Analysis of Statically Indeterminate Structures by the Force Method

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In this chapter we will apply the *force* or *flexibility* method to analyze statically indeterminate trusses, beams, and frames. At the end of the chapter we will present a method for drawing the influence line for a statically indeterminate beam or frame.

10.1 Statically Indeterminate Structures

Recall from Sec. 2–4 that a structure of any type is classified as *statically indeterminate* when the number of unknown reactions or internal forces exceeds the number of equilibrium equations available for its analysis. In this section we will discuss the merits of using indeterminate structures and two fundamental ways in which they may be analyzed. Realize that most of the structures designed today are statically indeterminate. This indeterminacy may arise as a result of added supports or members, or by the general form of the structure. For example, reinforced concrete buildings are almost always statically indeterminate since the columns and beams are poured as continuous members through the joints and over supports.

Advantages and Disadvantages. Although the analysis of a statically indeterminate structure is more involved than that of a statically determinate one, there are usually several very important reasons for choosing this type of structure for design. Most important, for a given loading the maximum stress and deflection of an indeterminate structure are generally *smaller* than those of its statically determinate counterpart. For example, the statically indeterminate, fixed-supported beam in Fig. 10–1*a* will be subjected to a maximum moment of $M_{\max} = PL/8$, whereas the same beam, when simply supported, Fig. 10–1*b*, will be subjected to twice the moment, that is, $M_{\max} = PL/4$. As a result, the fixed-supported beam has one fourth the deflection and one half the stress at its center of the one that is simply supported.

Another important reason for selecting a statically indeterminate structure is because it has a tendency to redistribute its load to its redundant supports in cases where faulty design or overloading occurs. In these cases, the structure maintains its stability and collapse is prevented. This is particularly important when *sudden* lateral loads, such as wind or earthquake, are imposed on the structure. To illustrate, consider again the fixed-end beam in Fig. 10–1*a*. As **P** is increased, the beam's material at the walls and at the center of the beam begins to *yield* and forms localized "plastic hinges," which causes the beam to deflect as if it were hinged or pin connected at these points. Although the deflection becomes large, the walls will develop horizontal force and moment reactions that will hold the beam and thus prevent it from totally collapsing. In the case of the simply supported beam, Fig. 10–1*b*, an excessive load **P** will cause the "plastic hinge" to form only at the center of the beam, and due to the large vertical deflection, the supports will not develop the horizontal force and moment reactions that may be necessary to prevent total collapse.

Although statically indeterminate structures can support a loading with thinner members and with increased stability compared to their statically determinate counterparts, there are cases when these advantages may instead become disadvantages. The cost savings in material must be compared with the added cost necessary to fabricate the structure, since oftentimes it becomes more costly to construct the supports and joints of an indeterminate structure compared to one that is determinate. More important, though, because statically indeterminate structures have redundant support reactions, one has to be very careful to prevent differential displacement of the supports, since this effect will introduce internal stress in the structure. For example, if the wall at one end of the fixed-end beam in Fig. 10–1*a* were to settle, stress would be developed in the beam because of this "forced" deformation. On the other hand, if the beam were simply supported or statically determinate, Fig. 10–1*b*, then any settlement of its end would not cause the beam to deform, and therefore no stress would be developed in the beam. In general, then, any deformation, such as that caused by relative support displacement, or changes in member lengths caused by temperature or fabrication errors, will introduce additional stresses in the structure, which must be considered when designing indeterminate structures.

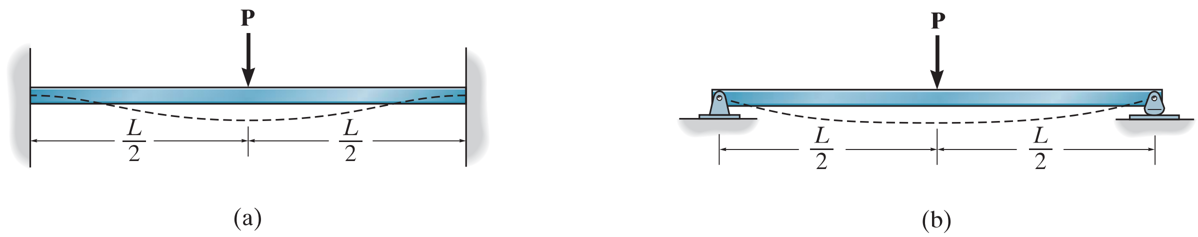

Fig. 10–1

Methods of Analysis. When analyzing any indeterminate structure, it is necessary to satisfy *equilibrium*, *compatibility*, and *force-displacement* requirements for the structure. *Equilibrium* is satisfied when the reactive forces hold the structure at rest, and *compatibility* is satisfied when the various segments of the structure fit together without intentional breaks or overlaps. The *force-displacement* requirements depend upon the way the material responds; in this text we have assumed linear elastic response. In general there are two different ways to satisfy these requirements when analyzing a statically indeterminate structure: the *force* or *flexibility method*, and the *displacement* or *stiffness method*.

Force Method. The force method was originally developed by James Clerk Maxwell in 1864 and later refined by Otto Mohr and Heinrich Müller-Breslau. This method was one of the first available for the analysis of statically indeterminate structures. Since compatibility forms the basis for this method, it has sometimes been referred to as the *compatibility method* or the *method of consistent displacements*. This method consists of writing equations that satisfy the *compatibility* and *force-displacement requirements* for the structure in order to determine the redundant *forces*. Once these forces have been determined, the remaining reactive forces on the structure are determined by satisfying the equilibrium requirements. The fundamental principles involved in applying this method are easy to understand and develop, and they will be discussed in this chapter.

Displacement Method. The displacement method of analysis is based on first writing force-displacement relations for the members and then satisfying the *equilibrium requirements* for the structure. In this case the *unknowns* in the equations are *displacements*. Once the displacements are obtained, the forces are determined from the compatibility and force-displacement equations. We will study some of the classical techniques used to apply the displacement method in Chapters 11 and 12. Since almost all present day computer software for structural analysis is developed using this method we will present a matrix formulation of the displacement method in Chapters 14, 15, and 16.

Each of these two methods of analysis, which are outlined in Fig. 10–2, has particular advantages and disadvantages, depending upon the geometry of the structure and its degree of indeterminacy. A discussion of the usefulness of each method will be given after each has been presented.

	Unknowns	Equations Used for Solution	Coefficients of the Unknowns
Force Method	Forces	Compatibility and Force Displacement	Flexibility Coefficients
Displacement Method	Displacements	Equilibrium and Force Displacement	Stiffness Coefficients

Fig. 10–2

10.2 Force Method of Analysis: General Procedure

Perhaps the best way to illustrate the principles involved in the force method of analysis is to consider the beam shown in Fig. 10–3*a*. If its free-body diagram were drawn, there would be four unknown support reactions; and since three equilibrium equations are available for solution, the beam is indeterminate to the first degree. Consequently, one additional equation is necessary for solution. To obtain this equation, we will use the principle of superposition and consider the *compatibility of displacement* at one of the supports. This is done by choosing one of the support reactions as "redundant" and temporarily removing its effect on the beam so that the beam then becomes statically determinate and stable. This beam is referred to as the *primary structure.* Here we will remove the restraining action of the rocker at B. As a result, the load $\mathbf{P}$ will cause B to be displaced downward by an amount Δ_B as shown in Fig. 10–3*b*. By superposition, however, the unknown reaction at B, i.e., $\mathbf{B_y}$, causes the beam at B to be displaced Δ'_{BB} upward, Fig. 10–3*c*. Here the first letter in this double-subscript notation refers to the point (B) where the deflection is specified, and the second letter refers to the point (B) where the unknown reaction acts. Assuming positive displacements act upward, then from Figs. 10–3*a* through 10–3*c* we can write the necessary compatibility equation at the rocker as

$$(+\uparrow) \qquad 0 = -\Delta_B + \Delta'_{BB}$$

Let us now denote the displacement at B caused by a *unit load* acting in the direction of $\mathbf{B_y}$ as the *linear flexibility coefficient* f_{BB}, Fig. 10–3*d*. Using the same scheme for this double-subscript notation as above, f_{BB} is the deflection at B caused by a unit load at B. Since the material behaves in a linear-elastic manner, a force of $\mathbf{B_y}$ acting at B, instead of the unit load, will cause a proportionate increase in f_{BB}. Thus we can write

$$\Delta'_{BB} = B_y f_{BB}$$

When written in this format, it can be seen that the linear flexibility coefficient f_{BB} is a *measure of the deflection per unit force,* and so its units are m/N, ft/lb, etc. The compatibility equation above can therefore be written in terms of the unknown B_y as

$$0 = -\Delta_B + B_y f_{BB}$$

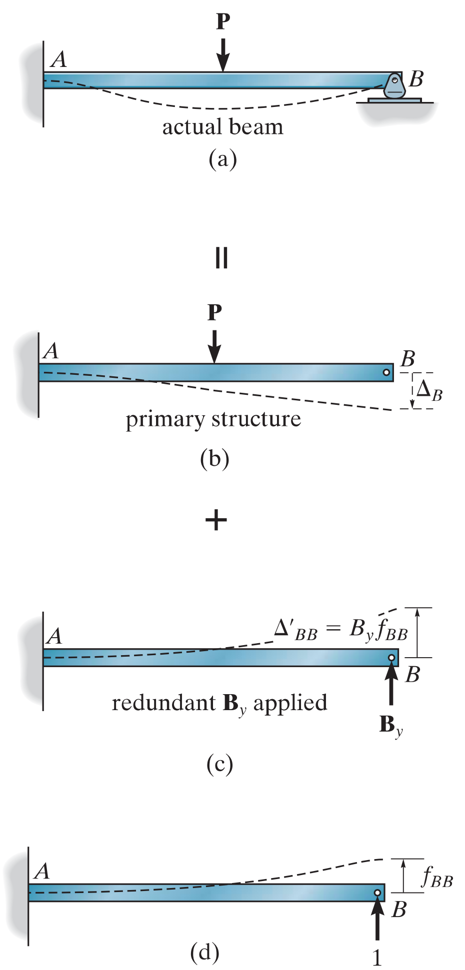

Fig. 10–3

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Using the methods of Chapter 8 or 9, or the deflection table on the inside front cover of the book, the appropriate load-displacement relations for the deflection Δ_B, Fig. 10–3*b*, and the flexibility coefficient f_{BB}, Fig. 10–3*d*, can be obtained and the solution for B_y determined, that is, $B_y = \Delta_B / f_{BB}$. Once this is accomplished, the three reactions at the wall A can then be found from the equations of equilibrium.

As stated previously, the choice of the redundant is *arbitrary*. For example, the moment at A, Fig. 10–4*a*, can be determined *directly* by removing the capacity of the beam to support a moment at A, that is, by replacing the fixed support by a pin. As shown in Fig. 10–4*b*, the rotation at A caused by the load **P** is θ_A, and the rotation at A caused by the redundant $\mathbf{M_A}$ at A is θ'_{AA}, Fig. 10–4*c*. If we denote an *angular flexibility coefficient* α_{AA} as the angular displacement at A caused by a unit couple moment applied to A, Fig. 10–4*d*, then

$$\theta'_{AA} = M_A \alpha_{AA}$$

Thus, the angular flexibility coefficient measures the angular displacement per unit couple moment, and therefore it has units of rad/N · m or rad/lb · ft, etc. The compatibility equation for rotation at A therefore requires

$$(\curvearrowleft +) \qquad 0 = \theta_A + M_A \alpha_{AA}$$

In this case, $M_A = -\theta_A / \alpha_{AA}$, a negative value, which simply means that $\mathbf{M_A}$ acts in the opposite direction to the unit couple moment.

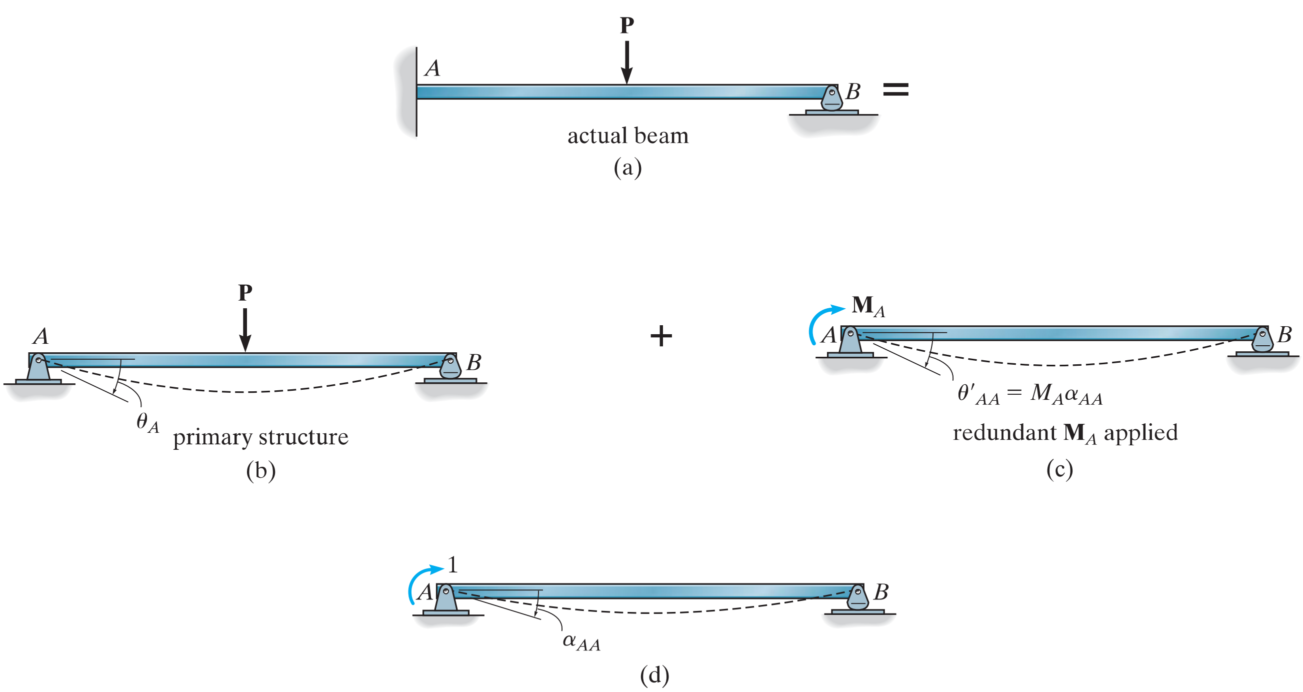


Fig. 10–4

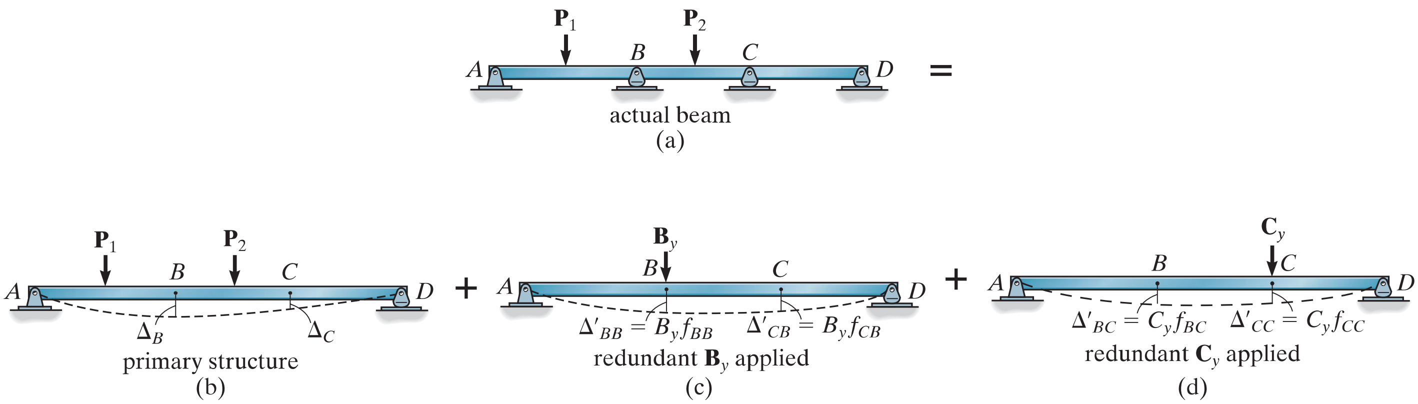
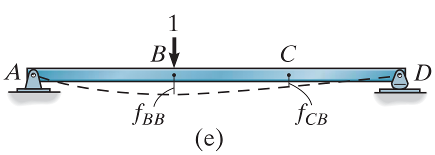
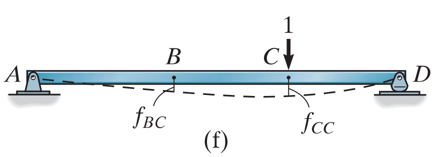


Fig. 10–5





A third example that illustrates application of the force method is given in Fig. 10–5*a*. Here the beam is indeterminate to the second degree and therefore two compatibility equations will be necessary for the solution. We will choose the vertical forces at the roller supports, *B* and *C*, as redundants. The resultant statically determinate beam deflects as shown in Fig. 10–5*b* when the redundants are removed. Each redundant force, which is *assumed* to act downward, deflects this beam as shown in Fig. 10–5*c* and 10–5*d*, respectively. Here the flexibility coefficients* f_{BB} and f_{CB} are found from a unit load acting at *B*, Fig. 10–5*e*; and f_{CC} and f_{BC} are found from a unit load acting at *C*, Fig. 10–5*f*. By superposition, the compatibility equations for the deflection at *B* and *C*, respectively, are

$$(+\downarrow) \qquad 0 = \Delta_B + B_y f_{BB} + C_y f_{BC}$$

$$(+\downarrow) \qquad 0 = \Delta_C + B_y f_{CB} + C_y f_{CC} \qquad (10\text{–}1)$$

Once the load-displacement relations are established using the methods of Chapter 8 or 9, these equations may be solved simultaneously for the two unknown forces B_y and C_y.

Having illustrated the application of the force method of analysis by example, we will now discuss its application in general terms and then we will use it as a basis for solving problems involving trusses, beams, and frames. For all these cases, however, realize that since the method depends on superposition of displacements, it is necessary that *the material remain linear elastic when loaded.* Also, recognize that *any* external reaction or internal loading at a point in the structure can be directly determined by first releasing the capacity of the structure to support the loading and then writing a compatibility equation at the point. See Example 10–4.

*f_{BB} is the deflection at *B* caused by a unit load at *B*; f_{CB} the deflection at *C* caused by a unit load at *B*.

Procedure for Analysis

The following procedure provides a general method for determining the reactions or internal loadings of statically indeterminate structures using the force or flexibility method of analysis.

Principle of Superposition

Determine the number of degrees n to which the structure is indeterminate. Then specify the n unknown redundant forces or moments that must be removed from the structure in order to make it statically determinate and stable. Using the principle of superposition, draw the statically indeterminate structure and show it to be equal to a series of corresponding statically *determinate* structures. The primary structure supports the same external loads as the statically indeterminate structure, and each of the other structures added to the primary structure shows the structure loaded with a separate redundant force or moment. Also, sketch the elastic curve on each structure and indicate symbolically the displacement or rotation at the point of each redundant force or moment.

Compatibility Equations

Write a compatibility equation for the displacement or rotation at each point where there is a redundant force or moment. These equations should be expressed in terms of the unknown redundants and their corresponding flexibility coefficients obtained from unit loads or unit couple moments that are collinear with the redundant forces or moments.

Determine all the deflections and flexibility coefficients using the table on the inside front cover or the methods of Chapter 8 or 9.* Substitute these load-displacement relations into the compatibility equations and solve for the unknown redundants. In particular, if a numerical value for a redundant is negative, it indicates the redundant acts opposite to its corresponding unit force or unit couple moment.

Equilibrium Equations

Draw a free-body diagram of the structure. Since the redundant forces and/or moments have been calculated, the remaining unknown reactions can be determined from the equations of equilibrium.

It should be realized that once all the support reactions have been obtained, the shear and moment diagrams can then be drawn, and the deflection at any point on the structure can be determined using the same methods outlined previously for statically determinate structures.

*It is suggested that if the M/EI diagram for a beam consists of simple segments, the moment-area theorems or the conjugate-beam method be used. Beams with complicated M/EI diagrams, that is, those with many curved segments (parabolic, cubic, etc.), can be readily analyzed using the method of virtual work or by Castigliano's second theorem.

10.3 Maxwell's Theorem of Reciprocal Displacements; Betti's Law

When Maxwell developed the force method of analysis, he also published a theorem that relates the flexibility coefficients of any two points on an elastic structure—be it a truss, a beam, or a frame. This theorem is referred to as the theorem of reciprocal displacements and may be stated as follows: *The displacement of a point B on a structure due to a unit load acting at point A is equal to the displacement of point A when the unit load is acting at point B, that is,* $f_{BA} = f_{AB}$.

Proof of this theorem is easily demonstrated using the principle of virtual work. For example, consider the beam in Fig. 10–6. When a real unit load acts at A, assume that the internal moments in the beam are represented by m_A. To determine the flexibility coefficient at B, that is, f_{BA}, a virtual unit load is placed at B, Fig. 10–7, and the internal moments m_B are computed. Then applying Eq. 9–18 yields

$$f_{BA} = \int \frac{m_B m_A}{EI} dx$$

Likewise, if the flexibility coefficient f_{AB} is to be determined when a real unit load acts at B, Fig. 10–7, then m_B represents the internal moments in the beam due to a real unit load. Furthermore, m_A represents the internal moments due to a virtual unit load at A, Fig. 10–6. Hence,

$$f_{AB} = \int \frac{m_A m_B}{EI} dx$$

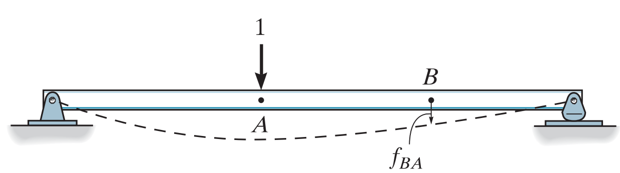


Fig. 10–6

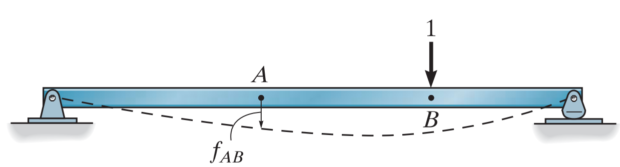


Fig. 10–7

Both integrals obviously give the same result, which proves the theorem. The theorem also applies for reciprocal rotations, and may be stated as follows: *The rotation at point B on a structure due to a unit couple moment acting at point A is equal to the rotation at point A when the unit couple moment is acting at point B.* Furthermore, using a unit force and unit couple moment, applied at separate points on the structure, we may also state: *The rotation in radians at point B on a structure due to a unit load acting at point A is equal to the displacement at point A when a unit couple moment is acting at point B.*

As a consequence of this theorem, some work can be saved when applying the force method to problems that are statically indeterminate to the second degree or higher. For example, only one of the two flexibility coefficients f_{BC} or f_{CB} has to be calculated in Eqs. 10–1, since $f_{BC} = f_{CB}$. Furthermore, the theorem of reciprocal displacements has applications in structural model analysis and for constructing influence lines using the Müller-Breslau principle (see Sec. 10–10).

When the theorem of reciprocal displacements is formalized in a more general sense, it is referred to as *Betti's law.* Briefly stated: The virtual work δU_{AB} done by a system of forces $\Sigma \mathbf{P}_B$ that undergo a displacement caused by a system of forces $\Sigma \mathbf{P}_A$ is equal to the virtual work δU_{BA} caused by the forces $\Sigma \mathbf{P}_A$ when the structure deforms due to the system of forces $\Sigma \mathbf{P}_B$ In other words, $\delta U_{AB} = \delta U_{BA}$. The proof of this statement is similar to that given above for the reciprocal-displacement theorem.

10.4 Force Method of Analysis: Beams

The force method applied to beams was outlined in Sec. 10–2. Using the "procedure for analysis" also given in Sec. 10–2, we will now present several examples that illustrate the application of this technique.

These bridge girders are statically indeterminate since they are continuous over their piers.

EXAMPLE 10.1

Determine the reaction at the roller support B of the beam shown in Fig. 10–8*a*. EI is constant.

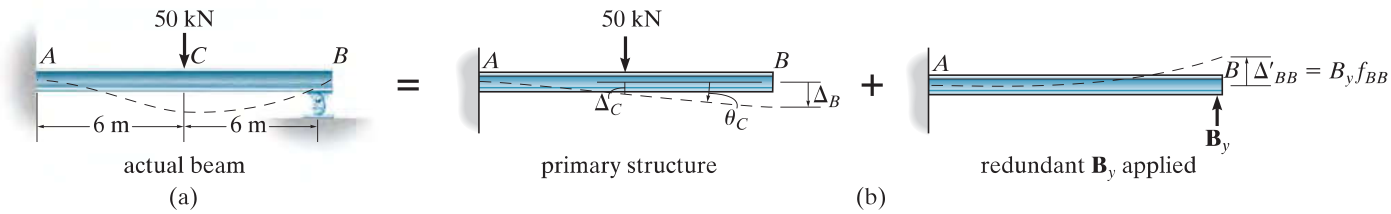


Fig. 10–8

SOLUTION

Principle of Superposition. By inspection, the beam is statically indeterminate to the first degree. The redundant will be taken as $\mathbf{B}_y$ so that this force can be determined *directly*. Figure 10–8*b* shows application of the principle of superposition. Notice that removal of the redundant requires that the roller support or the constraining action of the beam in the direction of $\mathbf{B}_y$ be removed. Here we have assumed that $\mathbf{B}_y$ acts upward on the beam.

Compatibility Equation. Taking positive displacement as upward, Fig. 10–8*b*, we have

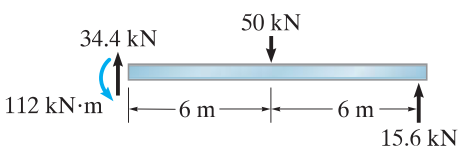
$$(+\uparrow) \qquad 0 = -\Delta_B + B_y f_{BB} \qquad (1)$$

The terms Δ_B and f_{BB} are easily obtained using the table on the inside front cover. In particular, note that $\Delta_B = \Delta_C + \theta_C(6\text{ m})$. Thus,

$$\Delta_B = \frac{P(L/2)^3}{3EI} + \frac{P(L/2)^2}{2EI}\left(\frac{L}{2}\right)$$

$$= \frac{(50\text{ kN})(6\text{ m})^3}{3EI} + \frac{(50\text{ kN})(6\text{ m})^2}{2EI}(6\text{ m}) = \frac{9000\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

$$f_{BB} = \frac{PL^3}{3EI} = \frac{1(12\text{ m})^3}{3EI} = \frac{576\text{ m}^3}{EI}\uparrow$$



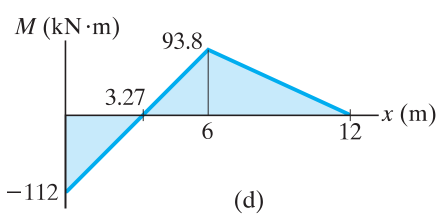
(c)

Substituting these results into Eq. (1) yields

$$(+\uparrow) \qquad 0 = -\frac{9000}{EI} + B_y\left(\frac{576}{EI}\right) \qquad B_y = 15.6\text{ kN} \qquad \textit{Ans.}$$

If this reaction is placed on the free-body diagram of the beam, the reactions at A can be obtained from the three equations of equilibrium, Fig. 10–8*c*.

Having determined all the reactions, the moment diagram can be constructed as shown in Fig. 10–8*d*.



(d)

EXAMPLE 10.2

Draw the shear and moment diagrams for the beam shown in Fig. 10–9*a*. The support at *B* settles 1.5 in. Take $E = 29(10^3)$ ksi, $I = 750\text{ in}^4$.

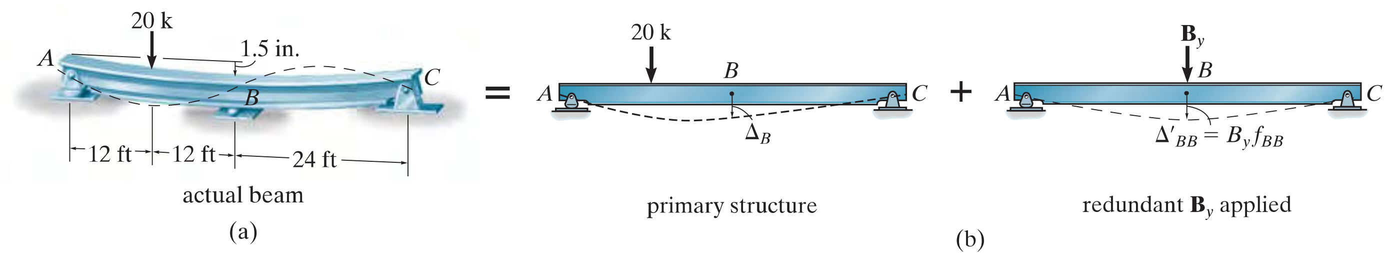


Fig. 10–9

SOLUTION

Principle of Superposition. By inspection, the beam is indeterminate to the first degree. The center support *B* will be chosen as the redundant, so that the roller at *B* is removed, Fig. 10–9*b*. Here $\mathbf{B}_y$ is assumed to act downward on the beam.

Compatibility Equation. With reference to point *B* in Fig. 10–9*b*, using units of inches, we require

$$(+\downarrow) \qquad 1.5\text{ in.} = \Delta_B + B_y f_{BB} \qquad (1)$$

We will use the table on the inside front cover. Note that for Δ_B the equation for the deflection curve requires $0 < x < a$. Since $x = 24$ ft, then $a = 36$ ft. Thus,

$$\Delta_B = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2) = \frac{20(12)(24)}{6(48)EI}[(48)^2 - (12)^2 - (24)^2]$$

$$= \frac{31{,}680\text{ k}\cdot\text{ft}^3}{EI}$$

$$f_{BB} = \frac{PL^3}{48EI} = \frac{1(48)^3}{48EI} = \frac{2304\text{ k}\cdot\text{ft}^3}{EI}$$

Substituting these values into Eq. (1), we get

$$1.5\text{ in.}\,(29(10^3)\text{ k/in}^2)(750\text{ in}^4)$$

$$= 31{,}680\text{ k}\cdot\text{ft}^3(12\text{ in./ft})^3 + B_y(2304\text{ k}\cdot\text{ft}^3)(12\text{ in./ft})^3$$

$$B_y = -5.56\text{ k}$$

The negative sign indicates that $\mathbf{B}_y$ acts *upward* on the beam.

EXAMPLE 10.2 (Continued)

Equilibrium Equations. From the free-body diagram shown in Fig. 10–9*c* we have

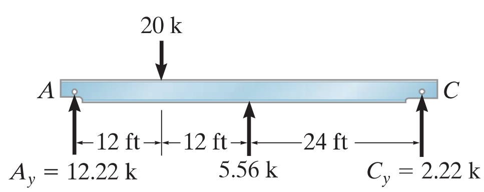
$$\circlearrowleft + \Sigma M_A = 0; \qquad -20(12) + 5.56(24) + C_y(48) = 0$$

$$C_y = 2.22 \text{ k}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 20 + 5.56 + 2.22 = 0$$

$$A_y = 12.22 \text{ k}$$

Using these results, verify the shear and moment diagrams shown in Fig. 10–9*d*.



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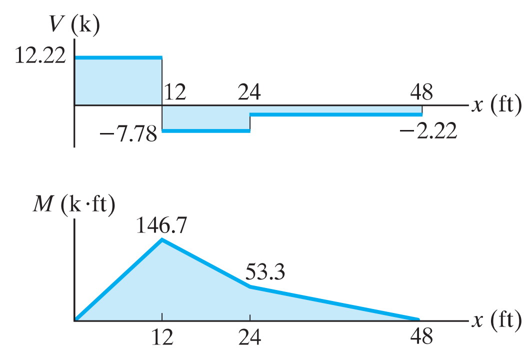


(d)

EXAMPLE 10.3

Draw the shear and moment diagrams for the beam shown in Figure 10–10*a*. *EI* is constant. Neglect the effects of axial load.

SOLUTION

Principle of Superposition. Since axial load is neglected, the beam is indeterminate to the second degree. The two end moments at *A* and *B* will be considered as the redundants. The beam's capacity to resist these moments is removed by placing a pin at *A* and a rocker at *B*. The principle of superposition applied to the beam is shown in Fig. 10–10*b*.

Compatibility Equations. Reference to points *A* and *B*, Fig. 10–10*b*, requires

(↺+) $$0 = \theta_A + M_A\alpha_{AA} + M_B\alpha_{AB} \quad (1)$$

(↻+) $$0 = \theta_B + M_A\alpha_{BA} + M_B\alpha_{BB} \quad (2)$$

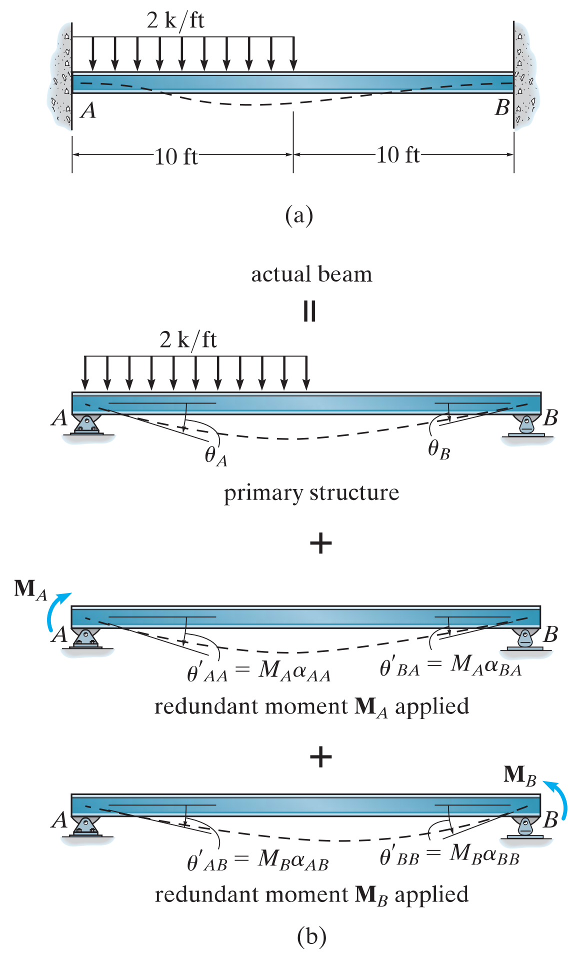


Fig. 10–10

EXAMPLE 10.3 (Continued)

The required slopes and angular flexibility coefficients can be determined using the table on the inside front cover. We have

$$\theta_A = \frac{3wL^3}{128EI} = \frac{3(2)(20)^3}{128EI} = \frac{375}{EI}$$

$$\theta_B = \frac{7wL^3}{384EI} = \frac{7(2)(20)^3}{384EI} = \frac{291.7}{EI}$$

$$\alpha_{AA} = \frac{ML}{3EI} = \frac{1(20)}{3EI} = \frac{6.67}{EI}$$

$$\alpha_{BB} = \frac{ML}{3EI} = \frac{1(20)}{3EI} = \frac{6.67}{EI}$$

$$\alpha_{AB} = \frac{ML}{6EI} = \frac{1(20)}{6EI} = \frac{3.33}{EI}$$

Note that $\alpha_{BA} = \alpha_{AB}$, a consequence of Maxwell's theorem of reciprocal displacements.

Substituting the data into Eqs. (1) and (2) yields

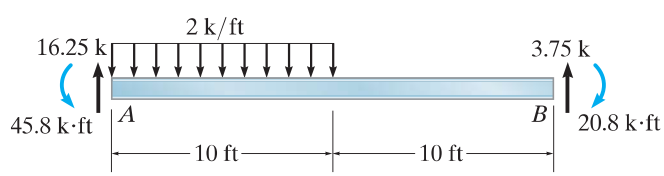
$$0 = \frac{375}{EI} + M_A\left(\frac{6.67}{EI}\right) + M_B\left(\frac{3.33}{EI}\right)$$

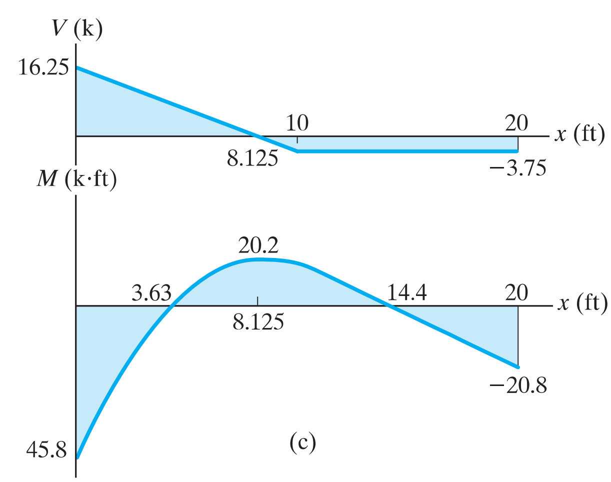
$$0 = \frac{291.7}{EI} + M_A\left(\frac{3.33}{EI}\right) + M_B\left(\frac{6.67}{EI}\right)$$

Canceling EI and solving these equations simultaneously, we have

$$M_A = -45.8 \text{ k} \cdot \text{ft} \qquad M_B = -20.8 \text{ k} \cdot \text{ft}$$

Using these results, the end shears are calculated, Fig. 10–10*c*, and the shear and moment diagrams plotted.





(c)

EXAMPLE 10.4

Determine the reactions at the supports for the beam shown in Fig. 10–11*a*. *EI* is constant.

SOLUTION

Principle of Superposition. By inspection, the beam is indeterminate to the first degree. Here, for the sake of illustration, we will choose the internal moment at support B as the redundant. Consequently, the beam is cut open and end pins or an internal hinge are placed at B in order to release *only* the capacity of the beam to resist moment at this point, Fig. 10–11*b*. The internal moment at B is applied to the beam in Fig. 10–11*c*.

Compatibility Equations. From Fig. 10–11*a* we require the relative rotation of one end of one beam with respect to the end of the other beam to be zero, that is,

$$(\curvearrowleft +) \qquad \theta_B + M_B\alpha_{BB} = 0$$

where

$$\theta_B = \theta'_B + \theta''_B$$

and

$$\alpha_{BB} = \alpha'_{BB} + \alpha''_{BB}$$

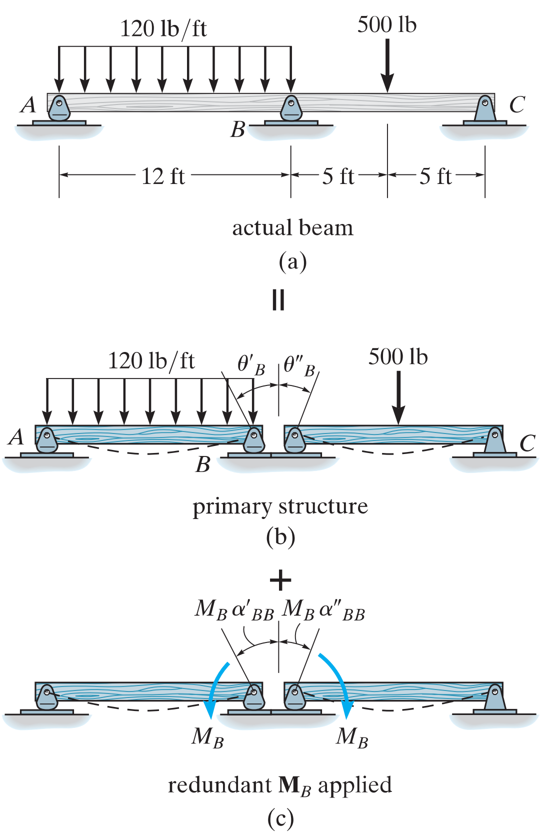


Fig. 10–11

EXAMPLE 10.4 (Continued)

The slopes and angular flexibility coefficients can be determined from the table on the inside front cover, that is,

$$\theta'_B = \frac{wL^3}{24EI} = \frac{120(12)^3}{24EI} = \frac{8640 \text{ lb}\cdot\text{ft}^2}{EI}$$

$$\theta''_B = \frac{PL^2}{16EI} = \frac{500(10)^2}{16EI} = \frac{3125 \text{ lb}\cdot\text{ft}^2}{EI}$$

$$\alpha'_{BB} = \frac{ML}{3EI} = \frac{1(12)}{3EI} = \frac{4 \text{ ft}}{EI}$$

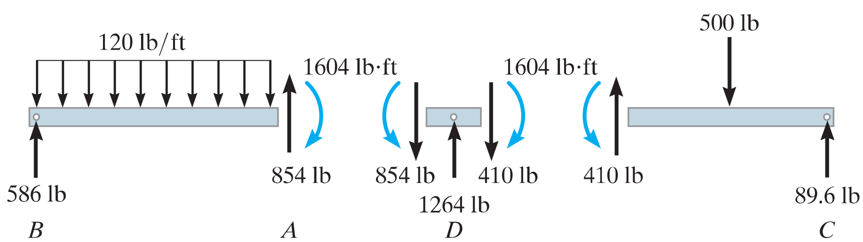
$$\alpha''_{BB} = \frac{ML}{3EI} = \frac{1(10)}{3EI} = \frac{3.33 \text{ ft}}{EI}$$

Thus

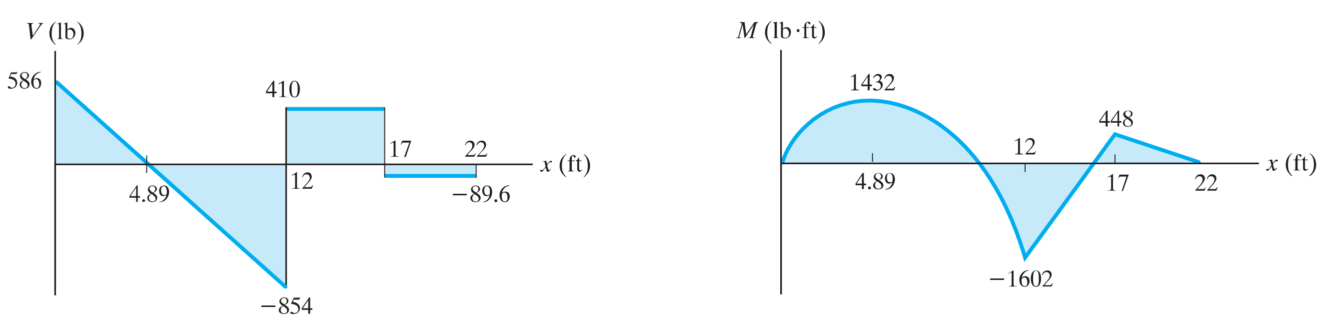
$$\frac{8640 \text{ lb}\cdot\text{ft}^2}{EI} + \frac{3125 \text{ lb}\cdot\text{ft}^2}{EI} + M_B\left(\frac{4 \text{ ft}}{EI} + \frac{3.33 \text{ ft}}{EI}\right) = 0$$

$$M_B = -1604 \text{ lb}\cdot\text{ft}$$

The negative sign indicates M_B acts in the opposite direction to that shown in Fig. 10–11*c*. Using this result, the reactions at the supports are calculated as shown in Fig. 10–11*d*. Furthermore, the shear and moment diagrams are shown in Fig. 10–11*e*.



(d)



(e)

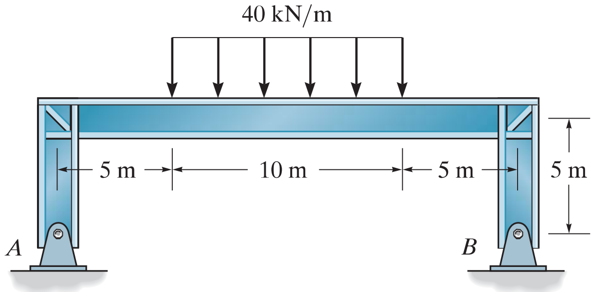
10.5 Force Method of Analysis: Frames

The force method is very useful for solving problems involving statically indeterminate frames that have a single story and unusual geometry, such as gabled frames. Problems involving multistory frames, or those with a high degree of indeterminacy, are best solved using the slope-deflection, moment-distribution, or the stiffness method discussed in later chapters.

The following examples illustrate the application of the force method using the procedure for analysis outlined in Sec. 10–2.

EXAMPLE 10.5

The frame, or bent, shown in the photo is used to support the bridge deck. Assuming EI is constant, a drawing of it along with the dimensions and loading is shown in Fig. 10–12a. Determine the support reactions.



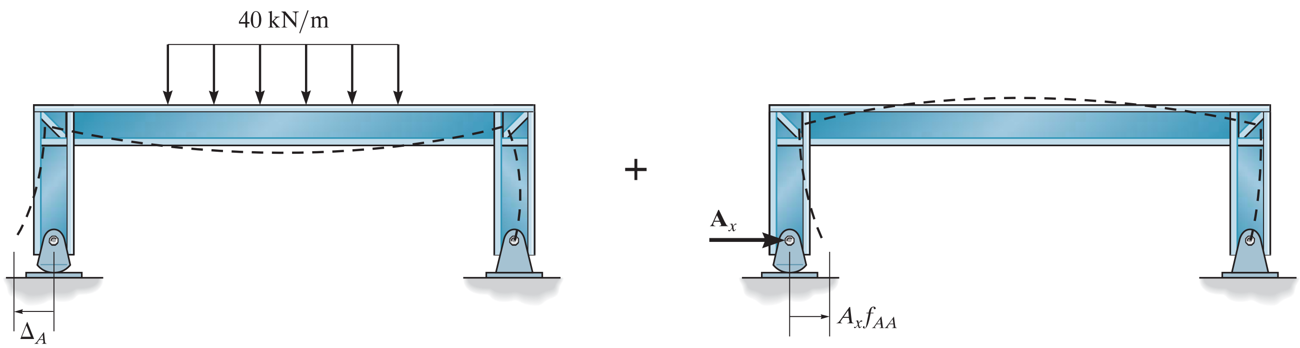
(a)

Fig. 10–12

SOLUTION

Principle of Superposition. By inspection the frame is statically indeterminate to the first degree. We will choose the horizontal reaction at A to be the redundant. Consequently, the pin at A is replaced by a rocker, since a rocker will not constrain A in the horizontal direction. The principle of superposition applied to the idealized model of the frame is shown in Fig. 10–12b. Notice how the frame deflects in each case.

EXAMPLE 10.5 (Continued)



Primary structure Redundant force $\mathbf{A}_x$ applied

(b)

Compatibility Equation. Reference to point A in Fig. 10–12b requires

$$(\stackrel{+}{\rightarrow}) \qquad 0 = \Delta_A + A_x f_{AA} \qquad (1)$$

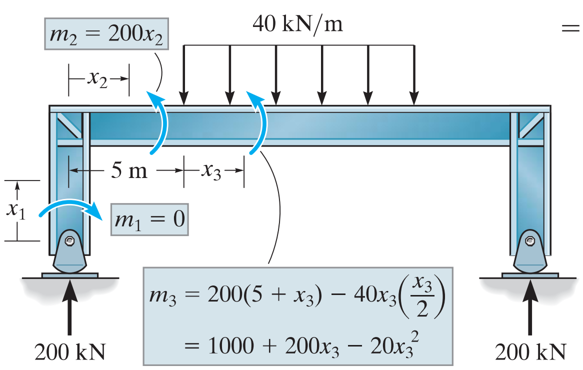
The terms Δ_A and f_{AA} will be determined using the method of virtual work. Because of symmetry of geometry *and* loading we need only three x coordinates. These and the internal moments are shown in Figs. 10–12c and 10–12d. It is important that each x coordinate be the *same* for both the real and virtual loadings. Also, the positive directions for **M** and **m** must be the *same*.

For Δ_A we require application of real loads, Fig. 10–12c, and a virtual unit load at A, Fig. 10–12d. Thus,

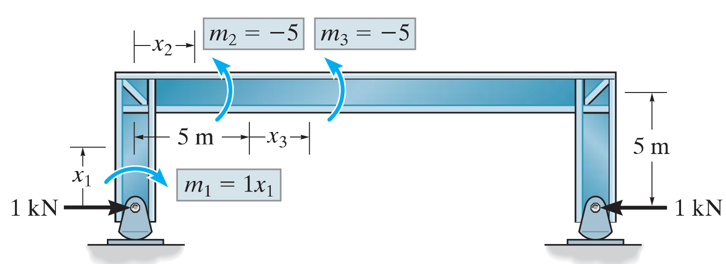
$$\Delta_A = \int_0^L \frac{Mm}{EI}dx = 2\int_0^5 \frac{(0)(1x_1)dx_1}{EI} + 2\int_0^5 \frac{(200x_2)(-5)dx_2}{EI}$$

$$+ 2\int_0^5 \frac{(1000 + 200x_3 - 20x_3^2)(-5)dx_3}{EI}$$

$$= 0 - \frac{25\,000}{EI} - \frac{66\,666.7}{EI} = -\frac{91\,666.7}{EI}$$



(c)



(d)

For f_{AA} we require application of a real unit load and a virtual unit load acting at A, Fig. 10–12d. Thus,

$$f_{AA} = \int_0^L \frac{mm}{EI}dx = 2\int_0^5 \frac{(1x_1)^2 dx_1}{EI} + 2\int_0^5 (5)^2 dx_2 + 2\int_0^5 (5)^2 dx_3$$

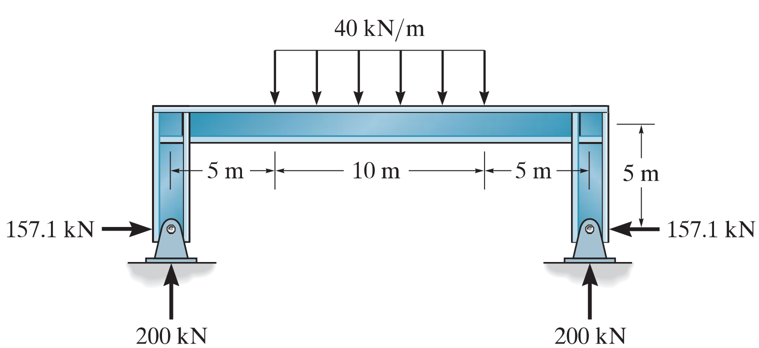
$$= \frac{583.33}{EI}$$

Substituting the results into Eq. (1) and solving yields

$$0 = \frac{-91\,666.7}{EI} + A_x\left(\frac{583.33}{EI}\right)$$

$$A_x = 157 \text{ kN} \qquad \textit{Ans.}$$

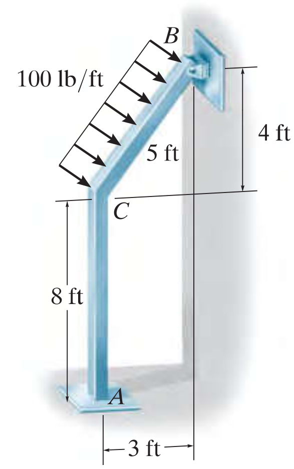
Equilibrium Equations. Using this result, the reactions on the idealized model of the frame are shown in Fig. 10–12e.



(e)

EXAMPLE 10.6

Determine the moment at the fixed support A for the frame shown in Fig. 10–13a. EI is constant.



(a)

Fig. 10–13

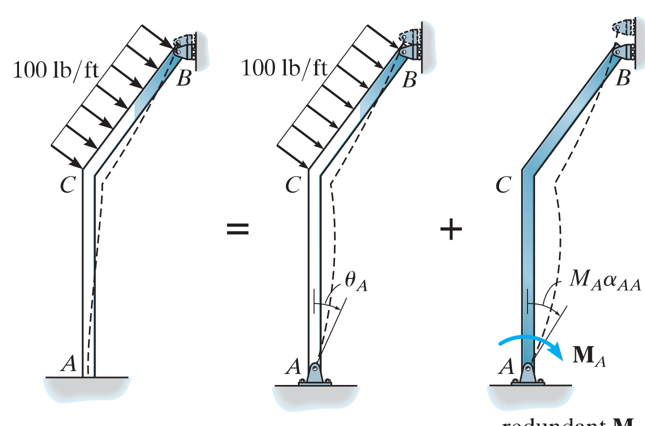
SOLUTION

Principle of Superposition. The frame is indeterminate to the first degree. A direct solution for $\mathbf{M}_A$ can be obtained by choosing this as the redundant. Thus the capacity of the frame to support a moment at A is removed and therefore a pin is used at A for support. The principle of superposition applied to the frame is shown in Fig. 10–13b.

Compatibility Equation. Reference to point A in Fig. 10–13b requires

$$(\curvearrowleft +) \qquad 0 = \theta_A + M_A\alpha_{AA} \qquad (1)$$

As in the preceding example, θ_A and α_{AA} will be computed using the method of virtual work. The frame's x coordinates and internal moments are shown in Figs. 10–13c and 10–13d.



(b)

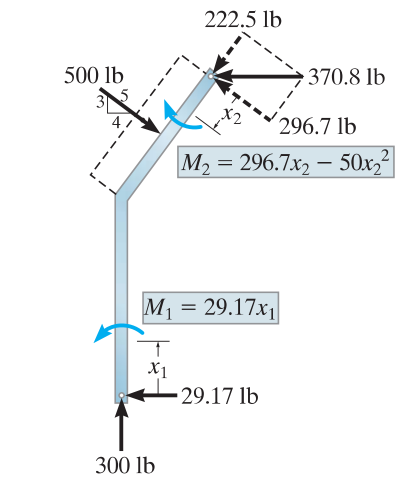
For θ_A we require application of the real loads, Fig. 10–13*c*, and a virtual unit couple moment at A, Fig. 10–13*d*. Thus,

$$\theta_A = \sum \int_0^L \frac{M m_\theta \, dx}{EI}$$

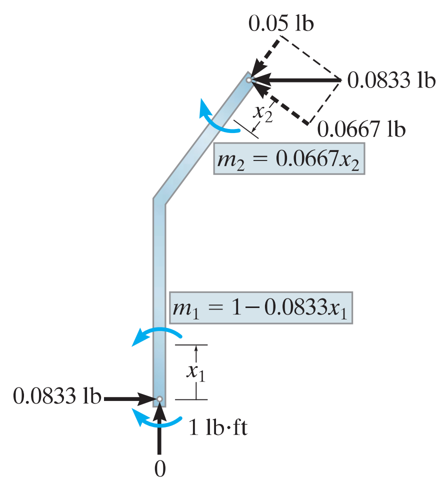
$$= \int_0^8 \frac{(29.17x_1)(1 - 0.0833x_1)\, dx_1}{EI}$$

$$+ \int_0^5 \frac{\left(296.7x_2 - 50x_2^2\right)(0.0667x_2)\, dx_2}{EI}$$

$$= \frac{518.5}{EI} + \frac{303.2}{EI} = \frac{821.8}{EI}$$



(c)



(d)

For α_{AA} we require application of a real unit couple moment and a virtual unit couple moment acting at A, Fig. 10–13*d*. Thus,

$$\alpha_{AA} = \sum \int_0^L \frac{m_\theta m_\theta}{EI} dx$$

$$= \int_0^8 \frac{(1 - 0.0833x_1)^2\, dx_1}{EI} + \int_0^5 \frac{(0.0667x_2)^2\, dx_2}{EI}$$

$$= \frac{3.85}{EI} + \frac{0.185}{EI} = \frac{4.04}{EI}$$

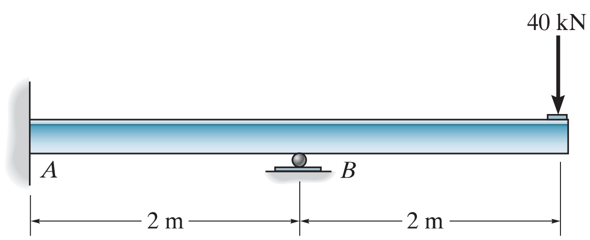
Substituting these results into Eq. (1) and solving yields

$$0 = \frac{821.8}{EI} + M_A\left(\frac{4.04}{EI}\right) \qquad M_A = -204 \text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$

The negative sign indicates $\mathbf{M}_A$ acts in the opposite direction to that shown in Fig. 10–13*b*.

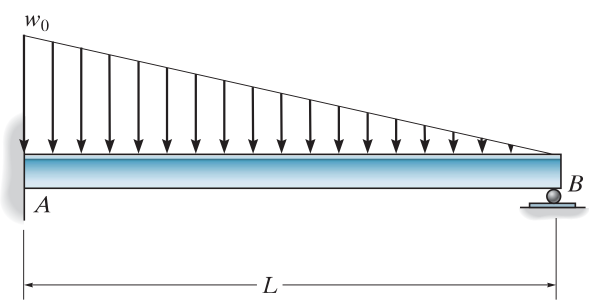
FUNDAMENTAL PROBLEMS

F10–1. Determine the reactions at the fixed support at A and the roller at B. EI is constant.



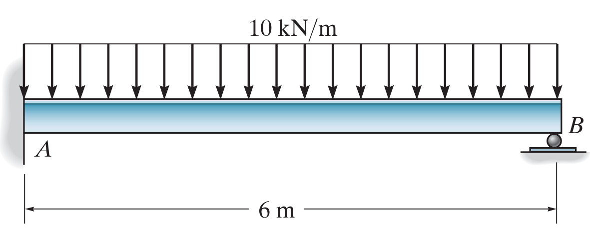
F10–1

F10–2. Determine the reactions at the fixed supports at A and the roller at B. EI is constant.



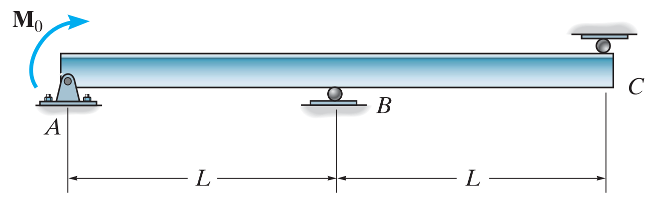
F10–2

F10–3. Determine the reactions at the fixed support at A and the roller at B. Support B settles 5 mm. Take $E = 200$ GPa and $I = 300(10^6)\ \text{mm}^4$.



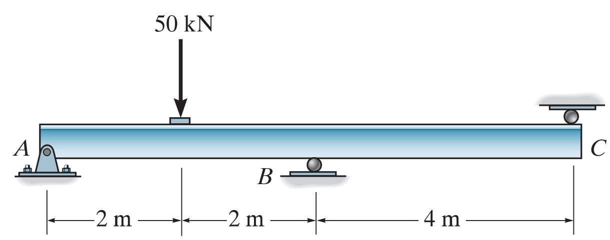
F10–3

F10–4. Determine the reactions at the pin at A and the rollers at B and C.



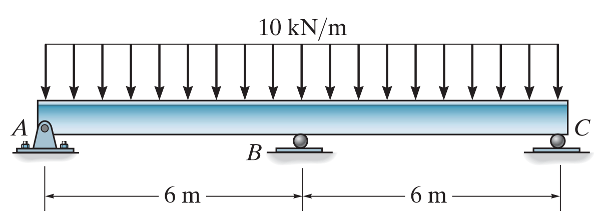
F10–4

F10–5. Determine the reactions at the pin A and the rollers at B and C on the beam. EI is constant.



F10–5

F10–6. Determine the reactions at the pin at A and the rollers at B and C on the beam. Support B settles 5 mm. Take $E = 200$ GPa, $I = 300(10^6)\ \text{mm}^4$.



F10–6