THEORY OF STRUCTURE II

CHAPTER 2

INFLUENCE LINES FOR INDETERMINATE STRUCTURES

INFLUENCE LINES FOR ARCHES

INFLUENCE LINES FOR THREE-HINGED ARCHES

Influence Lines for Horizontal Reaction, Moment, Radial Shear and Normal Thrust – General Case

In this section, we shall consider the action of moving loads over an arch. For this, influence line diagrams for horizontal reaction (H), bending moment (M), normal thrust (N), and radial shear (S) will be discussed here.

Let us consider a typical three-hinged arch with two hinges at A and B and the third hinge at the crown at C as shown in Figure 3.10 (a).

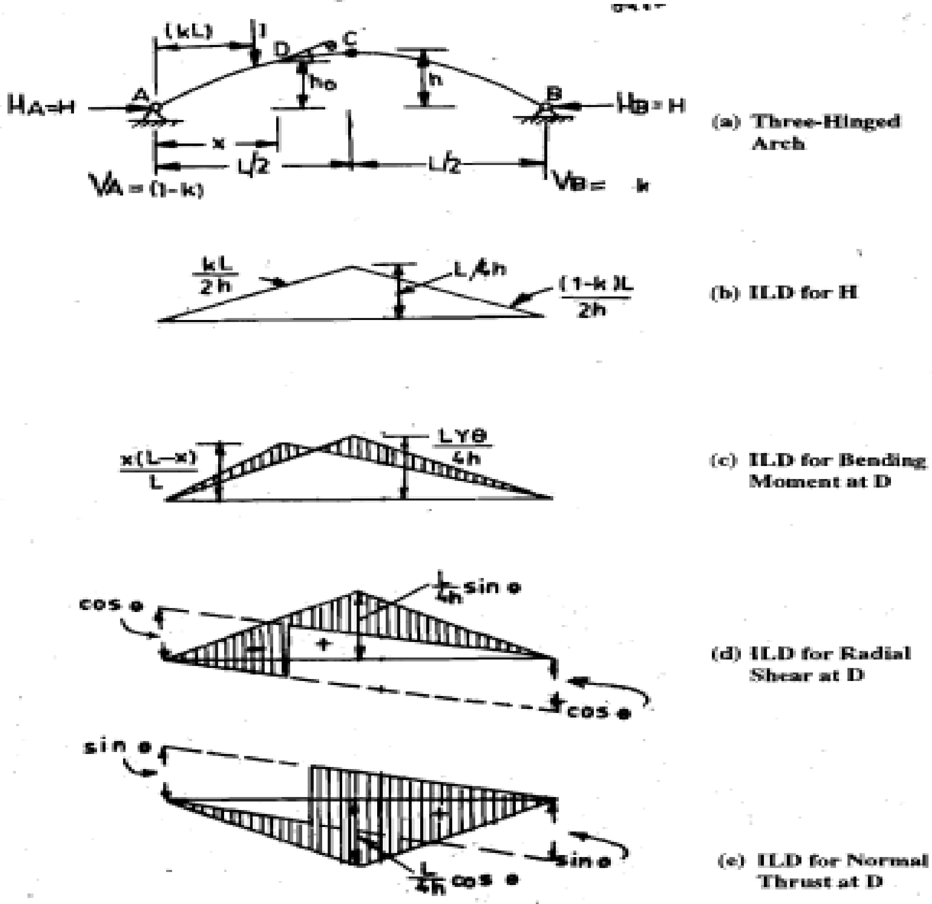


Figure 3.10

Analysis and Influence Line for Horizontal Reaction, H

Consider a unit load acting at a distance "kL" from "A" between A and C. From statics, the vertical reactions can be determined.

Taking moments about A, we get,

$$V_B \times L = 1 \times kL$$

Thus,

Therefore, $V_A = 1 - V_B = 1 - k$.

Also, $\sum H = 0$; we get, $H_A = -H_B = \underset{\rightarrow}{H}$

Now, to determine H, let us take moments about the central hinge at C, we get, $\left(V_B \times \frac{L}{2}\right) - (H \times h) = 0$

$$\therefore \quad H = \frac{V_B \times L}{2h} = \frac{kL}{2h} \qquad \text{(ii)}$$

Similarly, when unit load is between C and B, then taking moment of all forces on the left of unit load about C,

we get, $\left(V_A \times \frac{L}{2}\right) - (H \times h) = 0$

$$\therefore \quad H = \frac{V_A \times L}{2h} = \frac{(1-k)\,L}{2h} \qquad \text{(iii)}$$

From both values of H obtained in Eq. (ii) and (iii), we observe that it is directly related to position of unit load which is kL in this case. The variation of value of H can be obtained by assigning different values to kL. For example, at A, we have, $kL = 0$; $H = 0$ at A. Also, at B, we have, $kL = l$; therefore, substituting this value in Eq. (iii), we get, $H = 0$. As can be seen, the maximum value for H from both the equations is at $\frac{L}{2}$, i.e. at C and is $H = \frac{L}{4h}$.

This variation has been drawn in Figure 3.10 (b) and represents the influence line for horizontal reaction H.

Analysis and Influence Line for Bending Moment, *M*

Consider a point D on the arch whose horizontal distance is x from A and vertical height is y_D from base.

Then, by taking moment of all forces on the right of D about D, we can write

$$M_D = V_B \times (L - x) - H \times y_D \qquad \text{(iv)}$$

Similarly, when unit load is between C and B:

$$M_D = (V_A \times x) - (H \times y_D) \qquad \text{(v)}$$

where $V_B \times (L - x)$ is in fact the "beam bending moment" (μ) at D and $H \times y_D$ is the moment caused by the horizontal thrust of the arch.

Thus, M_D comprises two parts and can be written in general form as given below :

$$M_D = \mu - (H \times y_D) \qquad \text{(vi)}$$

where μ is the bending moment at the point under consideration if AB is considered as a simple straight beam and y_D is the ordinate of that point

$\mu = \frac{x\,(L - x)}{L}$ for unit load at D.

So, we can conveniently draw the variation for moment at D from Eq. (vi) by first drawing the ILD of beam bending moment for D and then superimposing on it the ILD for Hy shown in Figure 3.10 (b). This is shown in Figure 3.10 (c) wherein the net moment ($\mu - Hy_D$) is shown hatched.

Analysis and Influence Line for Radial Shear, *S*

The radial shear at any section D comprises two parts, namely the shear caused by the vertical loads and the shear caused by horizontal reaction H.

Consider the free body diagram shown in Figure 3.7 (d).

It can be seen that by resolving forces V_A and H along the normal to the tangent at D, the radial shear at D works out to be as follows :

$$S = S' \cos\theta - H \sin\theta \qquad \text{(vii)}$$

Here θ is the inclination of arch axis to the horizontal and S' is the beam shear at D.

So, here again the IL for "radial shear" can be obtained easily in two parts. The first part represents the shear in a simple beam which is multiplied by cos θ. The second part is the influence line for H shown in Figure 3.10 (b) multiplied by sin θ. Once these two are combined as per Eq. (vii), we get IL for radial shear at D and is shown in Figure 3.10 (d). The hatched portion is the resultant diagram.

Anaylsis and Influence Line for Normal Thrust, N

The normal thrust is the axial force acting along the arch axis at any point. In order to understand its concept, focus on Figure 3.7 (d), wherein in a free body diagram, all the forces and their components along the tangent to arch at point D are shown. The components of V_A and H along the tangent give us 'normal thrust' as follows :

$$N = S' \sin \theta + H \cos \theta \qquad \text{(viii)}$$

Thus, normal thrust also has two parts which can be combined to obtain its influence line. It is a compressive force and this is characteristic of an arch. The influence line for 'normal thrust' is shown in Figure 3.10 (e). The hatched portion gives the net values.

To Determine Maximum Bending Moment due to a Concentrated Rolling Load

Let us now consider the application of concepts derived earlier (Section 1.5 : Unit 1) through a numerical example and see how influence line concept can be used to locate given loads to obtain maximum values.

Example 3.6

We have a three hinged parabolic arch of 25 m span with a central rise of 5 m. Let us assume a load of 100 kN rolling over from left to right [Figure 3.11 (a)] and we have to find maximum SF and BM at a section 8 m from A.

Solution

Following the concepts enumerated in Section 3.4.1, the IL for bending moment at D is shown in Figure 3.11 (b).

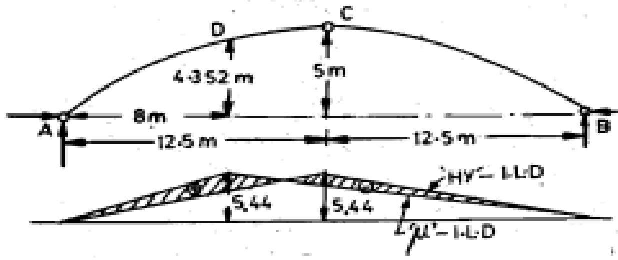


Figure 3.11 : Three-hinged Arch and ILD for BM at D (Shaded)

We have seen in Section 3.4.1,

$$M_D = \mu - (H \times y_D)$$

In this case, $$y_D = \frac{4 \times 5}{25 \times 25} \times 8 \times (25 - 8) = 4.352 \text{ m}$$

We know $\mu = \frac{x(L-x)}{L} = \frac{8 \times (25-8)}{25} = 5.44$ (with apex below D)

and $H \times y_D = \frac{L}{4h} \times 4.352 = \frac{25}{4 \times 5} \times 4.352 = 5.44$ (with apex below C)

The maximum positive BM at D occurs, as can be seen from the influence line, when the load is on the section. Thus,

$$\therefore \quad M_{max\,(Positive)} \text{ at D} = (100 \times 5.44) - \left(100 \times \frac{5.44 \times 8}{12.5}\right)$$

$$= 100 \times 1.96 = 196 \text{ kN m}$$

Similarly, maximum negative BM at D will occur, as can be seen from the influence line, when load is at the crown, i.e. at C. Thus,

$$\therefore \quad M_{max\,(Negative)} \text{ at D} = 100\left[(-5.44) + \frac{5.44 \times 12.5}{17}\right]$$

$$= 100 \times (-1.44) = -144 \text{ kN m}$$

The above results can also be obtained by simple analysis of the arch without using the influence line diagram. This is explained below.

(i) Maximum positive BM occurs when load is at the section D itself.

In this case, $V_B \times 25 = 100 \times 8$; $\therefore V_B = 32$ kN

$$\therefore \quad V_A = 100 - 32 = 68 \text{ kN}$$

Take moments about C, we get,

$$M_C = (V_B \times 12.5) - (H \times 5) = 0$$

$$= (32 \times 12.5) - (H \times 5) = 0$$

$$\therefore \quad H = \frac{32 \times 12.5}{5} = 80 \text{ kN}$$

Therefore, maximum positive BM at section D,

$$M^+_{D\,(max)} = (V_A \times 8) - (H \times y_D)$$

$$= (68 \times 8) - (80 \times 4.352)$$

$$= 544 - 348 = 196 \text{ kN}$$

This is the same result which we obtained from influence line approach earlier.

(ii) Maximum negative BM occurs when load is at the crown at C.

In this case, $V_A = V_B = \frac{100}{2} = 50$

$$M_C = (50 \times 12.5) - (H \times 5) = 0$$

$$\therefore \quad H = 125 \text{ kN}$$

Therefore, maximum negative BM at section D,

$$M^-_{D\,(max)} = (V_A \times 8) - (H \times 4.352)$$

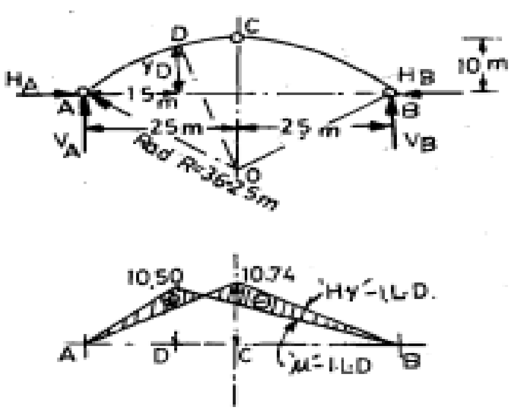
$$= (50 \times 8) - (125 \times 4.352)$$

$$= 400 - 544 = -144 \text{ kN}$$

This is also the same result as obtained by influence line approach earlier.

Example 3.7

Consider a three-hinged circular arch of rise 10 m and span of 50 m, with a load of 100 kN travelling from A to B. We have to determine maximum horizontal thrust and maximum (negative) and (positive) BM at 15 m from A.



(a) Three-Hinged Circular Arch

(b) ILD for BM at D (Shaded)

Figure 3.12 : Three-hinged Circular Arch and ILD for BM at D (Shaded)

Solution

(i) Let us first determine the radius of the arch from the first principle (Figure 3.12)

We have, $$10\,(2R - 10) = 25 \times 25$$

$$\therefore \quad R = 36.25 \text{ m}$$

Also, $$(36.25 - 10 + y_D)^2 + 10^2 = 36.25^2$$

$$\therefore \quad y_D = 8.59 \text{ m}$$

(ii) As already discussed, the ILD for BM (μ) at D can be drawn [Figure 3.12 (b)] and also the ILD for $H\,y_D$ is drawn.

The maximum positive BM will occur when load is at D itself. For this situation, we determine all reactions.

Taking moments about A, we get,

$$(V_B \times 50) - (100 \times 15) = 0$$

$$\therefore \quad V_B = 30 \text{ kN}$$

Thus, $$\therefore \quad V_A = 100 - 30 = 70 \text{ kN}$$

Taking moment about C, we get

$$(V_B \times 25) - (H \times y_C) = 0$$

$$\therefore \quad y_C = 10 \text{ m}$$

We also know, $\therefore \quad H = \dfrac{30 \times 25}{10} = 75 \text{ kN}$

$$\therefore \quad M_D = (V_B \times 35) - (H \times y_D)$$

$$= (30 \times 35) - (75 \times 8.59) = 1050 - 644.25 = \mathbf{405.75 \text{ kN m}}$$

The same value could also be obtained from IL diagram as shown below

$$M_D = \left[10.5 - \frac{10.74}{25} \times 15\right] \times 100 = \mathbf{405.75 \text{ kN m}}$$

(iii) The maximum negative BM will develop when load is at crown C.

In this case, $V_A = V_B = 50$ kN

and $M_C = (50 \times 25) - (H \times 10) = 0$

$\therefore\ H = 125$ kN m

Thus, $\therefore\ M_{D\,(\text{negative})} = (V_A \times 15) - (125 \times 8.59) = -$ **324 kN m**

The same result could also be obtained from IL diagram as shown below :

$$M_{D\,(\text{negative})} = \left[10.74 - \frac{10.5 \times 25}{35}\right] \times 100 = -\mathbf{324\ kN\ m}$$

Example 3.8

Consider a three-hinged parabolic arch of 40 m span and rise of 6 m with inclined concentrated loads as shown in Figure 3.13. We have to determine the horizontal thrust H and the bending moments under the concentrated loads.

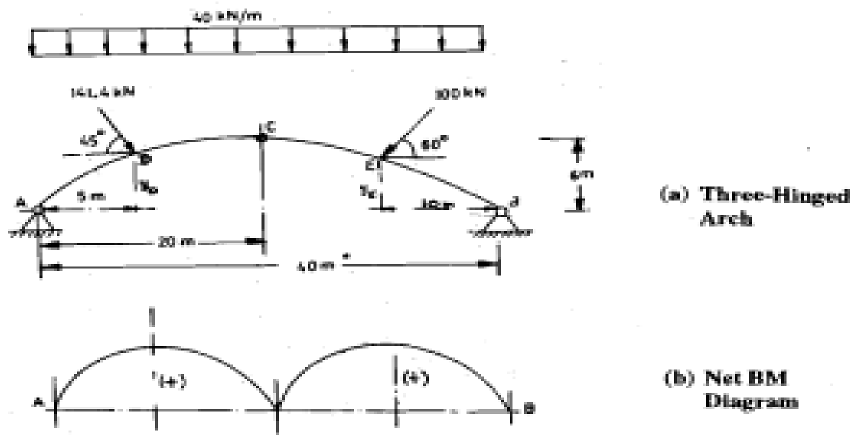


Figure 3.13 : Three Hinged Arch under Concentrated and Uniformly Distributed Load and its Net BMD

Solution

(a) The arch geometry is to be first determined to find ordinates at D and E

$$y = \frac{4 \times h}{L^2} \times x \times (L - x) = \frac{4 \times 6}{40 \times 40} \times (40x - x^2) = \frac{3}{200}(40x - x^2)$$

at $x = 5$ m : $y_D = \frac{3}{200}[(40 \times 5) - 5^2] = \frac{3}{200} \times 175 = 2.63$ m

Similarly, $y_E = 4.5$ m

Also we know that

$$\frac{dy}{dx} = \frac{4h}{L^2}(L - 2x)$$

$$\therefore\ \tan\theta = \left(\frac{dy}{dx}\right)_{x = 5\ \text{m}} = \frac{4 \times 6}{40 \times 40}[40 - (2 \times 5)] = 0.45$$

Thus, $\therefore\ \theta_D = 24.22^\circ$

Finally, we get,

$$\sin\theta = 0.4104$$
$$\cos\theta = 0.9119$$

Similarly, for $x = 30$ m, i.e. point E, we get, $\theta_E = -16.70^o$

In the same way, we get,

$$\sin\theta = 0.2873$$

$$\cos\theta = 0.9578$$

(b) The components of the concentrated loads can also be determined from trigonometry as given below.

For load at D,

Horizontal component $= 141.4 \times \cos 45^o = 100 \text{ kN} \rightarrow$

Vertical component $= 141.4 \times \sin 45^o = 100 \text{ kN} \downarrow$

Similarly, components of 100 kN at E are as follows :

Horizontal component $= 100 \cos 60^o = 50 \text{ kN} \leftarrow$

Vertical component $= 100 \sin 60^o = 86.66 \text{ kN} \downarrow$

(c) In order to determine reactions at supports, let us consider moment of all forces about A, we get,

$$M_A = (100 \times 2.63) + (100 \times 5) + (86.66 \times 30) - (50 \times 4.5) - (V_B \times 40) = 0$$

Thus, we get, $V_B = \mathbf{878.4\ kN}$

$$\therefore \quad V_A = (40 \times 40) + 100 + 86.66 - 878.4$$

and $V_A = \mathbf{908.20\ kN}$

(d) To determine horizontal thrust H; take moments about central hinge C of all forces on left of C, we get

$$(908.20 \times 20) - (100 \times 15) - \{100 \times (6 - 2.63)\} - (H \times 6) = 0$$

Thus, we get $H = \mathbf{1387.8\ kN}$

(e) We have now to determine net moments at D and E.

$$M_D = (908.2 \times 5) - \left(40 \times 5 \times \frac{5}{2}\right) - (1387.8 \times 2.63)$$

$= +$ **391.1 kN m** (sagging) [Figure 3.13 (b)].

$$M_E = (878.4 \times 10) - \left(40 \times 10 \times \frac{10}{2}\right) - (1387.8 \times 4.50)$$

$= +$ **538.9 kN m** (sagging) [Figure 3.13 (b)].

Similarly, if required the bending moment at other points can also be found, e.g. M_x in zone AD

$$M_x = (908.2 \times x) - \left(40 \times \frac{x^2}{2}\right) - \left(1387.8 \times \frac{4 \times 6}{1600} \times x\,(40 - x)\right) = 75.52\,x + 0.817\,x^2$$

The net bending moment diagram is shown in Figure 3.13 (b).

Absolute Maximum Bending Moment

Consider the three-hinged parabolic arch ACB of span L and central rise h. Consider point D on the arch which is at x from A [Figure 3.14 (a)].

The equation for parabola is

$$y = \frac{4h}{L^2}(x)(L - x)$$

The moment at D, as has already been discussed will be as follows :

$$M_D = \mu - (H \times y_D)$$

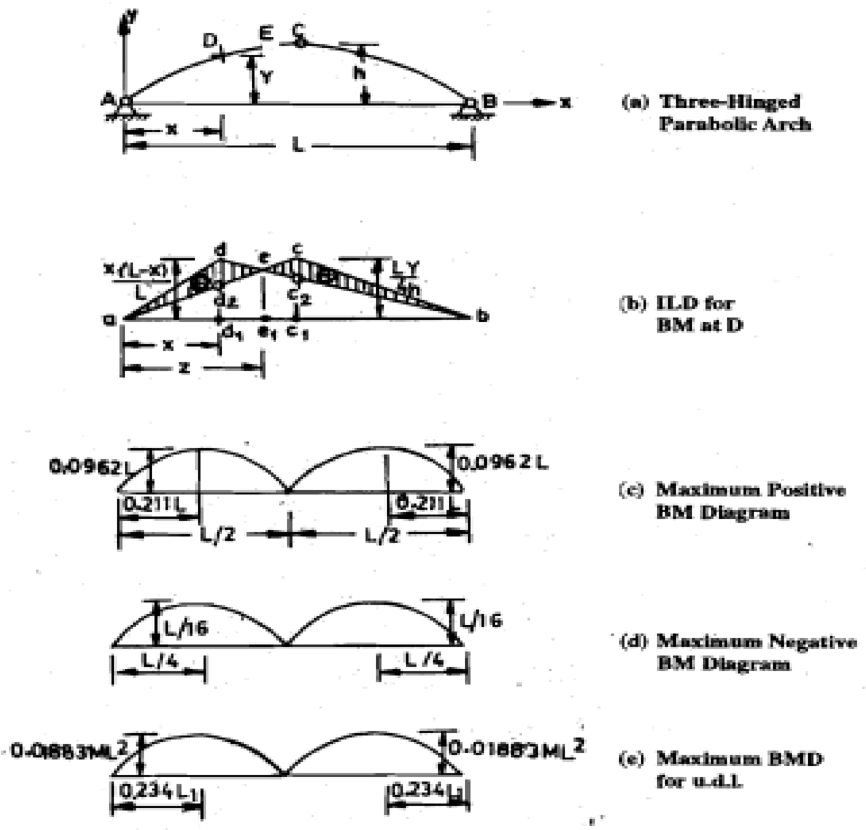


Figure 3.14

This IL for moment at D has been shown in Figure 3.14 (b) in which the ordinate

$$cc_1 = \frac{L}{4h} \times y_C$$

Substituting value of y_C from Eq. (i), we get,

$$cc_1 = \frac{x}{L}(L - x) \qquad \text{(ii)}$$

Thus, we find cc_1 is same as dd_1 which is the ordinate for diagram of μ.

$$\therefore \quad dd_1 = cc_1 = \frac{x}{L}(L - x)$$

An inspection of IL for moment at D clearly indicates that *maximum positive moment* will occur when the load is at section D itself.

The positive moment ordinate, $dd_2 = dd_1 - d_1 d_2$

$$= \frac{x}{L}(L - x) - \frac{x(L - x)}{L}\left(\frac{x \times 2}{L}\right)$$

$$\therefore \quad dd_2 = \frac{x(L - x)}{L^2}(L - 2x) \qquad \text{(iii)}$$

In order to obtain absolute maximum positive moment anywhere on arch, we differentiate Eq. (iii) with respect to x, and equate to zero. This results in a quadratic equation, which on solution gives two values of x, i.e. $x_1 = 0.211\ L$ and $x_2 = 0.789\ L$ which are symmetrically situated points.

Substituting for $x_1 = 0.211\ L$ in Eq. (iii), we get max positive moment $= 0.0962\ L$. The maximum positive bending moment diagram is shown in Figure 3.14 (c).

Similarly, maximum negative moment will occur when unit load is at

$$cc_2 = cc_1 - c_1c_2$$

$$= \frac{x(L-x)}{L} - \frac{x(L-x)}{L}\left(\frac{L/2}{L-x}\right)$$

$$cc_2 = \frac{x(L-2x)}{2L} \qquad \text{(iv)}$$

On differentiation with respect to x and equating to zero, we get the point where absolute maximum negative moment will occur, and it is

$$x = \frac{L}{4}$$

On substituting in Eq. (iv), we get,

$$\therefore \quad \text{Absolute maximum negative moment} = \frac{L}{16}$$

This is shown in Figure 3.14 (d).

The maximum BM diagram for a uniformly distributed load (w) is obtained by multiplying it by the area of the positive/negative (net moment) ILD. The principal values are shown in Figure 3.14 (e).

Example 3.9

Let us examine the case of a three hinged parabolic arch of 4 m rise and 20 m span. Consider a concentrated load of 100 kN travelling from A to B. We are to determine maximum positive and negative moments at a section which is 5 m away from A and then find out absolute maximum bending moment anywhere in the arch ACB (Figure 3.15).

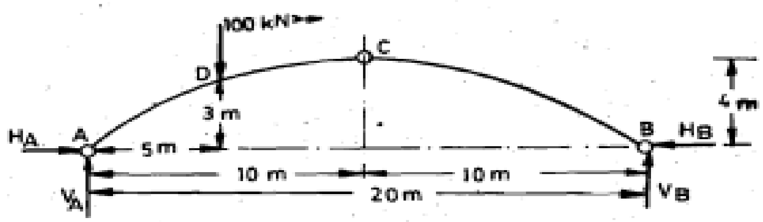


Figure 3.15

Solution

We have, $$y = \frac{4h}{L^2}x(L-x)$$

At $x = 5$, we get, $$y_{x=5} = \frac{4 \times 4}{20 \times 20} \times 5\,(20-5) = 3\ \text{m}$$

We have just seen in Section 3.3.5 [Figure 3.14 (b)] that *maximum positive moment* occurs when load is at the section itself. For this position of load, i.e. when 100 kN load is at D, we get,

$$V_B = \frac{100 \times 5}{20} = 25\ \text{kN}$$

$$\therefore \quad V_A = 75\ \text{kN}$$

Taking moment about C,

$$25 \times 10 = H_B \times 4$$

$$\therefore \quad H_B = \frac{25 \times 10}{4} = 62.5\ \text{kN}$$

Hence, $M_D = (25 \times 15) - (62.5 \times 3)$

$= 375 - 187.5 = 187.5$ kN m (maximum positive moment)

Similarly, maximum negative moment occurs when load is at C.

For this position;

$$V_A = V_B = 50 \text{ kN}$$

and $\therefore\ H \times 4 = 50 \times 10$

$$H = \frac{500}{4} = 125 \text{ kN}$$

Hence, $M_D = (50 \times 5) - (125 \times 3)$

$= 250 - 375 = -125$ kN (maximum negative moment)

Using the results obtained in Section 3.3.5 [Figure 3.14 (c)], we know that the *absolute maximum positive moment* occurs at section $x = 0.211\ L$. Thus,

$x = 0.211 \times 20 = 4.22$ m from supports A and B and its magnitude would be $(0.0962\ L \times W)$.

Thus, *absolute maximum positive BM* $= 0.0962 \times 20 \times 100 = 192.4$ kN m.

Now referring to Figure 3.14 (d), we know that *absolute maximum negative moment* occurs at section $x = 0.25\ L$. Thus,

$x = 0.25 \times 20 = 5$ m from supports A and B and its magnitude would be $(0.0625\ L \times W)$.

Thus, *absolute maximum negative BM* $= 100 \times \left(-\frac{20}{16}\right) = -125$ kN m