THEORY OF STRUCTURE II

CHAPTER 1

1. ANALYSIS OF INDETERMINATE STRUCTURES

1.1. DISPLACEMENT METHOD

MOMENT DISTRIBUTION METHOD

In the previous lesson we discussed the slope-deflection method. In slope deflection analysis, the unknown displacements (rotations and translations) are related to the applied loading on the structure. The slope-deflection method results in a set of simultaneous equations of unknown displacements. The number of simultaneous equations will be equal to the number of unknowns to be evaluated. Thus one needs to solve these simultaneous equations to obtain displacements and beam end moments. Today, simultaneous equations could be solved very easily using a computer. Before the advent of electronic computing, this really posed a problem as the number of equations in the case of multistory building is quite large. The moment-distribution method proposed by Hardy Cross in 1932, actually solves these equations by the method of successive approximations. In this method, the results may be obtained to any desired degree of accuracy. Until recently, the moment-distribution method was very popular among engineers. It is very simple and is being used even today for preliminary analysis of small structures. It is still being taught in the classroom for the simplicity and physical insight it gives to the analyst even though stiffness method is being used more and more. Had the computers not emerged on the scene, the moment-distribution method could have turned out to be a very popular method. In this lesson, first moment-distribution method is developed for continuous beams with unyielding supports.

18.2 Basic Concepts

In moment-distribution method, counterclockwise beam end moments are taken as positive. The counterclockwise beam end moments produce clockwise moments on the joint Consider a continuous beam *ABCD* as shown in Fig.18.1a. In this beam, ends *A* and *D* are fixed and hence, $\theta_A = \theta_D = 0$ .Thus, the deformation of this beam is completely defined by rotations θ_B and θ_C at joints *B* and *C* respectively. The required equation to evaluate θ_B and θ_C is obtained by considering equilibrium of joints *B* and *C*. Hence,

$$\sum M_B = 0 \quad \Rightarrow M_{BA} + M_{BC} = 0 \quad (18.1a)$$
$$\sum M_C = 0 \quad \Rightarrow M_{CB} + M_{CD} = 0 \quad (18.1b)$$

According to slope-deflection equation, the beam end moments are written as

$$M_{BA} = M_{BA}^F + \frac{2EI_{AB}}{L_{AB}}(2\theta_B)$$

$\frac{4EI_{AB}}{L_{AB}}$ is known as stiffness factor for the beam *AB* and it is denoted by k_{AB} . M_{BA}^F is the fixed end moment at joint *B* of beam *AB* when joint *B* is fixed. Thus,

$$M_{BA} = M_{BA}^F + K_{AB}\theta_B$$

$$M_{BC} = M_{BC}^F + K_{BC}\left(\theta_B + \frac{\theta_C}{2}\right)$$

$$M_{CB} = M_{CB}^F + K_{CB}\left(\theta_C + \frac{\theta_B}{2}\right)$$

$$M_{CD} = M_{CD}^F + K_{CD}\theta_C \quad (18.2)$$

In Fig.18.1b, the counterclockwise beam-end moments M_{BA} and M_{BC} produce a clockwise moment M_B on the joint as shown in Fig.18.1b. To start with, in moment-distribution method, it is assumed that joints are locked i.e. joints are prevented from rotating. In such a case (vide Fig.18.1b), $\theta_B = \theta_C = 0$, and hence

$$M_{BA} = M_{BAF}$$

$$M_{BC} = M_{BCF}$$

$$M_{CB} = M_{CBF}$$

$$M_{CD} = M_{CD}^F \quad (18.3)$$

Since joints *B* and *C* are artificially held locked, the resultant moment at joints *B* and *C* will not be equal to zero. This moment is denoted by M_B and is known as the unbalanced moment.

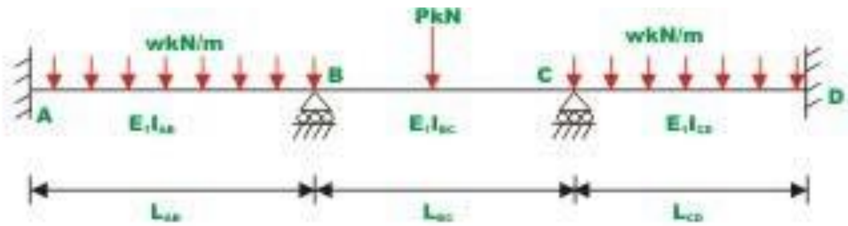

Fig. 18.1a Continuous Beam

Fig. 18.1b Continuous beam with fixed joints.

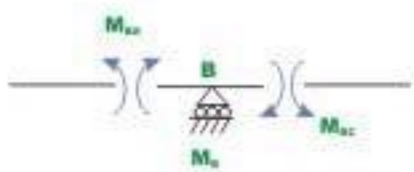

Fig. 18.1c Free - body diagram of joints B

Thus,

$$M_B = M_{BAF} + M_{BCF}$$

In reality joints are not locked. Joints *B* and *C* do rotate under external loads. When the joint *B* is unlocked, it will rotate under the action of unbalanced moment M_B. Let the joint B rotate by an angle θ_{B1}, under the action of M_B. This will deform the structure as shown in Fig.18.1d and introduces distributed moment M_{BA}^d, M_{BC}^d in the span BA and BC respectively as shown in the figure. The unknown distributed moments are assumed to be positive and hence act in counterclockwise direction. The unbalanced moment is the algebraic sum of the fixed end moments and act on the joint in the clockwise direction. The unbalanced moment restores the equilibrium of the joint B. Thus,

$$\sum M_B = 0, \quad M_{BA}^d + M_{BC}^d + M_B = 0 \qquad (18.4)$$

The distributed moments are related to the rotation θ_{B1} by the slope-deflection equation.

$$M_{BA}^{d} = K_{BA}\theta_{B1}$$

$$M_{BC}^{d} = K_{BC}\theta_{B1} \tag{18.5}$$

Substituting equation (18.5) in (18.4), yields

$$\theta_{B1}(K_{BA} + K_{BC}) = -M_B$$

$$\theta_{B1} = -\frac{M_B}{K_{BA} + K_{BC}}$$

In general,

$$\theta_{B1} = -\frac{M_B}{\sum K} \tag{18.6}$$

where summation is taken over all the members meeting at that particular joint. Substituting the value of θ_{B1} in equation (18.5), distributed moments are calculated. Thus,

$$M_{BA}^{d} = -\frac{K_{BA}}{\sum K} M_B$$

$$M_{BC}^{d} = -\frac{K_{BC}}{\sum K} M_B \tag{18.7}$$

The ratio $\frac{K_{BA}}{\sum K}$ is known as the distribution factor and is represented by DF_{BA}.

Thus,

$$M_{BA}^{d} = -DF_{BA}.M_B$$

$$M_{BC}^{d} = -DF_{BC}.M_B \tag{18.8}$$

The distribution moments developed in a member meeting at *B*, when the joint *B* is unlocked and allowed to rotate under the action of unbalanced moment M_B is equal to a distribution factor times the unbalanced moment with its sign reversed.

As the joint *B* rotates under the action of the unbalanced moment, beam end moments are developed at ends of members meeting at that joint and are known as distributed moments. As the joint *B* rotates, it bends the beam and beam end moments at the far ends (i.e. at *A* and *C*) are developed. They are known as carry over moments. Now consider the beam *BC* of continuous beam *ABCD*.

When the joint *B* is unlocked, joint *C* is locked. Joint *B* rotates by θ_{B1} under the action of unbalanced moment M_B (vide Fig. 18.1e). Now from slope deflection equations

$$M_{BC}^{d} = K_{BC}\theta_B$$

$$M_{BC} = \frac{1}{2}K_{BC}\theta_B$$

$$M_{CB} = \frac{1}{2}M_{BC}^{d} \qquad (18.9)$$

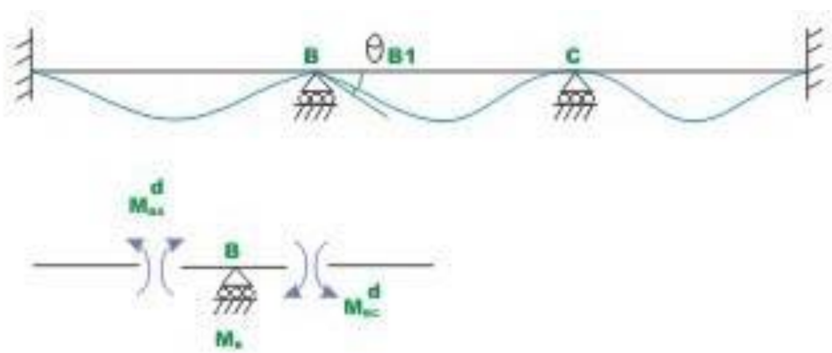

Fig. 18.1d Joint B is unlocked keeping C locked.

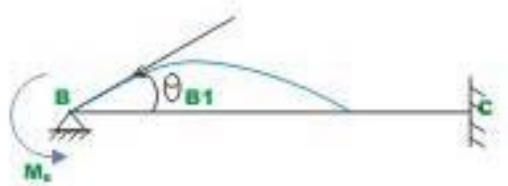

Fig.18.1e Carry - over moment

The carry over moment is one half of the distributed moment and has the same sign. With the above discussion, we are in a position to apply moment distribution method to statically indeterminate beam. Few problems are solved here to illustrate the procedure. Carefully go through the first problem, wherein the moment-distribution method is explained in detail.

Example 18.1

A continuous prismatic beam *ABC* (see Fig.18.2a) of constant moment of inertia is carrying a uniformly distributed load of 2 kN/m in addition to a concentrated load of 10 kN. Draw bending moment diagram. Assume that supports are unyielding.

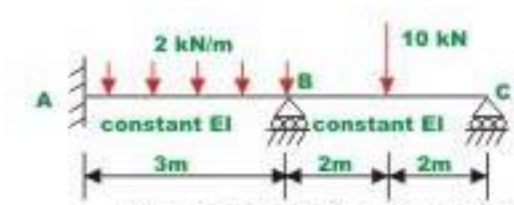

Fig. 18.2a Example 18.1

Solution

Assuming that supports *B* and C are locked, calculate fixed end moments developed in the beam due to externally applied load. Note that counterclockwise moments are taken as positive.

$$M_{AB}^{F} = \frac{wL_{AB}^{2}}{12} = \frac{2\times 9}{12} = 1.5 \text{ kN.m}$$

$$M_{BA}^{F} = -\frac{wL_{AB}^{2}}{12} = -\frac{2\times 9}{12} = -1.5 \text{ kN.m}$$

$$M_{BC}^{F} = \frac{Pab^{2}}{L_{BC}^{2}} = \frac{10\times 2\times 4}{16} = 5 \text{ kN.m}$$

$$M_{CB}^{F} = -\frac{Pa^{2}b}{L_{BC}^{2}} = -\frac{10\times 2\times 4}{16} = -5 \text{ kN.m} \qquad (1)$$

Before we start analyzing the beam by moment-distribution method, it is required to calculate stiffness and distribution factors.

$$K_{BA} = \frac{4EI}{3}$$

$$K_{BC} = \frac{4EI}{4}$$

At B: $\sum K = 2.333EI$

$$DF_{BA} = \frac{1.333EI}{2.333EI} = 0.571$$

$$DF_{BC} = \frac{EI}{2.333EI} = 0.429$$

At C: $\sum K = EI$

$DF_{CB} = 1.0$

Note that distribution factor is dimensionless. The sum of distribution factor at a joint, except when it is fixed is always equal to one. The distribution moments are developed only when the joints rotate under the action of unbalanced moment. In the case of fixed joint, it does not rotate and hence no distribution moments are developed and consequently distribution factor is equal to zero.

In Fig.18.2b the fixed end moments and distribution factors are shown on a working diagram. In this diagram B and C are assumed to be locked.

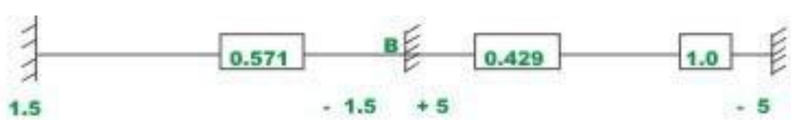

Fig. 18.2b

Now unlock the joint C. Note that joint C starts rotating under the unbalanced moment of 5 kN.m (counterclockwise) till a moment of -5 kN.m is developed (clockwise) at the joint. This in turn develops a beam end moment of +5 kN.m (M_{CB}). This is the distributed moment and thus restores equilibrium. Now joint C is relocked and a line is drawn below +5 kN.m to indicate equilibrium. When joint *C* rotates, a carry over moment of +2.5 kN.m is developed at the B end of member *BC*.These are shown in Fig.18.2c.

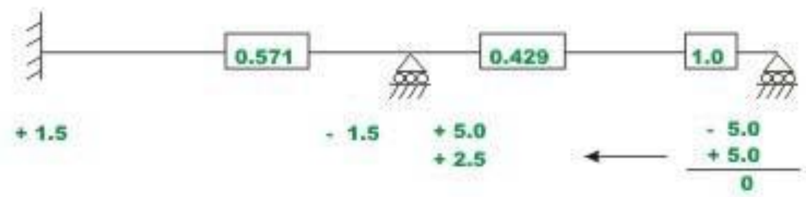

Fig. 18.2c

When joint B is unlocked, it will rotate under an unbalanced moment equal to algebraic sum of the fixed end moments(+5.0 and -1.5 kN.m) and a carry over moment of +2.5 kN.m till distributed moments are developed to restore equilibrium. The unbalanced moment is 6 kN.m. Now the distributed moments M_{BC} and M_{BA} are obtained by multiplying the unbalanced moment with the corresponding distribution factors and reversing the sign. Thus, $M_{BC} = -2.574$ kN.m and $M_{BA} = -3.426$ kN.m. These distributed moments restore the equilibrium of joint *B*. Lock the joint *B*. This is shown in Fig.18.2d along with the carry over moments.

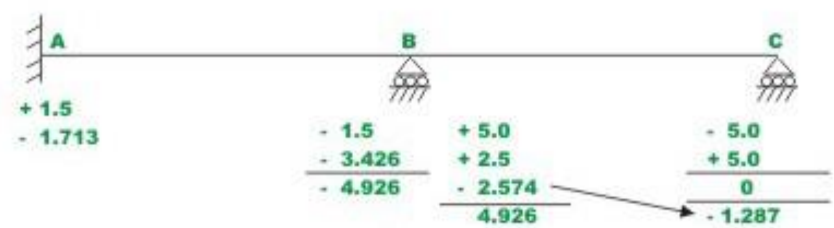

Fig. 18.2d

Now, it is seen that joint *B* is balanced. However joint *C* is not balanced due to the carry over moment -1.287 kN.m that is developed when the joint *B* is allowed to rotate. The whole procedure of locking and unlocking the joints *C* and *B* successively has to be continued till both joints B and C are balanced simultaneously. The complete procedure is shown in Fig.18.2e.

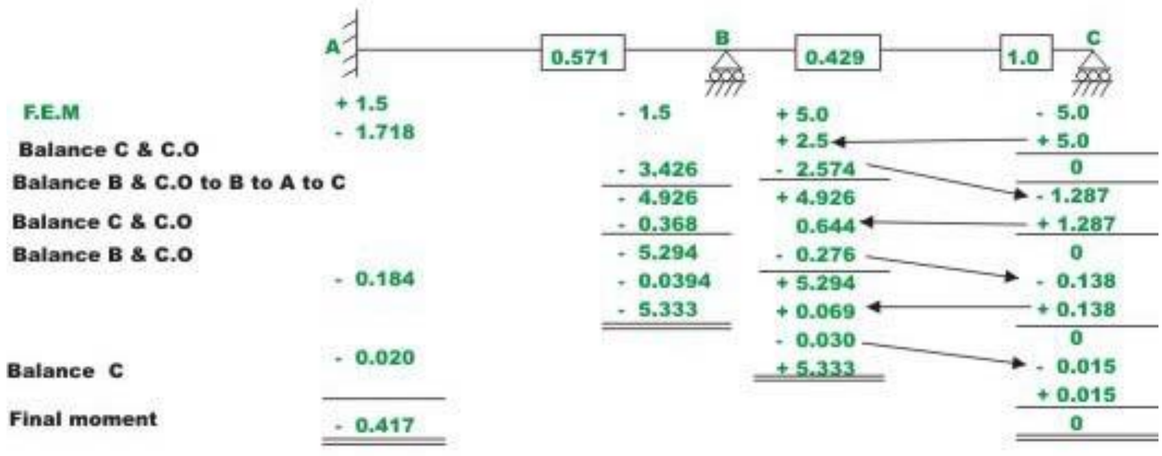

Fig. 18.2e Moment - distribution method : Computation

The iteration procedure is terminated when the change in beam end moments is less than say 1%. In the above problem the convergence may be improved if we leave the hinged end *C* unlocked after the first cycle. This will be discussed in the next section. In such a case the stiffness of beam BC gets modified. The above calculations can also be done conveniently in a tabular form as shown in Table 18.1. However; the above working method is preferred in this course.

Table 18.1 Moment-distribution for continuous beam ABC

Joint	A	B		C
Member	AB	BA	BC	CB
Stiffness	1.333EI	1.333EI	EI	EI
Distribution factor		0.571	0.429	1.0
FEM in kN.m	+1.5	-1.5	+5.0	-5.0
Balance joints C ,B and C.O.	-1.713	-3.426	+2.5 -2.579	+5.0 0
		-4.926	+4.926	-1.287
Balance C and C.O.			+0.644	1.287

Balance B and C.O.		-0.368	-0.276	-0.138
Balance C	-0.184	-5.294	+5.294	0.138
C.O.			+0.069	0
Balance B and C.O.	-0.02	-0.039	-0.030	-0.015
Balance C				+0.015
Balanced moments in kN.m	-0.417	-5.333	+5.333	0

Modified stiffness factor when the far end is hinged

As mentioned in the previous example, alternate unlocking and locking at the hinged joint slows down the convergence of moment-distribution method. At the hinged end the moment is zero and hence we could allow the hinged joint C in the previous example to rotate freely after unlocking it first time. This necessitates certain changes in the stiffness parameters. Now consider beam *ABC* as shown in Fig.18.2a. Now if joint *C* is left unlocked then the stiffness of member *BC* changes. When joint *B* is unlocked, it will rotate by θ_{B1} under the action of unbalanced moment M_B.The support *C* will also rotate by θ_{C1}as it is free to rotate. However, moment$M_{CB} = 0$. Thus

$$M_{CB} = K_{BC}\theta_C + \frac{K_{BC}}{2}\theta_B \qquad (18.7)$$

But, $M_{CB} = 0$

$$\Rightarrow \theta_C = -\frac{\theta_B}{2} \qquad (18.8)$$

Now,

$$M_{BC} = K_{BC}\theta_B + \frac{K_{BC}}{2}\theta_C \qquad (18.9)$$

Substituting the value of θ_C in eqn. (18.9),

$$M_{BC} = K_{BC}\theta_B - \frac{K_{BC}}{4}\theta_B = \frac{3}{4}K_{BC}\theta_B \quad (18.10)$$

$$M_{BC} = K_{BC}^{R}\theta_B \quad (18.11)$$

The K_{BC}^{R} is known as the reduced stiffness factor and is equal to $\frac{3}{4}K_{BC}$ .Accordingly distribution factors also get modified. It must be noted that there is no carry over to joint *C* as it was left unlocked.

Example 18.2

Solve the previous example by making the necessary modification for hinged end *C*.

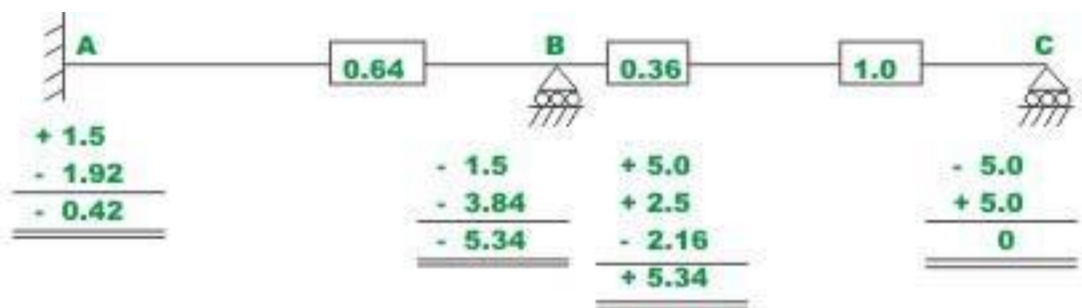

Fig. 18.3 Example 18.2

Fixed end moments are the same. Now calculate stiffness and distribution factors.

$$K_{BA} = 1.333EI, K_{BC} = \frac{3}{4}EI = 0.75EI$$

Joint B: $\sum K = 2.083,\quad D_{BA}^{F} = 0.64,\quad D_{BC}^{F} = 0.36$

Joint C: $\sum K = 0.75EI,\quad D_{CB}^{F} = 1.0$

All the calculations are shown in Fig.18.3a

Please note that the same results as obtained in the previous example are obtained here in only one cycle. All joints are in equilibrium when they are unlocked. Hence we could stop moment-distribution iteration, as there is no unbalanced moment anywhere.

Example 18.3

Draw the bending moment diagram for the continuous beam *ABCD* loaded as shown in Fig.18.4a.The relative moment of inertia of each span of the beam is also shown in the figure.

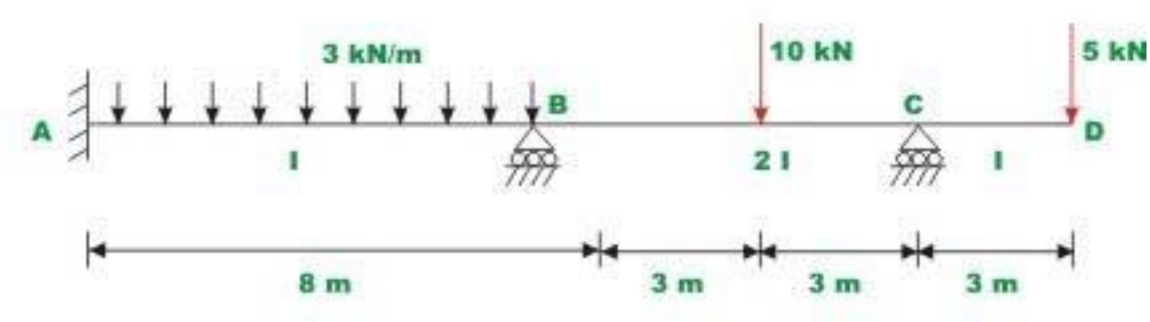

Fig. 18.4a Example 18.3

Solution

Note that joint *C* is hinged and hence stiffness factor *BC* gets modified. Assuming that the supports are locked, calculate fixed end moments. They are

$$M_{AB}^F = 16 \text{ kN.m}$$

$$M_{BA}^F = -16 \text{ kN.m}$$

$$M_{BC}^F = 7.5 \text{ kN.m}$$

$$M_{CB}^F = -7.5 \text{ kN.m} \text{ , and}$$

$$M_{CD}^F = 15 \text{ kN.m}$$

In the next step calculate stiffness and distribution factors

$$K_{BA} = \frac{4EI}{8}$$

$$K_{BC} = \frac{3}{4}\frac{8EI}{6}$$

$$K_{CB} = \frac{8EI}{6}$$

At joint B:

$$\sum K = 0.5EI + 1.0EI = 1.5EI$$

$$D_{BA}^{F} = \frac{0.5EI}{1.5EI} = 0.333$$

$$D_{BC}^{F} = \frac{1.0EI}{1.5EI} = 0.667$$

At C:

$$\sum K = EI, D_{CB}^{F} = 1.0$$

Now all the calculations are shown in Fig.18.4b

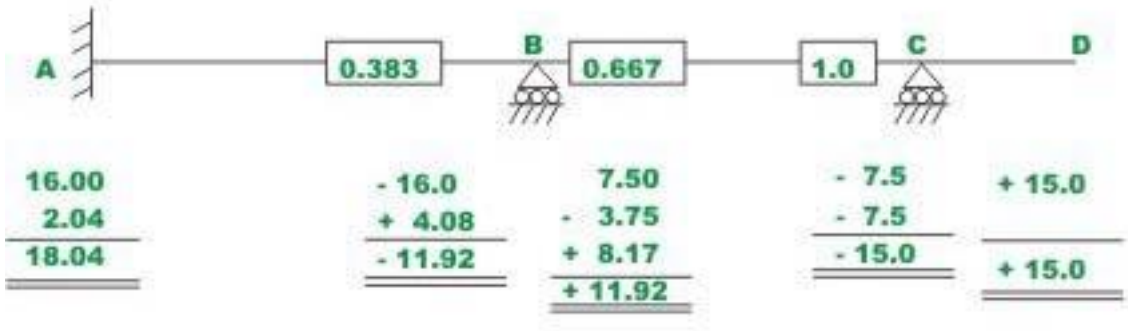

Fig. 18.4b Computation

This problem has also been solved by slope-deflection method (see example 14.2).The bending moment diagram is shown in Fig.18.4c.

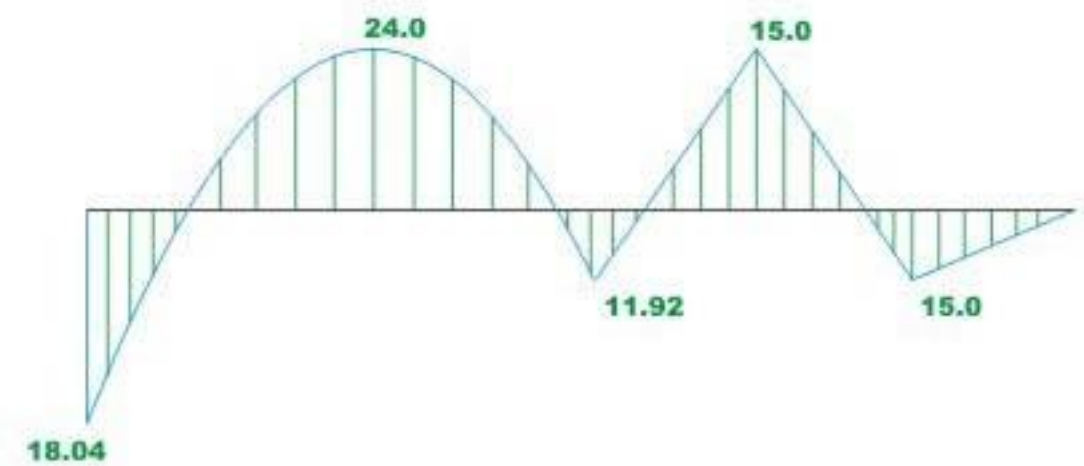

Fig. 18.4c Bending - moment diagram

Summary

An introduction to the moment-distribution method is given here. The momentdistribution method actually solves these equations by the method of successive approximations. Various terms such as stiffness factor, distribution factor, unbalanced moment, distributing moment and carry-over-moment are defined in this lesson. Few problems are solved to illustrate the moment-distribution method as applied to continuous beams with unyielding supports.