THEORY OF STRUCTURE II

CHAPTER 3

INTRODUCTION TO MATRICS METHOD ANALYSIS

INTRODUCTION

After the introduction of high-speed computers, there has been a revolution in structural analysis, not only in the computational methods but also in the fundamental theorems. Since digital computers are ideally suitable for automatic computations of matrix algebra, it was found desirable to formulate the entire structural analysis in matrix notation. Matrix methods of structural analysis are based on the concept of replacing the actual structure by an equivalent analytical model consisting of discrete structural elements having known properties which can be expressed in matrix form. Matrices are useful in expressing structural theory and in producing an efficient means for carrying out numerical calculations.

Two methods have been formulated in matrix structural analysis: the flexibility and stiffness methods. It will not be possible in this textbook to develop the two matrix methods to sufficient depth. The methods are developed to the level of manual computation.

FORCE AND DISPLACEMENT MEASUREMENTS

It is evident that the overall description of the behaviour of a structure is accomplished through the dual consideration of force and displacement components at designated points. There are a number of ways of measuring a force applied to a structure or its displacement at designated points in a prescribed direction. Such points are commonly known as *node points*. The first step in the analysis of structures is to idealise the actual structure into a mathematical model which consists of distinct structural elements interconnected through node points. In this text the word *force* includes moment.

Figure 7.1

To designate the forces and displacements at the nodes of a given structure, a *coordinate system* is used to identify these measurements. For the frame shown in Fig. 7.1, for example, the system consists of four arbitrary coordinates which are identified by four numbered arrows shown at the specific nodes or joints. The forces are listed in column matrix [*P*] and is referred to as a *force vector* and represents an ordered array of force measurements. For instance, the force vector for the frame of Fig. 7.1 is represented by

$$[P] = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \qquad [7.1]$$

Likewise, the coordinate displacement vector, having the same significance as in the force vector may be expressed as

$$[\Delta] = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \qquad [7.2]$$

In a similar manner, the forces and displacements at the nodes of a given element may be designated by listing in column matrices [*P*] and [Δ], respectively. For the beam element of Fig. 7.2, for example, with direct forces at

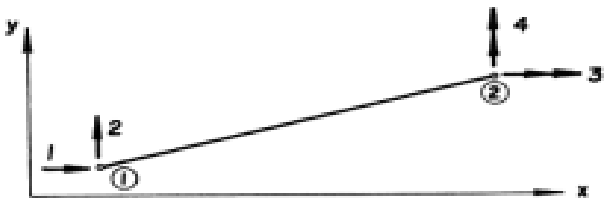

Figure 7.2

node 1 and moments at node 2, the force vector is written as

$$[P] = \begin{bmatrix} P_{x1} \\ P_{y1} \\ M_{x2} \\ M_{y2} \end{bmatrix} \qquad [7.3]$$

and the displacement vector as

$$[\Delta] = \begin{bmatrix} u_1 \\ v_1 \\ \theta_{x2} \\ \theta_{y2} \end{bmatrix} \qquad [7.4]$$

A necessary step in the formation of the force and displacement vectors is the establishment of the node points and their location with respect to coordinate axes. At this stage it is necessary to define two sets of orthogonal coordinate systems. The first set is that of the structure, known as the *global axes*, and consists of a single coordinate system. The second set is that of the members or elements, known as the *local axes*, and consists of one coordinate system for each member. Since the members are in general differently oriented within a structure, these axes originating at member ends will usually be differently oriented from one element to the next. Global and local coordinates are illustrated in Fig. 7.3(a) for trusses and in Fig. 7.3(b) for frames.

When forces are applied to structures, displacements occur. Alternatively, when displacements are prescribed, node forces are necessary to produce them. The relationships that exist between applied forces and displacements play an important role in structural analysis. The force and displacement characteristics of a structure are usually described under definitions of *flexibility* and *stiffness* coefficients. The flexibility and stiffness coefficients depend on the force-displacement properties of the structure and the coordinate system used.

A simple illustration of such relationships is obtained by considering a linear elastic spring shown in Fig. 7.4. Single coordinate is indicated for the force and displacement measurements. The force P will stretch the spring thereby producing a displacement Δ at the end of the spring. The relationship between P and Δ can be expressed as

$$\Delta = fP \qquad [7.5]$$

In [7.5], f is the *flexibility coefficient* of the spring and is defined as the value of the displacement at node 1. In general, a flexibility coefficient is the value of the displacement at a point of the structure, in a given direction, due to a unit force applied at a second point in a second direction.

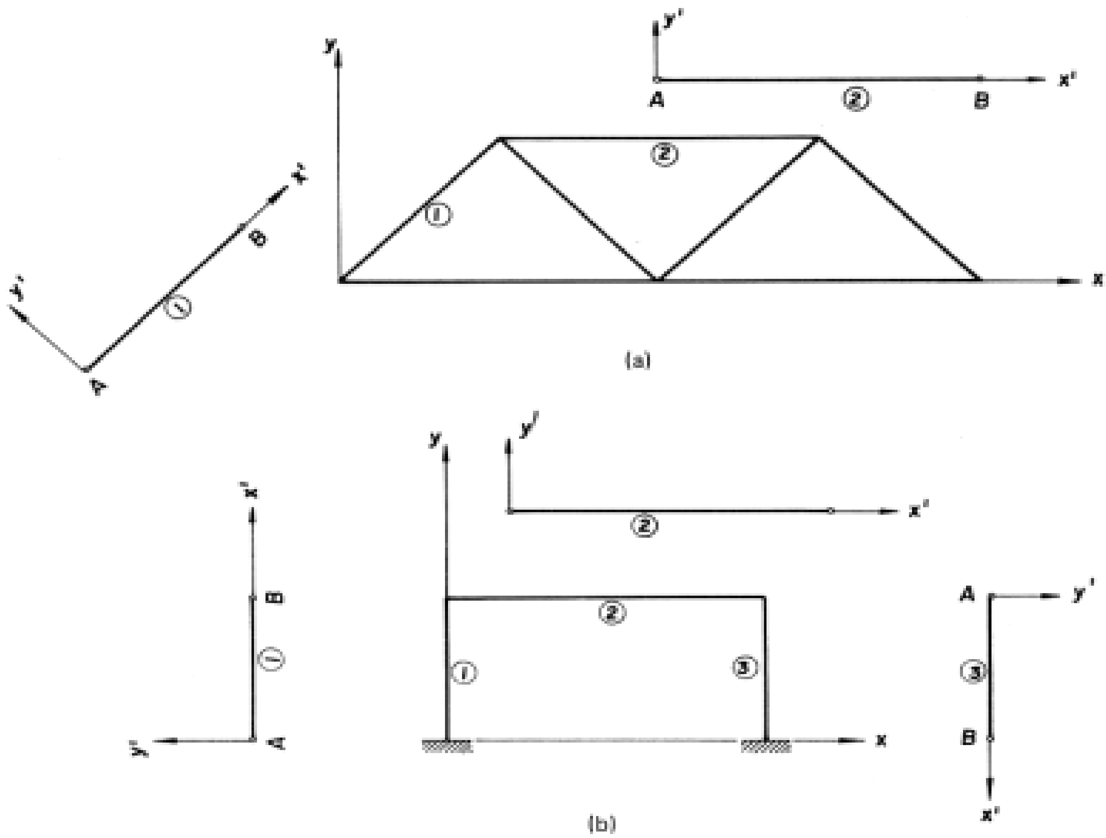

Figure 7.3

An alternative way is to establish a relationship between the force P and the displacement Δ for the spring of Fig. 7.4. The force P required to produce a displacement Δ units is determined from

$$P = k\Delta \qquad [7.6]$$

In [7.6], k is the *stiffness coefficient* of the spring and is defined as the value of the force required at coordinate 1 to produce a unit displacement at 1. In general, a stiffness coefficient is the value of the force at a point of the structure, in a given direction, due to unit displacement applied at a second point in a second direction.

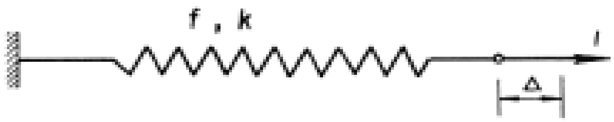

Figure 7.4

Comparison of [7.5] and [7.6] reveals that the flexibility and the stiffness of the spring are *inverse* to one another.

$$f = \frac{1}{k} = k^{-1}$$
$$k = \frac{1}{f} = f^{-1} \qquad [7.7]$$

Now consider a more general case consisting of an elastic structure, supported against rigid-body motion, and subjected to loads $P_1, P_2, \ldots, P_n$ acting at nodes $1, 2, \ldots, n$. The corresponding set of displacements is represented by Δ_1, $\Delta_2, \ldots, \Delta_n$. For linearly elastic systems, the principle of superposition is applicable. Therefore, the displacement Δ_i at node i is given by

$$\Delta_i = f_{i1}P_1 + f_{i2}P_2 + \ldots + f_{in}P_n \qquad [7.8]$$

or more generally,

$$\Delta_i = \sum_{j=1}^{j=n} f_{ij}P_j \qquad [7.9]$$

By definition, f_{ij} is the displacement produced at node i due to a unit load at node j ($P_j = 1$). The coefficients f_{ij}, which are the displacements due to unit loads, are known as flexibility coefficients.

In general, for n nodes, there will be n such displacements which may be written in a single matrix equation

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} \qquad [7.10]$$

and which can be written in compact matrix form as

$$[\Delta] = [F][P] \qquad [7.11]$$

where $[\Delta]$ is the column displacement matrix, $[F]$ is a square flexibility matrix and $[P]$ is a column load matrix (load vector). This equation is of the same type as [2.17].

Using matrix operation, one can solve the set of algebraic equations represented in [7.10] for forces in terms of displacements. In matrix notation

$$[P] = [F]^{-1}[\Delta] \qquad [7.12]$$

METHODS OF STRUCTURAL ANALYSIS

where $[F]^{-1}$ is the inverse of matrix $[F]$. It is noted that [7.12] has the same form as [7.6] since it expresses forces in terms of displacements. Consequently,

$$[F]^{-1} = [K] \qquad [7.13]$$

where $[K]$ is the stiffness matrix which is the inverse of the flexibility matrix. Thus

$$[P] = [K]\,[\Delta] \qquad [7.14]$$

The expanded form of [7.14] is

$$\begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n \end{bmatrix} \qquad [7.15]$$

By definition, k_{ij} is the force required at node i to produce a unit displacement at node j only (zero displacements at all other nodes).

Flexibility coefficients for linear elastic behaviour have the property of reciprocity which may be expressed analytically as

$$f_{ij} = f_{ji} \qquad [7.16]$$

This equation defines symmetry of $[F]$. Since $[F]$ is symmetrical the inverse of a symmetric matrix will also become symmetrical. Therefore, [7.13] guarantees that the stiffness matrix $[K]$ will likewise be symmetrical. Consequently,

$$k_{ij} = k_{ji} \qquad [7.17]$$

To illustrate these matrices consider a simple cantilever beam of uniform cross section shown in Fig. 7.5(a). To determine the flexibility matrix, the influence coefficients must be determined by applying unit loads to the free end.

Due to axial load $N = 1$ (Fig. 7.5(b))

$$\delta_n = \frac{L}{EA}$$

$$\delta_{vn} = 0 \qquad [7.18]$$

$$\theta_n = 0$$

INTRODUCTION TO MATRIX ANALYSIS

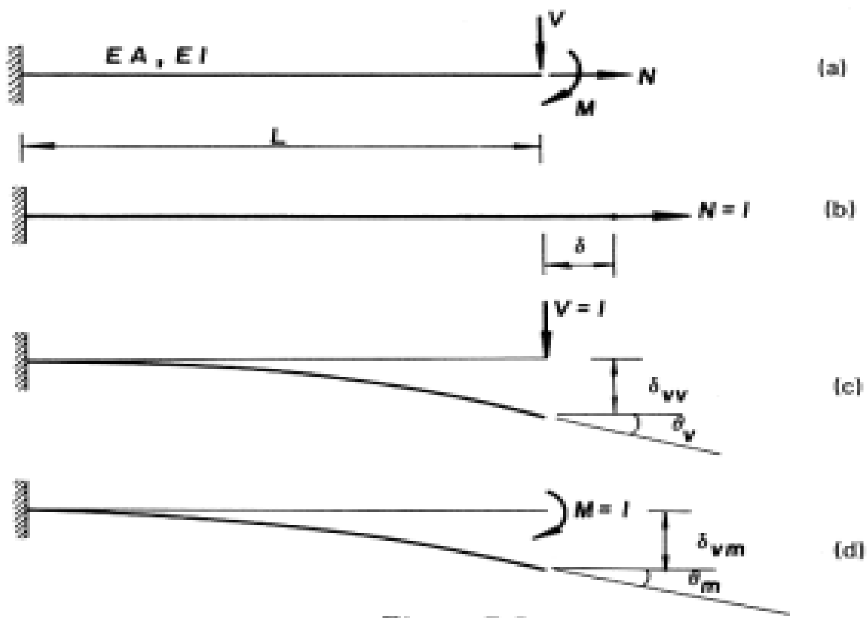

Figure 7.5

Due to vertical load $V = 1$ (Fig. 7.5(c))

$$\delta_n = 0$$

$$\delta_{vv} = \frac{L^3}{3EI} \quad [7.19]$$

$$\theta_v = \frac{L^2}{2EI}$$

Due to moment $M = 1$ (Fig. 7.5(d))

$$\delta_n = 0$$

$$\delta_{vm} = \frac{L^2}{2EI} \quad [7.20]$$

$$\theta_m = \frac{L}{EI}$$

The above results may be written in matrix form as

$$\begin{bmatrix} \delta_n \\ \delta_v \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{bmatrix} N \\ V \\ M \end{bmatrix} \quad [7.21]$$

or, when written in compact matrix form

$$[\Delta] = [F]\,[P] \qquad [7.11]$$

In a similar manner the stiffness matrix may be determined by unit displacements as shown in Fig. 7.6.

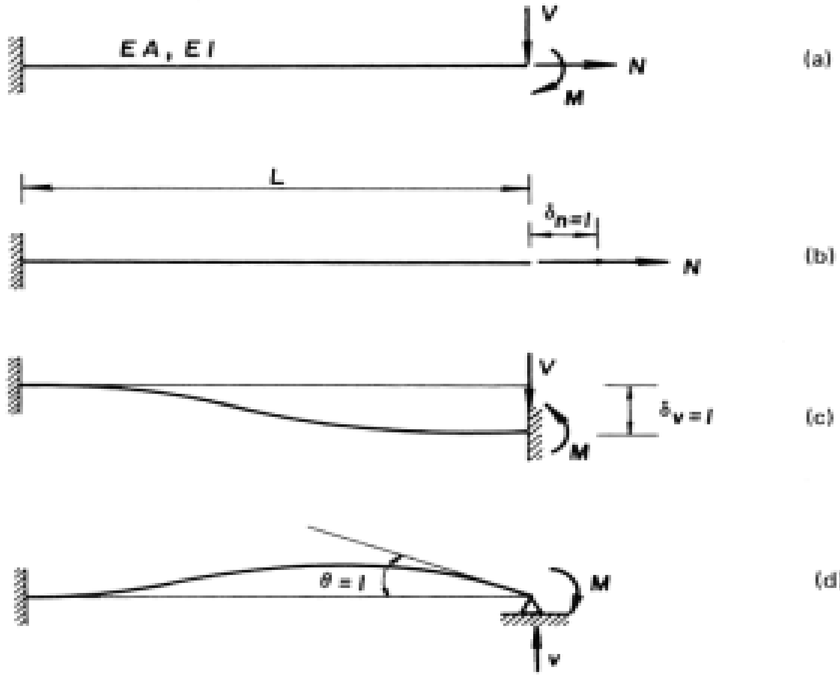

Figure 7.6

Due to unit axial displacement (Fig. 7.6(b))

$$N = \frac{EA}{L} \qquad [7.22]$$

Due to unit vertical displacement (Fig. 7.6(c))

$$V = \frac{12EI}{L^3} \qquad [7.23]$$

$$M = -\frac{6EI}{L^2}$$

Due to unit rotation (Fig. 7.6(d))

$$V = -\frac{6EI}{L^2}$$

$$M = \frac{4EI}{L} \qquad [7.24]$$

INTRODUCTION TO MATRIX ANALYSIS

The above results may be written in matrix form as

$$\begin{bmatrix} N \\ V \\ M \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} \delta_n \\ \delta_v \\ \theta \end{bmatrix} \quad [7.25]$$

which may be written in compact matrix form as

$$[P] = [K]\,[\Delta] \quad [7.26]$$

The results may be checked by matrix multiplication

$$[F]\,[K] = [K]\,[F] = [I] \quad [7.27]$$

It is noted that both $[F]$ and $[K]$ are symmetric matrices which is the consequence of the reciprocal theorem.

THE FLEXIBILITY METHOD

The basic theory of the flexibility method is developed in this section, and the concepts are clarified by numerical examples. The development of the method rests on the basic principles of equilibrium of forces, compatibility and linear force-displacement relationships.

Consider a structure, which is idealised into a model consisting of distinct structural elements interconnected through node points, under the action of generalised external forces applied at the nodes $P_1, P_2, \ldots P_n$. These may be conveniently represented by a column matrix or force vector $[P]$

$$[P] = \{P_1, P_2 \ldots P_n\} \quad [7.28]$$

Let it be assumed that the structure consists of m redundants which are forces to be determined, that is

$$[X] = \{X_1, X_2 \ldots X_m\} \quad [7.29]$$

which are the redundant forces or reactions. If such redundants are removed, the structure becomes determinate and the internal forces are determined from conditions of equilibrium alone. In an indeterminate structure, the internal forces must also satisfy compatibility in addition to equilibrium. In dealing with an indeterminate structure with m redundants, the redundants are treated as additional loads on the statically determinate structure. It is assumed that the structure is composed of an assemblage of j simple elements. Internal forces exist in the structure at the node points. If the internal force members are represented by the vector $[S]$ where

$$[S] = \{S_1 \quad S_2 \quad \dots \quad S_j\} \qquad [7.30]$$

then, $[S]$ can be related to the applied loads $[P]$ and $[X]$ as

$$\begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_j \end{bmatrix} = [B_0] \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} + [B_1] \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X \end{bmatrix} \qquad [7.31(a)]$$

which is written in compact notation as

$$[S] = [B_0]\,[P] + [B_1]\,[X] \qquad [7.31(b)]$$

or using partitioned matrices

$$[S] = [B_0 \,\vdots\, B_1] \begin{bmatrix} P \\ -- \\ X \end{bmatrix} \qquad [7.31(c)]$$

where, in general, $[B_0]$ and $[B_1]$ are rectangular matrices whose elements are obtained from equilibrium conditions of the structure. For example, if P_i is taken as a unit load with all other loads including $[X]$ held at zero, the internal forces in the structure represent the coefficients corresponding to the ith column in the $[B_0]$ matrix. Likewise, the internal forces which result from a unit load X_j with all others held at zero represent the coefficients corresponding to the jth column of the $[B_1]$ matrix.

To formulate the compatibility condition, the principle of least work will be utilised which may be stated as: *The true values of the redundant forces are those which make the strain energy U of the strained structure a minimum.*

The strain energy is given as

$$U = \frac{1}{2}[S_1 \quad S_2 \quad \dots \quad S_j] \begin{bmatrix} F_1 & & & & \\ & F_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & F_j \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_j \end{bmatrix} \qquad [7.32]$$

which is written in compact matrix form as

$$U = \frac{1}{2}[S]^T [F] \{S\} \qquad [7.33]$$

In order to obtain the strain energy U in terms of the unknown $\{X\}$, substitute [7.31(b)] into [7.33]. In doing so, note that the transpose $\{S\}^T$ may be written from [7.31(b)] as

$$[S] = \begin{bmatrix} P \\ -- \\ X \end{bmatrix}^T [B_0 \mid B_1]^T$$

$$= [P \mid X]\,[B_0 \mid B_1]^T \qquad [7.34]$$

Substituting [7.31(b)] and [7.34] into [7.33]

$$U = \frac{1}{2}\,[P \mid X]\,[H] \begin{bmatrix} P \\ -- \\ X \end{bmatrix} \qquad [7.35]$$

where

$$[H] = [B_0 \mid B_1]^T\,[F]\,[B_0 \mid B_1] \qquad [7.36]$$

Since [P] and [X] are the applied and redundant forces, respectively, it is convenient to partition [H] to conform to the load vectors, thus,

$$U = \frac{1}{2}\,[P \mid X] \begin{bmatrix} H_{pp} & \vdots & H_{px} \\ \cdots & & \cdots \\ H_{xp} & \vdots & H_{xx} \end{bmatrix} \begin{bmatrix} P \\ \cdots \\ X \end{bmatrix} \qquad [7.37]$$

After expanding [7.37]

$$U = \frac{1}{2}\,([P]\,[H_{pp}]\,[P] + [P]\,[H_{px}]\,[X] + [X]\,[H_{xp}]\,[P] + [X]\,[H_{xx}]\,[X]) \qquad [7.38]$$

Utilising the theorem of least work and noting that the [H] matrix is symmetric gives

$$\frac{\partial U}{\partial X} = [H_{xp}]\;[P] + [H_{xx}]\,[X] = 0 \qquad [7.39]$$

from which the redundants are determined as

$$[X] = -[H_{xx}]^{-1}\,[H_{xp}]\,[P] \qquad [7.40]$$

Solving for [X] from [7.40] all internal forces can be determined from [7.31].

Summarising, the essential steps in applying the flexibility method to lead to the solution of structural problems may be stated as follows:

1. Idealise the structural problem to be analysed
2. Specify the redundant forces and identify the internal member forces

3. Find the $[B_0]$ matrix for unit values of external forces; only one external force must act at a time with all other forces held at zero
4. Find the $[B_1]$ matrix for unit values of redundant forces; only one redundant force must act at a time with all other forces held at zero
5. Find the flexibility matrix $[F]$ for all members following the sequential order of the member forces in $[B_0]$ and $[B_1]$
6. Formulate the $[H]$ matrix of [7.36]
7. Calculate the redundant forces $[X]$ from [7.40]
8. Calculate the internal forces $[S]$ from [7.31]

EXAMPLE 7.1 *Using the flexibility matrix method determine the bar forces in the truss with double diagonal system shown in Fig. 7.7(a). The area of all top chord is twice the area of all remaining members.*

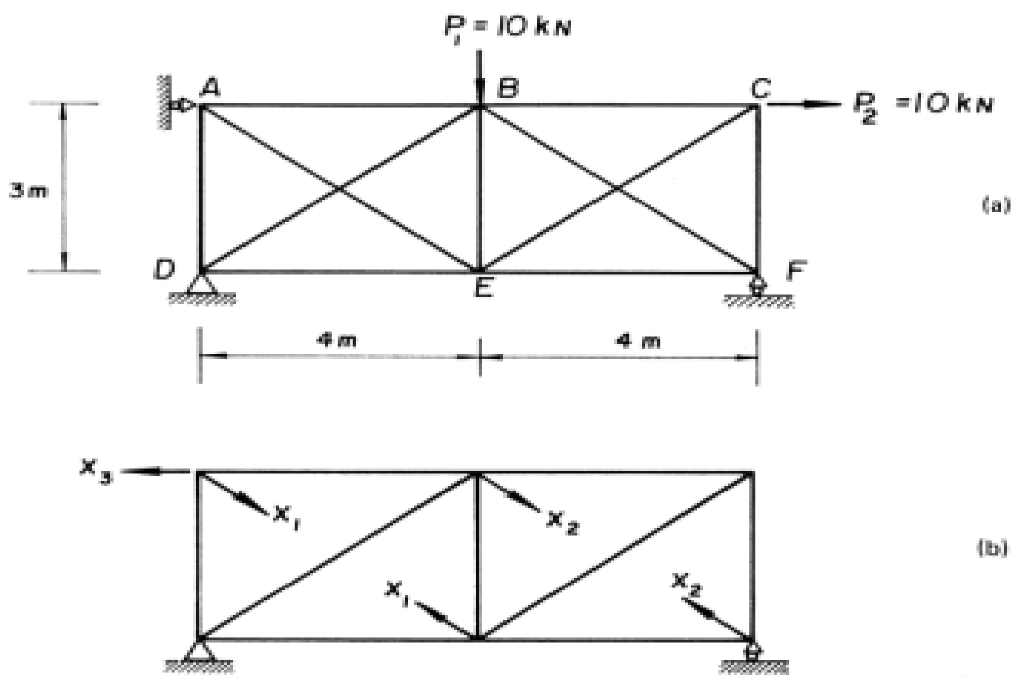

Figure 7.7

As can be easily seen, the truss is redundant to the second degree. For the selection of the redundant members several choices exist. Here members AE and BE and the reaction at A are taken as the redundants, then the truss is reduced to a determinate one as shown in Fig. 7.7(b).

To determine the $[B_0]$ matrix, P_1 and P_2 are set unit values one at a time with all other forces including the redundants held at zero, then calculate the internal forces in all members for each case. Thus,

$$
\begin{array}{c}
\\
[B_0] =
\end{array}
\begin{array}{cc}
P_1 = 1 & P_2 = 1 \\
\left[\begin{array}{cc}
0 & 0 \\
-0.667 & +0.500 \\
-0.500 & -0.375 \\
0 & 0 \\
+0.667 & 0.500 \\
0 & 0 \\
-0.833 & +0.625 \\
+0.833 & +0.625 \\
-0.500 & -0.375 \\
0 & 0 \\
0 & 0
\end{array}\right] &
\end{array}
\begin{array}{c}
\text{Member} \\
AB \\ BC \\ CF \\ FE \\ ED \\ DA \\ DB \\ CE \\ BE \\ AE \\ BF
\end{array}
$$

Similarly to determine the $[B_1]$ matrix the redundants X_1, X_2 and X_3 are set unit values one at a time with all other forces including the applied loads held at zero. The internal forces in each case are

$$
\begin{array}{c}
\\
[B_1] =
\end{array}
\begin{array}{ccc}
X_1 = 1 & X_2 = 1 & X_3 = 1 \\
\left[\begin{array}{ccc}
-0.80 & 0 & 1.0 \\
0 & -0.8 & +0.5 \\
0 & -0.6 & +0.375 \\
0 & -0.8 & 0 \\
-0.8 & 0 & -0.5 \\
-0.6 & 0 & 0 \\
1.0 & 0 & -0.625 \\
0 & 1.0 & -0.625 \\
-0.6 & -0.6 & 0.375 \\
1.0 & 0 & 0 \\
0 & 1.0 & 0
\end{array}\right] & &
\end{array}
\begin{array}{c}
\text{Member} \\
AB \\ BC \\ CF \\ FE \\ ED \\ DA \\ DB \\ CE \\ BE \\ AE \\ BF
\end{array}
$$

The flexibility matrix for the members is

$$
[F] = \frac{1}{EA}
\begin{bmatrix}
2 & & & & & & & & & & \\
& 2 & & & & & & & & & \\
& & 3 & & & & & & & & \\
& & & 4 & & & & & & & \\
& & & & 4 & & & & & & \\
& & & & & 3 & & & & & \\
& & & & & & 5 & & & & \\
& & & & & & & 5 & & & \\
& & & & & & & & 3 & & \\
& & & & & & & & & 5 & \\
& & & & & & & & & & 5
\end{bmatrix}
$$

From [7.36]

$$[H] = [B_0 \mid B_1]^T [F] [B_0 \mid B_1]$$

substituting and carrying out the matrix multiplications gives

$$[H] = \frac{1}{EA}\left[\begin{array}{cc|ccc} 11.111 & 1.792 & -5.400 & 7.031 & -3.125 \\ 1.792 & 6.25 & 2.200 & 3.675 & -5.250 \\ \hline -5.400 & 2.200 & 16.000 & 1.080 & -3.800 \\ 7.031 & 3.675 & 1.080 & 16.000 & -5.275 \\ -3.125 & -5.250 & -3.800 & -5.275 & 8.250 \end{array}\right]$$

$$= \left[\begin{array}{c|c} H_{pp} & H_{px} \\ \hline H_{xp} & H_{xx} \end{array}\right]$$

The redundants are determined using [7.40]

$$[X] = \begin{bmatrix} 16.000 & 1.080 & -3.800 \\ 1.080 & 16.000 & -5.275 \\ -3.800 & -5.275 & 8.250 \end{bmatrix}^{-1} \begin{bmatrix} 5.400 & 2.200 \\ 7.031 & 3.675 \\ -3.125 & -5.250 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Solving for the redundants,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 4.617 \\ -3.743 \\ 9.883 \end{bmatrix}$$

The bar forces are determined using [7.31]

$$\begin{bmatrix} N_{AB} \\ N_{BC} \\ N_{CF} \\ N_{FE} \\ N_{ED} \\ N_{DA} \\ N_{DB} \\ N_{CE} \\ N_{BE} \\ N_{AE} \\ N_{BF} \end{bmatrix} = \begin{bmatrix} 6.197 \\ 6.265 \\ -2.799 \\ 2.997 \\ 3.054 \\ -2.759 \\ -3.654 \\ 4.661 \\ -5.558 \\ 4.598 \\ 3.747 \end{bmatrix}$$

INTRODUCTION TO MATRIX ANALYSIS

EXAMPLE 7.2 *Determine the shear force and bending moment values in the continuous beam of Fig. 7.8 using the flexibility matrix method.*

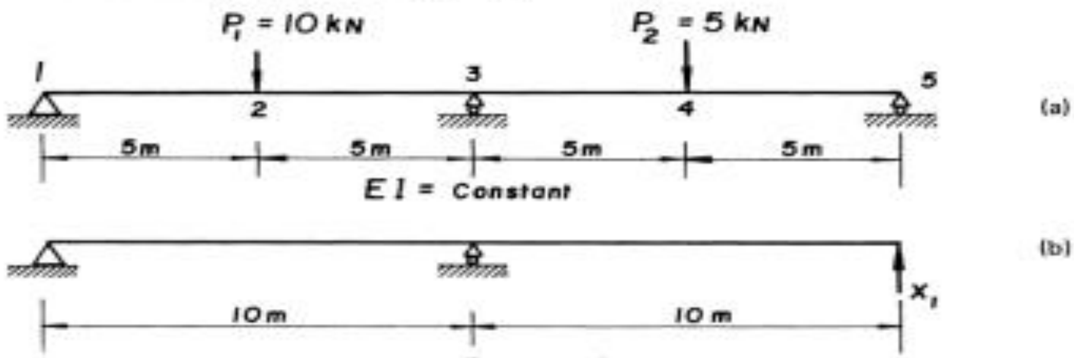

Figure 7.8

The beam is indeterminate to the first degree and the reaction at mode 5 is chosen as the redundant as shown in Fig. 7.8(b).

To determine the $[B_0]$ matrix, P_1 and P_2 are set unit values one at a time, thus

$$[B_0] = \begin{array}{c} P_1 = 1 \quad P_2 = 1 \\ \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0 \\ -0.5 & -0.5 \\ 2.5 & -2.5 \\ 0 & 1 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{array}$$

Similarly the $[B_1]$ matrix is determined setting $X_1 = 1$, thus

$$[B_1] = \begin{array}{c} X_1 = 1 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \\ -1 \\ 10 \\ -1 \\ 5 \end{bmatrix} \end{array} \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{array}$$

The member flexibility matrix from [7.20] neglecting the axial deformation is

$$[F] = \frac{0.833}{EI}\begin{bmatrix} \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & & & \\ & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & & \\ & & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & \\ & & & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} \end{bmatrix}$$

Using [7.36] to solve for the [H] matrix

$$[H] = [B_0 \mid B_1]^{T}[F][B_0 \mid B_1]$$

$$[H] = \frac{0.833}{EI}\left[\begin{array}{cc:c} 25.0 & -37.5 & 75.0 \\ -37.5 & 150.0 & -325.0 \\ \hdashline 75.0 & -325.0 & 800.0 \end{array}\right]$$

$$= \left[\begin{array}{c:c} H_{pp} & H_{px} \\ \hdashline H_{xp} & H_{xx} \end{array}\right]$$

The redundant is determined using [7.40]. Hence,

$$X_1 = -\frac{1}{800}\,[75 \quad -325]\begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= 1.09375 \text{ kN}$$

The shear force and bending moment values at the indicated nodes are obtained using [7.31].

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{bmatrix} = \left[\begin{array}{cc:c} 0.5 & -0.5 & 1 \\ 0 & 0 & 0 \\ -0.5 & -0.5 & 1 \\ 2.5 & -2.5 & 5 \\ 0 & 1 & -1 \\ 0 & -5 & 10 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{array}\right]\left[\begin{array}{c} 10 \\ 5 \\ \hdashline 1.09375 \end{array}\right] = \begin{bmatrix} 3.594 \\ 0 \\ -6.406 \\ 17.969 \\ 3.906 \\ -14.063 \\ -1.094 \\ 5.469 \end{bmatrix}$$

INTRODUCTION TO MATRIX ANALYSIS

THE STIFFNESS METHOD

As in the flexibility method, the stiffness method considers a structure as an assemblage of individual members. The connecting members are called *node points*. The fundamental difference in using the stiffness method is that the individual displacements of the nodes are taken as the unknowns. In the stiffness method the number of unknowns to be determined is the same as the degree of kinematic indeterminateness.

In this method, the first step is to derive the stiffness matrix for a component member by relating the member forces to the member deformations. In a similar manner the nodal forces must be related to the nodal displacements by the total stiffness matrix obtained from an assemblage of the stiffness matrices of the individual members. Finally, from equilibrium conditions the nodal forces obtained from the unknown nodal displacements must balance the externally applied nodal forces to find the total solution; that is, determining all unknown displacements, reactions and member forces. In developing the stiffness method, the same coordinate systems are used that were employed in the flexibility method.

Member Stiffnesses

The relationship between the forces acting at the nodes P_i and their corresponding nodal displacements forms the stiffness matrix approach. This relationship is given in matrix notation by [7.14] and in its generalised form by [7.15].

Consider a prismatic axial rod element m the ends of which are denoted as i and j as shown in Fig. 7.9.

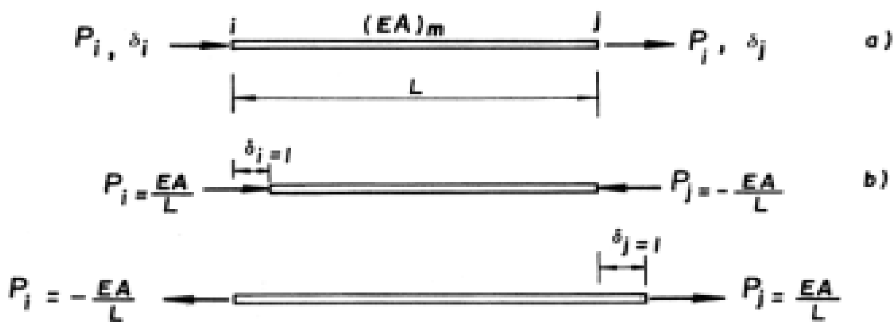

Figure 7.9

The relationship between the axial forces and the corresponding displacements of the rod is

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix} \qquad [7.41]$$

The coefficients of the stiffness matrix are found by considering two distinct displacement states. The first state is to let the nodal coordinate displacement $\delta_i = 1$ as shown in Fig. 7.9(b) while holding the others at zero. Imposing equilibrium on the forces gives

$$P_i = -P_j = \left(\frac{EA}{L}\right)_m \qquad [7.42]$$

The second state is similar, but distinct from the first. Following the same procedure as in the first state provides

$$P_j = -P_i = \left(\frac{EA}{L}\right)_m \qquad [7.43]$$

Combining the results given by [7.42] and [7.43] into a single matrix equation yields the force–displacement relationship of an axial rod element

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \left(\frac{EA}{L}\right)_m \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix} \qquad [7.44]$$

Consider a prismatic beam element shown in Fig. 7.10. Using the same procedure used in obtaining [7.23] to [7.25] the force–displacement relationship for the given nodal coordinate system may be determined by assigning unit values to the displacements as shown in Fig. 7.10(b) and (c). The coefficients are shown for unit displacements at the end i; similar

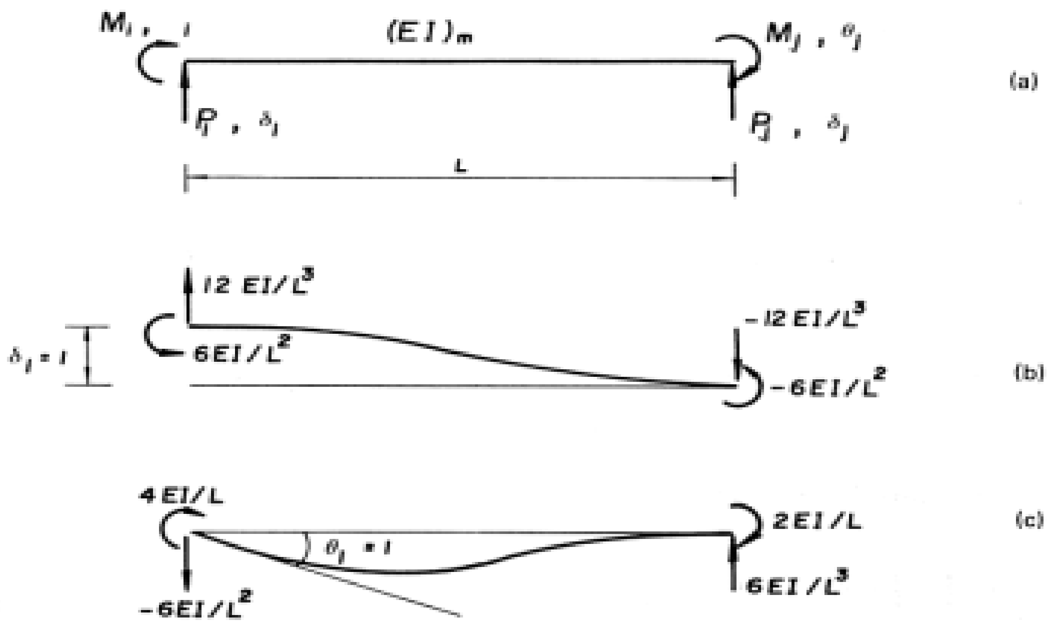

Figure 7.10

coefficients are obtained for unit displacements at end j. Thus

$$\begin{bmatrix} P_i \\ M_i \\ P_j \\ M_j \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{bmatrix} \delta_i \\ \theta_i \\ \delta_j \\ \theta_j \end{bmatrix} \qquad [7.45]$$

Transformation Matrices

If the properties of an element is known in terms of *local* axes, the transformations of these forces and displacements to the *global* coordinates is a necessary step in stiffness matrix formulation. Figure 7.11 shows member ij described by the two coordinate systems. The local coordinates are shown as x' and y' and global coordinates as x and y. In this text, the forces displacements and stiffness matrices with respect to local axes are identified by primes. The prime is omitted when written with respect to global axes.

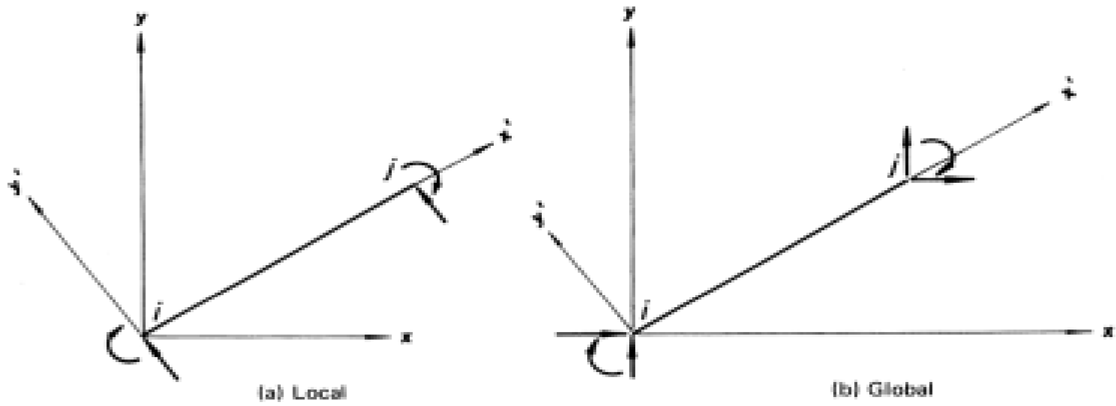

Figure 7.11

Referring to Fig. 7.11, the relationship between the quantities in the local and global axes for flexural members is established as

$$\begin{bmatrix} P'_{xi} \\ P'_{yi} \\ M'_i \\ P'_{xj} \\ P'_{yj} \\ M'_j \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} P_{xi} \\ P_{yi} \\ M_i \\ P_{xj} \\ P_{yj} \\ M_j \end{bmatrix} \qquad [7.46]$$

and for axial members, after omitting M and θ, the relationship is

$$\begin{bmatrix} P'_{xi} \\ P'_{yi} \\ P'_{xj} \\ P'_{yj} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} P_{xi} \\ P_{yi} \\ P_{xj} \\ P_{yj} \end{bmatrix} \qquad [7.47]$$

Equations [7.46] and [7.47] may be written in compact matrix form

$$[P'] = [T]\,[P] \qquad [7.48]$$

where $[T]$ is the *rotational transformation matrix*, which is a function of the direction cosines between the two sets of axes, for the particular system shown. Solving [7.48] for $[P]$

$$\begin{aligned}[P] &= [T^{-1}]\,[P'] \\ &= [T]^{\mathrm{T}}\,[P']\end{aligned} \qquad [7.49]$$

Such a matrix is called an *orthogonal matrix*, which may be defined as a square matrix having an inverse equal to its transpose.

If the displacements are denoted by $[\delta]$ then it follows that

$$[\delta'] = [T]\,[\delta] \qquad [7.50]$$

The transformation matrix $[T]$ may be applied to obtain the stiffness matrix in global coordinates. From the definition of stiffness, that is $[P] = [k]\,[\delta]$, it follows that

$$[P'] = [k']\,[\delta'] \qquad [7.51]$$

Substituting [7.48] and [7.50] into [7.51] and noting that $T^T = T^{-1}$ for orthogonal matrices

$$[T]\,[P] = [k']\,[T]\,[\delta]$$

or

$$\begin{aligned}[P] &= [T^{-1}]\,[k']\,[T]\,[\delta] \\ &= [T]^T\,[k']\,[T]\,[\delta] \\ &= [k]\,[\delta]\end{aligned} \qquad [7.52]$$

Hence, the transformed stiffness matrix is given by

$$[k] = [T]^T\,[k']\,[T] \qquad [7.53]$$

Using the relationship derived above, the stiffness matrix for axial members

(Fig. 7.11) in global coordinates will be

$$[k] = \left(\frac{EA}{L}\right)_m \left[\begin{array}{cc|cc} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ \hline -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{array}\right] \qquad [7.54]$$

where $\lambda = \cos\alpha$ and $\mu = \sin\alpha$. Similarly, for flexural members

$$[k] = \left(\frac{EI}{L^3}\right)_m \left[\begin{array}{ccc|ccc} 12\mu^2 & & & \text{Symmetric} & & \\ -12\lambda\mu & 12\lambda^2 & & & & \\ 6L\mu & -6L\lambda & 4L^2 & & & \\ \hline -12\mu^2 & 12\lambda\mu & -6L\mu & 12\mu^2 & & \\ 12\lambda\mu & -12\lambda^2 & 6L\lambda & -12\lambda\mu & 12\lambda^2 & \\ 6L\mu & -6L\lambda & 2L^2 & -6L\mu & 6L\lambda & 4L^2 \end{array}\right] \qquad [7.55]$$

Assembly of Element Matrices

It is important to form the total assemblage nodal stiffness matrix of a structure from the stiffness matrices of the separate structural elements. This involves only simple additions when all element stiffness matrices have been expressed in the same global coordinate system.

Consider the axial member system shown in Fig. 7.12 with a total of three possible joint displacements one for each node. The members have individual stiffness constants $(EA/L)_1$ and $(EA/L)_2$ as shown in the figure.

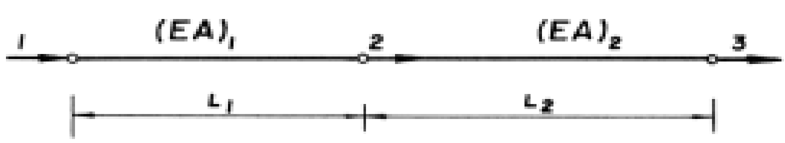

Figure 7.12

The order of the stiffness matrix for the assemblage will be 3 × 3. The individual member stiffness matrices are:

$$[k_1] = \left(\frac{EA}{L}\right)_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

$$[k_2] = \left(\frac{EA}{L}\right)_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} 3 \\ 4 \end{matrix} \qquad [7.56]$$

The assembled stiffness matrix for the complete system can be formed by superposition of the individual element stiffnesses contributing to each nodal point. Thus, the assembled stiffness matrix for the system shown in Fig. 7.12 becomes

$$[K] = \begin{bmatrix} \left(\frac{EA}{L}\right)_1 & -\left(\frac{EA}{L}\right)_1 & 0 \\ -\left(\frac{EA}{L}\right)_1 & \left(\frac{EA}{L}\right)_1 + \left(\frac{EA}{L}\right)_2 & -\left(\frac{EA}{L}\right)_2 \\ 0 & -\left(\frac{EA}{L}\right)_1 & \left(\frac{EA}{L}\right)_2 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \qquad [7.57]$$

In a similar manner, it may be concluded that the order of the global stiffness matrix of a system is equal to the total number of degrees of freedom of the system. The order of the matrix may be expressed as the sum of the unknown displacements f and the prescribed (support) displacements s. After reordering the rows and columns to separate the elements corresponding to the supports from the remainder, the rearranged stiffness matrix may be written as

$$\begin{bmatrix} P_f \\ P_s \end{bmatrix} = \begin{bmatrix} K_{ff} & K_{fs} \\ K_{sf} & K_{ss} \end{bmatrix} \begin{bmatrix} \Delta_f \\ \Delta_s \end{bmatrix} \qquad [7.58]$$

Method of Solution

Expanding [7.58] and noting that the support displacements, $\{\Delta_s\} = 0$

$$[P_f] = [K_{ff}][\Delta_f] \qquad [7.59(a)]$$

$$[P_s] = [K_{sf}][\Delta_f] \qquad [7.59(b)]$$

The vectors of all unknown nodal displacements (at unsupported nodes) are obtained from [7.59(a)]

$$[\Delta_f] = [K_{ff}]^{-1}[P_f] \qquad [7.60]$$

When $[\Delta_f]$ has been found from [7.60], the support reactions by substitution of the results in [7.59(b)] will be

$$[P_s] = [K_{sf}][K_{ff}]^{-1}[P_f] \qquad [7.61]$$

The internal force in any element m may be obtained by substituting the calculated degrees of freedom for that element, designated by $[\Delta_m]$, into the element stiffness matrix $[k_m]$. Thus, the joint force component acting on that element becomes

$$[P_m] = [k_m][\Delta_m] \qquad [7.62]$$

INTRODUCTION TO MATRIX ANALYSIS

For the case of an axial member shown in Fig. 7.11 the member force, denoted by P_m, is found to be

$$P_m = P_{xj}\cos\alpha + P_{yj}\sin\alpha \qquad [7.63]$$

but

$$P_{xj} = \frac{EA}{L}((\delta_{xj} - \delta_{xi})\cos^2\alpha + (\delta_{yj} - \delta_{yi})\cos\alpha\sin\alpha)$$

$$P_{yj} = \frac{EA}{L}((\delta_{xj} - \delta_{xi})\cos\alpha\sin\alpha + (\delta_{yj} - \delta_{yi})\sin^2\alpha) \qquad [7.64]$$

Rearranging and writing in matrix form, [7.64] may be written as

$$P_m = \left(\frac{EA}{L}\right)_m [\cos\alpha \quad \sin\alpha] \begin{bmatrix} \delta_{xj} - \delta_{xi} \\ \delta_{yj} - \delta_{yi} \end{bmatrix} \qquad [7.65]$$

Similarly, for the case of a beam element [7.45] and the true nodal displacements are used and the internal forces for the beam element taken to be the shear and bending moments are

$$\begin{bmatrix} V_i \\ M_i \end{bmatrix} = \left(\frac{EI}{L^3}\right)_m \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \end{bmatrix} \begin{bmatrix} \delta_{yi} \\ \theta_i \\ \delta_{yi} \\ \theta_j \end{bmatrix} \qquad [7.66]$$

When these are external loads acting between the joints of a beam element the concept of equivalent loads must be adopted. The member action is then computed by adding the effects of the member end deformation to the fixed-end actions produced by the loads. In a similar manner, the support reactions are computed by adding the fixed-end effects of the loads. Thus [7.14] may be written as

$$[P] = [K][\Delta] - [P^F] \qquad [7.67]$$

where $[P^F]$ is the load vector of the fixed-end actions.

EXAMPLE 7.3 *Determine the bar forces, using the stiffness matrix method, of the truss shown in Fig. 7.13. EA = constant.*

Member data for the truss

Member	Member ends		Member properties		Direction cosines	
	i	*j*	*A*	*L*	*cos α*	*sin α*
1	A	B	*A*	*L*	0	1
2	A	C	*A*	1.155*L*	–0.5	–0.866
3	A	D	*A*	1.4146*L*	0.707	–0.707

METHODS OF STRUCTURAL ANALYSIS

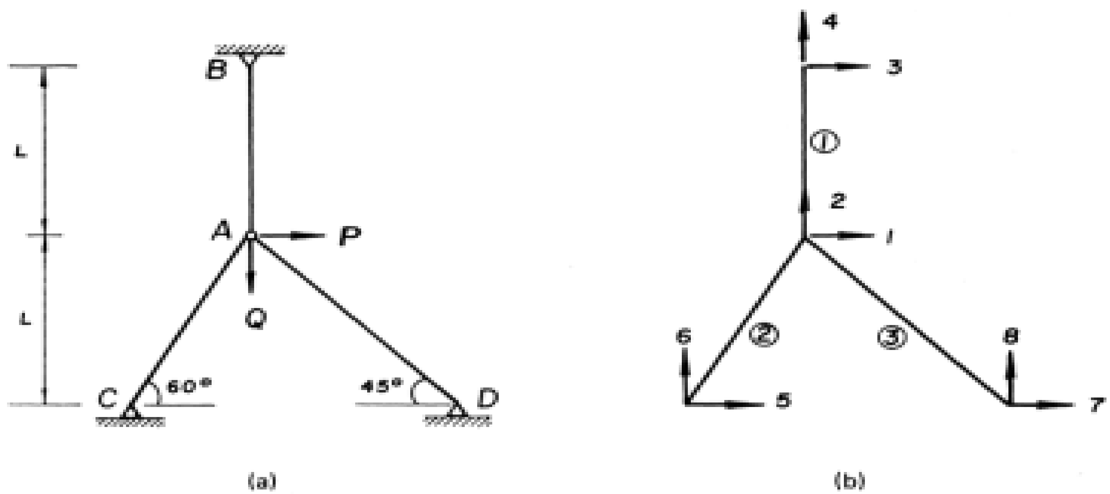

Figure 7.13

The member stiffnesses oriented in global coordinate system are obtained from [7.54].

Member AB

$$[k_1] = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Member AC

$$[k_2] = \frac{EA}{1.155L}\begin{bmatrix} 0.25 & 0.433 & -0.25 & -0.433 \\ 0.433 & 0.75 & -0.433 & -0.75 \\ -0.25 & -0.433 & 0.25 & 0.433 \\ -0.433 & -0.75 & 0.433 & 0.75 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

$$k_3 = \frac{EA}{1.414L}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 7 \\ 8 \end{matrix}$$

After assemblage of the element stiffness matrices, the global stiffness equation becomes

$$\begin{bmatrix} P \\ -Q \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L} \left[\begin{array}{cc:cccccc} 0.570 & 0.021 & 0 & 0 & -0.217 & -0.375 & -0.354 & 0.354 \\ 0.021 & 2.00 & 0 & -1.0 & -0.375 & -0.650 & 0.354 & -0.354 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 1.0 & 0 & 0 & 0 & 0 \\ -0.217 & -0.375 & 0 & 0 & 0.217 & 0.375 & 0 & 0 \\ -0.375 & -0.650 & 0 & 0 & 0.375 & 0.650 & 0 & 0 \\ -0.354 & 0.354 & 0 & 0 & 0 & 0 & 0.354 & -0.354 \\ 0.354 & -0.354 & 0 & 0 & 0 & 0 & -0.354 & 0.354 \end{array}\right] \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \end{bmatrix}$$

Note that the displacements Δ_3 to Δ_8 are restrained and the elements corresponding to Δ_1 and Δ_2 are placed at the top left of the global stiffness matrix. Hence the reduced stiffness matrix given by [7.59(a)] is written as

$$\begin{bmatrix} P \\ -Q \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.570 & 0.021 \\ 0.021 & 2.00 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$$

The unknown displacements are obtained by applying [7.60],

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{L}{EA} \begin{bmatrix} 1.755 & -0.018 \\ -0.018 & 0.500 \end{bmatrix} \begin{bmatrix} P \\ -Q \end{bmatrix}$$

Having found the unknown displacements, the reactions are calculated by applying [7.61]. Thus

$$\begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Cx} \\ P_{Cy} \\ P_{Dx} \\ P_{Dy} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1.0 \\ -0.217 & -0.375 \\ -0.375 & -0.650 \\ -0.354 & 0.354 \\ 0.354 & -0.354 \end{bmatrix} \begin{bmatrix} 1.755 & -0.018 \\ -0.018 & 0.500 \end{bmatrix} \begin{bmatrix} P \\ -Q \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0.018 & -0.500 \\ -0.373 & -0.184 \\ -0.646 & 0.318 \\ -0.628 & 0.183 \\ 0.628 & -0.183 \end{bmatrix} \begin{bmatrix} P \\ -Q \end{bmatrix}$$

The internal member forces are obtained by using Eq. [7.65]. Thus

$$P_m = \frac{EA}{L}[\cos\alpha \quad \sin\alpha]\begin{bmatrix}\Delta_{xj} - \Delta_{xi}\\ \Delta_{yj} - \Delta_{yi}\end{bmatrix}$$

Since $\Delta_{xj} = \Delta_{yj} = 0$ because of support conditions, the member forces are:

Member AB

$$P_{AB} = [0 \quad 1]\begin{bmatrix}-1.755P - 0.018Q\\ 0.018P + 0.50Q\end{bmatrix}$$

$$= 0.018P + 0.50Q$$

Member AC

$$P_{AC} = \frac{1}{1.155}[-0.5 \quad -0.866]\begin{bmatrix}-1.755P - 0.018Q\\ 0.018P + 0.50Q\end{bmatrix}$$

$$= 0.746P + 0.367Q$$

Member AD

$$P_{AD} = \frac{1}{1.414}[0.707 \quad -0.707]\begin{bmatrix}-1.755P - 0.018Q\\ 0.018Q + 0.50Q\end{bmatrix}$$

$$= -0.887P - 0.259Q$$

EXAMPLE 7.4 *Find the support rotations and the support reactions of the continuous beam shown in Fig. 7.14.*

The equivalent fixed-end actions are shown in Fig. 7.14(b).

The member stiffnesses are (see [7.45]):

Member AB

$$[k_1] = \frac{EI}{512}\begin{bmatrix}12 & -48 & -12 & -48\\ -48 & 256 & 48 & 128\\ -12 & 48 & 12 & 48\\ -48 & 128 & 48 & 256\end{bmatrix}\begin{matrix}1\\2\\3\\4\end{matrix}$$

INTRODUCTION TO MATRIX ANALYSIS

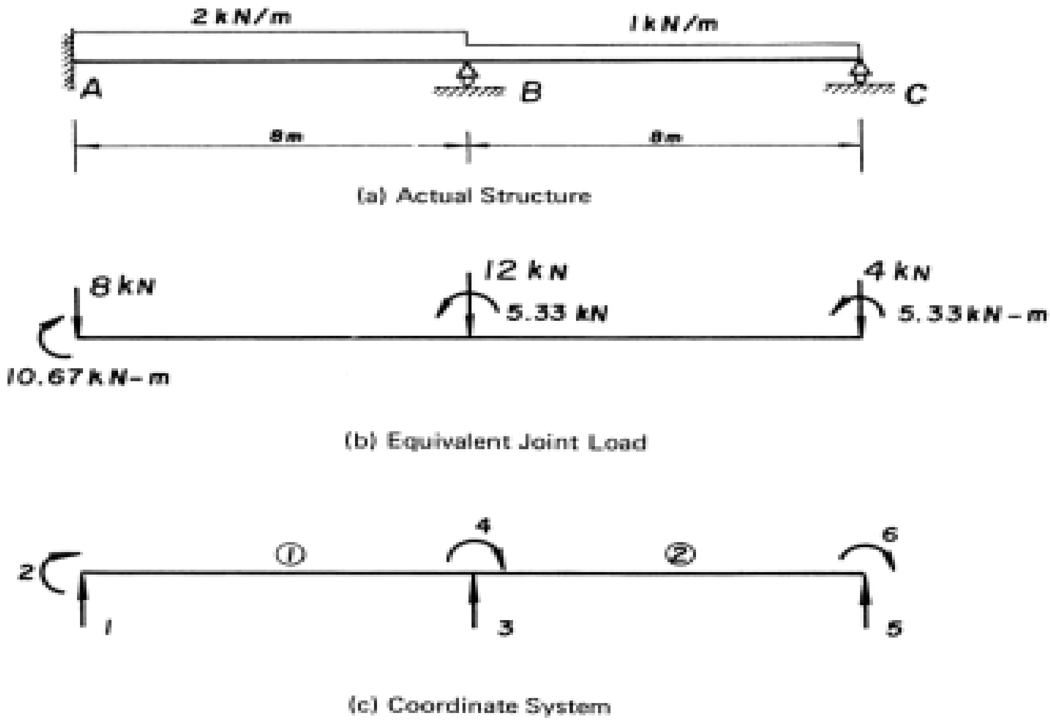

Figure 7.14

Member BC

$$[k_2] = \frac{EI}{512}\begin{bmatrix} 12 & -48 & -12 & -48 \\ -48 & 256 & 48 & 128 \\ -12 & 48 & 12 & 48 \\ -48 & 128 & 48 & 256 \end{bmatrix}\begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

The assembled stiffness matrix is

$$[K] = \frac{EI}{512}\begin{bmatrix} 12 & -48 & -12 & -48 & 0 & 0 \\ -48 & 256 & 48 & 128 & 0 & 0 \\ -12 & 48 & 24 & 0 & -12 & -48 \\ -48 & 128 & 0 & 512 & 48 & 128 \\ 0 & 0 & -12 & 48 & 12 & 48 \\ 0 & 0 & -48 & 128 & 48 & 256 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Note that the prescribed displacements are $\delta_1 = \delta_2 = \delta_5 = 0$ and δ_4 and δ_6 are the unknown displacements. Thus the assembled stiffness matrix must be rearranged by moving the rows and columns associated to δ_4 and δ_6 to the top left of the stiffness matrix. Thus

$$[K] = \frac{EI}{512}\begin{bmatrix} 512 & 128 & -48 & 128 & 0 & 48 \\ 128 & 256 & 0 & 0 & -48 & 48 \\ -48 & 0 & 12 & -48 & -12 & 0 \\ 128 & 0 & -48 & 256 & 48 & 0 \\ 0 & -48 & -12 & 48 & 24 & 12 \\ 48 & 48 & 0 & 0 & -12 & 12 \end{bmatrix}\begin{matrix} 4 \\ 6 \\ 1 \\ 2 \\ 3 \\ 5 \end{matrix}$$

$$= \left[\begin{array}{c:c} K_{ff} & K_{fs} \\ \hdashline K_{sf} & K_{ss} \end{array}\right]$$

The submatrix $[K_{ff}]$ is

$$[K_{ff}] = \frac{EI}{512}\begin{bmatrix} 512 & 128 \\ 128 & 256 \end{bmatrix}$$

and its inverse is

$$[K_{ff}]^{-1} = \frac{1}{224EI}\begin{bmatrix} 256 & -128 \\ -128 & 512 \end{bmatrix}$$

The load vector for the beam shown in Fig. 7.14(b) is

$$[P] = \begin{bmatrix} -8 \\ 10.67 \\ -12 \\ -5.33 \\ -4 \\ -5.33 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Rearranging the vector in conformity with the stiffness matrix is

$$[P] = \left[\begin{array}{c} -5.33 \\ -5.33 \\ \hdashline -8 \\ 10.67 \\ -12 \\ -4 \end{array}\right]\begin{matrix} 4 \\ 6 \\ 1 \\ 2 \\ 3 \\ 5 \end{matrix}$$

INTRODUCTION TO MATRIX ANALYSIS

The unknown rotations at support B and C are computed using [7.60]

$$[\Delta_f] = \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = [K_{ff}]\,[P_f]$$

$$= \frac{1}{224EI}\begin{bmatrix} 256 & -128 \\ -128 & 512 \end{bmatrix}\begin{bmatrix} -5.33 \\ -5.33 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -3.05 \\ -9.14 \end{bmatrix}$$

The support reactions due to the displacement contributions are determined by using [7.59(b)]. After adding the fixed-end effects of the loads, the support reactions will be

$$[P_s] = K_{sf}[\Delta_f] - [P^F]$$

$$\begin{bmatrix} R_A \\ M_A \\ R_B \\ R_C \end{bmatrix} = \frac{EI}{512}\begin{bmatrix} -48 & 0 \\ 128 & 0 \\ 0 & -48 \\ 48 & 48 \end{bmatrix}\frac{1}{EI}\begin{bmatrix} -3.05 \\ -9.14 \end{bmatrix} - \begin{bmatrix} -8 \\ 10.67 \\ -12 \\ -4 \end{bmatrix} = \begin{bmatrix} 8.29 \\ -11.43 \\ 12.85 \\ 2.86 \end{bmatrix}$$