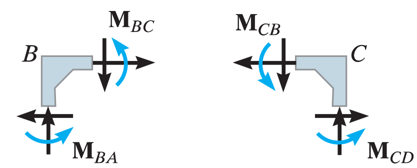
Equilibrium Equations. The preceding six equations contain eight unknowns. The remaining two equilibrium equations come from moment equilibrium at joints B and C, Fig. 11–16b. We have

$$M_{BA} + M_{BC} = 0 \qquad (7)$$
$$M_{CB} + M_{CD} = 0 \qquad (8)$$



(b)

To solve these eight equations, substitute Eqs. (2) and (3) into Eq. (7) and substitute Eqs. (4) and (5) into Eq. (8). We get

$$0.833EI\theta_B + 0.25EI\theta_C = 80$$
$$0.833EI\theta_C + 0.25EI\theta_B = -80$$

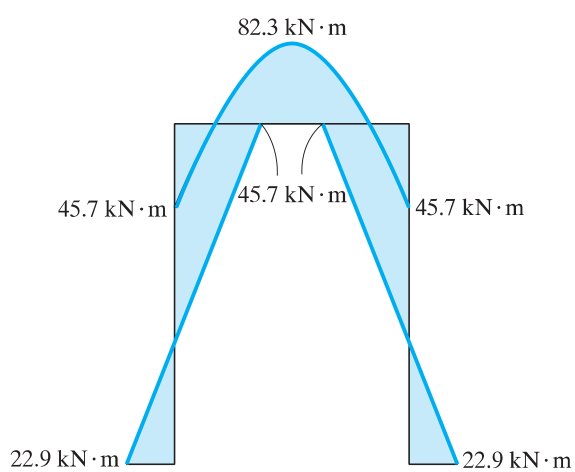
Solving simultaneously yields

$$\theta_B = -\theta_C = \frac{137.1}{EI}$$

which conforms with the way the frame deflects as shown in Fig. 11–16a. Substituting into Eqs. (1)–(6), we get

$$M_{AB} = 22.9 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{BA} = 45.7 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{BC} = -45.7 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{CB} = 45.7 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{CD} = -45.7 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{DC} = -22.9 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

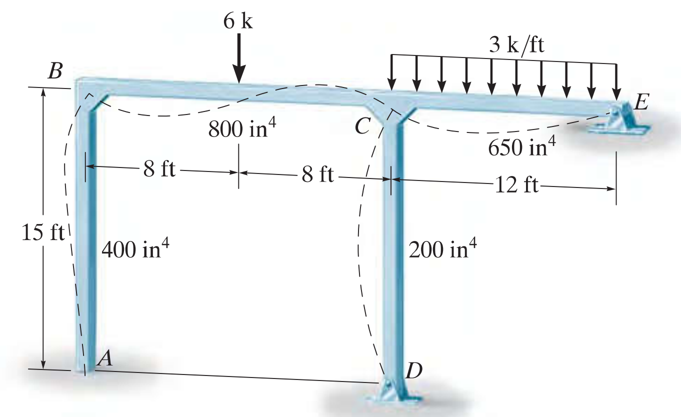
Using these results, the reactions at the ends of each member can be determined from the equations of equilibrium, and the moment diagram for the frame can be drawn, Fig. 11–16c.



(c)

EXAMPLE 11.6

Determine the internal moments at each joint of the frame shown in Fig. 11–17*a*. The moment of inertia for each member is given in the figure. Take $E = 29(10^3)$ ksi.



(a)

Fig. 11–17

SOLUTION

Slope-Deflection Equations. Four spans must be considered in this problem. Equation 11–8 applies to spans AB and BC, and Eq. 11–10 will be applied to CD and CE, because the ends at D and E are pinned.

Computing the member stiffnesses, we have

$$k_{AB} = \frac{400}{15(12)^4} = 0.001286 \text{ ft}^3 \qquad k_{CD} = \frac{200}{15(12)^4} = 0.000643 \text{ ft}^3$$

$$k_{BC} = \frac{800}{16(12)^4} = 0.002411 \text{ ft}^3 \qquad k_{CE} = \frac{650}{12(12)^4} = 0.002612 \text{ ft}^3$$

The FEMs due to the loadings are

$$(\text{FEM})_{BC} = -\frac{PL}{8} = -\frac{6(16)}{8} = -12 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = \frac{PL}{8} = \frac{6(16)}{8} = 12 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{CE} = -\frac{wL^2}{8} = -\frac{3(12)^2}{8} = -54 \text{ k}\cdot\text{ft}$$

Applying Eqs. 11–8 and 11–10 to the frame and noting that $\theta_A = 0$, $\psi_{AB} = \psi_{BC} = \psi_{CD} = \psi_{CE} = 0$ since no sidesway occurs, we have

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2[29(10^3)(12)^2](0.001286)[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 10740.7\theta_B \qquad (1)$$

$$M_{BA} = 2[29(10^3)(12)^2](0.001286)[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 21\,481.5\theta_B \tag{2}$$

$$M_{BC} = 2[29(10^3)(12)^2](0.002411)[2\theta_B + \theta_C - 3(0)] - 12$$

$$M_{BC} = 40\,277.8\theta_B + 20\,138.9\theta_C - 12 \tag{3}$$

$$M_{CB} = 2[29(10^3)(12)^2](0.002411)[2\theta_C + \theta_B - 3(0)] + 12$$

$$M_{CB} = 20\,138.9\theta_B + 40\,277.8\theta_C + 12 \tag{4}$$

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{CD} = 3[29(10^3)(12)^2](0.000643)[\theta_C - 0] + 0 \tag{5}$$

$$M_{CD} = 8055.6\theta_C$$

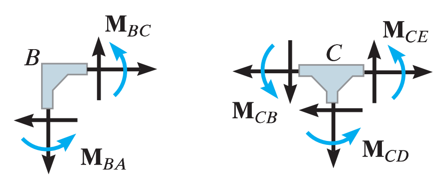
$$M_{CE} = 3[29(10^3)(12)^2](0.002612)[\theta_C - 0] - 54$$

$$M_{CE} = 32\,725.7\theta_C - 54 \tag{6}$$

Equations of Equilibrium. These six equations contain eight unknowns. Two moment equilibrium equations can be written for joints B and C, Fig. 11–17b. We have

$$M_{BA} + M_{BC} = 0 \tag{7}$$

$$M_{CB} + M_{CD} + M_{CE} = 0 \tag{8}$$



(b)

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), and Eqs. (4)–(6) into Eq. (8). This gives

$$61\,759.3\theta_B + 20\,138.9\theta_C = 12$$

$$20\,138.9\theta_B + 81\,059.0\theta_C = 42$$

Solving these equations simultaneously yields

$$\theta_B = 2.758(10^{-5})\text{ rad} \qquad \theta_C = 5.113(10^{-4})\text{ rad}$$

These values, being clockwise, tend to distort the frame as shown in Fig. 11–17a. Substituting these values into Eqs. (1)–(6) and solving, we get

$$M_{AB} = 0.296\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BA} = 0.592\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BC} = -0.592\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CB} = 33.1\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CD} = 4.12\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CE} = -37.3\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

11.5 Analysis of Frames: Sidesway

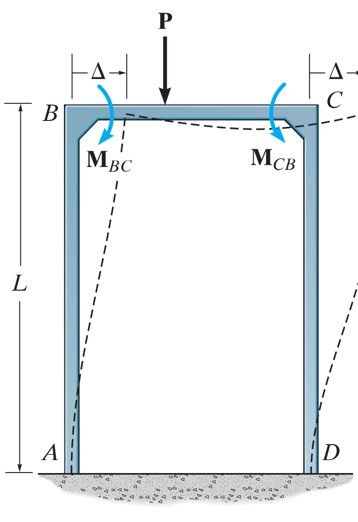
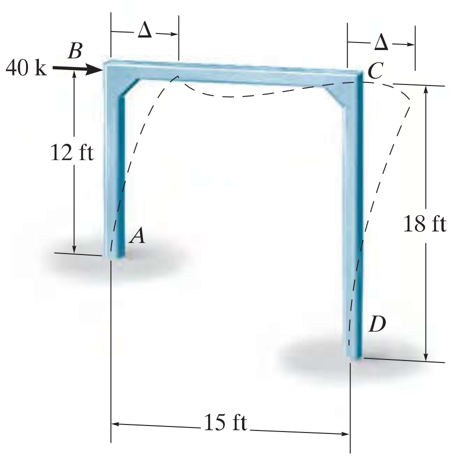


Fig. 11–18

A frame will sidesway, or be displaced to the side, when it or the loading acting on it is nonsymmetric. To illustrate this effect, consider the frame shown in Fig. 11–18. Here the loading **P** causes *unequal* moments $\mathbf{M}_{BC}$ and $\mathbf{M}_{CB}$ at the joints B and C, respectively. $\mathbf{M}_{BC}$ tends to displace joint B to the right, whereas $\mathbf{M}_{CB}$ tends to displace joint C to the left. Since $\mathbf{M}_{BC}$ is larger than $\mathbf{M}_{CB}$, the net result is a sidesway Δ of both joints B and C to the right, as shown in the figure.* When applying the slope-deflection equation to each column of this frame, we must therefore consider the column rotation ψ (since $\psi = \Delta/L$) as unknown in the equation. As a result an extra equilibrium equation must be included for the solution. In the previous sections it was shown that unknown *angular displacements* θ were related by joint *moment equilibrium equations*. In a similar manner, when unknown joint *linear displacements* Δ (or span rotations ψ) occur, we must write *force equilibrium equations* in order to obtain the complete solution. The unknowns in these equations, however, must only involve the internal *moments* acting at the ends of the columns, since the slope-deflection equations involve these moments. The technique for solving problems for frames with sidesway is best illustrated by examples.

EXAMPLE 11.7



(a)

Fig. 11–19

Determine the moments at each joint of the frame shown in Fig. 11–19*a*. *EI* is constant.

SOLUTION

Slope-Deflection Equations. Since the ends A and D are fixed, Eq. 11–8 applies for all three spans of the frame. Sidesway occurs here since both the applied loading and the geometry of the frame are non-symmetric. Here the load is applied directly to joint B and therefore no FEMs act at the joints. As shown in Fig. 11–19*a*, both joints B and C are assumed to be displaced an *equal amount* Δ. Consequently, $\psi_{AB} = \Delta/12$ and $\psi_{DC} = \Delta/18$. Both terms are positive since the cords of members AB and CD "rotate" clockwise. Relating ψ_{AB} to ψ_{DC}, we have $\psi_{AB} = (18/12)\psi_{DC}$. Applying Eq. 11–8 to the frame, we have

$$M_{AB} = 2E\left(\frac{I}{12}\right)\left[2(0) + \theta_B - 3\left(\frac{18}{12}\psi_{DC}\right)\right] + 0 = EI(0.1667\theta_B - 0.75\psi_{DC}) \quad (1)$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)\left[2\theta_B + 0 - 3\left(\frac{18}{12}\psi_{DC}\right)\right] + 0 = EI(0.333\theta_B - 0.75\psi_{DC}) \quad (2)$$

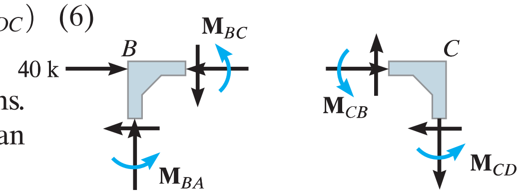
$$M_{BC} = 2E\left(\frac{I}{15}\right)[2\theta_B + \theta_C - 3(0)] + 0 = EI(0.267\theta_B + 0.133\theta_C) \quad (3)$$

*Recall that the deformation of all three members due to shear and axial force is neglected.

$$M_{CB} = 2E\left(\frac{I}{15}\right)[2\theta_C + \theta_B - 3(0)] + 0 = EI(0.267\theta_C + 0.133\theta_B) \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{18}\right)[2\theta_C + 0 - 3\psi_{DC}] + 0 = EI(0.222\theta_C - 0.333\psi_{DC}) \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{18}\right)[2(0) + \theta_C - 3\psi_{DC}] + 0 = EI(0.111\theta_C - 0.333\psi_{DC}) \quad (6)$$



(b)

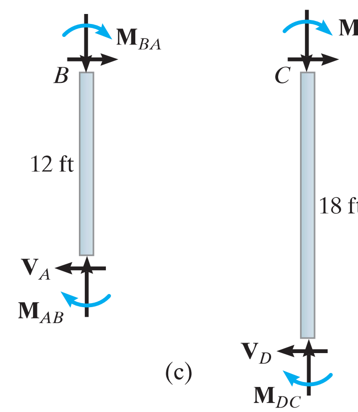
Equations of Equilibrium. The six equations contain nine unknowns. Two moment equilibrium equations for joints B and C, Fig. 11–19*b*, can be written, namely,

$$M_{BA} + M_{BC} = 0 \quad (7)$$

$$M_{CB} + M_{CD} = 0 \quad (8)$$

Since a horizontal displacement Δ occurs, we will consider summing forces on the *entire frame* in the x direction. This yields

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 40 - V_A - V_D = 0$$



(c)

The horizontal reactions or column shears V_A and V_D can be related to the internal moments by considering the free-body diagram of each column separately, Fig. 11–19*c*. We have

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{12}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC} + M_{CD}}{18}$$

Thus,

$$40 + \frac{M_{AB} + M_{BA}}{12} + \frac{M_{DC} + M_{CD}}{18} = 0 \quad (9)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), Eqs. (4) and (5) into Eq. (8), and Eqs. (1), (2), (5), (6) into Eq. (9). This yields

$$0.6\theta_B + 0.133\theta_C - 0.75\psi_{DC} = 0$$

$$0.133\theta_B + 0.489\theta_C - 0.333\psi_{DC} = 0$$

$$0.5\theta_B + 0.222\theta_C - 1.944\psi_{DC} = -\frac{480}{EI}$$

Solving simultaneously, we have

$$EI\theta_B = 438.81 \qquad EI\theta_C = 136.18 \qquad EI\psi_{DC} = 375.26$$

Finally, using these results and solving Eqs. (1)–(6) yields

$M_{AB} = -208 \text{ k}\cdot\text{ft}$ *Ans.*

$M_{BA} = -135 \text{ k}\cdot\text{ft}$ *Ans.*

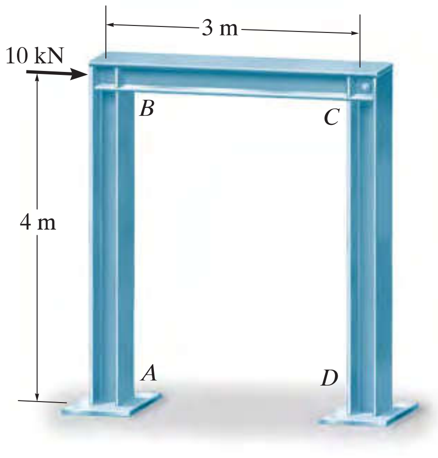
$M_{BC} = 135 \text{ k}\cdot\text{ft}$ *Ans.*

$M_{CB} = 94.8 \text{ k}\cdot\text{ft}$ *Ans.*

$M_{CD} = -94.8 \text{ k}\cdot\text{ft}$ *Ans.*

$M_{DC} = -110 \text{ k}\cdot\text{ft}$ *Ans.*

EXAMPLE 11.8



(a)

Fig. 11–20

Determine the moments at each joint of the frame shown in Fig. 11–20*a*. The supports at *A* and *D* are fixed and joint *C* is assumed pin connected. *EI* is constant for each member.

SOLUTION

Slope-Deflection Equations. We will apply Eq. 11–8 to member *AB* since it is fixed connected at both ends. Equation 11–10 can be applied from *B* to *C* and from *D* to *C* since the pin at *C* supports zero moment. As shown by the deflection diagram, Fig. 11–20*b*, there is an unknown linear displacement Δ of the frame and unknown angular displacement θ_B at joint *B*.* Due to Δ, the cord members *AB* and *CD* rotate clockwise, $\psi = \psi_{AB} = \psi_{DC} = \Delta/4$. Realizing that $\theta_A = \theta_D = 0$ and that there are no FEMs for the members, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

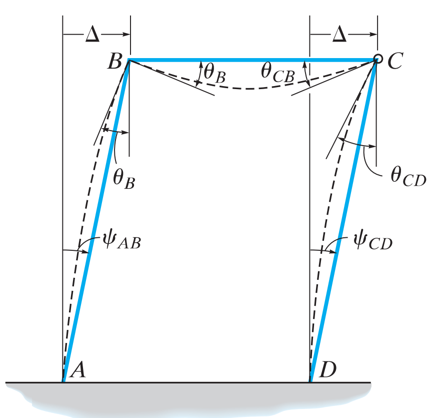
$$M_{AB} = 2E\left(\frac{I}{4}\right)[2(0) + \theta_B - 3\psi] + 0 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{4}\right)(2\theta_B + 0 - 3\psi) + 0 \qquad (2)$$

$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{BC} = 3E\left(\frac{I}{3}\right)(\theta_B - 0) + 0 \qquad (3)$$

$$M_{DC} = 3E\left(\frac{I}{4}\right)(0 - \psi) + 0 \qquad (4)$$



(b)

Equilibrium Equations. Moment equilibrium of joint *B*, Fig. 11–20*c*, requires

$$M_{BA} + M_{BC} = 0 \qquad (5)$$

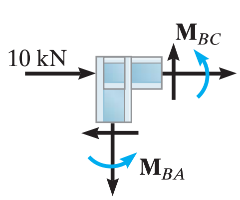
If forces are summed for the *entire frame* in the horizontal direction, we have

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 10 - V_A - V_D = 0 \qquad (6)$$

As shown on the free-body diagram of each column, Fig. 11–20*d*, we have

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{4}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC}}{4}$$



(c)

*The angular displacements θ_{CB} and θ_{CD} at joint *C* (pin) are not included in the analysis since Eq. 11–10 is to be used.

Thus, from Eq. (6),

$$10 + \frac{M_{AB} + M_{BA}}{4} + \frac{M_{DC}}{4} = 0 \qquad (7)$$

Substituting the slope-deflection equations into Eqs. (5) and (7) and simplifying yields

$$\theta_B = \frac{3}{4}\psi$$

$$10 + \frac{EI}{4}\left(\frac{3}{2}\theta_B - \frac{15}{4}\psi\right) = 0$$

Thus,

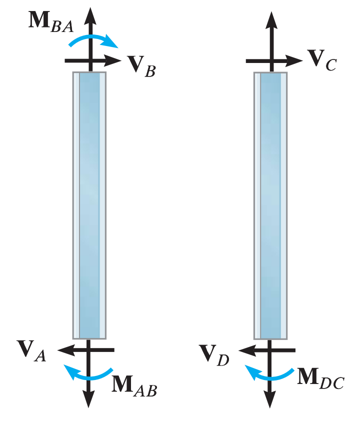
$$\theta_B = \frac{240}{21EI} \qquad \psi = \frac{320}{21EI}$$

Substituting these values into Eqs. (1)–(4), we have

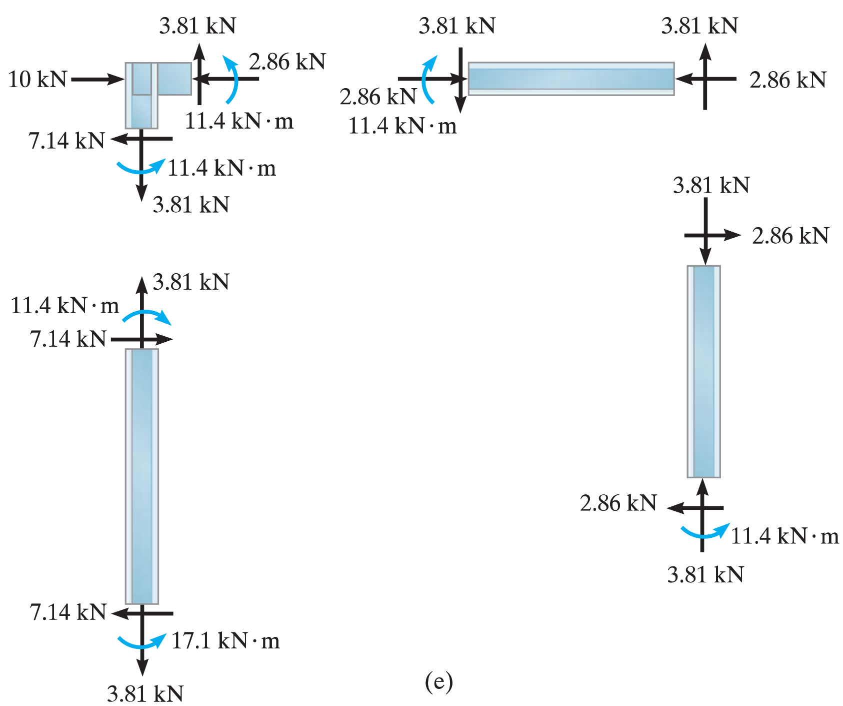
$M_{AB} = -17.1\ \text{kN}\cdot\text{m}, \quad M_{BA} = -11.4\ \text{kN}\cdot\text{m}$ *Ans.*

$M_{BC} = 11.4\ \text{kN}\cdot\text{m}, \quad M_{DC} = -11.4\ \text{kN}\cdot\text{m}$ *Ans.*

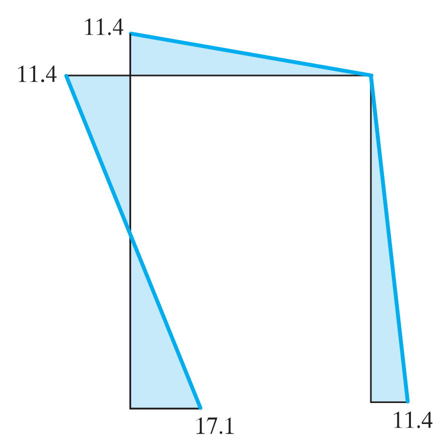
Using these results, the end reactions on each member can be determined from the equations of equilibrium, Fig. 11–20*e*. The moment diagram for the frame is shown in Fig. 11–20*f*.



(d)

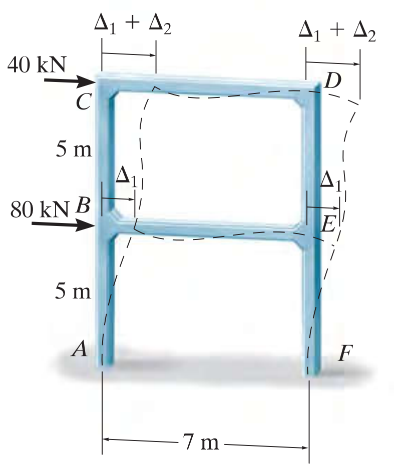


(e)



(f)

EXAMPLE 11.9



(a)

Fig. 11–21

Explain how the moments in each joint of the two-story frame shown in Fig. 11–21*a* are determined. *EI* is constant.

SOLUTION

Slope-Deflection Equation. Since the supports at A and F are fixed, Eq. 11–8 applies for all six spans of the frame. No FEMs have to be calculated, since the applied loading acts at the joints. Here the loading displaces joints B and E an amount Δ_1, and C and D an amount $\Delta_1 + \Delta_2$. As a result, members AB and FE undergo rotations of $\psi_1 = \Delta_1/5$ and BC and ED undergo rotations of $\psi_2 = \Delta_2/5$.

Applying Eq. 11–8 to the frame yields

$$M_{AB} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_B - 3\psi_1] + 0 \quad (1)$$

$$M_{BA} = 2E\left(\frac{I}{5}\right)[2\theta_B + 0 - 3\psi_1] + 0 \quad (2)$$

$$M_{BC} = 2E\left(\frac{I}{5}\right)[2\theta_B + \theta_C - 3\psi_2] + 0 \quad (3)$$

$$M_{CB} = 2E\left(\frac{I}{5}\right)[2\theta_C + \theta_B - 3\psi_2] + 0 \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{7}\right)[2\theta_C + \theta_D - 3(0)] + 0 \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{7}\right)[2\theta_D + \theta_C - 3(0)] + 0 \quad (6)$$

$$M_{BE} = 2E\left(\frac{I}{7}\right)[2\theta_B + \theta_E - 3(0)] + 0 \quad (7)$$

$$M_{EB} = 2E\left(\frac{I}{7}\right)[2\theta_E + \theta_B - 3(0)] + 0 \quad (8)$$

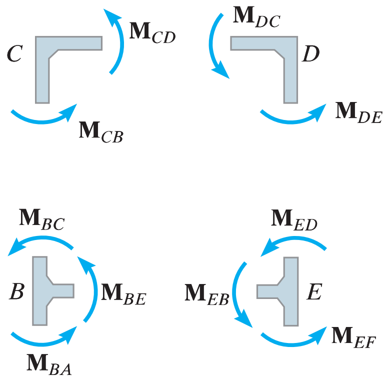
$$M_{ED} = 2E\left(\frac{I}{5}\right)[2\theta_E + \theta_D - 3\psi_2] + 0 \quad (9)$$

$$M_{DE} = 2E\left(\frac{I}{5}\right)[2\theta_D + \theta_E - 3\psi_2] + 0 \quad (10)$$

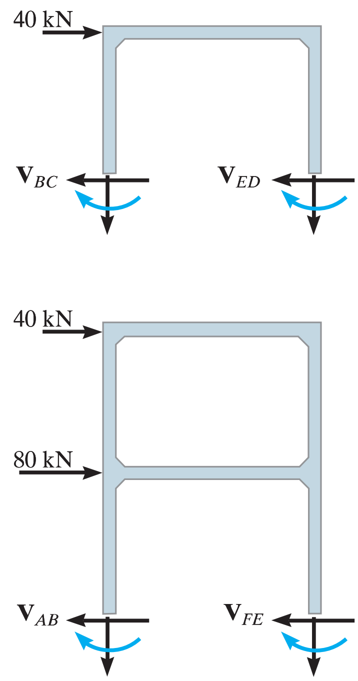
$$M_{FE} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_E - 3\psi_1] + 0 \quad (11)$$

$$M_{EF} = 2E\left(\frac{I}{5}\right)[2\theta_E + 0 - 3\psi_1] + 0 \quad (12)$$

These 12 equations contain 18 unknowns.



(b)



(c)

Equilibrium Equations. Moment equilibrium of joints B, C, D, and E, Fig. 11–21b, requires

$$M_{BA} + M_{BE} + M_{BC} = 0 \tag{13}$$

$$M_{CB} + M_{CD} = 0 \tag{14}$$

$$M_{DC} + M_{DE} = 0 \tag{15}$$

$$M_{EF} + M_{EB} + M_{ED} = 0 \tag{16}$$

As in the preceding examples, the shear at the base of all the columns for any story must balance the applied horizontal loads, Fig. 11–21c. This yields

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 40 - V_{BC} - V_{ED} = 0$$

$$40 + \frac{M_{BC} + M_{CB}}{5} + \frac{M_{ED} + M_{DE}}{5} = 0 \tag{17}$$

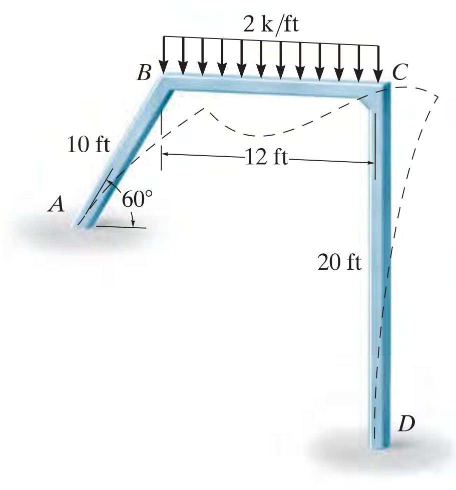
$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 40 + 80 - V_{AB} - V_{FE} = 0$$

$$120 + \frac{M_{AB} + M_{BA}}{5} + \frac{M_{EF} + M_{FE}}{5} = 0 \tag{18}$$

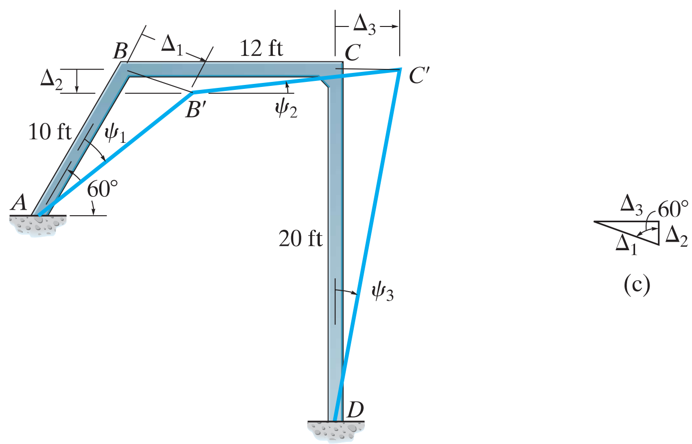
Solution requires substituting Eqs. (1)–(12) into Eqs. (13)–(18), which yields six equations having six unknowns, ψ_1, ψ_2, θ_B, θ_C, θ_D, and θ_E. These equations can then be solved simultaneously. The results are resubstituted into Eqs. (1)–(12), which yields the moments at the joints.

EXAMPLE 11.10

Determine the moments at each joint of the frame shown in Fig. 11–22*a*. *EI* is constant for each member.



(a)



(b)

(c)

Fig. 11–22

SOLUTION

Slope-Deflection Equations. Equation 11–8 applies to each of the three spans. The FEMs are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{2(12)^2}{12} = -24 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{2(12)^2}{12} = 24 \text{ k}\cdot\text{ft}$$

The sloping member AB causes the frame to sidesway to the right as shown in Fig. 11–22*a*. As a result, joints B and C are subjected to both rotational *and* linear displacements. The linear displacements are shown in Fig. 11–22*b*, where B moves Δ_1 to B' and C moves Δ_3 to C'. These displacements cause the members' cords to rotate ψ_1, ψ_3 (clockwise) and $-\psi_2$ (counterclockwise) as shown.* Hence,

$$\psi_1 = \frac{\Delta_1}{10} \qquad \psi_2 = -\frac{\Delta_2}{12} \qquad \psi_3 = \frac{\Delta_3}{20}$$

As shown in Fig. 11–22*c*, the three displacements can be related. For example, $\Delta_2 = 0.5\Delta_1$ and $\Delta_3 = 0.866\Delta_1$. Thus, from the above equations we have

$$\psi_2 = -0.417\psi_1 \qquad \psi_3 = 0.433\psi_1$$

Using these results, the slope-deflection equations for the frame are

*Recall that distortions due to axial forces are neglected and the arc displacements BB' and CC' can be considered as straight lines, since ψ_1 and ψ_3 are actually very small.

11

$$M_{AB} = 2E\left(\frac{I}{10}\right)[2(0) + \theta_B - 3\psi_1] + 0 \tag{1}$$

$$M_{BA} = 2E\left(\frac{I}{10}\right)[2\theta_B + 0 - 3\psi_1] + 0 \tag{2}$$

$$M_{BC} = 2E\left(\frac{I}{12}\right)[2\theta_B + \theta_C - 3(-0.417\psi_1)] - 24 \tag{3}$$

$$M_{CB} = 2E\left(\frac{I}{12}\right)[2\theta_C + \theta_B - 3(-0.417\psi_1)] + 24 \tag{4}$$

$$M_{CD} = 2E\left(\frac{I}{20}\right)[2\theta_C + 0 - 3(0.433\psi_1)] + 0 \tag{5}$$

$$M_{DC} = 2E\left(\frac{I}{20}\right)[2(0) + \theta_C - 3(0.433\psi_1)] + 0 \tag{6}$$

These six equations contain nine unknowns.

Equations of Equilibrium. Moment equilibrium at joints B and C yields

$$M_{BA} + M_{BC} = 0 \tag{7}$$

$$M_{CD} + M_{CB} = 0 \tag{8}$$

The necessary third equilibrium equation can be obtained by summing moments about point O on the entire frame, Fig. 11–22d. This eliminates the unknown normal forces $\mathbf{N}_A$ and $\mathbf{N}_D$, and therefore

$\curvearrowleft + \Sigma M_O = 0;$

$$M_{AB} + M_{DC} - \left(\frac{M_{AB} + M_{BA}}{10}\right)(34) - \left(\frac{M_{DC} + M_{CD}}{20}\right)(40.78) - 24(6) = 0$$

$$-2.4M_{AB} - 3.4M_{BA} - 2.04M_{CD} - 1.04M_{DC} - 144 = 0 \tag{9}$$

Substituting Eqs. (2) and (3) into Eq. (7), Eqs. (4) and (5) into Eq. (8), and Eqs. (1), (2), (5), and (6) into Eq. (9) yields

$$0.733\theta_B + 0.167\theta_C - 0.392\psi_1 = \frac{24}{EI}$$

$$0.167\theta_B + 0.533\theta_C + 0.0784\psi_1 = -\frac{24}{EI}$$

$$-1.840\theta_B - 0.512\theta_C + 3.880\psi_1 = \frac{144}{EI}$$

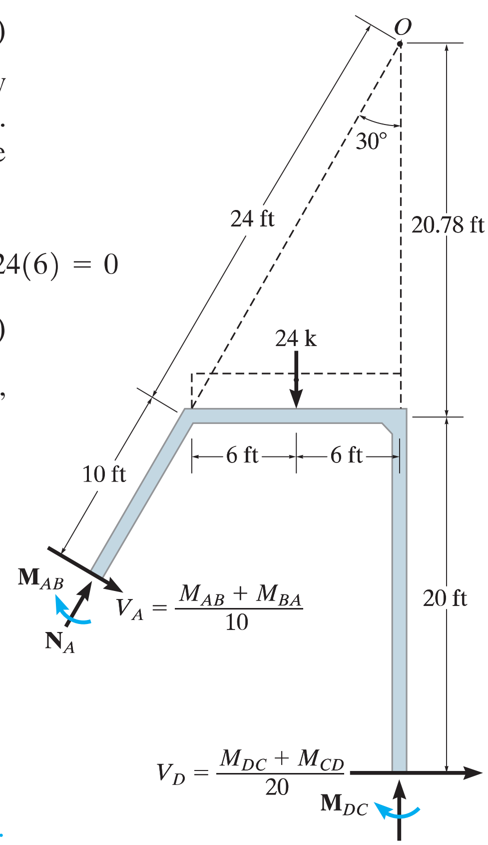
Solving these equations simultaneously yields

$$EI\theta_B = 87.67 \qquad EI\theta_C = -82.3 \qquad EI\psi_1 = 67.83$$

Substituting these values into Eqs. (1)–(6), we have

$M_{AB} = -23.2 \text{ k}\cdot\text{ft} \quad M_{BC} = 5.63 \text{ k}\cdot\text{ft} \quad M_{CD} = -25.3 \text{ k}\cdot\text{ft}$ *Ans.*

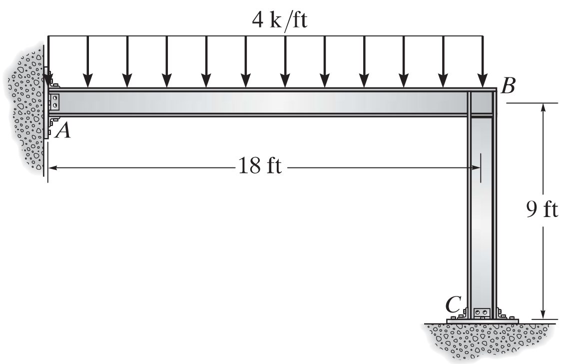
$M_{BA} = -5.63 \text{ k}\cdot\text{ft} \quad M_{CB} = 25.3 \text{ k}\cdot\text{ft} \quad M_{DC} = -17.0 \text{ k}\cdot\text{ft}$ *Ans.*



(d)

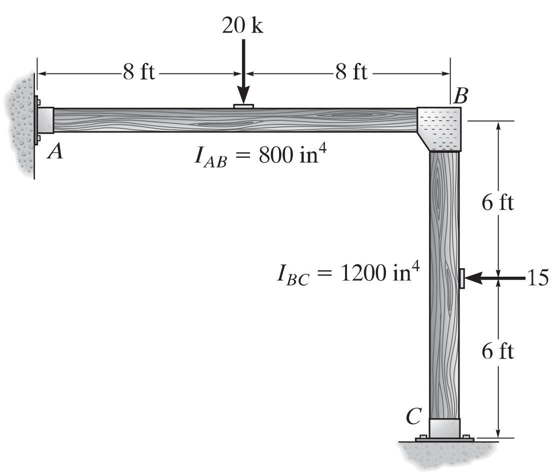
PROBLEMS

11–13. Determine the moments at A, B, and C, then draw the moment diagram for each member. Assume all joints are fixed connected. EI is constant.



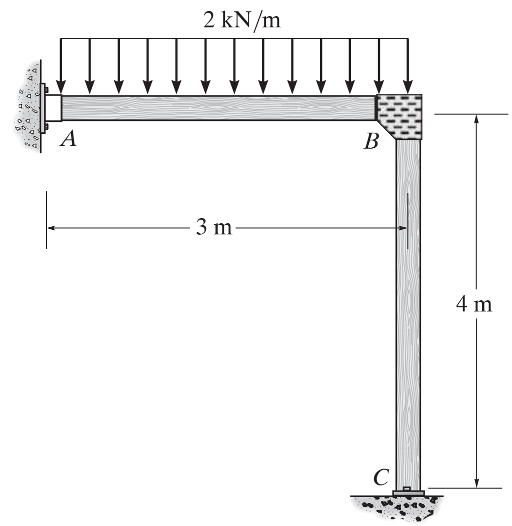
Prob. 11–13

11–14. Determine the moments at the supports, then draw the moment diagram. The members are fixed connected at the supports and at joint B. The moment of inertia of each member is given in the figure. Take $E = 29(10^3)$ ksi.



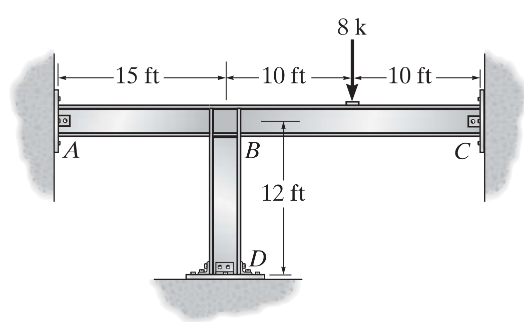
Prob. 11–14

11–15. Determine the moment at B, then draw the moment diagram for each member of the frame. Assume the support at A is fixed and C is pinned. EI is constant.



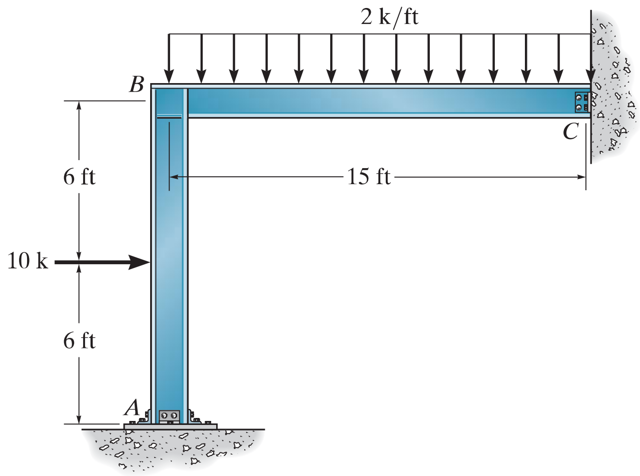
Prob. 11–15

***11–16.** Determine the moments at B and D, then draw the moment diagram. Assume A and C are pinned and B and D are fixed connected. EI is constant.



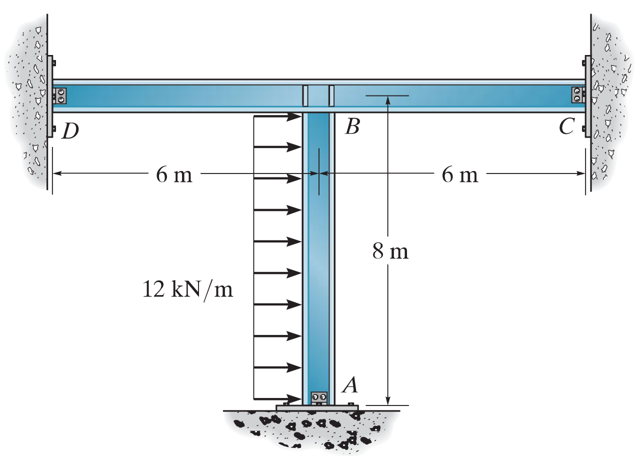
Prob. 11–16

11–17. Determine the moment that each member exerts on the joint at B, then draw the moment diagram for each member of the frame. Assume the support at A is fixed and C is a pin. EI is constant.



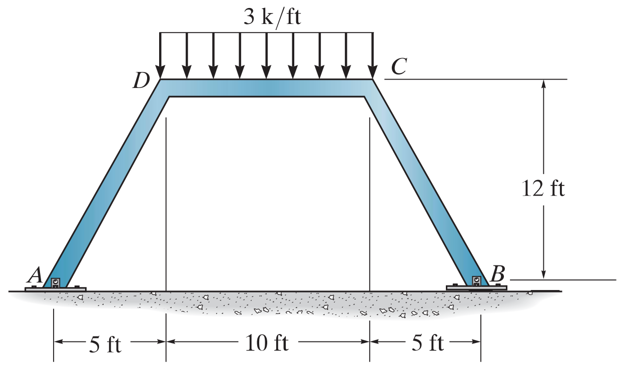
Prob. 11–17

11–18. Determine the moment that each member exerts on the joint at B, then draw the moment diagram for each member of the frame. Assume the supports at A, C, and D are pins. EI is constant.



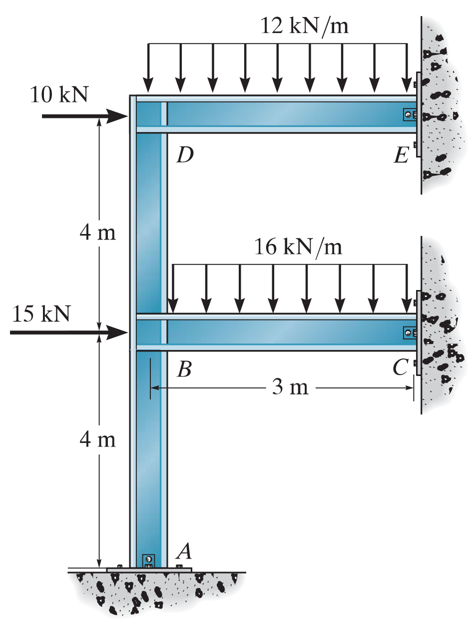
Prob. 11–18

11–19. Determine the moment at joints D and C, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins. EI is constant.



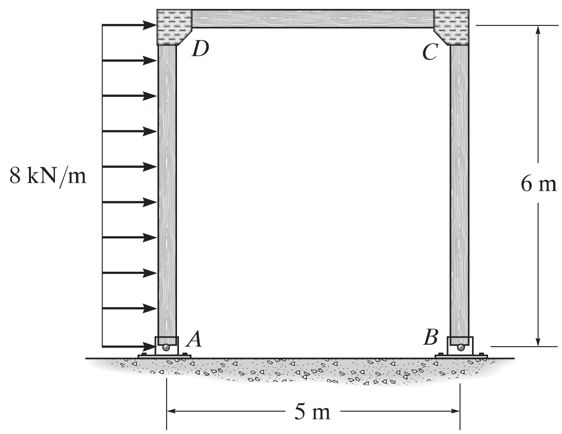
Prob. 11–19

***11–20.** Determine the moment that each member exerts on the joints at B and D, then draw the moment diagram for each member of the frame. Assume the supports at A, C, and E are pins. EI is constant.



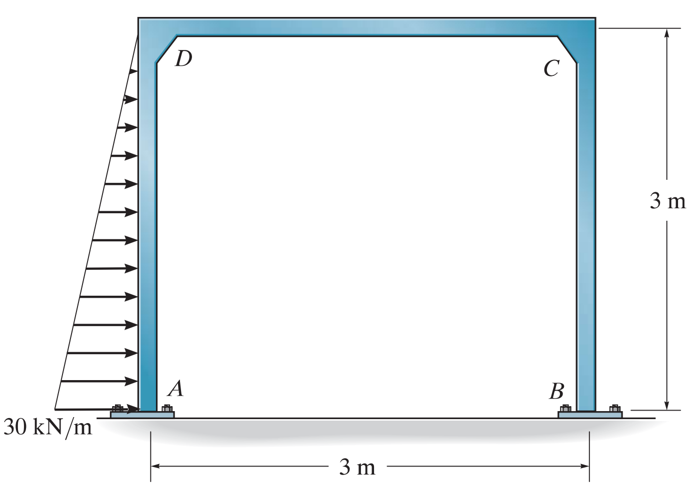
Prob. 11–20

11–21. Determine the moment at joints C and D, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins. EI is constant.



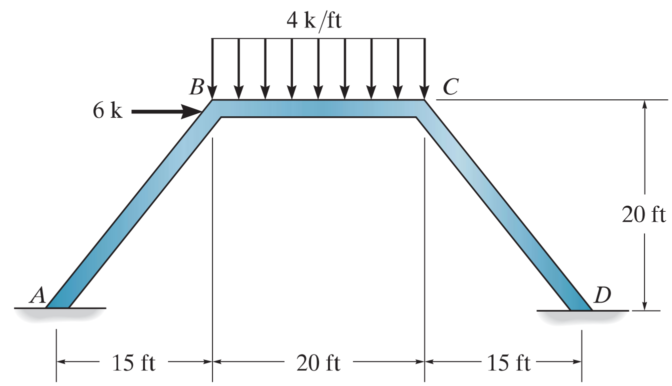
Prob. 11–21

11–22. Determine the moment at joints A, B, C, and D, then draw the moment diagram for each member of the frame. Assume the supports at A and B are fixed. EI is constant.



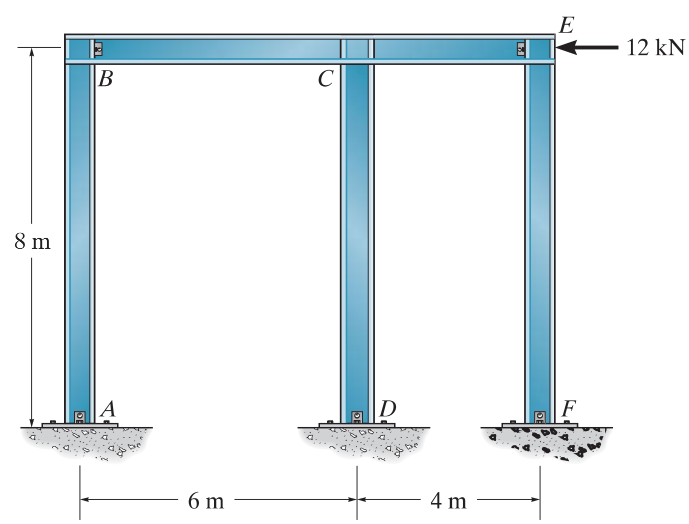
Prob. 11–22

11–23. Determine the moments acting at the supports A and D of the battered-column frame. Take $E = 29(10^3)$ ksi, $I = 600$ in^4.



Prob. 11–23

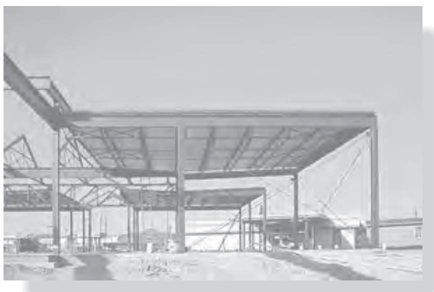
***11–24.** Wind loads are transmitted to the frame at joint E. If A, B, E, D, and F are all pin connected and C is fixed connected, determine the moments at joint C and draw the bending moment diagrams for the girder BCE. EI is constant.

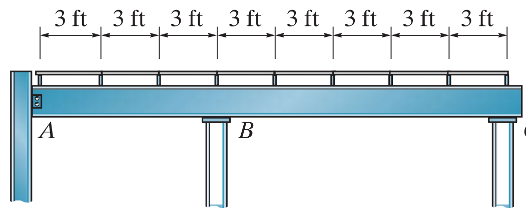


Prob. 11–24

PROJECT PROBLEM

11–1P. The roof is supported by joists that rest on two girders. Each joist can be considered simply supported, and the front girder can be considered attached to the three columns by a pin at A and rollers at B and C. Assume the roof will be made from 3 in.-thick cinder concrete, and each joist has a weight of 550 lb. According to code the roof will be subjected to a snow loading of 25 psf. The joists have a length of 25 ft. Draw the shear and moment diagrams for the girder. Assume the supporting columns are rigid.





Project Prob. 11–1P

CHAPTER REVIEW

The unknown displacements of a structure are referred to as the degrees of freedom for the structure. They consist of either joint displacements or rotations.

The slope-deflection equations relate the unknown moments at each joint of a structural member to the unknown rotations that occur there. The following equation is applied twice to each member or span, considering each side as the "near" end and its counterpart as the far end.

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

For Internal Span or End Span with Far End Fixed

This equation is only applied once, where the "far" end is at the pin or roller support.

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

Only for End Span with Far End Pinned or Roller Supported

Once the slope-deflection equations are written, they are substituted into the equations of moment equilibrium at each joint and then solved for the unknown displacements. If the structure (frame) has sidesway, then an unknown horizontal displacement at each floor level will occur, and the unknown column shears must be related to the moments at the joints, using both the force and moment equilibrium equations. Once the unknown displacements are obtained, the unknown reactions are found from the load-displacement relations.

The girders of this concrete building are all fixed connected, so the statically indeterminate analysis of the framework can be done using the moment distribution method.

Displacement Method of Analysis: Moment Distribution

12

The moment-distribution method is a displacement method of analysis that is easy to apply once certain elastic constants have been determined. In this chapter we will first state the important definitions and concepts for moment distribution and then apply the method to solve problems involving statically indeterminate beams and frames. Application to multistory frames is discussed in the last part of the chapter.

12.1 General Principles and Definitions

The method of analyzing beams and frames using moment distribution was developed by Hardy Cross, in 1930. At the time this method was first published it attracted immediate attention, and it has been recognized as one of the most notable advances in structural analysis during the twentieth century.

As will be explained in detail later, moment distribution is a method of successive approximations that may be carried out to any desired degree of accuracy. Essentially, the method begins by assuming each joint of a structure is fixed. Then, by unlocking and locking each joint in succession, the internal moments at the joints are "distributed" and balanced until the joints have rotated to their final or nearly final positions. It will be found that this process of calculation is both repetitive and easy to apply. Before explaining the techniques of moment distribution, however, certain definitions and concepts must be presented.

12

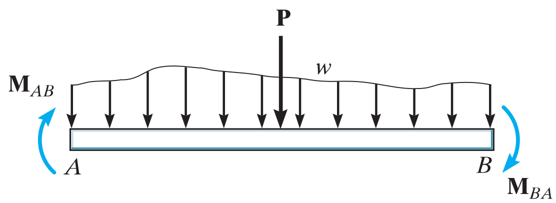


Fig. 12–1

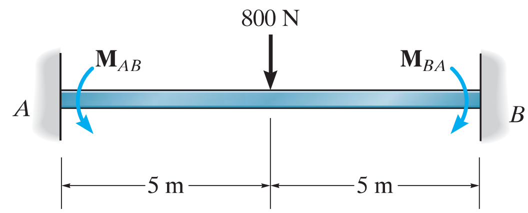


Fig. 12–2

Sign Convention. We will establish the same sign convention as that established for the slope-deflection equations: *Clockwise moments* that act *on the member* are considered *positive*, whereas *counterclockwise moments* are *negative*, Fig. 12–1.

Fixed-End Moments (FEMs). The moments at the "walls" or fixed joints of a loaded member are called *fixed-end moments*. These moments can be determined from the table given on the inside back cover, depending upon the type of loading on the member. For example, the beam loaded as shown in Fig. 12–2 has fixed-end moments of $\text{FEM} = PL/8 = 800(10)/8 = 1000\ \text{N}\cdot\text{m}$. Noting the action of these moments *on the beam* and applying our sign convention, it is seen that $M_{AB} = -1000\ \text{N}\cdot\text{m}$ and $M_{BA} = +1000\ \text{N}\cdot\text{m}$.

Member Stiffness Factor. Consider the beam in Fig. 12–3, which is pinned at one end and fixed at the other. Application of the moment **M** causes the end A to rotate through an angle θ_A. In Chapter 11 we related M to θ_A using the conjugate-beam method. This resulted in Eq. 11–1, that is, $M = (4EI/L)\,\theta_A$. The term in parentheses

$$\boxed{K = \frac{4EI}{L}} \tag{12–1}$$

Far End Fixed

is referred to as the *stiffness factor* at A and can be defined as the amount of moment M required to rotate the end A of the beam $\theta_A = 1$ rad.

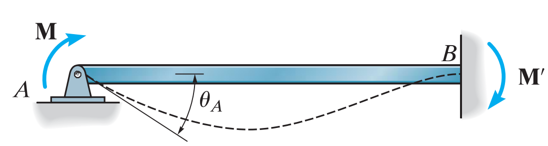


Fig. 12–3

Joint Stiffness Factor. If several members are fixed connected to a joint and each of their far ends is fixed, then by the principle of superposition, the *total stiffness factor* at the joint is the sum of the member stiffness factors at the joint, that is, $K_T = \Sigma K$. For example, consider the frame joint A in Fig. 12–4a. The numerical value of each member stiffness factor is determined from Eq. 12–1 and listed in the figure. Using these values, the total stiffness factor of joint A is $K_T = \Sigma K = 4000 + 5000 + 1000 = 10\,000$. This value represents the amount of moment needed to rotate the joint through an angle of 1 rad.

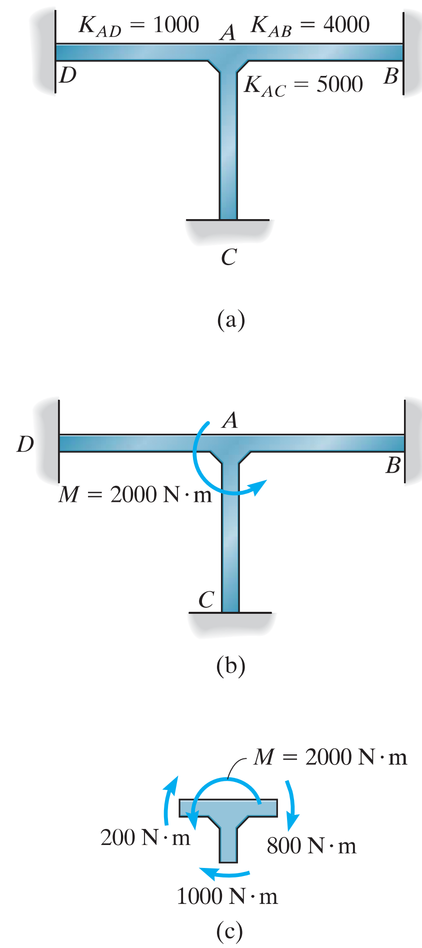


Fig. 12–4

Distribution Factor (DF). If a moment **M** is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint. That fraction of the total resisting moment supplied by the member is called the *distribution factor* (DF). To obtain its value, imagine the joint is fixed connected to n members. If an applied moment **M** causes the joint to rotate an amount θ, then each member i rotates by this same amount. If the stiffness factor of the ith member is K_i, then the moment contributed by the member is $M_i = K_i\theta$. Since equilibrium requires $M = M_1 + M_n = K_1\theta + K_n\theta = \theta\Sigma K_i$ then the distribution factor for the ith member is

$$\mathrm{DF}_i = \frac{M_i}{M} = \frac{K_i\theta}{\theta\Sigma K_i}$$

Canceling the common term θ, it is seen that the distribution factor for a member is equal to the stiffness factor of the member divided by the total stiffness factor for the joint; that is, in general,

$$\boxed{\mathrm{DF} = \frac{K}{\Sigma K}} \tag{12–2}$$

For example, the distribution factors for members AB, AC, and AD at joint A in Fig. 12–4a are

$$\mathrm{DF}_{AB} = 4000/10\,000 = 0.4$$
$$\mathrm{DF}_{AC} = 5000/10\,000 = 0.5$$
$$\mathrm{DF}_{AD} = 1000/10\,000 = 0.1$$

As a result, if $M = 2000$ N · m acts at joint A, Fig. 12–4b, the equilibrium moments exerted by the members on the joint, Fig. 12–4c, are

$$M_{AB} = 0.4(2000) = 800 \text{ N}\cdot\text{m}$$
$$M_{AC} = 0.5(2000) = 1000 \text{ N}\cdot\text{m}$$
$$M_{AD} = 0.1(2000) = 200 \text{ N}\cdot\text{m}$$

The statically indeterminate loading in bridge girders that are continuous over their piers can be determined using the method of moment distribution.

Member Relative-Stiffness Factor. Quite often a continuous beam or a frame will be made from the same material so its modulus of elasticity E will be the *same* for all the members. If this is the case, the common factor $4E$ in Eq. 12–1 will *cancel* from the numerator and denominator of Eq. 12–2 when the distribution factor for a joint is determined. Hence, it is *easier* just to determine the member's *relative-stiffness factor*

$$K_R = \frac{I}{L} \qquad \text{Far End Fixed} \tag{12–3}$$

and use this for the computations of the DF.

Carry-Over Factor. Consider again the beam in Fig. 12–3. It was shown in Chapter 11 that $M_{AB} = (4EI/L)\,\theta_A$ (Eq. 11–1) and $M_{BA} = (2EI/L)\,\theta_A$ (Eq. 11–2). Solving for θ_A and equating these equations we get $M_{BA} = M_{AB}/2$. In other words, the moment $\mathbf{M}$ at the pin induces a moment of $\mathbf{M}' = \frac{1}{2}\mathbf{M}$ at the wall. The carry-over factor represents the fraction of $\mathbf{M}$ that is "carried over" from the pin to the wall. Hence, in the case of a beam with *the far end fixed,* the carry-over factor is $+\frac{1}{2}$. The plus sign indicates both moments act in the same direction.

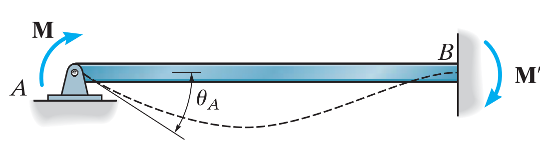


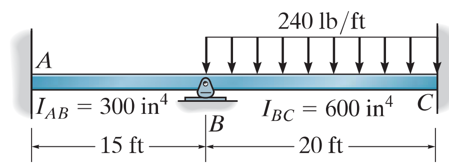
Fig. 12–3

12.2 Moment Distribution for Beams

Moment distribution is based on the principle of successively locking and unlocking the joints of a structure in order to allow the moments at the joints to be distributed and balanced. The best way to explain the method is by examples.

Consider the beam with a constant modulus of elasticity E and having the dimensions and loading shown in Fig. 12–5*a*. Before we begin, we must first determine the distribution factors at the two ends of each span. Using Eq. 12–1, $K = 4EI/L$, the stiffness factors on either side of B are

$$K_{BA} = \frac{4E(300)}{15} = 4E(20)\ \text{in}^4/\text{ft} \qquad K_{BC} = \frac{4E(600)}{20} = 4E(30)\ \text{in}^4/\text{ft}$$



(a)

Fig. 12–5

Thus, using Eq. 12–2, $\text{DF} = K/\Sigma K$, for the ends connected to joint B, we have

$$\text{DF}_{BA} = \frac{4E(20)}{4E(20) + 4E(30)} = 0.4$$

$$\text{DF}_{BC} = \frac{4E(30)}{4E(20) + 4E(30)} = 0.6$$

At the walls, joint A and joint C, the distribution factor depends on the member stiffness factor and the "stiffness factor" of the wall. Since in theory it would take an "infinite" size moment to rotate the wall one radian, the wall stiffness factor is infinite. Thus for joints A and C we have

$$\text{DF}_{AB} = \frac{4E(20)}{\infty + 4E(20)} = 0$$

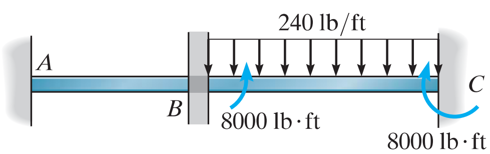
$$\text{DF}_{CB} = \frac{4E(30)}{\infty + 4E(30)} = 0$$

Note that the above results could also have been obtained if the relative stiffness factor $K_R = I/L$ (Eq. 12–3) had been used for the calculations. Furthermore, as long as a *consistent* set of units is used for the stiffness factor, the DF will always be dimensionless, and at a joint, except where it is located at a fixed wall, the sum of the DFs will always equal 1.

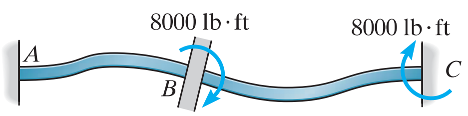
Having computed the DFs, we will now determine the FEMs. Only span BC is loaded, and using the table on the inside back cover for a uniform load, we have

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000\ \text{lb}\cdot\text{ft}$$

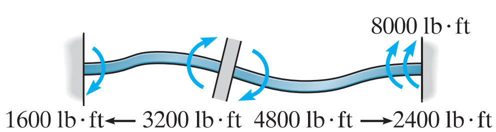
$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000\ \text{lb}\cdot\text{ft}$$



joint B held fixed
(b)



correction moment applied to joint B
(c)



moment at B distributed
(d)

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	0
FEM			−8000	8000
Dist,CO	1600 ←	3200	4800 →	2400
ΣM	1600	3200	−3200	10 400

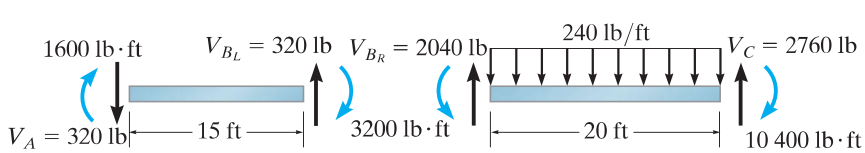
(e)

We begin by assuming joint B is fixed or locked. The fixed-end moment at B then holds span BC in this fixed or locked position as shown in Fig. 12–5*b*. This, of course, does not represent the actual equilibrium situation at B, since the moments on *each side* of this joint must be equal but opposite. To correct this, we will apply an equal, but opposite moment of 8000 lb · ft to the joint and allow the joint to rotate freely, Fig. 12–5*c*. As a result, portions of this moment are distributed in spans BC and BA in accordance with the DFs (or stiffness) of these spans at the joint. Specifically, the moment in BA is $0.4(8000) = 3200$ lb · ft and the moment in BC is $0.6(8000) = 4800$ lb · ft. Finally, due to the released rotation that takes place at B, these moments must be "carried over" since moments are developed at the far ends of the span. Using the carry-over factor of $+\frac{1}{2}$, the results are shown in Fig. 12–5*d*.

This example indicates the basic steps necessary when distributing moments at a joint: Determine the unbalanced moment acting at the initially "locked" joint, unlock the joint and apply an equal but opposite unbalanced moment to correct the equilibrium, distribute the moment among the connecting spans, and carry the moment in each span over to its other end. The steps are usually presented in tabular form as indicated in Fig. 12–5*e*. Here the notation Dist, CO indicates a line where moments are distributed, then carried over. In this particular case only *one cycle* of moment distribution is necessary, since the wall supports at A and C "absorb" the moments and no further joints have to be balanced or unlocked to satisfy joint equilibrium. Once distributed in this manner, the moments at each joint are summed, yielding the final results shown on the bottom line of the table in Fig. 12–5*e*. Notice that joint B is now in equilibrium. Since M_{BC} is negative, this moment is applied to span BC in a counterclockwise sense as shown on free-body diagrams of the beam spans in Fig. 12–5*f*. With the end moments known, the end shears have been computed from the equations of equilibrium applied to each of these spans.

Consider now the same beam, except the support at C is a rocker, Fig. 12–6*a*. In this case only *one member* is at joint C, so the distribution factor for member CB at joint C is

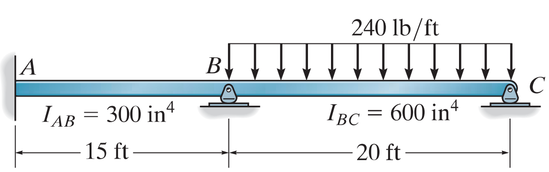
$$\mathrm{DF}_{CB} = \frac{4E(30)}{4E(30)} = 1$$



(f)

Fig. 12–5

The other distribution factors and the FEMs are the same as computed previously. They are listed on lines 1 and 2 of the table in Fig. 12–6*b*. Initially, we will assume joints B and C are locked. We begin by unlocking joint C and placing an equilibrating moment of $-8000\text{ lb}\cdot\text{ft}$ at the joint. The entire moment is distributed in member CB since $(1)(-8000)\text{ lb}\cdot\text{ft} = -8000\text{ lb}\cdot\text{ft}$. The arrow on line 3 indicates that $\frac{1}{2}(-8000)\text{ lb}\cdot\text{ft} = -4000\text{ lb}\cdot\text{ft}$ is carried over to joint B since joint C has been allowed to rotate freely. Joint C is now *relocked*. Since the total moment at C is *balanced,* a line is placed under the $-8000\text{-lb}\cdot\text{ft}$ moment. We will now consider the unbalanced $-12\,000\text{-lb}\cdot\text{ft}$ moment at joint B. Here for equilibrium, a $+12\,000\text{-lb}\cdot\text{ft}$ moment is applied to B and this joint is unlocked such that portions of the moment are distributed into BA and BC, that is, $(0.4)(12\,000) = 4800\text{ lb}\cdot\text{ft}$ and $(0.6)(12\,000) = 7200\text{ lb}\cdot\text{ft}$ as shown on line 4. Also note that $+\frac{1}{2}$ of these moments must be carried over to the fixed wall A and roller C since joint B has rotated. Joint B is now *relocked.* Again joint C is unlocked and the unbalanced moment at the roller is distributed as was done previously. The results are on line 5. Successively locking and unlocking joints B and C will essentially diminish the size of the moment to be balanced until it becomes negligible compared with the original moments, line 14. Each of the steps on lines 3 through 14 should be thoroughly understood. Summing the moments, the final results are shown on line 15, where it is seen that the final moments now satisfy joint equilibrium.



(a)

Joint	A	B		C	
Member	AB	BA	BC	CB	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
			−4000 ←	−8000	3
	2400 ←	4800	7200 →	3600	4
			−1800 ←	−3600	5
	360 ←	720	1080 →	540	6
			−270 ←	−540	7
	54 ←	108	162 →	81	8
			−40.5 ←	−81	9
	8.1 ←	16.2	24.3 →	12.2	10
			−6.1 ←	−12.2	11
	1.2 ←	2.4	3.6 →	1.8	12
			−0.9 ←	−1.8	13
		0.4	0.5		14
ΣM	2823.3	5647.0	−5647.0	0	15

(b)

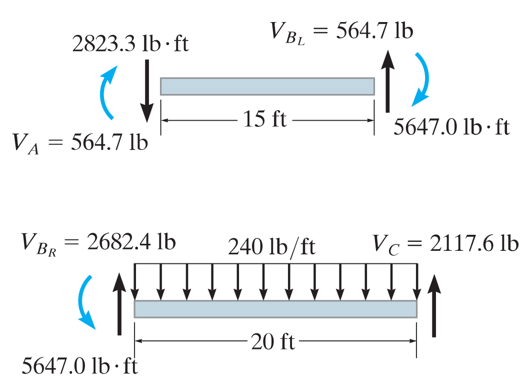
Fig. 12–6

Rather than applying the moment distribution process successively to each joint, as illustrated here, it is also possible to apply it to all joints at the *same time*. This scheme is shown in the table in Fig. 12–6*c*. In this case, we start by fixing all the joints and then balancing and distributing the fixed-end moments at both joints *B* and *C*, line 3. Unlocking joints *B* and *C* simultaneously (joint *A* is always fixed), the moments are then carried over to the end of each span, line 4. Again the joints are relocked, and the moments are balanced and distributed, line 5. Unlocking the joints once again allows the moments to be carried over, as shown in line 6. Continuing, we obtain the final results, as before, listed on line 24. By comparison, this method gives a slower convergence to the answer than does the previous method; however, in many cases this method will be more efficient to apply, and for this reason we will use it in the examples that follow. Finally, using the results in either Fig. 12–6*b* or 12–6*c*, the free-body diagrams of each beam span are drawn as shown in Fig. 12–6*d*.

Although several steps were involved in obtaining the final results here, the work required is rather methodical since it requires application of a series of arithmetical steps, rather than solving a set of equations as in the slope deflection method. It should be noted, however, that the

Joint	*A*	*B*		*C*	
Member	*AB*	*BA*	*BC*	*CB*	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
Dist.		3200	4800	−8000	3
CO	1600		−4000	2400	4
Dist.		1600	2400	−2400	5
CO	800		−1200	1200	6
Dist.		480	720	−1200	7
CO	240		−600	360	8
Dist.		240	360	−360	9
CO	120		−180	180	10
Dist.		72	108	−180	11
CO	36		−90	54	12
Dist.		36	54	−54	13
CO	18		−27	27	14
Dist.		10.8	16.2	−27	15
CO	5.4		−13.5	8.1	16
Dist.		5.4	8.1	−8.1	17
CO	2.7		−4.05	4.05	18
Dist.		1.62	2.43	−4.05	19
CO	0.81		−2.02	1.22	20
Dist.		0.80	1.22	−1.22	21
CO	0.40		−0.61	0.61	22
Dist.		0.24	0.37	−0.61	23
ΣM	2823	5647	−5647	0	24

(c)



(d)

Fig. 12–6

fundamental process of moment distribution follows the same procedure as any displacement method. There the process is to establish load-displacement relations at each joint and then satisfy joint equilibrium requirements by determining the correct angular displacement for the joint (compatibility). Here, however, the equilibrium and compatibility of rotation at the joint is satisfied *directly,* using a "moment balance" process that incorporates the load-deflection relations (stiffness factors). Further simplification for using moment distribution is possible, and this will be discussed in the next section.

Procedure for Analysis

The following procedure provides a general method for determining the end moments on beam spans using moment distribution.

Distribution Factors and Fixed-End Moments

The joints on the beam should be identified and the stiffness factors for each span at the joints should be calculated. Using these values the distribution factors can be determined from $\text{DF} = K/\Sigma K$. Remember that $\text{DF} = 0$ for a fixed end and $\text{DF} = 1$ for an *end* pin or roller support.

The fixed-end moments for each loaded span are determined using the table given on the inside back cover. Positive FEMs act clockwise on the span and negative FEMs act counterclockwise. For convenience, these values can be recorded in tabular form, similar to that shown in Fig. 12–6*c*.

Moment Distribution Process

Assume that all joints at which the moments in the connecting spans must be determined are *initially locked*. Then:

1. Determine the moment that is needed to put each joint in equilibrium.
2. Release or "unlock" the joints and distribute the counterbalancing moments into the connecting span at each joint.
3. Carry these moments in each span over to its other end by multiplying each moment by the carry-over factor $+\frac{1}{2}$.

By repeating this cycle of locking and unlocking the joints, it will be found that the moment corrections will diminish since the beam tends to achieve its final deflected shape. When a small enough value for the corrections is obtained, the process of cycling should be stopped with no "carry-over" of the last moments. Each column of FEMs, distributed moments, and carry-over moments should then be added. If this is done correctly, moment equilibrium at the joints will be achieved.

12

EXAMPLE 12.1

Determine the internal moments at each support of the beam shown in Fig. 12–7*a*. *EI* is constant.

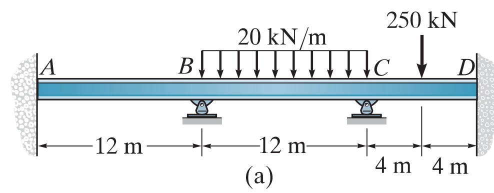


Fig. 12–7

SOLUTION

The distribution factors at each joint must be computed first.* The stiffness factors for the members are

$$K_{AB} = \frac{4EI}{12} \qquad K_{BC} = \frac{4EI}{12} \qquad K_{CD} = \frac{4EI}{8}$$

Therefore,

$$\text{DF}_{AB} = \text{DF}_{DC} = 0 \qquad \text{DF}_{BA} = \text{DF}_{BC} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

$$\text{DF}_{CB} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.4 \quad \text{DF}_{CD} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.6$$

The fixed-end moments are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = \frac{-20(12)^2}{12} = -240 \text{ kN}\cdot\text{m} \qquad (\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CD} = -\frac{PL}{8} = \frac{-250(8)}{8} = -250 \text{ kN}\cdot\text{m} \qquad (\text{FEM})_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250 \text{ kN}\cdot\text{m}$$

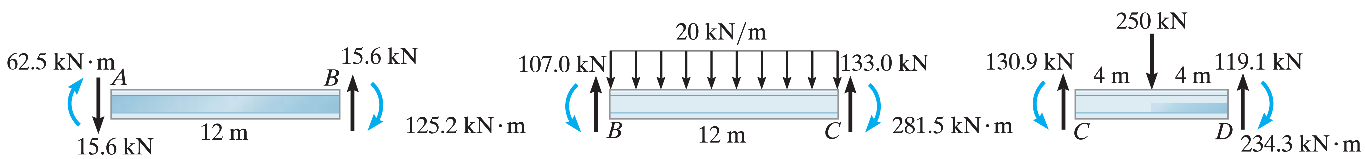
Starting with the FEMs, line 4, Fig. 12–7*b*, the moments at joints *B* and *C* are distributed *simultaneously*, line 5. These moments are then carried over *simultaneously* to the respective ends of each span, line 6. The resulting moments are again simultaneously distributed and carried over, lines 7 and 8. The process is continued until the resulting moments are diminished an appropriate amount, line 13. The resulting moments are found by summation, line 14.

Placing the moments on each beam span and applying the equations of equilibrium yields the end shears shown in Fig. 12–7*c* and the bending-moment diagram for the entire beam, Fig. 12–7*d*.

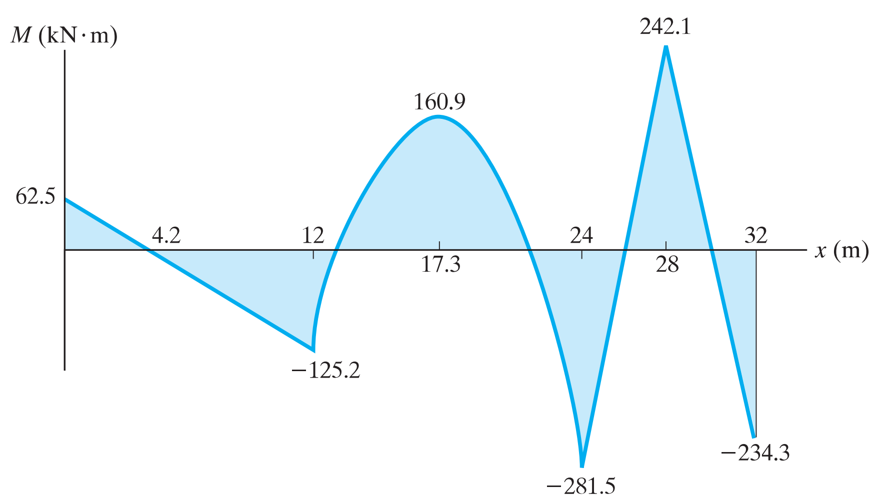
*Here we have used the stiffness factor $4EI/L$; however, the relative stiffness factor I/L could also have been used.

Joint	A	B		C		D	1
Member	AB	BA	BC	CB	CD	DC	2
DF	0	0.5	0.5	0.4	0.6	0	3
FEM			−240	240	−250	250	4
Dist.		120	120	4	6		5
CO	60		2	60		3	6
Dist.		−1	−1	−24	−36		7
CO	−0.5		−12	−0.5		−18	8
Dist.		6	6	0.2	0.3		9
CO	3		0.1	3		0.2	10
Dist.		−0.05	−0.05	−1.2	−1.8		11
CO	−0.02		−0.6	−0.02		−0.9	12
Dist.		0.3	0.3	0.01	0.01		13
ΣM	62.5	125.2	−125.2	281.5	−281.5	234.3	14

(b)



(c)



(d)

12

EXAMPLE 12.2

Determine the internal moment at each support of the beam shown in Fig. 12–8*a*. The moment of inertia of each span is indicated.

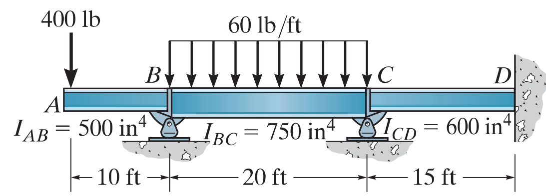


Fig. 12–8

SOLUTION

In this problem a moment does not get distributed in the overhanging span AB, and so the distribution factor $(\mathrm{DF})_{BA} = 0$. The stiffness of span BC is based on $4EI/L$ since the pin rocker is not at the far end of the beam. The stiffness factors, distribution factors, and fixed-end moments are computed as follows:

$$K_{BC} = \frac{4E(750)}{20} = 150E \qquad K_{CD} = \frac{4E(600)}{15} = 160E$$

$$\mathrm{DF}_{BC} = 1 - (\mathrm{DF})_{BA} = 1 - 0 = 1$$

$$\mathrm{DF}_{CB} = \frac{150E}{150E + 160E} = 0.484$$

$$\mathrm{DF}_{CD} = \frac{160E}{150E + 160E} = 0.516$$

$$\mathrm{DF}_{DC} = \frac{160E}{\infty + 160E} = 0$$

Due to the overhang,

$$(\mathrm{FEM})_{BA} = 400\text{ lb}(10\text{ ft}) = 4000\text{ lb}\cdot\text{ft}$$

$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{60(20)^2}{12} = -2000\text{ lb}\cdot\text{ft}$$

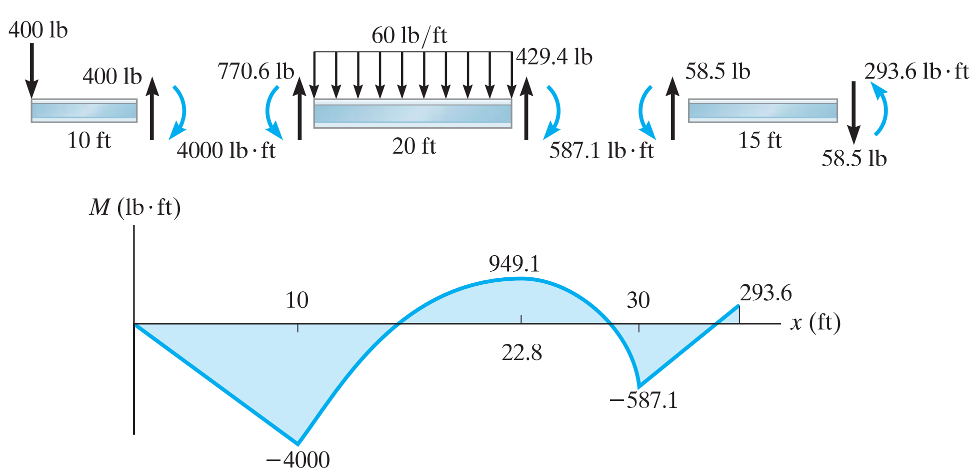
$$(\mathrm{FEM})_{CB} = \frac{wL^2}{12} = \frac{60(20)^2}{12} = 2000\text{ lb}\cdot\text{ft}$$

These values are listed on the fourth line of the table, Fig. 12–8*b*. The overhanging span requires the internal moment to the left of B to be $+4000\text{ lb}\cdot\text{ft}$. Balancing at joint B requires an internal moment of $-4000\text{ lb}\cdot\text{ft}$ to the right of B. As shown on the fifth line of the table $-2000\text{ lb}\cdot\text{ft}$ is added to BC in order to satisfy this condition. The distribution and carry-over operations proceed in the usual manner as indicated.

Since the internal moments are known, the moment diagram for the beam can be constructed (Fig. 12–8c).

Joint	*B*		*C*		*D*
Member		*BC*	*CB*	*CD*	*DC*
DF	0	1	0.484	0.516	0
FEM	4000	−2000	2000		
Dist.		−2000	−968	−1032	
CO		−484	−1000		−516
Dist.		484	484	516	
CO		242	242		258
Dist.		−242	−117.1	−124.9	
CO		−58.6	−121		−62.4
Dist.		58.6	58.6	62.4	
CO		29.3	29.3		31.2
Dist.		−29.3	−14.2	−15.1	
CO		−7.1	−14.6		−7.6
Dist.		7.1	7.1	7.6	
CO		3.5	3.5		3.8
Dist.		−3.5	−1.7	−1.8	
CO		−0.8	−1.8		−0.9
Dist.		0.8	0.9	0.9	
CO		0.4	0.4		0.4
Dist.		−0.4	−0.2	−0.2	
CO		−0.1	−0.2		−0.1
Dist.		0.1	0.1	0.1	
ΣM	4000	−4000	587.1	−587.1	−293.6

(b)



(c)

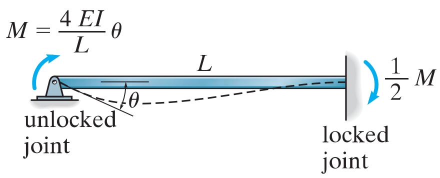


Fig. 12–9

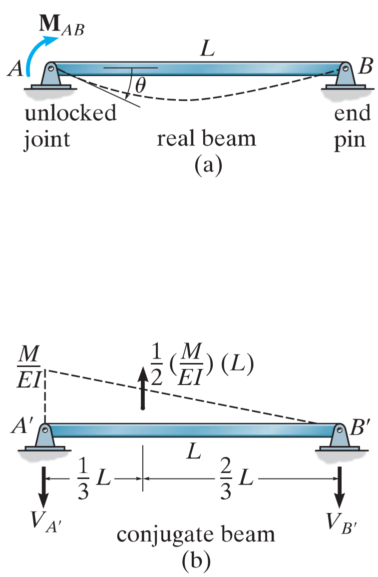


Fig. 12–10

12.3 Stiffness-Factor Modifications

In the previous examples of moment distribution we have considered each beam span to be constrained by a fixed support (locked joint) at its far end when distributing and carrying over the moments. For this reason we have computed the stiffness factors, distribution factors, and the carry-over factors based on the case shown in Fig. 12–9. Here, of course, the stiffness factor is $K = 4EI/L$ (Eq. 12–1), and the carry-over factor is $+\frac{1}{2}$.

In some cases it is possible to modify the stiffness factor of a particular beam span and thereby simplify the process of moment distribution. Three cases where this frequently occurs in practice will now be considered.

Member Pin Supported at Far End. Many indeterminate beams have their far end span supported by an end pin (or roller) as in the case of joint B in Fig. 12–10a. Here the applied moment $\mathbf{M}$ rotates the end A by an amount θ. To determine θ, the shear in the conjugate beam at A' must be determined, Fig. 12–10b. We have

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad V'_A(L) - \frac{1}{2}\left(\frac{M}{EI}\right)L\left(\frac{2}{3}L\right) = 0$$

$$V'_A = \theta = \frac{ML}{3EI}$$

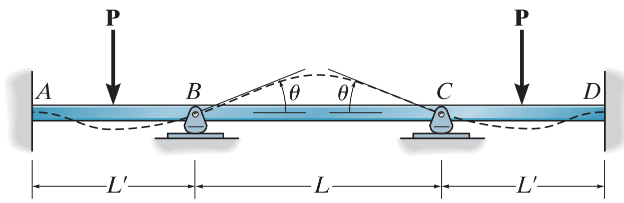
or

$$M = \frac{3EI}{L}\theta$$

Thus, the stiffness factor for this beam is

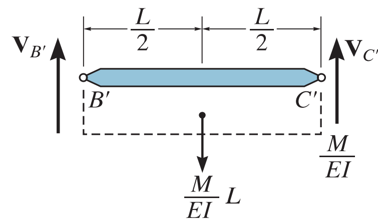
$$\boxed{\begin{array}{l} K = \dfrac{3EI}{L} \\ \text{Far End Pinned} \\ \text{or Roller Supported} \end{array}} \qquad (12\text{–}4)$$

Also, note that *the carry-over factor is zero,* since the pin at B does not support a moment. By comparison, then, *if the far end was fixed supported, the stiffness factor $K = 4EI/L$ would have to be modified by $\frac{3}{4}$ to model the case of having the far end pin supported*. If this modification is considered, the moment distribution process is simplified since the end pin does *not* have to be unlocked–locked successively when distributing the moments. Also, since the end span is pinned, the fixed-end moments for the span are computed using the values in the right column of the table on the inside back cover. Example 12–4 illustrates how to apply these simplifications.



real beam

(a)



conjugate beam

(b)

Fig. 12–11

Symmetric Beam and Loading. If a beam is symmetric with respect to both its loading and geometry, the bending-moment diagram for the beam will also be symmetric. As a result, a modification of the stiffness factor for the center span can be made, so that moments in the beam only have to be distributed through joints lying on either half of the beam. To develop the appropriate stiffness-factor modification, consider the beam shown in Fig. 12–11*a*. Due to the symmetry, the internal moments at B and C are equal. Assuming this value to be $\mathbf{M}$, the conjugate beam for span BC is shown in Fig. 12–11*b*. The slope θ at each end is therefore

$$\circlearrowright + \Sigma M_{C'} = 0; \qquad -V_{B'}(L) + \frac{M}{EI}(L)\left(\frac{L}{2}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{2EI}$$

or

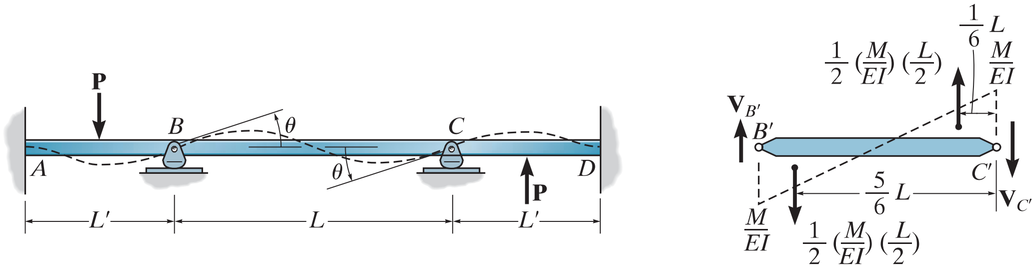
$$M = \frac{2EI}{L}\theta$$

The stiffness factor for the center span is therefore

$$\boxed{\begin{array}{c} K = \dfrac{2EI}{L} \\ \text{Symmetric Beam and Loading} \end{array}} \qquad (12\text{–}5)$$

Thus, moments for only half the beam can be distributed provided the stiffness factor for the center span is computed using Eq. 12–5. *By comparison, the center span's stiffness factor will be one half that usually determined using* $K = 4EI/L$.

12



real beam

(a)

conjugate beam

(b)

Fig. 12–12

Symmetric Beam with Antisymmetric Loading. If a symmetric beam is subjected to antisymmetric loading, the resulting moment diagram will be antisymmetric. As in the previous case, we can modify the stiffness factor of the center span so that only one half of the beam has to be considered for the moment-distribution analysis. Consider the beam in Fig. 12–12*a*. The conjugate beam for its center span *BC* is shown in Fig. 12–12*b*. Due to the antisymmetric loading, the internal moment at *B* is equal, but opposite to that at *C*. Assuming this value to be **M**, the slope θ at each end is determined as follows:

$$\circlearrowleft + \Sigma M_{C'} = 0; \quad -V_{B'}(L) + \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{6EI}$$

or

$$M = \frac{6EI}{L}\theta$$

The stiffness factor for the center span is, therefore,

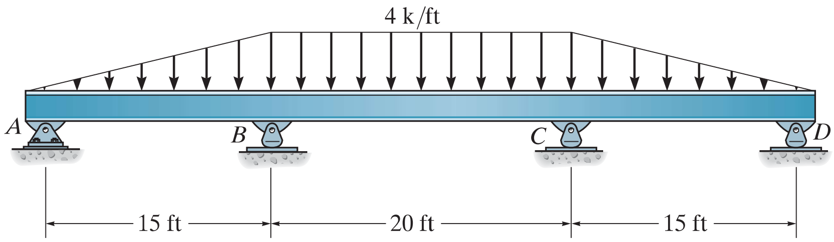
$$K = \frac{6EI}{L} \qquad (12\text{–}6)$$

Symmetric Beam with Antisymmetric Loading

Thus, when the stiffness factor for the beam's center span is computed using Eq. 12–6, the moments in only half the beam have to be distributed. *Here the stiffness factor is one and a half times as large as that determined using* $K = 4EI/L$.

EXAMPLE 12.3

Determine the internal moments at the supports for the beam shown in Fig. 12–13*a*. *EI* is constant.



(a)

Fig. 12–13

SOLUTION

By inspection, the beam and loading are symmetrical. Thus, we will apply $K = 2EI/L$ to compute the stiffness factor of the center span BC and therefore use only the left half of the beam for the analysis. The analysis can be shortened even further by using $K = 3EI/L$ for computing the stiffness factor of segment AB since the far end A is pinned. Furthermore, the distribution of moment at A can be skipped by using the FEM for a triangular loading on a span with one end fixed and the other pinned. Thus,

$$K_{AB} = \frac{3EI}{15} \qquad \text{(using Eq. 12–4)}$$

$$K_{BC} = \frac{2EI}{20} \qquad \text{(using Eq. 12–5)}$$

$$\text{DF}_{AB} = \frac{3EI/15}{3EI/15} = 1$$

$$\text{DF}_{BA} = \frac{3EI/15}{3EI/15 + 2EI/20} = 0.667$$

$$\text{DF}_{BC} = \frac{2EI/20}{3EI/15 + 2EI/20} = 0.333$$

$$(\text{FEM})_{BA} = \frac{wL^2}{15} = \frac{4(15)^2}{15} = 60 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{4(20)^2}{12} = -133.3 \text{ k}\cdot\text{ft}$$

Joint	A	B	
Member	AB	BA	BC
DF	1	0.667	0.333
FEM Dist.		60 48.9	−133.3 24.4
ΣM	0	108.9	−108.9

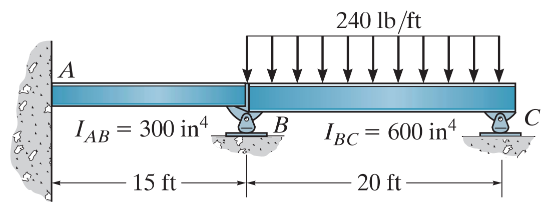
(b)

These data are listed in the table in Fig. 12–13*b*. Computing the stiffness factors as shown above considerably reduces the analysis, since only joint B must be balanced and carry-overs to joints A and C are not necessary. Obviously, joint C is subjected to the same internal moment of 108.9 k · ft.

12

EXAMPLE 12.4

Determine the internal moments at the supports of the beam shown in Fig. 12–14*a*. The moment of inertia of the two spans is shown in the figure.



(a)

Fig. 12–14

SOLUTION

Since the beam is roller supported at its far end C, the stiffness of span BC will be computed on the basis of $K = 3EI/L$. We have

$$K_{AB} = \frac{4EI}{L} = \frac{4E(300)}{15} = 80E$$

$$K_{BC} = \frac{3EI}{L} = \frac{3E(600)}{20} = 90E$$

Thus,

$$\mathrm{DF}_{AB} = \frac{80E}{\infty + 80E} = 0$$

$$\mathrm{DF}_{BA} = \frac{80E}{80E + 90E} = 0.4706$$

$$\mathrm{DF}_{BC} = \frac{90E}{80E + 90E} = 0.5294$$

$$\mathrm{DF}_{CB} = \frac{90E}{90E} = 1$$

Further simplification of the distribution method for this problem is possible by realizing that a *single* fixed-end moment for the end span BC can be used. Using the right-hand column of the table on the inside back cover for a uniformly loaded span having one side fixed, the other pinned, we have

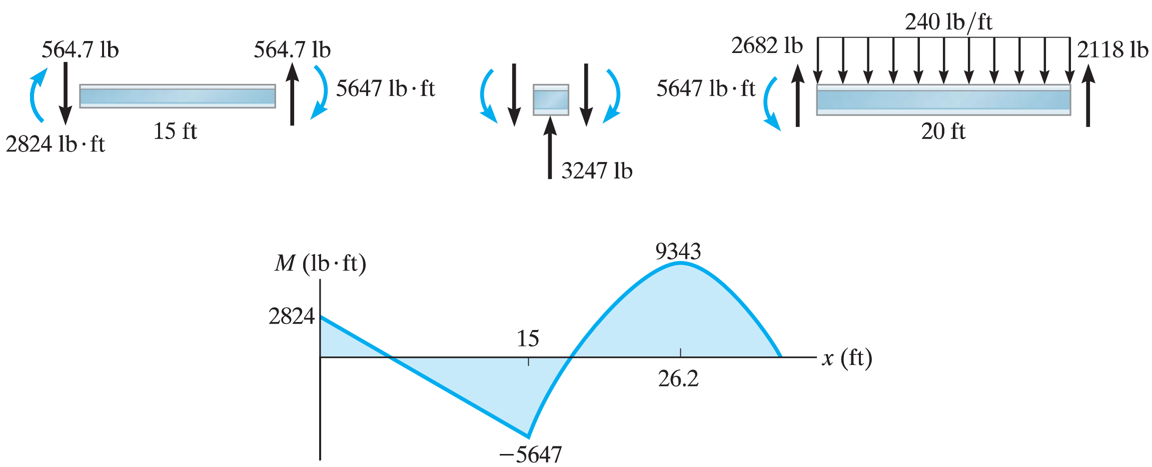
$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{8} = \frac{-240(20)^2}{8} = -12\,000 \text{ lb} \cdot \text{ft}$$

The foregoing data are entered into the table in Fig. 12–14*b* and the moment distribution is carried out. By comparison with Fig. 12–6*b*, this method considerably simplifies the distribution.

Using the results, the beam's end shears and moment diagrams are shown in Fig. 12–14*c*.

Joint	*A*	*B*		*C*
Member	*AB*	*BA*	*BC*	*CB*
DF	0	0.4706	0.5294	1
FEM Dist.		 5647.2	−12 000 6352.8	
CO	2823.6			
ΣM	2823.6	5647.2	−5647.2	0

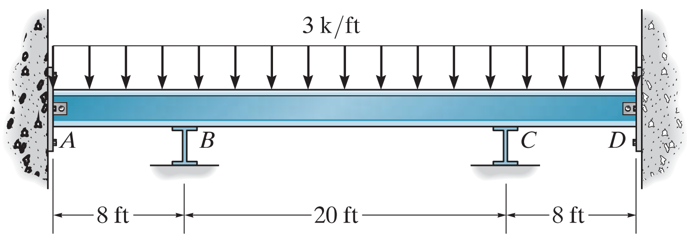
(b)



(c)

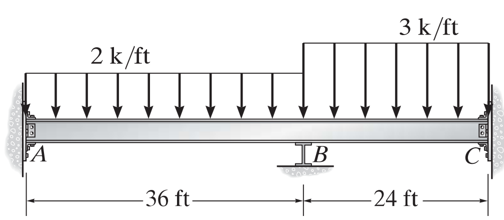
PROBLEMS

12–1. Determine the moments at B and C. EI is constant. Assume B and C are rollers and A and D are pinned.



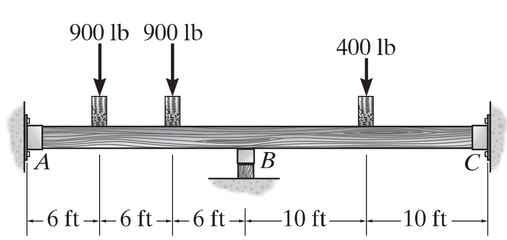
Prob. 12–1

12–2. Determine the moments at A, B, and C. Assume the support at B is a roller and A and C are fixed. EI is constant.



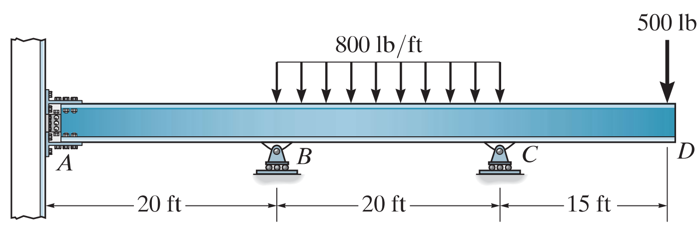
Prob. 12–2

12–3. Determine the moments at A, B, and C, then draw the moment diagram. Assume the support at B is a roller and A and C are fixed. EI is constant.



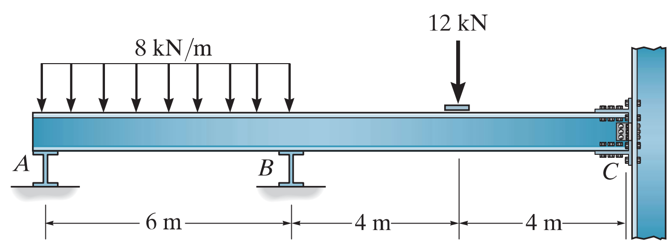
Prob. 12–3

***12–4.** Determine the reactions at the supports and then draw the moment diagram. Assume A is fixed. EI is constant.



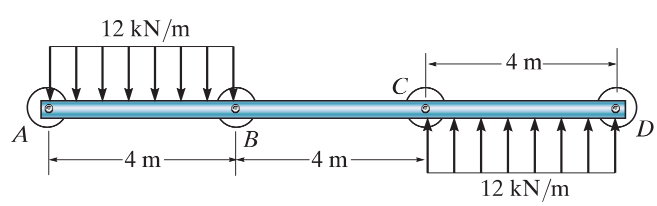
Prob. 12–4

12–5. Determine the moments at B and C, then draw the moment diagram for the beam. Assume C is a fixed support. EI is constant.



Prob. 12–5

12–6. Determine the moments at B and C, then draw the moment diagram for the beam. All connections are pins. Assume the horizontal reactions are zero. EI is constant.



Prob. 12–6