Module 12

Yield Line Analysis for Slabs

Lesson 30

Basic Principles, Theory and One-way Slabs

Instructional Objectives:

At the end of this lesson, the student should be able to:

- state the limitations of elastic analysis of reinforced concrete slabs,
- explain the meaning of yield lines,
- explain the basic principle of yield line theory,
- state the assumptions of yield line analysis,
- state the rules for predicting yield patterns and locating the axes of rotation of slabs with different plan forms and boundaries,
- state the upper and lower bound theorems,
- explain the two methods i.e., (i) method of segmental equilibrium and (ii) method of virtual work,
- explain if the yield line analysis is a lower or upper bound method,
- analyse one-way slab problems to determine the location of yield lines and determine the collapse load applying the theoretical formulations of (i) method of segmental equilibrium and (ii) method of virtual work.

12.30.1 Introduction

The limit state of collapse method of design of beams and slabs considers the actual inelastic behaviour of slabs when subjected to the factored loads. Accordingly, it is desirable that the structural analysis of beams and slabs has to be done taking the inelastic behaviour into account. Though the coefficients in Annex D-1 of IS 456 to determine the bending moments for the design of two-way slabs are based on inelastic analysis, the code also recommends the use of linear elastic theory for the structural analysis in cl. 22.1. Moreover, IS 456 further stipulates the use of coefficients of moments and shears for continuous beams given in Tables 12 and 13 of cl. 22.5 in lieu of rigorous elastic analysis. These coefficients of beams are also applicable for the design of one-way slabs, based on linear elastic theory. Thus, there are inconsistencies between the methods of analysis and design.

The above discussion clearly indicates the need of adopting the inelastic analysis or collapse limit state analysis for all structures. However, there are sufficient justifications for adopting the inelastic analysis for slabs as evident from the following limitations of the elastic analysis of slabs:

(i) Slab panels are square or rectangular.

(ii) One-way slab panels must be supported along two opposite sides only; the other two edges remain unsupported.

(iii) Two-way slab panels must be supported along two pairs of opposite sides, supports remaining unyielding.

(iv) Applied loads must be uniformly distributed.

(v) Slab panels must not have large opening.

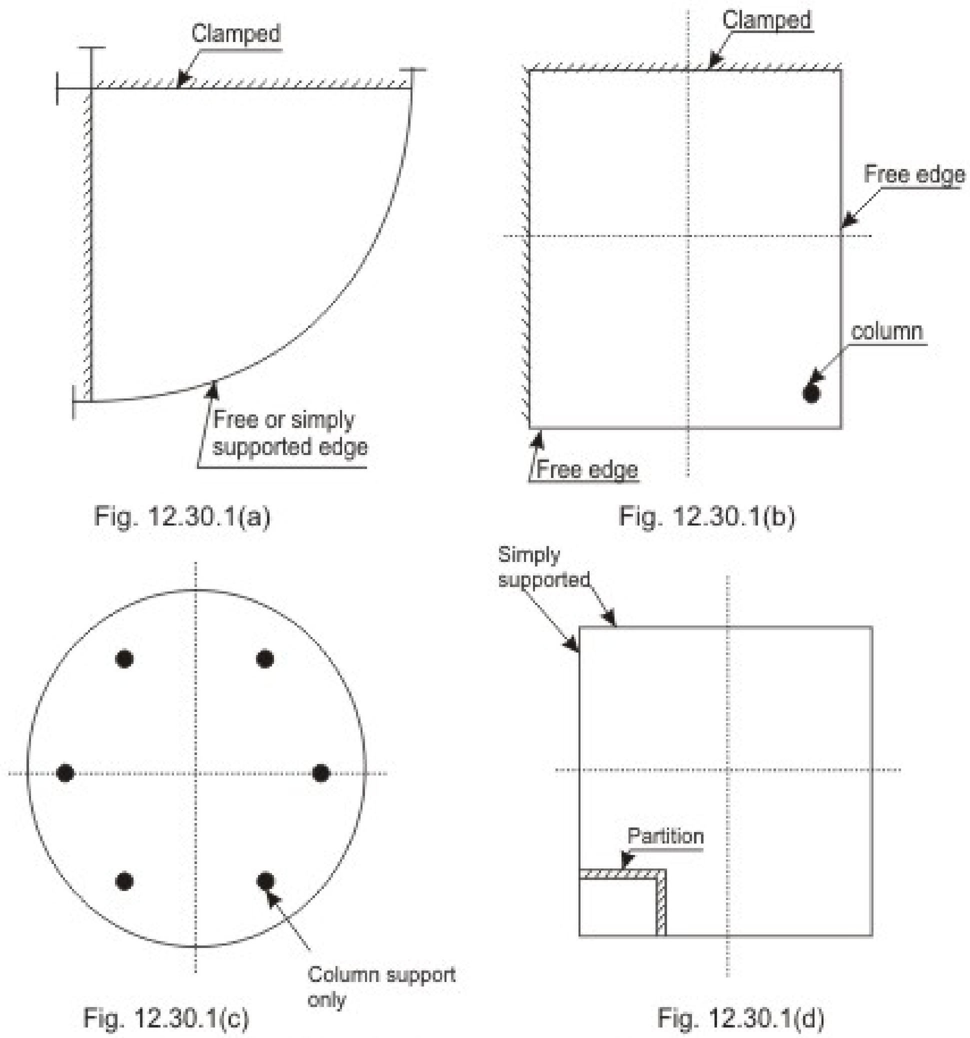


Fig. 12.30.1(a) Fig. 12.30.1(b) Fig. 12.30.1(c) Fig. 12.30.1(d)

Fig. 12.30.1: Different types of slabs

Therefore, for slabs of triangular, circular and other plan forms, for loads other than uniformly distributed, for support conditions other than those specified above and for slabs with large openings (Figs.12.30.1a to d); the collapse limit state analysis has been found to be a powerful and versatile method.

Inelastic or limit analysis is similar to the plastic analysis of continuous steel beams which is based on formation of plastic hinges to form a mechanism of collapse. However, full plastic analysis of reinforced concrete beams and frames is tedious and time consuming. One important advantage of the reinforced concrete slabs over the reinforced concrete beams and frames is that the slabs are mostly under-reinforced. This gives a large rotation capacity of slabs, which may be taken as the presence of sufficient ductility.

12.30.2 Yield Line Theory

The yield line theory, thus, is an ultimate or factored load method of analysis based on bending moment on the verge of collapse. At collapse loads, the slab begins to crack as they are mostly under-reinforced, with the yielding of reinforcement at points of high bending moment. With the propagation of cracks the yield lines are developed gradually. Finally, a mechanism is formed when the slab collapses due to uncontrolled rotation of members. Yield lines, therefore, are lines of maximum yielding moments of the reinforcement of slab. The essence is to find out the locations of the appropriate yield lines.

Yield line analysis, though first proposed by Ingerslev in 1923 (vide, “The strength of rectangular slabs”, by A. Ingerslev, J. of Institution of Structural Engineering, London, Vol. 1, No.1, 1923, pp. 3-14), Johansen is more known for his large extension of the analysis (please refer to (i) Brutlinieteorier, Jul. Gjellerups Forlag, Copenhagen, 1943, by K.W. Johansen, English translation, Cement and Concrete Association London 1962; and (ii) Pladeformler, 2d ed., Polyteknisk Forening, Copenhagen, 1949, by K.W. Johansen, English translation,” Yield Line formulas for slabs, Cement and Concrete Association, London, 1972). Its importance is reflected in the recommendation of the use of this method of analysis of slabs in the Note of cl. 24.4 of IS 456. It is to note that only under-reinforced bending failure is considered in this theory ignoring the effects due to shear, bond and deflection. Effect of in-plane forces developed is also ignored.

12.30.3 Basic Principles

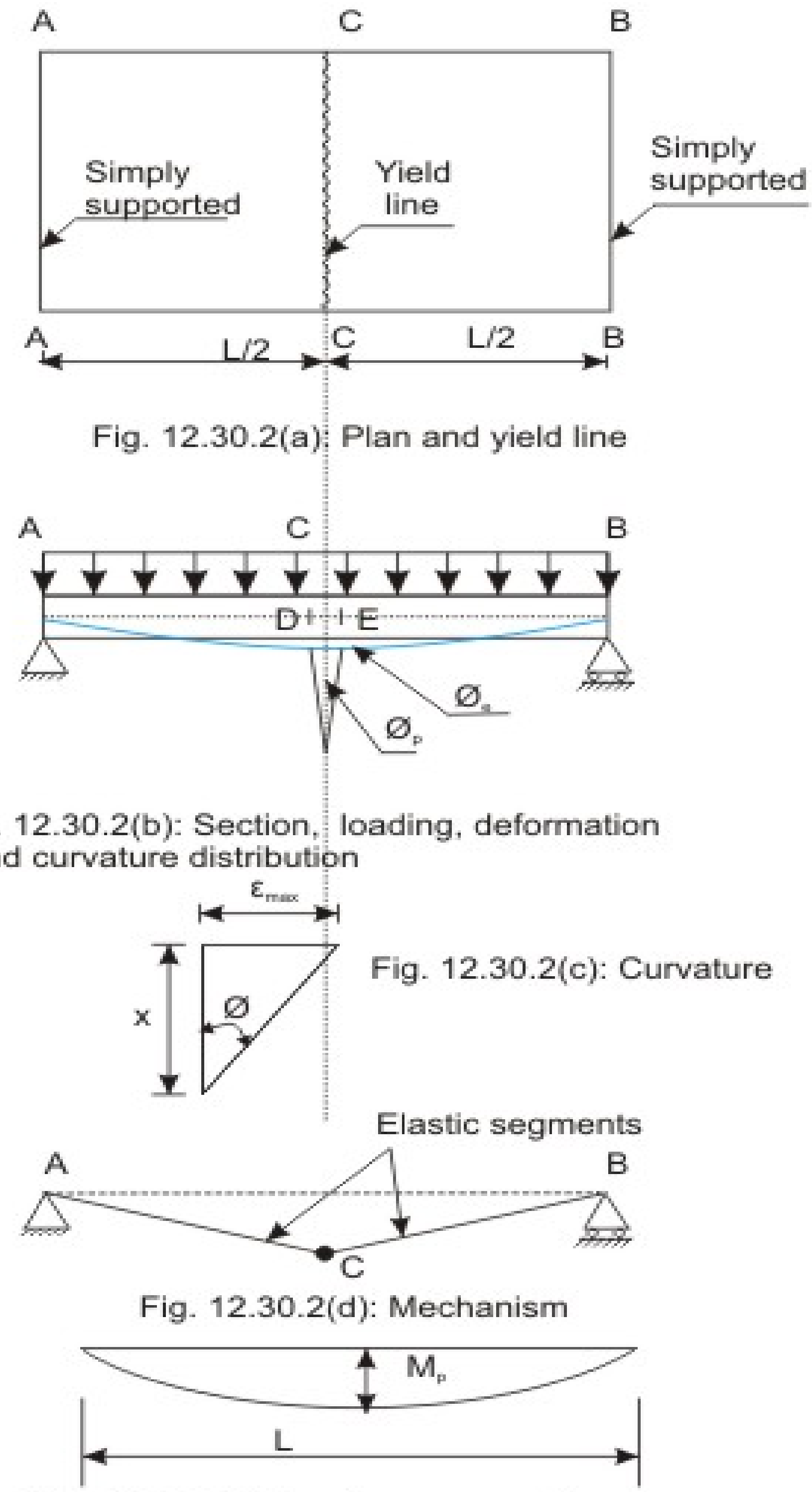


Fig. 12.30.2(a): Plan and yield line

Fig. 12.30.2(b): Section, loading, deformation and curvature distribution

Fig. 12.30.2(c): Curvature

Fig. 12.30.2(d): Mechanism

Fig. 12.30.2(e): Bending moment diagram

Fig. 12.30.2: Simply supported one-way slab

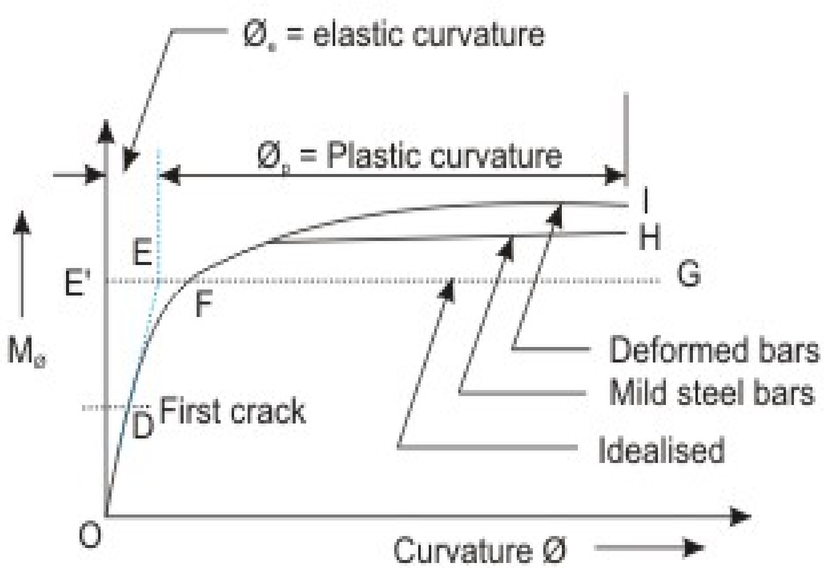


Fig. 12.30.3: Moment-curvature diagram of under-reinforced slabs

As mentioned earlier, reinforced concrete slabs are mostly under-reinforced and they fail in flexure. Figures 12.30.2a and b show such a reinforced concrete simply supported one-way slab subjected to short term loading. The maximum bending moment is along CC, at a distance of *L/2* from the supports, where the deflection and curvature will be the maximum. The curvature ϕ is expressed by (Fig. 12.30.2c):

$$\phi = \epsilon_{max}/x \qquad (12.1)$$

where ϵ_{max} is the compressive strain in concrete in the outermost fibre and x is the depth of the neutral axis. Figure 12.30.2b shows the plastic curvature ϕ_p distributed over a short length DE adjacent to the failure section and the ignored elastic curvatures ϕ_e elsewhere on the slab.

Figure 12.30.3 shows the schematic moment-curvature diagrams of the slab of Fig.12.30.2a. It is evident from Fig.12.30.3 that up to the point D, when the crack first appears anywhere along the line CC of Figs.12.30.2a, the line OD is elastic. Thereafter, the slope of the line changes gradually with the progress of cracks. Accordingly, the stiffness of the slab is reduced. The reinforcement starts yielding at point F, when the ultimate strength is almost reached. The two paths FH for the mild steel reinforcing bars and FI for the deformed bars show marginal increase in moment capacities. Two points H and I are the respective failure points showing larger disproportionate deformations and curvatures when the maximum moment capacity i.e., the resistance of the cross-section M_p is reached. It may be noted that at these points (H and I), the plastic curvatures ϕ_p

are much larger than the elastic curvature ϕ_e. Neglecting these marginal moment capacities in the regions beyond F, the moment curvature diagram is idealised as OEG. The extended zone of increasing curvature at nearly constant moment and considering ϕ_p as much larger than ϕ_e, it is justified to neglect the elastic curvature ϕ_e totally. This amounts to shifting the point E to E′ in Fig. 12.30.3.

The first crack starting anywhere along CC of the one-way slab of Fig.12.30.2a, after reaching the maximum moment capacity M_p, proceeds forming plastic hinges. In the process, the crack line or yield line CC is formed when the slab collapses forming a mechanism. It is worth mentioning that even at collapse; the two segments AC and BC remain elastic and plane.

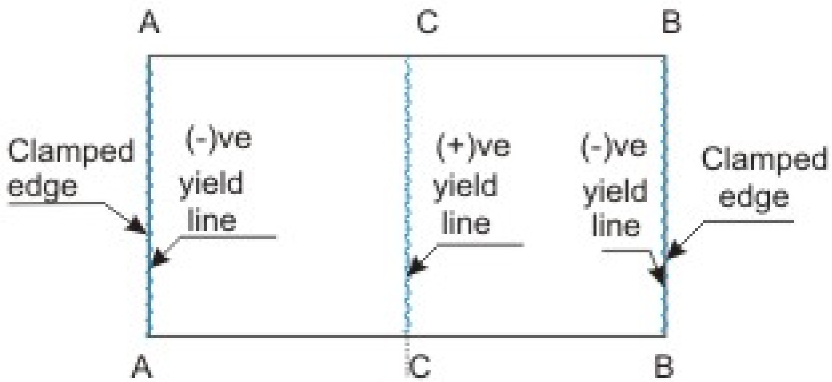


Fig. 12.30.4(a): Plan and yield lines

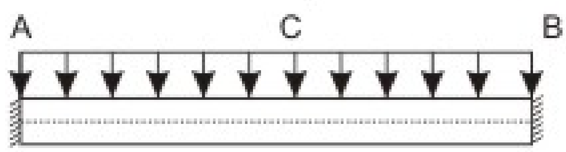


Fig. 12.30.4(b): Section and loading

Fig. 12.30.4: Clamped-clamped one-way slabs

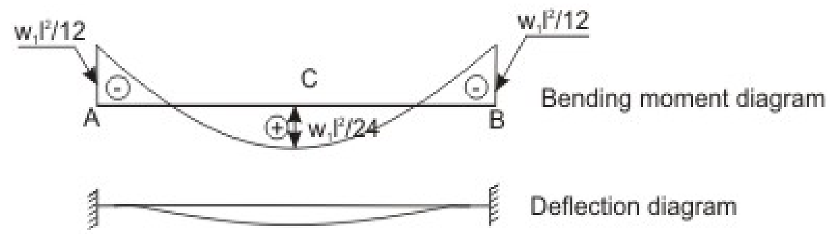


Fig. 12.30.4(c):
Elastic bending moment and deflection diagrams (load = w_1 and $M_p > w_1l^2/12$)

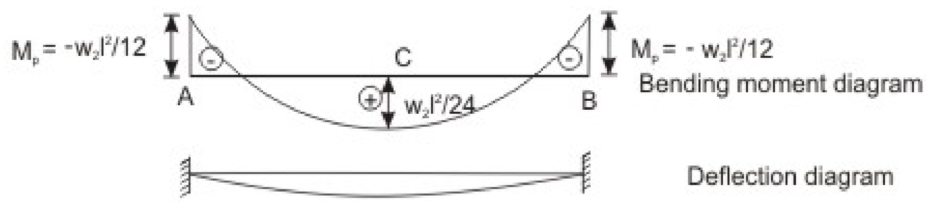


Fig. 12.30.4(d): Bending moment and deflection diagrams (load = w_2 and $M_p = w_2l^2/12$)

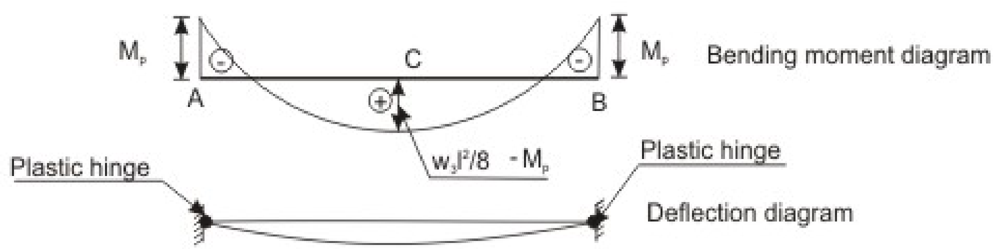


Fig. 12.30.4(e): Bending moment and deflection diagrams (load = w_3 and $M_p < w_3l^2/12$)

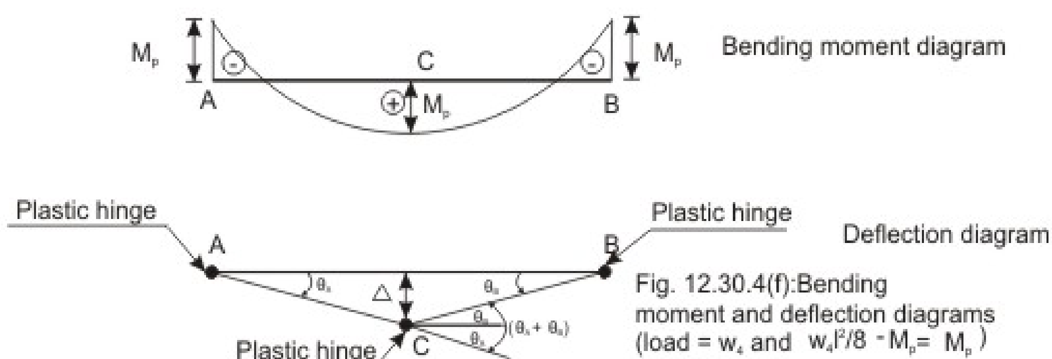


Fig. 12.30.4(f):Bending moment and deflection diagrams (load = w_4 and $w_4l^2/8 - M_p = M_p$) - Mechanism

Fig. 12.30.4: Clamped-clamped one-way slabs with increasing load $w_1 < w_2 < w_3 < w_4$

It us thus seen that only one yield line is needed to form a mechanism with two real hinges for the collapse of the statically determinate one-way slab of Fig.12.30.2a. In case of statically indeterminate slab, the clamped-clamped slab of Fig.12.30.4a, however, can sustain the loads without collapse even after the formation of one or more yield lines. Figures 12.30.4c to f show the bending moment and deflection diagrams of the slab with increasing loads $w_1 < w_2 < w_3 < w_4$. It is known that during the elastic range, the negative moment at the clamped ends ($w_1 l^2/12$) is twice of the positive moment at mid-span ($w_1 l^2/24$), as shown in Fig.12.30.4c. Assuming that the slab has equal reinforcement for positive and negative moments, the highly stressed clamped ends start yielding first. Though the support line hinges rotate, the restraining moments continue to act till the mid-span moment becomes equal to the moment at the supports. Accordingly, a third hinge is formed (Fig.12.30.4f). Now, the three hinges form the mechanism and the slab collapses showing large deflection and curvature. The moment diagram just before the collapse is shown in Fig.12.30.4f.

It is clear from the above discussion that such mechanism is possible with the bending moment diagram of Fig.12.30.4f, if the slab is having adequate reinforcement to resist equal moments at the support and mid-span. The elastic bending moment ratio of 1:2 between the mid-span and support could be increased to 1:1 by the redistribution of moment, which depends on the reinforcement provided in the supports and mid-span sections and not on the elastic bending moment diagram shown in Fig.12.30 4c.

12.30.4 Assumptions

The following are the assumptions of the yield line analysis of reinforced concrete slabs.

1. The steel reinforcement is fully yielded along the yield lines at collapse. Rotation following yield is at constant moment.

2. The slab deforms plastically at collapse and is separated into segments by the yield lines. The individual segments of the slab behave elastically.

3. The elastic deformations are neglected and plastic deformations are only considered. The entire deformations, therefore, take place only along the yield lines. The individual segments of the slab remain plane even in the collapse condition.

4. The bending and twisting moments are uniformly distributed along the yield lines. The maximum values of the moments depend on the capacities of the section based on the amount of reinforcement provided in the section.

5. The yield lines are straight lines as they are the lines of intersection between two planes.

12.30.5 Rules for Yield Lines

The first requirement of the yield line analysis is to assume possible yield patterns and locate the axes of rotation.

It has been observed that assuming the possible yield patterns and locating the axes of rotation are simple to establish for statically determinate or indeterminate (simply supported or clamped) one-way slabs. For other cases, however, suitable guidelines are needed for drawing the yield lines and locating the axes of rotation.

It is worth mentioning that other cases of two-way slabs will have sufficient number of real or plastic hinges to form a mechanism while they will be on the verge of collapse. The yield lines will divide the slabs into a number of segments, which will rotate as rigid bodies about the respective axes of rotation. The axes of rotations will be located along the lines of support or over columns, if provided as point supports. The yield line between two adjacent slab segments is a straight line, as the intersection of two-plane surfaces is always a straight line. The yield line should contain the point of intersection, if any, of the two axes of rotation of two adjacent segments as such point of intersection is common to the two planes.

The two terms, positive and negative yield lines, are used in the analysis to designate the yield lines for positive bending moments having tension at the bottom and negative bending moments having tension at the top of the slab, respectively.

The following are the guidelines for predicting the yield lines and axes of rotation:

1. Yield lines between two intersecting planes are straight lines.

2. Positive yield line will be at the mid-span of one-way simply supported slabs.

3. Negative yield lines will occur at the supports in addition to the positive yield lines at the mid-span of one-way continuous slabs.

4. Yield lines will occur under point loads and they will be radiating outward from the point of application of the point loads.

5. Yield line between two slab segments should pass through the point of intersection of the axes of rotation of the adjacent slab segments.

6. Yield lines should end at the boundary of the slab or at another yield line.

7. Yield lines represent the axes of rotation.

8. Supported edges of the slab will also act as axes of rotation. However, the fixed supports provide constant resistance to rotation having negative yield lines at the supported edges. On the other hand, axes of rotation at the simply supported edges will not provide any resistance to rotation of the segment.

9. Axis of rotation will pass over any column support, if provided, whose orientation will depend on other considerations.

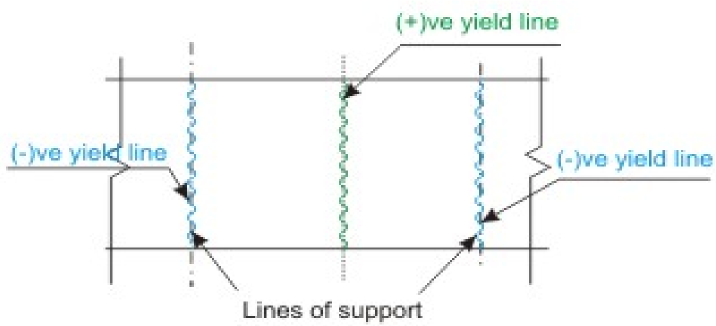


Fig. 12.30.5(a): One-way continuous slab

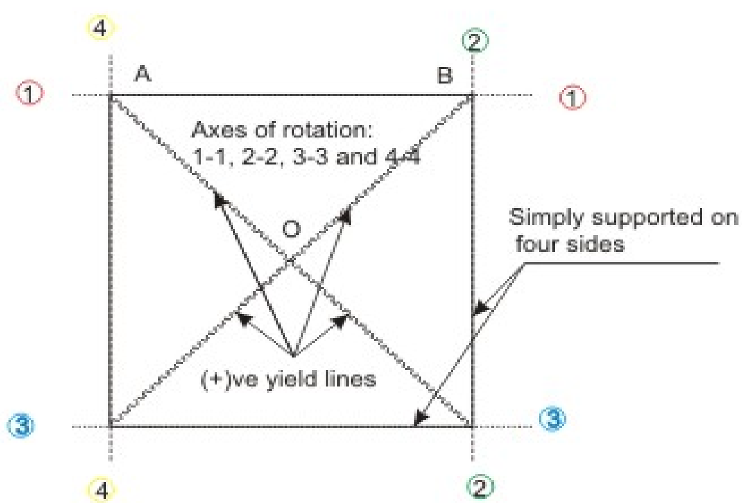


Fig. 12.30.5(b): Two-way simply supported square slab

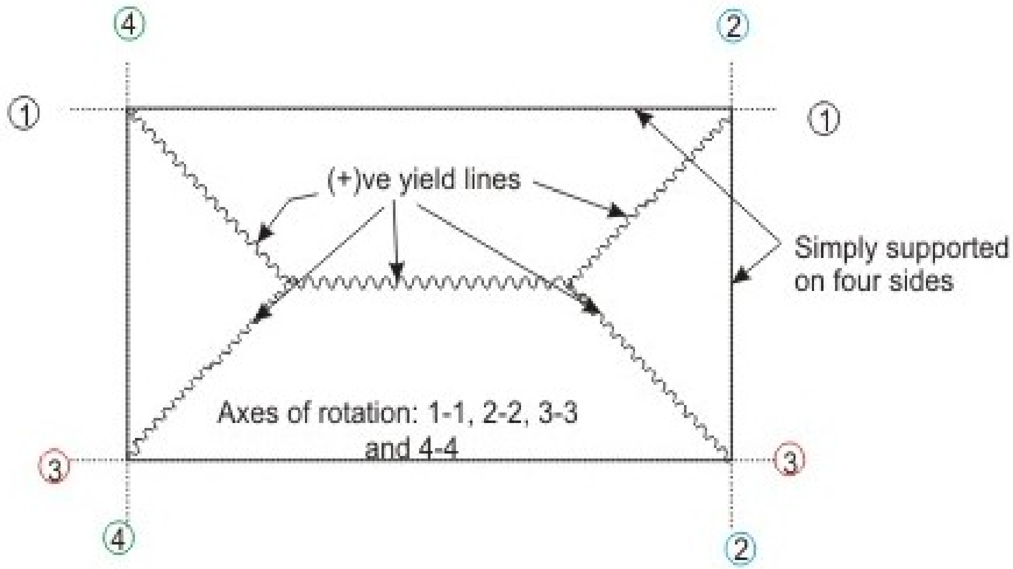


Fig 12.30.5(c): Two-way rectangular slab with simply supported edges

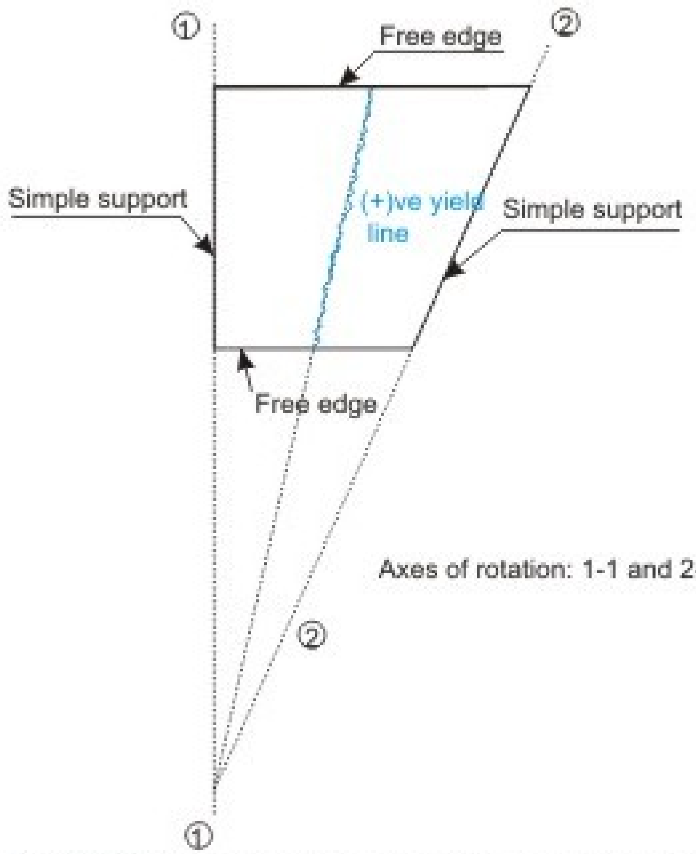


Fig. 12.30.5(d): Slab with non-parallel supported edges

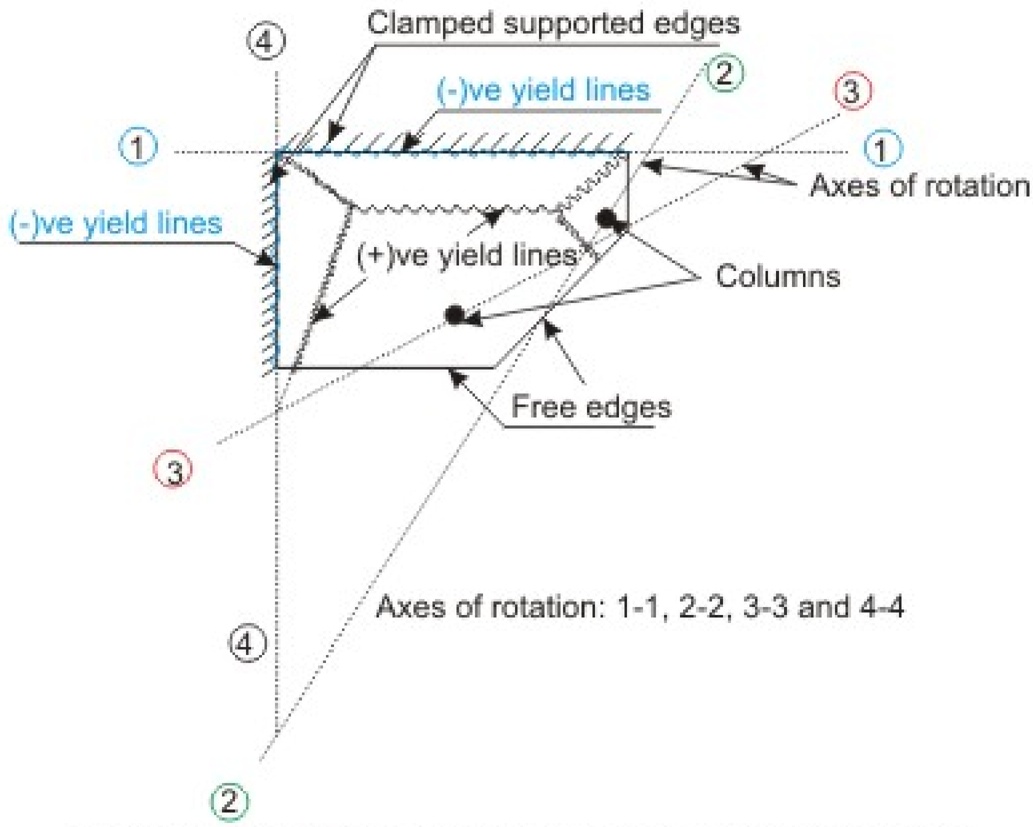
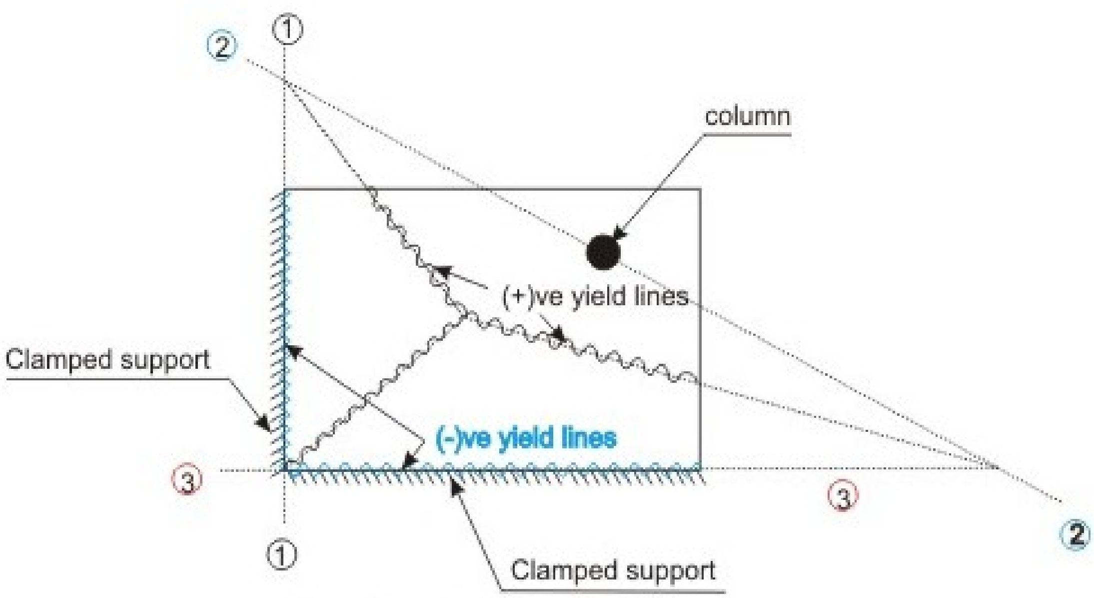


Fig. 12.30.5(e): Slabs on two supported edges and two columns

Fig. 12.30.5: Typical yield patterns



Axes of rotation: 1-1, 2-2 and 3-3

Fig 12.30.5(f): Slab on two supported edges and one column

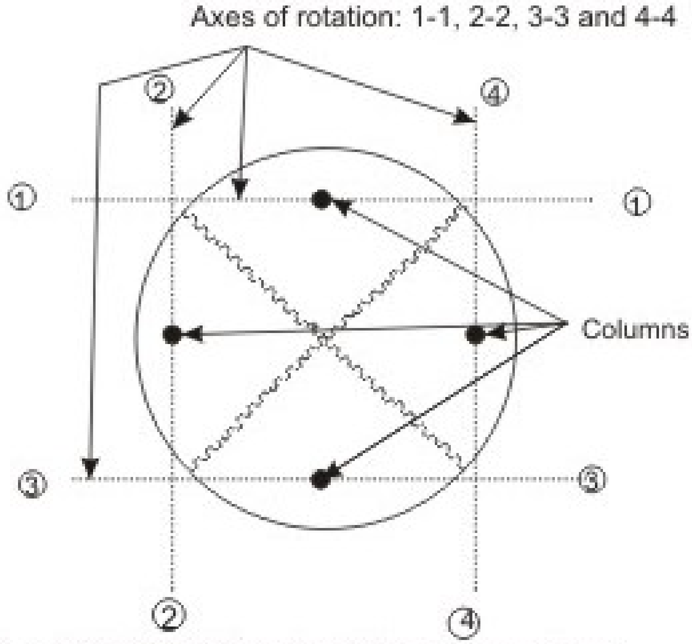


Fig. 12.30.5(g): Circular slab on four columns

Some examples are taken up to illustrate the applications of the guidelines for predicting the possible yield patterns in Figs.12.30.5a to g.

12.30.6 Upper and Lower Bound Theorems

According to the general theory of structural plasticity, the collapse load of a structure lies in between the upper bound and lower bound of the true collapse load. Therefore, the solution employing the theory of plasticity should ensure that lower and upper bounds converge to the unique and correct values of the collapse load.

The statements of the two theorems applied to slabs are given below:

(A) Lower bound theorem: The lower bound of the true collapse load is that external load for which a distribution of moments can be found satisfying the requirements of equilibrium and boundary conditions so that the moments at any location do not exceed the yield moment.

(B) Upper bound theorem: The upper bound of the true collapse load is that external load for which the internal work done by the slab for a small increment of displacement, assuming that moment at every plastic hinge is equal to the yield moment and satisfying the boundary conditions, is equal to the external work done by that external load for the same amount of small increment of displacement.

Thus, the collapse load satisfying the lower bound theorem is always lower than or equal to the true collapse load. On the other hand, the collapse load satisfying the upper bound theorem is always higher than or equal to the true collapse load.

The yield line analysis is an upper bound method in which the predicted failure load of a slab for given moment of resistance (capacity) may be higher than the true value. Thus, the solution of the upper bound method (yield line analysis) may result into unsafe design if the lowest mechanism could not be chosen. However, it has been observed that the prediction of the most probable true mechanism in slab is not difficult. Thus, the solution is safe and adequate in most of the cases. However, it is always desirable to employ a lower bound method, which is totally safe from the design point of view.

12.30.7 Methods of Analysis

After predicting the general yield pattern and locating the axes of rotation, the specific pattern and locations of axes of rotation and the collapse load for the slab can be determined by one of the two methods given below:

(1) Method of segmental equilibrium

(2) Method of virtual work.

The two methods are briefly explained below.

(1) Method of segmental equilibrium

In this method, equilibrium of the individual slab segments causing the collapse forming the required mechanism is considered to arrive at a set of simultaneous equations. The solutions of the simultaneous equations give the values of geometrical parameters for finalising the yield pattern and the relation between the load capacity and resisting moment.

Thus, in segmental equilibrium method, each segment of the slab is studied as a free body (Fig.12.30.5b) which is in equilibrium at incipient failure under the action of the applied loads, moment along the yield lines, and reactions or shear along the support lines. Since, yield moments are principal moments, twisting moments are zero along the yield lines. Further, in most of the cases, shear forces are also zero. Thus, with reference to Fig.12.30.5b, the vector sum of moments along yield lines AO and OB (segment AOB) is equal to moments of the loads on the segment AOB about the axis of rotation 1-1 just before the collapse.

It should be specially mentioned that equilibrium of a slab segment should not be confused with the general equilibrium equation of the true equilibrium method, which is lower bound. Strip method of slab design, developed by A. Hillerborg, (vide "Equilibrium theory for reinforced concrete slabs" (in Swedish), Betong, vol.4 No. 4, 1956, pp. 171-182) is a true equilibrium method resulting in a lower bound solution of the collapse load, which is safe and preferable too. The governing equilibrium equation for a small slab element of lengths *dx* and *dy* is

$$\frac{\partial^2 Mx}{\partial x^2} + 2\frac{\partial^2 Mxy}{\partial x \partial y} + \frac{\partial^2 My}{\partial y^2} = -w \qquad (12.2)$$

where w is the external load per unit area; Mx and My are the bending moments per unit width in x and y directions, respectively; and Mxy is the twisting moment. However, strip method of analysis is beyond the scope of this course. For more information about strip method, the reader may refer to Chapter 15 of "Design of concrete structures" by A.H. Nilson, Tata-McGraw – Hill Publishing Company Limited, New Delhi.

(2) Method of virtual work

This method is based on the principle of virtual work. After predicting the possible yield pattern and the axes of rotation, the slab, which is in equilibrium with the moments and loads on the structure, is given an infinitesimal increase in load to cause the structure further deflection. The principle of virtual work method is that the external work done by the loads to cause a small virtual deflection should be equal to the internal work done by the yield moments to cause the rotation in accommodating the virtual deflection. The relation between the applied loads and the ultimate resisting moments of the slab is obtained by equating the internal and external works. As the elastic deflections and rotations are small compared to the plastic deformations and rotations, they are neglected in the governing equation. Further, the compatibility of deflection must be maintained. The work equation is written as follows:

$$\sum w \Delta = \sum M \theta l \qquad (12.3)$$

where

w = collapse load,

Δ = vertical deflection through which the collapse load w moves,

M = moment capacity of the section per unit length,

θ = rotation of the slab segment satisfying the compatibility of deflection, and

l = length of the yield line.

As mentioned earlier, both the methods of segmental equilibrium and virtual work are upper bound methods. Therefore, the collapse load obtained by either method of yield line analysis will be at the higher end of the true collapse load. Accordingly, each of the two methods should be developed to get the correct solution for predicted mechanism. However, the true collapse load will be obtained only if the correct mechanism has been predicted.

Thus, the solution of any of the two upper bound methods has two essential parts: (1) predicting the correct yield pattern, and (2) determining the geometric parameters that define the exact location and orientation of the yield pattern and solving for the relation between applied load and resisting moments.

12.30.8 Analysis of One-Way Slab

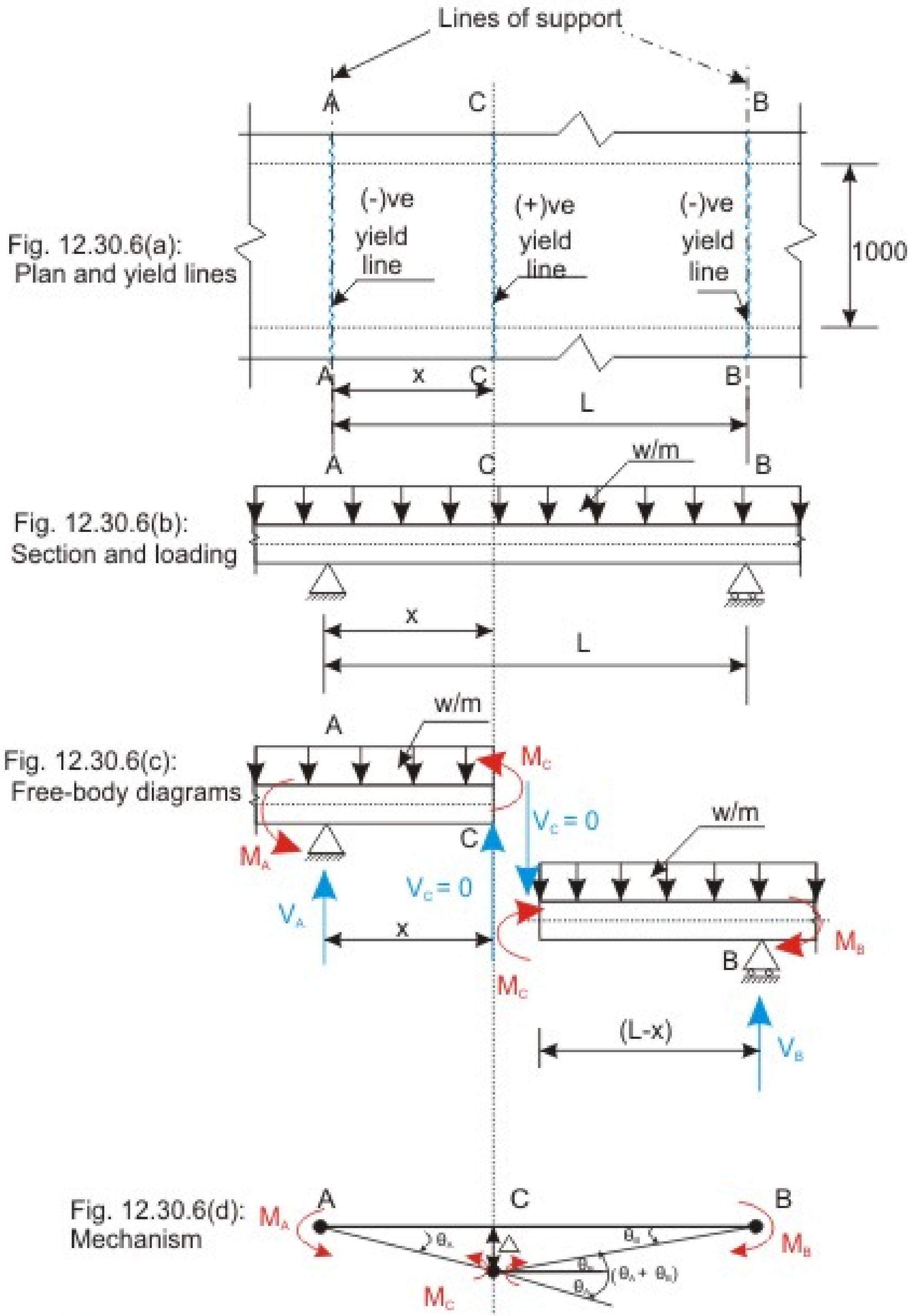


Fig. 12.30.6: One-way continuous slab under uniformly distributed loads

Figures12.30.6a and b show one-way continuous slab whose one span of width one metre is considered. The factored moments of resistance are M_A and M_B at the continuous supports A and B where negative yield lines have formed and M_C at C where the positive yield line has formed. All the moments of resistance are for one metre width. Figure 12.30.6c shows the free body diagrams of the two segments of slab, which are in equilibrium. We now apply the method of segmental equilibrium in the free body diagrams of Fig.12.30.6c.

(1) Method of segmental equilibrium

The line CC, where the positive yield line has formed is at a distance of x from AA. The shear $V = 0$ at C as the bending moment is the maximum (positive) there. Here there are two unknowns x, the locations of the positive yield line and w, the collapse load, which are determined from the two equations of equilibrium $\sum V = 0$ and $\sum M = 0$.

From the statical analysis, we know that the vertical reactions V_A and V_B at A and B, respectively are, (assuming $M_A > M_B$):

$$V_A = (wL/2) + (M_A - M_B) / L \qquad (12.4)$$
$$V_B = (wL/2) - (M_A - M_B) / L \qquad (12.5)$$

Now, $\sum V = 0$ gives: $V_A - wx = 0$
(12.6)

Substituting the expression of V_A from Eq. 12.4 in Eq. 12.6, we have, $wL/2 + (M_A - M_B) / L - wx = 0$, which gives:

$$w = -2 (M_A - M_B) / L (L-2x) \qquad (12.7)$$

$\sum M = 0$ gives:

$$V_A x - M_A - wx^2/2 - M_c = 0$$
(12.8)

Substituting the expression of V_A from Eq. 12.4 in Eq. 12.8, we have,

$$\{wL/2 + (M_A - M_B) / L\}x - M_A - wx^2/2 - M_C = 0$$
(12.9)

Substituting the expression of w from Eq. 12.7 into Eq. 12.9, we have,

$$(M_B - M_A) x^2 + 2 (M_A + M_C) Lx - (M_A + M_C) L^2 = 0 \qquad (12.10)$$

Equation 12.10 will give the values of x for known values of M_A, M_B and M_C. Equation12.7 will give the value of w after getting the value of x from Eq. 12.10.

(2) Method of virtual work

The work equation (Eq.12.3) is written equating the total work done by the collapse loads during the rotation of slab segments, maintaining the deflection compatibility to the total internal work done by bending and twisting moments on all the yield lines. Figure12.30.6d presents the rigid body rotations of the two slab segments. The left segment AACC rotates by θ_A in the clockwise direction and the right segment BBCC rotates by θ_B in the anticlockwise direction, while maintaining the deflection Δ compatible, as shown in Fig.12.30.6d.

$$\text{External work done} = wx\,(\Delta/2) + w\,(L\text{-}x)\,\Delta/2 = wL\Delta/2 \qquad (12.11)$$

$$\text{Internal work done} = +\,M_A\,\theta_A + M_B\,\theta_B + M_C\,(\theta_A + \theta_B) \qquad (12.12)$$

From Fig.12.30.6d:

$$\theta_A = \Delta/x \text{ and } \theta_B = \Delta/(L\text{-}x) \qquad (12.13)$$

Substituting the values of θ_A and θ_B from Eq.12.13 into Eq.12.12, we have:

$$\text{Internal work done} = M_A\,(\Delta/x) + M_B\,(\Delta\,/L\text{-}x) + M_C\,(\Delta/x + \Delta/L\text{-}x) \qquad (12.14)$$

Equating the works, we have form Eqs.12.11 and 12.14, $wL\ \ \Delta/2 = M_A\,(\Delta/x) + M_B\,(\Delta/L\text{-}x) + M_C\,(\Delta/x + \Delta/L\text{-}x)$

or, $$w = 2M_A / Lx + 2M_B / L\,(L\text{-}x) + 2\,M_C / x\,(L\text{-}x) \qquad (12.15)$$

In the method of segmental equilibrium, we have two equations (Eqs.12.7 and 12.9) for the two unknowns x and w. In the method virtual work, however, we have only one equation (Eq.12.15) to determine the two unknowns x and w. Accordingly; we generate another equation with the help of differential calculus. Since the method of analysis is upper bound, we have to consider the minimum value of w satisfying Eq.12.15. This can be obtained by differentiating w of Eq.12.15 with respect to x and putting $dw/dx = 0$. Hence, we have:

$$dw/dx = (2M_A / L)\,(-1/x^2) + (2\,M_B/L)\{1/(L\text{-}x)^2\} + 2\,M_C\,[(1/L\text{-}x)\,(-1/x^2) + (1/x)\{1/(L\text{-}x)^2\}] = 0 \qquad (12.16)$$

After arranging the coefficients of x^2, x and constant terms, we have:

$$(M_B - M_A)\, x^2 + 2\,(M_A + M_C)\, Lx - (M_A + M_C)\, L^2 = 0 \quad (12.10)$$

This is the same equation (Eq.12.10), as we obtained in the method of segmental equilibrium.

In the method of virtual work, therefore, we get the values of x from the same equation, Eq.12.10, and then we get w from Eq.12.15.

We consider two special cases of simply supported and clamped slabs from the above equations.

Case (i): Simply Supported Slab

Here, we have $M_A = M_B = 0$. We get the values of x and w from Eqs.12.10 and 12.7, respectively, by the method of segmental equilibrium and from Eqs.12.10 and 12.15, respectively, by the method of virtual work.

Equation 12.10 becomes $2\, M_C\, Lx - M_C\, L^2 = 0$, when $M_A = M_B = 0$, which gives $x = L/2$.

However, Eq. 12.7 gives division by zero for the simply supported case, when $M_A = M_B = 0$ and $x = L/2$. So, we use Eq.12.8 ($\sum M = 0$) when $M_A = M_B = 0$ and $x = L/2$. This gives $w = 8M_C / L^2$.

In the method of virtual work, Eq.12.10 (the same as in the method of segmental equilibrium) gives $x = L/2$. Thereafter, Eq.12.15 is used for determining w, which gives (when $M_A = M_B = 0$ and $x = L/2$),

$$w = 8M_C / L^2$$

Thus, we get the following values of x and w for the simply supported slab

$$x = L/2 \text{ and } w = 8M_C / L^2 \quad (12.17)$$

Case (ii) Clamped Slab with $M_A = M_B = 2M_C$

Here, we get $x = L/2$ from Eq.12.10. Then, we use Eq.12.8 as Eq.12.7 involves zero by zero case, as explained in case (i) above. This gives $w = 24\, M_C / L^2$ (when $M_A = M_B = 2M_C$ and $x = L/2$). These values are by the method of segmental equilibrium.

Similarly, by the method of virtual work, Eq.12.10 gives $x = L/2$. Then, Eq.12.15 gives, $w = 24\, M_C / L^2$.

Thus, we get for the clamped slab, when $M_A = M_B = 2\, M_{C,}$

$$x = L/2 \text{ and } w = 24\, M_C / L^2 \qquad (12.18)$$

It is clear from the two cases that there will be positive yield line at the centre of the slab when it is simply supported and there will be two negative yield lines at the two supports in addition to the positive yield line at the centre of the slab when the slab is clamped at both ends. The collapse loads are: $w = 8\, M_C / L^2$ and $24\, M_C / L^2$ for the simply supported and clamped slabs, respectively. However, it is worth mentioning that the formation of those specific yield lines and the respective collapse loads are possible only if the slab is designed with adequate positive and negative reinforcement, as assumed to get the solution.

In the practical cases of continuous one-way slabs, the negative moments over the supports may be anywhere in the range of $wl^2/8$ and $wl^2/12$. Moreover, it is not essential that the negative moment of resistances at the two ends should be equal. Depending on the values of M_A, M_B and M_C as per the reinforcement provided, the values of x and w shall be determined using the respective equations employing either the method of segmental analysis or the method of virtual work.

We now take up numerical problems in the next section for the purpose of illustration.

12.30.9 Numerical Problems

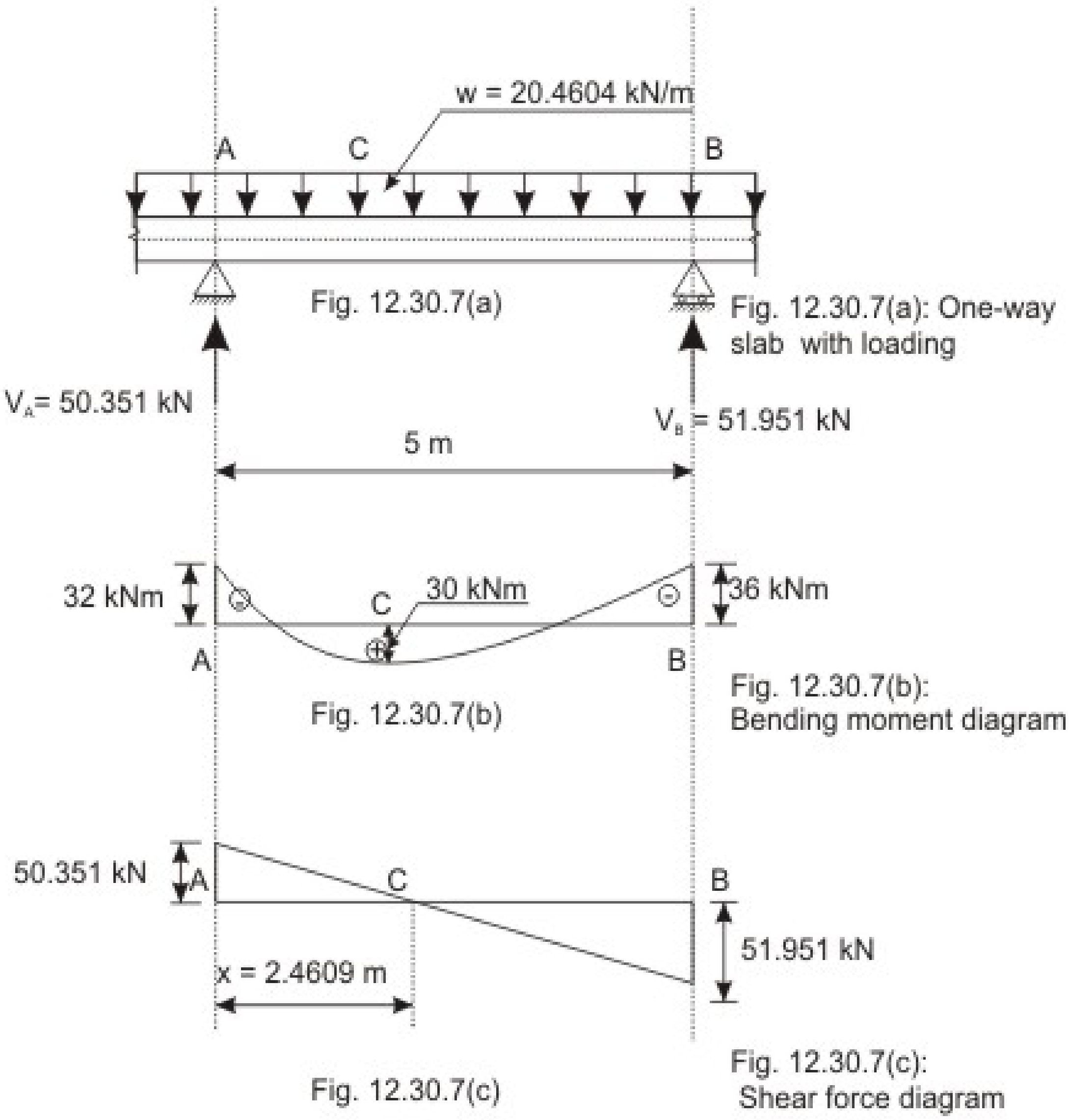


Fig. 12.30.7: Problem 1

Problem 1. The factored moment capacities of the one-way continuous reinforce concrete slab of Figs. 12.30.7 are M_A = 32 kNm, M_B = 36 kNm and M_C = 30 kNm. The span of the slab is 5 m. Determine the location of plastic hinges and the collapse load employing methods of segmental equilibrium and virtual work.

Solution 1:

(A) Method of segmental equilibrium

We have two equations (Eqs.12.10 and 12.7) to determine the location of positive yield line (distance x) and the collapse load (Figs.12.30.6a to d).

$$(M_B - M_A)\, x^2 + 2\,(M_A + M_C)\, Lx - (M_A + M_C)\, L^2 = 0 \qquad (12.10)$$

Using the given values of M_A = 32 kNm, M_B = 36 kNm and M_C = 30 kNm in Eq.12.10, we have

$$4\,x^2 + 2(62)\,(5)\,x - (62)\,(25) = 0$$

or,

$$x^2 + 155\,x - 387.5 = 0 \text{ gives } x = 2.4609 \text{ m.}$$

$$w = -\,2\,(M_A - M_B)\,/\,L\,(L\text{-}2x) \qquad (12.7)$$

Using the values of M_A = 32 kNm, M_B = 36 kNm and x = 2.4609 m in Eq.12.7, we have, w = 20.4604 kN/m.

Therefore, the locations of the positive yield line is at a distance of 2.4609 m from the left support at A, the negative yield lines are at the supports A and B, and the collapse load = 20.4604 kN/m.

(B) Method of virtual work

In this method, we have Eqs.12.10 and 12.15 to determine x and w (Figs.12.30.6a to d). In the method of segmental equilibrium (A) above, Eq.12.10 gives: x = 2.4609 m.

$$w = 2\,M_A/Lx + 2\,M_B\,/\,L(L\text{-}x) + 2\,M_C\,/x\,(L\text{-}x) \qquad (12.15)$$

Using M_A = 32 kNm, M_B = 36 kNm, M_C = 30 kNm and x = 2.4609 m, in Eq.12.15 above, we get w = 20.4749 kN/m. Thus, we get the same location of positive yield line i.e., at x = 2.4609 m from the left support A, negative yield lines are at the support A and B and the collapse load w = 20.4749 kN/m .

The marginal difference of the two collapse loads is due to the truncation error in the calculation. Otherwise, both the values should be the same. The bending moment and shear force diagrams are shown in Figs.12.30.7b and c, respectively.

12.30.10 Practice Questions and Problems with Answers

Q.1: Name five limitations of elastic analysis of reinforced concrete slabs.

A.1: Second paragraph of sec.12.30.1

Q.2: What are yield lines and what is yield line theory?

A.2: First paragraph of sec.12.30.2

Q.3: Explain the basic principles of yield line theory.

A.3: Sec.12.30.3

Q.4: State the assumptions of yield line theory.

A.4: Sec.12.30.4

Q.5: What are the guidelines to draw the possible yield patterns and locate the axes of rotations?

A.5: The last paragraph of sec.12.30.5

Q.6: State upper and lower bound theorems.

A.6: Sec.12.30.6

Q.7: Explain the two methods of yield line analysis.

A.7: Sec.12.30.7

Q.8: Compare the locations of positive and negative yield lines and the values of the respective collapse loads of one 3 m × 6 m slab supported along 6 m direction (Figs.12.30.6a) carrying a total factored load of 20 kN/m for the three cases: (i) the slab is simply supported, (ii) the slab is clamped and (iii) the positive and negative reinforcements are identical to have the equal resistance for the continuous slab. Discuss the results.

A.8: The moments M_C at the mid-span for one simply supported slab is $wl^2/8$ = 90 kNm and for another clamped supported slab $M_C = wl^2$ /24= 30 kNm. The support moments for clamped slab $M_A = M_B = -wl^2/12$ = - 60 kNm. For the third case the magnitude of M_A, M_B and M_C are equal for a continuous slab. So, $M_A = M_B$ = - 45 kNm and M_C = + 45 kNm. Method of virtual work is employed.

Case (i): From Eq. 12.17, we have the distance of positive yield line x = 6/2 = 3 m and $w = 8\ M_C / L^2$ = 8 (90) / 6(6) = 20 kN/m. There are no negative yield lines as the slab is simply supported.

Case (ii): From Eq. 12.18, we have the distance of positive yield line x = 6/2 = 3 m and $w = 24\ M_C / L^2$ = 24 (30) / 6(6) = 20 kN/m. The negative yield lines of the clamped slab are at the two sides AA and BB.

Case (iii): Employing method of virtual work

$$(M_B - M_A)x^2 + 2(M_A + M_C)Lx - (M_A + M_C)L^2 = 0 \quad (12.10)$$

$$w = 2M_A / Lx + 2M_B/L\,(L-x) + 2M_C / x\,(L-x) \quad (12.15)$$

Using $M_A = M_B = M_C$ = 45 kNm in Eq.12.10, we have, $2x - L = 0$ or $x = L/2$ = 6/2 = 3 m.

Using x = 3 m and $M_A = M_B = M_C$ = 45 kNm in Eq. 12.15, we have w = 90{1/6(3) + 1/6(3) + 1/3(3)} = 20 kN/m. The negative yield lines are over the supports of continuous slab i.e., along AA and BB.

Discussion of results: All the three slabs have x =3 m and w = 20 kN/m. However, the third case is the most economic as the depth of the slab may be provided for M = 45 kNm and the areas of positive and negative steel are the same. The simply supported slab has no negative moment and the clamped slab has large difference between the positive and negative moments and accordingly, wide variation is seen in the magnitudes of positive and negative steel.

12.30.11 References

1. Reinforced Concrete Limit State Design, 6th Edition, by Ashok K. Jain, Nem Chand & Bros, Roorkee, 2002.
2. Limit State Design of Reinforced Concrete, 2nd Edition, by P.C. Varghese, Prentice-Hall of India Pvt. Ltd., New Delhi, 2002.
3. Advanced Reinforced Concrete Design, by P.C. Varghese, Prentice-Hall of India Pvt. Ltd., New Delhi, 2001.
4. Reinforced Concrete Design, 2nd Edition, by S. Unnikrishna Pillai and Devdas Menon, Tata McGraw-Hill Publishing Company Limited, New Delhi, 2003.

5. Limit State Design of Reinforced Concrete Structures, by P. Dayaratnam, Oxford & I.B.H. Publishing Company Pvt. Ltd., New Delhi, 2004.
6. Reinforced Concrete Design, 1st Revised Edition, by S.N. Sinha, Tata McGraw-Hill Publishing Company. New Delhi, 1990.
7. Reinforced Concrete, 6th Edition, by S.K. Mallick and A.P. Gupta, Oxford & IBH Publishing Co. Pvt. Ltd. New Delhi, 1996.
8. Behaviour, Analysis & Design of Reinforced Concrete Structural Elements, by I.C.Syal and R.K.Ummat, A.H.Wheeler & Co. Ltd., Allahabad, 1989.
9. Reinforced Concrete Structures, 3rd Edition, by I.C.Syal and A.K.Goel, A.H.Wheeler & Co. Ltd., Allahabad, 1992.
10. Textbook of R.C.C, by G.S.Birdie and J.S.Birdie, Wiley Eastern Limited, New Delhi, 1993.
11. Design of Concrete Structures, 13th Edition, by Arthur H. Nilson, David Darwin and Charles W. Dolan, Tata McGraw-Hill Publishing Company Limited, New Delhi, 2004.
12. Concrete Technology, by A.M.Neville and J.J.Brooks, ELBS with Longman, 1994.
13. Properties of Concrete, 4th Edition, 1st Indian reprint, by A.M.Neville, Longman, 2000.
14. Reinforced Concrete Designer's Handbook, 10th Edition, by C.E.Reynolds and J.C. Steedman, E & FN SPON, London, 1997.
15. Indian Standard Plain and Reinforced Concrete – Code of Practice (4th Revision), IS 456: 2000, BIS, New Delhi.
16. Design Aids for Reinforced Concrete to IS: 456 – 1978, BIS, New Delhi.

12.30.12 Test 30 with Solutions

Maximum Marks = 50 Maximum Time = 30 minutes

Answer all questions.

TQ.1: Explain the basic principles of yield line theory.

A.TQ.1: Sec.12.30.3

[10 Marks]

TQ.2: State the assumptions of yield line theory.

A.TQ.2: Sec.12.30.4

[10 Marks]

TQ.3: What are the guidelines to draw the possible yield patterns and locate the axes of rotations?

A.TQ.3: The last paragraph of sec.12.30.5

[10 Marks]

TQ.4: Determine the location of plastic hinges and collapse load of the one-way slab supported at 4 m c/c (Fig.12.30.6a) having the positive and negative reinforcements to have the factored moments of resistance M_A = 25 kNm, M_B = 35 kNm and M_C = 30 kNm. Use method of virtual work.

[20 Marks]

A.TQ.4: $(M_B - M_A)x^2 + 2(M_A + M_C)Lx - (M_A + M_C)L^2 = 0$ (12.10)

$w = 2M_A / Lx + 2M_B/L\ (L\text{-}x) + 2M_C / x\ (L\text{-}x)$ (12.15)

Using M_A = 25 kNm, M_B = 35 kNm and M_C = 30 kNm in Eq. 12.10, we have, x = 1.9165 m and using the values of M_A, M_B, M_C and x in Eq. 12.15, we get w = 29.9478 kN/m^2.

12.30.13 Summary of this Lesson

This lesson explains the basic principle of yield line analysis which is required to remove the inconsistency between the elastic analysis and the design by limit state method considering inelastic behaviour. Moreover, the limitations of elastic analysis of slab are mentioned. The upper and lower bound theorems are explained to show that the two methods of yield line analysis viz. (i) method of segmental equilibrium and (ii) method of virtual work are upper bound methods. The governing equations of both the methods are derived. The first requirement of the yield line analysis is to predict the possible yield pattern and locate the axes of rotation. Suitable guidelines are given for the same as the correctness of the solution depends on the prediction of true yield line pattern and location of the axes of rotation. Numerical problems are solved by both methods of analysis.