Module 12

Yield Line Analysis for Slabs

Lesson
32
Two-way Rectangular, Square, Triangular and Circular Slabs

Instructional Objectives:

At the end of this lesson, the student should be able to:

- analyse rectangular slabs simply supported at three edges and free at the other edge considering the two possible yield patterns, employing (i) the method of segmental equilibrium and (ii) the method of virtual work,
- analyse square slab with forking yield patterns when the corners are having inadequate reinforcement,
- predict yield lines of fan pattern for slabs in case this may be a possibility,
- analyse the fan pattern of yield lines to determine the collapse loads of triangular and circular slabs with different support conditions,
- analyse the fan pattern of yield lines to determine the collapse loads of circular slabs clamped along the circumference and having a column support at the centre.

12.32.1 Introduction

Rectangular / square slabs may have different yield patterns depending on the support conditions and type of loads. Simply supported slabs at three edges and free at the other edge may have two types of yield patterns depending on the ratio of moment resisting capacities and the aspect ratio. This lesson first takes up such slabs to determine the condition for selecting a particular one out of the two possible yield patterns. The case of a square slab having forking yield pattern is explained when the corner reinforcement is inadequate. Several cases of triangular and circular slabs with or without a central column support are taken up to explain the yield lines of fan pattern.

All the expressions are derived by employing the method of either segmental equilibrium or virtual work. In some cases both the methods are taken up to compare the values.

12.32.2 Rectangular Slabs Simply Supported at Three Edges and Free at the Other Edge Considering Yield Pattern 1

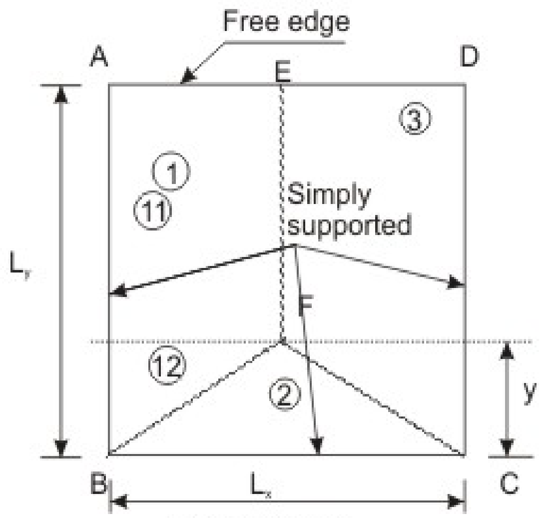


Fig. 12.32.1(a):
Slab with yield pattern 1

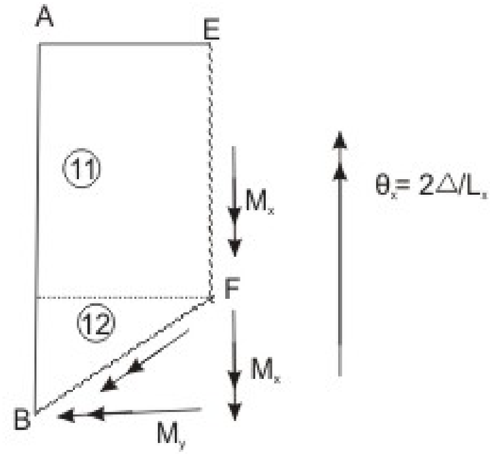


Fig. 12.32.1(b):
Free body diagram of segment 1

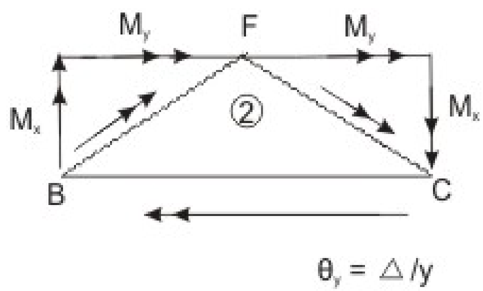


Fig. 12.32.1(c): Free body diagram of segment 2

Fig. 12.32.1: Two-way slab - simply supported at three edges and free at other edge - yield pattern 1

Figure 12.31.4c of Lesson 31 shows the yield pattern 1 of such slabs involving one additional unknown *y*. So, there are two unknowns – *y* and *w* to be determined. The yield pattern divides the slab into three segments marked by 1, 2 and 3 (Fig.12.32.1a). The slab carrying uniformly distributed load of *w* kN/m^2, undergoes deflection of Δ at point E. The free body diagrams of segments 1 and 2 are shown in Figs.12.32.1b and c, respectively. Due to symmetry, segments 1 and 3 are identical. Segment 1 is subdivided into two parts as 11 and 12. We employ the method of segmental equilibrium first.

(A) Method of segmental equilibrium

Here, the nodal forces are zero since the moment capacities of three intersecting yield lines are identical. The equilibrium equation of segment 1 is developed taking the moments of loads and moments of segment 1 about AB, and equating the same to zero. Thus, we get: $(L_x/2)\ (L_y - y)\ w\ (L_x / 4) + (L_x /4)y\ w\ (L_x / 6) - M_x L_y = 0$,

or $\quad w = (24\ M_x L_y) / \{3Lx^2 (L_y - y) + L_x^2\ y\}$
(12.41)

Similarly, the equilibrium equation of segment 2 is developed by taking moment of loads and moments of segment 2 about the base BC and equating the same to zero, which gives: $(L_x / 2)\ y\ w\ (y/3) - M_y L_x = 0$

or $\quad w = 6\ M_y / y^2$
(12.42)

Equating the expression of w from Eqs.12.41 and 12.42, we get:
$24\ M_x L_y / \{3L_x^2 (L_y - y) + L_x^2\ y\} = 6M_y / y^2$

or $\quad 4\ M_x L_y y^2 + 2M_y L_x^2\ y - 3\ M_y L_x^2\ L_y = 0$
(12.43)

The solution of Eq. 12.43 is

$y = [-2\ M_y L_x^2 + \{4\ M_y^2\ L_x^4 + 48\ M_x M_y L_x^2\ L_y^2\}^{1/2}] / 8M_x L_y$
(12.44)

After getting the value of y from Eq. 12.44, the value of the collapse load w is obtained from either Eq. 12.41 or Eq. 12.42. The condition that $y < L_y$ gives:
$(-2)(M_y/M_x)\ L_x^2 + \{(4)\ (M_y/M_x)^2\ L_x^4 + 48(M_y / M_x)\ L_x^2\ L_y^2\} < 8L_y^2$

or $\quad M_y / M_x < 4\ (L_y/L_x)^2$
(12.45)

(B) The method of virtual work

Referring to Figs.12.32.1a, b and c, total external work done by the loads of three segments TEW is as follows: $TEW = 2w\ [(L_x/2)\ (L_y\text{-}y)\ (\Delta/2) + (L_x/2)y\ (\Delta/3)]$

or $\quad TEW = w\ L_x\ (3L_y - y)\ (\Delta/6)$
(12.46)

Total internal work done by the yield moments TIW is: $TIW = 2\ M_x\ (L_y)\ \theta_x + M_y\ (y)\ \theta_y$.
Using $\theta_x = 2\Delta / L_x$ and $\theta_y = \Delta/y$, we have

$$TIW = \Delta \{4M_x (L_y/L_x) + M_y (L_x/y)\} \quad (12.47)$$

From the two equations of *TEW* and *TIW* (Eqs. 12.46 and 12.47), we have

$$w = (24\ M_x\ L_y\ y + (M_y\ L_x^2) / \{L_x^2\ (3y\ L_y - y^2)\} \quad (12.48)$$

This is the only equation in the method of virtual work to determine y and the collapse load w. Differentiating w with respect to y and equating that to zero to get the lowest w, we have: $L_x^2\ (3y\ L_y - y^2)\ (24\ M_x\ L_y) - (24\ M_x\ L_y\ y + 6M_y\ L_x^2)\ L_x^2\ (3L_y - 2y) = 0$

or $$4\ M_x\ L_y\ y^2 + 2\ M_y\ L_x^2\ y - 3M_y\ L_x^2\ L_y = 0 \quad (12.43)$$

Thus, the method of virtual work gives the same equation (Eq. 12.43) as that obtained by the method of segmental equilibrium. After getting the value of y from Eq. 12.44, the value of the collapse load w is obtained from Eq. 12.48.

12.32.3 Rectangular Slabs Simply Supported at Three Edges and Free at the Other Edge Considering Yield Pattern 2

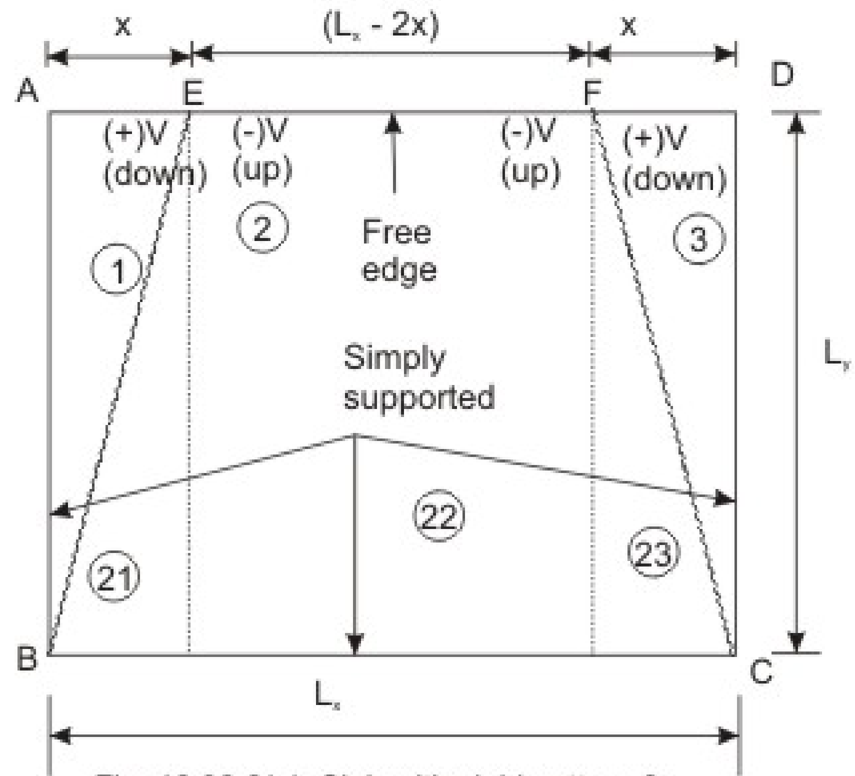


Fig. 12.32.2(a): Slab with yield pattern 2

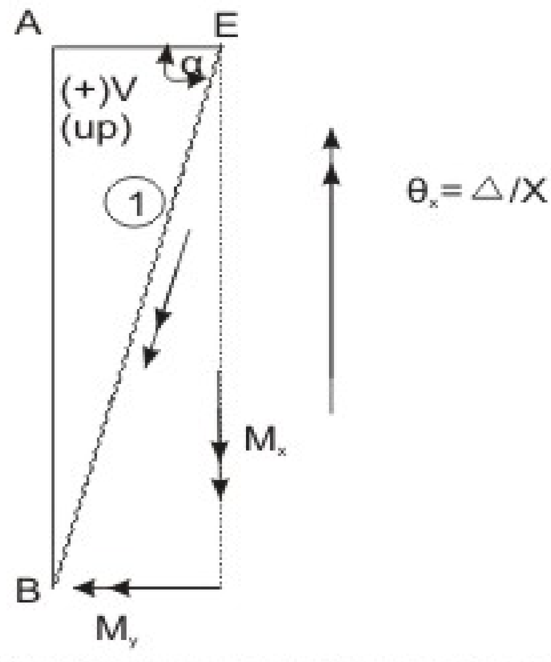


Fig. 12.32.2(b): Free body diagram of segment 1

Fig. 12.32.2: Two-way slab - simply supported at three edges and free at other edge - yield pattern 2

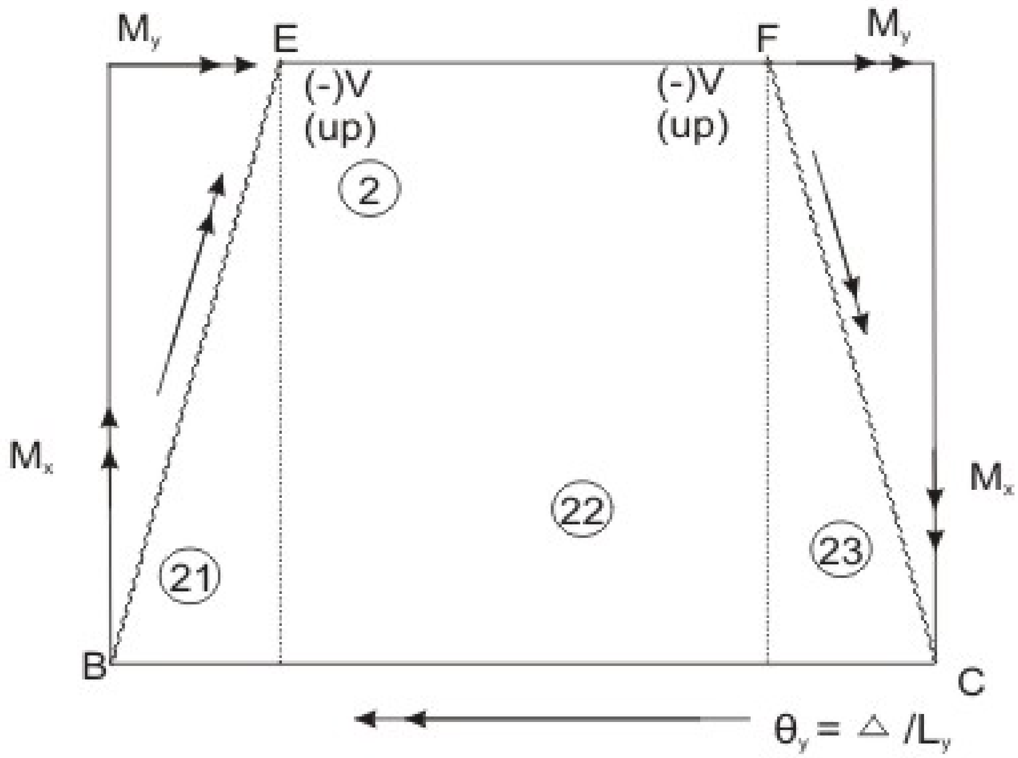


Fig. 12.32.2(c): Free body diagram of segment 2

Fig. 12.32.2: Two-way slab - simply supported at three edges and free at other edge - yield pattern 2

Figure 12.31.4d of Lesson 31 shows that yield pattern 2 of such slabs involving one additional unknown *x*. So, there are two unknowns – *x* and *w* to determine. The yield pattern divides the slab into three segments marked by 1, 2 and 3. The slab, carrying the uniformly distributed load of *w* kN/m^2, undergoes deflection of Δ along EF (Fig.12.32.2a). Free body diagrams of segments 1 and 2 are shown in Figs.12.32.2b and c, respectively. Due to symmetry, segments 1 and 3 are identical. Segment 2 is further subdivided into three parts 21, 22 and 23, of which sub-segments 21 and 23 are also symmetrical. We consider the method of segmental equilibrium first.

(A) Method of segmental equilibrium

The nodal forces on the left and right of point E of the yield line BE are (+) *V*, acting downward and (-) *V*, acting upward, respectively. Similarly, nodal forces on the left and right of point F of yield line CF are (-) *V*, acting upward and (+) *V*, acting downward, respectively. The magnitude of the nodal force, as given in Eq.12.23 of Lesson 31, is

$$V = M_y \cot \alpha = M_y \text{ (AE/AB)} = M_y (x/L_y) \quad (12.49)$$

We now develop the equilibrium equation of segment 1 taking moments of loads and moments of segment 1 about AB and equating the same to zero.

$w\,(x/2)\,L_y\,(x/3) + Vx - M_x L_y = 0$, which gives,

$$w = 6\,(M_x L_y^2 - M_y x^2) / (x^2 L_y^2) \qquad (12.50)$$

Similarly, the equilibrium equation of segment 2 is developed by taking moment of loads and forces of segment 2 about BC and equating the same to zero.

$2\,w\,(x/2)\,(L_y)\,(L_y/3) + w\,(L_x - 2x)\,L_y\,(L_y/2) - 2V L_y - 2M_y\,x = 0$, which gives,

$$w = (24\,M_y\,x) / \{ L_y^2\,(3L_x - 4x)\} \qquad (12.51)$$

Equating the two expressions of w from Eqs.12.50 and 12.51, we have,

$$\frac{6(M_x L_y^2 - M_y x^2)}{x^2 L_y^2} = \frac{24 M_y x}{L_y^2 (3L_x - 4x)}$$

or $$3(M_y / M_x)\,L_x\,x^2 + 4\,L_y^2\,x - 3L_y^2\,L_x = 0 \qquad (12.52)$$

The solution of the above equations is:

$$x = [-\,4L_y^2 + \{16\,L_y^4 + 36\,(M_y / M_x)\,L_x^2\,L_y^2\}^{1/2}] / \{6\,L_x\,(M_y / M_x)\} \qquad (12.53)$$

From Fig. 12.32.2a, it is evident that $x < L_x / 2$. So, we get from Eq. 12.53 for the condition that $x < L_x/2$,

$-\,4\,L_y^2 + \{16\,L_y^4 + 36\,(M_y / M_x)\,L_x^2\,L_y^2\}^{1/2} \langle\ (L_x / 2)\,(6\,L_x)\,(M_y / M_x)$, which finally gives

$$(M_y / M_x) > (4/3)\,(L_y / L_x)^2 \qquad (12.54)$$

Therefore, Eq. 12.54 shall be used to confirm if yield pattern 2 is possible or not. After getting the value of x from Eq. 12.53 (the solution of Eq. 12.52), the collapse load w is determined either from Eq. 12.50 or from Eq. 12.51.

(B) Method of virtual work

Referring to Figs. 12.32.2a, b and c, the total external work done by the loads of three segments TEW is as follows: $TEW = 2\,(W1) + 2\,(W21) + W22 = 2\{(x/2)\,L_y\,w\,(\Delta/3)\} + 2\,\{(x/2)\,L_y\,w\,(\Delta/3)\} + (L_x - 2x)\,L_y\,w\,(\Delta/2)$

or $\quad TEW = w\,L_y\,(\Delta/6)\,(3\,L_x - 2x)$ (12.55)

The total internal work done by the yield moments TIW is: $TIW = 2\,M_x\,L_y\,\theta_x + 2\,M_y\,x\,\theta_y$. Using $\theta_x = \Delta/x$ and $\theta_y = \Delta/L_y$, we have: $TIW = 2M_x\,L_y\,(\Delta/x) + 2\,M_y x\,(\Delta/L_y)$,

or $\quad TIW = \Delta\{2M_x\,L_y/x + 2\,M_y\,x/L_y)$ (12.56)

From the two expressions of TEW and TIW (Eqs. 12.55 and 12.56), we have: $w\,(L_y/6)\,(3L_x - 2x) = 2M_x\,(L_y/x) + 2M_y\,(x/L_y)$,

or $\quad w = \{12\,M_x\,L_y^2 + 12\,M_y\,x^2\} / \{L_y^2\,(3x\,L_x - 2x^2)$ (12.57)

The above is the only equation to determine x and the collapse load w in the method of virtual work. Differentiating w with respect to x and setting that to zero shall give the lowest load. Hence, we have: $L_y^2\,(3\,x\,L_x - 2x^2)\,(24\,M_y\,x) - (12\,M_x\,L_y^2 + 12\,M_y\,x^2)\,L_y^2\,(3L_x - 4x) = 0$,

or $\quad 3\,(M_y / M_x)\,L_x\,x^2 + 4\,L_y\,x^2 - 3L_y^2\,L_x = 0$ (12.52)

Thus, the method of virtual work gives the same equation (Eq.12.52) as that obtained by the method of segmental equilibrium. After getting the value of y from Eq.12.52 (or Eq.12.53, the solution of Eq.12.52), the collapse load w is determined from Eq. 12.57.

12.32.4 Special Cases for Predicting Yield Patterns

Let us examine the two conditions of Eqs.12.45 and 12.54 for determining the correct yield pattern.

Equation 12.45 specifies that for the yield pattern 1, $(M_y / M_x) < 4\,(L_y / L_x)^2$, while Eq. 12.54 specifies that for the yield pattern 2, $M_y/M_x > 4/3\,(L_y/L_x)^2$. However, for slabs when both the conditions are satisfied, it appears that both the yield patterns are possible. In such cases, both the yield patterns should be considered and the one giving the lowest load (either 1 or 2) is the correct yield pattern. We explain the above taking two specific cases of square slabs having simply supported edges on three sides and free at the other side, as shown in Fig.12.32.1a, with two different ratios of M_y / M_x.

(i) Case 1: $L_y / L_x = 1$ and $M_y / M_x = 1.5$

Here, (M_y/M_x) is less than $4\,(L_y/L_x)^2$ (= 4). So, yield pattern 1 is possible. Similarly, (M_y/M_x) is greater than $(4/3)\,(L_y/L_x)^2$ (= 4/3). So, yield pattern 2 is also possible. Therefore, we should consider both the yield patterns and select the one giving the lowest value of the collapse load.

For the yield pattern 1, the value of the collapse load is obtained from any of the three equations (Eqs.12.41, 12.42 and 12.48), after determining the value of y from Eq.12.43. The results are as follows.

$$4\, M_x L_y y^2 + 2M_y L_x^2\, y - 3\, M_y L_x^2 L_y = 0 \tag{12.43}$$

Using $M_y = 1.5\, M_x$ and $L_y = L_x$, we have

$$4y^2 + 3L_x\, y - 4.5\, L_x^2 = 0 \tag{12.58}$$

from which $y = 0.75\, L_x$ (12.59)

Using the value of y in the three equations, we get the value of w as:

From Eq.12.41: $$w = \frac{24 M_x L_y}{3L_x^2 (L_y - y) + (L_x^2 y)} = 16 (M_x / L_x^2) \tag{12.60}$$

From Eq. 12.42: $$w = 6M_y / y^2 = 16\,(M_x / L_x^2) \tag{12.60}$$

From Eq. 12.48: $$w = \frac{24 M_x L_y y + 6 M_y L_x^2}{L_x^2 (3yL_y - y^2)} = 16 (M_x / L_x^2) \tag{12.60}$$

We now consider yield pattern 2 in which the unknown distance x is obtained from Eq. 12.52, which is

$$3\,(M_y / M_x)\, L_x x^2 + 4\, L_y^2\, x - 3L_y^2 L_x = 0 \tag{12.52}$$

When M_y = 1.5 M_x and L_y = L_x, the above equation becomes

$$4.5\, x^2 + 4\, L_x\, x - 3\, L_x^2 = 0 \quad (12.61)$$

which gives $x = 0.4852\, L_x$ (12.62)

Using the value of x in Eqs. 12.50, 12.51 and 12.57, we get the value of w as given below.

From Eq. 12.50: $$w = \frac{6(M_x L_y^2 - M_y x^2)}{x^2 L_y^2} = 16.488(M_x / L_x^2) \quad (12.63)$$

From Eq. 12.51: $$w = \frac{24(M_y)x}{L_y^2(3L_x - 4x)} = 16.488(M_x / L_x^2) \quad (12.63)$$

From Eq. 12.57: $$w = \frac{12M_x L_y^2 + 12M_y x^2}{L_y^2(3xL_x - 2x^2)} = 16.488(M_x / L_x^2) \quad (12.63)$$

Computing the values of w from Eqs. 12.60 and 12.63, it is evident that yield pattern 1 is the correct yield pattern.

(ii) Case 2: L_y / L_x =1 and M_y/M_x = 3.5

Here also both the conditions of Eqs. 12.45 and 12.54 are satisfied. So, we consider them separately. Proceeding in the same manner, we determine the values of y and w for yield pattern 1, and x and w for yield pattern 2. Only the final results are given below.

(a) For the yield pattern 1, $y = 0.966\, L_x$ (12.64)

and $w = 22.487\, (M_x / L_x^2)$ (12.65)

(b) For the yield pattern 2, $x = 0.377\, L_x$ (12.66)

and $w = 21.222\ (M_x / L_x^2)$
(12.67)

Thus, in this case the yield pattern 2 is the correct yield pattern.

12.32.5 Square Slabs with Forking Yield Pattern

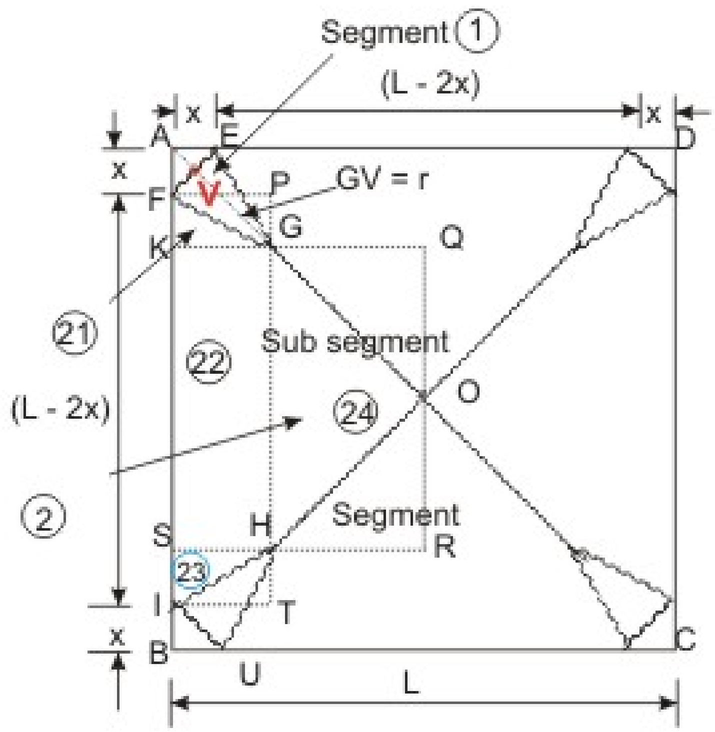


Fig. 12.32.3(a): Forking yield pattern

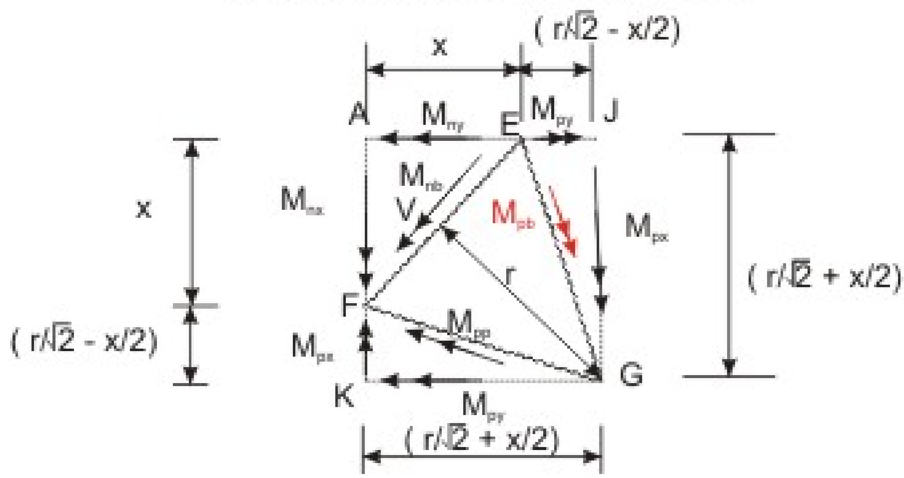


Fig. 12.32.3(b): Free body diagram of segment 1

Fig. 12.32.3: Square slab with forking yield pattern

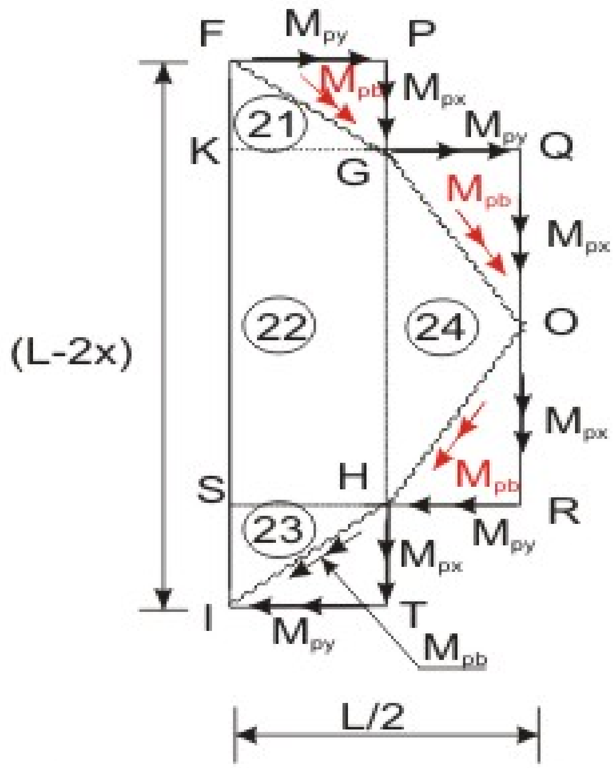


Fig. 12.32.3(c): Free body diagram of segment 2

Fig. 12.32.3: Square slab with forking yield pattern

Yield lines of two-way slabs having inadequate corner reinforcement fork out before they reach the corners, as shown in Fig.12.31.4e of Lesson 31. We analyse a square slab having simply supported edges in all four sides subjected to uniformly distributed load *w* kN/m^2 and having yield pattern as shown in Fig.12.32.3a. Yield lines divide the slab into symmetrical segments, of which we consider segments 1 and 2 employing the method of segmental equilibrium.

The yield lines have two unknown parameters *x* and *r* (Fig.12.32.3a), where *x* is the distance AE and *r* is the perpendicular distance GV from point G to the yield line EF. In segment 1, EF is the negative yield line, and EG and FG are positive yield lines. Figure 12.32.3b shows the free body diagram of segment 1. The negative and positive moments are represented by M_{nb} and M_{pb}, respectively. These moments are resolved into their respective components M_{nx}, M_{ny}, M_{px} and M_{py} along the sides of the slab, as shown in the figure. Segment 2 is shown in Fig.12.32.3c which is bounded by yield lines FG, GO, OH and HI, and the side FI. All the yield lines of segment 2 are having positive moment M_{pb} which are resolved as M_{px} and M_{py} along the sides of the square slab. Segment 2 is further subdivided into four sub-segments marked by 21, 22, 23 and 24 of which sub-segments 21 and 23 are symmetrical.

Before we take up the equilibrium of the two segments, let us determine the dimensions of different lengths of the two segments. With reference to Figs.12.32.3a, b and c,

AE = AF = BI = BU = x (assumed)
GV = r (assumed)
AK = KG = GJ = AJ = $\{(x/\sqrt{2}) + r\}/\sqrt{2} = r/\sqrt{2} + x/2$
FK = EJ = AJ – AE = $r/\sqrt{2} - x/2$

We further assume that $M_{nx} = M_{ny} = M_n$ and $M_{px} = M_{py} = M_p$.

The area of triangle EFG = (1/2) (EF) (FG) = $x\,r/\sqrt{2}$.

Taking moments of the loads and moments of segment 1 about EF and equating the same to zero, we get: w (area EFG) (r /3) + (- M_{nx} AF – M_{ny} AE + M_{py} EJ – M_{px} JG – M_{py} GK + M_{px} KF) $\cos 45^0 = 0$

or $$r = \{6(M_n + M_p)/w\}^{1/2} \quad (12.68)$$

We now consider the equilibrium of segment 2 taking moment of loads and moment about FI. For easy understanding, the contributions of different sub-segments are computed separately and then added with the contribution of the moments. The contributions of different sub-segments and intermediate addition are:

(i) For the sub-segment 21:

$$(1/2)(\mathrm{FK})(\mathrm{KG})w(\mathrm{KG}/3) = (1/2)\left(\frac{r}{\sqrt{2}} - \frac{x}{2}\right)\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 (w/3)$$

(ii) For the sub-segment 22:

$$(\mathrm{KS})(\mathrm{KG})^2(w/2) = \left\{\mathrm{L} - 2\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)\right\}\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 (w/2)$$

(iii) For the sub-segment 24: (1/2) (GH) $\{(L/2) - \mathrm{KG}\}$ $\{\mathrm{KG} + \{(L/2 - \mathrm{KG})/3\}$ w

$$= \frac{w}{2}\left\{L - 2\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)\right\}\left\{\frac{L}{2} - \left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)\right\}\left[\left(\frac{\mathrm{r}}{\sqrt{2}} + \frac{x}{2}\right) + \left\{\frac{L}{2} - \left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)\right\}/3\right]$$

$$= \frac{w}{2}\left\{\frac{L^2}{2} - L\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right) - L\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right) + 2\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2\right\}\left[\left(\frac{\mathrm{r}}{\sqrt{2}} + \frac{x}{2}\right) + \left\{\frac{L}{2} - \frac{r}{\sqrt{2}} - \frac{x}{2}\right\}/3\right]$$

$$= \left\{ \frac{wL^2}{4} - wL\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right) + w\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \right\} \left[\frac{\mathrm{r}}{\sqrt{2}} + \frac{x}{2} + \frac{L}{6} - \frac{r}{3\sqrt{2}} - \frac{x}{6} \right]$$

$$= \left\{ \frac{wL^2}{4} - wL\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right) + w\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \right\} \left\{ \frac{\sqrt{2}r}{3} + \frac{x}{3} + \frac{L}{6} \right\}$$

$$= \left\{ \frac{wL^2}{4} - wL\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right) \right\} \left\{ \frac{\sqrt{2}r}{3} + \frac{x}{3} + \frac{L}{6} \right\} + w\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \left\{ \frac{\sqrt{2}r}{3} + \frac{x}{3} + \frac{L}{6} \right\}$$

$$= \frac{wL^2}{4}\left(\frac{\sqrt{2}r}{3} + \frac{x}{3}\right) + \frac{wL^3}{24} - wL\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)\left\{ \frac{\sqrt{2}r}{3} + \frac{x}{3} + \frac{L}{6} \right\} + w\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \left\{ \frac{\sqrt{2}r}{3} + \frac{x}{3} + \frac{L}{6} \right\}$$

(iv) The addition of {2 (sub-segment 21) + (sub-segment 22)}

$$= \left(\frac{r}{\sqrt{2}} - \frac{x}{2}\right)\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \left(\frac{w}{3}\right) + \left\{ L - 2\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right) \right\}\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \left(\frac{w}{2}\right)$$

$$= w\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \left(\frac{r}{3\sqrt{2}} - \frac{x}{6} + \frac{L}{2} - \frac{r}{\sqrt{2}} - \frac{x}{2}\right)$$

$$= \mathrm{w}\left(\frac{\mathrm{r}}{\sqrt{2}} + \frac{\mathrm{x}}{2}\right)^2 \left(-\frac{\sqrt{2}\,\mathrm{r}}{3} - \frac{2\,\mathrm{x}}{3} + \frac{\mathrm{L}}{2}\right)$$

(v) Therefore, {2 (sub-segment 21) + (sub-segment 22) + (sub-segment 24)}

$$= w\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \left(-\frac{\sqrt{2}\,r}{3} - \frac{2\,x}{3} + \frac{L}{2} + \frac{\sqrt{2}\,r}{3} + \frac{x}{3} + \frac{L}{6}\right) + \frac{wL^2}{4}\left(\frac{\sqrt{2}\,r}{3} + \frac{x}{3}\right) + \frac{wL^3}{24} - wL\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)\left\{\frac{\sqrt{2}\,r}{3} + \frac{x}{3} + \frac{L}{6}\right\}$$

$$= w\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \left(-\frac{x}{3} + \frac{2L}{3}\right) + \frac{wL^2 r}{6\sqrt{2}} + \frac{wL^2 x}{12} + \frac{wL^3}{24} - wL\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)\left(\frac{\sqrt{2}\,r}{3} + \frac{x}{2} + \frac{L}{6}\right)$$

$$= w\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)^2 \left(-\frac{x}{3}\right) + \mathrm{w}\left(\frac{r}{\sqrt{2}} + \frac{x}{2}\right)\left(\frac{\sqrt{2}\,rL}{3\sqrt{2}} + \frac{xL}{3} - \frac{\sqrt{2}\,rL}{3} - \frac{xL}{3} - \frac{L^2}{6}\right) + \frac{wL^2 r}{6\sqrt{2}} + \frac{wL^2 x}{12} + \frac{wL^3}{24}$$

$$= -\frac{wr^2 x}{6} - \frac{wx^3}{12} - \frac{wrx^2}{3\sqrt{2}} + \frac{wL^3}{24}$$

(vi) Including the contributions of moment, the equilibrium equation of segment 2 becomes

$$-w\frac{r^2x}{6}-\frac{wx^3}{12}-\frac{wrx^2}{3\sqrt{2}}+\frac{wL^3}{24}-M_p(L-2x)=0$$
(12.69)

Using the expression of r from Eq.12.68 in the above equation, we have: - w $6(M_n + M_p)\ x / 6w - wx^3/12 - (w\ x^2/3\surd 2)\ \{6(M_n + M_p)/w\}^{1/2} + (w\ L^3/24) - M_p\ (L\text{-}2x) = 0$

or $\quad w\ x^3/ 12 + x^2\{w\ (M_n + M_p) / 3\}^{1/2} + (M_n - M_p)x + M_pL - wL^3/24 = 0$
(12.70)

Setting $dw/dx = 0$, Eq.12.70 finally gives:

$$wx^2/4 + 2x\{\ (M_n + M_p)\ w/3\}^{1/2} + (M_n\text{-}M_p) = 0$$
(12.71)

which has the solution of x as

$$x = 2\{(M_n + 7M_p) / 3\ w\}^{1/2} - 4\ \{(M_n + M_p) / 3w\}^{1/2}$$
(12.72)

Thus, the three unknowns x, r and w are determined from the three equations, Eqs. 12.68, 12.70 and 12.72.

We now consider two cases below: (i) with adequate corner reinforcement and (ii) with inadequate corner reinforcement.

Case (i): When the corner reinforcement is adequate

The value of $x = 0$ when the corner reinforcement is adequate as there will be no forking of the yield line. Putting $x = 0$ in Eq. 12.70, we have: $M_pL - wL^3 / 24 = 0$

or $\quad w = 24\ M_p / L^2$
(12.73)

Using $x = 0$ in Eq. 12.72, we get: $M_n + 7M_p = 4\ (M_n + M_p)$, which gives:

$$M_n = M_p$$
(12.74)

Using $w = 24\ M_p /L^2$ and $M_n = M_p$ from Eqs 12.73 and 12.74 in Eq. 12.68, we have:

$$r = L / \surd 2$$
(12.75)

Case (ii): When the corner reinforcement is inadequate

In this case, $M_n = 0$ and only M_p is present. Putting $M_n = 0$ in Eq. 12.71, we have:

$$x = 2(7M_p / 3w)^{1/2} - 4 (M_p /3w)^{1/2} = 0.746 (M_p/w)^{1/2} \quad (12.76)$$

Using $M_n = 0$ and substituting the value of x from Eq.12.76 in Eq.12.70, we have:

$$w x^3/12 + x^2 (w M_p/3)^{1/2} - M_p x + M_p L - wL^3 / 24 = 0$$

or $$wL^3 - 24 M_p L + 9.3624 M_p (M_p/w)^{1/2} = 0 \quad (12.77)$$

The numerical solution of w of Eq.12.77 is:

$$w = 22 M /L^2 \quad (12.78)$$

The two values of w of the two cases (i) and (ii) reveal the reduction of w by about 8.33 per cent (= 24 – 22 / 0.24) when the corner reinforcement is such that $M_n = 0$. Therefore, it is justified to reduce the load carrying capacity by ten per cent rather than involving in the complicated analysis.

Further, if the negative moment carrying capacity M_n is larger than M_p, the value of x becomes negative. In that case, the yield pattern is the same as that with adequate corner reinforcement.

12.32.6 Yield Lines of Fan Pattern

Triangular or circular slabs subjected to uniformly distributed loads or point loads may have yield lines of fan pattern. We explain below different cases of such slabs to estimate the collapse loads.

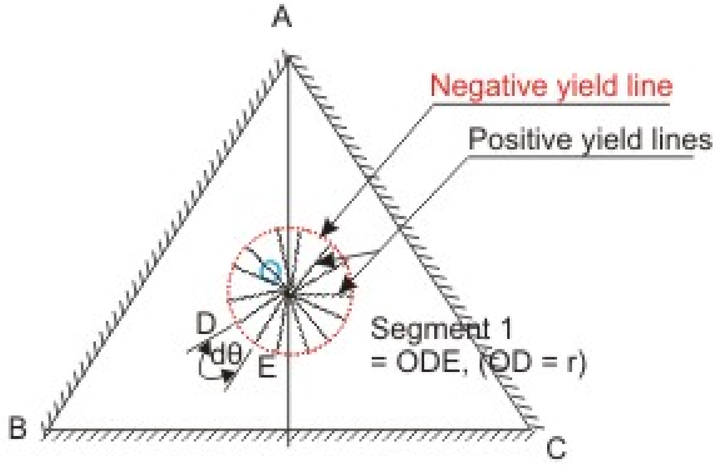
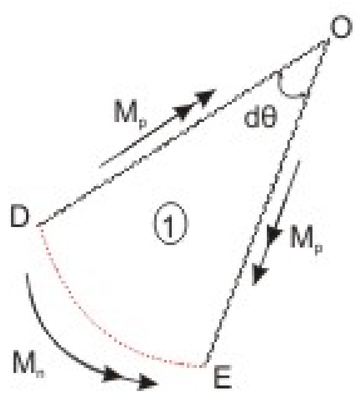


Fig. 12.32.4(a): Triangular slab with a point load P at O



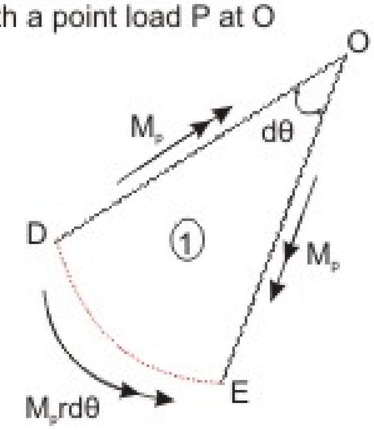


Fig. 12.32.4(b): Free body diagram of segment 1

Fig. 12.32.4(c): Resultant of two M_p-s

Fig. 12.32.4: Yield line of fan pattern

(A) Triangular slab subjected to a point load

Figure 12.32.4a shows a triangular slab clamped along the three edges and subjected to a point load P away from the edges and corners. Negative yield line of approximately circular pattern is formed with positive yield lines radiating outward from the point of application of the load. Assuming the resisting moment capacities of M_p and M_n per unit length for positive and negative moments, respectively, we consider one segment ODE, as shown in Fig.12.32.4b. The angle DOE is $d\theta$ and the resultant of the positive moments M_p along DO and OE, is $Mp\ rd\theta$ along DE, as shown in Fig.12.32.4c. The resultant $M_p\ rd\theta$ is in the same direction of the negative moment $M_n\ rd\theta$ along DE. The fractional part of the total load P acting on the segment is $P\ d\theta / 2\pi$.

Taking moment of the load and moments of the segment DOE about DE and equating the same to zero, we have: $(M_p + M_n)\, r\, d\theta - P(d\theta / 2\pi)\, r = 0$, which gives:

$$P = 2\pi (M_p + M_n) \tag{12.79}$$

If we assume $M_n = kM_p$, Eq.12.79 gives:

$$P = 2\pi M_p (1 + k) \tag{12.80}$$

When $M_n = M_p$ i.e., k =1, we have:

$$P = 4\pi M_p \tag{12.81}$$

Equations12.79 to 12.81 reveal that the collapse load P is independent of the radius of the fan patterns of yield lines. Thus, at the collapse load, the triangular slab clamped along three edges may have complete fan pattern of yield lines of any radius without any change of the collapse load. However, it is necessary that the boundaries must have the resisting moment capacities M_n at all points.

(B) Simply supported circular slabs subjected to uniformly distributed load w kN/m^2

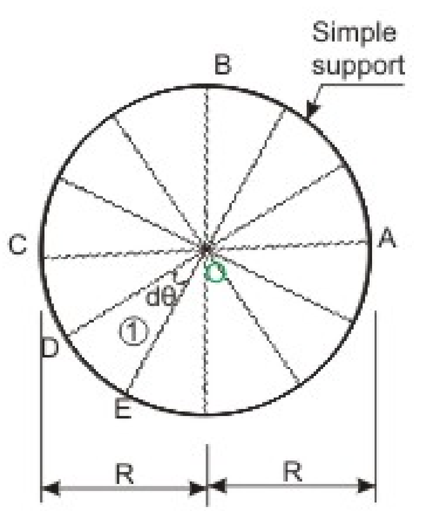


Fig. 12.32.5(a): Circular slab with yield lines

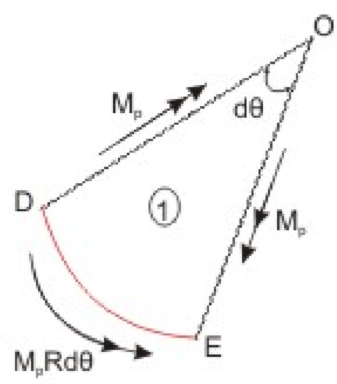


Fig. 12.32.5(b):
Free body diagram of segment 1

Fig. 12.32.5: Simply supported circular slab with uniformly distributed load

Figure 12.32.5a shows the positive yield lines of radiating type having the moment resistance capacity of M_p per unit length. Figure 12.32.5b shows the free body diagram of the segment DOE making an angle $d\theta$ at the center of the slab.

Equating the moment of the load and moment about DE to zero, we have: $M_pR\, d\theta = (1/2)\,(R\, d\theta)\, R\;\; w\,(R/3)$, which gives:

$$w = 6\, M_p \,/\, R^2 \tag{12.82}$$

(C) Clamped circular slab subjected to uniformly distributed load w kN/m^2

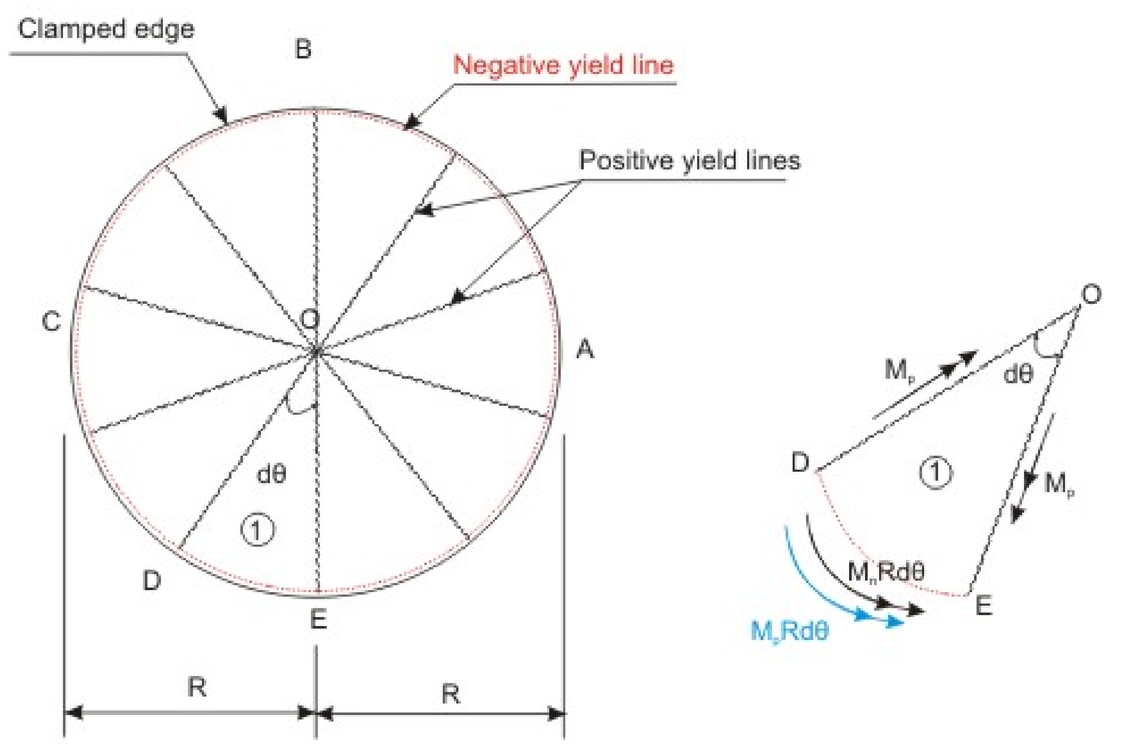


Fig. 12.32.6(a): Circular slab with clamped edges

Fig. 12.32.6(b): Free body diagram of segment 1

Fig. 12.32.6: Fixed supported circular slab subjected to uniformly distributed load

Figure 12.32.6a shows the circular slab clamped at the periphery having negative yield lines along the periphery of moment resisting capacity of M_n per unit length. Positive yield lines radiating from the centre of the slab are also shown in the figure having moment resisting capacity of M_p per unit length. Figure 12.32.6b shows the free body diagram of one segment DOE where $d\theta$ is

the angle made by the yield lines DO and OE at the centre. As explained earlier, the resultant of the two positive moments M_p of magnitude $M_pR\ d\theta$ and the negative moment M_n are in the same direction as shown in the figure.
Equating the moment of load and moments of segment DOE about DE to zero, we have: $M_p\ r\ d\theta + M_n\ r\ d\theta - (1/2)\ (rd\theta)\ r\ w\ (r/3) = 0$. This gives:

$$w = 6(M_n + M_p) / R^2 \tag{12.83}$$

Assuming $M_n = kM_p$, Eq.12.83 gives:

$$w = 6\ M_p\ (1 + k) / R^2 \tag{12.84}$$

(D) Circular slabs clamped along the circumference and having a column support at the centre

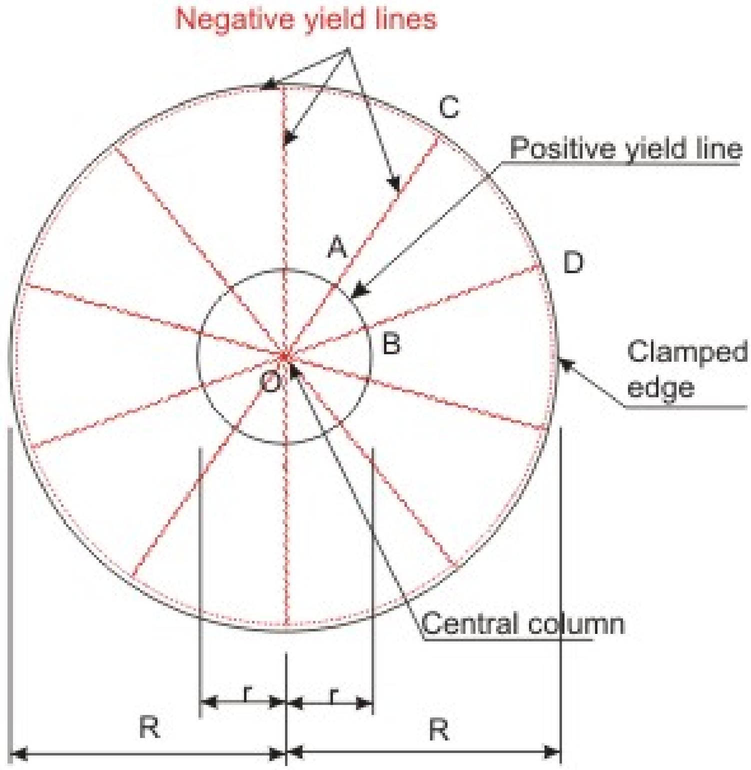


Fig. 12.32.7(a): Slab with yield lines

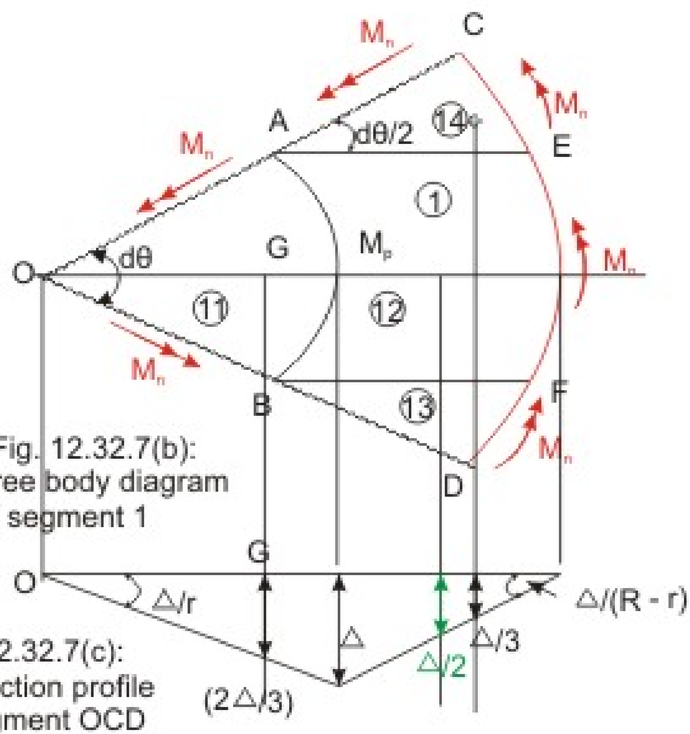


Fig. 12.32.7(b): Free body diagram of segment 1

Fig. 12.32.7(c): Deflection profile of segment OCD

Fig. 12.32.7: Circular slab with clamped edge and a central column under uniformly distributed load

In many practical examples of circular slabs, a column support is provided at the centre. The yield pattern of a circular slab clamped along the periphery and having a central column support, subjected to uniformly distributed load of w kN/m^2 is shown in Fig.12.32.7a.

Negative yield lines of moment resisting capacity M_n are along the periphery and radiating from the centre of the slab. Positive yield line is shown in approximately circular pattern having a radius of r. Figure 12.32.7b shows the free body diagram of a segment OCD where $d\theta$ is the angle made by the yield lines OC and OD at the centre.

Figure 12.32.7c shows the deflection profile of the segment OCD having a deflection of Δ along AB. The segment is further divided into four sub-segments 11, 12, 13 and 14, of which sub-segments 13 and 14 are symmetrical. The deflection of the centroid of the sub-segment 11 (OAB) is 2 Δ/3 at G and the deflection of the centroid of the sub-segments 13 and 14 is Δ/3. The deflection of the centroid of the sub-segment 12 is Δ/2. We employ the method of virtual work here. The external work done by the loads of the four sub-segments are presented in Table 12.1 giving the respective area of the sub-segment and the deflection at the centroid. By summing them considering that the work done by the load of segment ACE is the same as that of segment BDF, the total external work done *TEW* is as follows:

$$TEW = (w\,\Delta d\theta)\,\{r^2/3 + r\,(R-r)/2 + (R-r)^2/6\} \tag{12.85}$$

The internal works done by different positive and negative yield lines of the segment OCD are furnished in Table 12.2 giving the respective moment and rotation. The total internal work (*TIW*) is then obtained by summing them considering that the work done by the moment of yield line AC is the same as that of yield line BD. Thus, the expression of *TIW* is,

$$TIW = \Delta\{M_n\,(AB)/r + 2\,M_n\,(CE)/(R-r) + M_n\,(CD)/(R-r) + M_p\,(AB)/r + M_p\,(AB)/(R-r)\} \tag{12.86}$$

Table 12.1: Total External Work (*TEW*) by loads of segment OCD

Sl. No.	Sub-Segment	Area	Deflection at the centroid	External work done
1.	OAB (11)	$(r/2)\,r\,d\theta$	$2\,\Delta/3$	$w\Delta\,d\theta\,(r^2/3)$
2.	AEFB (12)	$r\,d\theta\,(R-r)$	$\Delta/2$	$(w\Delta\,d\theta)\,\{r\,(R-r)/2\}$
3.	ACE and BFD (13 and 14)	$(R-r)\,(R-r)\,(d\theta/2)$	$\Delta/3$	$(w\Delta\,d\theta)\,\{(R-r)^2/6\}$
Adding, we get: $TEW = (w\Delta\,d\theta)\,\{(r^2/3) + r\,(R-r)/2 + (R-r)^2/6\}$ … Eq. (12.85)				

Table 12.2: Total Internal Work (*TIW*) by positive and negative yield lines of segment OCD

Sl. No.	Yield line	Moment	Rotation	Internal work done
1.	OA and OB	$M_n\ (AB)$	Δ/r	$M_n\ (AB)\ (\Delta/r)$
2.	AC and BD	$2\ M_n\ (CE)$	$\Delta/(R-r)$	$2\ M_n\ (CE)\ \{\Delta/(R-r)\}$
3.	CD	$M_n\ (CD)$	$\Delta/(R-r)$	$M_n\ (CD)\ \{\Delta/(R-r)\}$
4.	AB	$M_p\ (AB)$	$\{\Delta/r + \Delta/(R-r)\}$	$M_p\ (AB)\ \{\Delta/r + \Delta/(R-r)\}$
Adding, we get: $TIW = \Delta\{M_n\ (AB)/r + 2M_n\ (CE)/(R-r) + M_n(CD)/(R-r) + M_p(AB)\ /r + M_p\ (AB)/(R-r)\}$ **… Eq. (12.86)**				

Equating the two works *TEW* and *TIW* form Eqs.12.85 and 12.86, and assuming that $M_n = M_p = M$, we get: $w\ d\theta\ \{\ r^2/3 + r(R\text{-}r)\ /\ 2 + (R-r)^2/6\} = M\ \{\ 2(AB)/\ r = (2\ CE + CD + AB)\ /\ (R-r)\}$

or $\quad w\ d\theta\{\ R\ (R+r)\ /\ 6\} = M\{2\ (d\theta) + 2R\ (d\theta)\ /\ (R\text{-}r)\}$

or $w = \{12M\ (2\ R-r)\}\ /\ \{R\ (R^2-r^2)\}$
(12.87)

Setting $dw/dr = 0$, we have the most critical layout of the mechanism and then

$$r^2 - 4\ R\ r + R^2 = 0 \tag{12.88}$$

which has a solution,

$$r = 0.2679\ R \tag{12.89}$$

Using the value of r from Eq. 12.89 in Eq. 12.87, we have

$$w = 22.39\ (M\ /\ R^2) \tag{12.90}$$

or $$M = w\ R^2\ /\ 22.39 \tag{12.91}$$

12.32.7 Practice Questions and Problems with Answers

Q.1: Determine the collapse load for the two possible yield patterns of a rectangular slab, simply supported at three edges and free at the other edge, when subjected to uniformly distributed loads employing (i) method of segmental equilibrium and (ii) method of virtual work.

A.1: secs. 12.32.2 and 3

Q.2: When do the yield lines of square slabs fork before reaching the corners? What should be the approach to estimate the uniformly distributed collapse load in such cases?

A.2: sec. 12.32.5

Q.3: Determine the collapse point load in a triangular slab clamped along the three edges.

A.3: sec. 12.32.6 – part A

Q.4: Determine the uniformly distributed collapse load of a circular slab clamped along the periphery.

A.4: sec. 12.32.6 – part C

Q.5: Determine the uniformly distributed collapse load of a circular slab having clamped support along the periphery and a column support at the centre of the slab.

A.5: sec.12.32.6 – part D

12.32.8 References

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12.32.9 Test 32 with Solutions

Maximum Marks = 50, Maximum Time = 30 minutes

Answer all questions.

TQ.1: Choose the correct answer for each of the following statements.

(4 × 5 = 20 Marks)

(A) Rectangular slabs simply supported at three edges and free at the other edge will have yield pattern 1 (Fig.12.32.1a),

(i) when $(M_x / M_y) < 4\,(L_y / L_x)$
(ii) when $(M_y / M_x) < 4\,(L_y / L_x)^2$
(iii) when $(M_x / M_y) < 4\,(L_y / L_x)^2$
(iv) when $(M_y / M_x) > 4\,(L_y / L_x)^2$

A.TQ.1 (A): (ii).

(B) Nodal forces are to be considered when

(i) the slab is subjected to vertical loads at the corners of the slab
(ii) the slab is one-way
(iii) the slab is polygonal having more than five corner node points

(iv) when the yield line meets a free edge

A.TQ.1 (B): (iv)

(C) Rectangular slabs simply supported at three edges and free at the other edge will have yield pattern 2 (Fig.12.32.2a),

(i) when $(M_y / M_x) > (4/3)\ (L_y / L_x)^2$
(ii) when $(M_x / M_y) > (4/3)\ (L_y / L_x)^2$
(iii) when $(M_x / M_y) = (4/3)\ (L_y / L_x)^2$
(iv) when $(M_y / M_x) = (4/3)\ (L_y / L_x)^2$

A.TQ.1 (C): (i)

(D) Yield lines of fan pattern will occur in

(i) any slab having a central cut out
(ii) any slab subjected to unsymmetrical loading
(iii) any slab having point load
(iv) any slab having torsional moment

A.TQ.1 (D): (iii)

TQ.2: Determine the uniformly distributed collapse load of a circular slab having clamped support along the periphery and a column support at the centre of the slab.

(30 marks)

A.TQ.2: sec.12.32.6 – part D

12.32.10 Summary of this Lesson

This lesson explains two different yield patterns of two-way slabs simply supported on three edges and free at the other edge. The expressions of determining the respective collapse loads are derived both by the method of segmental equilibrium and method of virtual work. The forking out type of yield pattern of square / rectangular slabs having inadequate corner reinforcement is explained. Slabs subjected to point loads or circular slabs having yield lines of fan pattern are taken up to determine the collapse loads. Practical circular slabs supported at the periphery and with a central column support is also taken up in this lesson.