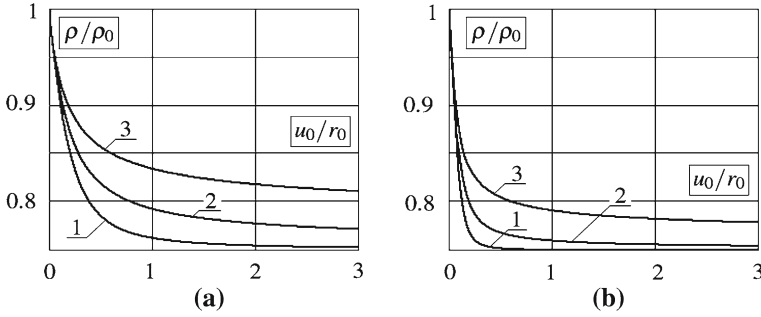


Fig. 9.9 Dependence of density of a medium near the inner surface on radial displacement: **a** $\alpha_0 = 0.5$, **b** $\alpha_0 = 2.5$

The inverse dependence allows us to determine the density as a function of a given displacement. Typical diagrams of the inverse dependence obtained by numerical calculations are shown in Fig. 9.9. The coefficient of internal friction α in the state of dense packing in Fig. 9.9a is 0.5, and its value in Fig. 9.9b is 2.5. In both diagrams the ultimate density ρ_* of dilatancy is equal to $0.75\rho_0$; diagram for $n = 0.5$ in Eq. (9.15) is denoted by number 1, diagrams for $n = 1$ and $n = 2$ by numbers 2 and 3, respectively. At $\alpha_0 = \sqrt{3} \in (0.5, 2.5)$ the parameter $\vartheta(\rho_0)$ changes its sign. As was found for the case of small strains, this leads to a qualitative change in the field of displacements.

Figure 9.10 shows the results of numerical solution of Eq. (9.59) by the Euler method with recalculation of the second order of accuracy after resolving this equation with respect to the derivative by the Newton–Raphson method [27]. The Raphson correction turned out to be necessary in the case of $n \leq 1$, where the usual Newton method does not converge because of the discontinuity of the derivative $d\alpha/d\rho$. In calculations $n = 1$, $\rho_* = 0.75\rho_0$; the relation u_0/r_0 is equal to 0.5, 1, and 2 (curves 1, 2, and 3, respectively); $\alpha_0 = 0.5$ in Fig. 9.10a, and $\alpha_0 = 2.5$ in Fig. 9.10b. An analysis of results shows that the medium dilatancy occurs mainly near the inner surface of the layer, and the limiting density ρ_* of a material is reached only as $u_0 \rightarrow \infty$.

The stresses in a layer can be found via strains by Eq. (9.57) with the tensor ϵ being replaced by the tensor $\mathbf{h}$:

$$\sigma_r = A p, \quad \sigma_\varphi = \sigma_\psi = B p,$$

$$A = \frac{4\alpha^2}{3} f - 1, \quad B = -\frac{2\alpha^2}{3} f - 1, \quad f = \frac{h_r - h_\varphi}{h_r + 2h_\varphi} = \frac{\ln(r R'/R)}{\ln(R^2 R'/r^2)}.$$

In these formulas, the hydrostatic pressure is the solution of the differential equation of equilibrium (9.58) in the Eulerian variables

Fig. 9.10 Distribution of density ρ in a spherical layer:
a $\alpha_0 = 0.5$, **b** $\alpha_0 = 2.5$

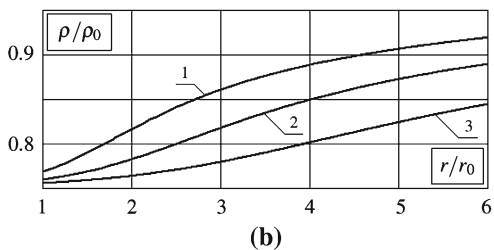
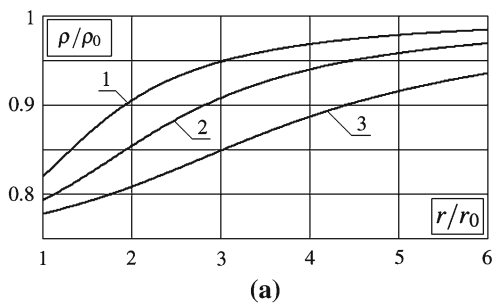
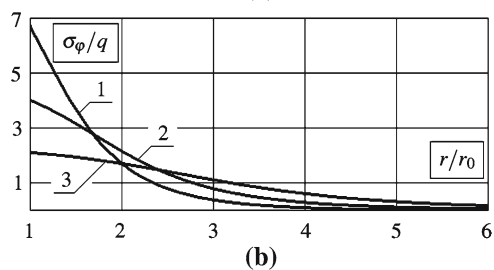
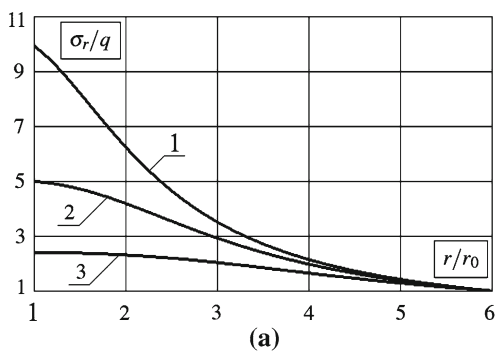


Fig. 9.11 Distribution of stresses σ_r **(a)** and σ_φ **(b)** in a spherical layer



$$\frac{d(Ap)}{dR} + \frac{2(A-B)p}{R} = 0,$$

which was solved numerically taking into account the boundary condition on the outer surface of a layer by the Crank–Nicholson difference scheme [18]:

$$\frac{A_j p_j - A_{j-1} p_{j-1}}{R_j - R_{j-1}} + 2 \frac{(A_j - B_j) p_j + (A_{j-1} - B_{j-1}) p_{j-1}}{R_j + R_{j-1}} = 0.$$

The scheme allows the nodal values of hydrostatic pressure to be calculated by explicit formulas, with subsequent recalculation of stresses. The obtained distributions of σ_r and σ_φ inside the layer for $\alpha_0 = 2.5$ are plotted in Fig. 9.11a, b respectively. The relation u_0/r_0 is equal to 0.5, 1, and 2 (curves 1, 2, and 3). For $\alpha_0 = 0.5$ the corresponding curves are smoother: on curve 1 the absolute values of the stresses σ_r and σ_φ on the inner surface of a layer are approximately three times greater than the corresponding values on the outer surface.

The described computational algorithms were tested through comparisons of the calculated results with exact solutions for small strains of a medium.

In a cylindrical layer, the principal values of the small strain tensor are calculated as $\varepsilon_r = du/dr$, $\varepsilon_\varphi = u/r$, and $\varepsilon_z = 0$. Hence, Eq. (9.56) reduces to the nonlinear differential equation

$$\frac{2\alpha}{\sqrt{3}} \sqrt{\left(\frac{du}{dr}\right)^2 - \frac{u}{r} \frac{du}{dr} + \frac{u^2}{r^2}} = \frac{du}{dr} + \frac{u}{r}, \quad (9.60)$$

whose general solution $u = C_2/r^\vartheta$ with the constant $C_2 > 0$ and

$$\vartheta = \frac{1 + 2\alpha^2/3 - 2\alpha\sqrt{1 - \alpha^2/3}}{1 - 4\alpha^2/3}$$

makes sense only if $\alpha \leq \sqrt{3}$. For $\alpha > \sqrt{3}$ plane deformation of a granular medium is impossible, in contrast to axial expansion, because particle jamming occurs. The value of ϑ satisfies the condition $\vartheta \leq 1$ which ensures a nonnegativeness of right-hand side of (9.60). The constant $C_2 = u_0 r_0^\vartheta$ is determined by virtue of the boundary condition on the inner surface.

In accordance with (9.57), the stresses in a layer are

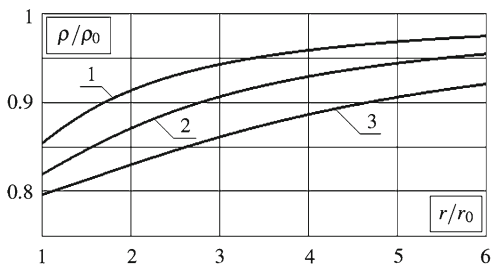
$$\frac{\sigma_r}{p} = -\frac{2\alpha^2}{3} \frac{1 + 2\vartheta}{1 - \vartheta} - 1, \quad \frac{\sigma_\varphi}{p} = \frac{2\alpha^2}{3} \frac{2 + \vartheta}{1 - \vartheta} - 1, \quad \frac{\sigma_z}{p} = -\frac{2\alpha^2}{3} - 1.$$

Integrating the equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0$$

and taking into account the boundary condition at $r = r_1$, we find

Fig. 9.12 Distribution of density ρ in a cylindrical layer



$$p = \frac{\beta p_0}{2 \mathfrak{x} \sqrt{1 - \mathfrak{x}^2/3}} \left(\frac{r_1}{r} \right)^\beta, \quad \beta = 2 \mathfrak{x} \frac{\sqrt{1 - \mathfrak{x}^2/3} - \mathfrak{x}}{1 - 4 \mathfrak{x}^2/3}. \quad (9.61)$$

The power index β monotonically increases from zero to two as the internal friction parameter $\mathfrak{x}$ changes from zero to $\sqrt{3}$. Therefore, the absolute values of stresses in a layer always decrease along the radius.

For finite strains of a medium, it is difficult to obtain the exact solution of the problem in explicit form. In this case the displacements and stresses can be found by numerical integration of the first-order ordinary differential equations. With allowance for cylindrical symmetry, the principal values of the Hencky logarithmic tensor are $h_r = \ln(dR/dr)$, $h_\varphi = \ln(R/r)$, and $h_\xi = 0$. The dilatancy Eq. (9.56) takes the form

$$\mathfrak{x} \sqrt{\left(\ln \frac{r}{R} \frac{dR}{dr} \right)^2 + \left(\ln \frac{dR}{dr} \right)^2 + \left(\ln \frac{R}{r} \right)^2} = \sqrt{\frac{3}{2}} \ln \left(\frac{R}{r} \frac{dR}{dr} \right).$$

Here the parameter $\mathfrak{x}$ depends on the medium density

$$\rho = \rho_0 \frac{r}{R} \frac{dr}{dR}.$$

To resolve this equation with respect to the derivative, one can apply, for instance, the Newton–Raphson method with subsequent integration by the Euler method.

The differential equation of equilibrium with respect to hydrostatic pressure is written in the form

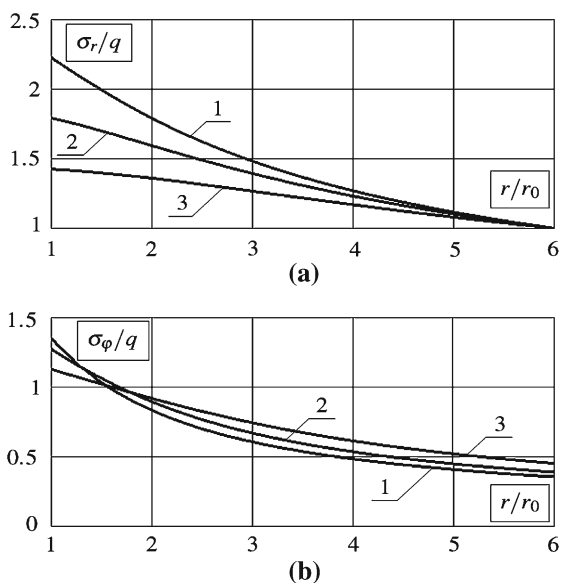
$$\frac{d(Ap)}{dR} + \frac{(A-B)p}{R} = 0,$$

where

$$A = \frac{2 \mathfrak{x}^2}{3} \frac{\ln(r(R')^2/R)}{\ln(RR'/r)} - 1, \quad B = -\frac{2 \mathfrak{x}^2}{3} \frac{\ln(r^2 R'/R^2)}{\ln(RR'/r)} - 1$$

are functions used for determining the stresses: $\sigma_r = Ap$ and $\sigma_\varphi = Bp$. As in the case of a spherical layer, this equation can be solved numerically by using the

Fig. 9.13 Distribution of stresses σ_r (a) and σ_φ (b) in a cylindrical layer



Crank–Nicholson scheme. The computational algorithm was tested on the basis of the exact solution of (9.61) within the framework of the theory of small strains.

A typical distribution of density in a cylindrical layer, which was obtained in calculations with $\alpha_0 = 0.5$, $n = 1$, $u_0/r_0 = 0.5$, 1, and 2 (curves 1, 2, and 3, respectively), is plotted in Fig. 9.12. Figure 9.13 shows the distribution of the stresses σ_r and σ_φ inside the layer for the same values of the problem parameters as in Fig. 9.12. It follows from the analysis of the results obtained that the process of medium dilatancy accompanied by relaxation of tangential stresses (transition of the stress state of a medium to the hydrostatic state) in a cylindrical layer occurs slower than in a spherical layer, as the cavity radius is changed by the same value. This is caused by the constraint of the degree of freedom of the particles motion in the ζ direction; for this reason, the shear intensity in the case of cylindrical symmetry is always lower than in the case of spherically symmetric motion.

To conclude, we should note that the exact and approximate solutions obtained can be used for testing algorithms of numerical implementation of mathematical models of mechanics of a granular medium.

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Chapter 10

Rotational Degrees of Freedom of Particles

Abstract On the basis of a mathematical model of the Cosserat continuum and a generalized model, that describes the different resistance of a material with respect to tension and compression, the influence of rotational motion of particles onto the stress-strain state of a granular material is studied. It is shown that a couple-stress elastic medium has the resonance frequency, coinciding with the frequency of natural oscillations of rotational motion of the particles. The solution of the problem of uniform shear of a granular material, having rotational degrees of freedom, is analyzed in the framework of linear and nonlinear models.

10.1 A Model of the Cosserat Continuum

One hundred years celebrated in 2009 from the date of appearance of the first monograph by brothers Eugene and François Cosserat [11] on a continuum with internal rotations (the Cosserat continuum), which initiated the development of mechanics of continua with microstructure. Further development of this theory refers to the sixties of twentieth century [3, 24, 31, 33] among others. On the state of the art in this research area is reported, for example, in [12].

A mathematical model of the Cosserat continuum, taking into account the microstructure of a material, is used to describe the stress-strain state of composites, powder, granular, micropolar and liquid-crystal materials, and also to construct nonclassical models for thin-walled structures such as rods, plates, and shells, see [5–8, 22, 48] and the references within. A simplification of this model is the reduced Cosserat model where rotational degrees of freedom are taken into account, but the couple stresses are not considered, at first it was suggested in [46]. Current researches on modeling of nanoscale structures show that the Cosserat model results from a limit transition in discrete molecular-dynamic models with an unbounded increase in a number of particles [10, 13, 36]. Therefore, in the near future this model will find wide application.

One of the main factors constraining the analysis of models of non-classical media is a lack of information about the material constants. This, in turn, opposes the adaptation of such models into practice of calculating dynamic characteristics of granular materials. First publications about experimental determination of the elasticity parameters in media with microstructure appeared in 1970–1980 years. Parameters for some materials such as bone tissue, highly porous polymer materials and composites were obtained in [15, 16, 29].

A fundamental difference between the model of the couple-stress elasticity theory and the classical one is that the former implicitly includes the small parameter that characterizes the size of particles in the microstructure. As a result, in order to obtain correct numerical solutions, computations must be performed on a grid whose mesh is smaller than the characteristic size of particles. To solve 2D and 3D dynamic problems, parallel algorithms can be efficiently used because they make it possible to distribute the computational load between multiple nodes of a cluster. The use of distributed computing allows to do much more fine grids, thereby increasing the accuracy of numerical solutions.

In the model of the Cosserat continuum, along with translational motion which is defined by the velocity vector $\mathbf{v}$, independent rotations of particles with the angular velocity vector $\boldsymbol{\omega}$ are considered and, along with the stress tensor $\boldsymbol{\sigma}$ with nonsymmetric components, the nonsymmetric tensor $\mathbf{m}$ of couple stresses is introduced.

Assume that vectors $\mathbf{x}(t)$ and $\mathbf{x}(t + \Delta t)$ correspond to positions of a point of a deformable material at close instants of time. Since in the time interval Δt the displacement of a point is approximately equal to $\Delta t \mathbf{v}$, for increments we have the approximate equality

$$\begin{aligned} d\mathbf{x}(t + \Delta t) &= (\boldsymbol{\delta} + \Delta t \nabla \mathbf{v})^* d\mathbf{x}(t) \\ &\approx \left(\boldsymbol{\delta} + \Delta t \frac{(\nabla \mathbf{v})^* - \nabla \mathbf{v}}{2} \right) \left(\boldsymbol{\delta} + \Delta t \frac{(\nabla \mathbf{v})^* + \nabla \mathbf{v}}{2} \right) d\mathbf{x}(t) \\ &\approx \left(\boldsymbol{\delta} + \Delta t \frac{(\nabla \mathbf{v})^* + \nabla \mathbf{v}}{2} \right) \left(\boldsymbol{\delta} + \Delta t \frac{(\nabla \mathbf{v})^* - \nabla \mathbf{v}}{2} \right) d\mathbf{x}(t). \end{aligned}$$

The expression in the first parentheses in the right-hand side of this equality defines the symmetric strain rate tensor $2\mathbf{e} = (\nabla \mathbf{v})^* + \nabla \mathbf{v}$ and the expression in the second parentheses defines the antisymmetric tensor $2\boldsymbol{\omega} = (\nabla \mathbf{v})^* - \nabla \mathbf{v}$ of velocity of rotational motion of a particle. The motion of a material in the time interval Δt turns out to be the superposition of small strain and small rotation. In an arbitrary Cartesian coordinate system the antisymmetric tensor $\boldsymbol{\omega}$ is defined by the matrix

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & v_{1,2} - v_{2,1} & v_{1,3} - v_{3,1} \\ v_{2,1} - v_{1,2} & 0 & v_{2,3} - v_{3,2} \\ v_{3,1} - v_{1,3} & v_{3,2} - v_{2,3} & 0 \end{pmatrix},$$

which is related to the angular velocity vector $\boldsymbol{\omega} = (1/2)\nabla \times \mathbf{v}$ with coordinates $\omega_1, \omega_2, \omega_3$. This relationship takes place in a usual (momentless) continuum.

This formula is not valid when taking into account independent rotations of particles which rotate relative to a binding material. In this case, by the theorem of velocity summation in a complex motion, the vector of angular velocity of a particle is equal to the sum of angular velocities of translational and relative motion.

The complete system of equations of the model of the Cosserat medium consists of the motion equations, the kinematic relationships, and the generalized law of the linear elasticity theory of the following form [24, 33]:

$$\begin{aligned} \rho \dot{\mathbf{v}} &= \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g}, \quad j \dot{\boldsymbol{\omega}} = \nabla \cdot \mathbf{m} - 2 \boldsymbol{\sigma}^a + j \mathbf{q}, \\ \dot{\mathbf{A}} &= \nabla \mathbf{v} + \boldsymbol{\omega}, \quad \dot{\mathbf{M}} = \nabla \boldsymbol{\omega}, \\ \boldsymbol{\sigma} &= \left(k - \frac{2\mu}{3} \right) (\delta : \mathbf{A}^s) \delta + 2\mu \mathbf{A}^s + 2\alpha \mathbf{A}^a, \\ \mathbf{m} &= \left(\kappa - \frac{2\eta}{3} \right) (\delta : \mathbf{M}^s) \delta + 2\eta \mathbf{M}^s + 2\beta \mathbf{M}^a. \end{aligned} \quad (10.1)$$

Here $\mathbf{A}$ and $\mathbf{M}$ are the strain and curvature tensors, respectively, which vanish in the natural (unstressed) state; $\mathbf{g}$ and $\mathbf{q}$ are the vectors of body forces and moments; j is a special dynamic characteristic of a material equal to the product of the moment of inertia of a particle about the axis passing through its center of gravity and the number of particles in a unit volume (inertia of rotation); $k, \mu, \alpha, \kappa, \eta, \beta$ are the phenomenological elasticity coefficients for an isotropic medium. The superscripts s and a denote the symmetric and antisymmetric parts of tensor. Where needed, the corresponding vector is identified with an antisymmetric part. In particular, the motion equations involve the vector $\boldsymbol{\sigma}^a$ with components σ_i^a in the Cartesian system, which corresponds to the tensor

$$\boldsymbol{\sigma}^a = \frac{\boldsymbol{\sigma} - \boldsymbol{\sigma}^*}{2} = \frac{1}{2} \begin{pmatrix} 0 & \sigma_{12} - \sigma_{21} & \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} & 0 & \sigma_{23} - \sigma_{32} \\ \sigma_{31} - \sigma_{13} & \sigma_{32} - \sigma_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_3^a & \sigma_2^a \\ \sigma_3^a & 0 & -\sigma_1^a \\ -\sigma_2^a & \sigma_1^a & 0 \end{pmatrix}.$$

To calculate the antisymmetric tensor of rotational velocity $\boldsymbol{\omega}$ in terms of the angular velocity vector in the Cartesian system with the orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ being its basis, we can use the formula of vector product of a tensor and a vector:

$$\boldsymbol{\omega} = \delta_{il} \mathbf{e}_i \mathbf{e}_l \times (\omega_h \mathbf{e}_h) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Using the third equation of the system (10.1), we can obtain the following equalities for the symmetric and antisymmetric parts of the strain tensor:

$$\dot{\mathbf{A}}^s = \frac{(\nabla \mathbf{v})^* + \nabla \mathbf{v}}{2} = \mathbf{e}, \quad \dot{\mathbf{A}}^a = \boldsymbol{\omega} - \frac{(\nabla \mathbf{v})^* - \nabla \mathbf{v}}{2}.$$

Thus, the power of internal stresses is given by

$$\sigma^* : \dot{\mathbf{A}} = \sigma^* : \dot{\mathbf{A}}^s + \sigma^* : \dot{\mathbf{A}}^a = \sigma^s : \mathbf{e} + \sigma^a \cdot (2\boldsymbol{\omega} - \nabla \times \mathbf{v}).$$

The last term in this formula is the doubled scalar product of the vector σ^a and the vector of relative rotation. The convolution $\mathbf{m} : \dot{\mathbf{M}}^*$ makes an additional contribution to the power due to internal moments.

If particles of a material are the regular spheres of mass m_0 and radius r_0 , then the parameter j is calculated by the formula $j = 2m_0 r_0^2 N/5$ (N is the number of particles per unit volume). The density of a material with the porous space taken into account is $\rho = m_0 N$. Hence,

$$j = \frac{2\rho r_0^2}{5}, \quad r_0 = \sqrt{\frac{5j}{2\rho}}.$$

The fundamental difference between the Cosserat model and the model of classical elasticity theory is that the first one implicitly involves a small parameter having the dimension of length. Hence, it is natural to expect that in the framework of this model stress fields vary considerably at a distance of order of r_0 . We can suppose that for the numerical solution of dynamic problems on the basis of the model of the Cosserat medium one must use sufficiently fine grids such that the mesh size is much less than r_0 . Otherwise the accuracy of a solution is unsatisfactory to analyze thin small-scale effects.

Let us consider the case of a plane strain state in more detail. In this case the model describes the behavior of a material, consisting of extended particles of cylindrical form with the axis directed along the x_3 axis. The angular velocity vector has the single nonzero projection ω_3 . The stress-strain state of a material is defined by the tensors

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 0 & 0 & m_{13} \\ 0 & 0 & m_{23} \\ m_{31} & m_{32} & 0 \end{pmatrix},$$

$$\dot{\mathbf{A}} = \begin{pmatrix} v_{1,1} & v_{2,1} - \omega_3 & 0 \\ v_{1,2} + \omega_3 & v_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \dot{\mathbf{M}} = \begin{pmatrix} 0 & 0 & \omega_{3,1} \\ 0 & 0 & \omega_{3,2} \\ 0 & 0 & 0 \end{pmatrix}.$$

The complete system of equations (10.1) with volume forces and moments taken into account in the plane case holds

$$\begin{aligned} \rho \dot{v}_1 &= \sigma_{11,1} + \sigma_{21,2} + \rho g_1, \\ \rho \dot{v}_2 &= \sigma_{12,1} + \sigma_{22,2} + \rho g_2, \\ \dot{\sigma}_{11} &= \left(k + \frac{4\mu}{3}\right)v_{1,1} + \left(k - \frac{2\mu}{3}\right)v_{2,2}, \\ \dot{\sigma}_{22} &= \left(k - \frac{2\mu}{3}\right)v_{1,1} + \left(k + \frac{4\mu}{3}\right)v_{2,2}, \\ \dot{\sigma}_{33} &= \left(k - \frac{2\mu}{3}\right)(v_{1,1} + v_{2,2}), \end{aligned} \tag{10.2}$$

$$\begin{aligned}
\dot{\sigma}_{12} &= (\mu - \alpha) v_{1,2} + (\mu + \alpha) v_{2,1} - 2\alpha \omega_3, \\
\dot{\sigma}_{21} &= (\mu + \alpha) v_{1,2} + (\mu - \alpha) v_{2,1} + 2\alpha \omega_3, \\
j\dot{\omega}_3 &= m_{13,1} + m_{23,2} + \sigma_{12} - \sigma_{21} + jq_3, \\
\dot{m}_{23} &= (\eta + \beta) \omega_{3,2}, \quad \dot{m}_{32} = (\eta - \beta) \omega_{3,2}, \\
\dot{m}_{31} &= (\eta - \beta) \omega_{3,1}, \quad \dot{m}_{13} = (\eta + \beta) \omega_{3,1}.
\end{aligned}$$

This system can be represented in the symmetric form [50]

$$\mathbf{A} \mathbf{U}_t = \mathbf{B}^1 \mathbf{U}_{x_1} + \mathbf{B}^2 \mathbf{U}_{x_2} + \mathbf{Q} \mathbf{U} + \mathbf{G} \quad (10.3)$$

in terms of the vector-function

$$\mathbf{U} = (v_1, v_2, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{21}, \omega_3, m_{23}, m_{32}, m_{31}, m_{13}).$$

The matrix-coefficients $\mathbf{A}$, $\mathbf{B}^1$, and $\mathbf{B}^2$ of the system (10.3) are symmetric and the matrix $\mathbf{Q}$ is antisymmetric:

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_2 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & b_1 & 0 \end{pmatrix}, \\
\mathbf{B}^i &= \begin{pmatrix} 0 & 0 & i_1 & 0 & 0 & 0 & i_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i_2 & 0 & i_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i_2 & 0 & 0 & i_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here for brevity we introduce the notations

$$\begin{aligned} i_1 &= 2 - i, \quad i_2 = i - 1 \quad i = 1, 2, \\ a_1 &= \frac{k + \mu/3}{3k\mu}, \quad a_2 = -\frac{k - 2\mu/3}{6k\mu}, \quad a_3 = \frac{\mu + \alpha}{4\mu\alpha}, \quad a_4 = -\frac{\mu - \alpha}{4\mu\alpha}, \\ b_1 &= \frac{\eta + \beta}{4\eta\beta}, \quad b_2 = -\frac{\eta - \beta}{4\eta\beta}. \end{aligned}$$

One can show that if the inequalities

$$k > 0, \quad \mu > 0, \quad \alpha > 0, \quad \eta + \beta > 0,$$

providing non-negativity of the elastic energy, are fulfilled then the matrix $\mathbf{A}$ is positive definite, i.e. the system (10.3) is hyperbolic in the sense of Friedrichs.

For this system the energy conservation law

$$(\mathbf{U}\mathbf{A}\mathbf{U})_t = (\mathbf{U}\mathbf{B}^1\mathbf{U})_{x_1} + (\mathbf{U}\mathbf{B}^2\mathbf{U})_{x_2} + 2\mathbf{U}\mathbf{G}$$

holds, where the doubled total energy in terms of $\mathbf{A}$ and $\mathbf{M}$ is

$$\begin{aligned} \mathbf{U}\mathbf{A}\mathbf{U} &= \rho(v_1^2 + v_2^2) + j\omega_3^2 + \left(k + \frac{4\mu}{3}\right)(\Lambda_{11}^2 + \Lambda_{22}^2) + 2\left(k - \frac{2\mu}{3}\right)\Lambda_{11}\Lambda_{22} \\ &\quad + (\mu + \alpha)(\Lambda_{12}^2 + \Lambda_{21}^2) + 2(\mu - \alpha)\Lambda_{12}\Lambda_{21} + (\eta + \beta)(M_{13}^2 + M_{23}^2). \end{aligned}$$

Flows of energy have the form

$$\mathbf{U}\mathbf{B}^1\mathbf{U} = \sigma_{11}v_1 + \sigma_{12}v_2 + m_{13}\omega_3, \quad \mathbf{U}\mathbf{B}^2\mathbf{U} = \sigma_{21}v_1 + \sigma_{22}v_2 + m_{23}\omega_3.$$

Solutions of the system satisfy the integral estimates (6.18) and (6.20) which provide that the Cauchy problem and the boundary-value problems with dissipative boundary

conditions are well-posed and the domain of dependence of a solution is bounded (velocity of disturbance propagation is finite).

Among dissipative boundary conditions are, in particular, boundary conditions in terms of velocities

$$v_1 = \bar{v}_1, \quad v_2 = \bar{v}_2, \quad \omega_3 = \bar{\omega}_3$$

and conditions in terms of stresses

$$\sigma_{11} v_1 + \sigma_{21} v_2 = \bar{p}_1, \quad \sigma_{12} v_1 + \sigma_{22} v_2 = \bar{p}_2, \quad m_{13} v_1 + m_{23} v_2 = \bar{q}_3,$$

where $\mathbf{v}$ is the outward normal vector.

The characteristic properties of the system (10.3) are described by the equation

$$\det(c \mathbf{A} - v_1 \mathbf{B}^1 - v_2 \mathbf{B}^2) = 0, \quad v_1^2 + v_2^2 = 1.$$

Its positive roots (velocities of longitudinal, transverse, and rotational waves) are

$$c_p = \sqrt{\frac{k + 4\mu/3}{\rho}}, \quad c_s = \sqrt{\frac{\mu + \alpha}{\rho}}, \quad c_\omega = \sqrt{\frac{\eta + \beta}{j}}.$$

In addition, there are the negative roots $-c_p$, $-c_s$, $-c_\omega$ and the zero root of multiplicity six.

The complete system of left eigenvectors of the matrix $\mathbf{A}^{-1}(v_1 \mathbf{B}^1 + v_2 \mathbf{B}^2)$, which is necessary for numerical solution of the boundary-value problems with the help of the shock-capturing method presented in Sect. 6.3, can be taken, for example, in the following form:

$$\begin{pmatrix} \pm \rho v_1 c_p & \pm \rho v_2 c_p & v_1^2 & v_2^2 & 0 & v_1 v_2 & v_1 v_2 & 0 & 0 & 0 & 0 & 0 \\ \mp \rho v_2 c_s & \pm \rho v_1 c_s & -v_1 v_2 & v_1 v_2 & 0 & v_1^2 & -v_2^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pm j c_\omega & v_2 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_1 & 0 & 0 & v_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & a_1 v_2 & a_2 v_2 & a_2 v_2 & -a_4 v_1 & -a_3 v_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 v_1 & -a_1 v_1 & -a_2 v_1 & a_3 v_2 & a_4 v_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_2 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first six vectors of this system correspond to the eigenvalues $\pm c_p$, $\pm c_s$, and $\pm c_\omega$ and the remaining six ones correspond to the eigenvalue $c = 0$.

To demonstrate the influence of the scale parameter in the Cosserat continuum model, we present a particular solution of the problem on simple shear of a body of a material in the $x_1 x_2$ plane with constant shear velocity $\dot{\chi} > 0$. In this problem

the projection $v_2 = \dot{\chi} x_1$ of the velocity vector is nonzero and the angular velocity of particles depends on time only. Taking into account that normal stresses σ_{ii} and moments are zero, the system (10.2) yields

$$j\dot{\omega}_3 = \sigma_{12} - \sigma_{21}, \quad \dot{\sigma}_{12} = (\mu + \alpha)\dot{\chi} - 2\alpha\omega_3, \quad \dot{\sigma}_{21} = (\mu - \alpha)\dot{\chi} + 2\alpha\omega_3.$$

Its remaining equations turn into identities. Hence, by differentiation with respect to time we obtain the equation

$$j\ddot{\omega}_3 = 2\alpha\dot{\chi} - 4\alpha\omega_3.$$

A solution of this equation

$$\omega_3(t) = \dot{\chi} \sin^2 \sqrt{\frac{\alpha}{j}} t,$$

satisfying the initial condition $\omega_3(0) = 0$, has an oscillatory nature. It shows that in the domain of shear the characteristic oscillations of rotational motion of particles arise. The oscillation period $T = \pi\sqrt{j/\alpha}$ essentially depends on the parameter α and tends to infinity as $\alpha \rightarrow 0$. With decreasing j , as the scale parameter r_0 of a structure decreases, frequency of characteristic oscillations increases.

In the spatial case the model is reduced to the hyperbolic system

$$\mathbf{A} \mathbf{U}_t = \sum_{i=1}^3 \mathbf{B}^i \mathbf{U}_{x_i} + \mathbf{Q} \mathbf{U} + \mathbf{G} \quad (10.4)$$

with symmetric matrix-coefficients $\mathbf{A}$ and $\mathbf{B}^i$, containing the elasticity parameters of a material, antisymmetric matrix $\mathbf{Q}$ and given vector $\mathbf{G}$ of body forces and moments. Here the vector-function $\mathbf{U}$ involves 24 unknown functions (components of the velocity vector and the angular velocity vector as well as components of the non-symmetric stress tensor and the couple-stress tensor):

$$\mathbf{U} = (v_1, v_2, v_3, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{32}, \sigma_{31}, \sigma_{13}, \sigma_{12}, \sigma_{21}, \\ \omega_1, \omega_2, \omega_3, m_{11}, m_{22}, m_{33}, m_{23}, m_{32}, m_{31}, m_{13}, m_{12}, m_{21}).$$

The system of equations (10.4) has the expanded form:

$$\begin{aligned} \rho \dot{v}_i &= \sigma_{1i,1} + \sigma_{2i,2} + \sigma_{3i,3} + \rho g_i, \\ a_1 \dot{\sigma}_{ii} + a_2 (\dot{\sigma}_{ll} + \dot{\sigma}_{hh}) &= v_{i,i}, \\ a_3 \dot{\sigma}_{il} + a_4 \dot{\sigma}_{li} &= v_{l,i} - \omega_h, \\ a_4 \dot{\sigma}_{il} + a_3 \dot{\sigma}_{li} &= v_{i,l} + \omega_h, \\ j\dot{\omega}_i &= m_{1i,1} + m_{2i,2} + m_{3i,3} + \sigma_{lh} - \sigma_{hl} + jq_i, \\ b_1 \dot{m}_{ii} + b_2 (\dot{m}_{ll} + \dot{m}_{hh}) &= \omega_{i,i}, \\ b_3 \dot{m}_{il} + b_4 \dot{m}_{li} &= \omega_{l,i}, \\ b_4 \dot{m}_{il} + b_3 \dot{m}_{li} &= \omega_{i,l}. \end{aligned} \quad (10.5)$$

This system includes 24 equations. For brevity, we use the notations:

$$i, l, h = 1, 2, 3, \quad i \neq l \neq h, \quad l = i + 1 \bmod 3, \quad h = l + 1 \bmod 3,$$

$$a_1 = \frac{k + \mu/3}{3k\mu}, \quad a_2 = -\frac{k - 2\mu/3}{6k\mu}, \quad a_3 = \frac{\mu + \alpha}{4\mu\alpha}, \quad a_4 = -\frac{\mu - \alpha}{4\mu\alpha},$$

$$b_1 = \frac{\kappa + \eta/3}{3\kappa\eta}, \quad b_2 = -\frac{\kappa - 2\eta/3}{6\kappa\eta}, \quad b_3 = \frac{\eta + \beta}{4\eta\beta}, \quad b_4 = -\frac{\eta - \beta}{4\eta\beta}.$$

The matrix $\mathbf{A}$ is positive definite provided that its diagonal blocks

$$\begin{pmatrix} a_1 & a_2 & a_2 \\ a_2 & a_1 & a_2 \\ a_2 & a_2 & a_1 \end{pmatrix}, \quad \begin{pmatrix} a_3 & a_4 \\ a_4 & a_3 \end{pmatrix}, \quad \begin{pmatrix} b_1 & b_2 & b_2 \\ b_2 & b_1 & b_2 \\ b_2 & b_2 & b_1 \end{pmatrix}, \quad \begin{pmatrix} b_3 & b_4 \\ b_4 & b_3 \end{pmatrix}$$

are positive definite. According to the Sylvester criterion, this condition restricts the admissible values of parameters of the material:

$$k, \mu, \alpha > 0, \quad \kappa, \eta, \beta > 0. \quad (10.6)$$

If the inequalities (10.6) are fulfilled, the potential energy of elastic deformation is a positive definite quadratic form and the system (10.4) is hyperbolic in the sense of Friedrichs. The characteristic properties of this system are described by the equation

$$\det \left(c \mathbf{A} - \sum_{i=1}^3 v_i \mathbf{B}^i \right) = 0, \quad \sum_{i=1}^3 v_i^2 = 1.$$

Analyzing the characteristic properties, we can show that in the spatial case there exist four types of elastic waves propagating in an infinite medium, namely, longitudinal, transverse and rotational waves with velocities c_p, c_s, c_ω , respectively, as in the plane problem, and also torsional waves with velocity c_m . These velocities are

$$c_p = \sqrt{\frac{k + 4\mu/3}{\rho}}, \quad c_s = \sqrt{\frac{\mu + \alpha}{\rho}}, \quad c_m = \sqrt{\frac{\kappa + 4\eta/3}{j}}, \quad c_\omega = \sqrt{\frac{\eta + \beta}{j}}. \quad (10.7)$$

In this case twelve vectors associated with nonzero eigenvalues $\pm c_p, \pm c_s, \pm c_m$, and $\pm c_\omega$ (among them, $\pm c_s$ and $\pm c_\omega$ are of multiplicity two) and twelve vectors for $c = 0$ form the complete system of eigenvectors. Obtained in explicit form the eigenvectors are used in the numerical implementation of one-dimensional schemes at steps of the splitting method, however, they are not presented here because of cumbersome expressions.

In the reduced Cosserat model additional rotational degrees of freedom are taken into account but couple stresses are absent. Three out of eight material constants of the linear Cosserat model are zero here. The system of constitutive equations can

also be written in form (10.4) with the vector-function

$$\mathbf{U} = (v_1, v_2, v_3, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{32}, \sigma_{31}, \sigma_{13}, \sigma_{12}, \sigma_{21}, \omega_1, \omega_2, \omega_3).$$

In the general case, the boundary-value problem for the hyperbolic system (10.4) subject to the initial conditions $\mathbf{U}(0, \mathbf{x}) = \mathbf{U}^0(\mathbf{x})$ and the dissipative boundary conditions is well-posed (see [17, 37]). The fulfillment of dissipative conditions for any two vector-functions $\mathbf{U}$ and $\tilde{\mathbf{U}}$ ensures that the inequality

$$(\tilde{\mathbf{U}} - \mathbf{U}) \sum_{i=1}^n v_i \mathbf{B}^i (\tilde{\mathbf{U}} - \mathbf{U}) \leq 0$$

holds at the points of boundary. In expanded form this inequality looks as follows:

$$(\tilde{v}_h - v_h)(\tilde{\sigma}_{lh} - \sigma_{lh}) v_l + (\tilde{\omega}_h - \omega_h)(\tilde{m}_{lh} - m_{lh}) v_l \leq 0.$$

In particular, among the dissipative conditions are the conditions in terms of velocities and in terms of stresses that are formulated within the framework of the model of a couple-stress medium:

$$\mathbf{v} = \bar{\mathbf{v}}, \quad \boldsymbol{\omega} = \bar{\boldsymbol{\omega}}; \quad \sigma_{lh} v_l = \bar{p}_h, \quad m_{lh} v_l = \bar{q}_h.$$

Here $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ are the given vectors of the external forces and moments specified on the boundary. If the matching conditions for the boundary and initial values of the given functions are not fulfilled on the boundary of the domain or if these functions are discontinuous, then discontinuous solutions with shock waves appear. Such solutions can be found from the strong discontinuity equations (see [43])

$$\left(c \mathbf{A} + \sum_{i=1}^n v_i \mathbf{B}^i \right) [[\mathbf{U}]] = 0, \quad (10.8)$$

where $[[\mathbf{U}]]$ is a jump of the solution on the discontinuity front, c is a velocity of front in the direction of normal vector. It follows from (10.8) that the shock waves of small amplitude in a couple-stress medium can propagate only with the velocities (10.7), and scalar products of the left eigenvectors and the vector $\mathbf{U}$ remain continuous on the wave fronts.

Under numerical analysis of problems an important role plays by artificial boundary conditions that appear due to the symmetry of the stress-strain state. They enable one to reduce considerably the computational domain. These conditions follow from the invariance of the system of equations relative to simple transformations that take the artificial boundaries (for example, symmetry planes) to themselves. Note that the angular velocity vector $\boldsymbol{\omega}$ is actually a second-rank tensor, while the moment tensor $\mathbf{m}$ is a third-rank tensor. The formulas for reducing the rank of these tensors have the form

$$\begin{aligned}
\omega_1 &= \omega_{32}, & \omega_2 &= \omega_{13}, & \omega_3 &= \omega_{21}, \\
m_{11} &= m_{132}, & m_{22} &= m_{213}, & m_{33} &= m_{321}, \\
m_{23} &= m_{221}, & m_{32} &= m_{313}, & m_{31} &= m_{332}, \\
m_{13} &= m_{121}, & m_{12} &= m_{113}, & m_{21} &= m_{232}.
\end{aligned}$$

Therefore, for example, under transformation of the mirror reflection $x_2 \mapsto -x_2$ the sign of ω_1 , ω_3 , m_{11} , m_{22} , m_{33} , m_{31} , and m_{13} changes because the index 2 appears only ones in their tensor components. These values are equal to zero on the plane x_2 . In order to solve 3D problems concerning elastic waves originated as a result of the action of a concentrated impulsive source, the boundary conditions of symmetry are presented in [38] for various types of loading.

If on the surface of the half-space $x_1 > 0$ the normal force $\sigma_{11} = -\bar{p}_1 \delta(\mathbf{x}) \delta(t)$, concentrated at the origin, acts, then the planes $x_2 = 0$ and $x_3 = 0$ are the symmetry planes (here δ is the Dirac delta function). The system of equations (10.5) is invariant under the mirror reflection $x_2 \mapsto -x_2$, therefore, the components of the tensors that change their sign under this transformation vanish on the plane $x_2 = 0$:

$$v_2 = 0, \quad \sigma_{21} = \sigma_{23} = 0, \quad \omega_1 = \omega_3 = 0, \quad m_{22} = 0. \quad (10.9)$$

The plane $x_3 = 0$ goes to itself under the mirror reflection $x_3 \mapsto -x_3$, therefore, on this plane

$$v_3 = 0, \quad \sigma_{31} = \sigma_{32} = 0, \quad \omega_1 = \omega_2 = 0, \quad m_{33} = 0. \quad (10.10)$$

If the concentrated tangential force $\sigma_{12} = -\bar{p}_2 \delta(\mathbf{x}) \delta(t)$ is applied at the point $\mathbf{x} = \mathbf{0}$, then $x_3 = 0$ is a symmetry plane and $x_2 = 0$ is an antisymmetry plane. The transformation $x_2 \mapsto -x_2$, under which the plane $x_2 = 0$ goes to itself, changes the sign of the solution: $\mathbf{U} \mapsto -\mathbf{U}$. Hence, on this plane

$$v_1 = v_3 = 0, \quad \sigma_{22} = 0, \quad \omega_2 = 0, \quad m_{21} = m_{23} = 0. \quad (10.11)$$

On the plane $x_3 = 0$ the conditions (10.10) are fulfilled.

Under the action of the concentrated torsional moment $m_{11} = -\bar{q}_1 \delta(\mathbf{x}) \delta(t)$, which torques the particle about the axis x_1 , both mirror reflections $x_2 \mapsto -x_2$ and $x_3 \mapsto -x_3$ change the sign of the solution: $\mathbf{U} \mapsto -\mathbf{U}$. Therefore, the conditions (10.11) are fulfilled on the antisymmetry plane $x_2 = 0$, and the similar conditions

$$v_1 = v_2 = 0, \quad \sigma_{33} = 0, \quad \omega_3 = 0, \quad m_{31} = m_{32} = 0 \quad (10.12)$$

are fulfilled on the plane $x_3 = 0$.

If the concentrated rotational moment $m_{12} = -\bar{q}_2 \delta(\mathbf{x}) \delta(t)$ acts on the boundary, the mirror reflection $x_3 \mapsto -x_3$ changes the sign of the solution and the transformation $x_2 \mapsto -x_2$ does not change this sign. Hence, the conditions (10.9) are fulfilled on the symmetry plane $x_2 = 0$, and the conditions (10.12) are fulfilled on the antisymmetry plane $x_3 = 0$. It can be shown that boundary conditions (10.9)–(10.12) are dissipative; so, their statement provides the correctness of a problem.

Such conditions are used in the next chapter for numerical solution of the problems with concentrated forces.

In the one-dimensional case, when the unknown functions depend only on time and one of the spatial variables (for example, on x_1), the system of equations (10.5) divides into four independent subsystems which describe:

- the plane longitudinal waves

$$\rho \dot{v}_1 = \sigma_{11,1}, \quad \dot{\sigma}_{11} = \left(k + \frac{4\mu}{3}\right) v_{1,1}, \quad \dot{\sigma}_{22} = \dot{\sigma}_{33} = \left(k - \frac{2\mu}{3}\right) v_{1,1}; \quad (10.13)$$

- the torsional waves

$$\begin{aligned} j \dot{\omega}_1 &= m_{11,1} + \sigma_{23} - \sigma_{32}, \quad \dot{\sigma}_{32} = -\dot{\sigma}_{23} = 2\alpha \omega_1, \\ \dot{m}_{11} &= \left(\kappa + \frac{4\eta}{3}\right) \omega_{1,1}, \quad \dot{m}_{22} = \dot{m}_{33} = \left(\kappa - \frac{2\eta}{3}\right) \omega_{1,1}; \end{aligned} \quad (10.14)$$

- the transverse waves (shear waves) with rotation of particles

$$\begin{aligned} \rho \dot{v}_2 &= \sigma_{12,1}, \quad j \dot{\omega}_3 = m_{13,1} + \sigma_{12} - \sigma_{21}, \\ \dot{\sigma}_{12} &= (\mu + \alpha) v_{2,1} - 2\alpha \omega_3, \quad \dot{\sigma}_{21} = (\mu - \alpha) v_{2,1} + 2\alpha \omega_3, \\ \dot{m}_{13} &= (\eta + \beta) \omega_{3,1}, \quad \dot{m}_{31} = (\eta - \beta) \omega_{3,1}. \end{aligned} \quad (10.15)$$

One more subsystem describing the transverse waves is obtained from (10.15) by changing the indices.

The subsystem (10.13) is reduced to the equation: $\ddot{v}_1 = c_p^2 v_{1,11}$. General solution of this subsystem is expressed by the d'Alembert formula, according to which the longitudinal waves propagate with the velocities $\pm c_p$, as in the classical elasticity theory, and have no dispersion.

The subsystem for torsional waves (10.14) is reduced to the telegraph equation relative to the angular velocity:

$$\ddot{\omega}_1 = c_m^2 \omega_{1,11} - \frac{4\alpha}{j} \omega_1.$$

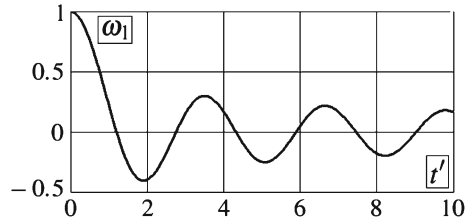
The corresponding dispersion equation

$$c = \frac{\pi v c_m}{\sqrt{\pi^2 v^2 - \alpha/j}}$$

(c is the group velocity and v is the cyclic frequency) determines a particular solution in the form of a monochromatic wave:

$$\omega_1 = C_1 \exp\left(2\pi i v \left(t - \frac{x_1}{c}\right)\right),$$

Fig. 10.1 Dependence of ω_1 on dimensionless time at the point $x_1 = 0$



where C_1 is a complex constant. The solution of the telegraph equation obtained as $c \rightarrow \infty$, which is independent of x_1 , describes the uniform oscillatory rotation of the medium particles in the process of the homogeneous shear with the oscillation period $T = \pi\sqrt{j/\alpha}$. Changing the scale of time and of spatial variable, we can write the telegraph equation in the dimensionless form

$$\omega_{1,\xi\zeta} = -\omega_1, \quad \xi = \left(t + \frac{x_1}{c_m}\right)\sqrt{\frac{\alpha}{j}}, \quad \zeta = \left(t - \frac{x_1}{c_m}\right)\sqrt{\frac{\alpha}{j}}.$$

General solution of this equation can be represented by the integral formula, a little more complicated than the d'Alembert formula:

$$\begin{aligned} \omega_1(\xi, \zeta) = & \int_0^\xi J_0(-2\sqrt{(\xi - \vartheta)\zeta}) df_1(\vartheta) \\ & + \int_0^\zeta J_0(-2\sqrt{\xi(\zeta - \vartheta)}) dh_1(\vartheta) + (f_1(0) + h_1(0)) J_0(-2\sqrt{\xi\zeta}), \end{aligned}$$

where J_0 is the zero-order Bessel function of the first kind, and f_1 and h_1 are the arbitrary continuously differentiable functions that determine the unknown solution on characteristics as in the Goursat problem [21]

$$\omega_1(\xi, 0) = f_1(\xi) + h_1(0), \quad \omega_1(0, \zeta) = f_1(0) + h_1(\zeta).$$

Let us consider the particular solution corresponding to $f_1(\xi) = h_1(\zeta) = 1/2$, in which the dimensionless angular velocity ω_1 is constant and equal to unity on the fronts of the torsional waves $x_1 = \pm c_m t$. Figure 10.1 shows the diagram of ω_1 as a function of the dimensionless time at the point $x_1 = 0$. It is seen that the particle at the point $x_1 = 0$ executes individual oscillations with a period approximately equal to π . In terms of the dimensional variables, this corresponds to the period of oscillations equal to T under a homogeneous shear of a medium. The following dimensionless variables are used: time $t' = \pi t/T$ and spatial coordinate $x'_1 = \pi x_1/(c_m T)$. Figure 10.2 illustrates the distribution of angular velocity between the fronts of torsional waves in the dimensionless variables at the times 5, 10, and 15.

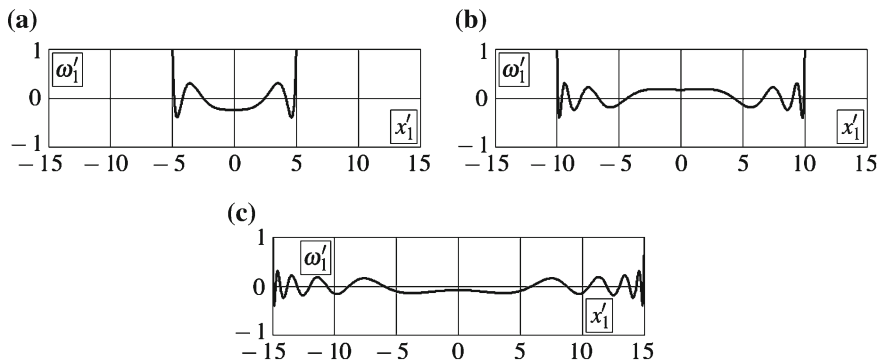


Fig. 10.2 Solution of the telegraph equation for various dimensionless times: **a** $t' = 5$, **b** $t' = 10$, **c** $t' = 15$

Note that, at the initial time $t = 0$, the solution has a smooth non-oscillating profile. The analysis shows that for $t > 0$ a wavelike rotational motion of the particles in the plane of the torsional waves fronts is excited, and the characteristic wavelength is approximately $c_m T$.

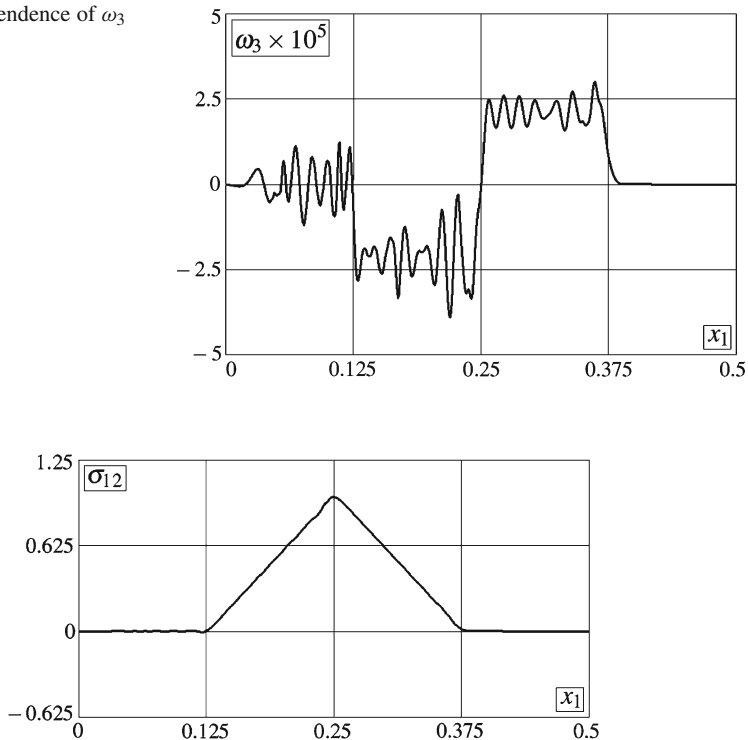
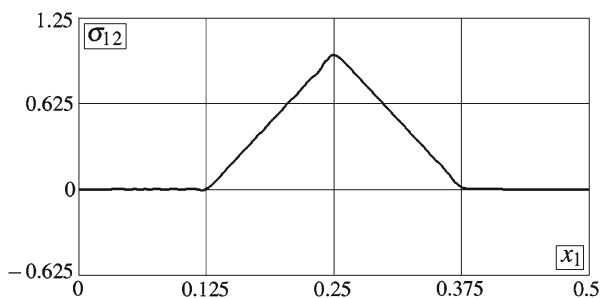
The subsystem (10.15), which describes the propagation of the transverse waves (shear waves) with rotation, is reduced to the equations

$$\ddot{v}_2 = c_s^2 v_{2,11} - \frac{2\alpha}{\rho} \omega_{3,1}, \quad \ddot{\omega}_3 = c_\omega^2 \omega_{3,11} - \frac{4\alpha}{j} \omega_3 + \frac{2\alpha}{j} v_{2,1}.$$

For this subsystem the dispersion equation takes the form [13, 33]:

$$\left(1 - \frac{c_s^2}{c^2}\right) \left(\pi^2 v^2 \left(1 - \frac{c_\omega^2}{c^2}\right) - \frac{\alpha}{j}\right) = \frac{\alpha^2}{\rho j c^2}.$$

Such waves also possess dispersion. In [42] numerical analysis of Eqs. (10.15) is carried out using the Neumann–Richtmyer finite-difference scheme [32]. The computations are performed for different scales of the microstructure of a material in the one-dimensional problem about the action of periodic Λ -shaped impulse of tangential stress on the boundary of an elastic medium. In Fig. 10.3 the graph of dependence of the angular velocity ω_3 on the longitudinal coordinate is shown. The impulse increases linearly in the time interval t_0 and then decreases linearly in the identical interval. On this graph the right-hand part of the oscillating curve, adjacent to the rise-up portion of a wave, corresponds to increasing stress (loading wave), the middle part corresponds to decreasing stress (unloading wave), and the left-hand part corresponds to free rotational oscillations which take place behind the descending part of a wave. The graphs of tangential stress σ_{12} and nonzero couple stress m_{13} are shown in Figs. 10.4 and 10.5. The maximal tangential stress in an impulse is assumed to be one. Distributions correspond to the instant $t = 0.68$ ms, the time of action of the impulse is 0.45 ms. In computations the following elasticity para-

Fig. 10.3 Dependence of ω_3 on x_1 **Fig. 10.4** Dependence of σ_{12} on x_1 

meters for polyurethane foam are used [29]: $\rho = 340 \text{ kg/m}^3$, $k = 485$, $\mu = 104$, $\alpha = 4.33 \text{ MPa}$, $\kappa = 3.87$, $\eta = 40$, $\beta = 5.3 \text{ N}$. The moment of inertia of particles, related to the characteristic size of inhomogeneity, varies: $j = 4.4 \times 10^{-4} \text{ kg/m}$ in Figs. 10.3, 10.4, 10.5, $j = 1.76 \times 10^{-3} \text{ kg/m}$ in Fig. 10.6, and $j = 4.4 \times 10^{-3} \text{ kg/m}$ in Fig. 10.7. In this range of the values of j the oscillation period in the problem on a simple shear varies from $T = 31.7$ to $T = 63.3 \mu\text{s}$ and $T = 100 \mu\text{s}$. The number of characteristic oscillations of the angular velocity varies according to this period.

The results show that at fixed t the angular velocity ω_3 and the couple stresses m_{13} and m_{31} are oscillating functions with the characteristic wavelength $c_s T$.

According to the general theory of hyperbolic systems for the model (10.1), perturbations propagate with the finite velocities (10.7). Furthermore, the perturbations corresponding to transverse waves automatically generate the waves of rotational motion, and vice versa, the perturbations corresponding to rotational motion lead to the formation of transverse waves. On such waves, as on the waves corresponding to the torsional motion of particles, there are oscillations in the solution, which is a distinctive feature of the Cosserat medium as compared to the classical linear elasticity theory. Another difference is that in a moment medium there is an eigenfrequency of the acoustic resonance of a material which does not depend on the sizes of the region

Fig. 10.5 Dependence of m_{13} on x_1

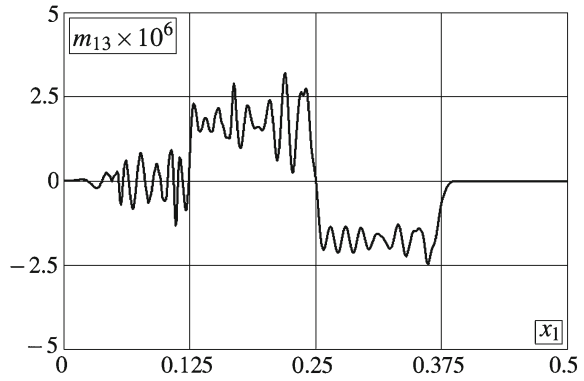


Fig. 10.6 Dependence of ω_3 on x_1 : $j = 1.76 \times 10^{-3} \text{ kg/m}$

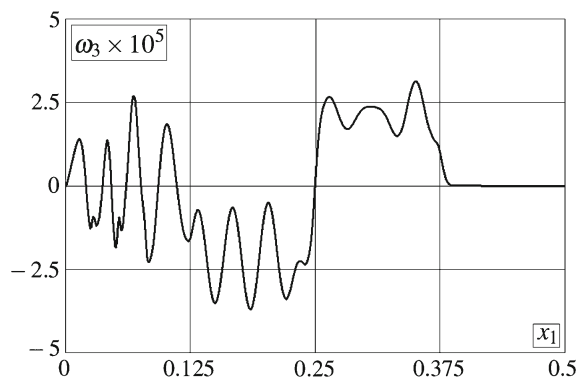
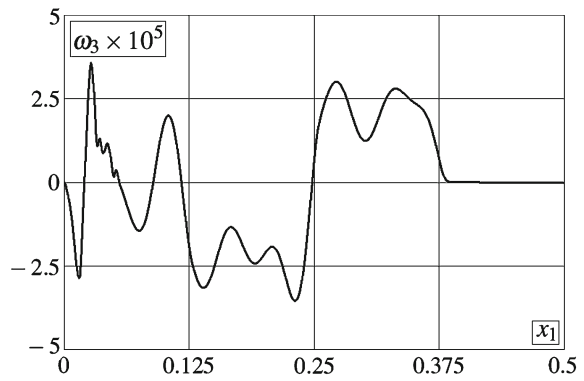


Fig. 10.7 Dependence of ω_3 on x_1 : $j = 4.4 \times 10^{-3} \text{ kg/m}$



studied and manifests itself only under certain perturbation conditions [45, 52]. Indeed, in the case of a homogeneous stress state, the subsystem (10.15) leads to the classical resonance equation

$$j\ddot{\omega}_3 = -4\alpha\omega_3 + 2\alpha\dot{\chi}, \quad (10.16)$$

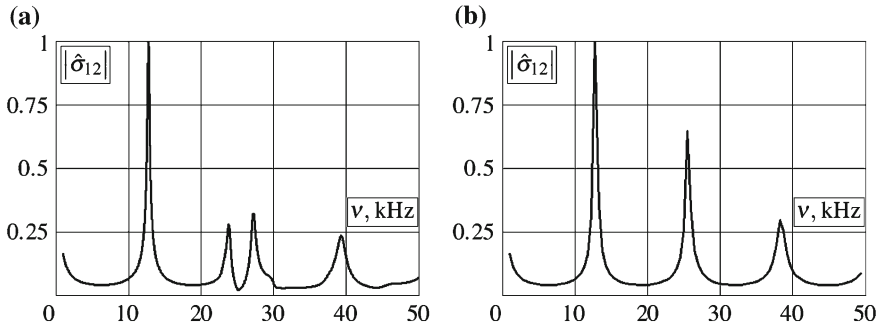


Fig. 10.8 Characteristic spectral curves of the tangential stress for heavy oil in a rock: **a** the Cosserat medium, **b** a momentless viscoelastic medium

from which it follows that, if the shear angle $\chi(t)$ changes under a harmonic law at the frequency $\nu_* = 1/T$ equal to the frequency of natural oscillations of rotational motion, the amplitude of angular velocity of the particles increases infinitely.

Equation (10.16) describes the behavior of an infinitely thin plane elastic layer, whose lower surface is motionless and upper surface moves under a specified law. Figure 10.8 depicts the tangential stress spectral curves obtained by numerical solution of the problem on uniform cyclic shear of a viscoelastic layer of finite thickness H . The graphs correspond to the rigid fixed bottom side of the layer. The solution describes also the torsional oscillations of a cylindrical sample with one edge rigidly fixed. In this case, the tangential stress at the fixed edge depends linearly on the radius and, hence, is proportional to σ_{12} , and the linear velocity at the opposite edge is proportional to ν_2 . Figure 10.8a corresponds to the Cosserat medium, Fig. 10.8b depicts the same curve for an ordinary momentless viscoelastic medium. The analogous graphs for an ideal non-viscous media have a system of resonance peaks with infinite amplitudes. Viscosity is used as a smoothing parameter. The shear process is described by Eqs. (10.15) where, according to the Boltzmann viscoelasticity theory, the products of the medium parameters on the kinematic characteristics of deformation are replaced by convolutions of the relaxation kernels corresponding to these parameters on the same characteristics. The boundary conditions of the problem are taken as

$$\nu_2|_{x_1=0} = \bar{\nu} e^{2\pi i \nu t}, \quad \omega_3|_{x_1=0} = 0, \quad \nu_2|_{x_1=H} = \omega_3|_{x_1=H} = 0. \quad (10.17)$$

Here the x_1 axis is directed inside the layer. The system (10.15) is solved using a spectral method. After the Fourier transform, the following system of ordinary differential equations for the amplitudes

$$\begin{aligned} 4\pi^2 \nu^2 \rho \hat{\nu}_2 + (\hat{\mu} + \hat{\alpha}) \hat{\nu}_{2,11} - 2\hat{\alpha} \hat{\omega}_{3,1} &= 0, \\ 4(\pi^2 \nu^2 j - \hat{\alpha}) \hat{\omega}_3 + (\hat{\eta} + \hat{\beta}) \hat{\omega}_{3,11} + 2\hat{\alpha} \hat{\nu}_{2,1} &= 0 \end{aligned} \quad (10.18)$$

is obtained. The solution of this system is constructed in the explicit form taking into account boundary conditions (10.17). The amplitude of the tangential stress is determined through the solution (10.18) using the formula

$$2\pi \iota v \hat{\sigma}_{12} = (\hat{\mu} + \hat{\alpha}) \hat{v}_{2,1} - 2\hat{\alpha} \hat{\omega}_3.$$

To check the reliability of the results, we perform a numerical solution of the problem based on the approximating (10.18) spectral-difference scheme

$$\begin{aligned} 4\pi^2 v^2 \rho \hat{v}_2^k + (\mu + \alpha) \frac{\hat{v}_2^{k+1} - 2\hat{v}_2^k + \hat{v}_2^{k-1}}{\Delta x_1^2} - 2\alpha \frac{\hat{\omega}_3^k - \hat{\omega}_3^{k-1}}{\Delta x_1} &= 0, \\ 4(\pi^2 v^2 j - \hat{\alpha}) \hat{\omega}_3^k + (\eta + \beta) \frac{\hat{\omega}_3^{k+1} - 2\hat{\omega}_3^k + \hat{\omega}_3^{k-1}}{\Delta x_1^2} + 2\alpha \frac{\hat{v}_2^{k+1} - \hat{v}_2^k}{\Delta x_1} &= 0, \end{aligned} \quad (10.19)$$

where Δx_1 is the step of the grid. The calculations are performed by the matrix sweep method, and the results obtained on fine grids are almost the same.

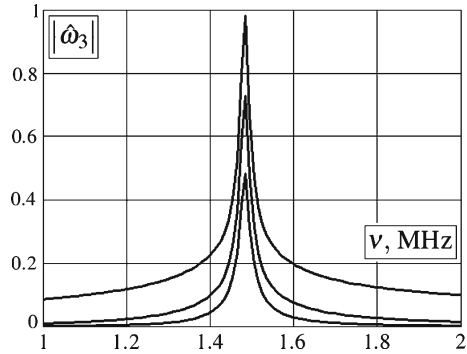
The phenomenological parameters of a medium were selected by the experimental data [9] for heavy oil in a rock at sufficiently low temperature where a material is in the solid phase. For this material $\rho = 1,114 \text{ kg/m}^3$, $j = 0.01 \text{ kg/m}$, $\mu = 966$, $\alpha = 52.2 \text{ MPa}$, $\eta + \beta = 12.51 \text{ N}$, and $H = 36.4 \text{ mm}$ are assumed. According to the Kelvin–Voigt theory used in the computations, the complex moduli are linear functions of frequency, in particular, $\hat{\mu} = \mu + 2\pi \iota v \mu'$. The imaginary parts are chosen so as to obtain the required smoothing of the solution.

A comparison of the graphs in Fig. 10.8 shows that in the Cosserat medium there is an additional resonance frequency of 23 kHz, close to the frequency of the rotational motion of the particles and independent of the layer thickness. This is confirmed by a great number of numerical experiments for different thicknesses. It appeared that a change in the thickness H of the layer leads to a displacement of the periodic system of the fundamental resonance frequencies which are approximately equal to $v_k = k c_s / (2H)$ ($k = 1, 2, \dots$), but the peak corresponding to the frequency v_* remains motionless.

Similar computations for polyurethane foam do not show a significant (compared to the fundamental frequencies) peak at the resonance frequency, which for this material has the same order of magnitude as for heavy oil. This fact is explained by the substantially less moment properties expressed by the values of parameters j and α . It is found that in materials with low moment properties the resonance of rotational motion of particles can be excited, for example, by the periodic variation of the rotational moment at the layer boundary. Spectral curves for synthetic polyurethane are presented in Fig. 10.9. Parameters of this material are $\rho = 590 \text{ kg/m}^3$, $j = 5.3 \times 10^{-6} \text{ kg/m}$, $k = 2.89$, $\mu = 1.03$, $\alpha = 0.115 \text{ GPa}$, $\kappa = 0.393$, $\eta = 4.1$, and $\beta = 0.13 \text{ N}$ (see [29]). The results are obtained by numerical solution of the system (10.18) with the boundary conditions

$$\sigma_{12}|_{x_1=0} = 0, \quad m_{13}|_{x_1=0} = \bar{m} e^{2\pi \iota v t}, \quad v_2|_{x_1=H} = \omega_3|_{x_1=H} = 0 \quad (10.20)$$

Fig. 10.9 Resonance peak of the angular velocity amplitude on the spectral curve for synthetic polyurethane



using the spectral-difference method (10.19). The presented graphs correspond to $H = 10$ cm, but they are almost independent of the layer thickness. The curves are related to different levels: the upper one corresponds to the layer boundary where the periodic perturbations are excited, the middle and lower ones correspond to the levels distant by a quarter and half-thickness of the layer deep from the boundary. Analysis shows that the amplitude of the angular velocity has a single resonance peak at the frequency $\nu_* = 1.48$ MHz within the range of interest, the height of the peak is determined by the imaginary part of the parameter $\hat{\eta} + \hat{\beta}$ and decreases almost linearly with increasing depth.

Qualitatively similar results in the problem with boundary conditions (10.20) are obtained in computations performed for the parameters of polyurethane foam and heavy oil.

To confirm the numerical results analytically, we consider the system (10.18) with $\nu = \nu_*$ for an inviscid moment medium:

$$\frac{4\alpha\rho}{j}\hat{\nu}_2 + (\mu + \alpha)\hat{\nu}_{2,11} - 2\alpha\hat{\omega}_{3,1} = 0, \quad (\eta + \beta)\hat{\omega}_{3,11} + 2\alpha\hat{\nu}_{2,1} = 0. \quad (10.21)$$

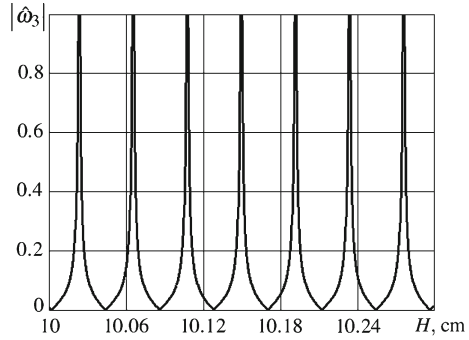
For this system, the characteristic equation has a multiple root equal to zero with a multiplicity of two. The fundamental system of solutions consists of the vectors

$$\begin{pmatrix} -i(\eta + \beta)\varepsilon \\ 2\alpha \end{pmatrix} e^{i\varepsilon x_1}, \quad \begin{pmatrix} i(\eta + \beta)\varepsilon \\ 2\alpha \end{pmatrix} e^{-i\varepsilon x_1}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} j/(2\rho) \\ x_1 \end{pmatrix},$$

where $\varepsilon = 2\sqrt{\frac{\alpha\rho(\eta + \beta)/j + \alpha^2}{(\mu + \alpha)(\eta + \beta)}}$. The general solution of the system (10.21) has the form

$$\begin{aligned} \hat{\nu}_2 &= -i(\eta + \beta)\varepsilon(C_2 e^{i\varepsilon x_1} - C_3 e^{-i\varepsilon x_1}) + \frac{jC_5}{2\rho}, \\ \hat{\omega}_3 &= 2\alpha(C_2 e^{i\varepsilon x_1} + C_3 e^{-i\varepsilon x_1}) + C_4 + C_5 x_1. \end{aligned}$$

Fig. 10.10 Magnitude of angular velocity amplitude versus layer thickness



By virtue of boundary conditions (10.20) on the surface $x_1 = 0$, the constants C_2 and C_3 are expressed in terms of C_4 and C_5 :

$$C_2 + C_3 = \frac{jC_4}{2\rho(\eta + \beta)}, \quad C_2 - C_3 = \frac{\iota C_5}{2\alpha\varepsilon} + \frac{\pi v_* \bar{m}}{\alpha\varepsilon(\eta + \beta)}.$$

The boundary conditions on the surface $x_1 = H$ lead to the system of equations

$$\begin{pmatrix} j\alpha\varepsilon \sin(\varepsilon H) & \rho(\eta + \beta) \cos(\varepsilon H) + j\alpha \\ j\alpha\varepsilon \cos(\varepsilon H) + \rho\varepsilon(\eta + \beta) & \rho(\eta + \beta)(\varepsilon H - \sin(\varepsilon H)) \end{pmatrix} \begin{pmatrix} C_4 \\ C_5 \end{pmatrix} = 2\pi\iota\rho v_* \bar{m} \begin{pmatrix} \cos(\varepsilon H) \\ -\sin(\varepsilon H) \end{pmatrix}, \quad (10.22)$$

which allows one to determine all constants. For $x_1 = 0$ the angular velocity amplitude is obtained from the formula

$$\hat{\omega}_3(0) = \frac{j\alpha C_4}{\rho(\eta + \beta)} + C_4.$$

From the system (10.22) it follows that for large values of H the constant C_5 has the order of magnitude of $1/H$ and C_4 has the order of magnitude of $\cot(\varepsilon H)$. Hence, the amplitude tends to infinity, if εH is close to πn ($n = 1, 2, \dots$).

Thus, in the problem with boundary conditions (10.20) ignoring the viscosity of a medium, the frequency v_* is resonant only for layers of strictly defined thickness. The increment in the thickness of the resonating layer $\Delta H = \pi/\varepsilon$ depends on the parameters of a material. For synthetic polyurethane, the value $\Delta H = 0.42$ mm is comparable to the characteristic microstructure parameter $r = 0.15$ mm. A curve of dependence of the magnitude $\hat{\omega}_3(0)$ on thickness of a layer is given in Fig. 10.10. It is evident that the asymptotic formula for the quantity ΔH is satisfied with high accuracy.

10.2 Computational Results

To demonstrate the efficiency of the computational technique proposed in Chaps. 6 and 8, numerical computations of plane and spatial problems of the Cosserat elasticity theory were performed on clusters [40, 41, 45, 51, 52].

The results of 2D computations of the elastic waves propagation in a rectangular body are presented in Figs. 10.11, 10.12, 10.13, 10.14, 10.15. All sides of the body, excepting the left one, are nonreflecting boundaries. On the left side distributed periodic load of Λ -shaped impulses of tangential stress σ_{12} is given. Considering that the problem is linear, the maximal stress for any impulse is assumed to be one again. The area of application of impulses takes half a side in its central part. Such load can be generated, for example, through a rigid plate glued to a sample with the help of a periodically repeating transverse impact. As a result of impulsive action, a sequence of loading and unloading waves propagates over a material. These waves are domains of smooth variation of a solution with clearly defined fronts. This is seen by the level curves of tangential stress σ_{12} in Fig. 10.11 (for $t = 26$ and $t = 78 \mu\text{s}$). The range of variation of stress from -0.25 to 0.75 corresponds to shades of gray. In the first case (Fig. 10.11a) a single wave induced by the first loading impulse on the boundary is observed, in the second case (Fig. 10.11b) we have three waves caused by three impulses. Notice that contrary to the one-dimensional solution, where tangential stress is positive everywhere (see Fig. 10.4), in the plane problem areas of negative value arise because of the lateral unloading.

In Fig. 10.12 for angular velocity ω_3 of particles and in Fig. 10.13 for couple stress m_{13} the oscillations, whose characteristic scale can be estimated with the help of a solution of the problem on simple shear again, are observed for the same instants of time. Here the values of levels of angular velocity vary in the range from -6×10^{-5} to 6×10^{-5} and the values of levels of couple stress vary from -1.7×10^{-7} to 1.7×10^{-7} .

In Figs. 10.14 and 10.15 the similar results for normal stress σ_{11} and linear velocity v_1 are presented. The level curves of stress lie in the range from -0.4 to 0.4 and the level curves of velocity lie in the range from -2.7×10^{-7} to 2.7×10^{-7} . The corresponding waves are generated at the points which belong to the boundary of the area of application of a load on the left side of the body. Analysis shows that these waves are related to intensive rotational motion of particles in a neighbourhood of points at which load change sharply. They are severely damped waves moving inwards a body in the vertical direction with velocity close to that of longitudinal waves and gradually leaving the domain of solution of the problem through horizontal sides.

The computations were performed on a square of side 0.1 m for synthetic polyurethane (mechanical parameters of this material were presented in Sect. 10.1) using 10 processors of the cluster. The half-period of action of an impulse, at the center of which tangential stress reaches its maximum, is 1 ms . The time between neighbouring impulses is 1 ms as well. The characteristic scale of the microstructure of a material is $r_0 = 0.15 \text{ mm}$. Velocities of elastic waves are $c_p = 2,687$,

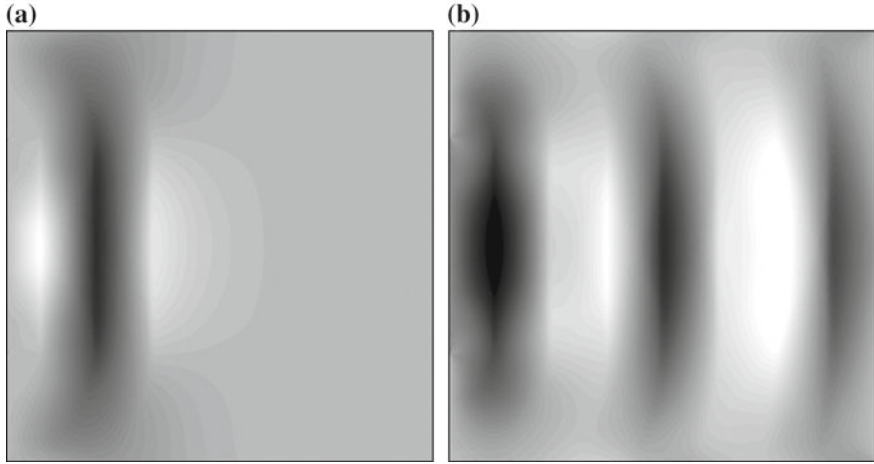


Fig. 10.11 The action of Λ -shaped impulses of tangential stress (2D case): level curves of σ_{12} ; **a** $t = 26 \mu\text{s}$, **b** $t = 78 \mu\text{s}$

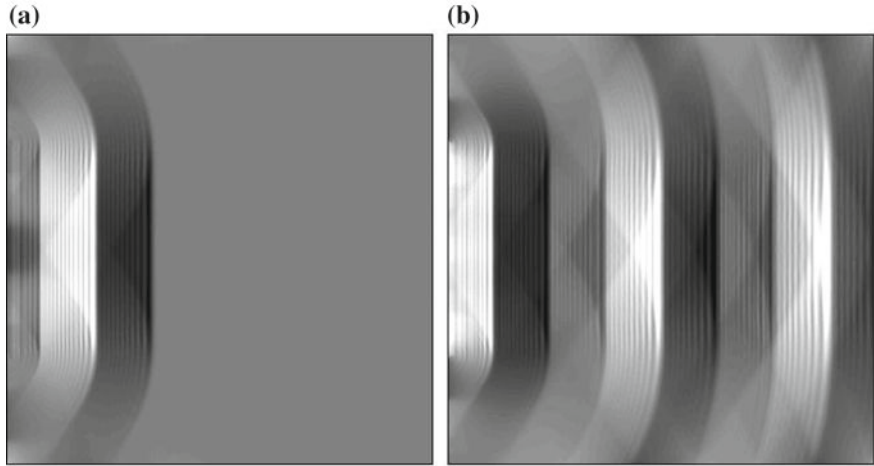


Fig. 10.12 The action of Λ -shaped impulses of tangential stress (2D case): level curves of ω_3 ; **a** $t = 26 \mu\text{s}$, **b** $t = 78 \mu\text{s}$

$c_s = 1,395$, and $c_\omega = 893$ m/s. The uniform difference grid, used in computations, consists of $1,000 \times 1,000$ meshes with a mesh size of 0.1 mm (this value is less than r_0). On coarser grids calculations with satisfactory accuracy may not be performed. For example, on a grid consisting of 500×500 meshes an understated number of oscillations of angular velocity and couple stresses within any wave is obtained.

Many authors noticed [13, 26] that in modern reference literature there is no reliable data on the values of phenomenological parameters of the Cosserat contin-

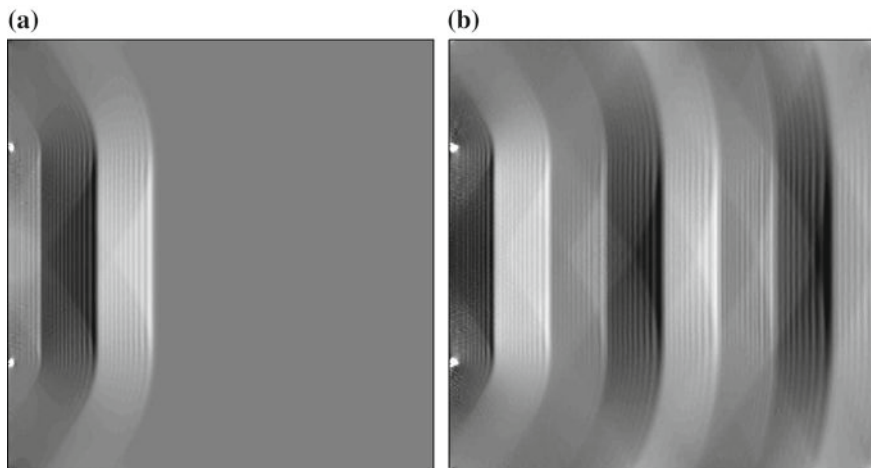


Fig. 10.13 The action of Λ -shaped impulses of tangential stress (2D case): level curves of m_{13} ; **a** $t = 26 \mu\text{s}$, **b** $t = 78 \mu\text{s}$

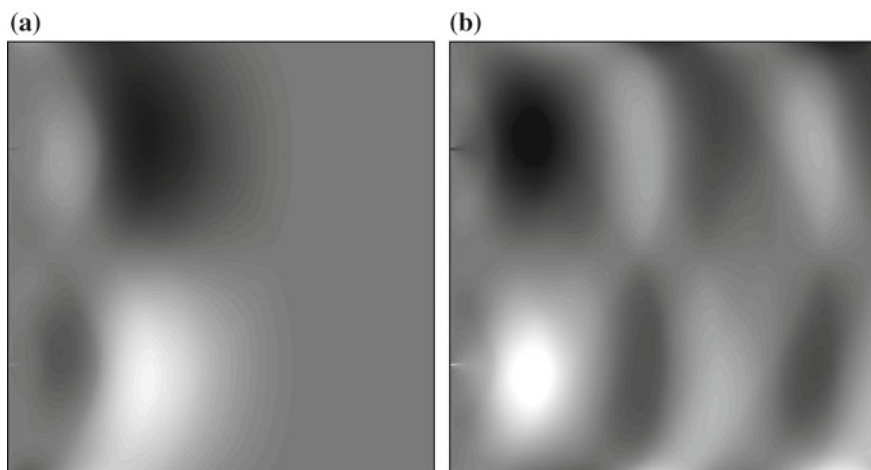


Fig. 10.14 The action of Λ -shaped impulses of tangential stress (2D case): level curves of σ_{11} ; **a** $t = 26 \mu\text{s}$, **b** $t = 78 \mu\text{s}$

uum for actual natural or artificial materials with microstructure. The difficulties are primarily related to the fact that it is impossible to determine these parameters with reasonable accuracy by means of standard static experiments. It is necessary to select appropriate schemes of tests where specific features of deformation of the Cosserat medium are exhibited. The appearance of natural oscillations of the rotational motion of particles behind the front of a shear wave seems to be one of these features.

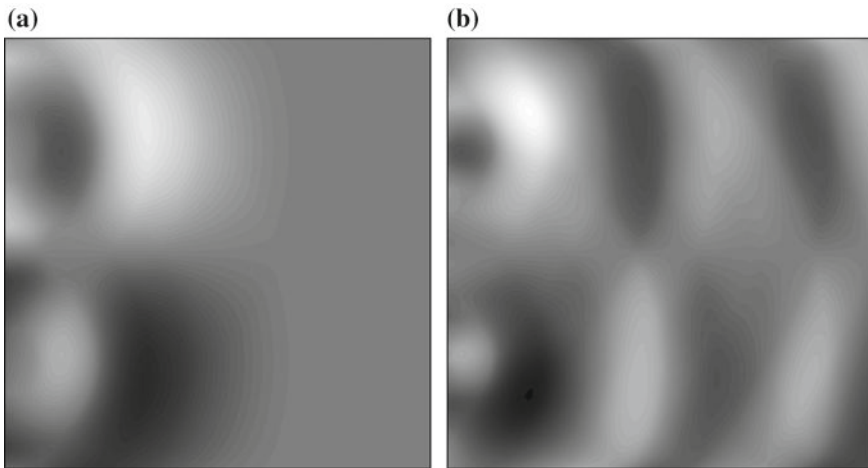


Fig. 10.15 The action of Λ -shaped impulses of tangential stress (2D case): level curves of v_1 ; **a** $t = 26 \mu\text{s}$, **b** $t = 78 \mu\text{s}$

Similar computations show that under extending an impulse with respect to time (when going from a Λ -shaped impulse to a Π -shaped one) the areas of natural oscillations of angular velocity and couple stresses are localized on fronts of transverse waves.

In Figs. 10.16 and 10.17 the results of numerical solution of Lamb's problem on the action of concentrated force on a surface of a half-space for plane strain are presented. These results were obtained on a grid consisting of $1,000 \times 500$ meshes. Computations were performed on cluster of ICM SB RAS (Krasnoyarsk). In the first case impulsive load is applied at the center of the upper boundary of computational domain in the normal direction and in the second case it is applied along a tangent to the boundary. Level curves of angular velocity ω_3 and stresses σ_{11} , σ_{12} , and σ_{33} are shown from left to right, respectively. All waves characteristic for the solution of Lamb's problem in the framework of the classical elasticity theory [35, 47] are clearly distinguished on level curves. Among them are incident longitudinal and transverse waves with circular fronts, two transverse waves in the form of symmetric straight-line segments tangent to the semicircle of smaller radius which constitute the so-called conical wave, the Rayleigh surface waves rapidly damped with depth shown by bright points on the boundary following an incident transverse wave. The distinction is that in the Cosserat model of an elastic medium a solution has clearly pronounced oscillatory nature. Stress oscillations of moderate amplitude are superimposed on significantly inhomogeneous fields which vary rapidly near wave fronts. In order to oscillations of stresses become more visible, isolines such that the values of their levels are in the range from one tenth of minimal stress to one tenth of maximal one are shown. Obviously, such representation of results smears the wave fronts which in fact are more narrow. Conversely, the amplitude of values

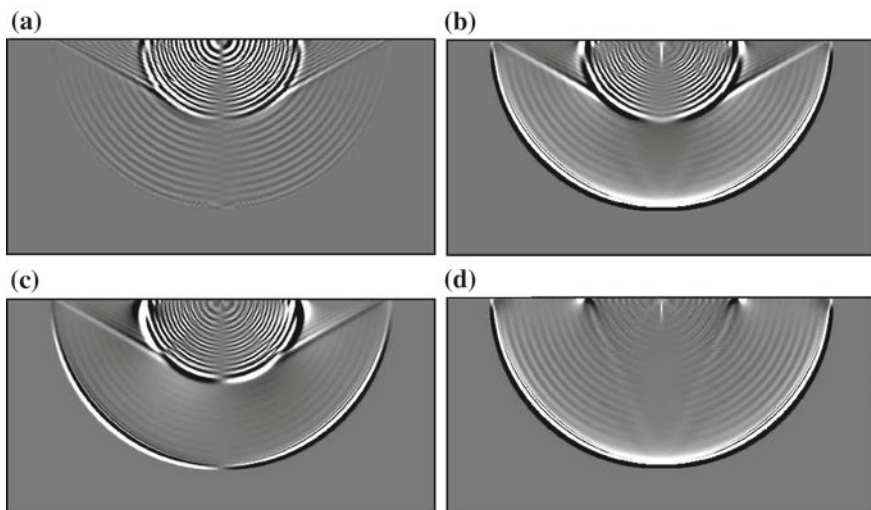


Fig. 10.16 Lamb's problem for normal load (2D case): **a** level curves of angular velocity ω_3 , **b** level curves of stress σ_{11} , **c** level curves of stress σ_{12} , **d** level curves of stress σ_{33}

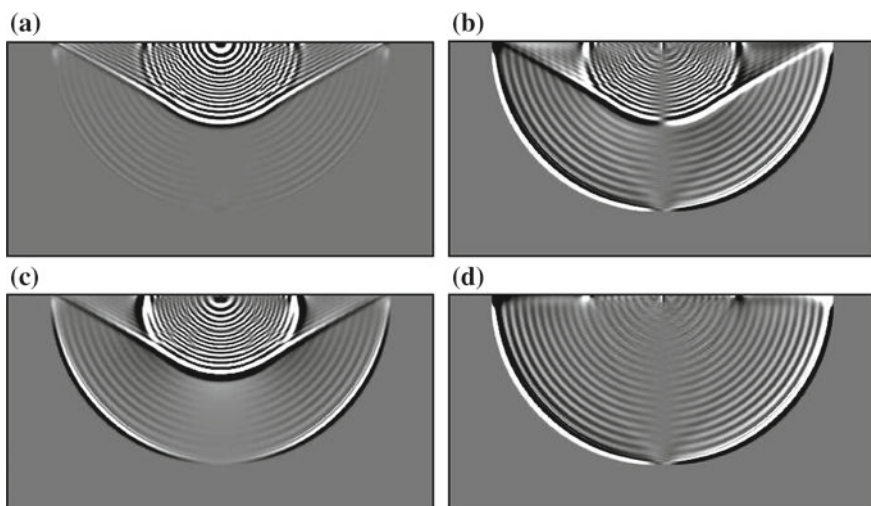


Fig. 10.17 Lamb's problem for tangential load (2D case): **a** level curves of angular velocity ω_3 , **b** level curves of stress σ_{11} , **c** level curves of stress σ_{12} , **d** level curves of stress σ_{33}

of angular velocity on longitudinal and transverse waves essentially coincides with that of oscillations, therefore the corresponding level curves are taken in the range from the minimal value to the maximal one.

In addition, contrary to the classical elasticity theory, in the Cosserat medium the Rayleigh surface waves have considerable dispersion [13, 27, 30] hence, in fact

wave packets rather than solitary waves move along the boundary of a half-space in opposite directions from the point of application of a force. This one can see from variation of the colour spectrum in corresponding areas.

The similar computations were performed for 3D case. Let us consider the Lamb problem on instantaneous action of concentrated forces and moments at the half-space surface. With regard to the symmetry conditions (see Sect. 10.1), the computation domain is a quarter of the half-space. In fact, the computations were performed on a cube with the side 0.01 m for synthetic polyurethane with the microstructure particles of the size $r = 0.15$ mm. Velocities of elastic waves (10.7) are $c_p = 2,687$, $c_s = 1,395$, $c_m = 1,050$, and $c_\omega = 893$ m/s. The uniform difference grid is used which consists of $200 \times 200 \times 200$ meshes with a mesh size of 0.05 mm. The computations cannot be performed with adequate accuracy using a coarser grid because the size of the meshes becomes comparable with the size of the medium particles. On the artificially defined faces of the cube, the symmetry conditions (10.9)–(10.12) and the nonreflecting boundary conditions, that model the free wave passage, were imposed. The computations were performed on 64 processors of the cluster MVS-100K of JSCC RAS (Moscow), every processor solved a part of the problem on the subgrid of the size $50 \times 50 \times 50$ meshes. Running time of 200 steps is about 18 h.

Figure 10.18 shows the level surfaces of the normal stress σ_{11} (Fig. 10.18a) and of the moment m_{23} (Fig. 10.18b) for Lamb's problem on the action of a concentrated normal force $\sigma_{11} = -\bar{p}_1 \delta(\mathbf{x}) \delta(t)$, applied at one of the vertices of the computation domain. Here we can see the incident longitudinal and transverse waves, conical wave, and the Rayleigh surface wave. Presented figures correspond to different points of time: $t = 3.25 \mu\text{s}$ above and $t = 6.5 \mu\text{s}$ below. The loading scheme and the seismograms of the incident waves for this problem are represented in Fig. 10.19. These results are processed in the SeisView computer system. More precisely, the displacement in the direction of the axis x_1 (Fig. 10.19b) and the angular displacement of the particles in the plane $x_1 x_2$ (Fig. 10.19c) along a trace passing through the point of application of the force on the axis x_2 are shown. On the seismograms one can see four waves propagating with the velocities c_p , c_s , c_m , and c_ω (the points where the waves appear at the domain boundary are marked by triangles). We also see the oscillations; additional computations showed that they depend on the characteristic size of the particles.

Figure 10.20 presents the results of the numerical solution of similar problem on the action of a concentrated tangential force $\sigma_{12} = -\bar{p}_2 \delta(\mathbf{x}) \delta(t)$. Figure 10.20a, b shows the level surfaces of the tangential stress σ_{12} and of the moment m_{13} , respectively. The loading scheme and the seismograms of the displacement in the direction of the axis x_2 (Fig. 10.21b) and the angular displacement in the plane $x_1 x_2$ (Fig. 10.21c) along the same trace are shown. As in the previous variant of loading, the oscillatory nature of the solution is clearly visible.

In Fig. 10.22 the level surfaces of the moment m_{11} and of the stress σ_{23} for the problem on the action of a concentrated torsional moment $m_{11} = -\bar{q}_1 \delta(\mathbf{x}) \delta(t)$ are presented. Next, the loading scheme (Fig. 10.23a) and the seismograms of the displacement in the direction of the axis x_3 (Fig. 10.23b) and the angular displacement in the plane $x_2 x_3$ (Fig. 10.23c) for this problem are shown.

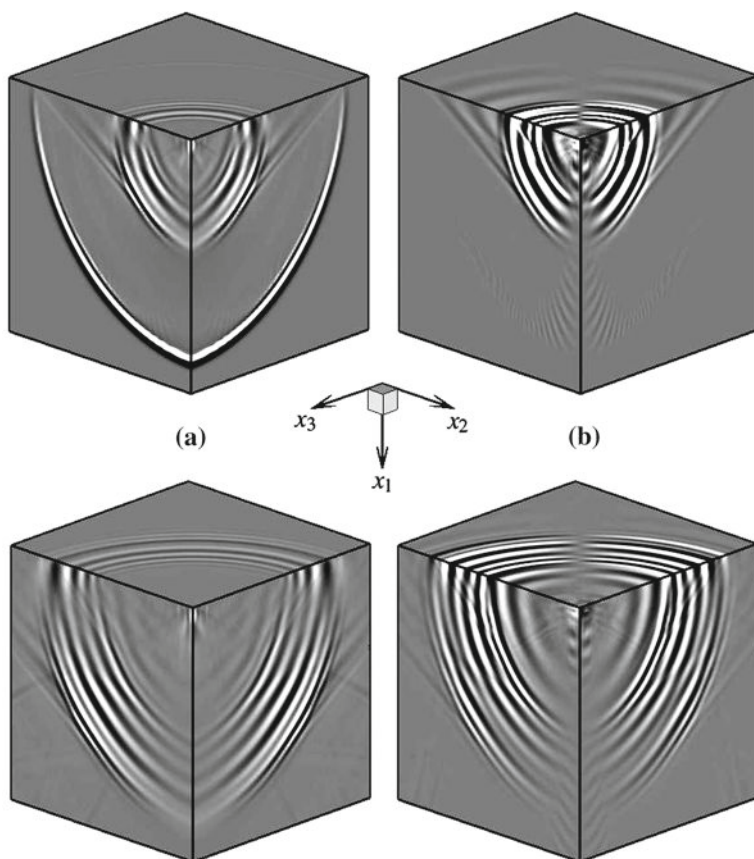


Fig. 10.18 Lamb's problem for the action of normal stress (3D case): **a** level surfaces of stress σ_{11} , **b** level surfaces of moment m_{23} ; $t = 3.25 \mu\text{s}$ (above) and $t = 6.5 \mu\text{s}$ (below)

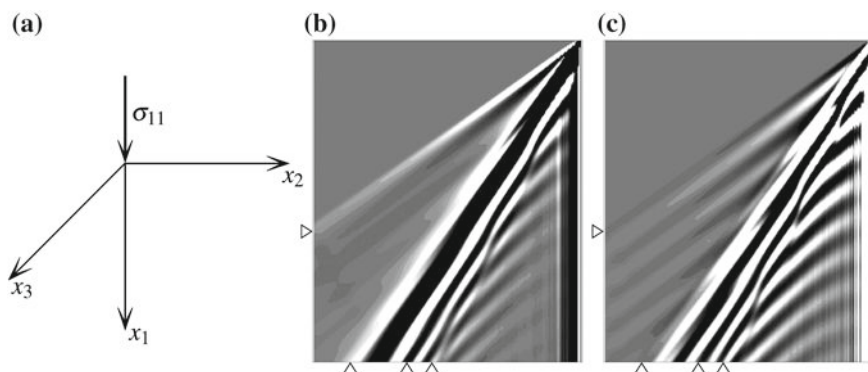


Fig. 10.19 Lamb's problem for the action of normal stress (3D case): **a** loading scheme, **b** seismogram of displacement along the axis x_1 , **c** seismogram of angular displacement in the plane $x_1 x_2$

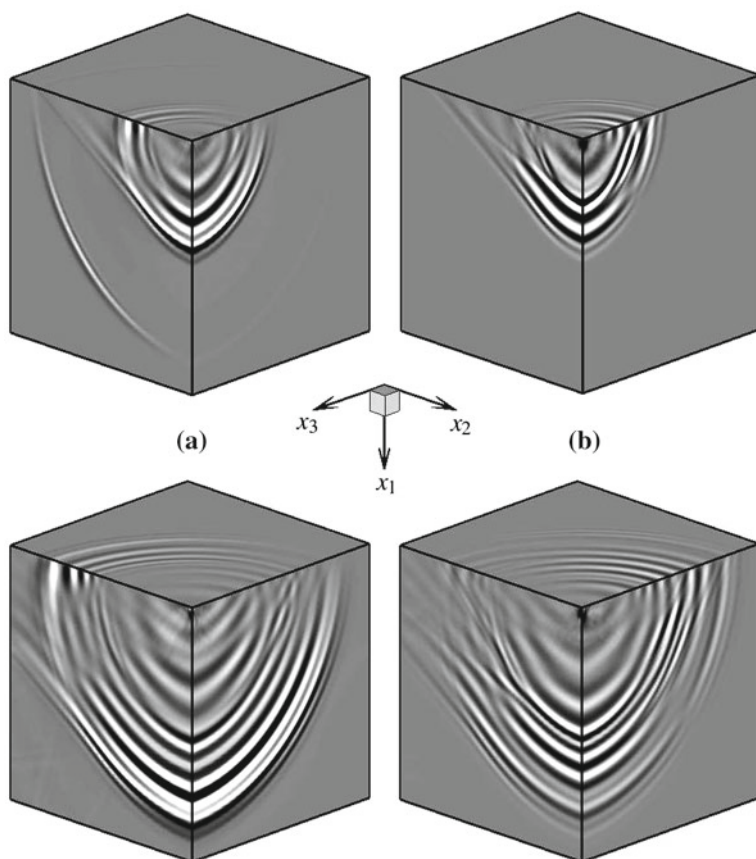


Fig. 10.20 Lamb's problem for the action of tangential stress (3D case): **a** level surfaces of stress σ_{12} , **b** level surfaces of moment m_{13} ; $t = 3.25 \mu\text{s}$ (above) and $t = 6.5 \mu\text{s}$ (below)

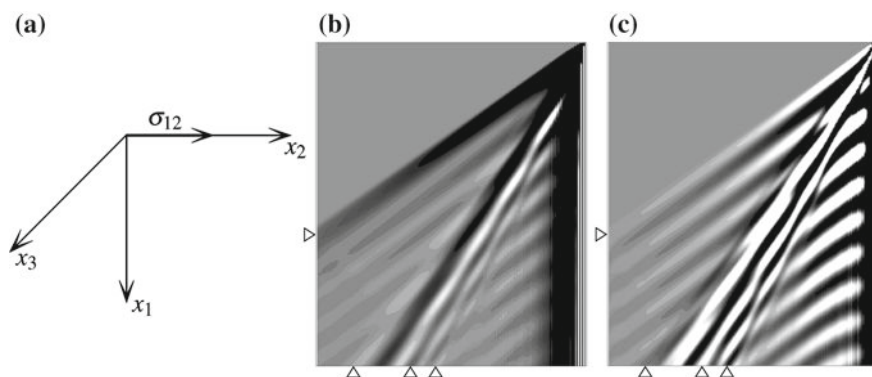


Fig. 10.21 Lamb's problem for the action of tangential stress (3D case): **a** loading scheme, **b** seismogram of displacement along the axis x_2 , **c** seismogram of angular displacement in the plane $x_2 x_3$

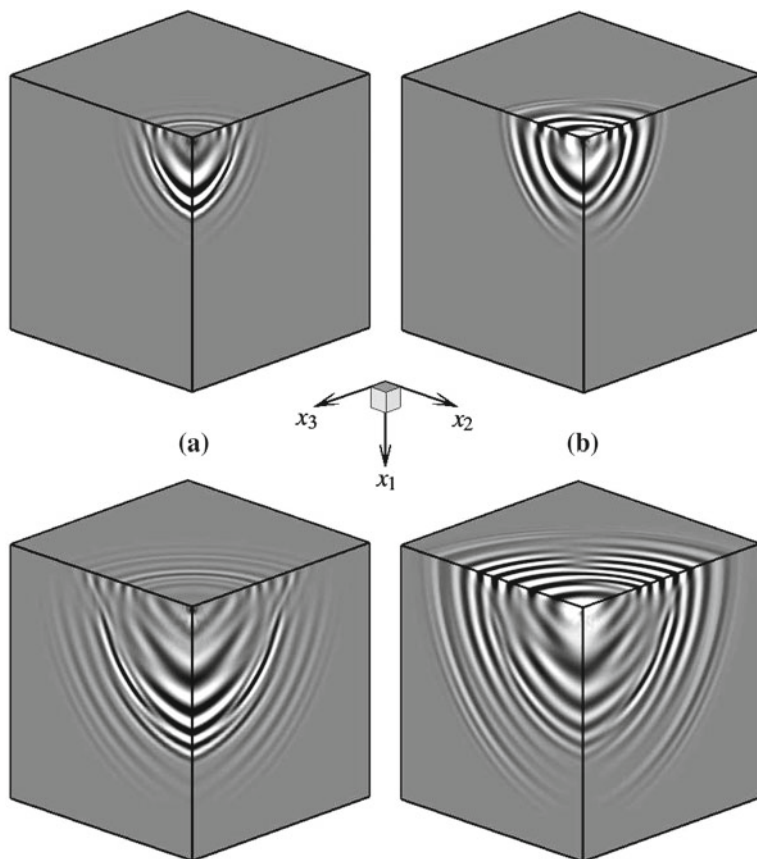


Fig. 10.22 Lamb's problem for the action of torsional moment (3D case): **a** level surfaces of moment m_{11} , **b** level surfaces of stress σ_{23} ; $t = 3.25 \mu s$ (above) and $t = 6.5 \mu s$ (below)

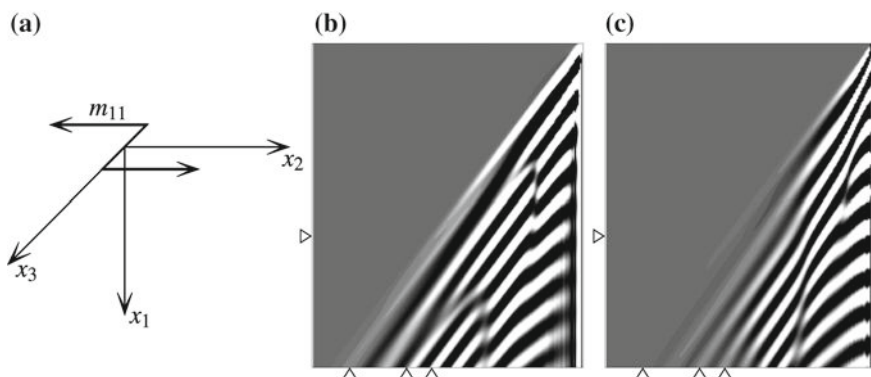


Fig. 10.23 Lamb's problem for the action of torsional moment (3D case): **a** loading scheme, **b** seismogram of displacement along the axis x_3 , **c** seismogram of angular displacement of particles in the plane $x_2 x_3$

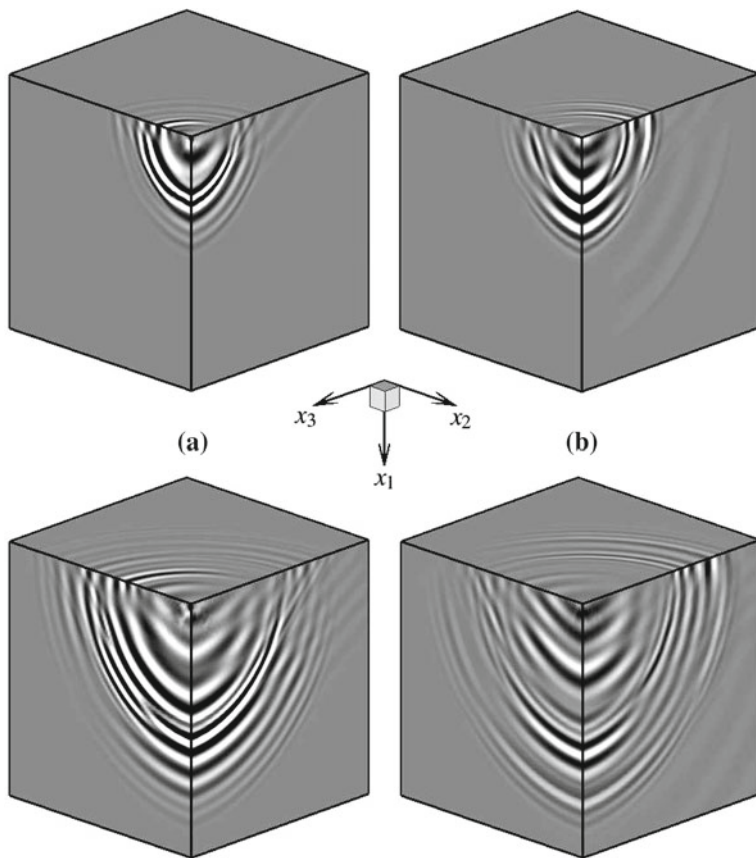


Fig. 10.24 Lamb's problem for the action of rotational moment (3D case): **a** level surfaces of moment m_{12} , **b** level surfaces of stress σ_{13} ; $t = 3.25 \mu\text{s}$ (*above*) and $t = 6.5 \mu\text{s}$ (*below*)

Figure 10.24 corresponds to the case of the action of a concentrated rotational moment $m_{12} = -\bar{q}_2 \delta(\mathbf{x}) \delta(t)$ on the boundary of the half-space. The level surfaces of the moment m_{12} and of the stress σ_{13} are represented in Fig. 10.24a, b, respectively. The corresponding loading scheme (Fig. 10.25a), the seismograms of the displacement in the direction of the axis x_3 (Fig. 10.25b) and the angular displacement in the plane $x_1 x_3$ (Fig. 10.25c) for this problem are represented, too. Note that in Figs. 10.23 and 10.25 the longitudinal wave is almost absent and the front of the transverse wave is strongly smeared.

Results of computations for the problem on the periodic action of a concentrated normal load $\sigma_{11} = -\bar{p}_1 \delta(\mathbf{x}) \sin(2\pi \nu t)$ with the frequency ν are shown in Fig. 10.26. Here one can see the level surfaces of the stress σ_{11} for the frequency of external action equal to the frequency ν_* of natural oscillations of rotational motion of particles (in the center), and also for $\nu = 0.5 \nu_*$ (*above*) and $\nu = 1.5 \nu_*$ (*below*).

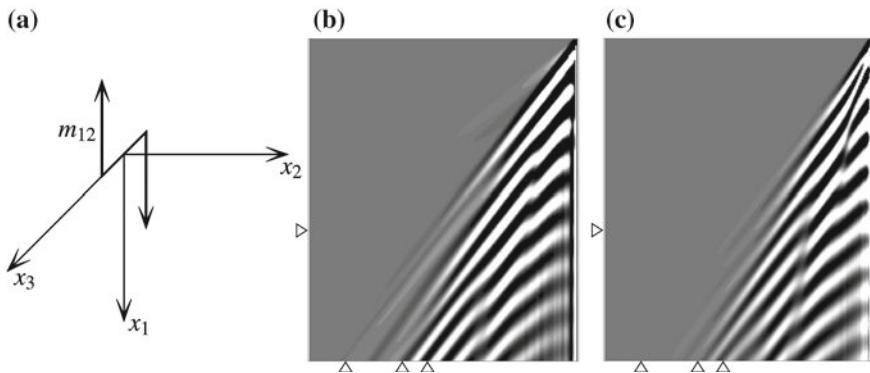


Fig. 10.25 Lamb's problem for the action of rotational moment (3D case): **a** loading scheme, **b** seismogram of displacement along the axis x_3 , **c** seismogram of angular displacement of particles in the plane $x_1 x_3$

As before $v_* = 1/T$, where $T = \pi\sqrt{j/\alpha}$ is the oscillation period for particles. Figure 10.26a corresponds to the time moment $t = 1.55 \mu\text{s}$, and Fig. 10.26b corresponds to the moment $t = 3.1 \mu\text{s}$. Unlike previous problems, the lower boundary of computational domain is fixed here, therefore the reflected waves propagate from this boundary inside the domain. Computations were performed for foamy polyurethane on 64 processors of the cluster MVS-100K of JSCC RAS, the total size of files at each time step is 320 MB, the running time for 200 time steps is about 10 h.

Numerical results for 3D problem on the action of a concentrated rotational moment $m_{12} = -\bar{q}_2 \delta(\mathbf{x}) \delta(2\pi \nu t)$, periodic by time, on the surface of elastic half-space are represented in Fig. 10.27. The corresponding loading scheme is shown in Fig. 10.25a. Computations were performed for synthetic polyurethane, the running time for 400 time steps is about 19 h. In Fig. 10.27 one can see the level surfaces of the angular velocity ω_2 for the nonresonance frequency of external action $\nu = 1.5 \nu_*$ and for the resonance frequency $\nu = \nu_*$ (from left to right) at different moments of time. The level surfaces correspond to the same range of values ω_2 . The maximum amplitude of oscillations of the angular velocity is achieved at the point of load application, and the wavelength depends essentially on the frequency. Comparison of the graphs shows that for the frequency ν_* of external action, equal to the natural frequency of the rotational motion of particles, the growth of amplitude with time occurs and a more smooth decay of oscillations with increasing the distance from the point of load application, characteristic of the acoustic resonance, takes place (Fig. 10.27b).

The analogous computations showed that the variants of specifying the periodic by time and spatially concentrated normal or tangential stress and also the linear or angular velocity do not lead to noticeable resonance excitation of a medium.

Thus, computations of 3D problems have confirmed the main qualitative difference of the wave field in the Cosserat continuum as compared with the classical elasticity theory, which consists in the appearance of oscillations of the rotational

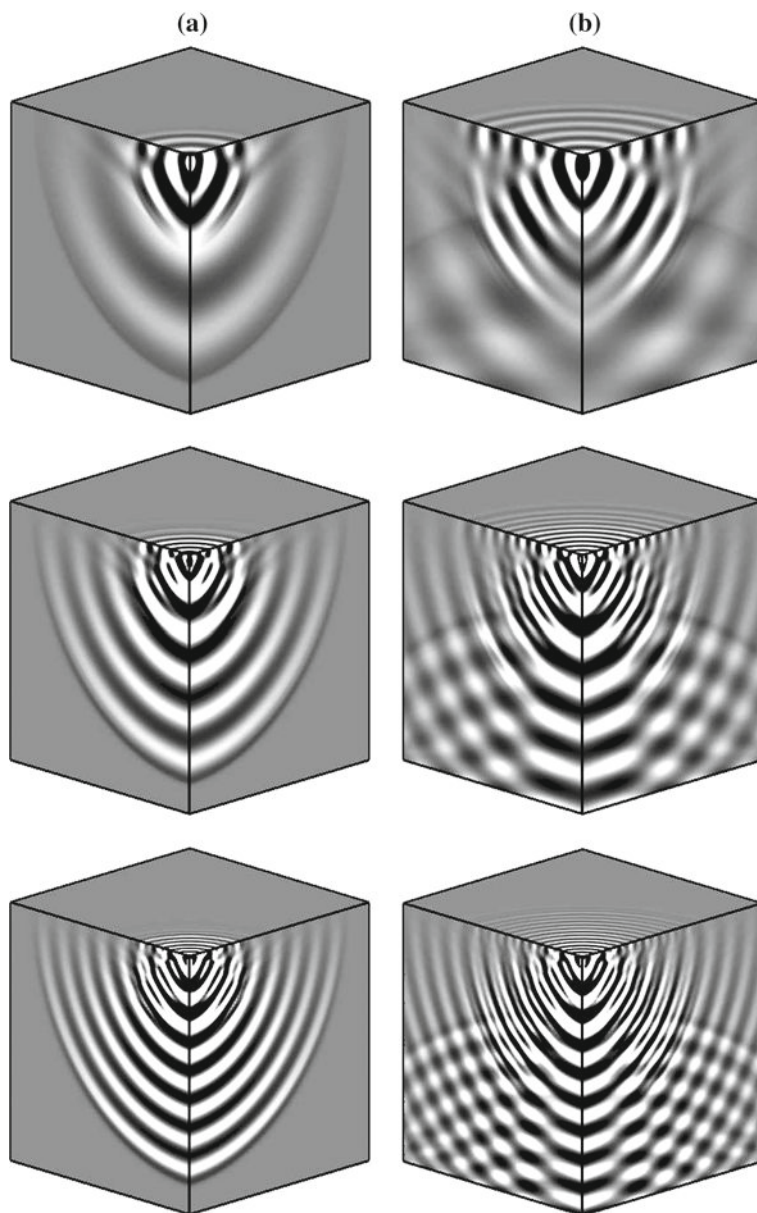


Fig. 10.26 Problem on the periodic action of the concentrated normal stress (3D case): level surfaces of stress σ_{11} for $\nu = 0.5 \nu_*$, $\nu = \nu_*$, $\nu = 1.5 \nu_*$ (from *above* to *below*); **a** $t = 1.55 \mu\text{s}$, **b** $t = 3.1 \mu\text{s}$

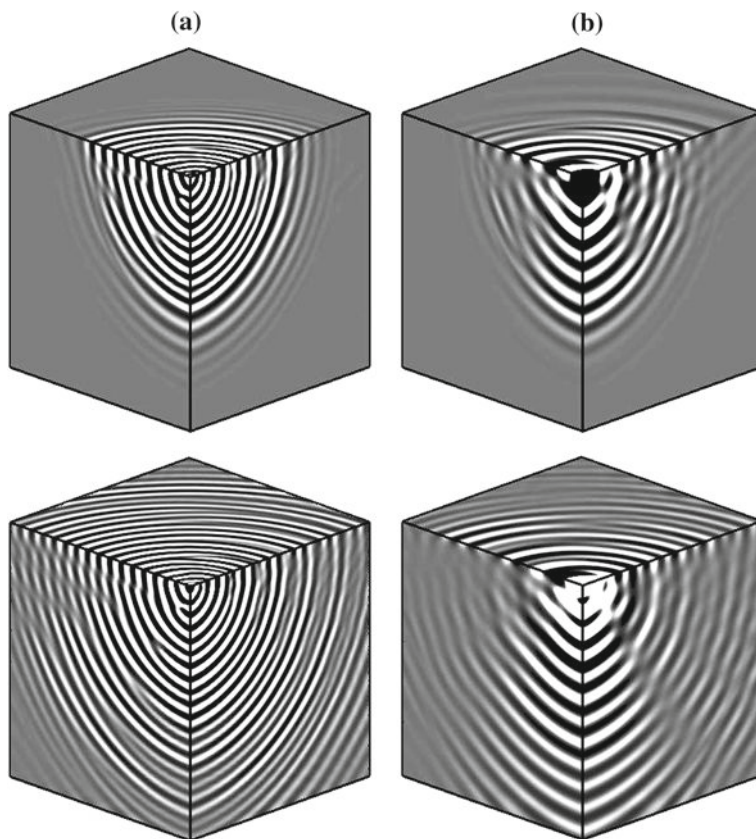


Fig. 10.27 Problem on the periodic action of the concentrated rotational moment (3D case): **a** level surfaces of angular velocity ω_2 for nonresonance frequency, **b** level surfaces of angular velocity ω_2 for resonance frequency; $t = 6.5 \mu\text{s}$ (above), $t = 13 \mu\text{s}$ (below)

motion of particles on the wave fronts. Comparative calculations with different values of scale of the microstructure of a material were performed, in which a direct proportional dependence of the period of natural oscillations from this scale was found. The results of the analysis of the oscillation processes show that the Cosserat medium possesses the eigenfrequency of acoustic resonance, which appears under certain conditions of perturbation and depends only on the inertial properties of the microstructure particles and the elasticity parameters of the material.

The numerical solution of the problems considered in this section cannot be obtained with sufficient accuracy using modern personal computer due to the limitations of the random access memory and processor performance. The results can be used as a methodological basis for designing experiments for determining the phenomenological parameters of the constitutive equations of the Cosserat continuum, which is a problem that still defies solution. Using the obtained solutions and the

integral representation of the solution in terms of Green's function, one can model arbitrary distributed forces applied at the surface of a homogeneous elastic half-space.

10.3 Generalization of the Model

Some variants of the constitutive equations of a granular material with independent rotation of particles were considered in [20, 49]. In cohesive granular materials under some special types of loading, such as vibrations, the independent translational motion of particles relative to the binder should be considered along with the rotational motion. In this case the resonant excitation of a material due to the translational degrees of freedom can be generated. Fundamentals of the theory describing the relative translational motion of particles in structurally inhomogeneous media are developed in [34].

Now we construct the model of a granular material with different tensile and compressive properties where rotational degrees of freedom of particles are taken into account [39]. To this end, in the space of tensor pairs $(\mathbf{A}, \mathbf{M})$ we consider the cone $\mathbf{C}$ with vertex at the origin which involves all possible loosened states. The cone $\mathbf{K}$ in the space of tensor pairs $(\boldsymbol{\sigma}, \mathbf{m})$, conjugate to $\mathbf{C}$, is a cone of admissible stresses in an ideal material with absolutely rigid particles. The constitutive relationships for such a material are written in the subdifferential form

$$(\boldsymbol{\sigma}, \mathbf{m}) \in \partial\Phi(\mathbf{A}, \mathbf{M}), \quad (\mathbf{A}, \mathbf{M}) \in \partial\Psi(\boldsymbol{\sigma}, \mathbf{m}). \quad (10.23)$$

Here $\Phi(\mathbf{A}, \mathbf{M}) = \delta_{\mathbf{C}}(\mathbf{A}, \mathbf{M})$ and $\Psi(\boldsymbol{\sigma}, \mathbf{m}) = \delta_{\mathbf{K}}(\boldsymbol{\sigma}, \mathbf{m})$ (potentials of stresses and strains) are the indicator functions of $\mathbf{C}$ and $\mathbf{K}$.

The form of constitutive relationships equivalent to (10.23) for a material with rigid particles is reduced to the system of inclusions with the complementing condition

$$(\boldsymbol{\sigma}, \mathbf{m}) \in \mathbf{K}, \quad (\mathbf{A}, \mathbf{M}) \in \mathbf{C}, \quad \boldsymbol{\sigma}^* : \mathbf{A} + \mathbf{m}^* : \mathbf{M} = 0. \quad (10.24)$$

By a definition of dual cones

$$\begin{aligned} \boldsymbol{\sigma}^* : \tilde{\mathbf{A}} + \mathbf{m}^* : \tilde{\mathbf{M}} &\leq 0, \quad (\tilde{\mathbf{A}}, \tilde{\mathbf{M}}) \in \mathbf{C}, \\ \tilde{\boldsymbol{\sigma}}^* : \mathbf{A} + \tilde{\mathbf{m}}^* : \mathbf{M} &\leq 0, \quad (\tilde{\boldsymbol{\sigma}}^*, \tilde{\mathbf{m}}^*) \in \mathbf{K}, \end{aligned}$$

hence, variational inequalities involving arbitrary admissible variations of generalized stresses and strains hold as well:

$$\begin{aligned} \boldsymbol{\sigma}^* : (\tilde{\mathbf{A}} - \mathbf{A}) + \mathbf{m}^* : (\tilde{\mathbf{M}} - \mathbf{M}) &\leq 0, \\ (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma})^* : \mathbf{A} + (\tilde{\mathbf{m}} - \mathbf{m})^* : \mathbf{M} &\leq 0. \end{aligned} \quad (10.25)$$

To take into account elastic and plastic strains of particles, we construct the model according to the rheological scheme shown in Fig. 6.1. We assume that $\mathbf{a}$ and $\mathbf{b}$ are the

elastic compliance tensors of rank four with the help of which the linear constitutive equations (10.1) of the Cosserat continuum are written in the form resolved with respect to strains and torsions. We also assume that $\mathbf{s}$ and $\mathbf{n}$ are the conditional stress and conditional couple stress tensors, respectively, such that $\mathbf{a} : \mathbf{s}^* = \mathbf{\Lambda} - \mathbf{\Lambda}^p$, $\mathbf{b} : \mathbf{n}^* = \mathbf{M} - \mathbf{M}^p$ (the superscript “ p ” is related to a plastic hinge). As before, elastic moduli $\mathbf{a}/(1 - \varsigma)$ and $\mathbf{b}/(1 - \varsigma)$ are associated with a spring involved in the scheme in series with a rigid contact, and moduli $\mathbf{a}/\varsigma$ and $\mathbf{b}/\varsigma$ are associated with a spring parallel to a rigid contact, $\varsigma \in (0, 1]$ is a parameter of regularization of the model. According to the rheological scheme of a material,

$$\frac{\mathbf{a} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}^c)^*}{\varsigma} = \frac{\mathbf{a} : (\boldsymbol{\sigma}^c)^*}{1 - \varsigma} + \mathbf{\Lambda}^c, \quad \frac{\mathbf{b} : (\mathbf{m} - \mathbf{m}^c)^*}{\varsigma} = \frac{\mathbf{b} : (\mathbf{m}^c)^*}{1 - \varsigma} + \mathbf{M}^c.$$

From here one can find the generalized strains $(\mathbf{\Lambda}^c, \mathbf{M}^c)$ of a rigid contact, the substitution of which into the second inequality of (10.25) gives

$$(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^c)^* : \mathbf{a} : (\boldsymbol{\sigma}^c - (1 - \varsigma)\boldsymbol{\sigma})^* + (\tilde{\mathbf{m}} - \mathbf{m}^c)^* : \mathbf{b} : (\mathbf{m}^c - (1 - \varsigma)\mathbf{m})^* \geq 0.$$

The tensors $\mathbf{a}$ and $\mathbf{b}$ are symmetric with respect to index pairs, in addition, they are positive definite. Hence, the obtained variational inequality defines a projection onto the cone $\mathbf{K}$ with respect to the Euclidean norm $|\boldsymbol{\sigma}, \mathbf{m}|^2 = |\boldsymbol{\sigma}|_a^2 + |\mathbf{m}|_b^2$. Since a projection onto a cone is positively homogeneous, we have

$$(\boldsymbol{\sigma}^c, \mathbf{m}^c) = ((1 - \varsigma)(\boldsymbol{\sigma}, \mathbf{m}))^\pi = (1 - \varsigma)(\boldsymbol{\sigma}, \mathbf{m})^\pi.$$

Thus,

$$\begin{aligned} \mathbf{a} : \mathbf{s}^* = \mathbf{\Lambda} - \mathbf{\Lambda}^p &= \frac{\mathbf{a}}{\varsigma} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}^c)^* = \frac{1}{\varsigma} \mathbf{a} : \boldsymbol{\sigma}^* - \frac{1 - \varsigma}{\varsigma} \mathbf{a} : (\boldsymbol{\sigma}^\pi)^*, \\ \mathbf{b} : \mathbf{n}^* = \mathbf{M} - \mathbf{M}^p &= \frac{\mathbf{b}}{\varsigma} : (\mathbf{m} - \mathbf{m}^c)^* = \frac{1}{\varsigma} \mathbf{b} : \mathbf{m}^* - \frac{1 - \varsigma}{\varsigma} \mathbf{b} : (\mathbf{m}^\pi)^*, \end{aligned}$$

hence,

$$(\mathbf{s}, \mathbf{n}) = \frac{1}{\varsigma} (\boldsymbol{\sigma}, \mathbf{m}) - \frac{1 - \varsigma}{\varsigma} (\boldsymbol{\sigma}, \mathbf{m})^\pi. \quad (10.26)$$

Equation (10.26) is a simple generalization of Eq. (6.6), obtained in Sect. 6.1, to the case of the Cosserat medium. Repeating the above considerations, we can prove that

$$(\boldsymbol{\sigma}, \mathbf{m}) = \varsigma (\mathbf{s}, \mathbf{n}) + (1 - \varsigma) (\mathbf{s}, \mathbf{n})^\pi.$$

Assume that $\mathbf{F}$ is a convex subset of the space of generalized stresses $(\boldsymbol{\sigma}, \mathbf{m})$ and its boundary defines the yield surface of a material. We formulate the law of plastic flow as the principle of maximum of velocity of the energy dissipation

$$(\tilde{\sigma} - \sigma)^* : \dot{\mathbf{A}}^p + (\tilde{\mathbf{m}} - \mathbf{m})^* : \dot{\mathbf{M}}^p \leq 0, \quad (\sigma, \mathbf{m}), (\tilde{\sigma}, \tilde{\mathbf{m}}) \in \mathbf{F}.$$

Eliminating $\mathbf{A}^p$ and $\mathbf{M}^p$, we have

$$(\tilde{\sigma} - \sigma)^* : (\mathbf{a} : \dot{\mathbf{s}}^* - \dot{\mathbf{A}}) + (\tilde{\mathbf{m}} - \mathbf{m})^* : (\mathbf{b} : \dot{\mathbf{n}}^* - \dot{\mathbf{M}}) \geq 0. \quad (10.27)$$

Now assume that $\mathbf{U}$ is the vector-function consisting of nonzero components of velocity vectors and stress tensors in the following order: $(\mathbf{v}, \mathbf{s}, \boldsymbol{\omega}, \mathbf{n})$. Assume also that $\mathbf{V}$ is the vector-function obtained from $\mathbf{U}$ by replacement of tensors $\mathbf{s}$ and $\mathbf{n}$ by $\boldsymbol{\sigma}$ and $\mathbf{m}$, respectively. Using the relationship (10.27) instead of the linear constitutive equations in the system (10.1), we reduce the model to the variational inequality (6.8), introduced in Sect. 6.1 to describe the behaviour of a momentless elastic-plastic granular material. This inequality must be combined with nonlinear equations of the form (6.9) which follow from (10.26) and relate the vector-functions $\mathbf{U}$ and $\mathbf{V}$. In general terms, the constructed model is similar to that of a momentless elastic-plastic granular material, however, it is of a greater dimension. In the case of the plane strain state the inequality (6.8) for the Cosserat medium contains 12 unknown functions instead of 6, and in spatial problems it contains 24 functions instead of 9. The matrix-coefficients of the differential operator are of block-diagonal structure as before. The matrix $\mathbf{A}$ involves density of a material, moment of inertia, and coefficients of elastic compliance tensors as factors of time derivatives. This matrix is symmetric and positive definite. The matrices $\mathbf{B}^i$ and $\mathbf{Q}$ consist of zeros and ones, besides, $\mathbf{B}^i$ are symmetric and $\mathbf{Q}$ is antisymmetric. Thus, the differential operator in the inequality is hyperbolic in the sense of Friedrichs. In the absence of restrictions, when the cone $\mathbf{C}$ consists of zero only and the cone $\mathbf{K}$ and the set $\mathbf{F}$ both coincide with the space of stresses, the variational inequality (6.8) is reduced to the system of equations (10.4) of spatial deformation of the Cosserat elastic medium.

Let us consider, for example, how looks like the system of constitutive equations in expanded form for 3D case. The vector-function $\mathbf{U}$ is as follows:

$$\mathbf{U} = (v_1, v_2, v_3, s_{11}, s_{22}, s_{33}, s_{23}, s_{32}, s_{31}, s_{13}, s_{12}, s_{21}, \\ \omega_1, \omega_2, \omega_3, n_{11}, n_{22}, n_{33}, n_{23}, n_{32}, n_{31}, n_{13}, n_{12}, n_{21}).$$

The vector $\mathbf{V}$ is obtained from $\mathbf{U}$ by the replacement of the components of tensors $\mathbf{s}$ and $\mathbf{n}$ by those of $\boldsymbol{\sigma}$ and $\mathbf{m}$, respectively. The matrix-coefficients $\mathbf{A}$, $\mathbf{B}^i$, and $\mathbf{Q}$ of the differential operator in the variational inequality (6.8) for the Cosserat continuum are constructed from the expanded form of the system of equations in a Cartesian coordinate system:

$$\begin{aligned}
\rho \dot{v}_1 &= \sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} + \rho g_1, \\
\rho \dot{v}_2 &= \sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} + \rho g_2, \\
\rho \dot{v}_3 &= \sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} + \rho g_3, \\
a_1 \dot{s}_{11} + a_2 \dot{s}_{22} + a_3 \dot{s}_{33} &= v_{1,1} - \dot{\Lambda}_{11}^p, \\
a_2 \dot{s}_{11} + a_1 \dot{s}_{22} + a_3 \dot{s}_{33} &= v_{2,2} - \dot{\Lambda}_{22}^p, \\
a_2 \dot{s}_{11} + a_2 \dot{s}_{22} + a_1 \dot{s}_{33} &= v_{3,3} - \dot{\Lambda}_{33}^p, \\
a_3 \dot{s}_{23} + a_4 \dot{s}_{32} &= v_{3,2} - \omega_1 - \dot{\Lambda}_{23}^p, \\
a_4 \dot{s}_{23} + a_3 \dot{s}_{32} &= v_{2,3} + \omega_1 - \dot{\Lambda}_{32}^p, \\
a_3 \dot{s}_{31} + a_4 \dot{s}_{13} &= v_{1,3} - \omega_2 - \dot{\Lambda}_{31}^p, \\
a_4 \dot{s}_{31} + a_3 \dot{s}_{13} &= v_{3,1} + \omega_2 - \dot{\Lambda}_{13}^p, \\
a_3 \dot{s}_{12} + a_4 \dot{s}_{21} &= v_{2,1} - \omega_3 - \dot{\Lambda}_{12}^p, \\
a_4 \dot{s}_{12} + a_3 \dot{s}_{21} &= v_{1,2} + \omega_3 - \dot{\Lambda}_{21}^p, \\
j \dot{\omega}_1 &= m_{11,1} + m_{21,2} + m_{31,3} + \sigma_{23} - \sigma_{32} + j q_1, \\
j \dot{\omega}_2 &= m_{12,1} + m_{22,2} + m_{32,3} + \sigma_{31} - \sigma_{13} + j q_2, \\
j \dot{\omega}_3 &= m_{13,1} + m_{23,2} + m_{33,3} + \sigma_{12} - \sigma_{21} + j q_3, \\
b_1 \dot{n}_{11} + b_2 \dot{n}_{22} + b_3 \dot{n}_{33} &= \omega_{1,1} - \dot{M}_{11}^p, \\
b_2 \dot{n}_{11} + b_1 \dot{n}_{22} + b_3 \dot{n}_{33} &= \omega_{2,2} - \dot{M}_{22}^p, \\
b_2 \dot{n}_{11} + b_2 \dot{n}_{22} + b_1 \dot{n}_{33} &= \omega_{3,3} - \dot{M}_{33}^p, \\
b_3 \dot{n}_{23} + b_4 \dot{n}_{32} &= \omega_{3,2} - \dot{M}_{23}^p, \\
b_4 \dot{n}_{23} + b_3 \dot{n}_{32} &= \omega_{2,3} - \dot{M}_{32}^p, \\
b_3 \dot{n}_{31} + b_4 \dot{n}_{13} &= \omega_{1,3} - \dot{M}_{31}^p, \\
b_4 \dot{n}_{31} + b_3 \dot{n}_{13} &= \omega_{3,1} - \dot{M}_{13}^p, \\
b_3 \dot{n}_{12} + b_4 \dot{n}_{21} &= \omega_{2,1} - \dot{M}_{12}^p, \\
b_4 \dot{n}_{12} + b_3 \dot{n}_{21} &= \omega_{1,2} - \dot{M}_{21}^p.
\end{aligned}$$

Here the coefficients a_i and b_i are the same as for the system of equations (10.5). The matrix $\mathbf{A}$ is positive definite, if its diagonal blocks, consisting of these coefficients, are positive definite.

As noticed above, up to now the problem on determining phenomenological elasticity parameters for the Cosserat linear continuum has not been solved completely. Introducing into the model additional parameters, similar to the interior friction parameter or the yield point of a material, makes the problem much more complicated. First, the structure of the cones $\mathbf{C}$ and $\mathbf{K}$ and of the set $\mathbf{F}$ in the spaces of non-symmetric tensors of strains and stresses is still unclear. As a possible variant of a solution of this part of the problem we consider an extremely simple construction of the cone $\mathbf{K}$ of admissible stresses constructed according to the principle of the von Mises–Schleicher cone for an usual momentless granular medium.

Taking into account the symmetry of material properties with respect to the rotational degrees of freedom, which are defined by the tensors $\mathbf{\Lambda}^a$ and $\mathbf{M}$, one can find that the section of the cone $\mathbf{C}$ by hyperplane $\mathbf{\Lambda}^s = \mathbf{0}$ is the subspace, i.e. it simultane-

ously contains the points $(\mathbf{A}, \mathbf{M})$ and $(-\mathbf{A}, -\mathbf{M})$. The equation of the cone surface is defined as dilatational equation which describes volumetric expansion of a densely packed granular medium in the process of forming. If a medium is isotropic then this equation is written in terms of invariants of the tensors $\mathbf{A}$ and $\mathbf{M}$, and it must be symmetric relative to the variations of the signs of $\mathbf{A}^a$ and $\mathbf{M}$.

Supposing that dilatational expansion of the volume of a medium element happens only at the expense of shears and rotations of particles, we accept the equation of the boundary of $\mathbf{C}$ in the following form:

$$\mathfrak{x}\gamma_s(\mathbf{A}) + \varpi\gamma_a(\mathbf{A}) = \theta(\mathbf{A}). \quad (10.28)$$

Here $\mathfrak{x}$ and ϖ are the phenomenological parameters of a material, $\theta(\mathbf{A}) = \delta : \mathbf{A}^s$ is the specific volume change, $\gamma_s(\mathbf{A}) = \sqrt{2} \mathbf{A}'^s : \mathbf{A}'^s$ is the shear intensity which is expressed by the symmetric part of the deviator $\mathbf{A}' = \mathbf{A} - \theta(\mathbf{A}) \delta/3$ of strain tensor, $\gamma_a(\mathbf{A}) = \sqrt{-2} \mathbf{A}^a : \mathbf{A}^a$ is the intensity of relative rotations.

If we assume that a material consists of absolutely rigid ball-shaped particles of small radius, then the transition from a loosened state to a dense packing state, corresponding to the vertex of the cone $\mathbf{C}$, can be described only in terms of the symmetric component of the strain tensor $\mathbf{A}$. The antisymmetric part of the tensor, which involves independent rotations of particles in the additive form, may be arbitrary since in the packed state the angles of relative rotations of particles are of no significance. The curvature tensor may be arbitrary as well, hence, for this case $\varpi = 0$.

In compliance with Eq. (10.28),

$$\mathbf{C} = \{(\mathbf{A}, \mathbf{M}) \mid \mathfrak{x}\gamma_s(\mathbf{A}) + \varpi\gamma_a(\mathbf{A}) \leq \theta(\mathbf{A})\}.$$

By the definition of conjugate cone, the cone $\mathbf{K}$ includes various pairs of tensors $(\boldsymbol{\sigma}, \mathbf{m})$ for which the linear function $-\boldsymbol{\sigma}^* : \tilde{\mathbf{A}} - \mathbf{m}^* : \tilde{\mathbf{M}}$ takes the minimal value over the whole $\mathbf{C}$ at the point $\tilde{\mathbf{A}} = \tilde{\mathbf{M}} = \mathbf{0}$, i.e. in the vertex of the cone. Consequently, according to the Kuhn–Tucker theorem (Sect. 3.4), the point $(0, 0) \in \mathbf{C}$ is the point of unconstrained minimum of the Lagrangian

$$L(\mathbf{A}, \mathbf{M}) = -\boldsymbol{\sigma}^* : \mathbf{A} - \mathbf{m}^* : \mathbf{M} + \lambda(\mathfrak{x}\gamma_s(\mathbf{A}) + \varpi\gamma_a(\mathbf{A}) - \theta(\mathbf{A}))$$

with the Lagrange multiplier $\lambda \geq 0$. As the functions $\gamma_s(\mathbf{A})$ and $\gamma_a(\mathbf{A})$ could not be differentiated at zero, then the minimum condition of Lagrangian is necessary to use in subdifferential form: $(0, 0) \in \partial L(0, 0)$. In accordance with the Moreau–Rockafellar theorem about subdifferential of the sum of convex functions

$$\boldsymbol{\sigma} \in \lambda(\mathfrak{x}\partial\gamma_s(\mathbf{0}) + \varpi\partial\gamma_a(\mathbf{0}) - \delta), \quad \mathbf{m} = \mathbf{0}. \quad (10.29)$$

The subdifferentials $\gamma_s(\mathbf{A})$ and $\gamma_a(\mathbf{A})$ at the point $\mathbf{A} = \mathbf{0}$ are easy to calculate directly reasoning from the definition. As a result we have:

$$\begin{aligned}\partial\gamma_s(\mathbf{0}) &= \{\boldsymbol{\sigma} \mid \tau_s(\boldsymbol{\sigma}) \leq 1, \boldsymbol{\sigma}^a = \mathbf{0}, p(\boldsymbol{\sigma}) = 0\}, \\ \partial\gamma_a(\mathbf{0}) &= \{\boldsymbol{\sigma} \mid \tau_a(\boldsymbol{\sigma}) \leq 1, \boldsymbol{\sigma}^s = \mathbf{0}\}.\end{aligned}$$

Here $\tau_s(\boldsymbol{\sigma}) = \sqrt{\boldsymbol{\sigma}'^s : \boldsymbol{\sigma}'^s / 2}$ and $\tau_a(\boldsymbol{\sigma}) = \sqrt{-\boldsymbol{\sigma}^a : \boldsymbol{\sigma}^a / 2}$ are the intensities of tangential stresses, $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p(\boldsymbol{\sigma}) \boldsymbol{\delta}$ is the deviator of the tensor $\boldsymbol{\sigma}$. The expression for $\partial\gamma_a(\mathbf{0})$ issues from the chain of relationships received by the Cauchy–Bunyakovskii inequality:

$$-\boldsymbol{\sigma}^a : \boldsymbol{\Lambda}^a \leq \sqrt{-\frac{\boldsymbol{\sigma}^a : \boldsymbol{\sigma}^a}{2}} \sqrt{-2 \boldsymbol{\Lambda}^a : \boldsymbol{\Lambda}^a} = \tau_a(\boldsymbol{\sigma}) \gamma_a(\boldsymbol{\Lambda}) \leq \gamma_a(\boldsymbol{\Lambda}).$$

The expression for $\partial\gamma_s(\mathbf{0})$ is received by analogy.

For any tensor $\boldsymbol{\sigma}$, which satisfies the inclusion (10.29), the following conditions are fulfilled:

$$\tau_s(\boldsymbol{\sigma}) \leq \varkappa \lambda, \quad \tau_a(\boldsymbol{\sigma}) \leq \varpi \lambda, \quad p(\boldsymbol{\sigma}) = \lambda,$$

therefore the cone, conjugate to $\mathbf{C}$, is

$$\mathbf{K} = \{(\boldsymbol{\sigma}, \mathbf{m}) \mid \tau_s(\boldsymbol{\sigma}) \leq \varkappa p(\boldsymbol{\sigma}), \tau_a(\boldsymbol{\sigma}) \leq \varpi p(\boldsymbol{\sigma}), \mathbf{m} = \mathbf{0}\}.$$

We suppose the relationships (10.25) in expanded form. According to the Kuhn–Tucker theorem, the first of these variational inequalities is a necessary and sufficient condition for unconstrained minimum of the Lagrangian

$$L(\tilde{\mathbf{A}}, \tilde{\mathbf{M}}) = -\boldsymbol{\sigma}^* : \tilde{\mathbf{A}} - \mathbf{m}^* : \tilde{\mathbf{M}} + \lambda(\varkappa \gamma_s(\tilde{\mathbf{A}}) + \varpi \gamma_a(\tilde{\mathbf{A}}) - \theta(\tilde{\mathbf{A}})),$$

in which the Lagrange multiplier $\lambda \geq 0$ satisfies the complementing equation

$$\lambda(\varkappa \gamma_s(\mathbf{A}) + \varpi \gamma_a(\mathbf{A}) - \theta(\mathbf{A})) = 0.$$

Minimum is achieved at the point $\tilde{\mathbf{A}} = \mathbf{A}$, $\tilde{\mathbf{M}} = \mathbf{M}$, therefore the condition of minimum takes the form of $(\mathbf{0}, \mathbf{0}) \in \partial L(\mathbf{A}, \mathbf{M})$:

$$\boldsymbol{\sigma} \in \lambda(\varkappa \partial\gamma_s(\mathbf{A}) + \varpi \partial\gamma_a(\mathbf{A}) - \boldsymbol{\delta}), \quad \mathbf{m} = \mathbf{0}.$$

For $\gamma_s(\mathbf{A}) > 0$ and $\gamma_a(\mathbf{A}) > 0$ the subdifferentials included in the last inclusion can be calculated according to the common differentiation rules. For $\gamma_s(\mathbf{A}) = 0$ and $\gamma_a(\mathbf{A}) = 0$ they are respectively equal to $\partial\gamma_s(\mathbf{0})$ and $\partial\gamma_a(\mathbf{0})$. That is why the constitutive relationships (10.25) are resulted in the following system of conditions:

(1) if $\gamma_s(\mathbf{A}) \gamma_a(\mathbf{A}) > 0$, then

$$\frac{\boldsymbol{\sigma}^s}{p(\boldsymbol{\sigma})} = \frac{2\varkappa}{\gamma_s(\mathbf{A})} \left(\boldsymbol{\Lambda}^s - \frac{1}{3} \theta(\mathbf{A}) \boldsymbol{\delta} \right) - \boldsymbol{\delta}, \quad \frac{\boldsymbol{\sigma}^a}{p(\boldsymbol{\sigma})} = \frac{2\varpi}{\gamma_a(\mathbf{A})} \boldsymbol{\Lambda}^a, \quad \mathbf{m} = \mathbf{0}; \quad (10.30)$$

- (2) if $\gamma_s(\mathbf{A}) = 0$, then the deviator of the symmetric part of the stress tensor can be an arbitrary tensor which satisfies the inequality $\tau_s(\boldsymbol{\sigma}^s) \leq \mathfrak{x} p(\boldsymbol{\sigma})$;
- (3) if $\gamma_a(\mathbf{A}) = 0$, then the antisymmetric part of the stress tensor can be an arbitrary tensor which satisfies the condition $\tau_a(\boldsymbol{\sigma}^a) \leq \varpi p(\boldsymbol{\sigma})$.

In this system the hydrostatic pressure $p(\boldsymbol{\sigma}) = \lambda \geq 0$, with the help of which the spherical part of the stress tensor is calculated, always stays indefinite.

For the conditions (1)–(3) it is possible to use geometric interpretation which is analogous to the interpretation of constitutive relationships of the theory of perfect plasticity with singular yield surface. If the point $(\mathbf{A}, \mathbf{M})$ is inside the cone $\mathbf{C}$, then by virtue of the system of equations (10.30) and the complementing equation the stress tensor and the couple stress tensor are equal to zero. This variant corresponds to the loosened state of a medium. If in the neighborhood of the boundary point $(\mathbf{A}, \mathbf{M})$ the conical surface is smooth, then by virtue of (10.30) the vector $(\boldsymbol{\sigma}, \mathbf{m})$ is directed to outer normal vector. This variant describes limit state of a medium. Finally, if $(\mathbf{A}, \mathbf{M})$ is a singular point of the conical surface, then the vector $(\boldsymbol{\sigma}, \mathbf{m})$ can take any position of the normal fan which is set by the conditions of (2) and (3). Specifically, if $\mathbf{A} = \mathbf{M} = \mathbf{0}$, i.e. if a medium is in the state of density packing, then it can be an arbitrary vector of the conjugate cone $\mathbf{K}$.

The second variational inequality (10.25) allows to get the analogous system of conditions for the definition of tensors, which describe the medium deformation, by means of the stress tensors and the couple stress tensors. By virtue of this inequality the linear function $-\tilde{\boldsymbol{\sigma}}^* : \mathbf{A} - \tilde{\mathbf{m}}^* : \mathbf{M}$ takes the minimum value on the cone $\mathbf{K}$ at the point $(\boldsymbol{\sigma}, \mathbf{m}) \in \mathbf{K}$. Taking into account that $\mathbf{m} = \mathbf{0}$ everywhere in $\mathbf{K}$, the corresponding Lagrangian function can be written in the following form:

$$L(\tilde{\boldsymbol{\sigma}}) = -\tilde{\boldsymbol{\sigma}}^* : \mathbf{A} + \lambda_s (\tau_s(\tilde{\boldsymbol{\sigma}}) - \mathfrak{x} p(\tilde{\boldsymbol{\sigma}})) + \lambda_a (\tau_a(\tilde{\boldsymbol{\sigma}}) - \varpi p(\tilde{\boldsymbol{\sigma}})),$$

where λ_s and λ_a are the nonnegative multipliers which satisfy the equations

$$\lambda_s (\tau_s(\boldsymbol{\sigma}) - \mathfrak{x} p(\boldsymbol{\sigma})) = 0, \quad \lambda_a (\tau_a(\boldsymbol{\sigma}) - \varpi p(\boldsymbol{\sigma})) = 0.$$

The necessary and sufficient condition for the minimum of the function $L(\tilde{\boldsymbol{\sigma}})$ in subdifferential form $\mathbf{0} \in \partial L(\boldsymbol{\sigma})$ after using the Moreau–Rockafellar theorem is reduced to the following inclusion:

$$\mathbf{A} \in \lambda_s \left(\partial \tau_s(\boldsymbol{\sigma}) + \frac{\mathfrak{x} \boldsymbol{\delta}}{3} \right) + \lambda_a \left(\partial \tau_a(\boldsymbol{\sigma}) + \frac{\varpi \boldsymbol{\delta}}{3} \right).$$

For $\tau_s(\boldsymbol{\sigma}) = 0$ and $\tau_a(\boldsymbol{\sigma}) = 0$ the subdifferentials are calculated by the following formulae:

$$\partial \tau_s(\boldsymbol{\sigma}) = \{ \mathbf{A} \mid \gamma_s(\mathbf{A}) \leq 1, \mathbf{A}^a = \mathbf{0}, \theta(\mathbf{A}) = 0 \},$$

$$\partial \tau_a(\boldsymbol{\sigma}) = \{ \mathbf{A} \mid \gamma_a(\mathbf{A}) \leq 1, \mathbf{A}^s = \mathbf{0} \}.$$

It can be shown with the help of the Cauchy–Bunyakovskii inequality. In the case of strictly positive values the functions $\tau_s(\sigma)$ and $\tau_a(\sigma)$ are differentiable. Thus, the constitutive relationships (10.25) are equivalent to the following system of conditions:

(1) if $\tau_s(\sigma) \tau_a(\sigma) > 0$, then

$$\mathbf{A}'^s = \frac{\lambda_s}{2 \tau_s(\sigma)} \sigma'^s, \quad \mathbf{A}'^a = \frac{\lambda_a}{2 \tau_a(\sigma)} \sigma'^a, \quad \theta(\mathbf{A}) = \lambda_s \varkappa + \lambda_a \varpi; \quad (10.31)$$

(2) if $\tau_s(\sigma) = 0$, then the deviator of the symmetric part of the tensor $\mathbf{A}$ can be an arbitrary tensor which satisfies the inequality $\gamma_s(\mathbf{A}'^s) \leq \lambda_s$;

(3) if $\tau_a(\sigma) = 0$, then the antisymmetric part of $\mathbf{A}$ can be an arbitrary tensor which satisfies the condition $\gamma_a(\mathbf{A}'^a) \leq \lambda_a$.

Under these conditions the multipliers $\lambda_s \geq 0$ and $\lambda_a \geq 0$ remain indefinite, and the tensor $\mathbf{M}$ of curvature and torsion can be an arbitrary nonsymmetric tensor.

The conditions (1)–(3) describe different states of a granular medium. For example, in the loosened state, when $\sigma = \mathbf{0}$, the tensor $\mathbf{A}$ of strains and relative rotations of the particles satisfies the inequalities $\gamma_s(\mathbf{A}'^s) \leq \lambda_s$, $\gamma_a(\mathbf{A}'^a) \leq \lambda_a$, from which follows that

$$\varkappa \gamma_s(\mathbf{A}) + \varpi \gamma_a(\mathbf{A}) \leq \lambda_s \varkappa + \lambda_a \varpi = \theta(\mathbf{A}).$$

In this case a pair of $(\mathbf{A}, \mathbf{M})$ can be an arbitrary element of the cone $\mathbf{C}$.

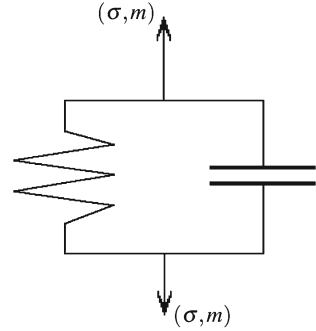
In the packed state of a medium, when σ is an inner point of section of the cone $\mathbf{K}$ by hyperplane $\mathbf{m} = \mathbf{0}$, the Lagrange multipliers λ_s and λ_a by virtue of complementing equations become equal to zero, therefore the tensor $\mathbf{A}$ is also equal to zero. The equality to zero of this tensor means that there are no deformations and relative rotations of the particles in a material. At the same time the tensor $\mathbf{M}$ can be arbitrary: the packed state is possible after the compression of a preliminary loosened medium, therefore the arbitrary curvature and torsion of elements are assumed.

Such conditions and equations describe the behaviour of an ideal material, which is absolutely rigid in compression, nonresistant to deformation and relative rotation of particles in the loosening state. For more complicated media, that show elastic, viscous and plastic properties, the constitutive relationships can be obtained with the help of a rheological method. As usual, firstly, the rheological scheme, imitating uniaxial tension-compression, must be constructed in which along with elastic springs, viscous dampers and plastic hinges a rigid contact takes part. As a simple example let us construct the model of a material, which rheological scheme is parallel connection of an elastic spring and a rigid contact (Fig. 10.28).

Such a material is characterized by minimal number of phenomenological parameters: by six parameters of elasticity of the Cosserat continuum $k, \mu, \alpha, \kappa, \eta, \beta$, and two dilatancy parameters $\varkappa$ and ϖ . At the parallel connection the strains of rheological elements are the same and the stresses are added, then

$$\sigma = \sigma^e + \sigma^c, \quad \mathbf{m} = \mathbf{m}^e + \mathbf{m}^c, \quad (10.32)$$

Fig. 10.28 Rheological scheme of a couple stress medium with elastic particles



where σ^e and m^e are the elastic stress tensors satisfying the constitutive equations (10.1), and σ^c and m^c are the stress tensors in a rigid contact which are connected with $\mathbf{A}$ and $\mathbf{M}$ by the (10.25).

Derivable relationships can be used for the description of the deformation process of a medium which consists of very large number of absolutely rigid particles connected between themselves by compliant elastic springs. In the case of inhomogeneous stress state there can be stagnant zones which move as absolutely rigid bodies, and loosening zones in which elastic deformations and relative rotations of the particles are possible.

In the framework of a medium with rheological scheme, shown in Fig. 10.28, let us analyze the state of shear in the plane $x_1 x_2$ of a weightless medium body which is under the influence of given pressure $p_0 > 0$ in the x_3 direction in the absence of external forces and moments. This state is realized in the device of uniform shear, i.e. the experimental facility for the analysis of the dilatancy of granular materials. Linear with respect to spatial variables field of velocities in the uniform shear is set by the following equations:

$$v_1 = 0, \quad v_2 = \dot{\chi} x_1, \quad v_3 = \dot{\Delta} x_3, \quad \omega_3 = \dot{\varphi}.$$

Here $\chi(t)$ is the angle of shear, $\Delta(t)$ is the axial strain, $\varphi(t)$ is the rotation angle of particles. The tensor $\mathbf{M}$ is identically zero, the tensor $\mathbf{A}$ is calculated according to the following formula

$$\mathbf{A} = \begin{pmatrix} 0 & \chi - \varphi & 0 \\ \varphi & 0 & 0 \\ 0 & 0 & \Delta \end{pmatrix}.$$

The invariants of this tensor are equal to

$$\gamma_s(\mathbf{A}) = \sqrt{\frac{4}{3} \Delta^2 + \chi^2}, \quad \gamma_a(\mathbf{A}) = |\chi - 2\varphi|, \quad \theta(\mathbf{A}) = \Delta.$$

Firstly, we consider a medium with $\varpi = 0$, in which the relative rotation of the particles does not influence on the dilatancy change of a volume. From Eq. (10.28) it follows that the volume is linearly dependent on the shear angle in accordance with the formula (9.9). This dependence makes sense only for the sufficiently small angles $\chi \ll 1$ in the limits of justifiability of the geometrically linear theory and only if $\mathfrak{x} < \sqrt{3}/2$. If the value of this parameter is greater, the shear is impossible. The jamming of particles occurs because of the constrained conditions do not ensure the volume expansion which is sufficient for the dilatancy process. For possible values of $\mathfrak{x}$ the tensor components which are different from zero by virtue of the constitutive equations (10.1) and (10.30) are equal to

$$\begin{aligned}\sigma_{11} &= \sigma_{22} = \left(k - \frac{2\mu}{3}\right)\Delta - \lambda\left(1 + \frac{2\mathfrak{x}^2}{3}\right), \\ \sigma_{33} &= \left(k + \frac{4\mu}{3}\right)\Delta - \lambda\left(1 - \frac{4\mathfrak{x}^2}{3}\right), \\ \sigma_{12} &= (\mu + \alpha)\chi - 2\alpha\varphi + \lambda\sqrt{1 - \frac{4\mathfrak{x}^2}{3}}, \\ \sigma_{21} &= (\mu - \alpha)\chi + 2\alpha\varphi + \lambda\sqrt{1 - \frac{4\mathfrak{x}^2}{3}}.\end{aligned}$$

The Lagrange multiplier $\lambda > 0$ is defined by the condition $\sigma_{33} = -p_0$:

$$\lambda = \frac{(k + 4\mu/3)\Delta + p_0}{1 - 4\mathfrak{x}^2/3}.$$

Under the constant shear rate $\dot{\chi}$ the equations of translational motion, which are included in the system (10.1), are fulfilled automatically. The couple stresses are equal to zero, therefore the equation of rotary motion is reduced to the following form:

$$j\ddot{\varphi} = \sigma_{12} - \sigma_{21} = -4\alpha\varphi + 2\alpha\chi. \quad (10.33)$$

The general solution of this equation is described by harmonic oscillations of angular velocity of the particles with the period $T = \pi\sqrt{j/\alpha}$. The particular solution which satisfies the initial conditions $\varphi(0) = \dot{\varphi}(0) = 0$ is

$$\varphi(t) = \frac{\dot{\chi}}{2} \left(t - \frac{T}{2\pi} \sin \frac{2\pi t}{T} \right).$$

The analysis of this solution shows that neglecting the rotary inertia forces in (10.1) is impossible even in the case of slow shear of a medium.

If the shear rate is changeable then the equations of translational motion are fulfilled approximately in two special cases: if there is quasistatic process, when $\dot{\chi}$ changes slowly enough, and if the length of a body is small in the x_3 direction. When the dependence $\chi(t)$ is set by harmonic function with the period T , in compli-

ance with Eq. (10.33) the resonance occurs. Thus, a medium has its own resonance frequency which is independent of the size of a body.

For a medium with two dilatancy parameters of ($\varpi \neq 0$) it is impossible to construct the exact solution of the problem on uniform shear, even if shear rate is constant. The equations of translational motion are fulfilled only approximately in the same cases. The dilatancy equation (10.28) allows to express the following:

$$\Delta = \frac{\varpi |\chi - 2\varphi| + \mathfrak{x} \sqrt{4\varpi^2 (\chi - 2\varphi)^2/3 + (1 - 4\mathfrak{x}^2/3) \chi^2}}{1 - 4\mathfrak{x}^2/3}. \quad (10.34)$$

The formulae for calculation of stresses became more complicated, in particular

$$\begin{aligned} \sigma_{11} = \sigma_{22} &= \left(k - \frac{2\mu}{3}\right) \Delta - \lambda \left(1 + \frac{2}{3} \frac{\mathfrak{x} \Delta}{\sqrt{4\mathfrak{x}^2/3 + \chi^2}}\right), \\ \sigma_{12} &= (\mu + \alpha) \chi - 2\alpha \varphi + \lambda \varpi \operatorname{sgn}(\chi - 2\varphi), \\ \sigma_{21} &= (\mu - \alpha) \chi + 2\alpha \varphi - \lambda \varpi \operatorname{sgn}(\chi - 2\varphi). \end{aligned}$$

The differential equation of rotational motion

$$j\ddot{\varphi} = -4\alpha \varphi + 2\alpha \chi + 2\lambda \varpi \operatorname{sgn}(\chi - 2\varphi)$$

becomes nonlinear as the Lagrange multiplier

$$\lambda = \frac{(k + 4\mu/3) \Delta + p_0}{1 - 4\mathfrak{x} \Delta / \sqrt{3} (4\Delta^2 + 3\chi^2)} > 0,$$

which is determined from the condition $\sigma_{33} = -p_0$, depends on Δ and therefore on φ . It refers to the equations with discontinuous right side [14]. It is possible to show that the sufficient conditions of existence of a solution of the Cauchy problem are fulfilled for this equation.

For constant shear rate the equation of rotational motion was solved numerically in dimensionless form

$$\psi'' = -\psi - \lambda \left(\frac{\varpi}{\alpha}\right) \operatorname{sgn} \psi, \quad \psi = \chi - 2\varphi, \quad \psi(0) = 0, \quad \psi'(0) = \chi',$$

where prime means the dimensionless time derivative onto $t' = 2\pi t/T$, with the help of explicit scheme of the second order of accuracy. The graphs in Fig. 10.29a demonstrate how the volume of a medium depends on the shear angle for $\varpi = 0.25$, the graphs in Fig. 10.29b correspond to the case of $\varpi = 0.5$. Upper curves are obtained by $\mathfrak{x} = 0.5$, middle—by $\mathfrak{x} = 0.25$, and lower—by $\mathfrak{x} = 0$. Other parameters of the equation are the following: $(k + 4\mu/3)/\alpha = 144$, $p_0 = 0$, $\chi' = 0.01$.

In Fig. 10.30 the graphs which characterize the dependence of relative angle of particles rotation on time for the same values of parameters are presented. It is easy to find the compliance between Figs. 10.29 and 10.30: the curves with smaller number

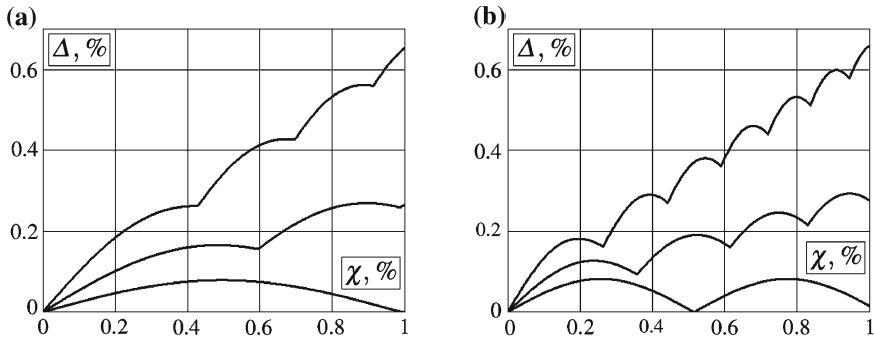


Fig. 10.29 The diagrams $\Delta(\chi)$: **a** $\omega = 0.25$, **b** $\omega = 0.5$

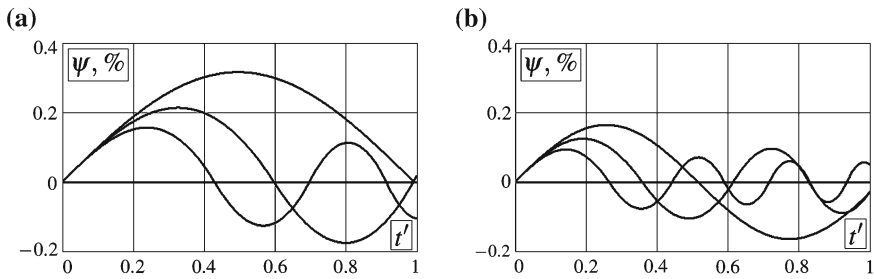


Fig. 10.30 The diagrams $\psi(t')$: **a** $\omega = 0.25$, **b** $\omega = 0.5$

of waves on each graph relate to smaller value of ω , and with larger number of waves relate to larger value of the ω . As indicated above, in the case of $\omega = 0$ the dependence $\Delta(\chi)$ is linear, and the dimensionless oscillation period of rotation angle is equal to 2π . Hence, accounting the relative rotation of particles in the dilatancy equation (10.28) leads to multiple decreasing the period of oscillations.

A series of computations for different values of the model parameters showed that in a medium, where the rotational motion of particles has influence on the volume changing, the dilatancy curve is a wavy line. Characteristic length of a wave on this line is proportional to the oscillation period of the particles in rotational motion, which in the compressed state is essentially dependent on the dilatancy parameters ω , ω and significantly differs from the oscillation period in a loose medium.

10.4 Finite Strains of a Medium With Rotating Particles

The purpose of this and next sections is to reduce the equations of elastic deformation of a granular material under finite strains to the self-consistent system of conservation laws [17, 19]. Such form of equations guarantees the thermodynamic correctness of the model and allows to write it as a symmetric system, provided with a simple proof

of the uniqueness and continuous dependence of the solution of the Cauchy problem and the boundary-value problems with dissipative boundary conditions on initial data. The Cosserat model is generalized to the case of finite strains, for example, in [3, 25].

The translational motion of particles of an elastic medium with microstructure under finite strains is described according to usual form $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$ which connects Lagrangian and Eulerian vectors of centers of mass in each fixed moment of time. Let $\mathbf{v} = \dot{\mathbf{x}}$ be the vector of linear velocity, and $\mathbf{q}(\boldsymbol{\xi}, t)$ be the orthogonal tensor of rotational motion of a particle

$$\mathbf{q} \cdot \mathbf{q}^* = \delta, \quad \det \mathbf{q} = +1, \quad \dot{\mathbf{q}} \cdot \mathbf{q}^* + \mathbf{q} \cdot \dot{\mathbf{q}}^* = \mathbf{0},$$

so that $\boldsymbol{\omega} = \dot{\mathbf{q}} \cdot \mathbf{q}^*$ is the tensor of angular velocity.

As the measure of deformation of infinitely small element of a medium let us take the tensor $\boldsymbol{\Lambda} = \nabla_{\boldsymbol{\xi}} \mathbf{x} \cdot \mathbf{q}$ which has the following property: by motion of a medium as a rigid body when the tensor of distortion $\mathbf{x}_{\boldsymbol{\xi}} = (\nabla_{\boldsymbol{\xi}} \mathbf{x})^*$ coincides with the tensor $\mathbf{q}$ of particles rotation, it is equal to δ , that corresponds to the undeformed state [44]. In addition, this tensor satisfies the following equation

$$\dot{\boldsymbol{\Lambda}} \cdot \mathbf{q}^* = \nabla_{\boldsymbol{\xi}} \mathbf{v} + \nabla_{\boldsymbol{\xi}} \mathbf{x} \cdot \boldsymbol{\omega}, \quad (10.35)$$

linear approximation of which exactly corresponds to kinematic equation for the strain rate tensor in the geometrically linear Cosserat model. Indeed, for small strains and rotations of the particles

$$\begin{aligned} \mathbf{q} &\approx \delta + \Delta t \boldsymbol{\omega}, \quad \nabla_{\boldsymbol{\xi}} \mathbf{x} \approx \delta + \Delta t \nabla_{\boldsymbol{\xi}} \mathbf{v}, \\ \boldsymbol{\Lambda} &\approx (\delta + \Delta t \nabla_{\boldsymbol{\xi}} \mathbf{v})(\delta + \Delta t \boldsymbol{\omega}) \approx \delta + \Delta t (\nabla_{\boldsymbol{\xi}} \mathbf{v} + \boldsymbol{\omega}). \end{aligned}$$

Therefore, for $\boldsymbol{\Lambda}$ the approximate equation $\dot{\boldsymbol{\Lambda}} = \nabla_{\boldsymbol{\xi}} \mathbf{v} + \boldsymbol{\omega}$ is valid which exactly coincides with the equation in (10.1).

Besides, it can be shown that $\boldsymbol{\Lambda}$ is the invariant tensor which does not change by the rotation of current configuration. Actually, if $\mathbf{o}$ is the orthogonal transformation of rotation, then

$$d\mathbf{x}' = \mathbf{o} \cdot d\mathbf{x} = \mathbf{o} \cdot \mathbf{x}_{\boldsymbol{\xi}} \cdot d\boldsymbol{\xi} = \mathbf{o} \cdot \mathbf{q} \cdot \boldsymbol{\Lambda}^* \cdot d\boldsymbol{\xi},$$

and, thus, $\mathbf{x}'_{\boldsymbol{\xi}} = \mathbf{q}' \cdot (\boldsymbol{\Lambda}')^*$, hence, $\boldsymbol{\Lambda}' = \nabla_{\boldsymbol{\xi}} \mathbf{x}' \cdot \mathbf{q}'$, where $\mathbf{q}' = \mathbf{o} \cdot \mathbf{q}$, $\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}$.

Direct examination shows that the tensor $\mathbf{q} \cdot \nabla_{\boldsymbol{\xi}} \mathbf{x}$, in which the order of multipliers is changed, is not the invariant tensor and it is unsuitable for the description of deformation of a medium by Lagrangian coordinates.

Let $\mathbf{x}_{\boldsymbol{\xi}} = \mathbf{q}_e \cdot \mathbf{d}$ be the polar decomposition of the distortion tensor in product of the orthogonal $\mathbf{q}_e$ and symmetric $\mathbf{d}$ tensors. By construction, the tensor $\boldsymbol{\Lambda} = \mathbf{d} \cdot \mathbf{q}_r$ takes into account both intrinsic deformation of a medium, which is described by symmetric part of this decomposition, and relative rotation of the particles which is

characterized by the tensor $\mathbf{q}_r = \mathbf{q}_e^* \cdot \mathbf{q}$, and the tensor of particles rotation $\mathbf{q} = \mathbf{q}_e \cdot \mathbf{q}_r$ is represented in the form of superposition of relative and transfer rotations. If the particle in the natural state of a medium returns to initial position after complete revolution around fixed axis, then the tensor $\mathbf{q}$ is equal to the unit tensor. Hence, by this description the complete revolution of particle does not lead to changing of strain state which is typical, for example, for micropolar media being the large ensembles of magnetized particles in the external magnetic field.

Fundamentally different variant of the model is suggested in [25]. In this variant, a measure of rotational motion of particles is the time integral of angular velocity and, as consequence, the strain state after a complete revolution differs from the natural state. In [1, 2, 4, 23] to describe the rotational motion of long molecules of nematic liquid crystals the vector-director $\mathbf{v}_0$ is used, indicating the direction of the axis, associated with the particle. With this approach, one of the equations is a nonlinear algebraic equation $|\mathbf{v}_0|^2 = 1$, preventing the reduction of the system to a symmetric form, which is necessary for theoretical justification of a model.

In the Cartesian coordinate system the tensor of angular velocity is identified with angular velocity vector, the coordinates of which are numbers ω_i . Tensor components are linked to the vector components by the following formulae

$$\omega_{kj} = \varepsilon_{ijk} \omega_i, \quad \omega_i = \frac{1}{2} \varepsilon_{ijk} \omega_{kj},$$

where ε_{ijk} is the discriminant Levi–Civita tensor. The last formula is for calculation of vector corresponding antisymmetric part of an arbitrary second-rank tensor.

The integral laws of conservation of impulse, moment of impulse and energy take the following form:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \rho_0 \mathbf{v} d\Omega &= \int_{\Gamma} \mathbf{v} \cdot \boldsymbol{\sigma} d\Gamma + \int_{\Omega} \mathbf{f} d\Omega, \\ \frac{\partial}{\partial t} \int_{\Omega} (\mathbf{J} \cdot \boldsymbol{\omega} + \rho_0 \mathbf{x} \times \mathbf{v}) d\Omega &= \int_{\Gamma} \mathbf{x} \times (\mathbf{v} \cdot \boldsymbol{\sigma}) d\Gamma + \int_{\Omega} (\mathbf{x} \times \mathbf{f} + \mathbf{g}) d\Omega, \\ \frac{\partial}{\partial t} \int_{\Omega} \left(\rho_0 \frac{\mathbf{v} \cdot \mathbf{v}}{2} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + \Phi \right) d\Omega &= \int_{\Gamma} \mathbf{v} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v} - \mathbf{h}) d\Gamma \\ &\quad + \int_{\Omega} (\mathbf{v} \cdot \mathbf{f} + \boldsymbol{\omega} \cdot \mathbf{g} + Q) d\Omega. \end{aligned} \quad (10.36)$$

Here Ω is an arbitrary domain with smooth boundary Γ , which is separated in the initial (undeformed) medium state, ρ_0 is the initial density, $\mathbf{J}$ is the symmetric and positive-definite inertia tensor, $\boldsymbol{\sigma}$ is the nonsymmetric Piola–Kirchhoff stress tensor, Φ is the internal energy per unit volume, $\mathbf{h}$ is the vector of heat flow, $\mathbf{f}$ and $\mathbf{g}$ are the volume densities of mass forces and moments, Q is the intensity of internal heat source.

In the process of medium motion the domain Ω , which consists of material particles, changes to the deformed state Ω_t , material mass is persisted $\rho_0 d\Omega = \rho d\Omega_t$, hence, the density changes by the law $\rho = \rho_0 / \det \mathbf{x}_\xi$, and the inertia tensor of particles which are contained per unit volume is transformed by the law $\mathbf{J}_t = (\rho/\rho_0)\mathbf{J}$. The inertia tensor $\mathbf{J}$, referred to the initial state, changes depending on time according to the equality $\mathbf{J} = \mathbf{q} \cdot \mathbf{J}_0 \cdot \mathbf{q}^*$, which can be justified turning to co-moving system of coordinates which is connected with the rotating particle. Time differentiation gives the following equation for inertia tensor which is used in [25]:

$$\dot{\mathbf{J}} = \dot{\mathbf{q}} \cdot \mathbf{J}_0 \cdot \mathbf{q}^* + \mathbf{q} \cdot \mathbf{J}_0 \cdot \dot{\mathbf{q}}^* = \boldsymbol{\omega} \cdot \mathbf{J} - \mathbf{J} \cdot \boldsymbol{\omega}.$$

For continuous motions the integral conservation laws are equivalent to differential equations, which can be obtained from (10.36) by the Green formula:

$$\begin{aligned} \rho_0 \dot{\mathbf{v}} &= \nabla_\xi \cdot \boldsymbol{\sigma} + \mathbf{f}, \quad \frac{\partial}{\partial t} (\mathbf{J} \cdot \boldsymbol{\omega}) = 2 (\boldsymbol{\sigma}^* \cdot \nabla_\xi \mathbf{x})^a + \mathbf{g}, \\ \dot{\boldsymbol{\phi}} &= \boldsymbol{\sigma}^* : (\nabla_\xi \mathbf{v} + \nabla_\xi \mathbf{x} \cdot \boldsymbol{\omega}) - \nabla_\xi \cdot \mathbf{h} + Q. \end{aligned} \quad (10.37)$$

By derivation of (10.37) the equality $\boldsymbol{\omega} \cdot \dot{\mathbf{J}} \cdot \boldsymbol{\omega} = 0$ (here $\boldsymbol{\omega}$ is the vector) which is direct consequence of kinematic equation for the inertia tensor was appreciably used.

The last equation of the system (10.37) for reversible processes, which thermodynamic parameters of a state are the deformation measure $\mathbf{A}$ and the entropy S , is decomposed taking into account (10.35) into the constitutive equation

$$\boldsymbol{\sigma} \cdot \mathbf{q} = \frac{\partial \Phi}{\partial \mathbf{A}} \quad (10.38)$$

and the heat production equation

$$T \dot{S} = -\nabla_\xi \cdot \mathbf{h} + Q, \quad (10.39)$$

where $T = \partial \Phi / \partial S$ is the absolute temperature.

If a medium is isotropic then inertia tensor is spherical: $\mathbf{J}_0 = j_0 \boldsymbol{\delta}$, and internal energy is the function of tensor invariants and entropy. As the system of functionally independent invariants it is possible to take three invariants of the symmetric part $\mathbf{A}^s = (\mathbf{A} + \mathbf{A}^*)/2$ of $\mathbf{A}$:

$$I_1^s = \boldsymbol{\delta} : \mathbf{A}^s, \quad I_2^s = \boldsymbol{\delta} : (\mathbf{A}^s)^2, \quad I_3^s = \boldsymbol{\delta} : (\mathbf{A}^s)^3,$$

and quadratic invariant $I_2^a = 2 |\mathbf{A}^a|^2$ of the asymmetric part $\mathbf{A}^a = (\mathbf{A} - \mathbf{A}^*)/2$. By this choice the constitutive equation (10.38) takes the following form:

$$\boldsymbol{\sigma} \cdot \mathbf{q} = a_1 \boldsymbol{\delta} + 2 a_2 \mathbf{A}^s + 3 a_3 (\mathbf{A}^s)^2 + 2 \alpha \mathbf{A}^a, \quad (10.40)$$

where $a_k = \partial\Phi/\partial I_k^s$ ($k = 1, 2, 3$) and $\alpha = \partial\Phi/\partial I_2^a$ are the functions of state. In the theory of small strains of the Cosserat continuum with couple stresses equal to zero (so-called reduced continuum) the elastic potential is defined by the following formula

$$\Phi = \left(k - \frac{2\mu}{3}\right) \frac{(I_1^s)^2}{2} + \mu I_2^s + \alpha I_2^a - 3k I_1^s$$

with constants k , μ and α . Hence, $a_1 = (k - 2\mu/3)(I_1^s - 3) - 2\mu$, $a_2 = \mu$ and $a_3 = 0$.

For simplicity let us consider adiabatic variant of a model with the absence of external heat sources, mass forces and moments supposing that the internal energy is a strongly convex function of variable $\mathbf{A}$. This supposition is completely justified because potential of stresses in the linear theory, which is equal to the quadratic part of decomposition of the function Φ in the Taylor series:

$$\Phi(\mathbf{A}, S) = \frac{1}{2} (\mathbf{A} - \delta)^* : \frac{\partial^2 \Phi(\delta, S_0)}{\partial \mathbf{A}^2} : (\mathbf{A} - \delta)^* + \dots,$$

satisfies the property of positiveness. For an isotropic medium this property is fulfilled by virtue of the following inequalities:

$$k > 0, \quad \mu > 0, \quad \alpha > 0. \quad (10.41)$$

Let $\boldsymbol{\tau}$ be the stress tensor dual to the tensor $\mathbf{A}$, and $\Psi(\boldsymbol{\tau}, S) = \boldsymbol{\tau}^* : \mathbf{A} - \Phi(\mathbf{A}, S)$ is the Legendre transformation of internal energy which is also the strongly convex function of $\boldsymbol{\tau}$. The constitutive equation (10.38) can be written in reversed form:

$$\mathbf{A} \cdot \mathbf{q}^* = \frac{\partial \Psi(\boldsymbol{\tau}, S)}{\partial \boldsymbol{\tau}} \cdot \mathbf{q}^* = \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, S)}{\partial \boldsymbol{\sigma}}.$$

The left part of obtained equality is equal to the transposed tensor of distortion, hence the following equation is valid:

$$\frac{\partial}{\partial t} \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, S)}{\partial \boldsymbol{\sigma}} = \nabla_{\xi} \mathbf{v}. \quad (10.42)$$

Closed mathematical model of the elastic medium, which considers rotational degrees of freedom, is constructed by the equations of motion (10.37), the constitutive equation (10.42), the ordinary differential equation $\dot{\mathbf{q}} = \boldsymbol{\omega} \cdot \mathbf{q}$ for rotation tensor and the equation of the entropy constancy which follows from (10.39) within the framework of made suppositions. In the Cartesian coordinates the model is reduced to the following system:

$$\rho_0 \dot{v}_j = \sigma_{ij,i}, \quad \frac{\partial}{\partial t} \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, S)}{\partial \sigma_{ij}} = v_{j,i},$$

$$\begin{aligned} \frac{\partial}{\partial t} (J_{ij} \omega_j) &= \varepsilon_{ijk} \sigma_{lk} \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, S)}{\partial \sigma_{lj}}, \\ \dot{q}_{ij} &= \varepsilon_{ikl} \omega_k q_{lj}, \quad \dot{S} = 0. \end{aligned} \quad (10.43)$$

In the spatial case the system consists of 25 equations which are written relative to 25 unknown functions: v_i , σ_{ij} , ω_i , q_{ij} and S . The components of inertia tensor, which are included in this system, can be expressed through primary unknown functions by the following formulae: $J_{ij} = J_{0kl} q_{ik} q_{jl}$.

10.5 Finite Strains of the Cosserat Medium

In more complete mathematical model, which is geometrical nonlinear generalization of the model of the Cosserat continuum, except the deformations characterized by tensor $\mathbf{A}$, curvature of elements is taking into account. For that we inject the second-rank tensor $\mathbf{M}$ which is equal to zero in natural (undeformed) state of the medium and which kinematics is described by the equation $\dot{\mathbf{M}} = \nabla_{\xi} \boldsymbol{\omega}$. In the integral conservation laws (10.36) the additional terms, caused by couple stresses which are characterized by the tensor $\mathbf{m}$, appear:

$$\int_{\Gamma} \mathbf{v} \cdot \mathbf{m} d\Gamma, \quad \int_{\Gamma} \mathbf{v} \cdot \mathbf{m} \cdot \boldsymbol{\omega} d\Gamma.$$

As a result, the equation of rotational motion takes the next form:

$$\frac{\partial}{\partial t} (\mathbf{J} \cdot \boldsymbol{\omega}) = \nabla_{\xi} \mathbf{m} + 2 (\boldsymbol{\sigma}^* \cdot \nabla_{\xi} \mathbf{x})^a + \mathbf{g}. \quad (10.44)$$

The change of internal energy is described by more common equation than the equation in (10.37):

$$\dot{\Phi} = \boldsymbol{\sigma}^* : (\nabla_{\xi} \mathbf{v} + \nabla_{\xi} \mathbf{x} \cdot \boldsymbol{\omega}) + \mathbf{m}^* : \mathbf{M} - \nabla_{\xi} \cdot \mathbf{h} + Q. \quad (10.45)$$

The tensors $\mathbf{A}$, $\mathbf{M}$ and the entropy S are set by independent thermodynamic parameters of elastic state of a medium. As the consequence of reversibility of deformation process, taking into account the formula (10.35) the constitutive equations of the following form can be obtained:

$$\boldsymbol{\sigma} \cdot \mathbf{q} = \frac{\partial \Phi}{\partial \mathbf{A}}, \quad \mathbf{m} = \frac{\partial \Phi}{\partial \mathbf{M}}.$$

In the case of an isotropic material the internal energy depends only on the considered invariants of tensor $\mathbf{A}$, the invariants of $\mathbf{M}$:

$$J_1^s = \delta : \mathbf{M}^s, \quad J_2^s = \delta : (\mathbf{M}^s)^2, \quad J_3^s = \delta : (\mathbf{M}^s)^3, \quad J_2^a = 2 |\mathbf{M}^a|^2,$$

and on the entropy. The combined invariants of these tensors can not be considered as arguments because $\mathbf{A}$ relates to the polar tensors and $\mathbf{M}$ relates to axial ones [33]. Thus, in the case of an isotropic medium the constitutive equation (10.40) and the equation

$$\mathbf{m} = b_1 \delta + 2 b_2 \mathbf{M}^s + 3 b_3 (\mathbf{M}^s)^2 + 2 \beta \mathbf{M}^a, \quad b_k = \frac{\partial \Phi}{\partial J_k^s}, \quad \beta = \frac{\partial \Phi}{\partial J_2^a},$$

take place. In the linear Cosserat theory $b_1 = (\kappa - 2\eta/3)J_1^s$, $b_2 = \eta$, $b_3 = 0$, the constants κ , η and β of a material satisfy the inequalities

$$\kappa > 0, \quad \eta > 0, \quad \beta > 0,$$

which guarantee together with the inequalities (10.41) the positivity of quadratic stress potential

$$\Phi = \left(k - \frac{2\mu}{3}\right) \frac{(I_1^s)^2}{2} + \mu I_2^s + \alpha I_2^a - 3k I_1^s + \left(\kappa - \frac{2\eta}{3}\right) \frac{(J_1^s)^2}{2} + \eta J_2^s + \beta J_2^a.$$

In general case the function Φ is a strongly convex function of variables $\mathbf{A}$ and $\mathbf{M}$, hence the Legendre transform allows to reverse the constitutive equations

$$\mathbf{A} \cdot \mathbf{q}^* = \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, \mathbf{m}, S)}{\partial \boldsymbol{\sigma}}, \quad \mathbf{M} = \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, \mathbf{m}, S)}{\partial \mathbf{m}}, \quad (10.46)$$

where $\Psi(\boldsymbol{\tau}, \mathbf{m}, S) = \boldsymbol{\tau}^* : \mathbf{A} + \mathbf{m}^* : \mathbf{M} - \Phi(\mathbf{A}, \mathbf{M}, S)$ is a strongly convex function of $\boldsymbol{\tau}$ and $\mathbf{m}$.

For adiabatic processes in the absence of mass forces, moments and heat sources, the equations of motion, constitutive equations, the equations for the rotation tensor and for the entropy, which are obtained by this method, form the closed mathematical model. In projections onto axes of the Cartesian coordinate system the equations of this model have the simplest form:

$$\begin{aligned} \rho_0 \dot{v}_i &= \sigma_{ji,j}, \quad \frac{\partial}{\partial t} \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, \mathbf{m}, S)}{\partial \sigma_{ij}} = v_{j,i}, \\ \frac{\partial}{\partial t} (J_{ij} \omega_j) &= m_{ji,j} + \varepsilon_{ijk} \sigma_{lk} \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, \mathbf{m}, S)}{\partial \sigma_{lj}}, \\ \frac{\partial}{\partial t} \frac{\partial \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, \mathbf{m}, S)}{\partial m_{ij}} &= \omega_{j,i}, \quad \dot{q}_{ij} = \varepsilon_{ikl} \omega_k q_{lj}, \quad \dot{S} = 0. \end{aligned} \quad (10.47)$$

Taking into account the formulae $J_{ij} = J_{0kl} q_{ik} q_{jl}$, the system (10.47) is formed by 34 equations for 34 unknown functions: v_i , σ_{ij} , ω_i , m_{ij} , q_{ij} and S .

It is fundamentally important that the systems (6.43) and (10.47) are thermodynamically self-consistent systems of conservation laws in the next sense: it is possible to find the generating potentials L^0 and L^j which allow to rewrite them as

$$\frac{\partial}{\partial t} \frac{\partial L^0(\mathbf{D}\mathbf{U})}{\partial \mathbf{U}} = \frac{\partial}{\partial \xi_j} \frac{\partial L^j(\mathbf{U})}{\partial \mathbf{U}} + \mathbf{F}, \quad \frac{\partial \mathbf{D}}{\partial t} = \mathbf{G}. \quad (10.48)$$

Here $\mathbf{U}$ is n -dimensional vector ($n = 15$ and 24 , respectively) which consists of the unknown functions except the entropy S , included in (10.48) as a parameter, and the components q_{ij} of rotational tensor, $\mathbf{D}$ is the orthogonal quadratic matrix of the same dimension, $\mathbf{F}(\mathbf{D}, \mathbf{U})$ and $\mathbf{G}(\mathbf{D}, \mathbf{U})$ are the given vector-functions. The elements of matrix $\mathbf{D}$, which are different from zero and one, are the components of q_{ij} . The vector-functions $\mathbf{F}$ and $\mathbf{G}$ are easy to define by the form of equations.

For the system (10.43) vector $\mathbf{D}\mathbf{U}$ consists of the components v_i , $\sigma_{ik} q_{kj}$ and $\omega_i q_{ij}$, generating potentials are equal to

$$L^0(\mathbf{D}\mathbf{U}) = \rho_0 \frac{v_i v_i}{2} + \frac{1}{2} (\mathbf{q}^* \cdot \boldsymbol{\omega})_i J_{0ij} (\mathbf{q}^* \cdot \boldsymbol{\omega})_j + \Psi(\boldsymbol{\sigma} \cdot \mathbf{q}, S), \quad L^j(\mathbf{U}) = v_i \sigma_{ij},$$

where $\boldsymbol{\omega}$ is used in a vector form, but not in a tensor one. For the system (10.47) vector $\mathbf{D}\mathbf{U}$ consists of v_i , $\sigma_{ik} q_{kj}$, $\omega_i q_{ij}$ and m_{ij} . The set of arguments in the potential L^0 is extended by m_{ij} , and the additional term $\omega_i m_{ij}$ is added to L^j .

The equation for entropy is not included into the system (10.48) because the complementary conservation law, which is equivalent to this equation, follows from this system automatically:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\mathbf{U} \frac{\partial L^0(\mathbf{D}\mathbf{U})}{\partial \mathbf{U}} - L^0(\mathbf{D}\mathbf{U}) \right) &= \frac{\partial}{\partial \xi_j} \left(\mathbf{U} \frac{\partial L^j(\mathbf{U})}{\partial \mathbf{U}} - L^j(\mathbf{U}) \right) \\ &+ \mathbf{U} \mathbf{G} - \frac{\partial L^0(\mathbf{D}\mathbf{U})}{\partial \mathbf{U}} \mathbf{D}^{-1} \mathbf{F} \mathbf{U}. \end{aligned} \quad (10.49)$$

The validity of complementary conservation law can be tested by means of the differentiation formula:

$$\frac{\partial L^0(\mathbf{D}\mathbf{U})}{\partial t} = \frac{\partial L^0(\mathbf{D}\mathbf{U})}{\partial \mathbf{U}} \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{D}^{-1} \frac{\partial \mathbf{D}}{\partial t} \mathbf{U} \right). \quad (10.50)$$

The system of equations (10.48), (10.49) has the divergence form and can be used for correct description of the generalized solutions with shock waves. The relations of strong discontinuity for this system can be written in the next form:

$$c \left[\left[\frac{\partial L^0(\mathbf{D}\mathbf{U})}{\partial \mathbf{U}} \right] \right] + \left[\left[\frac{\partial L^j(\mathbf{U})}{\partial \mathbf{U}} \right] \right] v_j = 0, \quad c [\mathbf{D}] = 0,$$

$$c \left[\left[\mathbf{U} \frac{\partial L^0(\mathbf{D}\mathbf{U})}{\partial \mathbf{U}} - L^0(\mathbf{D}\mathbf{U}) \right] \right] + \left[\left[\mathbf{U} \frac{\partial L^j(\mathbf{U})}{\partial \mathbf{U}} - L^j(\mathbf{U}) \right] \right] v_j = 0, \quad (10.51)$$

where c is the Lagrangian velocity of the wave front in normal direction, the square brackets mean a jump of function under transition through the discontinuity surface. The inequality $[[S]] \geq 0$ and some additional conditions of the discontinuity stability (see, for example, [28, 43]) must be added to the system (10.51). However, these conditions can be completely analyzed only by using the specific form of the thermodynamic potentials Φ and Ψ .

Applying the differentiation formula (10.50) to the derivative $\partial L^0(\mathbf{D}\mathbf{U})/\partial \mathbf{U}$ it is possible to reduce the system (10.48) to the symmetric form:

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{A} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{D} \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{B}^j \end{pmatrix} \frac{\partial}{\partial \xi_j} \begin{pmatrix} \mathbf{D} \\ \mathbf{U} \end{pmatrix} + \begin{pmatrix} \mathbf{G} \\ \mathbf{H} \end{pmatrix},$$

$$\mathbf{A} = \frac{\partial^2 L^0(\mathbf{D}\mathbf{U})}{\partial \mathbf{U}^2}, \quad \mathbf{B}^j(\mathbf{U}) = \frac{\partial^2 L^j(\mathbf{U})}{\partial \mathbf{U}^2}, \quad \mathbf{H} = \mathbf{F} - \mathbf{A} \mathbf{D}^{-1} \mathbf{G} \mathbf{U}. \quad (10.52)$$

The matrices $\mathbf{A}$ and $\mathbf{B}^j$ are symmetric, moreover, by virtue of strong convexity of the potential $L^0(\mathbf{D}\mathbf{U})$ the matrix $\mathbf{A}$ is positively defined, therefore the system of equations (10.48) is hyperbolic in the sense of Friedrichs.

Let us suppose that matrixes-coefficients of the system (10.52) and vectors $\mathbf{G}$ and $\mathbf{H}$ satisfy the Lipschitz condition with respect to $\mathbf{D}$ and $\mathbf{U}$. Then for the difference of two solutions of such system defined in the space-time domain of type of a truncated cone with the conical surface satisfying the Hamilton–Jacobi inequality, a priori estimate

$$\|(\mathbf{D}', \mathbf{U}') - (\mathbf{D}, \mathbf{U})\|(t_1) \leq \|(\mathbf{D}', \mathbf{U}') - (\mathbf{D}, \mathbf{U})\|(t_0) \exp(C_6(t_1 - t_0)) \quad (10.53)$$

is valid, where C_6 is a constant which depends on both solutions and its first derivatives by time and space. The energy norm is calculated as the integral over the section Ω_t of a conical domain by hyperplane $t = \text{const}$ (tr is the trace of matrix):

$$\|(\mathbf{D}, \mathbf{U})\|^2(t) = \frac{1}{2} \int_{\Omega_t} (\text{tr} \mathbf{D}^* \mathbf{D} + \mathbf{U} \mathbf{A} \mathbf{U}) d\Omega.$$

Consequently, the solution of the Cauchy problem

$$\mathbf{D}|_{t=t_0} = \mathbf{D}^0(\xi), \quad \mathbf{U}|_{t=t_0} = \mathbf{U}^0(\xi)$$

is unique in a conical domain and continuously depends on initial data. Moreover, the inequality (10.53) shows the limitations of the domains of dependence and influence of solutions. The method of derivation of such estimates and the method of constructing special domains of conical type are represented in Sect. 6.2.

The analogous estimate is valid for truncated cones which are adjacent to the boundary of domain, if there a dissipative boundary conditions are set. The dissipativity means that

$$(\mathbf{U}' - \mathbf{U}) \mathbf{B}^j (\mathbf{U}) (\mathbf{U}' - \mathbf{U}) v_j \leq 0$$

for any two of vector-functions satisfying the boundary conditions. From the integral estimate in this case follows the uniqueness and continuous dependence on initial data of the solution of boundary-value problems. Taking into account the structure of the matrices $\mathbf{B}^j$, it is possible to show that the dissipativity condition for the Cosserat model leads to the following inequality:

$$v_i (\sigma'_{ij} - \sigma_{ij})(v'_j - v_j) + v_i (m'_{ij} - m_{ij})(\omega'_i - \omega_i) \leq 0. \quad (10.54)$$

To dissipative conditions relate the conditions in terms of velocities $v_i = \bar{v}_i$, $\omega_i = \bar{\omega}_i$, and in terms of stresses $\sigma_{ij} v_i = \bar{\sigma}_j$, $m_{ij} v_i = \bar{m}_j$ as well as a combined variants of these conditions, for example, if the angular velocity vector $\bar{\omega}_i$ and the stress vector $\bar{\sigma}_i$ are given or the velocity vector $\bar{v}_i$ and the couple stress vector $\bar{m}_i$ are set. Mixed boundary conditions are also possible when normal velocities and tangential stresses are given or, on the contrary, normal stresses and tangential velocities are set.

For the model of an elastic medium with rotating particles the second term in the left-hand side of the inequality (10.54) is absent, therefore the dissipative boundary conditions are formulated in terms of the projections of velocity vector and components of nonsymmetric stress tensor in the same way as in the classical theory of elasticity.

Notice that the model of a nonlinear elastic medium can be obtained from the model which takes into account rotational degrees of freedom by limiting $\mathbf{J} \rightarrow \mathbf{0}$. In this case the size of particles of a structurally inhomogeneous material tends to zero and thus the classical continuum is obtained. The equation of rotational motion leads to the condition of the symmetry of tensor $\boldsymbol{\sigma}^* \cdot \nabla_{\xi} \mathbf{x}$ which means the symmetry of the Cauchy stress tensor. However, if $\mathbf{J} \rightarrow \mathbf{0}$ then the system of equations (10.43) degenerates, losing its hyperbolicity just as if the material density in classical elasticity tends to zero. Hence, the question of reducing the equations of the nonlinear elasticity theory to thermodynamically self-consistent system of conservation laws, which is solved in [18, 19], does not have a simple solution within the framework of a more general model.

As an example let us consider the process of plane shear for the isotropic Cosserat elastic medium which is described by the equations

$$x_1 = \xi_1, \quad x_2 = \xi_2 + \chi \xi_1, \quad x_3 = \xi_3. \quad (10.55)$$

If the rate of shear $\dot{\chi}$ is constant then homogeneous stress-strain state is realized because the inertial forces of translational motion are absent. The couple stresses turn out to be equal to zero, consequently the system of equations (10.47) coincides with the system (10.43). Matrices of the tensors are equal to

$$\nabla_{\xi} \mathbf{x} = \begin{pmatrix} 1 & \chi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\omega} = \dot{\varphi} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the absence of relative rotation, when the rotation angle φ coincides with the rotation angle of translational motion φ_e , the matrix

$$\mathbf{A} = \nabla_{\xi} \mathbf{x} \cdot \mathbf{q} = \begin{pmatrix} \chi \sin \varphi + \cos \varphi & \chi \cos \varphi - \sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is symmetric, hence $\tan \varphi_e = \chi/2$. In this case

$$\boldsymbol{\sigma}^* \cdot \nabla_{\xi} \mathbf{x} = \mathbf{q} \cdot (a_1 \boldsymbol{\delta} + 2a_2 \mathbf{A}^s + 3a_3 (\mathbf{A}^s)^2 - 2\alpha \mathbf{A}^a) \cdot \mathbf{A} \cdot \mathbf{q}^* \quad (10.56)$$

is also symmetric matrix, hence the equation of rotational motion, which includes only its antisymmetric part, takes the form: $j_0 \ddot{\varphi}_e = 0$. Obviously, it is fulfilled only if a medium does not have rotational inertia. In inertial media $\varphi = \varphi_e + \varphi_r$, where φ_r is the angle of relative rotation which is considered as small in comparison with φ_e . In this case the next equations are valid:

$$\sin \varphi \approx \sin \varphi_e + \varphi_r \cos \varphi_e, \quad \cos \varphi \approx \cos \varphi_e - \varphi_r \sin \varphi_e,$$

$$\sqrt{\chi^2 + 4} \mathbf{q} \approx \begin{pmatrix} 2 - \chi & 0 & 0 \\ \chi & 2 & 0 \\ 0 & 0 & \sqrt{\chi^2 + 4} \end{pmatrix} - \varphi_r \begin{pmatrix} \chi & 2 & 0 \\ -2 & \chi & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\sqrt{\chi^2 + 4} \mathbf{A} \approx \begin{pmatrix} \chi^2 + 2 & \chi & 0 \\ \chi & 2 & 0 \\ 0 & 0 & \sqrt{\chi^2 + 4} \end{pmatrix} - \varphi_r \begin{pmatrix} -\chi & \chi^2 + 2 & 0 \\ -2 & \chi & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The substitution of these approximations into (10.56) with following calculation of antisymmetric part of the matrix $\boldsymbol{\sigma}^* \cdot \nabla_{\xi} \mathbf{x}$ leads to the equation of rotational motion:

$$j_0 \ddot{\varphi}_r = -a_0 \varphi_r + \frac{4 j_0 \chi \dot{\chi}^2}{(\chi^2 + 4)^2},$$

$$a_0 = (\alpha - a_2)(\chi^2 + 4) - \left(a_1 + \frac{3}{2} a_3 (\chi^2 + 2)\right) \sqrt{\chi^2 + 4}. \quad (10.57)$$

In the shear state $I_1^s = \sqrt{\chi^2 + 4} + 1$, therefore in the case of quadratic stress potential

$$a_1 = \left(k - \frac{2\mu}{3}\right) \sqrt{\chi^2 + 4} - 2\left(k + \frac{\mu}{3}\right), \quad a_2 = \mu, \quad a_3 = 0,$$

$$a_0 = \left(\alpha - k - \frac{\mu}{3} \right) (\chi^2 + 4) + 2 \left(k + \frac{\mu}{3} \right) \sqrt{\chi^2 + 4}.$$

For small χ the parameter a_0 is constant and equal to 4α . In this case the general solution of the homogeneous equation (10.57)

$$\varphi_r(t) = C_7 \sin \frac{2\pi t}{T_0} + C_8 \cos \frac{2\pi t}{T_0}$$

describes cyclical natural oscillations with the period $T_0 = \pi \sqrt{j_0/\alpha}$.

For the most of known materials with moment properties the value of $k + \mu/3$ is greater than α . Therefore the parameter a_0 , which is positive under small shear, changes its sign. It happens when

$$\chi_* = \frac{2\sqrt{2\alpha(k + \mu/3) - \alpha^2}}{k + \mu/3 - \alpha}.$$

By achieving this value, the oscillatory regime of particles rotation in a layer changes by smooth nonoscillatory motion.

The obtained solution outlines the basic qualitative deformation feature of a medium with microstructure in comparison with a usual elastic medium. The process of shear in such a medium under small shear angles is accompanied by a natural oscillations of rotational motion.

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