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FLUID MECHANICS

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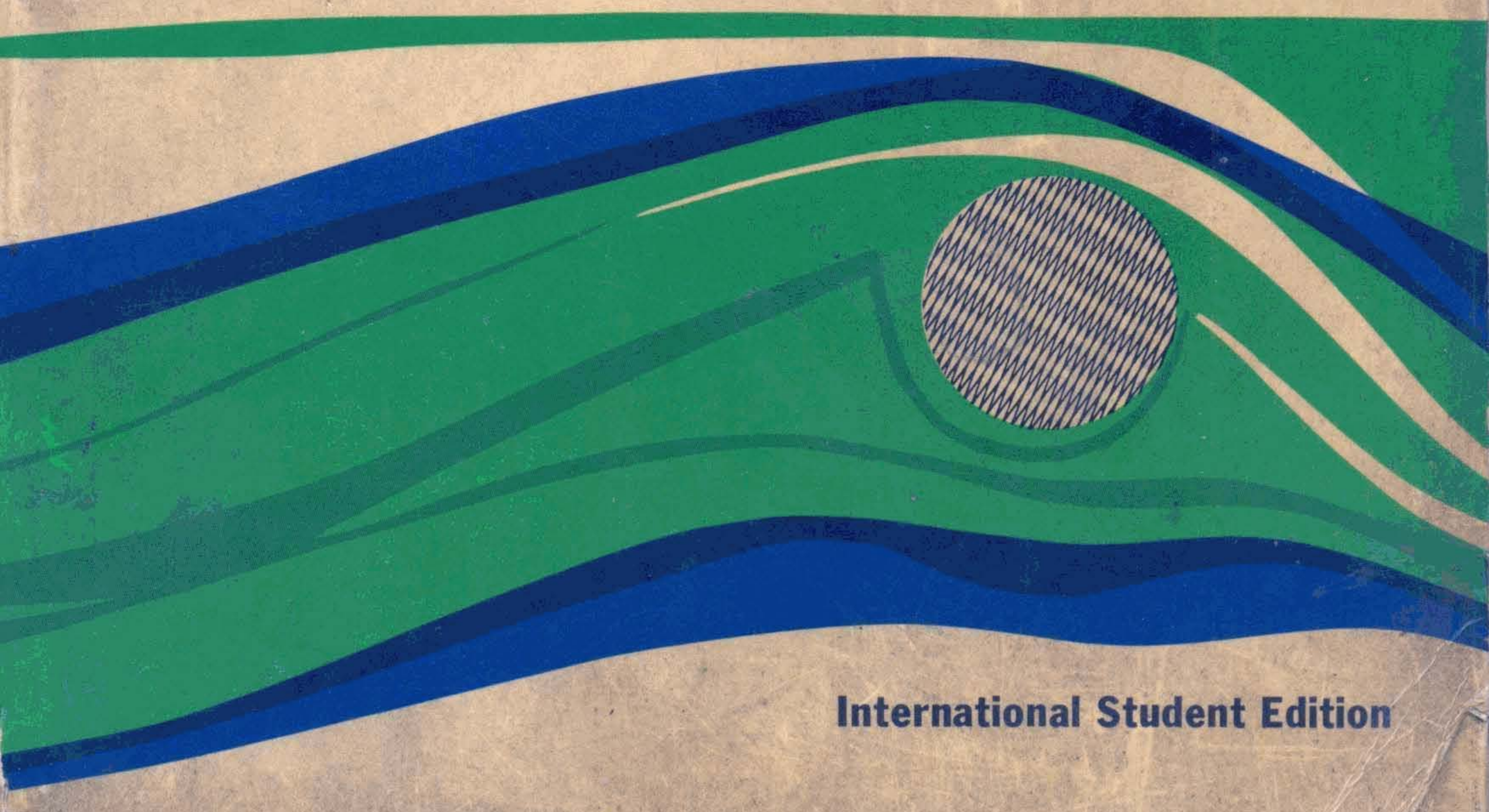

International Student Edition

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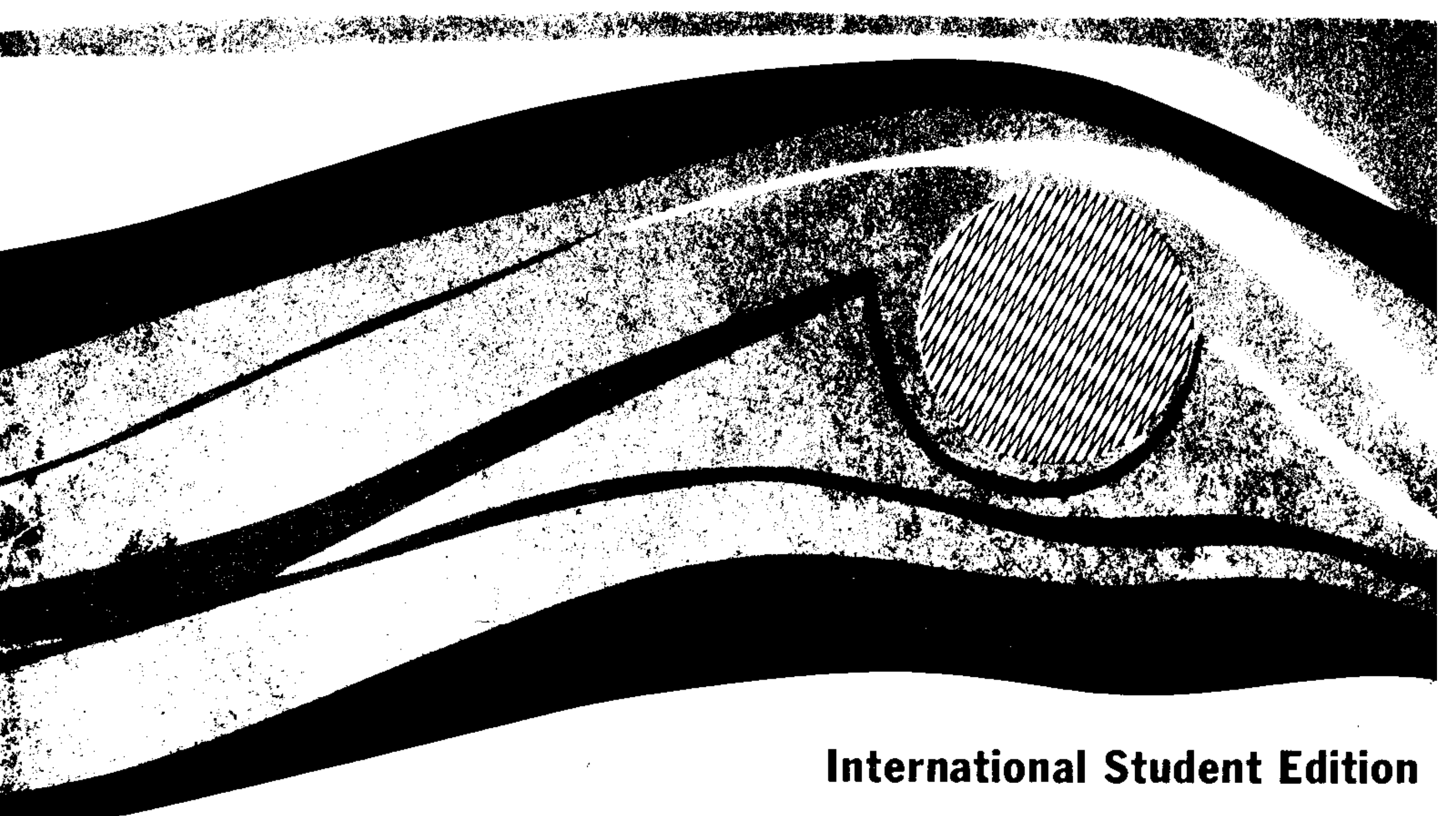

International Student Edition

Colored Schlieren photograph of a $22\frac{1}{2}°$ half-angle wedge, Mach number 2.75, with angle of attack of 5°. Shock-wave interaction with wind-tunnel boundary layer shown at top. (*Aeronautical and Astronautical Engineering Laboratories, University of Michigan.*)

FLUID MECHANICS

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THIRD EDITION

INTERNATIONAL STUDENT EDITION

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PREFACE

Several important changes in emphasis have been made in this revision. The most extensive change is in the handling of compressible flow. In general, there is no fixed pattern for the election of thermodynamics before fluid mechanics throughout the engineering colleges. The treatment of compressible fluids should not repeat an appreciable amount of work normally covered in thermodynamics but should either introduce this work or supplement it. Owing to the limited class time in a course on fluids, thermodynamic topics have been restricted to perfect gases with constant specific heats. The treatment of losses conforms generally to thermodynamic concepts. These changes have caused minor changes in the fluid properties treatment, major changes in fluid concepts and basic equations, and a new treatment of the chapter on compressible flow.

As the first courses in statics and dynamics are now being taught with vectors in many schools, they have been introduced where appropriate. Most of the fluid treatment is one-dimensional and hence neither requires nor benefits from vectors. In two- and three-dimensional flow, however, they are used for derivations of continuity, momentum, and Euler's equation. The chapter on dimensional analysis has been strengthened and moved forward to Chapter 4 for greater emphasis. The chapter on fluid statics has been shortened somewhat, and the viscous effects treatment, Chapter 5, has been shortened, with compressible examples and applications removed to Chapter 6.

Ideal-fluid flow has been expanded to cover three-dimensional flow cases, plus additional two-dimensional examples. The chapter on turbomachinery has been broadened to include compressible examples, and fluid measurements now include optical measurements.

Division of the material into two parts, fundamentals and applications, has been retained because of its wide acceptance in the second edition. The treatment is more comprehensive than needed for a first course, and the instructor should select those topics he wishes to stress. A three-semester-hour course could normally include most of the first five chapters, plus portions of Chapters 6 and 7, with selected topics from Part Two.

Most of the problems have been completely rewritten and range from very simple ones to those requiring further development of theory.

The author wishes to acknowledge the help he has received from his colleague Gordon Van Wylen for the many stimulating discussions of the thermodynamic aspects of fluid flow, from the reviewers who have added greatly to the text by their frank evaluations of the requirements of a first text on fluids, from the McGraw-Hill Book Company representatives for their understanding and full cooperation, and from Miss Pauline Bentley and Evelyn Streeter for their wholehearted aid in preparing the manuscript and in reading proof. The author is deeply greateful for their help.

V. L. Streeter

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APPENDIXES

PART ONE

Fundamentals of Fluid Mechanics

In the first three chapters of Part One, the properties of fluids, fluid statics, and the underlying framework of concepts, definitions, and basic equations for fluid dynamics are discussed. Dimensionless parameters are next introduced, including dimensional analysis and dynamic similitude. Chapter 5 deals with real fluids and the introduction of experimental data into fluid-flow calculations. Compressible flow of both real and frictionless fluids is then treated, and the final chapter on fundamentals deals with two- and three-dimensional ideal-fluid flow. The theory has been illustrated with elementary applications throughout Part One.

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FLUID PROPERTIES AND DEFINITIONS

Fluid mechanics is one of the engineering sciences that form the basis for all engineering. The subject branches out into various specialties such as aerodynamics, hydraulic engineering, marine engineering, gas dynamics, and rate processes. It deals with the statics, kinematics, and dynamics of fluids, since the motion of a fluid is caused by unbalanced forces exerted upon it. Available methods of analysis stem from the application of the following principles, concepts, and laws: Newton's laws of motion, the first and second laws of thermodynamics, the principle of conservation of mass, equations of state relating fluid properties, Newton's law of viscosity, mixing-length concepts, and restrictions caused by the presence of boundaries.

In fluid-flow calculations, viscosity and density are the fluid properties most generally encountered; they play the principal roles in open- and closed-channel flow and in flow around immersed bodies. Surface-tension effects are of importance in the formation of droplets, in flow of small jets, and in situations where liquid-gas-solid or liquid-liquid-solid interfaces occur, as well as in the formation of capillary waves. The property of vapor pressure, accounting for changes of phase from liquid to gas, becomes important when reduced pressures are encountered. In this chapter fluid properties are discussed, as well as units and dimensions and concepts of the continuum.

1.1. Definition of a Fluid. A fluid is a substance that deforms continuously when subjected to a shear stress, no matter how small that shear stress may be. A shear force is the force component tangent to a surface, and this force divided by the area of the surface is the average shear stress over the area. Shear stress at a point is the limiting value of shear force to area as the area is reduced to the point.

In Fig. 1.1 a substance is placed between two closely spaced parallel plates, so large that conditions at their edges may be neglected. The lower plate is fixed, and a force F is applied to the upper plate, which exerts a shear stress F/A on any substance between the plates. A is the area of the upper plate. When the force F causes the upper plate to move with a steady (nonzero) velocity, no matter how small the magnitude of F, one may conclude that the substance between the two plates is a fluid.

The fluid in immediate contact with a solid boundary has the same velocity as the boundary, i.e., there is no slip at the boundary.[1] The fluid in the area *abcd* flows to the new position *ab'c'd* with each fluid particle moving parallel to the plate and the velocity u varying uniformly from zero at the stationary plate to U at the upper plate. Experiments show

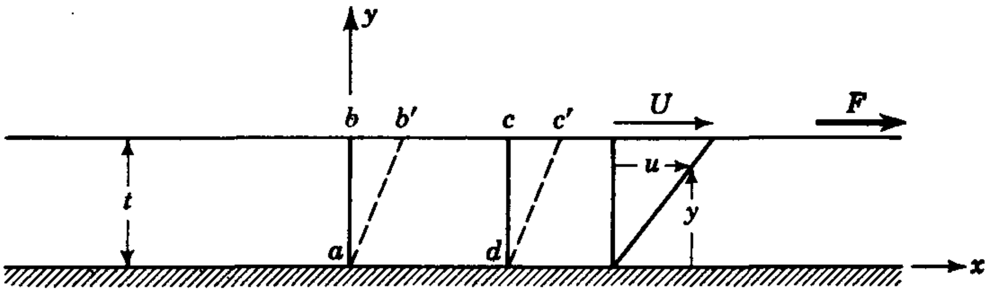


FIG. 1.1. Deformation resulting from application of constant shear force.

that other quantities being held constant, F is directly proportional to A and to U and is inversely proportional to t. In equation form

$$F = \mu \frac{AU}{t}$$

in which μ is the proportionality factor and includes the effect of the particular fluid. If $\tau = F/A$ for the shear stress,

$$\tau = \mu \frac{U}{t}$$

The ratio U/t is the angular velocity of line *ab*, or it is the *rate of angular deformation* of the fluid, i.e., the rate of decrease of angle *bad*. The angular velocity may also be written du/dy, as both U/t and du/dy express the velocity change divided by the distance over which the change occurs. However, du/dy is more general as it holds for situations in which the angular velocity and shear stress change with y. The velocity gradient du/dy may also be visualized as the rate at which one layer moves relative to an adjacent layer. In differential form,

$$\tau = \mu \frac{du}{dy} \tag{1.1.1}$$

is the relation between shear stress and rate of angular deformation for

[1] S. Goldstein, "Modern Developments in Fluid Dynamics," vol. II, pp. 676–680, Oxford University Press, London, 1938.

one-dimensional flow of a fluid. The proportionality factor μ is called the *viscosity* of the fluid, and Eq. (1.1.1) is *Newton's law of viscosity.*

A plastic substance cannot fulfill the definition of a fluid because it has an initial yield shear stress that must be exceeded to cause a continuous deformation. An elastic substance placed between the two plates would deform a certain *amount* proportional to the force, but not continuously at a definite rate. A complete vacuum between the plates would not result in a constant final rate, but in an ever-increasing rate. If sand were placed between the two plates, dry friction would require a *finite* force to cause a continuous motion. Thus sand will not satisfy the definition of a fluid.

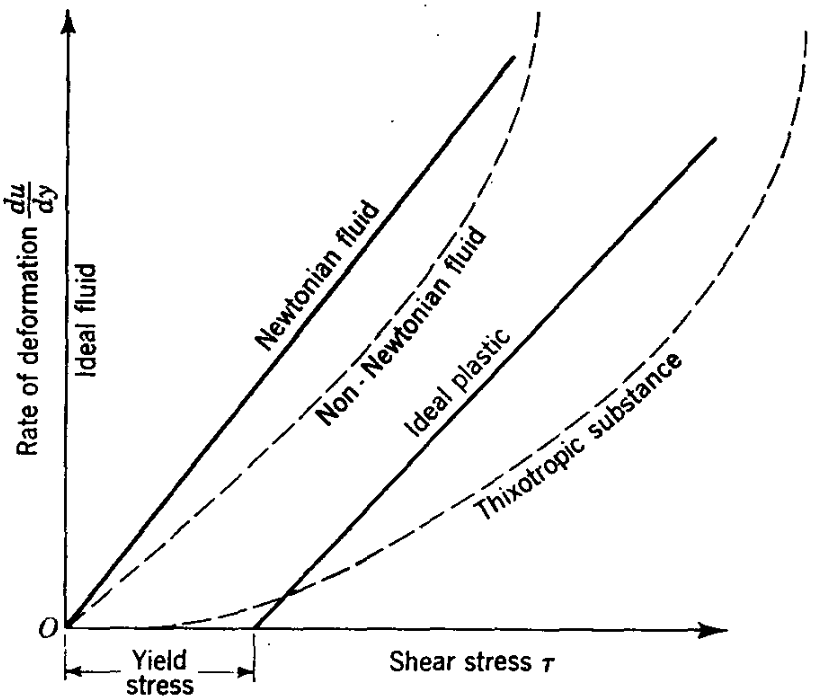


FIG. 1.2. Rheological diagram.

Fluids may be classified as Newtonian or non-Newtonian. In Newtonian fluid there is a linear relation between the magnitude of applied shear stress and the resulting rate of deformation [μ constant in Eq. (1.1.1)], as shown in Fig. 1.2. In non-Newtonian fluid there is a nonlinear relation between the magnitude of applied shear stress and the rate of angular deformation. An *ideal plastic* has a definite yield stress and a constant linear relation of τ to du/dy. A *thixotropic* substance, such as printer's ink, has a viscosity that is dependent upon the immediately prior angular deformation of the substance and has a tendency to take a set when at rest.

Gases and thin liquids tend toward Newtonian fluids, while thick liquids may be non-Newtonian. Tar is an example of a very viscous liquid that cannot sustain a shear stress while at rest. Its rate of deformation is so slow that it will apparently sustain a load, such as a stone placed on its free surface. However, after a day the stone will have penetrated into the tar.

For purposes of analysis, the assumption is frequently made that a fluid is nonviscous. With zero viscosity the shear stress is always zero, regardless of the motion of the fluid. If the fluid is also considered to be incompressible it is then called an *ideal* fluid, and plots as the ordinate in Fig. 1.2.

1.2. Force and Mass Units. The unit of force adopted in this text is the pound (lb). Two units of mass are employed, the slug and the pound mass (lb_m). Since thermodynamic properties are generally tabulated on a pound-mass basis, they are listed accordingly, but the example problems generally convert to the slug.

The pound of force is defined in terms of the pull of gravity, at a specified (standard) location, on a given mass of platinum. At standard gravitation, $g = 32.174$ ft/sec², the body having a pull of gravity of one pound has a mass of one pound mass. By writing Newton's second law of motion in the form

$$\mathbf{F} = \frac{m}{g_0}\mathbf{a} \tag{1.2.1}$$

and applying it to this object falling freely in a vacuum at standard conditions

$$1 \text{ lb} = 1\,\frac{\text{lb}_m}{g_0}\,32.174\,\frac{\text{ft}}{\text{sec}^2}$$

it is clear that

$$g_0 \equiv 32.174\,\frac{\text{lb}_m\text{-ft}}{\text{lb-sec}^2} \tag{1.2.2}$$

Whenever the pound mass is used in this text, it is labeled lb_m. The pound force is written lb. The number g_0 is a constant, independent of location of application of Newton's law and dependent only on the units pound, pound mass, foot, and second. At any other location than standard gravity, the mass of a body remains constant but the weight (force or pull of gravity) varies:

$$W = M\,(\text{lb}_m)\,\frac{g}{g_0} \tag{1.2.3}$$

For example, where $g = 31.0$ ft/sec²,

$$10 \text{ lb}_m \text{ weighs } 31.0 \times \frac{10}{32.174} = 9.635 \text{ lb}$$

The slug is a derived unit of mass, defined as the amount of mass that is accelerated one foot per second per second by a force of one pound. For these units the constant g_0 is unity, i.e., 1 slug-ft/lb-sec². Since fluid mechanics is so closely tied to Newton's second law, the slug may be defined as

$$1 \text{ slug} \equiv 1 \frac{\text{lb-sec}^2}{\text{ft}} \tag{1.2.4}$$

and the consistent set of units slug, pound, foot, second may be used without a dimensional constant g_0.

In the development of equations in this treatment, consistent units are assumed and the equations appear without the constant g_0. If the pound mass is to be used in dynamical equations, then g_0 must be introduced.

1.3. Viscosity. Of all the fluid properties, viscosity requires the greatest consideration in the study of fluid flow. The nature and characteristics of viscosity are discussed in this section as well as dimensions and conversion factors for both absolute and kinematic viscosity. Viscosity is that property of a fluid by virtue of which it offers resistance to shear stress. Newton's law of viscosity [Eq. (1.1.1)] states that for a given rate of angular deformation of fluid the shear stress is directly proportional to the viscosity. Molasses and tar are examples of highly viscous liquids; water and air have very small viscosities.

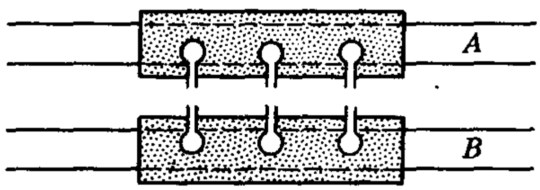


FIG. 1.3. Model for illustrating transfer of momentum.

The viscosity of a gas increases with temperature, but the viscosity of a liquid decreases with temperature. The variation in temperature trends may be explained upon examination of the causes of viscosity. The resistance of a fluid to shear depends upon its cohesion and upon its rate of transfer of molecular momentum. A liquid, with molecules much more closely spaced than a gas, has cohesive forces much larger than a gas. Cohesion appears to be the predominant cause of viscosity in a liquid, and since cohesion decreases with temperature, the viscosity does likewise. A gas, on the other hand, has very small cohesive forces. Most of its resistance to shear stress is the result of the transfer of molecular momentum.

As a rough model of the way in which momentum transfer gives rise to an apparent shear stress, consider two idealized railroad cars loaded with sponges and on parallel tracks, as in Fig. 1.3. Assume each car has a water tank and pump, arranged so that the water is directed by nozzles at right angles to the track. First, consider A stationary and B moving to the right, with the water from its nozzles striking A and being absorbed by the sponges. Car A will be set in motion owing to the component of the momentum of the jets which is parallel to the tracks, giving rise to an apparent shear stress between A and B. Now if A is pumping water back into B at the same rate, its action tends to slow down B, and equal and opposite apparent shear forces result. When A and B are both stationary or have the same velocity, the pumping does not exert an apparent shear stress on either car.

Within fluid there is always a transfer of molecules back and forth across any fictitious surface drawn in it. When one layer moves relative to an adjacent layer, the molecular transfer of momentum brings momentum from one side to the other so that an apparent shear stress is set up that resists the relative motion and tends to equalize the velocities of adjacent layers in a manner analogous to that of Fig. 1.3. The measure of the motion of one layer relative to an adjacent layer is du/dy.

Molecular activity gives rise to an apparent shear stress in gases which is more important than the cohesive forces, and since molecular activity increases with temperature, the viscosity of a gas also increases with temperature.

For ordinary pressures viscosity is independent of pressure and depends upon temperature only. For very great pressures gases and most liquids have shown erratic variations of viscosity with pressure.

A fluid at rest, or in motion so that no layer moves relative to an adjacent layer, will not have apparent shear forces set up, regardless of the viscosity, because du/dy is zero throughout the fluid. Hence, in the study of *fluid statics*, no shear forces can be considered because they do not occur in a static fluid, and the only stresses remaining are normal stresses, or pressures. This greatly simplifies the study of fluid statics, since any free body of fluid can have only gravity forces and normal surface forces acting on it.

The dimensions of viscosity are determined from Newton's law of viscosity [Eq. (1.1.1)]. Solving for the viscosity μ,

$$\mu = \frac{\tau}{du/dy}$$

Inserting dimensions F, L, T for force, length, and time,

$$\tau : FL^{-2} \qquad u : LT^{-1} \qquad y : L$$

μ is seen to have the dimensions $FL^{-2}T$. With the force dimension expressed in terms of mass by use of Newton's second law of motion, $F = MLT^{-2}$, the dimensions of viscosity may be expressed as $ML^{-1}T^{-1}$.

The English unit of viscosity (which has no special name) is 1 lb-sec/ft² or 1 slug/ft-sec (these are identical). The cgs unit of viscosity,[1] called

[1] The relation of the English unit to the poise may be established by converting from one system of units to the other. Consider a fluid that has a viscosity of 1 lb-sec/ft². After pounds are converted to dynes and feet to centimeters,

$$1\,\frac{\text{lb-sec}}{\text{ft}^2} = \frac{454 \times 980}{(30.48)^2} \times 1\,\frac{\text{dyne-sec}}{\text{cm}^2} = 479 \text{ poise}$$

The English unit is much larger. Hence, to convert from the poise to the English unit, divide by 479; to convert from the English unit to the poise, multiply by 479.

the *poise*, is 1 dyne-sec/cm² or 1 gm/cm-sec. The *centipoise* is one one-hundredth of a poise. Water at 68°F has a viscosity of 1.002 centipoise.

Kinematic Viscosity. The viscosity μ is frequently referred to as the *absolute* viscosity or the *dynamic* viscosity to avoid confusing it with the *kinematic* viscosity ν, which is the ratio of viscosity to mass density,

$$\nu = \frac{\mu}{\rho} \tag{1.3.1}$$

The kinematic viscosity occurs in many applications, e.g., the Reynolds number, which is VD/ν. The dimensions of ν are L^2T^{-1}. The English unit, 1 ft²/sec, has no special name; the cgs unit, called the *stoke*, is 1 cm²/sec.†

To convert to the English unit of viscosity from the English unit of kinematic viscosity, it is necessary to multiply by the mass density in slugs per cubic foot. To change to the poise from the stoke, it is necessary to multiply by the mass density in grams per cubic centimeter, which is numerically equal to the specific gravity.

Example 1.1: A liquid has a viscosity of 0.05 poise and a specific gravity of 0.85. Calculate (*a*) the viscosity in English units; (*b*) the kinematic viscosity in stokes; and (*c*) the kinematic viscosity in English units.

$$(a)\ \mu = \frac{0.05}{479} = 0.000105\ \frac{\text{slug}}{\text{ft-sec}}$$

$$(b)\ \nu = \frac{0.05}{0.85} = 0.0589 \text{ stoke}$$

$$(c)\ \nu = \frac{0.000105}{1.935 \times 0.85} \doteq 0.0000638\ \frac{\text{ft}^2}{\text{sec}}$$

Viscosity is practically independent of pressure and depends upon temperature only. The kinematic viscosity of liquids, and of gases at a given pressure, is substantially a function of temperature. Charts for the determination of absolute viscosity and kinematic viscosity are given in Appendix C, Figs. C.1 and C.2, respectively.

1.4. Continuum. In dealing with fluid-flow relationships on a mathematical or analytical basis, it is necessary to consider that the actual molecular structure is replaced by a hypothetical continuous medium, called the *continuum*. For example, velocity at a point in space is indefinite in a molecular medium, as it would be zero at all times except when a molecule occupied this exact point, and then it would be the

† The conversion from English unit to cgs is

$$1\ \frac{\text{ft}^2}{\text{sec}} = (30.48)^2 \times 1\ \frac{\text{cm}^2}{\text{sec}} = (30.48)^2 \text{ stokes}$$

The English unit is again much larger than the cgs unit; therefore, to convert from the stoke to the English unit, divide by (30.48)²; to convert from the English unit to the stoke, multiply by (30.48)².

velocity of the molecule and not the mean mass velocity of the particles in the neighborhood. This dilemma is avoided if one considers velocity at a point to be the average or mass velocity of all molecules surrounding the point, say, within a small sphere with radius large compared with the *mean distance between molecules.* With n molecules per cubic centimeter, the mean distance between molecules is of the order $n^{-\frac{1}{3}}$ cm. Molecular theory, however, must be used to calculate fluid properties (e.g., viscosity) which are associated with molecular motions, but continuum equations can be employed with the results of molecular calculations.

In rarefied gases, such as the atmosphere at 50 miles above sea level, the ratio of the mean free path[1] of the gas to a characteristic length for a body or conduit is used to distinguish the type of flow. The flow regime is called *gas dynamics* for very small values of the ratio, the next regime is called *slip flow*, and for large values of the ratio it is *free molecule flow*. In this text only the gas dynamics regime is studied.

The quantities density, specific volume, pressure, velocity, and acceleration are assumed to vary continuously throughout a fluid (or be constant).

1.5. Density, Specific Volume, Specific Weight, Specific Gravity, Pressure. The *density* ρ of a fluid is defined as its mass per unit volume. To define density at a point the mass Δm of fluid in a small volume ΔV surrounding the point is divided by ΔV and the limit is taken as ΔV becomes a value ϵ^3 in which ϵ is still large compared with the mean distance between molecules,

$$\rho = \lim_{\Delta V \to \epsilon^3} \frac{\Delta m}{\Delta V} \tag{1.5.1}$$

When mass is expressed in slugs, ρ is in slugs per cubic foot; when mass is expressed in pounds mass, then ρ is in pounds mass per cubic foot. These units are related by

$$\rho_{\text{slugs}} = \frac{\rho_{\text{lb}_m}}{32.174} \tag{1.5.2}$$

For water at standard pressure (14.7 lb/in.^{2}) and 75°F,

$$\rho = 1.935 \text{ slugs/ft}^3 \text{ or } 62.4 \text{ lb}_m/\text{ft}^3$$

The *specific volume* v_s is the reciprocal of the density ρ; i.e., it is the volume occupied by unit mass of fluid. Hence

$$v_s = \frac{1}{\rho} \tag{1.5.3}$$

The *specific weight* γ of a substance is its weight per unit volume. It

[1] The mean free path is the average distance a molecule travels between collisions.

changes with location,

$$\gamma = \rho_{slug}g = \frac{\rho_{lb_m}}{32.174} g \frac{\text{lb}}{\text{ft}^3} \tag{1.5.4}$$

depending upon gravity. It is a convenient property when dealing with fluid statics or with liquids with a free surface.

The *specific gravity* S of a substance is the ratio of its weight to the weight of an equal volume of water. It may also be expressed as a ratio of its density or specific weight to that of water.

The normal force pushing against a plane area, divided by the area, is the average *pressure*. The pressure at a point is the ratio of normal force to area as the area approaches a small value inclosing the point. Pressure has the units force/area and may be pounds per square inch or pounds per square foot. Pressure may also be expressed in terms of an equivalent length of a fluid column, as shown in Sec. 2.3.

1.6. Perfect Gas. In this treatment, thermodynamic relationships and compressible-fluid-flow cases have been limited generally to perfect gases. The perfect gas is defined in this section, and its various interrelationships with specific heats are treated in Sec. 6.1.

The perfect gas, as used herein, is defined as a substance that satisfies the *perfect-gas law*

$$pv_s = RT \tag{1.6.1}$$

and that has constant specific heats. p is the absolute pressure, v_s the specific volume, R the gas constant, and T the absolute temperature. The perfect gas must be carefully distinguished from the ideal fluid. An ideal fluid is frictionless and incompressible. The perfect gas has viscosity and can therefore develop shear stresses, and it is compressible according to Eq. (1.6.1).

Equation (1.6.1) is the equation of state for a perfect gas. It may be written

$$p = \rho RT \tag{1.6.2}$$

The units of R may be determined from the equation when the other units are known. For p in pounds per square foot, ρ in slugs per cubic foot, and T (°F + 459.6) in degrees Rankine (°R),

$$R: \frac{\text{lb}}{\text{ft}^2} \frac{\text{ft}^3}{\text{slug } ^\circ\text{R}} = \frac{\text{ft-lb}}{\text{slug } ^\circ\text{R}}$$

For ρ in pounds mass per cubic foot,

$$R: \frac{\text{lb}}{\text{ft}^2} \frac{\text{ft}^3}{\text{lb}_m\ ^\circ\text{R}} = \frac{\text{ft-lb}}{\text{lb}_m\ ^\circ\text{R}}$$

The magnitude of R in slug units is 32.174 times greater than in pound mass units. Values of R for several common gases are given in Table C.2.

Real gases at low pressure tend to obey the perfect-gas law. As the pressure increases, the discrepancy increases and becomes serious near the critical point. The perfect-gas law encompasses both Charles' law and Boyle's law. Charles' law states that for constant pressure the volume of a given mass of gas varies as its absolute temperature. Boyle's law (isothermal law) states that for constant temperature the density varies directly as the absolute pressure. The volume $V\!\!\!\!-$ of m mass units of gas is mv_s; hence

$$pV\!\!\!\!- = mRT \tag{1.6.3}$$

Certain simplifications result from writing the perfect-gas law on a mole basis. A pound-mole of gas is the number of pounds mass of gas equal to its molecular weight; e.g., a pound-mole of oxygen O_2 is 32 lb_m. With $\bar{v}_s$ the volume per mole, the perfect-gas law becomes

$$p\bar{v}_s = MRT \tag{1.6.4}$$

if M is the molecular weight. In general, if n is the number of moles of the gas in volume $V\!\!\!\!-$,

$$pV\!\!\!\!- = nMRT \tag{1.6.5}$$

since $nM = m$. Now, from Avogadro's law, equal volumes of gases at the same absolute temperature and pressure have the same number of molecules; hence their masses are proportional to the molecular weights. From Eq. (1.6.5) it is seen that MR must be constant, since $pV\!\!\!\!-/nT$ is the same for any perfect gas. The product MR is called the universal gas constant and has a value depending only upon the units employed. It is

$$MR = 1545 \frac{\text{ft-lb}}{\text{lb}_m\text{-mole °R}} \tag{1.6.6}$$

The gas constant R can then be determined from

$$R = \frac{1545}{M} \frac{\text{ft-lb}}{\text{lb}_m \text{ °R}} \tag{1.6.7}$$

or in slug units,

$$R = \frac{1545 \times 32.174}{M} \frac{\text{ft-lb}}{\text{slug °R}} \tag{1.6.8}$$

so that knowledge of molecular weight leads to the value of R. In Table C.2 of Appendix C molecular weights of some common gases are listed.

Additional relationships and definitions used in perfect-gas flow are introduced in Chaps. 3 and 6.

Example 1.2: A gas with molecular weight of 44 is at a pressure of 13.0 psia (pounds per square inch absolute) and a temperature of 60°F. Determine its density in slugs per cubic foot.

From Eq. (1.6.8)

$$R = \frac{1545 \times 32.174}{44} = 1129 \frac{\text{ft-lb}}{\text{slug °R}}$$

Then from Eq. (1.6.2)

$$\rho = \frac{p}{RT} = \frac{13.0 \times 144}{1129(460 + 60)} = 0.00319 \text{ slug/ft}^3$$

1.7. Bulk Modulus of Elasticity. In the preceding section the compressibility of a perfect gas is described by the perfect-gas law. For most purposes a liquid may be considered as incompressible, but for situations involving either sudden or great changes in pressure, its compressibility becomes important. Liquid compressibility (and gas also) becomes important also when temperature changes are involved (e.g., free convection). The compressibility of a liquid is expressed by its *bulk modulus of elasticity*. If the pressure of a unit volume of liquid is increased by dp, it will cause a volume decrease $-d\mathcal{V}$; the ratio $-dp/d\mathcal{V}$ is the bulk modulus of elasticity K. For any volume $\mathcal{V}$ of liquid

$$K = -\frac{dp}{d\mathcal{V}/\mathcal{V}} \tag{1.7.1}$$

Since $d\mathcal{V}/\mathcal{V}$ is dimensionless, K is expressed in the units of p. For water at ordinary temperatures and pressures $K = 300{,}000$ psi.

To gain some idea about the compressibility of water, consider the application of 100 psi pressure to 1 ft³ of water. When Eq. (1.7.1) is solved for $-d\mathcal{V}$,

$$-d\mathcal{V} = \frac{\mathcal{V}\,dp}{K} = \frac{1.0 \times 100}{300{,}000} = \frac{1}{3000} \text{ ft}^3$$

Hence, the application of 100 psi to water under ordinary conditions causes its volume to decrease by only 1 part in 3000. As a liquid is compressed, the resistance to further compression increases; therefore K increases with pressure. At 45,000 psi the value of K for water has doubled.

Example 1.3: A liquid compressed in a cylinder has a volume of 0.400 ft³ at 1000 psi and a volume of 0.396 ft³ at 2000 psi. What is its bulk modulus of elasticity?

$$K = -\frac{\Delta p}{\Delta\mathcal{V}/\mathcal{V}} = -\frac{2000 - 1000}{(0.396 - 0.400)/0.400} = 100{,}000 \text{ psi}$$

1.8. Vapor Pressure. Liquids evaporate because of molecules escaping from the liquid surface. The vapor molecules exert a partial pressure in the space, known as *vapor pressure*. If the space above the liquid is confined, after a sufficient time the number of vapor molecules striking the liquid surface and condensing are just equal to the number escaping in any interval of time, and equilibrium exists. Since this phenomenon depends upon molecular activity, which is a function of temperature, the vapor pressure of a given fluid depends upon temperature and increases with it. When the pressure above a liquid equals the vapor pressure of the liquid, boiling occurs. Boiling of water, for example, may occur at room temperature if the pressure is reduced sufficiently. At 68°F water has a vapor pressure of 0.339 psi, and mercury has a vapor pressure of 0.0000251 psi.

1.9. Surface Tension. Capillarity. At the interface between a liquid and a gas, a *film*, or special layer, seems to form on the liquid, apparently owing to the attraction of liquid molecules below the surface. It is a simple experiment to place a small needle on a quiet water surface and observe that it will be supported there by the film.

That property of the surface film to exert a tension is called the *surface tension* and is the force required to maintain unit length of the film in equilibrium. The surface tension of water varies from about 0.005 lb/ft at 68°F to 0.004 lb/ft at 212°F. Surface tensions of other liquids are given in Table 1.1.

TABLE 1.1. SURFACE TENSION OF COMMON LIQUIDS IN CONTACT WITH AIR AT 68°F

Liquid	*Surface tension,* σ, *lb/ft*
Alcohol, ethyl	0.00153
Benzene	0.00198
Carbon tetrachloride	0.00183
Kerosene	0.0016 to 0.0022
Water	0.00498
Mercury	
In air	0.0352
In water	0.0269
In vacuum	0.0333
Oil	
Lubricating	0.0024 to 0.0026
Crude	0.0016 to 0.0026

The action of surface tension is to increase the pressure within a droplet of liquid or within a small liquid jet. For a small spherical droplet of radius r the internal pressure p necessary to balance the tensile force due to the surface tension σ is calculated in terms of the forces which act on a hemispherical free body,[1]

$$p\pi r^2 = 2\pi r\sigma$$

or

$$p = \frac{2\sigma}{r}$$

[1] See Sec. 2.7.

For the cylindrical liquid jet of radius r, the pipe-tension equation applies,

$$p = \frac{\sigma}{r}$$

Both equations show that the pressure becomes large for a very small radius of droplet or cylinder.

Capillary attraction is caused by surface tension and by the relative value of adhesion between liquid and solid to cohesion of the liquid. A liquid that *wets* the solid has a greater adhesion than cohesion. The action of surface tension in this case is to cause the liquid to rise within a small vertical tube that is partially immersed in it. For liquids that do not wet the solid, surface tension tends to depress the meniscus in a small vertical tube. To avoid a correction for the effects of capillarity in

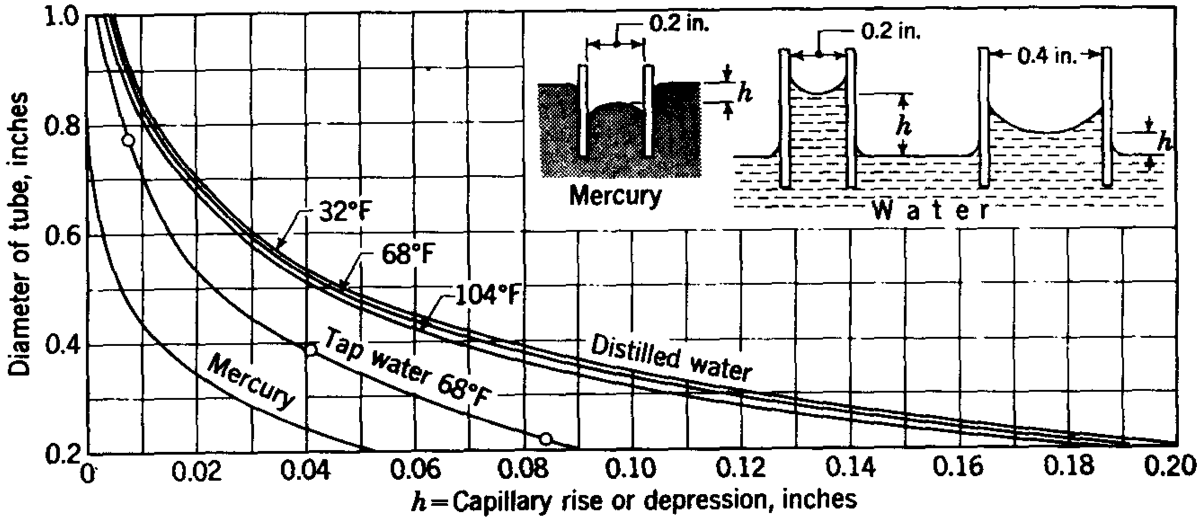


FIG. 1.4. Capillarity in circular glass tubes. (*By permission from "Hydraulics," by R. L. Daugherty, copyright* 1944, *McGraw-Hill Book Company, Inc.*)

manometers, a tube $\frac{1}{2}$ in. in diameter or larger should be used. When the contact angle between liquid and solid is known, the capillary rise may be computed for an assumed shape of the meniscus. Figure 1.4 shows the capillary rise for water and mercury in circular glass tubes in air.

PROBLEMS

1.1. Classify the substance that has the following rates of deformation and corresponding shear stresses:

du/dy, rad/sec........	0	1	3	5
τ, lb/ft^2..............	20	40	60	80

1.2. A Newtonian fluid is in the clearance between a shaft and a concentric sleeve. When a force of 100 lb is applied to the sleeve parallel to the shaft, the sleeve attains a speed of 2 ft/sec. If 500-lb force is applied, what speed will the sleeve attain? The temperature of the sleeve remains constant.

1.3. Classify the following substances (maintained at constant temperature):

(*a*) du/dy, rad/sec..	0	3	4	6	5	4
τ, lb/ft^2........	2	4	6	8	6	4

(*b*) du/dy, rad/sec..	0	0.5	1.1	1.8
τ, lb/ft^2........	0	2	4	6

(*c*) du/dy, rad/sec..	0	0.3	0.6	0.9	1.2
τ, lb/ft^2........	0	2	4	6	8

1.4. Determine the weight in pounds of 2 slugs mass at a location where $g = 31.7$ ft/sec^2.

1.5. When standard scale weights and a balance are used, a body is found to be equivalent in pull of gravity to two of the 1-lb scale weights at a location where $g = 31.5$ ft/sec^2. What would the body weigh on a correctly calibrated spring balance (for sea level) at this location?

1.6. Determine the value of proportionality constant g_0 needed for the following set of units: kip (1000 lb), slug, foot, second.

1.7. On another planet where standard gravity is 10 ft/sec^2, what would be the value of the proportionality constant g_0 in terms of the pound, pounds mass, foot, and second?

1.8. A correctly calibrated spring scale records the weight of a 51-lb$_m$ body as 17.0 lb at a location away from the earth. What is the value of g at this location?

1.9. A shear stress of 3 dynes/cm^2 causes a Newtonian fluid to have an angular deformation of 1 rad/sec. What is its viscosity in centipoises?

1.10. A plate, 0.001 in. distant from a fixed plate, moves at 2 ft/sec and requires a force of 0.04 lb/ft^2 to maintain this speed. Determine the fluid viscosity of the substance between the plates, in English units.

1.11. A 3.0-in.-diameter shaft slides at 0.4 ft/sec through a 6-in.-long sleeve with radial clearance of 0.01 in. (Fig. 1.5) when a 10.0-lb force is applied. Determine the viscosity of fluid between shaft and sleeve.

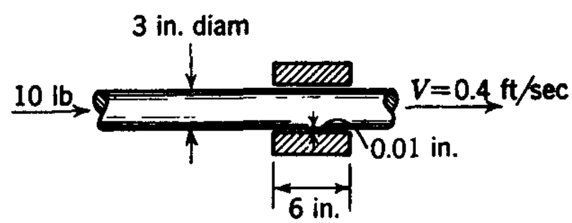


FIG. 1.5

1.12. A flywheel weighing 100 lb has a radius of gyration of 1 ft. When it is rotating 600 rpm, its speed reduces 1 rpm/sec owing to fluid viscosity between sleeve and shaft. The sleeve length is 2.0 in., shaft diameter 1.0 in., and radial clearance 0.002 in. Determine the fluid viscosity.

1.13. A fluid has a viscosity of 6 centipoises and a density of 50 lb$_m$/ft^3. Determine its kinematic viscosity in English units and in stokes.

1.14. A fluid has a specific gravity of 0.83 and a kinematic viscosity of 2 stokes. What is its viscosity in English units and in poises?

1.15. A body weighing 90 lb with a flat surface area of 1 ft^2 slides down a lubricated inclined plane making a 30° angle with the horizontal. For viscosity of 1 poise and body speed of 10 ft/sec, determine the lubricant film thickness.

1.16. What is the viscosity of gasoline at 100°F in poises?

1.17. Determine the kinematic viscosity of benzene at 60°F in stokes.

1.18. How much greater is the viscosity of water at 32°F than at 200°F? How much greater is its kinematic viscosity for the same temperature range?

1.19. What is the specific volume in cubic feet per pound mass and cubic feet per slug of a substance of specific gravity 0.75?

1.20. What is the relation between specific volume and specific weight?

1.21. The density of a substance is 2.94 gm/cm^3. What is its (*a*) specific gravity, (*b*) specific volume, and (*c*) specific weight?

1.22. A gas at 60°F and 20 psia has a volume of 4.0 ft^3 and a gas constant $R = 48$ ft-lb/lb_m °R. Determine the density and number of slugs of gas.

1.23. What is the specific weight of air at 40 psia and 120°F?

1.24. What is the density of water vapor at 6 psia and 48°F, in slugs per cubic foot?

1.25. A gas with molecular weight 48 has a volume of 4.0 ft^3 and a pressure and temperature of 2000 psfa and 600°R, respectively. What is its specific volume and specific weight?

1.26. 2.0 lb_m of hydrogen is confined in a volume of 1 ft^3 at −40°F. What is the pressure?

1.27. Express the bulk modulus of elasticity in terms of density change rather than volume change.

1.28. For constant bulk modulus of elasticity, how does the density of a liquid vary with the pressure?

1.29. What is the bulk modulus of a liquid that has a density increase of 0.01 per cent for a pressure increase of 1000 lb/ft^2?

1.30. For $K = 300{,}000$ psi for bulk modulus of elasticity of water what pressure is required to reduce its volume by 1 per cent?

1.31. A steel container expands in volume 1 per cent when the pressure within it is increased by 10,000 psi. At standard pressure, 14.7 psia, it holds 1000 lb_m water $\rho = 62.4\ lb_m/ft^3$. For $K = 300{,}000$ psi, when it is filled, how many pounds mass water need be added to increase the pressure to 10,000 psi?

1.32. What is the pressure within a droplet of water 0.002 in. in diameter at 68°F if the pressure outside the droplet is standard atmospheric pressure of 14.7 psi?

1.33. A small circular jet of mercury 0.002 in. in diameter issues from an opening. What is the pressure difference between the inside and outside of the jet when at 68°F?

1.34. Determine the capillary rise for distilled water at 32°F in a circular glass tube $\frac{1}{4}$ in. in diameter.

1.35. A fluid is a substance that

(*a*) always expands until it fills any container

(*b*) is practically incompressible
(*c*) cannot be subjected to shear forces
(*d*) cannot remain at rest under action of any shear force
(*e*) has the same shear stress at a point regardless of its motion

1.36. A 2.0-lb_m object weighs 1.90 lb on a spring balance. The value of g at this location is, in feet per second per second,

(*a*) 30.56 (*b*) 32.07 (*c*) 32.17 (*d*) 33.87 (*e*) none of these answers

1.37. At a location where $g = 30.00$ ft/sec^2, 2.0 slugs is equivalent to how many pounds mass?

(*a*) 60.0 (*b*) 62.4 (*c*) 64.35 (*d*) not equivalent units
(*e*) none of these answers

1.38. The weight, in pounds, of 3 slugs on a planet where $g = 10.00$ ft/sec^2 is

(*a*) 0.30 (*b*) 0.932 (*c*) 30.00 (*d*) 96.53 (*e*) none of these answers

1.39. Newton's law of viscosity relates

(*a*) pressure, velocity, and viscosity
(*b*) shear stress and rate of angular deformation in a fluid
(*c*) shear stress, temperature, viscosity, and velocity
(*d*) pressure, viscosity, and rate of angular deformation
(*e*) yield shear stress, rate of angular deformation, and viscosity

1.40. Viscosity has the dimensions

(*a*) $FL^{-2}T$ (*b*) $FL^{-1}T^{-1}$ (*c*) FLT^{-2} (*d*) FL^2T (*e*) FLT^2

1.41. Select the *incorrect* completion. Apparent shear forces

(*a*) can never occur when the fluid is at rest
(*b*) may occur owing to cohesion when the liquid is at rest
(*c*) depend upon molecular interchange of momentum
(*d*) depend upon cohesive forces
(*e*) can never occur in a frictionless fluid, regardless of its motion

1.42. Correct units for dynamic viscosity are

(*a*) dyne-sec^2/cm (*b*) gm/cm-sec^2 (*c*) gm-sec/cm (*d*) dyne-cm/sec^2 (*e*) dyne-sec/cm^2

1.43. Viscosity, expressed in poise, is converted to the English unit of viscosity by multiplication by

(*a*) $\frac{1}{479}$ (*b*) 479 (*c*) ρ (*d*) $1/\rho$ (*e*) none of these answers

1.44. The dimensions for kinematic viscosity are

(*a*) $FL^{-2}T$ (*b*) $ML^{-1}T^{-1}$ (*c*) L^2T^2 (*d*) L^2T^{-1} (*e*) L^2T^{-2}

1.45. In converting from the English unit of kinematic viscosity to the stoke, one multiplies by

(*a*) $\frac{1}{479}$ (*b*) $1/(30.48)^2$ (*c*) 479 (*d*) $(30.48)^2$ (*e*) none of these answers

1.46. The kinematic viscosity of kerosene at 90°F is, in square feet per second,

(*a*) 2×10^{-5} (*b*) 3.2×10^{-5} (*c*) 2×10^{-4} (*d*) 3.2×10^{-4}
(*e*) none of these answers

1.47. The kinematic viscosity of dry air at 25°F and 29.4 psia is, in square feet per second,

(*a*) 6.89×10^{-5} (*b*) 1.4×10^{-4} (*c*) 6.89×10^{-4} (*d*) 1.4×10^{-3}
(*e*) none of these answers

1.48. For $\mu = 0.60$ poise, sp gr $= 0.60$, ν, in stokes, is

(*a*) 2.78 (*b*) 1.0 (*c*) 0.60 (*d*) 0.36 (*e*) none of these answers

1.49. For $\mu = 2.0 \times 10^{-4}$ slug/ft-sec, the value of μ in pound-seconds per square foot is

(*a*) 1.03×10^{-4} (*b*) 2.0×10^{-4} (*c*) 6.21×10^{-4} (*d*) 6.44×10^{-3}
(*e*) none of these answers

1.50. For $\nu = 3 \times 10^{-4}$ stoke and $\rho = 0.8$ gm/cm³, μ, in slugs per foot-second, is

(*a*) 5.02×10^{-7} (*b*) 6.28×10^{-7} (*c*) 7.85×10^{-7} (*d*) 1.62×10^{-6}
(*e*) none of these answers

1.51. A perfect gas

(*a*) has zero viscosity (*b*) has constant viscosity
(*c*) is incompressible (*d*) satisfies $p\rho = RT$
(*e*) fits none of these statements

1.52. The molecular weight of a gas is 28. The value of R in foot-pounds per slug degree Rankine is

(*a*) 53.3 (*b*) 55.2 (*c*) 1545 (*d*) 1775 (*e*) none of these answers

1.53. The density of air at 40°F and 100 psia in slugs per cubic foot is

(*a*) 0.00017 (*b*) 0.0168 (*c*) 0.21 (*d*) 0.54 (*e*) none of these answers

1.54. How many pounds mass of carbon monoxide gas at 20°F and 30 psia is contained in a volume of 4.0 ft³?

(*a*) 0.00453 (*b*) 0.0203 (*c*) 0.652 (*d*) 2.175 (*e*) none of these answers

1.55. A container holds 2.0 lb_m air at 120°F and 120 psia. If 3.0 lb_m air is added and the final temperature is 240°F, the final pressure, in pounds per square inch absolute, is

(*a*) 300 (*b*) 362.2 (*c*) 600 (*d*) indeterminable (*e*) none of these answers

1.56. The bulk modulus of elasticity K for a gas at constant temperature T_0 is given by

(*a*) p/ρ (*b*) RT_0 (*c*) ρp (*d*) ρRT_0 (*e*) none of these answers

1.57. The bulk modulus of elasticity

(*a*) is independent of temperature
(*b*) increases with the pressure
(*c*) has the dimensions of $1/p$
(*d*) is larger when the fluid is more compressible
(*e*) is independent of pressure and viscosity

1.58. For 1000-psi increase in pressure the density of water has increased, in per cent, by about

(*a*) $\frac{1}{300}$ (*b*) $\frac{1}{30}$ (*c*) $\frac{1}{3}$ (*d*) $\frac{1}{2}$ (*e*) none of these answers

1.59. A pressure of 150 psi applied to 10 ft³ liquid causes a volume reduction of 0.02 ft³. The bulk modulus of elasticity, in pounds per square inch, is

(*a*) -750 (*b*) 750 (*c*) 7500 (*d*) 75,000 (*e*) none of these answers

1.60. Surface tension has the dimensions

(*a*) F (*b*) FL^{-1} (*c*) FL^{-2} (*d*) FL^{-3} (*e*) none of these answers

2

FLUID STATICS

The science of fluid statics will be treated in two parts: the study of pressure and its variation throughout a fluid and the study of pressure forces on finite surfaces. Special cases of fluids moving as solids are included in the treatment of statics because of the similarity of forces involved. Since there is no motion of a fluid layer relative to an adjacent layer, there are no shear stresses in the fluid. Hence, all free bodies in fluid statics have only normal pressure forces acting on them.

2.1. Pressure at a Point. The average pressure is calculated by dividing the normal force pushing against a plane area by the area. The pressure at a point is the limit of the ratio of normal force to area as the area approaches zero size at the point. At a point a fluid at rest has the same pressure in all directions. This means that an element δA of a very small area, free to rotate about its center when submerged in a fluid at rest, will have a force of constant magnitude acting on either side of it, regardless of its orientation.

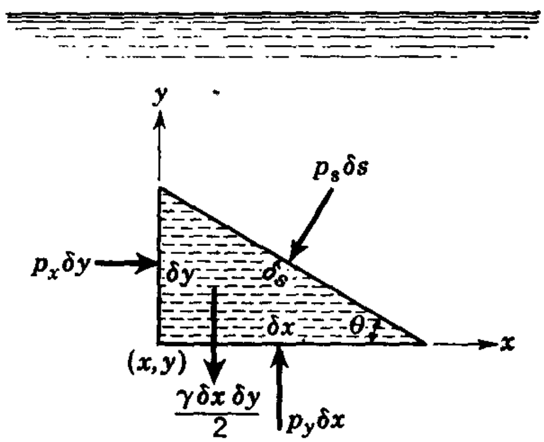


FIG. 2.1. Free-body diagram of wedge-shaped particle.

To demonstrate this, a small wedge-shaped free body of unit length is taken at the point (x,y) in a fluid at rest (Fig. 2.1). Since there can be no shear forces, the only forces are the normal surface forces and gravity. So, the equations of equilibrium in the x- and y-directions are, respectively,

$$p_x\,\delta y - p_s\,\delta s \sin\theta = 0$$

$$p_y\,\delta x - p_s\,\delta s \cos\theta - \gamma\frac{\delta x\,\delta y}{2} = 0$$

in which p_x, p_y, p_s are the average pressures on the three faces and γ is the specific weight of the fluid. Taking the limit as the free body is reduced to zero size, by allowing the inclined face to approach (x,y) maintaining the same angle θ, and using the geometric relations

$$\delta s \sin \theta = \delta y \qquad \delta s \cos \theta = \delta x$$

the equations simplify to

$$p_x \, \delta y - p_s \, \delta y = 0 \qquad p_y \, \delta x - p_s \, \delta x - \gamma \frac{\delta x \, \delta y}{2} = 0$$

The last term of the second equation is an infinitesimal of higher order of smallness and may be neglected. When divided by δy and δx, respectively, the equations may be combined,

$$p_s = p_x = p_y \tag{2.1.1}$$

Since θ is any arbitrary angle, this equation proves that the pressure is the same in all directions at a point in a static fluid. Although the proof was carried out for a two-dimensional case, it may be demonstrated for the three-dimensional case with the equilibrium equations for a small tetrahedron of fluid with three faces in the coordinate planes and the fourth face inclined arbitrarily.

If the fluid is in motion so that one layer moves relative to an adjacent layer, shear stresses occur and the normal stresses are, in general, no longer the same in all directions at a point. The pressure is then defined as the average of any three mutually perpendicular normal compressive stresses at a point,

$$p = \frac{p_x + p_y + p_z}{3}$$

In a fictitious fluid of zero viscosity, i.e., a frictionless fluid, no shear stresses can occur for any motion of the fluid, so at a point the pressure is the same in all directions.

2.2. Pressure Variations in a Static Fluid. The laws of variation of pressure in a static fluid may be developed by considering variations along a horizontal line and variations along a vertical line.

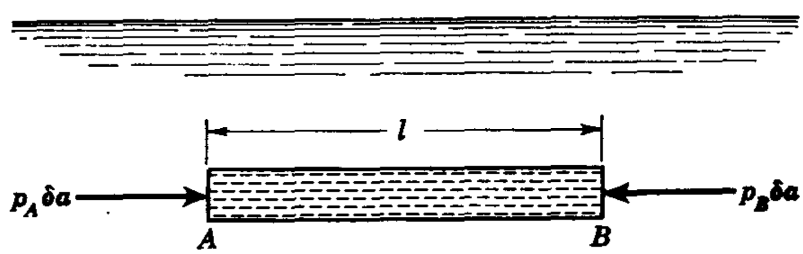


FIG. 2.2. Two points at same elevation in a static fluid.

Two points, A and B, in Fig. 2.2, are in a horizontal plane. On a cylindrical free body, with axis through the points and end areas normal to the axis and through the respective points, the only forces acting in an axial direction are $p_A\ \delta a$ and $p_B\ \delta a$, in which δa is the cross-sectional area of the cylinder. Therefore $p_A = p_B$, which proves that *two points in the same horizontal plane in a continuous mass of fluid at rest have the same pressure.* Although the proof was for two points that could be connected by a straight line through the fluid, it may be extended to such

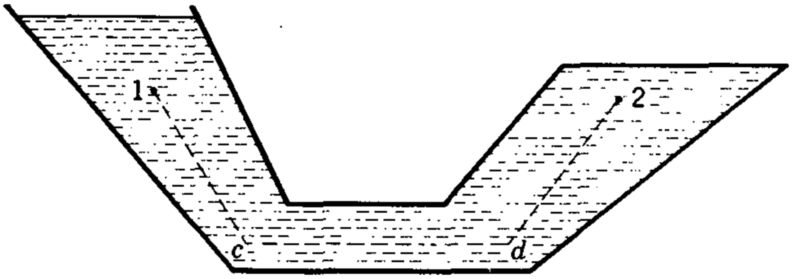


FIG. 2.3. Paths for considering variation of pressure in a fluid.

situations as points 1 and 2 in Fig. 2.3, when the variation of pressure in a vertical line is considered.

Basic Equation of Hydrostatics. Pressure Variation in an Incompressible Fluid. As there is no variation of pressure in a horizontal direction, the variation must occur in the vertical direction. Consider a free body of fluid (Fig. 2.4) consisting of a prism of cross-sectional area A, with axis vertical and height δy. The base is at elevation y from an arbitrary datum. The pressure at y is p and at $y + \delta y$ it is $p + (dp/dy)\delta y$. The weight of the free body is $\gamma A\ \delta y$, where γ is the specific weight of fluid at elevation y. Since no shear forces exist, the three forces shown in Fig. 2.4 must be in equilibrium, so

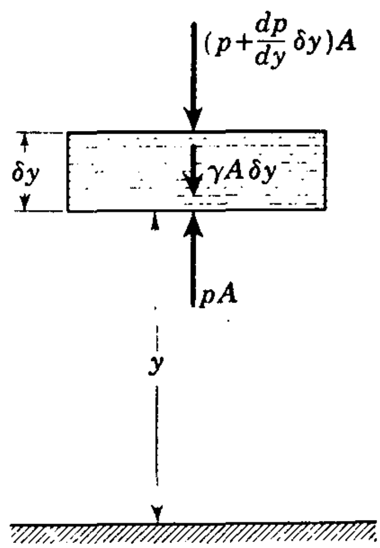


FIG. 2.4. Free-body diagram for vertical forces acting on a fluid element.

$$pA - \left(p + \frac{dp}{dy}\,\delta y\right)A - \gamma A\ \delta y = 0$$

When the equation is simplified and divided by the volume, $A\ \delta y$, as δy becomes very small,

$$dp = -\gamma\, dy \qquad (2.2.1)$$

This simple differential equation relates the change of pressure to specific weight and change of elevation, and holds for both compressible and incompressible fluids.

For fluids that may be considered incompressible, γ is constant, and Eq. (2.2.1), when integrated, becomes

$$p = -\gamma y + c$$

in which c is the constant of integration. The hydrostatic law of variation of pressure is frequently written in the form

$$p = \gamma h \tag{2.2.2}$$

in which h is measured vertically downward ($h = -y$) from a free liquid surface and p is the increase in pressure from that at the free surface. Equation (2.2.2) may be derived by taking as fluid free body a vertical column of liquid of finite height h with its upper surface in the free surface. This is left as an exercise for the student.

Example 2.1: An open tank contains 2 ft of water covered with 1 ft of oil, sp gr 0.83. Find the pressure at the interface and at the bottom of the tank.

At the interface, $h = 1$, $\gamma = 0.83 \times 62.4 = 51.7$ lb/ft³, and

$$p = \gamma h = 51.7 \text{ lb/ft}^2$$

At the bottom of the tank the pressure is that at the interface plus γh for the water, or

$$p = 51.7 + 2 \times 62.4 = 176.5 \text{ lb/ft}^2$$

Pressure Variation in a Compressible Fluid. When the fluid is a perfect gas at rest at constant temperature, from Eq. (1.6.2)

$$\frac{p}{\rho} = \frac{p_0}{\rho_0} \tag{2.2.3}$$

When the value of γ in Eq. (2.2.1) is replaced by ρg and ρ is eliminated between Eqs. (2.2.1) and (2.2.3),

$$dy = \frac{-p_0}{g\rho_0}\frac{dp}{p} \tag{2.2.4}$$

It must be remembered that if ρ is in pounds mass per cubic foot, then $\gamma = g\rho/g_0$ with $g_0 = 32.174$ lb$_m$-ft/lb-sec². If $p = p_0$ when $\rho = \rho_0$, integration between limits

$$\int_{y_0}^{y} dy = -\frac{p_0}{g\rho_0}\int_{p_0}^{p}\frac{dp}{p}$$

yields

$$y - y_0 = -\frac{p_0}{g\rho_0}\ln\frac{p}{p_0} \tag{2.2.5}$$

in which ln is the natural logarithm. Then

$$p = p_0 e^{-(y-y_0)/(p_0/g\rho_0)} \tag{2.2.6}$$

which is the equation for variation of pressure with elevation in an isothermal gas.

The atmosphere frequently is assumed to have a constant temperature gradient, expressed by

$$T = T_0 + \beta y \tag{2.2.7}$$

For the standard atmosphere $\beta = -0.00357°F/ft$ up to the stratosphere. The density may be expressed in terms of pressure and elevation from the perfect-gas law:

$$\rho = \frac{p}{RT} = \frac{p}{R(T_0 + \beta y)} \tag{2.2.8}$$

Substitution into $dp = -\rho g\, dy$ [Eq. (2.2.1)] permits the variables to be separated and p to be found in terms of y by integration.

Example 2.2: Assuming isothermal conditions to prevail in the atmosphere, compute the pressure and density at 5000 ft elevation if $p = 14.7$ psia, $\rho = 0.00238$ slug/ft³ at sea level.

From Eq. (2.2.6)

$$p = 14.7e^{-5000/(14.7\times 144/32.2\times 0.00238)} = 12.27 \text{ psia}$$

Then, from Eq. (2.2.3)

$$\rho = \frac{\rho_0}{p_0}p = \frac{0.00238}{14.7}12.27 = 0.00199 \text{ slug/ft}^3$$

When compressibility of a liquid in static equilibrium is taken into account, Eqs. (2.2.1) and (1.7.1) are utilized.

2.3. Units and Scales of Pressure Measurement. Pressure may be expressed with reference to any arbitrary datum. The usual ones are *absolute zero* and *local atmospheric pressure.* When a pressure is expressed as a difference between its value and a complete vacuum, it is called an *absolute pressure.* When it is expressed as a difference between its value and the local atmospheric pressure, it is called a *gage pressure.*

The *bourdon* gage (Fig. 2.5) is typical of the devices used for measuring gage pressures. The pressure element is a hollow, curved, flat, metallic tube, closed at one end, with the other end connected to the pressure to be measured. When the internal pressure is increased, the tube tends to straighten, pulling on a linkage to which is attached a pointer and causing the pointer to move. The dial reads zero when the inside and outside of the tube are at the same pressure, regardless of its particular value. The dial may be graduated to any convenient units, common ones being pounds per square inch, pounds per square foot, inches of mercury, and feet of water. Owing to the inherent construction of the gage, it measures pressure relative to the pressure of the medium surrounding the tube, which is the local atmosphere.

Figure 2.6 illustrates the data and the relationships of the common units of pressure measurement. Standard atmospheric pressure is the mean pressure at sea level, 29.92 in. mercury (rounded to 30 in. for slide-rule work). A pressure expressed in terms of a column of liquid refers to the force per unit area at the base of the column. The relation for variation of pressure with altitude in a liquid [Eq. (2.2.2)]

$$p = \gamma h \tag{2.3.1}$$

shows the relation between head h, in length of a fluid column of specific weight γ, and the pressure p. In consistent units, p is in pounds per

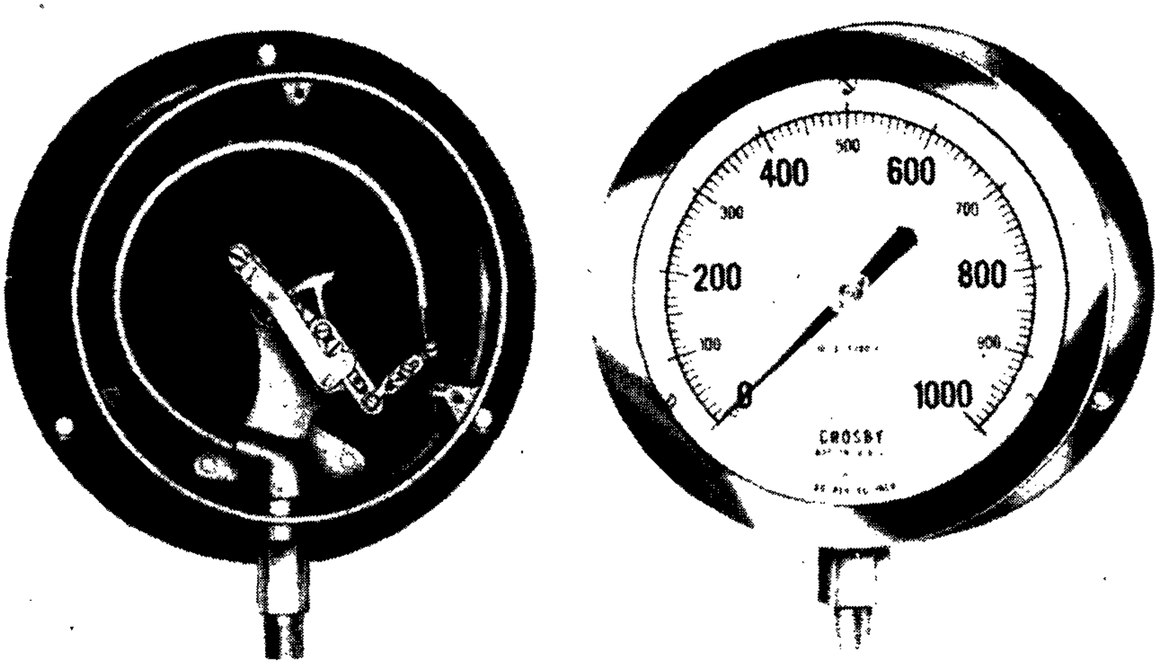


FIG. 2.5. Bourdon gage. (*Crosby Steam Gage and Valve Co.*)

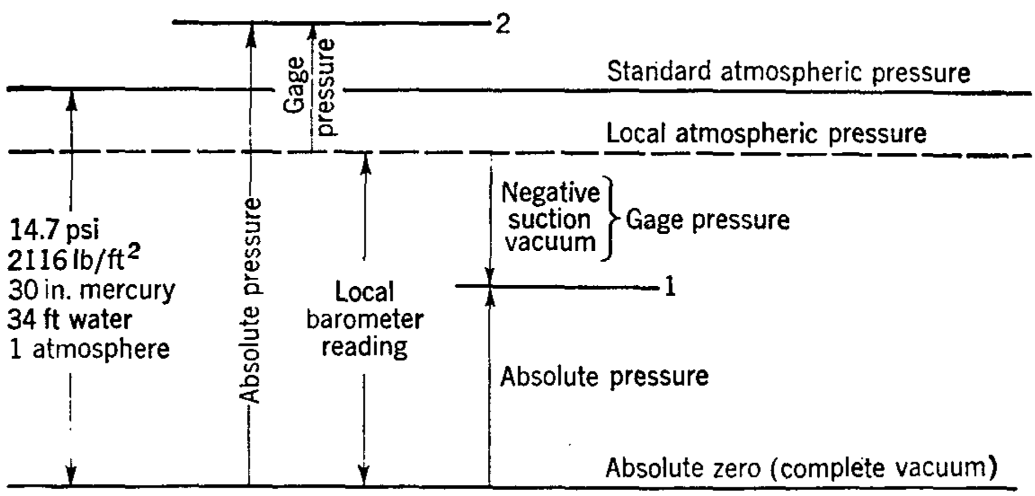


FIG. 2.6. Units and scales for pressure measurement.

square foot, γ in pounds per cubic foot, and h in feet. For water γ may be taken as 62.4 lb/ft³. With the specific weight of any liquid expressed as its specific gravity S times the specific weight of water, Eq. (2.3.1) becomes

$$p = 62.4Sh$$

When the pressure is desired in pounds per square inch, both sides of the

equation are divided by 144,

$$p_{\text{psi}} = \frac{62.4}{144} Sh = 0.433Sh \qquad (2.3.2)$$

in which h remains in feet.[1]

Local atmospheric pressure is measured by a mercury barometer (Fig. 2.7) or by an *aneroid* barometer which measures the difference in pressure between the atmosphere and an evacuated box or tube, in a manner analogous to the bourdon gage except that the tube is evacuated and sealed.

A mercury barometer consists of a glass tube sealed at one end, filled with mercury, and inverted so that the open end is submerged in mercury. It has a scale arranged so that the height of column R (Fig. 2.7) can be determined. The space above the mercury contains mercury vapor. If the pressure of the mercury vapor, h_v, is given in inches of mercury, the pressure at A may be expressed as

$$h_v + R'' = h_A \qquad \text{in. mercury}$$

Although h_v is a function of temperature, it is very small at usual atmospheric temperatures. The barometric pressure varies with location, i.e., elevation, and with weather conditions.

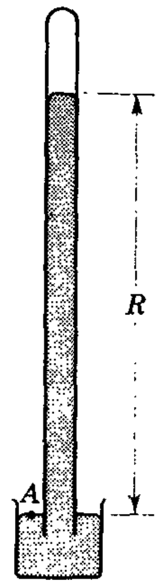


Fig. 2.7. Mercury barometer.

In Fig. 2.6 a pressure may be located vertically on the chart, which indicates its relation to absolute zero and to local atmospheric pressure. If the point is below the local-atmospheric-pressure line and is referred to gage datum, it is called *negative, suction,* or *vacuum.* For example, the pressure 18 in. mercury abs, as at 1, with barometer reading 29 in., may be expressed as -11 in. mercury, 11 in. mercury suction, or 11 in. mercury

[1] In Eq. (2.3.2) the standard atmospheric pressure may be expressed in pounds per square inch,

$$p_{\text{psi}} = 0.433 \times 13.6 \times \tfrac{30}{12} = 14.7$$

when $S = 13.6$ for mercury. When 14.7 is multiplied by 144, the standard atmosphere becomes 2116 lb/ft^2. Then 2116 divided by 62.4 yields 34 ft water. Any of these designations is for the standard atmosphere and may be called *one atmosphere,* if it is always understood that it is a standard atmosphere and is measured from absolute zero. These various designations of a standard atmosphere (Fig. 2.6) are equivalent and provide a convenient means of converting from one set of units to another. For example, to express 100 ft of water in pounds per square inch

$$\tfrac{100}{34}\ 14.7 = 43.3 \text{ psi}$$

since $\frac{100}{34}$ is the number of standard atmospheres and each standard atmosphere is 14.7 psi.

vacuum. It should be noted that

$$p_{\text{abs}} = p_{\text{bar}} + p_{\text{gage}}$$

To avoid any confusion, the convention is adopted throughout this text that a *pressure is gage unless specifically marked absolute,* with the exception of the *atmosphere,* which is an absolute pressure unit.

Example 2.3: Express 4 psi eight other customary ways. Barometer reading 28.5. in. mercury.

At point 2 in Fig. 2.6, other customary gage units are

1. $4 \times 144 = 576$ lb/ft²
2. $\dfrac{4}{14.7} \times 30 = 8.16$ in. mercury
3. $\dfrac{4}{14.7} \times 34 = 9.25$ ft water

With absolute units,

4. From 2, $8.16 + 28.5 = 36.66$ in. mercury abs
5. From 4, $\dfrac{36.66}{30} = 1.222$ atm
6. From 5, $1.222 \times 14.7 = 18.0$ psia
7. From 5, $1.222 \times 2116 = 2583$ lb/ft² abs
8. From 5, $1.222 \times 34 = 41.6$ ft water abs

The pressure conversion chart in Fig. 2.6 is most useful in working with pressure units and should be carefully studied.

2.4. Manometers. Manometers are devices that employ liquid columns for determining differences in pressure. The most elementary manometer, usually called a *piezometer,* is illustrated in Fig. 2.8*a*; it measures the pressure in a liquid when it is above zero gage. A glass tube is mounted vertically so that it is connected to the space within the container. Liquid rises in the tube until equilibrium is reached. The pressure is then given by the vertical distance h from the meniscus (liquid surface) to the point where the pressure is to be measured, expressed in feet of the liquid in the container. It is obvious that the piezometer would not work for negative gage pressures, because air would flow into the container through the tube. It is also impractical for measuring large pressures at A, since the vertical tube would need to be very long. If the specific gravity of the liquid is S, the pressure at A is hS ft of water.

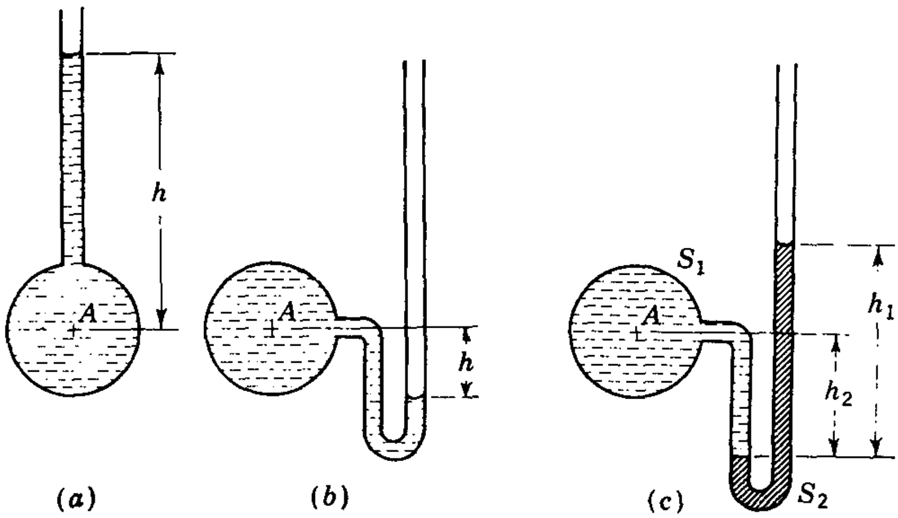
For measurement of small negative or positive gage pressures in a liquid the tube may take the form shown in Fig. 2.8*b*. With this arrangement the meniscus may come to rest below A, as shown. Since the pressure at the meniscus is zero gage and since pressure *decreases* with elevation,

$$h_A = -hS \qquad \text{ft of water}$$

For greater negative or positive gage pressures a second liquid of greater specific gravity is employed (Fig. 2.8c). It must be immiscible in the first fluid, which may now be a gas. If the specific gravity of the fluid at A is S_1 (based on water) and the specific gravity of the manometer liquid is S_2 the equation for pressure at A may be written, starting at either A or the upper meniscus, and proceeding through the manometer, thus

$$h_A + h_2S_1 - h_1S_2 = 0$$

in which h_A is the unknown pressure, expressed in feet of water, and h_1, h_2 are in feet. If A contains a gas, S_1 is generally so small that h_2S_1 may be neglected.



(a) (b) (c)

FIG. 2.8. Examples of simple manometers.

A general procedure may be followed in working all manometer problems:

a. Start at one end (or any meniscus if the circuit is continuous), and write the pressure there in an appropriate unit (say, feet of water) or in an appropriate symbol if it is unknown.

b. Add to this the change in pressure, in the same unit, from one meniscus to the next (plus if the next meniscus is lower, minus if higher). (For feet of water this is the product of the difference in elevation in feet and the specific gravity of the fluid.)

c. Continue until the other end of the gage (or the starting meniscus) is reached and equate the expression to the pressure at that point, known or unknown.

The expression will contain one unknown for a simple manometer or will give a difference in pressures for the differential manometer. In equation form,

$$h_0 - (y_1 - y_0)S_0 - (y_2 - y_1)S_1 - (y_3 - y_2)S_2 \\ - (y_4 - y_3)S_3 - \cdots - (y_n - y_{n-1})S_{n-1} = h_n$$

in which $y_0, y_1, \ldots, y_n$ are elevations of each meniscus in feet and $S_0, S_1, S_2, \ldots, S_{n-1}$ are specific gravities of the fluid columns. The above expression yields the answer in feet of water and may be converted to other units by use of the conversions in Fig. 2.6.

A differential manometer (Fig. 2.9) determines the difference in pressures at two points A and B, when the actual pressure at any point in the

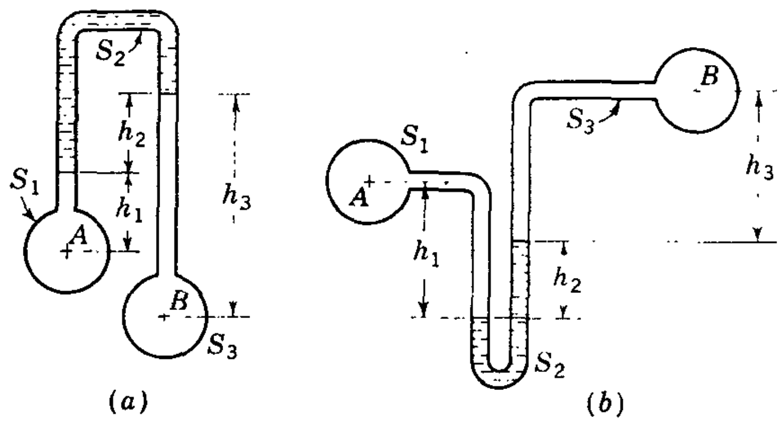


Fig. 2.9. Differential manometers.

system cannot be determined. Application of the procedure outlined above to Fig. 2.9a produces

$$h_A - h_1S_1 - h_2S_2 + h_3S_3 = h_B$$

or

$$h_A - h_B = h_1S_1 + h_2S_2 - h_3S_3 \qquad \text{ft of water}$$

Similarly, for Fig. 2.9b,

$$h_A + h_1S_1 - h_2S_2 - h_3S_3 = h_B$$

or

$$h_A - h_B = -h_1S_1 + h_2S_2 + h_3S_3$$

No formulas for particular manometers should be memorized. It is much more satisfactory to work them out from the general procedure for each case as needed.

Example 2.4: In Fig. 2.9a the liquids at A and B are water and the manometer liquid is oil, sp gr 0.80. $h_1 = 1.0$ ft, $h_2 = 0.50$ ft, $h_3 = 2.0$ ft. (a) Determine $p_A - p_B$ in pounds per square inch. (b) If $p_B = 10$ psia and the barometer reading is 29.5 in. mercury, find the gage pressure at A in pounds per square foot.

(a) $h_A - 1 \times 1 - 0.5 \times 0.8 + 2 \times 1 = h_B$

$h_A - h_B = 1 + 0.4 - 2 = -0.6$ ft water

and

$p_A - p_B = -0.6 \times 0.433 = -0.26$ psi

(b) $p_A = p_B - 0.26 = 10 - 0.26 = 9.74$ psia

$$\text{Local atmospheric pressure} = \frac{29.5}{30} \times 14.7 = 14.47 \text{ psi}$$

In Fig. 2.6,

$$p_A = 9.74 - 14.47 = -4.73 \text{ psi}$$

and

$$p_A = -4.73 \times 144 = 681 \text{ lb/ft}^2 \text{ vacuum}$$

A manometer may be calibrated to measure the volume of liquid in a reservoir, the procedure being given in the following example:

Example 2.5: On the vertical rod in Fig. 2.10*a* is to be laid off a scale that reads the volume V of liquid, in gallons, in the reservoir. Starting with manometer liquid up to *t-t* in both legs, when no liquid is in the reservoir or connecting tube, the distance R along the scale is desired for any depth y in the reservoir.

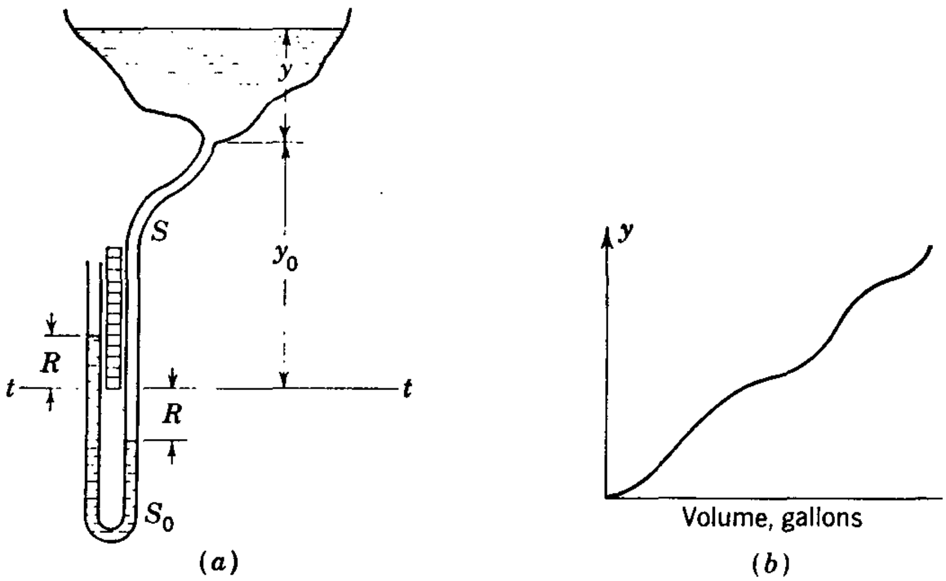


Fig. 2.10. Manometer used for measuring volume in tank.

Then, knowing the volume V in terms of y, as in Fig. 2.10*b*, the distance R is laid off and marked with the corresponding value of V, in gallons. Writing the equation for the manometer, starting at the surface of the reservoir,

$$0 + (y + y_0 + R)S - 2RS_0 = 0$$

or

$$R = \frac{y + y_0}{2(S_0/S) - 1}$$

which yields R in terms of y. For $V = 0$ and $y = 0$

$$R = \frac{y_0}{2(S_0/S) - 1}$$

a distance that is laid off on the scale from *t-t* and marked 0. Taking V as, say, 10,000 gal, y is determined from Fig. 2.10*b* and R is laid off from *t-t* and marked 10,000.

Micromanometers. Several types of manometers are on the market for the determining of very small differences in pressure or precise determining of large pressure differences. One type very accurately measures the differences in elevation of two menisci of a manometer. By means of small telescopes with horizontal cross hairs mounted along the tubes on a rack which is raised and lowered by a pinion and slow-

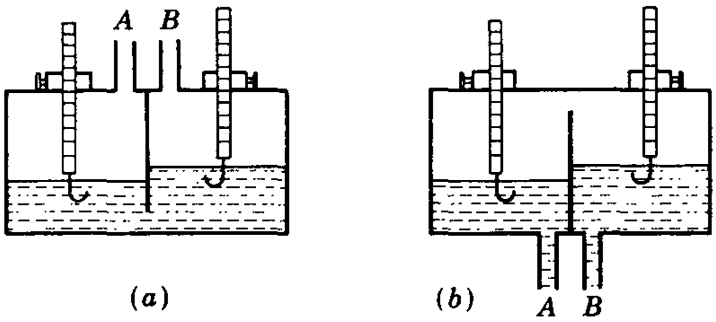


FIG. 2.11. Hook-gage micromanometers. (a) For gases; (b) for liquids.

motion screw so that the cross hairs may be set accurately, the difference in elevation of menisci (the gage difference) may be read with verniers.

The hook-gage micromanometer shown in Fig. 2.11 requires reservoirs several inches in diameter to accommodate the hooks. The one in Fig. 2.11a is for gas measurement, and that in Fig. 2.11b is for liquid measurement. A hook with a conical point is attached to a graduated rod that is moved vertically through a stuffing box by a rack and pinion. As the conical point is moved upward from below the liquid surface, it causes a slight curvature of the surface film before it penetrates it. By suitable lighting the hook may be set at the elevation where the surface-film reflection changes, with an accuracy of about 0.001 in. A vernier may be mounted on the rod, or a dial gage may be mounted against the upper end of the rod. When A and B are connected, both surfaces are at the same elevation; readings taken for this condition provide the "zero" for the gages.

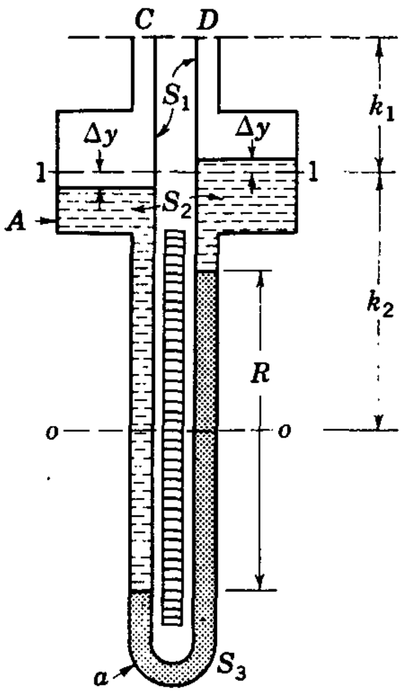


FIG. 2.12. Micromanometer using two gage liquids.

With two gage liquids, immiscible in each other and in the fluid to be measured, a large gage difference R (Fig. 2.12) may be produced for a small pressure difference. The heavier gage liquid fills the lower U-tube up to 0-0; then the lighter gage liquid is added to both sides, filling the larger reservoirs up to 1-1. The gas or liquid in the system fills the space above 1-1. When the pressure at C is slightly greater than at D, the menisci move as indicated in Fig. 2.12. The volume of liquid displaced in each reservoir equals the displacement in the U-tube, thus

$$\Delta y\, A = \frac{R}{2} a$$

in which A and a are the cross-sectional areas of reservoir and U-tube, respectively. The manometer equation may be written, starting at C, in feet of water,

$$h_C + (k_1 + \Delta y)S_1 + \left(k_2 - \Delta y + \frac{R}{2}\right) S_2 - RS_3 - \left(k_2 - \frac{R}{2} + \Delta y\right) S_2 - (k_1 - \Delta y)S_1 = h_D$$

in which S_1, S_2, and S_3 are the specific gravities as indicated in Fig. 2.12. After simplifying and substituting for Δy,

$$h_C - h_D = R\left[S_3 - S_2\left(1 - \frac{a}{A}\right) - S_1 \frac{a}{A}\right] \qquad (2.4.1)$$

The quantity in brackets is a constant for specified gage and fluids; hence, the pressure difference is directly proportional to R.

Example 2.6: In the micromanometer of Fig. 2.12 the pressure difference $p_C - p_D$ is wanted in pounds per square inch when air is in the system. $S_2 = 1.0$, $S_3 = 1.05$, $a/A = 0.01$, $R = 0.10$ in.

For air at standard conditions, 68°F, 30 in. mercury abs, $S_1 = 0.0765/62.4 = 0.00123$; then $S_1(a/A) = 0.0000123$, $S_3 - S_2(1 - a/A) = 1.05 - 0.99 = 0.06$. The term $S_1(a/A)$ may be neglected. Substituting into Eq. (2.4.1) produces

$$h_C - h_D = \frac{0.10}{12} \times 0.06 = 0.0005 \text{ ft water}$$

$$p_C - p_D = 0.0005 \times 0.433 = 0.00022 \text{ psi}$$

The inclined manometer (Fig. 2.13) is frequently used for measuring small differences in gas pressures. It is adjusted to read zero, by moving

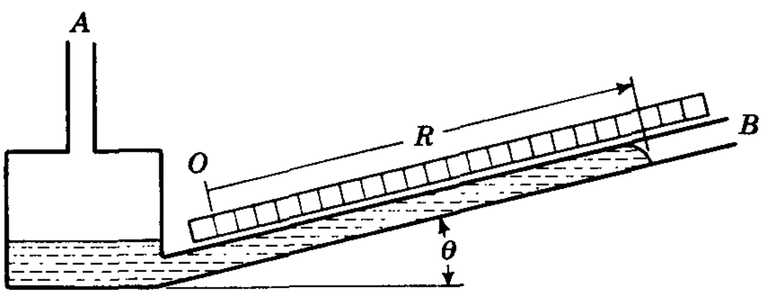


Fig. 2.13. Inclined manometer.

the inclined scale, when A and B are open. The inclined tube requires a greater displacement of the meniscus for given pressure difference than does a vertical tube, so the accuracy in reading the scale is greater in the former.

Surface tension causes a capillary rise in small tubes. If a U-tube is used with a meniscus in each leg, the surface tension effects cancel. The capillary rise is negligible in tubes with a diameter of 0.5 in. or greater.

2.5. Relative Equilibrium. In fluid statics the variation of pressure is simple to compute owing to the absence of shear stresses. For fluid motion such that no layer moves relative to an adjacent layer, the shear stress is also zero throughout the fluid. A fluid with a translation at uniform velocity still follows the laws of static variation of pressure. When a fluid is being accelerated so that no layer moves relative to an adjacent one, i.e., when the fluid moves as if it were a solid, no shear stresses occur and variation in pressure can be determined by writing the equation of motion for an appropriate free body. Two cases are of interest, a uniform linear acceleration and a uniform rotation about a

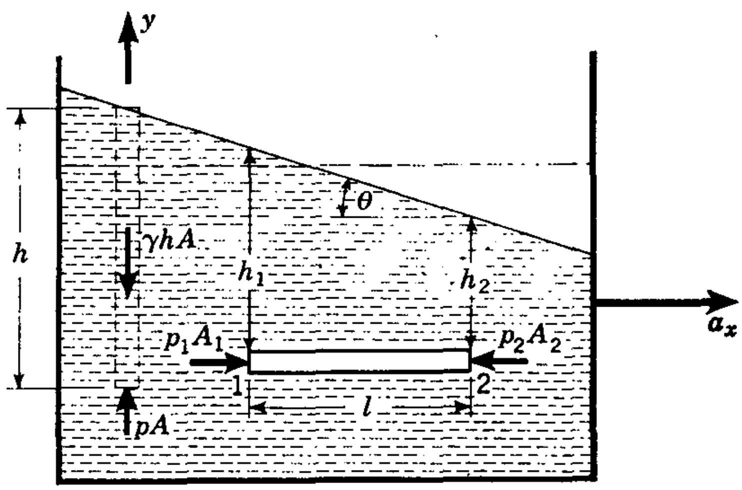


FIG. 2.14. Horizontal acceleration.

vertical axis. When moving thus, the fluid is said to be in *relative equilibrium.*

With very simple relations, equations for variation along single lines have been developed. These can then be combined to determine pressure differences between any two points.

Uniform Linear Acceleration. In an open container with liquid (Fig. 2.14) under uniform horizontal acceleration, the liquid adjusts itself so that it moves as a solid under the action of the accelerating force. To find the variation of pressure in the vertical direction, the vertical free body is used (Fig. 2.14) and the equation of motion in the vertical direction is utilized, $\Sigma f_y = ma_y$. Because the motion is that of a solid, no shear stresses occur in the liquid, and the only vertical forces are due to weight γhA and to pressure force pA at the base of the vertical prism. There is no acceleration in the y-direction; hence

$$pA - \gamma hA = 0$$

or $p = \gamma h$. The pressure variation along a vertical line is the same as for a static liquid.

In a prism of liquid considered as a free body normal to the direction of a_x but along a horizontal line, the pressure does not change, just as it does not change with a static liquid. Therefore, the effect of the acceleration a_x must be in the x-direction.

The equation of motion $\Sigma f_x = ma_x$ for the horizontal free body of Fig. 2.14 is

$$p_1 A - p_2 A = \frac{\gamma l A}{g} a_x \tag{2.5.1}$$

as the weight acts normal to x, and the normal forces on the periphery of the prism are normal to the x-direction. The mass is expressed in slugs as the weight in pounds divided by gravity. Equation (2.5.1) can be rewritten

$$\frac{p_1 - p_2}{\gamma l} = \frac{h_1 - h_2}{l} = \frac{a_x}{g} \tag{2.5.2}$$

in which h_1, h_2 are the distances to the free surface. The expression $(h_1 - h_2)/l$ is the slope of the free surface, $\tan\theta$. As

$$\tan\theta = \frac{a_x}{g} \tag{2.5.3}$$

is constant for constant a_x, the liquid surface is an inclined plane. Planes of constant pressure are parallel to the free surface.

If the vessel is filled with liquid and closed at the top, the liquid requires no preliminary adjustment period before moving as a solid when subjected to an acceleration. The planes of constant pressure are still given by Eq. (2.5.3). If the pressure is known at one point in the vessel, it can easily be computed for all other points. The shape of the container is unimportant so long as the fluid is connected.

Example 2.7: The tank in Fig. 2.15 is filled with oil, sp gr 0.8, and accelerated as shown. There is a small opening in the tank at A. Determine the pressure at B and C and the acceleration a_x required to make the pressure at B zero.

The planes of constant pressure have the slope

$$\tan\theta = \frac{a_x}{g} = \frac{16.1}{32.2} = 0.5$$

and at A the pressure is zero. The plane through A passes 1 ft vertically above B; hence

$$p_B = 1 \times 62.4 \times 0.8 = 49.9 \text{ lb/ft}^2$$

Similarly, C is vertically below the zero pressure plane a distance 4.75 ft, and

$$p_C = 4.75 \times 62.4 \times 0.8 = 237 \text{ lb/ft}^2$$

For zero pressure at B,

$$\tan \theta = \frac{4}{6} = \frac{a_x}{32.2}$$

and $a_x = \frac{2}{3} \times 32.2 = 21.47$ ft/sec².

For vertical acceleration a_y, the free surface (if one occurs) remains horizontal. The pressure is constant in horizontal planes. With a

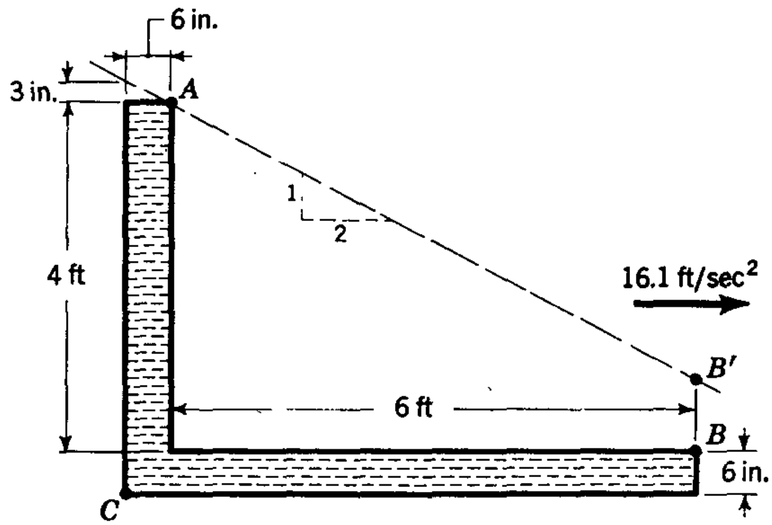


FIG. 2.15. Tank completely filled with liquid.

vertical circular cylinder of cross-sectional area A (Fig. 2.16) and height h as a free body and with the equation of motion written $\Sigma f_y = ma_y$,

$$p_2A - p_1A - \gamma hA = \frac{\gamma hA}{g} a_y$$

Simplified,

$$p_2 - p_1 = \gamma h \left(1 + \frac{a_y}{g}\right) \tag{2.5.4}$$

For example, if the container is dropped, $a_y = -g$ and $p_2 = p_1$ and the pressure is everywhere the same throughout the liquid.

Example 2.8: A cubical box, 2 ft on a side, half filled with oil, sp gr 0.90, is accelerated along an inclined plane at an angle of 30° with the horizontal, as shown in Fig. 2.17. Find the slope of free surface and the pressure along the bottom.

In the coordinate system as indicated in the figure,

$$a_x = 8.05 \cos 30° = 6.98 \text{ ft/sec}^2$$

and

$$a_y = 8.05 \sin 30° = 4.02 \text{ ft/sec}^2$$

If the pressure at the origin is p_0, the variation of pressure in the x-direction is [from Eq. (2.5.2)]

$$p = p_0 - \gamma x \frac{a_x}{g} = p_0 - 62.4 \times 0.90 \times \frac{6.98}{32.2} x = p_0 - 12.15x \qquad \text{lb/ft}^2$$

The pressure variation in the y-direction is [from Eq. (2.5.4)]

$$p = p_0 - \gamma h\left(1 + \frac{a_y}{g}\right) = p_0 - 62.4 \times 0.90\left(1 + \frac{4.02}{32.2}\right)y = p_0 - 63.1y$$

To find the slope of the lines of constant pressure, the expressions for p are equated,

$$y = \frac{12.15}{63.1}x = 0.1925x$$

$y/x = 0.1925$ is the slope of liquid surface, downward to the right. As $\tan^{-1} 0.1925 = 10°52'$, the surface then makes an angle of $40°52'$ with the bottom of the box. The depth parallel to a side is less on the right-hand side by 2 tan

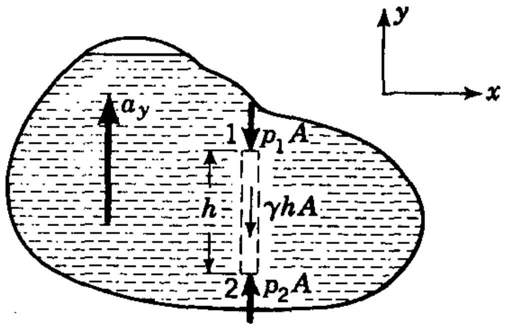


FIG. 2.16. Vertical acceleration.

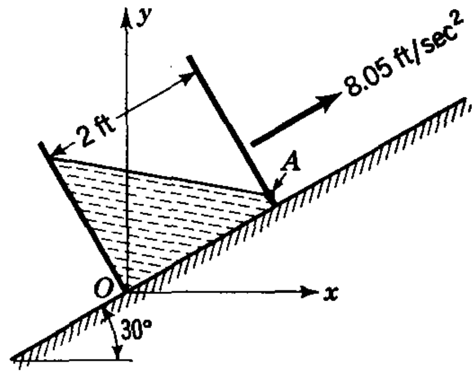


FIG. 2.17. Uniform acceleration along an inclined plane.

$40°52'$, or 1.73 ft. The total volume of oil is unchanged. Therefore, if s be the depth on the right-hand side,

$$2\left(\frac{1.73}{2} + s\right)2 = 4 \text{ ft}^3$$

or $s = 0.135$ ft. The point A on the free surface has the coordinates

$$x = 2 \cos 30° - 0.135 \sin 30° = 1.665 \text{ ft}$$

and

$$y = 2 \sin 30° + 0.135 \cos 30° = 1.117 \text{ ft}$$

The pressure there is zero, and when the expressions for change in pressure in the x- and y-directions are combined,

$$p = p_0 - 12.15x - 63.1y$$

After substituting for x, y, and p,

$$0 = p_0 - 12.15 \times 1.665 - 63.1 \times 1.117$$

or $p_0 = 90.73$ lb/ft². If t is the distance along the bottom from O, then

$$x = 0.866t \qquad \text{and} \qquad y = 0.50t$$

and

$$\begin{aligned} p &= 90.73 - 12.15 \times 0.866t - 63.5 \times 0.50t \\ &= 90.73 - 42.07t \qquad \text{lb/ft}^2 \end{aligned}$$

Uniform Rotation about a Vertical Axis. Rotation of a fluid, moving as a solid, about an axis is called *forced-vortex* motion. Every particle of fluid has the same angular velocity. This motion is to be distinguished from *free-vortex* motion, where each particle does not rotate but moves in a circular path with a speed varying inversely as the distance from the center. Free-vortex motion is discussed in Chaps. 7 and 8. A liquid in a container, when rotated about a vertical axis at constant angular velocity, moves as a solid after some time interval. No shear stresses exist in the liquid and the only acceleration that occurs is directed radially inward toward the axis of rotation. The equation of motion in the vertical direction on a free body shows that hydrostatic conditions prevail along any vertical line; hence, the pressure at any point in the liquid is given by the product of specific weight and vertical distance from the free surface.

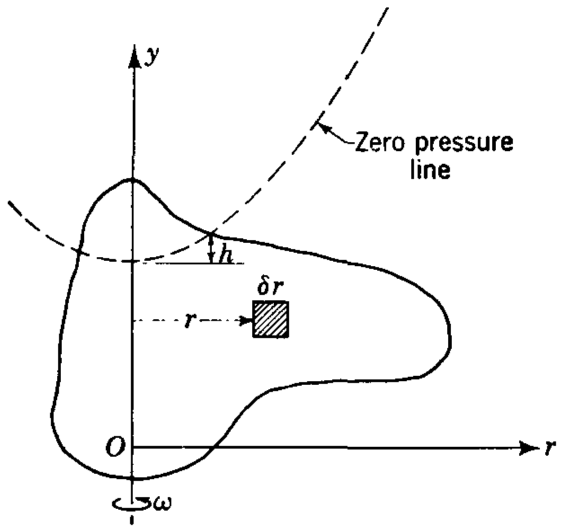


FIG. 2.18. Rotation of fluid about a vertical axis.

In the equation of motion tangent to the circular path of a particle, the acceleration is zero, and the pressure does not change along the path.

In the equation of motion in the radial (horizontal) direction (Fig. 2.18), with a free body of length δr and cross-sectional area δA, if the pressure at r be p, then, at the opposite face, the pressure is $p + (\partial p/\partial r)\delta r$. The acceleration is $-\omega^2 r$; hence

$$p\,\delta A - \left(p + \frac{\partial p}{\partial r}\,\delta r\right)\delta A = \frac{\delta A\ \delta r\ \gamma}{g}\,(-\omega^2 r)$$

After simplifying and dividing through by the volume of the element $\delta A\ \delta r$,

$$\frac{\partial p}{\partial r} = \frac{\gamma}{g}\,\omega^2 r$$

After integrating,

$$p = \frac{\gamma}{g}\,\omega^2\frac{r^2}{2} + c$$

in which c is the constant of integration If the value at the axis ($r = 0$) be p_0, then $c = p_0$, and

$$p = p_0 + \gamma\,\frac{\omega^2 r^2}{2g} \qquad (2.5.5)$$

When the particular horizontal plane for which $p_0 = 0$ is selected and

Eq. (2.5.5) is divided by γ,

$$h = \frac{p}{\gamma} = \frac{\omega^2 r^2}{2g} \tag{2.5.6}$$

which shows that the head, or vertical depth, varies as the square of the radius. The surfaces of equal pressure are paraboloids of revolution.

When a free surface occurs in a container that is being rotated, the fluid volume underneath the paraboloid of revolution is the original fluid volume. The shape of the paraboloid depends only upon the angular velocity ω.

For the case of a circular cylinder rotating about its axis (Fig. 2.19) the rise of liquid from its vertex to the wall of the cylinder is, from Eq. (2.5.6), $\omega^2 r_0^2/2g$. Since a paraboloid of revolution has a volume equal to one-half its circumscribing cylinder, the volume of the liquid above the horizontal plane through the vertex is

$$\pi r_0^2 \times \frac{1}{2}\frac{\omega^2 r_0^2}{2g}$$

When the liquid is at rest, this liquid is also above the plane through the vertex, to a uniform depth of

$$\frac{1}{2}\frac{\omega^2 r_0^2}{2g}$$

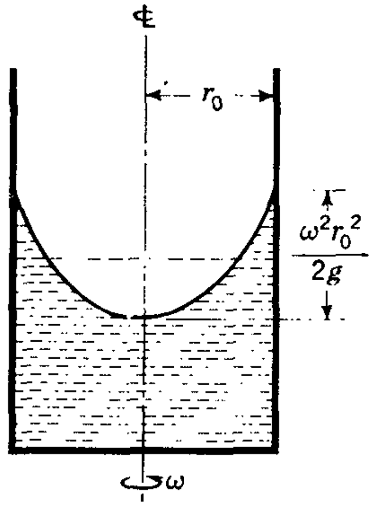


FIG. 2.19. Rotation of circular cylinder about its axis.

Hence, the liquid rises along the walls the same amount as the center drops, thereby permitting the vertex to be located when ω, r_0, and depth before rotation are given.

Example 2.9: A liquid, sp gr 1.2, is rotated at 200 rpm about a vertical axis. At one point, A, in the fluid 2 ft from the axis, the pressure is 10 psi. What is the pressure at a point B, 4 ft higher than A and 3 ft from the axis?

When Eq. (2.5.5) is written for the two points

$$p_A = p_{0A} + \gamma\frac{\omega^2 r_A^2}{2g} \qquad p_B = p_{0B} + \gamma\frac{\omega^2 r_B^2}{2g}$$

Then $\omega = 200 \times 2\pi/60 = 20.95$ rad/sec, $\gamma = 1.2 \times 62.4 = 74.8$ lb/ft³, $r_A = 2$ ft, $r_B = 3$ ft, $p_{0A} - p_{0B} = 4 \times 74.8 = 299$ lb/ft², $p_A = 1440$ lb/ft². When the second equation is subtracted from the first and the values are substituted,

$$p_A - p_B = p_{0A} - p_{0B} + \frac{\gamma\omega^2}{2g}(r_A^2 - r_B^2)$$

and

$$1440 - p_B = 299 + \frac{74.8}{64.4}\,20.95^2(2^2 - 3^2)$$

Hence, $p_B = 3691$ lb/ft², or 25.6 psi.

If a closed container with no free surface, or with a partially exposed free surface, is rotated uniformly about some vertical axis, an *imaginary* free surface can be constructed, consisting of a paraboloid of revolution of shape given by Eq. (2.5.6). The vertical distance from any point in the fluid to this free surface is the pressure head at the point.

Example 2.10: A straight tube 4 ft long, closed at the bottom and filled with water, is inclined 30° with the vertical and rotated about a vertical axis through its mid-point 8.02 rad/sec. Draw the paraboloid of zero pressure, and determine the pressure at the bottom and mid-point of the tube.

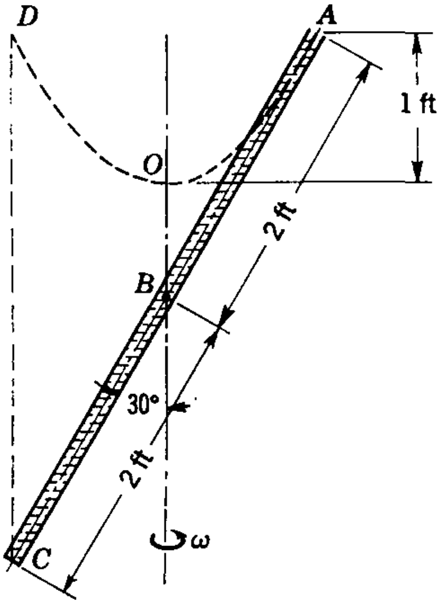


FIG. 2.20. Rotation of inclined tube of liquid about a vertical axis.

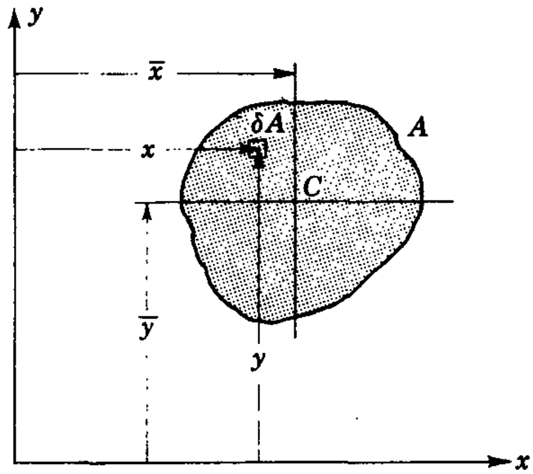


FIG. 2.21. Notation for determining line of action of a force.

In Fig. 2.20, the zero-pressure paraboloid passes through point A. If the origin is taken at the vertex, that is, $p_0 = 0$, Eq. (2.5.6) becomes

$$h = \frac{\omega^2 r^2}{2g} = \frac{\overline{8.02}^2}{64.4} (2 \sin 30°)^2 = 1.0 \text{ ft}$$

which locates the vertex at O, 1.0 ft below A. The pressure at the bottom of the tube is $\gamma \times \overline{CD}$, or

$$4 \cos 30° \times 62.4 = 216 \text{ lb/ft}^2$$

At the mid-point, $\overline{OB} = 0.732$ ft, and

$$p_B = 0.732 \times 62.4 = 45.6 \text{ lb/ft}^2$$

2.6. Forces on Plane Areas. In the preceding sections variations of pressure throughout a fluid have been considered. The distributed forces resulting from the action of fluid on a finite area may be conveniently replaced by a resultant force, in so far as external reactions to the force system are concerned. In this section the magnitude of resultant force and its line of action (pressure center) are determined by integration, by formula, and by use of the concept of the pressure prism.

Horizontal Surfaces. A plane surface in a horizontal position in a fluid at rest is subjected to a constant pressure. The magnitude of the force acting on one side of the surface is

$$\int p\,dA = p\int dA = pA$$

The elemental forces $p\,dA$ acting on dA are all parallel and in the same sense; therefore, a scalar summation of all such elements yields the magnitude of the resultant force. Its direction is normal to the surface, and *toward* the surface if p is *positive.* To find the line of action of the resultant, i.e., the point in the area where the moment of the distributed force about any axis through the point is zero, arbitrary xy-axes may be selected, as in Fig. 2.21. Then, since the moment of the resultant must equal the moment of the distributed force system about any axis, say the y-axis,

$$pAx' = \int_A xp\,dA$$

in which x' is the distance from the y-axis to the resultant. Since p is constant,

$$x' = \frac{1}{A}\int_A x\,dA = \bar{x}$$

in which $\bar{x}$ is the distance to the centroid of the area.[1] Hence, for a horizontal area subjected to static fluid pressure, the resultant passes through the centroid of the area.

Inclined Surfaces. In Fig. 2.22 a plane surface is indicated by its trace $A'B'$. It is inclined $\theta°$ from the horizontal. The intersection of the plane of the area and the free surface is taken as the x-axis. The y-axis is taken in the plane of the area, with origin O, as shown, in the free surface. The xy-plane portrays the arbitrary inclined area. The magnitude, direction, and line of action of the resultant force *due to the liquid*, acting on one side of the area, are sought.

For an element with area δA as a strip with thickness δy with long edges horizontal, the magnitude of force δF acting on it is

$$\delta F = p\,\delta A = \gamma h\,\delta A = \gamma y \sin\theta\,\delta A \qquad (2.6.1)$$

Since all such elemental forces are parallel, the integral over the area yields the magnitude of force F, acting on one side of the area,

$$F = \int p\,dA = \gamma \sin\theta \int y\,dA = \gamma\sin\theta\,\bar{y}A = \gamma\bar{h}A = p_G A \qquad (2.6.2)$$

with the relations from Fig. 2.22, $\bar{y}\sin\theta = \bar{h}$, and $p_G = \gamma\bar{h}$, the pressure

[1] See Appendix A.

at the centroid of the area. In words, the magnitude of force exerted on one side of a plane area submerged in a liquid is the product of the area and the pressure at its centroid. In this form, it should be noted, the presence of a free surface is unnecessary. Any means for determining the pressure at the centroid may be used. The sense of the force is to push against the area, if p_G is positive. As all force elements are normal to the surface, the line of action of the resultant is also normal to the surface. Any surface may be rotated about any axis through its centroid without

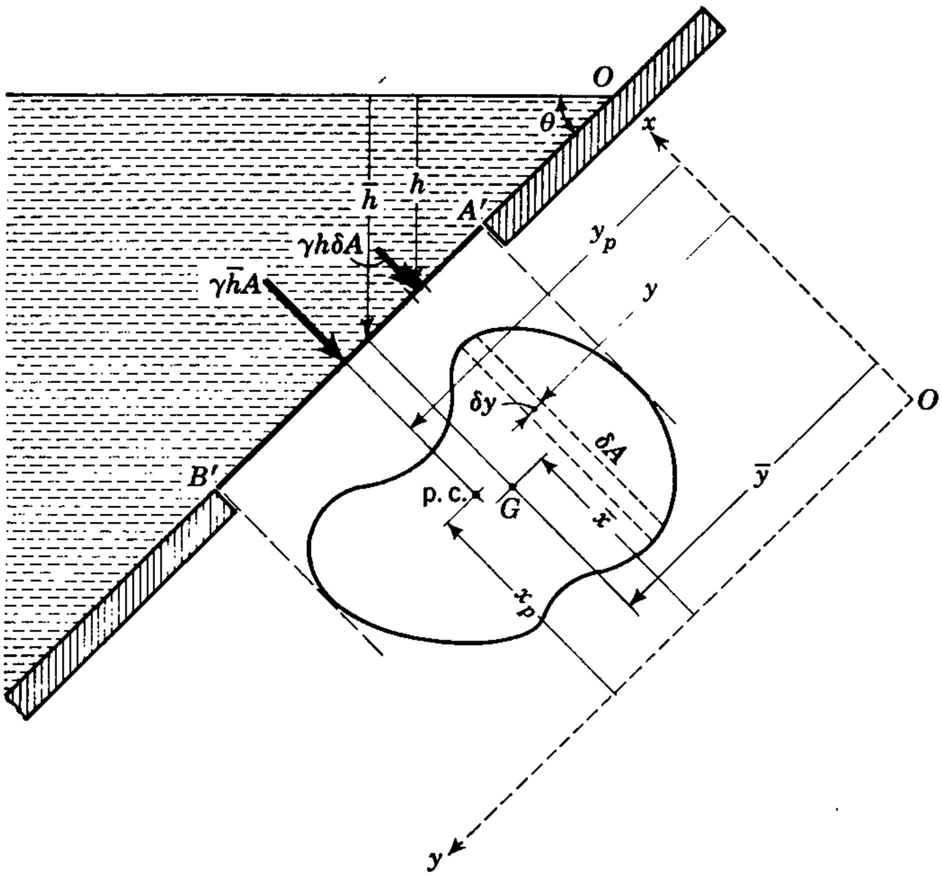


FIG. 2.22. Notation for force of liquid on one side of a plane inclined area.

changing the magnitude of the resultant, if the total area remains submerged in the static liquid.

Center of Pressure. The line of action of the resultant force has its piercing point in the surface at a point called the *pressure center*, with coordinates (x_p,y_p) (Fig. 2.22). Unlike that for the horizontal surface, the center of pressure of an inclined surface is not at the centroid. To find the pressure center, the moments of the resultant x_pF, y_pF are equated to the moment of the distributed forces about the y-axis and x-axis, respectively; thus

$$x_pF = \int_A xp\, dA \tag{2.6.3}$$

$$y_pF = \int_A yp\, dA \tag{2.6.4}$$

The area element in Eq. (2.6.3) should be $\delta x\ \delta y$, and not the strip shown in Fig. 2.22.

After solving for the coordinates of pressure center,

$$x_p = \frac{1}{F} \int_A xp\, dA \tag{2.6.5}$$

$$y_p = \frac{1}{F} \int_A yp\, dA \tag{2.6.6}$$

In many applications Eqs. (2.6.5) and (2.6.6) may be evaluated most conveniently through graphical integration; for simple areas they may be transformed into general formulas as follows:[1]

$$x_p = \frac{1}{\gamma \bar{y} A \sin\theta} \int_A x\gamma y \sin\theta\, dA = \frac{1}{\bar{y}A} \int_A xy\, dA = \frac{I_{xy}}{\bar{y}A} \tag{2.6.7}$$

In Eqs. (A.10), of Appendix A, and (2.6.7),

$$x_p = \frac{\bar{I}_{xy}}{\bar{y}A} + \bar{x} \tag{2.6.8}$$

When either of the centroidal axes, $x = \bar{x}$ or $y = \bar{y}$, is an axis of symmetry for the surface, $\bar{I}_{xy}$ vanishes and the pressure center lies on $x = \bar{x}$. Since $\bar{I}_{xy}$ may be either positive or negative, the pressure center may lie on either side of the line $x = \bar{x}$. To determine y_p by formula, with Eqs. (2.6.2) and (2.6.6),

$$y_p = \frac{1}{\gamma \bar{y} A \sin\theta} \int_A y\gamma y \sin\theta\, dA = \frac{1}{\bar{y}A} \int_A y^2\, dA = \frac{I_x}{\bar{y}A} \tag{2.6.9}$$

In the parallel-axis theorem for moments of inertia

$$I_x = I_G + \bar{y}^2 A$$

If I_x is eliminated from Eq. (2.6.9)

$$y_p = \frac{I_G}{\bar{y}A} + \bar{y} \tag{2.6.10}$$

or

$$y_p - \bar{y} = \frac{I_G}{\bar{y}A} \tag{2.6.11}$$

I_G is always positive; hence, $y_p - \bar{y}$ is always positive, and the pressure center is always below the centroid of the surface. It should be emphasized that $\bar{y}$ and $y_p - \bar{y}$ are distances in the plane of the surface.

Example 2.11: The triangular gate CDE (Fig. 2.23) is hinged along CD and is opened by a normal force P applied at E. It holds oil, sp gr 0.80, above it and

[1] See Appendix A.

is open to the atmosphere on its lower side. Neglecting the weight of the gate determine (*a*) the magnitude of force exerted on the gate, by integration and by Eq. (2.6.2); (*b*) the location of pressure center; (*c*) the force P necessary to open the gate.

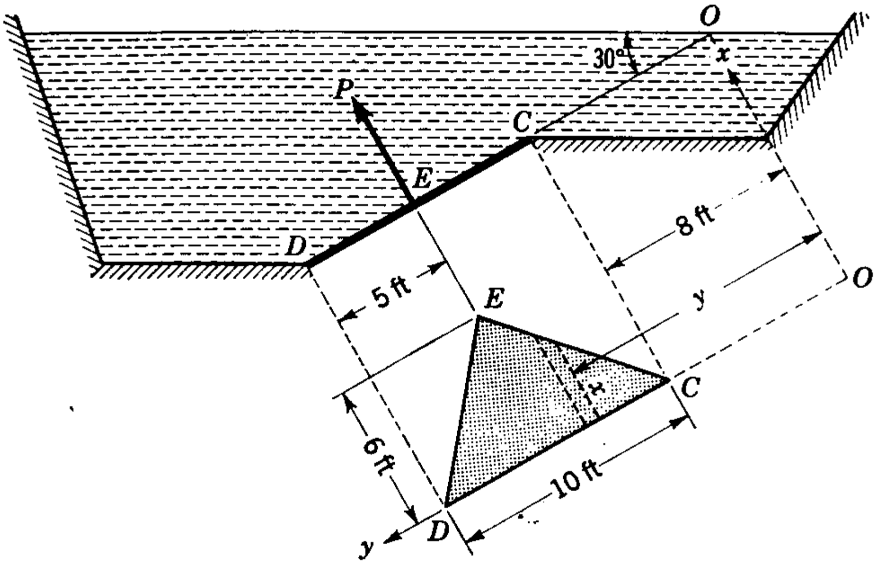


FIG. 2.23. Triangular gate.

a. By integration with reference to Fig. 2.23

$$F = \int_A p\,dA = \gamma \sin\theta \int yx\,dy = \gamma \sin\theta \int_8^{13} xy\,dy + \gamma \sin\theta \int_{13}^{18} xy\,dy$$

When $y = 8$, $x = 0$, and when $y = 13$, $x = 6$, with x varying linearly with y, thus

$$x = ay + b \qquad 0 = 8a + b \qquad 6 = 13a + b$$

in which the coordinates have been substituted to find x in terms of y. After solving for a and b,

$$a = \tfrac{6}{5} \qquad b = -\tfrac{48}{5}, \qquad x = \tfrac{6}{5}(y - 8)$$

Similarly $y = 13$, $x = 6$; $y = 18$, $x = 0$; and $x = \frac{6}{5}(18 - y)$. Hence

$$F = \gamma \sin\theta \frac{6}{5}\left[\int_8^{13} (y - 8)y\,dy + \int_{13}^{18} (18 - y)y\,dy\right]$$

After integrating and substituting for $\gamma \sin\theta$,

$$F = 62.4 \times 0.8 \times 0.50 \times \frac{6}{5}\left[\left(\frac{y^3}{3} - 4y^2\right)_8^{13} + \left(9y^2 - \frac{y^3}{3}\right)_{13}^{18}\right] = 9734.4 \text{ lb}$$

By Eq. (2.6.2)

$$F \doteq p_G A = \gamma \bar{y} \sin\theta\, A = 62.4 \times 0.80 \times 0.50 \times 30 \times 13 = 9734.4 \text{ lb}$$

b. With the axes as shown, $\bar{x} = 2.0$, $\bar{y} = 13$. In Eq. (2.6.8)

$$x_p = \frac{\bar{I}_{xy}}{\bar{y}A} + \bar{x}$$

$\bar{I}_{xy}$ is zero owing to symmetry about the centroidal axis parallel to the x-axis; hence $\bar{x} = x_p = 2.0$ ft. In Eq. (2.6.11),

$$y_p - \bar{y} = \frac{I_G}{\bar{y}A} = 2 \times \frac{1 \times 6 \times 5^3}{12 \times 13 \times 30} = 0.32 \text{ ft}$$

i.e., the pressure center is 0.32 ft below the centroid, measured in the plane of the area.

c. When moments about CD are taken and the action of the oil is replaced by the resultant,

$$P \times 6 = 9734.4 \times 2 \qquad P = 3244.8 \text{ lb}$$

The Pressure Prism. The concept of the pressure prism provides another means for determining the magnitude and location of the resultant force on an inclined plane surface. The volume of the pressure prism is the magnitude of the force and the resultant force passes through the centroid of the prism. The surface is taken as the base of the prism, and its altitude at each point is determined by the pressure γh laid off to an appropriate scale (Fig. 2.24). Since the pressure increases linearly with distance from the free surface, the upper surface of the prism is in a plane with its trace OM shown in Fig. 2.24. The force acting on an elemental area δA is

$$\delta F = \gamma h \, \delta A = \delta V \qquad (2.6.12)$$

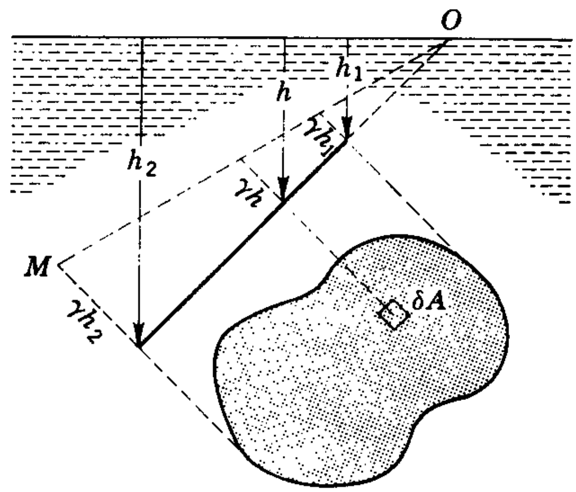


FIG. 2.24. Illustration of pressure prism.

which is an element of volume of the pressure prism. After integrating, $F = V$, the volume of the pressure prism equals the magnitude of the resultant force acting on one side of the surface.

Equations (2.6.5) and (2.6.6),

$$x_p = \frac{1}{V}\int_V x \, dV \qquad y_p = \frac{1}{V}\int_V y \, dV \qquad (2.6.13)$$

show that x_p, y_p are distances to the *centroid of the pressure* prism.[1] Hence, the line of action of the resultant passes through the centroid of the pressure prism. For some simple areas the pressure prism is more convenient than either integration or formula. For example, a rectangular area with one edge in the free surface has a wedge-shaped prism. Its centroid is one-third the altitude from the base; hence, the pressure center is one-third the altitude from its lower edge.

[1] Appendix A, Eq. (A.5).

Effect of Atmospheric Pressure on Forces on Plane Areas. In the discussion of pressure forces the pressure datum was not mentioned. The pressures were computed by $p = \gamma h$. in which h is the vertical distance below the free surface. Therefore, the datum taken was gage pressure zero, or the local atmospheric pressure. When the opposite side of the surface is open to the atmosphere, a force is exerted on it by the atmosphere equal to the product of the atmospheric pressure p_0 and the area, or p_0A, based on absolute zero as datum. On the liquid side the force is

$$\int(p_0 + \gamma h)dA = p_0A + \gamma\int h\,dA$$

The effect of the atmosphere p_0A acts equally on both sides and in no way contributes to the resultant force or its location.

So long as the same pressure datum is selected for all sides of a free body, the resultant can be determined by constructing a free surface at pressure zero on this datum and by using the above methods.

Fluid Pressure Forces in Relative Equilibrium. The magnitude of the force acting on a plane area in contact with a fluid accelerating as a rigid body may be obtained by integration over the surface,

$$F = \int p\,dA$$

The nature of the acceleration and orientation of the surface governs the particular variation of p over the surface. When the pressure varies linearly over the plane surface (linear acceleration), the magnitude of force is given by the product of pressure at the centroid and area since the volume of the pressure prism is given by p_GA. For nonlinear distributions the magnitude and line of action may be found by integration.

Example 2.12: *Forces on a Gravity Dam.* An application of pressure forces on plane areas is given in the design of a gravity dam. The maximum and minimum compressive stresses in the base of the dam are computed from the forces which act on the dam. Figure 2.25 shows a cross section through a concrete dam where the specific weight of concrete has been taken as 2.5γ and γ is the specific weight of water. A 1-ft section of dam is considered as a free body; the forces are due to the concrete, the water, the foundation pressure, and the hydrostatic uplift. The determination of amount of hydrostatic uplift is beyond the scope of this treatment, but will be assumed one-half the hydrostatic head at the upstream edge, decreasing linearly to zero at the downstream edge of the dam. Enough friction or shear stress must be developed at the base of the dam to balance the thrust due to the water, that is, $R_x = 5000\gamma$. The resultant upward force on the base equals the weight of the dam less the hydrostatic uplift, $R_y = 6750\gamma + 2625\gamma - 1750\gamma = 7625\gamma$ lb. The position of R_y is such that the free body is in equilibrium. For moments around O,

$$\Sigma M_o = 0 = R_yx - 5000\gamma \times 33.33 - 2625\gamma \times 5 - 6750\gamma \times 30 + 1750\gamma \times 23.33$$

and

$$x = 44.8 \text{ ft}$$

It is customary to assume that the foundation pressure varies linearly over the base of the dam, i.e., that the pressure prism is a trapezoid with a volume equal to R_y; thus

$$\frac{C_{\max} + C_{\min}}{2} 70 = 7625\gamma$$

in which $C_{\max}$, $C_{\min}$ are the maximum and minimum compressive stresses in pounds per square foot. The centroid of the pressure prism is at the point where

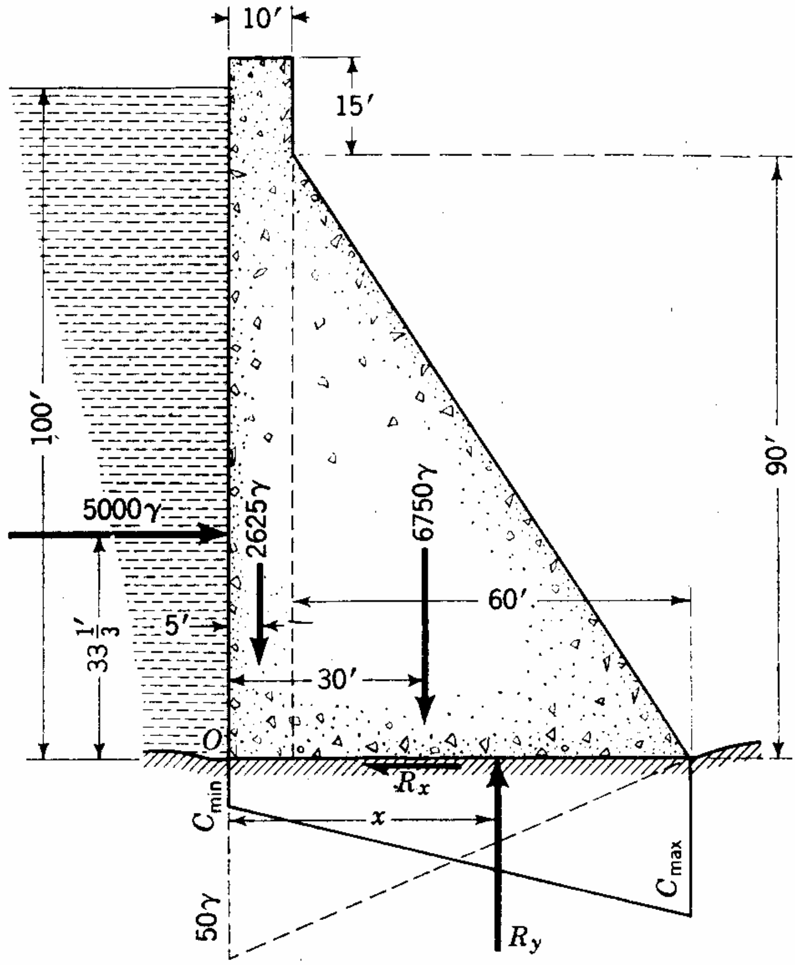


Fig. 2.25. Concrete gravity dam.

$x = 44.8$ ft. By taking moments about O to express the position of the centroid in terms of $C_{\max}$ and $C_{\min}$,

$$44.8 = \frac{C_{\min} 70 \times \frac{70}{2} + (C_{\max} - C_{\min})\frac{70}{2} \times \frac{2}{3}70}{(C_{\max} + C_{\min})\frac{70}{2}}$$

After simplifying,

$$C_{\max} = 11.75 C_{\min}$$

Then

$$C_{\max} = 210\gamma = 12{,}500 \text{ lb/ft}^2 \qquad C_{\min} = 17.1\gamma = 1067 \text{ lb/ft}^2$$

When the resultant falls within the middle third of the base of the dam, C_{min} will always be a compressive stress. Owing to the poor tensile properties of concrete, good design requires the resultant to fall within the middle third of the base.

2.7. Force Components on Curved Surfaces. When the elemental forces $p\ \delta A$ vary in direction, as in the case of a curved surface, they must be added as vector quantities; i.e., their components in three mutually perpendicular directions are added as scalars, and then the three components are added vectorially. With two horizontal components at right angles and with the vertical component, which are easily computed for a curved surface, the resultant can be determined. The lines of action of the components are readily determined, so the resultant and its line of action can be completely determined.

Horizontal Component of Force on a Curved Surface. The horizontal component of pressure force on a curved surface is equal to the pressure force

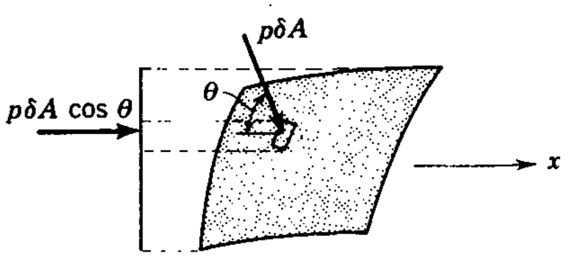


FIG. 2.26. Horizontal component of force on a curved surface.

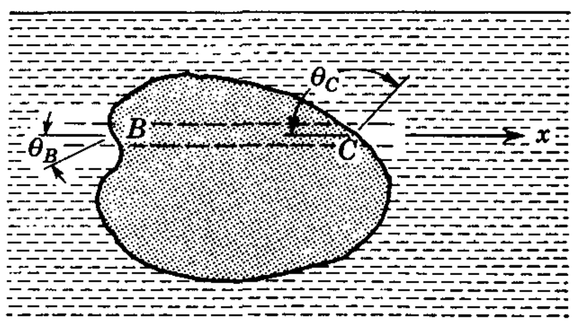


FIG. 2.27. Projections of area elements on opposite sides of a body.

exerted on a projection of the curved surface. The vertical plane of projection is normal to the direction of the component. The surface of Fig. 2.26 represents any three-dimensional surface, and δA an element of its area, with its normal making the angle θ with the negative x-direction. Then

$$\delta F_x = p\ \delta A \cos\theta$$

is the x-component of force exerted on one side of δA. Summing up the x-components of force over the surface,

$$F_x = \int_A p \cos\theta\, dA \tag{2.7.1}$$

Considering $\cos\theta\ \delta A$, it is the projection of δA onto a plane perpendicular to x. The element of force on the projected area is $p \cos\theta\ \delta A$, which is also in the x-direction. Projecting each element on a plane perpendicular to x is equivalent to projecting the curved surface as a whole onto the plane. Hence, the force acting on this projection of the curved surface is the horizontal component of force exerted on the curved surface, in the direction normal to the plane of projection. To find the horizontal component at right angles to the x-direction, the curved surface is projected onto a vertical plane parallel to x, and the force on the projection is determined.

When the horizontal component of pressure force on a closed body is to be found, the projection of the curved surface on a vertical plane is always zero, since on opposite sides of the body the area-element projections have opposite signs, as indicated in Fig. 2.27. Let a small cylinder of cross section δA with axis parallel to x intersect the closed body at B and C. If the element of area of the body cut by the prism at B is δA_B and at C is δA_C, then

$$\delta A_B \cos \theta_B = -\delta A_C \cos \theta_C = \delta A$$

as $\cos \theta_C$ is negative. Hence, with the pressure the same at each end of the cylinder,

$$p\, \delta A_B \cos \theta_B + p\, \delta A_C \cos \theta_C = 0$$

and similarly for all other area elements.

To find the line of action of a horizontal component of force on a curved surface, the resultant of the parallel force system composed of

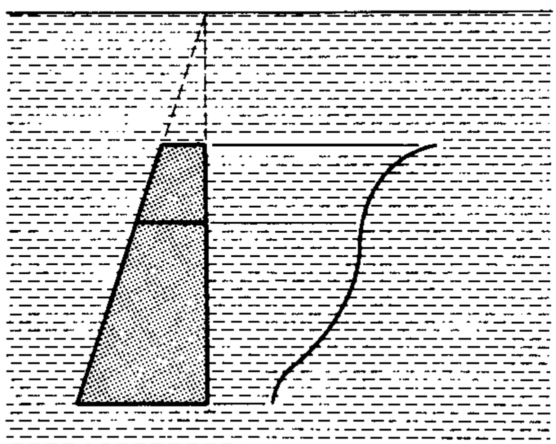


FIG. 2.28. Pressure prism for horizontal component of pressure

the force components from each area element is required. This is exactly the resultant of the force on the projected area, since the two force systems have identical pressure prisms, as indicated in Fig. 2.28. Hence, the pressure center is located on the projected area by the methods of Sec. 2.6.

Vertical Component of Force on a Curved Surface. The vertical component of pressure force on a curved surface is equal to the weight of liquid vertically above the curved surface and extending up to the free surface. The vertical component of force on a curved surface can be determined by summing up the vertical components of pressure force on elemental areas δA of the surface. In Fig. 2.29 an area element is shown with the force $p\ \delta A$ acting normal to it. (Let θ be the angle the normal to the area element makes with the vertical.) Then the vertical component of force acting on the area element is $p \cos \theta\ \delta A$, and the vertical component of force on the curved surface is given by

$$F_v = \int_A p \cos \theta \, dA \tag{2.7.2}$$

By replacing p by its equivalent γh, in which h is the distance from the area element to the free surface, and noting that $\cos \theta\ \delta A$ is the projection of δA on a horizontal plane, Eq. (2.7.2) becomes

$$F_v = \gamma \int_A h \cos \theta \, dA = \gamma \int_V dV \tag{2.7.3}$$

in which δV is the volume of the prism of height h and base $\cos \theta\ \delta A$,

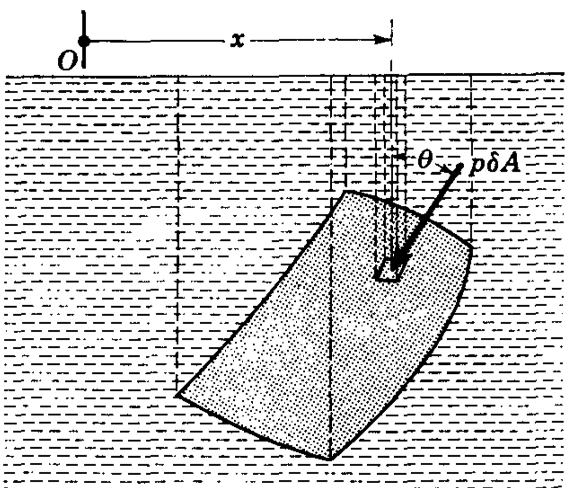


FIG. 2.29. Vertical component of force on a curved surface.

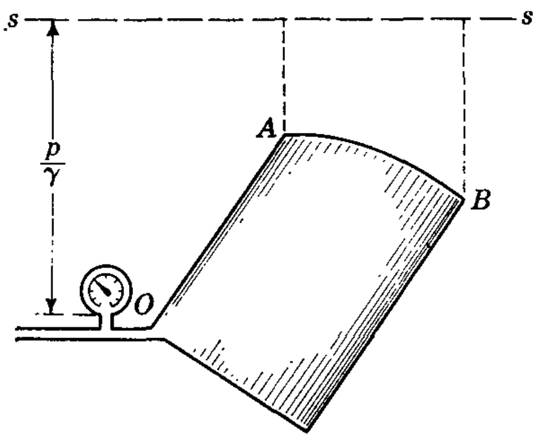


FIG. 2.30. Liquid with imaginary free surface.

or the volume of liquid vertically above the area element. Integrating,

$$F_v = \gamma V \tag{2.7.4}$$

When the liquid is below the curved surface (Fig. 2.30) and the pressure intensity is known at some point, e.g., O, an *imaginary* free surface s-s may be constructed p/γ above O, so that the product of specific weight and vertical distance to any point in the tank is the pressure at the point. The weight of the imaginary volume of liquid vertically above the curved surface is then the vertical component of pressure force on the curved surface. In the constructing of an imaginary free surface, the imaginary liquid must be of the same specific weight as the liquid in contact with the curved surface; otherwise, the pressure distribution over the surface will not be correctly represented. With an imaginary liquid above a surface, the pressure at a point on the curved surface is equal on both sides, but the elemental force components in the vertical direction are opposite in sign. Hence, the direction of the vertical force component is reversed when an imaginary fluid is above the surface. In some cases a confined liquid may be above the curved surface, and an imaginary liquid must be added (or subtracted) to determine the free surface.

The line of action of the vertical component is determined by equating moments of the elemental vertical components about a convenient axis with the moment of the resultant force. With the axis at O (Fig. 2.29),

$$F_v \bar{x} = \gamma \int_V x \, dV$$

in which $\bar{x}$ is the distance from O to the line of action. Then, since $F_v = \gamma V$,

$$\bar{x} = \frac{1}{V} \int_V x \, dV$$

the distance to the centroid of the volume. Therefore, the line of action of the vertical force passes through the centroid of the volume, real or imaginary, that extends above the curved surface up to the real or imaginary free surface.

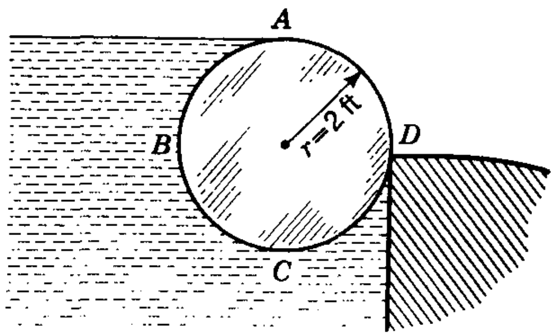


FIG. 2.31. Semifloating body.

Example 2.13: A cylindrical barrier (Fig. 2.31) holds water as shown. The contact between cylinder and wall is smooth. Considering a one-foot length of cylinder, determine (*a*) its weight and (*b*) the force exerted against the wall.

a. For equilibrium the weight of the cylinder must equal the vertical component of force exerted on it by the water. The vertical force on BCD is

$$F_{v_{BCD}} = \left(\frac{\pi r^2}{2} + 2r^2\right)\gamma = (2\pi + 8)\gamma$$

The vertical force on AB is

$$F_{v_{AB}} = -\left(r^2 - \frac{\pi r^2}{4}\right)\gamma = -(4 - \pi)\gamma$$

Hence, the weight per foot of length is

$$F_{v_{BCD}} + F_{v_{AB}} = (3\pi + 4)\gamma = 838 \text{ lb}$$

b. The force exerted against the wall is the horizontal force on ABC minus the horizontal force on CD. The horizontal components of force on BC and CD cancel since the projection of BCD on a vertical plane is zero. Hence,

$$F_H = F_{H_{AB}} = 2\gamma = 124.8 \text{ lb}$$

since the projected area is 2 ft² and the pressure at the centroid of the projected area is 62.4 lb/ft².

To find external reactions due to pressure forces, the action of the fluid may be replaced by the two horizontal components and one vertical component acting along their lines of action.

Tensile Stress in a Pipe. A circular pipe under the action of an internal pressure is in tension around its periphery. Assuming that no longitudinal stress occurs, the walls are in tension as shown in Fig. 2.32. A 1-in. section of pipe is considered, i.e., the ring between two planes normal to the axis and 1 in. apart. Taking one-half of this ring as a free body, the tensions per inch at top and bottom are, respectively, T_1, T_2, as shown in the figure. The horizontal component of force acts through the pressure center of the projected area and is $2pr$, in which p is the pressure at the center line in pounds per square inch and r is the pipe radius (internal) in inches.

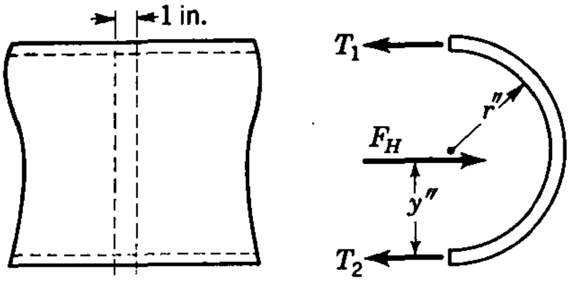


FIG. 2.32. Tensile stress in pipe.

For high pressures the pressure center may be taken at the pipe center; then $T_1 = T_2$, and

$$T = pr \tag{2.7.5}$$

in which T is the tensile force per inch. For wall thickness t in., the *tensile stress* S in the pipe wall is

$$S = \frac{T}{t} = \frac{pr}{t} \tag{2.7.6}$$

For larger variations in pressure between top and bottom of pipe the pressure center is computed, and two equations are needed,

$$T_1 + T_2 = 2pr$$
$$2rT_1 - 2pry = 0$$

in which the second equation is the moment equation about the lower end of the free body, neglecting the vertical component of force. Solving,

$$T_1 = py \qquad T_2 = p(2r - y)$$

in which y is in inches.

Example 2.14: A 4.0-in. ID steel pipe has a $\frac{1}{4}$-in. wall thickness. For an allowable tensile stress of 10,000 psi what is the maximum pressure?

$$S = \frac{pr}{t} = 10{,}000 = \frac{p2}{\frac{1}{4}}$$

and hence

$$p = 1250 \text{ psi}$$

2.8. Buoyant Force. The resultant force exerted on a body *by a static fluid* in which it is submerged or floating is called the *buoyant force*. The buoyant force always acts vertically upward. There can be no horizontal component of the resultant because the vertical projection of the submerged body or submerged portion of the floating body is always zero.

The buoyant force on a submerged body is the difference between the vertical component of pressure force on its underside and the vertical component of pressure force on its upper side. In Fig. 2.33 the upward

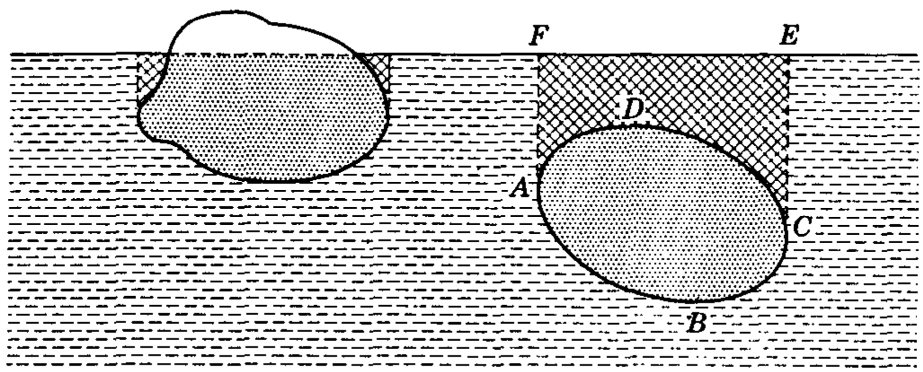


FIG. 2.33. Buoyant force on floating and submerged bodies.

force on the bottom is equal to the weight of liquid, real or imaginary, which is vertically above the surface ABC, indicated by the weight of liquid within $ABCEFA$. The downward force on the upper surface equals the weight of liquid $ADCEFA$. The difference between the two forces is a force, vertically upward, due to the weight of fluid $ABCD$ that is *displaced* by the solid. In equation form

$$F_B = \forall\gamma \tag{2.8.1}$$

in which F_B is the buoyant force, $\forall$ is the volume of fluid displaced, and γ is the specific weight of fluid. The same formula holds for floating bodies when $\forall$ is taken as the volume of liquid displaced. This is evident from inspection of the floating body in Fig. 2.33.

In Fig. 2.34*a*, the vertical force exerted on an element of the body in the form of a vertical prism of cross section δA is

$$\delta F_B = (p_2 - p_1)\,\delta A = \gamma h\,\delta A = \gamma\,\delta\forall$$

in which $\delta\forall$ is the volume of the prism. Integrating over the complete body,

$$F_B = \gamma \int_\forall d\forall = \gamma\forall$$

when γ is considered constant throughout the volume.

To find the line of action of the buoyant force, moments are taken about a convenient axis O and are equated to the moment of the resultant, thus,

$$\gamma \int x \, d\mathcal{V} = \gamma \mathcal{V} \bar{x}$$

or

$$\bar{x} = \frac{1}{\mathcal{V}} \int x \, d\mathcal{V}$$

in which $\bar{x}$ is the distance from the axis to the line of action. This equation yields the distance to the centroid of the volume; hence *the buoyant force acts through the centroid of the displaced volume of fluid.* This holds for

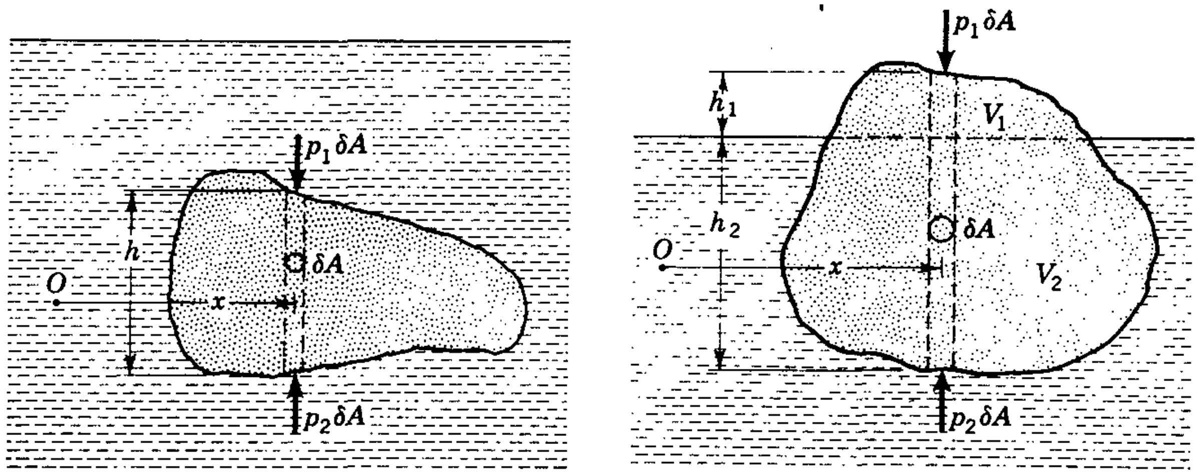


FIG. 2.34. Vertical force components on element of body.

both submerged and floating bodies. The centroid of the displaced volume of fluid is called the *center of buoyancy.*

When the body floats at the interface of a static two-fluid system (Fig. 2.34*b*) the buoyant force on a vertical prism of cross section δA is

$$\delta F_B = (p_2 - p_1)\delta A = (\gamma_2 h_2 + \gamma_1 h_1)\delta A$$

in which γ_1, γ_2 are the specific weights of the lighter and heavier fluids, respectively. Integrating over the area,

$$F_B = \gamma_2 \int h_2 \, dA + \gamma_1 \int h_1 \, dA = \gamma_2 \mathcal{V}_2 + \gamma_1 \mathcal{V}_1$$

$\mathcal{V}_1$ is the volume of lighter fluid displaced, and $\mathcal{V}_2$ is the volume of heavier fluid displaced. To locate the line of action of the buoyant force, moments are taken,

$$F_B \bar{x} = \gamma_1 \int x \, d\mathcal{V}_1 + \gamma_2 \int x \, d\mathcal{V}_2$$

or

$$\bar{x} = \frac{\gamma_1 \int x \, d\mathcal{V}_1 + \gamma_2 \int x \, d\mathcal{V}_2}{\gamma_1 \mathcal{V}_1 + \gamma_2 \mathcal{V}_2} = \frac{\gamma_1 \bar{x}_1 \mathcal{V}_1 + \gamma_2 \bar{x}_2 \mathcal{V}_2}{\gamma_1 \mathcal{V}_1 + \gamma_2 \mathcal{V}_2}$$

in which $\bar{x}_1$, $\bar{x}_2$ are distances to centroids of volumes $\mathcal{V}_1$, $\mathcal{V}_2$, respectively.

The resultant does not, in general, pass through the centroid of the whole volume.

In solving a statics problem involving submerged or floating objects, the object is generally taken as a free body, and a free-body diagram is drawn. The action of the fluid is replaced by the buoyant force. The weight of the object must be shown (acting through its center of gravity) as well as all other contact forces.

Weighing an odd-shaped object when suspended in two different fluids yields sufficient data to determine its weight, volume, specific weight, and specific gravity. Figure 2.35 shows two free-body diagrams for the same

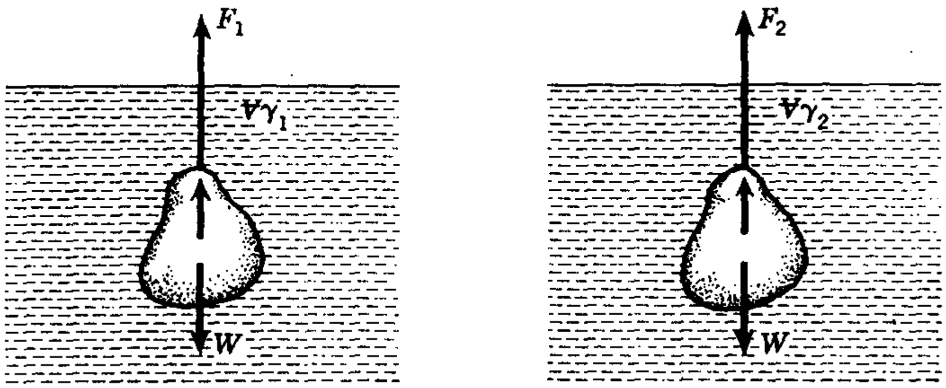


FIG. 2.35. Free-body diagram for body suspended in a fluid.

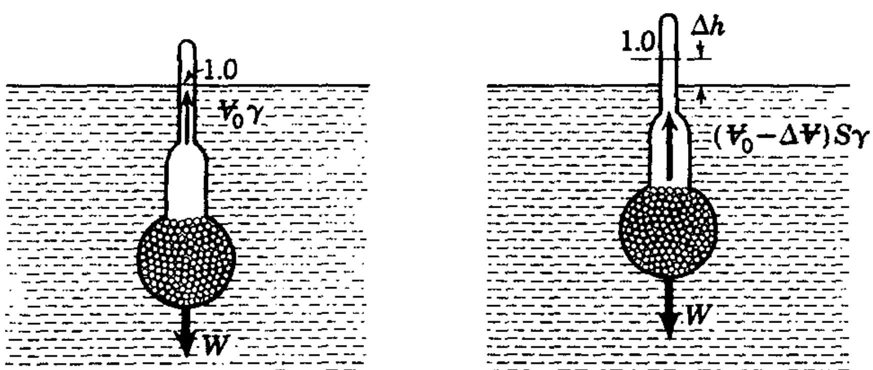


FIG. 2.36. Hydrometer, in water and in liquid of specific gravity S.

object suspended and weighed in two fluids. F_1, F_2 are the weights submerged; γ_1, γ_2 are the specific weights of the fluids. W and V, the weight and volume of the object, are desired.

The equations of equilibrium are written

$$F_1 + V\gamma_1 = W \qquad F_2 + V\gamma_2 = W$$

and solved

$$V = \frac{F_1 - F_2}{\gamma_2 - \gamma_1} \qquad W = \frac{F_1\gamma_2 - F_2\gamma_1}{\gamma_2 - \gamma_1}$$

A *hydrometer* uses the principle of buoyant force to determine specific gravities of liquids. Figure 2.36 shows a hydrometer in two liquids. It has a stem of prismatic cross section a. Considering the liquid on the left to be distilled water, $S = 1.00$, the hydrometer floats in equilibrium when

$$V_0\gamma = W \tag{2.8.2}$$

in which V_0 is the volume submerged, γ is the specific weight of water, and W is the weight of hydrometer. The position of the liquid surface is marked 1.00 on the stem to indicate unit specific gravity S. When the hydrometer is floated in another liquid, the equation of equilibrium becomes

$$(V_0 - \Delta V)S\gamma = W \tag{2.8.3}$$

in which $\Delta V = a\,\Delta h$. Solving for Δh, with Eqs. (2.8.2) and (2.8.3),

$$\Delta h = \frac{V_0}{a}\frac{S-1}{S} \tag{2.8.4}$$

from which the stem may be marked off to read specific gravities.

Example 2.15: A piece of ore weighing 7 lb in air was found to weigh 5.6 lb when submerged in water. What is its volume and specific gravity?

The buoyant force due to air may be neglected. From Fig. 2.35

$$7 = 5.6 + 62.4V \qquad V = 0.0224 \text{ ft}^3$$

$$S = \frac{7}{0.0224 \times 62.4} = 5$$

2.9. Stability of Floating and Submerged Bodies. A body floating in a static liquid has vertical stability. A small upward displacement decreases the volume of liquid displaced, resulting in an unbalanced downward force which tends to return the body to its original position. Similarly, a small downward displacement results in a greater buoyant force, which causes an unbalanced upward force.

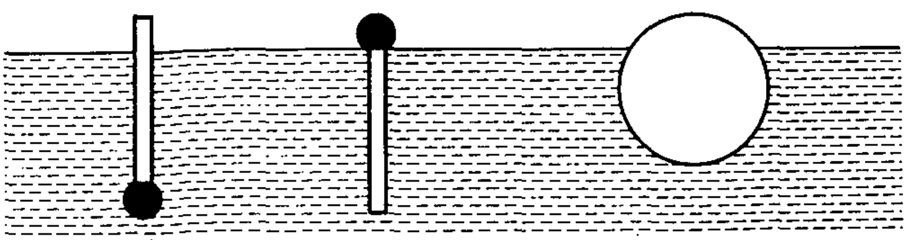


FIG. 2.37. Examples of (a) stable, (b) unstable, (c) neutral equilibrium.

A body has linear stability when a small linear displacement in any direction sets up restoring forces tending to return the body to its original position. It has rotational stability when a restoring couple is set up by any small angular displacement.

Methods for determining rotational stability are developed in the following discussion. A body may float in stable, unstable, or neutral equilibrium. When a body is in unstable equilibrium, any small angular displacement sets up a couple that tends to increase the angular displacement. With the body in neutral equilibrium, any small angular displacement sets up no couple whatever. Figure 2.37 illustrates the three cases of equilibrium: (*a*) a light piece of wood with a metal weight at its bottom is stable; (*b*) when the metal weight is at the top, the body is in equilibrium but any slight angular displacement causes the body to assume the position in *a*; (*c*) a homogeneous sphere or right-circular cylinder is in equilibrium for any angular rotation, i.e., no couple results from an angular displacement.

A submerged object is rotationally stable only when its center of gravity is below the center of buoyancy, as in Fig. 2.38*a*. When the object is

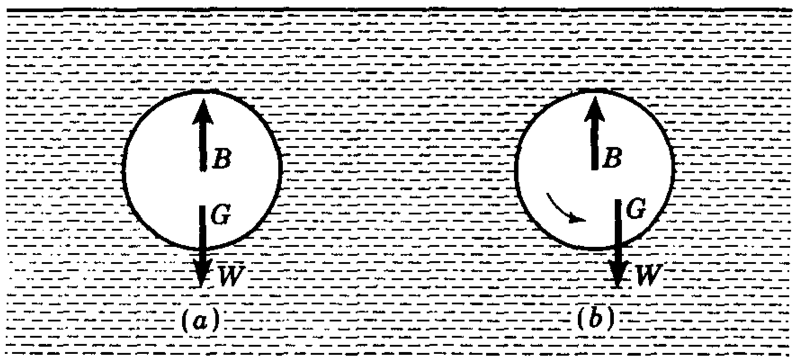


FIG. 2.38. Rotationally stable submerged body.

rotated in a counterclockwise direction as in Fig. 2.38*b*, the buoyant force and weight produce a couple in the clockwise direction.

Normally, when a body is too heavy to float, it submerges and goes down until it rests on the bottom. Although the specific weight of a liquid increases slightly with depth, the higher pressure tends to cause the liquid to compress the body or to penetrate into pores of solid substances, thus decreasing the buoyancy of the body. A ship, for example, is sure to go to the bottom once it is completely submerged, owing to compression of air trapped in various places within it.

Determination of Rotational Stability of Floating Objects. Any floating object with center of gravity below its center of buoyancy (centroid of displaced volume) floats in stable equilibrium, as in Fig. 2.37*a*. Certain floating objects, however, are in stable equilibrium when their center of gravity is above the center of buoyancy. The stability of prismatic bodies is first considered, followed by an analysis of general floating bodies for small angles of tip.

Figure 2.39*a* is a cross section of a body with all other parallel cross sections identical. The center of buoyancy is always at the centroid of the displaced volume, which is at the centroid of the cross-sectional area below liquid surface in this case. Hence, when the body is tipped, as in Fig. 2.39*b*, the center of buoyancy is at the centroid B' of the trapezoid $ABCD$; the buoyant force acts upward through B', and the weight acts downward through G, the center of gravity of the body. When the vertical through B' intersects the original center line above G, as at M, a restoring couple is produced, and the body is in stable equilibrium. The intersection of the buoyant force and the center line is called the *metacenter*, designated M. When M is above G, the body is stable; when

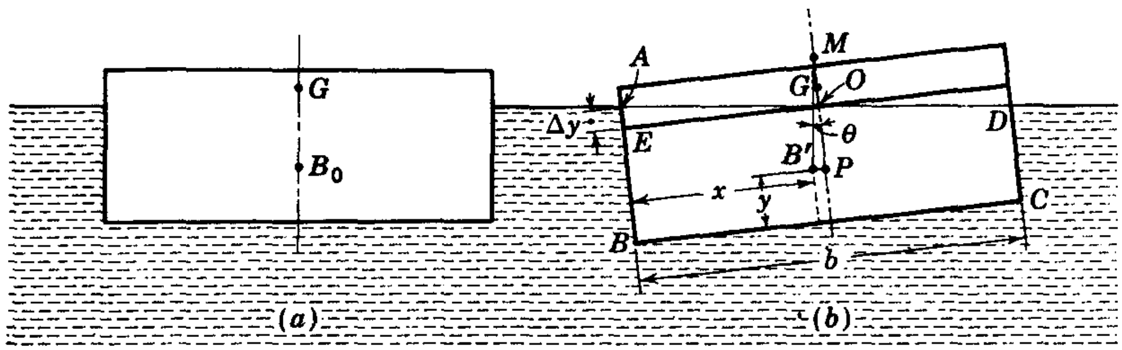


FIG. 2.39. Stability of prismatic body.

below G, it is unstable; and when at G, it is in neutral equilibrium. The distance $\overline{MG}$ is called the *metacentric height* and is a direct measure of the stability of the body. The restoring couple is

$$W\overline{MG} \sin \theta$$

in which θ is the angular displacement and W the weight of the body.

Example 2.16: In Fig. 2.39 a scow 20 ft wide and 60 ft long has a gross weight of 225 short tons (2000 lb). Its center of gravity is 1.0 ft above the water surface. Find the metacentric height and restoring couple when $\Delta y = 1.0$ ft.

The depth of submergence h in the water is

$$h = \frac{225 \times 2000}{20 \times 60 \times 62.4} = 6.0 \text{ ft}$$

The centroid in the tipped position is located with moments about AB and BC,

$$x = \frac{5 \times 20 \times 10 + 2 \times 20 \times \frac{1}{2} \times \frac{20}{3}}{6 \times 20} = 9.46 \text{ ft}$$

$$y = \frac{5 \times 20 \times \frac{5}{2} + 2 \times 20 \times \frac{1}{2} \times 5\frac{2}{3}}{6 \times 20} = 3.03 \text{ ft}$$

By similar triangles AEO and $B'PM$,

$$\frac{\Delta y}{b/2} = \frac{\overline{B'P}}{\overline{MP}}$$

$\Delta y = 1$, $b/2 = 10$, $\overline{B'P} = 10 - 9.46 = 0.54$ ft; then

$$\overline{MP} = \frac{0.54 \times 10}{1} = 5.40 \text{ ft}$$

G is 7.0 ft from the bottom; hence

$$\overline{GP} = 7.00 - 3.03 = 3.97 \text{ ft}$$

and

$$\overline{MG} = \overline{MP} - \overline{GP} = 5.40 - 3.97 = 1.43 \text{ ft}$$

The scow is stable since $\overline{MG}$ is positive; the righting moment is

$$W\overline{GM} \sin\theta = 225 \times 2000 \times 1.43 \times \frac{1}{\sqrt{101}} = 64{,}000 \text{ lb-ft}$$

Nonprismatic Cross Sections. For a floating object of variable cross section, such as a ship (Fig. 2.40a), a convenient formula may be developed for determination of metacentric height for very small angles of

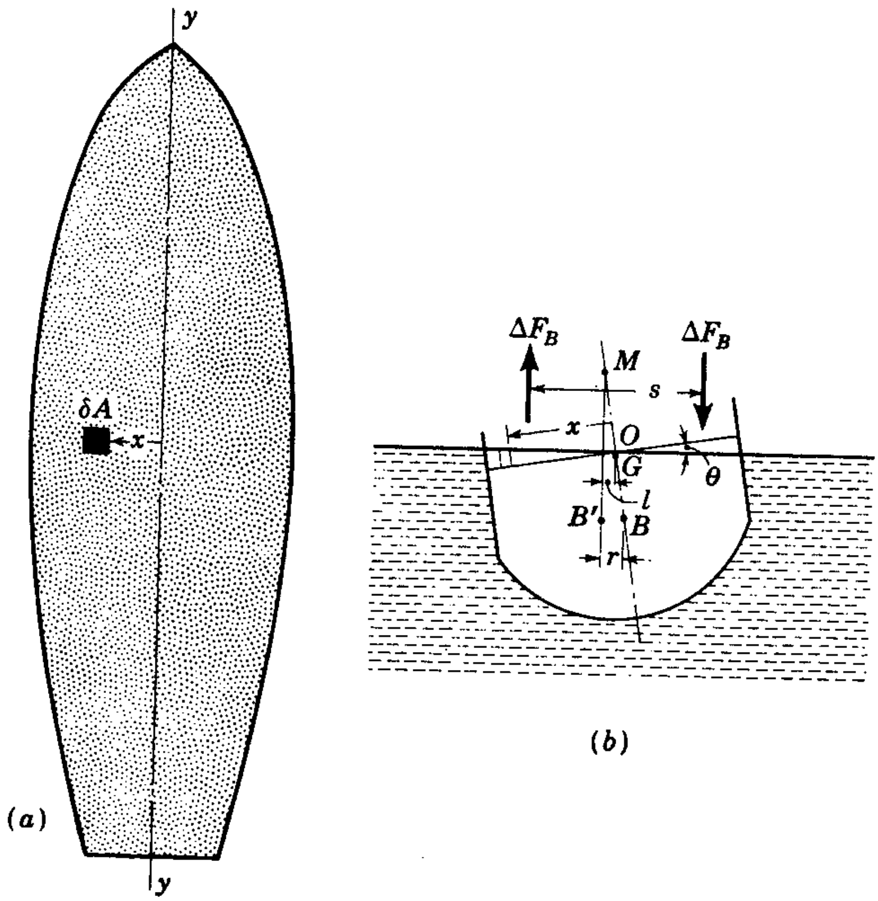


FIG. 2.40. Stability relations in body of variable cross section.

rotation θ. The horizontal shift in center of buoyancy r (Fig. 2.40b) is determined by the change in buoyant forces due to the wedge being submerged, which causes an upward force on the left, and by the other wedge decreasing the buoyant force by an equal amount ΔF_B on the right. The force system, consisting of the original buoyant force at B and the couple $\Delta F_B \times s$ due to the wedges, must have as resultant the equal buoyant force at B'. With moments about B to determine the shift r,

$$\Delta F_B \times s = Wr \tag{2.9.1}$$

The amount of the couple may be determined with moments about O, the center line of the body at the liquid surface. For an element of area δA on the horizontal section through the body at the liquid surface, an element of volume of the wedge is $x\theta\ \delta A$; the buoyant force due to this element is $\gamma x\theta\ \delta A$, and its moment about O is $\gamma\theta x^2\ \delta A$, in which θ is the small angle of tip in radians. By integrating over the complete original horizontal area at the liquid surface, the couple is determined to be

$$\Delta F_B \times s = \gamma\theta \int_A x^2\, dA = \gamma\theta I \tag{2.9.2}$$

in which I is the moment of inertia of the area

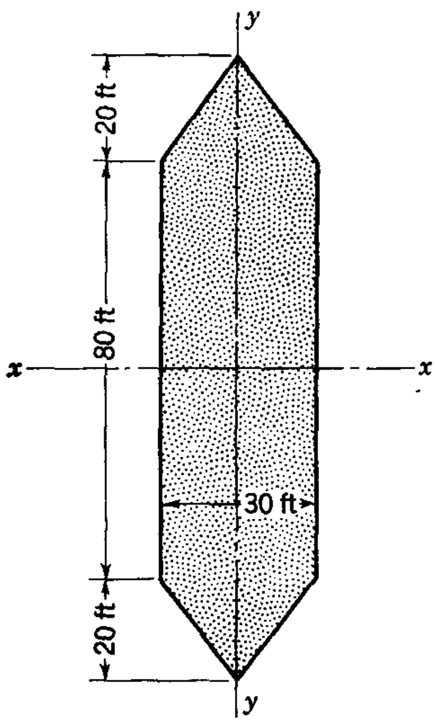


FIG. 2.41. Horizontal cross section of ship at water line.

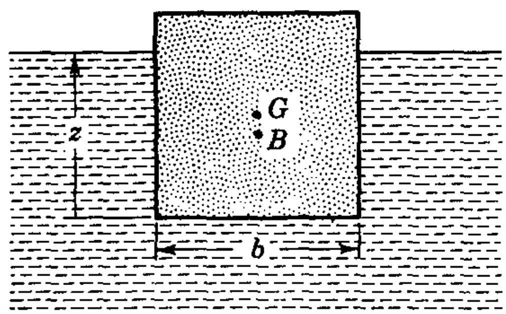


FIG. 2.42. Cube floating in liquid.

about the axis y–y (Fig. 2.40a). Substitution into Eq. (2.9.1) produces

$$\gamma\theta I = Wr = V\gamma r$$

in which V is the total volume of liquid displaced.

Since θ is very small,

$$\overline{MB} \sin\theta = \overline{MB}\theta = r$$

or

$$\overline{MB} = \frac{r}{\theta} = \frac{I}{V}$$

The metacentric height is then

$$\overline{MG} = \overline{MB} \mp \overline{GB}$$

or

$$\overline{MG} = \frac{I}{V} \mp \overline{GB} \tag{2.9.3}$$

The minus sign is used if G is above B, the plus sign if G is below B.

Example 2.17: A ship displacing 1000 tons has the horizontal cross section at water line shown in Fig. 2.41. Its center of buoyancy is 6.0 ft below water surface, and its center of gravity is 1.0 ft below water surface. Determine its metacentric height for rolling (about y–y-axis) and for pitching (about x–x-axis).

$$\overline{GB} = 5.0 \text{ ft} \qquad V = \frac{1000 \times 2000}{62.4} = 32{,}100 \text{ ft}^3$$

$$I_{y-y} = \tfrac{1}{12} \times 80 \times \overline{30^3} + 4 \times \tfrac{1}{12} \times 20 \times \overline{15^3} = 202{,}500 \text{ ft}^4$$

$$I_{x-x} = \tfrac{1}{12} \times 30 \times \overline{80^3} + 2 \times \tfrac{1}{36} \times 30 \times \overline{20^3} + 600 \times \overline{46.67^2} = 2{,}603{,}000 \text{ ft}^4$$

For rolling:

$$\overline{MG} = \frac{I}{V} - \overline{GB} = \frac{202{,}500}{32{,}100} - 5 = 1.32 \text{ ft}$$

For pitching:

$$\overline{MG} = \frac{I}{V} - \overline{GB} = \frac{2{,}603{,}000}{32{,}100} - 5 = 76.2 \text{ ft}$$

Example 2.18: A homogeneous cube of specific gravity S_c floats in a liquid of specific gravity S. Find the range of specific-gravity ratios S_c/S for it to float with sides vertical.

In Fig. 2.42, b is the length of one edge of the cube. The depth of submergence z is determined by application of the buoyant-force equation.

$$b^3\gamma S_c = b^2 z\gamma S$$

in which γ is the specific weight of water. Solving for depth of submergence,

$$z = b\frac{S_c}{S}$$

The center of buoyancy is $z/2$ from the bottom, and the center of gravity is $b/2$ from the bottom. Hence

$$\overline{GB} = \frac{b - z}{2} = \frac{b}{2}\left(1 - \frac{S_c}{S}\right)$$

After applying Eq. (2.9.3),

$$\overline{MG} = \frac{I}{V} - \overline{GB} = \frac{1}{12}\frac{b \times b^3}{zb^2} - \frac{b - z}{2}$$

or

$$\overline{MG} = \frac{b}{12}\frac{S}{S_c} - \frac{b}{2}\left(1 - \frac{S_c}{S}\right)$$

When $\overline{MG}$ equals zero, $S_c/S = 0.212,\ 0.788$. Substitution shows that $\overline{MG}$ is positive for

$$0 < \frac{S_c}{S} < 0.212 \qquad 0.788 < \frac{S_c}{S} < 1.0$$

Figure 2.43 is a graph of $\overline{MG}/b$ vs. S_c/S.

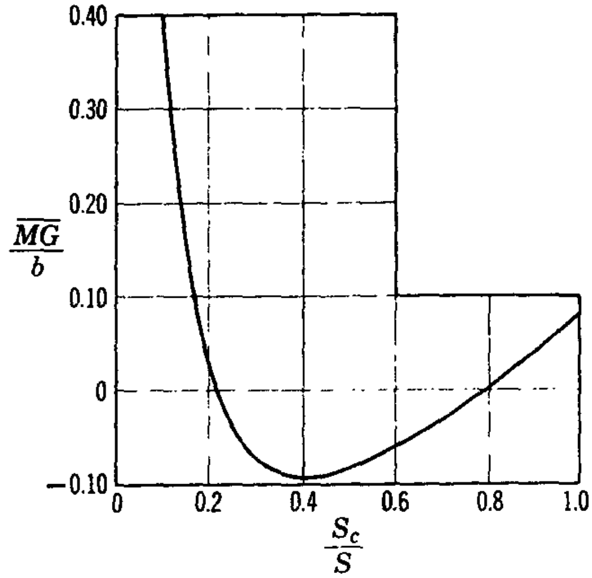


Fig. 2.43. Plot of S_c/S vs. $\overline{MG}/b$.

PROBLEMS

2.1. Prove that the pressure is the same in all directions at a point in a static fluid for the three-dimensional case.

2.2. The container of Fig. 2.44 holds water and air as shown. What is the pressure at A, B, C, and D in pounds per square foot?

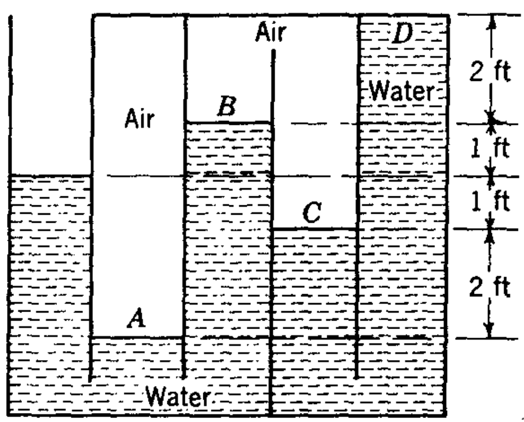


Fig. 2.44

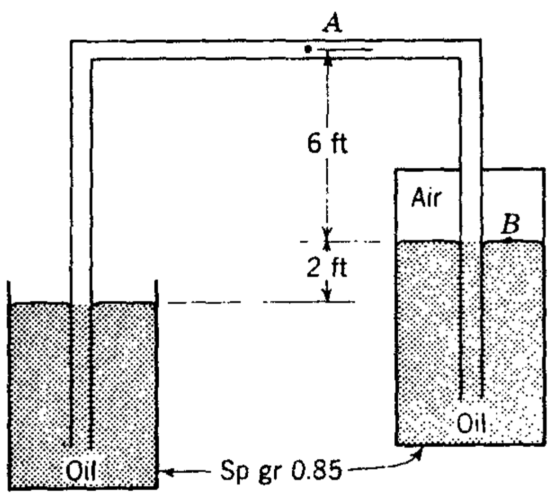


Fig. 2.45

2.3. The tube in Fig. 2.45 is filled with oil. Determine the pressure at A and B in feet of water.

2.4. Calculate the pressure at A, B, C, and D of Fig. 2.46 in pounds per square inch.

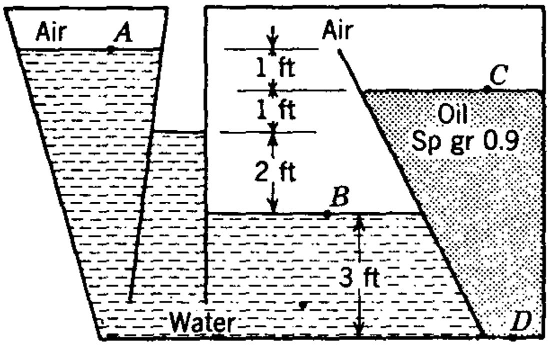


FIG. 2.46

2.5. Derive the law of variation of static pressure for an incompressible fluid by considering a free body of fluid that is an inclined right circular cylinder.

2.6. Derive the equations that give the pressure and density at any elevation in a static gas when conditions are known at one elevation and the temperature gradient β is known.

2.7. By a limiting process as $\beta \to 0$, derive the isothermal case from the results of Prob. 2.6.

2.8. By use of the results of Prob. 2.6, determine the pressure and density at 5000-ft elevation when $p = 14.5$ psia, $t = 68°F$, and $\beta = -0.003°F/ft$ at elevation 1000 ft for air.

2.9. For isothermal air at 40°F, determine the pressure and density at 10,000 ft when the pressure is 15 psia at sea level.

2.10. In isothermal air at 60°F what is the vertical distance for reduction of density by 10 per cent?

2.11. Express a pressure of 5 psi in: (*a*) inches of mercury, (*b*) feet of water, (*c*) feet of acetylene tetrabromide, sp gr 2.94.

2.12. A bourdon gage reads 2-psi suction, and the barometer is 29.5 in. mercury. Express the pressure in six other customary ways.

2.13. Express 3 atmospheres in feet of water gage. Barometer reading 29.2 in.

2.14. Bourdon gage A inside a pressure tank reads 10 psi. Another bourdon gage B outside the pressure tank, connected with the tank, reads 18 psi, and an aneroid barometer reads 30 in. mercury. What is the absolute pressure measured by A in inches of mercury?

2.15. Determine the heights of columns of water; kerosene, sp gr 0.83; and acetylene tetrabromide, sp gr 2.94, equivalent to 10 in. mercury.

2.16. For a reading $h = 16$ in. in Fig. 2.8*a* determine the pressure at A in pounds per square inch. The liquid has a specific gravity of 1.90.

2.17. Determine the reading h in Fig. 2.8*b* for $p_A = 2.5$ psi suction if the liquid is kerosene, sp gr 0.83

2.18. For $h = 6$ in. in Fig. 2.8*b* and barometer reading 29 in., with water the liquid, find p_A in feet of water absolute.

2.19. In Fig. 2.8*c* $S_1 = 0.86$, $S_2 = 1.0$, $h_2 = 8.3$ in., $h_1 = 17$ in. Find p_A in inches of mercury gage. If the barometer reading is 29.5 in., what is p_A in feet of water absolute?

2.20. Gas is contained in vessel A of Fig. 2.8c. With water the manometer fluid and $h_1 = 7$ in., determine the pressure at A in inches of mercury.

2.21. In Fig. 2.9a $S_1 = 1.0$, $S_2 = 0.95$, $S_3 = 1.0$, $h_1 = h_2 = 1.0$ ft, and $h_3 = 3.0$ ft. Compute $p_A - p_B$ in inches of water.

2.22. In Prob. 2.21 find the gage difference h_2 for $p_A - p_B = -10$ in. water.

2.23. In Fig. 2.9b $S_1 = S_3 = 0.83$, $S_2 = 13.6$, $h_1 = 16$ in., $h_2 = 8$ in., and $h_3 = 12$ in. (a) Find p_A if $p_B = 10$ psi. (b) For $p_A = 20$ psia and a barometer reading of 29.0 in. find p_B in feet of water gage.

2.24. Find the gage difference h_2 in Prob. 2.23 for $p_A = p_B$.

2.25. In Fig. 2.47, A contains water and the manometer fluid has a specific gravity of 2.94. When the left meniscus is at zero on the scale, $p_A = 4$ in. water. Find the reading of the right meniscus for $p_A = 1$ psi with no adjustment of the U-tube or scale.

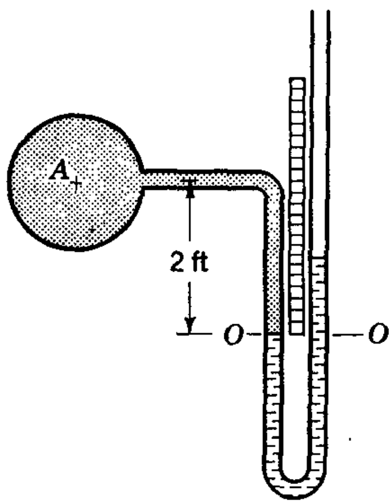


FIG. 2.47

2.26. A vertical gas pipe in a building contains gas, $\rho = 0.0016$ slug/ft^3 and $p = 3.0$ in. water gage in the basement. At the top of the building 800 ft higher, determine the gas pressure in inches water gage for two cases: (a) gas assumed incompressible and (b) gas assumed isothermal. Barometric pressure 34 ft water; $t = 70$°F.

2.27. In Fig. 2.12 determine R, the gage difference for a difference in gas pressure of 1 in. water. $S_2 = 1.0$; $S_3 = 1.05$; $a/A = 0.01$.

2.28. The inclined manometer of Fig. 2.13 reads zero when A and B are at the same pressure. The diameter of reservoir is 2.0 in., and that of the inclined tube $\frac{1}{4}$ in. For $\theta = 30°$, gage fluid sp gr 0.832, find $p_A - p_B$ in pounds per square inch as a function of gage reading R in feet.

2.29. A tank of liquid $S = 0.86$ is accelerated uniformly in a horizontal direction so that the pressure decreases within the liquid 1 psi/ft in the direction of motion. Determine the acceleration.

2.30. The free surface of a liquid makes an angle of 20° with the horizontal when accelerated uniformly in a horizontal direction. What is the acceleration?

2.31. In Fig. 2.48, $a_x = 8.05$ ft/sec^2, $a_y = 0$. Find the imaginary free liquid surface and the pressure at B, C, D, and E.

2.32. In Fig. 2.48, $a_x = 0$, $a_y = -16.1$ ft/sec^2. Find the pressure at B, C, D, and E.

2.33. In Fig. 2.48, $a_x = 8.05$ ft/sec^2, $a_y = 16.1$ ft/sec^2. Find the imaginary free surface and the pressure at B, C, D, and E.

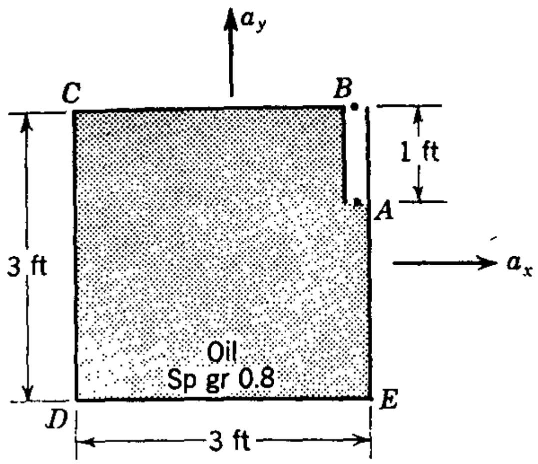


FIG. 2.48

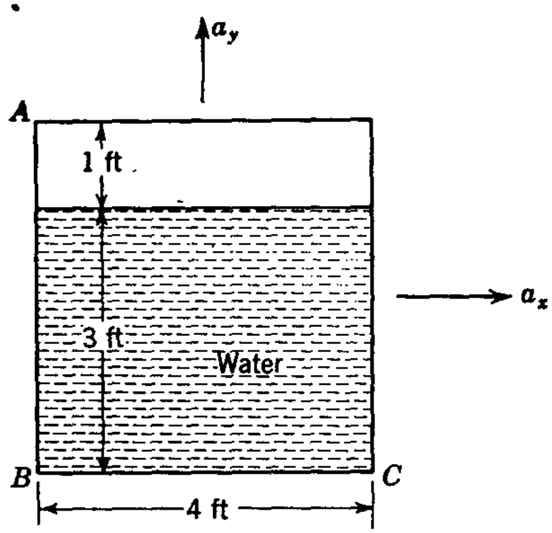


FIG. 2.49

2.34. In Fig. 2.49, $a_x = 32.2$ ft/sec^2, $a_y = 0$. Find the pressure at A, B, and C.

2.35. In Fig. 2.49, $a_x = 16.1$ ft/sec^2, $a_y = 16.1$ ft/sec^2. Find the pressure at A, B, and C.

2.36. A circular cross-sectioned tank of 6-ft depth and 4 ft diameter is filled with liquid and accelerated uniformly in a horizontal direction. If one-third of the liquid spills out, determine the acceleration.

2.37. Derive an expression for pressure variation in a constant-temperature gas undergoing an acceleration a_x in the x-direction.

2.38. The tube of Fig. 2.50 is filled with liquid, sp gr 2.40. When accelerated to the right 8.05 ft/sec^2, draw the imaginary free surface and determine the pressure at A. For $p_A = 8$ psi vacuum determine a_x.

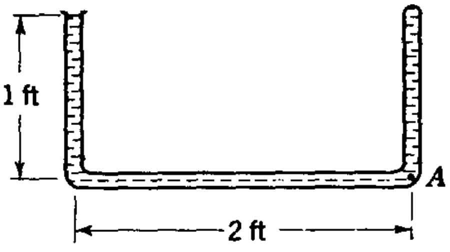


FIG. 2.50

2.39. A cubical box 3 ft on an edge, open at the top and half filled with water, is placed on an inclined plane making a 30° angle with the horizontal. The box alone weighs 100 lb and has a coefficient of friction with the plane of 0.30. Determine the acceleration of the box and the angle the free water surface makes with the horizontal.

2.40. Show that the pressure is the same in all directions at a point in a liquid moving as a solid.

2.41. A closed box contains two immiscible liquids. When accelerated uniformly in the x-direction, prove that the interface and zero pressure surface are parallel.

2.42. A vessel containing liquid, sp gr 1.2, is rotated about a vertical axis. The pressure at one point 2 ft radially from the axis is the same as at another point 4 ft from the axis and with elevation 2 ft higher. Calculate the rotational speed.

2.43. The U-tube of Fig. 2.50 is rotated about a vertical axis 6 in. to the right of A at such a speed that the pressure at A is zero gage. What is the rotational speed?

2.44. Locate the vertical axis of rotation and the speed of rotation of the U-tube of Fig. 2.50 so that the pressures of liquid at the mid-point of the U-tube and at A are both zero.

2.45. An incompressible fluid of density ρ moving as a solid rotates at speed ω about an axis inclined at $\theta°$ with the vertical. Knowing the pressure at one point in the fluid, how do you find the pressure at any other point?

2.46. A right circular cylinder of radius r_0 and height h_0 with axis vertical is open at the top and filled with liquid. At what speed must it rotate so that half the area of the bottom is exposed?

2.47. A liquid rotating about a *horizontal* axis as a solid has a pressure of 10 psi at the axis. Determine the pressure variation along a vertical line through the axis for density ρ and speed ω.

2.48. Prove by integration that a paraboloid of revolution has a volume equal to half its circumscribing cylinder.

2.49. A tank containing two immiscible liquids is rotated about a vertical axis. Prove that the interface has the same shape as the zero pressure surface.

2.50. A hollow sphere of radius r_0 is filled with liquid and rotated about its vertical axis at speed ω. Locate the circular line of maximum pressure.

2.51. A gas following the law $p\rho^{-n}$ = constant is rotated about a vertical axis as a solid. Derive an expression for pressure in a radial direction for speed ω, pressure p_0, and density ρ_0 at a point on the axis.

2.52. Determine the weight W that can be sustained by the 100-lb force acting on the piston of Fig. 2.51.

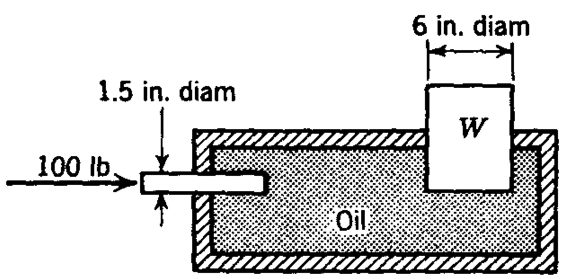


Fig. 2.51

2.53. Neglecting the weight of the container (Fig. 2.52), find (*a*) the force tending to lift the circular top *CD* and (*b*) the compressive load on the pipe wall at *A-A*.

2.54. Find the force of oil on the top surface *CD* of Fig. 2.52 if the liquid level in the open pipe is reduced by 4.0 ft.

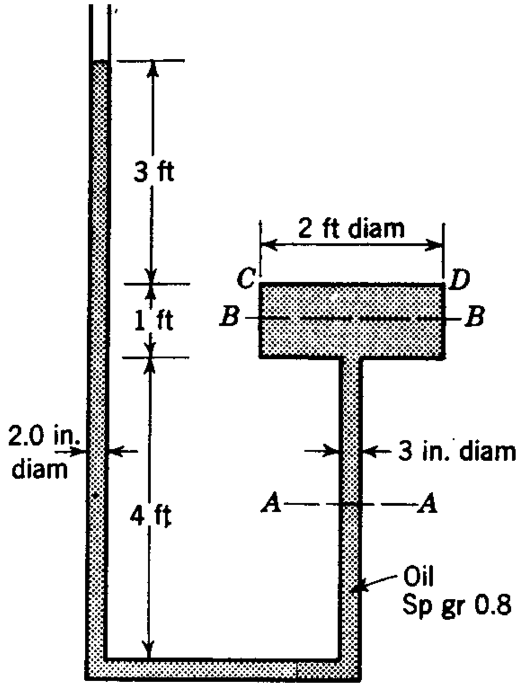


FIG. 2.52

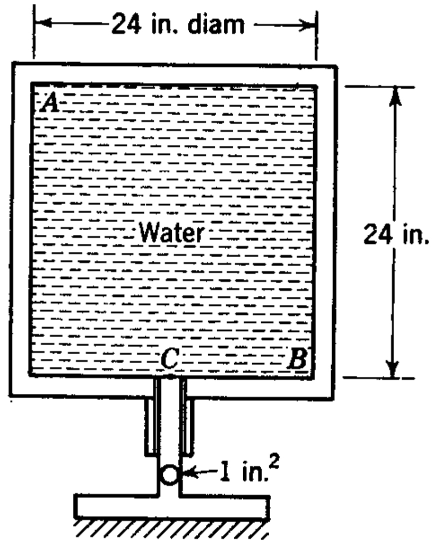


FIG. 2.53

2.55. The cylindrical container of Fig. 2.53 weighs 100 lb when empty. It is filled with water and supported on the piston. What force is exerted on the upper end of the cylinder? If an additional 100-lb weight were placed on the cylinder, how much would the water force against the top of the cylinder be increased?

2.56. A barrel 2 ft in diameter filled with water has a vertical pipe of 0.50 in. diameter attached to the top. Neglecting compressibility, how many pounds of water must be added to the pipe to exert a force of 2000 lb on the top of the barrel?

2.57. A right-angled triangular surface has a vertex in the free surface of a liquid (Fig. 2.54). Find the force on one side (*a*) by integration and (*b*) by formula.

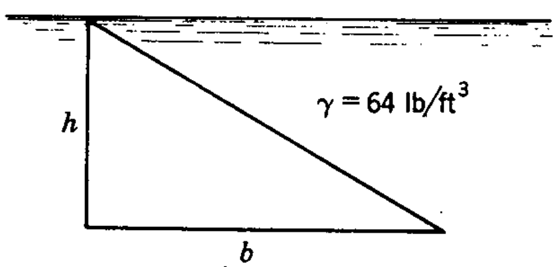


FIG. 2.54

2.58. Determine the magnitude of the force acting on triangle *ABC* of Fig. 2.55 (*a*) by integration and (*b*) by formula.

2.59. Find the moment about AB of the force acting on one side of the surface ABC of Fig. 2.55.

2.60. Locate a horizontal line below AB of Fig. 2.55 such that the magnitude of pressure force on the surface is equal above and below the line.

2.61. A cubical box 4 ft on an edge is open at the top and filled with water. When accelerated upward 8.05 ft/sec², find the magnitude of water force on one side of the box.

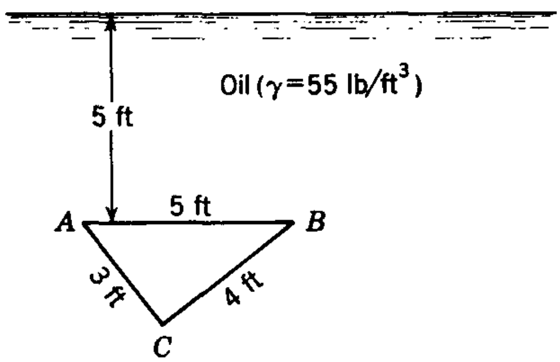


FIG. 2.55

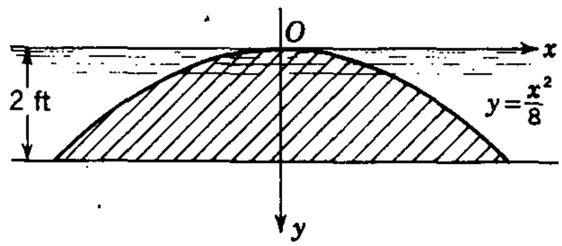


FIG. 2.56

2.62. Determine the force acting on one side of the vertical surface of Fig. 2.56.

2.63. Calculate the force exerted by water on one side of the vertical annular area shown in Fig. 2.57.

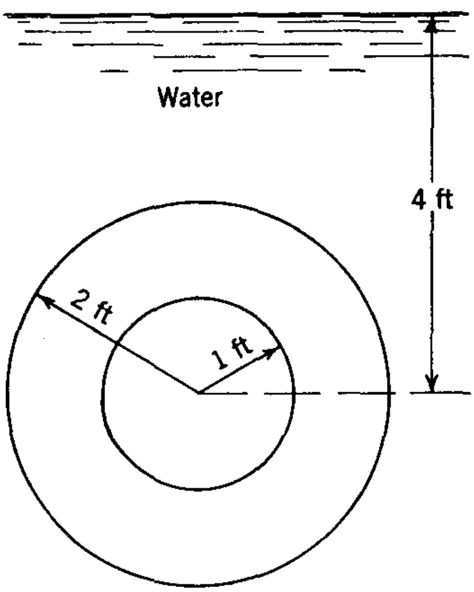


FIG. 2.57

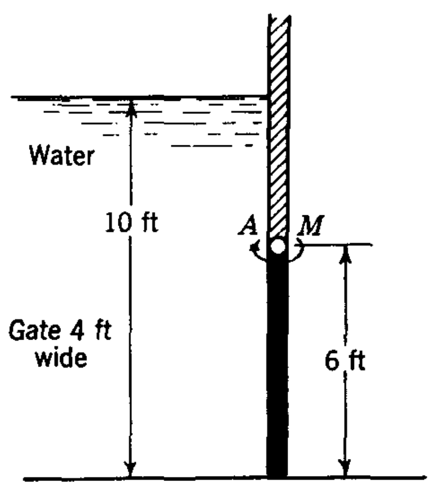


FIG. 2.58

2.64. Determine the moment at A required to hold the gate as shown in Fig. 2.58.

2.65. If there is water on the other side of the gate (Fig. 2.58) up to A, determine the resultant force due to water on both sides of the gate, including its line of action.

2.66. The shaft of the gate in Fig. 2.59 will fail at a moment of 100,000 lb-ft. Determine the maximum value of liquid depth h.

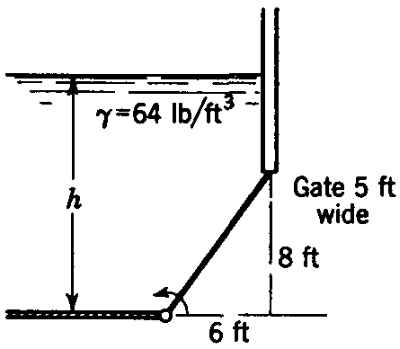


FIG. 2.59

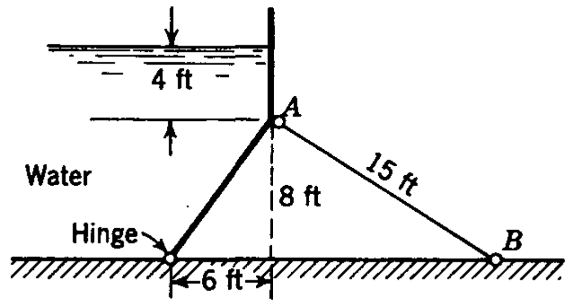


FIG. 2.60

2.67. The dam of Fig. 2.60 has a strut AB every 10 ft. Determine the compressive force in the strut, neglecting the weight of the dam.

2.68. Locate the distance the pressure center is below the liquid surface in the triangular area ABC of Fig. 2.55 by integration and by formula.

2.69. By integration locate the pressure center horizontally in Fig. 2.55.

2.70. By using the pressure prism, determine the resultant force and location for the triangle of Fig. 2.54.

2.71. By integration, determine the pressure center for Fig. 2.54.

2.72. Locate the pressure center for the annular area of Fig. 2.57.

2.73. Locate the pressure center for the gate of Fig. 2.58.

2.74. A vertical square area 4 by 4 ft is submerged in water with upper edge 2 ft below the surface. Locate a horizontal line on the surface of the square such that (a) the force on the upper portion equals the force on the lower portion and (b) the moment of force about the line due to the upper portion equals the moment due to the lower portion.

2.75. An equilateral triangle with one edge in a water surface extends downward at a 45° angle. Locate the pressure center in terms of the length of a side b.

2.76. In Fig. 2.59 develop the expression for y_p in terms of h.

2.77. Locate the pressure center of Fig. 2.56.

2.78. Locate the pressure center for the vertical area of Fig. 2.61.

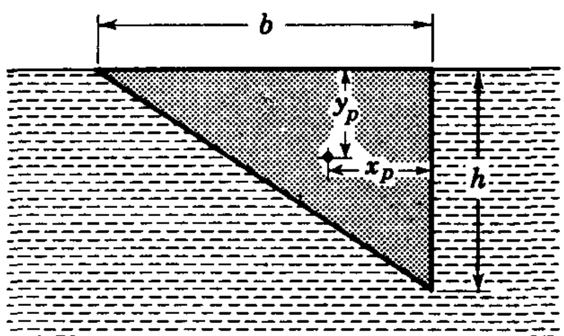


FIG. 2.61

2.79. The gate of Fig. 2.62 weighs 400 lb/ft normal to the paper. Its center of gravity is 1.5 ft from the left face and 2.0 ft above the lower face. It is hinged at O. Determine the water-surface position for the gate just to start to come up. (Water surface is below the hinge.)

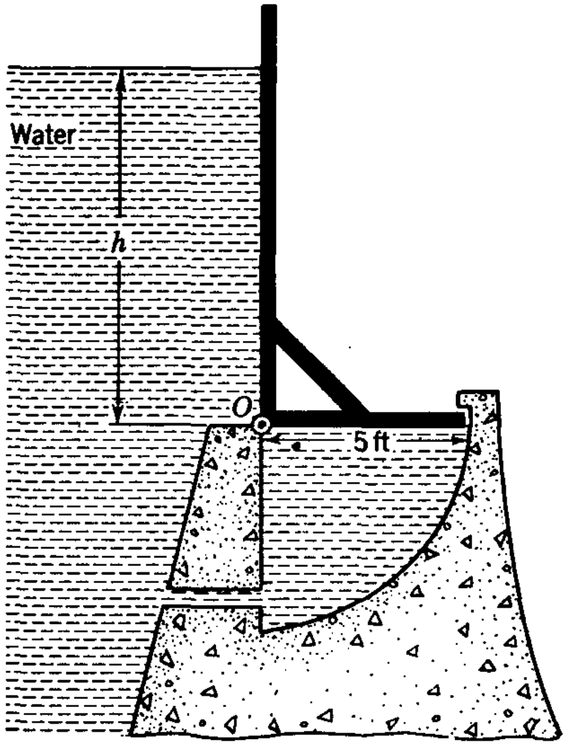


FIG. 2.62

2.80. Find h of Prob. 2.79 for the gate just to come up to the vertical position shown.

2.81. Determine the value of h and the force against the stop when this force is a maximum for the gate of Prob. 2.79.

2.82. Determine y of Fig. 2.63 so the flashboards will tumble when water reaches their top.

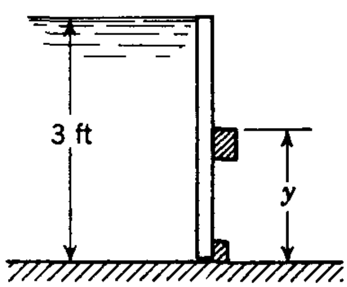


FIG. 2.63

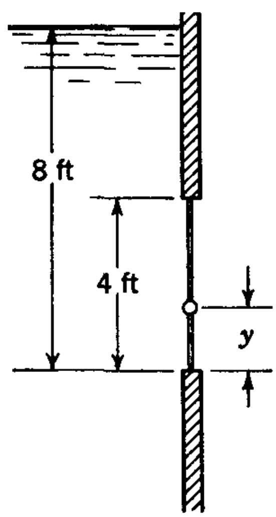


FIG. 2.64

2.83. Determine the hinge location y of the rectangular gate of Fig. 2.64 so that it will open when the liquid surface is as shown.

2.84. By use of the pressure prism, show that the pressure center approaches the centroid of an area as its depth of submergence is increased.

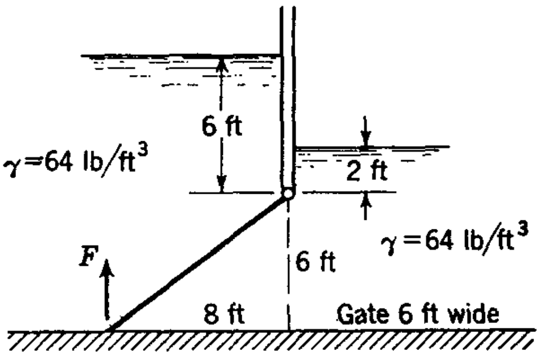


Fig. 2.65

2.85. (*a*) Find the magnitude and line of action of force on each side of the gate of Fig. 2.65. (*b*) Find the resultant force due to the liquid on both sides of the gate. (*c*) Determine F to open the gate if it is uniform and weighs 6000 lb.

2.86. For linear stress variation over the base of the dam of Fig. 2.66, (*a*) locate where the resultant crosses the base and (*b*) compute the maximum and minimum compressive stresses at the base. Neglect hydrostatic uplift.

2.87. Work Prob. 2.86 with the addition that the hydrostatic uplift varies linearly from 60 ft at A to zero at the toe of the dam.

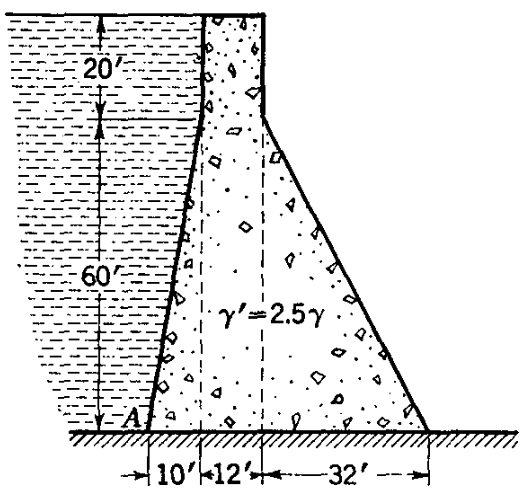


Fig. 2.66

Fig. 2.67

2.88. Find the moment M at O (Fig. 2.67) to hold the gate closed.

2.89. A cube 1 ft on an edge is filled with liquid, sp gr 0.65, and is accelerated downward 8.05 ft/sec². Find the resultant force on one side of the cube due to liquid pressure.

2.90. A cylinder 2 ft in diameter and 6 ft long is accelerated uniformly along its axis in a horizontal direction 16.1 ft/sec². It is filled with liquid, $\gamma = 50$ lb/ft³, and has a pressure along its axis of 10 psi before acceleration commences. Find the net force exerted against the liquid in the cylinder.

2.91. A closed cube, 1 ft on an edge, has a small opening at the center of its top. When it is filled with water and rotated uniformly about a vertical axis through its center at ω rad/sec, find the force on a side due to the water in terms of ω.

2.92. The gate shown in Fig. 2.68 is in equilibrium. Compute W, the weight of counterweight per foot of width, neglecting the weight of the gate. Is this gate in stable equilibrium?

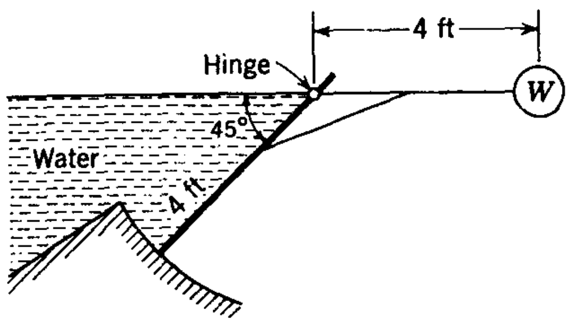


Fig. 2.68

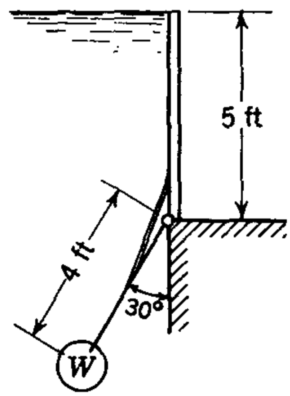


Fig. 2.69

2.93. The gate of Fig. 2.69 weighs 100 lb/ft normal to the page. It is in equilibrium as shown. Neglecting the weight of the arm and brace supporting the counterweight, (*a*) determine W and (*b*) determine whether the gate is in stable equilibrium. The weight is made of concrete, sp gr 2.50.

2.94. The plane gate (Fig. 2.70) weighs 500 lb per foot of length, with its center of gravity 6.0 ft from the hinge at O. (*a*) Find h as a function of θ for equilibrium of the gate. (*b*) Is the gate in stable equilibrium for any values of θ?

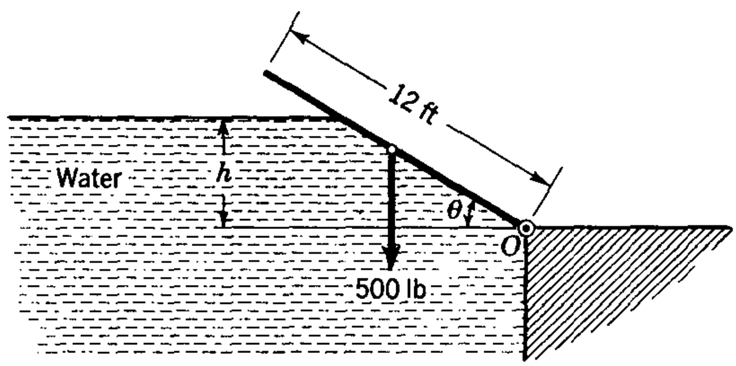


Fig. 2.70

2.95. A 16-ft-diameter pressure pipe carries liquid at 200 psi. What thickness pipe wall is required for maximum stress of 10,000 psi?

2.96. To obtain the same flow area, which pipe system requires the least steel: a single pipe or four pipes having half the diameter? The maximum allowable pipe wall stress is the same in each case.

2.97. A thin-walled hollow sphere 8 ft in diameter holds gas at 200 psi. For allowable stress of 6000 psi determine the minimum wall thickness.

2.98. A cylindrical container 6 ft high and 4 ft in diameter provides for pipe tension with two hoops a foot from each end. When filled with water, what is the tension in each hoop due to the water?

2.99. A 1-in.-diameter steel ball covers a $\frac{3}{4}$-in. hole in a pressure chamber where the pressure is 6000 psi. What force is required to lift the ball from the opening?

2.100. If the horizontal component of force on a curved surface did *not* equal the force on a projection of the surface onto a vertical plane, what conclusions could you draw regarding the propulsion of a boat (Fig. 2.71)?

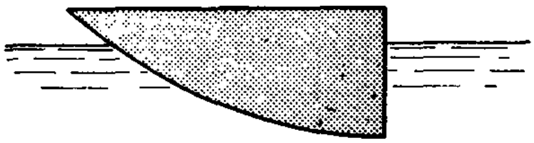


FIG. 2.71

2.101. (*a*) Determine the horizontal component of force acting on the radial gate (Fig. 2.72) and its line of action. (*b*) Determine the vertical component of force and its line of action. (*c*) What force F is required to open the gate, neglecting its weight?

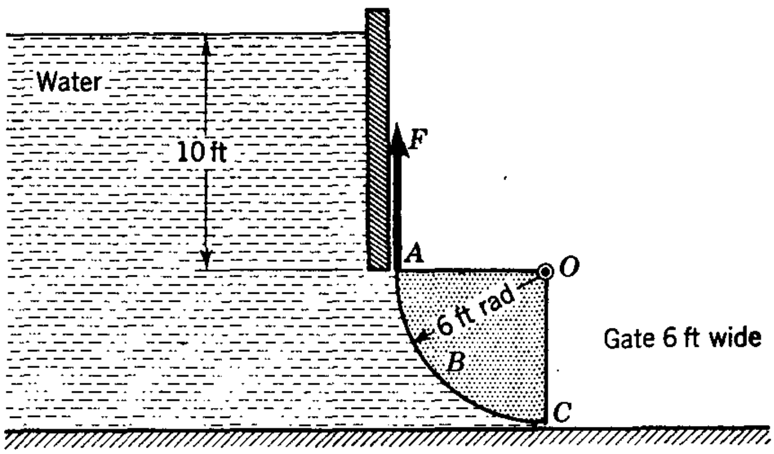


FIG. 2.72

2.102. Calculate the force F required to hold the gate of Fig. 2.73 in a closed position. $R = 2$ ft.

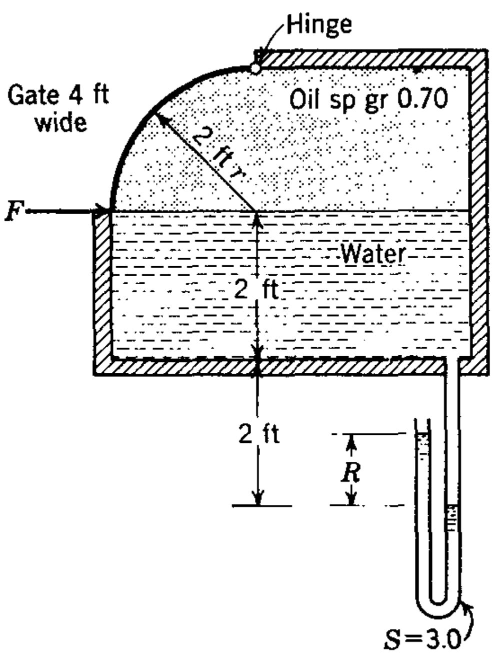


FIG. 2.73

2.103. Calculate the force F required to open or hold closed the gate of Fig. 2.73 when $R = 1.5$ ft.

2.104. What is R of Fig. 2.73 for no force F required to hold the gate closed or to open it?

2.105. Find the vertical component of force on the curved gate of Fig. 2.74, including its line of action.

2.106. Determine the moment M to hold the gate of Fig. 2.74, neglecting its weight.

2.107. Find the resultant force, including its line of action, acting on the outer surface of the first quadrant of a spherical shell of radius 2.0 ft with center at the origin. Its center is 3 ft below the water surface.

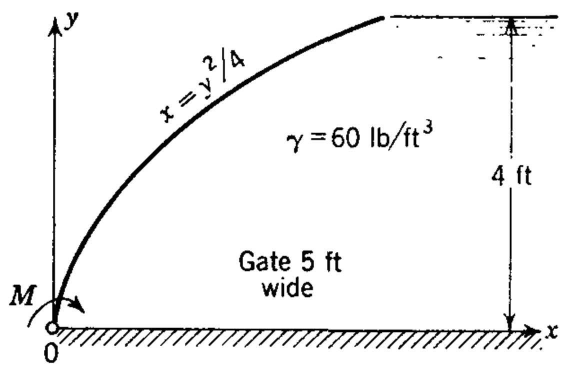


FIG. 2.74

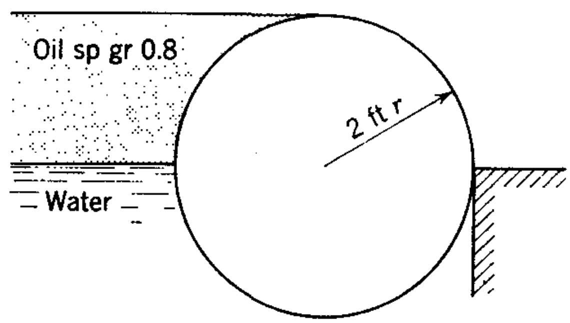


FIG. 2.75

2.108. The log holds the water as shown in Fig. 2.75. Determine (*a*) the force per foot pushing it against the dam, (*b*) the weight of the log per foot of length, and (*c*) its specific gravity.

2.109. The cylinder of Fig. 2.76 is filled with liquid as shown. Find (*a*) the horizontal component of force on AB per foot of length, including its line of action, and (*b*) the vertical component of force on AB per foot of length, including its line of action.

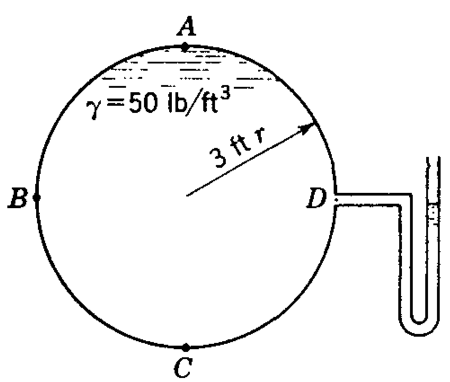


FIG. 2.76

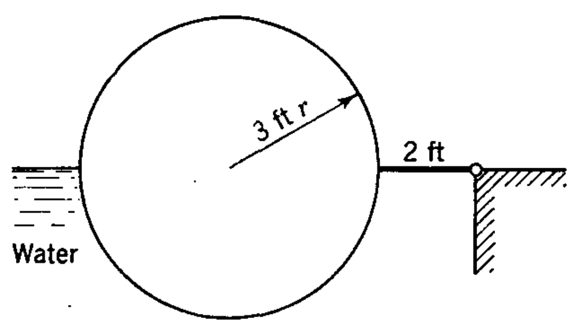


FIG. 2.77

2.110. The cylinder gate of Fig. 2.77 is made up from a circular cylinder and a plate, hinged at the dam. The gate position is controlled by pumping water into or out of the cylinder. The center of gravity of the empty gate is on the line of symmetry 4 ft from the hinge. It is in equilibrium when empty in the position shown. How many cubic feet of water must be added per foot of cylinder to hold the gate in its position when the water surface is raised 3 ft?

2.111. A hydrometer weighs 0.007 lb and has a stem 0.20 in. in diameter. Compute the distance between specific gravity markings 1.0 and 1.1.

2.112. Design a hydrometer to read specific gravities in the range from 0.80 to 1.10 when the scale is to be 2 in. long.

2.113. A sphere 1 ft in diameter, sp gr 1.4, is immersed in a liquid having a density varying with the depth y below the surface given by $\rho = 2 + 0.1y$. Determine the equilibrium position of the sphere in the liquid.

2.114. A cube, 2 ft on an edge, has its lower half of specific gravity 1.4 and upper half of specific gravity 0.6. It is submerged into a two-layered fluid, the lower of specific gravity 1.2 and the upper of specific gravity 0.9. Determine the height of the top of the cube above the interface.

2.115. Determine the density, specific volume, and volume of an object that weighs 3 lb in water and 4 lb in oil, sp gr 0.83.

2.116. Two cubes, of the same size, 27 ft^3, one of sp gr 0.80, the other of sp gr 1.1, are connected by a short wire and placed in water. What portion of the lighter cube is above the water surface, and what is the tension in the wire?

2.117. In Fig. 2.78 the hollow triangular prism is in equilibrium as shown when $z = 1$ ft and $y = 0$. Find the weight of prism per foot of length and z in terms of y for equilibrium. Both liquids are water. Determine the value of y for $z = 1.5$ ft.

2.118. How many pounds of concrete, $\gamma = 150$ lb/ft^3, must be attached to a beam having a volume of 4 ft^3 and specific gravity 0.65 to cause both to sink in water?

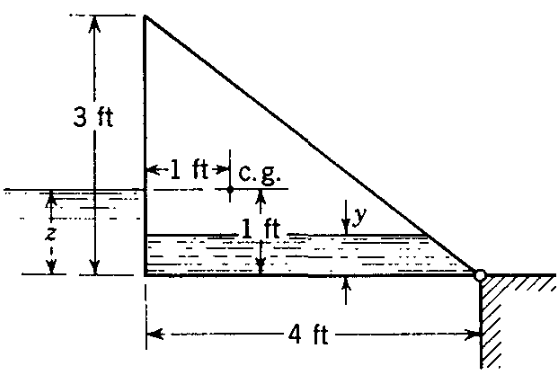


FIG. 2.78

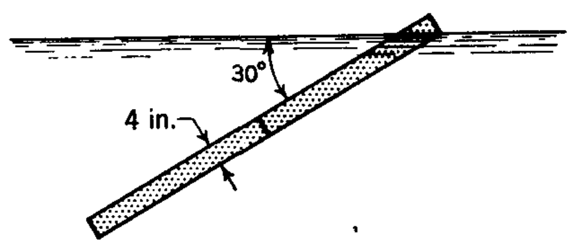


FIG. 2.79

2.119. Two beams, each 6 ft by 12 by 4 in., are attached at their ends and float as shown in Fig. 2.79. Determine the specific gravity of each beam.

2.120. A wooden cylinder 24 in. in diameter, sp gr 0.50, has a concrete cylinder 2 ft long of the same diameter, sp gr 2.50, attached to one end. Determine the lengths of wooden cylinder for the system to float in stable equilibrium with axis vertical.

2.121. What are the proportions r_0/h of a right circular cylinder of specific gravity S so that it will float in water with end faces horizontal in stable equilibrium?

2.122. Will a beam 10 ft long with square cross section, sp gr 0.75, float in stable equilibrium in water with two sides horizontal?

2.123. Determine the metacentric height of the torus shown in Fig. 2.80.

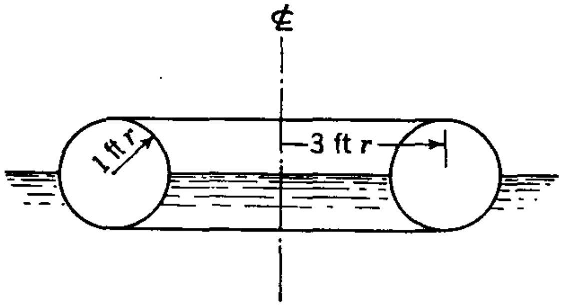


Fig. 2.80

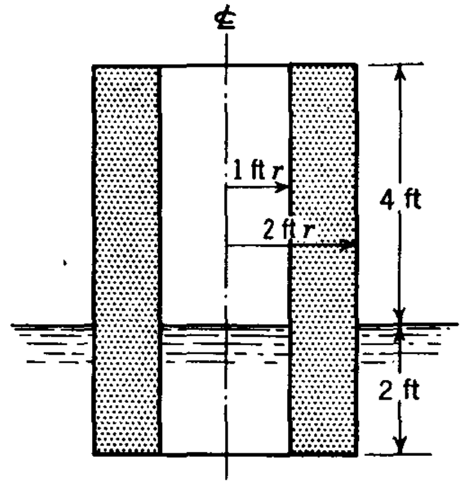


Fig. 2.81

2.124. Determine whether the thick-walled cylinder of Fig. 2.81 is stable in the position shown.

2.125. A spherical balloon 40 ft in diameter is open at the bottom and filled with hydrogen. For barometer reading of 28 in. mercury and 80°F, what is the total weight of the balloon and the load to hold it stationary?

2.126. The normal stress is the same in all directions at a point in a fluid

(*a*) *only* when the fluid is frictionless
(*b*) *only* when the fluid is frictionless and incompressible
(*c*) *only* when the fluid has zero viscosity and is at rest
(*d*) when there is no motion of one fluid layer relative to an adjacent layer
(*e*) regardless of the motion of one fluid layer relative to an adjacent layer

2.127. The pressure in the air space above an oil (sp gr 0.75) surface in a tank is 2 psi. The pressure 5.0 ft below the surface of the oil, in feet of water, is

(*a*) 7.0 (*b*) 8.37 (*c*) 9.62 (*d*) 11.16 (*e*) none of these answers

2.128. The pressure, in inches of mercury gage, equivalent to 8 in. of water plus 6 in. manometer fluid, sp gr 2.94, is

(*a*) 1.03 (*b*) 1.88 (*c*) 2.04 (*d*) 3.06 (*e*) none of these answers

2.129. The differential equation for pressure variation in a static fluid may be written (y measured vertically upward)

(*a*) $dp = -\gamma\, dy$ (*b*) $d\rho = -\gamma\, dy$ (*c*) $dy = -\rho\, dp$
(*d*) $dp = -\rho\, dy$ (*e*) $dp = -y\, d\rho$

2.130. In an isothermal atmosphere, the pressure

(*a*) remains constant
(*b*) decreases linearly with elevation
(*c*) increases exponentially with elevation
(*d*) varies in the same way as the density
(*e*) and density remain constant

2.131. Select the correct statement.

(*a*) Local atmospheric pressure is always below standard atmospheric pressure.
(*b*) Local atmospheric pressure depends upon elevation of locality only.
(*c*) Standard atmospheric pressure is the mean local atmospheric pressure at sea level.
(*d*) A barometer reads the difference between local and standard atmospheric pressure.
(*e*) Standard atmospheric pressure is 34 in. mercury abs.

2.132. Select the three pressures that are equivalent.

(*a*) 10.0 psi, 23.1 ft water, 4.91 in. mercury
(*b*) 10.0 psi, 4.33 ft water, 20.3 in. mercury
(*c*) 10.0 psi, 20.3 ft water, 23.1 in. mercury
(*d*) 4.33 psi, 10.0 ft water, 20.3 in. mercury
(*e*) 4.33 psi, 10.0 ft water, 8.83 in. mercury

2.133. 2 psi suction, with barometer reading 28 in. mercury, is the same as

(*a*) 4.08 in. mercury abs
(*b*) 4.08 in. mercury
(*c*) 4.62 ft water vacuum
(*d*) 32.08 in. mercury abs
(*e*) 36.42 ft water abs

2.134. With the barometer reading 29 in. mercury, 7.0 psia is equivalent to

(*a*) 0.476 atmosphere
(*b*) 0.493 atmosphere
(*c*) 7.9 psi suction
(*d*) 7.7 psi
(*e*) 13.8 in. mercury abs

2.135. In Fig. 2.8*b* the liquid is oil, sp gr 0.80. When $h = 2$ ft, the pressure at A may be expressed as

(*a*) -1.6 ft water abs
(*b*) 1.6 ft water
(*c*) 1.6 ft water suction
(*d*) 2.5 ft water vacuum
(*e*) none of these answers

2.136. In Fig. 2.8*c* air is contained in the pipe, water is the manometer liquid, and $h_1 = 2.0$ ft, $h_2 = 1.0$ ft. The pressure at A is

(*a*) 2.0 ft water abs
(*b*) 2.0 ft water vacuum
(*c*) 1.0 ft water
(*d*) 0.866 psi
(*e*) 0.433 psi

2.137. In Fig. 2.9*a*, $h_1 = 2.0$ ft, $h_2 = 1.0$ ft, $h_3 = 4.0$ ft, $S_1 = 0.80$, $S_2 = 0.65$, $S_3 = 1.0$. Then $h_B - h_A$ in feet of water is

(*a*) -3.05 (*b*) -1.75 (*c*) 3.05 (*d*) 6.25 (*e*) none of these answers

2.138. In Fig. 2.9*b*, $h_1 = 1.5$ ft, $h_2 = 1.0$ ft, $h_3 = 2.0$ ft, $S_1 = 1.0$, $S_2 = 3.0$, $S_3 = 1.0$. Then $p_A - p_B$ in pounds per square inch is

(*a*) -1.08 (*b*) 1.52 (*c*) 8.08 (*d*) 218 (*e*) none of these answers

2.139. A mercury-water manometer has a gage difference of 2.0 ft (difference in elevation of menisci). The difference in pressure, measured in feet of water, is

(*a*) 2.0 (*b*) 25.2 (*c*) 26.2 (*d*) 27.2 (*e*) none of these answers

2.140. In the inclined manometer of Fig. 2.13 the reservoir is so large that its surface may be assumed to remain at a fixed elevation. $\theta = 30°$. Used as a simple manometer for measuring air pressure, it contains water, and $R = 1.2$ ft. The pressure at A, in inches of water, is

(*a*) 7.2 (*b*) 7.2 vacuum (*c*) 12.5 (*d*) 14.4 (*e*) none of these answers

2.141. A closed cubical box, 2 ft on each edge, is half filled with water, the other half being filled with oil, sp gr 0.75. When accelerated vertically upward 16.1 ft/sec², the pressure difference between bottom and top, in pounds per square foot, is

(*a*) 187.2 (*b*) 163.8 (*c*) 109.0 (*d*) 54.6 (*e*) none of these answers

2.142. When the box of Prob. 2.141 is accelerated uniformly, in a horizontal direction parallel to one side, 16.1 ft/sec², the slope of the interface is

(*a*) 0 (*b*) $-\frac{1}{4}$ (*c*) $-\frac{1}{2}$ (*d*) -1 (*e*) none of these answers

2.143. When the minimum pressure in the box of Prob. 2.142 is zero gage, the maximum pressure in feet of water is

(*a*) 0.75 (*b*) 1.0 (*c*) 1.625 (*d*) 1.875 (*e*) 2.75

2.144. When a liquid rotates at constant angular velocity about a vertical axis as a rigid body, the pressure

(*a*) decreases as the square of the radial distance
(*b*) increases linearly as the radial distance
(*c*) decreases as the square of increase in elevation along any vertical line
(*d*) varies inversely as the elevation along any vertical line
(*e*) varies as the square of the radial distance

2.145. When a liquid rotates about a vertical axis as a rigid body so that points on the axis have the same pressure as points 2 ft higher and 2 ft from the axis, the angular velocity in radians per second is

(*a*) 8.02 (*b*) 11.34 (*c*) 64.4 (*d*) not determinable from data given (*e*) none of these answers

2.146. A right-circular cylinder, open at the top, is filled with liquid, sp gr 1.2, and rotated about its vertical axis at such speed that half the liquid spills out. The pressure at the center of the bottom is

(*a*) zero
(*b*) one-fourth its value when cylinder was full
(*c*) indeterminable; insufficient data
(*d*) greater than a similar case with water as liquid
(*e*) none of these answers

2.147. A forced vortex

(*a*) turns in an opposite direction to a free vortex
(*b*) always occurs in conjunction with a free vortex
(*c*) has the velocity decreasing with the radius
(*d*) occurs when fluid rotates as a solid
(*e*) has the velocity decreasing inversely with the radius

2.148. The magnitude of force on one side of a circular surface of unit area, with centroid 10 ft below a free water surface, is

(*a*) less than 10γ
(*b*) dependent upon orientation of the area
(*c*) greater than 10γ
(*d*) the product of γ and the vertical distance from free surface to pressure center
(*e*) none of the above

2.149. A rectangular surface 3 ft by 4 ft has the lower 3-ft edge horizontal and 6 ft below a free oil surface, sp gr 0.80. The surface is inclined 30° with the horizontal. The force on one side of the surface is

(*a*) 38.4γ (*b*) 48γ (*c*) 51.2γ (*d*) 60γ (*e*) none of these answers

2.150. The pressure center of the surface of Prob. 2.149 is vertically below the liquid surface

(*a*) 10.133 ft (*b*) 5.133 ft (*c*) 5.067 ft (*d*) 5.00 ft
(*e*) none of these answers

2.151. The pressure center is

(*a*) at the centroid of the submerged area
(*b*) the centroid of the pressure prism

(c) independent of the orientation of the area
(d) a point on the line of action of the resultant force
(e) always above the centroid of the area

2.152. What is the force exerted on the vertical annular area enclosed by concentric circles of radii 1.0 and 2.0 ft? The center is 3.0 ft below a free water surface. γ = sp wt.

(a) $3\pi\gamma$ (b) $9\pi\gamma$ (c) $10.25\pi\gamma$ (d) $12\pi\gamma$ (e) none of these answers

2.153. The pressure center for the annular area of Prob. 2.152 is below the centroid of the area

(a) 0 ft (b) 0.42 ft (c) 0.44 ft (d) 0.47 ft (e) none of these answers

2.154. A vertical triangular area has one side in a free surface, with vertex downward. Its altitude is h. The pressure center is below the free surface

(a) $h/4$ (b) $h/3$ (c) $h/2$ (d) $2h/3$ (e) $3h/4$

2.155. A vertical gate 4 ft by 4 ft holds water with free surface at its top. The moment about the bottom of the gate is

(a) 42.7γ (b) 57γ (c) 64γ (d) 85.3γ (e) none of these answers

2.156. The magnitude of the resultant force acting on both sides of the gate (Fig. 2.82) is

(a) 768γ (b) 1593γ (c) 1810γ (d) 3820γ (e) none of these answers

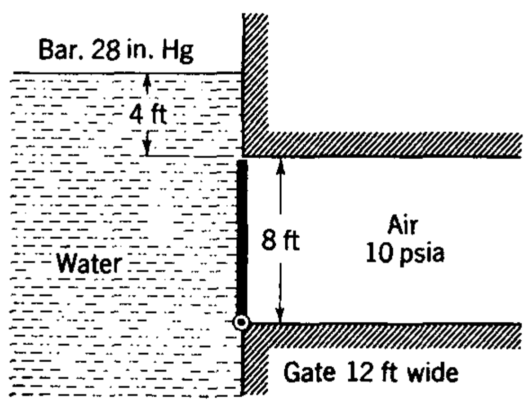


Fig. 2.82

2.157. The line of action of the resultant force on both sides of the gate in Fig. 2.82 is above the bottom of the gate

(a) 2.67 ft (b) 3.33 ft (c) 3.68 ft (d) 4.00 ft (e) none of these answers

2.158. Liquid in a cylinder 10 ft long is accelerated horizontally $20g$ ft/sec^2 along the axis of the cylinder. The difference in pressure intensities at the ends of the cylinder, in pounds per square foot, if γ = sp wt of liquid, is

(*a*) 20γ (*b*) 200γ (*c*) $20g\gamma$ (*d*) $200\gamma/g$ (*e*) none of these answers

2.159. The horizontal component of force on a curved surface is equal to the

(*a*) weight of liquid vertically above the curved surface
(*b*) weight of liquid retained by the curved surface
(*c*) product of pressure at its centroid and area
(*d*) force on a vertical projection of the curved surface
(*e*) scalar sum of all elemental horizontal components

2.160. A pipe 16 ft in diameter is to carry water at 200 psi. For an allowable tensile stress of 8000 psi, the thickness of pipe wall is

(*a*) 1.2 in. (*b*) 1.6 in. (*c*) 2.4 in. (*d*) 3.2 in. (*e*) none of these answers

2.161. The vertical component of pressure force on a submerged curved surface is equal to

(*a*) its horizontal component
(*b*) the force on a vertical projection of the curved surface
(*c*) the product of pressure at centroid and surface area
(*d*) the weight of liquid vertically above the curved surface
(*e*) none of the above answers

2.162. The vertical component of force on the quadrant of the cylinder AB (Fig. 2.83) is

(*a*) 224γ (*b*) 96.5γ (*c*) 81γ (*d*) 42.5γ (*e*) none of these answers

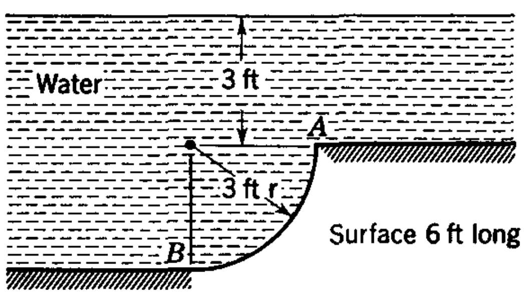


FIG. 2.83

2.163. The vertical component of force on the upper half of a horizontal right-circular cylinder, 3 ft in diameter and 10 ft long, filled with water, and with a pressure of 0.433 psi at the axis, is

(*a*) -458 lb (*b*) -331 lb (*c*) 124.8 lb (*d*) 1872 lb
(*e*) none of these answers

2.164. A cylindrical wooden barrel is held together by hoops at top and bottom. When the barrel is filled with liquid, the ratio of tension in the top hoop to tension in the bottom hoop, due to the liquid, is

(*a*) $\frac{1}{2}$ (*b*) 1 (*c*) 2 (*d*) 3 (*e*) none of these answers

2.165. A 1-in. ID pipe with $\frac{1}{8}$-in. wall thickness carries water at 250 psi. The tensile stress in the pipe wall, in pounds per square inch, is

(*a*) 125 (*b*) 250 (*c*) 500 (*d*) 2000 (*e*) none of these answers

2.166. A slab of wood 4 ft by 4 ft by 1 ft, sp gr 0.50, floats in water with a 400-lb load on it. The volume of slab submerged, in cubic feet, is

(*a*) 1.6 (*b*) 6.4 (*c*) 8.0 (*d*) 14.4 (*e*) none of these answers

2.167. The line of action of the buoyant force acts through the

(*a*) center of gravity of any submerged body
(*b*) centroid of the volume of any floating body
(*c*) centroid of the displaced volume of fluid
(*d*) centroid of the volume of fluid vertically above the body
(*e*) centroid of the horizontal projection of the body

2.168. Buoyant force is

(*a*) the resultant force on a body due to the fluid surrounding it
(*b*) the resultant force acting on a floating body
(*c*) the force necessary to maintain equilibrium of a submerged body
(*d*) a nonvertical force for nonsymmetrical bodies
(*e*) equal to the volume of liquid displaced

2.169. A body floats in stable equilibrium

(*a*) when its metacentric height is zero
(*b*) *only* when its center of gravity is below its center of buoyancy
(*c*) when $\overline{GB} - I/V$ is positive and G is above B
(*d*) when I/V is positive
(*e*) when the metacenter is above the center of gravity

2.170. A closed cubical metal box 3 ft on an edge is made of uniform sheet and weighs 1200 lb. Its metacentric height when placed in oil, sp gr 0.90, with sides vertical, is

(*a*) 0 ft (*b*) −0.08 ft (*c*) 0.62 ft (*d*) 0.78 ft (*e*) none of these answers

3

FLUID-FLOW CONCEPTS AND BASIC EQUATIONS

The statics of fluids, treated in the preceding chapter, is almost an exact science, specific weight (or density) being the only quantity that must be determined experimentally. On the other hand, the nature of *flow* of a real fluid is very complex. The basic laws describing the complete motion of a fluid are not easily formulated and handled mathematically, so recourse to experimentation is required. By an analysis based on mechanics, thermodynamics, and orderly experimentation, large hydraulic structures and efficient fluid machines have been produced.

This chapter introduces the concepts needed for analysis of fluid motion. The basic equations that enable us to predict fluid behavior are stated or derived: These are equations of motion, continuity, and momentum, and the first and second laws of thermodynamics as applied to steady flow of a perfect gas. The concepts of reversibility, irreversibility, and losses are first introduced. Viscous effects, the experimental determination of losses, and the dimensionless presentation of loss data are presented in Chap. 5 after dimensional analysis has been introduced in Chap. 4. In general, one-dimensional flow theory is developed in this chapter, with applications limited to incompressible cases where viscous effects do not predominate. Chapter 6 deals with compressible flow, and Chap. 7 with two- and three-dimensional flow.

3.1. The Concepts of Reversibility, Irreversibility, and Losses. A particular quantity of matter or a specified region in space may be designated as a *system*. All matter external to this system is referred to as its *surroundings*. A *closed system* refers to a specified mass and is limited by the boundaries of the mass. An example would be a pound mass of air contained in a cylinder. An *open system*, or *control volume*, refers to a definite, fixed region in space through which matter moves, an example being flow of air through a pipe.

A *process* may be defined as the path of the succession of states through which the system passes, such as the changes in velocity, elevation, pressure, density, temperature, etc. The expansion of air in a cylinder as the piston moves out and heat is transferred through the walls is an example of a process. Normally, the process causes some change in the surroundings, such as displacing it or transferring heat to or from its boundaries. When a process can be made to take place in such a manner that it can be *reversed*, i.e., made to return to its original state without a final change in either the system or its surroundings, it is said to be *reversible*. In any actual flow of a real fluid or change in a mechanical system, the effects of viscous friction, Coulomb friction, unrestrained expansion, hysteresis, etc., prohibit the process from being reversible. It is, however, an ideal to be strived for in design processes, and their efficiency is usually defined in terms of their nearness to reversibility.

When a certain process has a sole effect upon its surroundings that is equivalent to the raising of a weight, it is said to have done *work* on its surroundings. Any actual process is *irreversible*. The difference between the amount of work a substance can do by changing from one state to another state along a path reversibly and the actual work it produces for the same path is the *irreversibility* of the process. It may be defined in terms of work per unit mass or weight or work per unit time. Under certain conditions the irreversibility of a process is referred to as its *lost work*,[1] that is, the loss of ability to do work because of friction and other causes. In this treatise when *losses* are referred to, they mean irreversibility or lost work and do not mean an actual loss of energy.

Example 3.1: A hydroelectric power plant has a head (difference in elevation of headwater and tailwater) of 100 ft and a flow of 100 ft^3/sec of water through turbines, which rotate at 180 rpm. The torque in the turbine shaft is measured to be 28,700 lb-ft, and the horsepower output of the generator is 945. Determine the irreversibility, or losses, and the reversible work for the system. $g = 32.17$ ft/sec^2.

The potential energy of the water is 100 ft-lb/lb$_m$. Hence for perfect conversion the reversible work is 100 ft-lb/lb$_m$ or $100 \times 100 \times 62.4 = 6.24 \times 10^5$ ft-lb/sec. The work done on the shaft by the water is

$$T\omega = 28{,}700 \times \tfrac{180}{60} 2\pi = 5.41 \times 10^5 \text{ ft-lb/sec}$$

The irreversibility through the turbine is then

$$(6.24 - 5.41) \times 10^5 = 83{,}000 \text{ ft-lb/sec}$$

or

$$\frac{83{,}000}{100 \times 62.4} = 13.38 \text{ ft-lb/lb}_m$$

[1] The definitions of reversibility, irreversibility, and lost work just given are not complete; reference to a text on thermodynamics is advised for a full discussion of these concepts.

The irreversibility through the generator is

$$5.41 \times 10^5 - 945 \times 550 = 21{,}000 \text{ ft-lb/sec}$$

or

$$\frac{21{,}000}{100 \times 62.4} = 3.37 \text{ ft-lb/lb}_m$$

Efficiency of the turbine η_t is

$$\eta_t = 100 \times \frac{100 - 13.38}{100} = 86.6\%$$

and efficiency of the generator η_g is

$$\eta_g = \frac{86.62 - 3.37}{86.62} = 96.1\%$$

3.2. Types of Flow. Flow may be classified in many ways, such as turbulent, laminar; real, ideal; reversible, irreversible; steady, unsteady; uniform, nonuniform. In this and the following section various types of flow are distinguished.

Turbulent-flow situations are most prevalent in engineering practice. In turbulent flow the fluid particles (small molar masses) move in very irregular paths, causing an exchange of momentum from one portion of the fluid to another in a manner somewhat similar to the molecular momentum transfer described in Sec. 1.3, but on a much larger scale. The fluid particles can range in size from very small (say a few thousand molecules) to very large (thousands of cubic feet in a large swirl in a river or in an atmospheric gust). In a situation in which the flow could be either turbulent or nonturbulent (laminar), the turbulence sets up greater shear stresses throughout the fluid and causes more irreversibilities or losses. Also, in turbulent flow, the losses vary about as the square of the velocity, while in laminar flow, they vary as the first power of the velocity.

In *laminar* flow, fluid particles move along smooth paths in laminas, or layers, with one layer gliding smoothly over an adjacent layer. Laminar flow is governed by Newton's law of viscosity [Eq. (1.1.1) or extensions of it to three-dimensional flow], which relates shear stress to rate of angular deformation. In laminar flow, the action of viscosity damps out turbulent tendencies (see Sec. 5.3 for criteria for laminar flow). Laminar flow is not stable in situations involving combinations of low viscosity, high velocity, or large flow passages and breaks down into turbulent flow. An equation similar in form to Newton's law of viscosity may be written for turbulent flow:

$$\tau = \eta \frac{du}{dy} \tag{3.2.1}$$

The factor η, however, is not a fluid property alone but depends upon the fluid motion and the density. It is called the eddy viscosity.

In many practical flow situations both viscosity and turbulence contribute to the shear stress:

$$\tau = (\mu + \eta) \frac{du}{dy} \tag{3.2.2}$$

Experimentation is required for determination of this type of flow.

An *ideal fluid* is frictionless and incompressible and should not be confused with a perfect gas (Sec. 1.6). The assumption of an ideal fluid is helpful in analyzing flow situations involving large expanses of fluids, as in the motion of an airplane or a submarine. A frictionless fluid is nonviscous, and its flow processes are reversible.

The layer of fluid in the immediate neighborhood of an actual flow boundary that has had its velocity relative to the boundary affected by viscous shear is called the *boundary layer*. Boundary layers may be laminar or turbulent, depending generally upon their length, the viscosity, the velocity of the flow near them, and the boundary roughness.

Adiabatic flow is that flow of a fluid in which no heat is transferred to or from the fluid. *Reversible adiabatic* (frictionless adiabatic) flow is called *isentropic* flow.

Regardless of the nature of the flow, all flow situations are subject to the following relationships, which may be expressed in analytic form:

a. Newton's laws of motion must hold for every particle at every instant.

b. The continuity relationship, i.e., the law of conservation of mass.

c. The first and second laws of thermodynamics.

d. Boundary conditions, analytical statements that a real fluid has zero velocity relative to a boundary at a boundary or that ideal fluids cannot penetrate a boundary.

Other relations and equations may enter, such as an equation of state or Newton's law of viscosity.

3.3. Definitions. To proceed in an orderly manner into the analysis of fluid flow requires a clear understanding of the terminology involved. Several of the more important technical terms are defined and illustrated in this section.

Steady flow occurs when conditions at any point in the fluid do not change with the time. For example, if the velocity at a certain point is 10 ft/sec in the $+x$-direction in steady flow, it remains exactly that amount and in that direction indefinitely. This can be expressed as $\partial \mathbf{v}/\partial t = 0$, in which space ($x$, y, z coordinates of the point) is held constant. Likewise, in steady flow there is no change in density ρ, pressure p, or temperature T, with time at any point; thus

$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial p}{\partial t} = 0 \qquad \frac{\partial T}{\partial t} = 0$$

In turbulent flow, owing to the erratic motion of the fluid particles, there are always small fluctuations occurring at any point. The definition for steady flow must be generalized somewhat to provide for these fluctuations. To illustrate this, a plot of velocity against time, at some point in turbulent flow, is given in Fig. 3.1. When the temporal mean velocity

$$v_t = \frac{1}{t}\int_0^t v\,dt$$

indicated in the figure by the horizontal line, does not change with the time, the flow is said to be steady. The same generalization applies to density, pressure, temperature, etc., when they are substituted for v in the above formula.

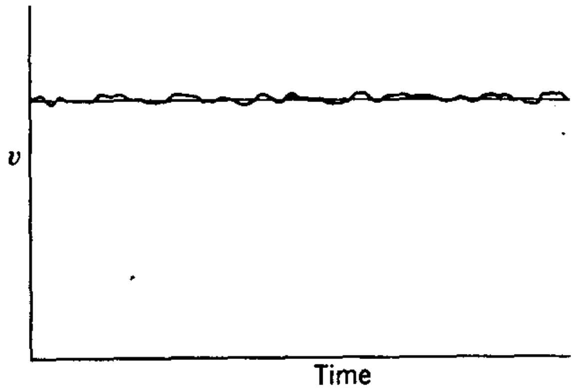


FIG. 3.1. Velocity at a point in steady turbulent flow.

The flow is *unsteady* when conditions at any point change with the time, $\partial \mathbf{v}/\partial t \neq 0$. Water being pumped through a fixed system at a constant rate is an example of steady flow. Water being pumped through a fixed system at an increasing rate is an example of unsteady flow.

Uniform flow occurs when at every point the velocity vector is identical (in magnitude and direction) for any given instant, or, in equation form, $\partial \mathbf{v}/\partial s = 0$, in which time is held constant and δs is a displacement in any direction. The equation states that there is no change in the velocity vector in any direction throughout the fluid at any one instant. It states nothing about the change in velocity at a point with time.

In flow of a real fluid in an open or closed conduit, the definition of uniform flow may also be extended in most cases even though the velocity vector at the boundary is always zero. When all parallel cross sections through the conduit are identical (i.e., when the conduit is prismatic) and the average velocity at each cross section is the same at any given instant, the flow is said to be *uniform.*

Flow such that the velocity vector varies from place to place at any instant ($\partial \mathbf{v}/\partial s \neq 0$) is *nonuniform* flow. A liquid being pumped through a long, straight pipe has uniform flow. A liquid flowing through a reducing section or through a curved pipe has nonuniform flow.

Examples of steady and unsteady flow and of uniform and nonuniform flow are: liquid flow through a long pipe at a constant rate is *steady uni-*

form flow; liquid flow through a long pipe at a decreasing rate is *unsteady uniform* flow; flow through an expanding tube at a constant rate is *steady nonuniform* flow; and flow through an expanding tube at an increasing rate is *unsteady nonuniform* flow.

One-dimensional flow neglects variations or changes in velocity, pressure, etc., transverse to the main flow direction. Conditions at a cross section are expressed in terms of average values of velocity, density, and other properties. Flow through a pipe, for example, may usually be characterized as one-dimensional. Many practical problems may be handled by this method of analysis, which is much simpler than two- and three-dimensional methods of analysis. In *two-dimensional* flow all particles are assumed to flow in parallel planes along identical paths in each of these planes; hence, there are no changes in flow normal to these planes. The flow net, developed in Chap. 7, is the most useful method for analysis of two-dimensional-flow situations. *Three-dimensional* flow is the most general flow in which the velocity components u, v, w in mutually perpendicular directions are functions of space coordinates and time x, y, z, and t. Methods of analysis are generally complex mathematically, and only simple geometrical flow boundaries may be handled.

A *streamline* is a continuous line drawn through the fluid so that it has the direction of the velocity vector at every point. There can be no flow across a streamline. Since a particle moves in the direction of the streamline at any instant, its displacement $\delta \mathbf{s}$, having components δx, δy, δz, has the direction of the velocity vector $\mathbf{q}$ that has components u, v, w in the x-, y-, z-directions, respectively. Then

$$\frac{\delta x}{u} = \frac{\delta y}{v} = \frac{\delta z}{w}$$

states that the corresponding components are proportional and hence that $\delta \mathbf{s}$ and $\mathbf{q}$ have the same direction. Expressing the displacements in differential form,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{3.3.1}$$

produces the differential equations of a streamline. Equations (3.3.1) are two independent equations. Any continuous line that satisfies them is a streamline.

In steady flow, since there is no change in direction of the velocity vector at any point, the streamline has a fixed inclination at every point and is, therefore, *fixed in space*. A particle always moves tangent to the streamline; hence, in steady flow the *path of a particle* is a streamline. In unsteady flow, since the direction of the velocity vector at any point may change with time, a streamline may shift in space from instant to instant.

A particle then follows one streamline one instant, another one the next instant, and so on, so that the path of the particle may have no resemblance to any given instantaneous streamline.

A dye, or smoke, is frequently injected into a fluid in order to trace its subsequent motion. The resulting dye, or smoke, trails are called *streak lines*. In steady flow a streak line is a streamline and the path of a particle.

Streamlines in two-dimensional flow may be obtained by inserting fine, bright particles (aluminum dust) into the fluid, brilliantly lighting one plane, and taking a photograph of the streaks made in a short time interval. Tracing on the picture continuous lines that have the direction of the streaks at every point portrays the streamlines for either steady or unsteady flow.

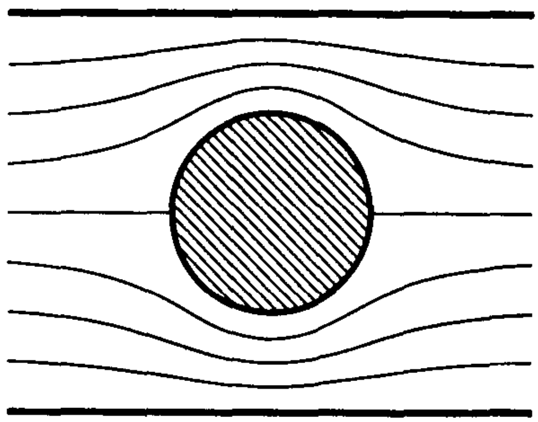


Fig. 3.2. Streamlines for steady flow around a cylinder between parallel walls.

In illustration of an incompressible two-dimensional flow, as in Fig. 3.2, the streamlines are drawn so that per unit time the volume flowing between adjacent streamlines is the same, if unit depth is considered normal to the plane of the figure. Hence, when the streamlines are closer together, the velocity must be greater, and vice versa. If v is the average velocity between two adjacent streamlines at some position where they are h apart, the flow rate Δq is

$$\Delta q = vh \tag{3.3.2}$$

At any other position on the chart where the distance between streamlines is h_1, the average velocity is $v_1 = \Delta q/h_1$. By increasing the number of streamlines drawn, i.e., by decreasing Δq, in the limiting case the velocity at a point is obtained.

A *stream tube* is the tube made by all the streamlines passing through a small, closed curve. In steady flow it is fixed in space and can have no flow through its walls because the velocity vector has no component normal to the tube surface.

Example 3.2: In two-dimensional, incompressible, steady flow around an airfoil the streamlines are drawn so that they are 1 in. apart at a great distance from the airfoil where the velocity is 120 ft/sec. What is the velocity near the airfoil where the streamlines are 0.75 in. apart?

The flow per unit width is the same at both positions; hence

$$120 \times 1 = v \times 0.75$$

and $v = 120/0.75 = 160$ ft/sec.

3.4. Continuity Equation. The continuity equation may take several forms, each appropriate for a certain class of problems, but they all derive from the general principle of *conservation of mass*. First, a general conservation of mass relation is developed, which states analytically that the net mass efflux from any control volume[1] is just equal to the time rate of decrease of mass within the control volume. The continuity equation applies to real fluids as well as to ideal fluids.

Consider a small finite volume element δV (Fig. 3.3*a*). An element of area $d\mathbf{A}$ of its surface may be expressed as a vector quantity. The vector is drawn normal to the area element, its length is proportional to the magnitude of the area element, and the sense is such that the vector is positive when drawn in the outward direction from the volume element. The fluid-velocity vector at some point in the area element is $\mathbf{v}$,

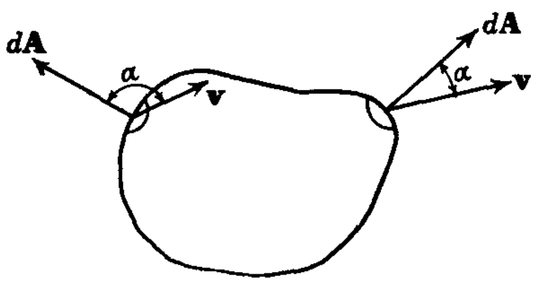


FIG. 3.3*a*. Notation for flow through a surface.

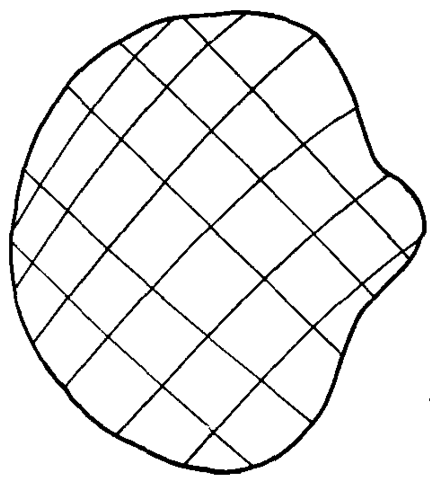


FIG. 3.3*b*. Decomposition of large volume into elements.

and the density is ρ. Then the rate of mass outflow through the area element is $\rho\mathbf{v} \cdot d\mathbf{A} = \rho v\, dA \cos\alpha$, as $v \cos\alpha$ is the component of velocity normal to the area element, and this is the component that accounts for flow through the area. Where the angle α is greater than 90°, mass flux is into the volume element. By integrating over the surface area of the small volume, the net mass efflux (mass outflow per unit time) is obtained $\int \rho\mathbf{v} \cdot d\mathbf{A}$. Since this is a small volume element, the density may be considered as given by its value at any point within the volume element. Then the time rate of decrease of mass within the element is $-(\partial/\partial t)(\rho\, \delta V)$ and conservation of mass takes the form

$$\int_{\text{area of element}} \rho\mathbf{v} \cdot d\mathbf{A} = -\frac{\partial}{\partial t}(\rho\, \delta V) \qquad (3.4.1)$$

To extend this relation to any size control volume (Fig. 3.3*b*) (remembering that control volumes are fixed in space), the volume is broken

[1] The control volume, as used here, is a fixed region in space through which matter flows.

down into a large number of very small control volume elements that completely comprise the volume. By applying Eq. (3.4.1) to each element and then summing up over all the elements, the left-hand side becomes the integral over the external surface of the control volume, because all internal surface elements occur in pairs that just cancel; i.e., flux out of one internal area element is the flux into the adjacent element. Hence,

$$\int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho\mathbf{v}\cdot d\mathbf{A} = -\int_{\substack{\text{control}\\ \text{volume}}} \frac{\partial}{\partial t}(\rho\, d\text{₮})\tag{3.4.2}$$

This is a rate equation that applies at any instant. Since the volume is fixed in space, ₮ is independent of t and

$$\int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho\mathbf{v}\cdot d\mathbf{A} = -\int_{\substack{\text{control}\\ \text{volume}}} \frac{\partial\rho}{\partial t}\, d\text{₮}\tag{3.4.3}$$

From this general law of conservation of mass, specific continuity equations may be derived.

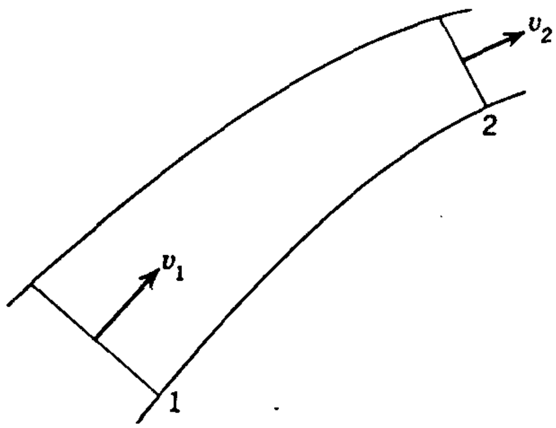


FIG. 3.4. Steady flow through a stream tube.

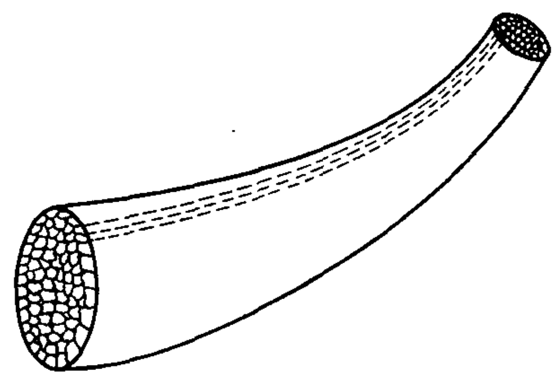


FIG. 3.5. Collection of stream tubes between fixed boundaries.

For steady flow, $\partial\rho/\partial t = 0$ and Eq. (3.4.3) becomes

$$\int_{\text{control volume area}} \rho\mathbf{v}\cdot d\mathbf{A} = 0\tag{3.4.4}$$

which states that the net mass rate of inflow into any control volume in steady flow must be zero. By applying Eq. (3.4.4) to a stream tube (Fig. 3.4), there is mass flow only through the cross sections 1 and 2; hence

$$\rho_1 v_1\,\delta A_1 = \rho_2 v_2\,\delta A_2$$

Summing up the mass flux over a collection of stream tubes (Fig. 3.5),

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \dot{m}\tag{3.4.5}$$

if ρ and V represent average density and velocity over the flow area A at each section, $\dot{m}$ is the mass per second flowing.

Example 3.3: A pipeline is carrying 0.50 lb_m/sec air. At section 1, where the diameter is 6.0 in., $p = 40$ psia, $t = 60°$, and at section 2, where the diameter is 8.0 in., $p = 30$ psia and $t = 80°F$. Find the velocity at each section.

$$\rho_1 = \frac{p_1}{RT_1} = \frac{40 \times 144}{53.3(460 + 60)} = 0.208 \text{ lb}_m/\text{ft}^3$$

and

$$\rho_2 = \frac{p_2}{RT_2} = \frac{30 \times 144}{53.3(460 + 80)} = 0.150 \text{ lb}_m/\text{ft}^3$$

From Eq. (3.4.5)

$$V_1 = \frac{\dot{m}}{\rho_1 A_1} = \frac{0.50}{0.208\pi/16} = 12.25 \text{ ft/sec}$$

$$V_2 = \frac{\dot{m}}{\rho_2 A_2} = \frac{0.50}{0.150\pi/9} = 9.56 \text{ ft/sec}$$

For incompressible flow, $\rho =$ constant and Eq. (3.4.3) becomes

$$\int \mathbf{v} \cdot d\mathbf{A} = 0 \tag{3.4.6}$$

which states that the net volume outflow per unit time is zero (this implies that the control volume is filled with fluid at all times). Applied to a collection of stream tubes, as in Fig. 3.5,

$$Q = V_1 A_1 = V_2 A_2 \tag{3.4.7}$$

in which Q, the *discharge*, is the volume per unit time flowing and V_1 and V_2 are the average velocities at cross sections 1 and 2, respectively.

Example 3.4: At section 1 of a pipe system carrying water the velocity is 3.0 ft/sec and the diameter is 2.0 ft. This same flow passes another section 2 where the diameter is 3.0 ft. Find the discharge and the velocity at section 2.

From Eq. (3.4.7)

$$Q = V_1 A_1 = 3.0\pi = 9.42 \text{ cfs}$$

and

$$V_2 = \frac{Q}{A_2} = \frac{9.42}{2.25\pi} = 1.33 \text{ ft/sec}$$

For two- and three-dimensional flow studies, differential expressions of the continuity equation are used. For three-dimensional cartesian coordinates, Eq. (3.4.3) is applied to the volume element $\delta x\, \delta y\, \delta z$ of Fig. 3.6 with center at (x,y,z) where the velocity components in the x, y, z-directions are u, v, w, respectively, and ρ is the density. Consider first the flux through the pair of faces normal to the x-direction. On the right-hand face the flux outward is

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right] \delta y\, \delta z$$

since both ρ and u are assumed to vary continuously throughout the fluid. In the expression, $\rho u\ \delta y\ \delta z$ is the mass flux through the center face normal to the x-axis. The second term is the rate of increase of mass flux with respect to x, multiplied by the distance $\delta x/2$ to the right-hand face. Similarly on the left-hand face the flux into the volume is

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y\ \delta z$$

since the step is $-\delta x/2$. The net flux out through these two faces is

$$\frac{\partial}{\partial x}(\rho u)\ \delta x\ \delta y\ \delta z$$

The other two directions yield similar expressions; hence the net outflow is

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\ \delta y\ \delta z$$

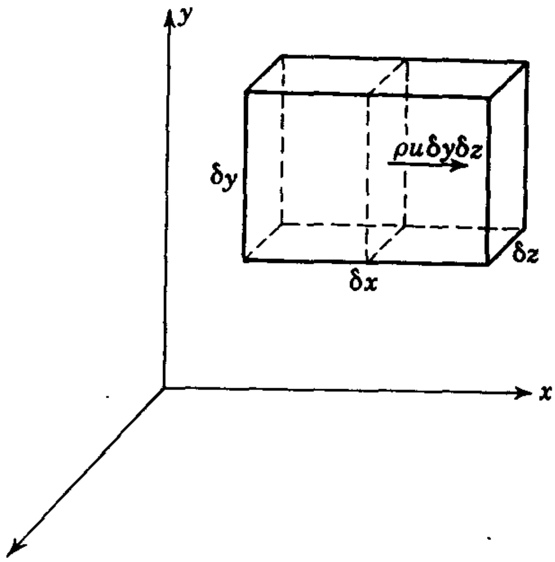


FIG. 3.6. Time rate of mass flow through a face.

which takes the place of the left-hand side of Eq. (3.4.3). The right-hand side of Eq. (3.4.3) becomes, for an element,

$$-\frac{\partial \rho}{\partial t}\ \delta x\ \delta y\ \delta z$$

By equating these two expressions and after dividing through by the volume element and taking the limit as $\delta x\ \delta y\ \delta z$ approaches zero, the continuity equation at a point becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = -\frac{\partial \rho}{\partial t} \tag{3.4.8}$$

which must hold for every point in the flow, steady or unsteady, compressible or incompressible. For incompressible flow, however, it simplifies to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.4.9}$$

Equations (3.4.8) and (3.4.9) may be compactly written in vector notation. By using fixed unit vectors in x, y, z-directions, **i**, **j**, **k**, respectively, the operator ∇ (pronounced "del") is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{3.4.10}$$

and the velocity vector $\mathbf{q}$ is given by

$$\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w \tag{3.4.11}$$

Then

$$\nabla \cdot (\rho \mathbf{q}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}\rho u + \mathbf{j}\rho v + \mathbf{k}\rho w)$$
$$= \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$$

because $\mathbf{i} \cdot \mathbf{i} = 1$, $\mathbf{i} \cdot \mathbf{j} = 0$, etc. Equation (3.4.8) becomes

$$\nabla \cdot (\rho \mathbf{q}) = -\frac{\partial \rho}{\partial t} \tag{3.4.12}$$

and Eq. (3.4.9) becomes

$$\nabla \cdot \mathbf{q} = 0 \tag{3.4.13}$$

The dot product $\nabla \cdot \mathbf{q}$ is called the *divergence* of the velocity vector $\mathbf{q}$. In words it is the net mass efflux at a point and must be zero for incompressible flow. See Sec. 7.2 for further discussion of the operator ∇.

In two-dimensional flow, generally assumed to be in planes parallel to the xy-plane, $w = 0$ and there is no change with respect to z, so $\partial/\partial z = 0$, which reduces the three-dimensional equations given for continuity.

Example 3.5: The velocity distribution for a two-dimensional incompressible flow is given by

$$u = -\frac{x}{x^2 + y^2} \qquad v = -\frac{y}{x^2 + y^2}$$

Show that it satisfies continuity.

In two dimensions the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Then

$$\frac{\partial u}{\partial x} = -\frac{1}{x^2 + y^2} + \frac{2x^2}{(x^2 + y^2)^2} \qquad \frac{\partial v}{\partial y} = -\frac{1}{x^2 + y^2} + \frac{2y^2}{(x^2 + y^2)^2}$$

and their sum does equal zero, satisfying continuity.

3.5. Euler's Equation of Motion along a Streamline. In addition to the continuity equation, other general controlling equations are Euler's equation, Bernoulli's equation, the momentum equations, and the first and second laws of thermodynamics. In this section Euler's equation is derived in differential form. In the following section it is integrated to obtain Bernoulli's equation. The first law of thermodynamics is then developed for steady flow, and some of the interrelations of the equations are explored, including an introduction to the second law of thermodynamics. In Chap. 7 Euler's equation is derived for general three-dimensional flow. Here it is restricted to steady flow along a streamline.

In Fig. 3.7 a prismatic-shaped fluid particle of mass $\rho\ \delta A\ \delta x$ is moving along a streamline in the $+s$-direction. To simplify the development of the equation of motion for this particle it is assumed that the viscosity is zero, or that the fluid is frictionless. This eliminates all shear forces from consideration, leaving as forces to take into consideration the body force due to the pull of gravity and surface forces on the end areas of the particle. The gravity force is $\rho g\ \delta A\ \delta s$. On the upstream face the pressure force is $p\ \delta A$ in the $+s$-direction; on the downstream face it is $[p + (\partial p/\partial s)\delta s]\ \delta A$ and acts in the $-s$-direction. Any forces on the sides of the element are normal to s and do not enter the equation. The body-force component in the s-direction is $-\rho g\ \delta A\ \delta s \cos\theta$. By substituting into Newton's second law of motion, $\Sigma f_s = \delta m\ a_s$,

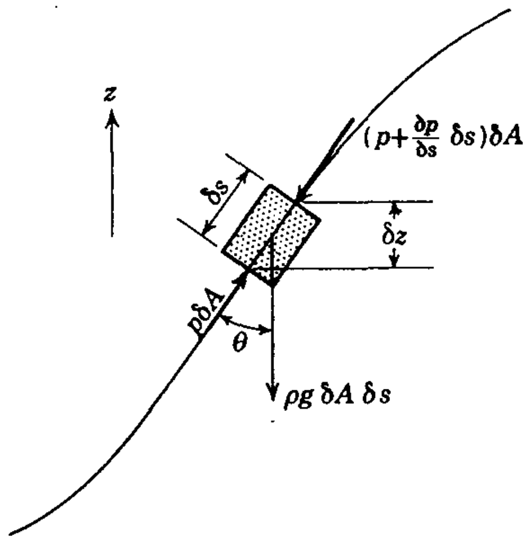


FIG. 3.7. Force components on a fluid particle in the direction of the streamline.

$$p\ \delta A - \left(p + \frac{\partial p}{\partial s}\ \delta s\right)\delta A - \rho g\ \delta A\ \delta s \cos\theta = \rho\ \delta A\ \delta s\ a_s$$

a_s is the acceleration of the fluid particle along the streamline. After dividing through by the mass of the particle, $\rho\ \delta A\ \delta s$, and simplifying,

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + g\cos\theta + a_s = 0 \tag{3.5.1}$$

δz is the increase in elevation of the particle for a displacement δs. From Fig. 3.7,

$$\frac{\delta z}{\delta s} = \cos\theta = \frac{\partial z}{\partial s}$$

The acceleration a_s is dv/dt. In general, if v depends upon s and time t, $v = v(s,t)$,

$$dv = \frac{\partial v}{\partial s}\,ds + \frac{\partial v}{\partial t}\,dt$$

s becomes a function of t in describing the motion of a particle, so one may divide by dt, yielding

$$a_s = \frac{dv}{dt} = \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} \tag{3.5.2}$$

To simplify the equation of motion the assumption is now made that the flow is steady, that is, $\partial v/\partial t = 0$. Since $ds/dt = v$,

$$a_s = v \frac{\partial v}{\partial s} \tag{3.5.3}$$

By use of this expression and that for $\cos \theta$, Eq. (3.5.1) becomes

$$\frac{1}{\rho} \frac{\partial p}{\partial s} + g \frac{\partial z}{\partial s} + v \frac{\partial v}{\partial s} = 0 \tag{3.5.4}$$

With ρ, p, v, and z not functions of t, but of s only, the partial differentials may be replaced by total differentials:

$$\frac{dp}{\rho} + g\,dz + v\,dv = 0 \tag{3.5.5}$$

This is Euler's equation of motion and requires three assumptions: (1) motion along a streamline, (2) frictionless fluid, and (3) steady flow. It may be integrated if ρ is known as a function of p or is a constant.

3.6. The Bernoulli Equation. Integration of Eq. (3.5.5) yields

$$gz + \frac{v^2}{2} + \int \frac{dp}{\rho} = \text{constant} \tag{3.6.1}$$

if ρ is a function of p only. The constant of integration (called the Bernoulli constant) in general varies from one streamline to another but remains constant along a streamline in steady, frictionless flow (with no pump or turbine involved). When ρ is some explicit function of p, such as $\rho = p\rho_0/p_0$ for isothermal flow, the integral can be evaluated.

By assuming that the fluid is incompressible, Eq. (3.6.1) becomes

$$gz + \frac{v^2}{2} + \frac{p}{\rho} = \text{constant} \tag{3.6.2}$$

This is *Bernoulli's* equation for incompressible flow. It is for *steady* flow of a *frictionless, incompressible* fluid *along a streamline.* These four assumptions are needed and must be kept in mind when applying this equation. Each term has the dimensions $(L/T)^2$ or the units ft²/sec², which is equivalent to ft-lb/slug:

$$\frac{\text{ft-lb}}{\text{slug}} = \frac{\text{ft-lb}}{\text{lb-sec}^2/\text{ft}} = \frac{\text{ft}^2}{\text{sec}^2}$$

as 1 slug $\equiv$ 1 lb-sec²/ft. Therefore Eq. (3.6.2) is energy per unit mass. By dividing it through by g,

$$z + \frac{v^2}{2g} + \frac{p}{\gamma} = \text{constant} \tag{3.6.3}$$

since $\gamma = \rho g$, or

$$z_1 + \frac{v_1^2}{2g} + \frac{p_1}{\gamma} = z_2 + \frac{v_2^2}{2g} + \frac{p_2}{\gamma} \tag{3.6.4}$$

it may now be interpreted as energy per unit weight, or ft-lb/lb. This form is particularly convenient for dealing with liquid problems with a free surface. By multiplying Eq. (3.6.2) by ρ

$$\gamma z + \frac{\rho v^2}{2} + p = \text{constant} \tag{3.6.5}$$

which is convenient for gas flow, since elevation changes are frequently unimportant and γz may be dropped out. In this form each term is ft-lb/ft^3 or energy per unit volume.

Each of the terms of Bernoulli's equation may be interpreted as a form of energy. In Eq. (3.6.2) the first term is potential energy per unit

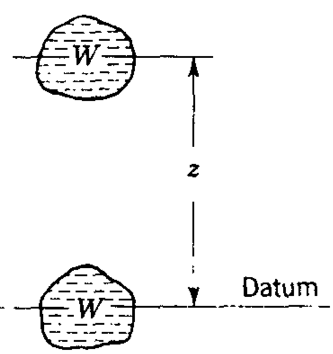


FIG. 3.8. Potential energy.

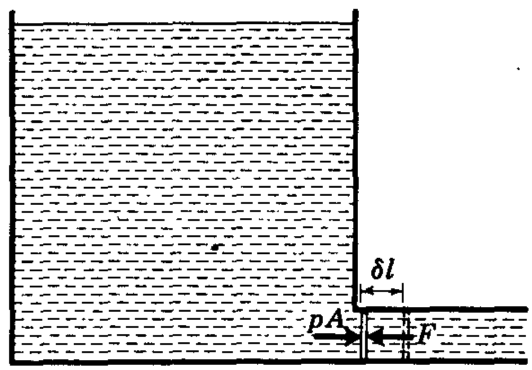


FIG. 3.9. Work done by sustained pressure force.

mass. With reference to Fig. 3.8 the work needed to lift W lb z ft is Wz. The mass of W lb weight is W/g slugs; hence the potential energy per slug is

$$\frac{Wz}{W/g} = gz$$

The next term, $v^2/2$, is interpreted as follows: Kinetic energy of a particle of mass is $\delta m\, v^2/2$. To place this on a unit mass basis, divide by δm; thus $v^2/2$ is ft-lb/slug kinetic energy.

The last term p/ρ is the *flow work* or *flow energy* per unit mass. Flow work is net work done by the fluid element on its surroundings while it is flowing. For example in Fig. 3.9, imagine a piston placed at the opening from the reservoir. The force on the piston would be pA. For flow through the length δl the work done on the piston is $pA\,\delta l$. The mass of fluid leaving the reservoir is $\rho A\,\delta l$; hence the work per unit mass is p/ρ. The three energy terms in Eq. (3.6.2) are referred to as the *available* energy.

Example 3.6: Show that the energy per unit mass is everywhere constant in a reservoir.

For any point A in the reservoir (Fig. 3.10) the energy is given by Eq. (3.6.2).

$$gz + \frac{v^2}{2} + \frac{p}{\rho} = gy + 0 + (H - y)\frac{\gamma}{\rho} = gH$$

Since y drops out of the equation, the energy per unit mass is gH for all locations.

By applying Eq. (3.6.4) to two points on a streamline,

$$z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g} \tag{3.6.6}$$

or

$$z_1 - z_2 + \frac{p_1 - p_2}{\gamma} + \frac{v_1^2 - v_2^2}{2g} = 0$$

This equation shows that it is the difference in potential energy, flow energy, and kinetic energy that actually has significance in the equation. Thus, $z_1 - z_2$ is independent of the particular elevation datum, as it is the difference in elevation of the two points. Similarly $(p_1/\gamma) - (p_2/\gamma)$ is the difference in pressure heads expressed in feet of the fluid flowing and is not altered by the particular pressure datum selected. Since the velocity terms are not linear, their datum is fixed.

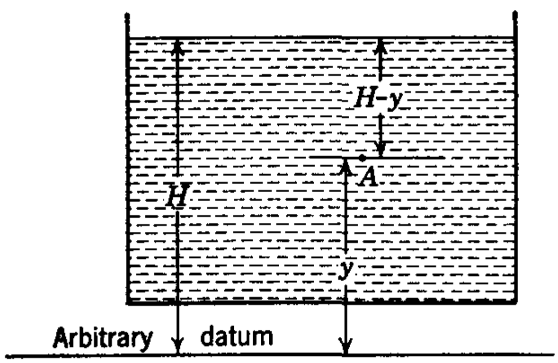


FIG. 3.10. Liquid reservoir.

Example 3.7: Water is flowing in an open channel at a depth of 4 ft and a velocity of 8.02 ft/sec. It then flows down a chute into another open channel, where the depth is 2 ft and the velocity is 40.1 ft/sec. Assuming frictionless flow, determine the difference in elevation of the channel floors.

If the difference in elevation of floors is y, then Bernoulli's equation from the upper water surface to the lower water surface may be written

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2$$

V_1 and V_2 are average velocities. With gage pressure zero as datum and the floor of the lower channel as elevation datum, then $z_1 = y + 4$, $z_2 = 2$, $V_1 = 8.02$, $V_2 = 40.1$, $p_1 = p_2 = 0$, and

$$\frac{(8.02)^2}{64.4} + 0 + y + 4 = \frac{(40.1)^2}{64.4} + 0 + 2$$

and $y = 22$ ft.

Kinetic-energy Correction Factor. In dealing with flow situations in open- or closed-channel flow, the so-called "one-dimensional" form of analysis is frequently used. The whole flow is considered to be one large stream tube with average velocity V at each cross section. The kinetic energy per unit weight given by $V^2/2g$, however, is not the average of $v^2/2g$ taken over the cross section. It is necessary to compute a correction factor α for $V^2/2g$, so that $\alpha V^2/2g$ is the average kinetic energy per unit weight passing the section. Referring to Fig. 3.11, the kinetic energy passing the cross section per unit time is

$$\gamma \int_A \frac{v^2}{2g} v\, dA$$

in which $\gamma v\, \delta A$ is the weight per unit time passing δA and $v^2/2g$ is the kinetic energy per unit weight. By equating this to the kinetic energy per unit time passing the section, in terms of $\alpha V^2/2g$

$$\alpha \frac{V^2}{2g} \gamma V A = \gamma \int_A \frac{v^3}{2g}\, dA$$

By solving for α, the *kinetic-energy correction factor*,

$$\alpha = \frac{1}{A} \int_A \left(\frac{v}{V}\right)^3 dA \tag{3.6.7}$$

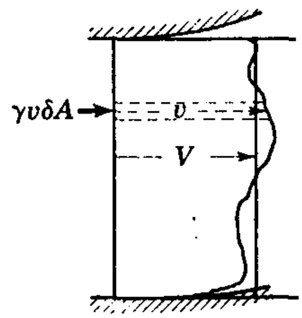


Fig. 3.11. Velocity distribution and average velocity.

Bernoulli's equation becomes

$$z_1 + \frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} \tag{3.6.8}$$

For laminar flow in a pipe, $\alpha = 2$, as shown in Sec. 5.2. For turbulent flow[1] in a pipe, α varies from about 1.01 to 1.10 and is usually neglected except for precise work.

Example 3.8: The velocity distribution in turbulent flow in a pipe is given approximately by Prandtl's one-seventh power law,

$$\frac{v}{v_{\max}} = \left(\frac{y}{r_0}\right)^{\frac{1}{7}}$$

with y the distance from the pipe wall and r_0 the pipe radius. Find the kinetic-energy correction factor.

The average velocity V is expressed by

$$\pi r_0^2 V = 2\pi \int_0^{r_0} rv\, dr$$

in which $r = r_0 - y$. By substituting for r and v,

$$\pi r_0^2 V = 2\pi v_{\max} \int_0^{r_0} (r_0 - y) \left(\frac{y}{r_0}\right)^{\frac{1}{7}} dy = \pi r_0^2 v_{\max} \frac{98}{120}$$

[1] V. L. Streeter, The Kinetic Energy and Momentum Correction Factors for Pipes and Open Channels of Great Width, *Civil Eng.*, vol. 12, no. 4, pp. 212–213, 1942.

or

$$V = \frac{98}{120} v_{max} \qquad \frac{v}{V} = \frac{120}{98} \left(\frac{y}{r_0}\right)^{\frac{1}{7}}$$

By substituting into Eq. (3.6.7)

$$\alpha = \frac{1}{\pi r_0^2} \int_0^{r_0} 2\pi r \left(\frac{120}{98}\right)^3 \left(\frac{y}{r_0}\right)^{\frac{3}{7}} dr = 2\left(\frac{120}{98}\right)^3 \frac{1}{r_0^2} \int_0^{r_0} (r_0 - y) \left(\frac{y}{r_0}\right)^{\frac{3}{7}} dy = 1.06$$

Modification of Assumptions Underlying Bernoulli's Equation. Under special conditions each of the four assumptions underlying Bernoulli's equation may be waived.

a. When all streamlines originate from a reservoir, where the energy content is everywhere the same, the constant of integration does not change from one streamline to another and points 1 and 2 for application of Bernoulli's equation may be selected arbitrarily, i.e., not necessarily on the same streamline.

b. In the flow of a gas, as in a ventilation system, where the change in pressure is only a small fraction (a few per cent) of the absolute pressure, the gas may be considered incompressible. Equation (3.6.6) may be applied, with an average specific weight γ.

c. For unsteady flow with gradually changing conditions, such as the emptying of a reservoir, Bernoulli's equation may be applied without appreciable error.

d. All real fluids have viscosity, and during flow, shear stresses result that cause the flow to be irreversible. Bernoulli's equation may be applied to a real fluid by adding a term to the equation that accounts for losses. By letting 1 be an upstream point and 2 a downstream point on a streamline, the available energy per unit weight at 1 equals the available energy per unit weight at 2 plus all the losses between the two points.

$$E_1 = E_2 + \text{losses}_{1-2} \tag{3.6.9}$$

This assumes no fluid machine such as a pump or turbine between the two points. Expanding Eq. (3.6.9),

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + \text{losses}_{1-2} \tag{3.6.10}$$

When a pump adds energy E_p per unit weight between the two points,

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + E_p = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + \text{losses}_{1-2} \tag{3.6.11}$$

For a turbine, replace E_p by $-E_T$, the energy per unit weight extracted by the turbine. The nature of the losses varies with the application, but experimental data are usually required.

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Example 3.9: (*a*) Determine the velocity of efflux from the nozzle in the wall of the reservoir of Fig. 3.12. (*b*) Find the discharge through the nozzle. Neglect losses.

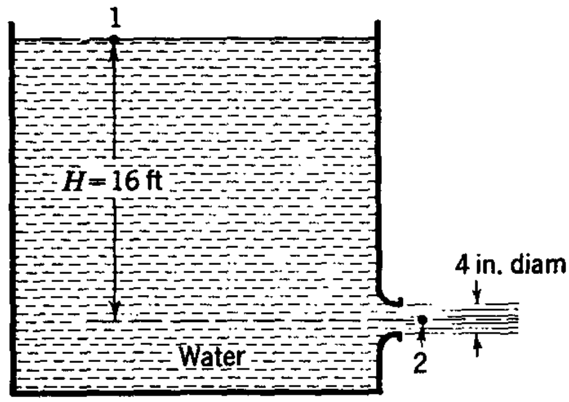

FIG. 3.12. Flow from a reservoir.

a. The jet issues as a cylinder with atmospheric pressure intensity around its periphery. The pressure along its center line is at atmospheric pressure for all practical purposes. Bernoulli's equation is applied between a point on the water surface and a point downstream from the nozzle,

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2$$

With the pressure datum as local atmospheric pressure, $p_1 = p_2 = 0$; with the elevation datum through point 2, $z_2 = 0$, $z_1 = H$. The velocity on the surface of the reservoir is zero (practically); hence

$$0 + 0 + H = \frac{V_2^2}{2g} + 0 + 0$$

and

$$V_2 = \sqrt{2gH} = \sqrt{2 \times 32.2 \times 16} = 32.08 \text{ ft/sec}$$

which states that the velocity of efflux is equal to the velocity of free fall from the surface of the reservoir. This is known as *Torricelli's theorem.*

b. The discharge Q is the product of velocity of efflux and area of stream,

$$Q = A_2V_2 = \frac{\pi}{36}\,32.08 = 2.80 \text{ cfs}$$

Equation (3.6.11) may be written on a unit mass basis:

$$gz_1 + \frac{p_1}{\rho} + \frac{V_1^2}{2} + E_p = gz_2 + \frac{p_2}{\rho} + \frac{V_2^2}{2} + \text{losses}_{1-2} \qquad (3.6.12)$$

E_p and losses are now per unit mass of fluid flowing.

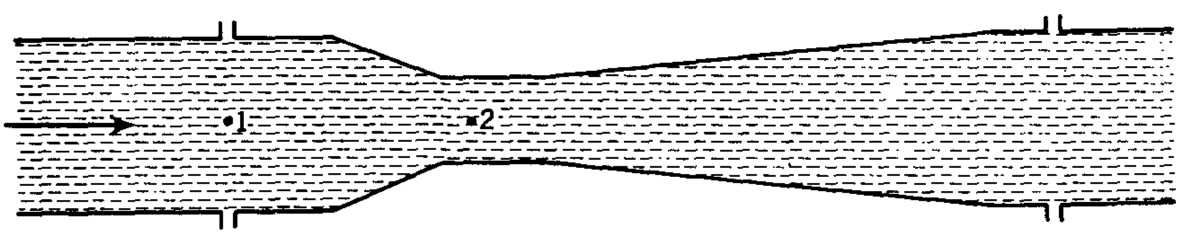

FIG. 3.13. Venturi meter.

Example 3.10: A venturi meter, consisting of a converging portion followed by a throat portion of constant diameter, and then a gradually diverging portion, is used to determine rate of flow in a pipe (Fig. 3.13). The diameter at section 1 is 6.0 in. and at section 2 is 4.0 in. Neglecting losses, find the discharge through the pipe when $p_1 - p_2 = 3$ psi and oil, sp gr 0.90, is flowing.

From the continuity equation, Eq. (3.4.7)

$$Q = A_1V_1 = A_2V_2 = \frac{\pi}{16}V_1 = \frac{\pi}{36}V_2$$

in which Q is the discharge (volume per unit time flowing). By applying Eq. (3.6.4) for $z_1 = z_2$,

$$p_1 - p_2 = 3 \times 144 = 432 \text{ lb/ft}^2 \qquad \gamma = 0.90 \times 62.4 = 56.16 \text{ lb/ft}^3$$

$$\frac{p_1 - p_2}{\gamma} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

or

$$\frac{432}{56.16} = \frac{Q^2}{\pi^2}\frac{1}{2g}[(36)^2 - (16)^2]$$

Solving for discharge, $Q = 2.20$ cfs (cubic feet per second).

Bernoulli's equation, with its four assumptions—(*a*) frictionless, (*b*) along a streamline, (*c*) steady, and (*d*) incompressible—is not a complete energy equation in the sense of the first law of thermodynamics.

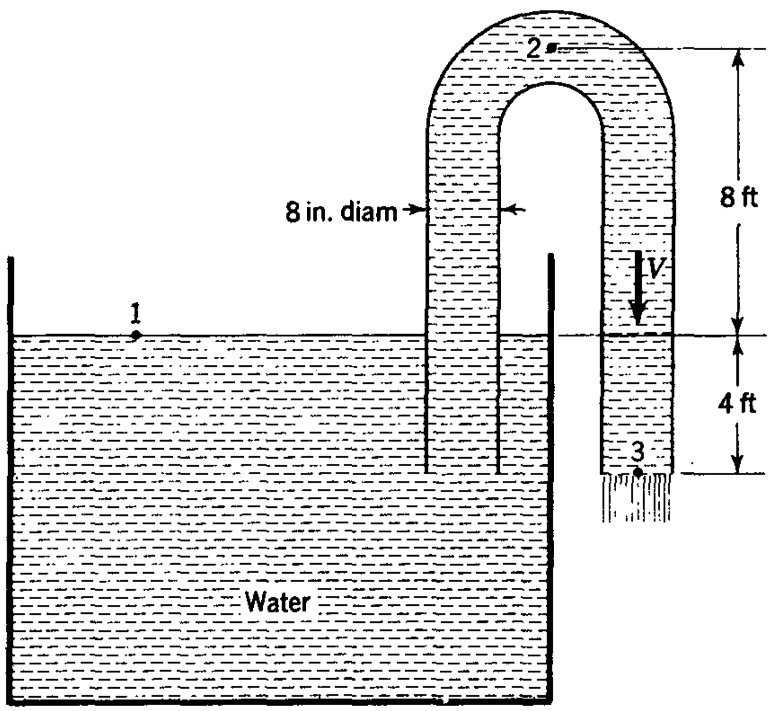

FIG. 3.14. Siphon.

It is an *available* energy equation, tabulating only those forms of energy that could be used to produce work, as through a turbine. When a corrective term is applied to the equation to permit it to be used with real, viscous fluids, as in Eq. (3.6.10), the available energy decreases in the downstream direction, owing to irreversibilities or losses. A plot showing how the available energy changes along a streamline is called the *energy grade line* (see Sec. 10.1). A plot of the two terms $z + p/\gamma$

along a streamline is called the *hydraulic grade line*. The energy grade line always slopes downward in real fluid flow, except at a pump or other source of energy. Reductions in energy grade line are referred to as head losses also.

Example 3.11: The siphon of Fig. 3.14 is filled with water and discharging at 2.80 cfs. Find the losses from point 1 to point 3 in terms of the velocity head $V^2/2g$. Find the pressure at point 2 if two-thirds of the losses occur between points 1 and 2.

Bernoulli's equation applied to points 1 and 3, with elevation datum at point 3 and gage pressure zero for pressure datum, is

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_3^2}{2g} + \frac{p_3}{\gamma} + z_3 + \text{losses}$$

or

$$0 + 0 + 4 = \frac{V_3^2}{2g} + 0 + 0 + \frac{KV_3^2}{2g}$$

in which the losses from 1 to 3 have been expressed as $KV_3^2/2g$. From the discharge

$$V_3 = \frac{Q}{A} = \frac{2.80}{\pi/9} = 8.02 \text{ ft/sec}$$

and $V_3^2/2g = 1.0$ ft. Hence $K = 3$ and the losses are $3V_3^2/2g$ or 3 ft-lb/lb.

Bernoulli's equation applied to points 1 and 2, with losses $2V_3^2/2g = 2.0$ ft, is

$$0 + 0 + 0 = 1 + \frac{p_2}{\gamma} + 8 + 2$$

The pressure at 2 is -11 ft of water, or 4.76 psi vacuum.

Example 3.12: The device shown in Fig. 3.15 is used to determine the velocity of liquid at point 1. It is a tube with its lower end directed upstream and its other leg vertical and open to the atmosphere. The impact of liquid against the opening 2 forces liquid to rise in the vertical leg to the height Δz above the free surface. Determine the velocity at 1.

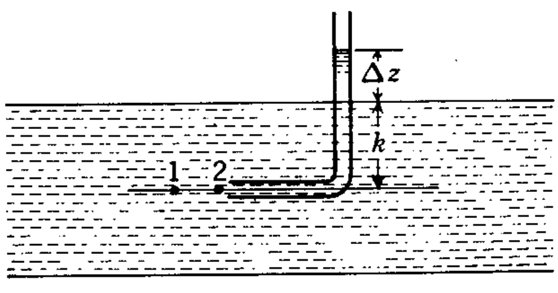

Fig. 3.15. Pitot tube.

Point 2 is a stagnation point, where the velocity of the flow is reduced to zero. This creates an impact pressure, called the dynamic pressure, which forces the fluid into the vertical leg. By writing Bernoulli's equation between points 1 and 2, neglecting losses, which are very small,

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + 0 = 0 + \frac{p_2}{\gamma} + 0$$

p_1/γ is given by the fluid above point 1 and equals k ft of fluid flowing. p_2/γ is given by the manometer as $k + \Delta z$, neglecting capillary rise. After substituting these values into the equation, $V_1^2/2g = \Delta z$ and

$$V_1 = \sqrt{2g\,\Delta z}$$

This is the pitot tube in a simple form.

Examples of compressible flow are given in Chap. 6.

3.7. Steady-flow Form of First Law of Thermodynamics. Entropy. The principle of conservation of energy may be applied to steady flow through a control volume. This approach permits a special form of the first law of thermodynamics to be developed. It is helpful in understanding the nature of losses when compared with Euler's equation.

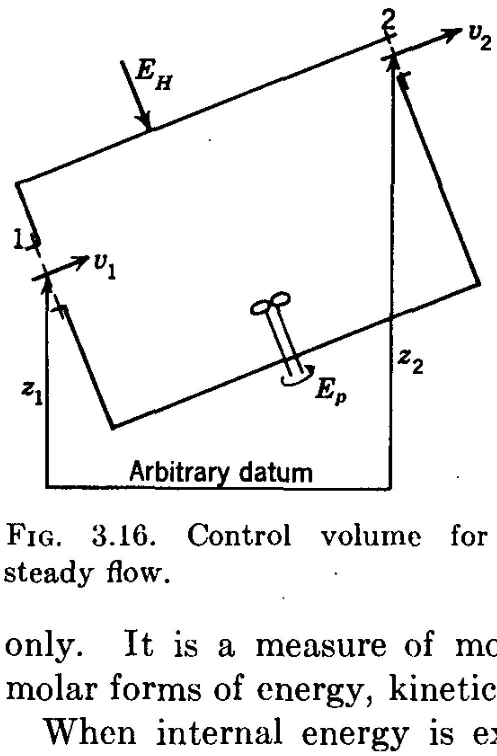

FIG. 3.16. Control volume for steady flow.

In the control volume of Fig. 3.16, contained between sections 1 and 2, an energy balance is taken that accounts for all work done, heat transferred, and energy brought into or out of the control volume. It is necessary to introduce the concept of internal energy, which is a fluid property. The internal energy comprises the molecular energy of the substance. In the absence of nuclear, electrical, surface-tension, and magnetic effects, the internal energy of a perfect gas may be shown to be a function of temperature only. It is a measure of molecular energy, as distinguished from the molar forms of energy, kinetic and potential.

When internal energy is expressed as u per unit mass, the kinetic, potential, and internal energy entering section 1 of Fig. 3.16 is

$$gz_1 + \frac{V_1^2}{2} + u_1$$

and similarly the energy per unit mass leaving at section 2 is

$$gz_2 + \frac{V_2^2}{2} + u_2$$

The flow work done per unit time at section 1 in forcing the fluid into the control volume is $p_1A_1V_1$, and the mass per unit time $\dot{m}$ is $\rho_1A_1V_1$; hence the flow work per unit mass is $p_1A_1V_1/\rho_1A_1V_1 = p_1/\rho_1$. Similarly the flow work at section 2 is p_2/ρ_2. Heat transfer to the control volume is Q_H per unit time. Per unit mass heat transfer q_H is

$$\frac{Q_H}{\dot{m}} = \frac{Q_H}{\rho_1A_1V_1} = q_H$$

This term is considered positive if heat is transferred into the control volume and negative if transferred to the surroundings.

The work done by the fluid within the control volume and transmitted out by a turning shaft, electric power lines, or other means is W per unit time, and the work per unit mass is

$$w = \frac{W}{\dot{m}} = \frac{W}{\rho_1 A_1 V_1}$$

When all the terms are assembled,

$$gz_1 + \frac{V_1^2}{2} + u_1 + q_H + \frac{p_1}{\rho_1} = gz_2 + \frac{V_2^2}{2} + u_2 + \frac{p_2}{\rho_2} + w \quad (3.7.1)$$

This is the first law of thermodynamics for steady flow. In case a pump is within the control volume, w becomes negative. This equation is valid for flow of real fluids, regardless of losses within the control volume.

It is informative to compare the Euler equation (3.5.5) with the first law when each is expressed in differential form, i.e., when sections 1 and 2 are close together. Equation (3.7.1) becomes

$$dq_H = g\,dz + V\,dV + d\frac{p}{\rho} + du + dw \quad (3.7.2)$$

Equation (3.5.5) is for a frictionless fluid without a work term. When a term for work done is included (as by an infinitesimal turbine),

$$g\,dz + V\,dV + \frac{dp}{\rho} + dw = 0 \quad (3.7.2a)$$

After this equation is subtracted from Eq. (3.7.2),

$$dq_H = d\frac{p}{\rho} + du - \frac{dp}{\rho} = du + pd\frac{1}{\rho} \quad (3.7.3)$$

Now, for reversible flow, entropy s per unit mass is defined by

$$ds = \left(\frac{dq_H}{T}\right)_{rev} \quad (3.7.4)$$

in which T is the absolute temperature. Entropy is shown to be a fluid property in texts on the subject. In this equation it may have the units Btu per slug per degree Rankine, or foot-pounds per slug per degree Rankine, as heat may be expressed in foot-pounds (1 Btu = 778 ft-lb). Since Eq. (3.7.3) is for a frictionless fluid (reversible), dq_H may be eliminated from Eqs. (3.7.3) and (3.7.4).

$$T\,ds = du + pd\frac{1}{\rho} \quad (3.7.5)$$

which is a very important thermodynamic relation. Although it was derived for a reversible process, since all terms are thermodynamic properties, it must also hold for irreversible flow cases as well. By use of Eq. (3.7.5) and various combinations of Euler's equation and the first law, a clearer understanding of entropy and losses is gained.

3.8. Interrelationships between the First Law and Euler's Equation. For a reversible flow, from Eq. (3.7.4),

$$T\,ds = dq_H$$

it is seen that the entropy increases if heat is added and it decreases if heat is transferred from the control volume. For the reversible, adiabatic case (i.e., isentropic flow) $dq_H = 0$ and $ds = 0$, so the entropy of the fluid per unit mass flowing remains constant.

To examine the relationships for flow of a real fluid, a loss term is included in Euler's equation in differential form, similar to Eq. (3.7.2*a*),

$$dw + \frac{dp}{\rho} + g\,dz + V\,dV = -d\text{ (losses)} \tag{3.8.1}$$

When this equation is subtracted from the first law [Eq. (3.7.2)],

$$d\text{ (losses)} = du + p\,d\frac{1}{\rho} - dq_H = T\,ds - dq_H \tag{3.8.2}$$

by use of Eq. (3.7.5). Now, for the adiabatic case ($dq_H = 0$), d (losses) $= T\,ds$ and it is seen that entropy always increases owing to irreversibilities. Also the adiabatic-flow process having the least change in entropy has the least losses and is most efficient. By rewriting Eq. (3.8.2)

$$T\,ds = dq_H + d\text{ (losses)} \tag{3.8.3}$$

it is seen that entropy can never decrease in adiabatic flow, and that it can decrease only when heat is transferred from the control volume. It can increase, however, owing to addition of heat, to irreversibilities, or to combinations of the two. Equation (3.8.3) is a consequence of the *second law of thermodynamics for steady flow.* In Eq. (3.8.1) account is taken only of losses in available energy. However, in Eq. (3.8.3), which now includes thermodynamic terms, it must include losses due to irreversible heat transfer, in order to satisfy Eq. (3.7.4).

For liquids $d(1/\rho) = 0$ and Eq. (3.8.2) becomes

$$d\text{ (losses)} = du - dq_H \tag{3.8.4}$$

Hence losses, due to viscous or turbulent shear, may show up as an increase in internal energy (i.e., increase in temperature) or may cause heat transfer from the control volume.

Two interesting flow cases[1] are (1) adiabatic flow of a real liquid through a horizontal pipeline and (2) adiabatic flow of a perfect gas through a horizontal pipeline. It is assumed that kinetic-energy changes along the pipes are unimportant. In the first case d (losses) $= du$ and the internal energy and temperature must rise in a downstream direction. In the second case, the first law [Eq. (3.7.1)] applied to the pipe at entrance and exit yields

$$\frac{p_1}{\rho_1} + u_1 = \frac{p_2}{\rho_2} + u_2$$

Each side of this equation is a combination of fluid properties and is also a fluid property. It is given the name *enthalpy* and symbol h. For a perfect gas h is a function of temperature only, and hence as $h_1 = h_2$, $T_1 = T_2$ and the flow must be isothermal. Therefore this is a case of adiabatic, isothermal flow, with du and dq_H equal to zero and

$$d \text{ (losses)} = T\,ds = p\,d\left(\frac{1}{\rho}\right)$$

The entropy must increase in both cases, and the losses in the second case cause a decrease in ρ. The term $d\,(1/\rho)$ is of the form of a work term and represents some of the energy of a unit mass of fluid in expanding its volume.

3.9. Linear Momentum Equation for Steady Flow through a Control Volume. The linear momentum equation is first derived for steady flow through a control volume for a given direction, the x-direction. In this form, with direction specified, it is a scalar equation. The result, however, is easily extended to the y- and z-directions and then to the general vector equation. Section 3.10 develops the linear momentum equation for unsteady flow through a control volume, and in Sec. 3.11 the steady-flow moment-of-momentum equation is developed.

Newton's second law of motion for a particle may be written for the x-component as

$$\delta f_x = \frac{d}{dt}(v_x\,\delta m) = \delta m\,\frac{dv_x}{dt} + v_x\,\frac{d\,\delta m}{dt} \tag{3.9.1}$$

δf_x is the resultant x-component of force on the particle. When the equation is applied to a given mass element as it moves through the control volume, δm is a constant and the last term drops out. The first term on the right may be expanded to

$$\delta f_x = \delta m\left(\frac{\partial v_x}{\partial s}\frac{ds}{dt} + \frac{\partial v_x}{\partial t}\right) \tag{3.9.2}$$

as in Eq. (3.5.2). For steady flow, $\partial v_x/\partial t = 0$. The mass element may

[1] These examples were furnished by Prof. Gordon Van Wylen.

conveniently be written as $\rho\ \delta Q\ \delta t$, the mass flowing by any section of a stream tube in steady flow. This yields

$$\delta f_x = \rho\ \delta Q \frac{\partial v_x}{\partial s}\, ds = \rho\ \delta Q\ \delta v_x \tag{3.9.3}$$

Now, by integrating along the stream tube from its entrance to its exit from the control volume (Fig. 3.17)

$$f_x = \rho\ \delta Q \int_{\text{in}}^{\text{out}} dv_x = \rho\ \delta Q\ (v_{x_{\text{out}}} - v_{x_{\text{in}}}) \tag{3.9.4}$$

This equation may be summed up for all stream tubes passing through the control volume, and since internal forces occur in equal and opposite pairs that cancel, they drop out of the expression, leaving the resultant x-component of force on the control volume due to both surface and body forces. Equation (3.9.4) becomes

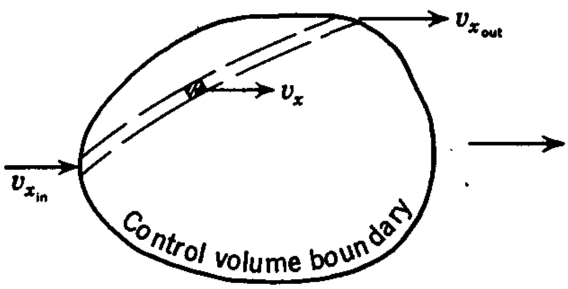

FIG. 3.17. Control volume for derivation of momentum equation.

$$F_x = \int \rho v_{x_{\text{out}}}\, dQ - \int \rho v_{x_{\text{in}}}\, dQ \tag{3.9.5}$$

in which the integrals are carried out over those portions of the control volume surface where $v_{x_{\text{out}}}$ and $v_{x_{\text{in}}}$ have positive nonzero values. By use of the notation of Fig. 3.3, $\delta Q = v \cos \alpha\, dA$ with α the angle between the normal to the surface area element and the velocity vector at the element, the two integrals of Eq. (3.9.5) may be combined:

$$F_x = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho v_x v \cos \alpha\, dA \tag{3.9.6}$$

In vector notation, this becomes

$$F_x = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho v_x \mathbf{v} \cdot d\mathbf{A} \tag{3.9.7}$$

The y- and z-components are

$$F_y = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho v_y \mathbf{v} \cdot d\mathbf{A} \qquad F_z = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho v_z \mathbf{v} \cdot d\mathbf{A} \tag{3.9.8}$$

By addition of the component equations vectorially, the general, steady linear momentum equation is obtained.

$$\mathbf{F} = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho \mathbf{v}(\mathbf{v} \cdot d\mathbf{A}) \tag{3.9.9}$$

Practical Formulations of Momentum Equation. In some applications ρ and v_x are constant over the inlet and outlet sections of the control volume. With V_x the average velocity over a section, Eq. (3.9.5) becomes

$$F_x = \rho Q(V_{x_{\text{out}}} - V_{x_{\text{in}}}) \tag{3.9.10}$$

since for steady flow $(\rho Q)_{\text{in}} = (\rho Q)_{\text{out}}$.

When the density is constant over the inlet section or the outlet section of the control volume, one of the integrals of Eq. (3.9.5) may be written (dropping subscripts)

$$\rho \int v_x \, dQ = \rho \beta V_x Q \tag{3.9.11}$$

in which β is called the *momentum correction factor* and V_x is the average x-component over the section. Since $Q = AV$, $dQ = v\,dA$, by solving for β

$$\beta = \frac{1}{A}\int_{\text{area}} \frac{v}{V}\frac{v_x}{V_x}\,dA = \frac{1}{A}\int_{\text{area}} \left(\frac{v}{V}\right)^2 dA \tag{3.9.12}$$

because $v_x/V_x = v/V$ from Fig. 3.18. The value of β is never less than 1.0. For laminar flow in a round tube, $\beta = \frac{4}{3}$. In turbulent flow in pipes,[1] β varies from about 1.01 to 1.05. When the momentum equation is applied, efforts are made to select the control volume so that the in and out sections have uniform velocity and $\beta = 1$.

The momentum equations for constant velocity over the sections, for the y- and z-directions, are

$$F_y = \rho Q(V_{y_{\text{out}}} - V_{y_{\text{in}}}) \tag{3.9.13}$$
$$F_z = \rho Q(V_{z_{\text{out}}} - V_{z_{\text{in}}}) \tag{3.9.14}$$

Adding Eqs. (3.9.10), (3.9.13), and (3.9.14) vectorially,

$$\mathbf{F} = \rho Q(\mathbf{V}_{\text{out}} - \mathbf{V}_{\text{in}}) \tag{3.9.15}$$

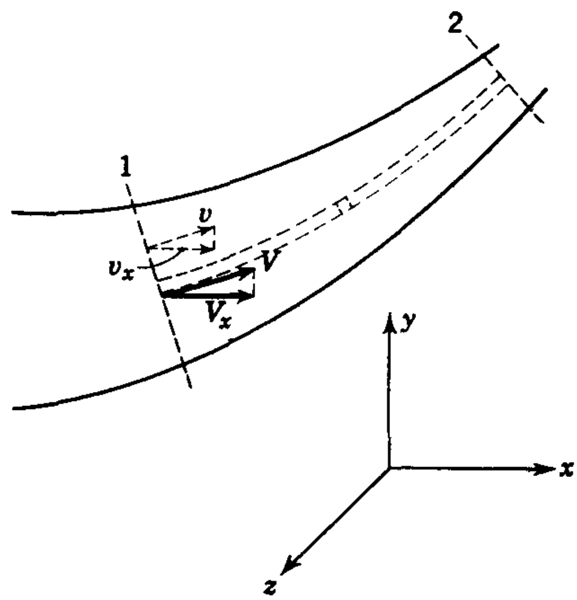

FIG. 3.18. Notation for momentum relationships.

Hence, the resultant force on a control volume in steady flow equals the product of ρQ (the mass of fluid per unit time having its momentum changed) and the velocity vector of leaving fluid minus the velocity vector of entering fluid.

[1] See footnote, p. 99.

Example 3.13: Determine the momentum correction factor for the velocity distribution in Example 3.8.

$$\beta = \frac{1}{\pi r_0^2}\int_0^{r_0} 2\pi r \left(\frac{120}{98}\right)^2 \left(\frac{y}{r_0}\right)^{\frac{2}{7}} dr$$
$$= 2\left(\frac{120}{98}\right)^2 \frac{1}{r_0^2}\int_0^{r_0} (r_0 - y)\left(\frac{y}{r_0}\right)^{\frac{2}{7}} dy = 1.02$$

Example 3.14: A jet of water 3 in. in diameter with a velocity of 120 ft/sec is discharged in a horizontal direction from a nozzle mounted on a boat. What force is required to hold the boat stationary?

The momentum in the jet requires a thrust, or unbalanced force [Eq. (3.9.10)], of

$$F = \rho Q(V_{x_{\text{out}}} - V_{x_{\text{in}}}) = 1.935\,\frac{\pi}{64}\,120(120 - 0) = 1370 \text{ lb}$$

This force must be applied to the boat, in the direction the jet is discharging, to hold it at rest.

A change in direction of a pipeline causes forces to be exerted on the line unless the bend or elbow is anchored in place. These forces are due to both static pressure in the line and dynamic reactions in the turning fluid stream. Expansion joints are placed in large pipelines to avoid stress in the pipe in an axial direction, whether caused by fluid or by temperature change. These expansion joints permit relatively free movement of the line in an axial direction and, hence, the static and dynamic forces must be provided for at the bends.

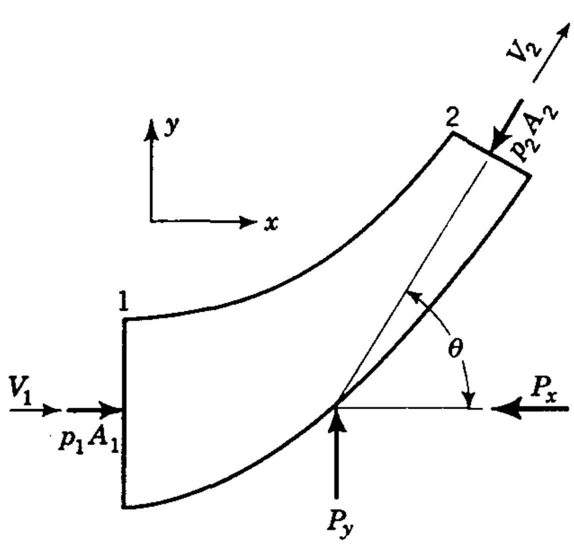

FIG. 3.19. Control volume for fluid within a reducing bend.

Example 3.15: The force components on a reducing elbow making a 60° turn in a horizontal plane are desired. At the entering section, $D_1 = 20$ ft, $V_1 = 50$ ft/sec, $p_1 = 40$ psi; at the exit section, $D_2 = 16$ ft. Water is flowing in the line and elbow losses are to be neglected.

With axes as in Fig. 3.19, $Q = V_1\pi D_1^2/4 = 15{,}710$ cfs. Then

$$V_2 = \frac{Q}{A_2} = \frac{15{,}710}{\pi(16)^2/4} = 78.1 \text{ ft/sec}$$

By using Bernoulli's equation,

$$z_1 + \frac{V_1^2}{2g} + \frac{p_1}{\gamma} = z_2 + \frac{V_2^2}{2g} + \frac{p_2}{\gamma}$$

Since $z_1 = z_2$,

$$\frac{(50)^2}{64.4} + \frac{40}{0.433} = \frac{(78.1)^2}{64.4} + \frac{p_2}{0.433}$$

and $p_2 = 15.8$ psi. Applying Eq. (3.9.10)

$$p_1A_1 - p_2A_2 \cos\theta - P_x = \rho Q(V_2 \cos\theta - V_1)$$

or

$$40 \times 144 \times 100\pi - 15.8 \times 144 \times 64\pi \times 0.500 - P_x = 1.935 \times 15{,}710(78.1 \times 0.500 - 50)$$

and $P_x = 1{,}915{,}000$ lb. Similarly for Eq. (3.9.13),

$$P_y - p_2A_2 \sin\theta = \rho QV_2 \sin\theta$$

or $P_y - 15.8 \times 144 \times 64\pi \times 0.866 = 1.935 \times 15{,}710 \times 78.1 \times 0.866$ and $P_y = 2{,}452{,}000$ lb. The force components exerted on the elbow are equal and opposite to P_x and P_y.

In this example gage pressures were used. If absolute pressures had been used a different answer would result. The forces would be those required to hold the elbow if it were surrounded by a complete vacuum.

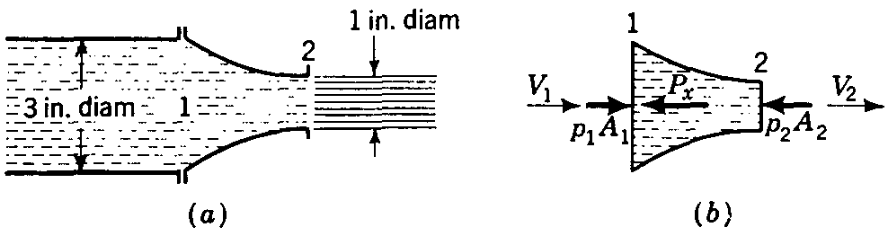

FIG. 3.20. Nozzle at the end of a pipe.

Example 3.16: Find the force exerted by the nozzle on the pipe of Fig. 3.20*a*. Neglect losses. The fluid is oil, sp gr 0.85, and $p_1 = 100$ psi.

To determine the discharge, Bernoulli's equation is written for the stream from section 1 to the downstream end of the nozzle, where the pressure is zero.

$$z_1 + \frac{V_1^2}{2g} + \frac{100}{0.85 \times 0.433} = z_2 + \frac{V_2^2}{2g} + 0$$

Since $z_1 = z_2$, and $V_2 = (D_1/D_2)^2V_1 = 9V_1$, after substituting,

$$\frac{V_1^2}{2g}(1 - 81) + \frac{100}{0.85 \times 0.433} = 0$$

and $V_1 = 14.78$ ft/sec, $V_2 = 133$ ft/sec, $Q = 14.78 \frac{\pi}{4}\left(\frac{1}{4}\right)^2 = 0.725$ cfs. Let P_x (Fig. 3.20*b*) be the force exerted on the free body of liquid by the nozzle; then, with Eq. (3.9.10),

$$100 \frac{\pi}{4} 9 - P_x = 1.935 \times 0.85 \times 0.725(133 - 14.78)$$

or $P_x = 565$ lb. The oil exerts a force on the nozzle of 565 lb to the right, and a tension force of 565 lb is exerted by the nozzle on the pipe.

The Momentum Theory for Propellers. The action of a propeller is to change the momentum of the fluid within which it is submerged and thus to develop a thrust that is used for propulsion. Propellers cannot be designed according to the momentum theory, although some of the relations governing them are made evident by its application. A propeller, with its slipstream and velocity distributions at two sections a fixed distance from it, is shown in Fig. 3.21. The propeller may be either (*a*) stationary in a flow as indicated or (*b*) moving to the left with a velocity V_1 through a stationary fluid since the relative picture is the same. The fluid is assumed to be frictionless and incompressible.

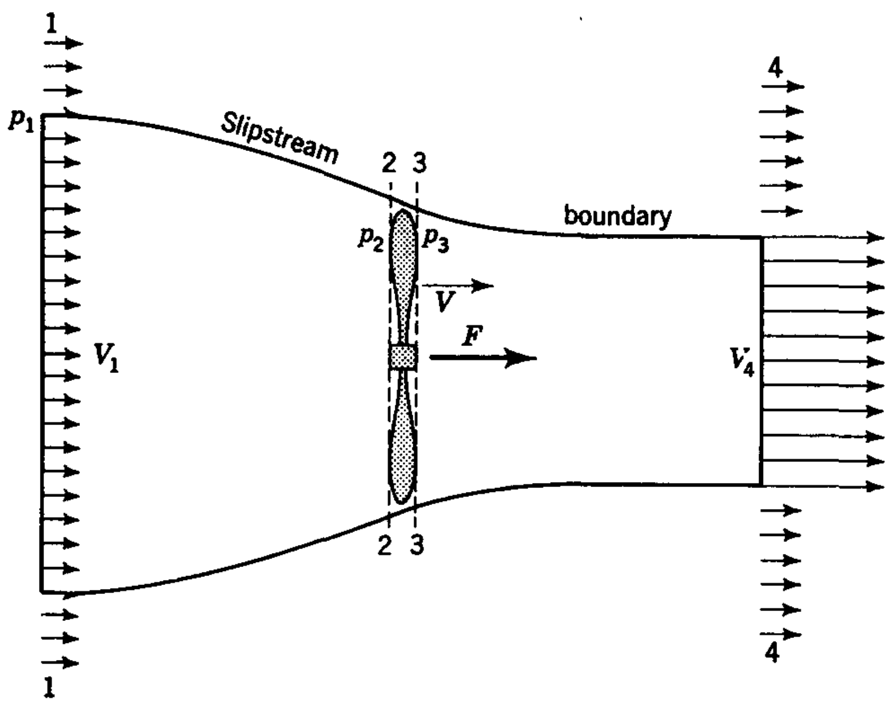

FIG. 3.21. Propeller in a fluid stream.

The flow is undisturbed at section 1 upstream from the propeller and is accelerated as it approaches the propeller, owing to the reduced pressure on its upstream side. In passing through the propeller, the fluid has its pressure increased, which further accelerates the flow and reduces the cross section at 4. The velocity V does not change across the propeller, from 2 to 3. The pressure intensities at 1 and 4 are those of the undisturbed fluid, which is also the pressure along the slipstream boundary.

When the momentum equation [Eq. (3.9.10)] is applied to the free body of fluid between sections 1 and 4 and the slipstream boundary, the only force, F, acting on it in the flow direction is that due to the propeller as shown, since the outer boundary of the free body is everywhere at the same pressure. Therefore,

$$F = \rho Q(V_4 - V_1) = (p_3 - p_2)A \tag{3.9.16}$$

in which A is the area swept over by the propeller blades. The force on the propeller must be equal and opposite to the force on the fluid. After substituting $Q = AV$ and simplifying,

$$\rho V(V_4 - V_1) = p_3 - p_2 \tag{3.9.17}$$

When Bernoulli's equation is written for the stream between sections 1 and 2 and between sections 3 and 4,

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_2 + \tfrac{1}{2}\rho V^2 \qquad p_3 + \tfrac{1}{2}\rho V^2 = p_4 + \tfrac{1}{2}\rho V_4^2$$

since $z_1 = z_2 = z_3 = z_4$. In solving for $p_3 - p_2$, with $p_1 = p_4$,

$$p_3 - p_2 = \tfrac{1}{2}\rho(V_4^2 - V_1^2) \tag{3.9.18}$$

By eliminating $p_3 - p_2$ in Eqs. (3.9.17) and (3.9.18),

$$V = \frac{V_1 + V_4}{2} \tag{3.9.19}$$

which shows that the velocity through the propeller area is the average of the velocities upstream and downstream from it.

The useful work done by a propeller moving through still fluid is the product of propeller thrust and velocity, i.e.,

$$\text{Power} = FV_1 = \rho Q(V_4 - V_1)V_1 \tag{3.9.20}$$

The power input is that required to increase the velocity of fluid from V_1 to V_4, or the useful work plus the kinetic energy per unit time remaining in the slipstream.

$$\text{Power input} = \rho \frac{Q}{2}(V_4^2 - V_1^2) = \rho Q(V_4 - V_1)V_1 + \rho \frac{Q}{2}(V_4 - V_1)^2 \tag{3.9.21}$$

With the ratio of Eqs. (3.9.20) and (3.9.21) used to obtain the theoretical efficiency e_t,

$$e_t = \frac{2V_1}{V_4 + V_1} = \frac{V_1}{V} \tag{3.9.22}$$

If $\Delta V = V_4 - V_1$ is the increase in slipstream velocity, substituting into Eq. (3.9.22) produces

$$e_t = \frac{V_1}{V_1 + \Delta V/2} \tag{3.9.23}$$

which shows that maximum efficiency is obtained with a propeller that increases the velocity of slipstream as little as possible, or for which $\Delta V/V_1$ is a minimum.

Owing to compressibility effects, the efficiency of an airplane propeller drops rapidly with speeds above 400 mph. Airplane propellers under optimum conditions have actual efficiencies close to the theoretical efficiencies, in the neighborhood of 85 per cent. Ship propeller efficiencies are less, around 60 per cent, owing to restrictions in diameter.

The windmill may be analyzed by application of the momentum relations. The jet has its speed reduced, and the diameter of slipstream is increased.

Example 3.17: An airplane traveling 250 mph through still air, $\gamma = 0.080$ lb/ft³, discharges 33,000 cfs through its two 7.0-ft-diameter propellers. Determine (*a*) the theoretical efficiency, (*b*) the thrust, (*c*) the pressure difference across the blades, and (*d*) the theoretical horsepower required.

a.

$$V_1 = \frac{250 \times 88}{60} = 366 \text{ ft/sec} \qquad V = \frac{16{,}500}{49\pi/4} = 428 \text{ ft/sec}$$

From Eq. (3.9.22)

$$e_t = \frac{V_1}{V} = \frac{366}{428} = 85.5\%$$

b. From Eq. (3.9.19)

$$V_4 = 2V - V_1 = 2 \times 428 - 366 = 490 \text{ ft/sec}$$

The thrust from both propellers is, from Eq. (3.9.16)

$$F = \frac{0.080}{32.2} \times 33{,}000(490 - 366) = 10{,}170 \text{ lb}$$

c. The pressure difference, from Eq. (3.9.17), is

$$p_3 - p_2 = \frac{0.080}{32.2} 428(490 - 366) = 132 \text{ lb/ft}^2$$

d. The theoretical horsepower is

$$\frac{FV_1}{550e_t} = \frac{10{,}170 \times 366}{550 \times 0.855} = 7920$$

Jet Propulsion. The propeller is one form of jet propulsion in that it creates a jet and by so doing has a thrust exerted upon it that is the propelling force. In jet engines, air (initially at rest) is taken into the engine and burned with a small amount of fuel; the gases are then ejected with a much higher velocity than in a propeller slipstream. The jet diameter is necessarily smaller than the propeller slipstream. For the mechanical energy only, the theoretical efficiency is given by the ratio of useful work to work input or by useful work divided by the sum of useful work and kinetic energy per unit time remaining in the jet. If the mass of fuel burned is neglected, the propelling force F [Eq. (3.9.16)] is

$$F = \rho Q V_{\text{abs}} \tag{3.9.24}$$

in which V_{abs} (Fig. 3.22) is the absolute velocity of fluid in the jet and ρQ is the mass per unit time being discharged. The useful work is FV_1, in which V_1 is the speed of the body. The kinetic energy per unit time being discharged in the jet is $\gamma Q\, V_{\text{abs}}^2/2g = \rho Q\, V_{\text{abs}}^2/2$, since γQ is the weight per unit time being discharged and $V_{\text{abs}}^2/2g$ is the kinetic energy per unit weight. Hence, the theoretical mechanical efficiency is

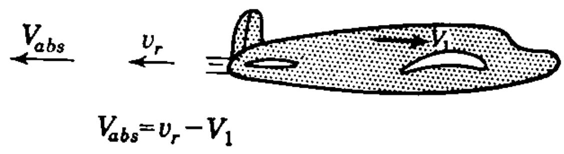

FIG. 3.22. Notation for jet propulsion.

$$e_t = \frac{\text{output}}{\text{output} + \text{loss}} = \frac{FV_1}{FV_1 + \rho Q V_{\text{abs}}^2/2}$$
$$= \frac{\rho Q V_{\text{abs}} V_1}{\rho Q V_{\text{abs}}\, V_1 + Q V_{\text{abs}}^2/2} = \frac{1}{1 + V_{\text{abs}}/2V_1} \tag{3.9.25}$$

which is the same expression as that for efficiency of the propeller. It is obvious that, other things being equal, V_{abs}/V_1 should be as small as possible. For a given speed V_1, the resistance force F is determined by the body and fluid in which it moves; hence, in Eq. (3.9.24) for V_{abs} to be very small, ρQ must be very large.

An example is the type of propulsion system to be used on a boat (Fig. 3.23). If the boat requires a force of 400 lb to move it through

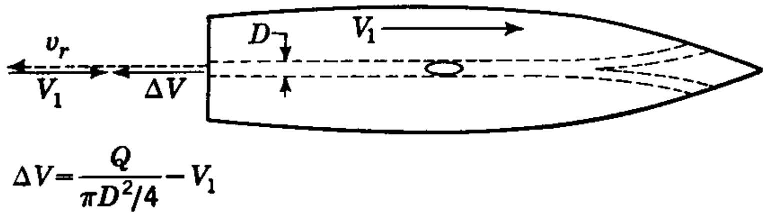

FIG. 3.23. Propulsion of boat with liquid jet.

water at 15 mph, first a method of jet propulsion can be considered in which water is taken in at the front of the boat and discharged out the rear by a 100 per cent efficient pumping system.

If a 6-in.-diameter jet pipe is used, $v_r = 16Q/\pi$ and the absolute velocity of the jet as it leaves the boat is $V_{\text{abs}} = (16Q/\pi) - V_1$. By substituting into Eq. (3.9.24) for $V_1 = 15$ mph $= 22$ ft/sec,

$$400 = 1.935Q\left(\frac{16Q}{\pi} - 22\right)$$

Hence, $Q = 8.89$ cfs, $V_{\text{abs}} = 23.2$, and the efficiency is

$$e_t = \frac{1}{1 + V_{\text{abs}}/2V_1} = \frac{1}{1 + 23.2/44} = 65.5\%$$

The horsepower required is

$$\frac{FV_1}{550e_t} = \frac{400 \times 22}{550 \times 0.655} = 24.4$$

With an 8-in.-diameter jet pipe, $v_r = 9Q/\pi$, $V_{abs} = (9Q/\pi) - 22$, and

$$400 = 1.935Q\left(\frac{9Q}{\pi} - 22\right)$$

so $Q = 13.14$ cfs, $V_{abs} = 15.72$, $e_t = 73.7$ per cent, and the horsepower required is 21.7.

With additional enlarging of the jet pipe and the pumping of more water with less velocity head, the efficiency can be further increased. The type of pump best suited for large flows at small head is the axial-flow propeller pump. Increasing the size of pump and jet pipe would increase weight greatly and take up useful space in the boat; the logical limit is to drop the propeller down below or behind the boat and thus eliminate the jet pipe, which is the usual propeller for boats. Jet propulsion of a boat by a jet pipe is practical, however, in very shallow water where a propeller would be damaged by striking bottom or other obstructions.

To take the weight of fuel into account in jet propulsion, let $\dot{m}_{air}$ be the mass of air per unit time and r the ratio of mass of fuel burned to mass of air. Then (Fig. 3.22), the propulsive force F is

$$F = \dot{m}_{air}V_{abs} + r\dot{m}_{air}v_r$$

The second term on the right is the mass of fuel per unit time multiplied by its change in velocity. By substituting $V_{abs} = v_r - V_1$ and rearranging,

$$F = \dot{m}_{air}[v_r(1 + r) - V_1] \tag{3.9.26}$$

Defining the mechanical efficiency again as the useful work divided by the sum of useful work and kinetic energy remaining,

$$e_t = \frac{FV_1}{FV_1 + \dot{m}_{air}(1 + r)(v_r - V_1)^2/2}$$

By use of Eq. (3.9.26)

$$e_t = \frac{1}{1 + \dfrac{(1 + r)[(v_r/V_1) - 1]^2}{2[(1 + r)(v_r/V_1) - 1]}} \tag{3.9.27}$$

The efficiency becomes unity for $v_1 = v_r$, as the combustion products are then brought to rest and no kinetic energy remains in the jet.

Example 3.18: An airplane consumes 1 lb_m fuel for each 20 lb_m air and discharges hot gases from the tail pipe at $v_r = 6000$ ft/sec. Determine the mechanical efficiency for airplane speeds of 1000 and 500 ft/sec.

For 1000 ft/sec $v_r/V_1 = \frac{6000}{1000} = 6$, $r = 0.05$. From Eq. (3.9.27),

$$e_t = \frac{1}{1 + \dfrac{(1 + 0.05)(6 - 1)^2}{2[6(1 + 0.05) - 1]}} = 0.287$$

For 500 ft/sec $v_r/V_1 = \frac{6000}{500} = 12$ and

$$e_t = \frac{1}{1 + \dfrac{(1 + 0.05)(12 - 1)^2}{2[12(1 + 0.05) - 1]}} = 0.154$$

Jet Propulsion of Aircraft or Missiles. Propulsion through air or water in each case is caused by reaction to the formation of a jet behind the body. The various means include the propeller, turbojet, turboprop, ram jet, and rocket motor, which are briefly described in the following paragraphs.

The momentum relations for a propeller determine that its theoretical efficiency increases as the speed of the aircraft increases and the absolute velocity of the slipstream decreases. As the speed of the blade tips approaches the speed of sound, however, compressibility effects greatly increase the drag on the blades and thus decrease the over-all efficiency of the propulsion system.

A *turbojet* is an engine consisting of a compressor, a combustion chamber, a turbine, and a jet pipe. Air is scooped through the front of the engine and is compressed, and fuel is added and burned with a great excess of air. The air and combustion gases then pass through a gas turbine that drives the compressor. Only a portion of the energy of the hot gases is removed by the turbine, since the only means of propulsion is the issuance of the hot gas through the jet pipe. The over-all efficiency of a jet engine increases with speed of the aircraft. Although there is very little information available on propeller systems near the speed of sound, it appears that the over-all efficiencies of the turbojet and propeller systems are about the same at the speed of sound.

The *turboprop* is a system combining thrust from a propeller with thrust from the ejection of hot gases. The gas turbine must drive both compressor and propeller. The proportion of thrust between the propeller and the jet may be selected arbitrarily by the designer.

The *ram jet* is a high-speed engine that has neither compressor nor turbine. The ram pressure of the air forces air into the front of the engine, where some of the kinetic energy is converted into pressure energy by enlarging the flow cross section. It then enters a combustion chamber, where fuel is burned, and the air and gases of combustion are ejected through a jet pipe. It is a supersonic device requiring very high speed for compression of the air. An intermittent ram jet was used by the Germans in the V-1 *buzz bomb.* Air is admitted through spring-closed flap valves in the nose. Fuel is ignited to build up pressure that closes the flap valves and ejects the hot gases as a jet. The ram pressure then opens the valves in the nose to repeat the cycle. The cyclic rate is around 40 per second. Such a device must be launched at high speed to initiate operation of the ram jet.

Rocket Motors. The *rocket motor* carries with it an oxidizing agent to mix with its fuel so that it develops a thrust that is independent of the medium through which it travels. On the contrary, a gas turbine can eject a mass many times the weight of fuel it carries, because it takes in air to mix with the fuel.

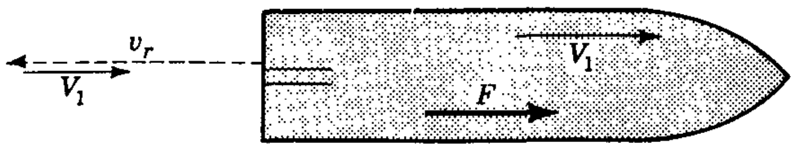

FIG. 3.24. Rocket notation.

The theoretical efficiency of a rocket motor (based on mechanical energy available) is shown to increase with rocket speed. E represents the mechanical energy available in the propellant per unit mass. When the propellant is ignited, its mechanical energy is converted into kinetic energy; $E = v_r^2/2$, in which v_r is the jet velocity relative to the rocket, or $v_r = \sqrt{2E}$ (Fig. 3.24). The force F exerted on the rocket depends on the rate of burning $\dot{m}$, in mass per unit time. According to the momentum equation [Eq. (3.9.10)]

$$F = \dot{m}v_r \tag{3.9.28}$$

since v_r is the final velocity minus the initial velocity. For rocket speed V_1 referred to axes fixed in the earth, the useful work is

$$FV_1 = \dot{m}v_rV_1 \tag{3.9.29}$$

The kinetic energy being used up per unit time is due to mass loss $\dot{m}V_1^2/2$ of the unburned propellant and to the burning $\dot{m}E$, or

$$\text{Mechanical-energy input per unit time} = \dot{m}\left(E + \frac{V_1^2}{2}\right) \tag{3.9.30}$$

The mechanical efficiency e is

$$e = \frac{(\dot{m})\,V_1\sqrt{2E}}{\dot{m}(E + V_1^2/2)} = 2\,\frac{v_r/V_1}{1 + (v_r/V_1)^2} \tag{3.9.31}$$

By taking the derivative of e with respect to v_r/V_1 and by equating to zero, $v_r/V_1 = 1$ for maximum efficiency, $e = 1$. In this case the absolute velocity of ejected gas is zero.

When the propulsive force on a rocket is greater than the resistance force, the rocket accelerates. Its mass is continuously reduced. To lift a large rocket off its launching pad, the thrust must exceed the weight W of rocket and fuel:

$$F = \dot{m}v_r > W \tag{3.9.32}$$

Example 3.19: (*a*) Determine the burning time for a rocket that initially weighs 1,000,000 lb, of which 75 per cent is fuel. It consumes fuel at a constant rate, and its initial thrust equals its weight. v_r = 12,000 ft/sec. (*b*) Considering g constant at 32.2 ft/sec² and the flight to be vertical without air resistance, find the speed of the rocket at burnout time and its height above sea level.

a. From Eq. (3.9.32)

$$W_0 = \dot{m}v_r = 1{,}000{,}000 = \dot{m}12{,}000$$

and $\dot{m}$ = 83.3 slugs/sec. The available mass of fuel is

$$\frac{0.75 \times 1{,}000{,}000}{32.2} = 23{,}280 \text{ slugs}$$

The burning time is

$$\frac{23{,}280}{83.3} = 279 \text{ sec}$$

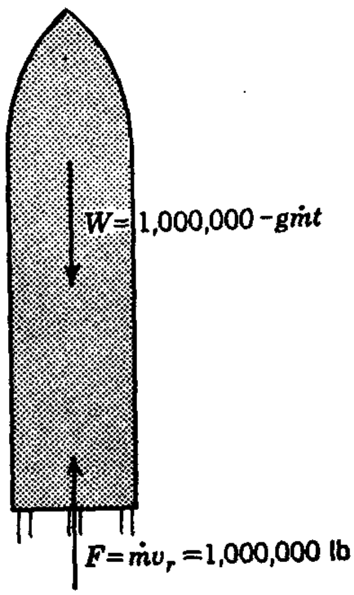

Fig. 3.25. Vertical rocket ascent.

b. The rocket thrust (Fig. 3.25) is constant at 1,000,000 lb. From Newton's second law of motion $F - W = (W/g)(dV/dt)$, with W decreasing at $83.3g$ lb/sec

$$1{,}000{,}000 - (1{,}000{,}000 - 83.3gt) = \frac{1{,}000{,}000 - 83.3gt}{g}\frac{dV}{dt}$$

or

$$\frac{dV}{g} = \frac{t\,dt}{373 - t}$$

After integrating, for $V = 0$, $t = 0$

$$\frac{V}{g} = -t - 373 \ln\left(1 - \frac{t}{373}\right)$$

When t = 279 sec, V = 7580 ft/sec = 5160 mph. The height reached is

$$z = \int_0^{279} V\,dt = g\int_0^{279}\left[-t - 373 \ln\left(1 - \frac{t}{373}\right)\right]\cdot dt$$
$$= 534{,}000 \text{ ft} = 101.2 \text{ miles}$$

Fixed and Moving Vanes. The theory of turbomachines is based on the relationships between jets and vanes. The mechanics of transfer of work and energy from fluid jets to moving vanes is studied as an application of the momentum principles. When a free jet impinges onto a smooth vane that is curved, as in Fig. 3.26, the jet is deflected, its momentum is changed, and a force is exerted on the vane. The jet is

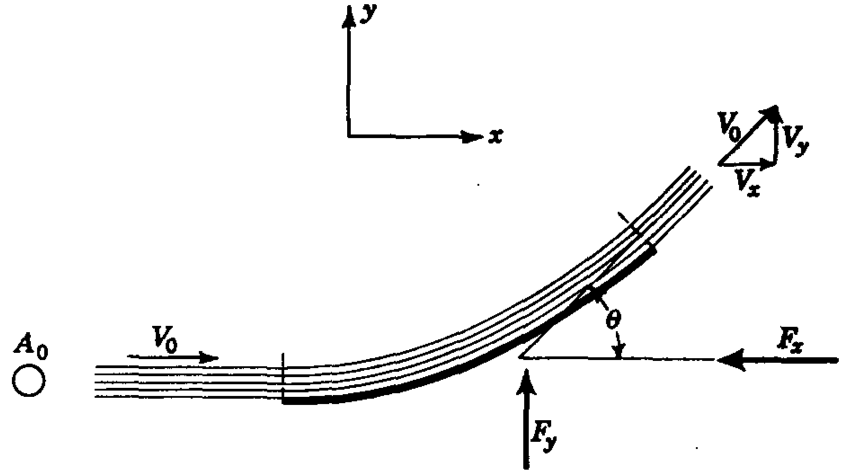

FIG. 3.26. Free jet impinging on a smooth, fixed vane.

assumed to flow onto the vane in a tangential direction, without shock; and furthermore, the frictional resistance between jet and vane is neglected. The velocity is assumed to be uniform throughout the jet upstream and downstream from the vane. Since the jet is open to the air, it has the same pressure intensity at each end of the vane. Neglecting the small change in elevation between ends, if any, application of Bernoulli's equation shows that the magnitude of the velocity is unchanged for *fixed* vanes.

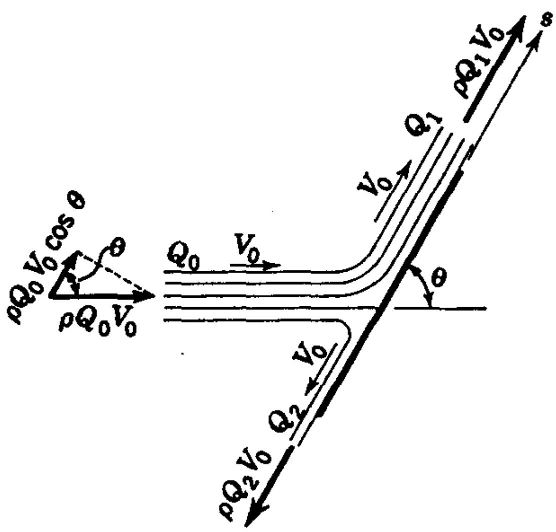

FIG. 3.27. Two-dimensional jet impinging on an inclined, fixed plane surface.

Example 3.20: Find the force exerted on a fixed vane when a jet discharging 2 cfs water at 150/ft sec is deflected through 135°.

By referring to Fig. 3.26 and by applying Eqs. (3.9.10) and (3.9.13), it is found that

$$-F_x = \rho Q(V_0 \cos\theta - V_0)$$
$$F_y = \rho Q V_0 \sin\theta$$

Hence,

$$F_x = -1.935 \times 2(150 \cos 135° - 150) = 990 \text{ lb}$$
$$F_y = 1.935 \times 2 \times 150 \sin 135° = 410 \text{ lb}$$

The force components on the fixed vane are then equal and opposite to F_x and F_y.

Example 3.21: Fluid issues from a long slot and strikes against a smooth inclined flat plate (Fig. 3.27). Determine the division of flow and the force exerted on the plate, neglecting energy loss due to impact.

As there are no changes in elevation or pressure before and after impact the magnitude of the velocity leaving is the same as the initial speed of jet. The division of flow Q_1, Q_2 can be computed by applying the momentum equation in the s-direction, parallel to the plate. No force is exerted on the fluid by the plate in this direction; hence, the final momentum component must equal the initial momentum component in the s-direction. By rewriting the momentum equation so that it contains two terms for final momentum,

$$F_s = 0 = \rho Q_1 V_0 - \rho Q_2 V_0 - \rho Q_0 V_0 \cos \theta$$

After simplifying

$$Q_1 - Q_2 = Q_0 \cos \theta$$

and with the continuity equation

$$Q_1 + Q_2 = Q_0$$

The two equations may be solved for Q_1 and Q_2,

$$Q_1 = \frac{Q_0}{2} (1 + \cos \theta)$$

$$Q_2 = \frac{Q_0}{2} (1 - \cos \theta)$$

The force F exerted on the plate must be normal to it. For the momentum equation normal to the plate

$$F = \rho Q_0 V_0 \sin \theta$$

Moving Vanes. Turbomachinery utilizes the forces resulting from the motion of fluid over moving vanes. No work can be done on or by a fluid that flows over a fixed vane. When vanes can be displaced, work

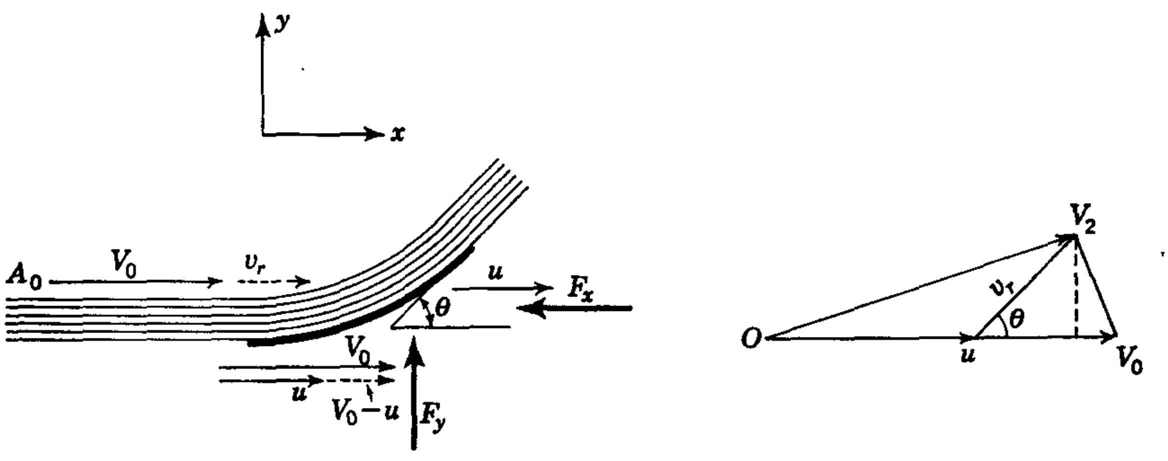

Fig. 3.28. Velocity relationships for a moving vane.

can be done either on the vane or on the fluid. A moving vane is shown in Fig. 3.28 with the fluid jet flowing onto it tangentially. The force components F_x, F_y exerted on the free body of fluid that is on the vane are determined from Eqs. (3.9.10) and (3.9.13). Since these equations contain terms with the difference in final and initial velocity components, either the absolute or the relative components may be used. In Fig. 3.28 the polar vector diagram shows both absolute and relative vectors. The relative velocity v_r is turned through the angle θ without change in magnitude. This vector, added to u, gives the final absolute velocity leaving the vane V_2.

The mass per unit time having its momentum changed is not the mass per unit time being discharged, as in the case of the single fixed vane. In each unit of time the vane is displaced a distance u; that is, the jet becomes longer each second. The mass per second that has its velocity (and, hence, its momentum) changed is that which overtakes the vane each second and flows onto it. In 1 sec the vane moves a distance u ft (Fig. 3.29). The fluid, however, moves V_0 ft, and thus v_r ft ride up

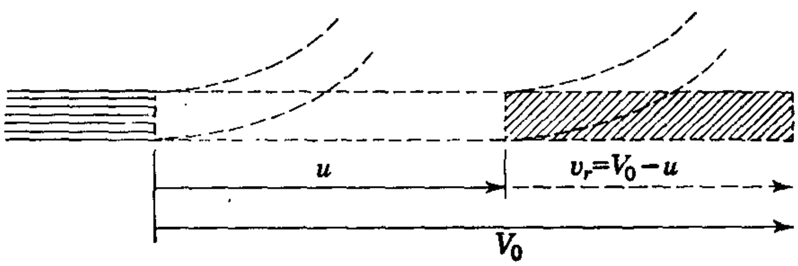

FIG. 3.29. Fluid overtaking vane in 1-sec period.

onto the vane in 1 sec. The volume per second overtaking the vane is v_rA_0, and the mass per second having its momentum changed is $\rho v_r A_0$.

The fluid velocity relative to the vane at entrance to the free body is v_r. The vane is assumed to be smooth; hence, this relative speed is maintained along its curved surface. At the exit the relative velocity makes the angle θ with the x-direction. To determine the absolute velocity leaving, the velocity of the vane u is added to the velocity of the fluid relative to the vane at its exit end (Fig. 3.28). The final absolute velocity then has the components, evident from the vector diagram, of

$$V_{x_{\text{out}}} = v_r \cos\theta + u$$
$$V_{y_{\text{out}}} = v_r \sin\theta$$

Example 3.22: Determine for the single moving vane of Fig. 3.30 the force components due to the water jet and the rate of work done on the vane.

The mass per second having its velocity changed is

$$1.935(120 - 60)\tfrac{3}{144} = 2.42 \text{ slugs/sec}$$

The final absolute velocity, from the vector diagram of Fig. 3.30, is

$$V_{x_{\text{out}}} = 60 - 60\cos 10° = 60(1 - 0.985) = 0.90 \text{ ft/sec}$$
$$V_{y_{\text{out}}} = 60 \sin 10° = 60 \times 0.174 = 10.44 \text{ ft/sec}$$

From the momentum equations

$$-F_x = 2.42(0.90 - 120) = -288 \text{ lb}$$
$$F_y = 2.42(10.44 - 0) = 25.3 \text{ lb}$$

The force components on the vane are 288 lb in the $+x$-direction and 25.3 lb in the $-y$-direction. The rate of work done is $F_x u$, or

$$288 \times 60 = 17{,}280 \text{ ft-lb/sec}$$

For the efficient development of power the single vane is not practical. With a *series of vanes,* usually on the periphery of a wheel, arranged so that one or another of the vanes deflects the entire jet as the vanes move almost tangent to the undeflected jet, the mass per second having its momentum changed becomes ρQ or $\rho V_0 A_0$, the total mass per second being discharged.

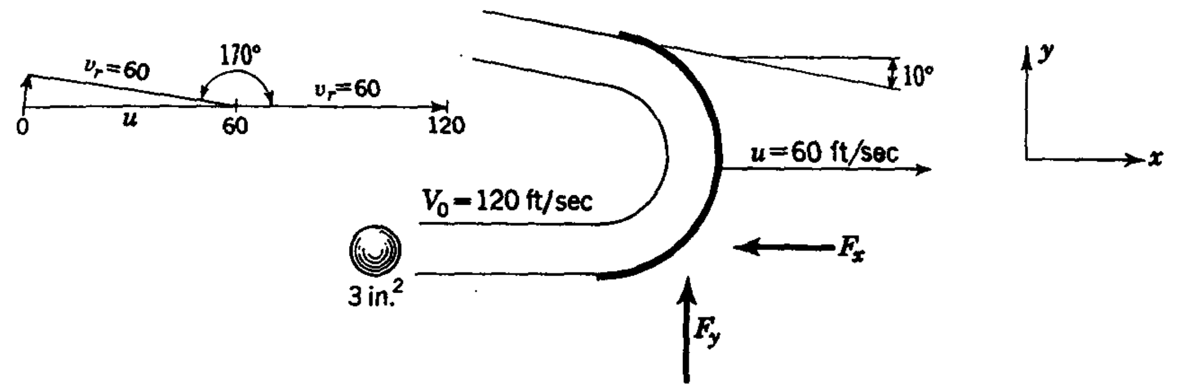

FIG. 3.30. Vector diagram for jet on moving vane.

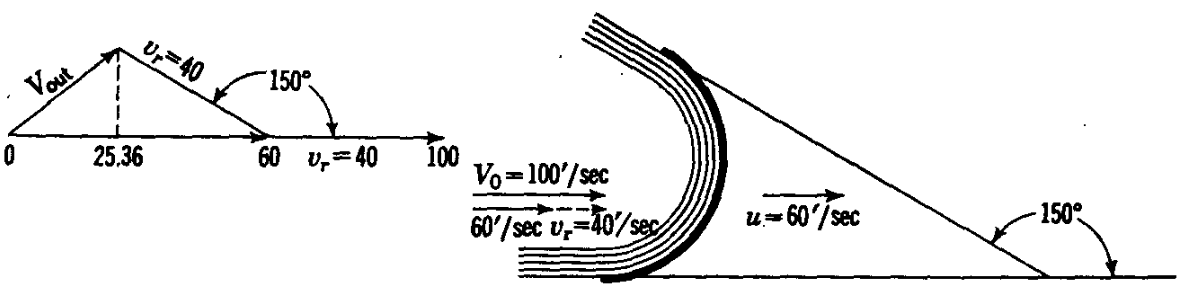

FIG. 3.31. Vector diagram for moving vane.

Example 3.23: Determine the horsepower that may be obtained from a series of vanes, curved through 150°, moving 60 ft/sec away from a 3.0 cfs water jet having a cross-sectional area of 0.030 ft². Draw the vector diagram and determine the energy remaining in the jet.

The jet velocity is 3/0.03 = 100 ft/sec. The vector diagram is shown in Fig. 3.31. The power is

$$P = -1.935 \times 3(25.36 - 100)60 = 26{,}000 \text{ ft-lb/sec}$$

and the horsepower

$$\text{hp} = \frac{26{,}000}{550} = 47.3$$

The absolute velocity component $V_{y_{out}}$ leaving the vane is

$$V_{y_{out}} = (100 - 60) \sin 30° = 20 \text{ ft/sec}$$

and the exit velocity head is

$$\frac{V^2}{2g} = \frac{(25.36)^2 + (20)^2}{64.4} = 16.2 \text{ ft-lb/lb}$$

The kinetic energy remaining in the jet, in foot-pounds per second, is

$$Q\gamma \frac{V_2^2}{2g} = 3 \times 62.4 \times 16.2 = 3030$$

The initial kinetic energy available was

$$3 \times 62.4 \times \frac{(100)^2}{64.4} = 29{,}030 \text{ ft-lb/sec}$$

which is the sum of the work done and the energy remaining per second.

When a vane, or series of vanes, moves toward a jet, work is done by the vane system on the jet, thereby increasing the energy of the fluid. Figure 3.32 illustrates this situation; the velocity leaving is 145.2 ft/sec as shown, and the velocity entering is 50 ft/sec.

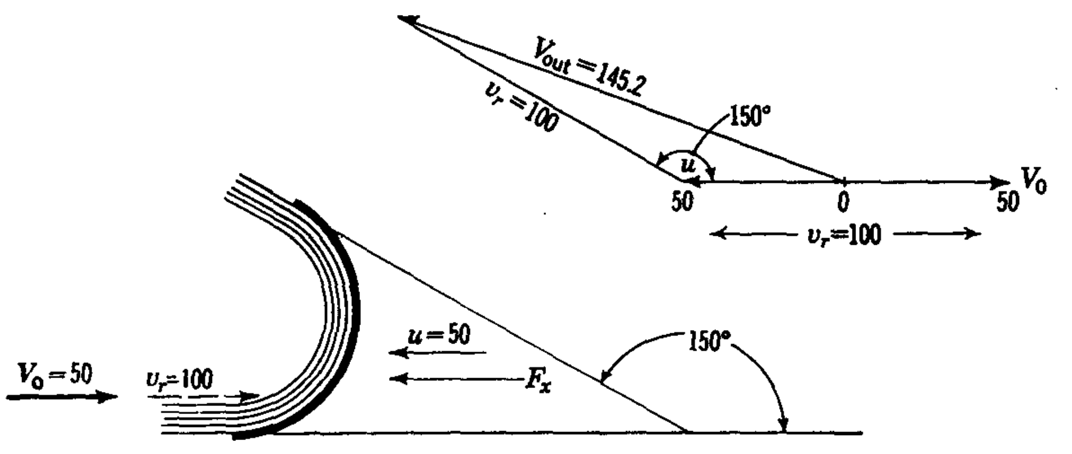

Fig. 3.32. Vector diagram for vane doing work on a jet.

In most instances losses must be determined by experiment. In the following two cases, application of the continuity, Bernoulli, and momentum equations permits the losses to be evaluated analytically.

Losses Due to Sudden Expansion in a Pipe. The losses due to sudden enlargement in a pipeline may be calculated with both the Bernoulli and momentum equations. For steady, incompressible, turbulent flow between sections 1 and 2 of the sudden expansion of Fig. 3.33*a*, the fluid may be taken as a free body (Fig. 3.33*b*) and the small shear force exerted on the walls between the two sections may be neglected. By assuming uniform velocity over the flow cross sections, which is approached in turbulent flow, application of Eq. (3.9.10) produces

$$p_1A_2 - p_2A_2 = \frac{Q\gamma}{g}(V_2 - V_1)$$

At section 1 the radial acceleration of fluid particles in the eddy along the surface is small, so generally a hydrostatic pressure variation occurs

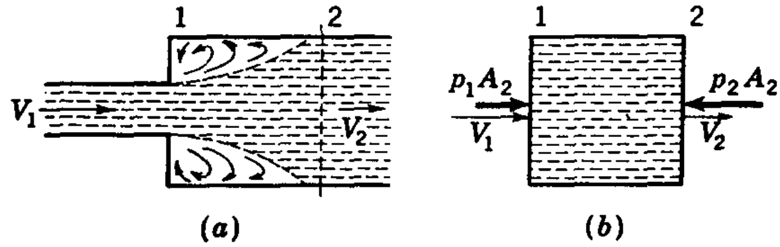

Fig. 3.33. Sudden expansion in a pipe.

across the section. The Bernoulli equation, applied to sections 1 and 2, with the loss term h_l, is (for $\alpha = 1$)

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + h_l$$

Solving for $(p_1 - p_2)/\gamma$ in each equation and equating the results,

$$\frac{Q}{A_2g}(V_2 - V_1) = \frac{V_2^2 - V_1^2}{2g} + h_l$$

As $Q/A_2 = V_2$,

$$h_l = \frac{(V_1 - V_2)^2}{2g} = \frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2 \qquad (3.9.33)$$

which indicates that the losses in turbulent flow are proportional to the square of the velocity.

Hydraulic Jump. The hydraulic jump is the second application of the basic equations to determine losses due to a turbulent-flow situation.

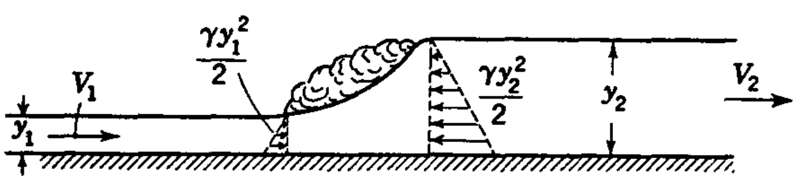

Fig. 3.34. Hydraulic jump in a rectangular channel.

Under proper conditions a rapidly flowing stream of liquid in an open channel suddenly changes to a slowly flowing stream with a larger cross-sectional area and a sudden rise in elevation of liquid surface. This phenomenon is known as the *hydraulic jump* and is an example of steady nonuniform flow. In effect, the rapidly flowing liquid jet expands (Fig. 3.34) and converts kinetic energy into potential energy and losses or irreversibilities. A roller develops on the inclined surface of the expanding liquid jet and draws air into the liquid. The surface of the jump is very rough and turbulent, the losses being greater as the jump height is greater. For small heights, the form of the jump changes to a standing wave (Fig. 3.35). The jump is discussed further in Sec. 11.4.

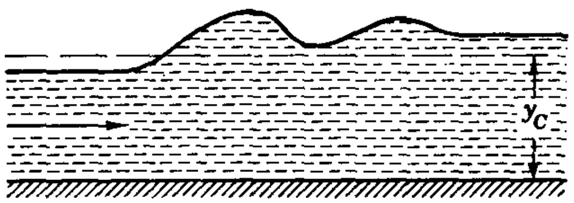

FIG. 3.35. Standing wave.

The relations among the variables for the hydraulic jump in a horizontal rectangular channel are easily obtained by use of the continuity, momentum, and Bernoulli equations. For convenience the width of channel is taken as unity. The continuity equation (Fig. 3.34) is

$$V_1 y_1 = V_2 y_2$$

The momentum equation is

$$\frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} = \frac{V_1 y_1 \gamma}{g} (V_2 - V_1)$$

and the Bernoulli equation (for points on the liquid surface)

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + h_j$$

in which h_j represents losses due to the jump. By eliminating V_2 in the first two equations,

$$y_2 = -\frac{y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2 + \frac{2V_1^2 y_1}{g}} \tag{3.9.34}$$

in which the plus sign has been taken before the radical (a negative y_2 has no physical significance). The depths y_1 and y_2 are referred to as *conjugate* depths. By solving the Bernoulli equation for h_j and eliminating V_1 and V_2,

$$h_j = \frac{(y_2 - y_1)^3}{4y_1 y_2} \tag{3.9.35}$$

The hydraulic jump, which is a very effective device for creating irreversibilities, is commonly used at the end of chutes or the bottom of spillways to destroy much of the kinetic energy in the flow. It is also an effective mixing chamber, because of the violent agitation that takes place in the roller. Experimental measurements of hydraulic jumps show that the equations yield the correct value of y_2 to within 1 per cent. The reasons the jump equations are not precise are due to neglect of shear stress on walls and to nonuniform velocity distribution.

Example 3.24: 120 cfs water per foot of width flows down a spillway onto a horizontal floor. The velocity is 50 ft/sec. Determine the depth of tail water required to cause a hydraulic jump and the losses in horsepower by the jump per foot of width.

$$y_1 = \tfrac{120}{50} = 2.4 \text{ ft}$$

By substituting into Eq. (3.9.34),

$$y_2 = -1.2 + \sqrt{(1.2)^2 + \frac{2 \times 50^2 \times 2.4}{32.2}} = 18.1 \text{ ft}$$

With Eq. (3.9.35),

$$\text{Losses} = \frac{(18.1 - 2.4)^3}{4 \times 2.4 \times 18.1} = 22.3 \text{ ft-lb/lb}$$

or

$$\text{hp/ft} = \frac{120 \times 62.4 \times 22.3}{550} = 304$$

3.10. Linear Momentum Equation for Unsteady Flow through a Control Volume. The unsteady-flow momentum equation is developed by finding the force component required for the unsteady portion of Eq. (3.9.1), which was neglected in the steady-flow derivation. Equation (3.9.1) in expanded form is

$$\delta f_x = \frac{\partial}{\partial s}(v_x\,\delta m)\frac{\delta s}{\delta t} + \frac{\partial}{\partial t}(v_x\,\delta m) \tag{3.10.1}$$

Attention is now focused on the last term. Its contribution to the x-component of resultant force is

$$\delta f'_x = \frac{\partial}{\partial t}(\rho v_x\,\delta \mathcal{V})$$

for a small element of volume $\delta\mathcal{V}$ in the control volume. By integrating throughout the control volume, one obtains

$$F'_x = \int_{\substack{\text{control}\\ \text{volume}}} \frac{\partial}{\partial t}(\rho v_x\,d\mathcal{V}) = \frac{\partial}{\partial t}\int_{\substack{\text{control}\\ \text{volume}}} \rho v_x\,d\mathcal{V} \tag{3.10.2}$$

in which F'_x is the contribution from the time rate of increase of momentum within the control volume.

By combining Eq. (3.10.2) with Eq. (3.9.7)

$$F_x = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho v_x \mathbf{v}\cdot d\mathbf{A} + \frac{\partial}{\partial t}\int_{\substack{\text{control}\\ \text{volume}}} \rho v_x\,d\mathcal{V} \tag{3.10.3}$$

which is the linear momentum equation for the x-component for unsteady flow through a control volume. The general vector equation obtained by adding vectorially the x-, y-, and z-components becomes

$$\mathbf{F} = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho\mathbf{v}(\mathbf{v} \cdot d\mathbf{A}) + \frac{\partial}{\partial t}\int_{\substack{\text{control}\\ \text{volume}}} \rho\mathbf{v}\, d\mathcal{V} \tag{3.10.4}$$

Example 3.25: Find the head H in the reservoir of Fig. 3.36 needed to accelerate the flow of oil, $S = 0.85$, at the rate of 0.5 ft/sec² when the velocity is 8.02 ft/sec. At 8.02-ft/sec steady flow the head is 20 ft.

The oil may be considered incompressible and to be moving uniformly in the pipeline. By applying Eq. (3.10.3), the first term is zero, as the momentum leaving equals the momentum entering per unit time. The second integral becomes

$$\frac{\partial}{\partial t}(\rho VAL) = \rho AL\frac{\partial V}{\partial t}$$

The friction force due to the walls of the pipe exerts a force just balanced by the 20 ft head at the upstream end, i.e., for steady conditions

$$f_{\text{friction}} = \gamma 20A$$

When the pipe is considered as the control volume, the momentum equation for the x-component yields

$$\gamma HA - \gamma 20A = \rho AL\frac{\partial V}{\partial t}$$

or

$$H - 20 = \frac{L}{g}\frac{\partial V}{\partial t} = \frac{1000}{32.2} \times 0.5 = 15.52 \text{ ft}$$

Hence, at 8.02-ft/sec velocity the level in the reservoir is 20 + 15.52 = 35.52 ft above the pipeline to cause the flow to accelerate at 0.5 ft/sec².

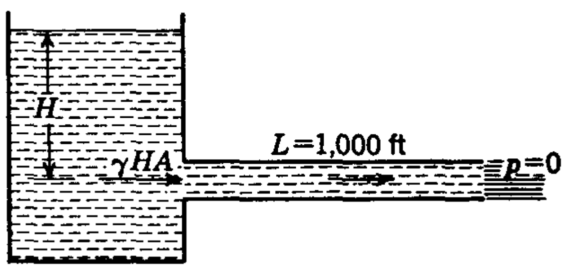

Fig. 3.36. Acceleration of liquid in a pipe.

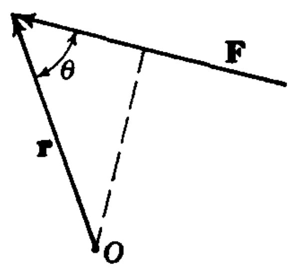

Fig. 3.37. Notation for moment of a vector.

3.11. The Moment-of-momentum Equation.

The moment of a force $\mathbf{F}$ about a point O, Fig. 3.37, is given by

$$\mathbf{F} \times \mathbf{r}$$

which is the cross, or vector, product of $\mathbf{F}$ and the position vector $\mathbf{r}$ of a point on the line of action of the vector from O. The cross product of two vectors is a vector at right angles to the plane defined by the first two vectors and with magnitude

$$Fr \sin \theta$$

which is the product of F and the shortest distance from O to the line of action of F. The sense of the final vector follows the right-hand rule. In Fig. 3.37 the force tends to cause a counterclockwise rotation around O. If this were a right-hand thread, it would tend to come up, so the vector is directed likewise up out of the paper. With the fingers of the right hand curled in the direction the force would tend to cause rotation, the thumb yields the direction, or sense, of the vector.

Since Eq. (3.10.4) represents the same vector $\mathbf{F}$ on either side of the equation, its vector product with the position vector $\mathbf{r}$ of a point O may be taken; thus

$$\mathbf{F} \times \mathbf{r} = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho\mathbf{v} \times \mathbf{r}(\mathbf{v} \cdot d\mathbf{A}) + \frac{\partial}{\partial t} \int_{\substack{\text{control}\\ \text{volume}}} \rho\mathbf{v} \times \mathbf{r}\, d\mathcal{V} \qquad (3.11.1)$$

The left-hand side of the equation is the *torque* exerted by the force, and the terms on the right-hand side represent the rate of change of *moment of momentum*. This is the general moment-of-momentum equation for unsteady flow through a control volume. It has great value in analyzing certain flow problems, such as in turbomachinery, where torques are more significant in the analysis than forces.

When Eq. (3.11.1) is applied to a case of flow in the xy-plane, with r the shortest distance to the tangential component of the velocity v_t, as in Fig. 3.38, and v_n the normal component of velocity,

$$F_t r = T_z = \int_{\substack{\text{area of}\\ \text{control}\\ \text{volume}}} \rho r v_t v_n\, dA + \frac{\partial}{\partial t} \int_{\substack{\text{control}\\ \text{volume}}} \rho r v_t\, d\mathcal{V} \qquad (3.11.2)$$

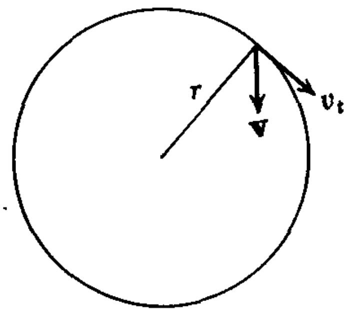

Fig. 3.38. Notation for two-dimensional flow.

in which T_z is the torque. A useful form of Eq. (3.11.2) for steady flow, which drops out the last term, is

$$T_z = \int \rho r v_t v_n\, dA_{\text{out}} - \int \rho r v_t v_n\, dA_{\text{in}} \qquad (3.11.3)$$

For complete circular symmetry, where r, ρ, v_t, and v_n are constant over the inlet and over the outlet, it takes the form

$$T_z = \rho Q[(rv_t)_{\text{out}} - (rv_t)_{\text{in}}] \qquad (3.11.4)$$

since $\int \rho v_n\, dA = \rho Q$, the same at inlet or outlet.

Example 3.26: A turbine discharging 400 cfs is to be designed so that a torque of 10,000 lb-ft is to be exerted on an impeller turning at 200 rpm that takes all the moment of momentum out of the fluid. At the outer periphery of the impeller, $r = 3.0$ ft. What must the tangential component of velocity be at this location?

Equation (3.11.4) is

$$T = \rho Q(rv_t)_{\text{in}}$$

in this case, since the outflow has $v_t = 0$. By solving for $v_{t_{\text{in}}}$

$$v_{t_{\text{in}}} = \frac{T}{\rho Q r} = \frac{10{,}000}{1.935 \times 400 \times 3} = 4.30 \text{ ft/sec}$$

Example 3.27: The sprinkler of Fig. 3.39 discharges 0.01 cfs through each nozzle. Neglecting friction, find its speed of rotation. The area of each nozzle opening is 0.001 ft².

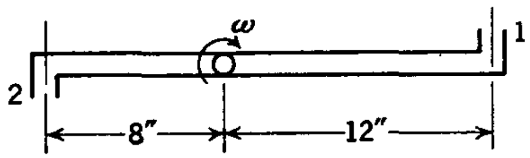

FIG. 3.39. Rotating jet system.

The fluid entering the sprinkler has no moment of momentum, and no torque is exerted on the system externally; hence the moment of momentum of fluid leaving must be zero. Let ω be the speed of rotation; then the moment of momentum leaving is

$$\rho Q_1 r_1 v_{t_1} + \rho Q_2 r_2 v_{t_2}$$

in which v_{t_1} and v_{t_2} are absolute velocities. Then

$$v_{t_1} = v_{r_1} - \omega r_1 = \frac{Q_1}{0.001} - \omega r_1 = 10 - \omega$$

and

$$v_{t_2} = v_{r_2} - \omega r_2 = 10 - \tfrac{2}{3}\omega$$

For moment of momentum to be zero

$$\rho Q(r_1 v_{t_1} + r_2 v_{t_2}) = 0$$

or

$$10 - \omega + \tfrac{2}{3}(10 - \tfrac{2}{3}\omega) = 0$$

and $\omega = 11.54$ rad/sec, $N = 110.2$ rpm.

PROBLEMS

3.1. A pump takes oil, sp gr 0.83, from a 2.0-in.-diameter pipe and returns it to a 2.0-in.-diameter pipe at the same elevation with a pressure increase of 20 psi. The quantity pumped is 0.50 cfs (cubic feet per second). The motor driving the pump delivers 3.50 hp to the pump shaft. Calculate the irreversibility of the pump in foot-pounds per pound mass and in foot-pounds per second. $g = 32.17$ ft/sec².

3.2. A pipeline leads from one water reservoir to another which has its water surface 20 ft lower. For a discharge of 1.0 cfs, determine the losses in foot-pounds per slug and in horsepower.

3.3. A blower delivers 10,000 cfm (cubic feet per minute) air, $\rho = 0.0024$ slugs/ft^3, at an increase in pressure of 4.0 in. water. It is 72 per cent efficient. Determine the irreversibility of the blower in foot-pounds per slug and in horsepower, and determine the torque in the shaft if the blower turns at 1800 rpm.

3.4. A three-dimensional velocity distribution is given by $u = -x$, $v = 2y$, $w = 2 - z$. Find the equation of the streamline through (1,1,1).

3.5. The irreversibilities in a pipeline amount to 20 ft-lb/lb$_m$ when the flow is 300 gpm (gallons per minute) and amount to 30 ft-lb/lb$_m$ when the flow is 450 gpm. What is the nature of the flow?

3.6. In flow of liquid through a pipeline the losses are 2 hp for average velocity of 4 ft/sec and 4 hp for 6 ft/sec. What is the nature of the flow?

3.7. When tripling the flow in a line causes the losses to increase by 7.64 times, how do the losses vary with velocity and what is the nature of the flow?

3.8. In two-dimensional flow around a circular cylinder (Fig. 3.2), the discharge between streamlines is 0.40 cfs/ft. At a great distance the streamlines are 0.20 in. apart, and at a point near the cylinder they are 0.12 in. apart. Calculate the magnitude of the velocity at these two points.

3.9. A pipeline carries oil, sp gr 0.83, at $V = 6$ ft/sec through 8.0-in. ID pipe. At another section the diameter is 6.0 in. Find the velocity at this section and the mass rate of flow in slugs per second.

3.10. Hydrogen is flowing in a 3.0-in.-diameter pipe at the mass rate of 0.40 lb$_m$/sec. At section 1 the pressure is 40 psia and $t = 40°$F. What is the average velocity?

3.11. A nozzle with base diameter of 3.0 in. and with $1\frac{1}{8}$-in.-diameter tip discharges 300 gpm. Find the velocity at the base and tip of nozzle.

3.12. An 18-ft-diameter pressure pipe has a velocity of 16 ft/sec. After passing through a reducing bend the flow is in a 16-ft-diameter pipe. If the losses vary as the square of the velocity, how much greater are they through the 16-ft pipe than through the 18-ft pipe per 1000 ft of pipe?

3.13. Does the velocity distribution of Prob. 3.4 for incompressible flow satisfy the continuity equation?

3.14. Does the velocity distribution

$$\mathbf{q} = \mathbf{i}(4 - x^2 + y) + \mathbf{j}(3 + 2y - z) + \mathbf{k}2z(x - 1)$$

satisfy continuity for incompressible flow.

3.15. Consider a cube with 1-ft edges parallel to the coordinate axes located in the first quadrant with one corner at the origin. By using the velocity distribution of Prob. 3.14, find the flow through each face and show that continuity is satisfied for the cube as a whole.

3.16. Find the flow (per foot in the z-direction) through each edge of the square with corners at (0,0), (0,1), (1,1), (1,0), due to

$$\mathbf{q} = \mathbf{i}2xy + \mathbf{j}(x^2 - y^2)$$

and show that continuity is satisfied.

3.17. Show that the velocity

$$\mathbf{q} = \mathbf{i}4\frac{x}{x^2 + y^2} + \mathbf{j}4\frac{y}{x^2 + y^2}$$

satisfies continuity at every point except the origin.

3.18. Problem 3.17 is a velocity distribution that is everywhere radial from the origin with magnitude $v_r = 4/r$. Show that the flow through each circle concentric with the origin (per foot in the z-direction) is the same.

3.19. Perform the operation $\nabla \cdot \mathbf{q}$ on the velocity vectors of Probs. 3.14, 3.16, and 3.17.

3.20. Does the velocity

$$\mathbf{q} = \mathbf{i} \ln x^2y^2 + \mathbf{j}\left(\frac{2y}{x} - \ln xt\right)$$

satisfy continuity?

3.21. A standpipe 16 ft in diameter and 40 ft high is filled with water. How much potential energy is in this water if the elevation datum is taken 10 ft below the base of the standpipe?

3.22. How much work could be obtained from the water of Prob. 3.21 if run through a 100 per cent efficient turbine that discharged into a reservoir with elevation 20 ft below the base of the standpipe?

3.23. What is the kinetic energy in foot-pounds per second of 200 gpm of oil, sp gr 0.80, discharging through a 1.0-in.-diameter nozzle?

3.24. By neglecting air resistance, determine the height a vertical jet of water will rise, with velocity 80.2 ft/sec.

3.25. If the water jet of Prob. 3.24 is directed upward 45° with the horizontal and air resistance is neglected, how high will it rise and what is the velocity at its high point?

3.26. Show that the work a liquid can do by virtue of its pressure $\int p\, dV$, in which V is the volume of liquid displaced.

3.27. What angle of jet α is required to reach the roof of the building of Fig. 3.40 with minimum jet velocity V_0 at the nozzle? What is the value of V_0?

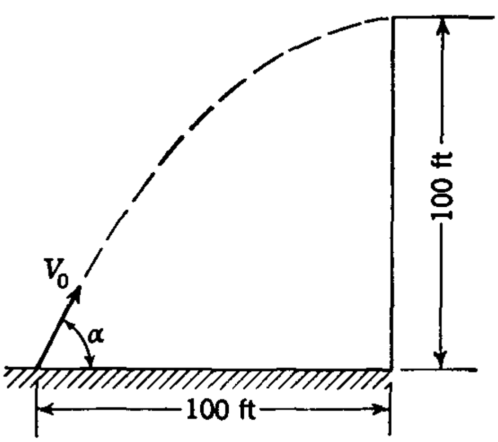

FIG. 3.40

3.28. For highly turbulent flow the velocity distribution in a pipe is given by

$$\frac{v}{v_{max}} = \left(\frac{y}{r_0}\right)^{\frac{1}{9}}$$

with y the wall distance and r_0 the pipe radius. Determine the kinetic-energy correction factor for this flow.

3.29. When the velocity over half a cross section is uniform at 40 per cent of the uniform velocity over the rest of the section, what is the kinetic-energy correction factor?

3.30. The velocity over half a cross section is V_0, and over the other half it is $-0.10V_0$. What is the kinetic-energy correction factor?

3.31. The velocity distribution in laminar flow in a pipe is given by $v = V_{max}[1 - (r/r_0)^2]$. Determine the average velocity and the kinetic-energy correction factor.

3.32. Water is flowing in a channel, as shown in Fig. 3.41. Neglecting all losses, determine the two possible depths of flow y_1 and y_2.

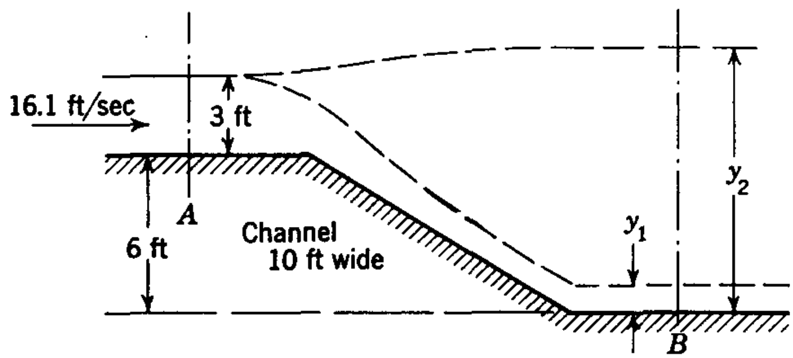

FIG. 3.41

3.33. High-velocity liquid, sp gr 1.20, flows up an inclined plane as shown in Fig. 3.42. Neglecting all losses, calculate the two possible depths of flow at section B.

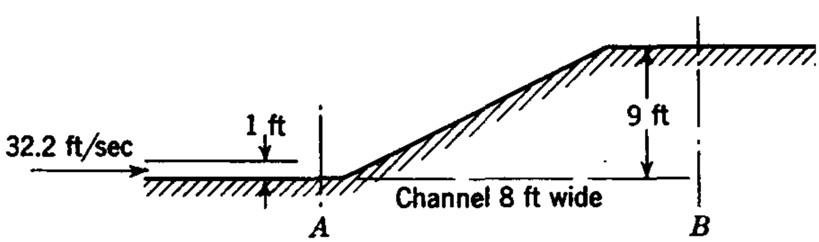

FIG. 3.42

3.34. If the losses from section A to section B of Fig. 3.41 are 2 ft-lb/lb, determine the two possible depths at section B.

3.35. In Fig. 3.42 losses are 8 hp per foot of width between sections A and B for water flowing. Determine the lower depth of flow at section B.

3.36. Neglecting all losses, in Fig. 3.41 the channel narrows in the drop to 5 ft wide at section B. For uniform flow across section B, determine the two possible depths of flow.

3.37. In Fig. 3.42 the channel changes in width from 4 ft at section A to 8 ft at section B. For losses of 1 ft-lb/lb between sections A and B, find the two possible depths at section B.

3.38. Some steam locomotives had scoops installed that took water from a tank between the tracks and lifted it into the water reservoir in the tender. To lift the water 12 ft with a scoop, neglecting all losses, what speed is required? (NOTE: Consider the locomotive stationary and the water moving toward it, to reduce to a steady-flow situation.)

3.39. In Fig. 3.43 oil discharges from a "two-dimensional" slot as indicated at A into the air. At B oil discharges from under a gate onto a floor. Neglecting all losses, determine the discharges of A and at B per foot of width. Why do they differ?

3.40. At point A in a pipeline carrying water the diameter is 4.0 ft, the pressure 10 psi, and the velocity 8.02 ft/sec. At point B, 6 ft higher than A, the diameter is 2.0 ft and the pressure 2 psi. Determine the direction of flow.

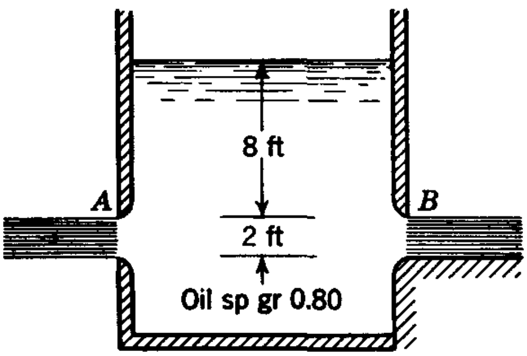

FIG. 3.43

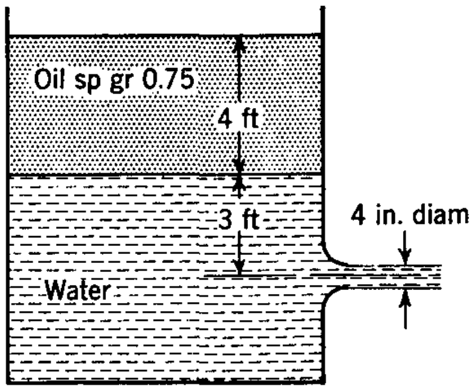

FIG. 3.44

3.41. Neglecting losses, determine the discharge in Fig. 3.44.

3.42. Neglecting losses, determine the discharge in Fig. 3.45.

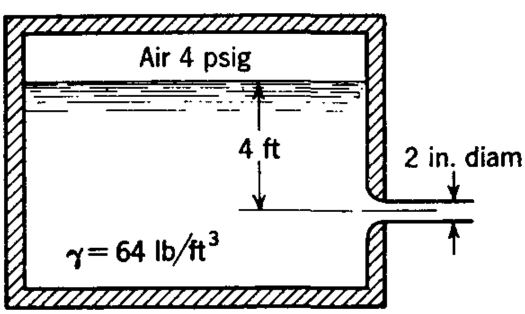

FIG. 3.45

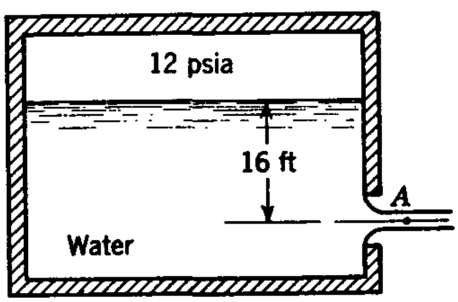

FIG. 3.46

3.43. For losses of 0.3 ft-lb/lb, find the velocity at A in Fig. 3.46. Barometer reading 29.5 in. mercury.

3.44. Neglecting losses in the converging section, calculate the discharge in Fig. 3.47.

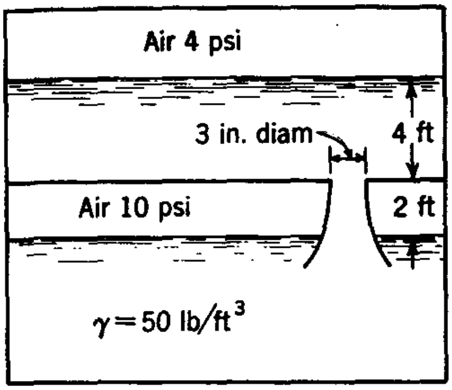

FIG. 3.47

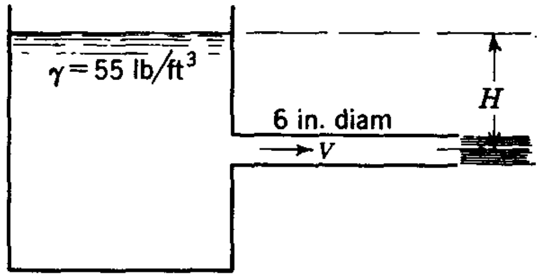

FIG. 3.48

3.45. The losses in Fig. 3.48 for $H = 16$ ft are $3V^2/2g$ ft-lb/lb. What is the discharge?

3.46. For flow of 705 gpm in Fig. 3.48, determine H for losses of $15V^2/2g$ ft-lb/lb.

3.47. For 1410-gpm flow and $H = 32$ ft in Fig. 3.48, calculate the losses through the system in velocity heads, $KV^2/2g$.

3.48. In Fig. 3.49 the losses up to section A are $4V_1^2/2g$ and the nozzle losses are $0.05V_2^2/2g$. Determine the discharge and the pressure at A. $H = 16$ ft.

3.49. For pressure at A of 5 psi in Fig. 3.49, with the losses in Prob. 3.48, determine the discharge and the head H.

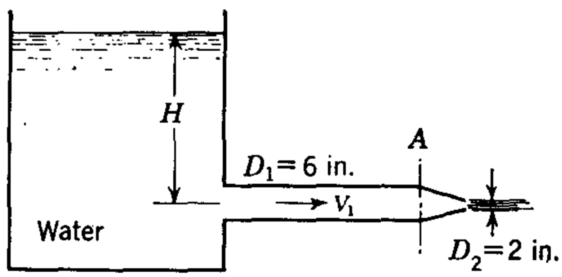

FIG. 3.49

3.50. Neglecting losses and surface-tension effects, derive an equation for the water surface of the jet of Fig. 3.50, r in terms of y/H.

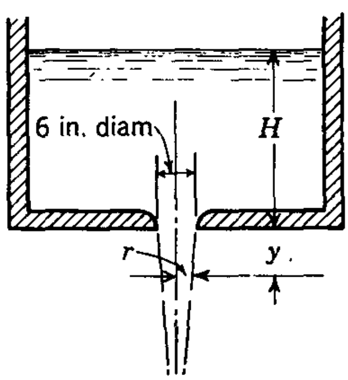

FIG. 3.50

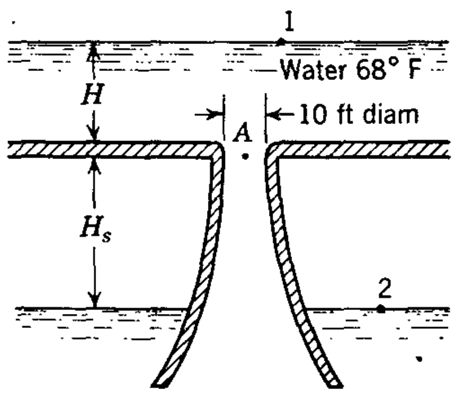

FIG. 3.51

3.51. For losses of $0.5V_A^2/2g$ between points A and 2 of Fig. 3.51, find H for the pressure at A to be equal to vapor pressure. Barometric pressure 34 ft water. $H_s = 10$ ft.

3.52. For $H = 30$ ft and losses from A downstream of $0.6V_A^2/2g$ in Fig. 3.51, determine H_s for vapor pressure at A. Barometric pressure 33 ft water.

3.53. In the siphon of Fig. 3.52, $h_1 = 3$ ft, $h_2 = 9$ ft, $D_1 = 10$ ft, $D_2 = 14$ ft, and the losses are $1.6V_2^2/2g$, with 10 per cent of the losses occurring before section 1. Find the discharge and the pressure at section 1.

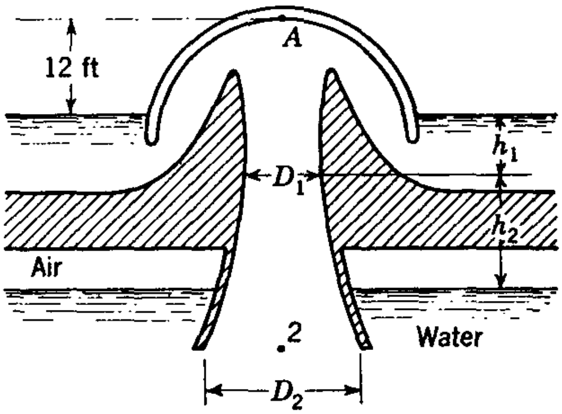

FIG. 3.52

3.54. Find the pressure at A of Prob. 3.53 if it is a stagnation point (velocity zero).

3.55. The siphon of Fig. 3.14 has a nozzle 6 in. long attached at section 3, reducing the diameter to 6 in. For no losses, compute the discharge, and the pressure at sections 2 and 3.

3.56. With exit velocity V_E in Prob. 3.55 and losses from 1 to 2 of $1.7V_2^2/2g$, from 2 to 3 of $0.9V_2^2/2g$ and through the nozzle $0.06V_E^2/2g$, calculate the discharge and the pressure at sections 2 and 3.

3.57. Determine the shaft horsepower for an 80 per cent efficient pump to discharge 1 cfs through the system of Fig. 3.53. The system losses, exclusive of pump losses, are $5V^2/2g$, and $H = 40$ ft.

3.58. The fluid horsepower ($Q\gamma H_p/550$) produced by the pump of Fig. 3.53 is 10. For $H = 60$ ft and system losses of $6V^2/2g$, determine the discharge and the pump head.

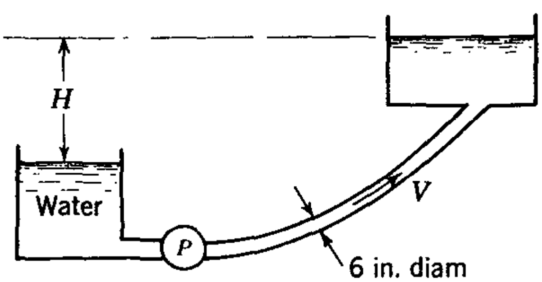

FIG. 3.53

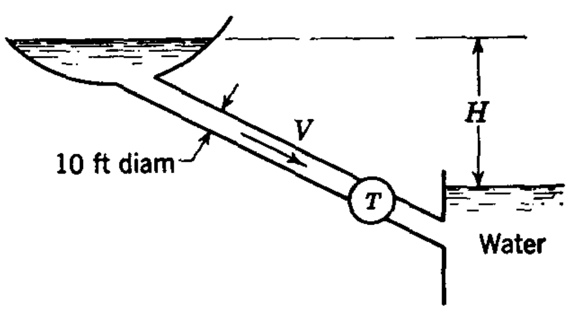

FIG. 3.54

3.59. If the over-all efficiency of the system and turbine in Fig. 3.54 is 80 per cent, what horsepower is produced for $H = 300$ ft and $Q = 1{,}000$ cfs?

3.60. Losses through the system of Fig. 3.54 are $4V^2/2g$, exclusive of the turbine. The turbine is 90 per cent efficient and runs at 200 rpm. To produce 1000 hp for $H = 400$ ft, determine the discharge and torque in the turbine shaft.

3.61. Neglecting losses, find the discharge through the venturi meter of Fig. 3.55.

3.62. With losses of $0.2V_1^2/2g$ between sections 1 and 2 of Fig. 3.55, calculate the flow in gallons per minute.

3.63. Neglecting losses in an 8- by 4-in.-diameter venturi meter carrying oil, sp gr 0.83, find the gage difference on a mercury-oil manometer for 600-gpm flow.

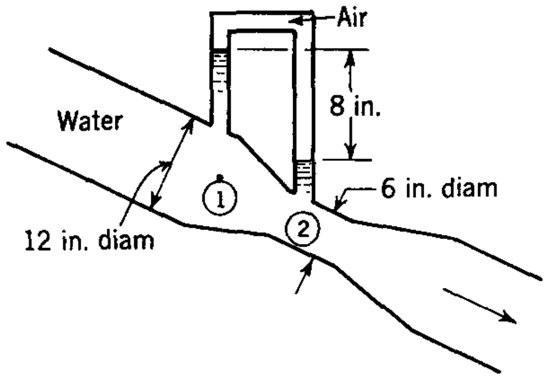

FIG. 3.55

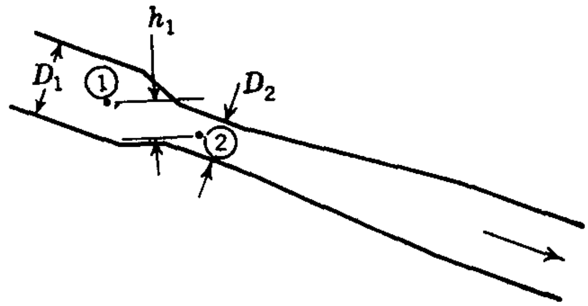

FIG. 3.56

3.64. In Fig. 3.56, $h_1 = 6$ in., $D_1 = 4$ in. and $D_2 = 3$ in. Oil, sp gr 0.85, is flowing. $p_1 = 16$ psi, and $p_2 = 12$ psi. Neglecting losses, find the flow in gallons per minute.

3.65. With losses of $0.05V_2^2/2g$ between sections 1 and 2 of Prob. 3.64, calculate the discharge.

3.66. In Fig. 3.57, for $R = 12$ in. and $V = 15$ ft/sec, determine the bourdon gage reading at A in pounds per square inch.

3.67. In Fig. 3.57 $p_A = 14$ psi and $R = 2$ ft. Determine V.

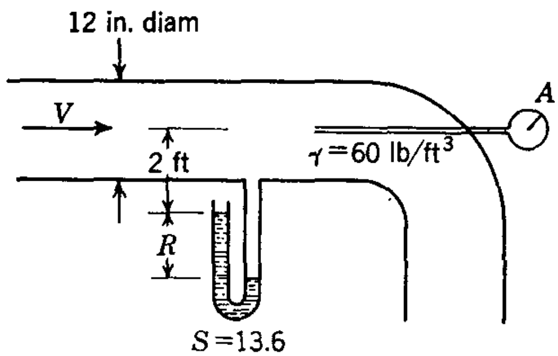

FIG. 3.57

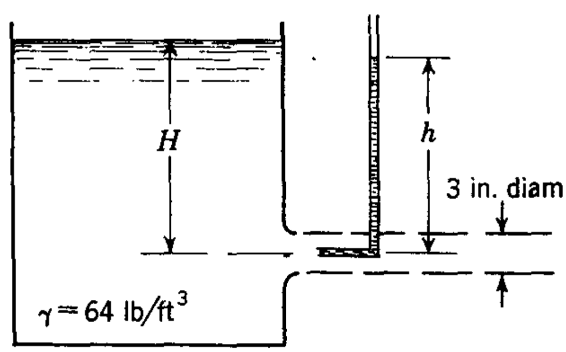

FIG. 3.58

3.68. In Fig. 3.58 $H = 16.0$ ft and $h = 15.7$ ft. Calculate the discharge and the losses in foot-pounds per pound and in horsepower.

3.69. Neglecting losses, calculate H in terms of R for Fig. 3.59.

3.70. For losses of $0.1H$ through the nozzle of Fig. 3.59, what is the gage difference R in terms of H?

3.71. A liquid flows through a long pipeline with losses of 4 ft-lb/lb per 100 ft of pipe. What is the slope of the hydraulic and energy grade lines?

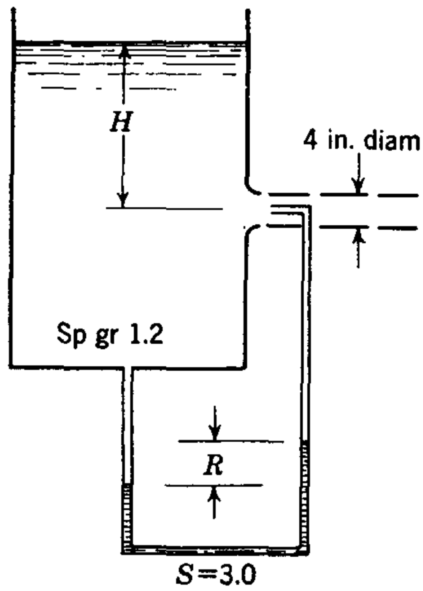

FIG. 3.59

3.72. In Fig. 3.60, 4 cfs water flows from section 1 to section 2 with losses of $0.4(V_1 - V_2)^2/2g$. $p_1 = 10$ psi. Compute p_2, and plot the energy and hydraulic grade lines through the diffuser.

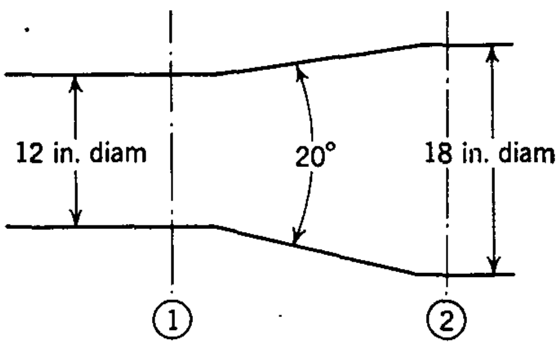

FIG. 3.60

3.73. In an isothermal, reversible flow at 200°F, 2 Btu/sec heat is added to 14 slug/sec flowing through a control volume. Calculate the entropy increase in foot-pounds per slug per degrees Rankine.

3.74. In isothermal flow of a real fluid through a pipe system the losses are 60 ft-lb/slug per 100 ft and 0.02 Btu/sec per 100 ft heat transfer from the fluid is required to hold the temperature at 40°F. What is the entropy change Δs in foot-pounds per slug per degree Rankine per 100 ft of pipe system for 10 lb_m/sec flowing?

3.75. In Example 3.19 of Sec. 3.9, to what height will the rocket glide?

3.76. Determine the momentum correction factor for the velocity distribution of Prob. 3.31.

3.77. Calculate the average velocity and momentum correction factor for the velocity distribution in a pipe

$$\frac{v}{v_{max}} = \left(\frac{y}{r_0}\right)^{1/n}$$

with y the wall distance and r_0 the pipe radius.

3.78. Determine the momentum correction factor for the velocity distribution of Prob. 3.29.

3.79. Determine the momentum correction factor for the velocity distribution of Prob. 3.30.

3.80. When the momentum correction factor is unity, prove that the velocity must be uniform over the cross section.

3.81. Determine the momentum per second passing an open-channel cross section carrying 1000 cfs water with velocity of 8 ft/sec.

3.82. What force F (Fig. 3.61) is required to hold the plate for oil flow, sp gr 0.83, for $V_0 = 40$ ft/sec.

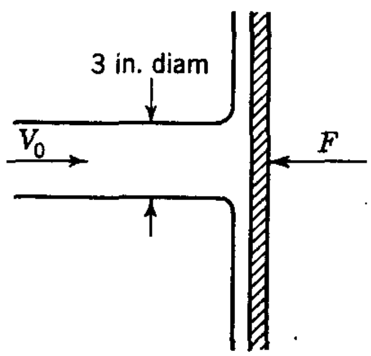

FIG. 3.61

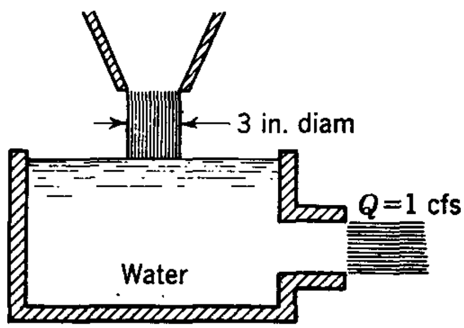

FIG. 3.62

3.83. How much is the apparent weight of the tank full of water (Fig. 3.62) increased by the steady jet flow into the tank?

3.84. Does a nozzle on a fire hose place the hose in tension or in compression?

3.85. When a jet from a nozzle is used to aid in maneuvering a fireboat, can more force be obtained by directing the jet against a solid surface such as a wharf than by allowing it to discharge into air?

3.86. Work Example 3.15 with the flow direction reversed, and compare results.

3.87. 25 ft³/sec of water flows through an 18-in.-diameter pipeline that contains a 90° bend. The pressure at the entrance to the bend is 10 psi. Determine the force components, parallel and normal to the approach velocity, required to hold the bend in place. Neglect losses.

3.88. Oil, sp gr 0.83, flows through a 90° expanding pipe bend from 18- to 24-in.-diameter pipe. The pressure at the bend entrance is 20 psi, and losses are to be neglected. For 20,000 gpm, determine the force components (parallel and normal to the approach velocity) necessary to support the bend.

3.89. Work Prob. 3.88 with elbow losses of $0.6V_1^2/2g$, with V_1 the approach velocity, and compare results.

3.90. A 4-in.-diameter steam line carries saturated steam at 1400 ft/sec velocity. Water is entrained by the steam at the rate of 0.3 lb/sec. What force is required to hold a 90° bend in place owing to the entrained water?

3.91. Neglecting losses, determine the x- and y-components of force needed to hold the tee (Fig. 3.63) in place.

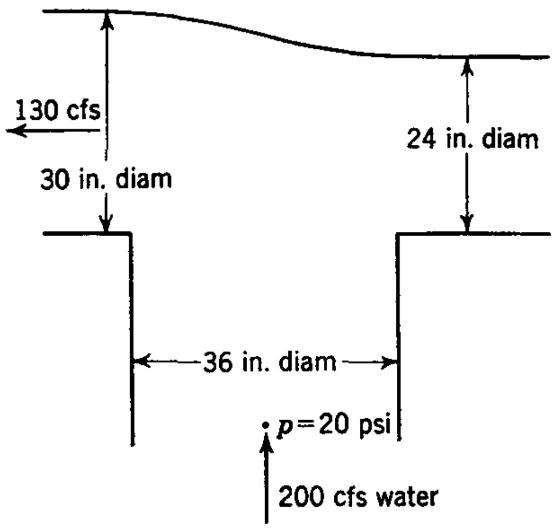

FIG. 3.63

3.92. Apply the momentum and energy equations to a windmill as if it were a propeller, noting that the slipstream is slowed down and expands as it passes through the blades. Show that the velocity through the plane of the blades is the average of the velocities in the slipstream at the downstream and upstream sections. By defining the theoretical efficiency (neglecting all losses) as the power output divided by the power available in an undisturbed jet having the area at the plane of the blades, determine the maximum theoretical efficiency of a windmill.

3.93. An airplane with propeller diameter of 8.0 ft travels through still air (ρ = 0.0022 slug/ft^3) at 180 mph. The speed of air through the plane of the propeller is 250 mph relative to the airplane. Calculate (*a*) the thrust on the plane, (*b*) the kinetic energy per second remaining in the slipstream, (*c*) the theoretical horsepower required to drive the propeller, (*d*) the propeller efficiency, and (*e*) the pressure difference across the blades.

3.94. A boat traveling at 30 mph has a 2-ft-diameter propeller that discharges 160 cfs through its blades. Determine the thrust on the boat, the theoretical efficiency of the propulsion system, and the horsepower input to the propeller.

3.95. A ship propeller has a theoretical efficiency of 60 per cent. If it is 4 ft in diameter and the ship travels 30 mph, what is the thrust developed and what is the theoretical horsepower required?

3.96. A jet-propelled airplane traveling 575 mph takes in 20 lb$_m$/sec air and discharges it at 5500 ft/sec relative to the airplane. Neglecting the weight of fuel, what thrust is produced?

3.97. A jet-propelled airplane travels 635 mph. It takes in 18 lb$_m$/sec air and uses 1 lb$_m$ fuel for each 12 lb$_m$ air. What thrust is developed when the exhaust gases have an absolute velocity of 5000 ft/sec?

3.98. What is the theoretical mechanical efficiency of the jet engine of Prob. 3.97?

3.99. A boat requires a 500-lb thrust to keep it moving at 16 mph. How many cubic feet per second water must be taken in and ejected through a 16-in. pipe to maintain this motion? What is the over-all efficiency if the pumping system is 60 per cent efficient?

3.100. In Prob. 3.99 what would be the required discharge if water were taken from a tank inside the boat and ejected from the stern through a 16-in. pipe?

3.101. Determine the size of jet pipe and the theoretical horsepower necessary to produce a thrust of 2000 lb on a boat moving 45 ft/sec when the propulsive efficiency is 68 per cent.

3.102. An airplane consumes 1 lb_m fuel for each 18 lb_m air and discharges hot gases from the tail pipe at $v_r = 5400$ ft/sec. What plane speed would be required to obtain a mechanical efficiency of 28 per cent?

3.103. What is the speed of a jet engine for zero thrust when the gas leaves at 5000 ft/sec relative to the plane and 1 lb_m of fuel is burned for each 12 lb_m of air?

3.104. In Fig. 3.64, a jet, $\rho = 2$ slugs/ft³, is deflected by a vane through 180°. Assume that the cart is frictionless and free to move in a horizontal direction. The cart weighs 200 lb. Determine the velocity and the distance traveled by the cart 10 sec after the jet is directed against the vane. $A_0 = 0.01$ ft²; $V_0 = 100$ ft/sec.

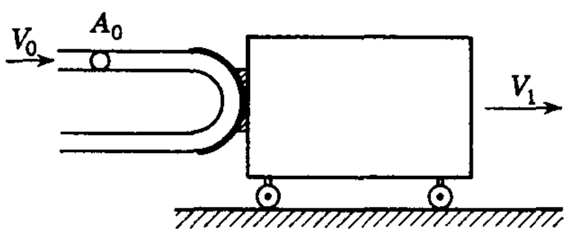

FIG. 3.64

3.105. A rocket burns 10 lb_m/sec fuel, ejecting hot gases at 8000 ft/sec relative to the rocket. How much thrust is produced at 500 and 1500 mph?

3.106. What is the mechanical efficiency of a rocket moving at 2000 ft/sec that ejects gas at 6000 ft/sec relative to the rocket?

3.107. Can a rocket travel faster than the velocity of ejected gas? What is the mechanical efficiency when it travels 12,000 ft/sec and the gas is ejected at 8000 ft/sec relative to the rocket? Is a positive thrust developed?

3.108. Neglecting air resistance, what velocity would a vertically directed rocket attain in 8 sec if it starts from rest, initially weighs 240 lb, burns 10 lb_m/sec, and ejects gas at $v_r = 6440$ ft/sec? Consider $g = 32.17$ ft/sec².

3.109. What height has the rocket of Prob. 3.108 attained at the end of 8 sec?

3.110. If the rocket of Prob. 3.108 has only 80 lb_m fuel, what is the maximum height it attains?

3.111. Draw the polar vector diagram for a vane, angle θ, doing work on a jet. Label all vectors.

3.112. Determine the resultant force exerted on the vane of Fig. 3.26. $A_0 =$

0.06 ft^2; $V_0 = 80$ ft/sec; $\theta = 60°$, $\gamma = 55$ lb/ft^3. How can the line of action be determined?

3.113. In Fig. 3.27, 40 per cent of the flow is deflected in one direction. What is the plate angle θ?

3.114. A flat plate is moving with velocity u into a jet, as shown in Fig. 3.65. Derive the expression for power required to move the plate.

3.115. At what speed u should the cart of Fig. 3.65 be given away from the jet in order to produce maximum work from the jet?

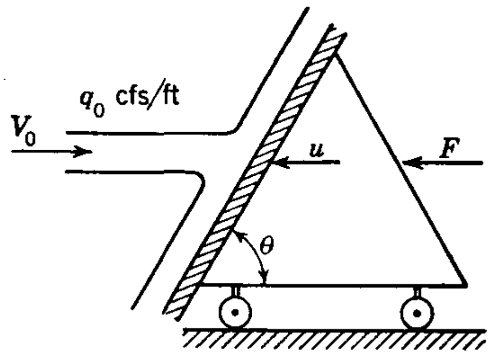

FIG. 3.65

3.116. At what speed u should the vane of Fig. 3.28 travel for maximum power from the jet?

3.117. Draw the polar vector diagram for the moving vane of Fig. 3.28 for $V_0 = 100$ ft/sec, $u = 60$ ft/sec, and $\theta = 120°$.

3.118. Draw the polar vector diagram for the moving vane of Fig. 3.28 for $V_0 = 120$ ft/sec, $u = -50$ ft/sec, and $\theta = 150°$.

3.119. What horsepower can be developed from (a) a single vane and (b) a series of vanes (Fig. 3.28) when $A_0 = 9$ in.^{2}, $V_0 = 270$ ft/sec, $u = 90$ ft/sec, and $\theta = 173°$, for water flowing?

3.120. Determine the blade angles θ_1 and θ_2 of Fig. 3.66 so that the flow enters the vane tangent to its leading edge and leaves with no x-component of absolute velocity.

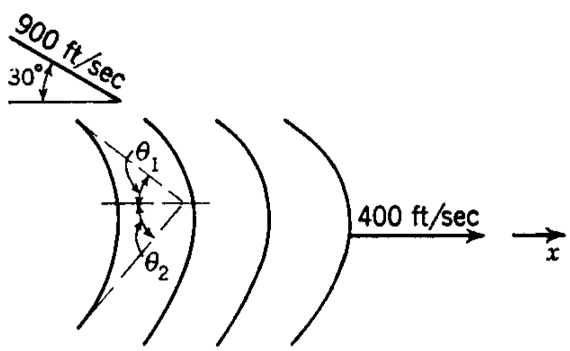

FIG. 3.66

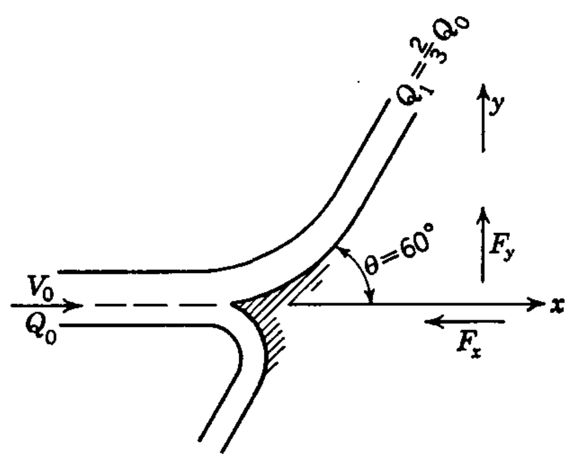

FIG. 3.67

3.121. Calculate the force components F_x, F_y needed to hold the stationary vane of Fig. 3.67. $Q_0 = 2$ cfs; $\rho = 2$ slugs/ft^3; $V_0 = 300$ ft/sec.

3.122. If the vane of Fig. 3.67 moves in the x-direction at $u = 40$ ft/sec, for $Q = 3$ cfs, $\rho = 1.935$ slugs/ft^3, $V_0 = 120$ ft/sec, what are the force components F_x, F_y?

3.123. What force components F_x, F_y are required to hold the "black box" of Fig. 3.68 stationary?

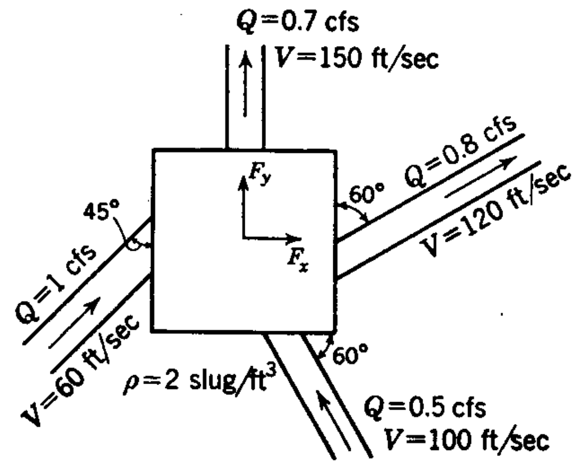

FIG. 3.68

3.124. Determine the vane angle required to deflect the absolute velocity of a jet 120° (Fig. 3.69).

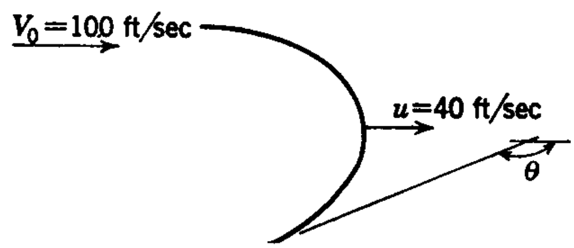

FIG. 3.69

3.125. In Prob. 3.38 for pickup of 2 cfs water at locomotive speed of 36 mph, what force is exerted parallel to the tracks?

3.126. Determine the irreversibility in foot-pounds per pound mass for 2 cfs flow of liquid, $\rho = 1.6$ slugs/ft^3, through a sudden expansion from a 12- to 24-in.-diameter pipe. $g = 30$ ft/sec^2.

3.127. Air flows through a 24-in.-diameter duct at $p = 20$ psia, $t = 40$°F, $V = 200$ ft/sec. The duct suddenly expands to 36 in. diameter. Considering the gas as incompressible, calculate the losses in foot-pounds per pound of air and the pressure difference in inches of water.

3.128. What are the losses when 200 cfs water discharges from a submerged 48-in.-diameter pipe into a reservoir?

3.129. Show that in the limiting case, as $y_1 = y_2$ in Eq. (3.9.34), the relation $V = \sqrt{gy}$ is obtained.

3.130. Derive the equation for depth y_1 needed before a hydraulic jump for it to reach y_2 and V_2.

3.131. A jump occurs in a 20-ft-wide channel carrying 600 cfs water at a depth of 1 ft. Determine y_2, V_2, and the losses in foot-pounds per pound and in horsepower.

3.132. Derive an expression for determining the initial depth y_1 before a jump when y_2 and V_1 are known.

3.133. Derive Eq. (3.9.35).

3.134. Assuming no losses through the gate of Fig. 3.70 and neglecting $V_0^2/2g$, for $y_0 = 16$ ft and $y_1 = 2$ ft, find y_2 and losses through the jump.

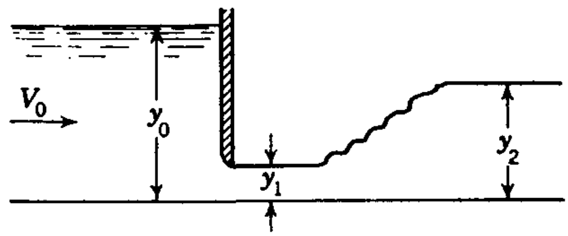

FIG. 3.70

3.135. Under the same assumption as in Prob. 3.134, for $y_1 = 1$ ft and $y_2 = 4$ ft, determine y_0.

3.136. Under the same assumptions as in Prob. 3.134, $y_0 = 20$ ft and $y_2 = 8$ ft. Find the discharge per foot.

3.137. For losses down the spillway of Fig. 3.71 of 10 ft-lb/lb and discharge per foot of 120 cfs, determine the floor elevation for the jump to occur.

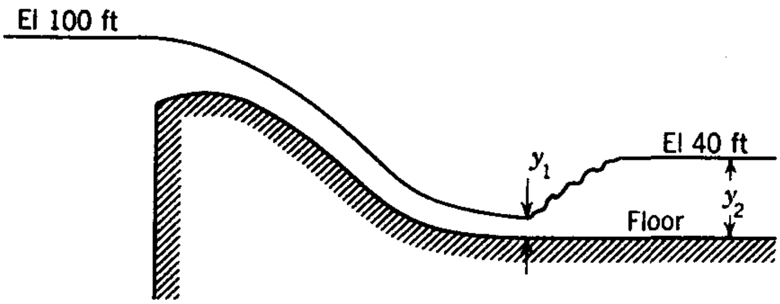

FIG. 3.71

3.138. Determine the depth after jump of a flow of kerosene, sp gr 0.83, with velocity 1 ft/sec and depth $\frac{1}{8}$ in.

3.139. Water is flowing through the pipe of Fig. 3.72 with velocity $V = 8.02$ ft/sec and losses of 8 ft-lb/lb up to section 1. When the obstruction at the end of the pipe is removed, calculate the acceleration of water in the pipe.

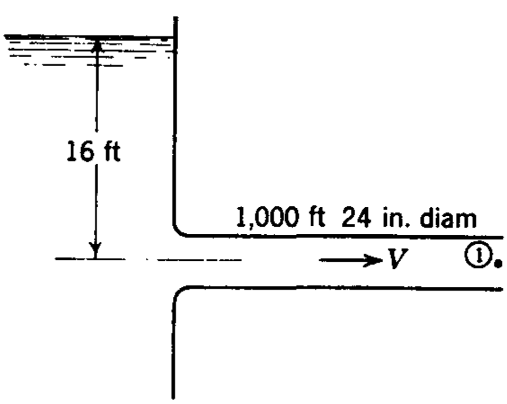

FIG. 3.72

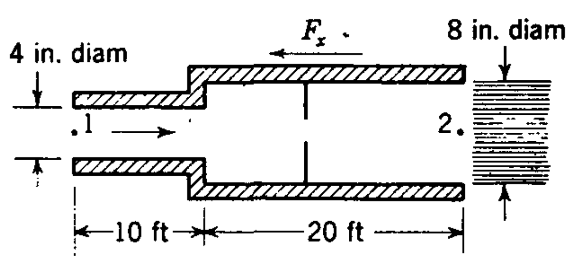

FIG. 3.73

3.140. Water fills the piping system of Fig. 3.73. At one instant $p_1 = 5$ psi, $p_2 = 0$, $V_1 = 10$ ft/sec, and the flow rate is increasing by 5000 gpm per minute. Find the force F_x required to hold the piping system stationary.

3.141. In a centrifugal pump 400 gpm water leaves an 8-in.-diameter impeller with a tangential velocity component of 30 ft/sec. It enters the impeller in a radial direction. For pump speed of 1200 rpm and neglecting all losses, determine the torque in the pump shaft, the horsepower input, and the energy added to the flow in foot-pounds per pound.

3.142. A water turbine at 240 rpm discharges 1200 cfs. To produce 50,000 hp, what must be the tangential component of velocity at the entrance to the impeller at $r = 6$ ft? All whirl is taken from the water when it leaves the turbine. Neglect all losses. What head is required for the turbine?

3.143. The symmetrical sprinkler of Fig. 3.74 has a total discharge of 20 gpm and is frictionless. Determine its rpm if the nozzle tips are $\frac{1}{4}$-in. diameter.

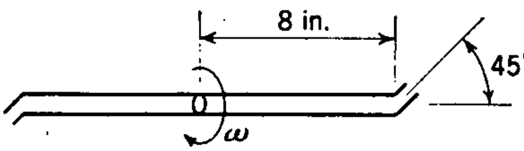

FIG. 3.74

3.144. If there is a torque resistance of 0.50 lb-ft in the shaft of Prob. 3.143, what is its speed of rotation?

3.145. For torque resistance of $0.01\omega^2$ in the shaft, determine the speed of rotation of the sprinkler of Prob. 3.143.

3.146. For a frictionless shaft in the sprinkler of Fig. 3.75 and equal flow through each nozzle ($v_r = 30$ ft/sec), find its speed of rotation.

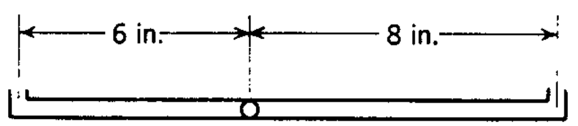

FIG. 3.75

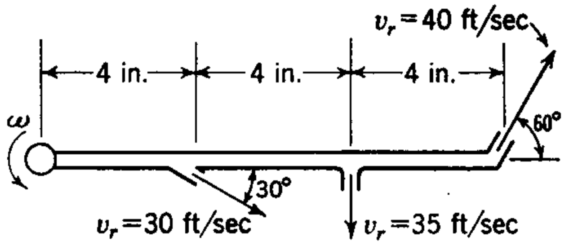

FIG. 3.76

3.147. For equal discharge through each of the nozzles of the sprinkler of Fig. 3.76 of 10 gpm and a frictionless shaft, determine its speed of rotation.

3.148. What torque would be required to hold the sprinkler of Prob. 3.147 stationary? Total flow 40 gpm water.

3.149. A *reversible* process requires that

(*a*) there be no heat transfer
(*b*) Newton's law of viscosity be satisfied
(*c*) temperature of system and surroundings be equal
(*d*) there be no viscous or Coloumb friction in the system
(*e*) heat transfer occurs from surroundings to system only

3.150. An *open system* implies

(*a*) the presence of a free surface
(*b*) that a specified mass is considered
(*c*) the use of a control volume

(*d*) no interchange between system and surroundings
(*e*) none of the above answers

3.151. A *control volume* refers to

(*a*) a fixed region in space
(*b*) a specified mass
(*c*) an isolated system
(*d*) a reversible process only
(*e*) a closed system

3.152. Which three of the following are synonymous?

1. losses
2. irreversibilities
3. energy losses
4. available energy losses
5. drop in hydraulic grade line

(*a*) 1, 2, 3 (*b*) 1, 2, 5 (*c*) 1, 2, 4 (*d*) 2, 3, 4 (*e*) 3, 4, 5

3.153. Irreversibility of the system of Fig. 3.77 is

(*a*) 9.2 hp (*b*) 36.8 hp (*c*) 8.45 ft (*d*) 11.55 ft (*e*) none of these answers

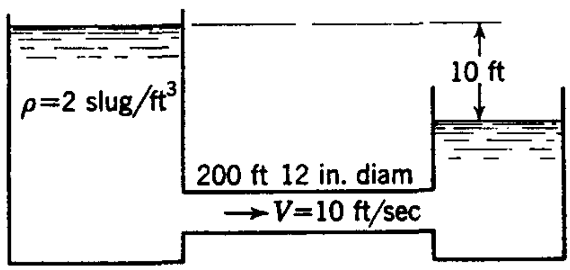

FIG. 3.77

3.154. Isentropic flow is

(*a*) irreversible adiabatic flow (*b*) perfect-gas flow
(*c*) ideal-fluid flow (*d*) reversible adiabatic flow
(*e*) frictionless reversible flow

3.155. One-dimensional flow is

(*a*) steady uniform flow
(*b*) uniform flow
(*c*) flow which neglects changes in a transverse direction
(*d*) restricted to flow in a straight line
(*e*) none of these answers

3.156. The continuity equation may take the form

(*a*) $Q = pAv$ (*b*) $\rho_1 A_1 = \rho_2 A_2$ (*c*) $p_1 A_1 v_1 = p_2 A_2 v_2$
(*d*) $\nabla \cdot \mathbf{p} = 0$ (*e*) $A_1 v_1 = A_2 v_2$

3.157. The first law of thermodynamics, for steady flow,

(*a*) accounts for all energy entering and leaving a control volume
(*b*) is an energy balance for a specified mass of fluid
(*c*) is an expression of the conservation of linear momentum
(*d*) is primarily concerned with heat transfer
(*e*) is restricted in its application to perfect gases

3.158. Entropy, for reversible flow, is defined by the expression

(*a*) $ds = du + p\, d(1/\rho)$ (*b*) $ds = T\, dq_H$ (*c*) $s = u + pv_s$
(*d*) $ds = dq_H/T$ (*e*) none of these answers

3.159. The equation d (losses) $= T\, ds$ is restricted to

(*a*) isentropic flow (*b*) reversible flow (*c*) adiabatic flow
(*d*) perfect-gas flow (*e*) none of these answers

3.160. In turbulent flow

(*a*) the fluid particles move in an orderly manner
(*b*) cohesion is more effective than momentum transfer in causing shear stress
(*c*) momentum transfer is on a molecular scale only
(*d*) one lamina of fluid glides smoothly over another
(*e*) the shear stresses are generally larger than in a similar laminar flow

3.161. The ratio $\eta = \tau/(du/dy)$ for turbulent flow is

(*a*) a physical property of the fluid
(*b*) dependent upon the flow and the density
(*c*) the viscosity divided by the density
(*d*) a function of temperature and pressure of fluid
(*e*) independent of the nature of the flow

3.162. Turbulent flow generally occurs for cases involving

(*a*) very viscous fluids
(*b*) very narrow passages or capillary tubes
(*c*) very slow motions
(*d*) combinations of (*a*), (*b*), and (*c*)
(*e*) none of these answers

3.163. In laminar flow

(*a*) experimentation is required for the simplest flow cases
(*b*) Newton's law of viscosity applies
(*c*) the fluid particles move in irregular and haphazard paths
(*d*) the viscosity is unimportant
(*e*) the ratio $\tau/(du/dy)$ depends upon the flow

3.164. An ideal fluid is

(*a*) very viscous
(*b*) one which obeys Newton's law of viscosity

(*c*) a useful assumption in problems in conduit flow
(*d*) frictionless and incompressible
(*e*) none of these answers

3.165. Which of the following must be fulfilled by the flow of any fluid, real or ideal?

1. Newton's law of viscosity
2. Newton's second law of motion
3. The continuity equation
4. $\tau = (\mu + \eta)\, du/dy$
5. Velocity at boundary must be zero relative to boundary
6. Fluid cannot penetrate a boundary

(*a*) 1, 2, 3 (*b*) 1, 3, 6 (*c*) 2, 3, 5 (*d*) 2, 3, 6 (*e*) 2, 4, 5

3.166. Steady flow occurs when

(*a*) conditions do not change with time at any point
(*b*) conditions are the same at adjacent points at any instant
(*c*) conditions change steadily with the time
(*d*) $\partial v/\partial t$ is constant
(*e*) $\partial v/\partial s$ is constant

3.167. Uniform flow occurs

(*a*) whenever the flow is steady
(*b*) when $\partial \bar{v}/\partial t$ is everywhere zero
(*c*) only when the velocity vector at any point remains constant
(*d*) when $\partial \bar{v}/\partial s = 0$
(*e*) when the discharge through a curved pipe of constant cross-sectional area is constant

3.168. Select the correct practical example of steady nonuniform flow:

(*a*) motion of water around a ship in a lake
(*b*) motion of a river around bridge piers
(*c*) steadily increasing flow through a pipe
(*d*) steadily decreasing flow through a reducing section
(*e*) constant discharge through a long, straight pipe

3.169. A streamline

(*a*) is the line connecting the mid-points of flow cross sections
(*b*) is defined for uniform flow only
(*c*) is drawn normal to the velocity vector at every point
(*d*) is always the path of a particle
(*e*) is fixed in space in steady flow

3.170. In two-dimensional flow around a cylinder the streamlines are 2 in. apart at a great distance from the cylinder, where the velocity is 100 ft/sec. At one point near the cylinder the streamlines are 1.5 in. apart. The average velocity there is

(*a*) 75 ft/sec (*b*) 133 ft/sec (*c*) 150 ft/sec (*d*) 200 ft/sec
(*e*) 300 ft/sec

3.171. An oil has a specific gravity of 0.80. Its density in slugs per cubic foot is

(*a*) 0.775 (*b*) 0.80 (*c*) 1.55 (*d*) 1.935 (*e*) 49.92

3.172. The continuity equation

(*a*) requires that Newton's second law of motion be satisfied at every point in the fluid
(*b*) expresses the relation between energy and work
(*c*) states that the velocity at a boundary must be zero relative to the boundary for a real fluid
(*d*) relates the momentum per unit volume for two points on a streamline
(*e*) relates mass rate of flow along a stream tube

3.173. Water has an average velocity of 10 ft/sec through a 24-in. pipe. The discharge through the pipe, in cubic feet per second, is

(*a*) 7.85 (*b*) 31.42 (*c*) 40 (*d*) 125.68 (*e*) none of these answers

3.174. The assumptions about flow required in deriving the equation $gz + v^2/2 + \int dp/\rho =$ constant are that it is

(*a*) steady, frictionless, incompressible, along a streamline
(*b*) uniform, frictionless, along a streamline, ρ a function of p
(*c*) steady, uniform, incompressible, along a streamline
(*d*) steady, frictionless, ρ a function of p, along a streamline
(*e*) none of these answers

3.175. The equation $z + p/\gamma + v^2/2g = C$ has the units of

(*a*) ft-lb/sec (*b*) lb (*c*) ft-lb/slug (*d*) ft-lb/ft^3 (*e*) ft-lb/lb

3.176. The work that a liquid is capable of doing by virtue of its sustained pressure is, in foot-pounds per pound,

(*a*) z (*b*) p (*c*) p/γ (*d*) $v^2/2g$ (*e*) $\sqrt{2gh}$

3.177. The velocity head is

(*a*) $v^2/2g$ (*b*) z (*c*) v (*d*) $\sqrt{2gh}$ (*e*) none of these answers

3.178. The kinetic-energy correction factor

(*a*) applies to the continuity equation
(*b*) has the units of velocity head

(c) is expressed by $\frac{1}{A}\int_A \left(\frac{v}{V}\right) dA$

(d) is expressed by $\frac{1}{A}\int_A \left(\frac{v}{V}\right)^2 dA$

(e) is expressed by $\frac{1}{A}\int_A \left(\frac{v}{V}\right)^3 dA$

3.179. The kinetic-energy correction factor for the velocity distribution given by Fig. 1.1 is

(a) 0 (b) 1 (c) $\frac{4}{3}$ (d) 2 (e) none of these answers

3.180. The equation $\Sigma F_x = \rho Q(V_{x_{out}} - V_{x_{in}})$ requires the following assumptions for its derivation:

1. Velocity constant over the end cross sections
2. Steady flow
3. Uniform flow
4. Compressible fluid
5. Frictionless fluid

(a) 1, 2 (b) 1, 5 (c) 1, 3 (d) 3, 5 (e) 2, 4

3.181. The momentum correction factor is expressed by

(a) $\frac{1}{A}\int_A \left(\frac{v}{V}\right) dA$ (b) $\frac{1}{A}\int_A \left(\frac{v}{V}\right)^2 dA$ (c) $\frac{1}{A}\int_A \left(\frac{v}{V}\right)^3 dA$

(d) $\frac{1}{A}\int_A \left(\frac{v}{V}\right)^4 dA$ (e) none of these answers

3.182. The momentum correction factor for the velocity distribution given by Fig. 1.1 is

(a) 0 (b) 1 (c) $\frac{4}{3}$ (d) 2 (e) none of these answers

3.183. The velocity over one-third of a cross section is zero and is uniform over the remaining two-thirds of the area. The momentum correction factor is

(a) 1 (b) $\frac{4}{3}$ (c) $\frac{3}{2}$ (d) $\frac{9}{4}$ (e) none of these answers

3.184. The magnitude of the resultant force necessary to hold a 6-in.-diameter 90° elbow under no-flow conditions when the pressure is 100 psi is, in pounds,

(a) 5644 (b) 3996 (c) 2822 (d) 0 (e) none of these answers

3.185. A 12-in.-diameter 90° elbow carries water with average velocity of 15 ft/sec and pressure of -5 psi. The force component in the direction of the approach velocity necessary to hold the elbow in place is, in pounds,

(a) -342 (b) 223 (c) 565 (d) 907 (e) none of these answers

3.186. A 3-in.-diameter 180° bend carries a liquid, $\rho = 2.0$, at 20 ft/sec at a pressure of zero gage. The force tending to push the bend off the pipe is, in pounds,

(*a*) 0 (*b*) 39.2 (*c*) 78.5 (*d*) 286.5 (*e*) none of these answers

3.187. The thickness of wall for a large high-pressure pipeline is determined by consideration of

(*a*) axial tensile stresses in the pipe
(*b*) forces exerted by dynamic action at bends
(*c*) forces exerted by static and dynamic action at bends
(*d*) circumferential pipe wall tension
(*e*) temperature stresses

3.188. Select from the following list the correct assumptions for analyzing flow of a jet that is deflected by a fixed or moving vane:

1. The momentum of the jet is unchanged.
2. The absolute speed does not change along the vane.
3. The fluid flows onto the vane without shock.
4. The flow from the nozzle is steady.
5. The cross-sectional area of jet is unchanged.
6. Friction between jet and vane is neglected.
7. The jet leaves without velocity.
8. The velocity is uniform over the cross section of the jet before and after contacting the vane.

(*a*) 1, 3, 4, 6 (*b*) 2, 3, 6, 7 (*c*) 3, 4, 5, 6 (*d*) 3, 4, 6, 8 (*e*) 3, 5, 6, 8

3.189. When a steady jet impinges on a fixed inclined plane surface

(*a*) the momentum in the direction of the approach velocity is unchanged
(*b*) no force is exerted on the jet by the vane
(*c*) the flow is divided into parts directly proportional to the angle of inclination of the surface
(*d*) the speed is reduced for that portion of the jet turned through more than 90° and increased for the other portion
(*e*) the momentum component is unchanged parallel to the surface

3.190. A jet with initial velocity of 100 ft/sec in the $+x$-direction is deflected by a fixed vane with a blade angle of 120°. The velocity components leaving the vane parallel to and normal to the approach velocity are

(*a*) $v_x = -50, v_y = 86.6$ (*b*) $v_x = 100, v_y = 0$
(*c*) $v_x = 50, v_y = 50$ (*d*) $v_x = 50, v_y = 86.6$
(*e*) $v_x = -86.6, v_y = 50$

3.191. An oil jet, sp gr 0.80, discharges 0.50 slug/sec onto a fixed vane that turns the flow through 90°. The speed of the jet is 100 ft/sec as it leaves the vane. The force component on the vane in the direction of the approach velocity is, in pounds,

(*a*) 70.7 (*b*) 50 (*c*) 40 (*d*) 35.35 (*e*) none of these answers

3.192. A water jet having a velocity of 120 ft/sec and cross-sectional area 0.05 ft² flows onto a vane moving 40 ft/sec in the same direction as the jet. The mass having its momentum changed per unit time, in slugs per second, is

(*a*) 4 (*b*) 7.74 (*c*) 11.61 (*d*) 15.48 (*e*) none of these answers

3.193. A jet having a velocity of 100 ft/sec flows onto a vane, angle $\theta = 150°$, having a velocity of 50 ft/sec in the same direction as the jet. The final absolute velocity components parallel and normal to the approach velocity are

(*a*) $v_x = 6.7$, $v_y = 25$ (*b*) $v_x = 24$, $v_y = 43.3$
(*c*) $v_x = -36.6$, $v_y = 50$ (*d*) $v_x = 14.65$, $v_y = 35.35$
(*e*) none of these answers

3.194. A vane moves toward a nozzle 30 ft/sec, and the jet issuing from the nozzle has a velocity of 40 ft/sec. The vane angle is $\theta = 90°$. The absolute velocity components of the jet as it leaves the vane, parallel and normal to the undisturbed jet, are

(*a*) $v_x = 10$, $v_y = 10$ (*b*) $v_x = -30$, $v_y = 10$
(*c*) $v_x = -30$, $v_y = 40$ (*d*) $v_x = -30$, $v_y = 70$
(*e*) none of these answers

3.195. A force of 60 lb is exerted upon a moving blade in the direction of its motion, $u = 55$ ft/sec. The horsepower obtained is

(*a*) 0.1 (*b*) 3 (*c*) 5.5 (*d*) 10 (*e*) none of these answers

3.196. A series of moving vanes, $u = 50$ ft/sec, $\theta = 90°$, intercepts a jet, $Q = 1$ cfs, $\rho = 1.5$ slugs/ft³, $V_0 = 100$ ft/sec. The work done on the vanes, in foot-pounds per second, is

(*a*) 1875 (*b*) 2500 (*c*) 3750 (*d*) 7500 (*e*) none of these answers

3.197. The horsepower available in a water jet of cross-sectional area 0.04 ft² and velocity 80.2 ft/sec is

(*a*) 1.13 (*b*) 36.35 (*c*) 39 (*d*) 72.7 (*e*) none of these answers

3.198. A ship moves through water at 30 ft/sec. The velocity of water in the slipstream behind the boat is 20 ft/sec, and the propeller diameter is 3.0 ft. The theoretical efficiency of the propeller is, in per cent,

(*a*) 0 (*b*) 60 (*c*) 75 (*d*) 86 (*e*) none of these answers

3.199. The thrust on the ship of Prob. 3.198, in pounds, is

(*a*) 1362 (*b*) 4090 (*c*) 5450 (*d*) 8180 (*e*) none of these answers

3.200. A rocket exerts a constant horizontal thrust of 40 lb on a missile for 3 sec. If the missile weighs 8 lb and starts from rest, its speed at the end of the period, neglecting the downward acceleration of gravity and reduction in weight of the rocket, is, in feet per second,

(*a*) 386 (*b*) 483 (*c*) 580 (*d*) 600 (*e*) none of these answers

3.201. What is the reduction in weight of the rocket of Prob. 3.200 if the jet leaves at 6000 ft/sec relative to the rocket?

(*a*) 0.02 lb (*b*) 0.04 lb (*c*) 0.32 lb (*d*) 0.64 lb (*e*) none of these answers

3.202. A glass tube with a 90° bend is open at both ends. It is inserted into a flowing stream of oil, sp gr 0.90, so that one opening is directed upstream and the other is directed upward. Oil inside the tube is 2 in. higher than the surface of flowing oil. The velocity measured by the tube is, in feet per second,

(*a*) 2.95 (*b*) 3.28 (*c*) 3.64 (*d*) 4.64 (*e*) none of these answers

3.203. In Fig. 9.6 the gage difference R' for $v_1 = 5$ ft/sec, $S = 0.08$, $S_0 = 1.2$, is, in feet,

(*a*) 0.39 (*b*) 0.62 (*c*) 0.78 (*d*) 1.17 (*e*) none of these answers

3.204. The theoretical velocity of oil, sp gr 0.75, flowing from an orifice in a reservoir under a head of 9.0 ft is, in feet per second,

(*a*) 18.1 (*b*) 24.06 (*c*) 32.1 (*d*) not determinable from data given (*e*) none of these answers

3.205. In which of the following cases is it possible for flow to occur from low pressure to high pressure?

(*a*) flow through a converging section
(*b*) adiabatic flow in a horizontal pipe
(*c*) flow of a liquid upward in a vertical pipe
(*d*) flow of air downward in a pipe
(*e*) impossible in a constant-cross-section conduit

3.206. The head loss in turbulent flow in a pipe

(*a*) varies directly as the velocity
(*b*) varies inversely as the square of the velocity
(*c*) varies inversely as the square of the diameter
(*d*) depends upon the orientation of the pipe
(*e*) varies approximately as the square of the velocity

3.207. The losses due to a sudden expansion is expressed by

(*a*) $\dfrac{V_1^2 - V_2^2}{2g}$ (*b*) $\dfrac{V_1 - V_2}{2g}$ (*c*) $\dfrac{V_2^2 - V_1^2}{g}$ (*d*) $\dfrac{(V_1 - V_2)^2}{g}$
(*e*) $\dfrac{(V_1 - V_2)^2}{2g}$

3.208. If all losses are neglected, the pressure at the summit of a siphon

(*a*) is a minimum for the siphon
(*b*) depends upon height of summit above upstream reservoir only
(*c*) is independent of the length of the downstream leg
(*d*) is independent of the discharge through the siphon
(*e*) is independent of the liquid density

3.209. The depth conjugate to $y = 1$ ft and $V = 20$ ft/sec is

(*a*) 2.32 ft (*b*) 4.5 ft (*c*) 5.0 ft (*d*) 5.5 ft (*e*) none of these answers

3.210. The depth conjugate to $y = 10$ ft and $V = 20$ ft/sec is

(*a*) 10.72 ft (*b*) 11.5 ft (*c*) 16.5 ft (*d*) 21.5 ft (*e*) none of these answers

3.211. The depth conjugate to $y = 10$ ft and $V = 1$ ft/sec is

(*a*) 0.06 ft (*b*) 1.46 ft (*c*) 5.06 ft (*d*) 10.06 ft (*e*) none of these answers

3.212. The continuity equation in ideal fluid flow

(*a*) states that the net rate of inflow into any small volume must be zero
(*b*) states that the energy is constant along a streamline
(*c*) states that the energy is constant everywhere in the fluid
(*d*) applies to irrotational flow only
(*e*) implies the existence of a velocity potential

4

DIMENSIONAL ANALYSIS AND DYNAMIC SIMILITUDE

Dimensionless parameters have aided materially in our understanding of fluid-flow phenomena. They permit limited experimental results to be applied to cases dealing with different physical dimensions and to fluids with different physical properties. As a means of formally determining dimensionless parameters, the process of dimensional analysis is introduced in this chapter. The concepts of dynamic similitude combined with careful selection and use of dimensionless parameters make possible the generalization of experimental data. In the following chapter, dealinging primarily with viscous effects, one parameter is highly significant, viz., Reynolds number. In Chap. 6, dealing with compressible flow, the Mach number is the most important dimensionless parameter. In Chap. 10, dealing with open channels, the Froude number has the greatest significance.

Many of the dimensionless parameters may be viewed as a ratio of a pair of fluid forces, the relative magnitude indicating the relative importance of one of the forces with respect to the other. For situations with several forces of the same magnitude, such as inertial, viscous, and gravitational forces, special techniques are required. After a discussion of dimensions, dimensional analysis, and dimensionless parameters, dynamic similitude and model studies are presented.

4.1. Dimensional Homogeneity and Dimensionless Ratios. The solving of practical design problems in fluid mechanics usually requires both theoretical developments and experimental results. By means of a grouping of significant quantities into dimensionless parameters it is possible to reduce the number of variables appearing and to make this compact result (equations or data plots) applicable to all similar situations,

If one were to write the equation of motion $\Sigma \mathbf{F} = m\mathbf{a}$ for a fluid particle. including all types of force terms that could act, such as gravity, pressure, viscous, elastic, and surface-tension forces, an equation of the sum of these forces equated to $m\mathbf{a}$, the inertial force, would result. As with all physical equations, each term must have the same dimensions, in this case, force. The division of each term of the equation by any one of the terms would make the equation dimensionless. For example, dividing through by the inertial force term would yield a sum of dimensionless parameters equated to unity. The relative size of any one parameter, compared with unity, would indicate its importance. If one were to divide the force equation through by a different term, say the viscous-force term, then another set of dimensionless parameters would result. Without experience in the flow case it is difficult to determine which parameters will be most useful.

The writing of such a force equation for a complex situation may not be feasible, and another process, *dimensional analysis*, is then used if one knows the pertinent quantities that enter into the problem.

In a given situation several of the forces may be of little significance, leaving perhaps two or three forces of the same order of magnitude. With three forces of the same order of magnitude, two dimensionless parameters are obtained; one set of experimental data on a geometrically similar model provides the relationships between parameters holding for all other similar flow cases.

4.2. Dimensions and Units. The dimensions of mechanics are force, mass, length, and time, related to Newton's second law of motion,

$$\mathbf{F} = cM\mathbf{a} \tag{4.2.1}$$

Force and mass units are discussed in Sec. 1.2. For all physical systems, it would probably be necessary to introduce two more dimensions, one dealing with electromagnetics and the other with thermal effects. For the compressible work in this text, it is unnecessary to include a thermal unit, as the equations of state link pressure, density, and temperature.

Newton's second law of motion in dimensional form is

$$F = MLT^{-2} \tag{4.2.2}$$

which shows that only three of the dimensions are independent. F is the force dimension, M the mass dimension, L the length dimension, and T the time dimension. One common system employed in dimensional analysis is the M, L, T-system. Table 4.1 is a listing of some of the quantities used in fluid flow, together with their symbols and dimensions.

4.3. The Π-Theorem. The Buckingham[1] Π-theorem proves that in a physical problem including n quantities in which there are m dimensions, the quantities may be arranged into $n - m$ independent dimensionless

[1] E. Buckingham, Model Experiments and the Form of Empirical Equations, *Trans. ASME*, vol. 37, pp. 263–296, 1915.

TABLE 4.1. DIMENSIONS OF PHYSICAL QUANTITIES USED IN FLUID MECHANICS

Quantity	Symbol	Dimensions (M,L,T)
Length	l	L
Time	t	T
Mass	M	M
Force	F	MLT^{-2}
Velocity	V	LT^{-1}
Acceleration	a	LT^{-2}
Area	A	L^2
Discharge	Q	L^3T^{-1}
Pressure	Δp	$ML^{-1}T^{-2}$
Gravity	g	LT^{-2}
Density	ρ	ML^{-3}
Specific weight	γ	$ML^{-2}T^{-2}$
Dynamic viscosity	μ	$ML^{-1}T^{-1}$
Kinematic viscosity	ν	L^2T^{-1}
Surface tension	σ	MT^{-2}
Bulk modulus of elasticity	K	$ML^{-1}T^{-2}$

parameters. Let A_1, A_2, A_3, . . . , A_n be the quantities involved, such as pressure, viscosity, velocity, etc. All the quantities are known to be essential to the solution, and hence some functional relation must exist.

$$F(A_1,A_2,A_3, \ldots ,A_n) = 0 \tag{4.3.1}$$

If Π_1, Π_2, etc., represent dimensionless groupings of the quantities A_1, A_2, A_3, etc., then with m dimensions involved, an equation of the form

$$f(\Pi_1,\Pi_2,\Pi_3, \ldots ,\Pi_{n-m}) = 0 \tag{4.3.2}$$

exists.

Proof of the Π-theorem may be found in Buckingham's paper. The method of determining the Π-parameters is to select m of the A-quantities, with different dimensions, that contain among them the m dimensions, and to use them as repeating variables together with one of the other A-quantities for each Π. For example, let A_1, A_2, A_3 contain M, L, and T, not necessarily in each one, but collectively. Then the first Π-parameter is made up as

$$\Pi_1 = A_1^{x_1}A_2^{y_1}A_3^{z_1}A_4 \tag{4.3.3}$$

the second one as

$$\Pi_2 = A_1^{x_2}A_2^{y_2}A_3^{z_2}A_5$$

and so on, until

$$\Pi_{n-m} = A_1^{x_{n-m}}A_2^{y_{n-m}}A_3^{z_{n-m}}A_n$$

In these equations the exponents are to be determined so that each Π is dimensionless. The dimensions of the A-quantities are substituted and the exponents M, L, and T are set equal to zero respectively. These produce three equations in three unknowns for each Π-parameter, so that the x, y, z exponents can be determined, and hence the Π-parameter.

If only two dimensions are involved, then two of the A-quantities are selected as repeating variables, and two equations in the two unknown exponents are obtained for each Π term.

In many cases the grouping of A-terms is such that the dimensionless arrangement is evident by inspection. The simplest case is that, when two quantities have the same dimensions, e.g., length, then the ratio of these two terms is the Π-parameter.

The procedure is best illustrated by several examples.

Example 4.1: The discharge through a horizontal capillary tube is thought to depend upon the pressure drop per unit length, the diameter, and the viscosity. Find the form of the equation.

The quantities are listed with their dimensions:

Quantity	Symbol	Dimensions
Discharge	Q	L^3T^{-1}
Pressure drop/length	$\Delta p/l$	$ML^{-2}T^{-2}$
Diameter	D	L
Viscosity	μ	$ML^{-1}T^{-1}$

Then

$$F\left(Q, \frac{\Delta p}{l}, D, \mu\right) = 0$$

Three dimensions are used, and with four quantities there will be one Π-parameter:

$$\Pi = Q^{x_1}\left(\frac{\Delta p}{l}\right)^{y_1} D^{z_1}\mu$$

By substituting in the dimensions,

$$\Pi = (L^3T^{-1})^{x_1}(ML^{-2}T^{-2})^{y_1}L^{z_1}ML^{-1}T^{-1} = M^0L^0T^0$$

The exponents of each dimension must be the same on both sides of the equation. With L first,

$$3x_1 - 2y_1 + z_1 - 1 = 0$$

and similarly for M and T

$$y_1 + 1 = 0$$
$$-x_1 - 2y_1 - 1 = 0$$

from which $x_1 = 1$, $y_1 = -1$, $z_1 = -4$, and

$$\Pi = \frac{Q\mu}{D^4\,\Delta p/l}$$

After solving for Q

$$Q = C \frac{\Delta p}{l} \frac{D^4}{\mu}$$

from which dimensional analysis yields no information about the numerical value of the dimensionless constant C. Experiment (or analysis) shows that it is $\pi/128$ [Eq. (5.2.6)].

When dimensional analysis is used, the variables in a problem must be known. In the last example if kinematic viscosity had been used in place of dynamic viscosity an incorrect formula would have resulted.

Example 4.2: A V-notch weir is a vertical plate with a notch of angle ϕ cut into the top of it and placed across an open channel. The liquid in the channel is backed up and forced to flow through the notch. The discharge Q is some function of the elevation H of upstream liquid surface above the bottom of the notch. In addition the discharge depends upon gravity and upon the velocity of approach V_0 to the weir. Determine the form of discharge equation.

A functional relationship

$$F(Q,H,g,V_0,\phi) = 0$$

is to be grouped into dimensionless parameters. ϕ is dimensionless, hence it is one Π-parameter. Only two dimensions are used, L and T. If g and H are the repeating variables

$$\Pi_1 = H^{x_1}g^{y_1}Q = L^{x_1}(LT^{-2})^{y_1}L^3T^{-1}$$
$$\Pi_2 = H^{x_2}g^{y_2}V_0 = L^{x_2}(LT^{-2})^{y_2}LT^{-1}$$

Then

$$x_1 + y_1 + 3 = 0 \qquad x_2 + y_2 + 1 = 0$$
$$-2y_1 - 1 = 0 \qquad -2y_2 - 1 = 0$$

and $x_1 = -\frac{5}{2}$, $y_1 = -\frac{1}{2}$, $x_2 = -\frac{1}{2}$, $y_2 = -\frac{1}{2}$

$$\Pi_1 = \frac{Q}{\sqrt{g}\,H^{\frac{5}{2}}} \qquad \Pi_2 = \frac{V_0}{\sqrt{gH}} \qquad \Pi_3 = \phi$$

or

$$f\left(\frac{Q}{\sqrt{g}\,H^{\frac{5}{2}}}, \frac{V_0}{\sqrt{gH}}, \phi\right) = 0$$

This may be written

$$\frac{Q}{\sqrt{g}\,H^{\frac{5}{2}}} = f_1\left(\frac{V_0}{\sqrt{gH}}, \phi\right)$$

in which both f, f_1 are unknown functions. After solving for Q

$$Q = \sqrt{g}\,H^{\frac{5}{2}} f_1\left(\frac{V_0}{\sqrt{gH}}, \phi\right)$$

Either experiment or analysis is required to yield additional information as to the function f_1.

If H and V_0 were selected as repeating variables in place of g and h,

$$\Pi_1 = H^{x_1}V_0^{y_1}Q = L^{x_1}(LT^{-1})^{y_1}L^3T^{-1}$$
$$\Pi_2 = H^{x_2}V_0^{y_2}g = L^{x_2}(LT^{-1})^{y_2}LT^{-2}$$

and

$$x_1 + y_1 + 3 = 0 \qquad x_2 + y_2 + 1 = 0$$
$$-y_1 - 1 = 0 \qquad -y_2 - 2 = 0$$

from which $x_1 = -2$, $y_1 = -1$, $x_2 = 1$, $y_2 = -2$; hence

$$\Pi_1 = \frac{Q}{H^2V_0} \qquad \Pi_2 = \frac{gH}{V_0^2} \qquad \Pi_3 = \phi$$

or

$$f\left(\frac{Q}{H^2V_0}, \frac{gH}{V_0^2}, \phi\right) = 0$$

Since any of the Π-parameters may be inverted or raised to any power without affecting their dimensionless status,

$$Q = V_0H^2f_2\left(\frac{V_0}{\sqrt{gH}}, \phi\right)$$

The unknown function f_2 has the same parameters as f_1, but it could not be the same function. The last form is not very useful, in general, because frequently V_0 may be neglected with V-notch weirs. This shows that a term of minor importance should not be selected as a repeating variable.

Another method of determining alternate sets of Π-parameters would be the arbitrary recombination of the first set. If four independent Π-parameters are known Π_1, Π_2, Π_3, Π_4, the term

$$\Pi_a = \Pi_1^{a_1}\Pi_2^{a_2}\Pi_3^{a_3}\Pi_4^{a_4}$$

with the exponents chosen at will would yield a new parameter. Then Π_a, Π_2, Π_3, Π_4 would constitute a new set. This procedure may be continued to find all possible sets.

Example 4.3: The losses per unit length of pipe $\Delta h/l$ in turbulent flow through a smooth pipe depend upon velocity V, diameter D, gravity g, dynamic viscosity μ, and density ρ. With dimensional analysis, determine the general form of the equation

$$F\left(\frac{\Delta h}{l}, V,D,\rho,\mu,g\right) = 0$$

Clearly, $\Delta h/l$ is a Π-parameter. If V, D, and ρ are repeating variables,

$$\Pi_1 = V^{x_1}D^{y_1}\rho^{z_1}\mu = (LT^{-1})^{x_1}L^{y_1}(ML^{-3})^{z_1}ML^{-1}T^{-1}$$
$$x_1 + y_1 - 3z_1 - 1 = 0$$
$$-x_1 \qquad -1 = 0$$
$$z_1 + 1 = 0$$
$$x_1 = -1 \qquad y_1 = -1 \qquad z_1 = -1$$

$$\Pi_2 = V^{x_2}D^{y_2}\rho^{z_2}g = (LT^{-1})^{x_2}L^{y_2}(ML^{-3})^{z_2}LT^{-2}$$

$$x_2 + y_2 - 3z_2 + 1 = 0$$
$$-x_2 \qquad -2 = 0$$
$$z_2 \qquad = 0$$
$$x_2 = -2 \qquad y_2 = 1 \qquad z_2 = 0$$
$$\Pi_1 = \frac{\mu}{VD\rho} \qquad \Pi_2 = \frac{gD}{V^2} \qquad \Pi_3 = \frac{\Delta h}{l}$$

or

$$f\left(\frac{VD\rho}{\mu}, \frac{V^2}{gD}, \frac{\Delta h}{l}\right) = 0$$

since the Π-quantities may be inverted if desired. The first parameter, $VD\rho/\mu$ is *Reynolds number*, one of the most important of the dimensionless parameters in fluid mechanics. The size of Reynolds number determines the nature of the flow. It is discussed in Sec. 5.3. After solving for $\Delta h/l$

$$\frac{\Delta h}{l} = f_1\left(\mathbf{R}, \frac{V^2}{gD}\right)$$

The usual formula employed is

$$\frac{\Delta h}{l} = f(\mathbf{R})\frac{1}{D}\frac{V^2}{2g}$$

Example 4.4: A fluid-flow situation depends upon the velocity V, the density ρ, several linear dimensions l, l_1, l_2, pressure drop Δp, gravity g, viscosity μ, surface tension σ, and bulk modulus of elasticity K. Apply dimensional analysis to these variables to find a set of Π-parameters.

$$F(V,\rho,l,l_1,l_2,\Delta p,g,\mu,\sigma,K) = 0$$

As three dimensions are involved, three repeating variables are selected. For complex situations, V, ρ, and l are generally helpful. There are seven Π-parameters:

$$\Pi_1 = V^{x_1}\rho^{y_1}l^{z_1}\Delta p \qquad \Pi_2 = V^{x_2}\rho^{y_2}l^{z_2}g \qquad \Pi_3 = V^{x_3}\rho^{y_3}l^{z_3}\mu$$
$$\Pi_4 = V^{x_4}\rho^{y_4}l^{z_4}\sigma \qquad \Pi_5 = V^{x_5}\rho^{y_5}l^{z_5}K \qquad \Pi_6 = l/l_1$$
$$\Pi_7 = l/l_2$$

By expanding the Π-quantities into dimensions,

$$\Pi_1 = (LT^{-1})^{x_1}(ML^{-3})^{y_1}L^{z_1}ML^{-1}T^{-2}$$
$$x_1 - 3y_1 + z_1 - 1 = 0$$
$$-x_1 \qquad -2 = 0$$
$$y_1 \qquad + 1 = 0$$
$$x_1 = -2 \qquad y_1 = -1 \qquad z_1 = 0$$
$$\Pi_2 = (LT^{-1})^{x_2}(ML^{-3})^{y_2}L^{z_2}LT^{-2}$$
$$x_2 - 3y_2 + z_2 + 1 = 0$$
$$-x_2 \qquad -2 = 0$$
$$y_2 \qquad = 0$$
$$x_2 = -2 \qquad y_2 = 0 \qquad z_2 = 1$$

$$\Pi_3 = (LT^{-1})^{x_3}(ML^{-3})^{y_3}L^{z_3}ML^{-1}T^{-1}$$

$$\begin{aligned} x_3 - 3y_3 + z_3 - 1 &= 0 \\ -x_3 \qquad\qquad -1 &= 0 \\ y_3 \qquad +1 &= 0 \end{aligned}$$

$$x_3 = -1 \qquad y_3 = -1 \qquad z_3 = -1$$

$$\Pi_4 = (LT^{-1})^{x_4}(ML^{-3})^{y_4}L^{z_4}MT^{-2}$$

$$\begin{aligned} x_4 - 3y_4 + z_4 \qquad &= 0 \\ -x_4 \qquad\qquad -2 &= 0 \\ y_4 \qquad +1 &= 0 \end{aligned}$$

$$x_4 = -2 \qquad y_4 = -1 \qquad z_4 = -1$$

$$\Pi_5 = (LT^{-1})^{x_5}(ML^{-3})^{y_5}L^{z_5}ML^{-1}T^{-2}$$

$$\begin{aligned} x_5 - 3y_5 + z_5 - 1 &= 0 \\ -x_5 \qquad\qquad -2 &= 0 \\ y_5 \qquad +1 &= 0 \end{aligned}$$

$$x_5 = -2 \qquad y_5 = -1 \qquad z_5 = 0$$

Hence

$$\Pi_1 = \frac{\Delta p}{\rho V^2} \qquad \Pi_2 = \frac{gl}{V^2} \qquad \Pi_3 = \frac{\mu}{Vl\rho} \qquad \Pi_4 = \frac{\sigma}{V^2\rho l}$$

$$\Pi_5 = \frac{K}{\rho V^2} \qquad \Pi_6 = \frac{l}{l_1} \qquad \Pi_7 = \frac{l}{l_2}$$

and

$$f\left(\frac{\Delta p}{\rho V^2}, \frac{gl}{V^2}, \frac{\mu}{Vl\rho}, \frac{\sigma}{V^2\rho l}, \frac{K}{\rho V^2}, \frac{l}{l_1}, \frac{l}{l_2}\right) = 0$$

It is convenient to invert some of the parameters and to take the square root of Π_5,

$$f_1\left(\frac{\Delta p}{\rho V^2}, \frac{V^2}{gl}, \frac{Vl\rho}{\mu}, \frac{V^2 l\rho}{\sigma}, \frac{V}{\sqrt{K/\rho}}, \frac{l}{l_1}, \frac{l}{l_2}\right) = 0$$

The first parameter, usually written $\Delta p/(\rho V^2/2)$, is the *pressure coefficient;* the second parameter is the *Froude* number $\mathbf{F}$; the third is *Reynolds* number $\mathbf{R}$; the fourth is the *Weber* number $\mathbf{W}$, and the fifth the *Mach* number $\mathbf{M}$. Hence

$$f_1\left(\frac{\Delta p}{\rho V^2}, \mathbf{F},\mathbf{R},\mathbf{W},\mathbf{M}, \frac{l}{l_1}, \frac{l}{l_2}\right) = 0$$

After solving for pressure drop

$$\Delta p = \rho V^2 f_2\left(\mathbf{F},\mathbf{R},\mathbf{W},\mathbf{M}, \frac{l}{l_1}, \frac{l}{l_2}\right)$$

in which f_1, f_2 must be determined from analysis or experiment. By selecting other repeating variables, a different set of Π-parameters could be obtained.

Example 4.5: The thrust due to any one of a family of geometrically similar airplane propellers is to be determined experimentally from a wind-tunnel test on a model. By means of dimensional analysis find suitable parameters for plotting test results.

The thrust F_T depends upon speed of rotation ω, speed of advance V_0, diameter D, air viscosity μ, density ρ, and speed of sound c. The function

$$f(F_T, V_0, D, \omega, \mu, \rho, c) = 0$$

is to be arranged into four dimensionless parameters, since there are seven quantities and three dimensions. Starting first, by selecting ρ, ω, and D as repeating variables,

$$\begin{aligned}
\Pi_1 &= \rho^{x_1}\omega^{y_1}D^{z_1}F_T = (ML^{-3})^{x_1}(T^{-1})^{y_1}L^{z_1}MLT^{-2} \\
\Pi_2 &= \rho^{x_2}\omega^{y_2}D^{z_2}V_0 = (ML^{-3})^{x_2}(T^{-1})^{y_2}L^{z_2}LT^{-1} \\
\Pi_3 &= \rho^{x_3}\omega^{y_3}D^{z_3}\mu = (ML^{-3})^{x_3}(T^{-1})^{y_3}L^{z_3}ML^{-1}T^{-1} \\
\Pi_4 &= \rho^{x_4}\omega^{y_4}D^{z_4}c = (ML^{-3})^{x_4}(T^{-1})^{y_4}L^{z_4}LT^{-1}
\end{aligned}$$

By writing the simultaneous equations in x_1, y_1, z_1, etc., as before and solving them,

$$\Pi_1 = \frac{F_T}{\rho\omega^2D^4} \qquad \Pi_2 = \frac{V_0}{\omega D} \qquad \Pi_3 = \frac{\mu}{\rho\omega D^2} \qquad \Pi_4 = \frac{c}{\omega D}$$

Solving for the thrust parameter

$$\frac{F_T}{\rho\omega^2D^4} = f_1\left(\frac{V_0}{\omega D}, \frac{\rho\omega D^2}{\mu}, \frac{c}{\omega D}\right)$$

Since the parameters may be recombined to obtain other forms, the second term is replaced by the product of the first and second terms, $VD\rho/\mu$ and the third term is replaced by the first term divided by the third term, V_0/c; thus

$$\frac{F_T}{\rho\omega^2D^4} = f_2\left(\frac{V_0}{\omega D}, \frac{V_0D\rho}{\mu}, \frac{V_0}{c}\right)$$

Of the dimensionless parameters, the first is probably of the most importance, since it relates speed of advance to speed of rotation. The second parameter is a Reynolds number and accounts for viscous effects. The last parameter, speed of advance divided by speed of sound, is a Mach number, which would be important for speeds near or higher than the speed of sound. Reynolds effects are usually small, so a plot of $F_r/\rho\omega^2D^4$ against $V_0/\omega D$ should be most informative.

The steps in a dimensional analysis may be summarized as follows:

1. Select the pertinent variables. This requires some knowledge of the process.
2. Write the functional relationships, e.g.,

$$f(V,D,\rho,\mu,c,H) = 0$$

3. Select the repeating variables. (Do not make the dependent quantity a repeating variable.)
4. Write the Π-parameters in terms of unknown exponents, e.g.,

$$\Pi_1 = V^{x_1}D^{y_1}\rho^{z_1}\mu = (LT^{-1})^{x_1}L^{y_1}(ML^{-3})^{z_1}ML^{-1}T^{-1}$$

5. For each of the Π-expressions write the equations of the exponents, so that the sum of the exponents of each dimension will be zero.
6. Solve the equations simultaneously.
7. Substitute back into the Π-expressions of step 4 the exponents to obtain the dimensionless Π-parameters.

8. Establish the functional relation

$$f_1(\Pi_1,\Pi_2,\Pi_3, \ldots ,\Pi_{n-m}) = 0$$

or solve for one of the Π's explicitly:

$$\Pi_2 = f(\Pi_1,\Pi_3, \ldots ,\Pi_{n-m})$$

9. Recombine, if desired, to alter the forms of the Π-parameters, keeping the same number of independent parameters.

4.4. Discussion of Dimensionless Parameters. The five dimensionless parameters, pressure coefficient, Reynolds number, Froude number, Weber number, and Mach number, are of importance in correlating experimental data. They are discussed in this section, with particular emphasis placed on the relation of pressure coefficient to the other parameters.

Pressure Coefficient. The pressure coefficient $\Delta p/(\rho V^2/2)$ is the ratio of pressure to dynamic pressure. When multiplied by area it is the ratio of pressure force to inertial force, as $(\rho V^2/2)A$ would be the force needed to reduce the velocity to zero. It may also be written as $\Delta h/(V^2/2g)$ by division by γ. For pipe flow the Darcy-Weisbach equation relates losses h_l to length of pipe L, diameter D, and velocity V by a dimensionless friction factor[1] f

$$h_l = f\frac{L}{D}\frac{V^2}{2g}$$

or

$$\frac{fL}{D} = \frac{h_l}{V^2/2g} = f_2\left(\mathbf{R},\mathbf{F},\mathbf{W},\mathbf{M}, \frac{l}{l_1}, \frac{l}{l_2}\right)$$

as fL/D is shown to be equal to the pressure coefficient (see Example 4.4). In pipe flow, gravity has no influence on losses; therefore **F** may be dropped out. Similarly surface tension has no effect and **W** drops out. For steady liquid flow compressibility is not important and **M** is dropped. l may refer to D, l_1 to roughness height projection ϵ in the pipe wall, and l_2 to their spacing ϵ'; hence

$$\frac{FL}{D} = f_2\left(\mathbf{R}, \frac{\epsilon}{D}, \frac{\epsilon'}{D}\right) \tag{4.4.1}$$

Pipe-flow problems are discussed in Chaps. 5, 6, and 10. If compressibility is important,

$$\frac{FL}{D} = f_2\left(\mathbf{R},\mathbf{M}, \frac{\epsilon}{D}, \frac{\epsilon'}{D}\right) \tag{4.4.2}$$

Compressible-flow problems are studied in Chap. 6.

[1] There are several friction factors in general use. This is the Darcy-Weisbach friction factor, which is four times the size of the *Fanning* friction factor, also called f.

With orifice flow, studied in Chap. 9, $V = C_v \sqrt{2gH}$,

$$\frac{H}{V^2/2g} = \frac{1}{C_v^2} = f_2\left(\mathbf{R},\mathbf{W},\mathbf{M}, \frac{l}{l_1}, \frac{l}{l_2}\right) \tag{4.4.3}$$

in which l may refer to orifice diameter and l_1 and l_2 to upstream dimensions. Viscosity and surface tension are unimportant for large orifices and low-viscosity fluids. Mach number effects may be very important for gas flow with large pressure drops, i.e., Mach numbers approaching unity.

In steady, uniform open-channel flow, discussed in Chap. 11, the Chézy formula relates average velocity V, slope of channel S, and hydraulic radius of cross section R (area of section divided by wetted perimeter) by

$$V = C\sqrt{RS} = C\sqrt{R\frac{\Delta h}{L}} \tag{4.4.4}$$

C is a coefficient depending upon size, shape, and roughness of channel. Then

$$\frac{\Delta h}{V^2/2g} = \frac{2gL}{R}\frac{1}{C^2} = f_2\left(\mathbf{F},\mathbf{R}, \frac{l}{l_1}, \frac{l}{l_2}\right) \tag{4.4.5}$$

since surface tension and compressible effects are usually unimportant.

The drag F on a body is expressed by $F = C_D A \rho V^2/2$, in which A is a typical area of the body, usually the projection of the body onto a plane normal to the flow. Then F/A is equivalent to Δp, and

$$\frac{F}{A\rho V^2/2} = C_D = f_2\left(\mathbf{R},\mathbf{F},\mathbf{M}, \frac{l}{l_1}, \frac{l}{l_2}\right) \tag{4.4.6}$$

The term $\mathbf{R}$ is related to *skin friction* drag due to viscous shear as well as to *form*, or *profile*, drag resulting from *separation* of the flow streamlines from the body; $\mathbf{F}$ is related to wave drag if there is a free surface; for large Mach numbers C_D may vary more markedly with $\mathbf{M}$ than with the other parameters; the length ratios may refer to shape or roughness of the surface.

Reynolds Number. Reynolds number $VD\rho/\mu$ is the ratio of inertial forces to viscous forces. It may also be viewed as a ratio of turbulent shear forces to viscous shear forces (Sec. 5.3). A "critical" Reynolds number distinguishes among flow regimes, such as laminar or turbulent flow in pipes, in the boundary layer, or around immersed objects. The particular value depends upon the situation. In compressible flow, the Mach number is generally more significant than the Reynolds number.

Froude Number. The Froude number V^2/gl, when multiplied and divided by ρA, is a ratio of dynamic (or inertial force) to weight. With free liquid surface flow the nature of the flow (rapid[1] or tranquil) depends upon whether the Froude number is greater or less than unity. It is useful in calculations of hydraulic jump, in design of hydraulic structures, and in ship design.

Weber Number. The Weber number $V^2 l\rho/\sigma$ is the ratio of inertial forces to surface-tension forces (evident when numerator and denominator are multiplied by l). It is important at gas-liquid or liquid-liquid interfaces and also where these interfaces are in contact with a boundary. Surface tension causes small (capillary) waves and droplet formation and has an effect on discharge of orifices and weirs at very small heads.

Mach Number. The speed of sound in a liquid is written $\sqrt{K/\rho}$, if K is the bulk modulus of elasticity (Secs. 1.7 and 6.2) or $c = \sqrt{kRT}$ (k is the specific heat ratio and T the absolute temperature, for a perfect gas). V/c or $V/\sqrt{K/\rho}$ is the Mach number. It is a measure of the ratio of inertial forces to elastic forces. By squaring V/c and multiplying by $\rho A/2$ in numerator and denominator, the numerator is the dynamic force and the denominator is the dynamic force at sonic flow. It may also be shown to be a measure of the ratio of kinetic energy of the flow to internal energy of the fluid. It is the most important correlating parameter when velocities are near or above local sonic velocities.

4.5. Similitude—Model Studies. Model studies of proposed hydraulic structures and machines are frequently undertaken as an aid to the designer. They permit visual observation of the flow and make possible the obtaining of certain numerical data, e.g., calibrations of weirs and gates, depths of flow, velocity distributions, forces on gates, efficiencies and capacities of pumps and turbines, pressure distributions, and losses.

If accurate quantitative data are to be obtained from a model study there must be dynamic similitude between model and prototype. This similitude requires (*a*) that there be exact geometric similitude, and (*b*) that the ratio of dynamic pressures at corresponding points be a constant. Part *b* may also be expressed as a kinematic similitude; i.e., the streamlines must be geometrically similar.

Geometric similitude extends to the actual surface roughness of model and prototype. If the model is one-tenth the size of the prototype in every linear dimension, then the height of roughness projections must be in the same ratio. For dynamic pressures to be in the same ratio at corresponding points in model and prototype, the ratios of the various types of forces must be the same at corresponding points. Hence, for strict dynamic similitude, the Mach, Reynolds, Froude, and Weber numbers must be the same in both model and prototype.

[1] Open-channel flow at depth y is *rapid* when the flow velocity is greater than the speed $\sqrt{gy}$ of an elementary wave in quiet liquid. *Tranquil* flow occurs when the flow velocity is less than $\sqrt{gy}$.

Strict fulfillment of these requirements is generally impossible of achievement, except with a 1:1 scale ratio. Fortunately, in many situations only two of the forces are of the same magnitude. Discussion of a few cases will make this clear.

Pipe Flow. In steady flow in a pipe viscous and inertial forces are the only ones of consequence; hence, when geometric similitude is observed, the same Reynolds number in model and prototype provides dynamic similitude. The various corresponding pressure coefficients are the same. For testing with fluids having the same kinematic viscosity in model and prototype, the product, VD, must be the same. Frequently this requires very high velocities in small models.

Open Hydraulic Structures. Structures such as spillways, stilling pools, channel transitions, and weirs generally have forces due to gravity (from changes in elevation of liquid surfaces) and inertial forces that are greater than viscous and turbulent shear forces. In these cases geometric similitude and the same value of Froude's number in model and prototype produce a good approximation to dynamic similitude; thus

$$\frac{V_m^2}{g_m l_m} = \frac{V_p^2}{g_p l_p}$$

Since gravity is the same the velocity ratio varies as the square root of the scale ratio $\lambda = l_p/l_m$,

$$V_p = V_m\sqrt{\lambda}$$

The corresponding times for events to take place (as time for passage of a particle through a transition) are related, thus

$$t_m = \frac{l_m}{V_m} \qquad t_p = \frac{l_p}{V_p}$$

and

$$t_p = t_m \frac{l_p}{l_m}\frac{V_m}{V_p} = t_m\sqrt{\lambda}$$

The discharge ratio Q_p/Q_m is

$$\frac{Q_p}{Q_m} = \frac{l_p^3/t_p}{l_m^3/t_m} = \lambda^{\frac{5}{2}}$$

Force ratios, e.g., on gates, F_p/F_m, are

$$\frac{F_p}{F_m} = \frac{\gamma h_p l_p^2}{\gamma h_m l_m^2} = \lambda^3$$

In a similar fashion other pertinent ratios may be derived so that model results can be interpreted as prototype performance.

Ship's Resistance. The resistance to motion of a ship through water is composed of pressure drag, skin friction, and wave resistance. Model studies are complicated by the three types of forces that are important, inertia, viscosity, and gravity. Skin-friction studies should be based on equal Reynolds numbers in model and prototype, but wave resistance depends upon the Froude number. To satisfy both requirements, model and prototype have to be the same size.

The difficulty is surmounted by using a small model and measuring the total drag on it when towed. The skin friction is then computed for the model and subtracted from the total drag. The remainder is stepped up to prototype size by Froude's law, and the prototype skin friction is computed and added to yield total resistance due to the water.

Hydraulic Machinery. Due to the moving parts in a hydraulic machine, an extra parameter is required to ensure that the streamline patterns are similar in model and prototype. This parameter must relate the throughflow (discharge) to the speed of moving parts. For geometrically similar machines if the vector diagrams of velocity entering or leaving the moving parts are similar, the units are *homologous;* i.e., for practical purposes dynamic similitude exists. The Froude number is unimportant, but the Reynolds number effects (called *scale effects* because it is impossible to maintain the same Reynolds number in homologous units) may cause a discrepancy of 2 or 3 per cent in efficiency between model and prototype. The Mach number is also of importance in axial-flow compressors and gas turbines.

PROBLEMS

4.1. Show that Eqs. (3.6.4), (3.7.5), and (3.9.15) are dimensionally homogeneous.

4.2. Arrange the following groups into dimensionless parameters:

(*a*) $\Delta p, \rho, V$ (*b*) ρ, g, V, F (*c*) $\mu, F, \Delta p, t$

4.3. By inspection, arrange the following groups into dimensionless parameters:

(*a*) a, l, t (*b*) ν, l, t (*c*) A, Q, ω (*d*) K, σ, A

4.4. Derive the unit of mass consistent with the units inches, minutes, tons.

4.5. In terms of M, L, T, determine the dimensions of radians, angular velocity, power, work, torque, and moment of momentum.

4.6. Find the dimensions of the quantities in Prob. 4.5 in the F, L, T-system.

4.7. Work Example 4.2 using Q and H as repeating variables.

4.8. Using the variables Q, D, $\Delta h/l$, ρ, μ, g as pertinent to smooth pipe flow, arrange them into dimensionless parameters with Q, ρ, μ as repeating variables.

4.9. If the shear stress τ is known to depend upon viscosity and rate of angular deformation du/dy in one-dimensional laminar flow, determine the form of Newton's law of viscosity by dimensional reasoning.

4.10. The variation of pressure Δp in static liquids is known to depend upon specific weight γ and elevation difference Δz. By dimensional reasoning determine the form of the hydrostatic law of variation of pressure.

4.11. Neglecting viscous and surface-tension effects, the velocity V of efflux of liquid from a reservoir is thought to depend upon the pressure drop Δp of the liquid and its density ρ. Determine the form of expression for V.

4.12. The buoyant force F_B on a body is thought to depend upon its volume submerged $V\!\!\!\!-$ and upon gravity g and fluid density ρ. Determine the form of the buoyant-force equation.

4.13. In a fluid rotated as a solid about a vertical axis with angular velocity ω, the pressure rise p in a radial direction depends upon speed ω, radius r, and fluid density ρ. Obtain the form of equation for p.

4.14. In Example 4.3, work out two other sets of dimensionless parameters by recombination of the dimensionless parameters given.

4.15. Find the dimensionless parameters of Example 4.4 using Δp, ρ, and l as repeating variables.

4.16. The Mach number **M** for flow of a perfect gas in a pipe depends upon the specific heat ratio k (dimensionless), the pressure p, the density ρ, and the velocity V. Obtain by dimensional reasoning the form of the Mach number expression.

4.17. Work out the scaling ratio for torque T on a disk of radius r that rotates in fluid of viscosity μ with angular velocity ω and clearance y between disk and fixed plate.

4.18. The velocity at a point in a model of a spillway for a dam is 4.3 ft/sec. For a ratio of prototype to model of 10:1 what is the velocity at the corresponding point in the prototype under similar conditions?

4.19. The power input to a pump depends upon discharge Q, head H, specific weight γ, and efficiency e. Find the expression for power by use of dimensional reasoning.

4.20. The torque delivered by a water turbine depends upon discharge Q, head H, specific weight γ, angular velocity ω, and efficiency e. Determine the form of equation for torque.

4.21. A model of a venturi meter has linear dimensions one-fourth those of the prototype. The prototype operates on water at 68°F, and the model on water at 200°F. For a throat diameter of 24 in. and a velocity at the throat of 20 ft/sec in the prototype, what discharge is needed through the model for similitude?

4.22. The drag F on a high-velocity projectile depends upon speed V of projectile, density of fluid ρ, acoustic velocity c, diameter of projectile D, and viscosity μ. Develop an expression for the drag.

4.23. The wave drag on a model of a ship is 2.35 lb at a speed of 8 ft/sec. For a prototype fifteen times as long what would be the corresponding speed and wave drag if the liquid is the same in each case?

4.24. A small spherical droplet of radius r_0 and density ρ_0 settles at velocity U in another liquid of density ρ and viscosity μ. Determine an expression for drag F on the droplet and for its terminal velocity U. (NOTE: Drag on an object at small Reynolds number is independent of density of fluid.)

4.25. The losses in a Y in a 48-in.-diameter pipe system carrying gas (ρ = 0.08 slug/ft^3, μ = 0.002 poise, V = 75 ft/sec) are to be determined by testing a model with water at 70°F. The laboratory has a water capacity of 1000 gpm. What model scale should be used, and how are the results converted into prototype losses?

4.26. A one-fifth scale model of a water pumping station piping system is to be tested to determine over-all head losses. Air at 80°F, 14 psia is available. For a prototype velocity of 1.0 ft/sec in a 14-ft-diameter section with water at 60°F, determine the air velocity and quantity needed and how losses determined from the model are converted to prototype losses.

4.27. Full-scale wind-tunnel tests of the lift and drag on hydrofoils for a boat are to be made. The boat will travel at 30 mph through water at 60°F. What velocity of air (p = 30 psia, t = 90°F) is required to determine the lift and drag? (NOTE: The lift coefficient C_L is dimensionless. Lift $= C_L A \rho V^2/2$.)

4.28. The resistance to ascent of a balloon is to be determined by studying the ascent of a 1:50 scale model in water. How would such a model study be conducted and the results converted to prototype behavior?

4.29. The moment exerted on a submarine by its rudder is to be studied with a 1:100 scale model in a water tunnel. If the torque measured on the model is 3.50 lb-ft for a tunnel velocity of 50 ft/sec, what are the corresponding torque and speed for the prototype?

4.30. For two hydraulic machines to be homologous they must (*a*) be geometrically similar; (*b*) have the same discharge coefficient when viewed as an orifice, $Q_1/(A_1\sqrt{2gH_1}) = Q_2/(A_2\sqrt{2gH_2})$; and (*c*) have the same ratio of peripheral speed to fluid velocity, $\omega D/(Q/A)$. Show that the scaling ratios may be expressed as Q/ND^3 = constant and $H/(ND)^2$ = constant.

4.31. By use of the scaling ratios of Prob. 4.30, determine the head and discharge of a 1:4 model of a centrifugal pump that produces 200 cfs at 96 ft head when turning 240 rpm. The model operates at 1200 rpm.

4.32. An *incorrect* arbitrary recombination of the Π-parameters

$$F\left(\frac{V_0}{\omega D}, \frac{\rho\omega D^2}{\mu}, \frac{c}{\omega D}\right) = 0$$

is

(*a*) $F\left(\frac{c}{V_0}, \frac{\rho C D}{\mu}, \frac{c}{\omega D}\right) = 0$

(*b*) $F\left(\frac{V_0}{\omega D}, \frac{\rho c D^2}{\mu}, \frac{c}{\omega D}\right) = 0$

(*c*) $F\left(\frac{V_0}{\omega D}, \frac{V_0 C \rho}{\omega \mu}, \frac{\rho c D}{\mu}\right) = 0$

(*d*) $F\left(\frac{V_0 \mu}{\omega^2 D^3 \rho}, \frac{V_0 \rho D}{\mu}, \frac{c}{\omega D}\right) = 0$

(*e*) none of these answers

4.33. The repeating variables in a dimensional analysis should

(*a*) include the dependent variable
(*b*) have two variables with the same dimensions if possible
(*c*) exclude one of the dimensions from each variable if possible
(*d*) include those variables not considered very important factors
(*e*) satisfy none of these answers

4.34. Select a common dimensionless parameter in fluid mechanics from the following:

(*a*) angular velocity (*b*) kinematic viscosity (*c*) specific gravity (*d*) specific weight (*e*) none of these answers

4.35. Select the quantity in the following that is *not* a dimensionless parameter:

(*a*) pressure coefficient (*b*) Froude number (*c*) Darcy-Weisbach friction factor (*d*) kinematic viscosity (*e*) Weber number

4.36. Which of the following has the form of a Reynolds number?

(*a*) ul/ν (*b*) $VD\mu/\rho$ (*c*) $u_w\nu/l$ (*d*) V/gD (*e*) $\Delta p/\rho V^2$

4.37. Reynolds number may be defined as the ratio of

(*a*) viscous forces to inertial forces
(*b*) viscous forces to gravity forces
(*c*) gravity forces to inertial forces
(*d*) elastic forces to pressure forces
(*e*) none of these answers

4.38. The pressure coefficient may take the form

(*a*) $\Delta p/\gamma H$ (*b*) $\Delta p/(\rho V^2/2)$ (*c*) $\Delta p/l\mu V$ (*d*) $\Delta p\rho/\mu^2 l^4$ (*e*) none of these answers

4.39. Select the correct answer. The pressure coefficient is a ratio of pressure forces to

(*a*) viscous forces
(*b*) inertial forces
(*c*) gravity forces
(*d*) surface-tension forces
(*e*) elastic-energy forces

4.40. How many Π-parameters are needed to express the function $F(a,V,t,\nu,L) = 0$?

(*a*) 5 (*b*) 4 (*c*) 3 (*d*) 2 (*e*) 1

4.41. Which of the following could be a Π-parameter of the function $F(Q,H,g,V_0,\phi) = 0$ when Q and g are taken as repeating variables?

(*a*) Q^2/gH^4 (*b*) V_0^2/g^2Q (*c*) $Q/g\phi^2$ (*d*) $Q/\sqrt{gH}$ (*e*) none of these answers

4.42. Select the situation in which inertial forces would be *unimportant:*

(*a*) flow over a spillway crest
(*b*) flow through an open-channel transition
(*c*) waves breaking against a sea wall
(*d*) flow through a long capillary tube
(*e*) flow through a half-opened valve

4.43. Which two forces are most important in laminar flow between closely spaced parallel plates:

(*a*) inertial, viscous (*b*) pressure, inertial (*c*) gravity, pressure (*d*) viscous, pressure (*e*) none of these answers

4.44. A dimensionless combination of Δp, ρ, l, Q is

(*a*) $\sqrt{\frac{\Delta p}{\rho}}\frac{Q}{l^2}$ (*b*) $\frac{\rho Q}{\Delta p l^2}$ (*c*) $\frac{pl}{\Delta p Q^2}$ (*d*) $\frac{\Delta p l Q}{\rho}$ (*e*) $\sqrt{\frac{\rho}{\Delta p}}\frac{Q}{l^2}$

4.45. What velocity of oil, $\rho = 1.6$ slugs/ft^3, $\mu = 0.20$ poise, must occur in a 1-in.-diameter pipe to be dynamically similar to 10 ft/sec water velocity at 68°F in a $\frac{1}{4}$-in.-diameter tube?

(*a*) 0.60 ft/sec (*b*) 9.6 ft/sec (*c*) 4.0 ft/sec (*d*) 60 ft/sec (*e*) none of these answers

4.46. The velocity at a point on a model dam crest was measured to be 2.5 ft/sec. The corresponding prototype velocity for $\lambda = 25$ is, in ft/sec,

(*a*) 62.5 (*b*) 12.5 (*c*) 0.5 (*d*) 0.10 (*e*) none of these answers

4.47. The height of a hydraulic jump in a stilling pool was found to be 4.0 in. in a model, $\lambda = 36$. The prototype jump height is

(*a*) 12 ft (*b*) 2 ft (*c*) not determinable from data given (*d*) less than 4 in. (*e*) none of these answers

4.48. A ship's model, scale 1:100, had a wave resistance of 2.5 lb at its design speed. The corresponding prototype wave resistance is, in lb,

(*a*) 2500 (*b*) 25,000 (*c*) 250,000 (*d*) 2,500,000 (*e*) none of these answers

4.49. A 1:5 scale model of a projectile has a drag coefficient of 3.5 at $\mathbf{M} = 2.0$. How many times greater would the prototype resistance be when fired at the same Mach number in air of the same temperature and half the density?

(*a*) 3.12 (*b*) 12.5 (*c*) 25 (*d*) 100 (*e*) none of these answers

4.50. If the capillary rise Δh of a liquid in a circular tube of diameter D depends upon surface tension σ, and specific weight γ, the formula for capillary rise could take the form,

(*a*) $\Delta h = \sqrt{\frac{\sigma}{\gamma}}F\left(\frac{\sigma}{\gamma D^2}\right)$ (*b*) $\Delta h = c\left(\frac{\sigma}{\gamma D^2}\right)^n$ (*c*) $\Delta h = cD\left(\frac{\sigma}{\gamma}\right)^n$

(*d*) $\Delta h = \sqrt{\frac{\gamma}{\sigma}}F\left(\frac{\gamma D^2}{\sigma}\right)$ (*e*) none of these answers

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5

VISCOUS EFFECTS—FLUID RESISTANCE

In Chap. 3 the basic equations used in the analysis of fluid-flow situations were discussed. The fluid was considered frictionless, or in some cases, losses were assumed or computed without probing into their underlying causes. This chapter deals with real fluids, i.e., with situations in which irreversibilities are important. Viscosity is the fluid property that causes shear stresses in a moving fluid; it is also one means by which irreversibilities or losses are developed. Without viscosity in a fluid there is no fluid resistance. Simple cases of laminar incompressible flow are first developed in this chapter, since in these cases the losses may be

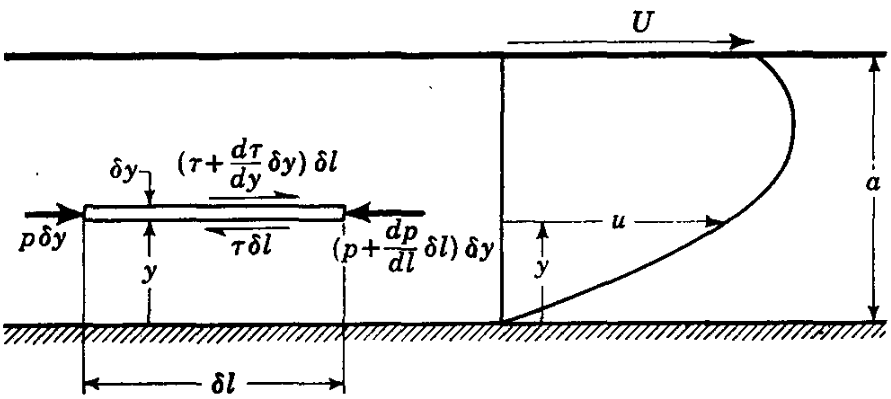


FIG. 5.1. Flow between parallel plates with upper plate in motion.

computed. The concept of Reynolds number, introduced in Chap. 4, is then further developed. Turbulent-flow shear relationships are introduced by use of the Prandtl mixing-length theory and are applied to turbulent velocity distributions. This is followed by boundary-layer concepts and by drag on immersed bodies. Resistance to steady, uniform, incompressible, turbulent flow is then examined for open and closed conduits, with a section devoted to open channels and to pipe flow. The chapter closes with a section on lubrication mechanics.

5.1. Laminar, Incompressible Flow between Parallel Plates. Flow between parallel plates when one plate moves with velocity U in its own plane is first developed. Flow between fixed parallel plates is a special case obtained by letting $U = 0$. In Fig. 5.1 the upper plate moves with velocity U in the l-direction and there is a pressure variation in the l-direction. The flow is analyzed by taking a thin lamina of unit width as a free body. The equation of motion for the lamina in steady motion in the l-direction is

$$p\,\delta y - \left(p + \frac{dp}{dl}\,\delta l\right)\delta y - \tau\,\delta l + \left(\tau + \frac{d\tau}{dy}\,\delta y\right)\delta l = 0$$

After dividing through by the volume of the element and after simplifying,

$$\frac{d\tau}{dy} = \frac{dp}{dl} \tag{5.1.1}$$

Since dp/dl is independent of y, this integrates at once with respect to y,

$$\int \frac{d\tau}{dy}\,dy = \frac{dp}{dl}\int dy + A$$

or

$$\tau = \frac{dp}{dl}\,y + A$$

The direction of the shear forces on the free body is that for the case in which u increases as y increases; hence, from Eq. (1.1.1)

$$\tau = \mu\,\frac{du}{dy}$$

After substituting for τ,

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dl}\,y + \frac{A}{\mu}$$

By integrating again with respect to y,

$$\int \frac{du}{dy}\,dy = \frac{1}{\mu}\frac{dp}{dl}\int y\,dy + \frac{A}{\mu}\int dy + B$$

or

$$u = \frac{1}{2\mu}\frac{dp}{dl}\,y^2 + \frac{Ay}{\mu} + B$$

in which A, B are constants of integration and may be selected to make the velocity of fluid at the boundary equal to the velocity of the boundary; that is, $u = U$ when $y = a$ and $u = 0$ when $y = 0$. Substitution in turn produces

$$U = \frac{1}{2\mu}\frac{dp}{dl}\,a^2 + \frac{Aa}{\mu} + B \qquad B = 0$$

After eliminating A and B,

$$u = \frac{Uy}{a} - \frac{1}{2\mu}\frac{dp}{dl}\,(ay - y^2) \tag{5.1.2}$$

For $dp/dl = 0$ there is no pressure drop, and the velocity has a straight-line distribution. When $U = 0$, the velocity distribution for flow between fixed parallel plates is obtained. The discharge is calculated with Eq. (5.1.2) by integration,

$$Q = \int_0^a u\,dy = \frac{Ua}{2} - \frac{1}{12\mu}\frac{dp}{dl}a^3 \tag{5.1.3}$$

The maximum velocity is generally at some point other than the midplane.

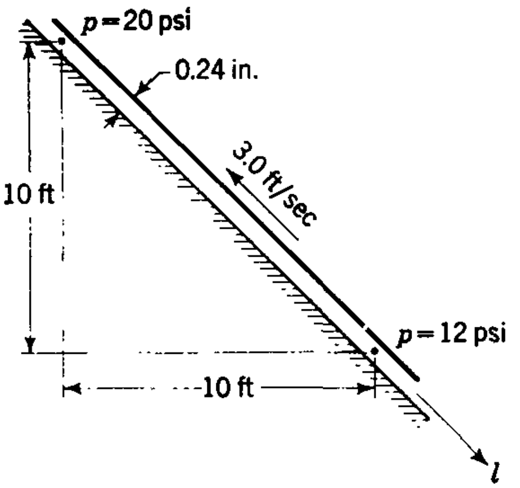


FIG. 5.2. Flow between inclined flat plates.

Example 5.1: In Fig. 5.2 one plate moves relative to the other as shown. $\mu = 0.80$ poise, $\rho = 1.7$ slugs/ft³. Determine the velocity distribution, the discharge, and the shear stress exerted on the upper plate.

In Eq. (5.1.2) dp/dl must be replaced by $d(p + \gamma z)/dl$ to account for the weight component. At the upper point

$$p + \gamma z = 20 \times 144 + 1.7 \times 32.2 \times 10 = 3426 \text{ lb/ft}^2$$

and at the lower point

$$p + \gamma z = 12 \times 144 = 1728 \text{ lb/ft}^2$$

to the same datum. Hence

$$\frac{d(p + \gamma z)}{dl} = \frac{1728 - 3427}{10\sqrt{2}} = -120 \text{ lb/ft}^3$$

From the figure $a = 0.24/12 = 0.02$ ft, $U = -3.0$ ft/sec, and from Eq. (5.1.2)

$$u = -\frac{3y}{0.02} + \frac{120}{2(0.80/479)}(0.02y - y^2)$$

After simplifying

$$u = 566y - 35{,}800y^2$$

the maximum velocity occurs where $du/dy = 0$, or $y = 0.0079$ ft. It is $u_{max} =$ 2.24 ft/sec, so the minimum velocity occurs at the upper plate.

The discharge is

$$Q = \int_0^{0.02} u\,dy = 283y^2 - 11{,}933y^3\Big]_0^{0.02} = 0.0177 \text{ cfs/ft}$$

and is downward.

To find the shear stress on the upper plate

$$\frac{du}{dy}\Big]_{y=0.02} = 566 - 71{,}600y\Big]_{y=0.02} = -866$$

and

$$\tau = \mu \frac{du}{dy} = \frac{0.80}{479}(-866) = -1.45 \text{ lb/ft}^2$$

This is the fluid shear at the plate; hence, the shear force on the plate is 1.45 lb/ft² resisting the motion of the plate.

Losses in Laminar Flow. An expression for the irreversibilities is developed for one-dimensional, incompressible, steady, laminar flow, in which the equation of motion and the principle of work and energy are utilized. There is no increase in kinetic energy in steady flow in a tube or between parallel plates. The pressure drop in horizontal flow, which represents work done on the fluid per unit volume, is converted into irreversibilities by the action of viscous shear. The losses in the length L are $Q\,\Delta p$ per unit time, in which Δp is the pressure drop.

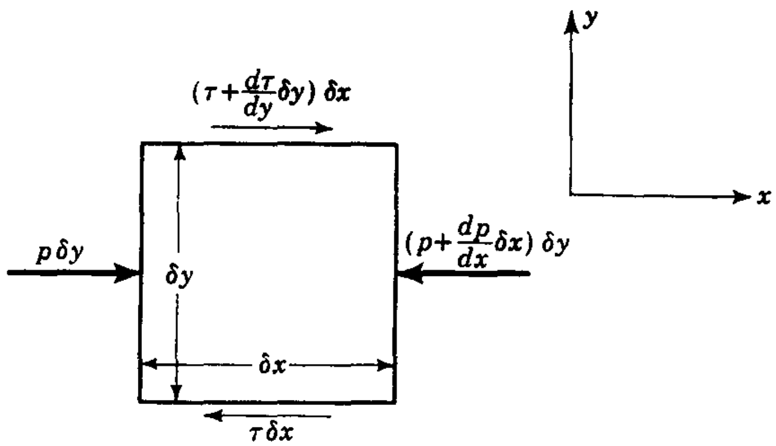


FIG. 5.3. Forces on a fluid element.

After examination of the work done on the fluid in one-dimensional flow, an expression for the losses can be developed. First, the equation of motion applied to an element (Fig. 5.3) relates the shear stress and pressure drop. There is no acceleration; hence, $\Sigma f_x = 0$, and

$$p\,\delta y - \left(p + \frac{dp}{dx}\,\delta x\right)\delta y - \tau\,\delta x + \left(\tau + \frac{d\tau}{dy}\,\delta y\right)\delta x = 0$$

After simplifying,

$$\frac{dp}{dx} = \frac{d\tau}{dy} \tag{5.1.4}$$

which implies that the rate of change of pressure in the x-direction must equal the rate of change of shear in the y-direction. Clearly, dp/dx is independent of y, and $d\tau/dy$ is independent of x.

The work done per unit time, or power input, to a fluid element (Fig. 5.4) for one-dimensional flow consists in the work done on the element by pressure and by shear stress, minus the work that the element does on the surrounding fluid, or

$$p\left(u + \frac{du}{dy}\frac{\delta y}{2}\right)\delta y - \left(p + \frac{dp}{dx}\delta x\right)\left(u + \frac{du}{dy}\frac{\delta y}{2}\right)\delta y + \tau u\,\delta x$$
$$+ \frac{d}{dy}(\tau u)\,\delta y\,\delta x - \tau u\,\delta x$$

After simplifying,

$$\frac{\text{Net power input}}{\text{Unit volume}} = \frac{d}{dy}(\tau u) - u\frac{dp}{dx} \tag{5.1.5}$$

By expanding Eq. (5.1.5) and substituting Eq. (5.1.4)

$$\frac{\text{Net power input}}{\text{Unit volume}} = \tau\frac{du}{dy} + u\frac{d\tau}{dy} - u\frac{dp}{dx} = \tau\frac{du}{dy} \tag{5.1.6}$$

With Newton's law of viscosity,

$$\frac{\text{Net power input}}{\text{Unit volume}} = \tau\frac{du}{dy} = \mu\left(\frac{du}{dy}\right)^2 = \frac{\tau^2}{\mu} \tag{5.1.7}$$

This power is used up by viscous friction and is converted into irreversibilities.

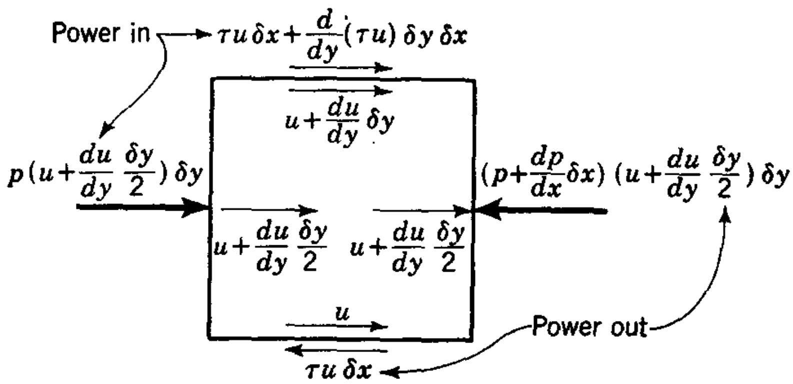


FIG. 5.4. Work done on a fluid element in one-dimensional motion.

By integrating the expression over a length L between two fixed parallel plates, with Eq. (5.1.2) for $U = 0$ and with Eq. (5.1.7),

$$\text{Net power input} = \int_0^a \mu\left(\frac{du}{dy}\right)^2 L\,dy = \mu L\int_0^a \left[\frac{1}{2\mu}\frac{dp}{dl}(2y - a)\right]^2 dy$$
$$= \left(\frac{dp}{dl}\right)^2 \frac{a^3 L}{12\mu}$$

By substituting for Q from Eq. (5.1.3) for $U = 0$,

$$\text{Losses} = \text{net power input} = -Q\frac{dp}{dl}L = Q\,\Delta p$$

in which Δp is the pressure drop in the length L. The expression for power input per unit volume [Eq. (5.1.7)] is also applicable to cases of laminar flow in a tube. The irreversibilities are greatest when du/dy is greatest. The distribution of shear stress, velocity, and losses is shown in Fig. 5.5 for a round tube.

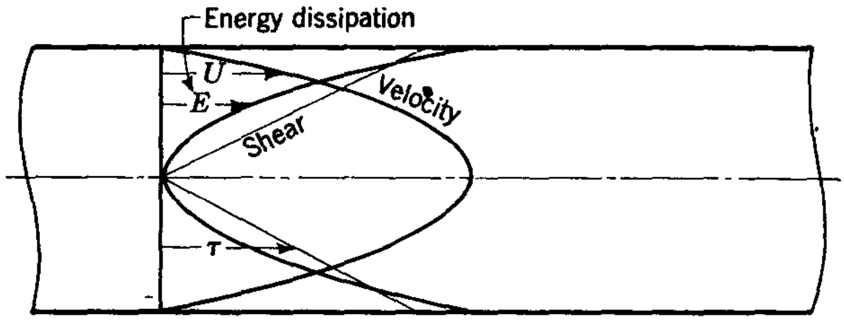


FIG. 5.5. Distribution of velocity, shear, and losses for a round tube.

5.2. Laminar Flow through Circular Tubes and Circular Annuli. *Flow through Circular Tubes.* For steady, incompressible, laminar flow through a straight, round tube, the velocity distribution, discharge, and pressure drop can be determined analytically. In a horizontal tube (Fig. 5.6) with a concentric cylinder of fluid as a free body, the flow is steady and, since the size of the cross section does not change, every particle of fluid moves without acceleration. Therefore, the summation of forces on the free body must equal zero. When the component of forces is taken

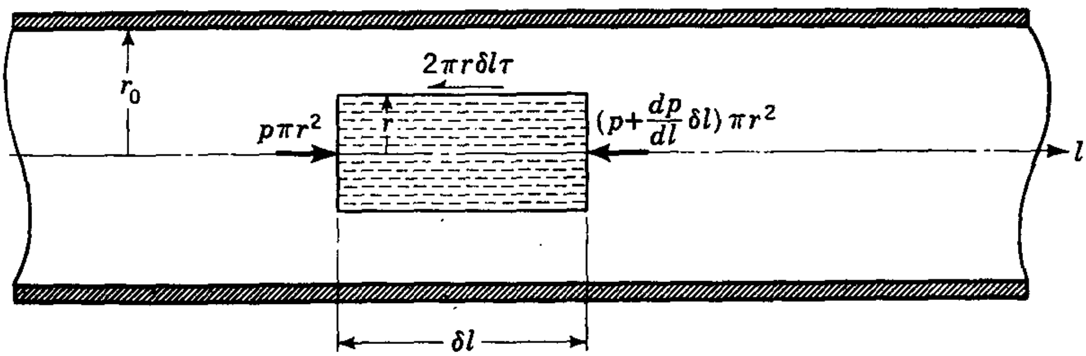


FIG. 5.6. Free-body diagram for steady flow through a round tube.

in the l-direction, there are normal pressure forces over the end areas and shear forces over the curved surface of the cylinder. In the figure,

$$p\pi r^2 - \left(p + \frac{dp}{dl}\,\delta l\right)\pi r^2 - 2\pi r\,\delta l\,\tau = 0$$

or, after dividing through by the volume $\pi r^2\,\delta l$ and simplifying,

$$\tau = -\frac{dp}{dl}\frac{r}{2} \tag{5.2.1}$$

The term dp/dl depends upon l only for a given flow. This equation shows that the shear stress is zero at the tube axis and increases linearly with r to its maximum value τ_0 at the wall of the tube. The pressure must decrease in the direction of flow in a horizontal tube in that pressure force is the only means available to overcome resistance to flow; the potential and kinetic energies remain constant. The term $-dp/dl$ is positive. Equation (5.2.1) holds for turbulent flow as well as for laminar flow since in deriving it no assumptions were made as to the nature of the flow.

For one-dimensional laminar flow the shear stress is related to the velocity by Newton's law of viscosity,

$$\tau = -\mu \frac{du}{dr}$$

into which the minus sign is introduced because du/dr is negative for the particular choice of coordinates; that is, u decreases as r increases. By substituting for τ in Eq. (5.2.1),

$$\frac{du}{dr} = \frac{1}{\mu}\frac{dp}{dl}\frac{r}{2}$$

The term $-dp/dl$ is the drop in pressure per unit length of tube and is not a function of r. By integrating with respect to r, if u and r are the only variables in the equation,

$$\int \frac{du}{dr}\,dr = \frac{1}{2\mu}\frac{dp}{dl}\int r\,dr + c$$

or

$$u = \frac{1}{4\mu}\frac{dp}{dl}r^2 + c$$

The velocity of a real fluid is always zero at a fixed boundary; hence, $u = 0$ for $r = r_0$. After substituting this boundary condition into the equation,

$$0 = \frac{1}{4\mu}\frac{dp}{dl}r_0^2 + c$$

To eliminate the constant of integration c, the difference between the last two equations is taken, so

$$u = -\frac{1}{4\mu}\frac{dp}{dl}(r_0^2 - r^2) \tag{5.2.2}$$

which is the equation for velocity distribution. The velocity varies parabolically, and the velocity distribution surface is a paraboloid of revolu-

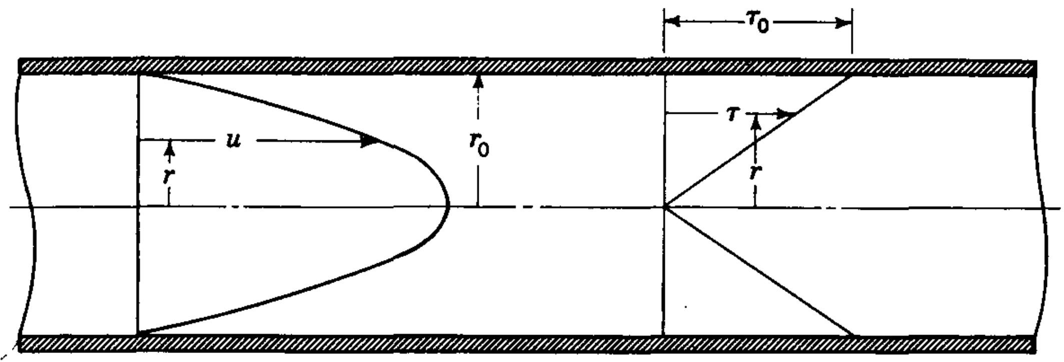


FIG. 5.7. Velocity distribution and shear-stress distribution in laminar flow in a round tube.

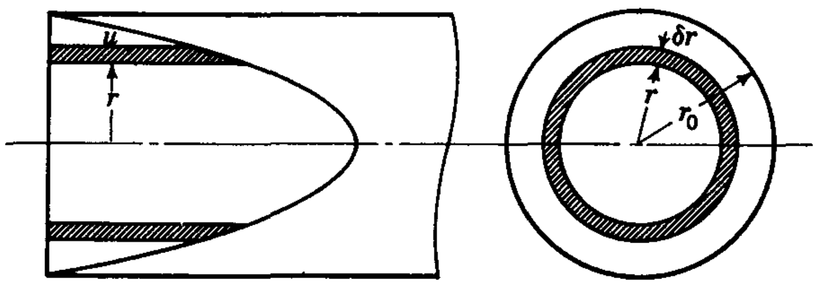


FIG. 5.8. Ring element of area used to compute discharge.

tion. It is shown, together with the shear-stress distribution, in Fig. 5.7. The maximum velocity u_{max} occurs at the axis and is

$$u_{max} = -\frac{dp}{dl}\frac{r_0^2}{4\mu} \tag{5.2.3}$$

The discharge is the quantity within the velocity distribution surface

$$Q = \int u\,dA = \int_0^{r_0} u 2\pi r\,dr$$

in which the ring element of area (Fig. 5.8) has been used. By substituting for u from Eq. (5.2.2) and performing the integration,

$$Q = -\frac{dp}{dl}\frac{\pi}{2\mu}\int_0^{r_0}(r_0^2 - r^2)r\,dr = -\frac{dp}{dl}\frac{\pi r_0^4}{8\mu} \tag{5.2.4}$$

The term $-dp/dl$ may be written $\Delta p/L$, in which Δp is the pressure *drop* in the length L. Equation (5.2.4) then becomes

$$Q = \frac{\Delta p\,\pi r_0^4}{8\mu L} \tag{5.2.5}$$

In terms of the tube diameter D,

$$Q = \frac{\Delta p\,\pi D^4}{128\mu L} \tag{5.2.6}$$

The average velocity V is $Q/\pi r_0^2$, or

$$V = \frac{\Delta p \, r_0^2}{8\mu L} \tag{5.2.7}$$

which is one-half of the maximum velocity.

Equation (5.2.6) can then be solved for pressure drop, which represents losses per unit volume,

$$\Delta p = \frac{128\mu L Q}{\pi D^4} \tag{5.2.8}$$

The losses are seen to vary directly as the viscosity, the length, and the discharge, and to vary inversely as the fourth power of the diameter.

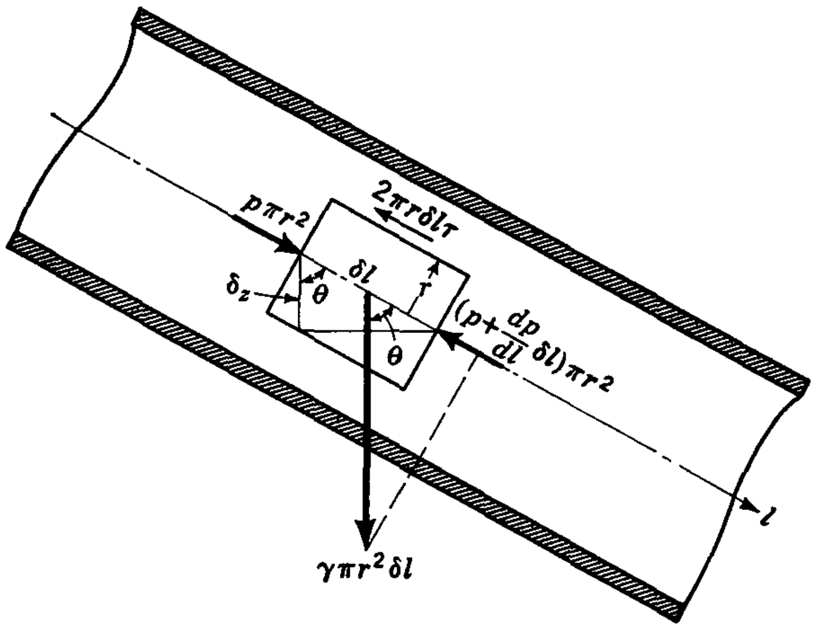


Fig. 5.9. Free-body diagram for steady flow through an inclined tube.

It should be noted that tube roughness does not enter into the equations. Equation (5.2.6) is known as the *Hagen-Poiseuille equation;* it was determined experimentally by Hagen in 1839 and independently by Poiseuille in 1840. The analytical derivation was made by Wiedemann in 1856.

The results as given by Eqs. (5.2.2) to (5.2.8) are not valid near the entrance of a pipe. If the flow enters the pipe from a reservoir through a well-rounded entrance, the velocity at first is almost uniform over the cross section. The action of wall shear stress (as the velocity must be zero at the wall) is to slow down the fluid near the wall. As a consequence of continuity the velocity must then increase in the central region. The transition length L' for the characteristic parabolic velocity distribution to develop is a function of the Reynolds number. Langhaar[1]

[1] H. L. Langhaar, Steady Flow in the Transition Length of a Straight Tube, *J. Appl. Mechanics*, vol. 9, pp. 55–58, 1942.

developed the theoretical formula

$$\frac{L'}{D} = 0.058\mathbf{R}$$

which agrees well with observation.

When the tube is inclined, as in Fig. 5.9, the losses can come from potential energy as well as from flow energy. An additional term comes into the equation due to the weight component $\gamma\pi r^2 \,\delta l \cos\theta$. If z is measured vertically upward, a change δz corresponds to a change $-\delta l$. In Fig. 5.9

$$\cos\theta = -\frac{\delta z}{\delta l} = -\frac{dz}{dl}$$

When the weight component is included in Eq. (5.2.1)

$$\tau = -\frac{d}{dl}(p + \gamma z)\frac{r}{2} \tag{5.2.9}$$

and Eq. (5.2.4) becomes

$$Q = -\frac{d}{dl}(p + \gamma z)\frac{\pi r_0^4}{8\mu} \tag{5.2.10}$$

The losses per unit volume per unit length of tube are $-d(p + \gamma z)/dl$.

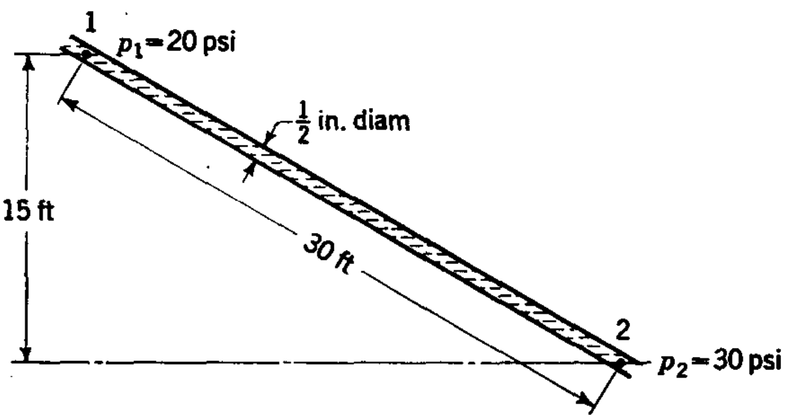


FIG. 5.10. Flow through an inclined tube.

Example 5.2: Determine the direction of flow through the tube shown in Fig. 5.10, in which $\gamma = 50$ lb/ft³, $\mu = 0.40$ poise. Find the quantity flowing in gallons per minute, and compute the Reynolds number for the flow.

At section 1

$$p + \gamma z = 20 \times 144 + 50 \times 15 = 3630 \text{ lb/ft}^2$$

and at section 2

$$p + \gamma z = 30 \times 144 = 4320 \text{ lb/ft}^2$$

if datum for z is taken through section 2. The flow is from 2 to 1 since the energy is greater at 2 (kinetic energy must be the same at both sections) than at 1.

To determine the quantity flowing, the expression is written

$$-\frac{d}{dl}(p + \gamma z) = -\frac{3630 - 4230}{30} = 23 \text{ lb/ft}^3$$

After substituting into Eq. (5.2.10),

$$Q = \frac{23\pi(\frac{1}{4})^4}{8 \times (12)^4(0.40/479)} = 0.00203 \text{ cfs}$$

By converting to gallons per minute,

$$Q = 0.00203 \times 7.46 \times 60 = 0.91 \text{ gpm}$$

The average velocity is $Q/\pi r_0^2$, or

$$V = \frac{0.00203}{\pi(\frac{1}{48})^2} = 1.486 \text{ ft/sec}$$

and the Reynolds number is (Sec. 4.4)

$$\mathbf{R} = \frac{VD\rho}{\mu} = \frac{1.486 \times 1 \times 50 \times 478}{24 \times 32.2 \times 0.40} = 115$$

If the Reynolds number had been above 2000, the Hagen-Poiseuille equation would no longer apply, as discussed in Sec. 5.3.

The kinetic-energy correction factor α [Eq. (3.6.7)] may be determined for laminar flow in a tube by use of Eqs. (5.2.2) and (5.2.3),

$$\frac{u}{V} = 2\frac{u}{u_{max}} = 2\left[1 - \left(\frac{r}{r_0}\right)^2\right] \tag{5.2.11}$$

By substituting into the expression for α,

$$\alpha = \frac{1}{A}\int\left(\frac{u}{V}\right)^3 dA = \frac{1}{\pi r_0^2}\int_0^{r_0}\left\{2\left[1 - \left(\frac{r}{r_0}\right)^2\right]\right\}^3 2\pi r\, dr = 2 \tag{5.2.12}$$

There is twice as much kinetic energy in the flow as in uniform flow at the same average velocity.

Flow through an Annulus. Steady laminar flow through the annular space between two concentric round tubes can be determined analytically. In place of the solid cylinder of Fig. 5.6, a cylindrical sleeve is taken as free body. The forces acting on it are shown in Fig. 5.11. Again the flow is steady, and the summation of forces on the free body in the axial direction must be zero. The equation may be written

$$p2\pi r\,\delta r - \left(p + \frac{dp}{dl}\,\delta l\right)2\pi r\,\delta r + 2\pi r\,\delta l\,\tau - 2\pi(r + \delta r)\,\delta l\left(\tau + \frac{d\tau}{dr}\,\delta r\right) = 0$$

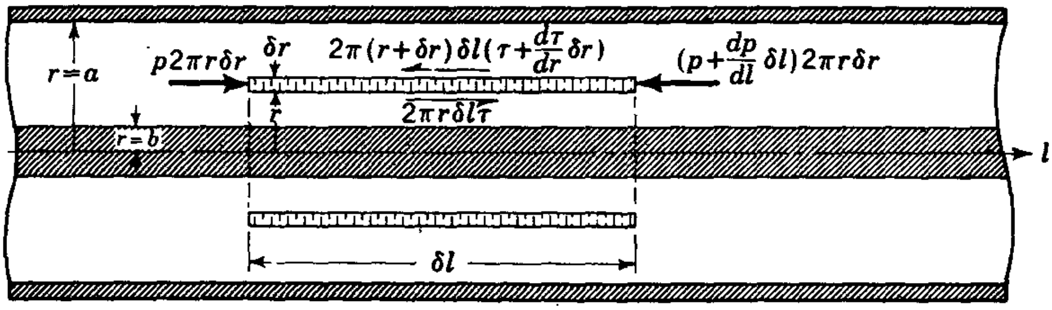


FIG. 5.11. Free-body diagram for flow through an annulus.

After dividing through by the volume of the element $2\pi r\ \delta r\ \delta l$ and dropping the term containing the infinitesimal,

$$\frac{dp}{dl} + \frac{d\tau}{dr} + \frac{\tau}{r} = 0$$

In this expression τ is a function of r only, and p is a function of l only. The last two terms may be combined, so

$$\frac{dp}{dl} + \frac{1}{r}\frac{d(\tau r)}{dr} = 0 \qquad (5.2.13)$$

Since dp/dl is not a function of r, the equation can be integrated with respect to r,

$$\frac{dp}{dl}\int r\,dr + \int \frac{d}{dr}(\tau r)\,dr = \frac{dp}{dl}\frac{r^2}{2} + \tau r = A$$

in which A is the constant of integration. By substituting for τ from $\tau = -\mu\, du/dr$ and multiplying through by dr/r, the equation can be integrated again,

$$\frac{1}{2}\frac{dp}{dl}\int r\,dr - u\int \frac{du}{dr}\,dr = A\int \frac{dr}{r} + B$$

or

$$\frac{dp}{dl}\frac{r^2}{4} - \mu u = A \ln r + B$$

B is the second constant of integration. The velocity must be zero at the outer wall, $u = 0$, $r = a$; and at the inner wall, $u = 0$, $r = b$. After substituting in turn,

$$\frac{dp}{dl}\frac{a^2}{4} = A \ln a + B \qquad \frac{dp}{dl}\frac{b^2}{4} = A \ln b + B$$

Eliminating the constants A, B in the three equations and solving for u,

$$u = -\frac{1}{4\mu}\frac{dp}{dl}\left(a^2 - r^2 + \frac{a^2 - b^2}{\ln b/a}\ln\frac{a}{r}\right) \qquad (5.2.14)$$

The discharge Q is

$$Q = \int_b^a 2\pi r u \, dr = -\frac{\pi}{8\mu}\frac{dp}{dl}\left[a^4 - b^4 - \frac{(a^2 - b^2)^2}{\ln a/b}\right] \tag{5.2.15}$$

For sloping tubes dp/dl may be replaced by $d(p + \gamma z)/dl$ as in Eq. (5.2.10).

5.3. Reynolds Number. Laminar flow is defined as flow in which the fluid moves in layers, or laminas, one layer gliding smoothly over an adjacent layer with only a molecular interchange of momentum. Any tendencies toward instability and turbulence are damped out by viscous shear forces that resist relative motion of adjacent fluid layers. Turbulent flow, however, has very erratic motion of fluid particles, with a violent transverse interchange of momentum. The nature of the flow, i.e., whether laminar or turbulent, and its relative position along a scale indicating the relative importance of turbulent to laminar tendencies are indicated by *Reynolds number*. The concept of Reynolds number and its interpretation are discussed in this section. In Sec. 3.5 an equation of motion was developed with the assumption that the fluid is frictionless, i.e., that the viscosity is zero. More general equations have been developed that include viscosity, by including shear stresses. These equations (Navier-Stokes) are complicated, nonlinear, partial differential equations for which no general solution has been obtained. In the last century Osborne Reynolds[1] studied these equations to try to determine when two different flow situations would be similar.

Two flow cases are said to be *dynamically similar* when

a. they are geometrically similar, i.e., corresponding linear dimensions have a constant ratio and

b. the corresponding streamlines are geometrically similar, or pressures at corresponding points have a constant ratio.

In considering two geometrically similar flow situations, Reynolds deduced that they would be dynamically similar if the general differential equations describing their flow were identical. By changing the units of mass, length, and time in one set of equations and determining the conditions that must be satisfied to make them identical to the original equations, Reynolds found that the dimensionless group $ul\rho/\mu$ must be the same for both cases. Of these, u is a characteristic velocity, l a characteristic length, ρ the mass density, and μ the viscosity. This group, or parameter, is now called the Reynolds number **R**,

$$\mathbf{R} = \frac{ul\rho}{\mu} \tag{5.3.1}$$

[1] O. Reynolds, An Experimental Investigation of the Circumstances Which Determine whether the Motion of Water Shall Be Direct or Sinuous, and of the Laws of Resistance in Parallel Channels, *Trans. Roy. Soc.* (*London*), vol. 174, 1883.

To determine the significance of the dimensionless group, Reynolds conducted his experiments on flow of water through glass tubes, illustrated in Fig. 5.12. A glass tube was mounted horizontally with one end in a tank and a valve on the opposite end. A smooth bellmouth entrance was attached to the upstream end, with a dye jet arranged so that a fine stream of dye could be ejected at any point in front of the bellmouth. Reynolds took the average velocity V as characteristic velocity and the diameter of tube D as characteristic length, so that $\mathbf{R} = VD\rho/\mu$.

For small flows the dye stream moved as a straight line through the tube, showing that the flow was laminar. As the flow rate increased, the Reynolds number increased, since D, ρ, μ were constant, and V was directly proportional to the rate of flow. With increasing discharge a

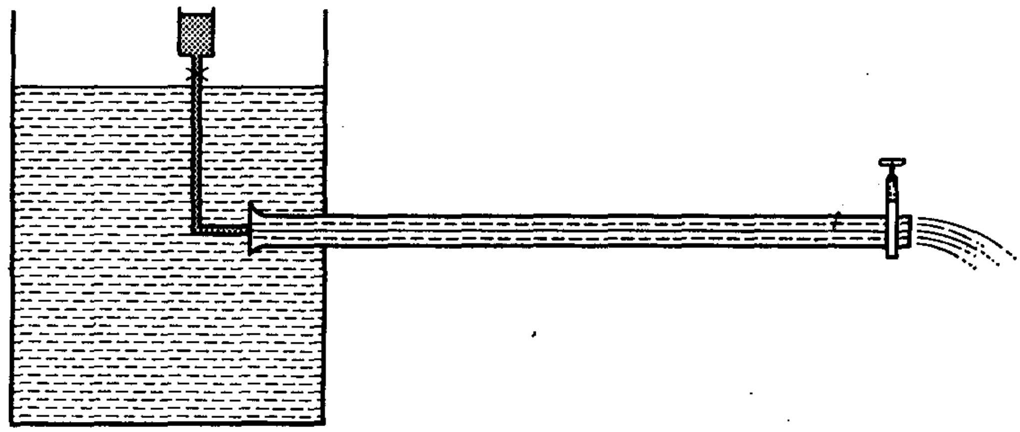


FIG. 5.12. Reynolds apparatus.

condition was reached at which the dye stream wavered and then suddenly broke up and was diffused throughout the tube. The flow had changed to turbulent flow with its violent interchange of momentum that had completely disrupted the orderly movement of laminar flow. By careful manipulation Reynolds was able to obtain a value $\mathbf{R} = 12{,}000$ before turbulence set in. A later investigator, using Reynolds' original equipment, obtained a value of 40,000 by allowing the water to stand in the tank for several days before the experiment and by taking precautions to avoid vibration of the water or equipment. These numbers, referred to as the *Reynolds upper critical numbers,* have no practical significance in that the ordinary pipe installation has irregularities that cause turbulent flow at a much smaller value of the Reynolds number.

Starting with turbulent flow in the glass tube, Reynolds found that it would always become laminar when the velocity was reduced to make $\mathbf{R}$ less than 2000. This is the *Reynolds lower critical number* for pipe flow and is of practical importance. With the usual piping installation, the flow will change from laminar to turbulent in the range of the Reynolds numbers from 2000 to 4000. For the purpose of this treatment it is assumed that the change occurs at $\mathbf{R} = 2000$. In laminar flow the losses are directly proportional to the average velocity, while in turbulent flow the losses are proportional to the velocity to a power varying from 1.7 to 2.0.

There are many Reynolds numbers in use today in addition to the one for straight, round tubes. For example, the motion of a sphere through a fluid may be characterized by $UD\rho/\mu$, in which U is the velocity of sphere, D is the diameter of sphere, and ρ and μ are the fluid density and viscosity.

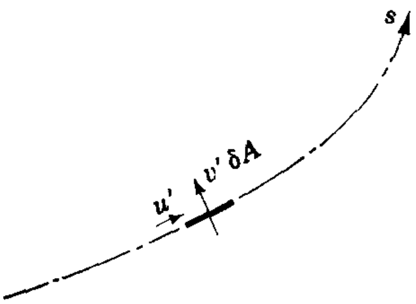


FIG. 5.13. Notation for shear stress due to turbulent flow.

The Reynolds number may be viewed as a ratio of shear stress τ_t due to turbulence to shear stress τ_v due to viscosity. By applying the momentum equation to the flow through an element of area δA (Fig. 5.13) the apparent shear stress due to turbulence can be determined. If v' is the velocity normal to δA and u' is the difference in velocity, or the velocity fluctuation, on the two sides of the area, then, with Eq. (3.9.10), the shear force δF acting is computed to be

$$\delta F = \rho v' \, \delta A \, u'$$

in which $\rho v' \, \delta A$ is the mass per second having its momentum changed and u' is the final velocity minus the initial velocity in the s-direction. By dividing through by δA, the shear stress τ_t due to turbulent fluctuations is obtained,

$$\tau_t = \rho u' v' \tag{5.3.2}$$

The shear stress due to viscosity may be written

$$\tau_v = \frac{\mu u'}{l} \tag{5.3.3}$$

in which u' is interpreted as the change in velocity in the distance l, measured normal to the velocity. Then the ratio,

$$\frac{\tau_t}{\tau_v} = \frac{v' l \rho}{\mu}$$

has the form of a Reynolds number.

Although this method of viewing the Reynolds number is not exact, it does indicate that for large Reynolds numbers the numerator is much more important than the denominator or that the viscous shear may be neglected because it is very small compared with the shear due to turbulence. On the other hand a small Reynolds number indicates that the denominator is much more important than the numerator, or that the viscous shear is much greater than turbulent shear.

The *nature* of a given flow of an incompressible fluid is characterized by its Reynolds number. For large values of **R** one or all of the terms in the numerator are large compared with the denominator. This implies a large expanse of fluid, high velocity, great density, extremely small viscosity, or combinations of these extremes. The numerator terms are related to *inertial* forces, or to forces set up by acceleration or deceleration of the fluid. The denominator term is the cause of viscous shear forces. Thus, the Reynolds number parameter may also be considered as a ratio of inertial to viscous forces. A large **R** indicates a highly turbulent flow with losses proportional to the square of the velocity. The turbulence may be *fine scale*, composed of a great many small eddies that rapidly convert mechanical energy into irreversibilities through viscous action; or it may be *large scale*, like the huge vortices and swirls in a river or gusts in the atmosphere. The large eddies generate smaller eddies, which in turn create fine-scale turbulence. Turbulent flow may be thought of as a smooth, possibly uniform flow, with a secondary flow superposed on it. A fine-scale turbulent flow has small fluctuations in velocity that occur with high frequency. The root-mean-square value of the fluctuations and the frequency of change of sign of the fluctuations are quantitative measures of turbulence. In general the intensity of turbulence increases as the Reynolds number increases.

For intermediate values of **R** both viscous and inertial effects are important, and changes in viscosity change the velocity distribution and the resistance to flow.

For the same **R**, two geometrically similar closed-conduit systems (one, say, twice the size of the other) will have the same ratio of losses to velocity head. The use of Reynolds number provides a means for using experimental results with one fluid for predicting results in a similar case with another fluid.

5.4. Prandtl Mixing Length. Velocity Distribution in Turbulent Flow. Pressure drop and velocity distribution for several cases of laminar flow were worked out in the preceding section. In this section the mixing-length theory of turbulence is developed, including its application to several flow situations. The apparent shear stress in turbulent flow is expressed by [Eq. (3.2.2)]

$$\tau = (\mu + \eta) \frac{du}{dy} \tag{5.4.1}$$

including direct viscous effects. Prandtl[1] has developed a most useful theory of turbulence called the *mixing-length* theory. In Sec. 5.3 the

[1] For an account of the development of turbulence theory the reader is referred to L. Prandtl, "Essentials of Fluid Dynamics," pp. 105–145, Hafner Publishing Company, New York, 1952.

shear stress τ, due to turbulence, was shown to be

$$\tau = \rho u'v' \tag{5.3.2}$$

in which u', v' are the velocity fluctuations at a point. In Prandtl's[1] theory, expressions for u' and v' are obtained in terms of a mixing-length distance l and the velocity gradient du/dy, in which u is the temporal mean velocity at a point and y is the distance normal to u, usually measured from the boundary. In a gas, one molecule, before striking another, travels an average distance known as the *mean free path* of the gas. Using this as an analogy (Fig. 5.14*a*), Prandtl assumed that a

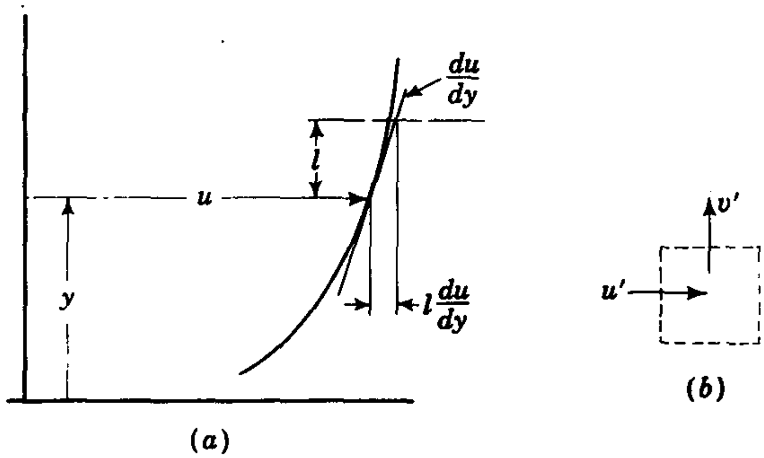


FIG. 5.14. Notation for mixing-length theory.

particle of fluid is displaced a distance l before its momentum is changed by the new environment. The fluctuation u' is then related to l by

$$u' \sim l \frac{du}{dy}$$

which means that the amount of the change in velocity depends upon the change in temporal mean velocity at two points distant l apart in the y-direction. From the continuity equation, he reasoned that there must be a correlation between u' and v' (Fig. 5.14*b*), so that v' is proportional to u',

$$v' \sim u' \sim l \frac{du}{dy}$$

By substituting for u' and v' in Eq. (5.3.2) and by letting l absorb the proportionality factor, the defining equation for mixing length is obtained:

$$\tau = \rho l^2 \left(\frac{du}{dy}\right)^2 \tag{5.4.2}$$

[1] L. Prandtl, Bericht über Untersuchungen zur ausgebildeten Turbulenz, *Z. angew. Math. u. Mech.*, vol. 5, no. 2, p. 136, 1925.

τ always acts in the sense that causes the velocity distribution to become more uniform. When Eq. (5.4.2) is compared with Eq. (3.2.1) it is found that

$$\eta = \rho l^2 \frac{du}{dy} \tag{5.4.3}$$

But η is not a fluid property as is dynamic viscosity. Rather, η depends upon the density, the velocity gradient and the mixing length l. In turbulent flow there is a violent interchange of globules of fluid except at a boundary, or very near to it, where this interchange is reduced to zero; hence, l must approach zero at a fluid boundary. The particular relationship of l to wall distance y is not given by Prandtl's derivation. Von Kármán[1] suggested, after considering similitude relationships in a turbulent fluid, that

$$l = \kappa \frac{du/dy}{d^2u/dy^2} \tag{5.4.4}$$

in which κ is a universal constant in turbulent flow, regardless of the boundary configuration or value of Reynolds number.

In turbulent flows, η, sometimes referred to as the *eddy viscosity*, is generally much larger than μ. It may be considered as a coefficient of momentum transfer, expressing the transfer of momentum from points where the concentration is high to points where it is lower. It is convenient to utilize a *kinematic eddy viscosity* $\epsilon = \eta/\rho$ which is a property of the flow alone and is analogous to kinematic viscosity.

The violent interchange of fluid globules in turbulence also tends to transfer any uneven concentration within the fluid, such as salinity, temperature, dye coloring, or sediment concentration. Studies[2] indicate that the transfer coefficient is roughly proportional to, but probably larger than, the eddy viscosity for turbulent diffusions of concentrations other than momentum.

If T is the temperature, H the heat transfer per unit area per unit time, and c_p the specific heat at constant pressure (Btu per unit of temperature per unit of mass), then

$$H = -c_p\eta \frac{\partial T}{\partial y} = -c_p\rho l^2 \frac{\partial u}{\partial y}\frac{\partial T}{\partial y} \tag{5.4.5}$$

in which $c_p\eta$ is the eddy conductivity. For transfer of material substances, such as salinity, dye, or sediment, if C is the concentration per unit volume (e.g., pounds of salt per cubic foot, number of particles per cubic foot) and c the rate of transfer per unit area per unit time (e.g.,

[1] Th. von Kármán, Turbulence and Skin Friction, *J. Aeronaut. Sci.*, vol. 1, no. 1, p. 1, 1934.

[2] See footnote, p. 189.

pounds of salt per square foot per second, number of sediment particles per square foot per second), then

$$c = -\epsilon_c \frac{\partial C}{\partial y} \tag{5.4.6}$$

and ϵ_c is proportional to ϵ.

Example 5.3: A tank of liquid containing fine solid particles of uniform size is agitated so that the kinematic eddy viscosity may be considered constant. If the fall velocity of the particles in still liquid is v_f and the concentration of particles is C_0 at $y = y_0$ (y measured from the bottom), find the distribution of solid particles vertically throughout the liquid.

By using Eq. (5.4.6) to determine the rate per second carried upward by turbulence per square foot of area at the level y, the amount per second falling across this surface by settling is equated to it for steady conditions. Those particles in the height v_f above the unit area will fall out in a second, i.e., Cv_f particles cross the level downward per second per square foot. From Eq. (5.4.6) $-\epsilon_c\, dC/dy$ particles are carried upward due to the turbulence and the higher concentration below; hence

$$Cv_f = -\epsilon_c \frac{dC}{dy}$$

or

$$\frac{dC}{C} = -\frac{v_f}{\epsilon_c}\, dy$$

After integrating

$$\ln C = -\frac{v_f}{\epsilon_c} y + \text{constant}$$

For $C = C_0$, $y = y_0$,

$$C = C_0 e^{-(v_f/\epsilon_c)(y-y_0)}$$

Velocity Distributions. Utilizing the mixing-length concept, turbulent velocity distributions are discussed for the flat plate, the pipe, and for spreading of a fluid jet. For turbulent flow over a smooth plane surface (such as the wind blowing over smooth ground) the shear stress in the fluid is constant, say τ_0. Equation (5.4.1) is applicable, but η approaches zero at the surface and μ becomes negligible away from the surface. If η is negligible for the film thickness $y = \delta$, in which μ predominates, Eq. (5.4.1) becomes

$$\frac{\tau_0}{\rho} = \frac{\mu}{\rho}\frac{u}{y} = \nu \frac{u}{y} \qquad y \leq \delta \tag{5.4.7}$$

The term $\sqrt{\tau_0/\rho}$ has the dimensions of a velocity and is called the shear-stress velocity u_*. Hence

$$\frac{u}{u_*} = \frac{u_* y}{\nu} \qquad y \leq \delta \tag{5.4.8}$$

shows a linear relation between u and y in the laminar film. For $y > \delta$,

μ is neglected, and Eq. (5.4.1) produces

$$\tau_0 = \rho l^2 \left(\frac{du}{dy}\right)^2 \tag{5.4.9}$$

Since l has the dimensions of a length and from dimensional considerations would be proportional to y (the only significant linear dimension), assume $l = \kappa y$. By substituting into Eq. (5.4.9) and rearranging,

$$\frac{du}{u_*} = \frac{1}{\kappa}\frac{dy}{y} \tag{5.4.10}$$

After integrating,

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln y + \text{constant} \tag{5.4.11}$$

It is to be noted that this value of u substituted in Eq. (5.4.4) also determines l proportional to y (d^2u/dy^2 is negative since the velocity gradient decreases as y increases). Equation (5.4.11) agrees well with experiment and, in fact, is also useful when τ is a function of y, because most of the velocity change occurs near the wall where τ is substantially constant. It is quite satisfactory to apply to turbulent flow in pipes.

Example 5.4: By integration of Eq. (5.4.11) find the relation between the average velocity V and the maximum velocity u_m in turbulent flow in a pipe.

When $y = r_0$, $u = u_m$, so

$$\frac{u}{u_*} = \frac{u_m}{u^*} + \frac{1}{\kappa} \ln \frac{y}{r_0}$$

The discharge $V\pi r_0^2$ is obtained by integrating the velocity distribution

$$V\pi r_0^2 = 2\pi \int_0^{r_0 - \delta} ur\,dr = 2\pi \int_\delta^{r_0} \left(u_m + \frac{u_*}{\kappa} \ln \frac{y}{r_0}\right)(r_0 - y)\,dy$$

The integration cannot be carried out to $y = 0$, since the equation holds in the turbulent zone only. The volume per second flowing in the laminar zone is so small that it may be neglected. Then

$$V = 2 \int_{\delta/r_0}^{1} \left(u_m + \frac{u_*}{\kappa} \ln \frac{y}{r_0}\right)\left(1 - \frac{y}{r_0}\right) d\left(\frac{y}{r_0}\right)$$

in which the variable of integration is y/r_0. By integrating,

$$V = 2\left\{u_m\left[\frac{y}{r_0} - \frac{1}{2}\left(\frac{y}{r_0}\right)^2\right] + \frac{u_*}{\kappa}\left[\frac{y}{r_0}\ln\frac{y}{r_0} - \frac{y}{r_0} - \frac{1}{2}\left(\frac{y}{r_0}\right)^2 \ln\frac{y}{r_0} + \frac{1}{4}\left(\frac{y}{r_0}\right)^2\right]\right\}_{\delta/r_0}^{1}$$

Since δ/r_0 is very small, such terms as δ/r_0 and $(\delta/r_0) \ln \delta/r_0$ are negligible ($\lim\limits_{x\to 0} x \ln x = 0$); so

$$V = u_m - \frac{3}{2}\frac{u_*}{\kappa}$$

or

$$\frac{u_m - V}{u_*} = \frac{3}{2\kappa}$$

In evaluating the constant in Eq. (5.4.11), following the methods of Bakhmeteff,[1] $u = u_w$, the "wall velocity," when $y = \delta$. According to Eq. (5.4.8)

$$\frac{u_w}{u_*} = \frac{u_* \delta}{\nu} = N \tag{5.4.12}$$

from which it is reasoned that $u^*\delta/\nu$ should have a critical value N at which flow changes from laminar to turbulent, since it is a Reynolds number in form. By substituting $u = u_w$ when $y = \delta$ into Eq. (5.4.11) and by using Eq. (5.4.12),

$$\frac{u_w}{u_*} = N = \frac{1}{\kappa} \ln \delta + \text{constant} = \frac{1}{\kappa} \ln \frac{N\nu}{u_*} + \text{constant}$$

After eliminating the constant

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{yu_*}{\nu} + N - \frac{1}{\kappa} \ln N$$

or

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{yu_*}{\nu} + A \tag{5.4.13}$$

in which $A = N - \dfrac{1}{\kappa} \ln N$ has been found experimentally by plotting u/u_* against $\ln yu_*/\nu$. For flat plates $\kappa = 0.417$, $A = 5.84$, but for smooth wall pipes Nikuradse[2] experiments yield $\kappa = 0.40$ and $A = 5.5$.

Prandtl has developed a convenient exponential velocity distribution formula for turbulent pipe flow,

$$\frac{u}{u_m} = \left(\frac{y}{r_0}\right)^n \tag{5.4.14}$$

in which n varies with Reynolds number. This empirical equation is valid only at some distance from the wall. For **R** less than 100,000, $n = 1/7$, and for greater values of **R**, n decreases. The velocity distribution equations, Eqs. (5.4.13) and (5.4.14), both have the fault of a nonzero value of du/dy at the center of the pipe.

Example 5.5: Find an approximate expression for mixing-length distribution in turbulent flow in a pipe from Prandtl's one-seventh-power law.

By applying Eq. (5.2.1) to the pipe wall, $\tau_0 = \dfrac{dp}{dl}\dfrac{r_0}{2}$. Dividing into Eq. (5.2.1) and using Eq. (5.4.2),

$$\tau = \tau_0 \left(1 - \frac{y}{r_0}\right) = \rho l^2 \left(\frac{du}{dy}\right)^2$$

[1] B. A. Bakhmeteff, "The Mechanics of Turbulent Flow," Princeton University Press, Princeton, N.J., 1941.

[2] J. Nikuradse, Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren. *VDI Forsch.* 356, 1932.

in solving for l,

$$l = \frac{u_* \sqrt{1 - y/r_0}}{du/dy}$$

From Eq. (5.4.14)

$$\frac{u}{u_m} = \left(\frac{y}{r_0}\right)^{\frac{1}{7}}$$

the approximate velocity gradient is obtained

$$\frac{du}{dy} = \frac{u_m}{r_0}\frac{1}{7}\left(\frac{y}{r_0}\right)^{-\frac{6}{7}}$$

and

$$\frac{l}{r_0} = \frac{u_*}{u_m} 7 \left(\frac{y}{r_0}\right)^{\frac{6}{7}} \sqrt{1 - y/r_0}$$

The dimensionless *velocity deficiency*, $(u_m - u)/u_*$, is a function of y/r_0 only for large Reynolds numbers (Example 5.4) whether the pipe surface is smooth or rough. From Eq. (5.4.11), by evaluating the constant for $u = u_m$ when $y = r_0$,

$$\frac{u_m - u}{u_*} = \frac{1}{\kappa} \ln \frac{r_0}{y} \tag{5.4.15}$$

For rough pipes, the velocity may be assumed to be u_w at the wall distance $y_w = m\epsilon$, in which ϵ is a typical height of the roughness projections and m is a form coefficient depending upon the nature of the roughness. By substituting into Eq. (5.4.15), then by eliminating u_m/u_* between the two equations

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{y}{\epsilon} + \frac{u_w}{u_*} - \frac{1}{\kappa} \ln m \tag{5.4.16}$$

in which the last two terms on the right-hand side are constant for a given type of roughness,

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{y}{\epsilon} + B \tag{5.4.17}$$

In Nikuradse's experiments with sand-roughened pipes constant-size sand particles (those passing a given screen and being retained on a slightly finer screen) were glued to the inside pipe walls. If ϵ represents the diameter of sand grains, experiment shows that $\kappa = 0.40$, $B = 8.48$.

Spreading of a Fluid Jet. A free jet of fluid issuing into a large space containing the same fluid otherwise at rest is acted upon by frictional forces between the jet and the surrounding fluid. The jet velocity reduces and additional fluid is set in motion in the axial direction. The pressure is substantially constant throughout the jet and surroundings so that the momentum in the axial direction remains constant. The turbulent mixing length within the jet can be taken as proportional to its breadth b (Fig. 5.15) $l = \alpha b$. Experiments show that $\alpha = \frac{1}{8}$. A conclusion from the constancy of momentum within the jet is that the maximum velocity (at the center line) varies inversely as the axial distance x along the jet. Both theory[1] and experiment show that the breadth varies linearly with axial distance, $b = x/8$. Turbulent shear forces reduce the jet velocity within the central cone, and equal turbulent shear forces act to increase velocity in the outer portions of the jet.

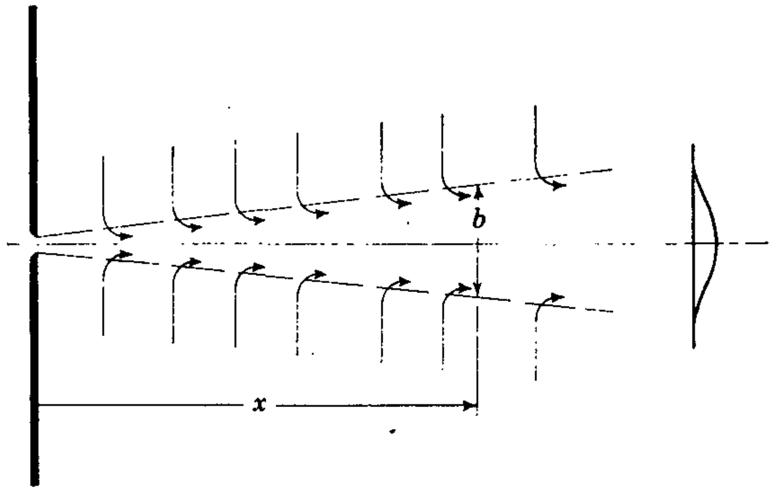


FIG. 5.15. Fluid jet issuing into same fluid medium.

5.5. Boundary-layer Concepts. In 1904 Prandtl[2] developed the concept of the boundary layer. It provides an important link between ideal fluid flow and real fluid flow. *For fluids having relatively small viscosity, the effect of internal friction in a fluid is appreciable only in a narrow region surrounding the fluid boundaries.* From this hypothesis, the flow outside of the narrow region near the solid boundaries may be considered as ideal flow or potential flow. Relations within the boundary-layer region may be computed from the general equations for viscous fluids, but use of the momentum equation permits the developing of approximate equations for boundary-layer growth and drag. In this section the boundary layer is described and the momentum equation applied to it. Two-dimensional flow along a flat plate is studied by means of the momentum relationships for both the laminar and the turbulent boundary layer. The phenomenon of separation of the boundary layer and formation of the wake is described.

Description of the Boundary Layer. When motion is started in a fluid having very small viscosity, the flow is essentially irrotational in the

[1] W. Tollmien, Berechnung turbulenter Ausbreitungsvorgange, *Z. angew. Math. u. Mech.*, vol. 6, p. 468, 1926.

[2] L. Prandtl, Über Flussigkeitsbewegung bei sehr kleiner Reibung, *Verhandl. III Intern. Math.-Kongr.*, Heidelberg, 1904.

first instants. Since the fluid at the boundaries has zero velocity relative to the boundaries, there is a steep velocity gradient from the boundary into the flow. This velocity gradient in a real fluid sets up near the boundary shear forces that reduce the flow relative to the boundary. That fluid layer which has had its velocity affected by the boundary shear is called the *boundary layer.* The velocity in the boundary layer approaches the velocity in the main flow asymptotically. The boundary layer is very thin at the upstream end of a streamlined body at rest in an otherwise uniform flow. As this layer moves along the body, the continual action of shear stress tends to slow down additional fluid particles, causing the thickness of the boundary layer to increase with distance from the upstream point. The fluid in the layer is also subjected to a pressure gradient, determined from the potential flow, that increases the momentum of the layer if the pressure decreases downstream and decreases its momentum if the pressure increases downstream (*adverse* pressure gradient). The flow outside the boundary layer may also bring momentum into the layer.

For smooth upstream boundaries, the boundary layer starts out as a *laminar boundary layer* in which the fluid particles move in smooth layers. As the thickness of the laminar boundary layer increases, it becomes unstable and finally transforms into a *turbulent boundary layer* in which the fluid particles move in haphazard paths, although their velocity has been reduced by the action of viscosity at the boundary. When the boundary layer has become turbulent, there is still a very thin layer next to the boundary that has laminar motion. It is called the *laminar sub-layer.*

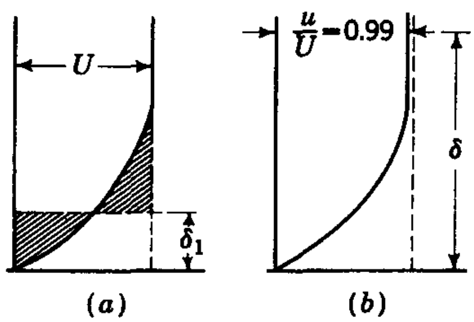


FIG. 5.16. Definitions of boundary-layer thickness.

Various definitions of boundary-layer thickness δ have been suggested. The most basic definition refers to the displacement of the main flow due to slowing down of fluid particles in the boundary zone. This thickness δ_1, called the *displacement thickness,* is expressed by

$$U\delta_1 = \int_0^\delta (U - u)\, dy \qquad (5.5.1)$$

in which δ is that value of y at which $u = U$. In Fig. 5.16*a*, the line $y = \delta_1$ is drawn so that the shaded areas are equal. This distance is, in itself, not the distance that is strongly affected by the boundary. In fact, that region is frequently taken as $3\delta_1$. Another definition, expressed by Fig. 5.16*b*, is the distance to the point where $u/U = 0.99$.

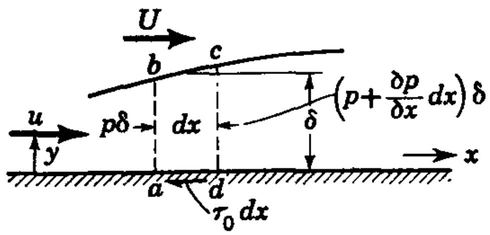


FIG. 5.17. Segment of boundary layer.

Momentum Equation Applied to the Boundary Layer. By following Von Kármán's method,[1] the principle of momentum may be applied directly to the boundary in steady flow. In a small segment of the layer (Fig. 5.17) where *abcd* is fixed, the resultant force in the x-direction must equal the net efflux of momentum across the surface of the element in unit time. The resultant force on the element is, for unit breadth,

$$-\tau_0\,dx - \frac{\partial p}{\partial x}\,dx\,\delta$$

The net mass outflow through *cd* and *ab* is

$$\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy\,dx$$

This mass must be entering through *bc* and, hence, brings into the element in unit time the momentum

$$U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy\,dx$$

The excess of momentum per unit time leaving *cd* over that entering *ab* is

$$\frac{\partial}{\partial x}\int_0^\delta \rho u^2\,dy\,dx$$

When the force and momentum terms are assembled and dx is divided out,

$$-\tau_0 - \frac{\partial p}{\partial x}\,\delta = \frac{\partial}{\partial x}\int_0^\delta \rho u^2\,dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u\,dy \qquad (5.5.2)$$

For a flat plate, $\partial p/\partial x = 0$, and U is constant. The equation reduces to

$$\tau_0 = \frac{\partial}{\partial x}\int_0^\delta \rho(U-u)u\,dy \qquad (5.5.3)$$

Two-dimensional Flow along a Flat Plate. Calculations of boundary-layer growth, in general, are very complex and require advanced mathematical treatment. As a simple example, the case of steady flow parallel to a flat plate is worked out by use of the momentum relationship.

Laminar Boundary Layer. Equation (5.5.3) may be written

$$\tau_0 = \frac{\partial}{\partial x}\int_0^h \rho(U-u)u\,dy \qquad (5.5.4)$$

[1] Th. von Kármán, On Laminar and Turbulent Friction, *Z. angew. Math. u. Mech.*, vol. 1, pp. 235–236, 1921.

in which h is greater than δ but is independent of x. This is permissible because the integrand is zero for $y > \delta$, as $u = U$. The momentum equation gives no information regarding the velocity distribution in the boundary layer. For an assumed distribution, which satisfies the boundary conditions $u = 0$, $y = 0$ and $u = U$, $y = \delta$, the boundary-layer thickness as well as the shear at the boundary can be determined. The velocity distribution is assumed to have the same form at each value of x,

$$\frac{u}{U} = F\left(\frac{y}{\delta}\right) = F(\eta) \qquad \eta = \frac{y}{\delta}$$

when δ is unknown. Prandtl assumed that

$$\frac{u}{U} = F = \frac{3}{2}\eta - \frac{\eta^3}{2} \qquad 0 \le y \le \delta$$

and

$$F = 1 \qquad \delta \le y$$

which satisfies the boundary conditions. Equation (5.5.4) may be rewritten

$$\tau_0 = \rho U^2 \frac{\partial \delta}{\partial x} \int_0^1 \left(1 - \frac{u}{U}\right) \frac{u}{U}\, d\eta$$

and

$$\tau_0 = \rho U^2 \frac{\partial \delta}{\partial x} \int_0^1 \left(1 - \frac{3}{2}\eta + \frac{\eta^3}{2}\right)\left(\frac{3}{2}\eta - \frac{\eta^3}{2}\right) d\eta = 0.139\rho U^2 \frac{\partial \delta}{\partial x}$$

At the boundary

$$\tau_0 = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{U}{\delta}\frac{\partial F}{\partial \eta}\bigg|_{\eta=0} = \mu \frac{U}{\delta}\frac{\partial}{\partial \eta}\left(\frac{3}{2}\eta - \frac{\eta^3}{2}\right)\bigg|_{\eta=0} = \frac{3}{2}\mu\frac{U}{\delta} \qquad (5.5.5)$$

In equating the two expressions for τ_0,

$$\frac{3}{2}\mu\frac{U}{\delta} = 0.139\rho U^2 \frac{\partial \delta}{\partial x}$$

By rearranging,

$$\delta\, d\delta = 10.78 \frac{\mu\, dx}{\rho U}$$

since δ is a function of x only in this equation. After integrating,

$$\frac{\delta^2}{2} = 10.78 \frac{\nu}{U} x + \text{constant}$$

If $\delta = 0$, for $x = 0$, the constant of integration is zero. In solving for δ/x,

$$\frac{\delta}{x} = 4.65\sqrt{\frac{\nu}{Ux}} = \frac{4.65}{\sqrt{\mathbf{R}_x}} \qquad (5.5.6)$$

in which $\mathbf{R}_x = Ux/\nu$ is a Reynolds number based on the distance x from the leading edge of the plate. This equation for boundary-layer thickness in laminar flow shows that δ increases as the square root of the distance from the leading edge.

After substituting the value of δ into Eq. (5.5.5),

$$\tau_0 = 0.322 \sqrt{\frac{\mu\rho U^3}{x}} \tag{5.5.7}$$

The shear stress varies inversely as the square root of x and directly as the three-halves power of the velocity. The drag on one side of the plate, of unit width, is

$$\text{Drag} = \int_0^l \tau_0\, dx = 0.644 \sqrt{\mu\rho U^3 l} \tag{5.5.8}$$

The selecting of other velocity distributions does not radically alter these results. The exact solution, worked out by Blasius from the general equations of viscous motion, yields the coefficients 0.332 and 0.664 for Eqs. (5.5.7) and (5.5.8), respectively.

The drag can be expressed in terms of a drag coefficient C_D times the stagnation pressure $\rho U^2/2$ and the area of plate l (per unit breadth),

$$\text{Drag} = C_D \frac{\rho U^2}{2} l$$

in which, for the laminar boundary layer,

$$C_D = \frac{1.328}{\sqrt{\mathbf{R}_l}} \tag{5.5.9}$$

and $\mathbf{R}_l = Ul/\nu$.

When the Reynolds number for the plate reaches a value between 500,000 and 1,000,000, the boundary layer becomes turbulent. Figure 5.18 indicates the growth and transition from laminar to turbulent boundary layer. The critical Reynolds number depends upon the initial turbulence of the fluid stream, upon the upstream edge of the plate, and upon the plate roughness.

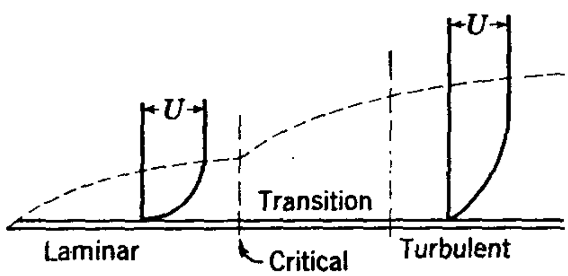


FIG. 5.18. Boundary-layer growth. (The vertical scale is greatly enlarged.)

Turbulent Boundary Layer. The momentum equation can be used to determine turbulent boundary-layer growth and shear stress along a smooth plate in a manner analogous to the treatment of the laminar boundary layer. The universal velocity-distribution law for smooth pipes, Eq. (5.4.13), provides the best basis but the calculations are involved. A simpler approach is to use Prandtl's *one-seventh-power* law. It is $u/u_{max} = (y/r_0)^{\frac{1}{7}}$, in which y is measured from the wall of the pipe and r_0 is the pipe radius. Applying it to flat plates produces

$$F = \frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} = \eta^{\frac{1}{7}}$$

and

$$\tau_0 = 0.0228\rho U^2 \left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} \tag{5.5.10}$$

in which the latter expression is the shear stress at the wall of a smooth plate with a turbulent boundary layer. With the same method as that used to calculate the laminar boundary layer,

$$\tau_0 = \rho U^2 \frac{d\delta}{dx} \int_0^1 (1 - \eta^{\frac{1}{7}})\eta^{\frac{1}{7}}\, d\eta = \frac{7}{72}\rho U^2 \frac{d\delta}{dx} \tag{5.5.11}$$

By equating the expressions for shear stress, the differential equation for boundary-layer thickness δ is obtained,

$$\delta^{\frac{1}{4}}\, d\delta = 0.234 \left(\frac{\nu}{U}\right)^{\frac{1}{4}} dx$$

After integrating, and then by assuming that the boundary layer is turbulent over the whole length of the plate so that the initial conditions $x = 0$, $\delta = 0$ can be used,

$$\delta^{\frac{5}{4}} = 0.292 \left(\frac{\nu}{U}\right)^{\frac{1}{4}} x$$

After solving for δ,

$$\delta = 0.37 \left(\frac{\nu}{U}\right)^{\frac{1}{5}} x^{\frac{4}{5}} = \frac{0.37x}{(Ux/\nu)^{\frac{1}{5}}} = \frac{0.37x}{\mathbf{R}_x^{\frac{1}{5}}} \tag{5.5.12}$$

The thickness increases more rapidly in the turbulent boundary layer. In it the thickness increases as $x^{\frac{4}{5}}$, but in the laminar boundary layer δ varies as $x^{\frac{1}{2}}$.

To determine the drag on a smooth, flat plate, δ is eliminated in Eqs. (5.5.10) and (5.5.12), and

$$\tau_0 = 0.029\rho U^2 \left(\frac{\nu}{Ux}\right)^{\frac{1}{5}} \tag{5.5.13}$$

The drag for unit width on one side of the plate is

$$\text{Drag} = \int_0^l \tau_0\, dx = 0.036\rho U^2 l \left(\frac{\nu}{Ul}\right)^{\frac{1}{5}} = \frac{0.036\rho U^2 l}{\mathbf{R}_l^{\frac{1}{5}}} \tag{5.5.14}$$

In terms of the drag coefficient

$$C_D = 0.072\mathbf{R}_l^{-\frac{1}{5}} \tag{5.5.15}$$

in which $\mathbf{R}_l$ is the Reynolds number based on the length of plate.

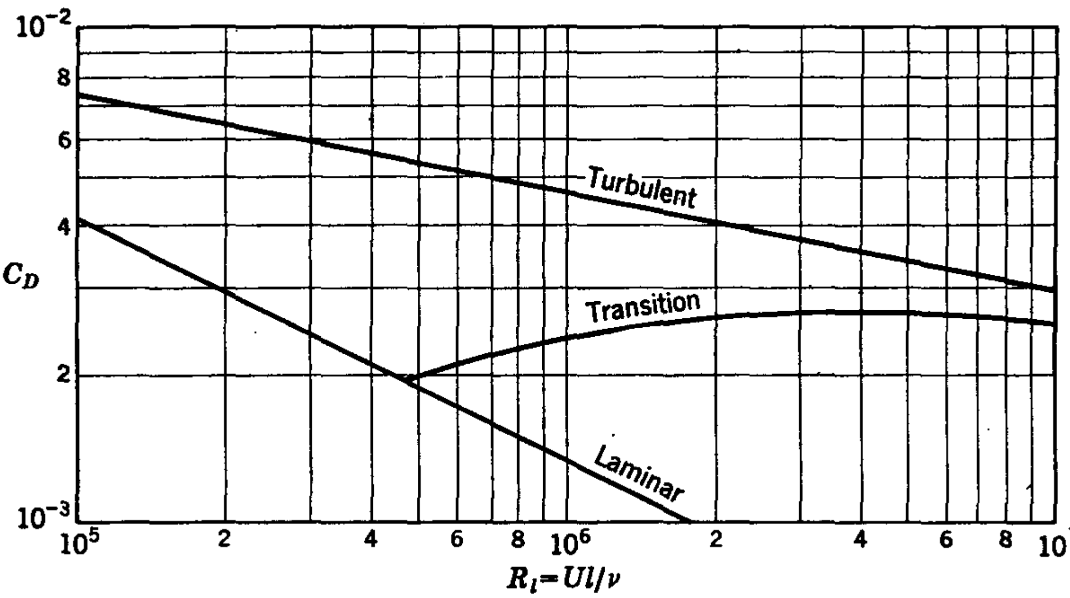
The above equations are valid only for the range in which the Blasius resistance equation holds. For larger Reynolds numbers in smooth-pipe flow, the exponent in the velocity-distribution law is reduced. At $\mathbf{R} = 400{,}000$, $n = \frac{1}{8}$, and for $\mathbf{R} = 4{,}000{,}000$, $n = \frac{1}{10}$. The drag law, Eq. (5.5.14), is valid for a range

$$5 \times 10^5 < \mathbf{R}_l < 10^7$$

Experiment shows that the drag is slightly higher than is predicted by Eq. (5.5.15),

$$C_D = 0.074\mathbf{R}_l^{-\frac{1}{5}} \tag{5.5.16}$$

The boundary layer is actually laminar along the upstream part of the



Laminar $C_D = \dfrac{1.328}{\sqrt{R_l}}$, transition $C_D = \dfrac{0.074}{R_l^{1/5}} - \dfrac{1700}{R_l}$, turbulent $C_D = \dfrac{0.074}{R_l^{1/5}}$

FIG. 5.19. The drag law for smooth plates.

plate. Prandtl[1] has subtracted the laminar portion, producing the equation

$$C_D = 0.074\mathbf{R}_l^{-\frac{1}{5}} - \frac{1700}{\mathbf{R}_l} \qquad 5 \times 10^5 < \mathbf{R}_l < 10^7 \tag{5.5.17}$$

In Fig. 5.19 a log-log plot of C_D vs. $\mathbf{R}_l$ shows the trend of the drag coefficients.

Use of the logarithmic velocity distribution for pipes produces

$$C_D = \frac{0.455}{(\log_{10} R_l)^{2.58}} \qquad 10^6 < \mathbf{R}_l < 10^9 \tag{5.5.18}$$

in which the constant term has been selected for best agreement with experimental results.

[1] L. Prandtl, Über den Reibungswiderstand strömender Luft, *Results Aerodynamic Test Inst.* (Gottingen), III. Lieferung, 1927.

Example 5.6: A smooth, flat plate 10 ft wide and 100 ft long is towed through still water at 68°F with a speed of 20 ft/sec. Determine the drag on one side of the plate and the drag on the first 10 ft of the plate.

For the whole plate

$$\mathbf{R}_l = \frac{100 \times 20 \times 1.935}{0.01/479} = 1.85 \times 10^8$$

From Eq. (5.5.18)

$$C_D = \frac{0.455}{[\log_{10}(1.85 \times 10^8)]^{2.58}} = \frac{0.455}{(8.2675)^{2.58}} = 0.00196$$

The drag on one side is

$$\text{Drag} = C_D b l \frac{\rho U^2}{2} = 0.00196 \times 10 \times 100 \times \frac{1.935}{2} \times \overline{20}^2 = 760 \text{ lb}$$

in which b is the plate width. If the critical Reynolds number occurs at 5×10^5, the length l_0 to the transition is

$$\frac{l_0 \times 20 \times 1.935}{0.01/479} = 5 \times 10^5 \qquad l_0 = 0.27 \text{ ft}$$

For the first 10 ft of the plate, $\mathbf{R}_l = 1.85 \times 10^7$, $C_D = 0.00274$, and

$$\text{Drag} = 0.00274 \times 10 \times 10 \times \frac{1.935}{2} \times \overline{20}^2 = 106 \text{ lb}$$

Calculation of the turbulent boundary layer over rough plates proceeds in similar fashion, starting with the rough-pipe tests using sand roughnesses. At the upstream end of the flat plate, the flow may be laminar; then, in the turbulent boundary layer where the boundary layer is still thin and the ratio of roughness height to boundary-layer thickness ϵ/δ is significant, the region of fully developed roughness occurs, and the drag is proportional to the square of the velocity. For long plates, this region is followed by a transition region where ϵ/δ becomes increasingly smaller, and eventually the plate becomes hydraulically smooth, i.e., the loss would not be reduced by reducing the roughness. Prandtl and Schlichting[1] have carried through these calculations, which are too complicated for reproduction here.

Separation. Wake. Along a flat plate the boundary layer continues to grow in the downstream direction, regardless of the length of the plate, when the pressure gradient remains zero. With the pressure decreasing in the downstream direction, as in a conical reducing section, the boundary layer tends to be reduced in thickness.

For *adverse* pressure gradients, i.e., with pressure increasing in the downstream direction, the boundary layer thickens rapidly. The adverse

[1] L. Prandtl and H. Schlichting, Das Widerstandsgesetz rauher Platten, *Werft, Reederei, Hafen*, p. 1, 1934. See also *NACA Tech. Mem.* 1218, part II.

gradient plus the boundary shear decrease the momentum in the boundary layer, and if they both act over a sufficient distance, they cause the boundary layer to come to rest. This phenomenon is called *separation.* Figure 5.20 illustrates this case. The boundary streamline must leave the boundary at the separation point, and downstream from this point the adverse pressure gradient causes backflow near the wall. This

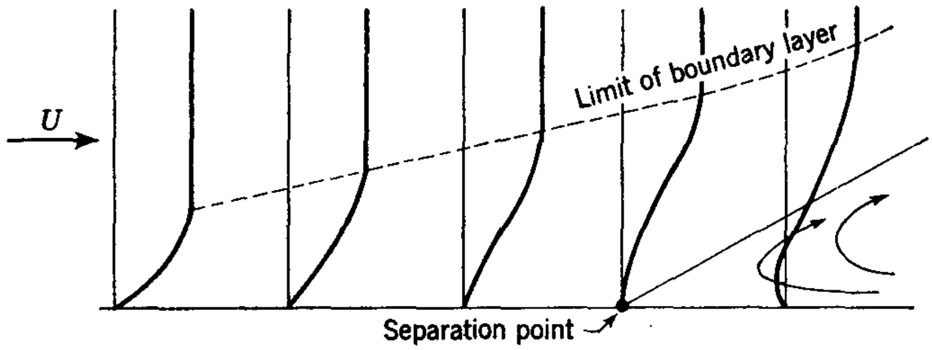


FIG. 5.20. Effect of adverse pressure gradient on boundary layer. Separation.

region downstream from the streamline that separates from the boundary is known as the *wake.* The effect of separation is to decrease the net amount of flow work that can be done by a fluid element on the surrounding fluid at the expense of its kinetic energy, with the net result that pressure recovery is incomplete and flow losses (drag) increase.

Streamlined bodies (Fig. 5.21) are designed so that the separation point occurs as far downstream along the body as possible. If separation can be avoided, the boundary layer remains thin, and the pressure is almost recovered downstream along the body. The only loss or drag is due to shear stress in the boundary layer, called *skin friction.* In the wake, the pressure is not recovered and a *pressure drag* results. Reduction of wake reduces the pressure drag on a body. In general, the drag is caused by both skin friction and pressure drag.

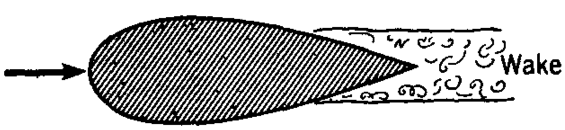


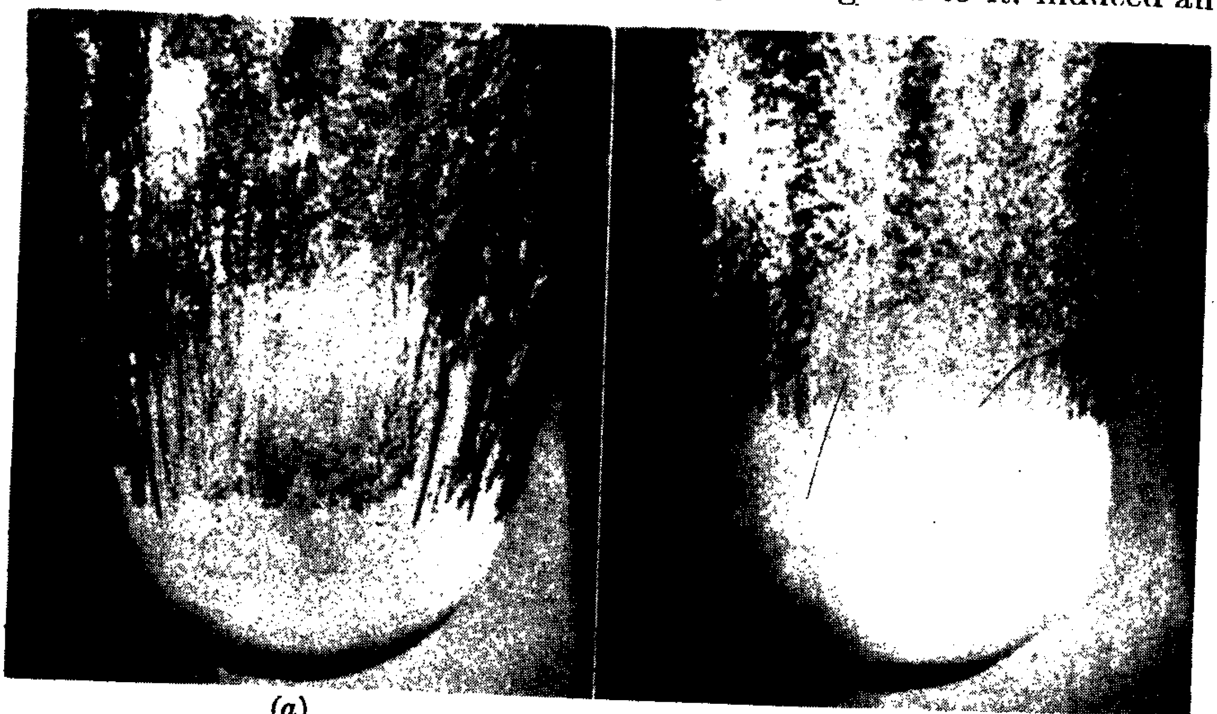
FIG. 5.21. Streamlined body.

Flow around a sphere is an excellent example of the effect of separation on drag. For very small Reynolds numbers, $VD/\nu < 1$, the flow is everywhere nonturbulent, and the drag is referred to as *deformation drag.* Stokes' law[1] gives the drag force for this case. For large Reynolds numbers, the flow may be considered potential flow except in the boundary layer and the wake. The boundary layer forms at the forward stagnation point and is generally laminar. In the laminar boundary layer, an adverse pressure gradient causes separation more readily than in a turbulent boundary layer, because of the small amount of momentum

[1] See Sec. 5.6.

brought into the laminar layer. If separation occurs in the laminar boundary layer, the location is farther upstream on the sphere than it is when the boundary layer becomes turbulent first and then separation occurs.

In Fig. 5.22 this is graphically portrayed by the photographs of the two spheres dropped into water at 25 ft/sec. In *a*, separation occurs in the laminar boundary layer that forms along the smooth surface and causes a very large wake with a resulting large pressure drag. In *b*, the nose of the sphere, roughened by sand glued to it, induced an



(*a*) (*b*)

FIG. 5.22. Shift in separation point due to induced turbulence. (*a*) 8.5-in. bowling ball, smooth surface, 25 ft/sec entry velocity into water. (*b*) Same except for 4-in.-diameter patch of sand on nose. (*Official U.S. Navy photograph made at Navy Ordnance Test Station, Pasadena Annex.*)

early transition to turbulent boundary layer before separation occurred. The high momentum transfer in the turbulent boundary layer delayed the separation so that the wake is substantially reduced, resulting in a total drag on the sphere less than half that occurring in *a*.

A plot of drag coefficient against Reynolds number, (Fig. 5.23) for smooth spheres shows that the shift to turbulent boundary layer (before separation) occurs by itself at a sufficiently high Reynolds number, as evidenced by the sudden drop in drag coefficient. The exact Reynolds number for the sudden shift depends upon the smoothness of the sphere and upon the turbulence in the fluid stream. In fact, the sphere is frequently used as a turbulence meter by determining the Reynolds number at which the drag coefficient is 0.30, a point located in the center of the sudden drop (Fig. 5.23). By use of the hot-wire anemometer, Dryden[1] has correlated the turbulence level of the fluid stream to the Reynolds number for the sphere at $C_D = 0.30$. The greater the turbulence of the fluid stream, the smaller the Reynolds number for shift in separation point.

5.6. Drag on Immersed Bodies. The principles of potential flow around bodies are developed in Chap. 7, and principles of the boundary layer, separation, and wake in the section preceding this one (Sec. 5.5).

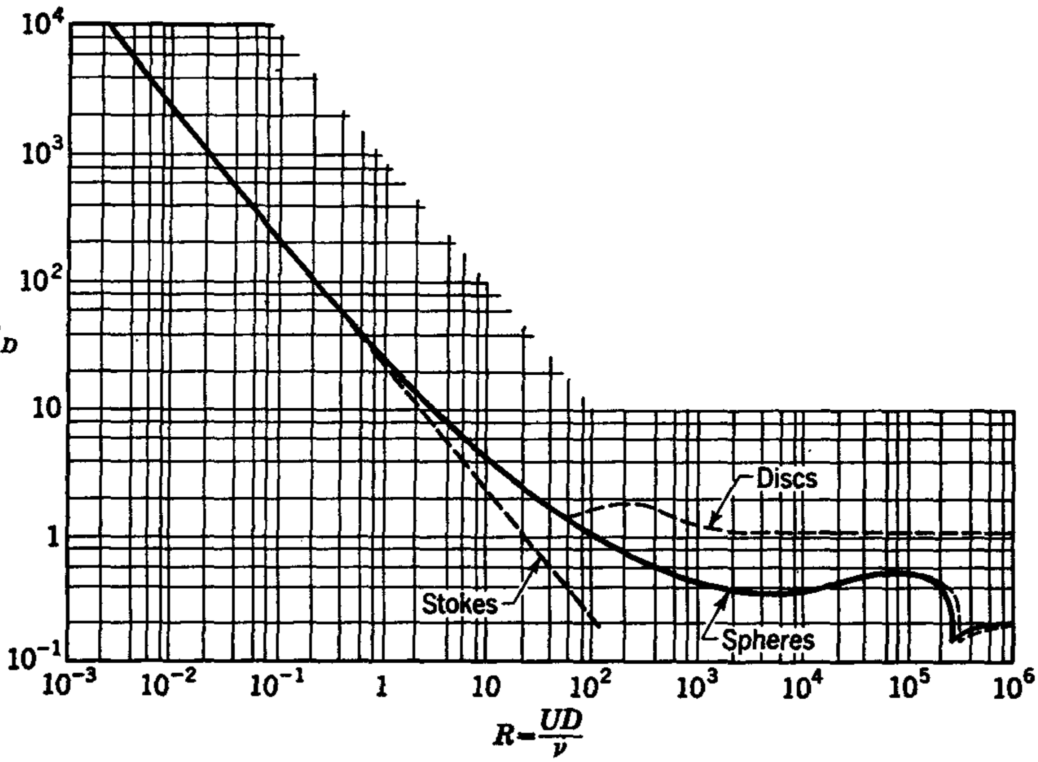


FIG. 5.23. Drag coefficients for spheres and circular disks.

In this section drag is defined, some experimental drag coefficients are listed, the effect of compressibility on drag is discussed, and Stokes' law is presented. Lift is defined and the lift and drag coefficients for an airfoil are given.

Drag is defined as the force component, parallel to the relative approach velocity, exerted on the body by the moving fluid. The drag-coefficient curves for spheres and circular disks are shown in Fig. 5.23. In Fig. 5.24 the drag coefficient for an infinitely long circular cylinder (two-dimensional case) is plotted against Reynolds number. This case also has the sudden shift in separation point as in the case of the sphere. In each case, the drag coefficient C_D is defined by

$$\text{Drag} = C_D A \frac{\rho U^2}{2}$$

in which A is the projected area of the body on a plane normal to the flow.

[1] H. Dryden, Reduction of Turbulence in Wind Tunnels, *NACA Tech. Rept.* 392, 1931.

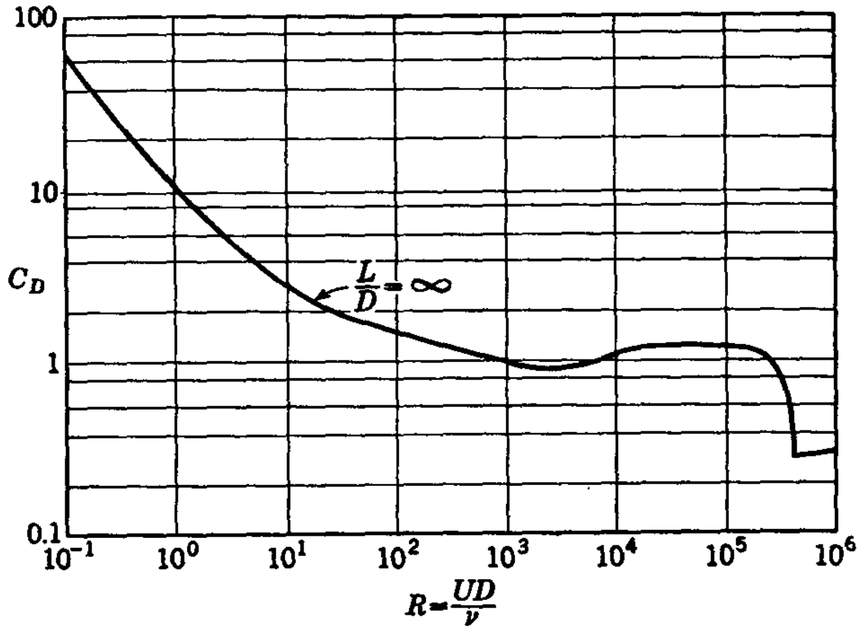


Fig. 5.24. Drag coefficients for circular cylinders.

Table 5.1. Typical Drag Coefficients for Various Cylinders in Two-dimensional Flow†

Body shape		C_D	Reynolds number
Circular cylinder	→	1.2	10^4 to 1.5×10^5
Elliptical cylinder	→ 2:1	0.6	4×10^4
		0.46	10^5
	→ 4:1	0.32	2.5×10^4 to 10^5
	→ 8:1	0.29	2.5×10^4
		0.20	2×10^5
Square cylinder	→	2.0	3.5×10^4
	→	1.6	10^4 to 10^5
Triangular cylinders	→ 120°	2.0	10^4
	→ 120°	1.72	10^4
	→ 90°	2.15	10^4
	→ 90°	1.60	10^4
	→ 60°	2.20	10^4
	→ 60°	1.39	10^4
	→ 30°	1.8	10^5
	→ 30°	1.0	10^5
Semitubular	→	2.3	4×10^4
	→	1.12	4×10^4

† Data from W. F. Lindsey, *NACA Tech. Rept.* 619, 1938.

In Table 5.1 typical drag coefficients are shown for several cylinders. In general, the values given are for the range of Reynolds number in which the coefficient changes little with Reynolds number.

A typical lift and drag curve for an airfoil section is shown in Fig. 5.25. Lift is the fluid-force component on a body at right angles to the relative approach velocity. The lift coefficient C_L is defined by

$$\text{Lift} = C_L A \frac{\rho U^2}{2}$$

in which A refers to the chord length times the wing length for lift and drag for airfoil sections.

Effect of Compressibility on Drag. To determine drag in gas flow the effects of compressibility, as expressed by the *Mach* number, are more important than Reynolds number. The Mach number $\mathbf{M}$ is defined as

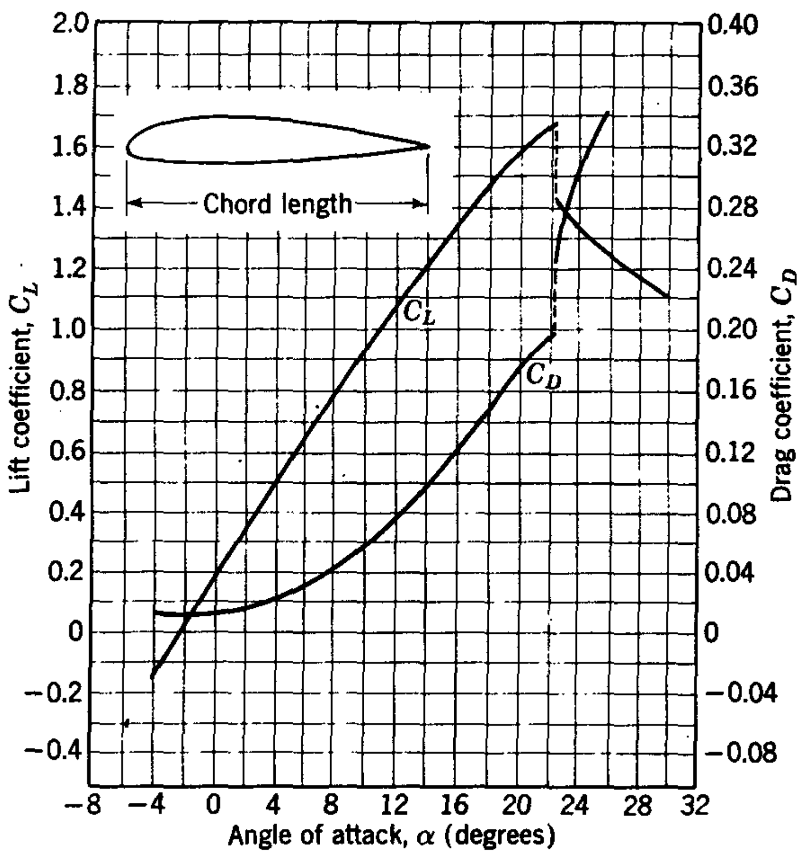


FIG. 5.25. Typical lift and drag coefficients for an airfoil.

the ratio of fluid velocity to velocity of sound in the fluid medium. When flow is at the critical velocity c, it has exactly the speed of the sound wave so small pressure waves cannot travel upstream. For this condition $\mathbf{M} = 1$. When $\mathbf{M}$ is greater than unity, the flow is supersonic; and when $\mathbf{M}$ is less than unity, it is subsonic.

Any small disturbance is propagated with the speed of sound, Sec. 6.2. For example, a disturbance in still air travels outward as a spherical pressure wave. When the source of the disturbance moves with a velocity less than c, as in Fig. 5.26*a*, the wave travels ahead of the disturbing body and gives the fluid a chance to adjust itself to the oncoming body. By the time the particle has moved a distance Vt, the disturbance wave has moved out as far as $r = ct$ from the point O. As the disturbing body moves along, new spherical waves are sent out, but in all subsonic cases

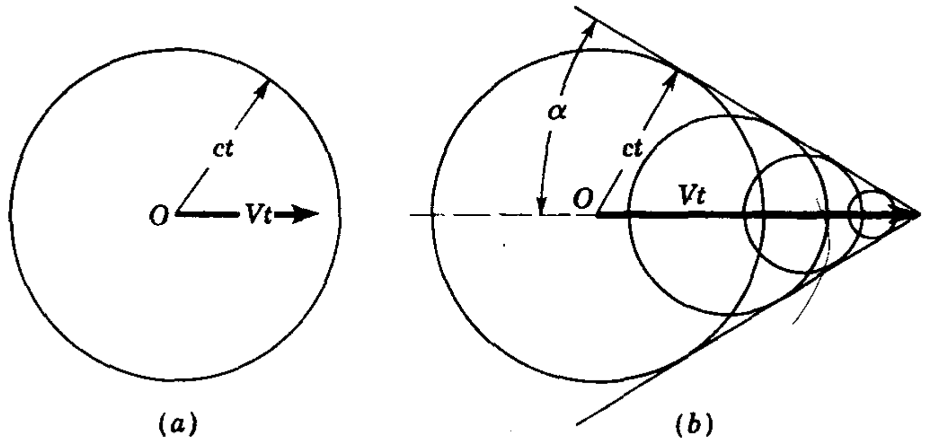


FIG. 5.26. Wave propagation produced by a particle moving at (*a*) subsonic velocity and (*b*) supersonic velocity.

they are contained within the initial spherical wave shown. In supersonic motion of a particle (Fig. 5.26*b*) the body moves faster than the spherical waves emitted from it, yielding a cone-shaped wave front with vertex at the body, as shown. The half angle of cone α is called the *Mach angle*,

$$\alpha = \sin^{-1}\frac{ct}{Vt} = \sin^{-1}\frac{c}{V}$$

The conical pressure front extends out behind the body and is called a *Mach wave*, Sec. 6.4. There is a sudden small change in velocity and pressure across a Mach wave.

The drag on bodies varies greatly with the Mach number and becomes relatively independent of the Reynolds number when compressibility effects become important. In Fig. 5.27 the drag coefficients for four projectiles are plotted against Mach number.

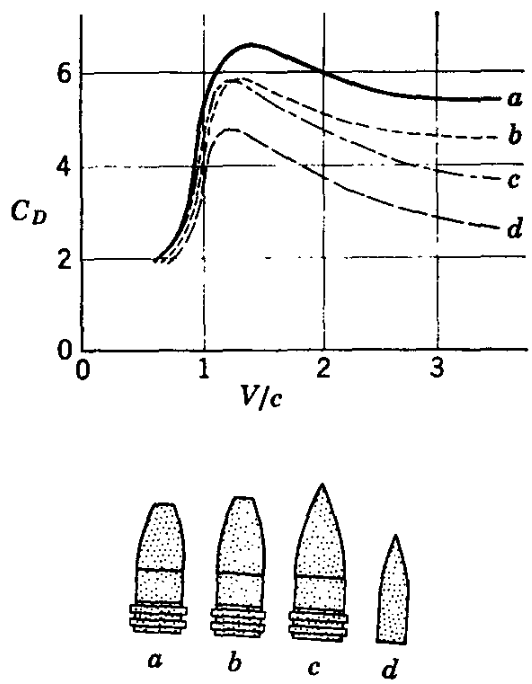


FIG. 5.27. Drag coefficients for projectiles as a function of Mach number. (*From L. Prandtl, "Abriss der Strömungslehre," Friedrig Vieweg und Söhne, Brunswick, Germany,* 1935.)

For low Mach numbers, a body should be rounded in front, with a blunt nose and a long, tapering afterbody for minimum drag. For high Mach numbers (0.7 and over), the drag rises very rapidly owing to formation of the vortices behind the projectile and to formation of the shock waves; the body should have a tapered nose or thin forward edge. As the Mach numbers increase, the curves tend to drop and to approach a constant value asymptotically. This appears to be due to the fact that the reduction of pressure behind the projectile is limited to absolute zero, and hence its contribution to the total drag tends to become constant. The pointed projectile creates a narrower shock front that tends to reduce the limiting value of the drag coefficient.

Stokes' Law. The flow of a viscous incompressible fluid around a sphere has been solved by Stokes[1] for values of Reynolds number UD/ν less than 1. The derivation is beyond the scope of this treatment; the results, however, are of value in such problems as the settling of dust particles. Stokes found the drag (force exerted on the sphere by flow of fluid around it) to be

$$\text{Drag} = 6\pi a\mu U$$

in which a is the radius of sphere and U the velocity of sphere relative to the fluid at a great distance. To find the terminal velocity for a sphere dropping through a fluid that is otherwise at rest, the buoyant force plus the drag force must just equal its weight, or

$$\tfrac{4}{3}\pi a^3\gamma + 6\pi a\mu U = \tfrac{4}{3}\pi a^3\gamma_s$$

in which γ is the specific weight of liquid and γ_s is the specific weight of the sphere. By solving for U, the terminal velocity is found to be

$$U = \frac{2}{9}\frac{a^2}{\mu}(\gamma_s - \gamma) \tag{5.6.1}$$

The straight-line portion of Fig. 5.23 represents Stokes' law.

5.7. Resistance to Turbulent Flow in Open and Closed Conduits. In steady turbulent incompressible flow in conduits of constant cross section (steady uniform flow) the wall shear stress varies closely proportional to the square of the velocity,

$$\tau_0 = \lambda \frac{\rho}{2} V^2 \tag{5.7.1}$$

in which λ is a dimensionless coefficient. For open channels and noncircular closed conduits the shear stress is not constant over the surface. In these cases, τ_0 is used as the average wall shear stress. Secondary flows[2] occurring in noncircular conduits act to equalize the wall shear stress. The wall shear-stress forces in steady flow are balanced either by pressure forces, by the axial weight component of fluid in the conduit, or by both forces (Fig. 5.28). The equilibrium expression, written in

[1] G. Stokes, *Trans. Cambridge Phil. Soc.*, vol. 8, 1845; vol. 9, 1851.

[2] Secondary flows, not wholly understood, are transverse components that cause the main central flow to spread out into corners or near walls.

the axial direction, is

$$(p_1 - p_2)A + \gamma A\,\Delta z = \tau_0 LP$$

in which $\Delta z = L \sin\theta$ and P is the *wetted perimeter* of the conduit, i.e., the portion of the perimeter where the wall is in contact with the fluid (free

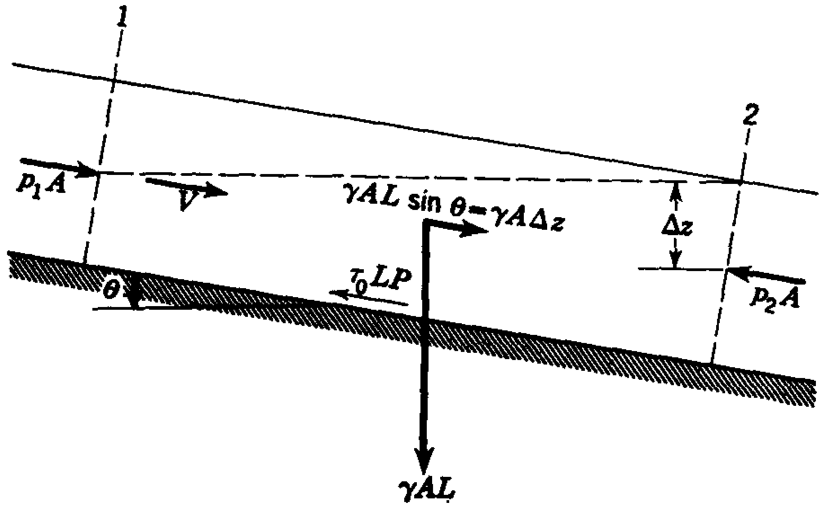


FIG. 5.28. Axial forces on free body of fluid in a conduit.

liquid surface excluded). The ratio A/P is called the *hydraulic radius* of the conduit R. If $p_1 - p_2 = \Delta p$,

$$\frac{\Delta p + \gamma\,\Delta z}{L} = \frac{\tau_0}{R} = \frac{\lambda\rho V^2}{2R} \tag{5.7.2}$$

or, when divided through by γ, if $h_f = (\Delta p + \gamma\,\Delta z)/\gamma$ be the losses per unit weight,

$$\frac{h_f}{L} = S = \frac{\lambda}{R}\frac{V^2}{2g}$$

in which S represents the losses per unit weight per unit length. After solving for V

$$V = \sqrt{\frac{2g}{\lambda}}\sqrt{RS} = C\sqrt{RS} \tag{5.7.3}$$

This is the Chézy formula, in which originally the Chézy coefficient C was thought to be a constant for any size conduit or wall surface condition. Various formulas for C are now generally used.

For pipes, when $\lambda = f/4$, and $R = D/4$ the Darcy-Weisbach equation is obtained,

$$h_f = f\frac{L}{D}\frac{V^2}{2g} \tag{5.7.4}$$

in which D is the inside pipe diameter. This equation may be applied to open channels in the form

$$V = \sqrt{\frac{8g}{f}}\sqrt{RS} \tag{5.7.5}$$

with values of f determined from pipe experiments.

5.8. Steady Uniform Flow in Open Channels. For incompressible, steady flow at constant depth in a prismatic open channel, the *Manning* formula is widely used. It can be obtained from the Chézy formula [Eq. (5.7.3)] by setting

$$C = \frac{1.49}{n} R^{\frac{1}{6}} \tag{5.8.1}$$

so

$$V = \frac{1.49}{n} R^{\frac{2}{3}} S^{\frac{1}{2}} \tag{5.8.2}$$

which is the Manning formula. V is the average velocity at a cross section, R the hydraulic radius (Sec. 5.7), and S the losses per unit weight per unit length of channel or the slope of the bottom of the channel. It is also the slope of the water surface, which is parallel to the channel bottom. The coefficient n was thought to be an absolute roughness coefficient, i.e., dependent upon surface roughness only, but actually depends upon the size and shape of channel cross section in some unknown manner. Values of the coefficient n, determined by many tests on actual canals, are given in Table 5.2. Equation (5.8.2) must have velocity in feet per second and R in feet for use with the values in Table 5.2.

TABLE 5.2. AVERAGE VALUES OF THE MANNING ROUGHNESS FACTOR FOR VARIOUS BOUNDARY MATERIALS

Boundary material	*Manning n*
Planed wood	0.012
Unplaned wood	0.013
Finished concrete	0.012
Unfinished concrete	0.014
Cast iron	0.015
Brick	0.016
Riveted steel	0.018
Corrugated metal	0.022
Rubble	0.025
Earth	0.025
Earth, with stones or weeds	0.035
Gravel	0.029

When Eq. (5.8.2) is multiplied by the cross-sectional area A, the Manning formula takes the form

$$Q = \frac{1.49}{n} A R^{\frac{2}{3}} S^{\frac{1}{2}} \tag{5.8.3}$$

When the cross-sectional area is known, any one of the other quantities can be obtained from Eq. (5.8.3) by direct solution.

Example 5.7: Determine the discharge for a trapezoidal channel (Fig. 5.29) with a bottom width $b = 8$ ft and side slopes 1 on 1. The depth is 6 ft, and the slope of the bottom is 0.0009. The channel has a finished concrete lining.

From Table 5.2, $n = 0.012$. The area is

$$A = 8 \times 6 + 6 \times 6 = 84 \text{ ft}^2$$

and the wetted perimeter is

$$P = 8 + 2 \times 6\sqrt{2} = 24.96$$

By substituting into Eq. (5.8.3),

$$Q = \frac{1.49}{0.012} 84 \left(\frac{84}{24.96}\right)^{\frac{2}{3}} (0.0009)^{\frac{1}{2}} = 703 \text{ cfs}$$

Trial solutions are required in some instances when the cross-sectional area is unknown. Expressions for both the hydraulic radius and the area contain the depth in a form that cannot be solved explicitly.

Example 5.8: What depth is required for 150 cfs flow in a rectangular planed-wood channel 5 ft wide with a bottom slope of 0.002?

If the depth is y, $A = 5y$, $P = 5 + 2y$, and $n = 0.012$. By substituting in Eq. (5.8.3),

$$150 = \frac{1.49}{0.012} 5y \left(\frac{5y}{5 + 2y}\right)^{\frac{2}{3}} (0.002)^{\frac{1}{2}}$$

After simplifying,

$$f(y) = y \left(\frac{y}{1 + 0.4y}\right)^{\frac{2}{3}} = 5.4$$

Assume $y = 4$ ft; then $f(y) = 5.332$. Assume $y = 4.05$; then $f(y) = 5.41$. The correct depth then is about 4.05 ft.

More general cases of open-channel flow are considered in Chap. 11.

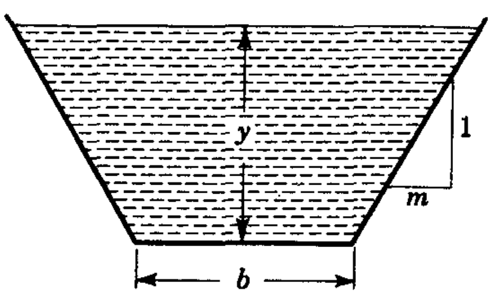


FIG. 5.29. Notation for trapezoidal cross section.

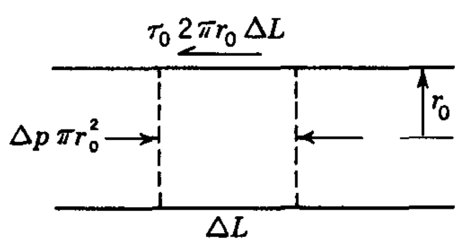


FIG. 5.30. Equilibrium conditions for steady flow in a pipe.

5.9. Steady, Incompressible Flow through Simple Pipe Systems.

Colebrook Formula. A force balance for steady flow (no acceleration) in a pipe (Fig. 5.30) yields

$$\Delta p\, \pi r_0^2 = \tau_0 2\pi r_0\, \Delta L$$

or simplifying,

$$\tau_0 = \frac{\Delta p}{\Delta L} \frac{r_0}{2} \tag{5.9.1}$$

which holds for laminar or turbulent flow. The Darcy-Weisbach equation (5.7.4) may be written

$$\Delta p = \gamma h_f = f \frac{\Delta L}{2r_0} \rho \frac{V^2}{2}$$

After eliminating Δp in the two equations and simplifying,

$$\sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{f}{8}}\, V \tag{5.9.2}$$

which relates wall shear stress, friction factor, and average velocity. The average velocity V may be obtained from Eq. (5.4.13) by integrating over the cross section. Substituting for V in Eq. (5.9.2.) and simplifying produces the equation for friction factor in smooth-pipe flow,

$$\frac{1}{\sqrt{f}} = A_s + B_s \ln (\mathbf{R} \sqrt{f}) \tag{5.9.3}$$

with the Nikuradse[1] data for smooth pipes, the equation becomes

$$\frac{1}{\sqrt{f}} = 0.86 \ln (\mathbf{R} \sqrt{f}) - 0.8 \tag{5.9.4}$$

For rough pipes in the complete turbulence zone,

$$\frac{1}{\sqrt{f}} = F_2\left(m, \frac{\epsilon'}{D}\right) + B_r \ln \frac{\epsilon}{D} \tag{5.9.5}$$

in which F_2 is, in general, a constant for a given form and spacing of the roughness elements. For the Nikuradse sand-grain roughness (Fig. 5.33) Eq. (5.9.5) becomes

$$\frac{1}{\sqrt{f}} = 1.14 - 0.86 \ln \frac{\epsilon}{D} \tag{5.9.6}$$

The roughness height ϵ for sand-roughened pipes may be used as a measure of the roughness of commercial pipes. If the value of f is known for a commercial pipe in the fully-developed wall turbulence zone, i.e., large Reynolds numbers and the loss proportional to the square of the velocity, the value of ϵ may be computed by Eq. (5.9.6) In the transition region where f depends upon both $\frac{\epsilon}{D}$ and $\mathbf{R}$ sand-roughened pipes produce different results than commercial pipes This is made evident by a graph based on Eqs. (5.9.4) and (5.9.6) with both sand-roughened and commercial-pipe-test results shown. By rearranging Eq. (5.9.6)

$$\frac{1}{\sqrt{f}} + 0.86 \ln \frac{\epsilon}{D} = 1.14$$

[1] J. Nikuradse, Gesetzmassigkeiten der turbulenten Strömung in glatten Rohren, *VDI Forsch.* 356, 1932.

and by adding 0.86 ln ϵ/D to each side of Eq. (5.9.4)

$$\frac{1}{\sqrt{f}} + 0.86 \ln \frac{\epsilon}{D} = 0.86 \ln \left(\mathrm{R} \sqrt{f} \frac{\epsilon}{D}\right) - 0.8$$

By selecting $1/\sqrt{f} + 0.86 \ln \epsilon/D$ as ordinate and $\ln (\mathrm{R}\sqrt{f}\ \epsilon/D)$ as abscissa (Fig. 5.31) smooth-pipe-test results plot as a straight line with slope +0.86 and rough-pipe-test results in the complete turbulence zone plot as the horizontal line. Nikuradse sand-roughness-test results plot along the dashed line in the transition region and commercial-pipe-test

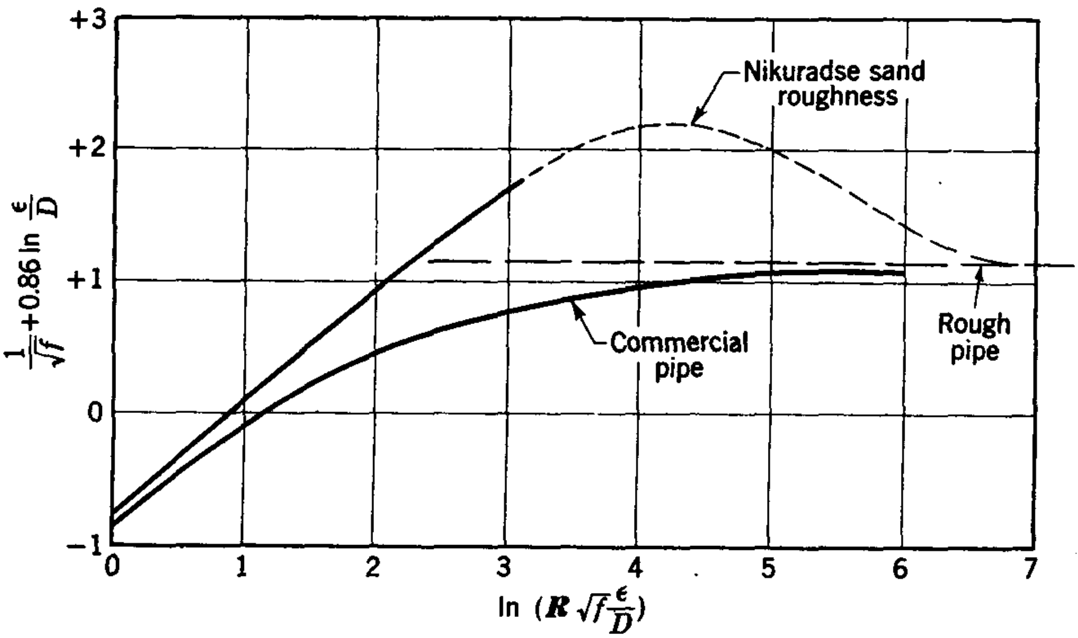


Fig. 5.31. Colebrook transition function.

results plot along the lower curved line. An empirical transition function for commercial pipes for the region between smooth pipes and the complete turbulence zone has been developed by Colebrook,[1]

$$\frac{1}{\sqrt{f}} = -0.86 \ln \left(\frac{\epsilon/D}{3.7} + \frac{2.51}{\mathrm{R}\sqrt{f}}\right) \tag{5.9.7}$$

which is the basis for the Moody diagram (Fig. 5.34).

Pipe Flow. In steady incompressible flow in a pipe the irreversibilities are expressed in terms of a head loss, or drop in *hydraulic grade line* (Sec. 10.1). The hydraulic grade line is p/γ above the center of the pipe, and if z is the elevation of the center of the pipe, then $z + p/\gamma$ is the elevation of a point on the hydraulic grade line. The locus of values of $z + p/\gamma$ along the pipeline gives the hydraulic grade line. Losses, or irreversi-

[1] C. F. Colebrook, Turbulent Flow in Pipes, with Particular Reference to the Transition Region between the Smooth and Rough Pipe Laws, *J. Inst. Civil Engs.* (London), vol. 11, pp. 133–156, 1938–1939.

bilities, cause this line to drop in the direction of flow. The Darcy-Weisbach equation (5.7.4)

$$h_f = f\frac{L}{D}\frac{V^2}{2g} \tag{5.7.4}$$

is generally adopted for pipe-flow calculations. h_f is the head loss, or drop in hydraulic grade line, in the pipe length L, having an inside diameter D and an average velocity V. h_f has the dimension length and is expressed in terms of foot-pounds per pound or feet. The friction factor f is a dimensionless factor that is required to make the equation produce the correct value for losses. All quantities in Eq. (5.7.4) except f may be measured experimentally. A typical setup is shown in Fig. 5.32. By measuring the discharge and inside diameter, the average velocity can be computed. The head loss h_f is measured by a differential manometer

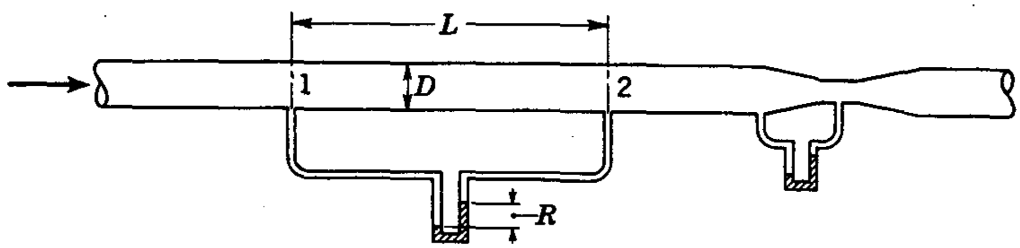


FIG. 5.32. Experimental arrangement for determination of head loss in a pipe.

attached to piezometer openings at sections 1 and 2, distance L apart. Experimentation shows the following to be true in turbulent flow:

a. The head loss varies directly as the length of the pipe.

b. The head loss varies almost as the square of the velocity.

c. The head loss varies almost inversely as the diameter.

d. The head loss depends upon the surface roughness of the interior pipe wall.

e. The head loss depends upon the fluid properties of density and viscosity.

f. The head loss is independent of the pressure.

The friction factor f must be selected in a manner so that Eq. (5.7.4) correctly yields the head loss; hence, f cannot be a constant but must depend upon velocity V, diameter D, density ρ, viscosity μ, and certain characteristics of the wall roughness that are signified by ϵ, ϵ', and m. These symbols are defined thus: ϵ is a measure of the *size* of the roughness projections and has the dimensions of a length; ϵ' is a measure of the *arrangement* or *spacing* of the roughness elements and also has the dimensions of a length; m is a form factor, depending upon the *shape* of the individual roughness elements, and is dimensionless. The term f, instead of being a simple constant, turns out to be a factor that depends upon seven quantities

$$f = f(V,D,\rho,\mu,\epsilon,\epsilon',m) \tag{5.9.8}$$

Since f is a dimensionless factor, it must depend upon the grouping of these quantities into dimensionless parameters. For *smooth* pipe $\epsilon = \epsilon' = m = 0$, leaving f dependent upon the first four quantities. They can be arranged in only one way to make them dimensionless, namely, $VD\rho/\mu$, which is the Reynolds number. For *rough* pipes the terms ϵ, ϵ' may be made dimensionless by dividing by D. Therefore, in general,

$$f = f\left(\frac{VD\rho}{\mu}, \frac{\epsilon}{D}, \frac{\epsilon'}{D}, m\right) \tag{5.9.9}$$

The proof of this relationship is left to experimentation. For smooth pipes a plot of all experimental results shows the functional relationship, subject to a scattering of ± 5 per cent. The plot of friction factor against Reynolds number on a log-log chart is called a *Stanton diagram.* Blasius[1] was the first to correlate the smooth-pipe experiments in turbulent flow. He presented the results by an empirical formula that is valid up to about $\mathbf{R} = 100{,}000$. The Blasius formula is

$$f = \frac{0.316}{\mathbf{R}^{\frac{1}{4}}} \tag{5.9.10}$$

In rough pipes the term ϵ/D is called the *relative roughness.* Nikuradse[2] proved the validity of the relative roughness concept by his tests on sand-roughened pipes. He used three sizes of pipes and glued sand grains (ϵ = diameter of the sand grains) of practically constant size to the interior walls so that he had the same values of ϵ/D for the different pipes. These experiments (Fig. 5.33) show that for one value of ϵ/D the f, $\mathbf{R}$ curve is smoothly connected regardless of the actual pipe diameter. These tests did not permit variation of ϵ'/D or m but proved the validity of the equation

$$f = f\left(\mathbf{R}, \frac{\epsilon}{D}\right)$$

for one type of roughness.

Because of the extreme complexity of naturally rough surfaces, most of the advances in understanding the basic relationships have been developed around experiments on artificially roughened pipes. Moody[3] has constructed one of the most convenient charts for determining friction factors in clean, commercial pipes. This chart, presented in Fig. 5.34, is the basis for pipe-flow calculations in this chapter. The chart

[1] H. Blasius, Das Aehnlichkeitsgesetz bei Reibungsvorgängen in Flüssigkeiten, *VDI Forsch.* 131, 1913.

[2] J. Nikuradse, Strömungsgesetze in rauhen Rohren, *VDI Forsch.* 361, 1933.

[3] L. F. Moody, Friction Factors for Pipe Flow, *Trans. ASME,* November, 1944.

is a Stanton diagram that expresses f as a function of relative roughness and Reynolds number. The values of absolute roughness of the commercial pipes are determined by experiment in which f and R are found and substituted into the Colebrook formula Eq. (5.9.7), which closely represents natural pipe trends. These are listed in the table in the lower left-hand corner of Fig. 5.34. The Colebrook formula provides the shape of the ϵ/D = constant curves in the transition region.

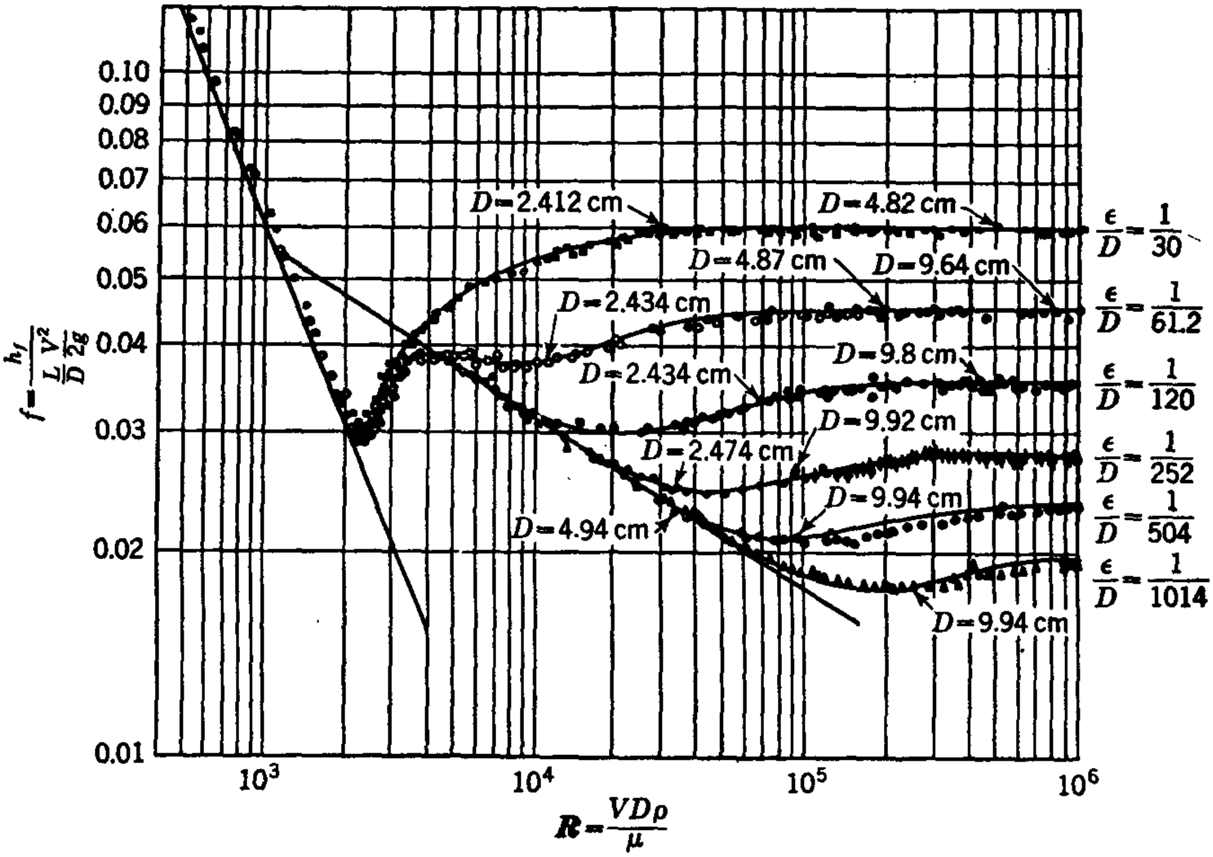


FIG. 5.33. Nikuradse's sand-roughened-pipe tests.

The straight line marked "laminar flow" is the Hagen-Poiseuille equation. Equation (5.2.7)

$$V = \frac{\Delta p \, r_0^2}{8\mu L}$$

may be transformed into Eq. (5.7.4) with $\Delta p = \gamma h_f$ and by solving for h_f,

$$h_f = \frac{V8\mu L}{\gamma r_0^2} = \frac{64\mu}{\rho D}\frac{L}{D}\frac{V}{2g} = \frac{64}{\rho DV/\mu}\frac{L}{D}\frac{V^2}{2g}$$

or

$$h_f = f\frac{L}{D}\frac{V^2}{2g} = \frac{64}{\mathrm{R}}\frac{L}{D}\frac{V^2}{2g} \tag{5.9.11}$$

from which

$$f = \frac{64}{\mathrm{R}} \tag{5.9.12}$$

This equation, which plots as a straight line with slope -1 on a log-log chart, may be used for the solution of laminar flow problems in pipes. It applies to all roughnesses, as the head loss in laminar flow is independent of wall roughness. The Reynolds critical number is about 2000, and the *critical zone*, where the flow may be either laminar or turbulent, is about 2000 to 4000.

It should be noted that the relative-roughness curves $\epsilon/D = 0.001$ and smaller approach the smooth-pipe curve for decreasing Reynolds numbers. This can be explained by the presence of a laminar film at the wall of the pipe that decreases in thickness as the Reynolds number increases. For certain ranges of Reynolds number in the transition zone, the film completely covers small roughness projections, and the pipe has a friction factor the same as that of a smooth pipe. For larger Reynolds numbers, projections protrude through the laminar film, and each projection causes extra turbulence that increases the head loss. For the zone marked "complete turbulence, rough pipes," the film thickness is negligible compared with the height of roughness projections, and each projection contributes fully to the turbulence. Viscosity does not affect the head loss in this zone, as evidenced by the fact that the friction factor does not change with Reynolds number. In this zone the loss follows the V^2 law, i.e., it varies directly as the square of the velocity.

Two auxiliary scales are given along the top of the Moody diagram. One is for water at 60°F, and the other is for air at standard atmospheric pressure and 60°F. Since the kinematic viscosity is constant in each case, the Reynolds number is a function of VD. For these two scales only, D must be expressed in inches.

Simple Pipe Problems. The three simple pipe-flow cases that are basic to solutions of the more complex problems are

	Given	*To find*
I.	Q, L, D, ν, ϵ	h_f
II.	h_f, L, D, ν, ϵ	Q
III.	h_f, Q, L, ν, ϵ	D

In each of these cases the Darcy-Weisbach equation, the continuity equation, and the Moody diagram are used to determine the unknown quantity.

In the first case the Reynolds number and the relative roughness are readily determined from the data given, and h_f is found by determining f from the Moody diagram and substituting into the Darcy-Weisbach equation.

Example 5.9: Determine the head loss due to the flow of 2000 gpm of oil, $\nu = 0.0001$ ft²/sec, through 1000 ft of 8-in.-diameter cast-iron pipe.

$$V = \frac{2000}{448\pi/9} = 12.8 \text{ ft/sec} \qquad \mathbf{R} = 12.8 \times \frac{2}{3} \times \frac{1}{0.0001} = 85{,}500$$

The relative roughness is $\epsilon/D = 0.00085/0.667 = 0.0013$. From Fig. 5.34, by interpolation, $f = 0.024$; hence

$$h_f = f\frac{L}{D}\frac{V^2}{2g} = 0.024 \times \frac{1000}{\frac{2}{3}}\frac{\overline{12.8}^2}{64.4} = 91.8 \text{ ft-lb/lb}$$

In the second case, V and f are unknowns, and the Darcy-Weisbach equation and Moody diagram must be used simultaneously to find their values. Since ϵ/D is known, a value of f may be assumed by inspection of the Moody diagram. Substitution of this trial f into the Darcy-Weisbach equation produces a trial value of V, from which a trial Reynolds number is computed. With the Reynolds number an improved value of f is found from the Moody diagram. When f has been found correct to two significant figures, the corresponding V is the value sought, and Q is determined by multiplying by the area.

Example 5.10: Water at 60°F flows through a 12-in.-diameter riveted-steel pipe, $\epsilon = 0.01$, with a head loss of 20 ft in 1000 ft. Determine the flow.

The relative roughness is $\epsilon/D = 0.01$, and from Fig. 5.34 a trial f is taken as 0.040. By substituting into Eq. (5.7.4),

$$20 = 0.040\frac{1000}{1}\frac{V^2}{64.4} \qquad V = 5.67 \text{ ft/sec}$$

and $VD'' = 68$ for use with the scale at the top of Fig. 5.34, which shows $f = 0.038$. With this f in place of 0.040 in the above equation, $V = 5.81$, $VD'' = 69.8$, and f remains 0.038. The discharge is

$$Q = 5.81\frac{\pi}{4} = 4.56 \text{ cfs} = 2044 \text{ gpm}$$

In the third case, with D unknown, there are three unknowns in Eq. (5.7.4), f, V, D; two in the continuity equation, V, D; and three in the Reynolds number equation, V, D, **R**. The relative roughness is also unknown. Using the continuity equation to eliminate the velocity in Eq. (5.7.4) and in the expression for **R**, simplifies the problem. Equation (5.7.4) becomes

$$h_f = f\frac{L}{D}\frac{Q^2}{2g(D^2\pi/4)^2}$$

or

$$D^5 = \frac{8LQ^2}{h_f g\pi^2}f = C_1 f \tag{5.9.13}$$

in which C_1 is the known quantity $8LQ^2/h_f g\pi^2$. As $VD^2 = 4Q/\pi$ from continuity,

$$\mathbf{R} = \frac{VD}{\nu} = \frac{4Q}{\pi\nu}\frac{1}{D} = \frac{C_2}{D} \tag{5.9.14}$$

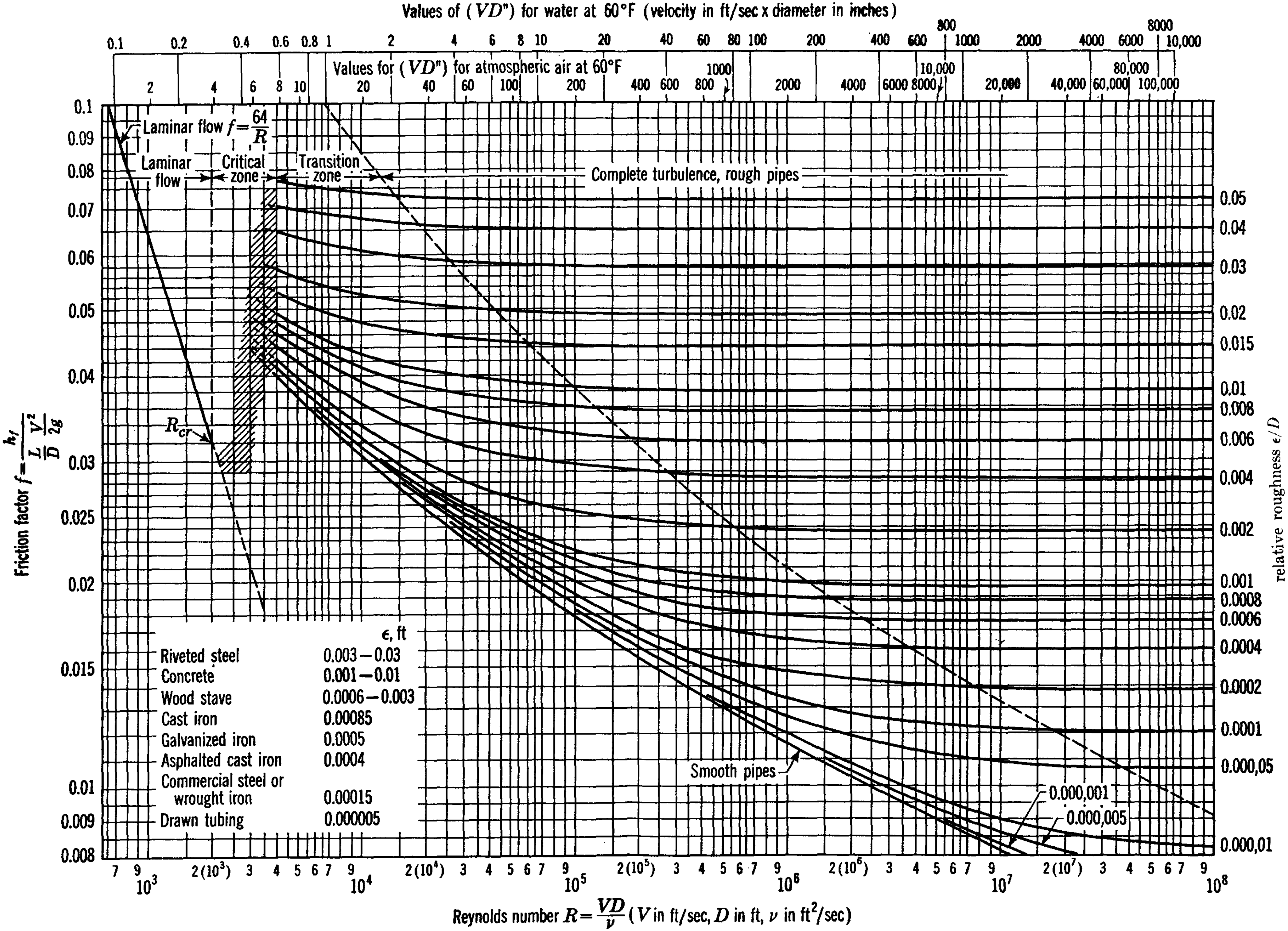


Fig. 5.34. Moody diagram.

in which C_2 is the known quantity $4Q/\pi\nu$. The solution is now effected by the following procedure:

1. Assume a value of f.
2. Solve Eq. (5.9.13) for D.
3. Solve Eq. (5.9.14) for **R**.
4. Find the relative roughness ϵ/D.
5. With **R** and ϵ/D, look up a new f from Fig. 5.34.
6. Use the new f, and repeat the procedure.
7. When the value of f does not change, all equations are satisfied and the problem is solved.

Normally only one or two trials are required. Since standard pipe sizes are usually selected, the next larger size of pipe from that given by the computation is taken. Nominal standard pipe sizes are $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{4}$, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, 5, 6, 8, 10, 12, 14, 16, 18, 24, and 30 in. The inside diameters are larger than the nominal up to 12 in. Above the 12-in. size the actual inside diameter depends upon the "schedule" of the pipe, and manufacturer's tables should be consulted. Throughout this chapter the nominal size is taken as the actual inside diameter.

Example 5.11: Determine the size of clean wrought-iron pipe required to convey 4000 gpm oil, $\nu = 0.0001$ ft²/sec, 10,000 ft with a head loss of 75 ft-lb/lb.

The discharge is

$$Q = \frac{4000}{448} = 8.93 \text{ cfs}$$

From Eq. (5.9.13)

$$D^5 = \frac{8 \times 10{,}000 \times \overline{8.93}^2}{75 \times 32.2 \times \pi^2} f = 267.0f$$

and from Eq. (5.9.14)

$$\mathbf{R} = \frac{4 \times 8.93}{\pi 0.0001} \frac{1}{D} = \frac{113{,}800}{D}$$

and from Fig. 5.34, $\epsilon = 0.00015$ ft.

If $f = 0.02$, $D = 1.398$ ft, $\mathbf{R} = 81{,}400$, $\epsilon/D = 0.00011$ and from Fig. 5.34, $f = 0.019$. In repeating the procedure, $D = 1.382$, $\mathbf{R} = 82{,}300$, $f = 0.019$. Therefore, $D = 1.382 \times 12 = 16.6$ in. If a 75-ft head loss is the maximum allowable, an 18-in. pipe is required.

In each of the cases considered, the loss has been expressed in feet of head or in foot-pounds per pound. For horizontal pipes, this loss shows up as a gradual reduction in pressure along the line. For nonhorizontal cases, Bernoulli's equation is applied to the two end sections of the pipe, and the loss term is included, thus

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_f \tag{5.9.15}$$

in which the kinetic-energy correction factors have been taken as unity. The upstream section is given the subscript 1 and the downstream section the subscript 2. The total head at section 1 is equal to the sum of the total head at section 2 and all the head losses between the two sections.

Example 5.12: In the preceding example, for $D = 16.6$ in., if the specific gravity is 0.85, $p_1 = 40$ psi, $z_1 = 200$ ft, and $z_2 = 50$ ft, determine the pressure at section 2.

In Eq. (5.9.15) $V_1 = V_2$; hence,

$$\frac{40}{0.85 \times 0.433} + 200 = \frac{p_2}{0.85 \times 0.433} + 50 + 75$$

and

$$p_2 = 67.6 \text{ psi}$$

Minor Losses. Those losses which occur in pipelines due to bends, elbows, joints, valves, etc., are called *minor losses.* This is a misnomer, because in many situations they are more important than the losses due to pipe friction considered in the preceding section, but it is the conventional name. In almost all cases the minor loss is determined by experiment. However, one important exception is the head loss due to a sudden expansion in a pipeline (Sec. 3.9).

Equation (3.9.33) may also be written

$$h_e = K\frac{V_1^2}{2g} = \left[1 - \left(\frac{D_1}{D_2}\right)^2\right]^2 \frac{V_1^2}{2g} \tag{5.9.16}$$

in which

$$K = \left[1 - \left(\frac{D_1}{D_2}\right)^2\right]^2 \tag{5.9.17}$$

From Eq. (5.9.16) it is obvious that the head loss varies as the square of the velocity. This is substantially true for all minor losses in turbulent flow. A convenient method of expressing the minor losses in flow is by means of the coefficient K, usually determined by experiment.

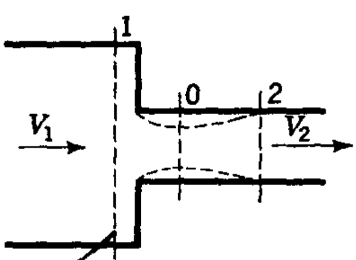


Fig. 5.35. Sudden contraction in a pipeline.

If the sudden expansion is from a pipe to a reservoir, $D_1/D_2 = 0$ and the loss becomes $V_1^2/2g$, that is, the complete kinetic energy in the flow is converted into thermal energy.

The head loss h_c due to a sudden contraction in the pipe cross section, illustrated in Fig. 5.35, is subject to the same analysis as the sudden expansion, provided that the amount of contraction of the jet is known. The process of converting pressure head into velocity head is very efficient; hence, the head loss from section 1 to the *vena contracta*[1] is small compared with the loss from section 0 to section 2, where velocity head

[1] The *vena contracta* is the section of greatest contraction of the jet.

is being reconverted into pressure head. By applying Eq. (3.9.33) to this expansion, the head loss is computed to be

$$h_c = \frac{(V_0 - V_2)^2}{2g}$$

With the continuity equation $V_0 C_c A_2 = V_2 A_2$, in which C_c is the contraction coefficient (i.e., the area of jet at section 0 divided by the area of section 0), the head loss is computed to be

$$h_c = \left(\frac{1}{C_c} - 1\right)^2 \frac{V_2^2}{2g} \tag{5.9.18}$$

The contraction coefficient for water C_c, determined by Weisbach,[1] is presented in the tabulation.

A_2/A_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C_c	0.624	0.632	0.643	0.659	0.681	0.712	0.755	0.813	0.892	1.00

The head loss at the entrance to a pipeline from a reservoir is usually taken as $0.5V^2/2g$, if the opening is square-edged. For well-rounded entrances, the loss is between $0.01V^2/2g$ and $0.05V^2/2g$ and may usually be neglected. For re-entrant openings, as with the pipe extending into the reservoir beyond the wall, the loss is taken as $1.0V^2/2g$, for thin pipe walls.

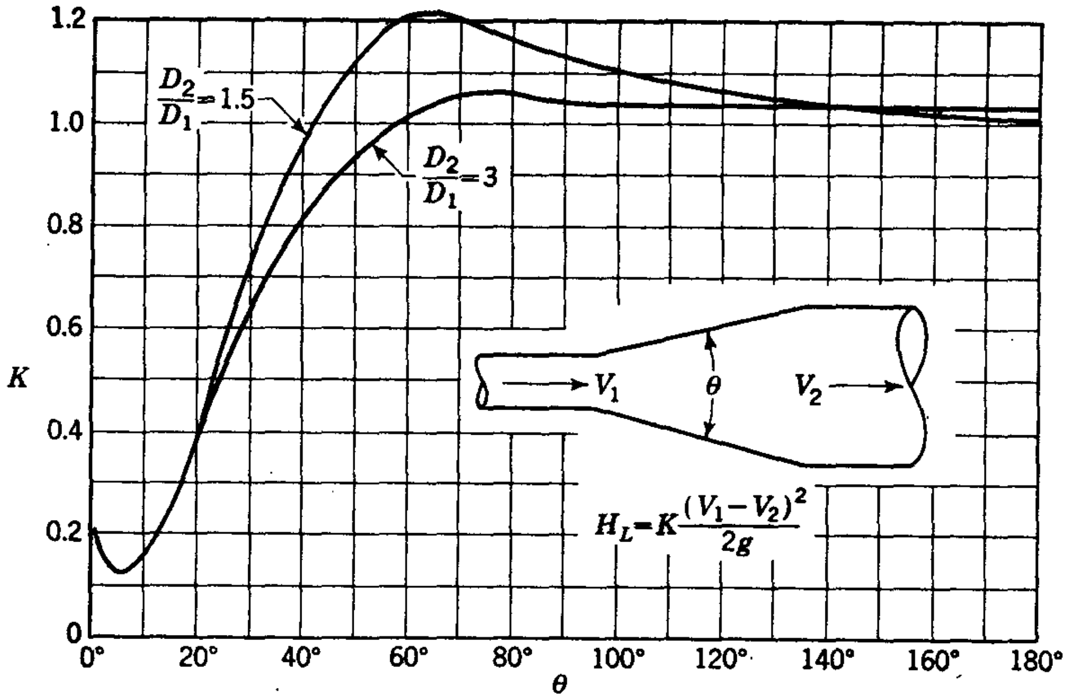


FIG. 5.36. Loss coefficients for gradual conical expansions.

[1] Julius Weisbach, "Die Experimental-Hydraulik," p. 133, J. S. Englehardt, Freiberg, 1855.

The head loss due to gradual expansions has been investigated experimentally by Gibson,[1] whose results are given in Fig. 5.36.

A summary of representative head loss coefficients K for typical fittings, published by the Crane Company,[2] is given in Table 5.3.

TABLE 5.3. HEAD LOSS COEFFICIENTS K FOR VARIOUS FITTINGS

	K
Globe valve (fully open)	10.0
Angle valve (fully open)	5.0
Swing check valve (fully open)	2.5
Gate valve (fully open)	0.19
Close return bend	2.2
Standard tee	1.8
Standard elbow	0.9
Medium sweep elbow	0.75
Long sweep elbow	0.60

Minor losses may be expressed in terms of the equivalent length of pipe L_e that has the same head loss in foot-pounds per pound for the same discharge, thus,

$$f\frac{L_e}{D}\frac{V^2}{2g} = K\frac{V^2}{2g}$$

in which K may refer to one minor head loss or to the sum of several losses. After solving for L_e,

$$L_e = \frac{KD}{f} \tag{5.9.19}$$

For example, if the minor losses in a 12-in. pipeline add to $K = 20$ and if $f = 0.020$ for the line, then to the actual length of line may be added $20 \times 1/0.020 = 1000$ ft, and this additional or equivalent length causes the same resistance to flow as the minor losses.

Example 5.13: Find the discharge through the pipeline in Fig. 5.37 for $H = 30$ ft, and determine the head loss H for $Q = 2.0$ cfs.

Bernoulli's equation app.ied to points 1 and 2, including all the losses, may be written

$$H + 0 + 0 = \frac{V^2}{2g} + 0 + 0 + \frac{1}{2}\frac{V^2}{2g} + f\frac{340}{\frac{1}{2}}\frac{V^2}{2g} + 2 \times 0.9\frac{V^2}{2g} + 10\frac{V^2}{2g}$$

After simplifying

$$H = \frac{V^2}{2g}(13.3 + 680f)$$

[1] A. H. Gibson, The Conversion of Kinetic to Pressure Energy in the Flow of Water through Passages Having Divergent Boundaries, *Engineering*, vol. 93, p. 205, 1912.

[2] Crane Company, "Flow of Fluids," *Tech. Paper* 409, May, 1942.

When the head is given, this problem is solved as the second type of simple pipe problem. If $f = 0.02$, then

$$30 = \frac{V^2}{2g}(13.3 + 680 \times 0.02)$$

and $V = 8.46$ ft/sec. $\epsilon/D = 0.00085/0.5 = 0.0017$; $VD'' = 8.46 \times 6 = 50.7$. From Fig. 5.34, $f = 0.023$. By solving again for the velocity, $V = 8.16$ ft/sec,

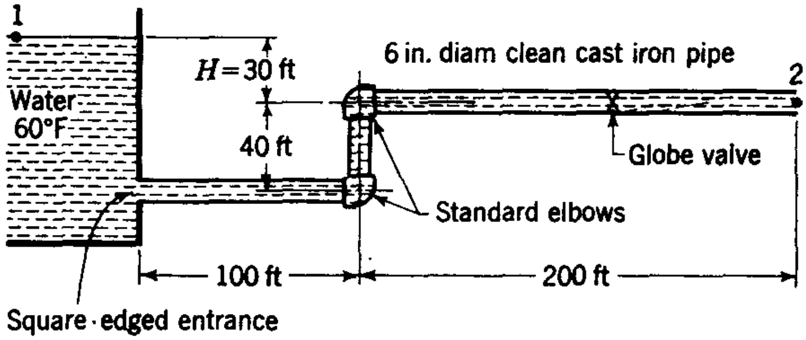


FIG. 5.37. Pipeline with minor losses.

$VD'' = 8.16 \times 6 = 49$, and f does not change. The discharge is

$$Q = 8.16\frac{\pi}{16} = 1.60 \text{ cfs}$$

For the second part, with Q known, the solution is straightforward,

$$V = \frac{2}{\pi} \times 16 = 10.18 \text{ ft/sec} \qquad VD'' = 61.1 \qquad f = 0.023$$

and

$$H = \frac{\overline{10.18^2}}{64.4}(13.3 + 680 \times 0.023) = 46.5 \text{ ft}$$

With equivalent lengths [Eq. (5.9.19)], the value of f is approximated, say $f = 0.020$. The sum of minor losses is $K = 13.3$, in which the kinetic energy at 2 is included as a minor loss,

$$L_e = \frac{13.3 \times 0.50}{0.02} = 332 \text{ ft}$$

Hence, the total length of pipe is $332 + 340 = 672$ ft. The first part of the problem is solved by this method,

$$30 = f\frac{L + L_e}{D}\frac{V^2}{2g} = f\frac{672}{0.50}\frac{V^2}{2g}$$

If $f = 0.02$, $V = 8.47$, $VD'' = 50.8$, $f = 0.023$; then $V = 7.9$, $VD'' = 47.4$, $f = 0.023$, $Q = 1.55$ cfs. Normally it is not necessary to use the new value of f in Eq. (5.9.19).

Minor losses may be neglected in those situations where they compose only 5 per cent or less of the head losses due to pipe friction. The friction factor, at best, is subject to about 5 per cent error, and it is meaningless to select values to more than two significant figures. In general, minor losses may be neglected when, on the average, there is a length of 1000 diameters between each minor loss.

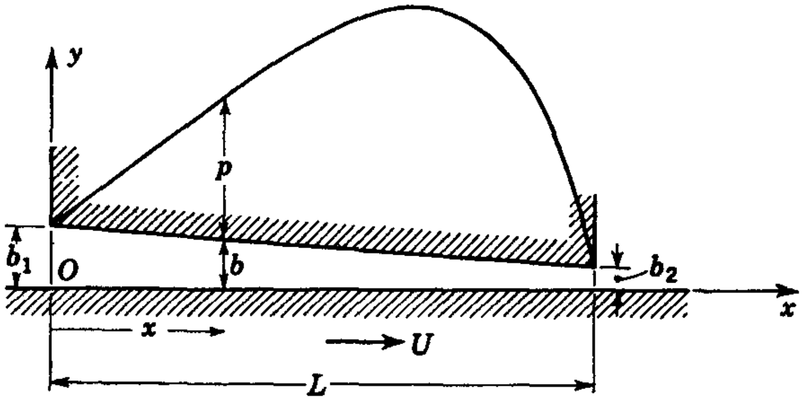


FIG. 5.38. Sliding bearing.

Compressible flow in pipes is treated in Chap. 6. Complex pipe-flow situations are treated in Chap. 10.

5.10. Lubrication Mechanics. The effect of viscosity on flow and its effect on head losses have been examined in the preceding sections of this chapter. A laminar-flow case of great practical importance is the hydrodynamic theory of lubrication. Simple aspects of this theory are developed in this section.

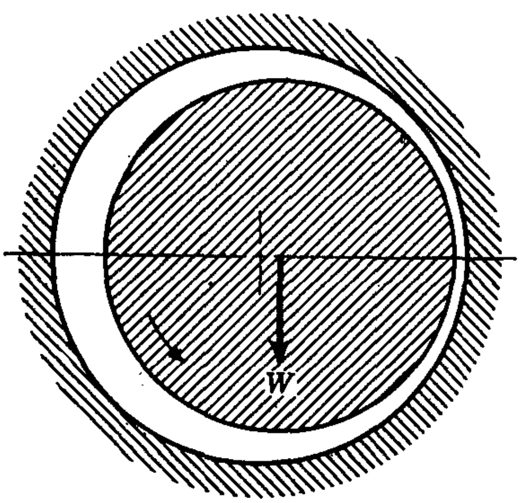


FIG. 5.39. Journal bearing.

Large forces are developed in small clearances when the surfaces are slightly inclined and one is in motion so that fluid is "wedged" into the decreasing space. The slipper bearing, which operates on this principle, is illustrated in Fig. 5.38. The journal bearing (Fig. 5.39) develops its force by the same action, except that the surfaces are curved.

The laminar-flow equations may be used to develop the theory of lubrication. The assumption is made that there is no flow out of the ends of the bearing, normal to the plane of Fig. 5.38. Starting with Eq. (5.1.4), which relates pressure drop and shear stress, the equation for the force P that the bearing will support is worked out, and the drag on the bearing is computed.

Substituting Newton's law of viscosity into Eq. (5.1.4) produces

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2} \tag{5.10.1}$$

Since the inclination of the upper portion of the bearing (Fig. 5.38) is very slight, it is assumed that the velocity distribution is the same as if the plates were parallel and that p is independent of y. Integrating Eq. (5.10.1) twice with respect to y, with dp/dx constant, produces

$$\frac{dp}{dx}\int dy = \mu \int \frac{d^2u}{dy^2}\,dy + A$$

or

$$\frac{dp}{dx}\,y = \mu\frac{du}{dy} + A$$

and the second time

$$\frac{dp}{dx}\int y\,dy = \mu\int\frac{du}{dy}\,dy + A\int dy + B$$

or

$$\frac{dp}{dx}\frac{y^2}{2} = \mu u + Ay + B$$

The constants of integration A, B are determined from the conditions $u = 0$, $y = b$; $u = U$, $y = 0$. Substituting in turn produces

$$\frac{dp}{dx}\frac{b^2}{2} = Ab + B \qquad \mu U + B = 0$$

Eliminating A and B and solving for u results in

$$u = \frac{y}{2\mu}\frac{dp}{dx}(y - b) + U\left(1 - \frac{y}{b}\right) \tag{5.10.2}$$

The discharge Q must be the same at each cross section. By integrating over a typical section, again with dp/dx constant,

$$Q = \int_0^b u\,dy = \frac{Ub}{2} - \frac{b^3}{12\mu}\frac{dp}{dx} \tag{5.10.3}$$

Now, since Q cannot vary with x, b may be expressed in terms of x, $b = b_1 - \alpha x$, in which $\alpha = (b_1 - b_2)/L$ and the equation is integrated with respect to x to determine the pressure distribution. Solving Eq. (5.10.3) for dp/dx produces

$$\frac{dp}{dx} = \frac{6\mu U}{(b_1 - \alpha x)^2} - \frac{12\mu Q}{(b_1 - \alpha x)^3} \tag{5.10.4}$$

By integrating,

$$\int\frac{dp}{dx}\,dx = 6\mu U\int\frac{dx}{(b_1 - \alpha x)^2} - 12\mu Q\int\frac{dx}{(b_1 - \alpha x)^3} + C$$

or

$$p = \frac{6\mu U}{\alpha(b_1 - \alpha x)} - \frac{6\mu Q}{\alpha(b_1 - \alpha x)^2} + C$$

In this equation Q and C are unknowns. Since the pressure must be the same, say zero, at the ends of the bearing, namely, $p = 0$, $x = 0$; $p = 0$, $x = L$, the constants may be determined,

$$Q = \frac{Ub_1b_2}{b_1 + b_2} \qquad C = -\frac{6\mu U}{\alpha(b_1 + b_2)}$$

With these values inserted, the equation for pressure distribution becomes

$$p = \frac{6\mu Ux(b - b_2)}{b^2(b_1 + b_2)} \tag{5.10.5}$$

This equation shows that p is positive between $x = 0$ and $x = L$ if $b > b_2$. It is plotted in Fig. 5.38 to show the distribution of pressure throughout the bearing. With this one-dimensional method of analysis the very slight change in pressure along a vertical line x = constant is neglected.

The total force P that the bearing will sustain, per unit width, is

$$P = \int_0^L p\,dx = \frac{6\mu U}{b_1 + b_2}\int_0^L \frac{x(b - b_2)dx}{b^2}$$

After substituting the value of b in terms of x and performing the integration,

$$P = \frac{6\mu UL^2}{(b_1 - b_2)^2}\left(\ln\frac{b_1}{b_2} - 2\,\frac{b_1 - b_2}{b_1 + b_2}\right) \tag{5.10.6}$$

The drag force D required to move the lower surface at speed U is expressed by

$$D = \int_0^L \tau\Big|_{y=0} dx = -\int_0^L \mu \frac{du}{dy}\Big|_{y=0} dx$$

By evaluating du/dy from Eq. (5.10.2), for $y = 0$,

$$\frac{du}{dy}\Big|_{y=0} = -\frac{b}{2\mu}\frac{dp}{dx} - \frac{U}{b}$$

With this value in the integral, along with the value of dp/dx from Eq. (5.10.4),

$$D = \frac{2\mu UL}{b_1 - b_2}\left(2\ln\frac{b_1}{b_2} - 3\,\frac{b_1 - b_2}{b_1 + b_2}\right) \tag{5.10.7}$$

The maximum load P is computed with Eq. (5.10.6) when $b_1 = 2.2b_2$. With this ratio,

$$P = 0.16\,\frac{\mu UL^2}{b_2{}^2} \qquad D = 0.75\,\frac{\mu UL}{b_2} \tag{5.10.8}$$

The ratio of load to drag for optimum load is

$$\frac{P}{D} = 0.21\,\frac{L}{b_2} \tag{5.10.9}$$

which can be very large since b_2 can be very small.

Example 5.14: A vertical turbine shaft carries a load of 80,000 lb on a thrust bearing consisting of 16 flat rocker plates, 3 in. by 9 in., arranged with their long dimensions radial from the shaft and with their centers on a circle of radius 1.5 ft. The shaft turns at 120 rpm; $\mu = 0.002$ lb-sec/ft². If the plates take the angle for maximum load, neglecting effects of curvature of path and radial lubricant flow, find (*a*) the clearance between rocker plate and fixed plate; (*b*) the torque loss due to the bearing.

a. Since the motion is considered straight-line,

$$U = 1.5 \times \tfrac{120}{60} 2\pi = 18.85 \text{ ft/sec}$$
$$L = 0.25 \text{ ft}$$

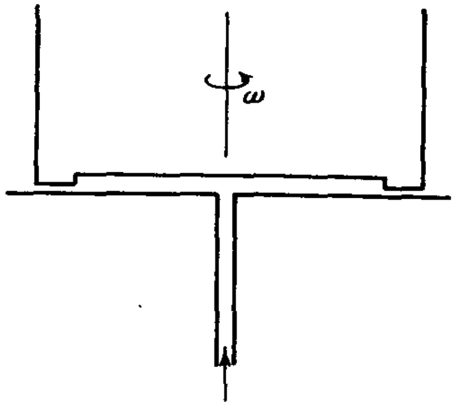


Fig. 5.40. Hydrostatic lubrication by high-pressure pumping of oil.

The load is 5000 lb for each plate, which is 5000/0.75 = 6667 lb for unit width. By solving for the clearance b_2, from Eq. (5.10.8),

$$b_2 = \sqrt{\frac{0.16\mu UL^2}{P}} = 0.4 \times 0.25 \sqrt{\frac{0.002 \times 18.85}{6667}} = 2.38 \times 10^{-4} \text{ ft} = 0.0029 \text{ in.}$$

(*b*) The drag due to one rocker plate is, per foot of width,

$$D = 0.75 \frac{\mu UL}{b_2} = \frac{0.75 \times 0.002 \times 18.85 \times 0.25}{2.38 \times 10^{-4}} = 29.6 \text{ lb}$$

For a 9-in. plate, $D = 29.6 \times 0.75 = 22.2$ lb. The torque loss due to the 16 rocker plates is

$$16 \times 22.2 \times 1.5 = 533 \text{ ft-lb}$$

Another form of lubrication, called *hydrostatic lubrication*,[1] has many important applications. It involves the continuous pumping of high-pressure oil under a step bearing, as illustrated in Fig. 5.40. The load may be lifted by the lubrication before rotation starts, which greatly reduces starting friction.

PROBLEMS

5.1. Derive Eq. (5.1.1) for the case of the plates making an angle θ with the horizontal, showing that in the equation p may be replaced by $p + \gamma z$. z is the change in elevation in length l.

5.2. Derive Eq. (5.1.3) for two fixed plates by starting with Eq. (5.1.1).

[1] For further information on hydrostatic lubrication see D. D. Fuller, Lubrication Mechanics, in "Handbook of Fluid Dynamics," ed. by V. L. Streeter, pp. **22**–21 to **22**–30, McGraw-Hill Book Company, Inc., New York, 1961.

5.3. Determine the formulas for shear stress on each plate and for the velocity distribution for flow in Fig. 5.1 when an adverse pressure gradient exists such that $Q = 0$.

5.4. In Fig. 5.1, with U positive as shown, find the expression for dp/dl such that the shear is zero at the fixed plate. What is the discharge for this case?

5.5. In Fig. 5.41*a*, $U = 2$ ft/sec. Find the rate at which oil is carried into the pressure chamber by the piston and the shear force and total force F acting.

5.6. Determine the force on the piston of Fig. 5.41*a* due to shear and the leakage from the pressure chamber for $U = 0$.

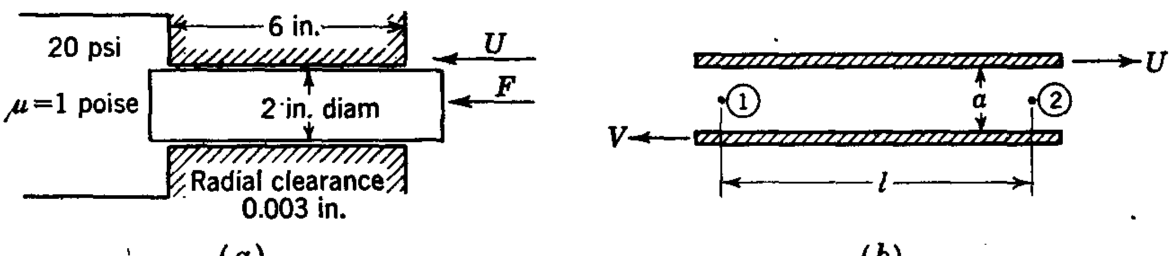


Fig. 5.41

5.7. Find F and U in Fig. 5.41*a* such that no oil is lost through the clearance from the pressure chamber.

5.8. Derive an expression for the flow past a fixed cross section of Fig. 5.41*b* for laminar flow between the two moving plates.

5.9. In Fig. 5.41*b*, for $p_1 = p_2 = 10$ psi, $U = 2V = 10$ ft/sec, $a = 0.005$ ft, $\mu = 0.5$ poise, find the shear stress at each plate.

5.10. Compute the kinetic-energy and momentum correction factors for laminar flow between fixed parallel plates.

5.11. Determine the formula for angle θ for fixed parallel plates so that laminar flow at constant pressure takes place.

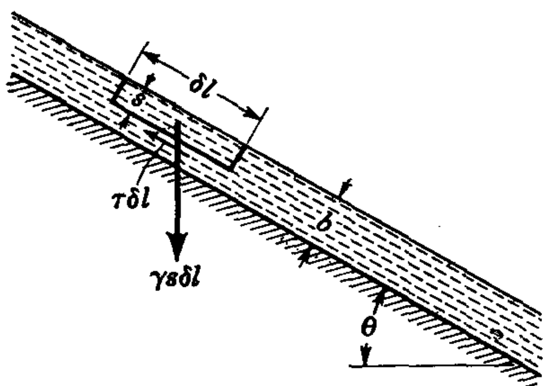


Fig. 5.42

5.12. With a free body, as in Fig. 5.42, for uniform flow of a thin lamina of liquid down an inclined plane, show that the velocity distribution is

$$u = \frac{\gamma}{2\mu}(b^2 - s^2)\sin\theta$$

and that the discharge per unit width is

$$Q = \frac{\gamma}{3\mu} b^2 \sin \theta$$

5.13. Derive the velocity distribution of Prob. 5.12 by inserting the condition that the shear at the moving plate must be zero from Eq. (5.1.2) when p is replaced by $p + \gamma z$.

5.14. In Fig. 5.43, $p_1 = 5$ psi, $p_2 = 8$ psi, $l = 4$ ft, $a = 0.005$ ft, $\theta = 30°$, $U = 3$ ft/sec, $\gamma = 50$ lb/ft^3, and $\mu = 0.8$ poise. Determine the force per square foot exerted on the upper plate and its direction.

5.15. For $\theta = 90°$ in Fig. 5.43, what speed U is required for no discharge? $\gamma = 55$ lb/ft^3, $a = 0.02$ ft, $p_1 = p_2$, and $\mu = 0.004$ lb-sec/ft^2.

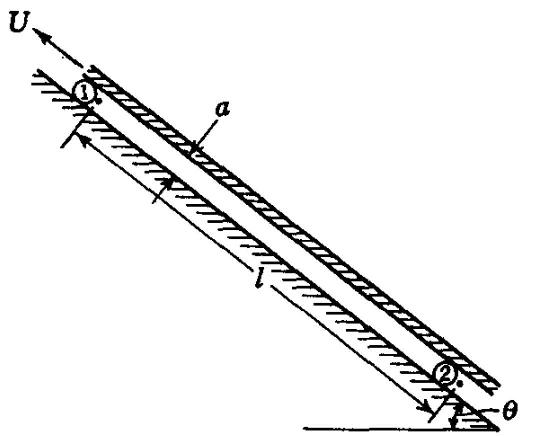


FIG. 5.43

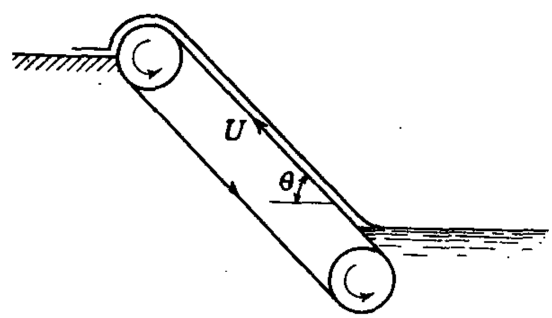


FIG. 5.44

5.16. The belt conveyor (Fig. 5.44) is of sufficient length that the velocity on the free liquid surface is zero. By considering only the work done by the belt on the fluid in shear, how efficient is this device in transferring energy to the fluid?

5.17. A film of fluid 0.005 ft thick flows down a fixed vertical surface with a surface velocity of 2 ft/sec. Determine the fluid viscosity. $\gamma = 60$ lb/ft^3.

5.18. Determine the momentum correction factor for laminar flow in a round tube.

5.19. What are the losses per pound per foot of tubing for flow of mercury at 60°F through 0.002 ft diameter at a Reynolds number of 1800?

5.20. Determine the shear stress at the wall of a $\frac{1}{16}$-in.-diameter tube when water at 50°F flows through it with a velocity of 1 ft/sec.

5.21. Determine the pressure drop per 100 ft of $\frac{1}{8}$-in. ID tubing for flow of liquid, $\mu = 60$ centipoises, sp gr $= 0.83$, at a Reynolds number of 20.

5.22. Glycerin at 80°F flows through a $\frac{3}{8}$-in.-diameter pipe with a pressure drop of 5 psi/ft. Find the discharge and the Reynolds number.

5.23. Calculate the diameter of vertical pipe needed for flow of liquid at a Reynolds number of 1800 when the pressure remains constant. $\nu = 1.5 \times 10^{-4}$ ft^2/sec.

5.24. Calculate the discharge of the system in Fig. 5.45, neglecting all losses except through the pipe.

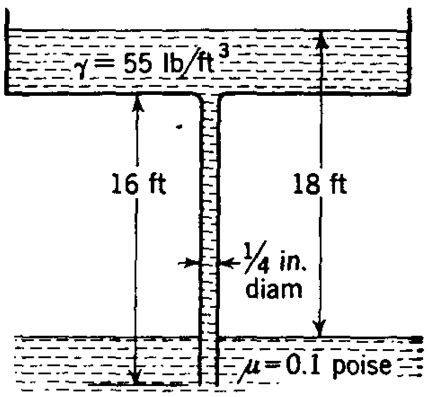


FIG. 5.45

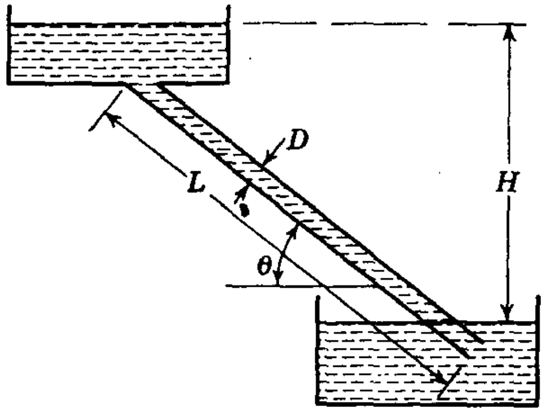


FIG. 5.46

5.25. In Fig. 5.46, $H = 30$ ft, $L = 60$ ft, $\theta = 30°$, $D = \frac{1}{2}$ in., $\gamma = 64$ lb/ft^3, and $\mu = 0.001736$ lb-sec/ft^2. Find the head loss per unit length of pipe and the discharge in gallons per minute.

5.26. In Fig. 5.46 and Prob. 5.25, find H if the velocity is 10 ft/sec.

5.27. At what distance r from the center of a tube of radius r_0 does the average velocity occur in laminar flow?

5.28. Determine the maximum wall shear stress for laminar flow in a tube of diameter D with fluid properties μ and ρ given.

5.29. Show that laminar flow between parallel plates may be used in place of flow through an annulus for 1 per cent accuracy if the clearance is no more than 4 per cent of the inner radius.

5.30. Oil, sp gr 0.85, $\mu = 0.50$ poise, flows through an annulus $a = 0.60$ in., $b = 0.30$ in. When the shear stress at the outer wall is 0.25 lb/ft^2, calculate (*a*) the pressure drop per foot for a horizontal system, (*b*) the discharge in gallons per hour, and (*c*) the axial force exerted on the inner tube per foot of length.

5.31. What is the Reynolds number for flow of 4000 gpm oil, sp gr 0.86, $\mu = 0.27$ poise, through an 18-in.-diameter pipe?

5.32. Calculate the flow of crude oil, sp gr 0.86, at 80°F in a $\frac{3}{8}$-in.-diameter tube to yield a Reynolds number of 700.

5.33. Determine the velocity of kerosene at 90°F in a 3-in. pipe to be dynamically similar to the flow of 6000 cfm air at 20 psia and 60°F through a 24-in. duct.

5.34. What is the Reynolds number for a sphere 0.004 ft in diameter falling through water at 100°F at 0.5 ft/sec?

5.35. Show that the power input for laminar flow in a round tube is $Q\,\Delta p$ by integration of Eq. (5.1.7).

5.36. By use of the one-seventh-power law of velocity distribution $u/u_{max} = (y/r_0)^{\frac{1}{7}}$, determine the mixing-length distribution l/r_0 in terms of y/r_0 from Eq. (5.4.4).

5.37. A fluid is agitated so that the kinematic eddy viscosity increases linearly from $y = 0$ at the bottom of the tank to 2.0 ft^2/sec at $y = 2$ ft. For uniform particles with fall velocities of 1 ft/sec in still fluid, find the concentration at $y = 1$ if it is 200/ft^3 at $y = 2$.

5.38. Plot a curve of $\epsilon/u_* r_0$ as a function of y/r_0 using Eq. (5.4.11) for velocity distribution in a pipe.

5.39. Find the value of y/r_0 in a pipe where the velocity equals the average velocity.

5.40. A 3-in.-diameter pipe discharges water (submerged) into a reservoir. The average velocity in the pipe is 40 ft/sec. At what distance is the velocity reduced to 1.0 ft/sec? $\Big\{$SUGGESTION: Assume a velocity distribution $u = u_m \left[1 - 3\left(\frac{r}{b}\right)^2 - 2\left(\frac{r}{b}\right)^3\right]$. The momentum per second is then $\frac{6\pi}{35}\rho b^2 u_m^2$.$\Big\}$

5.41. Estimate the skin-friction drag on an airship 400 ft long, average diameter 60 ft, with velocity of 80 mph traveling through air at 13 psia and 80°F.

5.42. The velocity distribution in a boundary layer is given by $u/U = 3(y/\delta) - 2(y/\delta)^2$. Show that the displacement thickness of the boundary layer is $\delta_1 = \delta/6$.

5.43. Using the velocity distribution $u/U = \sin \pi y/2\delta$, determine the equation for growth of the laminar boundary layer and for shear stress along a smooth, flat plate in two-dimensional flow.

5.44. Work out the equations for growth of the turbulent boundary layer, based on the exponential law $u/U = (y/\delta)^{\frac{1}{9}}$ and $f = 0.185/R^{\frac{1}{5}}$. ($\tau_0 = \rho f V^2/8$.)

5.45. Air at 70°F, 14.2 psia, flows along a smooth plate with a velocity of 100 mph. How long does the plate have to be to obtain a boundary-layer thickness of $\frac{1}{4}$ in.?

5.46. What is the terminal velocity of a 2-in.-diameter metal ball, sp gr 3.5, dropped in oil, sp gr 0.80, $\mu = 1$ poise?

5.47. At what speed must a 4-in. sphere travel through water at 50°F to have a drag of 1 lb?

5.48. A spherical balloon contains helium and ascends through air at 14 psia, 40°F. Balloon and pay load weigh 300 lb: What is its diameter to be able to ascend at 10 ft/sec? $C_D = 0.21$.

5.49. How many 100-ft-diameter parachutes ($C_D = 1.2$) should be used to drop a bulldozer weighing 11,000 lb at a terminal speed of 32 ft/sec through air at 14.5 psia, 70°F?

5.50. An object weighing 300 lb is attached to a circular disk and dropped from a plane. What diameter should the disk be to have the object strike the ground at 72 ft/sec? The disk is attached so that it is normal to direction of motion. $p = 14.7$ psia; $t = 70$°F.

5.51. A circular disk 10 ft in diameter is held normal to a 60-mph air stream ($\rho = 0.0024$ slug/ft^3). What force is required to hold it at rest.

5.52. A semitubular cylinder of 3-in. radius with concave side upstream is submerged in water flowing 2 ft/sec. Calculate the drag for a cylinder 24 ft long.

5.53. A projectile of the form of (a), Fig. 5.27, is 108 mm in diameter and travels at 3000 ft/sec through air. $\rho = 0.002$ slug/ft^3; $c = 1000$ ft/sec. What is its drag?

5.54. If an airplane 1 mile above the earth passes over an observer and the observer does not hear the plane until it has traveled 1.6 miles farther, what is its speed? Sound velocity is 1080 ft/sec. What is its Mach angle?

5.55. What is the ratio of lift to drag for the airfoil section of Fig. 5.25 for an angle of attack of 2°?

5.56. Determine the settling velocity of small metal spheres, sp gr 4.5, 0.004 in. diameter, in crude oil at 80°F.

5.57. How large a spherical particle of dust, sp gr 2.5, will settle in atmospheric air at 70°F in obedience to Stokes' law? What is the settling velocity?

5.58. The Chézy coefficient is 127 for flow in a rectangular channel 6 ft wide, 2 ft deep, with bottom slope of 0.0016. What is the discharge?

5.59. A rectangular channel 4 ft wide, Chézy $C = 60$, $S = 0.0064$, carries 40 cfs. Determine the velocity.

5.60. What is the value of the Manning roughness factor n in Prob. 5.59?

5.61. A rectangular, brick-lined channel 6 ft wide and 5 ft deep carries 210 cfs. What slope is required for the channel?

5.62. The channel cross section shown in Fig. 5.47 is made of unplaned wood and has a slope of 0.0009. What is the discharge?

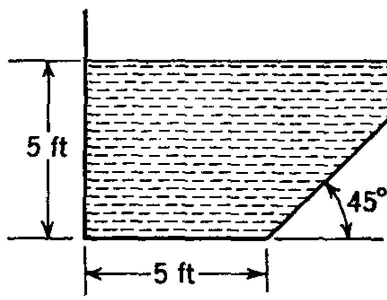


FIG. 5.47

5.63. A trapezoidal, unfinished concrete channel carries water at a depth of 6 ft. Its bottom width is 8 ft and side slope 1 horizontal to $1\frac{1}{2}$ vertical. For a bottom slope of 0.004 what is the discharge?

5.64. A trapezoidal channel with bottom slope 0.003, bottom width of 4 ft, and side slopes 2 horizontal to 1 vertical carries 220 cfs at a depth of 4 ft. What is the Manning roughness factor?

5.65. A trapezoidal earth canal, bottom width 8 ft and side slope 2 on 1 (2 horizontal to 1 vertical), is to be constructed to carry 280 cfs. The best velocity for nonscouring is 2.8 ft/sec with this material. What is the bottom slope required?

5.66. What diameter is required of a semicircular corrugated-metal channel to carry 50 cfs when its slope is 0.01?

5.67. A semicircular corrugated-metal channel 10 ft in diameter has a bottom slope of 0.004. What is its capacity when flowing full?

5.68. Calculate the depth of flow of 2000 cfs in a gravel trapezoidal channel with bottom width of 12 ft, side slopes of 3 horizontal to 1 vertical, and bottom slope of 0.001.

5.69. What is the velocity of flow of 260 cfs in a rectangular channel 12 ft wide? $S = 0.0049$; $n = 0.016$.

5.70. A trapezoidal channel, brick-lined, is to be constructed to carry 1200 cfs 5 miles with a head loss of 12 ft. The bottom width is 16 ft, the side slopes 1 on 1. What is the velocity?

5.71. How does the discharge vary with depth in Fig. 5.48?

5.72. How does the velocity vary with depth in Fig. 5.48?

5.73. Determine the depth of flow in Fig. 5.48 for discharge of 12 cfs. It is made of riveted steel with bottom slope 0.02.

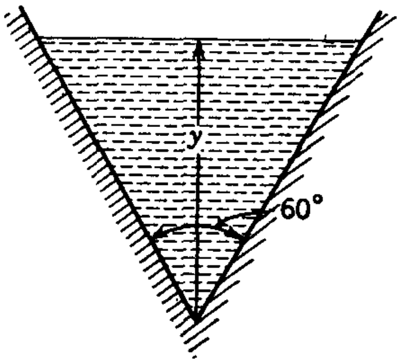


FIG. 5.48

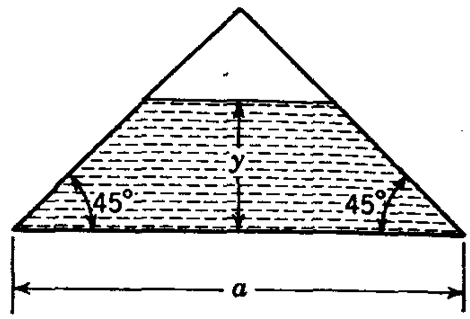


FIG. 5.49

5.74. Determine the depth y (Fig. 5.49) for maximum velocity for given n and S.

5.75. Determine the depth y (Fig. 5.49) for maximum discharge for given n and S.

5.76. A test on a 12-in.-diameter pipe with water showed a gage difference of 13 in. on a mercury-water manometer connected to two piezometer rings 400 ft apart. The flow was 8.24 cfs. What is the friction factor?

5.77. By using the Blasius equation for determination of friction factor, determine the horsepower per mile required to pump 3.0 cfs liquid, $\nu = 3.3 \times 10^{-4}$ ft²/sec, $\gamma = 55$ lb/ft³, through a 12-in. pipeline.

5.78. Determine the head loss per 1000 ft required to maintain a velocity of 14 ft/sec in a 0.50-in.-diameter pipe. $\nu = 4 \times 10^{-4}$ ft²/sec.

5.79. Fluid flows through a $\frac{1}{2}$-in.-diameter tube at a Reynolds number of 1600. The head loss is 30 ft in 100 ft of tubing. Calculate the discharge in gallons per minute.

5.80. What size galvanized-iron pipe is needed to be "hydraulically smooth" at $R = 3.5 \times 10^5$? (A pipe is said to be hydraulically smooth when it has the same losses as a smoother pipe under the same conditions.)

5.81. Above what Reynolds number is the flow through an 8-ft-diameter riveted steel pipe, $\epsilon = 0.01$, independent of the viscosity of the fluid?

5.82. Determine the absolute roughness of a 2-ft-diameter pipe that has a friction factor $f = 0.03$ for $R = 1{,}000{,}000$.

5.83. What diameter clean galvanized-iron pipe has the same friction factor for $R = 100{,}000$ as a 12-in.-diameter cast-iron pipe?

5.84. Under what conditions do the losses in a pipe vary as some power of the velocity greater than the second?

5.85. Why does the friction factor increase as the velocity decreases in laminar flow in a pipe?

5.86. Look up the friction factor for atmospheric air at 60°F traveling 80 ft/sec through a 3-ft-diameter galvanized pipe.

5.87. Water at 70°F is to be pumped through 1200 ft of 8-in.-diameter wrought-iron pipe at the rate of 1000 gpm. Compute the head loss and horsepower required.

5.88. 16,000 ft³/min atmospheric air at 90°F is conveyed 1000 ft through a 4-ft-diameter galvanized pipe. What is the head loss in inches of water?

5.89. 2.0 cfs oil, $\mu = 0.16$ poise, $\gamma = 53$ lb/ft^3, is pumped through a 12-in. pipeline of cast iron. If each pump produces 80 psi, how far apart may they be placed?

5.90. A 2.5-in.-diameter smooth pipe 500 ft long conveys 200 gpm water at 80°F from a water main, $p = 100$ psi, to the top of a building 85 ft above the main. What pressure can be maintained at the top of the building?

5.91. For water at 150°F, calculate the discharge for the pipe of Fig. 5.50.

5.92. In Fig. 5.50, how much power would be required to pump 160 gpm from a reservoir at the bottom of the pipe to the reservoir shown?

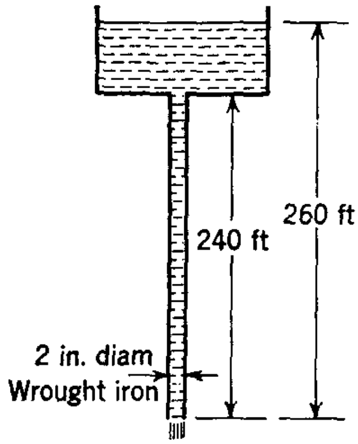


FIG. 5.50

5.93. A $\frac{1}{2}$-in.-diameter commercial steel pipe 40 ft long is used to drain an oil tank. Determine the discharge when the oil level in the tank is 6 ft above the exit end of the pipe. $\mu = 0.10$ poise; $\gamma = 50$ lb/ft^3.

5.94. Two liquid reservoirs are connected by 200 ft of 2-in.-diameter smooth tubing. What is the flow rate when the difference in elevation is 50 ft? $\nu = 0.001$ ft^2/sec.

5.95. For a head loss of 2-in. water in a length of 600 ft for flow of atmospheric air at 60°F through a 4-ft-diameter duct, $\epsilon = 0.003$ ft, calculate the flow in gallons per minute.

5.96. A gas of molecular weight 37 flows through a galvanized 24-in.-diameter duct at a pressure of 90 psia and 100°F. The head loss per 100 ft of duct is 2 in. water. What is the mass flow in slugs per hour?

5.97. What is the horsepower per mile required for a 70 per cent efficient blower to maintain the flow of Prob. 5.96?

5.98. 100 lb$_m$/min air is required to ventilate a mine. It is admitted through 2000 ft of 12-in.-diameter galvanized pipe. Neglecting minor losses, what head in inches of water does a blower have to produce to furnish this flow? $p = 14$ psia; $t = 90$°F.

5.99. In Fig. 5.46 $H = 60$ ft, $L = 500$ ft, $D = 2$ in., $\gamma = 55$ lb/ft^3, $\mu = 0.04$ poise, $\epsilon = 0.003$ ft. Find the pounds per second flowing.

5.100. In a process 10,000 lb/hr of distilled water at 70°F is conducted through a smooth tube between two reservoirs having a distance between them of 40 ft and a difference in elevation of 4 ft. What size tubing is needed?

5.101. What size of new cast-iron pipe is needed to transport 10 cfs water at 80°F 1 mile with head loss of 6 ft?

5.102. Two types of steel plate, having surface roughnesses of $\epsilon_1 = 0.0003$ ft and $\epsilon_2 = 0.001$ ft, have a cost differential of 10 per cent more for the smoother plate. With an allowable stress in each of 10,000 psi, which plate should be selected to convey 100 cfs water at 200 psi with a head loss of 6 ft/mile?

5.103. An old pipe 48 in. in diameter has a roughness of $\epsilon = 0.1$ ft. A $\frac{1}{2}$-in.-thick lining would reduce the roughness to $\epsilon = 0.0004$. How much in pumping costs would be saved per year per 1000 ft of pipe for water at 70°F with velocity of 8 ft/sec? The pumps and motors are 80 per cent efficient and power costs 1 cent per kilowatthour.

5.104. Calculate the diameter of new wood-stove pipe in excellent condition needed to convey 300 cfs water at 60°F with a head loss of 1 ft per 1000 ft of pipe.

5.105. Two oil reservoirs with difference in elevation of 12 ft are connected by 1000 ft of commercial steel pipe. What size must the pipe be to convey 1000 gpm? $\mu = 0.001$ slug/ft-sec; $\gamma = 55$ lb/ft^3.

5.106. 200 cfs air, $p = 16$ psia, $t = 90$°F, is to be delivered to a mine with a head loss of 3 n. water per 1000 ft. What size galvanized pipe is needed?

5.107. Compute the losses in foot-pounds per pound due to flow of 600 cfm air, $p = 14.7$ psia, $t = 70$°F, through a sudden expansion from 12- to 36-in. pipe. How much head would be saved by using a 10° conical diffuser?

5.108. Calculate the value of H in Fig. 5.51 for 6 cfs water at 60°F through commercial steel pipe. Include minor losses.

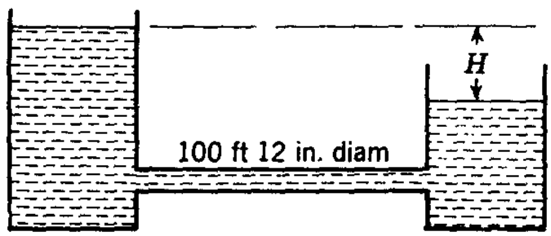


FIG. 5.51

5.109. In Fig. 5.51 for $H = 10$ ft, calculate the discharge of oil, $\gamma = 55$ lb/ft^3, $\mu = 0.07$ poise, through smooth pipe. Include minor losses.

5.110. If a valve is placed in the line in Prob. 5.109 and adjusted to reduce the discharge by one-half, what is K for the valve and what is its equivalent length of pipe at this setting?

5.111. A water line connecting two reservoirs at 70°F has 4000 ft of 24-in.-diameter steel pipe, three standard elbows, a globe valve, and a re-entrant pipe entrance. What is the difference in reservoir elevations for 20 cfs?

5.112. Determine the discharge in Prob. 5.111 if the difference in elevation is 40 ft.

5.113. Compute the losses in horsepower due to flow of 100 cfs water through a sudden contraction from 6- to 4-ft-diameter pipe.

5.114. What is the equivalent length of 2-in.-diameter pipe, $f = 0.022$, for (*a*) a re-entrant pipe entrance, (*b*) a sudden expansion from 2 to 4 in. diameter, (*c*) a globe valve and a standard tee?

5.115. Find H in Fig. 5.52 for 100-gpm oil flow, $\mu = 0.1$ poise, $\gamma = 60$ lb/ft^3, for the angle valve wide open.

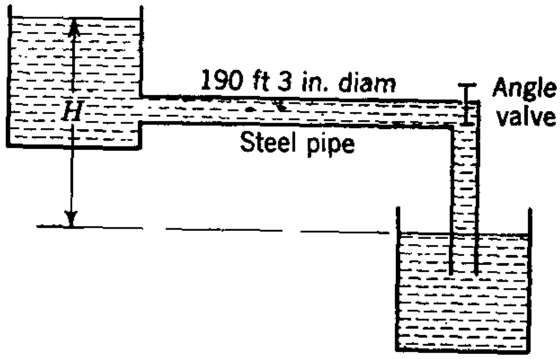


FIG. 5.52

5.116. Find K for the angle valve in Prob. 5.115 for flow of 60 gpm at the same H.

5.117. What is the discharge through the system of Fig. 5.52 for water at 80°F when $H = 16$ ft?

5.118. Compare the smooth-pipe curve on the Moody diagram with Eq. (5.9.4) for $\mathbf{R} = 10^5, 10^6, 10^7$.

5.119. Check the location of line $\epsilon/D = 0.0002$ on the Moody diagram with Eq. (5.9.7).

5.120. In Eq. (5.9.7) show that when $\epsilon = 0$, it reduces to Eq. (5.9.4) and that, when $\mathbf{R}$ is very large, it reduces to Eq. (5.9.6).

5.121. In Fig. 5.53 the rocker plate has a width of 1 ft. Calculate (*a*) the load the bearing will sustain, (*b*) the drag on the bearing. Assume no flow normal to the paper.

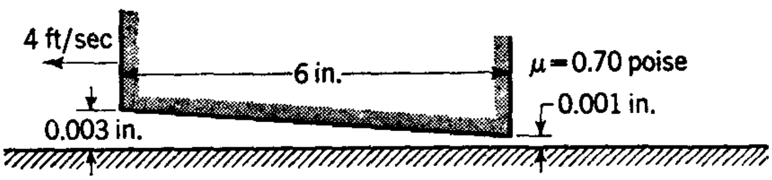


FIG. 5.53

5.122. Find the maximum pressure in the fluid of Prob. 5.121, and determine its location.

5.123. Determine the pressure center for the rocker plate of Prob. 5.121.

5.124. Show that a shaft concentric with a bearing can sustain no load.

5.125. The shear stress in a fluid flowing between two fixed parallel plates

(*a*) is constant over the cross section
(*b*) is zero at the plates and increases linearly to the mid-point
(*c*) varies parabolically across the section
(*d*) is zero at the midplane and varies linearly with distance from the midplane
(*e*) is none of these answers

5.126. The velocity distribution for flow between two fixed parallel plates

(*a*) is constant over the cross section
(*b*) is zero at the plates and increases linearly to the midplane
(*c*) varies parabolically across the section
(*d*) varies as the three-halves power of the distance from the mid-point
(*e*) is none of these answers

5.127. The discharge between two parallel plates, distant a apart, when one has the velocity U and the shear stress is zero at the fixed plate, is

(*a*) $Ua/3$ (*b*) $Ua/2$ (*c*) $2Ua/3$ (*d*) Ua (*e*) none of these answers

5.128. Fluid is in laminar motion between two parallel plates, with one plate in motion and is under the action of a pressure gradient so that the discharge through any fixed cross section is zero. The minimum velocity occurs at a point which is distant from the fixed plate

(*a*) $a/6$ (*b*) $a/3$ (*c*) $a/2$ (*d*) $2a/3$ (*e*) none of these answers

5.129. In Prob. 5.128 the value of the minimum velocity is

(*a*) $-3U/4$ (*b*) $-2U/3$ (*c*) $-U/2$ (*d*) $-U/3$ (*e*) $-U/6$

5.130. The relation between pressure and shear stress in one-dimensional laminar flow in the x-direction is given by

(*a*) $dp/dx = \mu\, d\tau/dy$ (*b*) $dp/dy = d\tau/dx$ (*c*) $dp/dy = \mu\, d\tau/dx$
(*d*) $dp/dx = d\tau/dy$ (*e*) none of these answers

5.131. The expression for power input per unit volume to a fluid in one-dimensional laminar motion in the x-direction is

(*a*) $\tau\, du/dy$ (*b*) τ/μ^2 (*c*) $\mu\, du/dy$ (*d*) $\tau(du/dy)^2$
(*e*) none of these answers

5.132. When liquid is in laminar motion at constant depth in flowing down an inclined plate (y measured normal to surface),

(*a*) the shear is zero throughout the liquid
(*b*) $d\tau/dy = 0$ at the plate
(*c*) $\tau = 0$ at the surface of the liquid
(*d*) the velocity is constant throughout the liquid
(*e*) there are no losses

5.133. The shear stress in a fluid flowing in a round pipe

(*a*) is constant over the cross section
(*b*) is zero at the wall and increases linearly to the center
(*c*) varies parabolically across the section
(*d*) is zero at the center and varies linearly with the radius
(*e*) is none of these answers

5.134. When the pressure drop in a 24-in.-diameter pipeline is 10 psi in 100 ft, the wall shear stress in pounds per square foot is

(*a*) 0 (*b*) 7.2 (*c*) 14.4 (*d*) 720 (*e*) none of these answers

5.135. In laminar flow through a round tube the discharge varies

(*a*) linearly as the viscosity
(*b*) as the square of the radius
(*c*) inversely as the pressure drop
(*d*) inversely as the viscosity
(*e*) as the cube of the diameter

5.136. When a tube is inclined, the term $-dp/dl$ is replaced by

(*a*) $-dz/dl$ (*b*) $-\gamma\, dz/dl$ (*c*) $-d(p + z)/dl$
(*d*) $-d(p + \rho z)/dl$ (*e*) $-d(p + \gamma z)/dl$

5.137. The upper critical Reynolds number is

(*a*) important from a design viewpoint
(*b*) the number at which turbulent flow changes to laminar flow
(*c*) about 2000
(*d*) not more than 2000
(*e*) of no practical importance in pipe-flow problems

5.138. The Reynolds number for pipe flow is given by

(*a*) VD/ν (*b*) $VD\mu/\rho$ (*c*) $VD\rho/\nu$ (*d*) VD/μ (*e*) none of these answers

5.139. The lower critical Reynolds number has the value

(*a*) 200 (*b*) 1200 (*c*) 12,000 (*d*) 40,000 (*e*) none of these answers

5.140. The Reynolds number for a 1.0-in.-diameter sphere moving 10 ft/sec through oil, sp gr 0.90, $\mu = 0.002$ lb-sec/ft^2, is

(*a*) 375 (*b*) 725 (*c*) 806 (*d*) 8700 (*e*) none of these answers

5.141. The Reynolds number for 10 cfs discharge of water at 68°F through a 12-in.-diamter pipe is

(*a*) 2460 (*b*) 980,000 (*c*) 1,178,000 (*d*) 14,120,000
(*e*) none of these answers

5.142. The Prandtl mixing length is

(*a*) independent of radial distance from pipe axis
(*b*) independent of the shear stress
(*c*) zero at the pipe wall
(*d*) a universal constant
(*e*) useful for computing laminar-flow problems

5.143. In a fluid stream of low viscosity

(*a*) the effect of viscosity does not appreciably increase the drag on a body
(*b*) the potential theory yields the drag force on a body
(*c*) the effect of viscosity is limited to a narrow region surrounding a body
(*d*) the deformation drag on a body always predominates
(*e*) the potential theory contributes nothing of value regarding flow around bodies

5.144. The lift on a body immersed in a fluid stream is

(*a*) due to buoyant force
(*b*) always in the opposite direction to gravity
(*c*) the resultant fluid force on the body

(*d*) the dynamic fluid-force component exerted on the body normal to the approach velocity
(*e*) the dynamic fluid-force component exerted on the body parallel to the approach velocity

5.145. The displacement thickness of the boundary layer is

(*a*) the distance from the boundary affected by boundary shear
(*b*) one-half the actual thickness of the boundary layer
(*c*) the distance to the point where $u/U = 0.99$
(*d*) the distance the main flow is shifted
(*e*) none of these answers

5.146. The shear stress at the boundary of a flat plate is

(*a*) $\partial p/\partial x$ (*b*) $\mu\, \partial u/\partial y|_{y=0}$ (*c*) $\rho\, \partial u/\partial y|_{y=0}$ (*d*) $\mu\, \partial u/\partial y|_{y=\delta}$
(*e*) none of these answers

5.147. Which of the following velocity distributions u/U satisfy the boundary conditions for flow along a flat plate? $\eta = y/\delta$.

(*a*) e^{η} (*b*) $\cos \pi\eta/2$ (*c*) $\eta - \eta^2$ (*d*) $2\eta - \eta^3$
(*e*) none of these answers

5.148. The drag coefficient for a flat plate is (D = drag)

(*a*) $2D/\rho U^2 l$ (*b*) $\rho U l/D$ (*c*) $\rho U l/2D$ (*d*) $\rho U^2 l/2D$
(*e*) none of these answers

5.149. The average velocity divided by the maximum velocity, as given by the one-seventh-power law, is

(*a*) $\frac{49}{120}$ (*b*) $\frac{1}{2}$ (*c*) $\frac{6}{7}$ (*d*) $\frac{98}{120}$ (*e*) none of these answers

5.150. The laminar-boundary-layer thickness varies as

(*a*) $1/x^{\frac{1}{2}}$ (*b*) $x^{\frac{1}{7}}$ (*c*) $x^{\frac{1}{2}}$ (*d*) $x^{\frac{6}{7}}$ (*e*) none of these answers

5.151. The turbulent-boundary-layer thickness varies as

(*a*) $1/x^{\frac{1}{5}}$ (*b*) $x^{\frac{1}{5}}$ (*c*) $x^{\frac{1}{2}}$ (*d*) $x^{\frac{4}{5}}$ (*e*) none of these answers

5.152. In flow along a rough plate, the order of flow type from upstream to downstream is

(*a*) laminar, fully developed wall roughness, transition region, hydraulically smooth
(*b*) laminar, transition region, hydraulically smooth, fully developed wall roughness
(*c*) laminar, hydraulically smooth, transition region, ully developed wall roughness
(*d*) laminar, hydraulically smooth, fully developed wall roughness, transition region
(*e*) laminar, fully developed wall roughness, hydraulically smooth, transition region

5.153. Separation is caused by

(*a*) reduction of pressure to vapor pressure
(*b*) reduction of pressure gradient to zero
(*c*) an adverse pressure gradient
(*d*) the boundary-layer thickness reducing to zero
(*e*) none of these answers

5.154. Separation occurs when

(*a*) the cross section of a channel is reduced
(*b*) the boundary layer comes to rest
(*c*) the velocity of sound is reached
(*d*) the pressure reaches a minimum
(*e*) a valve is closed

5.155. The wake

(*a*) is a region of high pressure
(*b*) is the principal cause of skin friction
(*c*) always occurs when deformation drag predominates
(*d*) always occurs after a separation point
(*e*) is none of these answers

5.156. Pressure drag results from

(*a*) skin friction
(*b*) deformation drag
(*c*) breakdown of potential flow near the forward stagnation point
(*d*) occurrence of a wake
(*e*) none of these answers

5.157. A body with a rounded nose and long, tapering tail is usually best suited for

(*a*) laminar flow
(*b*) turbulent subsonic flow
(*c*) supersonic flow
(*d*) flow at speed of sound
(*e*) none of these answers

5.158. A sudden change in position of the separation point in flow around a sphere occurs at a Reynolds number of about

(*a*) 1 (*b*) 300 (*c*) 30,000 (*d*) 3,000,000 (*e*) none of these answers

5.159. The effect of compressibility on the drag force is to

(*a*) greatly increase it near the speed of sound
(*b*) decrease it near the speed of sound
(*c*) cause it to asymptotically approach a constant value for large Mach numbers

(d) cause it to increase more rapidly than the square of the speed at high Mach numbers
(e) reduce it throughout the whole flow range

5.160. The terminal velocity of a small sphere settling in a viscous fluid varies as the

(a) first power of its diameter
(b) inverse of the fluid viscosity
(c) inverse square of the diameter
(d) inverse of the diameter
(e) square of the difference in specific weights of solid and fluid

5.161. The losses in open-channel flow generally vary as the

(a) first power of the roughness
(b) inverse of the roughness
(c) square of the velocity
(d) inverse square of the hydraulic radius
(e) velocity

5.162. The most simple form of open-channel-flow computation is

(a) steady uniform
(b) steady nonuniform
(c) unsteady uniform
(d) unsteady nonuniform
(e) gradually varied

5.163. In an open channel of great width the hydraulic radius equals

(a) $y/3$ (b) $y/2$ (c) $2y/3$ (d) y (e) none of these answers

5.164. The Manning roughness coefficient for finished concrete is

(a) 0.002 (b) 0.020 (c) 0.20 (d) dependent upon hydraulic radius (e) none of these answers

5.165. In turbulent flow a rough pipe has the same friction factor as a smooth pipe

(a) in the zone of complete turbulence, rough pipes
(b) when the friction factor is independent of Reynolds number
(c) when the roughness projections are much smaller than the thickness of the boundary layer
(d) everywhere in the transition zone
(e) when the friction factor is constant

5.166. The friction factor in turbulent flow in smooth pipes depends upon the following:

(a) V, D, p, L, μ
(b) Q, L, μ, ρ
(c) V, D, ρ, p, μ

(*d*) V, D, μ, ρ
(*e*) p, L, D, Q, V

5.167. In a *given* rough pipe, the losses depend upon

(*a*) f, V
(*b*) μ, ρ
(*c*) **R**
(*d*) Q only
(*e*) none of these answers

5.168. In the complete-turbulence zone, rough pipes,

(*a*) rough and smooth pipes have the same friction factor
(*b*) the laminar film covers the roughness projections
(*c*) the friction factor depends upon Reynolds number only
(*d*) the head loss varies as the square of the velocity
(*e*) the friction factor is independent of the relative roughness

5.169. The friction factor for flow of water at 60°F through a 2-ft-diameter cast-iron pipe with a velocity of 5 ft/sec is

(*a*) 0.013 (*b*) 0.017 (*c*) 0.019 (*d*) 0.021 (*e*) none of these answers

5.170. The procedure to follow in solving for losses when Q, L, D, ν, and ϵ are given is to

(*a*) assume an f, look up **R** on Moody diagram, etc.
(*b*) assume an h_f, solve for f, check against **R** on Moody diagram
(*c*) assume an f, solve for h_f, compute **R**, etc.
(*d*) compute **R**, look up f for ϵ/D, solve for h_f
(*e*) assume an **R**, compute V, look up f, solve for h_f

5.171. The procedure to follow in solving for discharge when h_f, L, D, ν, and ϵ are given is to

(*a*) assume an f, compute V, **R**, ϵ/D, look up f, and repeat if necessary
(*b*) assume an **R**, compute f, check ϵ/D, etc.
(*c*) assume a V, compute **R**, look up f, compute V again, etc.
(*d*) solve Darcy-Weisbach for V, compute Q
(*e*) assume a Q, compute V, **R**, look up f, etc.

5.172. The procedure to follow in solving for pipe diameter when h_f, Q, L, ν, and ϵ are given is to

(*a*) assume a D, compute V, **R**, ϵ/D, look up f, and repeat
(*b*) compute V from continuity, assume an f, solve for D
(*c*) eliminate V in **R** and Darcy-Weisbach, using continuity, assume an f, solve for D, **R**, look up f, and repeat
(*d*) assume an **R** and an ϵ/D, look up f, solve Darcy-Weisbach for V^2/D, and solve simultaneously with continuity for V and D, compute new **R**, etc.
(*e*) assume a V, solve for D, **R**, ϵ/D, look up f, and repeat

5.173. The losses due to a sudden contraction are given by

(a) $\left(\frac{1}{C_c^2} - 1\right)\frac{V_2^2}{2g}$ (b) $(1 - C_c^2)\frac{V_2^2}{2g}$ (c) $\left(\frac{1}{C_c} - 1\right)^2\frac{V_2^2}{2g}$
(d) $(C_c - 1)^2\frac{V_2^2}{2g}$ (e) none of these answers

5.174. The losses at the exit of a submerged pipe in a reservoir are

(a) negligible (b) $0.05(V^2/2g)$ (c) $0.5(V^2/2g)$ (d) $V^2/2g$
(e) none of these answers

5.175. Minor losses usually may be neglected when

(a) there are 100 ft of pipe between special fittings
(b) their loss is 5 per cent or less of the friction loss
(c) there are 500 diameters of pipe between minor losses
(d) there are no globe valves in the line
(e) rough pipe is used

5.176. The length of pipe ($f = 0.025$) in diameters, equivalent to a globe valve, is

(a) 40 (b) 200 (c) 300 (d) 400 (e) not determinable; insufficient data

5.177. The hydraulic radius is given by

(a) wetted perimeter divided by area
(b) area divided by square of wetted perimeter
(c) square root of area
(d) area divided by wetted perimeter
(e) none of these answers

5.178. The hydraulic radius of a 6-in. by 12-in. cross section is, in feet,

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$ (e) none of these answers

5.179. In the theory of lubrication the assumption is made that

(a) the velocity distribution is the same at all cross sections
(b) the velocity distribution at any section is the same as if the plates were parallel
(c) the pressure variation along the bearing is the same as if the plates were parallel
(d) the shear stress varies linearly between the two surfaces
(e) the velocity varies linearly between the two surfaces

5.180. A 4-in.-diameter shaft rotates at 240 rpm in a bearing with a radial clearance of 0.006 in. The shear stress in an oil film, $\mu = 0.1$ poise, is, in pounds per square foot,

(a) 0.15 (b) 1.75 (c) 3.50 (d) 16.70 (e) none of these answers

6

COMPRESSIBLE FLOW

In Chap. 5 viscous incompressible-fluid-flow situations were mainly considered. In this chapter on compressible flow, one new variable enters, the density, and one extra equation is available, the equation of state, which relates pressure and density. The other equations—continuity, momentum, and the first and second laws of thermodynamics—are also needed in the analysis of compressible-fluid-flow situations. In this chapter topics in steady one-dimensional flow of a perfect gas are discussed. The one-dimensional approach is limited to those applications in which the velocity and density may be considered constant over any cross section. When density changes are gradual and do not change by more than a few per cent, the flow may be treated as incompressible with the use of an average density.

The following topics are discussed in this chapter: perfect-gas relationships, speed of a sound wave, Mach number, isentropic flow, shock waves, Fanno and Rayleigh lines, adiabatic flow, flow with heat transfer, isothermal flow, high-speed flight, and the analogy between shock waves and open-channel waves.

6.1. Perfect-gas Relationships. In Sec. 1.6 [Eq. (1.6.2)] a perfect gas is defined as a fluid that has constant specific heats and that follows the law

$$p = \rho RT \tag{6.1.1}$$

in which p and T are the absolute pressure and absolute temperature, respectively, ρ is the density, and R the gas constant. In this section specific heats are defined, the specific heat ratio is introduced and related to specific heats and the gas constant, internal energy and enthalpy are related to temperature, entropy relations are established, and the isentropic and reversible polytropic processes are introduced.

In general, the specific heat at constant volume c_v is defined by

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v \tag{6.1.2}$$

in which u is the internal energy per unit mass. In words, c_v is the amount of internal energy increase required by a unit mass of gas to increase its temperature by one degree when its volume is held constant. In thermodynamic theory it is proved that u is a function only of temperature for a perfect gas.

The specific heat at constant pressure c_p is defined by

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p \tag{6.1.3}$$

in which h is the enthalpy per unit mass given by $h = u + p/\rho$. Since p/ρ is equal to RT and u is a function only of temperature for a perfect gas, h depends only on temperature. Many of the common gases, such as water vapor, hydrogen, oxygen, carbon monoxide, and air, have a fairly small change in specific heats over the temperature range 500 to 1000°R, and an intermediate value is taken for their use as perfect gases. Table C.2 of Appendix C lists some common gases with values of specific heats at 80°F.

For perfect gases Eq. (6.1.2) becomes

$$du = c_v \, dT \tag{6.1.4}$$

and Eq. (6.1.3) becomes

$$dh = c_p \, dT \tag{6.1.5}$$

Then, from

$$h = u + \frac{p}{\rho} = u + RT$$

differentiating

$$dh = du + R \, dT$$

and by substitution of Eqs. (6.1.4) and (6.1.5)

$$c_p = c_v + R \tag{6.1.6}$$

which is valid for any gas obeying Eq. (1.6.2) (even when c_p and c_v are changing with temperature). If c_p and c_v are given in heat units per unit mass (i.e., Btu per pound mass per degree Rankine or Btu per slug per degree Rankine), then R must be in heat units also (i.e., Btu per pound mass per degree Rankine or Btu per slug per degree Rankine). The conversion factor is 1 Btu = 778 ft-lb if it is desired to express units in the foot-pound-second system.

The *specific-heat ratio* k is defined as the ratio

$$k = \frac{c_p}{c_v} \tag{6.1.7}$$

By solving with Eq. (6.1.6)

$$c_p = \frac{k}{k-1} R \qquad c_v = \frac{R}{k-1} \tag{6.1.8}$$

Entropy Relationships. The internal energy change for a perfect gas is

$$u_2 - u_1 = c_v(T_2 - T_1) \tag{6.1.9}$$

and the enthalpy change

$$h_2 - h_1 = c_p(T_2 - T_1) \tag{6.1.10}$$

From Eq. (3.6.15)

$$T\,ds = du + p d\,\frac{1}{\rho} \tag{3.6.15}$$

which is a relationship among thermodynamic properties and must hold for all pure substances; the change in entropy s may be obtained

$$ds = \frac{du}{T} + \frac{p}{T}\,d\,\frac{1}{\rho} = c_v\,\frac{dT}{T} + R\rho d\,\frac{1}{\rho} \tag{6.1.11}$$

from Eqs. (6.1.4) and (6.1.1). After integrating,

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{\rho_1}{\rho_2} \tag{6.1.12}$$

By use of Eqs. (6.1.8) and (6.1.1), Eq. (6.1.12) becomes

$$s_2 - s_1 = c_v \ln \left[\frac{T_2}{T_1}\left(\frac{\rho_1}{\rho_2}\right)^{k-1}\right] \tag{6.1.13}$$

or

$$s_2 - s_1 = c_v \ln \left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{k}\right] \tag{6.1.14}$$

and

$$s_2 - s_1 = c_v \ln \left[\left(\frac{T_2}{T_1}\right)^{k}\left(\frac{p_2}{p_1}\right)^{1-k}\right] \tag{6.1.15}$$

These equations are forms of the second law of thermodynamics.

An *isentropic* process is a reversible adiabatic process. Equation (3.8.3)

$$T\,ds = dq_H + d\text{ (losses)} \tag{3.8.3}$$

shows that $ds = 0$ for an isentropic process, since there is no heat transfer; $dq_H = 0$; and there are no losses. Then, from Eq. (6.1.14) for $s_2 = s_1$

$$\frac{p_1}{\rho_1{}^k} = \frac{p_2}{\rho_2{}^k} \tag{6.1.16}$$

Equation (6.1.16) combined with the general gas law yields

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{(k-1)/k} = \left(\frac{\rho_2}{\rho_1}\right)^{k-1} \tag{6.1.17}$$

The enthalpy change for an isentropic process is

$$h_2 - h_1 = c_p(T_2 - T_1) = c_pT_1\left(\frac{T_2}{T_1} - 1\right) = c_pT_1\left[\left(\frac{p_2}{p_1}\right)^{(k-1)/k} - 1\right] \tag{6.1.18}$$

The *polytropic* process is defined by

$$\frac{p}{\rho^n} = \text{constant} \tag{6.1.19}$$

and is an approximation to certain actual processes in which p would plot substantially as a straight line against ρ on log-log paper. This relationship is frequently used to calculate the work when the polytropic process is reversible, by substitution into the relation $w = \int p\, d\mathcal{V}$. Heat transfer occurs in a reversible polytropic process except when $n = k$, the isentropic case.

Example 6.1: Express R in Btu per slug per degree Rankine for helium.

A conversion from 1 ft-lb/lb$_m$ °R to the Btu per slug per degree Rankine is made first. Since 1 Btu = 778 ft-lb and 1 slug = 32.17 lb$_m$,

$$\frac{1 \text{ ft-lb}}{\text{lb}_m\ °\text{R}} = 1 \frac{\text{Btu}}{778} \frac{32.17}{\text{slug }°\text{R}} = 0.0414 \frac{\text{Btu}}{\text{slug }°\text{R}}$$

Then, for helium, from Table C.2

$$R = 386 \frac{\text{ft-lb}}{\text{lb}_m\ °\text{R}} = 386 \times 0.0414 \frac{\text{Btu}}{\text{slug }°\text{R}} = 16.0 \frac{\text{Btu}}{\text{slug }°\text{R}}$$

Example 6.2: Compute the value of R from the values of k and c_p for air and check in Table C.2.

From Eq. (6.1.8)

$$R = \frac{k-1}{k} c_p = \frac{1.40 - 1.0}{1.40} \times 0.240 = 0.0686 \frac{\text{Btu}}{\text{lb}_m\ °\text{R}}$$

By converting from Btu to foot-pounds,

$$R = 0.0686 \times 778 = 53.3 \frac{\text{ft-lb}}{\text{lb}_m\ °\text{R}}$$

which checks the value in Table C.2.

Example 6.3: Compute the enthalpy change in 7.0 lb$_m$ of oxygen when the initial conditions are p_1 = 20 psia, t_1 = 50°F and final conditions p_2 = 80 psia, t_2 = 200°F.

Enthalpy is a function of temperature only. By use of Eq. (6.1.10), the enthalpy change per pound mass is

$$c_p(T_2 - T_1) = 0.219(200 - 50) = 32.9 \frac{\text{Btu}}{\text{lb}_m}$$

and the enthalpy change for 7.0 lb$_m$

$$H_2 - H_1 = 7.0 \times 32.9 = 230.3 \text{ Btu}$$

Example 6.4: Determine the entropy change in 4.0 slugs of water vapor when the initial conditions are p_1 = 6 psia, t_1 = 110°F and the final conditions are p_2 = 40 psia and t_2 = 38°F.

From Eq. (6.1.15) and Table C.2

$$s_2 - s_1 = 0.335 \ln \left[\left(\frac{460 + 38}{460 + 110} \right)^{1.33} \left(\frac{40}{6} \right)^{-0.33} \right] = -0.271 \frac{\text{Btu}}{\text{lb}_m \text{ °R}}$$

or

$$S_2 - S_1 = -0.271 \times 4.0 \times 32.17 = 34.9 \frac{\text{Btu}}{\text{°R}}$$

Example 6.5: A cylinder containing 3.5 lb_m nitrogen at 20 psia and 40°F is compressed isentropically to 45 psia. Find the final temperature and the work required.

From the principle of conservation of energy, the work done on the gas must equal its increase in internal energy, since there is no heat transfer in an isentropic process; i.e.,

$$u_2 - u_1 = c_v(T_2 - T_1) = \text{work}$$

By Eq. (6.1.17)

$$T_2 = T_1 \left(\frac{p_2}{p_1} \right)^{(k-1)/k} = (460 + 40) \left(\frac{45}{20} \right)^{(1.4-1)/1.4} = 630\text{°R}$$

and

$$\text{Work} = 0.177(630 - 500) \times 3.5 = 80.6 \text{ Btu}$$

Example 6.6: 3.0 slugs of air are involved in a reversible polytropic process in which the initial conditions $p_1 = 12$ psia, $t_1 = 60$°F change to $p_2 = 20$ psia, and volume $V = 1011$ ft³. Determine (*a*) the formula for the process, (*b*) the work done on the air, (*c*) the amount of heat transfer, and (*d*) the entropy change.

a.
$$\rho_1 = \frac{p_1}{RT_1} = \frac{12 \times 144}{53.3 \times 32.17(460 + 60)} = 0.00194 \frac{\text{slug}}{\text{ft}^3}$$

R was converted to foot-pounds per slug degree Rankine by multiplying by 32.17. Also

$$\rho_2 = \frac{3}{1011} = 0.002965 \frac{\text{slug}}{\text{ft}^3}$$

From Eq. (6.1.19)

$$\frac{p_1}{\rho_1{}^n} = \frac{p_2}{\rho_2{}^n}$$

$$n = \frac{\ln (p_2/p_1)}{\ln (\rho_2/\rho_1)} = \frac{\ln (\frac{20}{12})}{\ln (0.002965/0.00194)} = 1.20$$

hence

$$\frac{p}{\rho^{1.2}} = \text{constant}$$

describes the polytropic process.

b. Work of expansion is

$$W = \int_{V_1}^{V_2} p \, dV$$

This is the work done by the gas on its surroundings. Since

$$p_1 V_1{}^n = p_2 V_2{}^n = p V^n$$

by substituting into the integral,

$$W = p_1 V_1^n \int_{V_1}^{V_2} \frac{dV}{V^n} = \frac{p_2 V_2 - p_1 V_1}{1 - n} \frac{mR}{1 - n} (T_2 - T_1)$$

if m is the mass of gas. $V_2 = 1011$ ft³ and

$$V_1 = V_2 \left(\frac{p_2}{p_1}\right)^{1/n} = 1011 \left(\frac{20}{12}\right)^{1/1.2} = 1548 \text{ ft}^3$$

Then

$$W = \frac{20 \times 144 \times 1011 - 12 \times 144 \times 1548}{1 - 1.2} = -1{,}183{,}000 \text{ ft-lb}$$

Hence the work done on the gas is 1,183,000 ft-lb.

c. From the conservation of energy the heat added minus the work done by the gas must equal the increase in internal energy; i.e.,

$$Q_H - W = U_2 - U_1 = c_v m(T_2 - T_1)$$

First

$$T_2 = \frac{p_2}{\rho_2 R} = \frac{20 \times 144}{0.002965 \times 53.3 \times 32.17} = 566°\text{R}$$

Then

$$\begin{aligned} Q_H &= -\frac{1{,}183{,}000}{778} + 0.171 \times 32.17 \times 3(566 - 520) \\ &= -760 \text{ Btu} \end{aligned}$$

760 Btu were transferred from the mass of air.

d. From Eq. (6.1.14) the entropy change is computed:

$$s_2 - s_1 = 0.171 \ln \left[\frac{20}{12} \left(\frac{0.00194}{0.002965}\right)^{1.4}\right] = -0.01441 \frac{\text{Btu}}{\text{lb}_m \text{ °R}}$$

and

$$S_2 - S_1 = -0.01441 \times 3 \times 32.17 = -1.392 \frac{\text{Btu}}{\text{°R}}$$

A rough check on the heat transfer may be made by using Eq. (3.6.18), by using an average temperature $T = (520 + 566)/2 = 543$, and by remembering that the losses are zero in a reversible process.

$$Q_H = T(S_2 - S_1) = 543 \times (-1.392) = -756 \text{ Btu}$$

6.2. Speed of a Sound Wave. Mach Number. The speed of a small disturbance in a channel may be determined by application of the momentum equation and the continuity equation. The question is first raised as to whether a *stationary* small change in velocity, pressure, and density can occur in a channel. By referring to Fig. 6.1, the continuity equation can be written

$$\rho V A = (\rho + d\rho)(V + dV)A$$

in which A is the cross-sectional area of channel. The equation can be reduced to

$$\rho\, dV + V\, d\rho = 0$$

When the momentum equation [Eq. (3.9.10)] is applied to the control volume within the dotted lines,

$$pA - (p + dp)A = \rho VA(V + dV - V)$$

or

$$dp = -\rho V\, dV$$

If $\rho\, dV$ is eliminated between the two equations,

$$V^2 = \frac{dp}{d\rho} \tag{6.2.1}$$

So, a small disturbance or sudden change in conditions in steady flow can occur only when a particular velocity $V = \sqrt{dp/d\rho}$ exists in the channel. Now, if a uniform velocity $V = \sqrt{dp/d\rho}$ is assumed to the left in Fig. 6.1, the continuity and momentum equations apply as before, and the small disturbance is propagated through a fluid at rest. This is called the speed of sound c in the medium. The disturbance from a point source would cause a spherical wave to emanate, but at some distance from the source the wave front would be essentially linear or one-dimensional. Large disturbances may travel faster than the speed of sound, e.g., a bomb explosion.

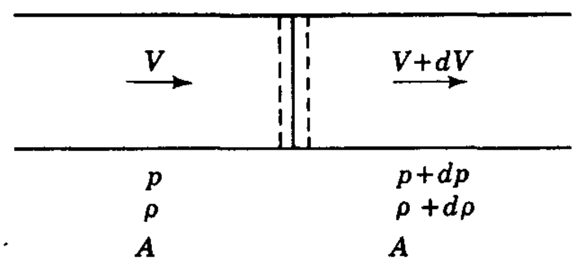


FIG. 6.1. Steady flow in prismatic channel with sudden small change in velocity, pressure, and density.

The equation for speed of sound

$$c = \sqrt{\frac{dp}{d\rho}} \tag{6.2.2}$$

may be expressed in several useful forms. The bulk modulus of elasticity can be introduced:

$$K = -\frac{dp}{d\mathcal{V}/\mathcal{V}}$$

in which $\mathcal{V}$ is the volume of fluid subjected to the pressure change dp. Since

$$\frac{d\mathcal{V}}{\mathcal{V}} = \frac{dv_s}{v_s} = -\frac{d\rho}{\rho}$$

K may be expressed as

$$K = \frac{\rho\, dp}{d\rho}$$

Then, from Eq. (6.2.2),

$$c = \sqrt{\frac{K}{\rho}} \tag{6.2.3}$$

This equation applies to liquids as well as gases.

Example 6.7: Carbon tetrachloride has a bulk modulus of elasticity of 163,000 psi and a density of 3.09 slugs/ft³. What is the speed of sound in the medium?

$$c = \sqrt{\frac{K}{\rho}} = \sqrt{163{,}000 \times \frac{144}{3.09}} = 2758 \text{ ft/sec}$$

The rapid thermodynamic changes resulting from passage of a sound wave are isentropic for all practical purposes. Then

$$p\rho^{-k} = \text{constant}, \qquad \frac{dp}{d\rho} = \frac{kp}{\rho}$$

and

$$c = \sqrt{\frac{kp}{\rho}} \tag{6.2.4}$$

or, from the perfect-gas law $p = \rho RT$,

$$c = \sqrt{kRT} \tag{6.2.5}$$

which shows that the speed of sound in a perfect gas is a function of its absolute temperature only. In flow of gas through a channel, the speed of sound generally changes from section to section as the temperature is changed by density changes and friction effects. In isothermal flow the speed of sound remains constant.

The Mach number has been defined as the ratio of velocity of a fluid to the local velocity of sound in the medium,

$$\mathbf{M} = \frac{V}{c} \tag{6.2.6}$$

Squaring the Mach number produces V^2/c^2, which may be interpreted as the ratio of kinetic energy of the fluid to its thermal energy, since kinetic energy is proportional to V^2 and thermal energy is proportional to T. The Mach number is a measure of the importance of compressibility. In an incompressible fluid K is infinite and $\mathbf{M} = 0$. For perfect gases

$$K = kp \tag{6.2.7}$$

when the compression is isentropic.

Example 6.8: What is the speed of sound in dry air at sea level when $t = 68°\text{F}$, and in the stratosphere when $t = -67°\text{F}$?

At sea level, from Eq. (6.2.5)

$$c = \sqrt{1.4 \times 32.2 \times 53.3(460 + 68)} = 1125 \text{ ft/sec}$$

and in the stratosphere

$$c = \sqrt{1.4 \times 32.2 \times 53.3(460 - 67)} = 972 \text{ ft/sec}$$

6.3. Isentropic Flow. Frictionless adiabatic, or isentropic, flow is an ideal that cannot be reached in the flow of real gases. It is approached, however, in flow through transitions, nozzles, and venturi meters where friction effects are minor, owing to the short distances traveled, and heat transfer is minor because the changes that a particle undergoes are slow enough to keep the velocity and temperature gradients small.[1] The performance of fluid machines is frequently compared with the performance assuming the flow were isentropic. In this section one-dimensional steady flow of a perfect gas through converging and converging-diverging ducts is studied.

Some very general results may be obtained by use of Euler's equation (3.5.5), neglecting elevation changes,

$$V\,dV + \frac{dp}{\rho} = 0 \tag{6.3.1}$$

and the continuity equation

$$\rho A V = \text{constant} \tag{6.3.2}$$

By differentiating ρAV, then dividing through by ρAV,

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0 \tag{6.3.3}$$

From Eq. (6.2.2) dp may be obtained and substituted into Eq. (6.3.1) yielding

$$V\,dV + c^2\frac{d\rho}{\rho} = 0 \tag{6.3.4}$$

By eliminating $d\rho/\rho$ in the last two equations and rearranging,

$$\frac{dA}{dV} = \frac{A}{V}\left(\frac{V^2}{c^2} - 1\right) = \frac{A}{V}(\mathbf{M}^2 - 1) \tag{6.3.5}$$

The assumptions underlying this equation are that the flow is steady and frictionless. No restrictions as to heat transfer have been imposed. Equation (6.3.5) shows that for subsonic flow ($\mathbf{M} < 1$), dA/dV is always negative; i.e., the channel area must decrease for increasing

[1] H. W. Liepmann and A. Roshko, "Elements of Gas Dynamics," p. 51, John Wiley & Sons, Inc., New York, 1957.

velocity. As dA/dV is zero for $\mathbf{M} = 1$ only, the velocity keeps increasing until the minimum section or throat is reached, and that is the only section at which sonic flow may occur. Also, for Mach numbers greater than unity (supersonic flow) dA/dV is positive and the area must increase for an increase in velocity. Hence to obtain supersonic steady flow from a fluid at rest in a reservoir, it must first pass through a converging duct and than a diverging duct.

When the analysis is restricted to isentropic flow, Eq. (6.1.16) may be written

$$p = p_1 \rho_1^{-k} \rho^k \tag{6.3.6}$$

After differentiating and substituting for dp in Eq. (6.3.1),

$$V\,dV + k\frac{p_1}{\rho_1^k}\rho^{k-2}\,d\rho = 0$$

Integration yields

$$\frac{V^2}{2} + \frac{k}{k-1}\frac{p_1}{\rho_1^k}\rho^{k-1} = \text{constant}$$

or

$$\frac{V_1^2}{2} + \frac{k}{k-1}\frac{p_1}{\rho_1} = \frac{V_2^2}{2} + \frac{k}{k-1}\frac{p_2}{\rho_2} \tag{6.3.7}$$

This equation is useful when expressed in terms of temperature; from $p = \rho RT$

$$\frac{V_1^2}{2} + \frac{k}{k-1}RT_1 = \frac{V_2^2}{2} + \frac{k}{k-1}RT_2 \tag{6.3.8}$$

For adiabatic flow from a reservoir where conditions are given by p_0, ρ_0, T_0, at any other section

$$\frac{V^2}{2} = \frac{kR}{k-1}(T_0 - T) \tag{6.3.9}$$

In terms of the local Mach number V/c, with $c^2 = kRT$,

$$\mathbf{M}^2 = \frac{V^2}{c^2} = \frac{2kR(T_0 - T)}{(k-1)kRT} = \frac{2}{k-1}\left(\frac{T_0}{T} - 1\right)$$

or

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\mathbf{M}^2 \tag{6.3.10}$$

From Eqs. (6.1.10) and (6.3.17), which now restrict the following equations to isentropic flow,

$$\frac{p_0}{p} = \left(1 + \frac{k-1}{2}\mathbf{M}^2\right)^{k/(k-1)} \tag{6.3.11}$$

and

$$\frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2}\mathbf{M}^2\right)^{1/(k-1)} \tag{6.3.12}$$

Flow conditions are termed critical at the throat section when the velocity there is sonic. Sonic conditions are marked with an asterisk. $\mathbf{M} = 1$; $c^* = V^* = \sqrt{kRT^*}$. By applying Eqs. (6.3.10) to (6.3.12) to the throat section for critical conditions (for $k = 1.4$ in the numerical portion),

$$\frac{T^*}{T_0} = \frac{2}{k+1} = 0.833 \qquad k = 1.40 \tag{6.3.13}$$

$$\frac{p^*}{p_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)} = 0.528 \qquad k = 1.40 \tag{6.3.14}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{1/(k-1)} = 0.634 \qquad k = 1.40 \tag{6.3.15}$$

These relations show that for air flow, the absolute temperature drops about 17 per cent from reservoir to throat, the critical pressure is 52.8 per cent of the reservoir pressure, and the density is reduced by about 37 per cent.

The variation of area with the Mach number for the critical case is obtained by use of the continuity equation and Eqs. (6.3.10) to (6.3.15). First

$$\rho A V = \rho^* A^* V^* \tag{6.3.16}$$

in which A^* is the minimum, or throat, area. Then

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V} \tag{6.3.17}$$

Now $V^* = c^* = \sqrt{kRT^*}$, and $V = c\mathbf{M} = \mathbf{M}\sqrt{kRT}$, so

$$\frac{V^*}{V} = \frac{1}{\mathbf{M}}\sqrt{\frac{T^*}{T}} = \frac{1}{\mathbf{M}}\sqrt{\frac{T^*}{T_0}}\sqrt{\frac{T_0}{T}} = \frac{1}{\mathbf{M}}\left\{\frac{1 + [(k-1)/2]\mathbf{M}^2}{(k+1)/2}\right\}^{\frac{1}{2}} \tag{6.3.18}$$

by use of Eqs. (6.3.13) and (6.3.10). In a similar manner

$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho} = \left\{\frac{1 + [(k-1)/2]\mathbf{M}^2}{(k+1)/2}\right\}^{1/(k-1)} \tag{6.3.19}$$

By substituting the last two equations into Eq. (6.3.17),

$$\frac{A}{A^*} = \frac{1}{\mathbf{M}}\left\{\frac{1 + [(k-1)/2]\mathbf{M}^2}{(k+1)/2}\right\}^{(k+1)/2(k-1)} \tag{6.3.20}$$

which yields the variation of area of duct in terms of Mach number. A/A^* is never less than unity, and for any value greater than unity there will be two values of Mach number, one less than and one greater than unity. For gases with $k = 1.40$, Eq. (6.3.20) reduces to

$$\frac{A}{A^*} = \frac{1}{\mathbf{M}}\left(\frac{5 + \mathbf{M}^2}{6}\right)^3 \qquad k = 1.40 \tag{6.3.21}$$

The maximum mass flow rate $\dot{m}_{max}$ can be expressed in terms of the throat area and reservoir conditions:

$$\dot{m}_{max} = \rho^* A^* V^* = \rho_0 \left(\frac{2}{k+1}\right)^{1/(k-1)} A^* \sqrt{\frac{kR2T_0}{k+1}}$$

by use of Eqs. (6.3.15) and (6.3.13). By replacing ρ_0 by p_0/RT_0,

$$\dot{m}_{max} = \frac{A^* p_0}{\sqrt{T_0}} \sqrt{\frac{k}{R}\left(\frac{2}{k+1}\right)^{(k+1)/(k-1)}} \tag{6.3.22}$$

For $k = 1.40$ this reduces to

$$\dot{m}_{max} = 0.686 \frac{A^* p_0}{\sqrt{RT_0}} \tag{6.3.23}$$

For $\dot{m}_{max}$ in slugs per second, with air, $R = 53.3 \times 32.2$ ft-lb/slug °R, A^* is in square feet, p_0 in pounds per square foot absolute, and T_0 in degrees Rankine. Equation (6.3.23) shows that the mass flow rate varies linearly as A^* and p_0 and varies inversely as the absolute temperature.

For subsonic flow throughout a converging-diverging duct, the velocity at the throat must be less than sonic velocity, or $\mathbf{M}_t < 1$ with subscript t indicating the throat section. The mass rate of flow $\dot{m}$ is obtained from

$$\dot{m} = \rho V A = A \sqrt{2p_0\rho_0 \frac{k}{k-1}\left(\frac{p}{p_0}\right)^{2/k}\left[1 - \left(\frac{p}{p_0}\right)^{(k-1)/k}\right]} \tag{6.3.24}$$

which is derived from Eqs. (6.3.9) and (6.3.6) and the perfect-gas law. This equation holds for any section and is applicable as long as the velocity at the throat is subsonic. It may be applied to the throat section, and for this section, from Eq. (6.3.14),

$$\frac{p_t}{p_0} \geq \left(\frac{2}{k+1}\right)^{k/(k-1)}$$

p_t is the throat pressure. When the equal sign is used in the expression, Eq. (6.3.24) reduces to Eq. (6.3.22).

For maximum mass flow rate, the flow downstream from the throat may be either supersonic or subsonic, depending upon the downstream pressure. After substituting Eq. (6.3.22) for $\dot{m}$ in Eq. (6.3.24) and simplifying,

$$\left(\frac{p}{p_0}\right)^{2/k}\left[1 - \left(\frac{p}{p_0}\right)^{(k-1)k}\right] = \frac{k-1}{2}\left(\frac{2}{k+1}\right)^{(k+1)/(k-1)}\left(\frac{A^*}{A}\right)^2 \tag{6.3.25}$$

A may be taken as the outlet area and p as the outlet pressure. For a given A^*/A (less than unity) there will be two values of p/p_0 between zero and unity, the upper value for subsonic flow through the diverging duct and the lower value for supersonic flow through the diverging duct. For all other pressure ratios less than the upper value complete isentropic flow is impossible and shock waves form in or just downstream from the diverging duct. They are briefly discussed in the following section.

Example 6.9: A preliminary design of a wind-tunnel duct to produce Mach number 3.0 at the exit is desired. The mass flow rate is 2.0 lb_m/sec at $p_0 = 12.0$ psia, $t_0 = 80°\text{F}$. Determine: (*a*) the throat area, (*b*) the outlet area, and (*c*) the velocity, pressure, temperature, and density at the outlet.

a. The throat area is determined from Eq. (6.3.23):

$$A^* = \frac{\dot{m}_{\max}\sqrt{RT_0}}{0.686p_0} = \frac{2.0}{32.17} \times \frac{\sqrt{53.3 \times 32.17(460 + 80)}}{0.686 \times 12 \times 144} = 0.0504 \text{ ft}^2$$

b. The area of outlet may be determined from Eq. (6.3.21):

$$A = \frac{A^*}{\mathbf{M}}\left(\frac{5 + \mathbf{M}^2}{6}\right)^3 = \frac{0.0504}{3}\left(\frac{5 + 3^2}{6}\right)^3 = 0.2137 \text{ ft}^2$$

c. From Eq. (6.3.11)

$$p = \frac{p_0}{[1 + (k - 1)\mathbf{M}^2/2]^{k/(k-1)}} = \frac{12}{[1 + (1.4 - 1)3^2/2]^{1.4/(1.4-1)}} = 0.326 \text{ psia}$$

From Eq. (6.3.12)

$$\rho = \frac{\rho_0}{[1 + (k - 1)\mathbf{M}^2/2]^{1/(k-1)}} = \frac{p_0}{RT_0[1 + (k - 1)\mathbf{M}^2/2]^{1/(k-1)}}$$

$$= \frac{12 \times 144}{53.3 \times 32.2 \times 540(1 + 0.2 \times 3^2)^{2.5}} = 0.000142 \text{ slug/ft}^3$$

From Eq. (6.3.10)

$$T = \frac{T_0}{1 + (k - 1)\mathbf{M}^2/2} = \frac{540}{1 + 0.2 \times 3^2} = 192.7°\text{R}$$

The velocity is

$$V = c\mathbf{M} = \sqrt{kRT}\,3 = 3\sqrt{1.4 \times 53.3 \times 32.17 \times 192.7} = 2040 \text{ ft/sec}$$

Example 6.10: A converging-diverging air duct has a throat cross section of 0.40 ft² and an exit cross section of 1.0 ft². Reservoir pressure is 30 psia, and temperature is 60°F. Determine the range of Mach numbers and the pressure range at the exit for isentropic flow. Find the maximum flow rate.

Equation (6.3.21)

$$\frac{A}{A^*} = 2.5 = \frac{1}{\mathbf{M}}\left(\frac{5 + \mathbf{M}^2}{6}\right)^3$$

when solved by trial yields $\mathbf{M} = 2.44$ and 0.24. Each of these values of Mach number at the exit is for critical conditions; hence the Mach number range for isentropic flow is 0 to 0.24 and the one value 2.44.

From Eq. (6.3.11)

$$\frac{p_0}{p} = (1 + 0.2\mathbf{M}^2)^{3.5}$$

for $\mathbf{M} = 2.44$, $p = 30/15.55 = 1.929$ psia, and for $\mathbf{M} = 0.24$, $p = 30/1.041 = 28.8$ psia. The downstream pressure range is then from 28.8 to 30 psia, and the isolated point 1.929 psia.

The maximum mass flow rate is determined from Eq. (6.3.23):

$$\dot{m}_{\max} = \frac{0.686 \times 0.40 \times 30 \times 144}{\sqrt{53.3 \times 32.17(460 + 60)}} = 1.26\,\frac{\text{slug}}{\text{sec}} = 40.6\,\frac{\text{lb}_m}{\text{sec}}$$

Example 6.11: A converging-diverging duct in an air line downstream from a reservoir has a 2.0-in.-diameter throat. Determine the mass rate of flow when $p_0 = 120$ psia, $t_0 = 90°\text{F}$, and $p_t = 80$ psia.

$$\rho_0 = \frac{p_0}{RT_0} = \frac{120 \times 144}{53.3 \times 32.17(460 + 90)} = 0.0183 \text{ slug/ft}^3$$

From Eq. (6.3.24)

$$\dot{m} = \frac{\pi}{144}\sqrt{2 \times 120 \times 144 \times 0.0183\,\frac{1.4}{1.4 - 1}\left(\frac{80}{120}\right)^{2/1.4}\left[1 - \left(\frac{80}{120}\right)^{0.4/1.4}\right]} = 0.254\,\frac{\text{slug}}{\text{sec}}$$

Tables which greatly simplify isentropic flow calculations are available in the books by Cambel and Jennings and by Shapiro et al., listed at the end of the chapter.

6.4. Shock Waves. In one-dimensional flow the only type of shock wave that can occur is a normal compression shock wave, as illustrated in Fig. 6.2. For a complete discussion of converging-diverging flow for all downstream pressure ranges[1] oblique shock waves must be taken into account as they occur at the exit. In the preceding section isentropic flow was shown to occur throughout a converging-diverging tube for a range of downstream pressures in which the flow was subsonic throughout and for one downstream pressure for supersonic flow through the diffuser (diverging portion). In this section the normal shock wave in a diffuser is studied, with isentropic flow throughout the tube, except for the shock-wave surface. The shock wave occurs in supersonic flow and reduces the flow to subsonic flow, as proved in the

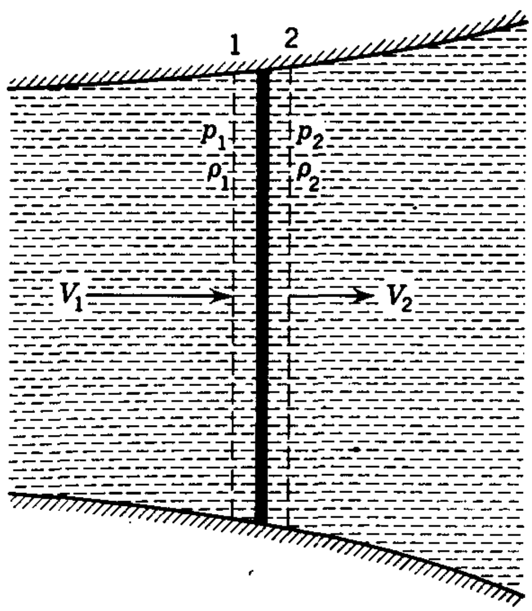


FIG. 6.2. Normal compression shock wave.

[1] H. W. Liepmann and A. Roshko, "Elements of Gas Dynamics," John Wiley & Sons, Inc., New York, 1957.

following section. It has very little thickness, of the order of the molecular mean free path of the gas. The controlling equations are (Fig. 6.2) for adiabatic flow

Continuity: $$G = \frac{\dot{m}}{A} = \rho_1 V_1 = \rho_2 V_2 \tag{6.4.1}$$

Energy: $$\frac{V_1^2}{2} + h_1 = \frac{V_2^2}{2} + h_2 = h_0 = \frac{V^2}{2} + \frac{k}{k-1}\frac{p}{\rho} \tag{6.4.2}$$

which are obtained from Eq. (3.6.10) for no change in elevation, no heat transfer, and no work done. $h = u + p/\rho = c_pT$ is the enthalpy, and h_0 is the value of stagnation enthalpy, i.e., its value in the reservoir or where the fluid is at rest. Equation (6.4.2) holds for real fluids and is valid both upstream and downstream from a shock wave. The momentum equation, (3.9.10) for a control volume between sections 1 and 2 becomes

$$(p_1 - p_2)A = \rho_2 A V_2^2 - \rho_1 A V_1^2$$

or

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \tag{6.4.3}$$

For given upstream conditions h_1, p_1, V_1, ρ_1, the three equations are to be solved for p_2, ρ_2, V_2. The equation of state for a perfect gas is also available for use, $p = \rho RT$. The value of p_2 is

$$p_2 = \frac{1}{k+1}[2\rho_1 V_1^2 - (k-1)p_1] \tag{6.4.4}$$

Once p_2 is determined by combination of the continuity and momentum equations

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_1 V_1 V_2 \tag{6.4.5}$$

V_2 is readily obtained. Finally ρ_2 is obtained from the continuity equation.

For given upstream conditions, with $\mathbf{M}_1 > 1$, the values of p_2, V_2, ρ_2, and $\mathbf{M}_2 = V_2/\sqrt{kp_2/\rho_2}$ exist and $\mathbf{M}_2 < 1$. By eliminating V_1 and V_2 among Eqs. (6.4.1), (6.4.2), and (6.4.3) the Rankine-Hugoniot equations are obtained:

$$\frac{p_2}{p_1} = \frac{[(k+1)/(k-1)](\rho_2/\rho_1) - 1}{[(k+1)/(k-1)] - \rho_2/\rho_1} \tag{6.4.6}$$

and

$$\frac{\rho_2}{\rho_1} = \frac{1 + [(k+1)/(k-1)]p_2/p_1}{[(k+1)/(k-1)] + p_2/p_1} = \frac{V_1}{V_2} \tag{6.4.7}$$

These equations, relating conditions on either side of the shock wave, take the place of the isentropic relation, Eq. (6.1.16), $p\rho^{-k}$ = constant.

From Eq. (6.4.2), the energy equation,

$$\frac{V^2}{2} + \frac{k}{k-1}\frac{p}{\rho} = \frac{c^{*2}}{2} + \frac{c^{*2}}{k-1} = \frac{k+1}{k-1}\frac{c^{*2}}{2} \tag{6.4.8}$$

since the equation holds for all points in adiabatic flow without change in elevation, and $c^* = \sqrt{kp^*/\rho^*}$ is the velocity of sound. After dividing Eq. (6.4.3) by Eq. (6.4.1),

$$V_1 - V_2 = \frac{p_2}{\rho_2 V_2} - \frac{p_1}{\rho_1 V_1}$$

and by eliminating p_2/ρ_2 and p_1/ρ_1 by use of Eq. (6.4.8),

$$V_1 - V_2 = (V_1 - V_2)\left[\frac{c^{*2}(k+1)}{2kV_1V_2} + \frac{k-1}{2k}\right] \tag{6.4.9}$$

which is satisfied by $V_1 = V_2$ (no shock wave) or by

$$V_1V_2 = c^{*2} \tag{6.4.10}$$

It may be written

$$\frac{V_1}{c^*}\cdot\frac{V_2}{c^*} = 1 \tag{6.4.11}$$

When V_1 is greater than c^*, the upstream Mach number is greater than unity and V_2 is less than c^*, so the final Mach number is less than unity, and vice versa. It is shown in the following section that the process can occur only from supersonic upstream to subsonic downstream.

By use of Eq. (6.1.14), together with Eqs. (6.4.4), (6.4.6), and (6.4.7), an expression for change of entropy across a normal shock wave may be obtained in terms of $\mathbf{M}_1$ and k. From Eq. (6.4.4)

$$\frac{p_2}{p_1} = \frac{1}{k+1}\left[\frac{2k\rho_1 V_1^2}{kp_1} - (k-1)\right] \tag{6.4.12}$$

Since $c_1^2 = kp_1/\rho_1$ and $\mathbf{M}_1 = V_1/c_1$, from Eq. (6.4.12),

$$\frac{p_2}{p_1} = \frac{2k\mathbf{M}_1^2 - (k-1)}{k+1} \tag{6.4.13}$$

Placing this value of p_2/p_1 in Eq. (6.4.7) yields

$$\frac{\rho_2}{\rho_1} = \frac{\mathbf{M}_1^2(k+1)}{2 + \mathbf{M}_1^2(k-1)}$$

Now, after substituting these pressure and density ratios into Eq. (6.1.14),

$$S_2 - S_1 = c_v \ln\left\{\frac{2k\mathbf{M}_1^2 - k + 1}{k+1}\left[\frac{2 + \mathbf{M}_1^2(k-1)}{\mathbf{M}_1^2(k+1)}\right]^k\right\} \tag{6.4.14}$$

By substitution of $\mathbf{M}_1 > 1$ into this equation for the appropriate value of k, the entropy may be shown to increase across the shock wave, showing that the normal shock may proceed from supersonic flow upstream to subsonic flow downstream. Substitution of values of $\mathbf{M}_1 < 1$ into Eq. (6.4.14) has no meaning, since Eq. (6.4.13) yields a negative value of the ratio p_2/p_1.

In the next section the shock wave is examined further by introduction of Fanno and Rayleigh lines.

Example 6.12: If a normal shock wave occurs in the flow of helium, $p_1 = 1$ psia, $t_1 = 40°\text{F}$, $V_1 = 4500$ ft/sec, find p_2, ρ_2, V_2, and t_2.

From Table C.2, $R = 386$, $k = 1.66$, and

$$\rho_1 = \frac{p_1}{RT_1} = \frac{1 \times 144}{386 \times 32.17(460 + 40)} = 0.0000232 \text{ slug/ft}^3$$

From Eq. (6.4.4)

$$p_2 = \frac{1}{1.66 + 1} [2 \times 0.0000232 \times (4500)^2 - (1.66 - 1) \times 144 \times 1]$$
$$= 317 \text{ lb/ft}^2 \text{ abs}$$

From Eq. (6.4.5)

$$V_2 = V_1 - \frac{p_2 - p_1}{\rho_1 V_1} = 4500 - \frac{317 - 144}{4500 \times 0.0000232} = 2840 \text{ ft/sec}$$

From Eq. (6.4.1)

$$\rho_2 = \rho_1 \frac{V_1}{V_2} = \frac{0.000746}{32.2} \times \frac{4500}{2840} = 0.0000367 \text{ slug/ft}^3$$

and

$$t_2 = T_2 - 460 = \frac{p_2}{\rho_2 R} - 460 = \frac{317}{0.0000367 \times 32.2 \times 386} - 460 = 235°\text{F}$$

6.5. Fanno and Rayleigh Lines. To examine more closely the nature of the flow change in the short distance across a shock wave, where the area may be considered constant, the continuity and energy equations are combined for steady, frictional, adiabatic flow. By considering upstream conditions fixed, that is, p_1, V_1, ρ_1, a plot may be made of all possible conditions at section 2, Fig. 6.2. The lines on such a plot for constant mass flow per unit area G are called *Fanno lines*. The most revealing plot is that of enthalpy against entropy, i.e., an hs diagram.

The entropy equation for a perfect gas, Eq. (6.1.14), is

$$s - s_1 = c_v \ln \left[\frac{p}{p_1} \left(\frac{\rho_1}{\rho} \right)^k \right] \tag{6.5.1}$$

The energy equation for adiabatic flow with no change in elevation, from Eq. (6.4.2) is

$$h_0 = h + \frac{V^2}{2} \tag{6.5.2}$$

and the continuity equation for no change in area, from Eq. (6.4.1), is

$$G = \rho V \tag{6.5.3}$$

The equation of state, linking h, p, and ρ, is

$$h = c_p T = \frac{c_p p}{R\rho} \tag{6.5.4}$$

By eliminating p, ρ, and V from the four equations,

$$s = s_1 + c_v \ln \left[\frac{\rho_1{}^k}{p_1}\frac{R}{c_p}\left(\frac{\sqrt{2}}{G}\right)^{k-1}\right] + c_v \ln [h(h_0 - h)^{(k-1)/2}] \quad (6.5.5)$$

which is shown on Fig. 6.3 (not to scale). To find the conditions for maximum entropy, Eq. (6.5.5) is differentiated with respect to h and ds/dh set equal to zero. By indicating by subscript a values at the maximum entropy point,

$$\frac{ds}{dh} = 0 = \frac{1}{h_a} - \frac{k-1}{2}\frac{1}{h_0 - h_a}$$

or

$$h_a = \frac{2}{k+1} h_0$$

After substituting this into Eq. (6.5.2) to find V_a,

$$h_0 = \frac{k+1}{2} h_a = h_a + \frac{V_a{}^2}{2}$$

and

$$V_a{}^2 = (k-1)h_a = (k-1)c_pT_a = (k-1)\frac{kR}{k-1}T_a = kRT_a = c_a{}^2 \quad (6.5.6)$$

Hence the maximum entropy at point a is for $\mathbf{M} = 1$, or sonic conditions. For $h > h_a$ the flow is subsonic, and for $h < h_a$ the flow is supersonic. The two conditions, before and after the shock, must lie on the proper Fanno line for the area at which the shock wave occurs. The momentum equation was not used to determine the Fanno line, so the complete solution is not determined yet.

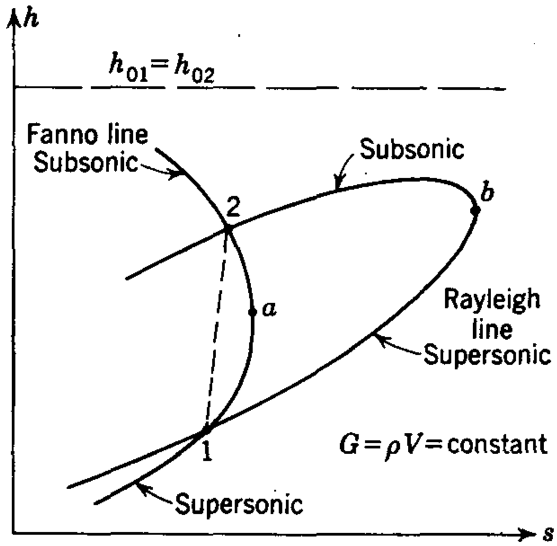


FIG. 6.3. Fanno and Rayleigh lines.

Rayleigh Line. Conditions before and after the shock must also satisfy the momentum and continuity equations. Assuming constant upstream conditions and constant area, Eqs. (6.5.1), (6.5.3), (6.5.4), and (6.4.1) are used to determine the *Rayleigh line.* Eliminating V in the continuity and momentum equations,

$$p + \frac{G^2}{\rho} = \text{constant} = B \quad (6.5.7)$$

Next, by eliminating p from this equation and the entropy equation,

$$s = s_1 + c_v \ln \frac{\rho_1^k}{p_1} + c_v \ln \frac{B - G^2/\rho}{\rho^k} \tag{6.5.8}$$

Enthalpy may be expressed as a function of ρ and upstream conditions, from Eq. (6.5.7):

$$h = c_p T = c_p \frac{p}{R\rho} = \frac{c_p}{R}\frac{1}{\rho}\left(B - \frac{G^2}{\rho}\right) \tag{6.5.9}$$

The last two equations determine s and h in terms of the parameter ρ and plot on the hs diagram as indicated in Fig. 6.3. This is a *Rayleigh line.* The value of maximum entropy is found by taking $ds/d\rho$ and $dh/d\rho$ from the equations, then by division and equating to zero, using subscript b for maximum point:

$$\frac{ds}{dh} = \frac{c_v}{c_p} R\rho_b \frac{G^2/[\rho_b(B - G^2/\rho_b)] - k}{2G^2/\rho_b - B} = 0$$

To satisfy this equation, the numerator must be zero and the denominator not zero. The numerator, set equal to zero, yields

$$k = \frac{G^2}{\rho_b(B - G^2/\rho_b)} = \frac{\rho_b^2 V_b^2}{\rho_b p_b}$$

or

$$V_b^2 = \frac{kp_b}{\rho_b} = c_b^2$$

that is, $\mathbf{M} = 1$. For this value the denominator is not zero. Again, as with the Fanno line, sonic conditions occur at the point of maximum entropy. Since the flow conditions must be on both curves, just before and just after the shock wave, it must suddenly change from one point of intersection to the other. The entropy cannot decrease, as no heat is being transferred from the flow, so the upstream point must be the intersection with least entropy. In all gases investigated the intersection in the subsonic flow has the greater entropy. Thus the shock occurs from supersonic to subsonic.

The Fanno and Rayleigh lines are of value in analyzing flow in constant-area ducts. These are treated in the following sections.

6.6. Adiabatic Flow with Friction in Conduits. Gas flow through a pipe or constant-area duct is analyzed in this section subject to the following assumptions:

1. Perfect gas (constant specific heats).
2. Steady, one-dimensional flow.
3. Adiabatic flow (no heat transfer through walls).
4. Constant friction factor over length of conduit.
5. Effective conduit diameter D is four times hydraulic radius (cross-sectioned area divided by perimeter).

6. Elevation changes are unimportant as compared with friction effects.
7. No work added to or extracted from the flow.

The controlling equations are continuity, energy, momentum, and the equation of state. The Fanno line, developed in Sec. 6.5 and shown in Fig. 6.3, was for constant area and used the continuity and energy equations; hence, it applies to adiabatic flow in a duct of constant area. A particle of gas at the upstream end of the duct may be represented by a point on the appropriate Fanno line for proper stagnation enthalpy h_0 and mass flow rate G per unit area. As the particle moves downstream, its properties change, owing to friction or irreversibilities such that the entropy always increases in adiabatic flow. Thus the point representing these properties moves along the Fanno line toward the maximum s point, where $\mathbf{M} = 1$. If the duct is fed by a converging-diverging nozzle, the flow may originally be supersonic; the velocity must then decrease downstream. If the flow is subsonic at the upstream end, the velocity must increase in the downstream direction.

For exactly one length of pipe, depending upon upstream conditions, the flow is just sonic ($\mathbf{M} = 1$) at the downstream end. For shorter lengths of pipe, the flow will not have reached sonic conditions at the outlet, but for longer lengths of pipe, there must be shock waves (and possibly *choking*) if supersonic and *choking* effects if subsonic. By *choking*, one means that the mass flow rate specified cannot take place in this situation and less flow will occur. The following chart indicates the trends in properties of a gas in adiabatic flow through a constant-area duct, as can be shown from the equations in this section.

Property	Subsonic flow	Supersonic flow
Velocity V	Increases	Decreases
Mach number $\mathbf{M}$	Increases	Decreases
Pressure p	Decreases	Increases
Temperature T	Decreases	Increases
Density ρ	Decreases	Increases
Stagnation enthalpy	Constant	Constant
Entropy	Increases	Increases

The gas cannot change gradually from subsonic to supersonic or vice versa in a constant-area duct.

The momentum equation must now include the effects of wall shear stress and is conveniently written for a segment of duct of length δx (Fig. 6.4):

$$pA - \left(p + \frac{dp}{dx}\,\delta x\right)A - \tau_0 \pi D\,\delta x = \rho V A\left(V + \frac{dv}{dx}\,\delta x - V\right)$$

Upon simplification,

$$dp + \frac{4\tau_0}{D}\,dx + \rho V\,dV = 0 \tag{6.6.1}$$

By use of Eq. (5.9.2) $\tau_0 = \rho f V^2/8$, in which f is the Darcy-Weisbach friction factor,

$$dp + \frac{f\rho V^2}{2D}\,dx + \rho V\,dV = 0 \tag{6.6.2}$$

For constant f, or average value over the length of reach, this equation may be transformed into an equation for x as a function of Mach number.

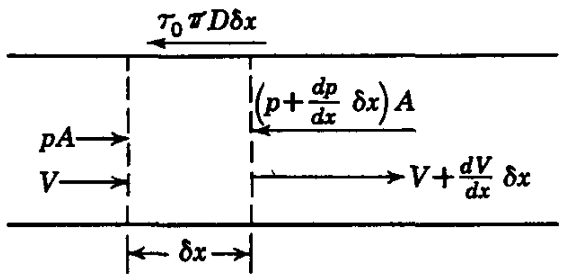


FIG. 6.4. Notation for application of momentum equation.

By dividing Eq. (6.6.2) by p,

$$\frac{dp}{p} + \frac{f}{2D}\frac{\rho V^2}{p}\,dx + \frac{\rho V}{p}\,dV = 0 \tag{6.6.3}$$

each term is now developed in terms of $\mathbf{M}$. By definition $V/c = \mathbf{M}$

$$V^2 = \mathbf{M}^2\frac{kp}{\rho} \tag{6.6.4}$$

or

$$\frac{\rho V^2}{p} = k\mathbf{M}^2 \tag{6.6.5}$$

for the middle term of the momentum equation. By rearranging Eq. (6.6.4)

$$\frac{\rho V}{p}\,dV = k\mathbf{M}^2\frac{dV}{V} \tag{6.6.6}$$

Now to express dV/V in terms of $\mathbf{M}$, from the energy equation

$$h_0 = h + \frac{V^2}{2} = c_pT + \frac{V^2}{2} \tag{6.6.7}$$

Differentiating,

$$c_p\,dT + V\,dV = 0 \tag{6.6.8}$$

By dividing through by $V^2 = \mathbf{M}^2kRT$,

$$\frac{c_p}{R}\frac{1}{k\mathbf{M}^2}\frac{dT}{T} + \frac{dV}{V} = 0$$

Since $c_p/R = k/(k - 1)$,

$$\frac{dT}{T} = -\mathbf{M}^2(k - 1)\frac{dV}{V} \tag{6.6.9}$$

Differentiating $V^2 = \mathbf{M}^2kRT$ and dividing by the equation,

$$2\frac{dV}{V} = 2\frac{d\mathbf{M}}{\mathbf{M}} + \frac{dT}{T} \tag{6.6.10}$$

Eliminating dT/T in Eqs. (6.6.9) and (6.6.10) and simplifying,

$$\frac{dV}{V} = \frac{d\mathbf{M}/\mathbf{M}}{[(k - 1)/2]\mathbf{M}^2 + 1} \tag{6.6.11}$$

which permits elimination of dV/V from Eq. (6.6.6), yielding

$$\frac{\rho V}{p}\,dV = \frac{k\mathbf{M}\,d\mathbf{M}}{[(k - 1)/2]\mathbf{M}^2 + 1} \tag{6.6.12}$$

And finally, to express dp/p in terms of $\mathbf{M}$, from $p = \rho RT$ and $G = \rho V$,

$$pV = GRT \tag{6.6.13}$$

By differentiation

$$\frac{dp}{p} = \frac{dT}{T} - \frac{dV}{V}$$

Equations (6.6.9) and (6.6.11) are used to eliminate dT/T and dV/V:

$$\frac{dp}{p} = -\frac{(k - 1)\mathbf{M}^2 + 1}{[(k - 1)/2]\mathbf{M}^2 + 1}\,\frac{d\mathbf{M}}{\mathbf{M}} \tag{6.6.14}$$

Equations (6.6.5), (6.6.12), and (6.6.14) are now substituted into the momentum equation (6.6.3). After rearranging,

$$\begin{aligned}\frac{f}{D}\,dx &= \frac{2(1 - \mathbf{M}^2)}{k\mathbf{M}^3\{[(k - 1)/2]\mathbf{M}^2 - 1\}}\,d\mathbf{M} \\ &= \frac{2}{k}\,\frac{d\mathbf{M}}{\mathbf{M}^3} - \frac{k + 1}{k}\,\frac{d\mathbf{M}}{\mathbf{M}\{[(k - 1)/2]\mathbf{M}^2 + 1\}}\end{aligned} \tag{6.6.15}$$

which may be integrated directly. By using the limits $x = 0$, $\mathbf{M} = \mathbf{M}_0$, $x = l$, $\mathbf{M} = \mathbf{M}$,

$$\frac{fl}{D} = -\frac{1}{k\mathbf{M}^2}\bigg]_{\mathbf{M}_0}^{\mathbf{M}} - \frac{k + 1}{2k}\ln\frac{\mathbf{M}^2}{[(k - 1)/2]\mathbf{M}^2 + 1}\bigg]_{\mathbf{M}_0}^{\mathbf{M}} \tag{6.6.16}$$

$$= \frac{1}{k}\left(\frac{1}{\mathbf{M}_0^2} - \frac{1}{\mathbf{M}^2}\right) + \frac{k + 1}{2k}\ln\left[\left(\frac{\mathbf{M}_0}{\mathbf{M}}\right)^2\frac{(k - 1)\mathbf{M}^2 + 2}{(k - 1)\mathbf{M}_0^2 + 2}\right] \tag{6.6.17}$$

For $k = 1.4$, this reduces to

$$\frac{fl}{D} = \frac{5}{7}\left(\frac{1}{\mathbf{M}_0^2} - \frac{1}{\mathbf{M}^2}\right) + \frac{6}{7}\ln\left[\left(\frac{\mathbf{M}_0}{\mathbf{M}}\right)^2\frac{\mathbf{M}^2 + 5}{\mathbf{M}_0^2 + 5}\right] \qquad k = 1.4 \tag{6.6.18}$$

If $\mathbf{M}_0$ is greater than 1, then $\mathbf{M}$ cannot be less than 1, and if $\mathbf{M}_0$ is less than 1, then $\mathbf{M}$ cannot be greater than 1. For the limiting condition $\mathbf{M} = 1$ and $k = 1.4$,

$$\frac{fL_{max}}{D} = \frac{5}{7}\left(\frac{1}{\mathbf{M}_0^2} - 1\right) + \frac{6}{7}\ln\frac{6\mathbf{M}_0^2}{\mathbf{M}_0^2 + 5} \qquad k = 1.4 \qquad (6.6.19)$$

Experiments by Keenan and Neumann[1] show apparent friction factors for supersonic flow of about half the value for subsonic flow.

Example 6.13: Determine the maximum length of 2.0-in. ID pipe, $f = 0.02$ for flow of air, when the Mach number at the entrance to the pipe is 0.30.

From Eq. (6.6.19)

$$\frac{0.02}{\frac{2}{12}} L_{max} = \frac{5}{7}\left(\frac{1}{0.3^2} - 1\right) + \frac{6}{7}\ln\frac{6 \times 0.30^2}{0.30^2 + 5}$$

$L_{max} = 44.17$ ft.

The pressure, velocity, and temperature may also be expressed in integral form in terms of the Mach number. To simplify the equations that follow they will be integrated from upstream conditions to conditions at $\mathbf{M} = 1$, indicated by p^*, V^*, and T^*. From Eq. (6.6.14)

$$\frac{p^*}{p_0} = \mathbf{M}_0\sqrt{\frac{(k-1)\mathbf{M}_0^2 + 2}{k+1}} \qquad (6.6.20)$$

From Eq. (6.6.11)

$$\frac{V^*}{V_0} = \frac{1}{\mathbf{M}_0}\sqrt{\frac{(k-1)\mathbf{M}_0^2 + 2}{k+1}} \qquad (6.6.21)$$

From Eqs. (6.6.9) and (6.6.11)

$$\frac{dT}{T} = -(k-1)\frac{\mathbf{M}\,d\mathbf{M}}{[(k-1)/2]\mathbf{M}^2 + 1}$$

which, when integrated, yields

$$\frac{T^*}{T_0} = \frac{(k-1)\mathbf{M}_0^2 + 2}{k+1} \qquad (6.6.22)$$

Example 6.14: A 4.0-in. ID pipe, $f = 0.010$, has air at 14.7 psia and at $t = 60°$F flowing at the upstream end with Mach number 3.0. Determine L_{max}, p^*, V^*, T^*, and values of p, V, T, and L at $\mathbf{M} = 2.0$

From Eq. (6.6.19)

$$\frac{0.01}{0.333} L_{max} = \frac{5}{7}\left(\frac{1}{9} - 1\right) + \frac{6}{7}\ln\frac{6 \times 3^2}{3^2 + 5}$$

[1] J. H. Keenan and E. P. Neumann, Measurements of Friction in a Pipe for Subsonic and Supersonic Flow of Air, *J. Appl. Mech.*, vol. 13, no. 2, p. A-91, 1946.

from which $L_{max} = 17.33$ ft. If the flow originated at $\mathbf{M} = 2$, the length L_{max} is given by the same equation:

$$\frac{0.01}{0.333} L_{max} = \frac{5}{7}\left(\frac{1}{4} - 1\right) + \frac{6}{7} \ln \frac{6 \times 2^2}{2^2 + 5}$$
$$L'_{max} = 10.14 \text{ ft}$$

Hence the length from the upstream section at $\mathbf{M} = 3$ to the section where $\mathbf{M} = 2$ is $17.33 - 10.14 = 7.19$ ft.

The velocity at the entrance is

$$V = \sqrt{kRT}\,\mathbf{M} = \sqrt{1.4 \times 53.3 \times 32.17(460 + 60)} \times 3 = 3354 \text{ ft/sec}$$

From Eqs. (6.6.20) to (6.6.22)

$$\frac{p^*}{14.7} = 3\sqrt{\frac{0.4 \times 3^2 + 2}{2.4}} = 4.581$$
$$\frac{V^*}{3354} = \frac{1}{3}\sqrt{\frac{0.4 \times 3^2 + 2}{2.4}} = 0.509$$
$$\frac{T^*}{520} = \frac{0.4 \times 3^2 + 2}{2.4} = \frac{7}{3}$$

So $p^* = 67.4$ psia, $V^* = 1707$ ft/sec, $T^* = 1213$°R. For $\mathbf{M} = 2$ the same equations are now solved for p'_0, V'_0, and T'_0:

$$\frac{67.4}{p'_0} = 2\sqrt{\frac{0.4 \times 2^2 + 2}{2.4}} = 2.45$$
$$\frac{1707}{V'_0} = \frac{1}{2}\sqrt{\frac{0.4 \times 2^2 + 2}{2.4}} = 0.6125$$
$$\frac{1213}{T'_0} = \frac{0.4 \times 2^2 + 2}{2.4} = \frac{3}{2}$$

So $p'_0 = 27.5$ psia, $V'_0 = 2790$ ft/sec, and $T'_0 = 809$°R.

6.7. Frictionless Flow through Ducts with Heat Transfer. The steady flow of a perfect gas (with constant specific heats) through a constant-area duct is considered in this section. Friction is neglected, and no work is done on or by the flow.

The appropriate equations for analysis of this case are

Continuity: $$G = \frac{\dot{m}}{A} = \rho V \qquad (6.7.1)$$

Momentum: $$p + \rho V^2 = \text{constant} \qquad (6.7.2)$$

Energy: $$q_H = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2} = c_p(T_2 - T_1) + \frac{V_2^2 - V_1^2}{2}$$
$$= c_p(T_{02} - T_{01}) \qquad (6.7.3)$$

T_{01} and T_{02} are the isentropic stagnation temperatures, i.e., the temperature produced at a section by bringing the flow isentropically to rest.

The Rayleigh line, obtained from the solution of momentum and continuity for a constant cross section by neglecting friction, is very helpful in examining the flow. First, by eliminating V in Eqs. (6.7.1) and (6.7.2),

$$p + \frac{G^2}{\rho} = \text{constant} \tag{6.7.4}$$

which is Eq. (6.5.7). Equations (6.5.8) and (6.5.9) express the entropy s and enthalpy h in terms of the parameter ρ for the assumptions of this section, as in Fig. 6.5.

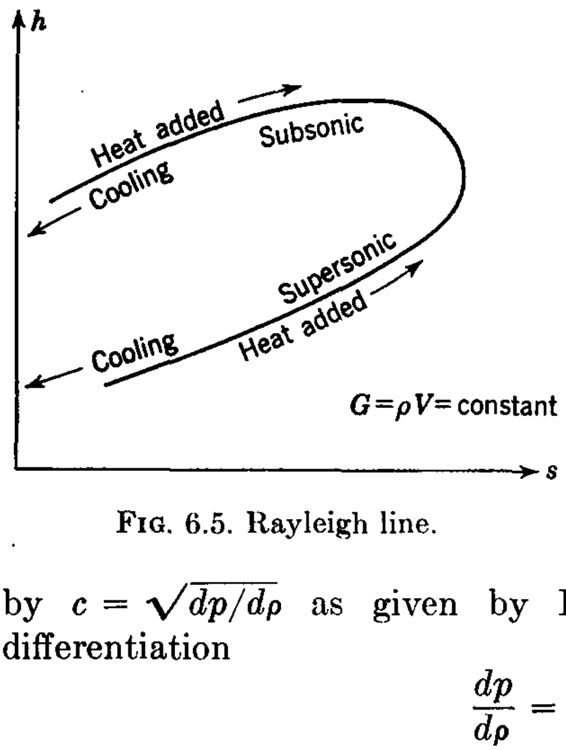


FIG. 6.5. Rayleigh line.

Since, by Eq. (3.8.3), for no losses, entropy can increase only when heat is added, the properties of the gas must change as indicated in Fig. 6.5, moving toward the maximum entropy point as heat is added. At the maximum s point there is no change in entropy for a small change in h, and isentropic conditions apply to the point. The speed of sound under isentropic conditions is given by $c = \sqrt{dp/d\rho}$ as given by Eq. (6.2.2). From Eq. (6.7.4), by differentiation

$$\frac{dp}{d\rho} = \frac{G^2}{\rho^2} = V^2$$

using Eq. (6.7.1). Hence at the maximum s point of the Rayleigh line $V = \sqrt{dp/d\rho}$ also and $\mathbf{M} = 1$, or sonic conditions, prevail. The addition of heat to supersonic flow causes the Mach number of the flow to decrease toward $\mathbf{M} = 1$, and if just the proper amount of heat is added, $\mathbf{M}$ becomes 1. If more heat is added, choking results and conditions at the upstream end are altered to reduce the mass rate of flow. The addition of heat to subsonic flow causes an increase in the Mach number toward $\mathbf{M} = 1$, and again, too much heat transfer causes choking with an upstream adjustment of mass flow rate to a smaller value.

From Eq. (6.7.3) it is noted that the increase in isentropic stagnation pressure is a measure of the heat added. From $V^2 = \mathbf{M}^2kRT$, $p = \rho RT$, and continuity,

$$pV = GRT$$

and

$$\rho V^2 = kp\mathbf{M}^2$$

Now, from the momentum equation

$$p_1 + kp_1\mathbf{M}_1^2 = p_2 + kp_2\mathbf{M}_2^2$$

and

$$\frac{p_1}{p_2} = \frac{1 + k\mathbf{M}_2^2}{1 + k\mathbf{M}_1^2} \tag{6.7.5}$$

By writing this equation for the limiting case $p_2 = p^*$ when $\mathbf{M}_2 = 1$,

$$\frac{p}{p^*} = \frac{1 + k}{1 + k\mathbf{M}^2} \tag{6.7.6}$$

with p the pressure at any point in the duct where $\mathbf{M}$ is the corresponding Mach number. For the subsonic case, with $\mathbf{M}$ increasing to the right (Fig. 6.5), p must decrease, and for the supersonic case, as $\mathbf{M}$ decreases toward the right, p must increase.

To develop the other pertinent relations, the energy equation (6.7.3) is used

$$c_pT_0 = \frac{kR}{k - 1}T_0 = \frac{kR}{k - 1}T + \frac{V^2}{2}$$

in which T_0 is the isentropic stagnation temperature and T the free stream temperature at the same section. By applying this to section 1, after dividing through by $kRT_1/(k - 1)$,

$$\frac{T_{01}}{T_1} = 1 + (k - 1)\frac{\mathbf{M}_1^2}{2} \tag{6.7.7}$$

and for section 2

$$\frac{T_{02}}{T_2} = 1 + (k - 1)\frac{\mathbf{M}_2^2}{2} \tag{6.7.8}$$

Dividing Eq. (6.7.7) by Eq. (6.7.8)

$$\frac{T_{01}}{T_{02}} = \frac{T_1}{T_2}\frac{2 + (k - 1)\mathbf{M}_1^2}{2 + (k - 1)\mathbf{M}_2^2} \tag{6.7.9}$$

The ratio T_1/T_2 is determined in terms of the Mach numbers as follows: From the perfect-gas law, $p_1 = \rho_1RT_1$, $p_2 = \rho_2RT_2$,

$$\frac{T_1}{T_2} = \frac{p_1}{p_2}\frac{\rho_2}{\rho_1} \tag{6.7.10}$$

From continuity $\rho_2/\rho_1 = V_1/V_2$, and by definition,

$$\mathbf{M}_1 = \frac{V_1}{\sqrt{kRT_1}} \qquad \mathbf{M}_2 = \frac{V_2}{\sqrt{kRT_2}}$$

so

$$\frac{V_1}{V_2} = \frac{\mathbf{M}_1}{\mathbf{M}_2}\sqrt{\frac{T_1}{T_2}}$$

and

$$\frac{\rho_2}{\rho_1} = \frac{\mathbf{M}_1}{\mathbf{M}_2}\sqrt{\frac{T_1}{T_2}} \tag{6.7.11}$$

Now, by substituting Eqs. (6.7.5) and (6.7.11) into Eq. (6.7.10) and simplifying,

$$\frac{T_1}{T_2} = \left(\frac{\mathbf{M}_1}{\mathbf{M}_2}\frac{1 + k\mathbf{M}_2^2}{1 + k\mathbf{M}_1^2}\right)^2 \tag{6.7.12}$$

With this equation substituted into Eq. (6.7.9),

$$\frac{T_{01}}{T_{02}} = \left(\frac{\mathbf{M}_1}{\mathbf{M}_2}\frac{1 + k\mathbf{M}_2^2}{1 + k\mathbf{M}_1^2}\right)^2 \frac{2 + (k-1)\mathbf{M}_1^2}{2 + (k-1)\mathbf{M}_2^2} \tag{6.7.13}$$

By applying this equation to the downstream section where $T_{02} = T_0^*$ and $\mathbf{M}_2 = 1$, by dropping the subscripts for the upstream section,

$$\frac{T_0}{T_0^*} = \frac{\mathbf{M}^2(k+1)[2 + (k-1)\mathbf{M}^2]}{(1 + k\mathbf{M}^2)^2} \tag{6.7.14}$$

All the necessary equations for determination of frictionless flow with heat transfer in a constant-area duct are now available. Heat transfer per unit mass is given by $q_H = c_p(T_0^* - T_0)$ for $\mathbf{M} = 1$ at the exit. Use of the equations is illustrated in the following example.

Example 6.15: Air at $V_1 = 300$ ft/sec, $p = 40$ psia, $t = 60°\text{F}$ flows into a 4.0-in.-diameter duct. How much heat transfer per unit mass is needed for sonic conditions at the exit? Determine pressure, temperature, and velocity at the exit and at the section where $\mathbf{M} = 0.70$.

$$\mathbf{M}_1 = \frac{V_1}{\sqrt{kRT_1}} = \frac{300}{\sqrt{1.4 \times 53.3 \times 32.17(460 + 60)}} = 0.268$$

The isentropic stagnation temperature at the entrance, from Eq. (6.7.7), is

$$T_{01} = T_1\left(1 + \frac{k-1}{2}\mathbf{M}_1^2\right) = 520(1 + 0.2 \times \overline{0.268}^2) = 527°\text{R}$$

The isentropic stagnation temperature at the exit, from Eq. (6.7.14), is

$$T_0^* = \frac{T_0(1 + k\mathbf{M}^2)^2}{(k+1)\mathbf{M}^2[2 + (k-1)\mathbf{M}^2]} = \frac{527(1 + 1.4 \times \overline{0.268}^2)^2}{2.4 \times 0.268^2(2 + 0.4 \times \overline{0.268}^2)} = 1827°\text{R}$$

The heat transfer per slug of air flowing is

$$q_H = c_p(T_0^* - T_{01}) = 0.24 \times 32.17(1827 - 527) = 10{,}050\,\frac{\text{Btu}}{\text{slug}}$$

The pressure at the exit, Eq. (6.7.6), is

$$p^* = p\,\frac{1 + k\mathbf{M}^2}{k+1} = \frac{40}{2.4}(1 + 1.4 \times \overline{0.268}^2) = 18.35 \text{ psia}$$

and the temperature, from Eq. (6.7.12),

$$T^* = T\left[\frac{1 + k\mathbf{M}^2}{(k+1)\mathbf{M}}\right]^2 = 520\left(\frac{1 + 1.4 \times 0.268^2}{2.4 \times 0.268}\right)^2 = 1520°\text{R}$$

At the exit,

$$V^* = c^* = \sqrt{kRT^*} = \sqrt{1.4 \times 53.3 \times 32.17 \times 1520} = 1910 \text{ ft/sec}$$

At the section where $\mathbf{M} = 0.7$, from Eq. (6.7.6),

$$p = p^* \frac{k+1}{1 + k\mathbf{M}^2} = \frac{18.35 \times 2.4}{1 + 1.4 \times 0.7^2} = 26.1 \text{ psia}$$

From Eq. (6.7.12)

$$T = T^*\left[\frac{(k+1)\mathbf{M}}{1 + k\mathbf{M}^2}\right]^2 = 1520\left(\frac{2.4 \times 0.7}{1 + 1.4 \times 0.7^2}\right)^2 = 1507°\text{R}$$

and

$$V = \mathbf{M}\sqrt{kRT} = 0.7\sqrt{1.4 \times 53.3 \times 32.17 \times 1507} = 1332 \text{ ft/sec}$$

The trends in flow properties are shown in the following table:

TRENDS IN FLOW PROPERTIES

	Heating		Cooling	
	$\mathbf{M} > 1$	$\mathbf{M} < 1$	$\mathbf{M} > 1$	$\mathbf{M} < 1$
Pressure p	Increases	Decreases	Decreases	Increases
Velocity V	Decreases	Increases	Increases	Decreases
Isentropic stagnation temperature T_0	Increases	Increases	Decreases	Decreases
Density ρ	Increases	Decreases	Decreases	Increases
Temperature T	Increases	Increases for $\mathbf{M} < 1/k$; Decreases for $\mathbf{M} > 1/k$	Decreases	Decreases for $\mathbf{M} < 1/k$; Increases for $\mathbf{M} > 1/k$

For curves and tables tabulating the various equations, consult the books by Cambel and Jennings, Shapiro, and Shapiro et al., listed in the references at the end of the chapter.

6.8. Steady, Isothermal Flow in Long Pipelines. In the analysis of isothermal flow of a perfect gas through long ducts, neither the Fanno nor Rayleigh lines are applicable, since the Fanno line applies to adiabatic flow and the Rayleigh line to frictionless flow. An analysis somewhat similar to those of the previous two sections is carried out to show the trend in properties with Mach number.

The appropriate equations are

Momentum [Eq. (6.6.3)]: $$\frac{dp}{p} + \frac{f}{2D}\frac{\rho V^2}{p}\,dx + \frac{\rho V}{p}\,dV = 0 \qquad (6.8.1)$$

Equation of state: $\dfrac{p}{\rho} = \text{constant} \qquad \dfrac{dp}{p} = \dfrac{d\rho}{\rho}$ (6.8.2)

Continuity: $\rho V = \text{constant} \qquad \dfrac{d\rho}{\rho} = -\dfrac{dV}{V}$ (6.8.3)

Energy [Eq. (6.7.7)]: $T_0 = T\left[1 + \dfrac{(k-1)}{2}\mathbf{M}^2\right]$ (6.8.4)

in which T_0 is the isentropic stagnation temperature at the section where the free-stream static temperature is T and the Mach number is $\mathbf{M}$.

Stagnation pressure [Eq. (6.3.11)]: $p_0 = p\left(1 + \dfrac{k-1}{2}\mathbf{M}^2\right)^{k/(k-1)}$ (6.8.5)

in which p_0 is the pressure (at the section of p and $\mathbf{M}$) obtained by reducing the velocity to zero isentropically.

From definitions and use of the above equations,

$$V = c\mathbf{M} = \sqrt{kRT}\,\mathbf{M} \qquad \frac{dV}{V} = \frac{d\mathbf{M}}{\mathbf{M}} = \frac{d\mathbf{M}^2}{2\mathbf{M}^2}$$

$$\frac{\rho V}{p}\,dV = \frac{V\,dV}{RT} = \frac{c^2}{RT}\,\mathbf{M}\,d\mathbf{M} = k\mathbf{M}\,d\mathbf{M}$$

$$\frac{\rho V^2}{p} = \frac{c^2\mathbf{M}^2}{RT} = k\mathbf{M}^2$$

By substituting into the momentum equation, using the relations,

$$\frac{dp}{p} = \frac{d\rho}{\rho} = -\frac{dV}{V} = -\frac{1}{2}\frac{d\mathbf{M}^2}{\mathbf{M}^2} = -\frac{k\mathbf{M}^2}{1-k\mathbf{M}^2}\frac{f\,dx}{2D} \tag{6.8.6}$$

The differential dx is positive in the downstream direction, so one may conclude that the trends in properties vary depending upon whether $\mathbf{M}$ is less than $1/\sqrt{k}$ or greater than $1/\sqrt{k}$. For $\mathbf{M} < 1/\sqrt{k}$, the pressure and density decrease and velocity and Mach number increase, with the opposite trends for $\mathbf{M} > 1/\sqrt{k}$; hence, the Mach number always approaches $1/\sqrt{k}$, in place of unity for adiabatic flow in pipelines.

To determine the direction of heat transfer, by differentiation of Eq. (6.8.4) then division by it, remembering that T is constant,

$$\frac{dT_0}{T_0} = \frac{k-1}{2+(k-1)\mathbf{M}^2}\,d\mathbf{M}^2 \tag{6.8.7}$$

By eliminating $d\mathbf{M}^2$ in this equation and Eq. (6.8.6),

$$\frac{dT_0}{T_0} = \frac{k(k-1)\mathbf{M}^4}{(1-k\mathbf{M}^2)[2+(k-1)\mathbf{M}^2]}\frac{f\,dx}{D} \tag{6.8.8}$$

which shows that the isentropic stagnation temperature increases for $\mathbf{M} < 1/\sqrt{k}$, indicating that heat is transferred to the fluid. For $\mathbf{M} > 1/\sqrt{k}$ heat transfer is from the fluid.

From Eqs. (6.8.5) and (6.8.6)

$$\frac{dp_0}{p_0} = \frac{2 - (k + 1)\mathbf{M}^2}{2 + (k - 1)\mathbf{M}^2} \frac{k\mathbf{M}^2}{k\mathbf{M}^2 - 1} \frac{f\,dx}{2D} \tag{6.8.9}$$

The following tabulation shows the trends of fluid properties.

TRENDS IN FLUID PROPERTIES FOR ISOTHERMAL FLOW

	$\mathbf{M} < 1/\sqrt{k}$ subsonic	$\mathbf{M} > 1/\sqrt{k}$ subsonic or supersonic
Pressure p	Decreases	Increases
Density ρ	Decreases	Increases
Velocity V	Increases	Decreases
Mach Number $\mathbf{M}$	Increases	Decreases
Stagnation temperature T_0	Increases	Decreases
Stagnation pressure p_0	Decreases	Increases for $\mathbf{M} < \sqrt{2/(k+1)}$ Decreases for $\mathbf{M} > \sqrt{2/(k+1)}$

By integration of the various Eqs. (6.8.6) in terms of $\mathbf{M}$, the change with Mach number is found. The last two terms yield

$$\frac{f}{D}\int_0^{L_{max}} dx = \frac{1}{k}\int_{\mathbf{M}}^{1/\sqrt{k}} \frac{(1 - k\mathbf{M}^2)}{\mathbf{M}^4}\, d\mathbf{M}^2$$

or

$$\frac{f}{D} L_{max} = \frac{1 - k\mathbf{M}^2}{k\mathbf{M}^2} + \ln (k\mathbf{M}^2) \tag{6.8.10}$$

in which L_{max}, as before, represents the maximum length of duct. For greater lengths choking occurs and the mass rate is decreased. To find the pressure change

$$\int_p^{p^{*t}} \frac{dp}{p} = -\frac{1}{2}\int_{\mathbf{M}}^{1/\sqrt{k}} \frac{d\mathbf{M}^2}{\mathbf{M}^2}$$

and

$$\frac{p^{*t}}{p} = \sqrt{k}\,\mathbf{M} \tag{6.8.11}$$

The superscript ^{*t} indicates conditions at $\mathbf{M} = 1/\sqrt{k}$, and $\mathbf{M}$ and p represent values at any upstream section.

Example 6.16: Helium enters a 4.0-in. ID pipe from a converging-diverging nozzle at $\mathbf{M} = 1.30$, $p = 2.0$ psia, $T = 400°$R. Determine for isothermal flow:

(*a*) the maximum length of pipe for no choking, (*b*) the downstream conditions, and (*c*) the length from the exit to the section where $\mathbf{M} = 1.0$. $f = 0.006$.

a. From Eq. (6.8.10) for $k = 1.66$

$$\frac{0.006L_{max}}{\frac{4}{12}} = \frac{1 - 1.66 \times \overline{1.3}^2}{1.66 \times \overline{1.3}^2} + \ln (1.66 \times \overline{1.3}^2)$$

$L_{max} = 21.54$ ft.

b. From Eq. (6.8.11)

$$p^{*t} = p\sqrt{k}\,\mathbf{M} = 2.0\sqrt{1.66}\,1.3 = 3.35 \text{ psia}$$

The Mach number at the exit is $1/\sqrt{1.66} = 0.756$. From Eqs. (6.8.6)

$$\int_V^{V^{*t}} \frac{dV}{V} = \frac{1}{2}\int_{\mathbf{M}}^{1/\sqrt{k}} \frac{d\mathbf{M}^2}{\mathbf{M}^2}$$

or

$$\frac{V^{*t}}{V} = \frac{1}{\sqrt{k}\,\mathbf{M}}$$

At the upstream section

$$V = \mathbf{M}\sqrt{kRT} = 1.3\sqrt{1.66 \times 386 \times 32.17 \times 400} = 3740 \text{ ft/sec}$$

and

$$V^{*t} = \frac{V}{\sqrt{k}\,\mathbf{M}} = \frac{3740}{\sqrt{1.66}\,1.3} = 2072 \text{ ft/sec}$$

c. From Eq. (6.8.10) for $\mathbf{M} = 1$

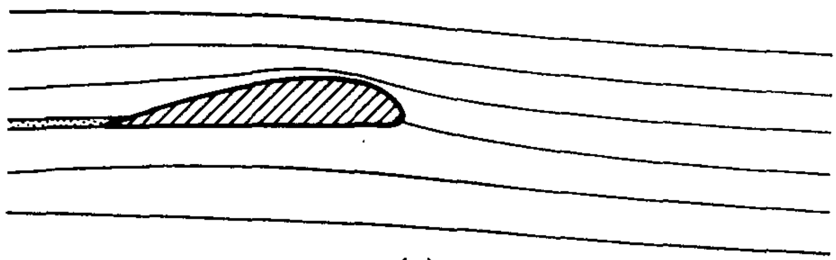
$$\frac{0.006}{\frac{4}{12}} L'_{max} = \frac{1 - 1.66}{1.66} + \ln 1.66$$

or $L'_{max} = 6.0$ ft. $\mathbf{M} = 1$ occurs 6.0 ft from the exit.

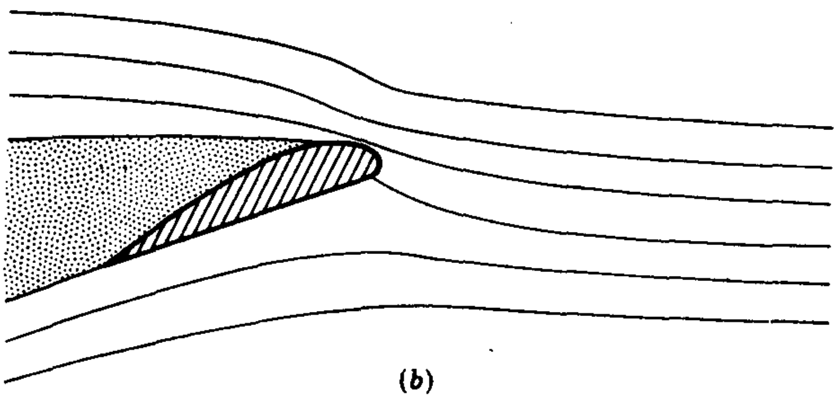
6.9. High-speed Flight. This section on high-speed flight deals with five aspects of the problem: effect of shock waves and stalling on airfoil lift and drag, sonic boom, wave drag, area rule, and aerodynamic heating. The last four of these topics are reproduced with minor changes from "Supplementary Notes, Aerodynamics and Gas Dynamics," Department of Mechanics, United States Military Academy, West Point, New York.

Effect of Shock Waves and Stalling on Airfoil Lift and Drag. The lift coefficient C_L of the usual airfoil profile in subsonic flow tends to increase almost linearly (Fig. 5.25) with the angle of attack. C_L reaches a maximum value between 1.2 and 1.8, which is limited by separation of the boundary layer (Sec. 5.5) from the upper airfoil surface. When the bounding streamline becomes detached, as in Fig. 6.6, the pressure over the airfoil in the detached region becomes about equal to the undisturbed pressure in the fluid stream. Since most of the lift is normally produced by underpressure on the upper surface rather than overpressure on the lower surface, the lift coefficient drops sharply. This formation of a large, turbulent zone over most of the upper surface is known as stalling, and is also accompanied by a sharp increase in drag coefficient. At small angles of attack (Fig. 6.6*a*) the flow separates near the trailing edge, but this does not materially affect the lift.

As the airfoil speed approaches that of the speed of sound in the air, compressibility effects become important. A thin airfoil in subsonic



(*a*)



(*b*)

FIG. 6.6. Airfoil in subsonic flow. (*a*) Flow without stalling; (*b*) flow with stalling.

flow has a lift coefficient that is related to the lift coefficient for incompressible flow C_{L0}, by the Prandtl-Glauert transformation

$$C_L = \frac{C_{L0}}{\sqrt{1 - \mathbf{M}_\infty^2}}$$

in which $\mathbf{M}_\infty$ is the Mach number of the approaching velocity relative to the airfoil. Hence, the lift coefficient increases as Mach number increases up to the transonic range. The drag coefficient increases greatly in this range (Fig. 5.27).

The transonic range is defined as the Mach-number range of approach velocity when both supersonic and subsonic flow occur around the airfoil (Fig. 6.7). By considering slowly increasing approach velocity (or speed of airfoil through still air), a region of supersonic flow first occurs over a small zone of the upper airfoil surface where the local velocity is highest. Oblique shock waves form and there is a decrease in lift coefficient and an increase in drag coefficient. The adverse pressure gradient across the shock waves undoubtedly affects the boundary layer and may seriously influence separation. At slightly larger Mach numbers, shock waves occur along the undersurface too, as in Fig. 6.7*b*. In Fig. 6.8, for some airfoils, the lift coefficient starts to decrease as the upper shock waves form (point *A*), and then starts to increase when the lower shock waves occur (point *B*).

For higher approach velocities in the transonic range, a detached shock wave forms ahead of the airfoil, with subsonic flow between it and the forward portion of the airfoil (Fig. 6.7*c*). For increasing $\mathbf{M}_\infty$ the detached shock wave approaches the airfoil leading edge. When it becomes attached, the flow is everywhere supersonic (Fig. 6.7*d*). Figure 6.7*e* indicates the shock-wave formation for the *hypersonic* range.

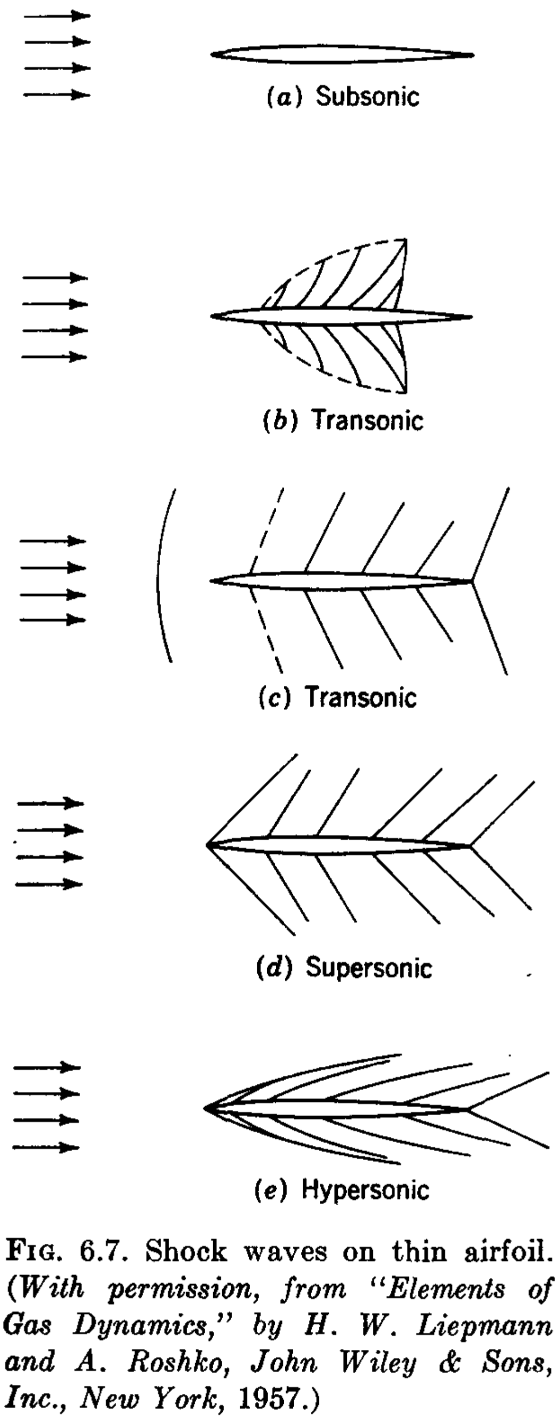


FIG. 6.7. Shock waves on thin airfoil. (*With permission, from "Elements of Gas Dynamics," by H. W. Liepmann and A. Roshko, John Wiley & Sons, Inc., New York,* 1957.)

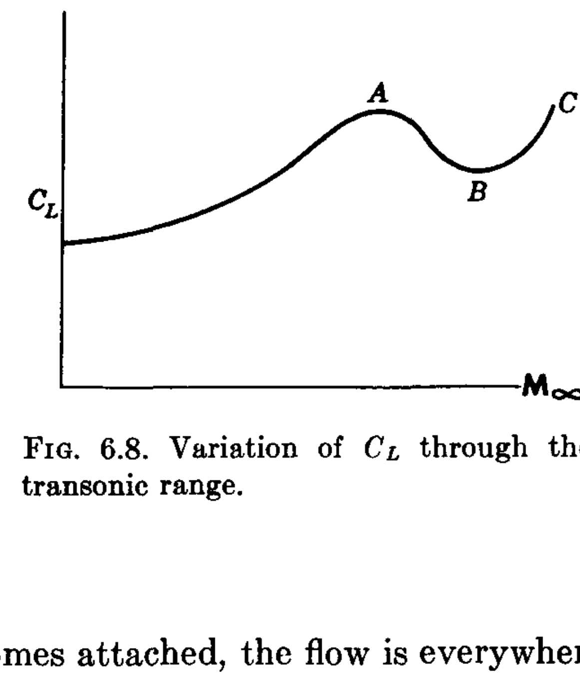


FIG. 6.8. Variation of C_L through the transonic range.

Sonic Boom. Sound is caused by a pressure wave striking the ear. A very loud sound, where a very great difference in pressure occurs across the wave, is interpreted by the ear as an explosion. This type of pressure wave is called a shock wave.

When an aircraft flies at a speed faster than sound, it creates shock waves in the air. Under certain atmospheric conditions these shock waves reach the ground and are heard as explosions, or "sonic booms." Most booms heard are the strong shock waves caused by aircraft accelerating from below to above the speed of sound in a dive. In so doing, many shock waves are formed on the aircraft, with the strongest (greatest pressure difference) occurring at the nose and tail. Then, as the pilot pulls out of the dive, the aircraft slows down and the shock wave continues on, striking the listener's ear and causing him to hear either one or two booms, depending on atmospheric conditions, direction of dive, etc. The loudness will depend on the aircraft speed, its rate of pull-out, and its altitude at the bottom of the dive. In low-altitude, level flight at supersonic speeds the boom will be heard, but not until after the aircraft has flown past the listener.

Wave Drag. The occurrence of shock waves is detrimental to the performance of an aircraft for two reasons. The sudden pressure increase through a shock wave produces an adverse pressure gradient in the boundary layer, promoting separation and the usual effects on lift and drag (a decrease and an increase, respectively). Also, additional drag results because energy is made unavailable by the shock waves.

The drag resulting from compressibility effects (called *wave drag*) begins to affect an aircraft at flight velocities slightly below the speed of sound owing to the presence of regions of supersonic flow on the aircraft surfaces at these speeds. The lowest Mach number at which such regions and the accompanying shock waves will occur is called the critical Mach number ($\mathbf{M}_{\text{crit}}$). The rapid increase in drag and decrease in lift and propeller efficiency occurring at about $\mathbf{M}_{\text{crit}} = 0.7$ convinced a large number of people in the 1930s that there existed a "sonic barrier," a limiting speed beyond which aircraft would never fly. The aircraft propulsion systems in use at that time simply could not produce sufficient thrust to accelerate past this velocity. Even by the early 1940s, aerodynamic refinements had extended this "drag divergence" speed only up to speeds in the vicinity of $\mathbf{M}_{\text{crit}} = 0.8$. Then in 1945, North American Aviation combined a sweptback wing and a jet engine and the sonic barrier was overcome.

Probably the two most effective methods for delaying compressibility effects on airfoils are the use of thin airfoils and sweepback. A sweptback wing is one whose mean chord line (see Fig. 6.9) is not perpendicular to the relative air velocity. To understand the physical concept of this design consider a uniform wing of infinite span with its leading edge swept back at an angle σ from the normal to the relative air velocity V. The flow normal to the leading edge has the velocity $V \cos \sigma$. The tangential velocity $V \sin \sigma$ of the original flow does not influence the lift on the wing but is important only in the determination of frictional stresses. Since only the normal component of velocity is significant, the effective Mach number is $\mathbf{M}\cos\sigma$. Therefore, even though the flight Mach number may be 1 or greater, the effective Mach number $\mathbf{M}\cos\sigma$ may, through sufficient sweepback, be made small enough to postpone and lessen the adverse effect of shocks (shock stall).

Sweepback does have two major disadvantages. First, the lift is decreased by reducing the normal component of velocity from V to $V\cos\sigma$, thus requiring larger wing areas. Second, severe structural problems are associated with sweptback wings, which must be made longer to provide additional area. The above disadvantages of sweepback are overcome in the transonic regime by the use of delta wings (for example, the F-102 or B-58), which do, however, possess problems of stability and control. On the other hand, the inherent advantages of sweptback wings can be utilized in the regime of supersonic flight. Here the velocities are sufficiently large that the wing area is relatively unimportant in the generation of lift ($L = C_L\frac{1}{2}\rho AV^2$) and strong, stubby wings of this design are sufficient for flight (examples include the F-101 and F-105).

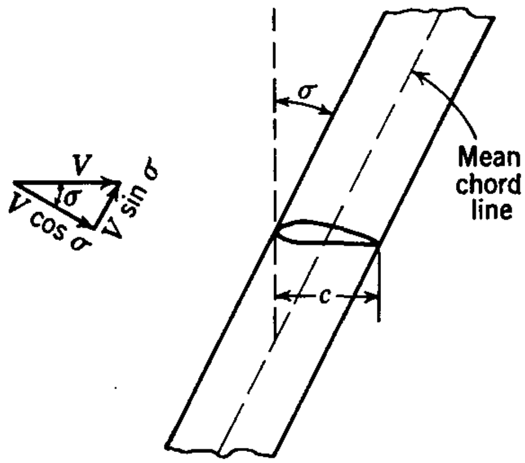


FIG. 6.9. Sweptback wing.

In addition to the major disadvantages of sweepback stated above, there is also a decrease in $dC_L/d\alpha$ at low speeds, thereby requiring a high angle of attack in landing. Also, difficulty in control occurs near the stall owing to a tendency for early stall at the tips. Vertical strips parallel to the direction of flight placed just inboard of the ailerons are useful in preventing the boundary-layer flow responsible for the tip-stall phenomenon.

Area Rule. Another method of reducing the drag rise which occurs as an aircraft enters the transonic zone is the use of the *area rule* in aircraft design. Experiments have shown that the drag rise in this zone is primarily a function of the axial distribution of the cross-sectional area of the aircraft normal to the air stream. In other words, if the change in cross-sectional area (slope of curve, Fig. 6.10) is gradual from nose to tail, the drag will be small. The addition of wing and tail to a body of revolution, however, causes an abrupt increase in the effective cross-sectional area (Fig. 6.10) and therefore a large transonic drag. The area rule prescribes that the entire aircraft be designed to provide a gradual change in area from nose to tail (solid line, Fig. 6.11). This could be accomplished by indentation of the fuselage at the wing and tail roots.

However, the volume requirements for engine, fuel, instruments, pay load, etc., remain the same, so the gradual change is effected by building up the body fore and aft of the wing, giving the aircraft a "coke-bottle" shape. By means of this rule, 90 per cent of the drag rise can be eliminated in the region of $1.00 < \mathbf{M} < 1.05$. At higher Mach numbers the drag on an aircraft using the area rule approaches the drag of a conventional aircraft.

Aerodynamic Heating. One of the problems of very high-speed (hypersonic) flight is aerodynamic heating. If heat conduction is neglected

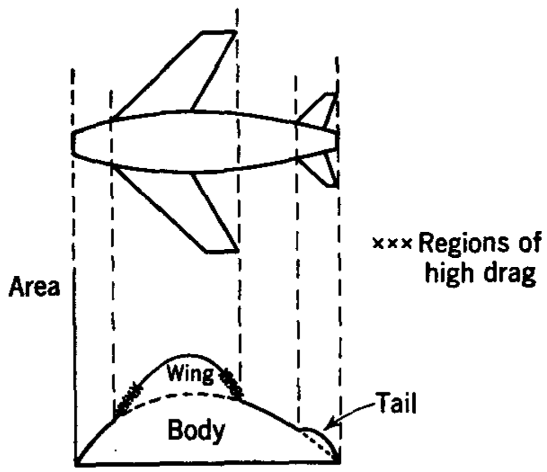


FIG. 6.10. Variation in cross-sectional area of airplane.

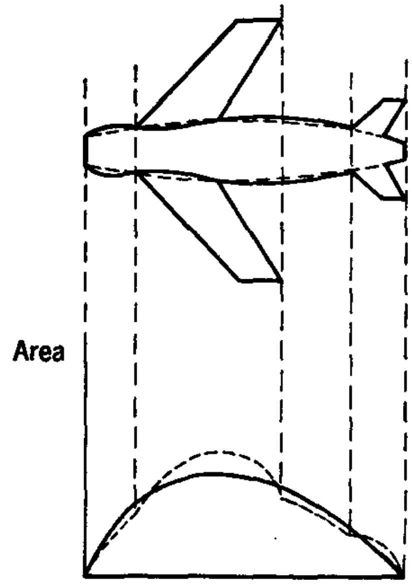


FIG. 6.11. Variation in area with area rule design.

(adiabatic flow), the temperature of a stagnation point is found from Eq. (6.6.7) by replacing h_0 by c_pT_0; then

$$T_0 = T + \frac{V^2}{2c_p} \tag{6.9.1}$$

in which T is the free-stream static temperature of the fluid at velocity V relative to a body immersed in the fluid. T_0 is the stagnation temperature. From Eq. (6.1.8) and $c = \sqrt{kRT}$

$$R = \frac{k-1}{k} c_p$$

and

$$c = \sqrt{c_p(k-1)T} \tag{6.9.2}$$

By eliminating c_p in Eqs. (6.9.1) and (6.9.2),

$$T_0 = T + \frac{V^2}{c^2}\frac{k-1}{2}T = T\left(1 + \frac{k-1}{2}\mathbf{M}^2\right) \tag{6.9.3}$$

For air, $k = 1.4$,

$$T_0 = T(1 + 0.2\mathbf{M}^2) \tag{6.9.4}$$

A Mach number of 5 results in a stagnation temperature six times the free-stream static temperature.

In a real fluid, because of viscous action, the velocity at a solid boundary is zero relative to the boundary, and it may be shown[1] that the frictional heating in the boundary layer causes about the same temperature rise as the adiabatic compression given by Eq. (6.9.3). The severity of this aerodynamic heating is one of the dominant considerations in all advanced aircraft design.

The aerodynamic heating problem can be solved in several ways, some of which are demonstrated in the design of current high-velocity missiles. One, the *heat-sink* approach, uses sufficient mass or coolant to absorb all the incoming heat without exceeding the temperature limits of the materials. A second method, *ablation*, is to make the leading edges out of a material that is a poor conductor so that the outer surface melts or sublimes while the inner surface remains cool. Both of these methods, particularly ablation, would not be suited for long flight (re-entry) times. Another attempt at a solution is by the use of transpiration, or "sweat cooling," in which a liquid, gas, or vapor is pumped through a porous surface, absorbing the heat and cooling by evaporation. This method works for flights of both long and short duration but has two principal disadvantages: (1) a materials problem and (2) a strong tendency to cause transition from laminar to turbulent boundary-layer flow, the latter having the undesirable characteristic of transferring several times as much heat to the surface as the former.

The heat-sink design is best exemplified by the blunt nose cones presently employed on intercontinental ballistic missiles (ICBM's). The blunt-nosed body is surrounded by a boundary layer of high-temperature air which will be partly dissociated (broken down into constituent elements or dissociation of a single element, e.g., $H_2 \rightleftharpoons 2H$) and, to a lesser extent, ionized and will be at temperatures of 15,000°F at ICBM velocities ($M \approx 24$). This hot gas can transfer heat to the body by convection and by radiation. The former is the prime contributor of heat, which the body can then dissipate by conduction and radiation. The convective heat transfer to the stagnation point of a blunt body is inversely proportional to the square root of the nose radius $1/\sqrt{r}$. Thus a large nose radius, i.e., blunt rather than pointed, results in less convective heat transfer from the air to the surface. This design also provides more nose-cone material immediately adjacent to the highest temperature (the stagnation point), thus facilitating the conduction of heat away from this point. Transition of the boundary layer from laminar to turbulent with the accompanying rise in convective heat

[1] H. Schlichting, "Boundary Layer Theory," 4th ed., chap. 15, McGraw-Hill Book Company, Inc., New York, 1961.

transfer to the nose cone is prevented by giving the nose cone a highly polished surface.

The use of the heat-sink design has a limitation imposed by the thermal conductivity of the heat-sink material. For the larger peak heat transfer associated with the highest speed missiles, the heat cannot be conducted away from the exposed face of the heat sink into the interior rapidly enough, and catastrophic melting occurs. This process of melting during re-entry is called ablation and is actually a type of self-regulating heat transfer by vaporization.

Slow satellite re-entry usually requires additional insulating material behind the ablating skin to keep inner surface temperatures within limits which can be tolerated by the pay load. However, in the re-entry of a ballistic nose cone, involving much more severe heat transfer, the surface of the ablating material recedes at the same rate that the heat penetrates into the interior. Ablating materials that have been or are being studied include pure plastics, plastics reinforced with organic or inorganic fibers, silica and other oxides, carbon or graphite, gypsum, magnesium nitride, and ceramics.

The problem of aerodynamic heating is still far from being completely solved and will constitute a challenge in the field of aerophysics for years to come. However, the progress made toward a solution in the few short years since the problem has become important is positive proof that better and better methods of reducing heat input and transferring more heat to the surrounding space will be discovered in the very near future.

6.10. Analogy of Shock Waves to Open-channel Waves. Both the oblique and normal shock waves in a gas have their counterpart in open-channel flow. An elementary surface wave has a speed in still liquid of $\sqrt{gy}$, in which y is the depth in a wide, open channel. When flow in the channel is such that $V = V_c = \sqrt{gy}$, the Froude number is unity and flow is said to be *critical*, i.e., a small disturbance cannot be propagated upstream. This is analogous to sonic flow at the throat of a tube, with Mach number unity. For liquid velocities greater than $V_c = \sqrt{gy}$ the Froude number is greater than unity and the velocity is supercritical, analogous to supersonic gas flow. Changes in depth are analogous to changes in density in gas flow.

The continuity equation in an open channel of constant width is

$$Vy = \text{constant}$$

and the continuity equation for compressible flow in a tube of constant cross section is

$$V\rho = \text{constant}$$

Compressible fluid density ρ and open-channel depth y are analogous.

The same analogy is also present in the energy equation. The energy equation for a horizontal open channel of constant width, neglecting friction, is

$$\frac{V^2}{2g} + y = \text{constant}$$

After differentiating

$$V\,dV + g\,dy = 0$$

By substitution from $V_c = \sqrt{gy}$ to eliminate g,

$$V\,dV + V_c^2\frac{dy}{y} = 0$$

which is to be compared with the energy equation for compressible flow [Eq. (6.3.4)]

$$V\,dV + c^2\frac{d\rho}{\rho} = 0 \tag{6.3.4}$$

The two critical velocities V_c and c are analogous, and, hence, y and ρ are analogous.

By applying the momentum equation to a small depth change in horizontal open-channel flow, and to a sudden density change in compressible flow, the density and the open-channel depth can again be shown to be analogous. In effect, the analogy is between the Froude number and the Mach number.

Analogous to the normal shock wave is the hydraulic jump, which causes a sudden change in velocity and depth, and a change in Froude number from greater than unity to less than unity. Analogous to the oblique shock and rarefaction waves in gas flow are oblique liquid waves produced in a channel by changes in the direction of the channel walls, or by changes in floor elevation.

A body placed in an open channel with flow at Froude number greater than unity causes waves on the surface that are analogous to shock and rarefaction waves on a similar (two-dimensional) body in a supersonic wind tunnel. Changes to greater depth are analogous to compression shock, and changes to lesser depth to rarefaction waves. Shallow water tanks, called ripple tanks, have been used to study supersonic flow situations.

PROBLEMS

6.1. 3 lb_m of a perfect gas, molecular weight 36, had its temperature increased 3.2°F when 2000 ft-lb of work was done on it in an insulated constant-volume chamber. Determine c_v and c_p.

6.2. A gas of molecular weight 48 has a $c_p = 0.372$. What is c_v for this gas?

6.3. Calculate the specific heat ratio k for Probs. 6.1 and 6.2.

6.4. The enthalpy of a gas is increased by 0.4 Btu/lb$_m$ °R when heat is added at constant pressure, and the internal energy is increased by 0.3 Btu/lb$_m$ °R when the volume is maintained constant and heat is added. Calculate the molecular weight.

6.5. Calculate the enthalpy change of 3 lb$_m$ carbon monoxide from $p_1 = 20$ psia, $T_1 = 40°F$ to $p_2 = 60$ psia, $T_2 = 340°F$

6.6. Calculate the entropy change in Prob. 6.5.

6.7. From Eq. (6.1.13) and the perfect-gas law, derive the equation of state for isentropic flow.

6.8. Compute the enthalpy change per slug for helium from $T_1 = 0°F$, $p_1 =$ 20 psia, to $T_2 = 100°F$ in an isentropic process.

6.9. In an isentropic process 3 lb$_m$ oxygen with a volume of 4.0 ft^3 at 60°F has its absolute pressure doubled. What is the final temperature?

6.10. Work out the expression for density change with temperature for a reversible polytropic process.

6.11. Hydrogen at 40 psia, 30°F, has its temperature increased to 100°F by a reversible polytropic process with $n = 1.20$. Calculate the final pressure.

6.12. A gas has a density decrease of 13 per cent in a reversible polytropic process when the temperature decreases from 115 to 40°F. Compute the exponent n for the process.

6.13. A projectile moves through water at 60°F at 3000 ft/sec. What is its Mach number?

6.14. If an airplane travels at 500 mph at sea level $p = 14.7$ psia, $t = 68°F$, and at the same speed in the stratosphere where $t = -67°F$, how much greater is the Mach number in the latter case?

6.15. What is the speed of sound through hydrogen at 100°F?

6.16. Derive the equation for speed of a small liquid wave in an open channel by using the methods of Sec. 6.2 for determination of speed of sound (Fig. 6.12):

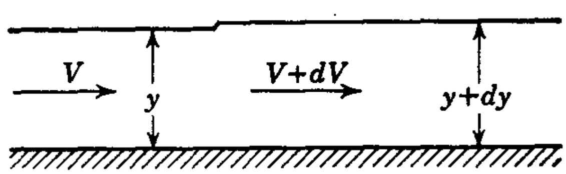


FIG. 6.12

6.17. By using the Euler equation with a loss term

$$V\,dV + \frac{dp}{\rho} + d\,(\text{losses}) = 0$$

the continuity equation $\rho V =$ constant, and $c = \sqrt{dp/d\rho}$, show that for subsonic flow in a pipe the velocity must increase in the downstream direction.

6.18. Isentropic flow of air occurs at a section of a pipe where $p = 40$ psia, $t = 90°F$, and $V = 430$ ft/sec. An object is immersed in the flow which brings the velocity to zero. What are the temperature and pressure at the stagnation point?

6.19. What is the Mach number for the flow of Prob. 6.18?

6.20. How does the temperature and pressure at the stagnation point in isentropic flow compare with reservoir conditions?

6.21. Air flows from a reservoir at 160°F, 80 psia. Assuming isentropic flow, calculate the velocity, temperature, pressure, and density at a section where $\mathbf{M} = 0.60$.

6.22. Oxygen flows from a reservoir $p_0 = 100$ psia, $t_0 = 80$°F, to a 6-in.-diameter section where the velocity is 600 ft/sec. Calculate the mass rate of flow (isentropic) and the Mach number, pressure, and temperature at the 6-in. section.

6.23. Helium discharges from a $\frac{1}{2}$-in.-diameter converging nozzle at its maximum rate for reservoir conditions of $p = 60$ psia, $t = 72$°F. What restrictions are placed on the downstream pressure? Calculate the mass flow rate and velocity of the gas at the nozzle.

6.24. Air in a reservoir at 400 psi, $t = 290$°F, flows through a 2-in.-diameter throat in a converging-diverging nozzle. For $\mathbf{M} = 1$ at the throat, calculate p, ρ, and T there.

6.25. What must be the velocity, pressure, density, temperature, and diameter at a cross section of the nozzle of Prob. 6.24 where $\mathbf{M} = 2.4$?

6.26. Nitrogen in sonic flow at a 1-in.-diameter throat section has a pressure of 10 psia, $t = 0$°F. Determine the mass flow rate.

6.27. What is the Mach number for Prob. 6.26 at a $1\frac{1}{2}$-in.-diameter section in supersonic and in subsonic flow?

6.28. What diameter throat section is needed for critical flow of 0.6 lb$_m$/sec carbon monoxide from a reservoir where $p = 300$ psia, $t = 100$°F?

6.29. A supersonic nozzle is to be designed for air flow with $\mathbf{M} = 3$ at the exit section, which is 6 in. in diameter and has a pressure of 1 psia and temperature of -120°F. Calculate the throat area and reservoir conditions.

6.30. In Prob. 6.29 calculate the diameter of cross section for $\mathbf{M} = 1.5$, 2.0, and 2.5.

6.31. For reservoir conditions $p_0 = 120$ psia, $t_0 = 120$°F, air flows through a converging-diverging tube with a 3.0-in.-diameter throat with a maximum Mach number of 0.80. Determine the mass rate of flow and the diameter, pressure, velocity, and temperature at the exit where $\mathbf{M} = 0.50$.

6.32. Calculate the exit velocity and the mass rate of flow of nitrogen from a reservoir $p = 60$ psia, $t = 50$°F, through a converging nozzle of 2 in. diameter discharging to atmosphere.

6.33. Reduce Eq. (6.3.25) to its form for air flow. Plot p/p_0 vs. A^*/A for the range of p/p_0 from 0.98 to 0.02.

6.34. By utilizing the plot of Prob. 6.33, find the two pressure ratios for $A^*/A = 0.50$.

6.35. In a converging-diverging duct in supersonic flow of hydrogen, the throat diameter is 2.0 in. Determine the pressure ratios p/p_0 in the converging and diverging ducts where the diameter is 2.5 in.

6.36. A shock wave occurs in a duct carrying air where the upstream Mach number is 2.0 and upstream temperature and pressure are 60°F and 2 psia. Calculate the Mach number, pressure, temperature, and velocity after the shock wave.

6.37. Show that entropy has increased across the shock wave of Prob. 6.36.

6.38. Conditions immediately before a normal shock wave in air flow are $p_u = 8$ psia, $t_u = 100°F$, $V_u = 1800$ ft/sec. Find $\mathbf{M}_u$, $\mathbf{M}_d$, p_d, and t_d, where the subscript d refers to conditions just downstream from the shock wave.

6.39. For $A = 0.16$ ft^2 in Prob. 6.38, calculate the entropy increase across the shock wave in Btu per second per degree Rankine.

6.40. Show, from the equations of Sec. 6.6, that temperature, pressure, and density decrease in real, adiabatic duct flow for subsonic conditions and increase for supersonic conditions.

6.41. What length of 2-in.-diameter insulated duct, $f = 0.012$, is needed when oxygen enters at $\mathbf{M} = 3.0$ and leaves at $\mathbf{M} = 2.0$?

6.42. Air enters an insulated pipe at $\mathbf{M} = 0.3$ and leaves at $\mathbf{M} = 0.7$. What portion of the duct length is required for the flow to occur at $\mathbf{M} = 0.5$?

6.43. Determine the maximum length, without choking, for the adiabatic flow of air in a 4-in.-diameter duct, $f = 0.025$, when upstream conditions are $t = 120°F$, $V = 700$ ft/sec, $p = 30$ psia. What are the pressure and temperature at the exit?

6.44. What minimum size insulated duct is required to transport 2 lb$_m$/sec nitrogen 1000 ft? The upstream temperature is 90°F, and the velocity there is 200 ft/sec. $f = 0.020$.

6.45. Find the upstream and downstream pressures in Prob. 6.44.

6.46. What is the maximum mass rate of flow of air from a reservoir, $t = 60°F$, through 19.15 ft of insulated 1-in.-diameter pipe, $f = 0.020$, discharging to atmosphere? $p = 14.7$ psia.

6.47. In frictionless oxygen flow through a duct the following conditions prevail at inlet and outlet: $V_1 = 300$ ft/sec; $t_1 = 60°F$; $\mathbf{M}_2 = 0.4$. Find the heat added per slug and the pressure ratio p_1/p_2.

6.48. In frictionless air flow through a 4-in.-diameter duct 0.4 lb$_m$/sec enters at $t = 30°F$, $p = 10$ psia. How much heat, in Btu per pound mass, can be added without choking the flow?

6.49. Frictionless flow through a duct with heat transfer causes the Mach number to decrease from 2 to 1.8. $k = 1.4$. Determine the temperature, velocity, pressure, and density ratios.

6.50. In Prob. 6.49 the duct is 2 in. square, $p_1 = 10$ psia, and $V_1 = 2000$ ft/sec. Calculate the mass rate of flow for air flowing.

6.51. How much heat must be transferred per pound mass to cause the Mach number to increase from 2 to 2.8 in a frictionless duct carrying air? $V_1 = 2000$ ft/sec.

6.52. Oxygen at $V_1 = 1600$ ft/sec, $p = 12$ psia, $t = 0°F$ flows in a 2-in.-diameter frictionless duct. How much heat transfer per pound mass is needed for sonic conditions at the exit?

6.53. Prove the density, pressure, and velocity trends given in Sec. 6.8 in the table of trends in flow properties.

6.54. Apply the first law of thermodynamics, Eq. (3.7.1), to isothermal flow of a perfect gas in a horizontal pipeline, and develop an expression for the heat added per slug flowing.

6.55. Air is flowing at constant temperature through a 3-in.-diameter horizontal pipe, $f = 0.02$. At the entrance $V_1 = 300$ ft/sec, $t = 100°$F, $p_1 = 30$ psia. What is the maximum pipe length for this flow, and how much heat is transferred to the air per pound mass?

6.56. Air at 60°F flows through a 1-in.-diameter pipe at constant temperature. At the entrance $V_1 = 200$ ft/sec, and at the exit $V_2 = 400$ ft/sec. $f = 0.016$. What is the length of the pipe?

6.57. If the pressure at the entrance of the pipe of Prob. 6.56 is 20 psia, what is the pressure at the exit and what is the heat transfer to the pipe per second?

6.58. Hydrogen enters a pipe from a converging nozzle at $\mathbf{M} = 1$, $p = 1$ psia, $t = 0°$F. Determine for isothermal flow the maximum length of pipe, in diameters, and the pressure change over this length. $f = 0.010$.

6.59. Oxygen flows at constant temperature of 68°F from a pressure tank, $p = 2000$ psia, through 10 ft of 0.01-ft ID tubing to another tank where $p = 1600$ psia. $f = 0.010$. Determine the mass rate of flow.

6.60. In isothermal flow of nitrogen at 90°F, 2 lb_m/sec is to be transferred 100 ft from a tank $p = 200$ psia to a tank $p = 160$ psia. What is the minimum size tubing, $f = 0.016$, that is needed?

6.61. Specific heat at constant volume, is defined by

(*a*) kc_p (*b*) $\left(\dfrac{\partial u}{\partial T}\right)_p$ (*c*) $\left(\dfrac{\partial T}{\partial u}\right)_v$ (*d*) $\left(\dfrac{\partial u}{\partial T}\right)_v$ (*e*) none of these answers

6.62. Specific heat at constant pressure, for a perfect gas, is *not* given by

(*a*) kc_v (*b*) $(\partial h/\partial T)_p$ (*c*) $(h_2 - h_1)/(T_2 - T_1)$
(*d*) $[\Delta u + \Delta(p/\rho)]/\Delta t$ (*e*) any of these answers

6.63. For a perfect gas, the enthalpy

(*a*) always increases owing to losses
(*b*) depends upon the pressure only
(*c*) depends upon the temperature only
(*d*) may increase while the internal energy decreases
(*e*) satisfies none of these answers

6.64. The following classes of substances may be considered perfect gases:

(*a*) ideal fluids
(*b*) saturated steam, water vapor, and air
(*c*) fluids with a constant bulk modulus of elasticity
(*d*) water vapor, hydrogen, and nitrogen at low pressure
(*e*) none of these answers

6.65. c_p and c_v are related by

(*a*) $k = c_p/c_v$ (*b*) $k = c_pc_v$ (*c*) $k = c_v/c_p$ (*d*) $c_p = c_v^k$
(*e*) none of these answers

6.66. If $c_p = 0.30$ Btu/lb_m °R and $k = 1.66$, in foot-pounds per slug degree Fahrenheit, c_v equals (*a*) 0.582 (*b*) 1452 (*c*) 4520 (*d*) 7500 (*e*) none of these answers

6.67. If $c_p = 0.30$ Btu/lb$_m$ °R and $k = 1.33$, the gas constant in Btu per pound mass per degree Rankine is

(*a*) 0.075 (*b*) 0.099 (*c*) 0.399 (*d*) 0.699 (*e*) none of these answers

6.68. $R = 62$ ft-lb/lb$_m$ °R and $c_p = 0.279$ Btu/lb$_m$ °F. The isentropic exponent k is

(*a*) 1.2 (*b*) 1.33 (*c*) 1.66 (*d*) 1.89 (*e*) none of these answers

6.69. The specific heat ratio is given by

(*a*) $\dfrac{1}{1 - R/c_p}$ (*b*) $1 + \dfrac{c_v}{R}$ (*c*) $\dfrac{c_p}{c_v} + R$ (*d*) $\dfrac{1}{1 - c_v/R}$
(*e*) none of these answers

6.70. The entropy change for a perfect gas is

(*a*) always positive
(*b*) a function of temperature only
(*c*) $(\Delta q_H/T)_{rev}$
(*d*) a thermodynamic property depending upon temperature and pressure
(*e*) a function of internal energy only

6.71. An isentropic process is always

(*a*) irreversible and adiabatic
(*b*) reversible and isothermal
(*c*) frictionless and adiabatic
(*d*) frictionless and irreversible
(*e*) none of these answers

6.72. The relation p = constant ρ^k holds only for those processes that are

(*a*) reversible polytropic
(*b*) isentropic
(*c*) frictionless isothermal
(*d*) adiabatic irreversible
(*e*) none of these answers

6.73. The reversible polytropic process is

(*a*) adiabatic frictionless
(*b*) given by p/ρ = constant
(*c*) given by $p\rho^k$ = constant
(*d*) given by p/ρ^n = constant
(*e*) none of these answers

6.74. A reversible polytropic process could be given by

(*a*) $\dfrac{T_1}{T_2} = \left(\dfrac{\rho_1}{\rho_2}\right)^{n-1}$ (*b*) $\dfrac{p_1}{p_2} = \left(\dfrac{\rho_2}{\rho_1}\right)^{n}$ (*c*) $\dfrac{T_1}{T_2} = \left(\dfrac{p_1}{p_2}\right)^{n-1}$

(*d*) $\dfrac{T_1}{T_2} = \left(\dfrac{\rho_1}{\rho_2}\right)^{(n-1)/n}$ (*e*) none of these answers

6.75. In a reversible polytropic process

(*a*) some heat transfer occurs
(*b*) the entropy remains constant
(*c*) the enthalpy remains constant
(*d*) the internal energy remains constant
(*e*) the temperature remains constant

6.76. The differential equation for energy in isentropic flow may take the form

(*a*) $dp + d(\rho V^2) = 0$
(*b*) $\dfrac{dV}{V} + \dfrac{d\rho}{\rho} + \dfrac{dA}{A} = 0$
(*c*) $2V\,dV + \dfrac{dp}{\rho} = 0$
(*d*) $V\,dV + \dfrac{dp}{\rho} = 0$
(*e*) none of these answers

6.77. Select the expression that does *not* give the speed of a sound wave:

(*a*) $\sqrt{kRT}$ (*b*) $\sqrt{p/\rho}$ (*c*) $\sqrt{dp/d\rho}$ (*d*) $\sqrt{kp/\rho}$ (*e*) $\sqrt{K/\rho}$

6.78. The speed of a sound wave in a gas is analogous to

(*a*) the speed of flow in an open channel
(*b*) the speed of an elementary wave in an open channel
(*c*) the change in depth in an open channel
(*d*) the speed of a disturbance traveling upstream in moving liquid
(*e*) none of these answers

6.79. The speed of sound in water, in feet per second, under ordinary conditions is about

(*a*) 460 (*b*) 1100 (*c*) 4600 (*d*) 11,000 (*e*) none of these answers

6.80. The speed of sound in an ideal gas varies directly as

(*a*) the density
(*b*) the absolute pressure
(*c*) the absolute temperature
(*d*) the bulk modulus of elasticity
(*e*) none of these answers

6.81. Select the correct statement regarding frictionless flow:

(*a*) In diverging conduits the velocity always decreases.
(*b*) The velocity is always sonic at the throat of a converging-diverging tube.
(*c*) In supersonic flow the area decreases for increasing velocity.
(*d*) Sonic velocity cannot be exceeded at the throat of a converging-diverging tube.
(*e*) At Mach zero the velocity is sonic.

6.82. In isentropic flow the temperature

(*a*) cannot exceed the reservoir temperature
(*b*) cannot drop, then increase again downstream
(*c*) is independent of the Mach number
(*d*) is a function of Mach number only
(*e*) remains constant in duct flow

6.83. The critical pressure ratio for isentropic flow of carbon monoxide is

(*a*) 0.528 (*b*) 0.634 (*c*) 0.833 (*d*) 1.0 (*e*) none of these answers

6.84. Select the correct statement regarding flow through a converging-diverging tube.

(*a*) When the Mach number at exit is greater than unity no shock wave has developed in the tube.
(*b*) When the critical pressure ratio is exceeded, Mach number at the throat is greater than unity.
(*c*) For sonic velocity at the throat, one and only one pressure or velocity can occur at a given section downstream.
(*d*) The Mach number at the throat is always unity.
(*e*) The density increases in the downstream direction throughout the converging portion of the tube.

6.85. In a normal shock wave in one-dimensional flow the

(*a*) velocity, pressure, and density increase
(*b*) pressure, density, and temperature increase
(*c*) velocity, temperature, and density increase
(*d*) pressure, density, and momentum per unit time increase
(*e*) entropy remains constant

6.86. A normal shock wave

(*a*) is reversible
(*b*) may occur in a converging tube
(*c*) is irreversible
(*d*) is isentropic
(*e*) is none of these answers

6.87. A normal shock wave is analogous to

(*a*) an elementary wave in still liquid
(*b*) the hydraulic jump
(*c*) open-channel conditions with $\mathbf{F} < 1$
(*d*) flow of liquid through an expanding nozzle
(*e*) none of these answers

6.88. Across a normal shock wave in a converging-diverging nozzle for adiabatic flow the following relationships are valid:

(*a*) continuity and energy equations, equation of state, isentropic relationship
(*b*) energy and momentum equations, equation of state, isentropic relationship
(*c*) continuity, energy, and momentum equations; equation of state
(*d*) equation of state, isentropic relationship, momentum equation, mass-conservation principle
(*e*) none of these answers

6.89. Across a normal shock wave there is an increase in

(*a*) p, $\mathbf{M}$, s (*b*) p, s; decrease in $\mathbf{M}$ (*c*) p; decrease in s, $\mathbf{M}$
(*d*) p, $\mathbf{M}$; no change in s (*e*) p, $\mathbf{M}$, T

6.90. A Fanno line is developed from the following equations:

(*a*) momentum and continuity
(*b*) energy and continuity
(*c*) momentum and energy
(*d*) momentum, continuity, and energy
(*e*) none of these answers

6.91. A Rayleigh line is developed from the following equations:

(*a*) momentum and continuity
(*b*) energy and continuity
(*c*) momentum and energy
(*d*) momentum, continuity, and energy
(*e*) none of these answers

6.92. Select the correct statement regarding a Fanno or Rayleigh line:

(*a*) Two points having the same value of entropy represent conditions before and after a shock wave.
(*b*) pV is held constant along the line.
(*c*) Mach number always increases with entropy.
(*d*) The subsonic portion of the curve is at higher enthalpy than the supersonic portion.
(*e*) Mach 1 is located at the maximum enthalpy point.

6.93. Choking in pipe flow means that

(*a*) a valve is closed in the line
(*b*) a restriction in flow area occurs
(*c*) the specified mass flow rate cannot occur
(*d*) shock waves always occur
(*e*) supersonic flow occurs somewhere in the line

6.94. In subsonic adiabatic flow with friction in a pipe

(*a*) V, $\mathbf{M}$, s increase; p, T, ρ decrease.
(*b*) p, V, $\mathbf{M}$, increase; T, ρ decrease.
(*c*) p, $\mathbf{M}$, s increase; V, T, ρ decrease.
(*d*) ρ, $\mathbf{M}$, s increase; V, T, p decrease.
(*e*) T, V, s increase; $\mathbf{M}$, p, ρ decrease.

6.95. In supersonic adiabatic flow with friction in a pipe

(*a*) V, $\mathbf{M}$, s increase; p, T, ρ decrease.
(*b*) p, T, s increase; ρ, V, $\mathbf{M}$ decrease.
(*c*) p, $\mathbf{M}$, s increase; V, T, ρ decrease.
(*d*) p, T, ρ, s increase; V, $\mathbf{M}$ decrease.
(*e*) p, ρ, s increase; V, $\mathbf{M}$, T decrease.

6.96. Select the correct statement regarding frictionless duct flow with heat transfer:

(*a*) Adding heat to supersonic flow increases the Mach number.
(*b*) Adding heat to subsonic flow increases the Mach number.
(*c*) Cooling supersonic flow decreases the Mach number.
(*d*) The Fanno line is valuable in analyzing the flow.
(*e*) The isentropic stagnation temperature remains constant along the pipe.

6.97. Select the correct trends in flow properties for frictionless duct flow with heat transferred to the pipe, $\mathbf{M} < 1$:

(*a*) p, V increase; ρ, T, T_0 decrease.
(*b*) V, T_0 increase; p, ρ decrease.
(*c*) p, ρ, T increase; V, T_0 decrease.
(*d*) V, T increase; p, ρ, T_0 decrease.
(*e*) T_0, V, ρ increase; p, T decrease.

6.98. Select the correct trends for cooling in frictionless duct flow, $\mathbf{M} > 1$:

(*a*) V increases; p, ρ, T, T_0 decrease.
(*b*) p, V increase; ρ, T, T_0 decrease.
(*c*) p, ρ, V increase; T, T_0 decrease.
(*d*) p, ρ increase; V, T, T_0 decrease.
(*e*) V, T, T_0 increase; p, ρ decrease.

6.99. In steady, isothermal flow in long pipelines, the significant value of **M** for determining trends in flow properties is

(*a*) $1/k$ (*b*) $1/\sqrt{k}$ (*c*) 1 (*d*) $\sqrt{k}$ (*e*) k

6.100. Select the correct trends in fluid properties for isothermal flow in ducts for $\mathbf{M} < 0.5$:

(*a*) V increases; $\mathbf{M}$, T_0, p, p_0, ρ decrease.
(*b*) V, $\mathbf{M}$ increase; T_0, p, p_0, ρ decrease.
(*c*) V, $\mathbf{M}$, T_0 increase; p, p_0, ρ decrease.
(*d*) V, T_0 increase; $\mathbf{M}$, p, p_0, ρ decrease.
(*e*) V, $\mathbf{M}$, p_0, T_0 increase; p, ρ decrease.

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7

IDEAL-FLUID FLOW

In the preceding chapters most of the relationships have been developed for one-dimensional flow, i.e., flow in which the average velocity at each cross section is used and variations across the section are neglected. Many design problems in fluid flow, however, require more exact knowledge of velocity and pressure distributions, such as in flow over curved boundaries along an airplane wing, through the passages of a pump or compressor, or over the crest of a dam. An understanding of two- and three-dimensional flow of a nonviscous, incompressible fluid provides the student with a much broader approach to many real fluid-flow situations. There are also analogies that permit the same methods to apply to flow through porous media.

In this chapter the principles of irrotational flow of an ideal fluid are developed and applied to elementary flow cases. After the flow requirements are established, the vector operator ∇ is introduced, Euler's equation is derived, and the velocity potential is defined. Euler's equation is then integrated to obtain Bernoulli's equation, and stream functions and boundary conditions are developed. Flow cases are then studied in three and two dimensions.

7.1. Requirements for Ideal-fluid Flow. The Prandtl hypothesis, Sec. 5.5, states that, for fluids of low viscosity, the effects of viscosity are appreciable only in a narrow region surrounding the fluid boundaries. For incompressible flow situations in which the boundary layer remains thin, ideal-fluid results may be applied to flow of a real fluid to a satisfactory degree of approximation. Converging or accelerating flow situations generally have thin boundary layers, but decelerating flow may have separation of the boundary layer and development of a large wake that is difficult to predict analytically.

An ideal fluid must satisfy the following requirements:

a. The continuity equation, Sec. 3.4, div $\mathbf{q} = 0$, or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

b. Newton's second law of motion at every point at every instant.

c. Neither penetration of fluid into nor gaps between fluid and boundary at any solid boundary.

If, in addition to requirements *a*, *b*, and *c*, the assumption of irrotational flow is made, the resulting fluid motion closely resembles real fluid motion for fluids of low viscosity, outside boundary layers.

Using the above conditions, the application of Newton's second law to a fluid particle leads to the Euler equation which, together with the assumption of irrotational flow, may be integrated to obtain the Bernoulli equation. The unknowns in a fluid-flow situation with given boundaries are velocity and pressure at every point. Unfortunately, in most cases it is impossible to proceed directly to equations for velocity and pressure distribution from the boundary conditions.

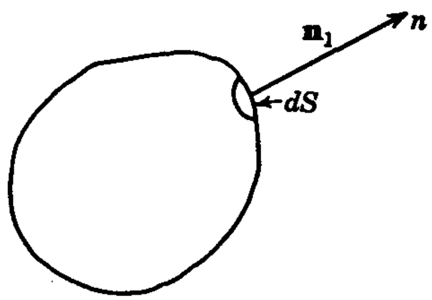


Fig. 7.1. Notation for unit normal $\mathbf{n}_1$ to area element dS.

7.2. The Vector Operator ∇. The vector operator ∇ (pronounced "del"), which may act on a vector as a scalar or vector product or may act on a scalar function, is most useful in developing ideal-fluid-flow theory.

Let U be the quantity acted upon by the operator. The operator ∇ is defined by

$$\nabla U = \lim_{V \to 0} \frac{1}{V} \int_S \mathbf{n}_1 U \, dS \tag{7.2.1}$$

U may be interpreted as $\cdot \, \mathbf{a}$, $\times \, \mathbf{a}$, where $\mathbf{a}$ is any vector, or as a scalar, say ϕ. Consider a small volume V with surface S and surface element dS. $\mathbf{n}_1$ is a unit vector in the direction of the outwardly drawn normal n of the surface element dS (Fig. 7.1). This definition of the operator is now examined to develop the concepts of gradient, divergence, and curl.

When U is a scalar, say ϕ, the *gradient* of ϕ is

$$\text{grad } \phi = \nabla \phi = \lim_{V \to 0} \frac{1}{V} \int_S \mathbf{n}_1 \phi \, dS \tag{7.2.2}$$

To interpret grad ϕ, the volume element is taken as a small prism of cross-sectional area dS, of height dn, with one end area in the surface $\phi(x,y,z) = c$ and the other end area in the surface

$$\phi + \left(\frac{\partial \phi}{\partial n}\right) dn = \text{constant}$$

(Fig. 7.2). As there is no change in ϕ in surfaces parallel to the end faces, by symmetry $\int \mathbf{n}_1 \phi \, dS$ over the curved surface of the element vanishes. Then

$$\int_S \mathbf{n}_1 \phi \, dS = \mathbf{n}_1 \left(\phi + \frac{\partial \phi}{\partial n} dn - \phi \right) dS$$

and the right-hand side of Eq. (7.2.2) becomes

$$\lim_{V \to 0} \frac{1}{dS\, dn} \frac{\partial \phi}{\partial n} dn\, dS = \mathbf{n}_1 \frac{\partial \phi}{\partial n}$$

and

$$\text{grad } \phi = \nabla \phi = \mathbf{n}_1 \frac{\partial \phi}{\partial n} \tag{7.2.3}$$

in which $\mathbf{n}_1$ is the unit vector, drawn normal to the surface over which ϕ is constant, positive in the direction of increasing ϕ. grad ϕ is a vector.

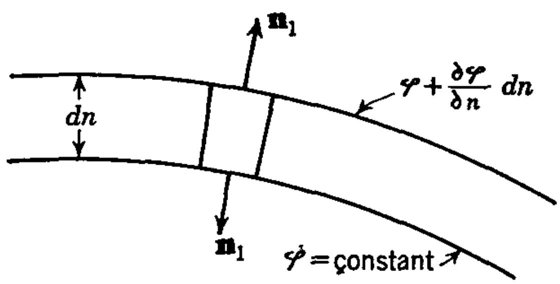


FIG. 7.2. Surfaces of constant scalar ϕ.

By interpreting U as the scalar (dot) product with ∇, the *divergence* is obtained. Let U be $\cdot\, \mathbf{q}$; then

$$\text{div } \mathbf{q} = \nabla \cdot \mathbf{q} = \lim_{V \to 0} \frac{1}{V} \int_S \mathbf{n}_1 \cdot \mathbf{q} \, dS \tag{7.2.4}$$

This expression has been used (in somewhat different form) in deriving the general continuity equation in Sec. 3.4. It is the volume flux per unit volume at a point and is a scalar.

The *curl* $\nabla \times \mathbf{q}$ is a more difficult concept that deals with the *vorticity* or rotation of a fluid element:

$$\text{curl } \mathbf{q} = \nabla \times \mathbf{q} = \lim_{V \to 0} \frac{1}{V} \int_S \mathbf{n}_1 \times \mathbf{q} \, dS \tag{7.2.5}$$

With reference to Fig. 7.3, $\mathbf{n}_1 \times \mathbf{q}$ is the velocity component tangent to the surface element dS at a point, since the vector product is a vector at right angles to the plane of the two constituent vectors, with magnitude $q \sin \theta$, as $n_1 = 1$. Then $\mathbf{n}_1 \times \mathbf{q}\, dS$ is an elemental vector that is the product of tangential velocity by the surface area element. Summed up over the surface, then divided by the volume, and the limit taken as $V \to 0$ yields the curl $\mathbf{q}$ at a point.

A special type of fluid motion is examined to demonstrate the connection between curl and rotation. Let a small circular cylinder of fluid be rotating about its axis as if it were a solid (Fig. 7.4), with angular velocity $\boldsymbol{\omega}$, which is a vector parallel to the axis of rotation. The radius of the cylinder is r and the length l. $\mathbf{n}_1 \times \mathbf{q}$ at every point on the curved surface is a vector parallel to the axis having the magnitude $q = \omega r$. Over the end areas the vector $\mathbf{n}_1 \times \mathbf{q}$ is equal and opposite at correspond-

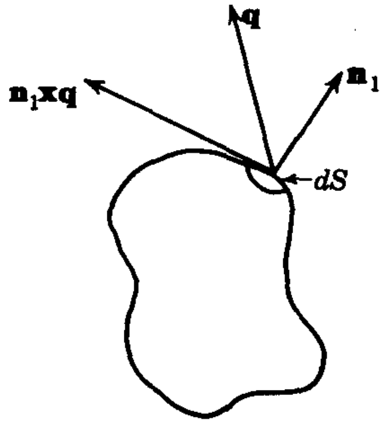


FIG. 7.3. Notation for curl of the velocity vector.

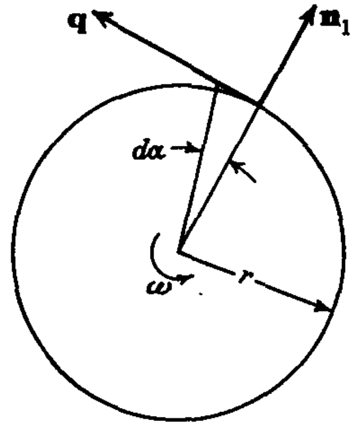


FIG. 7.4. Small fluid cylinder rotating as a solid.

ing points on each end and contributes nothing to the curl. Then, since $ds = lr\,d\alpha$,

$$\int_S \mathbf{n}_1 \times \mathbf{q}\,dS = \boldsymbol{\omega} \int_0^{2\pi} rlr\,d\alpha = 2\pi r^2 l\boldsymbol{\omega}$$

Equation (7.2.5) now yields

$$\text{curl}\,\mathbf{q} = \lim_{V\to 0} \frac{1}{\pi r^2 l}\,2\pi r^2 l\boldsymbol{\omega} = 2\boldsymbol{\omega}$$

showing that for solid-body rotation the curl of the velocity at a point is twice the rotation vector. If one considers the pure translation of a small element moving as a solid, then the curl $\mathbf{q}$ is always zero. As any rigid body motion is a combination of a translation and a rotation, it is noted that the curl of the velocity vector is always twice the rotation vector.

A fluid, however, not only may translate and rotate but may also deform. The definition of curl $\mathbf{q}$ applies, and hence the *rotation of a fluid* at a point is defined by

$$\boldsymbol{\omega} = \tfrac{1}{2}\,\text{curl}\,\mathbf{q} = \tfrac{1}{2}\nabla \times \mathbf{q} \tag{7.2.6}$$

When $\boldsymbol{\omega} = 0$ throughout certain portions of a fluid, the motion there is described as *irrotational*. The vorticity vector curl $\mathbf{q}$ has certain characteristics similar to the velocity vector $\mathbf{q}$. Vortex lines are everywhere tangent to the vorticity vector, and vortex tubes, comprised of the vortex lines through a small closed curve, follow certain continuity principles;

viz., the product of vorticity by area of the tube must remain constant along the vortex tube, or div (curl $\mathbf{q}$) $= \nabla \cdot (\nabla \times \mathbf{q}) = 0$.

The operator ∇ acts like a vector but must be applied to a scalar or a vector to have physical significance.

Scalar Components of Vector Relationships. Any vector may be decomposed into three components along mutually perpendicular axes, say the x, y, z-axes. The component is a scalar, as only magnitude and sign (sense) is needed to specify it; $f_x = -3$ indicates the x-component of a vector $\mathbf{f}$ acting in the $-x$-direction.

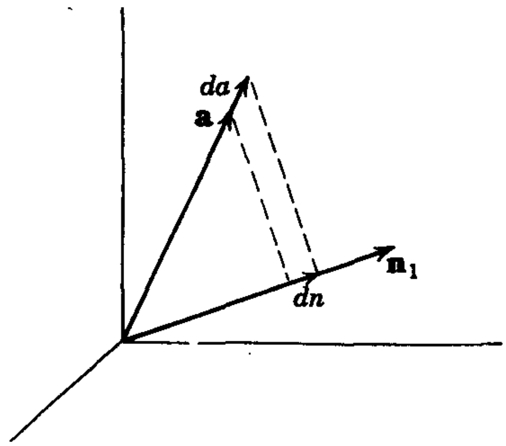


Fig. 7.5. Change of vector a corresponding to change in normal direction.

The vector may be expressed in terms of its scalar components by use of the fixed unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ parallel to the x, y, z-axes, respectively:

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z$$

The unit vectors combine as follows:

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \qquad \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k} \qquad \text{etc.}$$

The scalar product of two vectors $\mathbf{a} \cdot \mathbf{b}$ is

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (\mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z) \cdot (\mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z) \\ &= a_xb_x + a_yb_y + a_zb_z\end{aligned}$$

The vector product of two vectors $\mathbf{a} \times \mathbf{b}$ is

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (\mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z) \times (\mathbf{i}b_x + \mathbf{j}b_y + \mathbf{k}b_z) \\ &= \mathbf{i}(a_yb_z - a_zb_y) + \mathbf{j}(a_zb_x - a_xb_z) + \mathbf{k}(a_xb_y - a_yb_x)\end{aligned}$$

It is conveniently written in determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

To find the scalar components of $\nabla\phi$, first consider $\mathbf{a} \cdot \nabla\phi$ (Fig. 7.5) in which $\mathbf{a}$ is any vector. By Eq. (7.2.3)

$$\mathbf{a} \cdot \nabla\phi = \mathbf{a} \cdot \mathbf{n}_1 \frac{\partial\phi}{\partial n} = a \cos\theta \frac{\partial\phi}{\partial n}$$

as θ is the angle between $\mathbf{a}$ and $\mathbf{n}_1$ and $n_1 = 1$. A change da in magnitude of **a** corresponds to a change in **n**, given by $da \cos \theta = dn$, hence

$$a \cos \theta \frac{\partial \phi}{\partial n} = a \frac{\partial \phi}{\partial a}$$

and

$$\mathbf{a} \cdot \nabla \phi = a \frac{\partial \phi}{\partial a} \tag{7.2.7}$$

The scalar components of $\nabla \phi$ are

$$\mathbf{i} \cdot \nabla \phi = \frac{\partial \phi}{\partial x} \qquad \mathbf{j} \cdot \nabla \phi = \frac{\partial \phi}{\partial y} \qquad \mathbf{k} \cdot \nabla \phi = \frac{\partial \phi}{\partial z}$$

and

$$\nabla \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \tag{7.2.8}$$

The operator ∇, in terms of its scalar components, is

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \tag{7.2.9}$$

The scalar product, say $\nabla \cdot \mathbf{q}$, becomes

$$\begin{aligned} \nabla \cdot \mathbf{q} &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \end{aligned} \tag{7.2.10}$$

as in Sec. 3.4.

The vector product $\nabla \times \mathbf{q}$, in scalar components, is

$$\begin{aligned} \nabla \times \mathbf{q} &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (\mathbf{i}u + \mathbf{j}v + \mathbf{k}w) \\ &= \mathbf{i} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \mathbf{j} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + \mathbf{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \end{aligned} \tag{7.2.11}$$

The quantities in parentheses are vorticity components, which are twice the value of rotation components, ω_x, ω_y, ω_z, so

$$\nabla \times \mathbf{q} = \mathbf{i}2\omega_x + \mathbf{j}2\omega_y + \mathbf{k}2\omega_z \tag{7.2.12}$$

7.3. Euler's Equation of Motion. In Sec. 3.5 Euler's equation was derived for steady flow of a frictionless fluid along a streamline. The assumption is made here that the flow is frictionless, and a continuum is assumed. Newton's second law of motion is applied to a fluid particle of mass $\rho \delta V$. Three terms enter, the body force, the surface force, and mass times acceleration. Let **F** be the body force (such as gravity) per unit mass acting on the particle. Then $\mathbf{F}\rho\delta V$ is the body-force vector. The surface force, from the preceding section, is $-\int_S \mathbf{n}_1 p\, dS$ if the fluid is frictionless or nonviscous, so only normal forces act. The mass times acceleration term is $\rho\delta V\, d\mathbf{q}/dt$. After assembling these terms,

$$\mathbf{F}\rho\, \delta V - \int_S \mathbf{n}_1 p\, ds = \rho\, \delta V \frac{d\mathbf{q}}{dt}$$

Now, dividing through by the mass of the element and taking the limit as $\delta V \to 0$,

$$\mathbf{F} - \frac{1}{\rho} \lim_{\delta V \to 0} \frac{1}{V} \int_S \mathbf{n}_1 p\, dS = \frac{d\mathbf{q}}{dt}$$

By use of the operator ∇

$$\mathbf{F} - \frac{1}{\rho} \nabla p = \frac{d\mathbf{q}}{dt} \tag{7.3.1}$$

This is Euler's equation of motion in vector notation. By forming the scalar product of each term with **i**, then **j**, then **k**, the following scalar component equations are obtained

$$\begin{aligned} X - \frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{du}{dt} \\ Y - \frac{1}{\rho}\frac{\partial p}{\partial y} &= \frac{dv}{dt} \\ Z - \frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{dw}{dt} \end{aligned} \tag{7.3.2}$$

in which X, Y, Z are the body force components per unit mass. The acceleration terms may be expanded. In general $u = u(x,y,z,t)$, so (see Appendix B)

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial t} dt$$

For du/dt to be the acceleration component of a particle in the x-direction, the x, y, z-coordinates of the moving particle become functions of time, and du may be divided by dt, yielding

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt} \qquad w = \frac{dz}{dt}$$

and

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \tag{7.3.3}$$

Similarly

$$\frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \tag{7.3.4}$$

$$\frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \tag{7.3.5}$$

If the extraneous force is *conservative*, it may be derived from a potential ($\mathbf{F} = -\operatorname{grad}\Omega$):

$$X = -\frac{\partial \Omega}{\partial x} \qquad Y = -\frac{\partial \Omega}{\partial y} \qquad Z = -\frac{\partial \Omega}{\partial z} \tag{7.3.6}$$

In particular, if gravity is the only body force acting, $\Omega = gh$, with h a direction measured vertically upward; thus

$$X = -g\frac{\partial h}{\partial x} \qquad Y = -g\frac{\partial h}{\partial y} \qquad Z = -g\frac{\partial h}{\partial z} \tag{7.3.7}$$

Remembering that ρ is constant for an ideal fluid, substituting Eqs. (7.3.3) to (7.3.7) into Eqs. (7.3.2),

$$-\frac{1}{\rho}\frac{\partial}{\partial x}(p + \gamma h) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \tag{7.3.8}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial y}(p + \gamma h) = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \tag{7.3.9}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial z}(p + \gamma h) = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \tag{7.3.10}$$

The first three terms on the right-hand side of the equations are "convective acceleration" terms, depending upon changes of velocity with space. The last term is the "local acceleration," depending upon velocity change with time at a point.

Natural Coordinates in Two-dimensional Flow. Euler's equations in two dimensions are obtained from the general component equations by setting $w = 0$ and $\partial/\partial z = 0$; thus

$$-\frac{1}{\rho}\frac{\partial}{\partial x}(p + \gamma h) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} \tag{7.3.11}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial y}(p + \gamma h) = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} \tag{7.3.12}$$

By taking particular directions for the x- and y-axes, they may be reduced to a form that aids in understanding them. If the x-axis, called the s-axis, is taken parallel to the velocity vector at a point (Fig. 7.6), it is then tangent to the streamline through the point. The y-axis, called the n-axis, is drawn toward the center of curvature of the streamline. The velocity component u is v_s, and the component v is v_n. As v_n is zero at the point, Eq. (7.3.11) becomes

$$-\frac{1}{\rho}\frac{\partial}{\partial s}(p+\gamma h) = v_s\frac{\partial v_s}{\partial s} + \frac{\partial v_s}{\partial t} \qquad (7.3.13)$$

Although v_n is zero at the point (s,n), its rates of change with respect to s and t are not necessarily zero. Equation (7.3.12) becomes

$$-\frac{1}{\rho}\frac{\partial}{\partial n}(p+\gamma h) = v_s\frac{\partial v_n}{\partial s} + \frac{\partial v_n}{\partial t} \qquad (7.3.14)$$

By considering the velocity at s and at $s + \delta s$ along the streamline, v_n

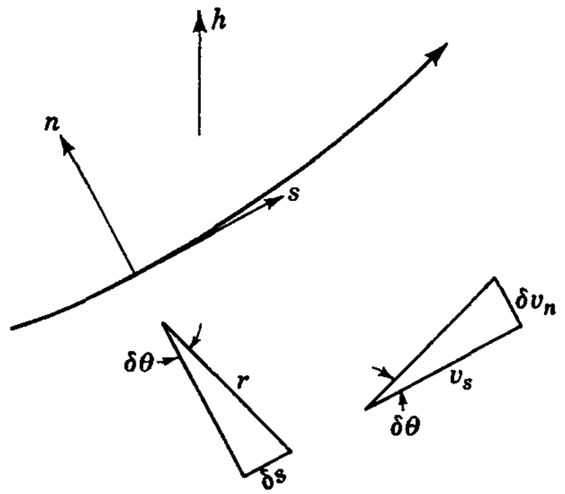


FIG. 7.6. Notation for natural coordinates.

changes from zero to δv_n. With r the radius of curvature of the streamline at s, from similar triangles (Fig. 7.6)

$$\frac{\delta s}{r} = \frac{\delta v_n}{v_s}$$

or

$$\frac{\partial v_n}{\partial s} = \frac{v_s}{r}$$

By substituting into Eq. (7.3.14)

$$-\frac{1}{\rho}\frac{\partial}{\partial n}(p+\gamma h) = \frac{v_s^2}{r} + \frac{\partial v_n}{\partial t} \qquad (7.3.15)$$

For steady flow of an incompressible fluid Eqs. (7.3.11) and (7.3.15) may be written

$$-\frac{1}{\rho}\frac{\partial}{\partial s}(p+\gamma h) = \frac{\partial}{\partial s}\left(\frac{v_s^2}{2}\right) \qquad (7.3.16)$$

and

$$-\frac{1}{\rho}\frac{\partial}{\partial n}(p+\gamma h) = \frac{v_s^2}{r} \qquad (7.3.17)$$

Equation (7.3.16) may be integrated with respect to s to produce Eq. (3.6.2), with the constant of integration varying with n, i.e., from one streamline to another. Equation (7.3.17) shows how pressure head varies across streamlines. With v_s and r known functions of n, Eq. (7.3.17) may be integrated.

Example 7.1: A container of liquid is rotated with angular velocity ω about a vertical axis as a solid. Determine the variation of pressure intensity in the liquid.

n is the radial distance, measured inwardly, $n = -r$, $dn = -dr$, and $v_s = \omega r$. By integrating Eq. (7.3.17)

$$-\frac{1}{\rho}(p + \gamma h) = -\int \frac{\omega^2 r^2 dr}{r} + \text{constant}$$

or

$$\frac{1}{\rho}(p + \gamma h) = \frac{\omega^2 r^2}{2} + \text{constant}$$

To evaluate the constant, if $p = p_0$ when $r = 0$ and $h = 0$, then

$$p = p_0 - \gamma h + \rho \frac{\omega^2 r^2}{2}$$

which shows that the pressure is hydrostatic along a vertical line and increases as the square of the radius. Integration of Eq. (7.3.16) shows that the pressure is constant for a given h and v_s, i.e., along a streamline.

7.4. Irrotational Flow. Velocity Potential. In this section it is shown that the assumption of irrotational flow leads to the existence of a velocity potential. By use of these relations and the assumption of a conservative body force, the Euler equations may be integrated.

The individual particles of a frictionless incompressible fluid initially at rest cannot be caused to rotate. This may be visualized by considering a small free body of fluid in the shape of a sphere. Surface forces act normal to its surface, since the fluid is frictionless, and therefore act through the center of the sphere. Similarly the body force acts at the mass center. Hence no torque can be exerted on the sphere, and it remains without rotation. Likewise, once an ideal fluid has rotation, there is no way of altering it, as no torque can be exerted on an elementary sphere of the fluid.

By assuming that the fluid has no rotation, i.e., it is irrotational, curl $\mathbf{q} = 0$, or from Eq. (7.2.11)

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \qquad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \tag{7.4.1}$$

These restrictions on the velocity must hold at every point (except special singular points or lines). The first equation is the irrotational condition for two-dimensional flow. It is the condition that the differential expression

$$u\,dx + v\,dy$$

is exact, say

$$u\,dx + v\,dy = -d\phi = -\frac{\partial \phi}{\partial x}\,dx - \frac{\partial \phi}{\partial y}\,dy \tag{7.4.2}$$

The minus sign is arbitrary. It is a convention that causes the value of ϕ to decrease in the direction of the velocity. By comparing terms in Eq. (7.4.2), $u = -\partial\phi/\partial x$, $v = -\partial\phi/\partial y$. This proves the existence, in two-dimensional flow, of a function ϕ such that its negative derivative with respect to any direction is the velocity component in that direction. It may also be demonstrated for three-dimensional flow. In vector form

$$\mathbf{q} = -\operatorname{grad}\phi = -\nabla\phi \tag{7.4.3}$$

is equivalent to

$$u = -\frac{\partial \phi}{\partial x} \qquad v = -\frac{\partial \phi}{\partial y} \qquad w = -\frac{\partial \phi}{\partial z} \tag{7.4.4}$$

The assumption of a velocity potential is equivalent to the assumption of irrotational flow, as

$$\operatorname{curl}(-\operatorname{grad}\phi) = -\nabla \times \nabla\phi = 0 \tag{7.4.5}$$

because $\nabla \times \nabla = 0$. This is shown from Eq. (7.4.4) by cross differentiation:

$$\frac{\partial u}{\partial y} = -\frac{\partial^2 \phi}{\partial x\,\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial^2 \phi}{\partial y\,\partial x}$$

proving $\partial v/\partial x = \partial u/\partial y$, etc.

Substitution of Eqs. (7.4.4) into the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

yields

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{7.4.6}$$

In vector form this is

$$\nabla \cdot \mathbf{q} = -\nabla \cdot \nabla\phi = -\nabla^2\phi = 0 \tag{7.4.7}$$

and is written $\nabla^2\phi = 0$. Equation (7.4.6) or (7.4.7) is the *Laplace equation*. Any function ϕ that satisfies the Laplace equation is a possible irrotational fluid-flow case. As there are an infinite number of solutions to the Laplace equation, each of which satisfies certain flow boundaries, the main problem is the selection of the proper function for the particular flow case.

Bceause ϕ appears to the first power in each term of Eq. (7.4.6), it is a linear equation, and the sum of two solutions is also a solution; e.g., if ϕ_1 and ϕ_2 are solutions of Eq. (7.4.6), then $\phi_1 + \phi_2$ is a solution; thus

$$\nabla^2\phi_1 = 0 \qquad \nabla^2\phi_2 = 0$$

then

$$\nabla^2(\phi_1 + \phi_2) = \nabla^2\phi_1 + \nabla^2\phi_2 = 0$$

Similarly if ϕ_1 is a solution, $C\phi_1$ is a solution if C is constant.

7.5. Integration of Euler's Equations. Bernoulli Equation. Equation (7.3.8) may be rearranged so that every term contains a partial derivative with respect to x. From Eq. (7.4.1)

$$v\frac{\partial u}{\partial y} = v\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\frac{v^2}{2} \qquad w\frac{\partial u}{\partial z} = w\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\frac{w^2}{2}$$

and from Eq. (7.4.4)

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t}$$

Making these substitutions into Eq. (7.3.8) and rearranging,

$$\frac{\partial}{\partial x}\left(\frac{p}{\rho} + gh + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} - \frac{\partial \phi}{\partial t}\right) = 0$$

As $u^2 + v^2 + w^2 = q^2$,

$$\frac{\partial}{\partial x}\left(\frac{p}{\rho} + gh + \frac{q^2}{2} - \frac{\partial \phi}{\partial t}\right) = 0 \tag{7.5.1}$$

Similarly for the y- and z-directions

$$\frac{\partial}{\partial y}\left(\frac{p}{\rho} + gh + \frac{q^2}{2} - \frac{\partial \phi}{\partial t}\right) = 0 \tag{7.5.2}$$

$$\frac{\partial}{\partial z}\left(\frac{p}{\rho} + gh + \frac{q^2}{2} - \frac{\partial \phi}{\partial t}\right) = 0 \tag{7.5.3}$$

The quantities within the parentheses are the same in Eqs. (7.5.1) to (7.5.3). Equation (7.5.1) states that the quantity is not a function of x, since the derivative with respect to x is zero. Similarly the other equations show that the quantity is not a function of y or z. Therefore it can be a function of t only, say $F(t)$:

$$\frac{p}{\rho} + gh + \frac{q^2}{2} - \frac{\partial \phi}{\partial t} = F(t) \tag{7.5.4}$$

In steady flow $\partial\phi/\partial t = 0$ and $F(t)$ becomes a constant E:

$$\frac{p}{\rho} + gh + \frac{q^2}{2} = E \tag{7.5.5}$$

The available energy is everywhere constant throughout the fluid. This is Bernoulli's equation for an irrotational fluid.

The pressure term may be separated into two parts, the hydrostatic pressure p_s and the dynamic pressure p_d, so that $p = p_s + p_d$. By inserting in Eq. (7.5.5),

$$gh + \frac{p_s}{\rho} + \frac{p_d}{\rho} + \frac{q^2}{2} = E$$

The first two terms may be written

$$gh + \frac{p_s}{\rho} = \frac{1}{\rho}(p_s + \gamma h)$$

with h measured vertically upward. The expression is a constant, since it expresses the hydrostatic law of variation of pressure. These two terms may be included in the constant E. After dropping the subscript on the dynamic pressure, there remains

$$\frac{p}{\rho} + \frac{q^2}{2} = E \tag{7.5.6}$$

This simple equation permits the variation in pressure to be determined if the velocity is known, or vice versa. Assuming both the velocity q_0 and the dynamic pressure p_0 to be known at one point,

$$\frac{p_0}{\rho} + \frac{q_0^2}{2} = \frac{p}{\rho} + \frac{q^2}{2}$$

or

$$p = p_0 + \frac{\rho q_0^2}{2}\left[1 - \left(\frac{q}{q_0}\right)^2\right] \tag{7.5.7}$$

Example 7.2: A submarine moves through water at a speed of 30 ft/sec. At a point A on the submarine 5 ft above the nose, the velocity of submarine relative to the water is 50 ft/sec. Determine the dynamic pressure difference between this point and the nose, and determine the difference in total pressure between the two points.

If the submarine is stationary and the water is moving past it, the velocity at the nose is zero, and the velocity at A is 50 ft/sec. By selecting the dynamic pressure at infinity as zero, from Eq. (7.5.6)

$$E = 0 + \frac{q_0^2}{2} = \frac{\overline{30}^2}{2} = 450$$

For the nose

$$\frac{p}{\rho} = E = 450 \qquad p = 450 \times 1.935 = 870 \text{ lb/ft}^2$$

For point A

$$\frac{p}{\rho} = E - \frac{q^2}{2} = 450 - \frac{\overline{50}^2}{2}$$

and

$$p = 1.935\left(\frac{\overline{30}^2}{2} - \frac{\overline{50}^2}{2}\right) = -1548\ \text{lb/ft}^2$$

Therefore the difference in dynamic pressure is

$$-1548 - 870 = -2418\ \text{lb/ft}^2$$

The difference in total pressure may be obtained by applying Eq. (7.5.5) to point A and to the nose n,

$$gh_A + \frac{p_A}{\rho} + \frac{q_A^2}{2} = gh_n + \frac{p_n}{\rho} + \frac{q_n^2}{2}$$

Hence

$$p_A - p_n = \rho\left(gh_n - gh_A + \frac{q_n^2 - q_A^2}{2}\right) = 1.935\left(-5g - \frac{\overline{50}^2}{2}\right) = -2740\ \text{lb/ft}^2$$

It may also be reasoned that the actual pressure difference varies by 5γ from the dynamic pressure difference since A is 5 ft above the nose, or $-2418 - 5 \times 62.4 = -2740\ \text{lb/ft}^2$.

7.6. Stream Functions. Boundary Conditions. Two stream functions are defined: one for two-dimensional flow, where all lines of motion are

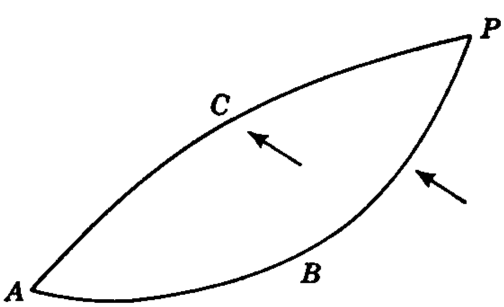


FIG. 7.7. Fluid region showing the positive flow direction used in the definition of a stream function.

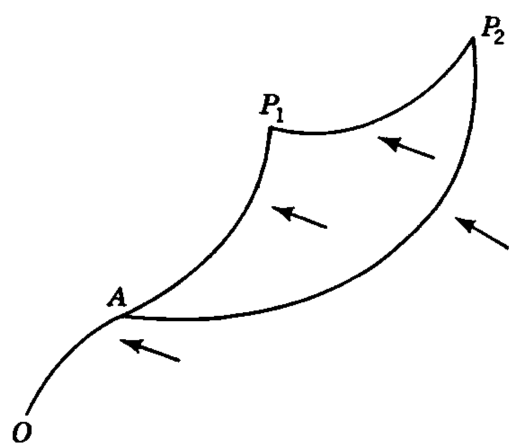


FIG. 7.8. Flow between two points in a fluid region.

parallel to a fixed plane, say the xy-plane, and the flow is identical in each of these planes, and the other for three-dimensional flow with axial symmetry, i.e., all flow lines are in planes intersecting the same line or axis, and the flow is identical in each of these planes.

Two-dimensional Stream Function. If A, P represent two points in one of the flow planes, e.g., the xy-plane (Fig. 7.7), and if the plane has unit thickness, the rate of flow across any two lines ACP, ABP must be the same, if the density is constant and no fluid is created or destroyed within the region, as a consequence of continuity. Now, if A is a fixed point and P a movable point, the flow rate across any line connecting the two points is a function of the position of P. If this function is ψ, and if it is taken as a sign convention that it denotes the flow rate from right to left as the observer views the line from A looking toward P, then

$$\psi = \psi(x,y)$$

is defined as the stream function.

If ψ_1, ψ_2 represent the values of stream function at points P_1, P_2 (Fig. 7.8), respectively, then $\psi_2 - \psi_1$ is the flow across P_1P_2 and is independent of the location of A. Taking another point O in the place of A changes the values of ψ_1, ψ_2 by the same amount, viz., the flow across OA. Then ψ is indeterminate to the extent of an arbitrary constant.

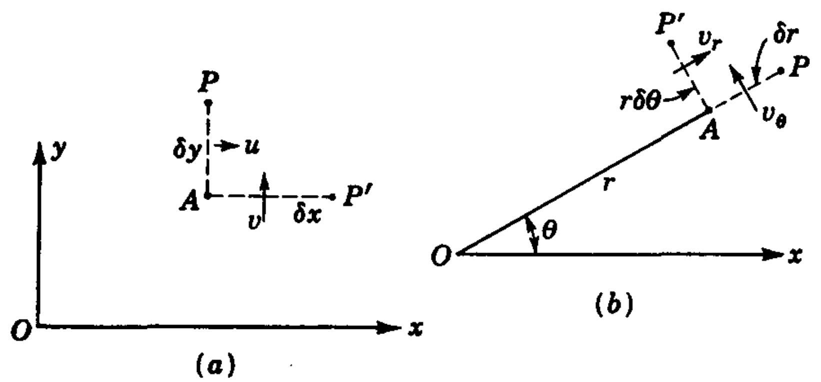


FIG. 7.9. Selection of path to show relation of velocity components to stream function.

The velocity components u, v in the x-, y-directions may be obtained from the stream function. In Fig. 7.9a, the flow $\delta\psi$ across $\overline{AP} = \delta y$, from right to left, is $-u\,\delta y$, or

$$u = -\frac{\delta\psi}{\delta y} = -\frac{\partial\psi}{\partial y} \tag{7.6.1}$$

and similarly

$$v = \frac{\delta\psi}{\delta x} = \frac{\partial\psi}{\partial x} \tag{7.6.2}$$

In words, the partial derivative of the stream function with respect to any direction gives the velocity component +90° (counterclockwise) to that direction. In plane polar coordinates

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = \frac{\partial\psi}{\partial r}$$

from Fig. 7.9b.

When the two points P_1, P_2 of Fig. 7.8 lie on the same streamline, $\psi_1 - \psi_2 = 0$ as there is no flow across a streamline. Hence, a streamline is given by ψ = constant. By comparing Eqs. (7.4.4) with Eqs. (7.6.1) and (7.6.2),

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{7.6.3}$$

By Eqs. (7.6.3) a stream function may be found for each velocity potential. If the velocity potential satisfies the Laplace equation, then the stream function also satisfies it. Hence, the stream function may be considered as velocity potential for another flow case.

Stokes' Stream Function for Axially Symmetric Flow. In any one of the planes through the axis of symmetry select two points A, P, such that A is fixed and P is variable. Draw a line connecting AP. The flow through the surface generated by rotating AP about the axis of symmetry is a function of the position of P. Let this function be $2\pi\psi$, and let the axis of symmetry be the x-axis of a cartesian system of reference. Then ψ is a function of x and $\hat{\omega}$, where

$$\hat{\omega} = \sqrt{y^2 + z^2}$$

is the distance from P to the x-axis. The surfaces ψ = constant are stream surfaces.

To find the relation between ψ and the velocity components u, v' parallel to the x-axis and the $\hat{\omega}$-axis (perpendicular to x-axis), respectively, a similar procedure is employed to that for two-dimensional flow. Let PP' be an infinitesimal step first parallel to $\hat{\omega}$ and then to x; i.e., $PP' = \delta\hat{\omega}$ and then $PP' = \delta x$. The resulting relations between stream function and velocity are given by

$$-2\pi\hat{\omega}\,\delta\hat{\omega}\,u = 2\pi\,\delta\psi \qquad \text{and} \qquad 2\pi\hat{\omega}\,\delta x\,v' = 2\pi\,\delta\psi$$

Solving for u, v',

$$u = -\frac{1}{\hat{\omega}}\frac{\partial \psi}{\partial \hat{\omega}}, \qquad v' = \frac{1}{\hat{\omega}}\frac{\partial \psi}{\partial x} \tag{7.6.4}$$

The same sign convention is used as in the two-dimensional case.

The relations between stream function and potential function are

$$\frac{\partial \phi}{\partial x} = \frac{1}{\hat{\omega}}\frac{\partial \psi}{\partial \hat{\omega}} \qquad \frac{\partial \phi}{\partial \hat{\omega}} = -\frac{1}{\hat{\omega}}\frac{\partial \psi}{\partial x} \tag{7.6.5}$$

In three-dimensional flow with axial symmetry ψ has the dimensions L^3T^{-1}, or volume per unit time.

The stream function is used for flow about bodies of revolution that are frequently expressed most readily in spherical polar coordinates. Let r be the distance from the origin and θ be the polar angle; the meridian angle is not needed because of axial symmetry. Referring to Fig. 7.10*a* and *b*,

$$2\pi r \sin\theta\,\delta r\,v_\theta = 2\pi\,\delta\psi$$
$$-2\pi r \sin\theta\,r\,\delta\theta\,v_r = 2\pi\,\delta\psi$$

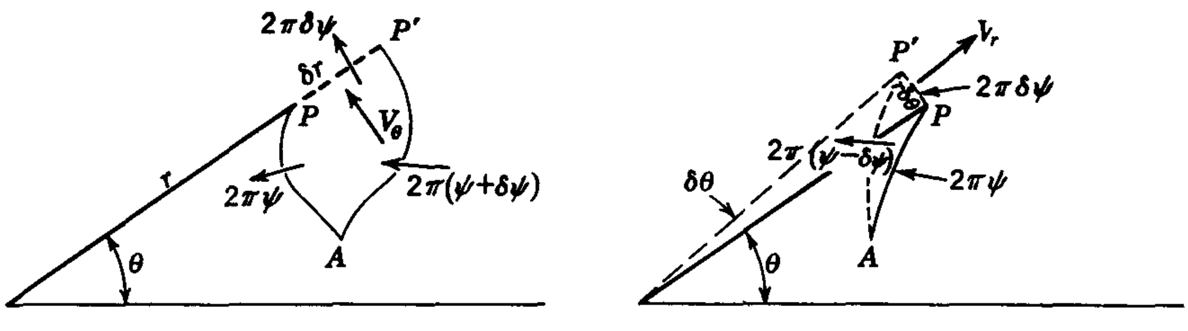
from which

$$v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \qquad v_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \tag{7.6.6}$$

and

$$\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} = r^2 \frac{\partial \phi}{\partial r} \qquad \frac{\partial \psi}{\partial r} = -\sin \theta \frac{\partial \phi}{\partial \theta} \tag{7.6.7}$$

These expressions are useful in dealing with flow about spheres, ellipsoids, and disks and through apertures.



(a) (b)

Fig. 7.10. Displacement of P to show the relation between velocity components and Stokes' stream function.

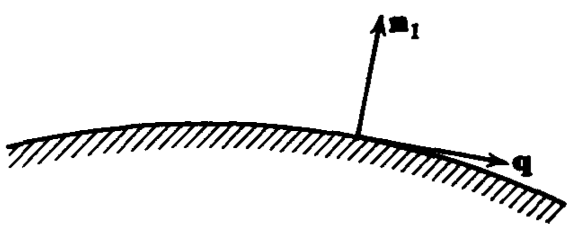


Fig. 7.11. Notation for boundary condition at a fixed boundary.

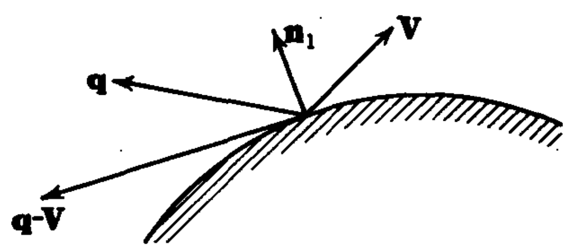


Fig. 7.12. Notation for boundary condition at a moving boundary.

Boundary Conditions. At a fixed boundary the velocity component normal to the boundary must be zero at every point on the boundary (Fig. 7.11):

$$\mathbf{q} \cdot \mathbf{n}_1 = 0 \tag{7.6.8}$$

$\mathbf{n}_1$ is a unit vector normal to the boundary. In scalar notation this is easily expressed in terms of the velocity potential

$$\frac{\partial \phi}{\partial n} = 0 \tag{7.6.9}$$

at all points on the boundary. For a moving boundary (Fig. 7.12), where the boundary point has the velocity $\mathbf{V}$, the fluid-velocity component normal to the boundary must equal the velocity of the boundary normal to the boundary; thus

$$\mathbf{q} \cdot \mathbf{n}_1 = \mathbf{V} \cdot \mathbf{n}_1 \tag{7.6.10}$$

or

$$(\mathbf{q} - \mathbf{V}) \cdot \mathbf{n}_1 = 0 \tag{7.6.11}$$

For two fluids in contact, a dynamical boundary condition is required; viz., the pressure must be continuous across the interface.

A stream surface in steady flow (fixed boundaries) satisfies the condition for a boundary and may be taken as a solid boundary.

7.7. The Flow Net. In two-dimensional flow the flow net is of great benefit; it is taken up in this section.

The line given by $\phi(x,y) =$ constant is called an *equipotential* line. It is a line along which the value of ϕ (the velocity potential) does not change. Since velocity v_s in any direction s is given by

$$v_s = -\frac{\partial \phi}{\partial s} = -\lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s}$$

and $\Delta\phi$ is zero for two closely spaced points on an equipotential line, the velocity vector has no component in the direction defined by the line through the two points. In the limit as $\Delta s \to 0$ this proves that there is no velocity component tangent to an equipotential line and, therefore, the velocity vector must be everywhere *normal* to an equipotential line (except at singular points where the velocity is zero or infinite).

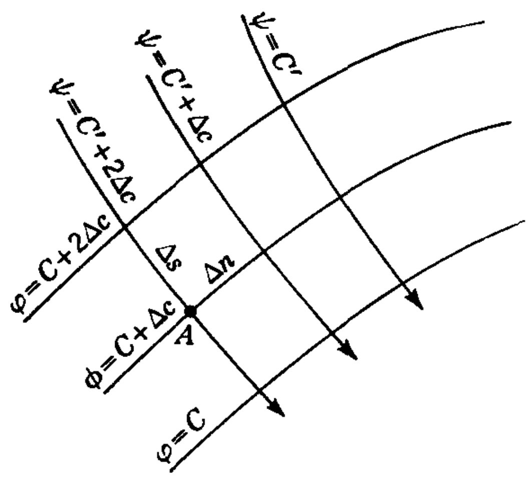


FIG. 7.13. Elements of a flow net.

The line $\psi(x,y) =$ constant is a streamline, and is everywhere tangent to the velocity vector. Streamlines and equipotential lines are therefore *orthogonal*, i.e., they intersect at right angles, except at singular points. A *flow net* is composed of a family of equipotential lines and a corresponding family of streamlines with the constants varying in arithmetical progression. It is customary to let the change in constant between adjacent equipotential lines and adjacent streamlines be the same, e.g., Δc. In Fig. 7.13, if the distance between streamlines be Δn and the distance between equipotential lines be Δs at some small region in the flow net, the approximate velocity v_s is then given in terms of the spacing of the equipotential lines [Eq. (7.4.4)]

$$v_s \approx -\frac{\Delta \phi}{\Delta s} = -\frac{-\Delta c}{\Delta s} = \frac{\Delta c}{\Delta s}$$

or in terms of the spacing of streamlines [Eqs. (7.6.1) and (7.6.2)]

$$v_s \approx \frac{\Delta\psi}{\Delta n} = \frac{\Delta c}{\Delta n}$$

These expressions are approximate when Δc is finite, but when Δc becomes very small the expressions become exact and yield velocity at a point. As both velocities referred to are the same, the equations show that $\Delta s = \Delta n$, or that the flow net consists of an orthogonal grid that reduces to perfect squares in the limit as the grid size approaches zero.

Once a flow net has been found by any means to satisfy the boundary conditions and to form an orthogonal net reducing to perfect squares in the limit as the number of lines is increased, the flow net is the only solution for the particular boundaries as uniqueness theorems in hydrodynamics prove. In steady flow when the boundaries are stationary, the boundaries themselves become part of the flow net as they are streamlines. The problem of finding the flow net to satisfy given fixed boundaries may be considered purely as a *graphical* exercise, i.e., the construction of an orthogonal system of lines that compose the boundaries and that reduce to perfect squares in the limit as the number of lines increase. This is one of the practical methods employed in two-dimensional-flow analysis, although it usually requires many attempts and much erasing.

Another practical method of obtaining a flow net for a particular set of fixed boundaries is the *electric analogy*. The boundaries in a model are formed out of strips of nonconducting material mounted on a flat nonconducting surface, and the end equipotential lines are formed out of a conducting strip, e.g., brass or copper. An electrolyte (conducting liquid) is placed at uniform depth in the flow space and a voltage potential applied to the two end conducting strips. By means of a probe and a voltmeter, lines with constant drop in voltage from one end are mapped out and plotted. These are equipotential lines. By reversing the process and making the flow boundaries out of conducting material and the end equipotential lines from nonconducting material, the streamlines are mapped.

The relaxation method[1] numerically determines the value of potential function at points throughout the flow, usually located at the intersections of a square grid. The Laplace equation is written as a difference equation, and it is shown that the value of potential function at a grid point is the average of the four values at the neighboring grid points. Near the boundaries special formulas are required. With values known at

[1] C.-S. Yih, Ideal-fluid Flow, p.4–67 in "Handbook of Fluid Dynamics," ed. by V. L. Streeter, McGraw-Hill Book Company, Inc., New York, 1961.

the boundaries, each grid point is computed based on the assumed values at the neighboring grid points, then these values are improved by repeating the process until the changes are within the desired accuracy. This method is particularly convenient for solution with high-speed digital computers.

Use of the Flow Net. After a flow net for a given boundary configuration has been obtained, it may be used for all irrotational flows with geometrically similar boundaries. It is necessary to know the velocity at a single point and the pressure at one point. Then, by use of the flow net, the velocity can be determined at every other point. Application of the Bernoulli equation [Eq. (7.5.7)] produces the dynamic pressure. If the velocity is known, e.g., at A (Fig. 7.13), Δn or Δs may be scaled from the adjacent lines. Then $\Delta c \cong \Delta n\, v_s \cong \Delta s\, v_s$. With the constant Δc determined for the whole grid in this manner, measurement of Δs or Δn at any other point permits the velocity to be computed there,

$$v_s \approx \frac{\Delta c}{\Delta s} = \frac{\Delta c}{\Delta n}$$

7.8. Three-dimensional Flow Cases. Because of space limitations only a few three-dimensional cases are considered. They are sources and sinks, the doublet, and uniform flow singly or combined.

Three-dimensional Sources and Sinks. A source in three-dimensional flow is a point from which fluid issues at a uniform rate in all directions. It is entirely fictitious, as there is nothing resembling it in nature. That does not, however, reduce its usefulness in obtaining flow patterns. The "strength" of the source m is the rate of flow passing through any surface enclosing the source.

As the flow is outward and is uniform in all directions, the velocity, a distance r from the source, is the strength divided by the area of the sphere through the point with center at the source, or

$$v_r = \frac{m}{4\pi r^2}$$

Since $v_r = -\partial\phi/\partial r$ and $v_\theta = 0$, hence $\partial\phi/\partial\theta = 0$, and the velocity potential can be found.

$$-\frac{\partial\phi}{\partial r} = \frac{m}{4\pi r^2}$$

and

$$\phi = \frac{m}{4\pi r} \tag{7.8.1}$$

A *negative source* is a *sink*. Fluid is assumed to flow uniformly into a sink and there disappear.

Three-dimensional Doublets. A *doublet*, or *double source*, is a combination of a source and a sink of equal strength, which are allowed to approach each other in such a manner that the product of their strength and the distance between them remains a constant in the limit.

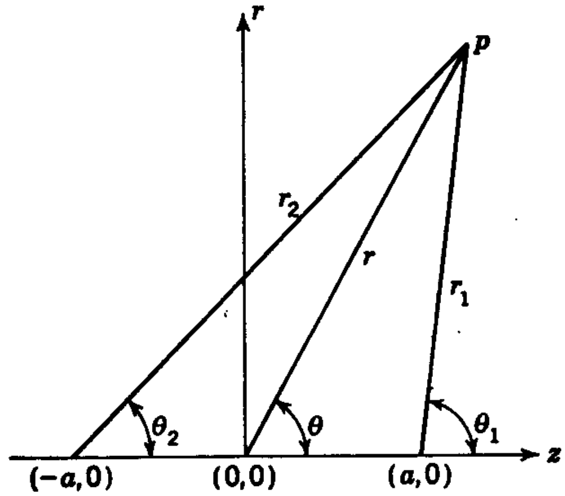


Fig. 7.14. Auxiliary coordinate systems used for Rankine body.

Referring to Fig. 7.14, a source of strength m is located at $(a,0)$ and a sink of the same strength at $(-a,0)$. Since each satisfies the Laplace equation, their sum also satisfies it:

$$\phi = \frac{m}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{7.8.2}$$

By the law of sines and Fig. 7.14,

$$\frac{r_1}{\sin\theta_2} = \frac{r_2}{\sin\theta_1} = \frac{2a}{\sin(\theta_1 - \theta_2)} = \frac{2a}{2\sin\frac{1}{2}(\theta_1 - \theta_2)\cos\frac{1}{2}(\theta_1 - \theta_2)}$$

as the angle between r_2 and r_1 at P is $\theta_1 - \theta_2$. Solving for $r_2 - r_1$,

$$r_2 - r_1 = \frac{a(\sin\theta_1 - \sin\theta_2)}{\sin\frac{1}{2}(\theta_1 - \theta_2)\cos\frac{1}{2}(\theta_1 - \theta_2)} = \frac{2a\cos\frac{1}{2}(\theta_1 + \theta_2)}{\cos\frac{1}{2}(\theta_1 - \theta_2)}$$

From Eq. (7.8.2)

$$\phi = \frac{m}{4\pi}\frac{r_2 - r_1}{r_1 r_2} = \frac{2am\cos\frac{1}{2}(\theta_1 + \theta_2)}{4\pi r_1 r_2\cos\frac{1}{2}(\theta_1 - \theta_2)} = \frac{\mu}{r_1 r_2}\frac{\cos\frac{1}{2}(\theta_1 + \theta_2)}{\cos\frac{1}{2}(\theta_1 - \theta_2)}$$

In the limit as a approaches zero, $\theta_2 = \theta_1 = \theta$, $r_2 = r_1 = r$, and

$$\phi = \frac{\mu}{r^2}\cos\theta \tag{7.8.3}$$

which is the velocity potential for a doublet[1] at the origin with axis in the positive x-direction. Equation (7.8.3) may be converted into the stream function by Eqs. (7.6.7). The stream function is

$$\psi = -\frac{\mu\hat{\omega}^2}{r^3} = -\frac{\mu \sin^2\theta}{r} \tag{7.8.4}$$

Streamlines and equipotential lines for the doublet are drawn in Fig. 7.15.

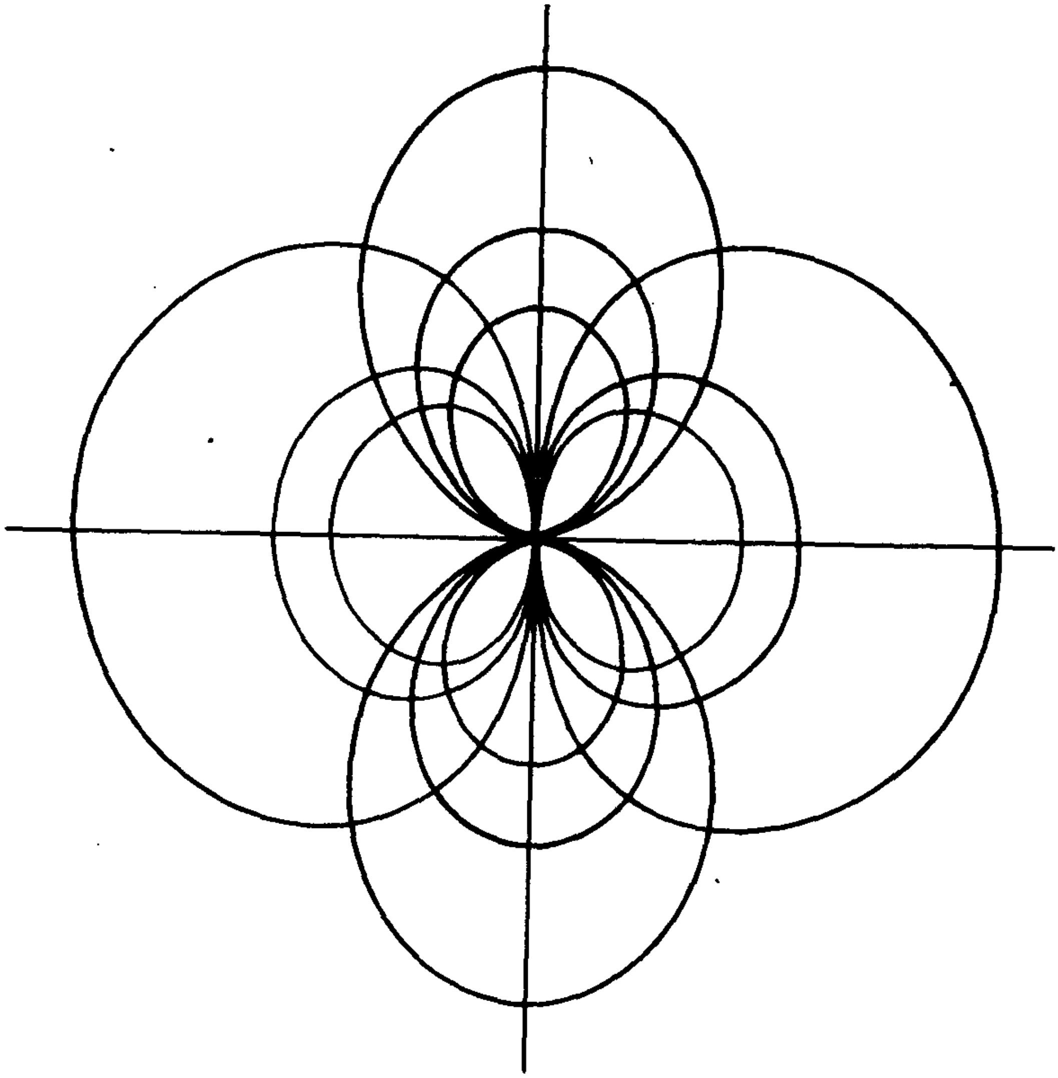


FIG. 7.15. Streamlines and equipotential lines for a three-dimensional doublet.

Source in a Uniform Stream. The radial velocity v_r due to a source at the origin

$$\phi = \frac{m}{4\pi r} \tag{7.8.1}$$

is

$$v_r = -\frac{\partial\phi}{\partial r} = \frac{m}{4\pi r^2}$$

[1] L. M. Milne-Thompson, "Theoretical Hydrodynamics," p. 414, Macmillan & Co., Ltd., London, 1938.

which, when multiplied by the surface area of the sphere concentric with it, gives the strength m. Since the flow from the source has axial symmetry, Stokes' stream function is defined. For spherical polar coordinates, from Eqs. (7.6.7),

$$\frac{\partial \psi}{\partial r} = -\sin\theta \frac{\partial \phi}{\partial \theta} \qquad \frac{\partial \psi}{\partial \theta} = r^2 \sin\theta \frac{\partial \phi}{\partial r}$$

with Eq. (7.8.1)

$$\frac{\partial \psi}{\partial r} = 0 \qquad \frac{\partial \psi}{\partial \theta} = -\frac{m}{4\pi}\sin\theta$$

Integrating,

$$\psi = \frac{m}{4\pi}\cos\theta \tag{7.8.5}$$

is the stream function for a source at the origin. Equipotential lines and streamlines are shown in Fig. 7.16 for constant increments of ϕ and ψ.

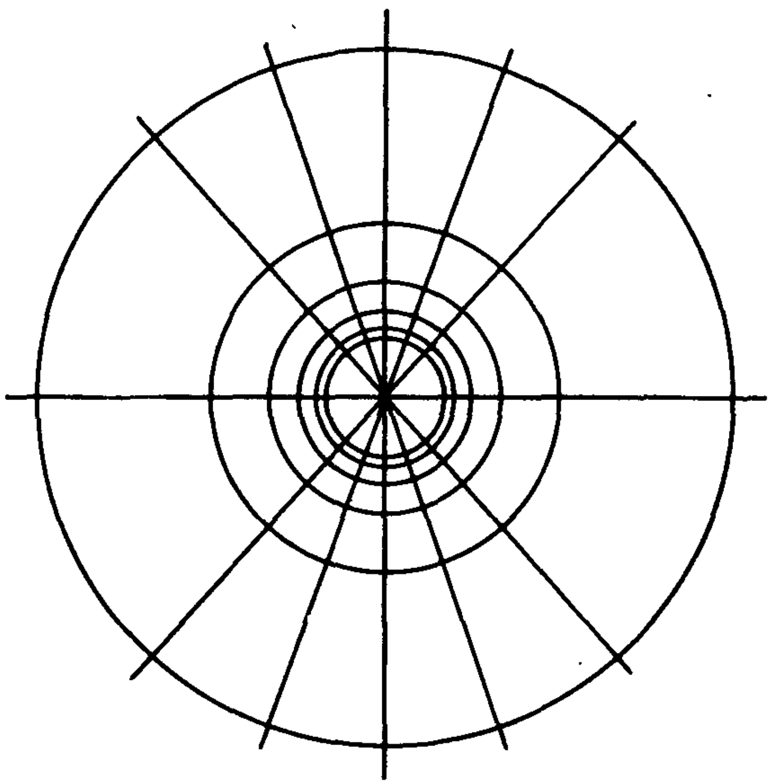


FIG. 7.16. Streamlines and equipotential lines for a source.

A uniform stream of fluid having a velocity U in the negative x-direction throughout space is given by

$$-\frac{\partial \phi}{\partial x} = -U \qquad \frac{\partial \phi}{\partial \bar{\omega}} = 0$$

Integrating,

$$\phi = Ux = Ur\cos\theta \tag{7.8.6}$$

The stream function is found in the same manner as above to be

$$\psi = \frac{U}{2}\,\tilde{\omega}^2 = \frac{Ur^2}{2}\sin^2\theta \tag{7.8.7}$$

The flow network is shown in Fig. 7.17.

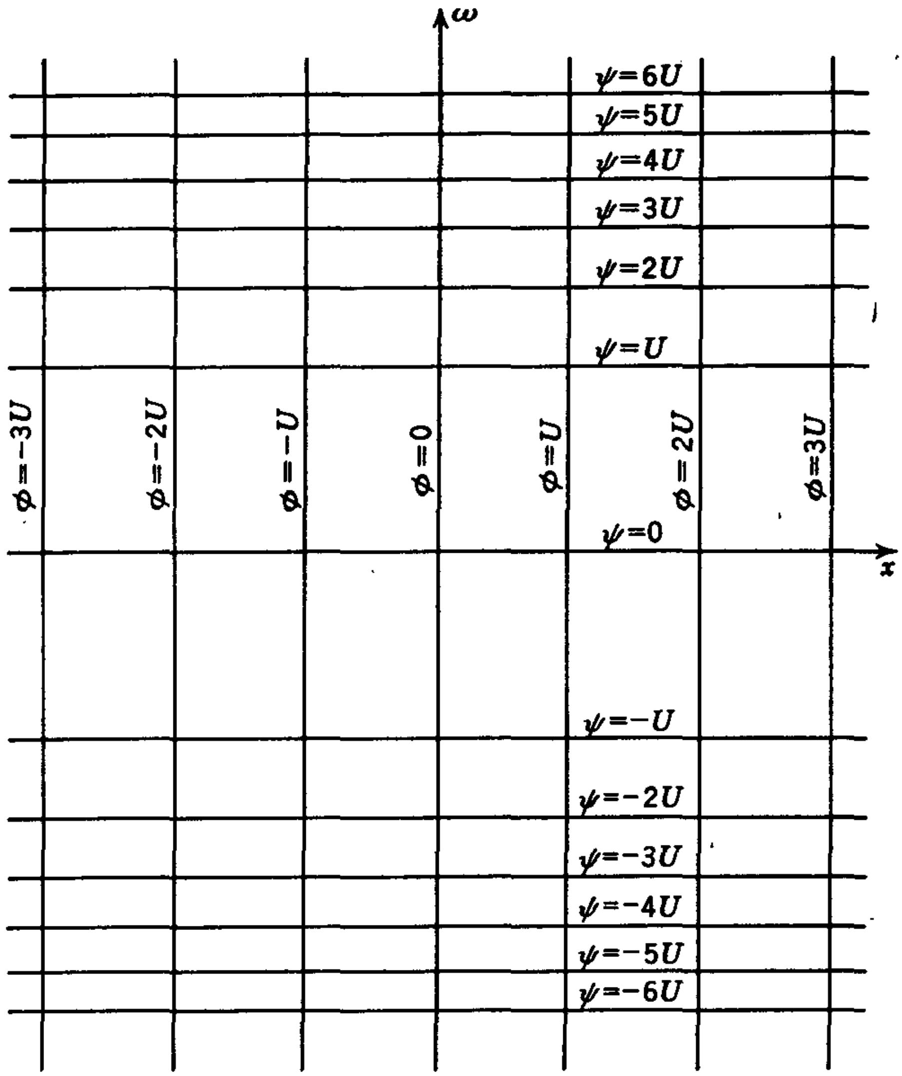


FIG. 7.17. Streamlines and equipotential lines for uniform flow in negative x-direction.

Combining the uniform flow and the source flow, which may be accomplished by adding the two velocity potentials and the two stream functions, gives

$$\begin{aligned} \phi &= \frac{m}{4\pi r} + Ur\cos\theta \\ \psi &= \frac{m}{4\pi}\cos\theta + \frac{Ur^2}{2}\sin^2\theta \end{aligned} \tag{7.8.8}$$

The resulting flow is everywhere the same as if the separate velocity vectors were added for each point in space.

A stagnation point is a point in the fluid where the velocity is zero. The conditions for stagnation point, where spherical polar coordinates are used and when the flow has axial symmetry, are

$$v_r = -\frac{\partial \phi}{\partial r} = 0 \qquad v_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

Use of these expressions with Eqs. (7.8.8) gives

$$\frac{m}{4\pi r^2} - U \cos \theta = 0 \qquad U \sin \theta = 0$$

which are satisfied by only one point in space, viz.,

$$\theta = 0 \qquad r = \sqrt{\frac{m}{4\pi U}}$$

Substituting this point back into the stream function gives $\psi = m/4\pi$,

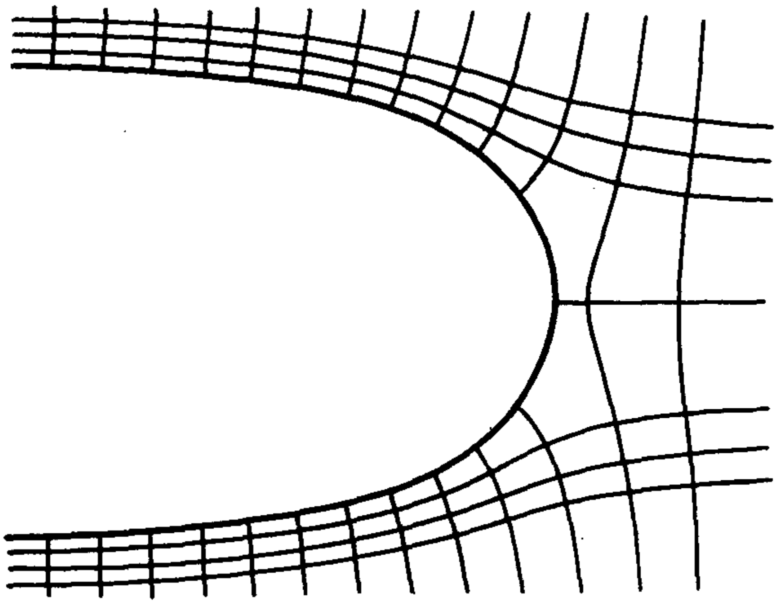


FIG. 7.18. Streamlines and equipotential lines for a half body.

which is the stream surface through the stagnation point. The equation of this surface is found from Eqs. (7.8.8):

$$\cos \theta + \frac{2\pi U}{m} r^2 \sin^2 \theta = 1 \tag{7.8.9}$$

The flow under consideration is steady, as the velocity potential does not change with the time. Therefore, any stream surface satisfies the conditions for a boundary: The velocity component normal to the stream surface in steady flow is always zero. Since stream surfaces through stagnation points usually split the flow, they are frequently the most interesting possible boundary. This stream surface is plotted in Fig. 7.18. Substituting $\tilde{\omega} = r \sin \theta$ in Eq. (7.8.9), the distance of a point (r,θ) from the x-axis is given by

$$\tilde{\omega}^2 = \frac{m}{2\pi U}(1 - \cos \theta)$$

which shows that $\hat{\omega}$ has a maximum value $\sqrt{m/\pi U}$ as θ approaches π, i.e., as r approaches infinity. Hence, $\hat{\omega} = \sqrt{m/\pi U}$ is an asymptotic surface to the dividing stream surface. Equation (7.8.9) may be expressed in the form

$$r = \frac{1}{2}\sqrt{\frac{m}{\pi U}} \sec \frac{\theta}{2} \tag{7.8.10}$$

from which the surface is easily plotted. Such a figure of revolution is called a *half body*, as it extends to negative infinity, surrounding the negative x-axis.

The pressure at any point, i.e., the dynamic pressure from Eq. (7.5.7) is

$$p = \frac{\rho}{2}(U^2 - q^2)$$

where the dynamic pressure at infinity is taken as zero. q is the speed at any point. Evaluating q from Eqs. (7.8.8),

$$q^2 = \left(\frac{\partial \phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2 = U^2 + \frac{m^2}{16\pi^2 r^2} - \frac{mU \cos \theta}{2\pi r^2}$$

and

$$p = \frac{\rho}{2} U^2 \left(\frac{m \cos \theta}{2\pi r^2 U} - \frac{m^2}{16\pi^2 r^4 U^2}\right) \tag{7.8.11}$$

from which the pressure can be found for any point except the origin, which is a singular point. Substituting Eq. (7.8.10) into Eq. (7.8.11), the pressure is given in terms of r for any point on the half body; thus

$$p = \frac{\rho}{2} U^2 \left(\frac{3m^2}{16\pi^2 r^4 U^2} - \frac{m}{2\pi r^2 U}\right) \tag{7.8.12}$$

This shows that the dynamic pressure approaches zero as r increases downstream along the body.

Source and Sink of Equal Strength in a Uniform Stream. Rankine Bodies. A source of strength m, located at $(a,0)$, has the velocity potential at any point P given by

$$\phi_1 = \frac{m}{4\pi r_1}$$

where r_1 is the distance from $(a,0)$ to P, as shown in Fig. 7.14. Similarly, the potential function for a sink of strength m at $(-a,0)$ is

$$\phi_2 = -\frac{m}{4\pi r_2}$$

Since both ϕ_1 and ϕ_2 satisfy the Laplace equation, their sum will also be a solution,

$$\phi = \frac{m}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad (7.8.13)$$

Because r_1, r_2 are measured from different points, this expression must be handled differently from the usual algebraic equation.

The stream functions for the source and sink may also be added to give the stream function for the combined flow

$$\psi = \frac{m}{4\pi}(\cos\theta_1 - \cos\theta_2) \quad (7.8.14)$$

The stream surfaces and equipotential surfaces take the form shown in Fig. 7.19, which is plotted from Eqs. (7.8.13) and (7.8.14) by taking constant values of ϕ and ψ.

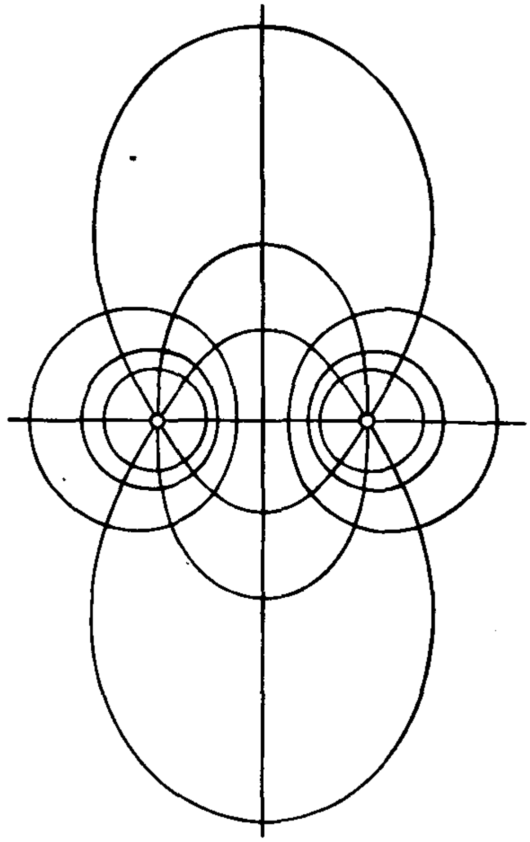


FIG. 7.19. Streamlines and equipotential lines for a source and sink of equal strength.

Superposing a uniform flow of velocity U in the negative x-direction, $\phi = Ux$, $\psi = \frac{1}{2}U\hat{\omega}^2$, the potential and stream functions for source and sink of equal strength in a uniform flow (in direction of source to sink) are

$$\begin{aligned}\phi &= Ux + \frac{m}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\\ &= Ux + \frac{m}{4\pi}\left[\frac{1}{\sqrt{(x-a)^2 + \hat{\omega}^2}} - \frac{1}{\sqrt{(x+a)^2 + \hat{\omega}^2}}\right] \quad (7.8.15)\end{aligned}$$

$$\psi = \tfrac{1}{2}Ur^2\sin^2\theta + \frac{m}{4\pi}(\cos\theta_1 - \cos\theta_2) \quad (7.8.16)$$

As any stream surface may be taken as a solid boundary in steady flow, the location of a closed surface for this flow case will represent flow of a uniform stream around a body. Examining the stream function, for $x > a$ and $\theta_1 = \theta_2 = \theta = 0$, $\psi = 0$. For $x < -a$ and $\theta_1 = \theta_2 = \theta = \pi$, $\psi = 0$. Therefore, $\psi = 0$ must be the dividing streamline, since the x-axis is the axis of symmetry. The equation of the dividing streamline is, from Eq. (7.8.16)

$$\hat{\omega}^2 + \frac{m}{2\pi U}(\cos\theta_1 - \cos\theta_2) = 0 \quad (7.8.17)$$

where $\hat{\omega} = r \sin \theta$ is the distance of a point on the dividing stream surface from the x-axis. Since $\cos \theta_1$ and $\cos \theta_2$ are never greater than unity, $\hat{\omega}$ cannot exceed $\sqrt{m/\pi U}$, which shows that the surface is closed and hence can be replaced by a solid body of exactly the same shape. By changing the signs of m and U the flow is reversed and the body should change end for end. From Eq. (7.8.17) it is seen that the equation is unaltered; hence, the body has symmetry with respect to the plane $x = 0$. It is necessarily a body of revolution because of axial symmetry of the equations.

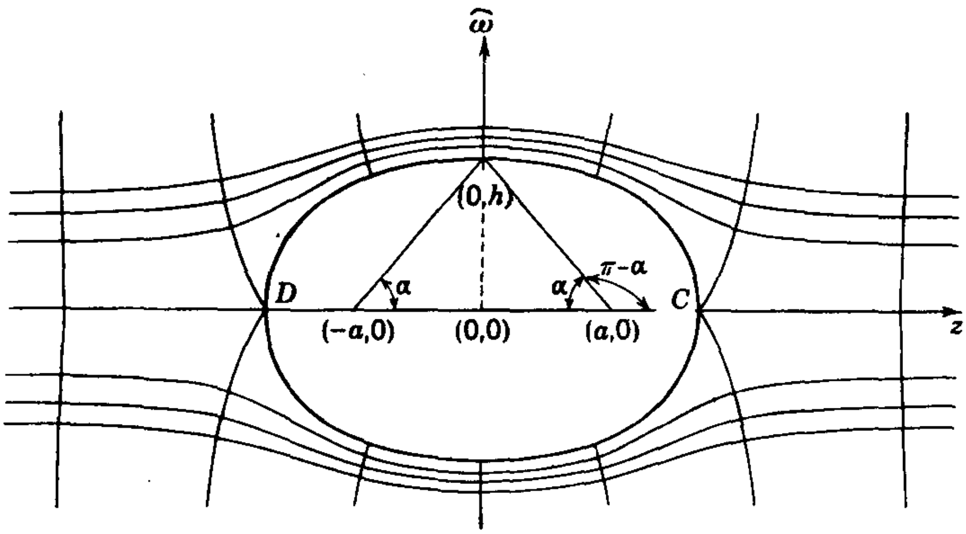


FIG. 7.20. Rankine body.

To locate the stagnation points C, D (Fig. 7.20), which must be on the x-axis, it is known that the velocity is along the x-axis (it is a streamline). From Eq. (7.8.15) the velocity potential ϕ_x for points on the x-axis is given by

$$\phi_x = \frac{ma}{2\pi} \frac{1}{x^2 - a^2} + Ux$$

since

$$r_1 = x - a \qquad r_2 = x + a$$

Differentiating with respect to x and setting the result equal to zero,

$$\frac{\partial \phi_x}{\partial x} = U - \frac{max_0}{\pi(x_0{}^2 - a^2)^2} = 0 \tag{7.8.18}$$

where x_0 is the x-coordinate of the stagnation point. This gives the point $C(x_0,0)$ (a trial solution). The half breadth h is determined as follows: From Fig. 7.20

$$\theta_1 = \pi - \alpha \qquad \theta_2 = \alpha$$

where

$$\cos \alpha = \frac{a}{\sqrt{h^2 + a^2}}$$

Substituting into Eq. (7.8.17),

$$h^2 = \frac{m}{\pi U} \frac{a}{\sqrt{h^2 + a^2}} \tag{7.8.19}$$

from which h may be determined (also by trial solution).

Eliminating m/U between Eqs. (7.8.18) and (7.8.19)

$$\frac{m}{U} = \frac{(x_0{}^2 - a^2)^2}{x_0} \frac{\pi}{a} = \frac{\pi}{a} h^2 \sqrt{h^2 + a^2}$$

the value of a may be obtained for a predetermined body (x_0, h, specified). Hence, U can be given any positive value and the pressure and velocity distribution can be determined.

In determining the velocity at points throughout the region it is convenient to find the velocity at each point due to each component of the flow, i.e., due to the source, the sink, and the uniform flow, separately, and add the components graphically or by $\hat{\omega}$- and x-components.

Bodies obtained from source-sink combinations with uniform flow are called *Rankine bodies.*

Translation of a Sphere in an Infinite Fluid. The velocity potential for a solid moving through an infinite fluid otherwise at rest must satisfy the following conditions:[1]

1. The Laplace equation, $\nabla^2\phi = 0$ everywhere except singular points.
2. The fluid must remain at rest at infinity; hence, the space derivatives of ϕ must vanish at infinity.
3. The boundary conditions at the surface of the solid must be satisfied.

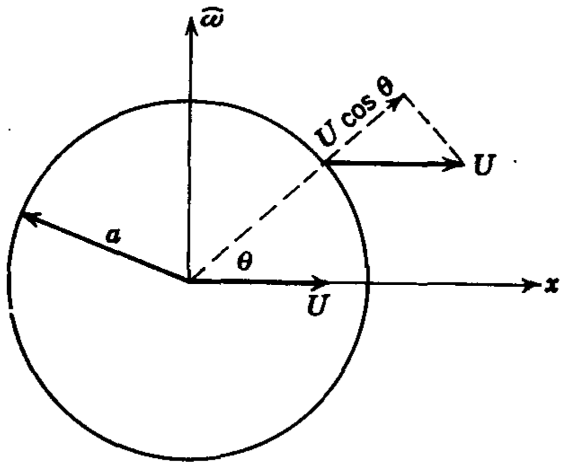


Fig. 7.21. Sphere translating in the positive x-direction.

For a sphere of radius a with center at the origin moving with velocity U in the positive x-direction, the velocity of the surface normal to itself is $U \cos \theta$, from Fig. 7.21. The fluid velocity normal to the surface is $-\partial\phi/\partial r$; hence the boundary condition is

$$-\frac{\partial \phi}{\partial r} = U \cos \theta$$

The velocity potential for the doublet [Eq. (7.8.3)]

$$\phi = \frac{\mu \cos \theta}{r^2}$$

[1] G. G. Stokes, "Mathematical and Physical Papers," vol. 1, pp. 38–43, Cambridge University Press, London, 1880.

satisfies $\nabla^2\phi = 0$ for any constant value of μ. Substituting it into the boundary condition

$$-\frac{\partial\phi}{\partial r} = \frac{2\mu}{r^3}\cos\theta = U\cos\theta$$

which is satisfied for $r = a$ if $\mu = Ua^3/2$. It may also be noted that the velocity components, $-\partial\phi/\partial r$ and $-(1/r)(\partial\phi/\partial\theta)$, are zero at infinity. Therefore,

$$\phi = \frac{Ua^3}{2r^2}\cos\theta \tag{7.8.20}$$

satisfies all the conditions for translation of a sphere in an infinite fluid. This case is one of unsteady flow, solved for the instant when the center of the sphere is at the origin. Because this equation has been specialized for a particular instant, the pressure distribution cannot be found from it by use of Eq. (7.5.7). Streamlines and equipotential lines for the sphere are shown in Fig. 7.22.

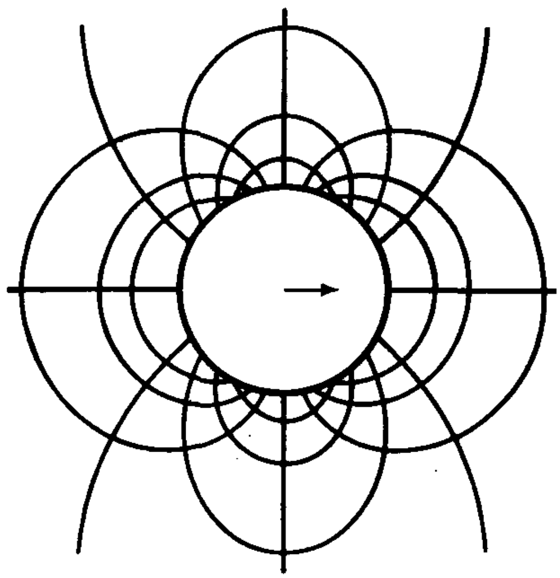


FIG. 7.22. Streamlines and equipotential lines for a sphere moving through fluid.

The stream function for this flow case is

$$\psi = -\frac{Ua^3}{2r}\sin^2\theta \tag{7.8.21}$$

Steady Flow of an Infinite Fluid around a Sphere. The unsteady-flow case in the preceding section may be converted into a steady-flow case by superposing upon the flow a uniform stream of magnitude U in the negative x-direction. To prove this, add $\phi = Ux = Ur\cos\theta$ to the potential function [Eq. (7.8.20)]; thus

$$\phi = \frac{Ua^3}{2r^2}\cos\theta + Ur\cos\theta \tag{7.8.22}$$

The stream function corresponding to this is

$$\psi = -\frac{Ua^3}{2r}\sin^2\theta + \frac{Ur^2}{2}\sin^2\theta \tag{7.8.23}$$

Then from Eq. (7.8.23), $\psi = 0$ when $\theta = 0$ and when $r = a$. Hence, the stream surface $\psi = 0$ is the sphere $r = a$, which may be taken as a solid, fixed boundary. Streamlines and equipotential lines are shown in Fig. 7.23. Perhaps mention should be made that the equations give a flow pattern for the interior portion of the sphere as well. No fluid passes through the surface of the sphere, however.

The velocity at any point on the surface of the sphere is

$$-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\bigg]_{r=a} = q = \tfrac{3}{2}U\sin\theta$$

The stagnation points are at $\theta = 0$, $\theta = \pi$. The maximum velocity $\frac{3}{2}U$

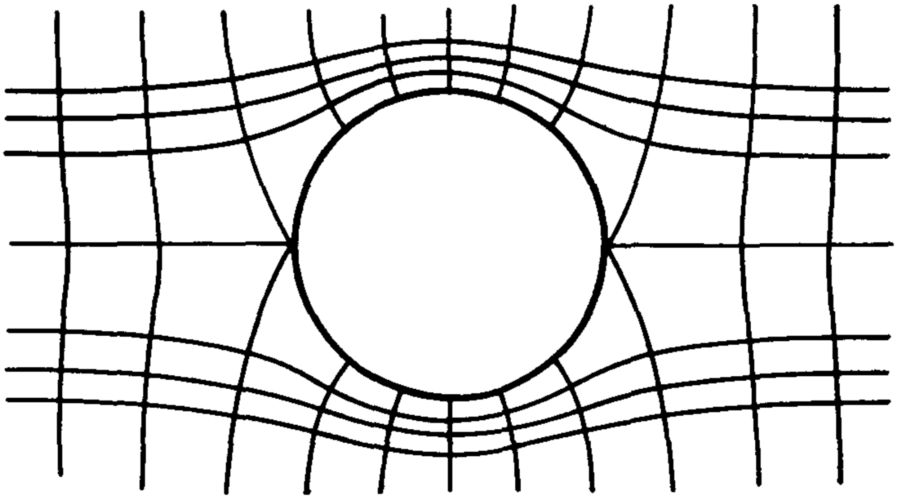


FIG. 7.23. Streamlines and equipotential lines for uniform flow about a sphere at rest.

occurs at $\theta = \pi/2$. The dynamic pressure distribution over the surface of the sphere is

$$p = \frac{\rho U^2}{2}\left(1 - \tfrac{9}{4}\sin^2\theta\right)$$

for dynamic pressure of zero at infinity.

7.9. Two-dimensional Flow Cases. Two simple flow cases that may be interpreted for flow along straight boundaries are first examined, then the source, vortex, doublet, uniform flow, and flow around a cylinder, with and without circulation, are discussed.

Flow around a Corner. The potential function

$$\phi = A(x^2 - y^2)$$

has as its stream function

$$\psi = 2Axy = Ar^2\sin 2\theta$$

in which r and θ are polar coordinates. It is plotted for equal increment changes in ϕ and ψ in Fig. 7.24. Conditions at the origin are not defined, as it is a stagnation point. As any of the streamlines may be taken as fixed boundaries, the plus axes may be taken as walls, yielding flow into a 90° corner. The equipotential lines are hyperbolas having axes coincident

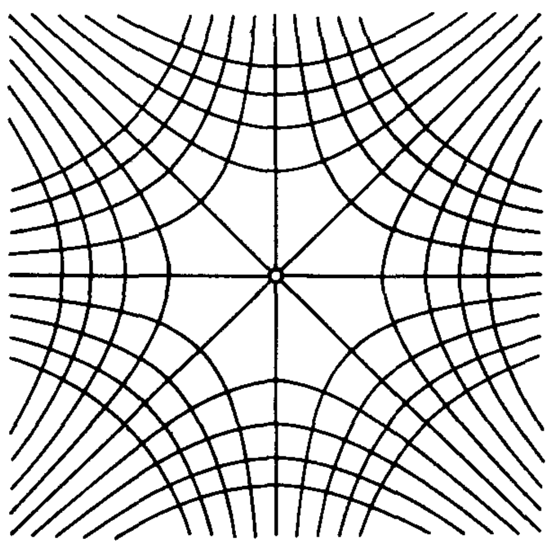


FIG. 7.24. Flow net for flow around 90° bend.

with the coordinate axes and asymptotes given by $y = \pm x$. The streamlines are rectangular hyperbolas, having $y = \pm x$ as axes and the coordinate axes as asymptotes. From the polar form of the stream function it is noted that the two lines $\theta = 0$ and $\theta = \pi/2$ are the streamline $\psi = 0$.

This case may be generalized to yield flow around a corner with angle α. By examining

$$\phi = Ar^{\pi/\alpha} \cos \frac{\pi\theta}{\alpha} \qquad \psi = Ar^{\pi/\alpha} \sin \frac{\pi\theta}{\alpha}$$

it is noted the streamline $\psi = 0$ is now given by $\theta = 0$ and $\theta = \alpha$. Two flow nets are shown in Fig. 7.25, for the cases $\alpha = 225°$ and $\alpha = 45°$.

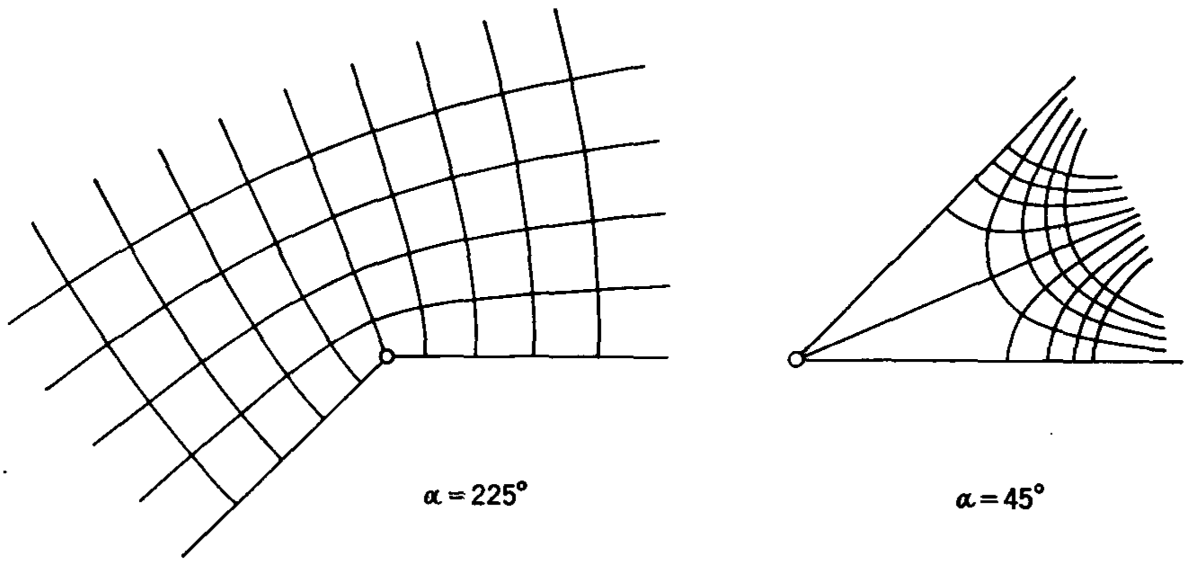


FIG. 7.25. Flow net for flow along two inclined surfaces.

Source. A line normal to the xy-plane, from which fluid is imagined to flow uniformly in all directions *at right angles* to it, is a source. It appears as a point in the customary two-dimensional flow diagram. The total flow per unit time per unit length of line is called the *strength* of the source. As the flow is in radial lines from the source, the velocity a distance r from the source is determined by the strength divided by the flow area of the cylinder, or $2\pi\mu/2\pi r$, in which the strength is $2\pi\mu$. Then, since by Eq. (7.4.4) the velocity in any direction is given by the negative derivative of the velocity potential with respect to the direction,

$$-\frac{\partial \phi}{\partial r} = \frac{\mu}{r} \qquad \frac{\partial \phi}{\partial \theta} = 0$$

and

$$\phi = -\mu \ln r$$

is the velocity potential, in which ln indicates the natural logarithm and r is the distance from the source. This value of ϕ satisfies the Laplace equation in two dimensions.

The streamlines are radial lines from the source, i.e.,

$$\frac{\partial \psi}{\partial r} = 0 \qquad -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\mu}{r}$$

From the second equation

$$\psi = -\mu\theta$$

Lines of constant ϕ (equipotential lines) and constant ψ are shown in Fig. 7.26. A *sink* is a negative source, a line into which fluid is flowing.

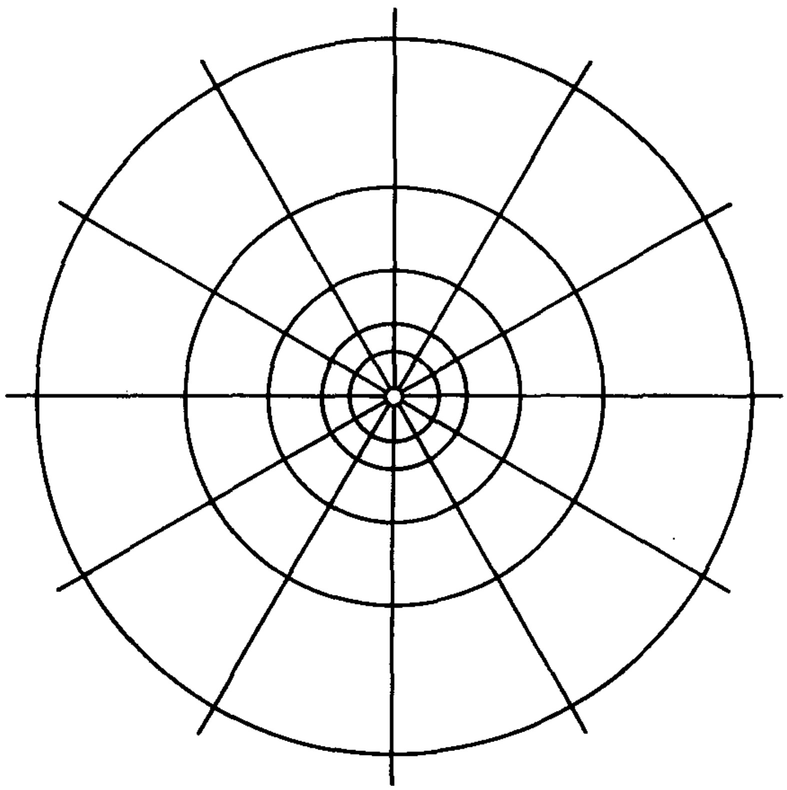


Fig. 7.26. Flow net for source or vortex.

Vortex. In examining the flow case given by selecting the stream function for the source as a velocity potential,

$$\phi = -\mu\theta \qquad \psi = \mu \ln r$$

which also satisfies the Laplace equation, it is seen that the equipotential lines are radial lines and the streamlines are circles. The velocity is in a tangential direction only, since $\partial\phi/\partial r = 0$. It is $q = -(1/r)\partial\phi/\partial\theta = \mu/r$, since $r\ \delta\theta$ is the length element in the tangential direction.

In referring to Fig. 7.27, the *flow along a closed curve* is called the *circulation.* The flow along an element of the curve is defined as the product of the length element δs of the curve and the component of the velocity tangent to the curve, $q \cos \alpha$. Hence the circulation Γ around a closed path C is

$$\Gamma = \int_C q \cos \alpha \, ds = \int_C \mathbf{q} \cdot d\mathbf{s}$$

The velocity distribution given by the equation $\phi = -\mu\theta$ is for the *vortex* and is such that the circulation around any closed path that contains the vortex is constant. The value of the circulation is the strength of the vortex. By selecting any circular path of radius r to determine the circulation, $\alpha = 0°$, $q = \mu/r$, and $ds = r\,d\theta$; hence,

$$\Gamma = \int_C q \cos \alpha \, ds = \int_0^{2\pi} \frac{\mu}{r} r \, d\theta = 2\pi\mu$$

At the point $r = 0$, $q = \mu/r$ goes to infinity; hence, this point is called a singular point. Figure 7.26 shows the equipotential lines and streamlines for the vortex.

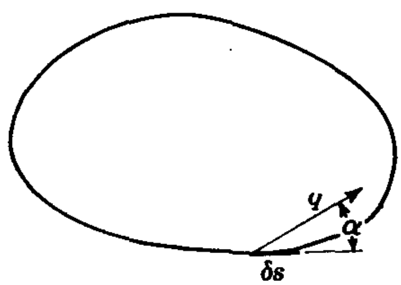


FIG. 7.27. Notation for definition of circulation.

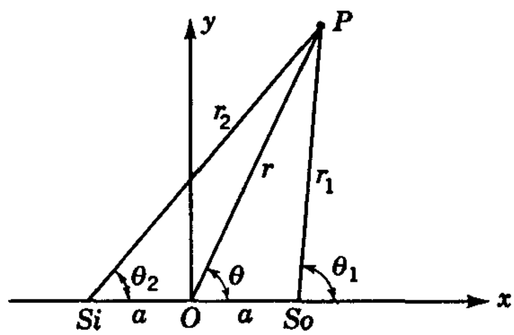


FIG. 7.28. Notation for derivation of two-dimensional doublet.

Doublet. The two-dimensional doublet is defined as the limiting case as a source and sink of equal strength approach each other so that the product of their strength and the distance between them remains a constant μ, called the *strength* of the doublet. The axis of the doublet is from the sink toward the source, i.e., the line along which they approach each other.

In Fig. 7.28 a source is located at $(a,0)$ and a sink of equal strength at $(-a,0)$. The velocity potential for both, at some point P, is

$$\phi = -m \ln r_1 + m \ln r_2$$

with r_1, r_2 measured from source and sink, respectively, to the point P. Thus, $2\pi m$ is the strength of source and sink. To take the limit as a approaches zero for $2am = \mu$ the form of the expression for ϕ must be altered. The terms r_1 and r_2 may be expressed in terms of the polar coordinates r, θ by the cosine law, as follows:

$$r_1^2 = r^2 + a^2 - 2ar\cos\theta = r^2\left[1 + \left(\frac{a}{r}\right)^2 - 2\frac{a}{r}\cos\theta\right]$$

$$r_2^2 = r^2 + a^2 + 2ar\cos\theta = r^2\left[1 + \left(\frac{a}{r}\right)^2 + 2\frac{a}{r}\cos\theta\right]$$

After rewriting the expression for ϕ, with these relations,

$$\phi = -\frac{m}{2}(\ln r_1^2 - \ln r_2^2) = -\frac{m}{2}\left\{\ln r^2 + \ln\left[1 + \left(\frac{a}{r}\right)^2 - 2\frac{a}{r}\cos\theta\right] - \ln r^2 - \ln\left[1 + \left(\frac{a}{r}\right)^2 + 2\frac{a}{r}\cos\theta\right]\right\}$$

By using the series expression,

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\begin{aligned}\phi = -\frac{m}{2}\Bigg\{&\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta - \frac{1}{2}\left[\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta\right]^2 \\ &+ \frac{1}{3}\left[\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\theta\right]^3 - \cdots - \left[\left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta\right] \\ &+ \frac{1}{2}\left[\left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta\right]^2 - \frac{1}{3}\left[\left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta\right]^3 + \cdots\Bigg\}\end{aligned}$$

After simplifying,

$$\phi = 2am\left[\frac{\cos\theta}{r} + \left(\frac{a}{r}\right)^2\frac{\cos\theta}{r} - \left(\frac{a}{r}\right)^4\frac{\cos\theta}{r} - \frac{4}{3}\left(\frac{a}{r}\right)^2\frac{\cos^3\theta}{r} + \cdots\right]$$

Now, if $2am = \mu$ and if the limit is taken as a approaches zero,

$$\phi = \frac{\mu\cos\theta}{r}$$

which is the velocity potential for a two-dimensional doublet at the origin, with axis in the $+x$-direction.

By using the relations

$$v_r = -\frac{\partial\phi}{\partial r} = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\partial\psi}{\partial r}$$

for the doublet

$$\frac{\partial\psi}{\partial\theta} = -\frac{\mu\cos\theta}{r} \qquad \frac{\partial\psi}{\partial r} = \frac{\mu}{r^2}\sin\theta$$

After integrating,

$$\psi = -\frac{\mu\sin\theta}{r}$$

is the stream function for the doublet. The equations in cartesian coordinates are

$$\phi = \frac{\mu x}{x^2 + y^2} \qquad \psi = -\frac{\mu y}{x^2 + y^2}$$

After rearranging,

$$\left(x - \frac{\mu}{2\phi}\right)^2 + y^2 = \frac{\mu^2}{4\phi^2} \qquad x^2 + \left(y + \frac{\mu}{2\psi}\right)^2 = \frac{\mu^2}{4\psi^2}$$

The lines of constant ϕ are circles through the origin with centers on the x-axis, and the streamlines are circles through the origin with centers on the y-axis, as shown in Fig. 7.29. The origin is a singular point where the velocity goes to infinity.

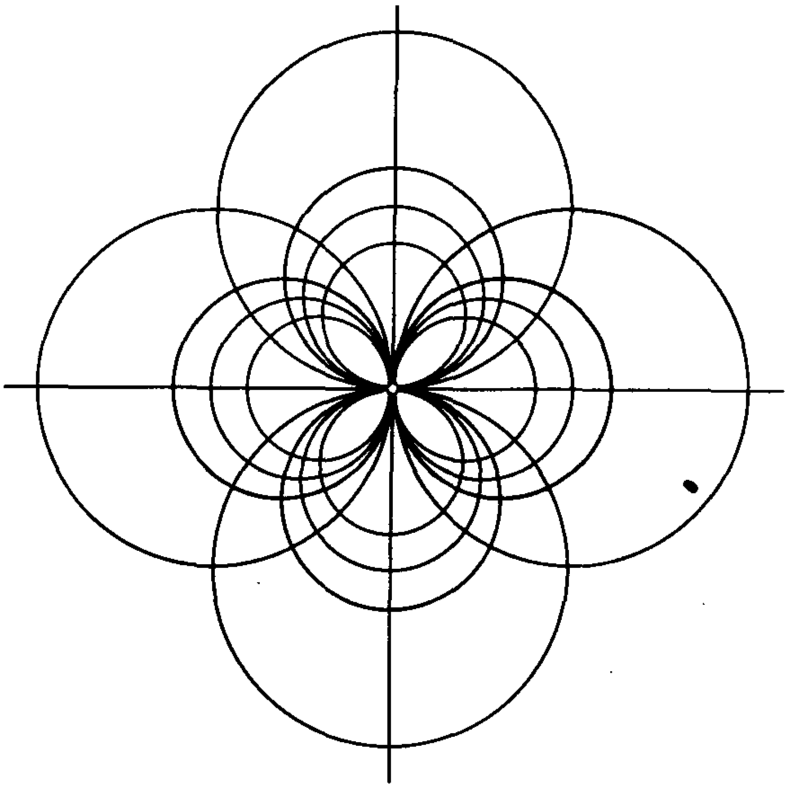


FIG. 7.29. Equipotential lines and streamlines for the two-dimensional doublet.

Uniform Flow. Uniform flow in the $-x$-direction, $u = -U$, is expressed by

$$\phi = Ux \qquad \psi = Uy$$

In polar coordinates,

$$\phi = Ur \cos \theta \qquad \psi = Ur \sin \theta$$

Flow around a Circular Cylinder. The addition of the flow due to a doublet and a uniform flow results in flow around a circular cylinder; thus

$$\phi = Ur \cos \theta + \frac{\mu \cos \theta}{r} \qquad \psi = Ur \sin \theta - \frac{\mu \sin \theta}{r}$$

As a streamline in steady flow is a possible boundary, the streamline $\psi = 0$ is given by

$$0 = \left(Ur - \frac{\mu}{r}\right) \sin \theta$$

which is satisfied by $\theta = 0, \pi$, or by the value of r that makes

$$Ur - \frac{\mu}{r} = 0$$

If this value is $r = a$, which is a circular cylinder, then

$$\mu = Ua^2$$

and the streamline $\psi = 0$ is the x-axis and the circle $r = a$. The potential and stream functions for uniform flow around a circular cylinder of radius a are, by substitution of the value of μ,

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta \qquad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

for the uniform flow in the $-x$-direction. The equipotential lines and streamlines for this case are shown in Fig. 7.30.

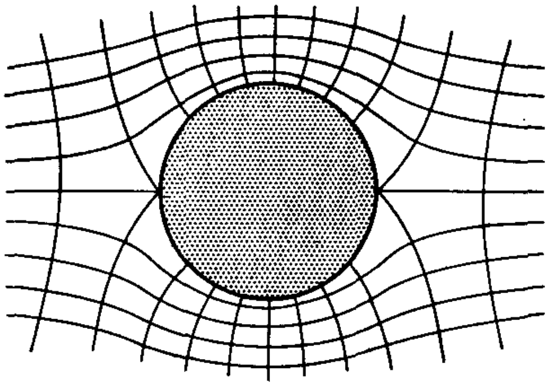


FIG. 7.30. Equipotential lines and streamlines for flow around a circular cylinder.

The velocity at any point in the flow can be obtained from either the velocity potential or the stream function. On the surface of the cylinder the velocity is necessarily tangential and is expressed by $\partial\psi/\partial r$ for $r = a$; thus

$$q\Big|_{r=a} = U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\Big|_{r=a} = 2U\sin\theta$$

The velocity is zero (stagnation point) at $\theta = 0, \pi$ and has maximum values of $2U$ at $\theta = \pi/2, 3\pi/2$. For the dynamic pressure zero at infinity, with Eq. (7.5.7) for $p_0 = 0$, $q_0 = U$,

$$p = \frac{\rho}{2}U^2\left[1 - \left(\frac{q}{U}\right)^2\right]$$

which holds for any point in the plane except the origin. For points on the cylinder

$$p = \frac{\rho}{2}U^2(1 - 4\sin^2\theta)$$

The maximum pressure, which occurs at the stagnation points, is $\rho U^2/2$; and the minimum pressure, at $\theta = \pi/2, 3\pi/2$, is $-3\rho U^2/2$. The points of zero dynamic pressure are given by $\sin\theta = \pm\frac{1}{2}$, or $\theta = \pm\pi/6, \pm 5\pi/6$.

A cylindrical pitot-static tube is made by providing three openings in a cylinder, at 0° and ±30°, as the difference in pressure between 0° and ±30° is the dynamic pressure $\rho U^2/2$.

The drag on the cylinder is shown to be zero by integration of the x-component of the pressure force over the cylinder; thus

$$\text{Drag} = \int_0^{2\pi} pa \cos\theta \, d\theta = \frac{\rho a U^2}{2} \int_0^{2\pi} (1 - 4\sin^2\theta)\cos\theta \, d\theta = 0$$

Similarly, the lift force on the cylinder is zero.

Flow around a Circular Cylinder with Circulation. The addition of a vortex to the doublet and the uniform flow results in flow around a circular cylinder with circulation,

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta$$

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln r$$

The streamline $\psi = (\Gamma/2\pi)\ln a$ is the circular cylinder $r = a$, and, at great distances from the origin, the velocity remains $u = -U$, showing

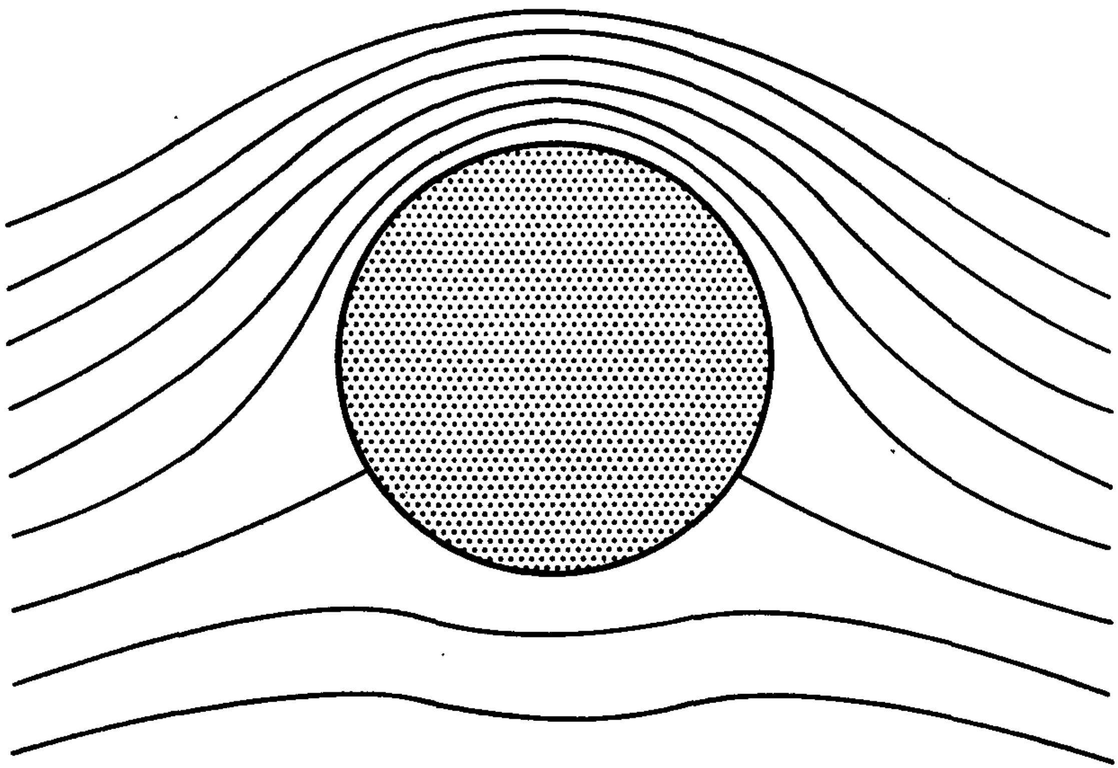


FIG. 7.31. Streamlines for flow around a circular cylinder with circulation.

that flow around a circular cylinder is maintained with addition of the vortex. Some of the streamlines are shown in Fig. 7.31.

The velocity at the surface of the cylinder, necessarily tangent to the cylinder, is

$$q = \left.\frac{\partial \psi}{\partial r}\right|_{r=a} = 2U \sin \theta + \frac{\Gamma}{2\pi a}$$

Stagnation points occur where $q = 0$; that is,

$$\sin \theta = -\frac{\Gamma}{4\pi U a}$$

When the circulation is $4\pi Ua$, the two stagnation points coincide at $r = a$, $\theta = -\pi/2$. For larger circulation, the stagnation point moves away from the cylinder.

The pressure at the surface of the cylinder is

$$p = \frac{\rho U^2}{2}\left[1 - \left(2 \sin \theta + \frac{\Gamma}{2\pi a U}\right)^2\right]$$

The drag again is zero. The lift. however, becomes

$$\text{Lift} = -\int_0^{2\pi} pa \sin \theta \, d\theta$$

$$= -\frac{\rho a U^2}{2}\int_0^{2\pi}\left[1 - \left(2 \sin \theta + \frac{\Gamma}{2\pi a U}\right)^2\right] \sin \theta \, d\theta = \rho U \Gamma$$

showing that the lift is directly proportional to the density of fluid, the approach velocity U, and the circulation Γ. This thrust, which acts at right angles to the approach velocity, is referred to as the *Magnus effect.* The Flettner rotor ship was designed to utilize this principle by the mounting of circular cylinders with axes vertical on a ship, and then mechanically rotating the cylinders to provide circulation. Air flowing around the rotors produces the thrust at right angles to the relative wind direction. The close spacing of streamlines along the upper side of Fig. 7.31 indicates that the velocity is high there and that the pressure must then be correspondingly low.

The airfoil develops its lift by producing a circulation around it due to its shape. It may be shown[1] that the lift is $\rho U \Gamma$ for any cylinder in two-dimensional flow. The angle of inclination of the airfoil relative to the approach velocity (angle of attack) greatly affects the circulation. For large angles of attack, the flow does not follow the wing profile, and the theory breaks down.

Example 7.3: A source with strength 6 cfs/ft and a vortex with strength 12 ft²/sec are located at the origin. Determine the equation for velocity potential and stream function. What are the velocity components at $x = 2$, $y = 3$?

[1] V. L. Streeter, "Fluid Dynamics," pp. 137–155, McGraw-Hill Book Company, Inc., New York, 1948.

The velocity potential for the source is

$$\phi = -\frac{6}{2\pi}\ln r$$

and the corresponding stream function is

$$\psi = -\frac{6}{2\pi}\theta$$

The velocity potential for the vortex is

$$\phi = -\frac{12}{2\pi}\theta$$

and the corresponding stream function is

$$\psi = \frac{12}{2\pi}\ln r$$

By adding the respective functions

$$\phi = -\frac{3}{\pi}(\ln r + 2\theta)$$

and

$$\psi = -\frac{3}{\pi}(\theta - 2\ln r)$$

The radial and tangential velocity components are

$$v_r = -\frac{\partial \phi}{\partial r} = \frac{3}{\pi r} \qquad v_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{6}{\pi r}$$

At (2,3), $r = \sqrt{2^2 + 3^2} = 3.61$, $v_r = 0.265$, $v_\theta = 0.53$.

PROBLEMS

7.1. Compute the gradient of the following two-dimensional scalar functions:

(a) $\phi = -\frac{1}{2}\ln(x^2 + y^2)$ (b) $\phi = Ux + Vy$ (c) $\phi = 2xy$

7.2. Compute the divergence of the gradients of ϕ found in Prob. 7.1.

7.3. Compute the curl of the gradients of ϕ found in Prob. 7.1.

7.4. For $\mathbf{q} = \mathbf{i}(x + y) + \mathbf{j}(y + z) + \mathbf{k}(x^2 + y^2 + z^2)$ find the components of rotation at (1,1,1).

7.5. Derive the equation of continuity for two-dimensional flow in polar coordinates by equating the net efflux from a small polar element to zero (Fig. 7.32). It is

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$$

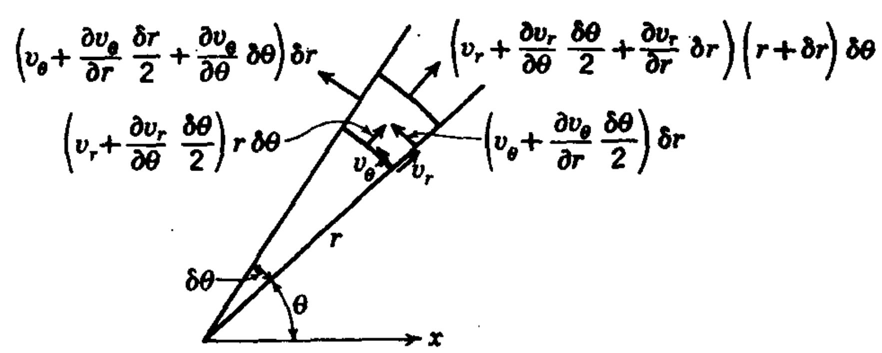


FIG. 7.32

7.6. The x-component of velocity is $u = x^2 + z^2$, and the y-component is $v = y^2 + z^2$. Find the simplest z-component of velocity that satisfies continuity.

7.7. A velocity potential in two-dimensional flow is $\phi = x + x^2 - y^2$. Find the stream function for this flow.

7.8. The two-dimensional stream function for a flow is $\psi = 9 + 3x - 4y + 7xy$. Find the velocity potential.

7.9. Derive the partial differential equations relating ϕ and ψ for two-dimensional flow in plane polar coordinates.

7.10. From the continuity equation in polar coordinates in Prob. 7.5, derive the Laplace equation in the same coordinate system.

7.11. Does the function $\phi = 1/r$ satisfy the Laplace equation in two dimensions? In three-dimensional flow is it satisfied?

7.12. By use of the equations developed in Prob. 7.9 find the two-dimensional stream function for $\phi = \ln r$.

7.13. Find the Stokes stream function for $\phi = 1/r$.

7.14. For the Stokes stream function $\psi = 13r^2 \sin^2 \theta$, find ϕ in cartesian coordinates.

7.15. In Prob. 7.14 what is the discharge between stream surfaces through the points $r = 1$, $\theta = 0$ and $r = 1$, $\theta = \pi/4$?

7.16. Write the boundary conditions for steady flow around a sphere, of radius a, at its surface and at infinity.

7.17. A circular cylinder of radius a has its center at the origin and is translating with velocity V in the y-direction. Write the boundary condition in terms of ϕ that is to be satisfied at its surface and at infinity.

7.18. A source of strength 40 cfs is located at the origin, and another source of strength 20 cfs is located at (1,0,0). Find the velocity components u, v, w at $(-1,0,0)$ and (1,1,1).

7.19. If the dynamic pressure is zero at infinity in Prob. 7.18, for $\rho = 2.00$ slugs/ft^3 calculate the dynamic pressure at $(-1,0,0)$ and (1,1,1).

7.20. A source of strength m at the origin and a uniform flow of 10 ft/sec are combined in three-dimensional flow so that a stagnation point occurs at (1,0,0). Obtain the velocity potential and stream function for this flow case.

7.21. By use of symmetry obtain the velocity potential for a three-dimensional sink of strength 50 cfs located 3 ft from a plane barrier.

7.22. Equations are wanted for flow of a uniform stream of 10 ft/sec around a Rankine body 4 ft long and 2 ft thick in a transverse direction.

7.23. A source of strength 10 cfs at (1,0,0) and a sink of the same strength at (−1,0,0) are combined with a uniform flow of 30 ft/sec in the $-x$-direction. Determine the size of Rankine body formed by this flow.

7.24. A sphere of radius 2 ft, with center at the origin, has a uniform flow of 20 ft/sec in the $-x$-direction flowing around it. At (4,0,0) the dynamic pressure is 100 lb/ft² and $\rho = 1.935$ slugs/ft³. Find the equation for pressure distribution over the surface of the sphere.

7.25. By integration over the surface of the sphere of Prob. 7.24 show that the drag on the sphere is zero.

7.26. In two-dimensional flow what is the nature of the flow given by $\phi = 7x + 2 \ln r$?

7.27. A source discharging 20 cfs/ft is located at (−1,0), and a sink of twice the strength is located at (2,0). For dynamic pressure at the origin of 200 lb/ft², $\rho = 1.8$ slugs/ft³, find the velocity and dynamic pressure at (0,1) and (1,1).

7.28. Select the strength of doublet needed to portray a uniform flow of 50 ft/sec around a cylinder of radius 2 ft.

7.29. Develop the equations for flow around a "Rankine cylinder" formed by a source, an equal sink, and a uniform flow.

7.30. In the Rankine cylinder of Prob. 7.29, if $2a$ is the distance between source and sink, their strength is $2\pi\mu$, and U is the uniform velocity, develop an equation for length of the body.

7.31. A circular cylinder 8 ft in diameter rotates at 600 rpm. When in an air stream, $\rho = 0.002$ slug/ft³, moving at 400 ft/sec, what is the lift force per foot of cylinder, assuming 90 per cent efficiency in developing circulation from the rotation?

7.32. An unsteady-flow case may be transformed into a steady-flow case

(*a*) regardless of the nature of the problem
(*b*) when two bodies are moving toward each other in an infinite fluid
(*c*) when an unsymmetrical body is rotating in an infinite fluid
(*d*) when a single body translates in an infinite fluid
(*e*) under no circumstances

7.33. Select the value of ϕ that satisfies continuity.

(*a*) $x^2 + y^2$ (*b*) $\sin x$ (*c*) $\ln (x + y)$ (*d*) $x + y$
(*e*) none of these answers

7.34. The units for Euler's equations of motion are given by

(*a*) force per unit mass
(*b*) velocity
(*c*) energy per unit weight
(*d*) force per unit weight
(*e*) none of these answers

7.35. Euler's equations of motion can be integrated when it is assumed that

(*a*) the continuity equation is satisfied
(*b*) the fluid is incompressible

(c) a velocity potential exists and the density is constant
(d) the flow is rotational and incompressible
(e) the fluid is nonviscous

7.36. Euler's equations of motion are a mathematical statement that at every point

(a) rate of mass inflow equals rate of mass outflow
(b) force per unit mass equals acceleration
(c) the energy does not change with the time
(d) Newton's third law of motion holds
(e) the fluid momentum is constant

7.37. In irrotational flow of an ideal fluid

(a) a velocity potential exists
(b) all particles must move in straight lines
(c) the motion must be uniform
(d) the flow is always steady
(e) the velocity must be zero at a boundary

7.38. A function ϕ that satisfies the Laplace equation

(a) must be linear in x and y
(b) is a possible case of rotational fluid flow
(c) does not necessarily satisfy the continuity equation
(d) is a possible fluid-flow case
(e) is none of these answers

7.39 If ϕ_1 and ϕ_2 are each solutions of the Laplace equation, which of the following is also a solution?

(a) $\phi_1 - 2\phi_2$ (b) $\phi_1\phi_2$ (c) ϕ_1/ϕ_2 (d) ϕ_1^2 (e) none of these answers

7.40. Select the relation that must hold if the flow is irrotational.

(a) $\partial u/\partial y + \partial v/\partial x = 0$ (b) $\partial u/\partial x = \partial v/\partial y$
(c) $\partial^2 u/\partial x^2 + \partial^2 v/\partial y^2 = 0$ (d) $\partial u/\partial y = \partial v/\partial x$
(e) none of these answers

7.41. The Bernoulli equation in steady ideal fluid flow states that

(a) the velocity is constant along a streamline
(b) the energy is constant along a streamline but may vary across streamlines
(c) when the speed increases, the pressure increases
(d) the energy is constant throughout the fluid
(e) the net flow rate into any small region must be zero

7.42. The Stokes stream function applies to

(a) all three-dimensional ideal-fluid-flow cases
(b) ideal (nonviscous) fluids only

(*c*) irrotational flow only
(*d*) cases of axial symmetry
(*e*) none of these cases

7.43. The Stokes stream function has the value $\psi = 1$ at the origin and the value $\psi = 2$ at (1,1,1). The discharge through the surface between these points is

(*a*) 1 (*b*) π (*c*) 2π (*d*) 4 (*e*) none of these answers

7.44. Select the relation that must hold in two-dimensional, irrotational flow.

(*a*) $\partial\phi/\partial x = \partial\psi/\partial y$ (*b*) $\partial\phi/\partial x = -\partial\psi/\partial y$ (*c*) $\partial\phi/\partial y = \partial\psi/\partial x$
(*d*) $\partial\phi/\partial x = \partial\phi/\partial y$ (*e*) none of these answers

7.45. The two-dimensional stream function

(*a*) is constant along an equipotential surface
(*b*) is constant along a streamline
(*c*) is defined for irrotational flow only
(*d*) relates velocity and pressure
(*e*) is none of these answers

7.46. In two-dimensional flow $\psi = 4$ ft²/sec at (0,2) and $\psi = 2$ ft²/sec at (0,1). The discharge between the two points is

(*a*) from left to right (*b*) 4π cfs/ft (*c*) 2 cfs/ft (*d*) $1/\pi$ cfs/ft
(*e*) none of these answers

7.47. The boundary condition for steady flow of an ideal fluid is that the

(*a*) velocity is zero at the boundary
(*b*) velocity component normal to the boundary is zero
(*c*) velocity component tangent to the boundary is zero
(*d*) boundary surface must be stationary
(*e*) continuity equation must be satisfied

7.48. An equipotential surface

(*a*) has no velocity component tangent to it
(*b*) is composed of streamlines
(*c*) is a stream surface
(*d*) is a surface of constant dynamic pressure
(*e*) is none of these answers

7.49. A source in two-dimensional flow

(*a*) is a point from which fluid is imagined to flow outward uniformly in all directions
(*b*) is a line from which fluid is imagined to flow uniformly in all directions at right angles to it
(*c*) has a strength defined as the speed at unit radius

(*d*) has streamlines that are concentric circles
(*e*) has a velocity potential independent of the radius

7.50. The two-dimensional vortex

(*a*) has a strength given by the circulation around a path enclosing the vortex
(*b*) has radial streamlines
(*c*) has a zero circulation around it
(*d*) has a velocity distribution that varies directly as the radial distance from the vortex
(*e*) creates a velocity distribution that has rotation throughout the fluid

PART TWO

Applications of Fluid Mechanics

In Part One the fundamental concepts and equations have been developed and illustrated by many examples and simple applications. Fluid resistance, dimensional analysis, compressible flow, and ideal fluid flow have been presented. In Part Two several of the important fields of application of fluid mechanics are explored: turbomachinery, measuring of flow, closed conduit, and open-channel flow.

8

TURBOMACHINERY

The turning of a fluid stream or the changing of the magnitude of its velocity requires that forces be applied. When a moving vane deflects a fluid jet and changes its momentum, forces are exerted between vane and jet and work is done by displacement of the vane. Turbomachines make use of this principle: The axial and centrifugal pumps, blowers, and compressors, by continuously doing work on the fluid, add to its energy; the impulse, Francis, and propeller turbines and steam and gas turbines continuously extract energy from the fluid and convert it into torque on a moving shaft; the fluid coupling and the torque converter, each consisting of a pump and a turbine built together, make use of the fluid to transmit power smoothly. The designing of efficient turbomachines utilizes both theory and experimentation. A good design of given size and speed may be readily adapted to other speeds and other geometrically similar sizes by application of the theory of scaled models, as outlined in Sec. 4.5.

Similarity relationships are first discussed in this chapter by consideration of homologous units and specific speed. Elementary cascade theory is next taken up, before considering the theory of turbomachines. Water turbines and pumps are next considered, followed by blowers, centrifugal compressors, and fluid couplings and torque converters. The chapter closes with a discussion of cavitation.

8.1. Homologous Units. Specific Speed. In utilizing scaled models in the designing of turbomachines, geometric similitude is required as well as geometrically similar velocity vector diagrams at entrance to or exit from the impellers. Viscous effects must, unfortunately, be neglected, as it is generally impossible to satisfy the two above conditions and have equal Reynolds numbers in model and prototype. Two geometrically similar units having similar velocity vector diagrams are *homologous*. They will also have geometrically similar streamlines.

The velocity vector diagram in Fig. 8.1 at exit from a pump impeller may be used to formulate the condition for similar streamline patterns.

The blade angle is β, u is the peripheral speed of the impeller at the end of the vane or blade, v is the velocity of fluid *relative* to the vane, and V is the absolute velocity leaving the impeller, the vector sum of u and v; V_r is the radial component of V and is proportional to the discharge; α is the angle which the absolute velocity makes with u, the tangential direction. According to geometric similitude, β must be the same for two units, and for similar streamlines α must also be the same in each case.

It is convenient to express the fact that α is to be the same in any of a series of turbomachines, called homologous units, by relating the speed of rotation N, the impeller diameter (or other characteristic

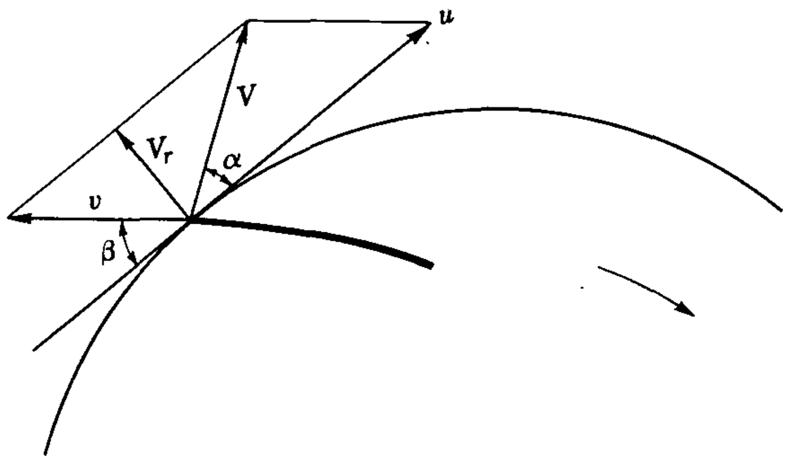


FIG. 8.1. Velocity vector diagram for exit from a pump impeller.

dimension) D, and the flow rate Q. For constant α, V_r is proportional to $V(V_r = V \sin \alpha)$ and u is proportional to V_r. Hence the conditions for constant α in a homologous series of units may be expressed as

$$\frac{V_r}{u} = \text{constant}$$

The discharge Q is proportional to V_rD^2, since any cross-sectional flow area is proportional to D^2. The speed of rotation N is proportional to u/D. By inserting these values

$$\frac{Q}{ND^3} = \text{constant} \tag{8.1.1}$$

expresses the condition in which geometrically similar units are homologous.

The discharge Q through homologous units may be related to head H and cross-sectional flow path A by the orifice formula

$$Q = C_dA\sqrt{2gH}$$

in which C_d, the discharge coefficient, varies slightly with Reynolds number and so actually causes a small change in efficiency with size in a homologous series. The change in discharge with Reynolds number is referred to as "scale effect." The smaller machines, having smaller hydraulic radii of passages, will have lower Reynolds numbers and correspondingly higher friction factors; hence they are less efficient. The change in efficiency from model to prototype may be from 1 to 4 per cent. However, in the homologous theory, the scale effect must be neglected, so an empirical correction for change in efficiency with size is used [see Eq. (8.5.1)]. As $A \sim D^2$, the discharge equation may be

$$\frac{Q}{D^2\sqrt{H}} = \text{constant} \tag{8.1.2}$$

After eliminating Q between Eqs. (8.1.1) and (8.1.2)

$$\frac{H}{N^2D^2} = \text{constant} \tag{8.1.3}$$

Equations (8.1.1) and (8.1.3) are most useful in determining performance characteristics for one unit from those of a homologous unit of different size and speed.[1]

Example 8.1: A prototype test of a mixed-flow pump with a 72-in.-diameter discharge opening, operating at 225 rpm, resulted in the following characteristics:

H_{ft}	Q_{cfs}	$e\%$	H_{ft}	Q_{cfs}	$e\%$
60	200	69	40	382	86.3
57.5	228	75	37.5	396	84.4
55	256	80	35	411	82
52.5	280	83.7	32.5	425	79
50	303	86	30	438	75
47.5	330	87.3	27.5	449	71
45	345	88	25	459	66.5
42.5	362	87.4			

[1] The homologous requirement Q/ND^3 is dimensionless; the other requirement (assuming geometric similitude) may be made dimensionless by retaining g. In $Q = CA\sqrt{2gH}$ the dimensionless ratio is $Q/A\sqrt{gH}$ or $Q/D^2\sqrt{gH}$. Elimination of Q between this relation and Q/ND^3 yields $H/(N^2D^2/g)$ as a second dimensionless requirement. The characteristic curve for a pump in dimensionless form is the plot of Q/ND^3 as abscissa against $H/(N^2D^2/g)$ as ordinate. This curve, obtained from tests on one unit of the series, then applies to all homologous units, and may be converted to the usual characteristic curve by selecting desired values of N and D. As power is proportional to γQH, the dimensionless power term is

$$\frac{\gamma}{\gamma} \times \frac{Q}{ND^3} \times \frac{H}{N^2D^2/g} = \frac{\text{power}}{\rho N^3D^5}$$

What size and synchronous speed of homologous pump should be used to produce 200 cfs at 60 ft head at point of best efficiency? Find the characteristic curves for this case.

Subscript 1 refers to the 72-in. pump. For best efficiency $H_1 = 45$, $Q_1 = 345$, $e = 88$ per cent. With Eqs. (8.1.1) and (8.1.3)

$$\frac{H}{N^2D^2} = \frac{H_1}{N_1^2D_1^2} \qquad \frac{Q}{ND^3} = \frac{Q_1}{N_1D_1^3}$$

or

$$\frac{60}{N^2D^2} = \frac{45}{(225)^2(72)^2} \qquad \frac{200}{ND^3} = \frac{345}{225(72)^3}$$

After solving for N and D,

$$N = 366 \qquad D = 51.1$$

The nearest synchronous speed (3600 divided by number of pairs of poles) is 360 rpm. To maintain the desired head of 60 ft, a new D is necessary. Its size may be computed:

$$D = \sqrt{\tfrac{60}{45}} \times \tfrac{225}{360} \times 72 = 52 \text{ in.}$$

The discharge at best efficiency is then

$$Q = \frac{Q_1ND^3}{N_1D_1^3} = 345 \times \frac{360}{225}\left(\frac{52}{72}\right)^3 = 208 \text{ cfs}$$

which is slightly more capacity than required. With $N = 360$ and $D = 52$, equations for transforming the corresponding values of H and Q for any efficiency may be obtained:

$$H = H_1\left(\frac{ND}{N_1D_1}\right)^2 = H_1\left(\frac{360}{225} \times \frac{52}{72}\right)^2 = 1.336H_1$$

and

$$Q = Q_1\frac{ND^3}{N_1D_1^3} = Q_1\frac{360}{225} \times \left(\frac{52}{72}\right)^3 = 0.605Q_1$$

The characteristics of the new pump are

H_{ft}	Q_{cfs}	e%	H_{ft}	Q_{cfs}	e%
80	121	69	53.5	231	86.3
76.7	138	75	50	239	84.4
73.4	155	80	46.7	248	82
70	169	83.7	43.4	257	79
66.7	183	86	40	264	75
63.5	200	87.3	36.7	271	71
60	208	88	33.4	277	66.5
56.7	219	87.4			

The efficiency of the 52-in. pump might be a fraction of a per cent less than that of the 72-in. pump, as the hydraulic radii of flow passages are smaller, so Reynolds number would be less.

Specific Speed. The specific speed of a homologous unit is a constant that is widely used in the selection of type of unit and in preliminary design. It is usually defined differently for a pump than for a turbine.

The specific speed N_s of a homologous series of pumps is defined as the speed of some one unit of the series of such a size that it delivers unit discharge at unit head. It is obtained as follows: By eliminating D in Eqs. (8.1.1) and (8.1.3), and rearranging

$$\frac{N\sqrt{Q}}{H^{\frac{3}{4}}} = \text{constant} \tag{8.1.4}$$

By definition of specific speed, the constant is N_s, the speed of a unit for $Q = 1$, $H = 1$:

$$N_s = \frac{N\sqrt{Q}}{H^{\frac{3}{4}}} \tag{8.1.5}$$

The specific speed of a series is usually defined for the point of best efficiency, i.e., for the speed, discharge, and head that is most efficient.

The specific speed of a homologous series of turbines is defined as the speed of a unit of the series of such a size that it produces unit horsepower with unit head. Since power P is proportional to QH,

$$\frac{P}{QH} = \text{constant} \tag{8.1.6}$$

The terms D and Q may be eliminated from Eqs. (8.1.1), (8.1.3), and (8.1.6) to produce

$$\frac{N\sqrt{P}}{H^{\frac{5}{4}}} = \text{constant} \tag{8.1.7}$$

For unit power and unit head the constant of Eq. (8.1.7) becomes the speed, or the specific speed, N_s, of the series, so

$$N_s = \frac{N\sqrt{P}}{H^{\frac{5}{4}}} \tag{8.1.8}$$

The specific speed of a unit required for a given discharge and head can be estimated from Eqs. (8.1.5) and (8.1.8). For pumps handling large discharges at low heads a high specific speed is indicated; for a high head turbine producing relatively low power (small discharge) the specific speed is low. Experience has shown that for best efficiency one particular type of pump or turbine is usually indicated for a given specific speed.

Centrifugal pumps have low specific speeds; mixed-flow pumps have medium specific speeds; and axial-flow pumps have high specific speeds. Impulse turbines have low specific speeds; Francis turbines have medium specific speeds; and propeller turbines have high specific speeds.

8.2. Elementary Cascade Theory. Turbomachines either do work on a fluid or extract work from it in a continuous manner by having it flow through a series of moving (and possibly fixed) vanes. By examination of flow through a series of similar blades or vanes, called a *cascade*, some of the requirements of an efficient system may be developed. Consider, first, flow through the simple fixed cascade system of Fig. 8.2. It is seen that the velocity vector representing the fluid has been turned through the angle θ by the presence of the cascade system. A force has been exerted on the fluid, but neglecting friction effects and turbulence, no work is done on the fluid. Section 3.9 deals with forces on a single vane.

FIG. 8.2. Simple cascade system.

Since turbomachines are rotational devices, the cascade system may be arranged symmetrically around the periphery of a circle, as in Fig. 8.3. If the fluid now approaches the fixed cascade in a radial direction, it has moment of momentum changed from zero to a value dependent upon the mass per unit time flowing, the tangential component of velocity V_t developed, and the radius, from Eq. (3.11.4),

$$T = \rho Q r V_t \tag{8.2.1}$$

Again, no work is done by the fixed-vane system.

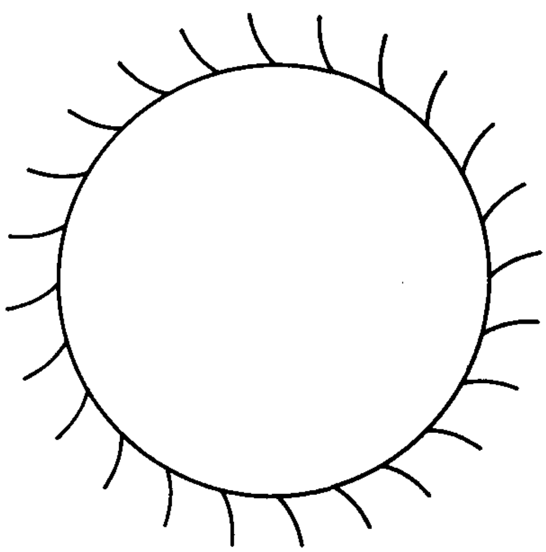


FIG. 8.3. Cascade arranged on the periphery of a circle.

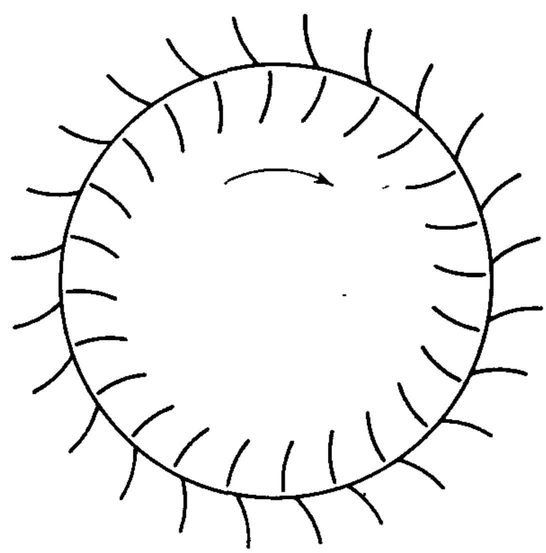


FIG. 8.4. Moving cascade within fixed cascade.

Consider now another series of vanes (Fig. 8.4) that are rotating within the fixed vane system at a speed ω. For efficient operation of the system it is important that the fluid flow onto the moving vanes with the least disturbance, i.e., in a tangential manner, as illustrated in Fig. 8.5. When the relative velocity is not tangent to the blade at its entrance, separation may occur, as shown in Fig. 8.6. The losses tend to increase rapidly (about as the square) with angle from the tangential and radically impair the efficiency of the machine. Separation also frequently occurs when the approaching relative velocity is tangential to the vane, owing to curvature of the vanes or to expansion of the flow passages, which causes the boundary layer to thicken and come to rest. These losses are called *shock* or *turbulence* losses. When the fluid exits from the moving cascade, it will generally have its velocity altered in both magnitude and direction, thereby changing its moment of momentum and either doing work on the cascade or having work done on it by the moving cascade. In the case of a turbine it is desired to have the fluid leave with no moment of

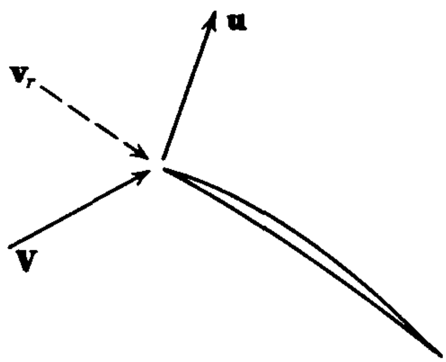


FIG. 8.5. Relative velocity tangent to blade.

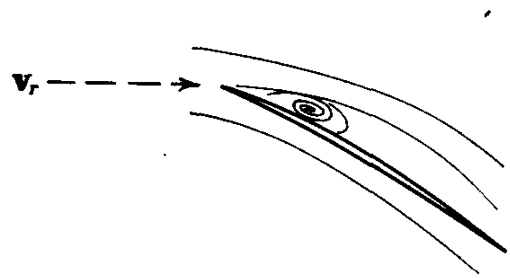


FIG. 8.6. Flow separation, or "shock," from blade with relative velocity not tangent to leading edge.

momentum. An old saying in turbine design is "have the fluid enter without shock and leave without velocity."

Turbomachinery design requires the proper arrangement and shaping of passages and vanes so that the purpose of the design can be most efficiently met. The particular design depends upon the purpose of the machine, the amount of work to be done per unit mass of fluid, and the fluid density.

8.3. Theory of Turbomachines. Turbines extract useful work from fluid energy; and pumps, blowers, and turbocompressors add energy to fluids by means of a runner consisting of vanes rigidly attached to a shaft. Since the only displacement of the vanes is in the tangential direction, work is done by the displacement of the tangential components of force on the runner. The radial components of force on the runner have no displacement in a radial direction and, hence, can do no work.

In turbomachine theory, friction is neglected and the fluid is assumed to have perfect guidance through the machine, i.e., an infinite number of thin vanes, so the relative velocity of the fluid is always tangent to the vane. This yields circular symmetry and permits the moment-of-momentum equation, Sec. 3.11, to take the simple form of Eq. (3.11.4), for steady flow,

$$T = \rho Q[(rv_t)_{\text{out}} - (rv_t)_{\text{in}}] \tag{8.3.1}$$

in which T is the torque acting on the fluid within the control volume

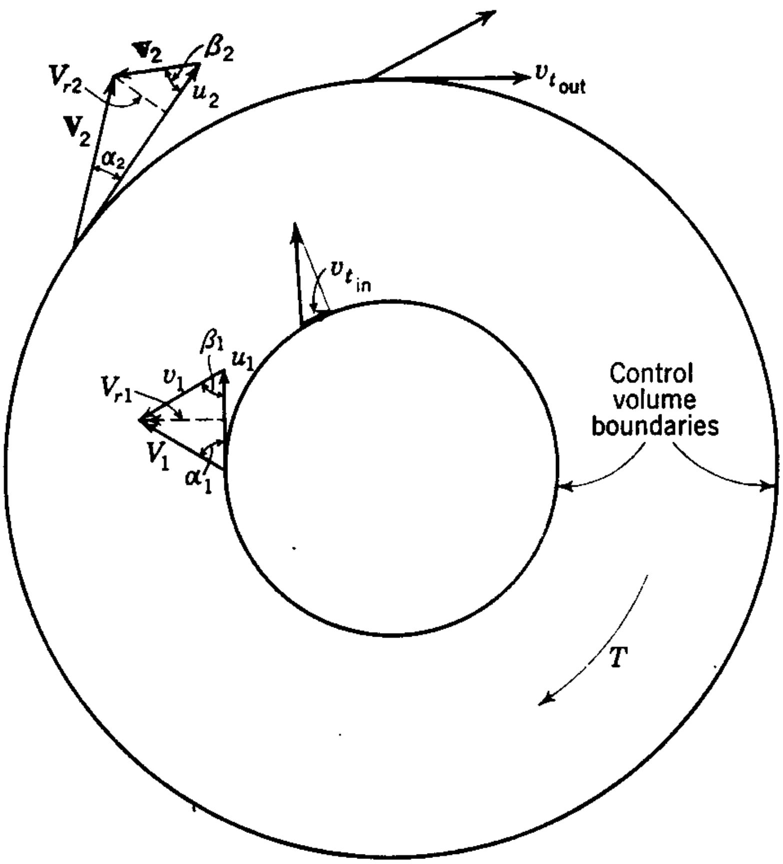


FIG. 8.7. Steady flow through control volume with circular symmetry.

(Fig. 8.7) and $\rho Q(rV_t)_{\text{out}}$ and $\rho Q(rv_t)_{\text{in}}$ represent the moment of momentum leaving and entering the control volume, respectively.

The polar vector diagram is generally used in studying vane relationships (Fig. 8.8), with subscript 1 for entering fluid and subscript 2 for

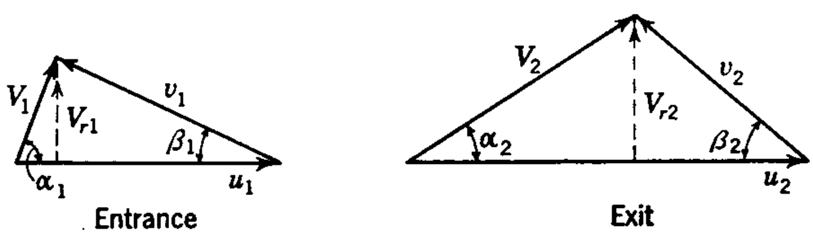


FIG. 8.8. Polar vector diagrams.

exiting fluid. V is the absolute fluid velocity, u the peripheral velocity of the runner, and v the fluid velocity relative to the runner. The absolute velocities V, u are laid off from 0, and the relative velocity connects them as shown. V_u is designated as the component of absolute velocity in the tangential direction. α is the angle the absolute velocity V makes with the peripheral velocity u, and β is the angle the relative velocity makes with $-u$, or it is the *blade angle*, as perfect guidance is assumed. V_r is the absolute velocity component normal to the periphery. In this notation Eq. (8.3.1) becomes

$$\begin{aligned} T &= \rho Q(r_2 V_2 \cos \alpha_2 - r_1 V_1 \cos \alpha_1) \\ &= \rho Q(r_2 V_{u2} - r_1 V_{u1}) = \dot{m}(r_2 V_{u2} - r_1 V_{u1}) \end{aligned} \tag{8.3.2}$$

The mass per unit time flowing is $\dot{m} = \rho Q = (\rho Q)_{\text{out}} = (\rho Q)_{\text{in}}$. In the form above, when T is positive, the fluid moment of momentum increases

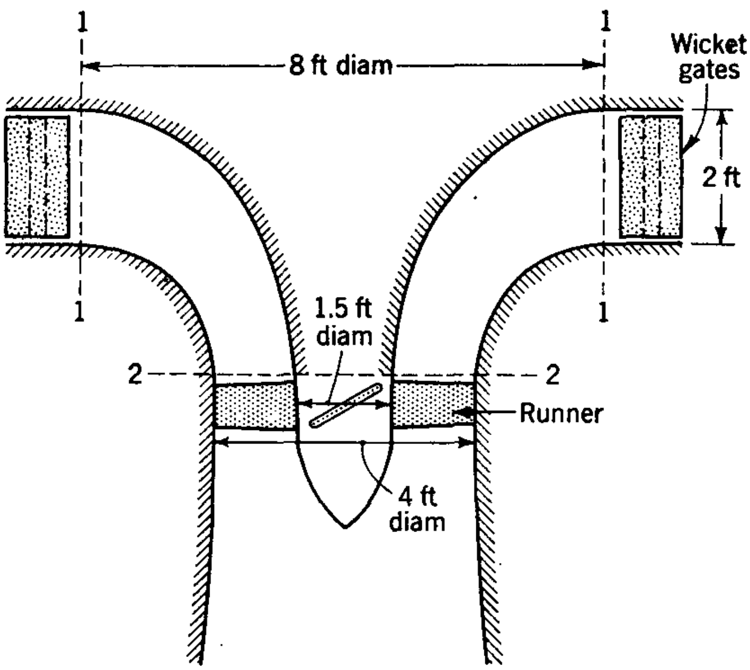


FIG. 8.9. Schematic view of propeller turbine.

through the control volume, as for a pump. For T negative moment of momentum of the fluid is decreased as for a turbine runner. When $T = 0$, as in passages where there are no vanes,

$$rV_u = \text{constant}$$

This is *free-vortex* motion, with the tangential component of velocity varying inversely with radius. It is discussed in Sec. 7.9 and compared with the forced vortex in Sec. 2.5.

Example 8.2: The wicket gates of Fig. 8.9 are turned so that the flow makes an angle of 45° with a radial line at section 1, where the speed is 8 ft/sec. Determine the magnitude of tangential velocity component V_u over section 2.

Since no torque is exerted on the flow between sections 1 and 2, the moment of momentum is constant and the motion follows the free-vortex law

$$V_u r = \text{constant}$$

At section 1

$$V_{u1} = 8 \cos 45° = 5.65 \text{ ft/sec}$$

Then

$$V_{u1} r_1 = 5.65 \times 4 = 22.6 \text{ ft}^2\text{/sec}$$

Across section 2

$$V_{u2} = \frac{22.6}{r}$$

at the hub $V_u = 22.6/0.75 = 30.1$ ft/sec, and at the outer edge $V_u = 22.6/2 = 11.3$ ft/sec.

Head and Energy Relations. By multiplying Eq. (8.3.2) by the rotational speed of runner ω,

$$\begin{aligned} T\omega &= \rho Q(\omega r_2 V_{u2} - \omega r_1 V_{u1}) \\ &= \rho Q(u_2 V_{u2} - u_1 V_{u1}) \end{aligned} \tag{8.3.3}$$

For no losses the power available from a turbine is $Q\,\Delta p = Q\gamma H$, in which H is the head on the runner, since Q is the weight per unit time and H the potential energy per unit weight. Similarly a pump runner produces work $Q\gamma H$ in which H is the pump head. The power exchange is

$$T\omega = Q\gamma H \tag{8.3.4}$$

By solving for H, using Eq. (8.3.3) to eliminate T,

$$H = \frac{u_2 V_{u2} - u_1 V_{u1}}{g} \tag{8.3.5}$$

For turbines the sign is reversed in Eq. (8.3.5).

For pumps the *actual* head H_p produced is

$$H_p = e_h H = H - H_L \tag{8.3.6}$$

and for turbines the actual head H_t is

$$H_t = \frac{H}{e_h} = H + H_L \tag{8.3.7}$$

in which e_h is the hydraulic efficiency of the machine and H_L represents all the internal fluid loss in the machine. The over-all efficiency of the machines is further reduced by bearing friction, by friction caused by fluid between runner and housing, and by leakage or flow that passes around the runner without going through it. These losses do not affect the head relations.

Pumps are generally designed so that the angular momentum of fluid entering the runner (impeller) is zero. Then

$$H = \frac{u_2 V_2 \cos \alpha_2}{g} \tag{8.3.8}$$

Turbines are designed so that the angular momentum is zero at the exit section of the runner for conditions at best efficiency; hence,

$$H = \frac{u_1 V_1 \cos \alpha_1}{g} \tag{8.3.9}$$

In writing Bernoulli's equation for a pump, with Eqs. (8.3.5) and (8.3.6) of this section

$$H_p = \left(\frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2\right) - \left(\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1\right) = \frac{u_2 V_2 \cos \alpha_2 - u_1 V_1 \cos \alpha_1}{g} - H_L \tag{8.3.10}$$

for which it is assumed that all streamlines through the pump have the same total energy. With the relations among the absolute velocity V, the velocity relative to the runner v, and the velocity of runner u, from the vector diagrams (Fig. 8.8) by the law of cosines,

$$u_1^2 + V_1^2 - 2u_1 V_1 \cos \alpha_1 = v_1^2$$
$$u_2^2 + V_2^2 - 2u_2 V_2 \cos \alpha_2 = v_2^2$$

After eliminating the absolute velocities V_1, V_2 in these relations and in Eq. (8.3.10)

$$H_L = \frac{u_2^2 - u_1^2}{2g} - \frac{v_2^2 - v_1^2}{2g} - \frac{p_2 - p_1}{\gamma} - (z_2 - z_1) \tag{8.3.11}$$

or

$$H_L = \frac{u_2^2 - u_1^2}{2g} - \left[\left(\frac{v_2^2}{2g} + \frac{p_2}{\gamma} + z^2\right) - \left(\frac{v_1^2}{2g} + \frac{p_1}{\gamma} + z_1\right)\right] \tag{8.3.12}$$

The losses are the difference in centrifugal head, $(u_2^2 - u_1^2)/2g$, and in the head change in the relative flow. For no loss, the increase in pressure head, from Eq. (8.3.11), is

$$\frac{p_2 - p_1}{\gamma} + z_2 - z_1 = \frac{u_2^2 - u_1^2}{2g} - \frac{v_2^2 - v_1^2}{2g} \tag{8.3.13}$$

With no flow through the runner, v_1, v_2 are zero, and the head rise is as expressed in the relative equilibrium relationships [Eq. (2.5.6)]. When flow occurs, the head rise is equal to the centrifugal head minus the difference in relative velocity heads.

For the case of a turbine, exactly the same equations result.

Example 8.3: A centrifugal pump with a 24-in.-diameter impeller runs at 1800 rpm. The water enters without whirl, and $\alpha_2 = 60°$. The actual head produced by the pump is 50 ft. Find its hydraulic efficiency when $V_2 = 20$ ft/sec.

From Eq. (8.3.8) the theoretical head is

$$H = \frac{u_2 V_2 \cos \alpha_2}{g} = \frac{1800 \times 2\pi \times 20 \times 0.50}{60 \times 32.2} = 58.6 \text{ ft}$$

The actual head is 50.0 ft; hence, the hydraulic efficiency is

$$e_h = \frac{50}{58.6} = 85.4 \text{ per cent}$$

8.4. Impulse Turbines. The impulse turbine is one in which all available energy of the flow is converted by a nozzle into kinetic energy at

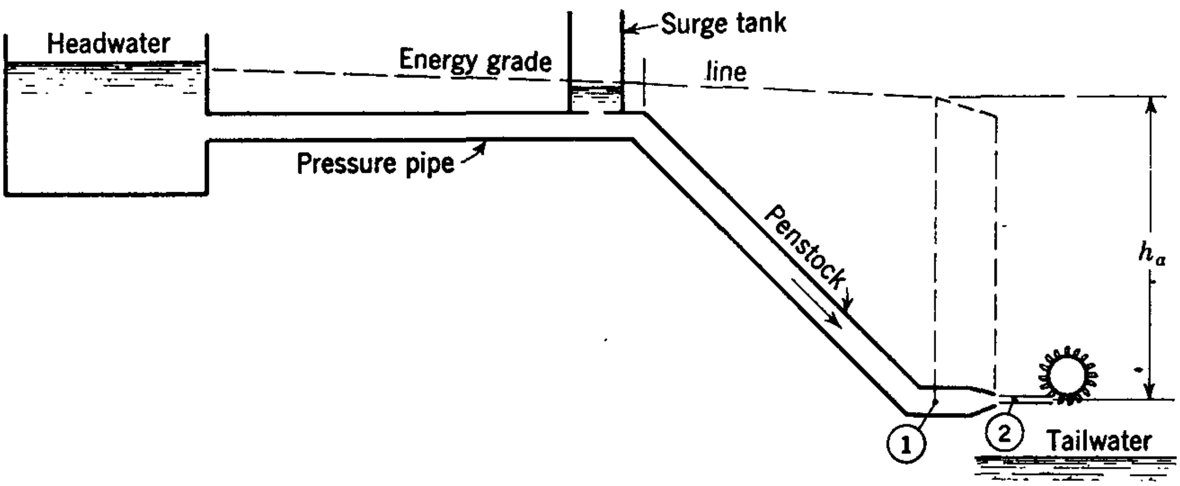


FIG. 8.10. Impulse turbine system.

atmospheric pressure before the fluid contacts the moving blades. Losses occur in flow from the reservoir through the pressure pipe (penstock) to the base of the nozzle, which may be computed from pipe friction data. At the base of the nozzle the available energy, or total head, is

$$h_a = \frac{p_1}{\gamma} + \frac{V_1^2}{2g} \tag{8.4.1}$$

from Fig. 8.10. With C_v the nozzle coefficient the jet velocity V_2 is

$$V_2 = C_v \sqrt{2gh_a} = C_v \sqrt{2g\left(\frac{p_1}{\gamma} + \frac{V_1^2}{2g}\right)} \tag{8.4.2}$$

The head lost in the nozzle is

$$h_a - \frac{V_2^2}{2g} = h_a - C_v^2 h_a = h_a(1 - C_v^2) \tag{8.4.3}$$

and the efficiency of the nozzle is

$$\frac{V_2^2/2g}{h_a} = \frac{C_v^2 h_a}{h_a} = C_v^2 \tag{8.4.4}$$

The jet, with velocity V_2, strikes double cupped buckets (Figs. 8.11 and 8.12) which split the flow and turn the relative velocity through the angle θ (Fig. 8.12).

FIG. 8.11. Southern California Edison, Big Creek 2A, 1948. $8\frac{1}{2}$-in.-diameter jet impulse buckets and disk in process of being reamed. 56,000 hp, 2200 ft head, 300 rpm. (*Allis-Chalmers Mfg. Co.*)

The x-component of momentum is changed by (Fig. 8.12)

$$F = \rho Q(v_r - v_r \cos\theta)$$

and the work done by the vanes is

$$Fu = \rho Q u v_r(1 - \cos\theta) \tag{8.4.5}$$

To maximize the work done, theoretically, $\theta = 180°$, and uv_r must be a maximum; i.e., $u(V_2 - u)$ must be a maximum. By differentiating with respect to u and equating to zero,

$$(V_2 - u) + u(-1) = 0$$

$u = V_2/2$. After making these substitutions into Eq. (8.4.5),

$$Fu = \rho Q \frac{V_2}{2}\left(V_2 - \frac{V_2}{2}\right)(1 - -1) = \gamma Q \frac{V_2^2}{2g} \tag{8.4.6}$$

which accounts for the total kinetic energy of the jet. The velocity diagram for these values shows that the absolute velocity leaving the vanes is zero.

Practically, when vanes are arranged on the periphery of a wheel (Fig. 8.11), it is necessary that the fluid retain enough velocity to move out of the way of the following bucket. Most of the practical impulse turbines are Pelton wheels. The jet is split in two, turned in a horizontal plane, and half is discharged from each side to avoid any unbalanced thrust on the shaft. There are losses due to the splitter and to friction between jet and bucket surface, which make the most economical speed somewhat less than $V_2/2$. It is expressed in terms of the *speed factor*

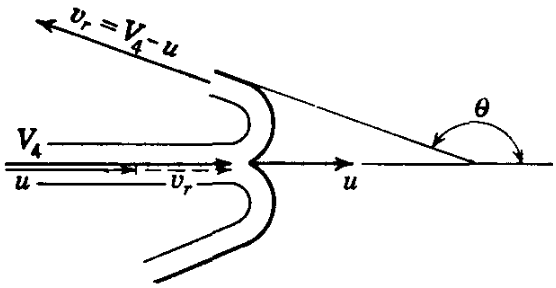


FIG. 8.12. Flow through bucket.

$$\phi = \frac{U}{\sqrt{2gh_a}} \tag{8.4.7}$$

For most efficient turbine operation ϕ has been found to be dependent upon specific speed as shown in the table.[1] The angle θ of the bucket

N_s	2	3	4	5	6	7
ϕ	0.47	0.46	0.45	0.44	0.433	0.425

is usually 173 to 176°. If the diameter of the jet is d and the diameter of the wheel D at the center line of the buckets, it has been found in practice that the diameter ratio D/d should be about $54/N_s$ for maximum efficiency.

In the majority of installations only one jet is used, which discharges horizontally against the lower periphery of the wheel as shown in Fig. 8.10. The wheel speed is carefully regulated for the generation of electrical power. A governor operates a needle valve that controls the jet discharge by changing its area. So V_0 remains practically constant for a wide range of positions of the needle valve.

[1] J. W. Daily, Hydraulic Machinery, in "Engineering Hydraulics," p. 943, ed. by H. Rouse, John Wiley & Sons, Inc., 1950.

The efficiency of the power conversion drops off rapidly with change in head (which changes V_0), as is evident when power is plotted against V_0 for constant u in Eq. (8.4.5). The wheel operates in atmospheric air although it is enclosed by a housing. It is therefore essential that the wheel be placed above the maximum flood water level of the river into which it discharges. The head from nozzle to tailwater is wasted. Because of their inefficiency at other than the design head and because of the wasted head, Pelton wheels usually are employed for high heads, e.g., from 600 ft to more than a mile. For high heads, the efficiency of the complete installation, from headwater to tailwater, may be in the high 80's.

Impulse wheels with a single nozzle are most efficient in the specific speed range of 2 to 6, when P is in horsepower, H is in feet, and N is in revolutions per minute. Multiple nozzle units are designed in the specific speed range of 6 to 12.

Example 8.4: A Pelton wheel is to be selected to drive a generator at 600 rpm. The water jet is 3 in. in diameter and has a velocity of 300 ft/sec. With the blade angle at 170°, the ratio of vane speed to initial jet speed at 0.47, and neglecting losses, determine (*a*) diameter of wheel to center line of buckets (vanes), (*b*) horsepower developed, and (*c*) kinetic energy per pound remaining in the fluid.

a. The peripheral speed of wheel is

$$u = 0.47 \times 300 = 141 \text{ ft/sec}$$

Then

$$\frac{600}{60} 2\pi \frac{D}{2} = 141 \qquad \text{or} \qquad D = 4.49 \text{ ft}$$

b. From Eq. (8.4.5) the power, in foot-pounds per second, is computed to be

$$1.935 \times \frac{\pi}{4} \times \frac{3^2}{144} \times 300(300 - 141)141[1 - (-0.9848)] = 1{,}270{,}000$$

and

$$\text{hp} = \frac{1{,}270{,}000}{550} = 2305$$

c. From Fig. 3.28, the absolute velocity components leaving the vane are determined to be

$$V_x = (300 - 141)(-0.9848) + 141 = -15.5 \text{ ft/sec}$$
$$V_y = (300 - 141)(0.1736) = 27.6 \text{ ft/sec}$$

The kinetic energy remaining in the jet is

$$\frac{\overline{15.5}^2 + \overline{27.6}^2}{64.4} = 15.6 \text{ ft-lb/lb}$$

Example 8.5: A small impulse wheel is to be used to drive a generator for 60-cycle power. The head is 300 ft, and the discharge 1.40 cfs. Determine the diameter of the wheel at the center line of the buckets and the speed of the wheel. $C_v = 0.98$ Assume efficiency of 80 per cent.

The power is

$$P = \frac{\gamma Q H e}{550} = \frac{62.4 \times 1.4 \times 300 \times 0.80}{550} = 38.2 \text{ hp}$$

Taking a trial value of N_s of 4,

$$N = \frac{N_s H^{\frac{5}{4}}}{\sqrt{P}} = \frac{4 \times \overline{300}^{\frac{5}{4}}}{38.2} = 809 \text{ rpm}$$

For 60-cycle power the speed must be 3600 divided by the number of pairs of poles in the generator. For five pairs of poles the speed would be $\frac{3600}{5} = 720$ rpm and for four pairs of poles $\frac{3600}{4} = 900$ rpm. The closer speed 720 is selected, although some engineers prefer an even number of pairs of poles in the generator. Then

$$N_s = \frac{N\sqrt{P}}{H^{\frac{5}{4}}} = \frac{720\sqrt{38.2}}{300^{\frac{5}{4}}} = 3.56 \text{ rpm}$$

For $N_s = 3.56$, take $\phi = 0.455$,

$$u = \phi\sqrt{2gH} = 0.455\sqrt{2 \times 32.2 \times 300} = 63.2 \text{ ft/sec}$$

and

$$\omega = \tfrac{720}{60} 2\pi = 75.4 \text{ rad/sec}$$

The peripheral speed u and D and ω are related:

$$u = \frac{\omega D}{2} \qquad D = \frac{2u}{\omega} = \frac{2 \times 63.2}{75.4} = 1.676 \text{ ft} = 20.1 \text{ in.}$$

The diameter d of the jet is obtained from the jet velocity V_2; thus

$$V_2 = C_v\sqrt{2gH} = 0.98\sqrt{2 \times 32.2 \times 300} = 136 \text{ ft/sec}$$

$$a = \frac{Q}{V_2} = \frac{1.40}{136} \times 144 = 1.482 \text{ in.}^2$$

and

$$d = \sqrt{\frac{4a}{\pi}} = \sqrt{\frac{1.482}{0.7854}} = 1.375 \text{ in.}$$

Hence the diameter ratio D/d is

$$\frac{D}{d} = \frac{20.1}{1.375} = 14.6$$

The desired diameter ratio for best efficiency is

$$\frac{D}{d} = \frac{54}{N_s} = \frac{54}{3.56} = 15.15$$

which is satisfactory. Hence the wheel diameter is 20.1 in. and speed 720 rpm

8.5. Reaction Turbines. In the *reaction* turbine a portion of the energy of the fluid is converted into kinetic energy by the fluid's passing through adjustable gates (Fig. 8.13) before entering the runner, and the remainder of the conversion takes place through the runner. All passages are filled with liquid including the passage (draft tube) from the runner to the downstream liquid surface. The static fluid pressure occurs on both sides of the vanes and, hence, does no work. The work done is entirely due to the conversion to kinetic energy.

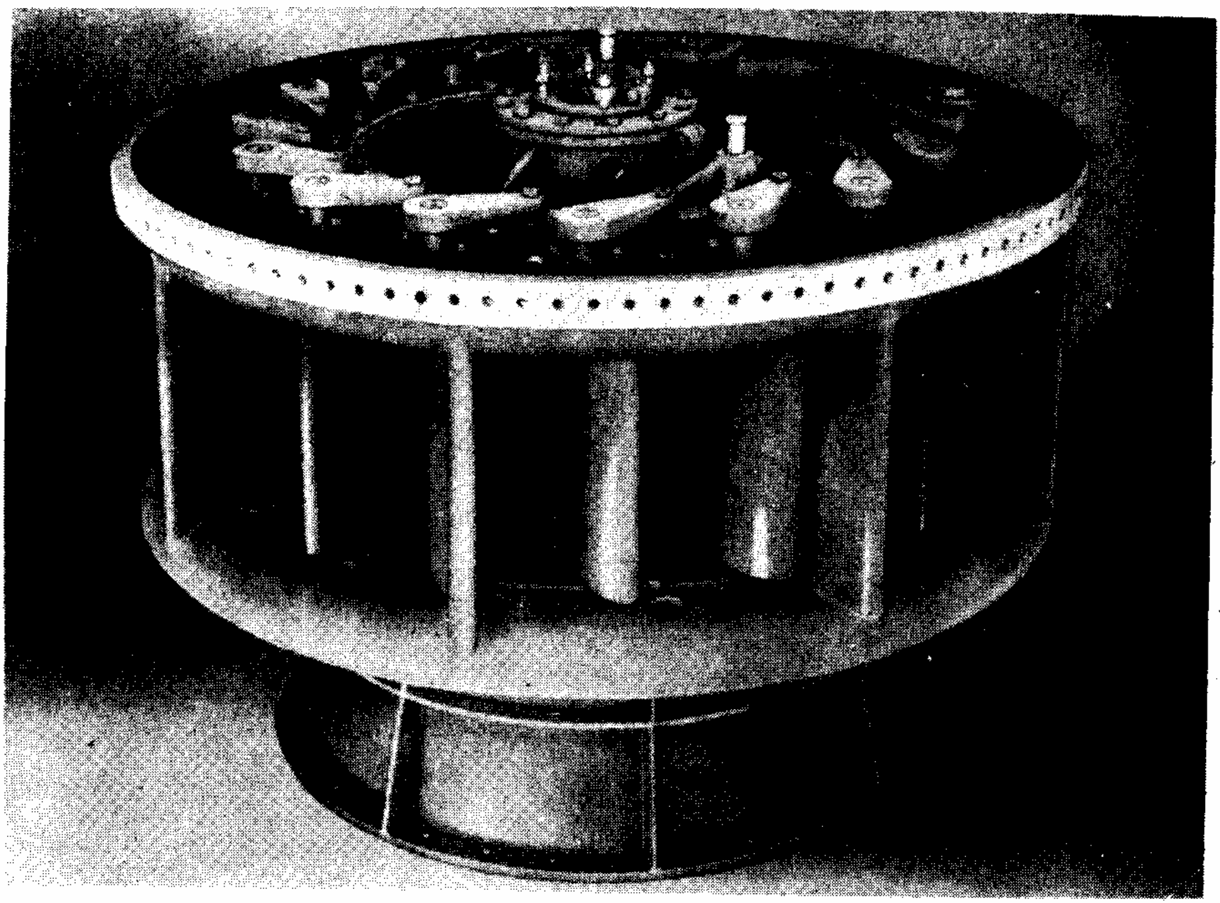


Fig. 8.13. Stay ring and wicket gates for reaction turbine. (*Allis-Chalmers Mfg. Co.*)

The reaction turbine is quite different from the impulse turbine discussed in Sec. 8.4. In an impulse turbine all the available energy of the fluid is converted into kinetic energy by a nozzle that forms a free jet. The energy is then taken from the jet by suitable flow through moving vanes. The vanes are partly filled, with the jet open to the atmosphere throughout its travel through the runner.

In contrast, in the reaction turbine the kinetic energy is appreciable as the fluid leaves the runner and enters the draft tube. The function of the draft tube is to reconvert the kinetic energy to flow energy by a gradual expansion of the flow cross section. Application of Bernoulli's equation between the two ends of the draft tube shows that the action of the tube is to reduce the pressure at its upstream end to less than atmospheric pressure, thus increasing the effective head across the runner to the difference in elevation between headwater and tail water, less losses.

By referring to Fig. 8.14, Bernoulli's equation from 1 to 2 yields

$$z_s + \frac{V_1^2}{2g} + \frac{p_1}{\gamma} = 0 + 0 + 0 + \text{losses}$$

The losses include friction plus velocity head loss at the exit from the draft tube, both of which are quite small; hence

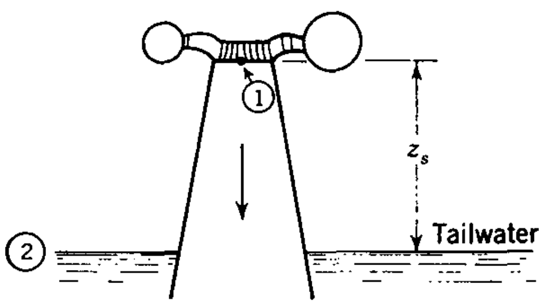


FIG. 8.14. Draft tube.

$$\frac{p_1}{\gamma} = -z_s - \frac{V_1^2}{2g} + \text{losses} \qquad (8.5.1)$$

shows that considerable vacuum is produced at section 1, which effectively increases the head across the turbine runner. The turbine setting may not be too high, or cavitation occurs in the runner and draft tube (see Sec. 8.9).

Example 8.6: A turbine has a velocity of 20 ft/sec at the entrance to the draft tube and a velocity of 4.0 ft/sec at its exit. For friction losses of 0.3 ft and a tailwater 16 ft below the entrance to the draft tube, find the pressure head at the entrance.

From Eq. (8.5.1)

$$\frac{p_1}{\gamma} = -16 - \frac{20^2}{2 \times 32.2} + \frac{4^2}{2 \times 32.2} + 0.3 = -21.7$$

as the kinetic energy at the exit from the draft tube is lost. Hence a suction head of 21.7 ft is produced by the presence of the draft tube.

There are two forms of the reaction turbine in common use, the *Francis* turbine (Fig. 8.15) and the *propeller* (axial-flow) turbine (Fig. 8.16). In both, all passages flow full, and energy is converted to useful work entirely by the changing of the moment of momentum of the liquid. The flow passes first through the wicket gates, which impart a tangential and a radially inward velocity to the fluid. A space between the wicket gates and the runner permits the flow to close behind the gates and move as a free vortex, without external torque being applied.

In the Francis turbine (Fig. 8.17) the fluid enters the runner so that the relative velocity is tangent to the leading edge of the vanes. The radial component is gradually changed to an axial component, and the tangential component is reduced as the fluid traverses the vane, so that at the runner exit the flow is axial with very little whirl (tangential component) remaining. The pressure has been reduced to less than atmospheric and most of the remaining kinetic energy is reconverted to flow energy by the time it discharges from the draft tube. The Francis turbine is best suited to medium-head installations from 80 to 600 ft and has an efficiency between 90 and 95 per cent for the larger installations. Francis turbines are designed in the specific speed range of 10 to 110 with best efficiency in the range 40 to 60

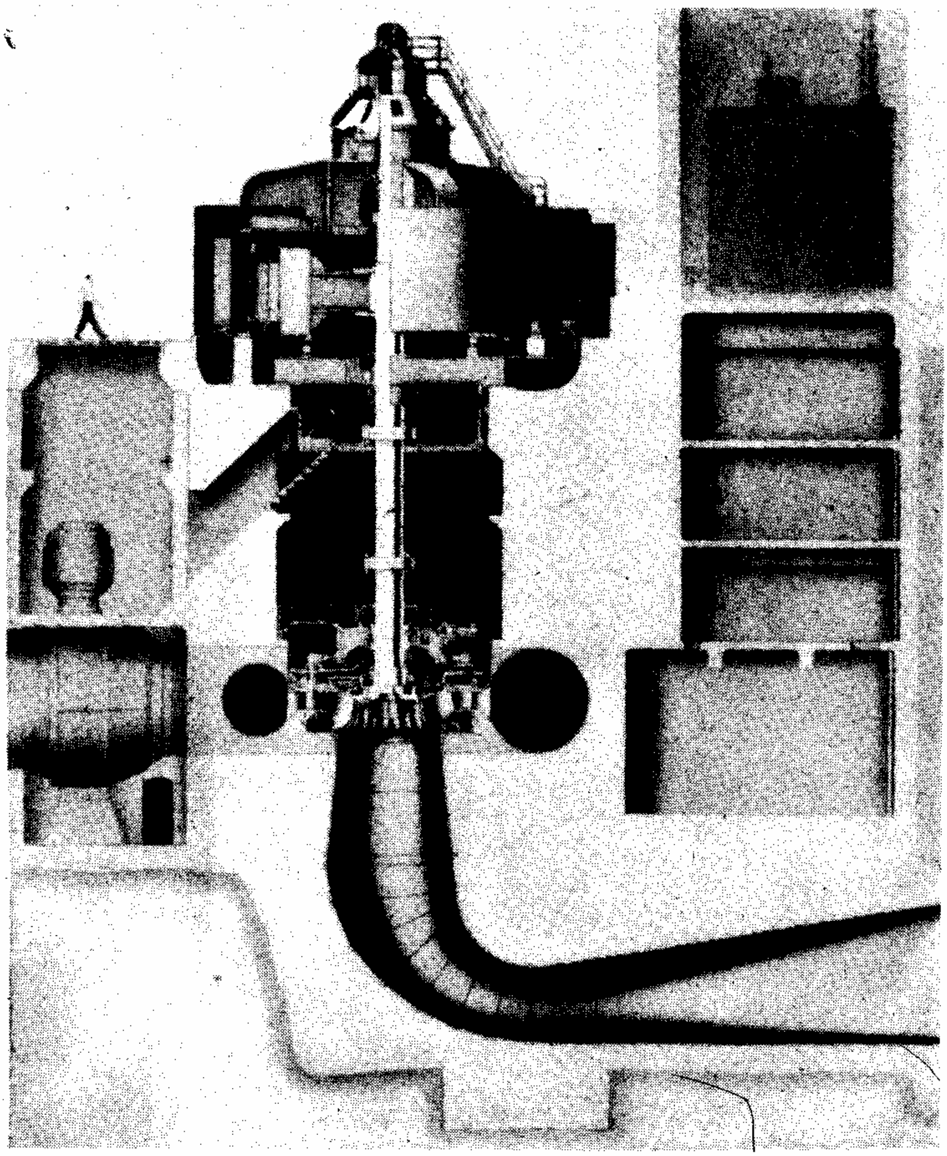


FIG. 8.15. Section through a hydroelectric unit installed and put in operation at Hoover Dam in 1952. The turbine is rated 115,000 hp at 180 rpm under 480 ft head. (*Allis-Chalmers Mfg. Co.*)

In the propeller turbine (Fig. 8.9), after passing through the wicket gates, the flow moves as a free vortex and has its radial component changed to axial component by guidance from the fixed housing. The moment of momentum is constant, and the tangential component of velocity is increased through the reduction in radius. The blades are few in number, relatively flat, with very little curvature, and placed so that the relative flow entering the runner is tangential to the leading edge of the blade. The relative velocity is high, as with the Pelton wheel, and changes slightly in traversing the blade. The velocity diagrams in Fig. 8.18 show how the tangential component of velocity is reduced. Propeller turbines are made with blades that pivot around

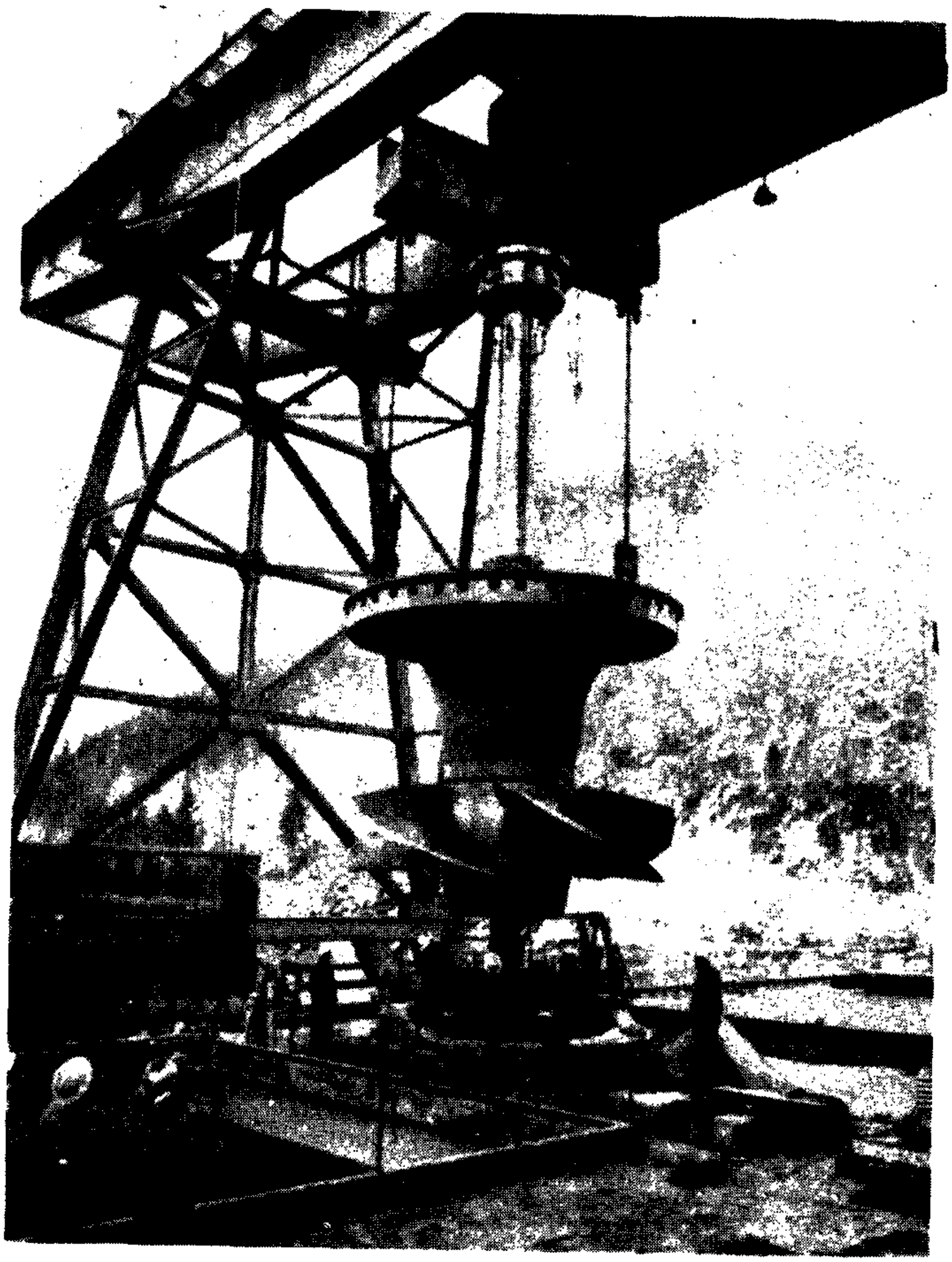


FIG. 8.16. Field view of installation of runner of 24,500 hp, 100 rpm, 41 ft head, Kaplan adjustable runner hydraulic turbine. Box Canyon Project, Public Utility District No. 1 of Pend Oreille County, Washington. Plant placed in operation in 1955. (*Allis-Chalmers Mfg. Co.*)

the hub, thus permitting the blade angle to be adjusted for different gate openings and for changes in head. They are particularly suited for low-head installations, up to 100 ft, and have top efficiencies around 94 per cent. Axial-flow turbines are designed in the specific speed range of 100 to 210 with best efficiency from 120 to 160.

The windmill is a form of axial-flow turbine. It has no fixed vanes to give an initial tangential component to the air stream and hence must impart the tangential component to the air with the moving vanes. The

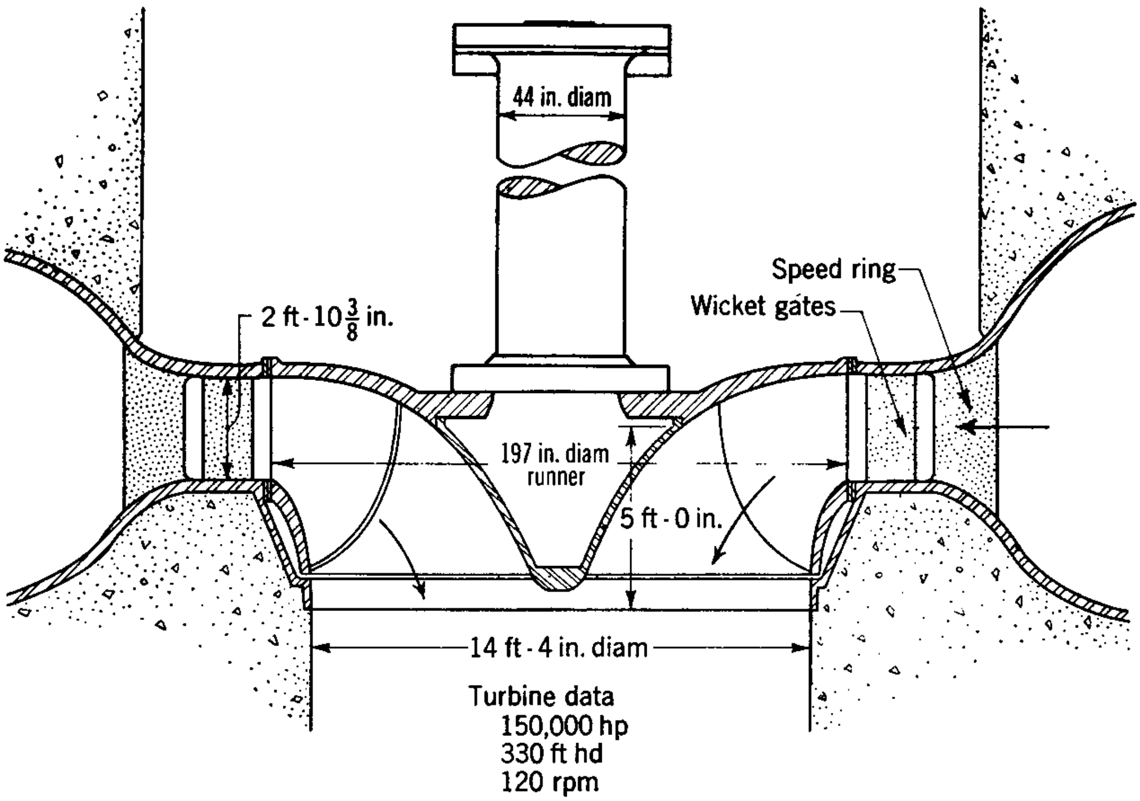


FIG. 8.17. Francis turbine for Grand Coulee, Columbia Basin Project. (*Newport News Shipbuilding and Dry Dock Co.*)

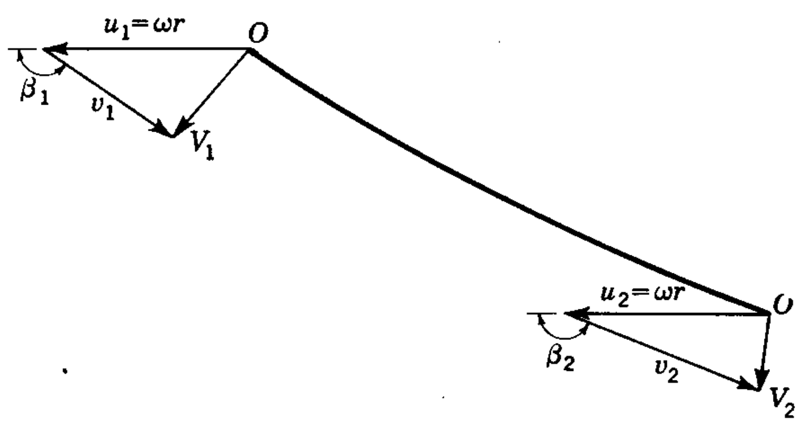


FIG. 8.18. Velocity diagram for entrance and exit of a propeller turbine, blade at fixed radial distance.

air stream expands in passing through the vanes with a reduction in its axial velocity.

Example 8.7: Assuming uniform axial velocity over section 2 of Fig. 8.9 and using the data of Example 8.2, determine the angle of the leading edge of the propeller at $r = 0.75$, 1.50, and 2.0 ft, for a propeller speed of 240 rpm.

At $r = 0.75$,

$$u = \tfrac{240}{60} \times 2\pi \times 0.75 = 18.88 \text{ ft/sec},$$
$$V_u = 30.1 \text{ ft/sec}$$

At $r = 1.50$,

$$u = \tfrac{240}{60} \times 2\pi \times 1.50 = 37.76 \text{ ft/sec},$$
$$V_u = 15.05 \text{ ft/sec}$$

At $r = 2.00$,

$$u = \tfrac{240}{60} \times 2\pi \times 2.00 = 50.2 \text{ ft/sec},$$
$$V_u = 11.3 \text{ ft/sec}$$

The discharge through the turbine is, from section 1,

$$Q = 2 \times 4\pi \times 2 \times 8 \cos 45° = 284.5 \text{ cfs}$$

Hence, the axial velocity at section 2 is

$$V_a = \frac{284.5}{\pi(2^2 - 0.75^2)} = 26.3 \text{ ft/sec}$$

Figure 8.19 shows the initial vane angle for the three positions.

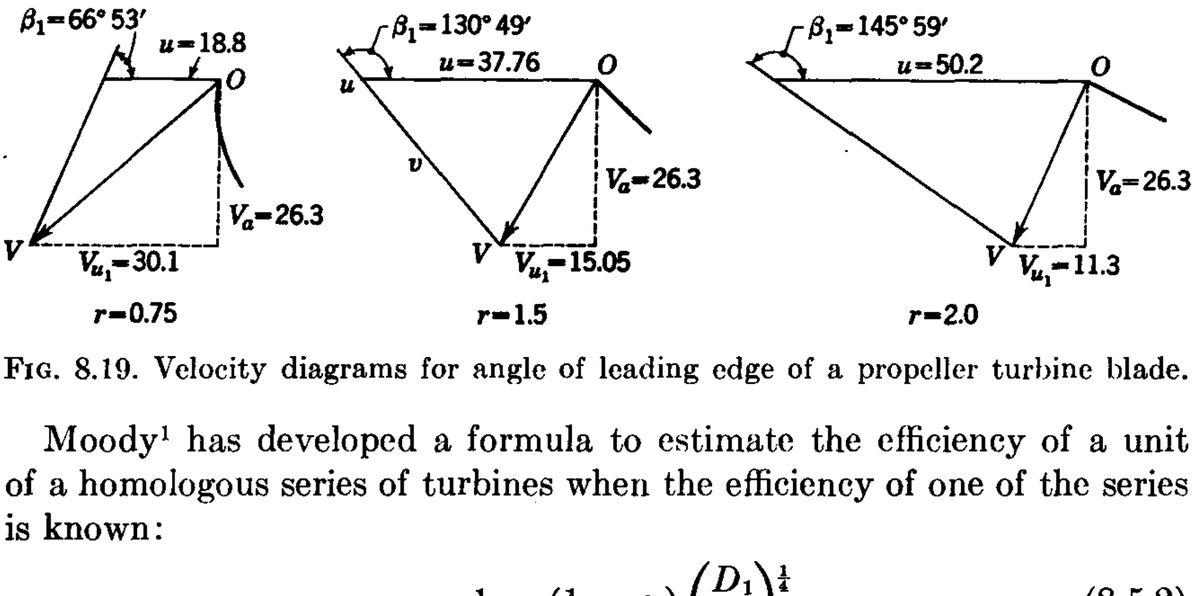


FIG. 8.19. Velocity diagrams for angle of leading edge of a propeller turbine blade.

Moody[1] has developed a formula to estimate the efficiency of a unit of a homologous series of turbines when the efficiency of one of the series is known:

$$e = 1 - (1 - e_1)\left(\frac{D_1}{D}\right)^{\frac{1}{4}} \tag{8.5.2}$$

in which e_1 and D_1 are usually efficiency and diameter of a model.

8.6. Pumps and Blowers. Pumps add energy to liquids and blowers to gases. The procedure for designing them is the same for both, except for those cases in which the density is appreciably increased. Turbo-pumps and -blowers are *radial-flow*, *axial-flow*, or a combination of the two, called *mixed-flow*. For high heads the radial (centrifugal) pump, frequently with two or more stages (two or more impellers in series), is best adapted. A double-suction general service centrifugal pump is

[1] Lewis F. Moody, The Propeller Type Turbine, *Trans. ASCE*, vol. 89, p. 628, 1926.

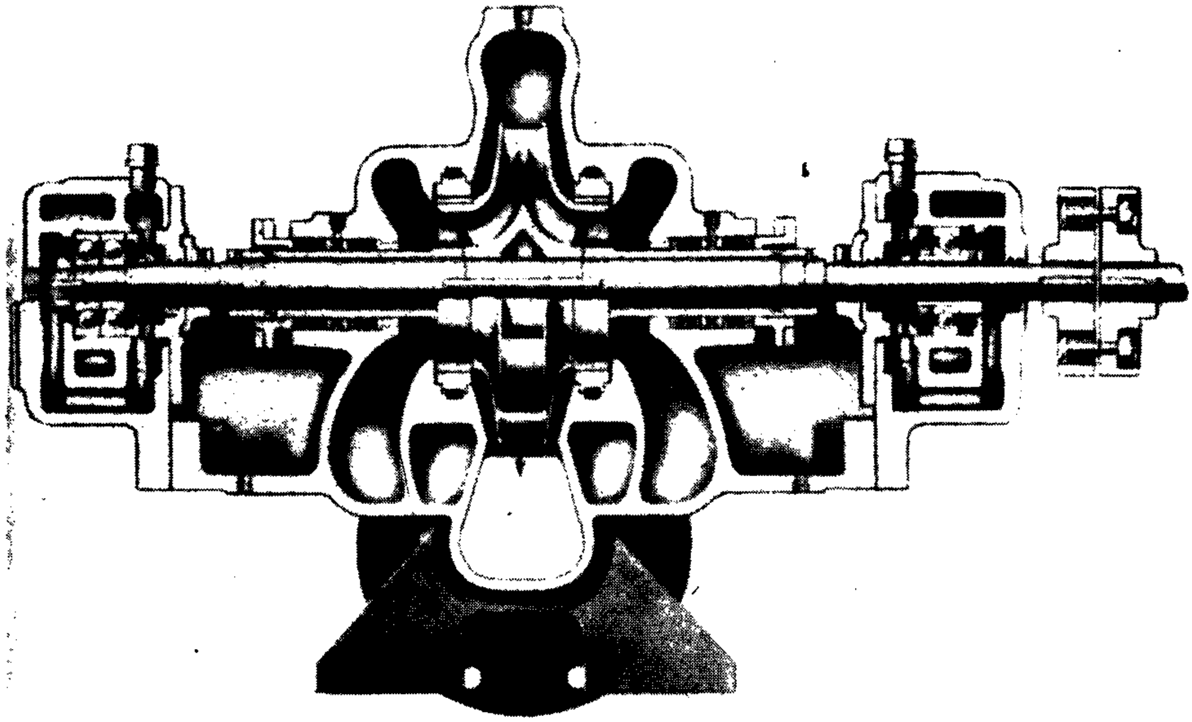


FIG. 8.20. Cross section of a single-stage double-suction centrifugal pump. (*Ingersoll-Rand Co.*)

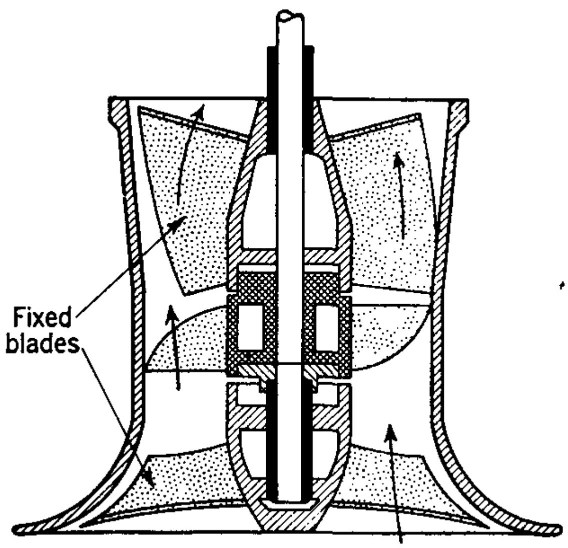


FIG. 8.21. Axial-flow pump. (*Ingersoll-Rand Co.*)

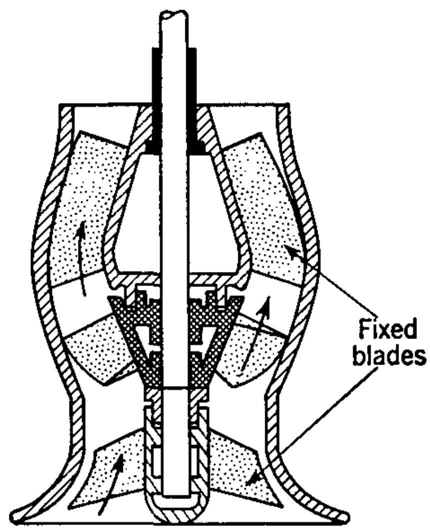


FIG. 8.22. Mixed-flow pump. (*Ingersoll-Rand Co.*)

shown in Fig. 8.20. For large flows under small heads the axial-flow pump or blower (Fig. 8.21) is best suited. The mixed-flow pump (Fig. 8.22) is used for medium head and medium discharge.

The equations developed in Sec. 8.2 apply just as well to pumps and blowers as to turbines. The usual centrifugal pump has a suction, or inlet, pipe leading to the center of the impeller, a radial outward-flow runner, as in Fig. 8.23, and a collection pipe or spiral casing that guides

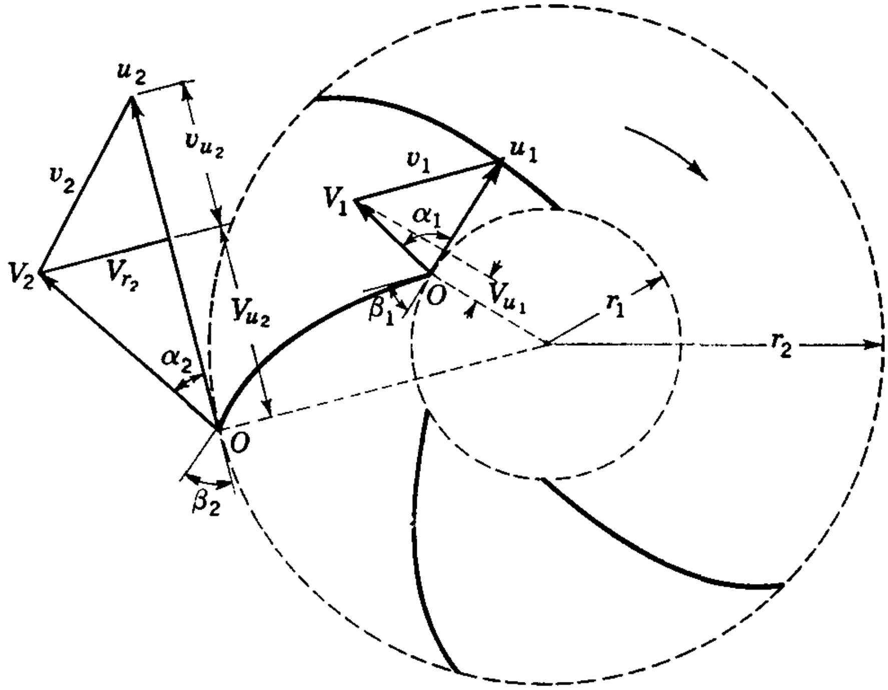


FIG. 8.23. Velocity relationships for flow through a centrifugal pump impeller.

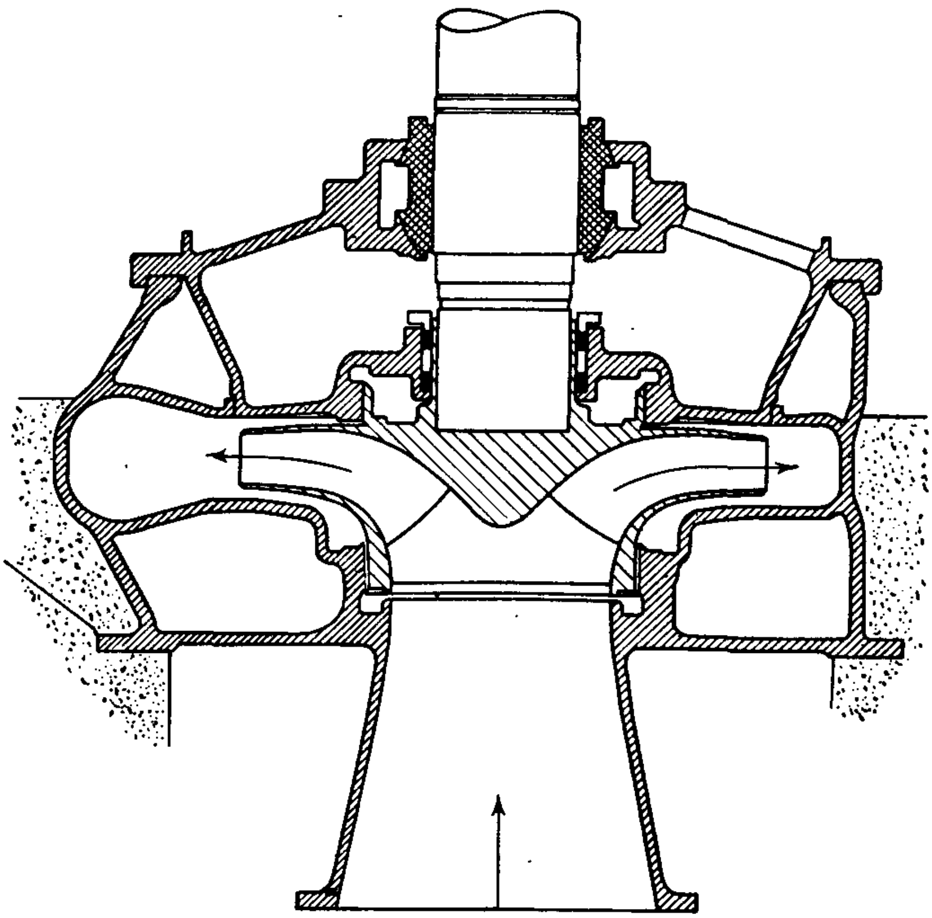


FIG. 8.24. Sectional elevation of Eagle Mountain and Hayfield pumps, Colorado River Aqueduct. (*Worthington Pump and Machinery Corp.*)

the fluid to the discharge pipe. Ordinarily, no fixed vanes are used, except for multistage units in which the flow is relatively small and the additional fluid friction is less than the additional gain in conversion of kinetic energy to pressure energy upon leaving the impeller.

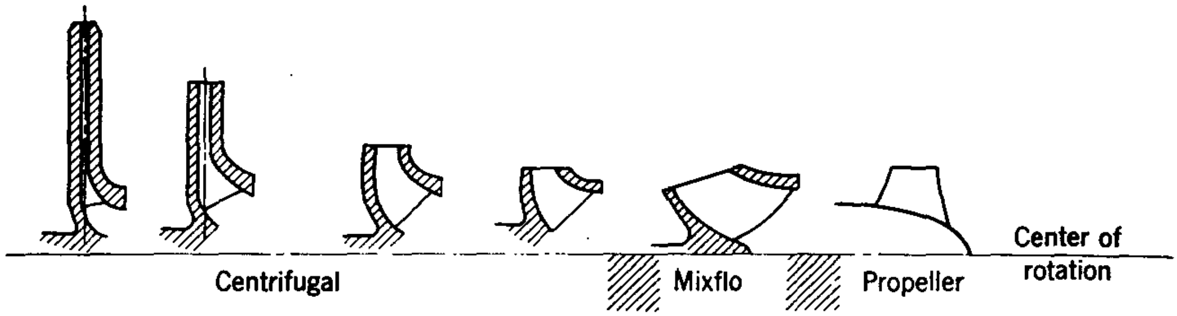


FIG. 8.25. Impeller types used in pumps and blowers. (*Worthington Pump and Machinery Corp.*)

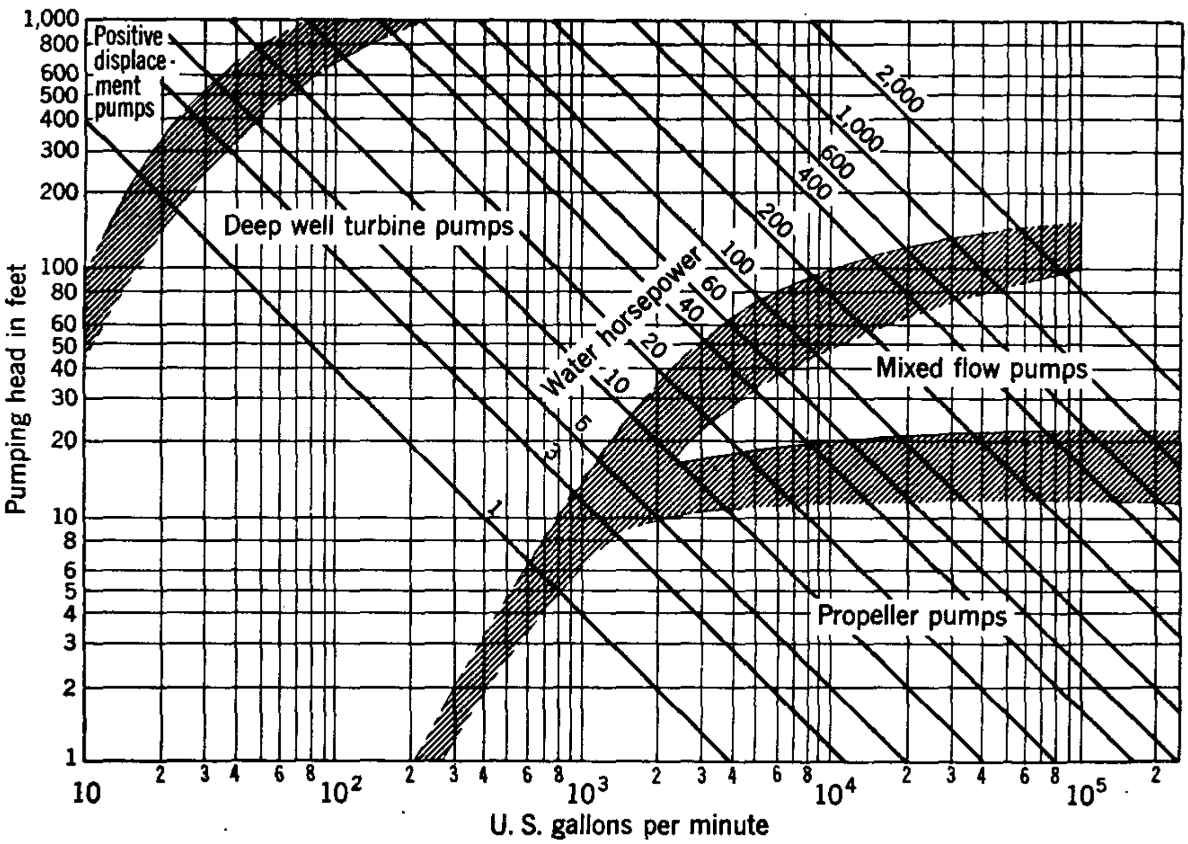


FIG. 8.26. Chart for selection of type of pump. (*Fairbanks, Morse & Co.*)

Figure 8.24 shows a sectional elevation of a large centrifugal pump. For lower heads and greater discharges (relatively) the impellers vary as shown in Fig. 8.25, from high head at left to low head at right with the axial-flow impeller. The specific speed increases from left to right. A chart for determining the types of pump for best efficiency is given in Fig. 8.26 for water.

Centrifugal and mixed-flow pumps are designed in the specific speed range 500 to 6500 and axial pumps from 5000 to 11,000; speed is expressed in revolutions per minute, discharge in gallons per minute, and head in feet.

Characteristic curves showing head, efficiency, and brake horsepower as a function of discharge for a typical centrifugal pump with backward-curved vanes are given in Fig. 8.27. Pumps are not as efficient as turbines, in general, owing to the inherently high losses that result from conversion of kinetic energy into flow energy.

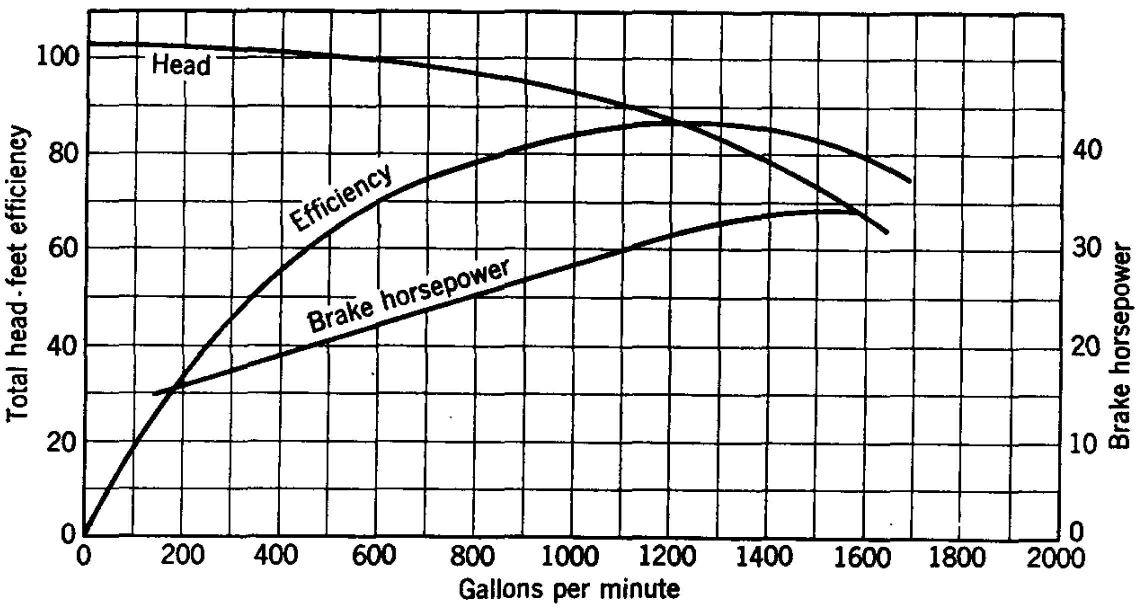


FIG. 8.27. Characteristic curves for typical centrifugal pump. 10-in. impeller, 1750 rpm. (*Ingersoll-Rand Co.*)

Theoretical Head-discharge Curve. A theoretical head-discharge curve may be obtained by use of Eq. (8.3.8) and the vector diagrams of Fig. 8.8. From the exit diagram of Fig. 8.8

$$V_2 \cos \alpha_2 = V_{u2} = u_2 - V_{r2} \cot \beta_2$$

From the discharge, if b_2 is the width of the impeller at r_2 and vane thickness is neglected,

$$Q = 2\pi r_2 b_2 V_{r2}$$

By eliminating V_{r2} and substituting these last two equations into Eq. (8.3.8),

$$H = \frac{u_2^2}{g} - \frac{u_2 Q \cot \beta_2}{2\pi r_2 b_2 g} \tag{8.6.1}$$

For a given pump and speed, H varies linearly with Q, as shown in Fig. 8.28. The usual design of centrifugal pump has $\beta_2 < 90°$, which gives decreasing head with increasing discharge. For blades radial at the exit, $\beta_2 = 90°$ and the theoretical head is independent of discharge. For blades curved forward, $\beta_2 > 90°$ and the head rises with discharge.

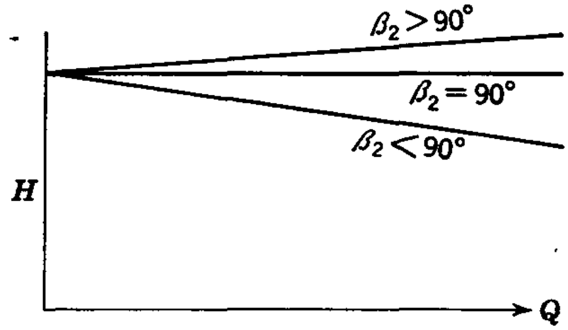


FIG. 8.28. Theoretical head-discharge curves.

Actual Head-discharge Curve. By subtracting head losses from the theoretical head-discharge curve, the actual head-discharge curve is obtained. The most important subtraction is not an actual loss, but a failure of the finite number of blades to impart the relative velocity with angle β_2 of the blades. Without perfect guidance (infinite number of blades) the fluid actually is discharged as if the blades had an angle β_2' which is less than β_2 (Fig. 8.29) for the same discharge. This inability of the blades to impart proper guidance reduces V_{u2} and hence decreases the actual head produced. This is called *circulatory flow* and is shown in Fig. 8.30. Fluid friction in flow through the fixed and moving passages causes losses that are proportional to the square of the discharge. They

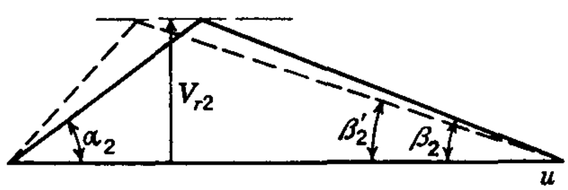


FIG. 8.29. Effect of circulatory flow.

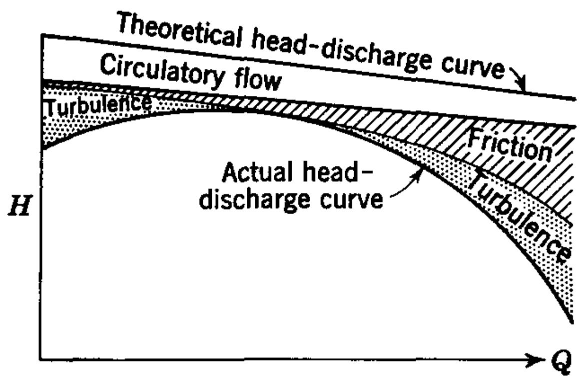


FIG. 8.30. Head-discharge relationships.

are shown in Fig. 8.30. The final head loss to consider is that of turbulence, the loss due to improper relative-velocity angle at the blade inlet. The pump can be designed for one discharge (at a given speed) at which the relative velocity is tangent to the blade at the inlet. This is the point of best efficiency, and shock or turbulence losses are negligible. For other discharges the loss varies about as the square of the discrepancy in approach angle, as shown in Fig. 8.30. The final lower line then represents the actual head-discharge curve. Shutoff head is usually about $u_2^2/2g$, or half of the theoretical shutoff head.

In addition to the head losses and reductions, pumps and blowers have torque losses due to bearing and packing friction and disk friction losses from the fluid between the moving impeller and housing. Internal leakage is also an important power loss, in that fluid which has passed through the impeller, with its energy increased, escapes through clearances and flows back to the suction side of the impeller.

Example 8.8: A centrifugal water pump has an impeller (Fig. 8.23) with $r_2 =$ 12 in., $r_1 = 4$ in., $\beta_1 = 20°$, $\beta_2 = 10°$. The impeller is 2 in. wide at $r = r_1$ and $\frac{3}{4}$ in. wide at $r = r_2$. For 1800 rpm, neglecting losses and vane thickness, determine (*a*) the discharge for shockless entrance when $\alpha_1 = 90°$; (*b*) α_2 and the

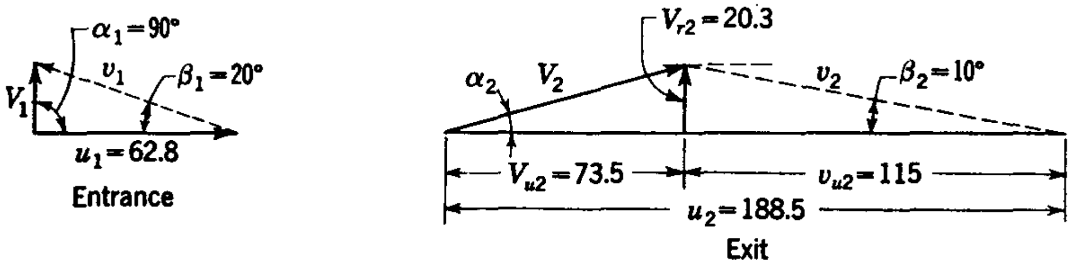


FIG. 8.31. Vector diagrams for entrance and exit of pump impeller.

theoretical head H; (*c*) the horsepower required; and (*d*) the pressure rise through the impeller.

a. The peripheral speeds are

$$u_1 = \frac{1800}{60} \times 2\pi \times \frac{1}{3} = 62.8 \text{ ft/sec} \qquad u_2 = 3u_1 = 188.5 \text{ ft/sec}$$

The vector diagrams are shown in Fig. 8.31. With u_1 and the angles α_1, β_1 known, the entrance diagram is determined, $V_1 = u_1 \tan 20° = 22.85$ ft/sec; hence

$$Q = 22.85 \times \pi \times \frac{2}{3} \times \frac{2}{12} = 7.97 \text{ cfs}$$

b. At the exit the radial velocity V_{r2} is

$$V_{r2} = \frac{7.97 \times 12}{2\pi \times 0.75} = 20.3 \text{ ft/sec}$$

By drawing u_2 (Fig. 8.31) and a parallel line, distance V_{r2} from it, the vector triangle is determined when β_2 is laid off. Thus

$$v_{u2} = 20.3 \cot 10° = 115 \qquad V_{u2} = 188.5 - 115 = 73.5$$

$$\alpha_2 = \tan^{-1} \frac{20.3}{73.5} = 15°26' \qquad V_2 = 20.3 \csc 15°26' = 76.2$$

From Eq. (8.3.8)

$$H = \frac{u_2 V_2 \cos \alpha_2}{g} = \frac{u_2 V_{u2}}{g} = \frac{188.5 \times 73.5}{32.2} = 430 \text{ ft}$$

c.

$$\text{hp} = \frac{Q\gamma H}{550} = \frac{7.97 \times 62.4 \times 430}{550} = 388$$

d. By applying Bernoulli's equation from the entrance to exit of the impeller, including the energy H added (elevation change across impeller may be neglected),

$$H + \frac{V_1^2}{2g} + \frac{p_1}{\gamma} = \frac{V_2^2}{2g} + \frac{p_2}{\gamma}$$

and

$$\frac{p_2 - p_1}{\gamma} = 430 + \frac{\overline{22.85^2}}{64.4} - \frac{\overline{76.2^2}}{64.4} = 348 \text{ ft}$$

or

$$p_2 - p_1 = 348 \times 0.433 = 151 \text{ psi}$$

8.7. Centrifugal Compressors. Centrifugal compressors operate according to the same principles as turbomachines for liquids. It is important for the fluid to enter the impeller without shock, i.e., with the relative velocity tangent to the blade. Work is done on the gas by rotation of the vanes, the moment-of-momentum equation relating torque to production of tangential velocity. At the impeller exit the high-velocity gas must have its kinetic energy converted in part to flow energy by suitable expanding flow passages. For adiabatic compression (no cooling of the gas) the actual work of compression w_a per unit mass is compared with the work w_{th} per unit mass to compress the gas to the same pressure isentropically. For cooled compressors the work w_{th} is based on the isothermal work of compression to the same pressure as the actual case. Hence

$$\eta = \frac{w_{th}}{w_a} \tag{8.7.1}$$

is the formula for efficiency of a compressor.

The efficiency formula for compression of a perfect gas is developed for the adiabatic compressor, assuming no internal leakage in the machine, i.e., no "short circuiting" of high-pressure fluid back to the low-pressure end of the impeller. Centrifugal compressors are usually multistage, with pressure ratios up to 3 across a single stage. From the moment-of-momentum equation (8.3.2) with inlet absolute velocity radial, $\alpha_1 = 90°$, the theoretical torque T_{th} is

$$T_{th} = \dot{m} V_{u2} u_2 \tag{8.7.2}$$

in which $\dot{m}$ is the mass per unit time being compressed, V_{u2} is the tangential component of the absolute velocity leaving the impeller, and r_2 is the impeller radius at exit. The actual applied torque T_a is greater than the theoretical torque by the torque losses due to bearing and packing friction plus disk friction; hence

$$T_{th} = T_a \eta_m \tag{8.7.3}$$

if η_m is the mechanical efficiency of the compressor.

In addition to the torque losses, there are irreversibilities due to flow through the machine. The actual work of compression through the adiabatic machine is obtained from the first law of thermodynamics, Eq. (3.7.1), neglecting elevation changes and replacing $u + p/\rho$ by h

$$-w_a = \frac{V_{2a}^2 - V_1^2}{2} + h_2 - h_1 \tag{8.7.4}$$

The isentropic work of compression may be obtained from Eq. (3.7.1) in differential form, neglecting the z-terms,

$$\begin{aligned} -dw_{th} &= V\,dV + d\frac{p}{\rho} + du \\ &= V\,dV + \frac{dp}{\rho} + pd\frac{1}{\rho} + du \end{aligned}$$

The last two terms are equal to $T\,ds$ from Eq. (3.7.5), which is zero for isentropic flow, so

$$-dw_{th} = V\,dV + \frac{dp}{\rho} \tag{8.7.5}$$

By integrating for p/ρ^k = constant between sections 1 and 2,

$$-w_{th} = \frac{V_{2th}^2 - V_1^2}{2} + \frac{k}{k-1}\left(\frac{p_2}{\rho_{2th}} - \frac{p_1}{\rho_1}\right) = \frac{V_{2th}^2 - V_1^2}{2} + c_pT_1\left[\left(\frac{p_2}{p_1}\right)^{(k-1)/k} - 1\right] \tag{8.7.6}$$

The efficiency may now be written as

$$\eta = \frac{-w_{th}}{-w_a} = \frac{[(V_{2th}^2 - V_1^2)/2] + c_pT_1[(p_2/p_1)^{(k-1)/k} - 1]}{[(V_{2a}^2 - V_1^2)/2] + c_p(T_{2a} - T_1)} \tag{8.7.7}$$

since $h = c_pT$. In terms of Eqs. (8.7.2) and (8.7.3)

$$-w_a = \frac{T_a\omega}{\dot{m}} = \frac{T_{th}\omega}{\eta_m\dot{m}} = \frac{V_{u2}r_2\omega}{\eta_m} = \frac{V_{u2}u_2}{\eta_m} \tag{8.7.8}$$

then

$$\eta = \frac{\eta_m}{V_{u2}u_2}\left\{c_pT_1\left[\left(\frac{p_2}{p_1}\right)^{(k-1)/k} - 1\right] + \frac{V_{2th}^2 - V_1^2}{2}\right\} \tag{8.7.9}$$

Use of this equation is made in the following example.

Example 8.9: An adiabatic turbocompressor has blades that are radial at the exit of its 6.0-in.-diameter impeller. It is compressing 1.0 lb_m/sec air at 14.0 psia, t = 60°F, to 42.0 psia. The entrance area is 0.07 ft², and the exit area 0.04 ft². $\eta = 0.75$; $\eta_m = 0.90$. Determine the rotational speed of the impeller and the actual temperature of air at the exit.

The density at the inlet is

$$\rho_1 = \frac{p_1}{RT_1} = \frac{14.0 \times 144}{53.3 \times 32.17(460 + 60)} = 0.00226 \text{ slug/ft}^3$$

and the velocity at the entrance is

$$V_1 = \frac{\dot{m}}{\rho_1 A_1} = \frac{1.0}{32.17} \times \frac{1}{0.00226} \times \frac{1}{0.07} = 196.4 \text{ ft/sec}$$

The theoretical density at the exit is

$$\rho_{2th} = \rho_1 \left(\frac{p_2}{p_1}\right)^{1/k} = 0.00226 \times 3^{1/1.4} = 0.00496 \text{ slug/ft}^3$$

and the theoretical velocity at the exit

$$V_{2th} = \frac{\dot{m}}{\rho_{2th} A_2} = \frac{1}{32.17} \times \frac{1}{0.00496} \times \frac{1}{0.04} = 156.5 \text{ ft/sec}$$

For radial vanes at the exit, $V_{u2} = u_2 = \omega r_2$. From Eq. (8.7.9)

$$u_2^2 = \frac{\eta_m}{\eta} \left\{ c_p T_1 \left[\left(\frac{p_2}{p_1}\right)^{(k-1)/k} - 1 \right] + \frac{V_{2th}^2 - V_1^2}{2} \right\}$$
$$= \frac{0.90}{0.75} \left\{ 0.24 \times 778 \times 32.17(460 + 60)[3^{(1.4-1)/1.4} - 1] + \frac{\overline{156.5}^2 - \overline{196.4}^2}{2} \right\}$$

and $u_2 = 1173$ ft/sec.
Then

$$\omega = \frac{u_2}{r_2} = \frac{1173}{\frac{3}{12}} = 4692 \text{ rad/sec}$$

and

$$N = \frac{\omega}{2\pi} 60 = \frac{4692}{2\pi} 60 = 44{,}800 \text{ rpm}$$

The theoretical work w_{th} is the term in the brackets in the expression for u_2^2. It is $-w_{th} = 1.147 \times 10^6$ ft-lb/slug. Then from Eq. (8.7.1)

$$w_a = \frac{w_{th}}{\eta} = -\frac{1.147 \times 10^6}{0.75} = -1.53 \times 10^6 \text{ ft-lb/slug}$$

Since the kinetic energy term is small, Eq. (8.7.4) may be solved for $h_2 - h_1$ and a trial solution effected,

$$h_2 - h_1 = c_p(T_{2a} - T_1) = 1.53 \times 10^6 + \frac{\overline{196.4}^2 - V_{2a}^2}{2}$$

As a first approximation, let $V_{2a} = V_{2th} = 156.5$, then

$$T_{2a} = 520 + \frac{1}{0.24 \times 778 \times 32.17} \left(1.53 \times 10^6 + \frac{\overline{196.4}^2 - \overline{156.5}^2}{2} \right) = 776°\text{R}$$

For this temperature the density at the exit is 0.00454 slug/ft³ and the velocity is 171 ft/sec. Insertion of this value in place of 156.5 reduces the temperature by about 1°; hence $T_{2a} = 775°$R.

8.8. Fluid Couplings and Fluid Torque Converters. The *fluid coupling* is a centrifugal pump and a turbine built together into a single housing to avoid losses by eliminating piping or channels that would otherwise be needed to connect them. There is no solid connection between the pump and the turbine (Fig. 8.32); the liquid, usually oil, transmits the torque by carrying moment of momentum across from pump to turbine. The coupling has two principal advantages: (*a*) smoothness of operation, since torsional vibrations are not transmitted through it; and (*b*) the full torque is not developed until the unit is up to speed, which is desirable for both electric motors and internal-combustion engines with heavy inertial loads.

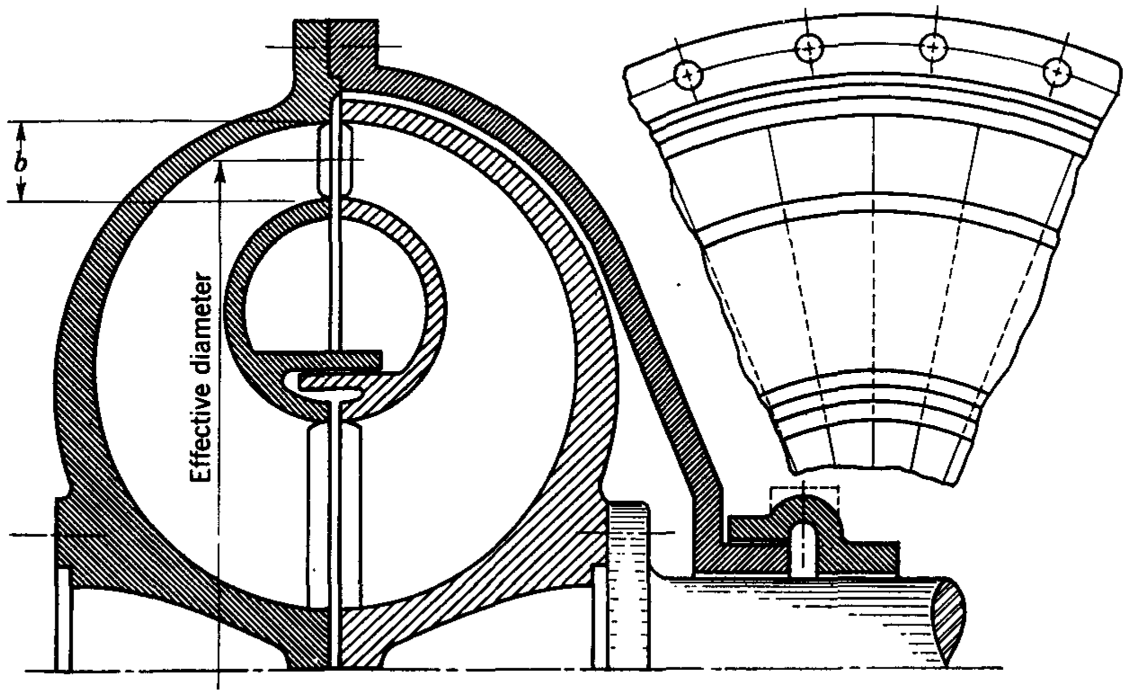


FIG. 8.32. Fluid coupling, Foettinger type. (*David Taylor Model Basin, U.S. Navy Dept.*)

Application of the moment-of-momentum equation [Eq. (8.3.1)] produces the relation between torque developed and change in moment of momentum for either the pump or the turbine. The torque must be the same for both when the operating conditions are steady, since there are no stationary members to absorb torque and no angular acceleration. When the coupling and a portion of the driven and driving shafts are considered as a free body, the angular acceleration is zero for steady conditions; hence, the summation of torques acting on the free body must be zero, and the torque in the driven shaft is exactly the same as the torque in the driving shaft.

In the operation of the coupling, consider that the driven member is first stationary and the driving member is rotating at its design speed. Liquid enters the pump near the shaft and is given moment of momentum as it traverses radially outward and flows into the turbine at its outer edge. The moment of momentum of the fluid is reduced to zero in the stationary turbine and the fluid exerts the torque supplied by the driving shaft. As the driven shaft comes up to speed, centrifugal action in the turbine creates a resistance to flow that reduces the amount of liquid pumped. No pumping takes place when both shafts rotate at the same speed and, hence, no torque is transmitted. Since the unit is symmetrical, the driven shaft when turning at greater speed than the driving shaft transmits a torque to the driving shaft that in effect provides a braking action. For normal steady operation there must always be a difference in speed, or *slip*, if torque is to be transmitted. The efficiency e is work out divided by work in, or

$$e = \frac{T\omega_t}{T\omega_p} = \frac{\omega_t}{\omega_p} = 1 - s \qquad (8.8.1)$$

in which T is the torque, ω_p the driving-shaft speed, ω_t the driven-shaft speed, and s the slip, or $(\omega_p - \omega_t)/\omega_p$.

The larger the diameter of coupling, the less the slip required to transmit a given torque and, hence, the greater the efficiency of the coupling. Since efficiency can thus be increased by simply increasing diameter, no effort is made to curve the vanes or round their leading edges to decrease turbulence or fluid shock upon entering the vanes. Efficiencies are above 95 per cent.

In some applications in which variable torque transmission is desired, the amount of oil in the coupling is varied.

The *fluid torque converter* (Fig. 8.33) is in many ways similar to a fluid coupling. It has a *fixed* vane system, however, that transmits torque to the earth, and it always operates completely filled with liquid.

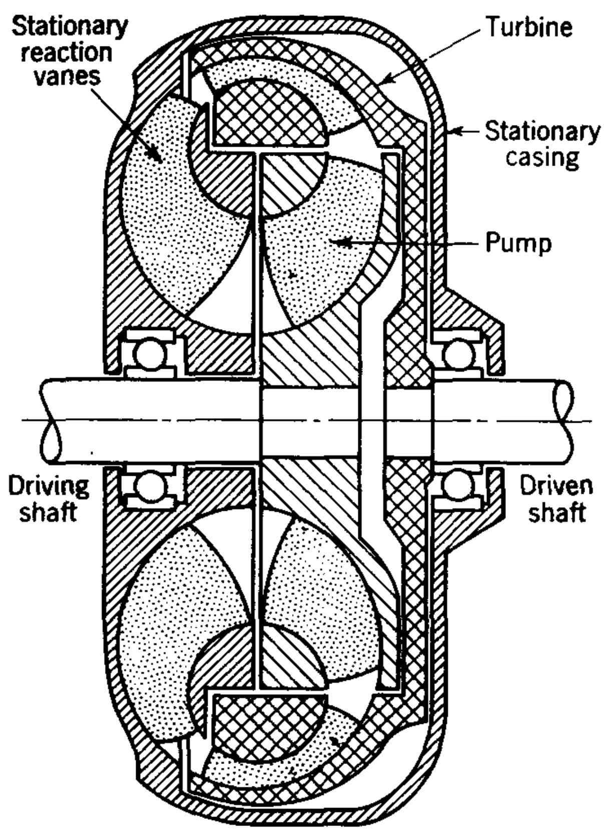


Fig. 8.33. Torque converter

For steady conditions of operation there is no angular acceleration, and the summation of all torques acting on the unit must be zero. Since there is a stationary vane system that requires an outside torque T_f to hold it fixed, the torques in the driving and driven shafts are no longer equal.

For example, if the stationary vanes are curved so that the liquid acts to rotate them in the sense opposite to that of the driving shaft, there is a torque multiplication with the torque in the driven shaft equal to the sum of torque in driving shaft and torque on the fixed vanes, and with a corresponding decrease in speed of driven shaft.

By the proper designing of the fixed vane system there may be either a torque multiplication or a torque division. Since no work is done on the fixed vanes, the work output to the driven shaft must equal the work

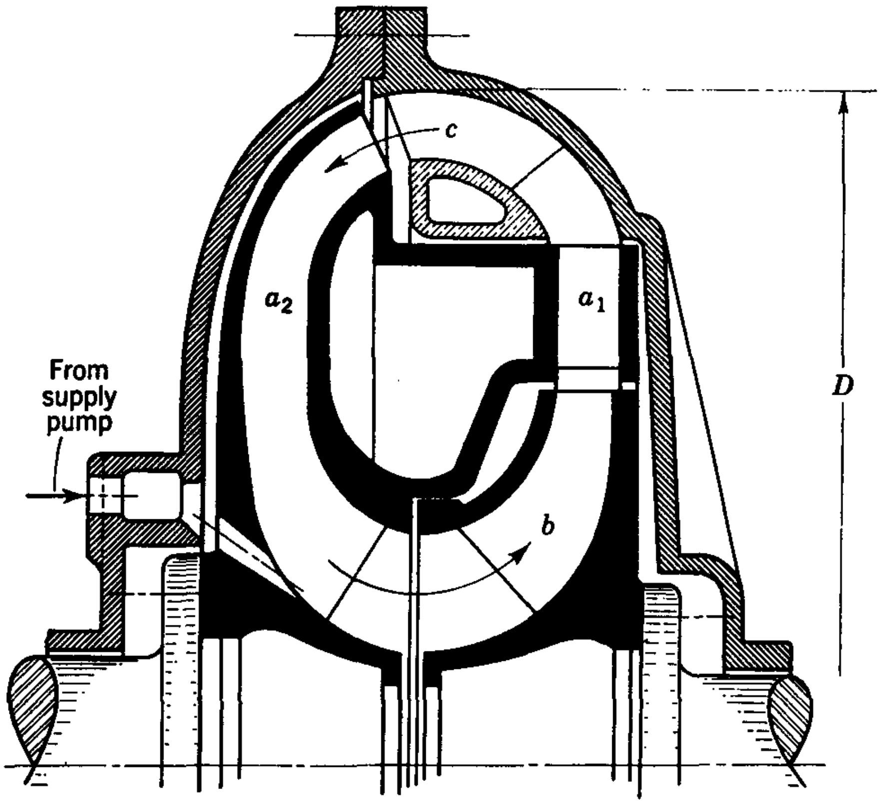


FIG. 8.34. Torque converter with two-stage turbine. (a_1) First-stage turbine; (a_2) second-stage turbine; (b) pump; (c) stationary guide vanes; (d) maximum diameter of circuit. (*David Taylor Model Basin, U.S. Navy Dept.*)

input to the driving shaft less the losses. A single-stage torque converter is shown in Fig. 8.33. Maximum efficiency is less than for a fluid coupling, usually between 80 and 90 per cent.

As in the case of a reaction turbine, where fixed guide vanes create moment of momentum that is reduced by moving vanes to create torque on a rotating shaft, the fixed vanes of a torque converter are curved to give the liquid moment of momentum. The pump adds to this moment of momentum, and the turbine, by proper design and a speed much less than the pump, takes the moment of momentum out of the liquid and thus has a large torque exerted on it. When large torque multiplication is desired, the torque converter usually is designed with two or more sets of turbine vanes with fixed vanes or pump vanes between them, as in Fig. 8.34.

By use of a *freewheeling* arrangement on the fixed guide vanes that permit them to rotate in one direction when torque is applied in that direction, the torque converter becomes a fluid coupling. When there is a large difference in speed of pump and turbine, torque conversion is required and the reaction (fixed) vanes have a torque exerted on them that holds them stationary. As the pump and turbine speeds become nearly the same, the turbine moves so that the fluid discharged from it causes the reaction vanes to freewheel, or take no part in the process, resulting in a fluid coupling with better efficiency than the torque converter.

8.9. Cavitation. When a liquid flows into a region where its pressure is reduced to vapor pressure, it boils and vapor pockets develop in the liquid. The vapor bubbles are carried along with the liquid until a region of higher pressure is reached, where they suddenly collapse. This process is called *cavitation*. If the vapor bubbles are near to (or in contact with) a solid boundary when they collapse, the forces exerted by the liquid rushing into the cavities create very high localized pressures that cause pitting of the solid surface. The phenomenon is accompanied by noise and vibrations that have been described as similar to gravel going through a centrifugal pump.

In a flowing liquid, the *cavitation parameter* σ is useful in characterizing the susceptibility of the system to cavitate. It is defined by

$$\sigma = \frac{p - p_v}{\rho V^2/2} \tag{8.9.1}$$

in which p is the absolute pressure at the point of interest, p_v is the vapor pressure of the liquid, ρ is the density of the liquid, and V is the undisturbed, or reference, velocity. The cavitation parameter is a form of pressure coefficient. In two geometrically similar systems, they would be equally likely to cavitate or would have the same degree of cavitation for the same value of σ. When $\sigma = 0$, the pressure is reduced to vapor pressure and boiling should occur.

Tests made on chemically pure liquids show that they will sustain high tensile stresses, of the order of thousands of pounds per square inch, which is in contradiction to the concept of cavities forming when pressure is reduced to vapor pressure. Since there is generally spontaneous boiling when vapor pressure is reached with commercial or technical liquids, it is generally accepted that nuclei must be present around which the vapor bubbles form and grow. The nature of the nuclei is not thoroughly understood, but they may be microscopic dust particles or other contaminants, which are widely dispersed through technical liquids.

Cavitation bubbles may form on nuclei, grow, then move into an area of higher pressure and collapse, all in a few thousandths of a second in flow within a turbomachine. In aerated water the bubbles have been photographed as they move through several oscillations, but this phenomenon does not seem to occur in nonaerated liquids. Surface tension of the vapor bubbles appears to be an important property accounting for the high-pressure pulses resulting from collapse of a vapor bubble. Recent experiments indicate pressures of the order of 200,000 psi based on the analysis of strain waves in a photoelastic specimen exposed to cavitation.[1] Pressures of this magnitude appear to be reasonable, in line with the observed mechanical damage caused by cavitation.

TABLE 8.1. WEIGHT LOSS IN MATERIALS USED IN HYDRAULIC MACHINES

Alloy	*Weight loss after 2 hr, mg*
Rolled stellite†	0.6
Welded aluminum bronze‡	3.2
Cast aluminum bronze§	5.8
Welded stainless steel (two layers, 17 Cr, 7% Ni)	6.0
Hot-rolled stainless steel (26 Cr, 13% Ni)	8.0
Tempered, rolled stainless steel (12% Cr)	9.0
Cast stainless steel (18 Cr, 8% Ni)	13.0
Cast stainless steel (12% Cr)	20.0
Cast manganese bronze	80.0
Welded mild steel	97.0
Plate steel	98.0
Cast steel	105.0
Aluminum	124.0
Brass	156.0
Cast iron	224.0

† This material is not suitable for ordinary use, in spite of its high resistance, because of its high cost and difficulty in machining.

‡ Ampco-Trode 200: 83 Cu, 10.3 Al, 5.8% Fe.

§ Ampco 20: 83.1 Cu, 12.4 Al, 4.1% Fe.

The formation and collapse of great numbers of bubbles on a surface subject that surface to intense local stressing, which appears to damage the surface by fatigue. Some ductile materials withstand battering for a period, called the incubation period, before damage is noticeable, while brittle materials may lose weight immediately. There may be certain electrochemical, corrosive, and thermal effects which hasten the deterioration of exposed surfaces. Rheingans[2] has collected a series

[1] G. W. Sutton, A Photoelastic Study of Strain Waves Caused by Cavitation, *J. Appl. Mech.*, vol. 24, part 3, pp. 340–348, 1957.

[2] W. J. Rheingans, Selecting Materials to Avoid Cavitation Damage, *Materials in Design Engineering*, 1958, pp. 102–106.

of measurements made by magnetostriction-oscillator tests, showing weight losses of various metals used in hydraulic machines.

Protection against cavitation should start with the hydraulic design of the system in order to avoid the low pressures if practicable. Otherwise, the use of special cavitation-resistant materials or coatings may be used. Small amounts of air entrained into water systems have markedly reduced cavitation damage, and recent studies indicate that cathodic protection is helpful.

FIG. 8.35. Cavitation damage to a Francis runner. (*Ingersoll-Rand Co.*)

The formation of vapor cavities decreases the useful channel space for liquid and thus decreases the efficiency of a fluid machine. Cavitation causes three undesirable conditions: lowered efficiency, damage to flow passages, and noise and vibrations. Curved vanes are particularly susceptible to cavitation on their convex sides and may have localized areas where cavitation causes pitting or failure, as in Fig. 8.35. All turbomachinery, ship propellers, and many hydraulic structures are subject to cavitation; hence, attention must be given to it in the designing of all of these.

A *cavitation index* σ' is useful in the proper selection of turbomachinery, and in its location with respect to suction or tail-water elevation. The minimum pressure in a pump or turbine generally occurs along the convex side of vanes near the suction side of the impeller. In Fig. 8.36, if e be the point of minimum pressure, Bernoulli's equation applied

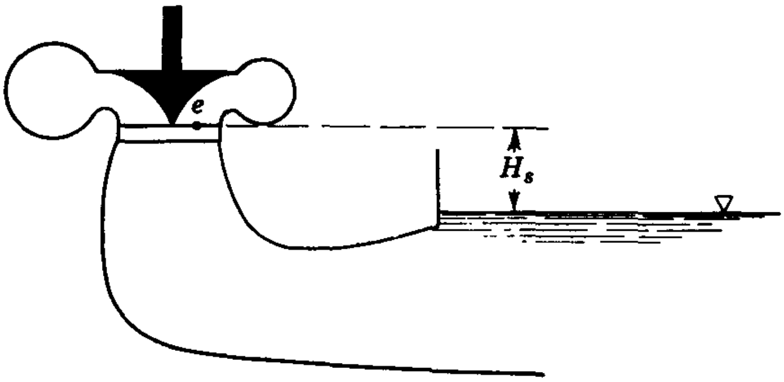


FIG. 8.36. Turbine or pump setting.

between e and the downstream liquid surface, neglecting losses between the two points, may be written

$$\frac{p_e}{\gamma} + \frac{V_e^2}{2g} + H_s = \frac{p_a}{\gamma} + 0 + 0$$

in which p_a is atmospheric pressure and p_e the absolute pressure. For cavitation to occur at e, the pressure must be equal to or less than p_v, the vapor pressure. If $p_e = p_v$,

$$\sigma' = \frac{V_e^2}{2gH} = \frac{p_a - p_v - \gamma H_s}{\gamma H} \tag{8.9.2}$$

is the ratio of energy available at e to total energy H, since the only energy is kinetic energy. The ratio σ' is a cavitation index or number. The critical value σ_c may be determined by a test on a model of the homologous series. For cavitationless performance, the suction setting H_s for an impeller installation must be so fixed that the resulting value of σ' is greater than that of σ_c.

Example 8.10: Tests on a pump model indicate a $\sigma_c = 0.10$. A homologous unit is to be installed at a location where $p_a = 13$ psi and $p_v = 0.50$ psi and is to pump water against a head of 80 ft. What is the maximum permissible suction head?

By solving Eq. (8.9.2) for H_s, and after substituting the values of σ_c, H, p_a, and p_v

$$H_s = \frac{p_a - p_v}{\gamma} - \sigma' H = \frac{13 - 0.50}{0.433} - 0.10 \times 80 = 20.8 \text{ ft}$$

The less the value of H_s, the greater the value of the plant σ', and the greater the assurance against cavitation.

PROBLEMS

8.1. By use of Eqs. (8.1.1) and (8.1.3) together with $P = \gamma QH$ for power, develop the homologous relationship for P in terms of speed and diameter.

8.2. A centrifugal pump is driven by an induction motor that reduces in speed as the pump load increases. A test determines several sets of values of N, Q, H for the pump. How is a characteristic curve of the pump for a constant speed determined from these data?

8.3. What is the specific speed of the pump of Example 8.1 at point of best efficiency?

8.4. Plot the dimensionless characteristic curve of the pump of Example 8.1. On this same curve plot several points from the characteristics of the new (52-in.) pump. Why are they not exactly on the same curve?

8.5. Determine the size and synchronous speed of a pump homologous to the 72-in. pump of Example 8.1 that will produce 120 cfs at 300 ft head at its point of best efficiency.

8.6. Develop the characteristic curve for a homologous pump of the series of Example 8.1 for 18-in. discharge and 1800 rpm.

8.7. A pump with an 8-in.-diameter impeller discharges 2000 gpm at 1140 rpm and 30 ft head at its point of best efficiency. What is its specific speed?

8.8. A hydroelectric site has a head of 300 ft and an average discharge of 400 cfs. For a generator speed of 200 rpm, what specific speed turbine is needed? Assume an efficiency of 92 per cent.

8.9. A model turbine, $N_s = 36$, with a 14-in.-diameter impeller develops 27 hp at a head of 44 ft and an efficiency of 86 per cent. What are the discharge and speed of the model?

8.10. What size and synchronous speed of homologous unit of Prob. 8.9 would be needed to discharge 600 cfs at 260 ft of head?

8.11. 800 cfs water flowing through the fixed vanes of a turbine has a tangential component of 6 ft/sec at a radius of 4 ft. The impeller, turning at 180 rpm, discharges in an axial direction. What torque is exerted on the impeller?

8.12. In Prob. 8.11, neglecting losses, what is the head on the turbine?

8.13. A generator with speed $N = 240$ rpm is to be used with a turbine at a site where $H = 400$ ft and $Q = 300$ cfs. Neglecting losses, what tangential component must be given to the water at $r = 3$ ft by the fixed vanes? What torque is exerted on the impeller? How much horsepower is produced?

8.14. A site for a Pelton wheel has a steady flow of 2 cfs with a nozzle velocity of 240 ft/sec. With a blade angle of 174°, and $C_v = 0.98$, for 60 cycle power, determine (*a*) the diameter of wheel, (*b*) the speed, (*c*) the horsepower, (*d*) the energy remaining in the water. Neglect losses.

8.15. An impulse wheel is to be used to develop 50 cycle/sec power at a site where $H = 400$ ft and $Q = 2.7$ cfs. Determine the diameter of the wheel and its speed. $C_v = 0.97$; $e = 82$ per cent.

8.16. At what angle should the wicket gates of a turbine be set to extract 12,000 hp from a flow of 900 cfs? The diameter of the opening just inside the wicket gates is 12 ft, and the height is 3 ft. The turbine runs at 200 rpm, and flow leaves the runner in an axial direction.

8.17. For a given setting of wicket gates how does the moment of momentum vary with the discharge?

8.18. Assuming constant axial velocity just above the runner of the propeller turbine of Prob. 8.16, calculate the tangential velocity components if the hub radius is 1 ft and the outer radius is 3 ft.

8.19. Determine the vane angles β_1, β_2 for entrance and exit from the propeller turbine of Prob. 8.18 so that no angular momentum remains in the flow. (Compute the angles for inner radius, outer radius, and mid-point.)

8.20. Neglecting losses, what is the head on the turbine of Prob. 8.16?

8.21. The hydraulic efficiency of a turbine is 95 per cent, and its theoretical head is 290 ft. What is the actual head required?

8.22. A turbine model test with 10-in.-diameter impeller showed an efficiency of 90 per cent. What efficiency could be expected from a 48-in.-diameter impeller?

8.23. A turbine draft tube (Fig. 8.37) expands from 6 to 18 ft diameter. At section 1 the velocity is 30 ft/sec for vapor pressure of 1 ft and barometric pressure of 32 ft of water. Determine h_s for incipient cavitation (pressure equal to vapor pressure at section 1).

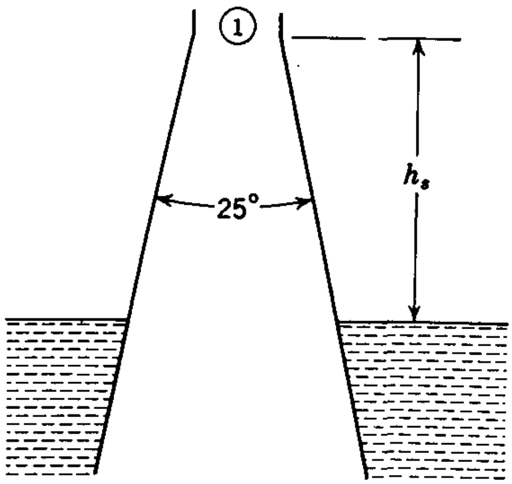


FIG. 8.37

8.24. Construct a theoretical head-discharge curve for the following specifications of a centrifugal pump: $r_1 = 2$ in., $r_2 = 4$ in., $b_1 = 1$ in., $b_2 = \frac{3}{4}$ in., 1200 rpm, and $\beta_2 = 30°$.

8.25. A centrifugal water pump (Fig. 8.23) has an impeller $r_1 = 2.5$ in., $b_1 = 1\frac{3}{8}$ in., $r_2 = 4.5$ in., $b_2 = \frac{3}{4}$ in., $\beta_1 = 30°$, $\beta_2 = 45°$ (b_1, b_2 are impeller width at r_1 and r_2, respectively). Neglect thickness of vanes. For 1800 rpm, calculate (*a*) the design discharge for no prerotation of entering fluid, (*b*) α_2 and the theoretical head at point of best efficiency, and (*c*) for hydraulic efficiency of 85 per cent and over-all efficiency of 78 per cent, the actual head produced, losses in foot-pounds per pound, and brake horsepower.

8.26. A centrifugal pump has an impeller with dimensions $r_1 = 3$ in., $r_2 = 6$ in., $b_1 = 2.0$ in., $b_2 = 1.25$ in., $\beta_1 = \beta_2 = 30°$. For a discharge of 2 cfs and shockless entry to vanes compute (*a*) the speed, (*b*) the head, (*c*) the torque, (*d*) the horsepower, and (*e*) the pressure rise across impeller. Neglect losses. $\alpha_1 = 90°$.

8.27. A centrifugal water pump with impeller dimensions $r_1 = 2$ in., $r_2 = 5$ in., $b_1 = 3.0$ in., $b_2 = 1$ in., $\beta_2 = 60°$ is to pump 5 cfs at 64 ft head. Determine (*a*) β_1, (*b*) the speed, (*c*) the horsepower, and (*d*) the pressure rise across the impeller. Neglect losses, and assume no shock at the entrance. $\alpha_1 = 90°$.

8.28. Select values of r_1, r_2, β_1, β_2, b_1, and b_2 of a centrifugal impeller to take 1 cfs water from a 4-in.-diameter suction line and increase its energy by 40 ft-lb/lb. $N = 1200$ rpm; $\alpha_1 = 90°$. Neglect losses.

8.29. A pump has blade angles $\beta_1 = \beta_2$; $b_1 = 2b_2 = 1.0$ in.; $r_1 = r_2/3 = 2$ in. For a theoretical head of 95.2 ft at a discharge at best efficiency of 1.052 cfs, determine the blade angles and speed of the pump. Neglect thickness of vanes and assume perfect guidance. (HINT: Write down every relation you know connecting β_1, β_2, b_1, b_2, r_1, r_2, u_1, u_2, H_{th}, Q, V_{r2}, V_{u2}, V_1, ω, and N from the two velocity vector diagrams, and by substitution reduce to one unknown.)

8.30. A mercury-water differential manometer, $R' = 26$ in., is connected from the 4-in.-diameter suction pipe to the 3-in.-diameter discharge pipe of a pump. The center line of the suction pipe is 1 ft below the discharge pipe. For $Q = 900$ gpm water, calculate the head developed by the pump.

8.31. The impeller for a blower (Fig. 8.38) is 18 in. wide. It has straight blades and turns at 1200 rpm. For 10,000 ft³/min air, $\gamma = 0.08$ lb/ft³, calculate (*a*) entrance and exit blade angles ($\alpha_1 = 90°$), (*b*) the head produced in inches of water, and (*c*) the theoretical horsepower required.

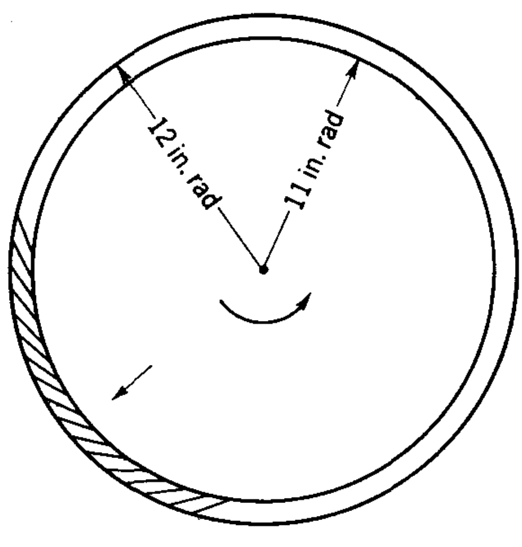


FIG. 8.38

8.32. An air blower is to be designed to produce 4-in. water pressure when operating at 3600 rpm. $\gamma = 0.07$ lb/ft³; $r_2 = 1.1r_1$; $\beta_2 = \beta_1$; width of impeller is 4 in.; $\alpha_1 = 90°$. Find r_1.

8.33. In Prob. 8.32 when $\beta_1 = 30°$, calculate the discharge in cubic feet per minute.

8.34. Develop the equation for efficiency of a cooled compressor,

$$\eta = \frac{\eta_m}{V_{u2}u_2}\left(\frac{V_{2th}^2 - V_1^2}{2} + \frac{p_1}{\rho_1}\ln\frac{p_2}{p_1}\right)$$

8.35. Find the rotational speed in Example 8.9 for a cooled compressor, using results of Prob. 8.34, with the actual air temperature at exit 60°F.

8.36. A fluid coupling transmits 60 hp when the driving shaft turns 1200 rpm, and the driven shaft speed is 1160 rpm. What is the torque in each shaft, and how efficient is the coupling?

8.37. What is the cavitation parameter at a point in flowing water where $t = 68°\text{F}$, $p = 2$ psia, and the velocity is 40 ft/sec?

8.38. A turbine with $\sigma_c = 0.08$ is to be installed at a site where $H = 200$ ft and a water barometer stands at 27.6 ft. What is the maximum permissible impeller setting above tail water?

8.39. Two units are homologous when they are geometrically similar and have

(*a*) similar streamlines
(*b*) the same Reynolds number
(*c*) the same efficiency
(*d*) the same Froude number
(*e*) none of these answers

8.40. The following two relationships are necessary for homologous units:

(*a*) H/ND^3 = constant; Q/N^2D^2 = constant
(*b*) $Q/D^2\sqrt{H}$ = constant; H/N^3D = constant
(*c*) P/QH = constant; H/N^2D^2 = constant
(*d*) $N\sqrt{Q}/H^{\frac{3}{2}}$ = constant; $N\sqrt{P}/H^{\frac{3}{4}}$ = constant
(*e*) none of these answers

8.41. The specific speed of a pump is defined as the speed of a unit

(*a*) of unit size with unit discharge at unit head
(*b*) of such a size that it requires unit power for unit head
(*c*) of such a size that it delivers unit discharge at unit head
(*d*) of such a size that it delivers unit discharge at unit power
(*e*) none of these answers

8.42. An impulse turbine

(*a*) always operates submerged
(*b*) makes use of a draft tube
(*c*) is most suited for low-head installations
(*d*) converts pressure head into velocity head throughout the vanes
(*e*) operates by initial complete conversion to kinetic energy

8.43. A Pelton wheel 24 in. in diameter turns at 400 rpm. Select from the following the head, in feet, best suited for this wheel:

(*a*) 7 (*b*) 30 (*c*) 120 (*d*) 170 (*e*) 480

8.44. A shaft transmits 200 hp at 600 rpm. The torque in pound-feet is

(*a*) 19.2 (*b*) 183 (*c*) 1750 (*d*) 3500 (*e*) none of these answers

8.45. What torque is required to give 100 cfs water a moment of momentum so that it has a tangential velocity of 10 ft/sec at a distance of 6 ft from the axis?

(*a*) 116 lb-ft (*b*) 1935 lb-ft (*c*) 6000 lb-ft (*d*) 11,610 lb-ft
(*e*) none of these answers

8.46. The moment of momentum of water is reduced by 20,000 lb-ft in flowing through vanes on a shaft turning 400 rpm. The horsepower developed on the shaft is

(*a*) 242 (*b*) 1522 (*c*) 14,500 (*d*) not determinable; insufficient data (*e*) none of these answers

8.47. Liquid moving with constant angular momentum has a tangential velocity of 4.0 ft/sec 10 ft from the axis of rotation. The tangential velocity 5 ft from the axis is, in feet per second,

(*a*) 2 (*b*) 4 (*c*) 8 (*d*) 16 (*e*) none of these answers

8.48. A reaction-type turbine discharges 1200 cfs under a head of 26 ft and with an over-all efficiency of 91 per cent. The horsepower developed is

(*a*) 3890 (*b*) 3540 (*c*) 3220 (*d*) 100 (*e*) none of these answers

8.49. The head developed by a pump with hydraulic efficiency of 80 per cent, for $u_2 = 100$ ft/sec, $V_2 = 60$ ft/sec, $\alpha_2 = 45°$, $\alpha_1 = 90°$, is

(*a*) 52.6 (*b*) 105.3 (*c*) 132 (*d*) 165 (*e*) none of these answers

8.50. Select the correct relationship for pump vector diagrams

(*a*) $\alpha_1 = 90°$; $v_1 = u_1 \cot \beta_1$
(*b*) $V_{u2} = u_2 - V_{r2} \cot \beta_2$
(*c*) $\omega_2 = r_2/u_2$
(*d*) $r_1 V_1 = r_2 V_{r2}$
(*e*) none of these answers

8.51. The cavitation parameter is defined by

(*a*) $\dfrac{p_v - p}{\rho V^2/2}$ (*b*) $\dfrac{p_{atm} - p_v}{\rho V^2/2}$ (*c*) $\dfrac{p - p_v}{\gamma V^2/2}$ (*d*) $\dfrac{p - p_v}{\rho V^2/2}$
(*e*) none of these answers

8.52. Cavitation is caused by

(*a*) high velocity (*b*) low barometric pressure (*c*) high pressure
(*d*) low pressure (*e*) low velocity

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9

FLUID MEASUREMENT

Fluid measurements include the determination of pressure, velocity, discharge, shock waves, density gradients, turbulence, and viscosity. There are many ways these measurements may be taken, e.g., direct, indirect, gravimetric, volumetric, electronic, electromagnetic, and optical. Direct measurements for discharge consist in the determination of the volume or weight of fluid that passes a section in a given time interval. Indirect methods of discharge measurement require the determination of head, difference in pressure, or velocity at several points in a cross section, and with these the computing of discharge. The most precise methods are the gravimetric or volumetric determinations, in which the weight or volume is measured by weigh scales or by a calibrated tank for a time interval that is measured by a stop watch.

Pressure and velocity measurement is first undertaken in this chapter, followed by optical-flow measurement, positive-displacement meters, rate meters, river-flow measurement, and electromagnetic flow devices, and concluding with turbulence and viscosity measurement.

9.1. Pressure Measurement. The measurement of pressure is required in many devices that determine the velocity of a fluid stream or its rate of flow, because of the relationship between velocity and pressure given by the Bernoulli equation. The static pressure of a fluid in motion is its pressure when the velocity is undisturbed by the measurement. Figure 9.1 indicates one method of measuring static pressure, the *piezometer* opening. When the flow is parallel, as indicated, the pressure variation is hydrostatic normal to the streamlines; hence, by measuring the pressure at the wall, the pressure at any other point in the cross section may be determined. The piezometer opening should be small, with length of opening at least twice its diameter, and should be normal to the surface, with no burrs at its edges because small eddies form and distort the measurement. A small amount of rounding of the opening is permissible. Any slight misalignment or roughness at the opening may cause errors in measurement; therefore, it is advisable to use several piezometer openings connected together into a *piezometer ring*. When the surface is rough in the vicinity of the opening the reading is unreliable. For small irregularities it may be possible to smooth the surface around the opening.

For rough surfaces, the *static tube* (Fig. 9.2) may be used. It consists of a tube that is directed upstream with the end closed. It has radial holes in the cylindrical portion downstream from the nose. The flow

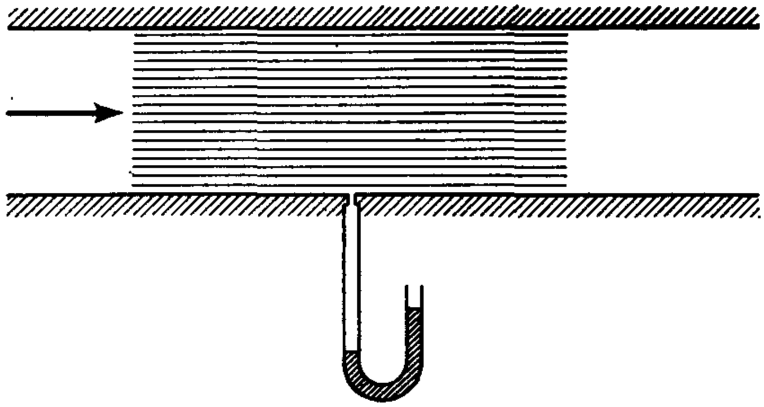

FIG. 9.1. Piezometer opening for measurement of static pressure.

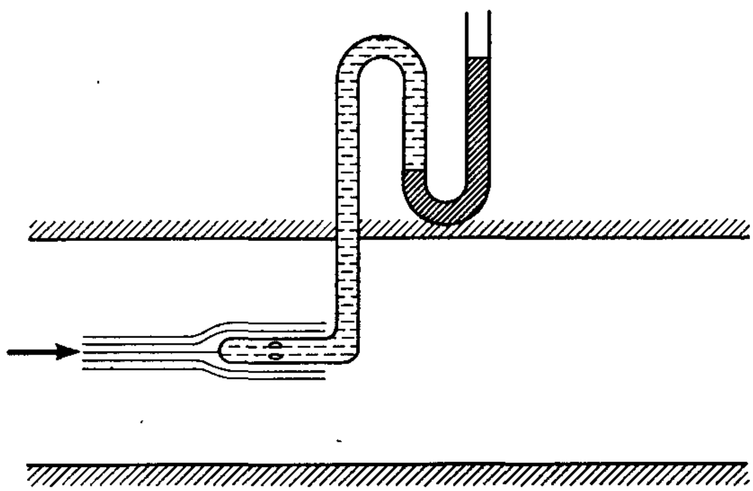

FIG. 9.2. Static tube.

is presumed to be moving by the openings as if it were undisturbed. There are disturbances, however, due to both the nose and the right-angled leg that is normal to the flow. The static tube should be calibrated, as it may read too high or too low. If it does not read true static pressure, the discrepancy Δh normally varies as the square of the velocity of flow by the tube; i.e.,

$$\Delta h = C \frac{v^2}{2g}$$

in which C is determined by towing the tube in still fluid where pressure and velocity are known or by inserting it into a smooth pipe that contains a piezometer ring.

The distribution of static pressure around the surface of a body may be determined by taking pressure readings from a series of piezometer openings, as in Fig. 9.3. With Bernoulli's equation the velocity distribution is determined from the pressure distribution.

Pressure may also be determined by making use of the *piezoelectric* properties of certain crystals, such as quartz or rochelle salt. Pressure

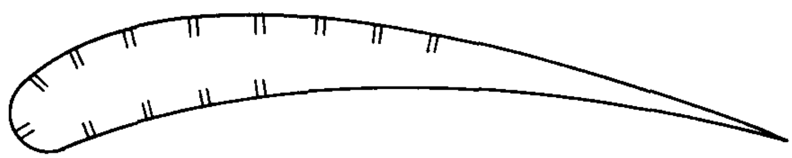

FIG. 9.3. Static pressure openings on a body submerged in a fluid.

exerts a strain on the crystals that liberates a small electric charge that can be measured by electronic means. Another method (Fig. 9.4) is the *capacitance* gage; the pressure distorts a diaphragm which varies the capacitance between plate and diaphragm.

9.2. Velocity Measurement. Since the determining of velocity at a number of points in a cross section permits the evaluating of discharge, the measuring of velocity is an important phase of measuring flow.

The *pitot tube* is one of the most accurate methods of measuring velocity. In Fig. 9.5 a glass tube with a right-angled bend is used to measure

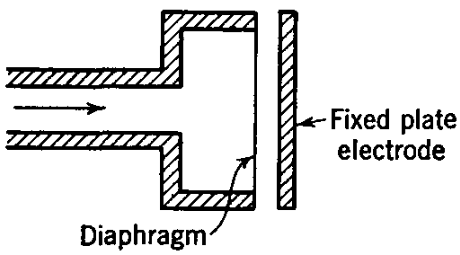

FIG. 9.4. Capacitance gage.

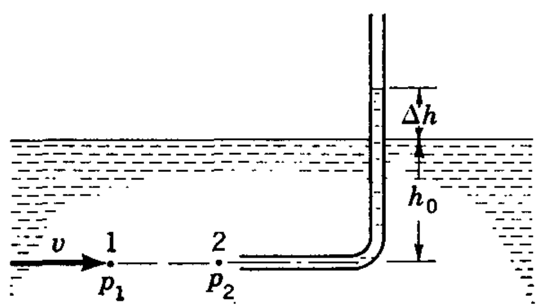

FIG. 9.5. Simple pitot tube.

the velocity v in an open channel. The tube opening is directed upstream so that the fluid flows into the opening until the pressure builds up within the tube sufficiently to withstand the impact of velocity against it. Directly in front of the opening the fluid is at rest. The streamline through 1 leads to the point 2, called the *stagnation point,* where the fluid is at rest, and there divides and passes around the tube. The pressure at 2 is known from the liquid column within the tube. Bernoulli's equation applied between points 1 and 2, produces

$$\frac{v^2}{2g} + \frac{p_1}{\gamma} = \frac{p_2}{\gamma} = h_0 + \Delta h$$

since both points are at the same elevation. As $p_1/\gamma = h_0$, the equation reduces to

$$\frac{v^2}{2g} = \Delta h \tag{9.2.1}$$

or

$$v = \sqrt{2g\,\Delta h} \tag{9.2.2}$$

Practically, it is very difficult to read the height Δh from a free surface.

The pitot tube measures the stagnation pressure, which is also referred to as the *total* pressure. The total pressure is composed of two parts, the static pressure h_0 and the *dynamic* pressure Δh, expressed in length of a column of the flowing fluid (Fig. 9.5). The dynamic pressure is related to velocity head by Eq. (9.2.1).

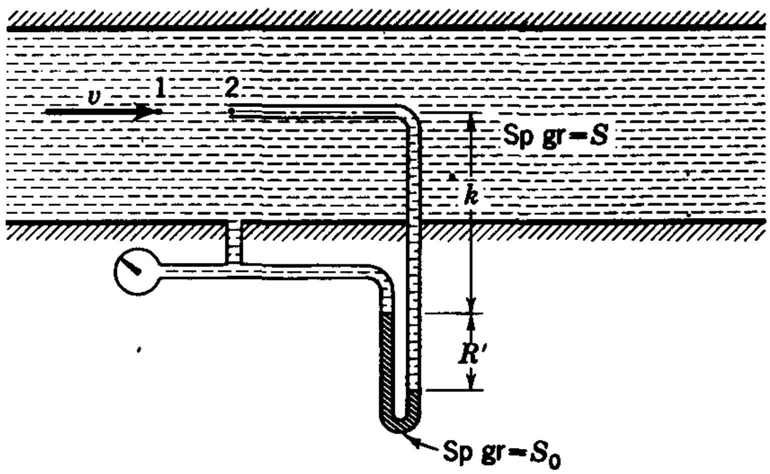

FIG. 9 6. Use of pitot tube and piezometer opening for measurement of velocity.

By combining the static-pressure measurement and the total-pressure measurement, i.e., measuring each and connecting to opposite ends of a differential manometer, the dynamic pressure head is obtained. Figure 9.6 illustrates one arrangement. Bernoulli's equation applied from 1 to 2 is

$$\frac{v^2}{2g} + \frac{p_1}{\gamma} = \frac{p_2}{\gamma} \tag{9.2.3}$$

The equation for the pressure through the manometer, in feet of water, is

$$\frac{p_1}{\gamma}S + kS + R'S_0 - (k + R')S = \frac{p_2}{\gamma}S$$

By simplifying,

$$\frac{p_2 - p_1}{\gamma} = R'\left(\frac{S_0}{S} - 1\right) \tag{9.2.4}$$

After substituting for $(p_2 - p_1)/\gamma$ in Eq. (9.2.3) and solving for v,

$$v = \sqrt{2gR'\left(\frac{S_0}{S} - 1\right)} \tag{9.2.5}$$

The static tube and pitot tube may be combined into one instrument, called a *pitot-static tube* (Fig. 9.7). Analyzing this system in a manner similar to that in Fig. 9.6 shows that the same relations hold; Eq. (9.2.5)

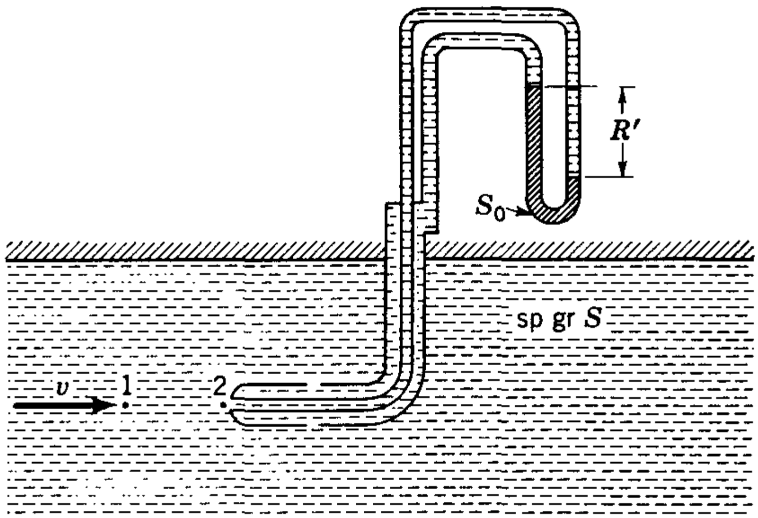

FIG. 9.7. Pitot-static tube.

expresses the velocity, but, owing to the uncertainty of the measurement of static pressure, a corrective coefficient C is applied,

$$v = C\sqrt{2gR'\left(\frac{S_0}{S} - 1\right)} \tag{9.2.6}$$

A particular form of pitot-static tube with a blunt nose, the *Prandtl tube*, has been designed so that the disturbances due to nose and leg cancel, leaving $C = 1$ in the equation. For other pitot-static tubes the constant C must be determined by calibration.

Velocity and Temperature Measurement in Compressible Flow. The pitot-static tube may be used for velocity determinations in compressible flow. In Fig. 9.7 the velocity reduction from free-stream velocity at 1 to zero at 2 takes place very rapidly without significant heat transfer, and friction plays a very small part, so the compression may be assumed to be isentropic. By applying Eq. (6.3.7) to points 1 and 2 of Fig. 9.7, with $V_2 = 0$,

$$\frac{V_1^2}{2} = \frac{k}{k-1}\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} = \frac{kR}{k-1}(T_2 - T_1) = c_pT_1\left(\frac{T_2}{T_1} - 1\right) \tag{9.2.7}$$

The substitution of c_p is from Eq. (6.1.8). By use of Eq. (6.1.17)

$$\frac{V_1^2}{2} = c_pT_1\left[\left(\frac{p_2}{p_1}\right)^{(k-1)/k} - 1\right] = c_pT_2\left[1 - \left(\frac{p_1}{p_2}\right)^{(k-1)/k}\right] \quad (9.2.8)$$

The static pressure p_1 may be obtained from the side openings of the pitot tube, and the stagnation pressure may be obtained from the impact opening leading to a simple manometer, or $p_2 - p_1$ may be found from the differential manometer. If the tube is not designed so that true static pressure is measured, it must be calibrated and the true static pressure computed.

Temperature measurement of undisturbed flow of a compressible gas must be made indirectly, by measurement of the velocity of flow and the stagnation temperature. By solving Eq. (9.2.7) for T_1,

$$T_1 = T_2 - \frac{V_1^2}{2c_p} \quad (9.2.9)$$

V_1 is obtained from pitot-static tube measurements. T_2, the true stagnation temperature, is difficult to obtain because of heat transfer to and from the temperature-sensing element. Devices[1] comprising a thermocouple, with shielding such that true stagnation temperature is measured, have been developed. Otherwise correction factors must be applied to the temperature readings, and the devices must be calibrated.

The Hot-wire Anemometer. Gas velocities are successfully measured with the *hot-wire anemometer.* A short length of fine platinum wire is heated by an electric current. The resistance to flow of electricity through the wire is a function of its temperature. Flow of a gas around the hot wire cools it and thus changes its resistance. By holding constant either the voltage across the wire or the current through the wire by a suitable circuit, the change in amperes or voltage, respectively, becomes a function of the speed of gas flow by the hot wire. It may be calibrated by placing it in a gas stream of known velocity. The hot-wire anemometer has a very quick response to changes in gas velocity and is the best practical means for measuring the rapid fluctuations caused by turbulence at a point.

In Figs. 9.8 and 9.9 circuits for the two systems are shown. A Wheatstone-bridge circuit is utilized for both, with the hot wire forming one resistance and R_1, R_2, R_3 the other resistances. In the constant-resistance circuit, the temperature of the wire is held constant and hence its resistance remains constant. The circuit is first adjusted so that the galvanometer reads zero. Then for a change in fluid flow over the wire, the variable resistance B is adjusted to bring the galvanometer reading

[1] A. Franz, Pressure and Temperature Measurements in Supercharger Investigations, *NACA Tech. Mem.* 953, 1940.

back to zero, and the voltmeter reading has changed. By calibration in a stream of known velocity, voltmeter reading is related to fluid velocity.

In the constant-voltage circuit (Fig. 9.9) the variable resistance B is first adjusted so that the galvanometer reads zero when the hot wire is exposed to fluid at rest. Flow over the wire then cools the wire and varies its resistance, causing a change in galvanometer reading. Calibration relates velocity to galvanometer reading.

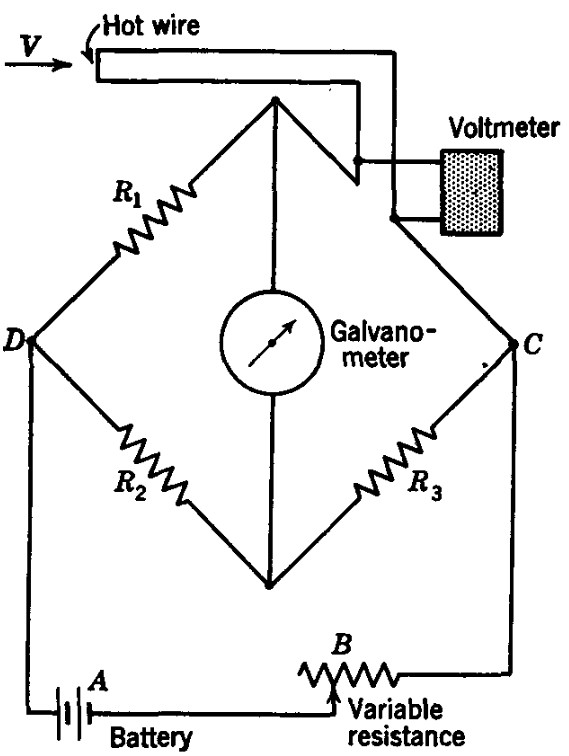

Fig. 9.8. Constant-resistance hot-wire anemometer.

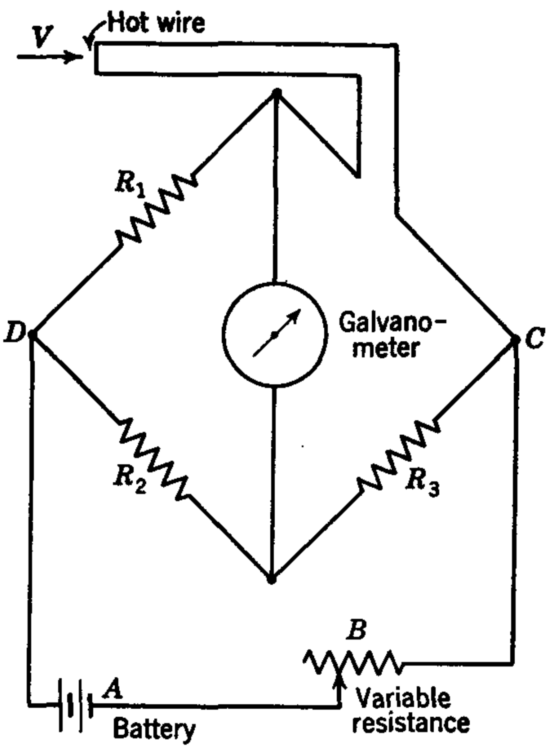

Fig. 9.9. Constant-voltage hot-wire anemometer.

The *current meter* (Fig. 9.10) is used to measure the velocity of liquid flow in open channels. The cups are shaped so that the drag varies with orientation, causing a relatively slow rotation. With an electrical circuit and headphones, an audible signal is detected for a fixed number of revolutions. The number of signals in a given time period is a function of the velocity. The meters are calibrated by towing them through liquid at known speeds. For measuring high-velocity flow a current meter with a propeller as rotating element is used, as it offers less resistance to the flow.

Air velocities are measured with cup-type or vane-(propeller) type anemometers (Fig. 9.11) which drive generators that indicate air velocity directly or drive counters that indicate the number of revolutions.

9.3. Optical Flow Measurement. Three optical flow-measuring devices are described and illustrated in this section. The principal advantage of the optical techniques is that the flow is undisturbed by the measurement. Each method is based on the principle that changes in density of a medium change the angle of refraction of light; i.e., the denser the medium, the larger the angle of refraction. The index of refraction n of a substance is defined as the ratio of the speed of light in

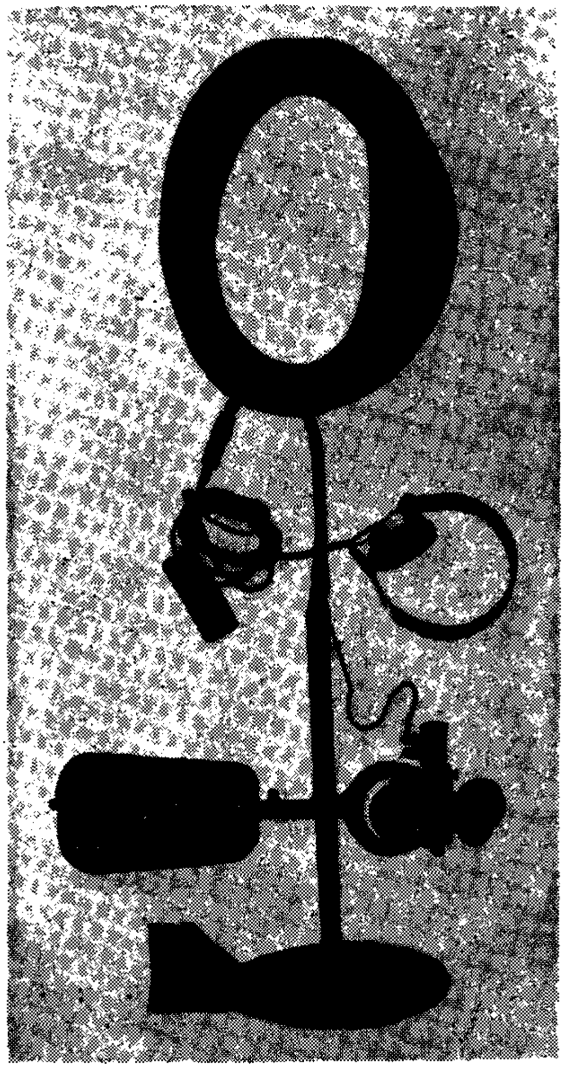

FIG. 9.10. Price current meter. (*W. and L. E. Gurley.*)

a vacuum to the speed of light through the medium; hence n is always greater than unity. The index of refraction varies with the wavelength of the light and tends to increase linearly with the density. The relationship between angle of incidence i, angle of refraction r, and the two values of index of refraction n_a, n_b (Fig. 9.12) is given by *Snell's law:*

$$n_a \sin i = n_b \sin r$$

When the light passes from a less dense medium to a more dense medium, the angle of refraction is less than the angle of incidence. If the index of refraction is very close to unity, as is the case for most gases, the empirical *Gladstone-Dale* equation may be used:

$$\frac{n-1}{\rho} = \frac{n_a - 1}{\rho_a} = \frac{n_b - 1}{\rho_b}$$

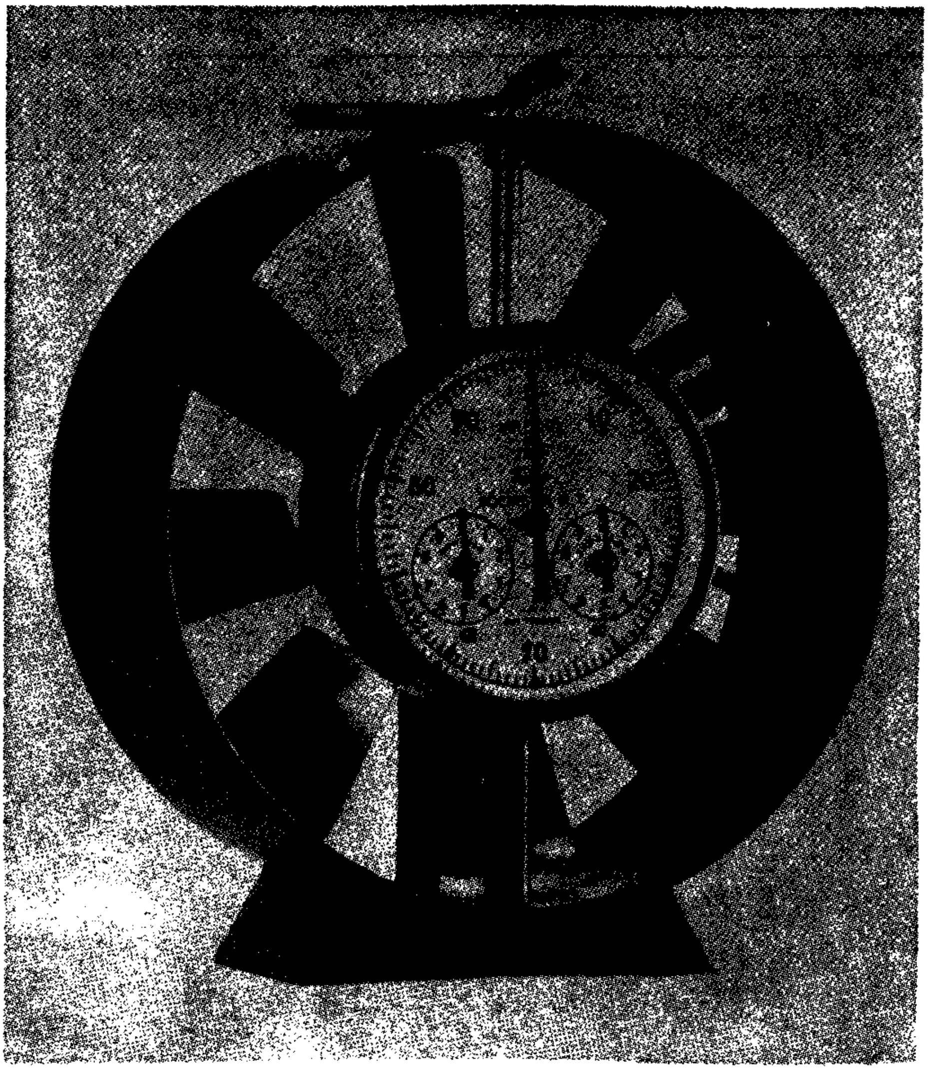

FIG. 9.11. Air anemometer. (*Taylor Instrument Co.*)

The Schlieren Method. The Schlieren system is illustrated in Fig. 9.13, where it is employed in flow across a two-dimensional test section. Light from a source is collimated by the first lens and passed through the test section to a second lens, which brings it to a focus and then projects it on a screen or photographic plate. At the focal point a knife-edge is introduced which cuts off some of the light. For no flow in the test section the screen is uniformly illuminated. If the density within the test section is altered slightly by flow around a model, the light rays will be refracted in varying amounts. Where it is refracted so the rays are intercepted by the knife-edge, less light strikes the screen, and where it is refracted in the opposite direction, more light strikes the screen. It is to be noted that this system portrays changes in density. Figure 9.14 shows a Schlieren photograph of a shock wave caused by propagation of an explosive wave due to detonation of a gas.

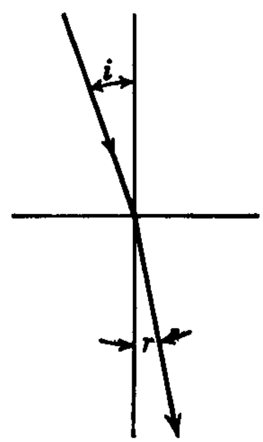

FIG. 9.12. Angles of incidence and refraction.

The Shadowgraph Method. The same setup as the Schlieren system may be used for shadowgraph studies, except that the knife-edge is not utilized. For a uniform change in density all light rays are refracted the same amount and illumination of the screen remains uniform and of

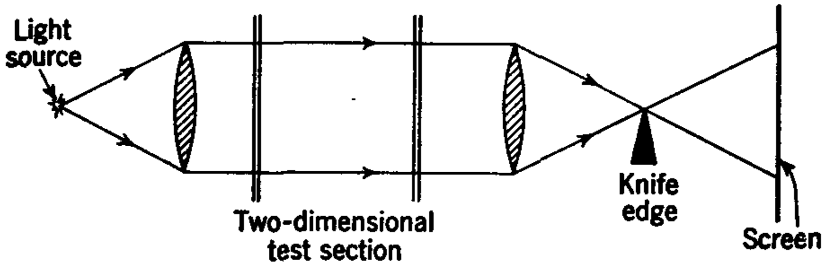

FIG. 9.13. Schlieren system.

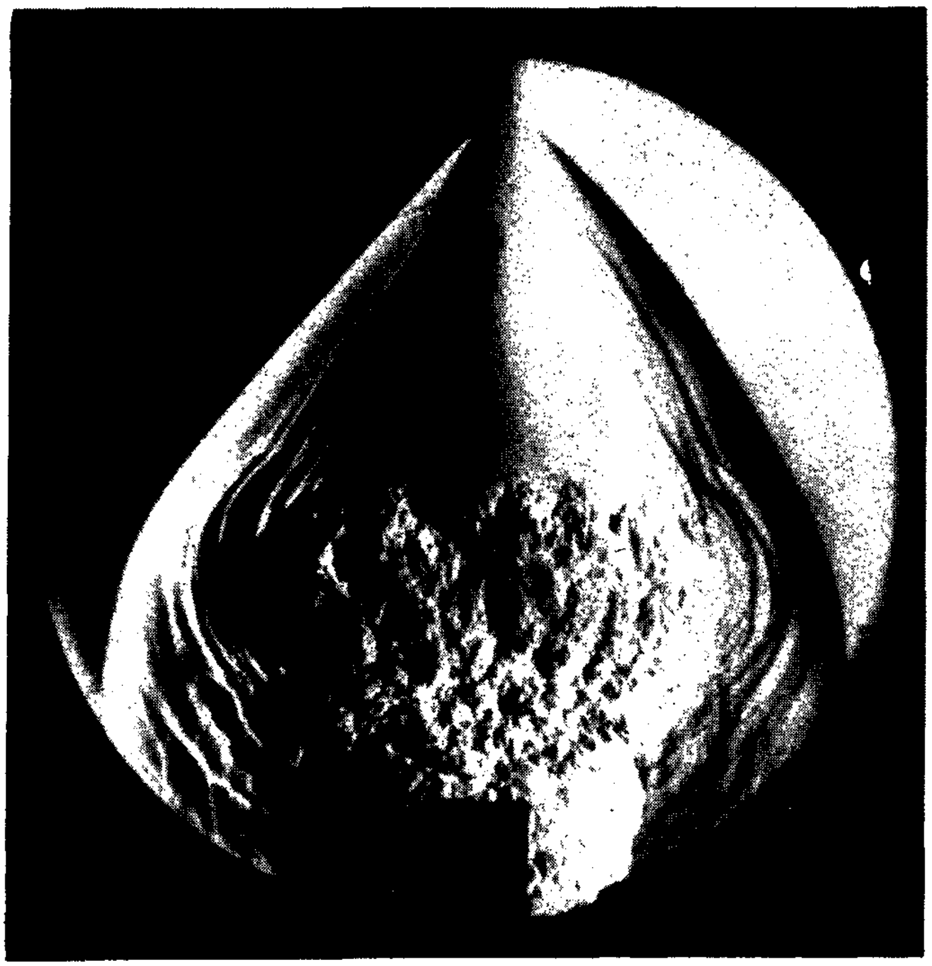

FIG. 9.14. Schlieren photograph of shock wave formed by ignition of explosive gas issuing from a tube. (*From doctoral thesis by W. P. Sommers, taken in Aeronautical and Astronautical Engineering Laboratories, The University of Michigan,* 1961.)

the same intensity. Changes in the density gradient, however, will cause uneven refraction of the light waves which will project a pattern on the screen. A shadowgraph view of a flame is shown in Fig. 9.15. Schlieren and shadowgraph methods are generally qualitative, in that they aid in visualization of the flow. The *interferometer* is used for quantitative measurement of density within the test section.

The Interferometer Method. The interferometer makes use of a phase shift in the wave motion of light to obtain a pattern from which density changes may be read. In the Mach-Zehnder interferometer (Fig. 9.16), light from a single source is split into two circuits, one including the test section, and is then recombined for projection onto a screen or photographic plate. The first mirror is silvered over half of its surface; hence it transmits half of the light and reflects half of the light, forming the two circuits. One circuit passes through the test section, and the other circuit through compensating plates. These circuits are recombined, as indicated in the figure, by one being transmitted and the other reflected by the second half-silvered mirror and then projected onto a screen.

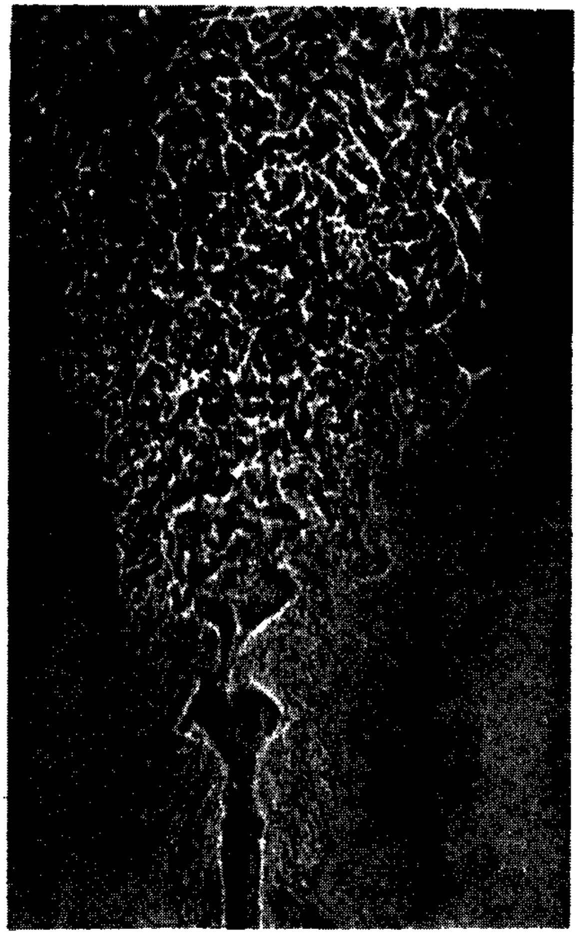

FIG. 9.15. Shadowgraph view of propane-air combustion. Flame is stabilized around $\frac{1}{16}$-in.-diameter spherical flame holder at bottom. Jet velocity 77 ft/sec. (*Willow Run Research Center, The University of Michigan.*)

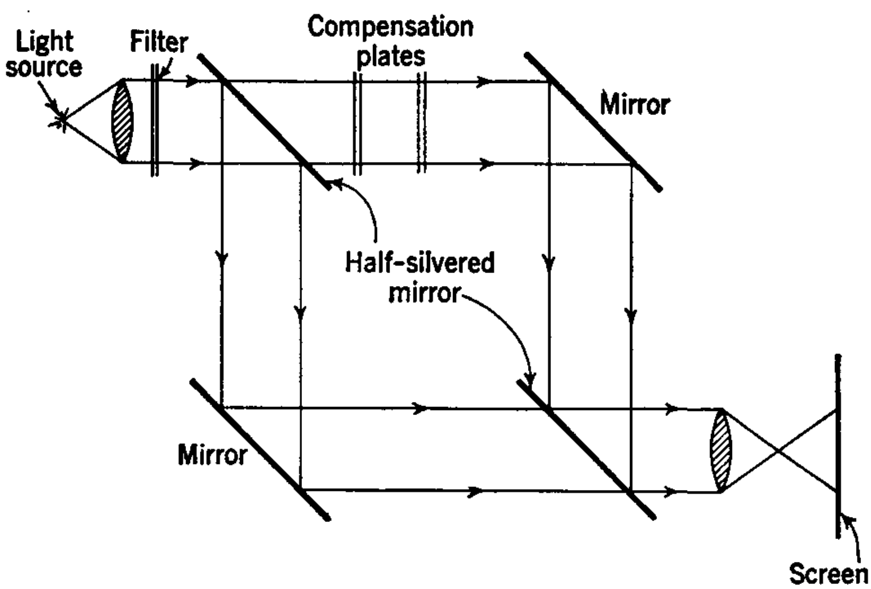

FIG. 9.16. Mach-Zehnder interferometer system.

When there is no flow through the test section and the fluid there has the same density as the surrounding fluid, the screen is uniformly bright, as both circuits have the same length and same light speed. Now, if the density within the test section is uniformly changed, the speed of transmission of light is changed and the two streams are out of phase. If the

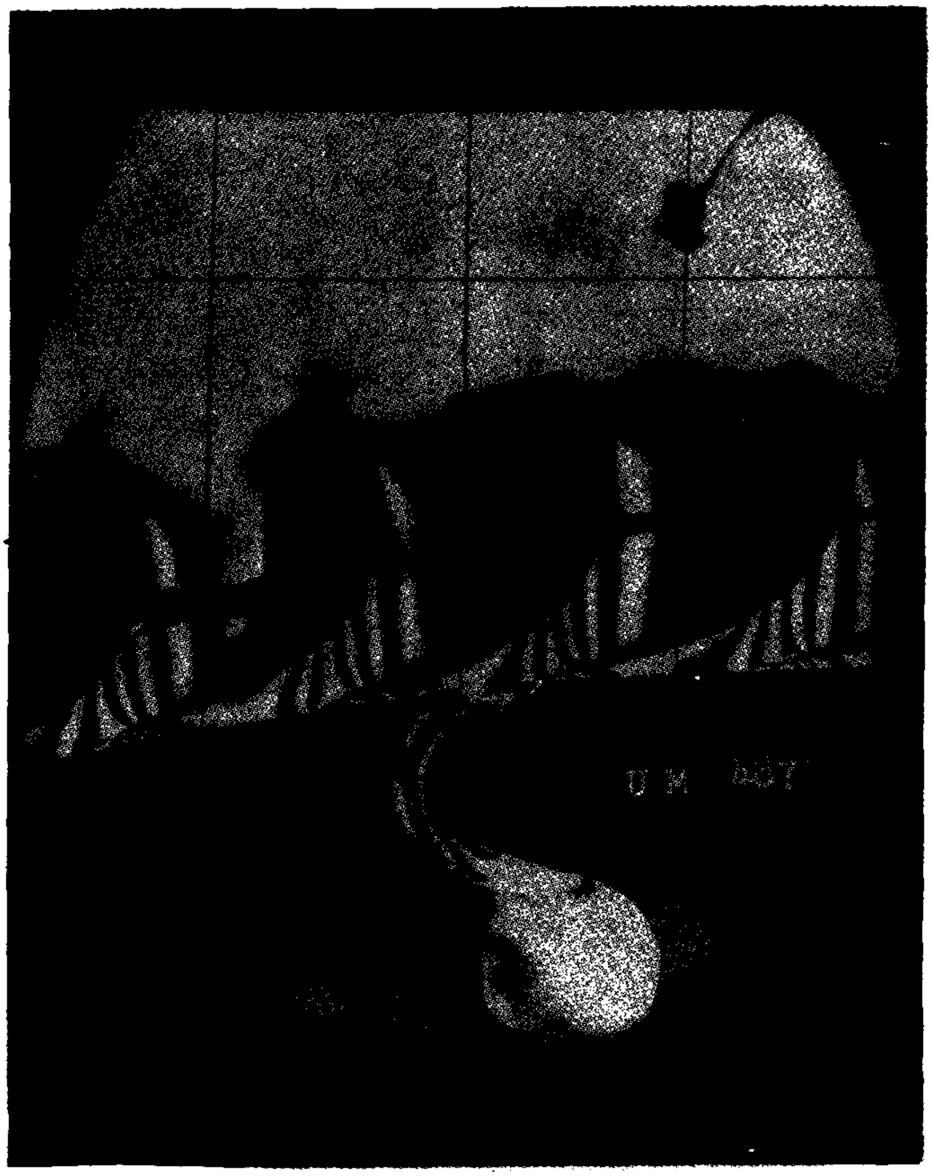

FIG. 9.17. Interferometer photograph of flow through nozzles and one bucket of model of a turbine. The density change across each band (black or white) is $\frac{1}{2}$ per cent. (*Photograph taken in Aeronautical and Astronautical Laboratories of The University of Michigan for the General Electric Co.*)

trough of one light wave coincides with the crest of the wave from the other circuit, the screen will be uniformly dark; hence the amount of light on the screen depends on the phase shift. With flow around a model in the test section, the zones of uniform density will appear as bands on the screen, as shown in Fig. 9.17.

9.4. Positive-displacement Meters. One volumetric meter is a positive-displacement meter that has pistons or partitions which are displaced by the flowing fluid and a counting mechanism that records the number of displacements in any convenient unit, such as gallons or cubic feet.

A common meter is the *disk* meter, or *wobble* meter (Fig. 9.18), used on many domestic water-distribution systems. The disk oscillates in a passageway so that a known volume of fluid moves through the meter for each oscillation. A stem normal to the disk operates a gear train,

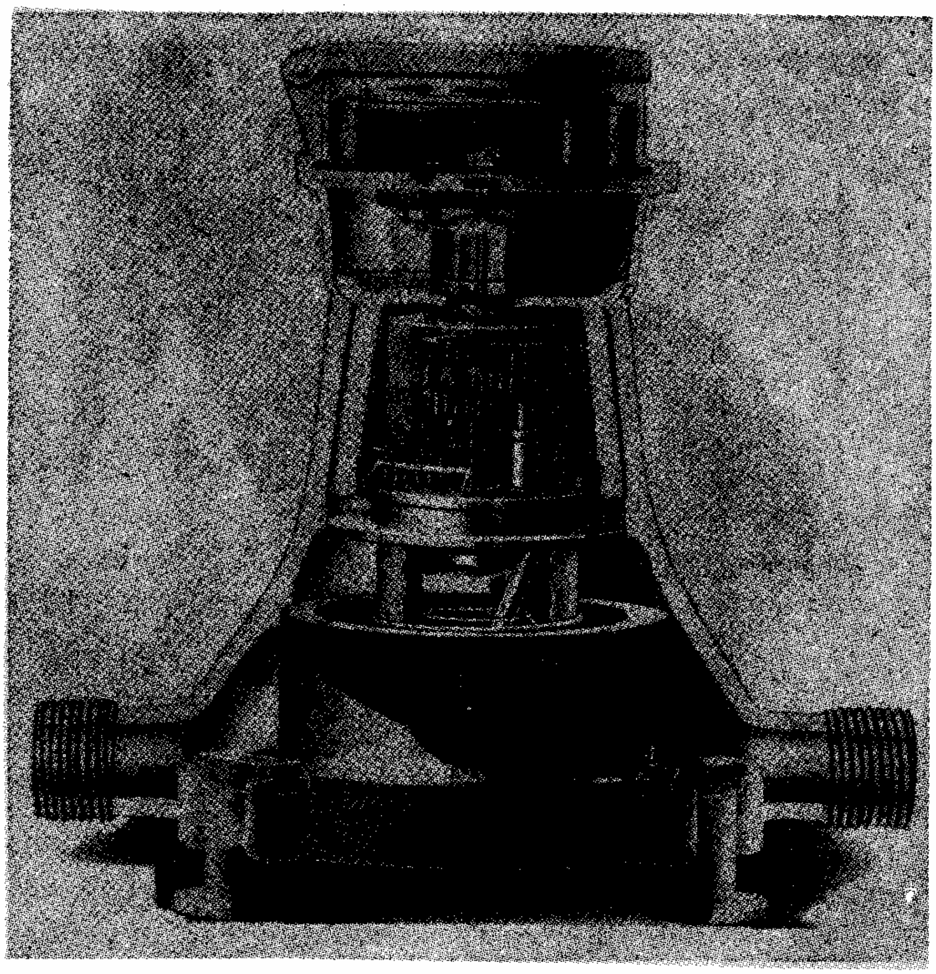

Fig. 9.18. Disk meter. (*Neptune Meter Co.*)

which in turn operates a counter. When in good condition, these meters are accurate to within 1 per cent. When worn, the error may be very large for small flows, such as those caused by a leaky faucet.

The flow of domestic gas at low pressure is usually measured by a volumetric meter with a traveling partition. The partition is displaced by gas inflow to one end of the chamber in which it operates, and then, by a change in valving, it is displaced to the opposite end of the chamber. The oscillations operate a counting mechanism.

Oil flow or high-pressure gas flow in a pipeline is frequently measured by a rotary meter in which cups or vanes move about an annular opening and displace a fixed volume of fluid for each revolution. Radial or axial pistons may be arranged so that the volume of a continuous flow through them is determined by rotations of a shaft.

Positive-displacement meters normally have no timing equipment that measures the rate of flow. The rate of steady flow may be determined with a stop watch to record the time for displacement of a given volume of fluid.

9.5. Rate Meters. A *rate meter* is a device that determines, generally by a single measurement, the quantity (weight or volume) per unit time that passes a given cross section. Included among rate meters are the *orifice*, *nozzle*, *venturi meter*, *rotameter*, *weir*, and *mass meter*, which are discussed in this section.

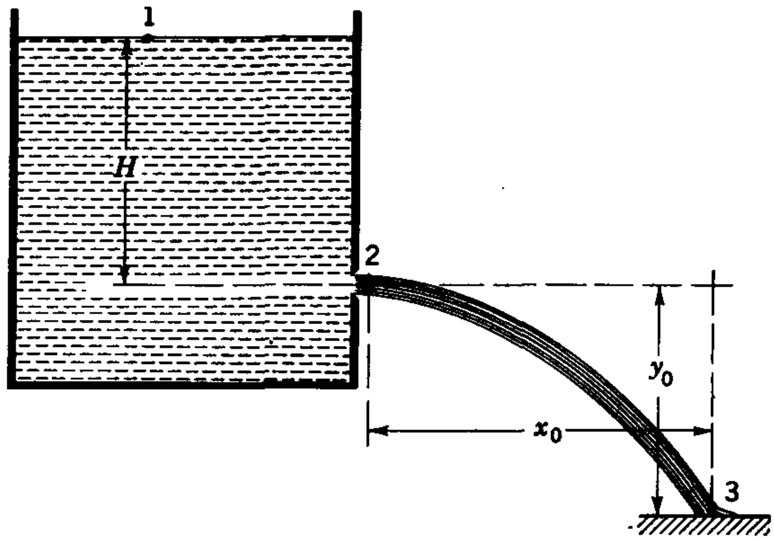

FIG. 9.19. Orifice in a reservoir.

Orifice in a Reservoir. An orifice may be used for measuring the rate of flow out of a reservoir or through a pipe. An orifice in a reservoir or tank may be in the wall or in the bottom. It is an opening, usually round, through which the fluid flows, as in Fig. 9.19. It may be square-edged, as shown, or rounded, as in Fig. 3.12. The area of the orifice is the area of the opening. With the square-edged orifice, the fluid jet contracts during a short distance of about one-half diameter downstream from the opening. That portion of the flow that approaches along the wall cannot make a right-angled turn at the opening, so it maintains a radial velocity component that reduces the jet area. The cross section where the contraction is greatest is called the *vena contracta*. The streamlines are parallel throughout the jet at this section, and the pressure is atmospheric.

The head on the orifice, H, is measured from the center of the orifice to the free surface. The head is assumed to be held constant. Bernoulli's equation applied from a point 1 on the free surface to the center of the *vena contracta*, point 2, with local atmospheric pressure as datum and point 2 as elevation datum, neglecting losses, is written

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2$$

By inserting the values,

$$0 + 0 + H = \frac{V_2^2}{2g} + 0 + 0$$

or

$$V_2 = \sqrt{2gH} \tag{9.5.1}$$

This is only the *theoretical* velocity because the losses between the two points were neglected. The ratio of the *actual* velocity V_a to the theoretical velocity V_t is called the *velocity coefficient* C_v, that is,

$$C_v = \frac{V_a}{V_t} \tag{9.5.2}$$

Hence

$$V_{2a} = C_v \sqrt{2gH} \tag{9.5.3}$$

The actual discharge from the orifice, Q_a, is the product of the actual velocity at the *vena contracta* and the area of the jet. The ratio of jet area A_2 at *vena contracta* to area of orifice A_0 is symbolized by another coefficient, called the *coefficient of contraction*, C_c,

$$C_c = \frac{A_2}{A_0} \tag{9.5.4}$$

The area at the *vena contracta* is C_cA_0. The actual discharge is thus

$$Q_a = C_vC_cA_0 \sqrt{2gH} \tag{9.5.5}$$

It is customary to combine the two coefficients into a *discharge coefficient* C_d,

$$C_d = C_vC_c \tag{9.5.6}$$

from which

$$Q_a = C_dA_0 \sqrt{2gH} \tag{9.5.7}$$

There is no way to compute the losses between points 1 and 2; hence, C_v must be determined experimentally. It varies from 0.95 to 0.99 for the square-edged or rounded orifice. For most orifices, such as the square-edged one, the amount of contraction cannot be computed, and test results must be used. There are several methods for obtaining one or more of the coefficients. By measuring area A_0, the head H, and the discharge Q_a (by gravimetric or volumetric means), C_d is obtained from Eq. (9.5.7). Determination of either C_v or C_c then permits determination of the other by Eq. (9.5.6). Several methods follow:

a. Trajectory method. By measuring the position of a point on the trajectory of the free jet downstream from the *vena contracta* (Fig. 9.19) the actual velocity V_a may be determined if air resistance is neglected. The x-component of velocity does not change; therefore, $V_a t = x_0$, in which t is the time for a fluid particle to travel from the *vena contracta* to the point 3. The time for a particle to drop a distance y_0 under the action of gravity when it has no initial velocity in that direction is expressed by $y_0 = gt^2/2$. After eliminating t in the two relations,

$$V_a = \frac{x_0}{\sqrt{2y_0/g}}$$

With V_{2t} determined by Eq. (9.5.1) the ratio $V_a/V_t = C_v$ is known.

b. Direct measuring of V_a. With a pitot tube placed at the *vena contracta*, the actual velocity V_a is determined.

c. Direct measuring of jet diameter. With outside calipers, the diameter of jet at the *vena contracta* may be approximated. This is not a precise measurement and, in general, is less satisfactory than the other methods.

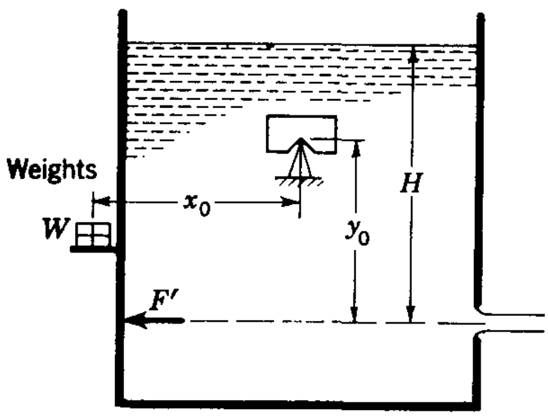

FIG. 9.20. Momentum method for determination of C_v and C_c.

d. Use of momentum equation. When the reservoir is of such size that it may be suspended on knife-edges, as in Fig. 9.20, it is possible to determine the force F that creates the momentum in the jet. With the orifice opening closed, the tank is leveled by adding or subtracting weights. With the orifice discharging, a force creates the momentum in the jet and an equal and opposite force F' acts against the tank. By addition of sufficient weights, W, the tank is again leveled. From the figure, $F' = Wx_0/y_0$. With the momentum equation,

$$\Sigma F_x = \frac{Q\gamma}{g}(V_{x\text{out}} - V_{x\text{in}})$$

or

$$\frac{Wx_0}{y_0} = \frac{Q_a \gamma V_a}{g}$$

as $V_{x\text{in}}$ is zero and V_a is the final velocity. Since the actual discharge is measured, V_a is the only unknown in the equation.

Losses in Orifice Flow. The head loss in flow through an orifice is determined by applying Bernoulli's equation with a loss term for the distance between points 1 and 2 (Fig. 9.19)

$$\frac{V_{1a}^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_{2a}^2}{2g} + \frac{p_2}{\gamma} + z_2 + \text{losses}$$

By substituting the values for this case,

$$\text{Losses} = H - \frac{V_{2a}^2}{2g} = H(1 - C_v^2) = \frac{V_{2a}^2}{2g}\left(\frac{1}{C_v^2} - 1\right) \qquad (9.5.8)$$

in which Eq. (9.5.3) has been used to obtain the losses in terms of H and C_v, or V_{2a} and C_v.

Example 9.1: A 3-in.-diameter orifice under a head of 16.0 ft discharges 2000 lb water in 32.6 sec. The trajectory was determined by measuring $x_0 = 15.62$ ft for a drop of 4.0 ft. Determine C_v, C_c, C_d, the head loss per unit weight, and the horsepower loss.

The theoretical velocity, V_{2t} is

$$V_{2t} = \sqrt{2gH} = \sqrt{64.4 \times 16} = 32.08 \text{ ft/sec}$$

The actual velocity is determined from the trajectory. The time to drop 4 ft is

$$t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2 \times 4}{32.2}} = 0.498 \text{ sec}$$

and the velocity is expressed by

$$x_0 = V_{2a}t \qquad V_{2a} = \frac{15.62}{0.498} = 31.4 \text{ ft/sec}$$

Then

$$C_v = \frac{V_{2a}}{V_{2t}} = \frac{31.4}{32.08} = 0.98$$

The actual discharge Q_a is

$$Q_a = \frac{2000}{62.4 \times 32.6} = 0.984 \text{ cfs}$$

With Eq. (9.5.7)

$$C_d = \frac{Q_a}{A_0\sqrt{2gH}} = \frac{0.984}{(\pi/64)\sqrt{64.4 \times 16}} = 0.625$$

Hence, from Eq. (9.5.6),

$$C_c = \frac{C_d}{C_v} = \frac{0.625}{0.98} = 0.638$$

The head loss, from Eq. (9.5.8), is

$$\text{Loss} = H(1 - C_v^2) = 16[1 - (0.98)^2] = 0.63 \text{ ft-lb/lb}$$

The horsepower loss is

$$\frac{0.63 \times 2000}{550 \times 32.6} = 0.070 \text{ hp}$$

The Borda mouthpiece (Fig. 9.21), a short, thin-walled tube about one diameter long that projects into the reservoir (re-entrant), permits application of the momentum equation, which yields one relation between C_v and C_d. The velocity along the wall of the tank is almost zero at all points; hence, the pressure distribution is hydrostatic. With the component of force exerted on the liquid by the tank parallel to the axis of the tube, there is an unbalanced force due to the opening, which

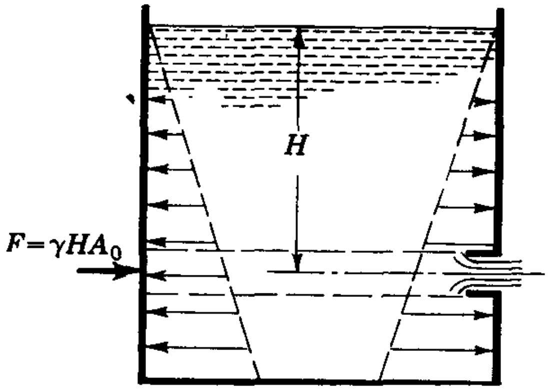

FIG. 9.21. The Borda mouthpiece.

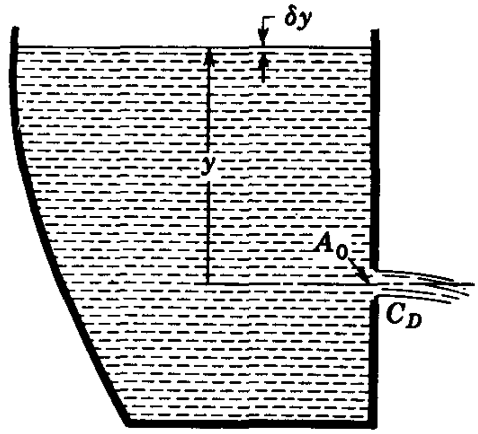

FIG. 9.22. Notation for falling head.

is $\gamma H A_0$. The final velocity is V_{2a}, the initial velocity is zero, and Q_a is the actual discharge. Then

$$\gamma H A_0 = Q_a \frac{\gamma}{g} V_{2a}$$

So

$$Q_a = C_d A_0 \sqrt{2gH} \qquad V_{2a} = C_v \sqrt{2gH}$$

by substituting for Q_a and V_{2a}, and simplifying,

$$1 = 2C_dC_v = 2C_v{}^2C_c$$

In the orifice situations considered, the liquid surface in the reservoir has been assumed to be held constant. An unsteady-flow case of some practical interest is that of determining the time to lower the reservoir surface a given distance. Theoretically, Bernoulli's equation applies only to steady flow, but if the reservoir surface drops slowly enough, the error from using Bernoulli's equation is negligible. The volume discharged from the orifice in time δt is $Q\ \delta t$, which must just equal the reduction in volume in the reservoir in the same time increment (Fig. 9.22), $A_R(-\delta y)$. in which A_R is the area of liquid surface at height y above the orifice. By equating the two expressions,

$$Q\ \delta t = -A_R\ \delta y$$

By solving for δt and integrating between the limits $y = y_1$, $t = 0$, and $y = y_2$, $t = t$,

$$t = \int_0^t dt = -\int_{y_1}^{y_2} \frac{A_R\,dy}{Q}$$

The orifice discharge Q is $C_d A_0 \sqrt{2gy}$. After substituting for Q,

$$t = -\frac{1}{C_d A_0 \sqrt{2g}} \int_{y_1}^{y_2} A_R y^{-\frac{1}{2}}\,dy$$

When A_R is known as a function of y, the integral can be evaluated. Consistent with other English units, t is in seconds. For the special case of a tank with constant cross section,

$$t = -\frac{A_R}{C_d A_0 \sqrt{2g}} \int_{y_1}^{y_2} y^{-\frac{1}{2}}\,dy = \frac{2A_R}{C_d A_0 \sqrt{2g}} (\sqrt{y_1} - \sqrt{y_2})$$

Example 9.2: A tank has a horizontal cross-sectional area of 20 ft² at the elevation of the orifice, and the area varies linearly with elevation so that it is 10 ft² at a horizontal cross section 10 ft above the orifice. For a 4-in.-diameter orifice, $C_d = 0.65$, compute the time in seconds to lower the surface from 8 ft to 4 ft above the orifice.

$$A_R = 20 - y \text{ ft}^2$$

and

$$t = -\frac{1}{0.65(\pi/36)\sqrt{64.4}} \int_8^4 (20 - y)y^{-\frac{1}{2}}\,dy = 51.3 \text{ sec}$$

Venturi Meter. The venturi meter is used to measure the rate of flow in a pipe. It is generally a casting (Fig. 9.23) consisting of an upstream section which is the same size as the pipe, has a bronze liner, and contains a piezometer ring for measuring static pressure; a converging conical section; a cylindrical throat with a bronze liner containing a piezometer ring; and a gradually diverging conical section leading to a cylindrical section the size of the pipe. A differential manometer is attached to the two piezometer rings. The size of a venturi meter is specified by the pipe and throat diameter; e.g., a 6-in. by 4-in. venturi meter fits a 6-in.-diameter pipe and has a 4-in.-diameter throat.

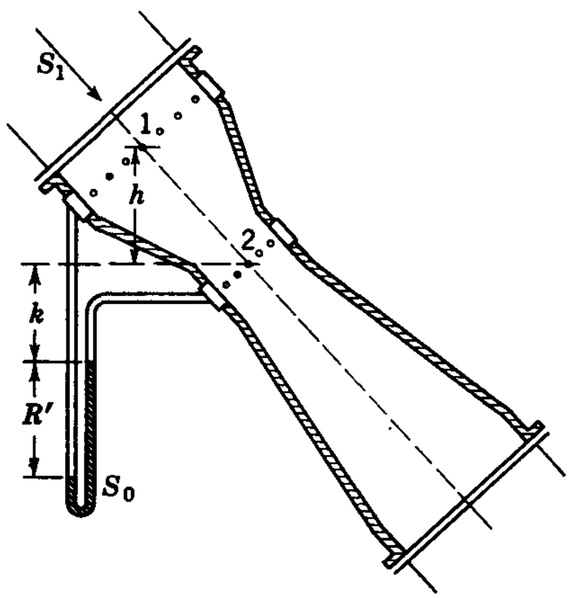

Fig. 9.23. Venturi meter.

For accurate results the venturi meter should be preceded by at least 10 diameters of straight pipe. In the flow from the pipe to the throat, the velocity is greatly increased and the pressure correspondingly decreased. The amount of discharge in incompressible flow is shown to be a function of the manometer reading.

The pressures at the upstream section and throat are *actual pressures,* and the velocities from Bernoulli's equation without a loss term are *theoretical velocities.* When losses are considered in Bernoulli's equation, the velocities are *actual velocities.* First, with the Bernoulli equation without a head-loss term, the theoretical velocity at the throat is obtained. Then by multiplying this by the velocity coefficient C_v, the actual velocity is obtained. The actual velocity times the actual area of the throat determines the actual discharge. From Fig. 9.23,

$$\frac{V_{1t}^2}{2g} + \frac{p_1}{\gamma} + h = \frac{V_{2t}^2}{2g} + \frac{p_2}{\gamma} \tag{9.5.9}$$

in which elevation datum is taken through point 2. V_1 and V_2 are average velocities at sections 1 and 2, respectively; hence, α_1, α_2 are assumed to be unity. With the continuity equation $V_1 D_1^2 = V_2 D_2^2$,

$$\frac{V_1^2}{2g} = \frac{V_2^2}{2g}\left(\frac{D_2}{D_1}\right)^4 \tag{9.5.10}$$

which holds for either the actual velocities or the theoretical velocities. Equation (9.5.9) may be solved for V_{2t},

$$\frac{V_{2t}^2}{2g}\left[1 - \left(\frac{D_2}{D_1}\right)^4\right] = \frac{p_1 - p_2}{\gamma} + h$$

and

$$V_{2t} = \sqrt{\frac{2g[h + (p_1 - p_2)/\gamma]}{1 - (D_2/D_1)^4}} \tag{9.5.11}$$

By introducing the velocity coefficient, $V_{2a} = C_v V_{2t}$,

$$V_{2a} = C_v \sqrt{\frac{2g[h + (p_1 - p_2)/\gamma]}{1 - D_2/D_1)^4}} \tag{9.5.12}$$

After multiplying by A_2, the actual discharge Q is determined to be

$$Q = C_v A_2 \sqrt{\frac{2g[h + (p_1 - p_2)/\gamma]}{1 - (D_2/D_1)^4}} \tag{9.5.13}$$

The gage difference R' may now be related to the pressure difference by writing the equation for the manometer. In feet of water (S_1 is the specific gravity of flowing fluid and S_0 the specific gravity of manometer liquid),

$$\frac{p_1}{\gamma} S_1 + (h + k + R')S_1 - R'S_0 - kS_1 = \frac{p_2}{\gamma} S_1$$

After simplifying,

$$h + \frac{p_1 - p_2}{\gamma} = R'\left(\frac{S_0}{S_1} - 1\right) \tag{9.5.14}$$

By substituting into Eq. (9.5.14),

$$Q = C_v A_2 \sqrt{\frac{2gR'[(S_0/S_1) - 1]}{1 - (D_2/D_1)^4}} \tag{9.5.15}$$

which is the venturi-meter equation for incompressible flow. The contraction coefficient is unity; hence, $C_v = C_d$. It should be noted that

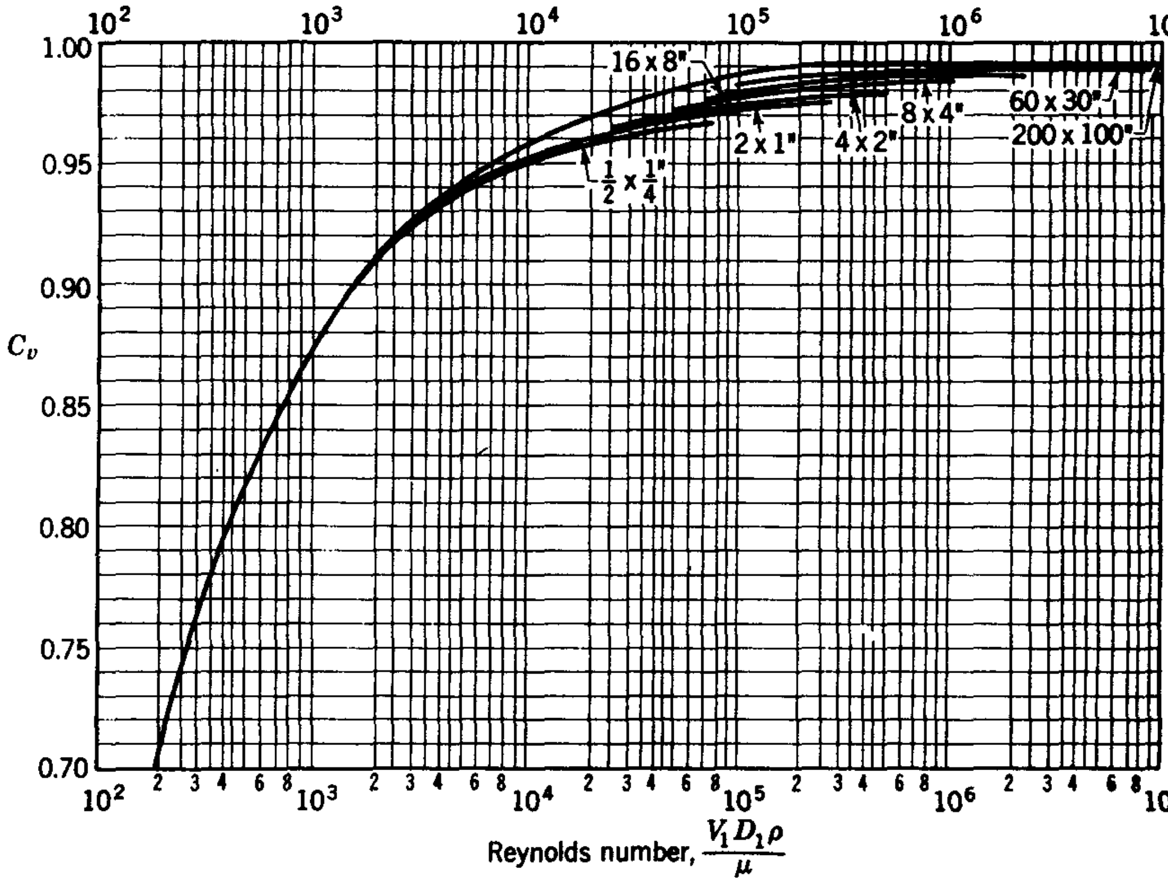

FIG. 9.24. Coefficient C_v for venturi meters (*"Fluid Meters: Their Theory and Application," American Society of Mechanical Engineers, 4th ed.*, 1937.)

h has dropped out of the equation. The discharge depends upon the gage difference R' regardless of the orientation of the venturi meter; whether it is horizontal, vertical, or inclined, exactly the same equation holds.

C_v is determined by calibration, i.e., by measuring the discharge and the gage difference and solving for C_v, which is usually plotted against the Reynolds number. Experimental results for venturi meters with throat diameters one-half the pipe diameters are given in Fig. 9.24. Where feasible, a venturi meter should be selected so that its coefficient is constant over the range of Reynolds numbers for which it is to be used.

The coefficient may be slightly greater than unity for venturi meters that are unusually smooth inside. This does not mean that there are no losses but results from neglecting the kinetic-energy correction factors α_1, α_2 in the Bernoulli equation. Generally α_1 is greater than α_2 since the reducing section acts to make the velocity distribution uniform across section 2.

The venturi meter has a low over-all loss, due to the gradually expanding conical section, which aids in reconverting the high kinetic energy at the throat into pressure energy. The loss is about 10 to 15 per cent of the head change between sections 1 and 2.

Venturi Meter for Compressible Flow. The theoretical mass flow rate through a venturi meter in compressible flow is given by Eq. (6.3.24) for isentropic flow through a converging-diverging duct when the throat velocity is less than sonic velocity. When multiplied by C_v, the velocity coefficient, it yields the actual mass flow rate $\dot{m}$. Equation (9.5.13), for incompressible flow, may be written in terms of mass flow rate

$$\dot{m} = C_v \rho_1 A_2 \sqrt{\frac{2\Delta p/\rho_1}{1 - (D_2/D_1)^4}} = C_v A_2 \sqrt{\frac{2\rho_1 \, \Delta p}{1 - (D_2/D_1)^4}} \qquad (9.5.16)$$

(h is dropped because it is negligible for gas flow). This equation may be modified by insertion of an *expansion factor* Y, so that it applies to compressible flow:

$$\dot{m} = C_v Y A_2 \sqrt{\frac{2\rho_1 \, \Delta p}{1 - (D_2/D_1)^4}} \qquad (9.5.17)$$

Y may be found by solving Eqs. (9.5.17) and (6.3.24) with coefficient C_v inserted and is shown to be a function of k, p_2/p_1, and A_2/A_1. Values of Y are plotted in Fig. 9.25 for $k = 1.40$; hence, by the use of Eq. (9.5.17) and Fig. 9.25 compressible flow may be computed for a venturi meter.

Flow Nozzle. The Verein Deutscher Ingenieure (VDI) flow nozzle, (Fig. 9.26) has no contraction of the jet other than that of the nozzle opening; therefore, the coefficient of contraction is unity. Equations (9.5.13) and (9.5.15) hold equally well for the flow nozzle. For a horizontal pipe ($h = 0$), Eq. (9.5.13) may be written

$$Q = CA_2 \sqrt{\frac{2\Delta p}{\rho}} \qquad (9.5.18)$$

in which

$$C = \frac{C_v}{\sqrt{1 - (D_2/D_1)^4}} \qquad (9.5.19)$$

and $\Delta p = p_1 - p_2$. The value of coefficient C in Fig. 9.26 is for use in Eq. (9.5.18). When the coefficient given in the figure is to be used, it is important that the dimensions shown shall have been closely adhered to, particularly in the location of the piezometer openings (two methods shown) for measuring pressure drop. At least 10 diameters of straight pipe should precede the nozzle.

The flow nozzle is less costly than the venturi meter. It has the disadvantage that the over-all losses are much higher because of the lack of guidance of jet downstream from the nozzle opening.

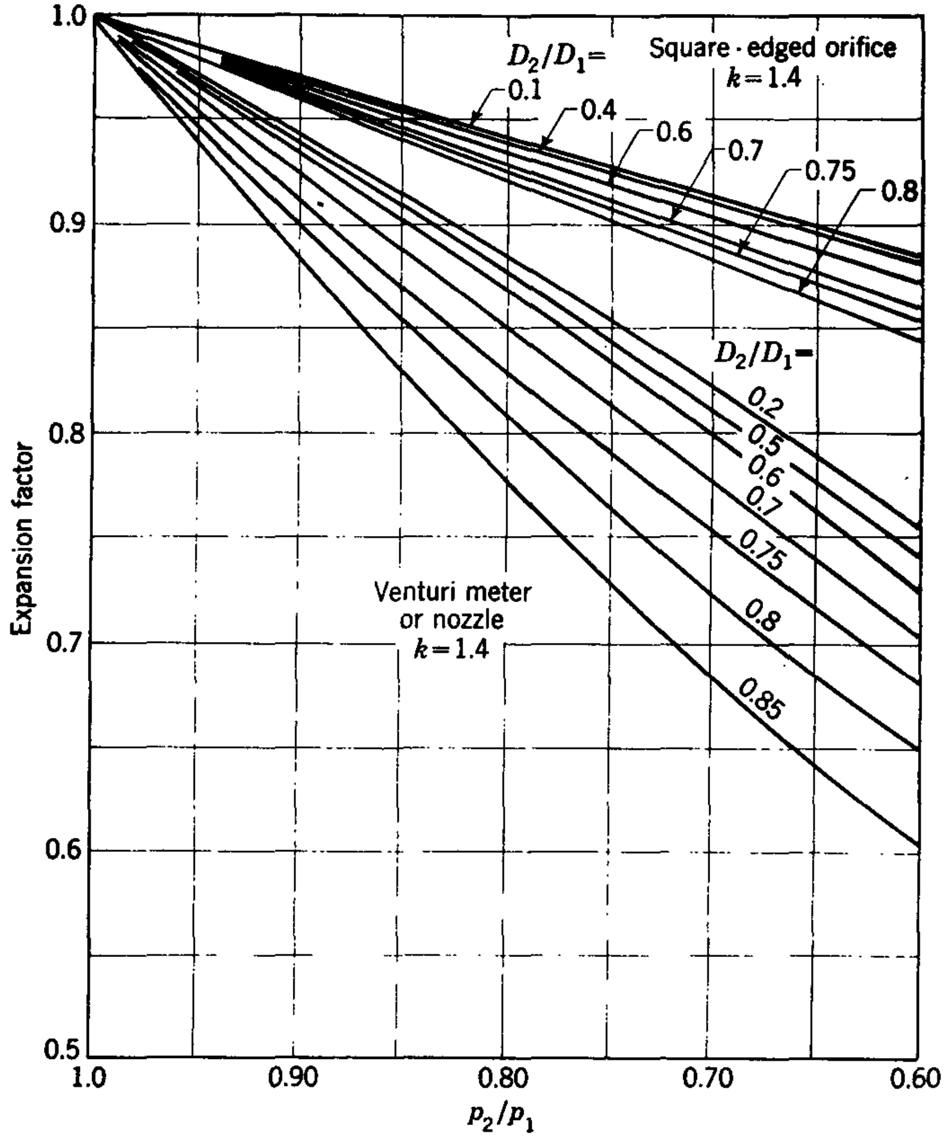

FIG. 9.25. Expansion factors.

Compressible flow through a nozzle is found by Eq. (9.5.17) and Fig. 9.25, if $k = 1.4$. For other values of specific-heat ratio k, Eq. (6.3.24) may be used and then modified with the velocity coefficient.

Example 9.3: Determine the flow through a 6-in.-diameter water line that contains a 4-in.-diameter flow nozzle. The mercury-water differential manometer has a gage difference of 10 in. Water temperature is 60°F.

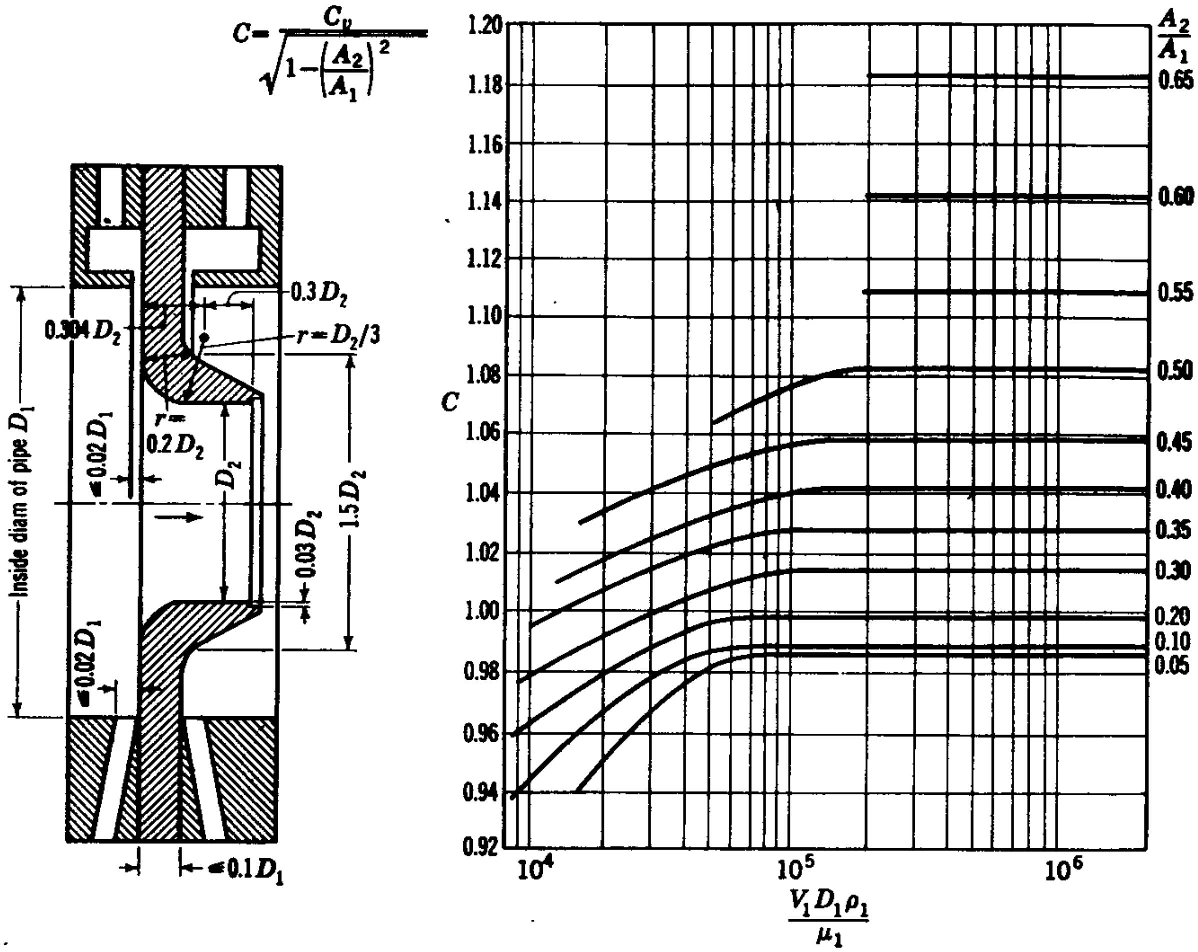

FIG. 9.26. VDI flow nozzle and discharge coefficients. (*NACA Tech. Mem.* 952, *Reference* 11.)

From the data given, $S_0 = 13.6$, $S_1 = 1.0$, $R' = \frac{10}{12} = 0.833$ ft, $A_2 = \pi/36 = 0.0873$ ft^2, $\rho = 1.935$ slugs/ft^3, $\mu = 0.01/479 = 2.09 \times 10^{-5}$ lb-sec/ft^2. By substituting Eq. (9.5.19) into Eq. (9.5.15),

$$Q = CA_2\sqrt{2gR'\left(\frac{S_0}{S_1} - 1\right)}$$

From Fig. 9.26, for $A_2/A_1 = (\frac{4}{6})^2 = 0.444$, assume that the horizontal region of the curves applies; hence, $C = 1.056$; then compute the flow and the Reynolds number.

$$Q = 1.056 \times 0.0873\sqrt{64.4 \times 0.833\left(\frac{13.6}{1.0} - 1.0\right)} = 2.39 \text{ cfs}$$

Then

$$V_1 = \frac{Q}{A_1} = \frac{2.39}{\pi/16} = 12.19 \text{ ft/sec}$$

and

$$\mathbf{R} = \frac{V_1 D_1 \rho}{\mu} = \frac{12.19 \times 1.935}{2 \times 2.09 \times 10^{-5}} = 565{,}000$$

The chart shows the value of C to be correct; therefore, the discharge is 2.39 cfs.

Orifice in a Pipe. The square-edged orifice in a pipe (Fig. 9.27) causes a contraction of the jet downstream from the orifice opening. For

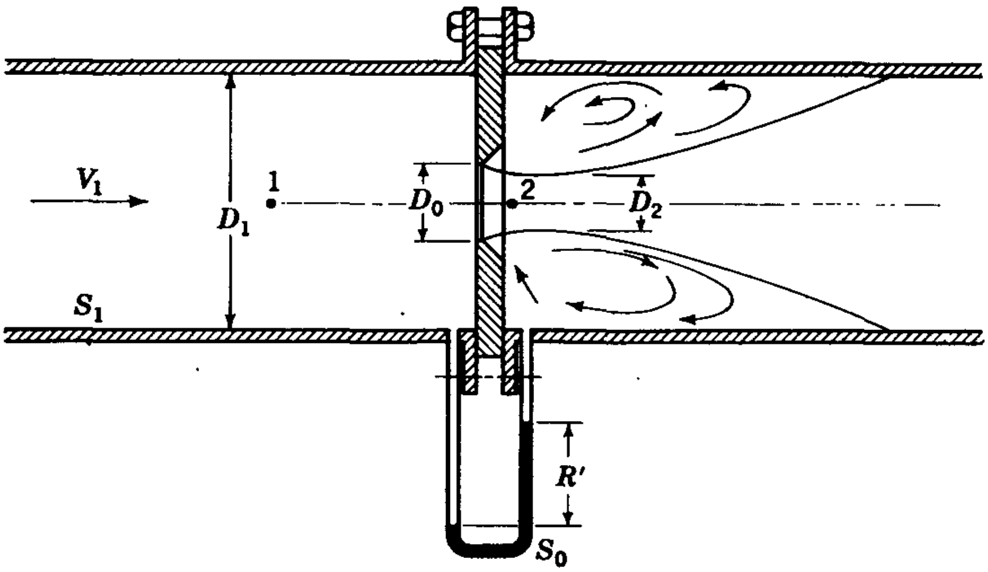

Fig. 9.27. Orifice in a pipe.

incompressible flow Bernoulli's equation applied from section 1 to the jet at its *vena contracta*, section 2, is

$$\frac{V_{1t}^2}{2g} + \frac{p_1}{\gamma} = \frac{V_{2t}^2}{2g} + \frac{p_2}{\gamma}$$

The continuity equation relates V_{1t} and V_{2t} with the contraction coefficient $C_c = A_2/A_0$,

$$V_1 \frac{\pi D_1^2}{4} = V_2 C_c \frac{\pi D_0^2}{4} \tag{9.5.20}$$

After eliminating V_{1t},

$$\frac{V_{2t}^2}{2g}\left[1 - C_c^2\left(\frac{D_0}{D_1}\right)^4\right] = \frac{p_1 - p_2}{\gamma}$$

and by solving for V_{2t},

$$V_{2t} = \sqrt{\frac{2g(p_1 - p_2)/\gamma}{1 - C_c^2(D_0/D_1)^4}}$$

By multiplying by C_v to obtain the actual velocity at the *vena contracta*,

$$V_{2a} = C_v \sqrt{\frac{2(p_1 - p_2)/\rho}{1 - C_c^2(D_0/D_1)^4}}$$

and, finally multiplying by the area of the jet, $C_c A_0$, produces the actual discharge Q,

$$Q = C_d A_0 \sqrt{\frac{2(p_1 - p_2)/\rho}{1 - C_c^2(D_0/D_1)^4}} \tag{9.5.21}$$

in which $C_d = C_v C_c$. In terms of the gage difference R', Eq. (9.5.21) becomes

$$Q = C_d A_0 \sqrt{\frac{2gR'[(S_0/S_1) - 1]}{1 - C_c^2(D_0/D_1)^4}} \tag{9.5.22}$$

Because of the difficulty in determining the two coefficients separately, a simplified formula is generally used, Eq. (9.5.18),

$$Q = CA_0 \sqrt{\frac{2\Delta p}{\rho}} \tag{9.5.23}$$

or its equivalent

$$Q = CA_0 \sqrt{2gR'\left(\frac{S_0}{S_1} - 1\right)} \tag{9.5.24}$$

Values of C are given in Fig. 9.28 for the VDI orifice.

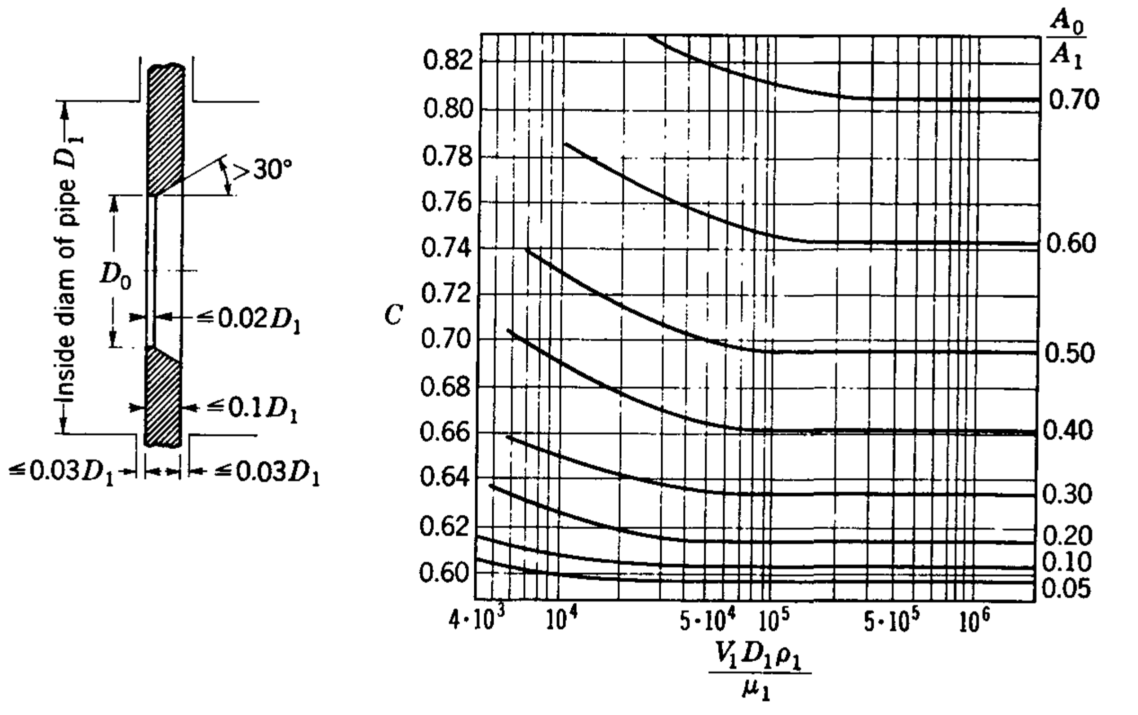

FIG. 9.28. VDI orifice and discharge coefficients. (*NACA Tech. Mem.* 952, *Reference* 11.)

Experimental values of expansion factor, for $k = 1.4$, are given in Fig. 9.25. Equation (9.5.23) for actual mass flow rate in compressible flow becomes

$$\dot{m} = CYA_0 \sqrt{2\rho_1 \Delta p} \tag{9.5.25}$$

Elbow Meter. The elbow meter for incompressible flow is one of the simplest flow-rate measuring devices. Piezometer openings on the inside and on the outside of the elbow are connected to a differential manometer. Because of centrifugal force at the bend, the difference in pressure intensities is related to the discharge. A straight calming length should precede the elbow, and, for accurate results, the meter should be calibrated in place.[1] As most pipelines have an elbow, it may be used as the meter. After calibration the results are as reliable as with a venturi meter or a flow nozzle.

Rotameter. The rotameter (Fig. 9.29) is a variable-area meter that consists of an enlarging transparent tube and a metering "float" (actually heavier than the liquid) that is displaced upward by the upward flow of fluid through the tube. The tube is graduated to read the flow directly. Notches in the float cause it to rotate and thus maintain a central position in the tube. The greater the flow, the higher the position that the float assumes.

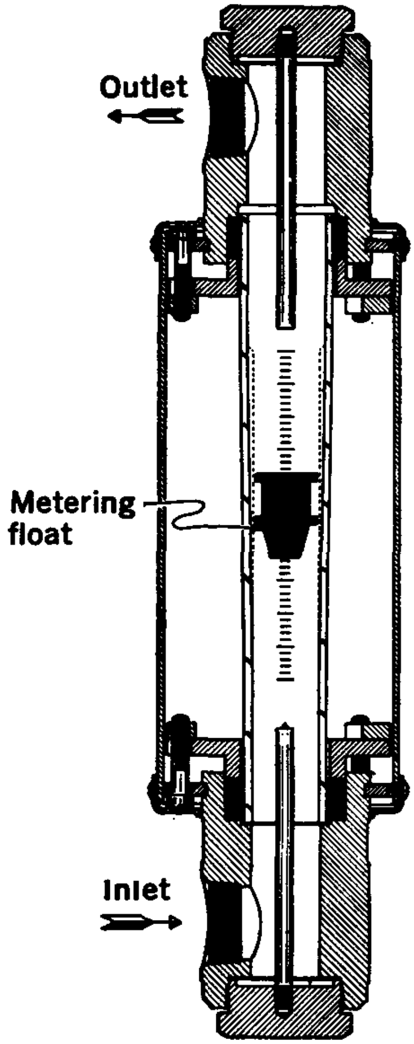

FIG. 9.29. Rotameter. (*Fischer & Porter Co.*)

Weirs. Open-channel flow may be measured by a *weir,* which is an obstruction in the channel that causes the liquid to back up behind it and to flow over it or through it. By measuring the height of upstream water surface, the rate of flow is determined. Weirs constructed from a sheet of metal or other material so that the jet, or *nappe,* springs free as it leaves the upstream face are called *sharp-crested* weirs. Other weirs such as the *broad-crested* weir support the flow in a longitudinal direction.

The sharp-crested, rectangular weir (Fig. 9.30) has a horizontal crest. The nappe is contracted at top and bottom as shown. An equation for discharge may be derived if the contractions are neglected. Without contractions the flow appears as in Fig. 9.31. The nappe has parallel streamlines with atmospheric pressure throughout.

Bernoulli's equation applied between 1 and 2 is

$$H + 0 + 0 = \frac{v^2}{2g} + H - y + 0$$

in which the velocity head at section 1 is neglected. By solving for v,

$$v = \sqrt{2gy}$$

[1] W. M. Lansford, The Use of an Elbow in a Pipe Line for Determining the Rate of Flow in a Pipe, *Univ. of Illinois Eng. Exp. Sta. Bull.* 289, December, 1936.

The theoretical discharge Q_t is

$$Q_t = \int v\,dA = \int_0^H vL\,dy = \sqrt{2g}\,L \int_0^H y^{\frac{1}{2}}\,dy = \tfrac{2}{3}\sqrt{2g}\,LH^{\frac{3}{2}}$$

in which L is the width of weir. Experiment shows that the exponent of H is correct but that the coefficient is too great. The contractions and

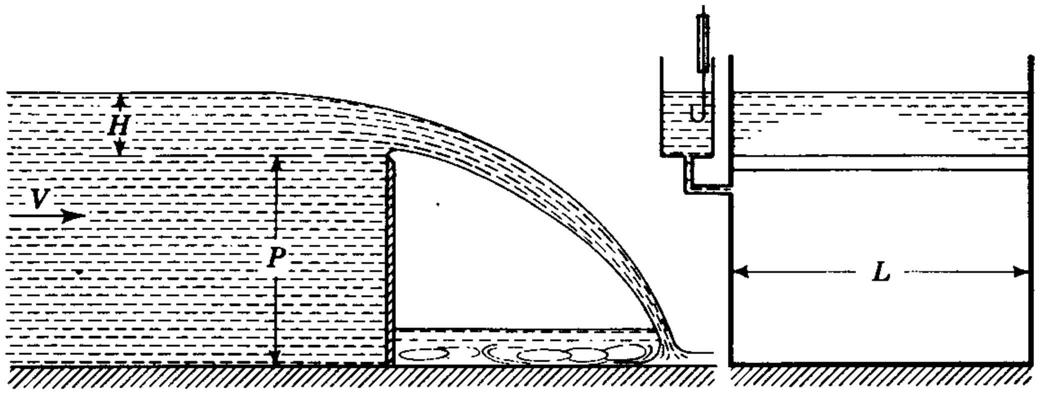

FIG. 9.30. Sharp-crested rectangular weir.

losses reduce the actual discharge to about 60 per cent of the theoretical, or

$$Q = 3.33LH^{\frac{3}{2}} \tag{9.5.26}$$

in which Q is in cubic feet per second, L and H are in feet.

When the weir does not extend completely across the width of the channel, it has *end contractions*, illustrated in Fig. 9.32. An empirical correction for the reduction of flow is accomplished by subtracting $0.1H$ from L for each end contraction. The weir in Fig. 9.30 is said to have its end contractions *suppressed.*

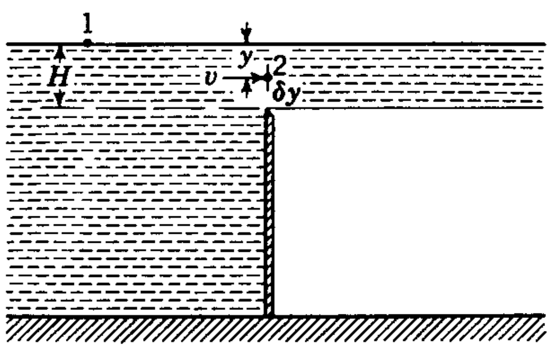

FIG. 9.31. Weir nappe without contractions.

The head H is measured upstream from the weir a sufficient distance to avoid the surface contraction. A hook gage mounted in a stilling pot connected to a piezometer opening determines the water-surface elevation from which the head is determined.

When the height of weir P (Fig. 9.30) is small, the velocity head at 1 cannot be neglected. A correction may be added to the head,

$$Q = CL\left(H + \alpha\frac{V^2}{2g}\right)^{\frac{3}{2}} \tag{9.5.27}$$

in which V is velocity and α is greater than unity, usually taken around 1.4, which accounts for the nonuniform velocity distribution. Equation

(9.5.27) must be solved for Q by trial since Q and V are both unknown. As a first trial, the term $\alpha V^2/2g$ may be neglected to approximate Q;

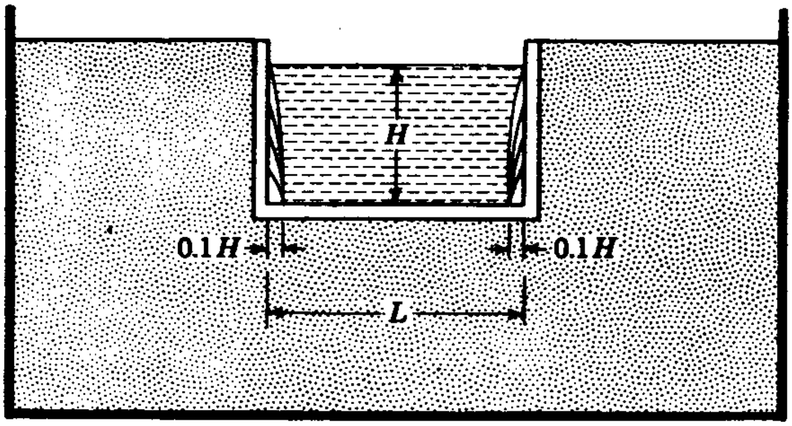

Fig. 9.32. Weir with end contractions.

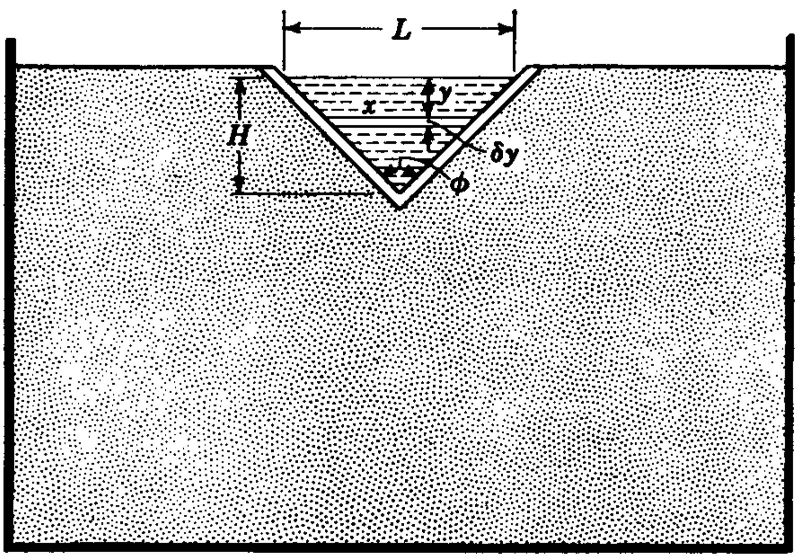

Fig. 9.33. V-notch weir.

then with this trial discharge a value of V is computed, since

$$V = \frac{Q}{L(P + H)}$$

For small discharges the V-notch weir is particularly convenient. Neglecting contraction of the nappe, the theoretical discharge is computed (Fig. 9.33) as follows:

The velocity at depth y is $v = \sqrt{2gy}$, and the theoretical discharge

$$Q_t = \int v\,dA = \int_0^H vx\,dy$$

By similar triangles, x may be related to y,

$$\frac{x}{H - y} = \frac{L}{H}$$

After substituting for v and x,

$$Q_t = \sqrt{2g}\,\frac{L}{H}\int_0^H y^{\frac{1}{2}}(H - y)\,dy = \frac{4}{15}\sqrt{2g}\,\frac{L}{H}\,H^{\frac{5}{2}}$$

By expressing L/H in terms of the angle ϕ of the V-notch,

$$\frac{L}{2H} = \tan\frac{\phi}{2}$$

Hence,

$$Q_t = \frac{8}{15}\sqrt{2g}\,\tan\frac{\phi}{2}\,H^{\frac{5}{2}}$$

The exponent in the equation is approximately correct, but the coefficient must be reduced by about 40 per cent. An approximate equation for a 90° V-notch weir is

$$Q = 2.50H^{2.50} \tag{9.5.28}$$

in which Q is in cubic feet per second and H is in feet. Experiments show that the coefficient is increased by roughening the upstream side of the

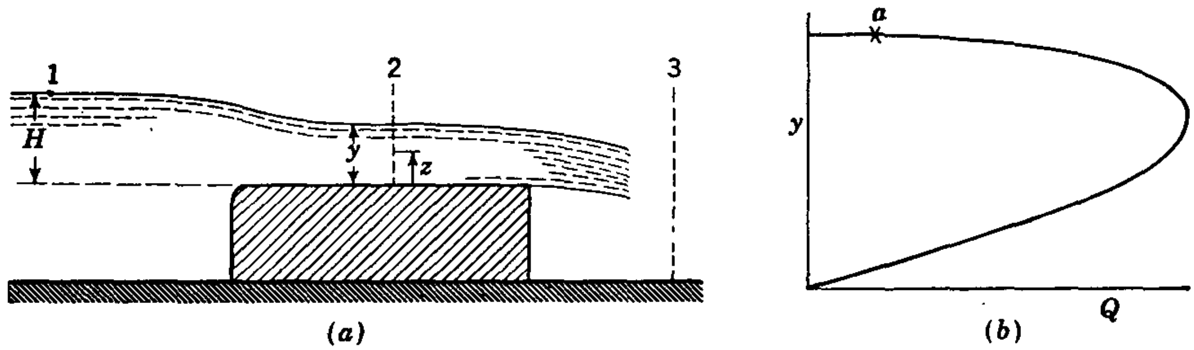

FIG. 9.34. Broad-crested weir.

weir plate, which causes the boundary layer to grow thicker. The greater amount of slow-moving liquid near the wall is more easily turned, and hence there is less contraction of the nappe.

The broad-crested weir (Fig. 9.34a) supports the nappe so that the pressure variation is hydrostatic at section 2. Bernoulli's equation applied between points 1 and 2 can be used to find the velocity v_2 at height z, neglecting the velocity of approach,

$$H + 0 + 0 = \frac{v_2^{\,2}}{2g} + z + (y - z)$$

In solving for v_2,

$$v_2 = \sqrt{2g(H - y)}$$

z drops out; hence, v_2 is constant at section 2. For a weir of width L normal to the plane of the figure, the theoretical discharge is

$$Q = v_2Ly = Ly\sqrt{2g(H - y)} \tag{9.5.29}$$

A plot of Q as abscissa against the depth y as ordinate, for constant H, is given in Fig. 9.34b. The depth is shown to be that which yields the maximum discharge, by the following reasoning:

A gate or other obstruction placed at section 3 of Fig. 9.34a can completely stop the flow by making $y = H$. Now, if a small flow is permitted to pass section 3 (holding H constant), the depth y becomes a little less than H, and the discharge is, for example, as shown by point a on the depth-discharge curve. By further lifting of the gate or obstruction at section 3 the discharge-depth relationship follows the upper portion of the curve until the maximum discharge is reached. Any additional removal of downstream obstructions, however, has no effect upon the discharge, because the velocity of flow at section 2 is $\sqrt{gy}$, which is exactly the speed that an elementary wave can travel in still liquid of depth y. Hence, the effect of any additional lowering of the downstream surface elevation cannot travel upstream to affect further the value of y, and the discharge occurs at the maximum value. This depth y, called the *critical depth*, is discussed in Sec. 11.4. The speed of an elementary wave is derived in Sec. 11.9.

By taking dQ/dy and with the result set equal to zero, for constant H,

$$\frac{dQ}{dy} = 0 = L\sqrt{2g(H - y)} + Ly\frac{1}{2}\frac{(-2g)}{\sqrt{2g(H - y)}}$$

and by solving for y,

$$y = \tfrac{2}{3}H$$

After inserting the value of H, that is, $3y/2$, into the equation for velocity v_2,

$$v_2 = \sqrt{gy}$$

After substituting the value of y into Eq. (9.5.29),

$$Q_t = 3.09LH^{\frac{3}{2}} \tag{9.5.30}$$

Experiments show that for a well-rounded upstream edge the discharge is

$$Q = 3.03LH^{\frac{3}{2}} \tag{9.5.31}$$

which is within 2 per cent of the theoretical value. The flow, therefore, adjusts itself to discharge at the maximum rate.

Viscosity and surface tension have a minor effect on the discharge coefficients of weirs. Therefore, a weir should be calibrated with the liquid that it will measure.

Mass Meter. Most rate meters determine the volumetric flow rate, which requires a separate determination of density before the mass flow rate can be found. By means of the moment-of-momentum principle, measurement of torque on an impeller may be related to the mass flow rate. Consider the impeller of Fig. 9.35, in which fluid flows into the impeller without prerotation, i.e., with $V_{u1} = 0$. The impeller has many blades, which are radial at the exit section, so that $V_{u2} = r_2\omega$, with ω the speed of rotation. Then, from Eq. (3.11.4)

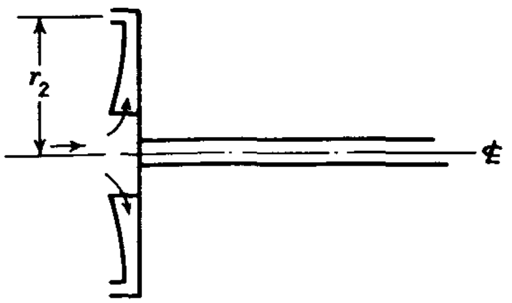

Fig. 9.35. Schematic view of impeller with radial closed blades for use as a mass meter.

$$T = \dot{m}V_{u2}r_2 = \dot{m}\omega r_2^2 \quad (9.5.32)$$

in which T is the torque applied and $\dot{m}$ is the mass per unit time being discharged. By determination of torque, speed, and radius the mass rate may be computed. The torque must be corrected for torque losses due to bearings and disk friction. Practical details of mass meters are discussed in the literature.[1]

9.6. Electromagnetic Flow Devices. If a magnetic field is set up across a nonconducting tube and a conducting fluid flows through the tube, an induced voltage is produced across the flow which may be measured if electrodes are embedded in the tube walls.[2] The voltage is a linear function of the volume rate passing through the tube. Either an alternating- or a direct-current field may be used, with a corresponding signal generated at the electrodes. A disadvantage of the method is the small signal received and the large amount of amplification needed. The device has been used to measure the flow in blood vessels.

9.7. Measurement of River Flow. Daily records of the discharge of rivers over long periods of time are essential to economic planning for utilization of their water resources or for protection against floods. The daily measurement of discharge by determining velocity distribution over a cross section of the river is costly. To avoid this and still obtain daily records, *control sections* are established where the river channel is stable, i.e., with little change in bottom or sides of the stream bed. The control section is frequently at a break in slope of the river bottom where it becomes steeper downstream.

A gage rod is mounted at the control section so that the elevation of water surface is determined by reading the water line on the rod; in some installations float-controlled recording gages keep a continuous record of river elevation. A *gage height-discharge* curve is established by taking

[1] V. A. Olando and F. B. Jennings, Momentum Principle Measures Mass Rate of Flow, *Trans. ASME*, vol. 76, p. 961, August, 1954. V. T. Li and S. Y. Lee, A Fast Responsive True Mass-rate Flowmeter, *Trans. ASME*, vol. 74, p. 835, July, 1953.

[2] H. G. Elrod, Jr., and R. R. Fouse, An Investigation of Electromagnetic Flowmeters, *Trans. ASME*, vol. 74, p. 589, May, 1952.

current-meter measurements from time to time as the river discharge changes and plotting the resulting discharge against the gage height.

With a stable control section the gage height-discharge curve changes very little, and current-meter measurements are infrequent. For unstable control sections the curve changes continuously, and discharge measurement must be made every few days to maintain an accurate curve.

Daily readings of gage height produce a daily record of the river discharge.

9.8. Measurement of Turbulence. Turbulence is a characteristic of the flow. It affects the calibration of measuring instruments and has an important effect upon heat transfer, evaporation, diffusion, and many other phenomena connected with fluid movement.

Turbulence is generally specified by two quantities, the *size* and the *intensity* of the fluctuations. In steady flow the temporal mean velocity components at a point are constant. If these mean values be $\bar{u}$, $\bar{v}$, $\bar{w}$ and the velocity components at an instant be u, v, w, the fluctuations are given by u', v', w', in

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$w = \bar{w} + w'$$

The root-mean-square of measured values of the fluctuations (Fig. 9.36) is a measure of the intensity of the turbulence. These are $\sqrt{\overline{u'^2}}$, $\sqrt{\overline{v'^2}}$, $\sqrt{\overline{w'^2}}$.

The size of the fluctuation is an average measure of the size of eddy, or vortex, in the flow. When two velocity measuring instruments (hot-wire anemometers) are placed adjacent to each other in a flow, the velocity fluctuations are correlated, i.e., they tend to change in unison. Separating these instruments reduces this correlation. The distance between instruments for zero correlation is a measure of the size of the fluctuation. Another method for determining turbulence is discussed in Sec. 5.5.

9.9. Measurement of Viscosity. This chapter on fluid measurement is concluded with a discussion of methods for determining viscosity. Viscosity may be measured in a number of ways: (*a*) by use of Newton's law of viscosity; (*b*) by use of the Hagen-Poiseuille equation; (*c*) by methods that require calibration with fluids of known viscosity.

By measurement of the velocity gradient du/dy and the shear stress τ, in Newton's law of viscosity [Eq. (1.1.1)],

$$\tau = \mu \frac{du}{dy} \tag{9.9.1}$$

the dynamic or absolute viscosity can be computed. This is the most basic method as it determines all other quantities in the defining equation for viscosity. By means of a cylinder that rotates at a known speed with respect to an inner concentric stationary cylinder, du/dy is determined. By measurement of torque on the stationary cylinder, the shear stress may be computed. The ratio of shear stress to rate of change of velocity expresses the viscosity.

A schematic view of a concentric-cylinder viscometer is shown in Fig. 9.37. When the

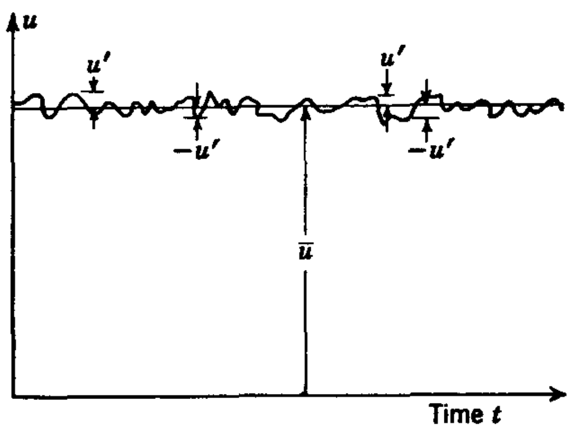

FIG. 9.36. Turbulent fluctuations in direction of flow.

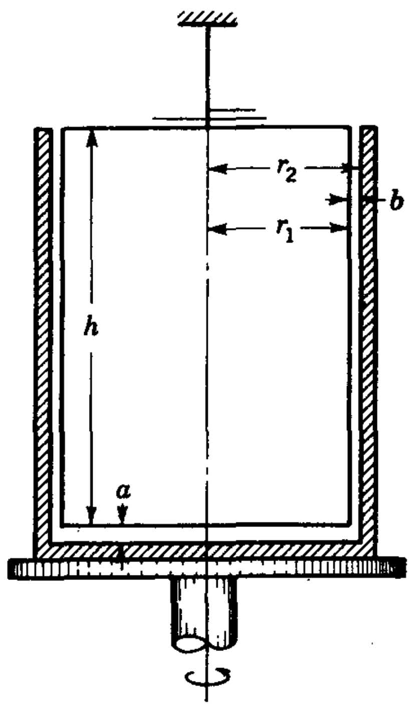

FIG. 9.37. Concentric-cylinder viscometer.

speed of rotation is N rpm and the radius is r_2 ft, the fluid velocity at the surface of the outer cylinder is $2\pi r_2 N/60$. With clearance b ft

$$\frac{du}{dy} = \frac{2\pi r_2 N}{60b}$$

The torque T_c on the inner cylinder is measured by a torsion wire from which it is suspended. By attaching a disk to the wire, its rotation may be determined by a fixed pointer. If the torque due to fluid below the bottom of the inner cylinder is neglected, the shear stress is

$$\tau = \frac{T_c}{2\pi r_1{}^2 h}$$

By substituting into Eq. (9.9.1) and solving for the viscosity,

$$\mu = \frac{15T_c b}{\pi^2 r_1{}^2 r_2 h N} \qquad (9.9.2)$$

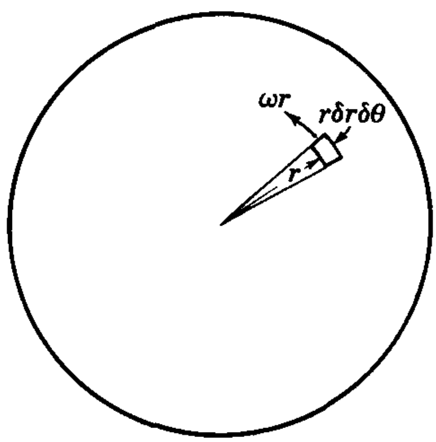

FIG. 9.38. Notation for determination of torque on a disk.

When the clearance a is so small that the torque contribution from the bottom is appreciable, it may be calculated in terms of the viscosity.

Referring to Fig. 9.38,

$$\delta T = r\tau\,\delta A = r\mu\frac{\omega r}{a}r\,\delta r\,\delta\theta$$

in which the velocity change is ωr in the distance a ft. By integrating over the circular area of the disk and letting $\omega = 2\pi N/60$,

$$T_d = \frac{\mu}{a}\frac{\pi}{30}N\int_0^{r_1}\int_0^{2\pi} r^3\,dr\,d\theta = \frac{\mu\pi^2}{a60}Nr_1{}^4 \tag{9.9.3}$$

The torque due to disk and cylinder must equal the torque T in the torsion wire, so

$$T = \frac{\mu\pi^2 N r_1{}^4}{a60} + \frac{\mu\pi^2 r_1{}^2 r_2 h N}{15b} = \frac{\mu\pi^2 N r_1{}^2}{15}\left(\frac{r_1{}^2}{4a} + \frac{r_2 h}{b}\right) \tag{9.9.4}$$

in which all quantities are known except μ. The flow between the surfaces must be laminar for Eqs. (9.9.2) to (9.9.4) to be valid.

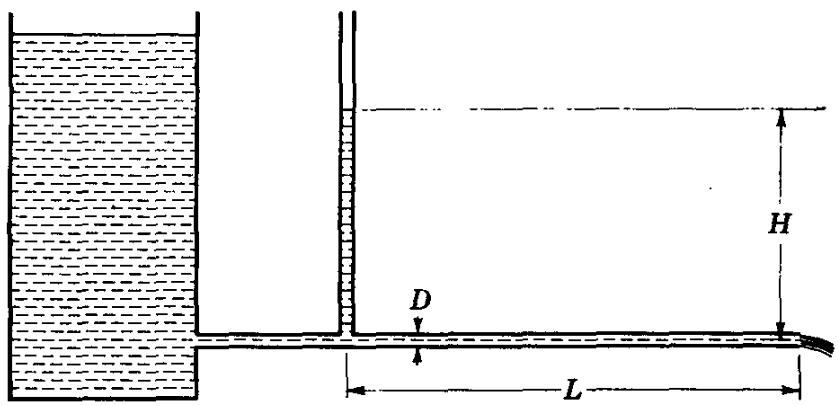

FIG. 9.39. Determination of viscosity by flow through a capillary tube.

The measurement of all quantities in the Hagen-Poiseuille equation, except μ, by a suitable experimental arrangement, is another basic method for determination of viscosity. A setup as in Fig. 9.39 may be used. Some distance is required for the fluid to develop its characteristic velocity distribution after it enters the tube; therefore, the head or pressure must be measured by some means at a point along the tube. The volume $V\!\!\!\!-$ of flow can be measured over a time t where the reservoir surface is held at a constant level. This yields Q, and, by determining γ, Δp may be computed. Then with L and D known, from Eq. (5.2.6)

$$\mu = \frac{\Delta p\pi D^4}{128QL}$$

Since it is difficult to measure the pressure in the tube and to determine its diameter and be sure it is uniform, an adaptation of the capillary tube for industrial purposes is the *Saybolt* viscometer (Fig. 9.40). A short capillary tube is utilized, and the time is measured for 60 cm^3 of fluid to flow through the tube under a falling head. The time in seconds is the Saybolt reading. This device measures kinematic viscosity, evident from a rearrangement of Eq. (5.2.6). When $\Delta p = \rho g h$, $Q = \text{vol.}/t$, and when the terms are separated that are the same regardless of the fluid,

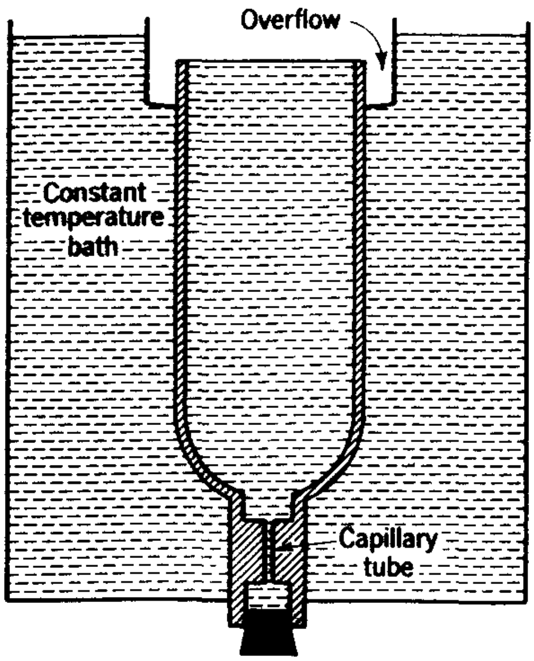

FIG. 9.40. Schematic view of Saybolt viscometer.

$$\frac{\mu}{\rho t} = \frac{g h \pi D^4}{128(\text{vol.})L} = C_1$$

Although the head h varies during the test, it varies over the same range for all liquids; and the terms on the right-hand side may be considered as a constant of the particular instrument. Since $\mu/\rho = \nu$, the kinematic viscosity is

$$\nu = C_1 t$$

which shows that the kinematic viscosity varies directly as the time t. The capillary tube is quite short, so the velocity distribution is not established. The flow tends to enter uniformly; and then, owing to viscous drag at the walls, to slow down there and speed up in the center region. A correction in the above equation is needed, which is of the form C/t; hence

$$\nu = C_1 t + \frac{C_2}{t}$$

The approximate relationship between viscosity and Saybolt seconds is expressed by

$$\nu = 0.0022t - \frac{1.80}{t}$$

in which ν is in stokes and t in seconds.

For measuring viscosity there are many other industrial methods that generally have to be calibrated for each special case to convert to the absolute units. One consists of several tubes containing "standard" liquids of known graduated viscosities with a steel ball in each of the tubes. The time for the ball to fall the length of the tube depends upon the viscosity of the liquid. By placing the test sample in a similar tube, its viscosity may be approximated by comparison with the other tubes.

PROBLEMS

9.1. A static tube (Fig. 9.2) indicates a static pressure that is 0.12 psi too low when liquid is flowing at 8 ft/sec. Calculate the correction to be applied to the indicated pressure for the liquid flowing at 14 ft/sec.

9.2. Four piezometer openings in the same cross section of a cast-iron pipe indicate the following pressures: 4.30, 4.26, 4.24, 3.7 psi for simultaneous readings. What value should be taken for the pressure?

9.3. A simple pitot tube (Fig. 9.5) is inserted into a small stream of flowing oil, $\gamma = 55$ lb/ft^3, $\mu = 0.65$ poise, $\Delta h = 2$ in., $h_0 = 5$ in. What is the velocity at point 1?

9.4. A stationary body immersed in a river has a maximum pressure of 10 psi exerted on it at a distance of 20 ft below the free surface. Calculate the river velocity at this depth.

9.5. From Fig. 9.6 derive the equation for velocity at 1.

9.6. In Fig. 9.6 air is flowing ($p = 16$ psia, $t = 40°$F) and water is in the manometer. For $R' = 1.2$ in., calculate the velocity of air.

9.7. In Fig. 9.6 air is flowing ($p = 16$ psia, $t_1 = 40°$F) and mercury is in the manometer. For $R' = 6$ in., calculate the velocity at 1 (*a*) for isentropic compression of air between 1 and 2 and (*b*) for air considered incompressible.

9.8. A pitot-static tube directed into a 12 ft/sec water stream has a gage difference of 1.47 in. on a water-mercury differential manometer. Determine the coefficient for the tube.

9.9. A pitot-static tube, $C = 1.32$, has a gage difference of 2.7 in. on a water-mercury manometer when directed into a water stream. Calculate the velocity.

9.10. A pitot-static tube of the Prandtl type has the following value of gage difference R' for the radial distance from center of a 3-ft-diameter pipe:

r, ft	0	0.3	0.6	0.9	1.2	1.48
R', in.	4	3.91	3.76	3.46	3.02	2.40

Water is flowing, and the manometer fluid has a specific gravity of 2.93. Calculate the discharge.

9.11. What would be the gage difference on a water-nitrogen manometer for flow of nitrogen at 600 ft/sec, using a pitot-static tube? The static pressure is 18 psia, and corresponding temperature 80°F True static pressure is measured by the tube.

9.12. Measurements in an air stream indicate that the stagnation pressure is 12 psia, the static pressure is 10 psia, and the stagnation temperature is 102°F. Determine the temperature and velocity of the air stream.

9.13. 0.1 lb$_m$/sec nitrogen flows through a 2-in.-diameter tube with stagnation temperature of 90°F and undisturbed temperature of 60°F. Find the velocity and static and stagnation pressures.

9.14. A disk meter has a volumetric displacement of 1.73 in.^{3} for one complete oscillation. Calculate the flow in gallons per minute for 173 oscillations per minute.

9.15. A disk water meter with volumetric displacement of 2.40 in.^{3} per oscillation requires 470 oscillations per minute to pass 5 gpm and 3840 oscillations per minute to pass 40 gpm. Calculate the per cent error, or slip, in the meter.

9.16. A volumetric tank 4 ft in diameter and 5 ft high was filled with oil in 16 min 32.4 sec. What is the average discharge in gallons per minute?

9.17. A weigh tank receives 13.6 lb liquid, sp gr 0.86, in 14.9 sec. What is the flow rate in gallons per minute?

9.18. Determine the equation for trajectory of a jet discharging horizontally from a small orifice with head of 16 ft and velocity coefficient of 0.96. Neglect air resistance.

9.19. An orifice of area 0.03 ft^2 in a vertical plate has a head of 3.6 ft of oil, sp gr 0.91. It discharges 1418 lb of oil in 79.3 sec. Trajectory measurements yield $X = 7.38$ ft, $Y = 4.025$ ft. Determine C_v, C_c, C_d.

9.20. Calculate Y, the maximum rise of a jet from an inclined plate (Fig. 9.41) in terms of H and α. Neglect losses.

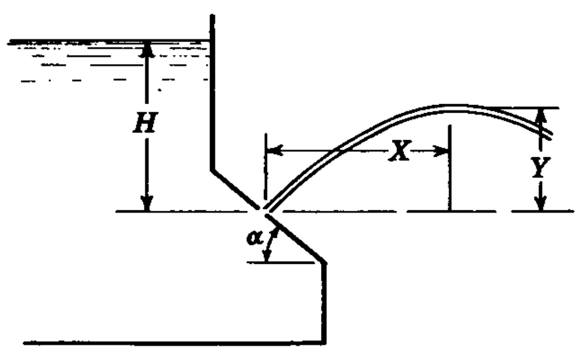

FIG. 9.41

9.21. In Fig. 9.41, for $\alpha = 45°$, $Y = 0.48H$. Neglecting air resistance of the jet, find C_v for the orifice.

9.22. Show that the locus of maximum points of the jet of Fig. 9.41 is given by

$$X^2 = 4Y(H - Y)$$

when losses are neglected.

9.23. A 3-in.-diameter orifice discharges 64 ft^3 liquid, sp gr 1.07, in 82.2 sec under a 9 ft head. The velocity at the *vena contracta* is determined by a pitot-static tube with coefficient 1.17. The manometer liquid is acetylene tetrabromide, sp gr 2.96, and the gage difference is $R' = 3.35$ ft. Determine C_v, C_c, and C_d.

9.24. A 4-in.-diameter orifice discharges 1.575 cfs water under a head of 9 ft. A flat plate held normal to the jet just downstream from the *vena contracta* requires a force of 69.7 lb to resist impact of the jet. Find C_d, C_v, and C_c.

9.25. Compute the discharge from the tank shown in Fig. 9.42.

9.26. For $C_v = 0.96$ in Fig. 9.42, calculate the losses in foot-pounds per pound and in foot-pounds per second.

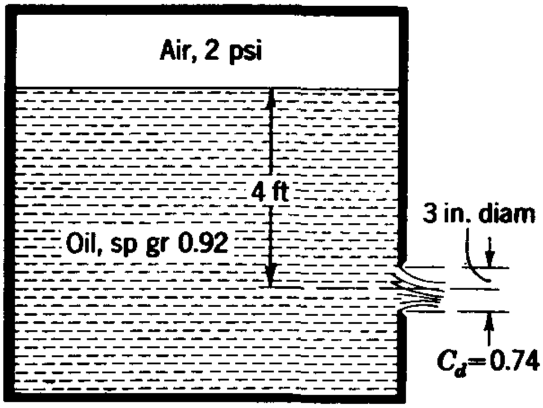

FIG. 9.42

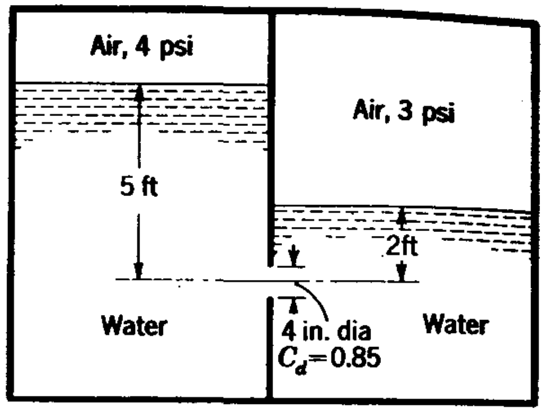

FIG. 9.43

9.27. Calculate the discharge through the orifice of Fig. 9.43.

9.28. For $C_v = 0.93$ in Fig. 9.43, determine the losses in foot-pounds per pound and in foot-pounds per second.

9.29. A 4-in.-diameter orifice discharges 1.60 cfs liquid under a head of 11.8 ft. The diameter of jet at the *vena contracta* is found by calipering to be 3.47 in. Calculate C_v, C_d, and C_c.

9.30. A Borda mouthpiece 3 in. in diameter has a discharge coefficient of 0.51. What is the diameter of the issuing jet?

9.31. A 3-in.-diameter orifice, $C_d = 0.82$, is placed in the bottom of a vertical tank that has a diameter of 4 ft. How long does it take to draw the surface down from 8 to 4 ft?

9.32. Select the size of orifice that permits a tank of horizontal cross section 16 ft² to have the liquid surface drawn down at the rate of 0.6 ft/sec for 11 ft head on the orifice. $C_d = 0.63$.

9.33. A 4-in.-diameter orifice in the side of a 6-ft-diameter tank draws the surface down from 8 to 4 ft above the orifice in 83.7 sec. Calculate the discharge coefficient.

9.34. Select a reservoir of such size and shape that the liquid surface drops 3 ft/min over a 10-ft distance for flow through a 4-in.-diameter orifice. $C_d = 0.74$.

9.35. In Fig. 9.44 the truncated cone has an angle $\theta = 30°$. How long does it take to draw the liquid surface down from $y = 12$ ft to $y = 4$ ft?

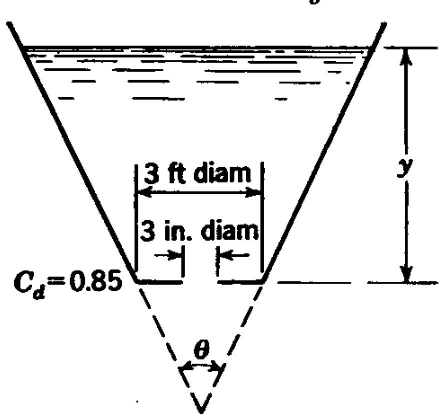

FIG. 9.44

9.36. Calculate the dimensions of a tank such that the surface velocity varies inversely as the distance from the center line of an orifice draining the tank. When the head is 1 ft, the velocity of fall of the surface is 0.1 ft/sec. 0.5-in.-diameter orifice, $C_d = 0.66$.

9.37. Determine the time required to raise the right-hand surface of Fig. 9.45 by 2 ft.

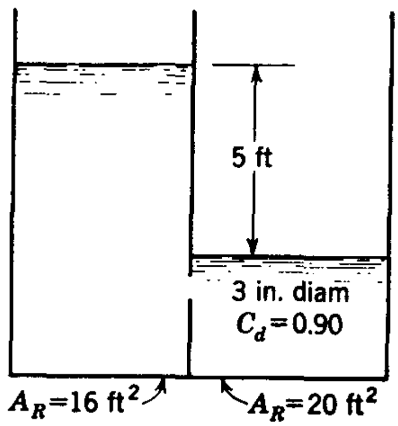

FIG. 9.45

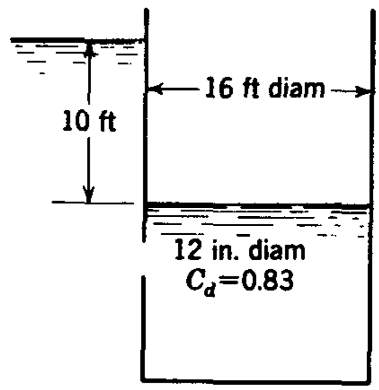

FIG. 9.46

9.38. How long does it take to raise the water surface of Fig. 9.46 6 ft? The left-hand surface is a large reservoir of constant water-surface elevation.

9.39. Show that for incompressible flow the losses per unit weight of fluid between the upstream section and throat of a venturi meter are $KV_2^2/2g$ if $K = [(1/C_v)^2 - 1][1 - (D_2/D_1)^4]$.

9.40. A 200- by 100-in. venturi meter carries water at 98°F. A water-air differential manometer has a gage difference of 2.4 in. What is the discharge?

9.41. What is the pressure difference between the upstream section and throat of a 6- by 3-in. horizontal venturi meter carrying 600 gpm water at 120°F?

9.42. A 12- by 6-in. venturi meter is mounted in a vertical pipe with the flow upward. 1000 gpm oil, sp gr 0.80, $\mu = 1$ poise, flows through the pipe. The throat section is 4 in. above the upstream section. What is $p_1 - p_2$?

9.43. Air flows through a venturi meter in a 2-in.-diameter pipe having a throat diameter of 1.25 in., $C_v = 0.97$. For $p_1 = 120$ psia, $t_1 = 60$°F, $p_2 = 90$ psia, calculate the mass per second flowing.

9.44. Oxygen, $p_1 = 40$ psia, $t_1 = 120$°F, flows through a 1- by $\frac{1}{2}$-in. venturi meter with a pressure drop of 15 psia. Find the mass per second flowing and the throat velocity.

9.45. Air flows through a 3-in.-diameter VDI flow nozzle in a 4-in. diameter pipe. $p_1 = 20$ psia, $t_1 = 40$°F, and a differential manometer with liquid, sp gr 1.37, has a gage difference of 2.7 ft when connected between the pressure taps. Calculate the mass rate of flow.

9.46. A 2.5-in.-diameter VDI nozzle is used to measure flow of water at 40°F in a 6-in.-diameter pipe. What gage difference on a water-mercury manometer is required for 200 gpm?

9.47. Determine the discharge in an 8-in.-diameter line with a 5-in.-diameter VDI orifice for water at 68°F when the gage difference is 12 in. on an acetylene tetrabromide (sp gr 2.94)-water differential manometer.

9.48. A $\frac{1}{2}$-in.-diameter VDI orifice is installed in a 1-in.-diameter pipe carrying nitrogen at $p_1 = 120$ psia, $t_1 = 120°F$. For a pressure drop of 23 psi across the orifice, calculate the mass flow rate.

9.49. Air at 14.7 psia, $t = 72°F$, flows through a 36-in.-square duct that contains an 18-in.-diameter square-edged orifice. With a 3 in. water head loss across the orifice, compute the flow in cubic feet per minute.

9.50. A 4-in.-diameter VDI orifice is installed in a 12-in.-diameter oil line, $\mu = 0.06$ poise, $\gamma = 52$ lb/ft³. An oil-air differential manometer is used. For a gage difference of 27 in. determine the flow rate in gallons per minute.

9.51. A rectangular sharp-crested weir 12 ft long with end contractions suppressed is 4 ft high. Determine the discharge when the head is 0.76 ft.

9.52. In Fig. 9.30, $L = 10$ ft, $P = 1.5$ ft, $H = 0.80$ ft. Estimate the discharge over the weir. $C = 3.33$.

9.53. A rectangular sharp-crested weir with end contractions is 4 ft long. How high should it be placed in a channel to maintain an upstream depth of 5 ft for 16 cfs flow?

9.54. Determine the head on a 60° V-notch weir for discharge of 6 cfs.

9.55. Tests on a 90° V-notch weir gave the following results: $H = 0.60$ ft, $Q = 0.685$ cfs; $H = 1.35$ ft, $Q = 5.28$ ft. Determine the formula for the weir.

9.56. A sharp-crested rectangular weir 2.5 ft long with end contractions suppressed and a 90° V-notch weir are placed in the same weir box, with the vertex of the 90° V-notch weir 6 in. below the rectangular weir crest. Determine the head on the V-notch weir (*a*) when the discharges are equal and (*b*) when the rectangular weir discharges its greatest amount above the discharge of the V-notch weir.

9.57. A broad-crested weir 5 ft high and 10 ft long has a well-rounded upstream corner. What head is required for a flow of 100 cfs?

9.58. A circular disk 6 in. in diameter has a clearance of 0.012 in. from a flat plate. What torque is required to rotate the disk 800 rpm when the clearance contains oil, $\mu = 0.8$ poise?

9.59. The concentric-cylinder viscometer (Fig. 9.37) has the following dimensions: $a = 0.012$ in.; $b = 0.02$ in.; $r_1 = 2.8$ in.; $h = 6.0$ in. The torque is 24 lb-in. when the speed is 160 rpm. What is the viscosity?

9.60. With the apparatus of Fig. 9.39, $D = 0.020$ in., $L = 36$ in., $H = 2.4$ ft, and 60 cm³ was discharged in 1 hr 20 min. What is the viscosity in poise? $\gamma = 52$ lb/ft³.

9.61. The piezoelectric properties of quartz are used to measure

(*a*) temperature (*b*) density (*c*) velocity (*d*) pressure
(*e*) none of these answers.

9.62. A static tube is used to measure

(*a*) the pressure in a static fluid
(*b*) the velocity in a flowing stream

(c) the total pressure
(d) the dynamic pressure
(e) the undisturbed fluid pressure

9.63. A piezometer opening is used to measure

(a) the pressure in a static fluid
(b) the velocity in a flowing stream
(c) the total pressure
(d) the dynamic pressure
(e) the undisturbed fluid pressure

9.64. The simple pitot tube measures the

(a) static pressure
(b) dynamic pressure
(c) total pressure
(d) velocity at the stagnation point
(e) difference in total and dynamic pressure

9.65. A pitot-static tube ($C = 1$) is used to measure air speeds. With water in the differential manometer and a gage difference of 3 in., the air speed for $\gamma = 0.0624\ \text{lb/ft}^3$, in feet per second, is

(a) 4.01 (b) 15.8 (c) 24.06 (d) 127 (e) none of these answers

9.66. The pitot-static tube measures

(a) static pressure
(b) dynamic pressure
(c) total pressure
(d) difference in static and dynamic pressure
(e) difference in total and dynamic pressure

9.67. The temperature of a known flowing gas may be determined from measurement of

(a) static and stagnation pressure only
(b) velocity and stagnation pressure only
(c) velocity and dynamic pressure only
(d) velocity and stagnation temperature only
(e) none of these answers

9.68. The velocity of a known flowing gas may be determined from measurement of

(a) static and stagnation pressure only
(b) static pressure and temperature only
(c) static and stagnation temperature only
(d) stagnation temperature and stagnation pressure only
(e) none of these answers

9.69. The hot-wire anemometer is used to measure

(*a*) pressure in gases
(*b*) pressure in liquids
(*c*) wind velocities at airports
(*d*) gas velocities
(*e*) liquid discharges

9.70. Snell's law relates

(*a*) pressure and density in optical measurements
(*b*) angle of incidence, angle of refraction, and index of refraction
(*c*) velocity, pressure, and piezoelectric properties of crystals
(*d*) density and index of refraction
(*e*) none of these answers

9.71. The Gladstone-Dale equation relates

(*a*) pressure and density in optical measurements
(*b*) angle of incidence, angle of refraction, and index of refraction
(*c*) velocity, pressure, and piezoelectric properties of crystals
(*d*) density and index of refraction
(*e*) none of these answers

9.72. The Schlieren optical system portrays

(*a*) temperature changes in gas flow
(*b*) pressure changes in gas flow
(*c*) density changes in gas flow
(*d*) density gradient changes in gas flow
(*e*) none of these answers

9.73. The shadowgraph optical system portrays

(*a*) temperature changes in gas flow
(*b*) pressure changes in gas flow
(*c*) density changes in gas flow
(*d*) density gradient changes in gas flow
(*e*) none of these answers

9.74. The interferometer optical system

(*a*) makes use of a knife-edge
(*b*) requires two light sources
(*c*) depends upon a phase shift in light wave motion
(*d*) splits the light from a single source into three circuits
(*e*) satisfies none of these answers

9.75. A piston-type displacement meter has a volume displacement of 2.15 in.^{3} per revolution of its shaft. The discharge in gallons per minute for 1000 rpm is

(*a*) 0.497 (*b*) 1.23 (*c*) 9.3 (*d*) 10.72 (*e*) none of these answers

9.76. Water for a pipeline was diverted into a weigh tank for exactly 10 min. The increased weight in the tank was 4765 lb. The average flow rate in gallons per minute was

(*a*) 66.1 (*b*) 57.1 (*c*) 7.95 (*d*) 0.13 (*e*) none of these answers

9.77. A rectangular tank with cross-sectional area of 90 ft² was filled to a depth of 4.00 ft by a steady flow of liquid for 12 min. The rate of flow in cubic feet per second was

(*a*) 0.50 (*b*) 30 (*c*) 31.2 (*d*) 224 (*e*) none of these answers

9.78. Which of the following measuring instruments is a rate meter?

(*a*) current meter (*b*) disk meter (*c*) hot-wire anemometer
(*d*) pitot tube (*e*) venturi meter

9.79. The actual velocity at the *vena contracta* for flow through an orifice from a reservoir is expressed by

(*a*) $C_v \sqrt{2gH}$ (*b*) $C_c \sqrt{2gH}$ (*c*) $C_d \sqrt{2gH}$ (*d*) $\sqrt{2gH}$
(*e*) $C_v V_a$

9.80. A fluid jet discharging from a 2-in.-diameter orifice has a diameter 1.75 in. at its *vena contracta*. The coefficient of contraction is

(*a*) 1.31 (*b*) 1.14 (*c*) 0.875 (*d*) 0.766 (*e*) none of these answers

9.81. The ratio of actual discharge to theoretical discharge through an orifice is

(*a*) C_cC_v (*b*) C_cC_d (*c*) C_vC_d (*d*) C_d/C_v (*e*) C_d/C_c

9.82. The losses in orifice flow are

(*a*) $\dfrac{1}{C_v^2}\left(\dfrac{V_{2a}^2}{2g} - 1\right)$ (*b*) $\dfrac{V_{2t}^2}{2g} - \dfrac{V_{2a}^2}{2g}$ (*c*) $H(C_v^2 - 1)$
(*d*) $H - V_{2t}^2/2g$ (*e*) none of these answers

9.83. For a liquid surface to lower at a constant rate, the area of reservoir A_R must vary with head y on the orifice, as

(*a*) $\sqrt{y}$ (*b*) y (*c*) $1/\sqrt{y}$ (*d*) $1/y$ (*e*) none of these answers

9.84. A 2-in.-diameter Borda mouthpiece discharges 0.268 cfs under a head of 9.0 ft. The velocity coefficient is

(*a*) 0.96 (*b*) 0.97 (*c*) 0.98 (*d*) 0.99 (*e*) none of these answers

9.85. The discharge coefficient for a 4-in. by 2-in. venturi meter at a Reynolds number of 200,000 is

(*a*) 0.95 (*b*) 0.96 (*c*) 0.97 (*d*) 0.98 (*e*) 0.99

9.86. Select the correct statement:

(*a*) The discharge through a venturi meter depends upon Δp only and is independent of orientation of the meter.
(*b*) A venturi meter with a given gage difference R' discharges at a greater rate when the flow is vertically downward through it than when the flow is vertically upward.
(*c*) For a given pressure difference the equations show that the discharge of gas is greater through a venturi meter when compressibility is taken into account than when it is neglected.
(*d*) The coefficient of contraction of a venturi meter is unity.
(*e*) The over-all loss is the same in a given pipeline whether a venturi meter or a nozzle with the same D_2 is used.

9.87. The expansion factor Y depends upon

(*a*) k, p_2/p_1, and A_2/A_1
(*b*) R, p_2/p_1, and A_2/A_1
(*c*) k, R, and p_2/p_1
(*d*) k, R, and A_2/A_1
(*e*) none of these answers

9.88. The discharge through a V-notch weir varies as

(*a*) $H^{-\frac{1}{2}}$ (*b*) $H^{\frac{1}{2}}$ (*c*) $H^{\frac{3}{2}}$ (*d*) $H^{\frac{5}{2}}$ (*e*) none of these answers

9.89. The discharge of a rectangular sharp-crested weir with end contractions is less than for the same weir with end contractions suppressed by

(*a*) 5% (*b*) 10% (*c*) 15% (*d*) no fixed percentage
(*e*) none of these answers

9.90. A mass meter, with radius at exit of impeller of 6 in. turns at 1200 rpm and has a torque applied of 36 lb-in. The mass per second flowing is, in pounds mass per second,

(*a*) 0.00796 (*b*) 0.096 (*c*) 0.478 (*d*) 3.08 (*e*) none of these answers

9.91. A homemade viscometer of the Saybolt type is calibrated by two measurements with liquids of known kinematic viscosity. For $\nu = 0.461$ stoke, $t = 97$ sec, and for $\nu = 0.18$ stoke, $t = 46$ sec. The coefficients C_1, C_2 in $\nu = C_1 t + C_2/t$ are

(*a*) $C_1 = 0.005$, $C_2 = -2.3$ (*b*) $C_1 = 0.0044$, $C_2 = 3.6$ (*c*) $C_1 = 0.0046$, $C_2 = 1.55$
(*d*) $C_1 = 0.00317$, $C_2 = 14.95$ (*e*) none of these answers

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10

CLOSED-CONDUIT FLOW

The basic procedures for solving problems in incompressible steady flow in closed conduits are presented in Sec. 5.9, where simple pipe-flow situations are discussed, including losses due to change in cross section or direction of flow. Compressible flow in ducts is treated in Secs. 6.6 to 6.8. Velocity distributions in turbulent flow are discussed in Sec. 5.4. This chapter deals with flow situations and applications more complex than those in Chap. 5. It is divided into two parts: the first part dealing with incompressible steady turbulent-flow situations, the second part introducing some of the methods of analyzing unsteady flow in pipes.

STEADY FLOW IN CONDUITS

10.1. Hydraulic and Energy Grade Lines. The concepts of *hydraulic* and *energy* grade lines are useful in analyzing more complex flow problems. If, at each point along a pipe system, the term p/γ is determined and plotted as a vertical distance above the center of the pipe, the locus of end points is the hydraulic grade line. More generally, the plot of the two terms

$$\frac{p}{\gamma} + z$$

for the flow, as ordinates, against length along the pipe as abscissas, produces the hydraulic grade line. The hydraulic grade line is the locus of heights to which liquid would rise in vertical glass tubes connected to piezometer openings in the line. When the pressure in the line is less than atmospheric, p/γ is negative and the hydraulic grade line is below the pipeline.

The energy grade line is a line joining a series of points marking the available energy in foot-pounds per pound for each point along the pipe as ordinate, plotted against distance along the pipe as the abscissa. It consists of the plot of

$$\frac{V^2}{2g} + \frac{p}{\gamma} + z \quad ,$$

for each point along the line. By definition, the energy grade line is always vertically above the hydraulic grade line a distance of $V^2/2g$, neglecting the kinetic-energy correction factor.

The hydraulic and energy grade lines are shown in Fig. 10.1 for a simple pipeline containing a square-edged entrance, a valve, and a nozzle at the end of the line. To construct these lines when the reservoir surface is given, it is necessary first to apply Bernoulli's equation from the reservoir to the exit, including all minor losses as well as pipe friction, and to solve for the velocity head $V^2/2g$. Then, to find the elevation of hydraulic grade line at any point, Bernoulli's equation is applied from

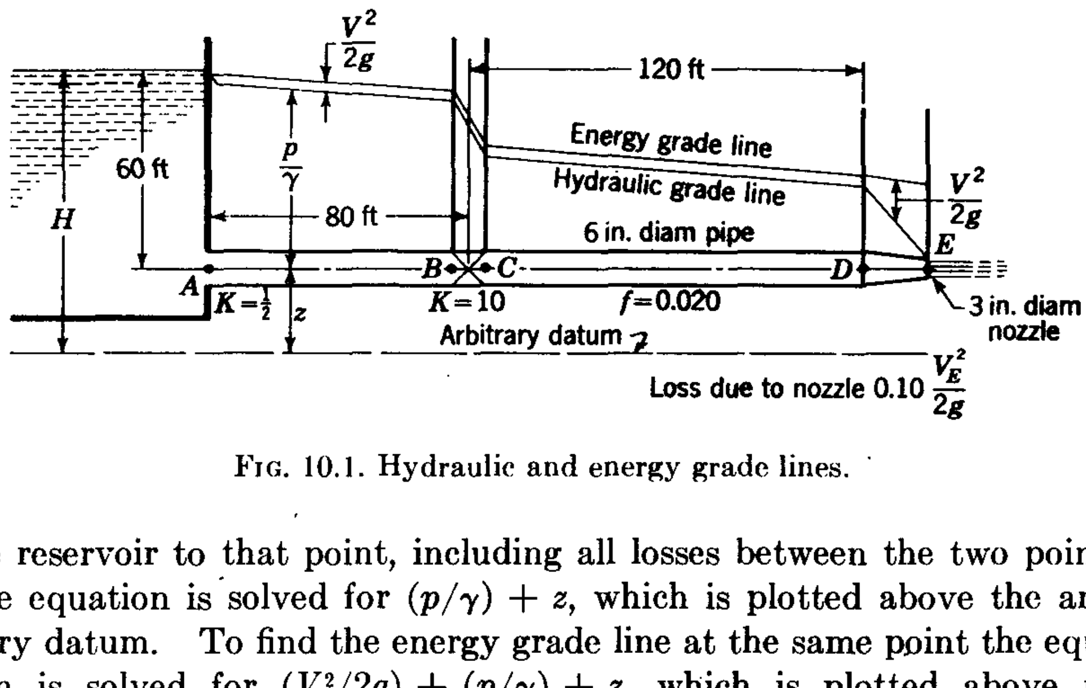


FIG. 10.1. Hydraulic and energy grade lines.

the reservoir to that point, including all losses between the two points. The equation is solved for $(p/\gamma) + z$, which is plotted above the arbitrary datum. To find the energy grade line at the same point the equation is solved for $(V^2/2g) + (p/\gamma) + z$, which is plotted above the arbitrary datum.

The reservoir surface is the hydraulic grade line and is also the energy grade line. At the square-edged entrance the energy grade line drops by $0.5V^2/2g$ because of the loss there, and the hydraulic grade line drops $1.5V^2/2g$. This is made obvious by applying Bernoulli's equation between the reservoir surface and a point just downstream from the pipe entrance:

$$H + 0 + 0 = \frac{V^2}{2g} + z + \frac{p}{\gamma} + 0.5\frac{V^2}{2g}$$

Solving for $z + p/\gamma$,

$$z + \frac{p}{\gamma} = H - 1.5\frac{V^2}{2g}$$

shows the drop of $1.5V^2/2g$. The head loss due to the sudden entrance does not actually occur at the entrance itself, but over a distance of 10 or more diameters of pipe downstream. It is customary to show it at the fitting.

Example 10.1: Determine the elevation of hydraulic and energy grade lines at points A, B, C, D, and E of Fig. 10.1.

If the arbitrary datum is selected as center line of the pipe, both grade lines start at elevation 60 ft. First, solving for the velocity head is accomplished by applying the Bernoulli equation from the reservoir to E,

$$60 + 0 + 0 = \frac{V_E^2}{2g} + 0 + 0 + \frac{1}{2}\frac{V^2}{2g} + 0.020 \times \frac{200}{0.50}\frac{V^2}{2g} + 10\frac{V^2}{2g} + 0.10\frac{V_E^2}{2g}$$

From the continuity equation, $V_E = 4V$. After simplifying,

$$60 = \frac{V^2}{2g}\left(16 + \frac{1}{2} + 8 + 10 + 16 \times 0.1\right) = 36.1\frac{V^2}{2g}$$

and $V^2/2g = 1.66$ ft. By applying Bernoulli's equation for the portion from the reservoir to A,

$$60 + 0 + 0 = \frac{V^2}{2g} + \frac{p}{\gamma} + z + 0.5\frac{V^2}{2g}$$

Hence the hydraulic gradient at A is

$$\left.\frac{p}{\gamma} + z\right|_A = 60 - 1.5\frac{V^2}{2g} = 60 - 1.5 \times 1.66 = 57.51 \text{ ft}$$

The energy grade line for A is

$$\frac{V^2}{2g} + z + \frac{p}{\gamma} = 57.51 + 1.66 = 59.17 \text{ ft}$$

For B,

$$60 + 0 + 0 = \frac{V^2}{2g} + \frac{p}{\gamma} + z + 0.5\frac{V^2}{2g} + 0.02 \times \frac{80}{0.5}\frac{V^2}{2g}$$

and

$$\left.\frac{p}{\gamma} + z\right|_B = 60 - (1.5 + 3.2)1.66 = 52.19 \text{ ft}$$

the energy grade line is at $52.19 + 1.66 = 53.85$ ft.

Across the valve the hydraulic grade line drops by $10V^2/2g$, or 16.6 ft. Hence, at C the energy and hydraulic grade lines are at 37.25 ft and 35.59 ft, respectively.

At point D

$$60 = \frac{V^2}{2g} + \frac{p}{\gamma} + z + \left(10.5 + 0.02 \times \frac{200}{0.50}\right)\frac{V^2}{2g}$$

and

$$\left.\frac{p}{\gamma} + z\right|_D = 60 - 19.5 \times 1.66 = 27.6 \text{ ft}$$

with the energy grade line at $27.6 + 1.66 = 29.26$ ft.

At point E the hydraulic grade line is at zero elevation, and the energy grade line is

$$\frac{V_E^2}{2g} = 16\frac{V^2}{2g} = 16 \times 1.66 = 26.6 \text{ ft}$$

The *hydraulic gradient* is the slope of the hydraulic grade line if the conduit is horizontal; otherwise, it is

$$-\frac{d(z + p/\gamma)}{dL}$$

The *energy gradient* is the slope of the energy grade line if the conduit is horizontal; otherwise, it is

$$\frac{d(z + p/\gamma + V^2/2g)}{dL}$$

In many situations involving long pipelines the minor losses may be neglected (when less than 5 per cent of the pipe friction losses) or they

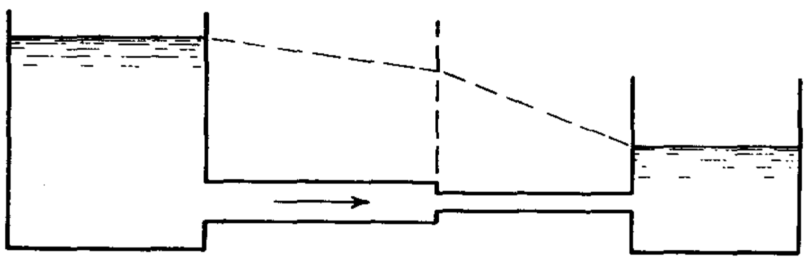


FIG. 10.2. Hydraulic grade line for long pipelines where minor losses are neglected or included as equivalent lengths of pipe.

may be included as equivalent lengths of pipe which are added to actual length in solving the problem. For these situations the value of the velocity head $V^2/2g$ is small compared with $f(L/D)V^2/2g$ and is neglected. The hydraulic grade line is then utilized, as shown in Fig. 10.2. No change in hydraulic grade line is shown for minor losses. For these situations with long pipes the hydraulic gradient becomes h_f/L, with h_f given by the Darcy-Weisbach equation,

$$h_f = f\frac{L}{D}\frac{V^2}{2g} \tag{10.1.1}$$

Flow (except through a pump) is always in the direction of decreasing energy grade line.

Pumps add energy to the flow, a fact which may be expressed in the Bernoulli equation by including as a *negative loss* or by stating the energy per unit weight added as a positive term on the upstream side of the equation. The hydraulic grade line rises sharply at a pump. Figure 10.3 shows the hydraulic and energy grade lines for a system with a pump and a siphon. The true slope of the grade lines can be shown only for horizontal lines.

Example 10.2: A pump with a shaft input of 10 hp and an efficiency of 70 per cent is connected in a water line carrying 3 cfs. The pump has a 6-in.-diameter suction line and a 4-in.-diameter discharge line. The suction line enters the pump 3 ft below the discharge line. For a suction pressure of 10 psi, calculate the pressure at the discharge flange and the rise in the hydraulic grade line across the pump.

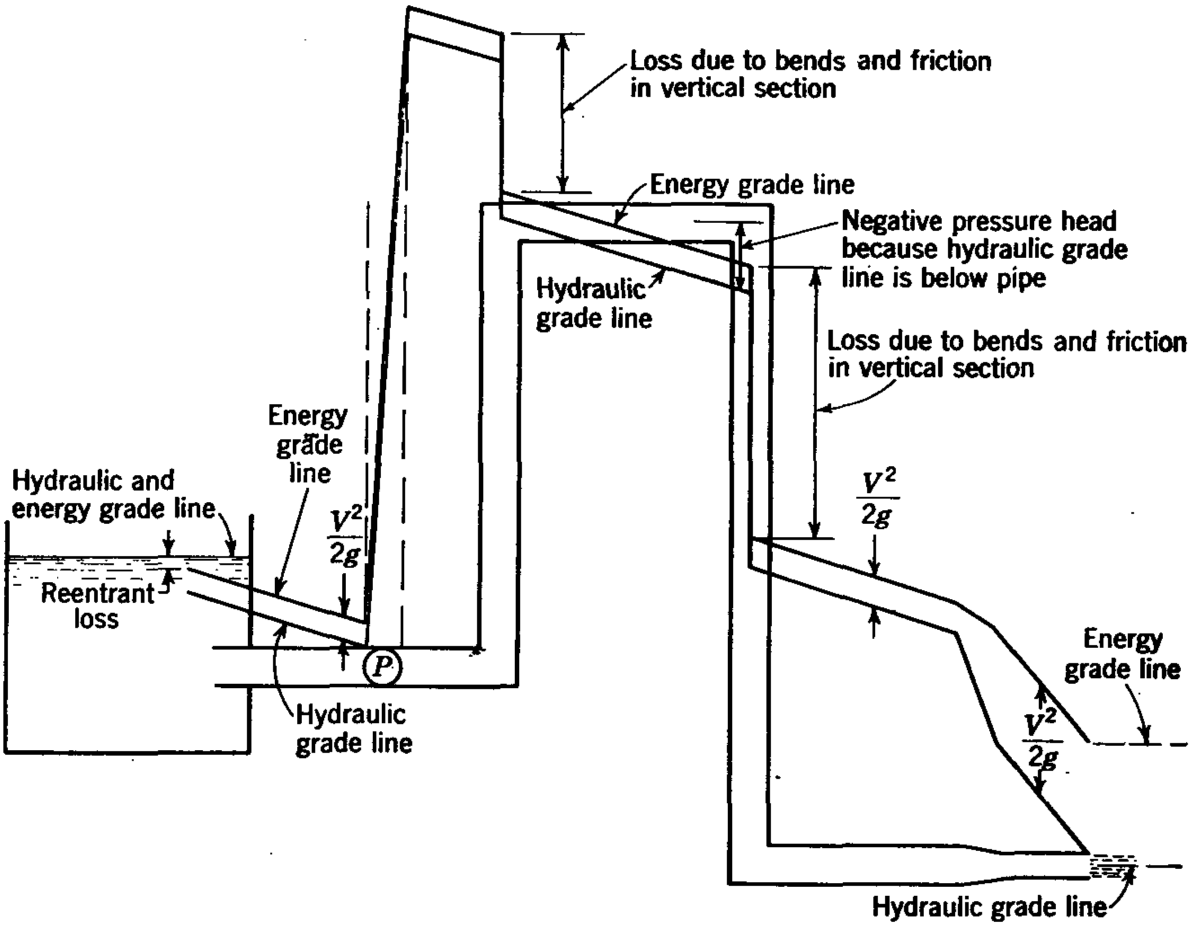


FIG. 10.3. Hydraulic and energy grade lines for a system with pump and siphon.

The energy added in foot-pounds per pound is symbolized by E,

$$\frac{Q\gamma E}{550} = 10 \times 0.7$$

or

$$E = \frac{550 \times 7}{3 \times 62.4} = 20.6 \text{ ft}$$

By applying Bernoulli's equation from suction flange to discharge flange,

$$\frac{V_s^2}{2g} + \frac{p_s}{\gamma} + 0 + 20.6 = \frac{V_d^2}{2g} + \frac{p_d}{\gamma} + 3$$

in which the subscripts s and d refer to the suction and discharge conditions, respectively. From the continuity equation $V_s = 3 \times 16/\pi = 15.3$ ft/sec,

$V_d = 3 \times 36/\pi = 34.4$ ft/sec. By solving for p_d,

$$\frac{p_d}{\gamma} = \frac{15.3^2}{64.4} + \frac{10}{0.433} + 20.6 - \frac{34.4^2}{64.4} - 3 = 25.94 \text{ ft}$$

and $p_d = 11.21$ psi. The rise in hydraulic grade line is

$$\left(\frac{p_d}{\gamma} + 3\right) - \frac{p_s}{\gamma} = 25.94 + 3 - \frac{10}{0.433} = 5.84 \text{ ft}$$

In this example almost all the energy was added in the form of kinetic energy, and the hydraulic grade line rises only 5.84 ft for a rise of energy grade line of 20.6 ft.

A turbine takes energy from the flow and causes a sharp drop in both the energy and the hydraulic grade lines. The energy removed per unit weight of fluid may be treated as a loss in computing grade lines.

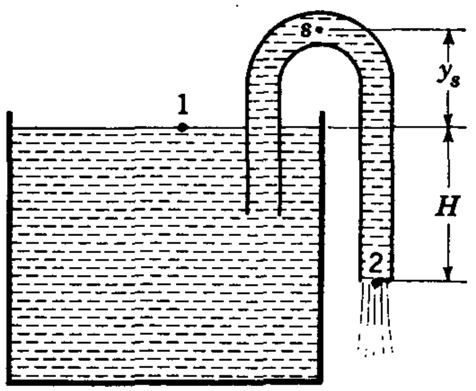


Fig. 10.4. Siphon.

10.2. The Siphon. A closed conduit, arranged as in Fig. 10.4, which lifts the liquid to an elevation higher than its free surface and then discharges it at a lower elevation is a *siphon*. It has certain limitations to its performance due to the low pressures that occur near the summit s.

Assuming that the siphon flows full, with a continuous liquid column throughout the siphon, the application of Bernoulli's equation for the portion from 1 to 2 produces the equation

$$H = \frac{V^2}{2g} + K\frac{V^2}{2g} + f\frac{L}{D}\frac{V^2}{2g}$$

in which K is the sum of all the minor-loss coefficients. After factoring out the velocity head,

$$H = \frac{V^2}{2g}\left(1 + K + \frac{fL}{D}\right) \qquad (10.2.1)$$

which is solved in the same fashion as the simple pipe problems of the first or second type. With the discharge known, the solution for H is straightforward, but the solution for velocity with H given is a trial solution started by assuming an f.

The pressure at the summit s is found by applying Bernoulli's equation for the portion between 1 and s after Eq. (10.2.1) is solved. It is

$$0 = \frac{V^2}{2g} + \frac{p_s}{\gamma} + y_s + K'\frac{V^2}{2g} + f\frac{L'}{D}\frac{V^2}{2g}$$

in which K' is the sum of the minor-loss coefficients between the two points and L' is the length of conduit upstream from s. By solving for the pressure,

$$\frac{p_s}{\gamma} = -y_s - \frac{V^2}{2g}\left(1 + K' + \frac{fL'}{D}\right) \tag{10.2.2}$$

which shows that the pressure is negative and that it decreases with y_s and $V^2/2g$. If the solution of the equation should be a value of p_s/γ equal to or less than the vapor pressure[1] of the liquid, then Eq. (10.2.1) is not valid because the vaporization of portions of the fluid column invalidates the incompressibility assumption used in deriving Bernoulli's equation.

Although Eq. (10.2.1) is not valid for this case theoretically there will be a discharge so long as y_s plus the vapor pressure is less than local atmospheric pressure expressed in length of the fluid column. When Eq. (10.2.2) yields a pressure less than vapor pressure at s, the pressure at s may be taken as vapor pressure. Then, with this pressure known, Eq. (10.2.2) is solved for $V^2/2g$, and the discharge is obtained therefrom. It is assumed that air does not enter the siphon at 2 and break at s the vacuum that produces the flow.

Practically a siphon does not work satisfactorily when the pressure intensity at the summit is close to vapor pressure. Air and other gases come out of solution at the low pressures and collect at the summit, thus reducing the length of the right-hand column of liquid that produces the low pressure at the summit. Large siphons that operate continuously have at the summits vacuum pumps to remove the gases.

The lowest pressure may not occur at the summit, but somewhere downstream from that point, because friction and minor losses may reduce the pressure more than the decrease in elevation increases pressure.

Example 10.3: Neglecting minor losses and considering the length of pipe equal to its horizontal distance, determine the point of minimum pressure in the siphon of Fig. 10.5.

When minor losses are neglected the kinetic-energy term $V^2/2g$ is usually neglected also. Then the hydraulic grade line is a straight line connecting the two liquid surfaces. Coordinates of two points on the line are $x = -100$ ft, $y = 10$ ft; $x = 141.4$ ft, $y = 20$ ft. The equation of the line is, by substitution into $y = mx + b$,

$$y = 0.0415x + 14.15 \text{ ft}$$

[1] A liquid boils when its pressure is reduced to its vapor pressure. The vapor pressure is a function of temperature for a particular liquid. Water has a vapor pressure of 0.203 ft of water abs at 32°F, 0.773 ft of water abs at 68°F, 6.630 ft of water abs at 140°F, and 33.91 ft of water abs at 212°F. See Sec. 1.8.

The minimum pressure occurs where the distance between hydraulic grade line and pipe is a maximum

$$\frac{p}{\gamma} = 0.001x^2 - 0.0415x - 14.15$$

To find minimum p/γ, set $d(p/\gamma)/dx = 0$, which yields $x = 20.75$, and $p/\gamma = -14.58$ ft of fluid flowing. The minimum point occurs where the slopes of the pipe and of the hydraulic grade line are equal.

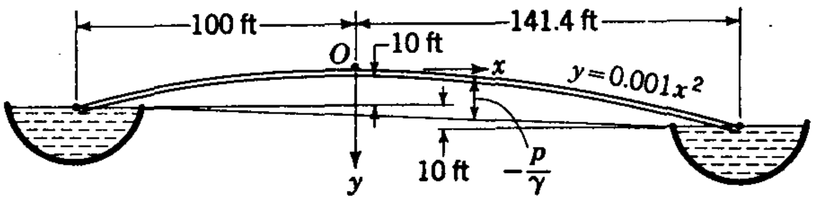


FIG. 10.5. Siphon connecting two reservoirs.

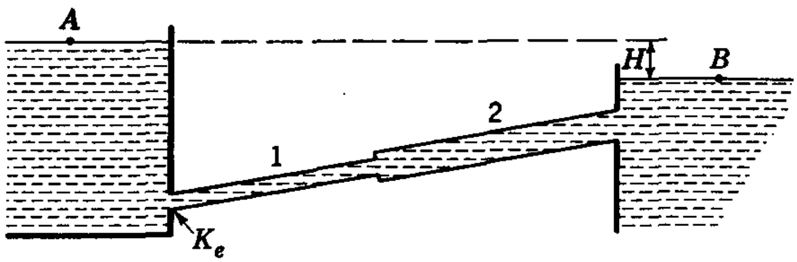


FIG. 10.6. Pipes connected in series.

10.3. Pipes in Series. When two pipes of different sizes or roughnesses are connected so that fluid flows through one pipe and then through the other pipe, they are said to be connected in series. A typical series-pipe problem, in which the head H may be desired for a given discharge or the discharge wanted for a given H, is illustrated in Fig. 10.6. By applying Bernoulli's equation from A to B, including all losses,

$$H + 0 + 0 = 0 + 0 + 0 + K_e\frac{V_1^2}{2g} + f_1\frac{L_1}{D_1}\frac{V_1^2}{2g} + \frac{(V_1 - V_2)^2}{2g} + f_2\frac{L_2}{D_2}\frac{V_2^2}{2g} + \frac{V_2^2}{2g}$$

in which the subscripts refer to the two pipes. The last item is the head loss at exit from pipe 2. With the continuity equation

$$V_1D_1^2 = V_2D_2^2$$

V_2 is eliminated from the equations, so

$$H = \frac{V_1^2}{2g}\left\{K_e + \frac{f_1L_1}{D_1} + \left[1 - \left(\frac{D_1}{D_2}\right)^2\right]^2 + \frac{f_2L_2}{D_2}\left(\frac{D_1}{D_2}\right)^4 + \left(\frac{D_1}{D_2}\right)^4\right\}$$

For known lengths and sizes of pipes this reduces to

$$H = \frac{V_1^2}{2g}(C_1 + C_2 f_1 + C_3 f_2) \tag{10.3.1}$$

in which C_1, C_2, C_3 are known. With the discharge given, the Reynolds number is readily computed, and the f's may be looked up in the Moody diagram. Then H is found by direct substitution. With H given, V_1, f_1, f_2 are unknowns in Eq. (10.3.1). By assuming values of f_1 and f_2 (they may be assumed equal), a trial V_1 is found from which trial Reynolds numbers are determined and values of f_1, f_2 looked up. Using these new values, a better V_1 is computed from Eq. (10.3.1). Since f varies so slightly with Reynolds number, the trial solution converges very rapidly. The same procedures apply for more than two pipes in series.

Example 10.4: In Fig. 10.6, $K_e = 0.5$, $L_1 = 1000$ ft, $D_1 = 2$ ft, $\epsilon_1 = 0.005$ ft, $L_2 = 800$ ft, $D_2 = 3$ ft, $\epsilon_2 = 0.001$ ft, $\nu = 0.00001$ ft²/sec, and $H = 20$ ft. Determine the discharge through the system.

With Bernoulli's equation,

$$20 = \frac{V_1^2}{2g}\left\{0.5 + f_1\frac{1000}{2} + \left[1 - \left(\frac{2}{3}\right)^2\right]^2 + f_2\frac{800}{3}\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^4\right\}$$

After simplifying,

$$20 = \frac{V_1^2}{2g}(1.01 + 500f_1 + 52.6f_2)$$

From $\epsilon_1/D_1 = 0.0025$, $\epsilon_2/D_2 = 0.00033$, and Fig. 5.34 values of f's are assumed for the complete turbulence range,

$$f_1 = 0.025 \qquad f_2 = 0.015$$

By solving for V_1, with these values, $V_1 = 9.49$ ft/sec, $V_2 = 4.21$ ft/sec,

$$\mathbf{R}_1 = \frac{9.49 \times 2}{0.00001} = 1{,}898{,}000 \qquad \mathbf{R}_2 = \frac{4.21 \times 3}{0.00001} = 1{,}263{,}000$$

and from Fig. 5.34, $f_1 = 0.025$, $f_2 = 0.016$. By solving for V_1 again, $V_1 = 9.46$, and $Q = 9.46\pi = 29.8$ cfs.

Equivalent Pipes. Series pipes may be solved by the method of equivalent lengths. Two pipe systems are said to be equivalent when the same head loss produces the same discharge in both systems. From Eq. (10.1.1)

$$h_{f1} = f_1\frac{L_1}{D_1}\frac{Q_1^2}{(D_1^2\pi/4)^2\,2g} = \frac{f_1 L_1}{D_1^5}\frac{8Q_1^2}{\pi^2 g}$$

and for a second pipe

$$h_{f2} = \frac{f_2 L_2}{D_2^5}\frac{8Q_2^2}{\pi^2 g}$$

For the two pipes to be equivalent,

$$h_{f1} = h_{f2} \qquad Q_1 = Q_2$$

After equating $h_{f1} = h_{f2}$ and simplifying,

$$\frac{f_1 L_1}{D_1^5} = \frac{f_2 L_2}{D_2^5}$$

By solving for L_2,

$$L_2 = L_1 \frac{f_1}{f_2}\left(\frac{D_2}{D_1}\right)^5 \qquad (10.3.2)$$

which determines the length of a second pipe to be equivalent to that of the first pipe. For example, to replace 1000 ft of 8-in. pipe with an equivalent length of 6-in. pipe, the values of f_1, f_2 must be approximated by selecting a discharge within the range intended for the pipes. Say $f_1 = 0.020$, $f_2 = 0.018$; then

$$L_2 = 1000\,\frac{0.020}{0.018}\left(\frac{6}{8}\right)^5 = 264 \text{ ft}$$

For these assumed conditions 264 ft of 6-in. pipe is equivalent to 1000 ft of 8-in. pipe.

Hypothetically two or more pipes composing a system may also be replaced by a pipe which has the same discharge for the same over-all head loss.

Example 10.5: Solve example 10.4 by means of equivalent pipes.

First, by expressing the minor losses in terms of equivalent lengths, for pipe 1

$$K_1 = 0.5 + [1 - (\tfrac{2}{3})^2]^2 = 0.809$$

$$L_{e1} = \frac{K_1 D_1}{f_1} = \frac{0.809 \times 2}{0.025} = 65 \text{ ft}$$

and for pipe 2

$$K_2 = 1$$

$$L_{e2} = \frac{K_2 D_2}{f_2} = \frac{1 \times 3}{0.015} = 200 \text{ ft}$$

The values of f_1, f_2 are selected for the fully turbulent range as an approximation. The problem is now reduced to 1065 ft of 2-ft pipe and 1000 ft of 3-ft pipe. By expressing the 3-ft pipe in terms of an equivalent length of 2-ft pipe, by Eq. (10.3.2)

$$L_e = 1000 \times \frac{0.015}{0.025}\left(\frac{2}{3}\right)^5 = 79 \text{ ft}$$

By adding to the 2-ft pipe, the problem is reduced to the simple pipe problem of finding the discharge through $1065 + 79 = 1144$ ft of 2-ft diameter pipe, $\epsilon = 0.005$ ft, for a head loss of 20 ft,

$$20 = f\,\frac{1144}{2}\,\frac{V^2}{2g}$$

With $f = 0.025$, $V = 9.5$ ft/sec, $\mathbf{R} = 9.5 \times 2/0.00001 = 1{,}900{,}000$. For $\epsilon/D = 0.0025$, $f = 0.025$, and $Q = 9.5\pi = 29.9$ cfs.

10.4. Pipes in Parallel. A combination of two or more pipes connected as in Fig. 10.7, so that the flow is divided among the pipes and then is joined again, is a *parallel-pipe* system. In series pipes the same fluid flows through all the pipes, and the head losses are cumulative; however, in parallel pipes the head losses are the same in any of the lines, and the discharges are cumulative.

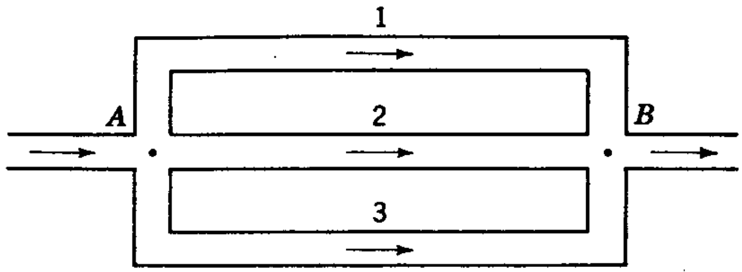


FIG. 10.7. Parallel-pipe system.

In analyzing parallel-pipe systems, it is assumed that the minor losses are added into the lengths of each pipe as equivalent lengths. From Fig. 10.7 the conditions to be satisfied are

$$h_{f1} = h_{f2} = h_{f3} = \frac{p_A}{\gamma} + z_A - \left(\frac{p_B}{\gamma} + z_B\right) \qquad Q = Q_1 + Q_2 + Q_3 \quad (10.4.1)$$

in which z_A, z_B are elevations of points A and B, and Q is the discharge through the approach pipe or the exit pipe.

Two types of problems occur: (1) with elevation of hydraulic grade line at A and B known, to find the discharge Q; (2) with Q known, to find the distribution of flow and the head loss. Sizes of pipe, fluid properties, and roughnesses are assumed to be known.

The first type is, in effect, the solution of simple pipe problems for discharge since the head loss is the drop in hydraulic grade line. These discharges are added to determine the total discharge.

The second type of problem is more complex, as neither the head loss nor the discharge for any one pipe is known. The recommended procedure is as follows:

1. Assume a discharge Q_1' through pipe 1.
2. Solve for h_f', using the assumed discharge.
3. Using h_{f1}', find Q_2', Q_3'.
4. With the three discharges for a common head loss, now assume that the given Q is split up among the pipes in the same proportion as Q_1', Q_2', Q_3'; thus

$$Q_1 = \frac{Q_1'}{\Sigma Q'} Q \qquad Q_2 = \frac{Q_2'}{\Sigma Q'} Q \qquad Q_3 = \frac{Q_3'}{\Sigma Q'} Q \qquad (10.4.2)$$

5. Check the correctness of these discharges by computing h_{f1}, h_{f2}, h_{f3} for the computed Q_1, Q_2, Q_3.

This procedure works for any number of pipes. By judicious choice of Q_1', obtained by estimating the per cent of the total flow through the system that should pass through pipe 1 (based on diameter, length, and roughness), Eq. (10.4.2) produces values that check within a few per cent, which is well within the range of accuracy of the friction factors.

Example 10.6: In Fig. 10.7, $L_1 = 3000$ ft, $D_1 = 1$ ft, $\epsilon_1 = 0.001$ ft; $L_2 = 2000$ ft, $D_2 = 8$ in., $\epsilon_2 = 0.0001$ ft; $L_3 = 4000$ ft, $D_3 = 16$ in., $\epsilon_3 = 0.0008$ ft; $\rho = 2.00$ slugs/ft³, $\nu = 0.00003$ ft²/sec, $p_A = 80$ psi, $z_A = 100$ ft, $z_B = 80$ ft. For a total flow of 12 cfs, determine flow through each pipe and the pressure at B.

Assume $Q_1' = 3$ cfs; then $V_1' = 3.82$, $\mathbf{R}_1' = 3.82 \times 1/0.00003 = 127{,}000$, $\epsilon_1/D_1 = 0.001$, $f_1' = 0.022$, and

$$h_{f1}' = 0.022 \times \frac{3000}{1.0}\frac{\overline{3.82^2}}{64.4} = 14.97 \text{ ft}$$

For pipe 2

$$14.97 = f_2' \frac{2000}{0.667}\frac{V_2'^2}{2g}$$

Then $\epsilon_2/D_2 = 0.00015$. Assume $f_2' = 0.020$; then $V_2' = 4.01$ ft/sec, $\mathbf{R}_2' = 4.01 \times \frac{2}{3} \times 1/0.00003 = 89{,}000$, $f_2' = 0.019$, $V_2' = 4.11$ ft/sec, $Q_2' = 1.44$ cfs.

For pipe 3

$$14.97 = f_3' \frac{4000}{1.333}\frac{V_3'^2}{2g}$$

Then $\epsilon_3/D_3 = 0.0006$. Assume $f_3' = 0.020$; then $V_3' = 4.01$ ft/sec, $\mathbf{R}_3' = 4.01 \times 1.333/0.00003 = 178{,}000$, $f_3' = 0.020$, $Q_3' = 5.60$ cfs.

The total discharge for the assumed conditions is

$$\Sigma Q' = 3.00 + 1.44 + 5.60 = 10.04 \text{ cfs}$$

Hence

$$Q_1 = \frac{3.00}{10.04} \times 12 = 3.58 \text{ cfs} \qquad Q_2 = \frac{1.44}{10.04} \times 12 = 1.72 \text{ cfs}$$

$$Q_3 = \frac{5.60}{10.04} \times 12 = 6.70 \text{ cfs}$$

Checking the values of h_1, h_2, h_3,

$$V_1 = \frac{3.58}{\pi/4} = 4.56 \qquad \mathbf{R}_1 = 152{,}000 \qquad f_1 = 0.021 \qquad h_{f1} = 20.4 \text{ ft}$$

$$V_2 = \frac{1.72}{\pi/9} = 4.93 \qquad \mathbf{R}_2 = 109{,}200 \qquad f_2 = 0.019 \qquad h_{f2} = 21.6 \text{ ft}$$

$$V_3 = \frac{6.70}{4\pi/9} = 4.80 \qquad \mathbf{R}_3 = 213{,}000 \qquad f_3 = 0.019 \qquad h_{f3} = 20.4 \text{ ft}$$

f_2 is about midway between 0.018 and 0.019. If 0.018 had been selected, h_2 would be 20.4 ft.

To find p_B

$$\frac{p_A}{\gamma} + z_A = \frac{p_B}{\gamma} + z_B + h_f$$

or

$$\frac{p_B}{\gamma} = \frac{80 \times 144}{62.4} + 100 - 80 - 20.8 = 183.5$$

in which the average head loss was taken. Then

$$p_B = \frac{183.5 \times 2 \times 32.2}{144} = 81.8 \text{ psi}$$

10.5. Branching Pipes. A simple *branching-pipe* system is shown in Fig. 10.8. In this situation the flow through each pipe is wanted when the reservoir elevations are given. The sizes and types of pipes and fluid

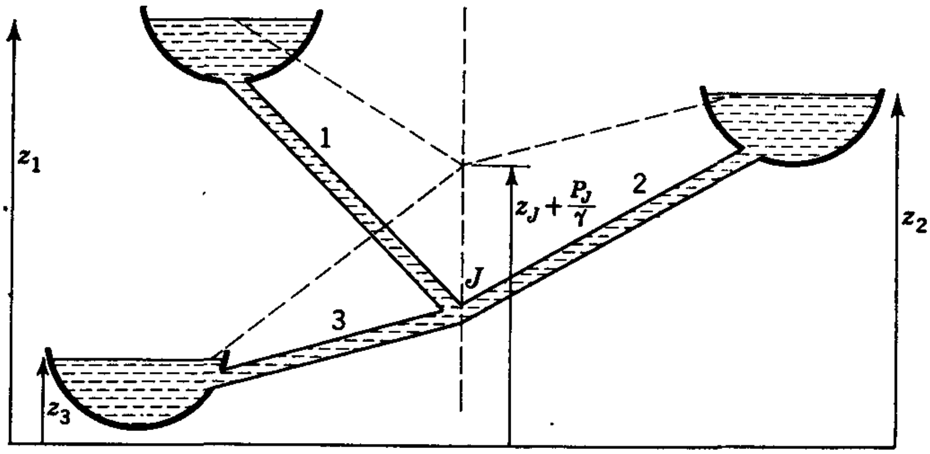


Fig. 10.8. Three interconnected reservoirs.

properties are assumed known. The Darcy-Weisbach equation must be satisfied for each pipe, and the continuity equation must be satisfied. It takes the form that the flow into the junction J must just equal the flow out of the junction. Flow must be out of the highest reservoir and into the lowest; hence, the continuity equation may be either of the following,

$$Q_1 = Q_2 + Q_3 \qquad Q_1 + Q_2 = Q_3$$

If the elevation of hydraulic grade line at the junction is above the elevation of the intermediate reservoir, flow is *into* it; but if the elevation of hydraulic grade line at J is below the intermediate reservoir, the flow is *out* of it. Minor losses may be expressed as equivalent lengths and added to the actual lengths of pipe.

The solution is effected by assuming an elevation of hydraulic grade line at the junction, then computing Q_1, Q_2, Q_3 and substituting into the continuity equation. If the flow into the junction is too great, a higher grade-line elevation, which will reduce the inflow and increase the outflow, is assumed.

Example 10.7: In Fig. 10.8, find the discharges for water at 60°F and with the following pipe data and reservoir elevations: $L_1 = 10{,}000$ ft, $D_1 = 3$ ft, $\epsilon_1/D_1 = 0.0002$; $L_2 = 2000$ ft, $D_2 = 1.5$ ft, $\epsilon_2/D_2 = 0.002$; $L_3 = 4000$ ft, $D_3 = 2$ ft, $\epsilon_3/D_3 = 0.001$; $z_1 = 100$ ft, $z_2 = 60$ ft, $z_3 = 30$ ft.

Assume $z_J + p_J/\gamma = 65$ ft; then

$$35 = f_1 \frac{10{,}000}{3} \frac{V_1^2}{2g} \qquad f_1 = 0.014 \qquad V_1 = 6.95 \text{ ft/sec} \qquad Q_1 = 49.1 \text{ cfs}$$

$$5 = f_2 \frac{2000}{1.5} \frac{V_2^2}{2g} \qquad f_2 = 0.024 \qquad V_2 = 3.17 \qquad Q_2 = 5.60$$

$$35 = f_3 \frac{4000}{2} \frac{V_3^2}{2g} \qquad f_3 = 0.020 \qquad V_3 = 7.51 \qquad Q_3 = 23.6$$

so the inflow is greater than the outflow by

$$49.1 - 5.60 - 23.6 = 19.9 \text{ cfs}$$

Assume $z_J + p_J/\gamma = 80$ ft; then

$$20 = f_1 \frac{10{,}000}{3} \frac{V_1^2}{2g} \qquad f_1 = 0.015 \qquad V_1 = 5.07 \qquad Q_1 = 35.8$$

$$20 = f_2 \frac{2000}{1.5} \frac{V_2^2}{2g} \qquad f_2 = 0.024 \qquad V_2 = 6.35 \qquad Q_2 = 11.33$$

$$50 = f_3 \frac{4000}{2} \frac{V_3^2}{2g} \qquad f_3 = 0.020 \qquad V_3 = 8.98 \qquad Q_3 = 28.2$$

The outflow is now greater by 3.73 cfs. Taking a straight line interpolation, $z_J + p_J/\gamma = 77.6$ ft, and

$$22.4 = f_1 \frac{10{,}000}{3} \frac{V_1^2}{2g} \qquad f_1 = 0.015 \qquad V_1 = 5.36 \qquad Q_1 = 37.83$$

$$17.6 = f_2 \frac{2000}{1.5} \frac{V_2^2}{2g} \qquad f_2 = 0.024 \qquad V_2 = 5.96 \qquad Q_2 = 10.65$$

$$47.6 = f_3 \frac{4000}{2} \frac{V_3^2}{2g} \qquad f_3 = 0.020 \qquad V_3 = 8.75 \qquad Q_3 = 27.50$$

and the outflow is 0.32 cfs greater. By extrapolating from the last two values, $z_J + p_J/\gamma = 77.4$, $Q_1 = 38.05$, $Q_2 = 10.60$, $Q_3 = 27.45$.

More complex branching-pipe problems are solved by a similar method of taking a trial solution. It is important that only one independent assumption be made; otherwise convergence of the solution would be haphazard. Figure 10.9 illustrates a four-reservoir problem with two junctions. By assuming the elevation of hydraulic grade line at one junction point, e.g., J_1, the flow through pipes 1 and 2 can be determined. Thus, by continuity, the flow between junctions is obtained, and the elevation of hydraulic grade line at J_2 is computed. The check on the assumption is to see whether the flows in pipes 3 and 4 satisfy continuity at J_2. If not, a new assumption at J_1 is made, and the process is repeated. The direction in which the shift is made is usually obvious.

In pumping from one reservoir to two or more other reservoirs, as in Fig. 10.10, the characteristics of the pump must be known. Assuming

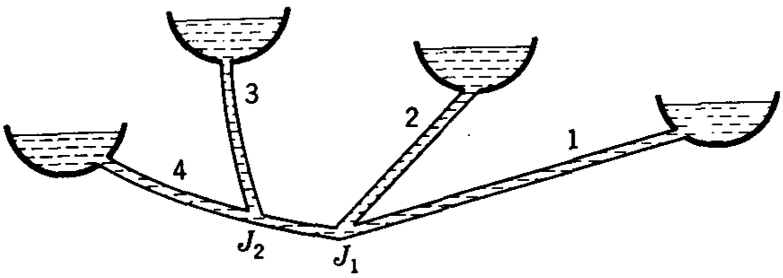


FIG. 10.9. Four reservoirs with two junctions.

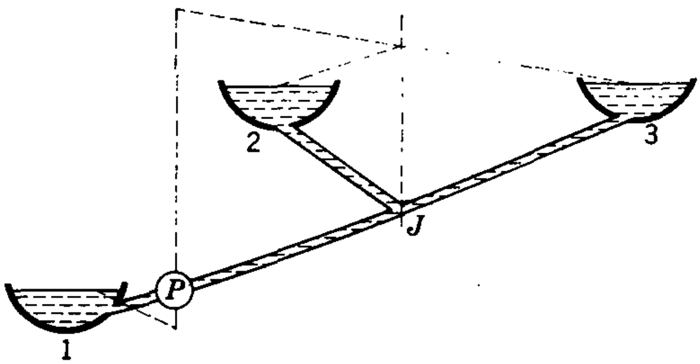


FIG. 10.10. Pumping from one reservoir to two other reservoirs.

that the pump runs at constant speed, its head depends upon the discharge. A suitable procedure is as follows:

1. Assume a discharge through the pump.
2. Compute the hydraulic-grade-line elevation at the suction side of the pump.
3. From the pump characteristic curve find the head produced, and add it to suction hydraulic grade line.
4. Compute drop in hydraulic grade line to the junction J, and determine elevation of hydraulic grade line there.
5. For this elevation, compute flow into reservoirs 2 and 3.
6. If flow into J equals flow out of J, the problem is solved. If flow into J is too great, assume less flow through the pump and repeat the procedure.

10.6. Networks of Pipes. Interconnected pipes through which the flow to a given outlet may come from several circuits are called a *network of pipes*, in many ways analogous to flow through electrical networks. Problems on these in general are complicated and require trial solutions in which the elementary circuits are balanced in turn until all conditions for the flow are satisfied.

The following conditions must be satisfied in a network of pipes:

a. The algebraic sum of the pressure drops around each circuit must be zero.

b. Flow into each junction must equal flow out of the junction.

c. The Darcy-Weisbach equation must be satisfied for each pipe, i.e., the proper relation between head loss and discharge must be maintained for each pipe.

The first condition states that the pressure drop between any two points in the circuit, e.g., A and G (Fig. 10.11) must be the same whether through the pipe AG or through $AFEDG$. The second condition is the continuity equation.

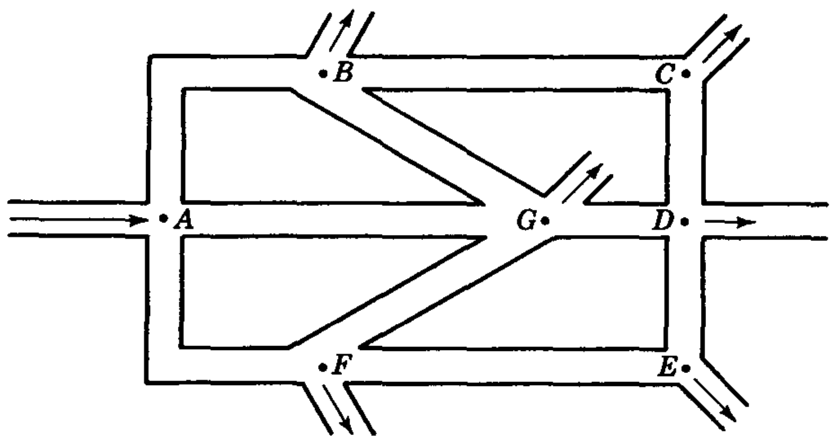


Fig. 10.11. Pipe network.

An exponential equation is usually developed to replace the Darcy-Weisbach equation. By expressing f as a function of V for a given pipe and a given fluid, the Darcy-Weisbach equation may be reduced to

$$h_f = rQ^n \tag{10.6.1}$$

Example 10.8: Determine the exponential formula for flow of water at 60°F through a 6-in.-diameter clean cast-iron pipe for the velocity range 2 to 6 ft/sec.

First, f is determined for 2 ft/sec and for 6 ft/sec. Using the Moody diagram, $\epsilon/D = 0.0017$. For 2 ft/sec, $f = 0.025$, and for 6 ft/sec, $f = 0.023$. Hence

$$f = 0.025 \qquad Q = \frac{2\pi}{16} = 0.392 \text{ cfs} \qquad f = 0.023 \qquad Q = \frac{6\pi}{16} = 1.180 \text{ cfs}$$

Substituting into

$$f = aQ^b$$

produces

$$0.025 = a(0.392)^b \qquad 0.023 = a(1.180)^b$$

By taking the ratio of the two equations,

$$\frac{0.025}{0.023} = \left(\frac{0.392}{1.180}\right)^b \qquad b = -0.076 \qquad a = 0.0233$$

Hence

$$f = 0.0233Q^{-0.076}$$

Substituting into the Darcy-Weisbach equation produces

$$\frac{h_f}{L} = \frac{fQ^2}{2gD^5(\pi/4)^2} = \frac{0.0233}{64.4(0.5)^5(0.7854)^2} Q^{1.924} = 0.0188Q^{1.924}$$

If the pipe is 1000 ft long, $h_f = 18.8Q^{1.924}$.

Since it is impractical to solve network problems analytically, methods of successive approximations are utilized. The Hardy Cross method[1] is one in which flows are assumed for each pipe so that continuity is satisfied at every junction. A correction to the flow in each circuit is then computed in turn and applied to bring the circuits into closer balance.

Minor losses are included as equivalent lengths in each pipe. With the head loss equation $h_f = rQ^n$, in which r and n are determined for each pipe, the method is applied as follows:

a. Assume the best distribution of flows that satisfies continuity by careful examination of the network.

b. Compute the head loss in each pipe $h = rQ^n$. Compute the net head loss around each elementary circuit: $\Sigma h = \Sigma rQ^n$ (should be zero for a balanced circuit).

c. Compute for each circuit: $\Sigma|nrQ^{n-1}|$ (all terms are considered positive).

d. Set up in each circuit a corrective flow ΔQ to balance the head in that circuit (for $\Sigma rQ^n \doteq 0$):

$$\Delta Q = \frac{\Sigma rQ^n}{\Sigma|nrQ^{n-1}|} \tag{10.6.2}$$

e. Compute the revised flows in each pipe, and repeat the procedure until the desired accuracy is obtained.

The solution is known to be correct when all the conditions are satisfied for each circuit. The source of the corrective term is obtained as follows:

For any pipe

$$Q = Q_0 + \Delta Q$$

in which Q is the correct discharge, Q_0 the assumed discharge, and ΔQ the correction. Then, for each pipe

$$h = rQ^n = r(Q_0 + \Delta Q)^n = r(Q_0{}^n + nQ_0{}^{n-1}\,\Delta Q + \cdots)$$

If ΔQ is small compared with Q_0, all terms of the series after the second one may be dropped. Now for a circuit,

$$\Sigma h = \Sigma rQ^n = \Sigma rQ_0{}^n + \Delta Q \Sigma rnQ_0{}^{n-1} = 0$$

[1] Hardy Cross, Analysis of Flow in Networks of Conduits or Conductors, *Univ. Illinois Bull.* 286, November, 1946.

in which ΔQ has been taken out of the summation as it is the same for all pipes in the circuit. After solving for ΔQ,

$$\Delta Q = \frac{\Sigma r Q_0{}^n}{\Sigma |rnQ_0{}^{n-1}|}$$

When ΔQ is applied to a circuit, it has the same sense in every pipe; i.e., it adds to flows in the counterclockwise direction and subtracts from flows

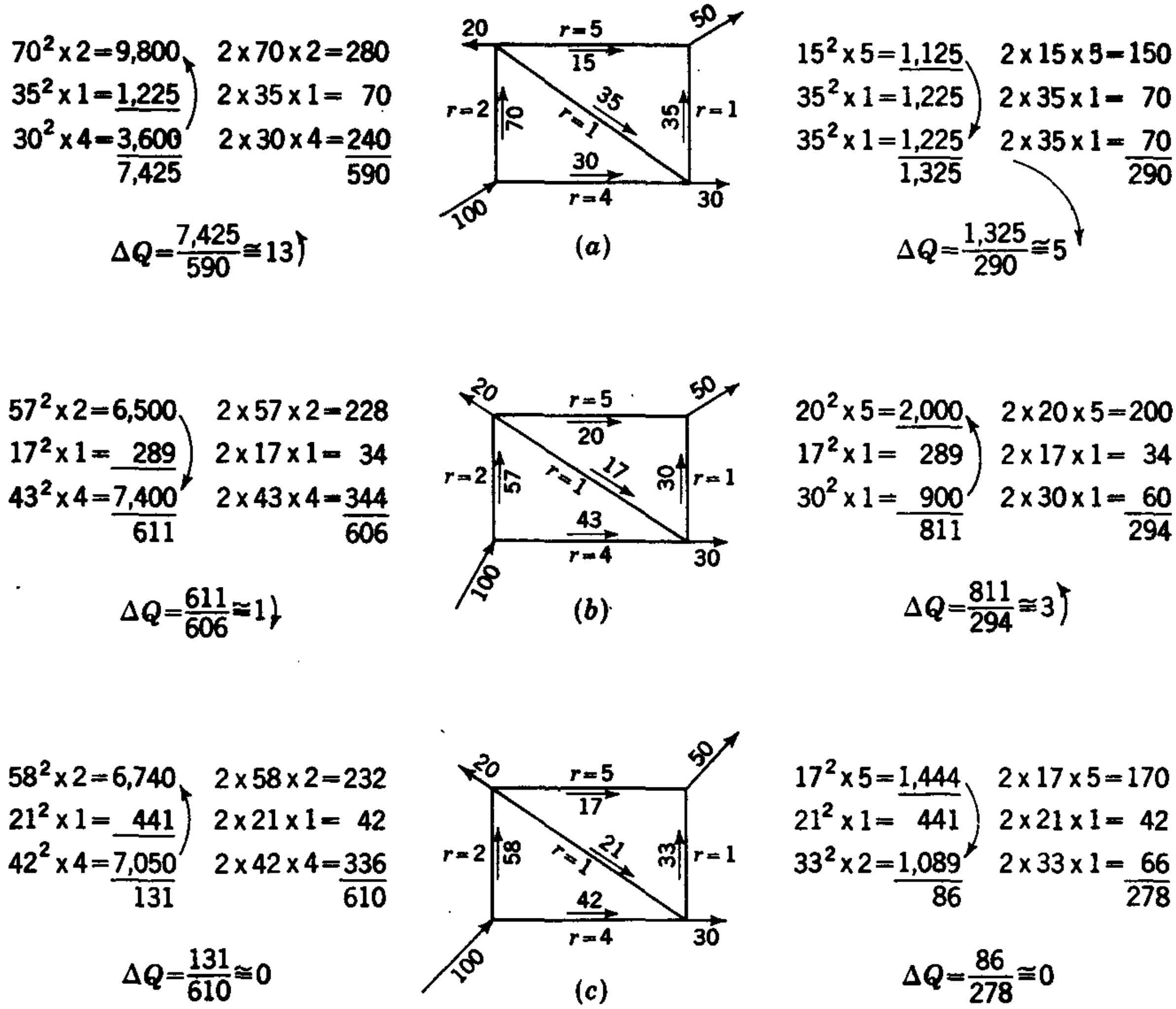


FIG. 10.12. Solution of flow distribution in a simple network.

in the clockwise direction. Since the ΔQ contains the sign change, the denominator of the correction term is the sum of the absolute terms.

The values of r occur in both numerator and denominator; hence, values proportional to the actual r may be used to find the distribution. Similarly, the apportionment of flows may be expressed as a per cent of the actual flows. To find a particular head loss, the actual values of r and Q must be used after the distribution has been determined.

Example 10.9: The distribution of flow through the network of Fig. 10.12 is desired for the inflows and outflows as given. For simplicity n has been given the value 2.0.

The assumed distribution is shown in diagram *a*. At the upper left the terms ΣrQ^n are computed for the lower circuit. By listing the clockwise terms first, an arrow at the right drawn to the larger terms shows the direction of the counterbalancing ΔQ. Next to the diagram on the left is the computation of $\Sigma|nrQ^{n-1}|$. Diagram *b* gives the distribution after both circuits have been corrected once. Diagram *c* shows the values correct to within about 1 per cent of the distribution, which is more accurate than the exponential equations for head loss.

10.7. Conduits with Noncircular Cross Sections. In this chapter so far, only circular pipes have been considered. For cross sections that are noncircular, the Darcy-Weisbach equation may be applied if the term D can be interpreted in terms of the section. The concept of the *hydraulic radius* R permits circular and noncircular sections to be treated in the same manner. The hydraulic radius is defined as the cross-sectional area divided by the *wetted perimeter*. Hence, for a circular section,

$$R = \frac{\text{area}}{\text{perimeter}} = \frac{\pi D^2/4}{\pi D} = \frac{D}{4} \tag{10.7.1}$$

and the diameter is equivalent to $4R$. Assuming that the diameter may be replaced by $4R$ in the Darcy-Weisbach equation, in the Reynolds number, and in the relative roughness,

$$h_f = f\frac{L}{4R}\frac{V^2}{2g} \qquad \mathbf{R} = \frac{V4R\rho}{\mu} \qquad \frac{\epsilon}{D} = \frac{\epsilon}{4R} \tag{10.7.2}$$

Noncircular sections may be handled in a similar manner. The Moody diagram applies as before. The assumptions in Eqs. (10.7.2) cannot be expected to hold for odd-shaped sections but should give reasonable values for square, oval, triangular, and similar types of sections.

Example 10.10: Determine the head loss in inches of water required for flow of 10,000 ft³/min of air at 60°F and 14.7 psia through a rectangular galvanized-iron section 2 ft wide, 1 ft high, and 200 ft long.

$$R = \frac{2}{6} = 0.333 \text{ ft} \qquad \frac{\epsilon}{4R} = 0.00038$$

$$V = \frac{10{,}000}{60 \times 2} = 83.3 \text{ ft/sec} \qquad V4R'' = 83.3 \times 4 \times 4 = 1330$$

$$f = 0.017$$

Then

$$h_f = f\frac{L}{4R}\frac{V^2}{2g} = \frac{0.017 \times 200}{4 \times 0.333}\frac{\overline{83.3}^2}{64.4} = 275 \text{ ft}$$

The specific weight of air is $\gamma = (14.7 \times 144)/(53.3 \times 520) = 0.0762$ lb/ft³. In inches of water the head loss is

$$\frac{275 \times 0.0762 \times 12}{62.4} = 4.04 \text{ in.}$$

10.8. Aging of Pipes. The Moody diagram, with the values of absolute roughness shown there, is for new, clean pipe. With use, pipes become rougher, owing to corrosion, incrustations, and deposition of material on the pipe walls. The speed with which the friction factor changes with time depends greatly on the fluid being handled. Colebrook and White[1] found that the absolute roughness ϵ increases linearly with time,

$$\epsilon = \epsilon_0 + \alpha t \tag{10.8.1}$$

in which ϵ_0 is the absolute roughness of the new surface. Tests on a pipe are required to determine α.

Example 10.11: An 18-in.-diameter wrought-iron pipe 12 years old has a pressure drop of 1.365 psi/1000 ft when carrying 7.08 cfs water at 60°F. Estimate the loss per thousand feet for 10 cfs water when the pipe is 20 years old.

When new, $\epsilon_0 = 0.00015$ from Fig. 5.34. At 12 years

$$V = \frac{7.08}{\pi(0.75)^2} = 4 \text{ ft/sec} \qquad h_f = \frac{1.365}{0.433} = 3.145$$

and

$$f = \frac{h_f}{(L/D)V^2/2g} = \frac{3.145}{(1000/1.5)(16/64.4)} = 0.019$$

from Fig. 5.34, $\epsilon/D = 0.00075$, $\epsilon = 0.00075 \times 1.5 = 0.00112$ ft. After computing α from Eq. (10.8.1),

$$\alpha = \frac{\epsilon - \epsilon_0}{t} = \frac{0.00112 - 0.00015}{12} = 0.000081 \text{ ft/year}$$

When 20 years old

$$\epsilon = 0.00015 + 20 \times 0.000081 = 0.0018 \text{ ft}$$

For 10 cfs, $V = 5.65$, $VD'' = 102$, $\epsilon/D = 0.0012$, $f = 0.021$, and

$$\Delta p = 0.021 \times \frac{1000}{1.5} \frac{\overline{5.65}^2}{64.4} \times 0.433 = 3.00 \text{ psi/1000 ft}$$

UNSTEADY FLOW IN CONDUITS

In general, unsteady-flow situations are more difficult to analyze than steady-flow situations. The Bernoulli equation is not applicable and the equation of motion leads to differential equations for the velocity or pressure as a function of time. Numerical and graphical methods are

[1] C. F. Colebrook and C. M. White, The Reduction of Carrying Capacity of Pipes with Age, *J. Inst. Civil Engs.* (*London*), 1937.

frequently resorted to, with the use of analog and digital computers to speed up the process of finding solutions. A few simple unsteady-flow situations are discussed as an introduction to the subject, because of its increasing importance in engineering.

10.9. Oscillation of Liquid in a U-tube. Three cases of oscillations of liquid in a simple U-tube are of interest: (*a*) frictionless liquid, (*b*) laminar resistance, and (*c*) turbulent resistance.

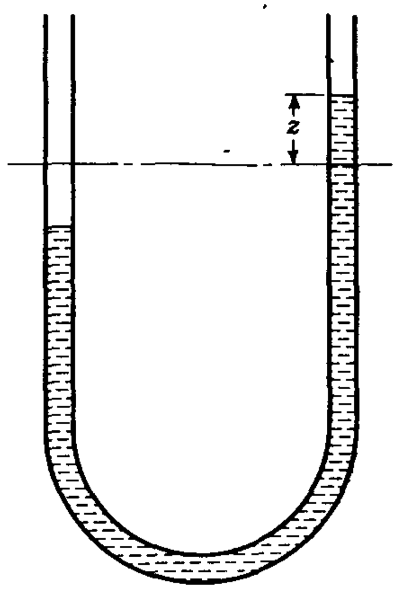


Fig. 10.13. Oscillation of liquid in a U-tube.

a. Frictionless Liquid. If no appreciable friction occurs within a U-tube, the equation of motion for the liquid is easily formulated as a differential equation. In Fig. 10.13 the line $z = 0$ is drawn through the equilibrium position of the menisci. The force accelerating the liquid column is due to the unbalanced weight of liquid $2zA\gamma$ acting to reduce z. The mass is $\gamma AL/g$ in which L is the length of total liquid column and A is the cross-sectional area of tube, both considered to be constant. From Newton's second law

$$2Az\gamma = -\frac{\gamma AL}{g}\frac{d^2z}{dt^2}$$

in which d^2z/dt^2 is the acceleration of liquid column. The minus sign is required because the acceleration term is negative when z is positive. After simplifying

$$\frac{d^2z}{dt^2} + \frac{2g}{L}z = 0 \qquad (10.9.1)$$

The general solution of this equation is

$$z = C_1 \cos\sqrt{\frac{2g}{L}}\,t + C_2 \sin\sqrt{\frac{2g}{L}}\,t$$

in which C_1 and C_2 are arbitrary constants of integration. The solution is readily checked by differentiating twice and substituting into the differential equation. To evaluate the constants, if $z = Z$ and $dz/dt = 0$ when $t = 0$, then $C_1 = Z$ and $C_2 = 0$, or

$$z = Z \cos\sqrt{\frac{2g}{L}}\,t \qquad (10.9.2)$$

This equation defines a simple harmonic motion of a meniscus, with a period for a complete oscillation of $2\pi\sqrt{L/2g}$. Velocity of the column may be obtained by differentiating z with respect to t.

Example 10.12: A frictionless fluid column 4.025 ft long has a speed of 4 ft/sec when $z = 1$ ft. Find (*a*) the maximum value of z, (*b*) the maximum speed, and (*c*) the period.

(*a*) By differentiating Eq. (10.9.2), after substituting for L,

$$\frac{dz}{dt} = -4Z \sin 4t$$

If t_1 is the time when $z = 1$ and $dz/dt = 4$,

$$1 = Z \cos 4t_1$$
$$-4 = -4Z \sin 4t_1$$

Dividing the second equation by the first equation

$$\tan 4t_1 = 1$$

or $4t_1 = 0.785$ radians, $t_1 = 0.196$ sec, $\sin 4t_1 = 0.707$, and $\cos 4t_1 = 0.707$. Then $Z = 1/\cos 4t_1 = 1/0.707 = 1.41$ ft, the maximum value of z.

(*b*) The maximum speed occurs when $\sin 4t = 1$, or $4Z \doteq 4 \times 1.41 = 5.64$ ft/sec.

(*c*) The period is

$$2\pi \sqrt{\frac{L}{2g}} = 1.571 \text{ sec}$$

b. Laminar Resistance. By making the assumption that the resistance to laminar flow in an unsteady situation is exactly the same as for a similar steady flow at the same velocity, the differential equation is easily obtained. Equation (5.2.7), when solved for head h causing velocity V is

$$h = \frac{32\mu LV}{\gamma D^2}$$

in which D is the tube diameter and μ the dynamic viscosity. With reference to Fig. 10.13, the force accelerating the column is $2zA\gamma$ as in the previous case. The resistance to motion is $hA\gamma$, the mass is $\gamma AL/g$, and the acceleration is d^2z/dt^2. If z is increasing in the figure,

$$-2zA\gamma - \frac{32\mu LA\gamma}{\gamma D^2}\frac{dz}{dt} = \frac{\gamma AL}{g}\frac{d^2z}{dt^2}$$

or

$$\frac{d^2z}{dt^2} + \frac{32\nu}{D^2}\frac{dz}{dt} + \frac{2g}{L}z = 0 \tag{10.9.3}$$

in which the kinematic viscosity ν has replaced $\mu g/\gamma$. The assumption has been made that $dz/dt = V$ and that $d^2z/dt^2 = dV/dt$, or that the column moves as a solid with velocity V.

By substitution

$$z = C_1e^{at} + C_2e^{bt}$$

can be shown to be the general solution of Eq. (10.9.3) provided that

$$a^2 + \frac{32\nu}{D^2}a + \frac{2g}{L} = 0$$

and

$$b^2 + \frac{32\nu}{D^2}b + \frac{2g}{L} = 0$$

C_1 and C_2 are arbitrary constants of integration that are determined by given values of z and dz/dt at a given time. To keep a and b distinct, since the equations defining them are identical, they are taken with opposite signs before the radical term in solution of the quadratics, thus

$$a = -\frac{16\nu}{D^2} + \sqrt{\left(\frac{16\nu}{D^2}\right)^2 - \frac{2g}{L}}$$

and

$$b = -\frac{16\nu}{D^2} - \sqrt{\left(\frac{16\nu}{D^2}\right)^2 - \frac{2g}{L}}$$

To simplify the formulas, if

$$m = \frac{16\nu}{D^2} \qquad n = \sqrt{\left(\frac{16\nu}{D^2}\right)^2 - \frac{2g}{L}}$$

then

$$z = C_1e^{-mt+nt} + C_2e^{-mt-nt}$$

When the initial condition is taken that $t = 0$, $z = 0$, $dz/dt = V_0$, then by substitution $C_1 = -C_2$, and

$$z = C_1e^{-mt}(e^{nt} - e^{-nt}) \tag{10.9.4}$$

Since

$$\frac{e^{nt} - e^{-nt}}{2} = \sinh nt$$

Eq. (10.9.4) becomes

$$z = 2C_1e^{-mt}\sinh nt$$

By differentiating with respect to t

$$\frac{dz}{dt} = 2C_1(-me^{-mt}\sinh nt + ne^{-mt}\cosh nt)$$

and after setting $dz/dt = V_0$ for $t = 0$

$$V_0 = 2C_1n$$

since $\sinh 0 = 0$ and $\cosh 0 = 1$. Then

$$z = \frac{V_0}{n} e^{-mt} \sinh nt \tag{10.9.5}$$

This equation gives the displacement z of one meniscus of the column as a function of time, starting with the meniscus at $z = 0$ when $t = 0$, and rising with velocity V_0.

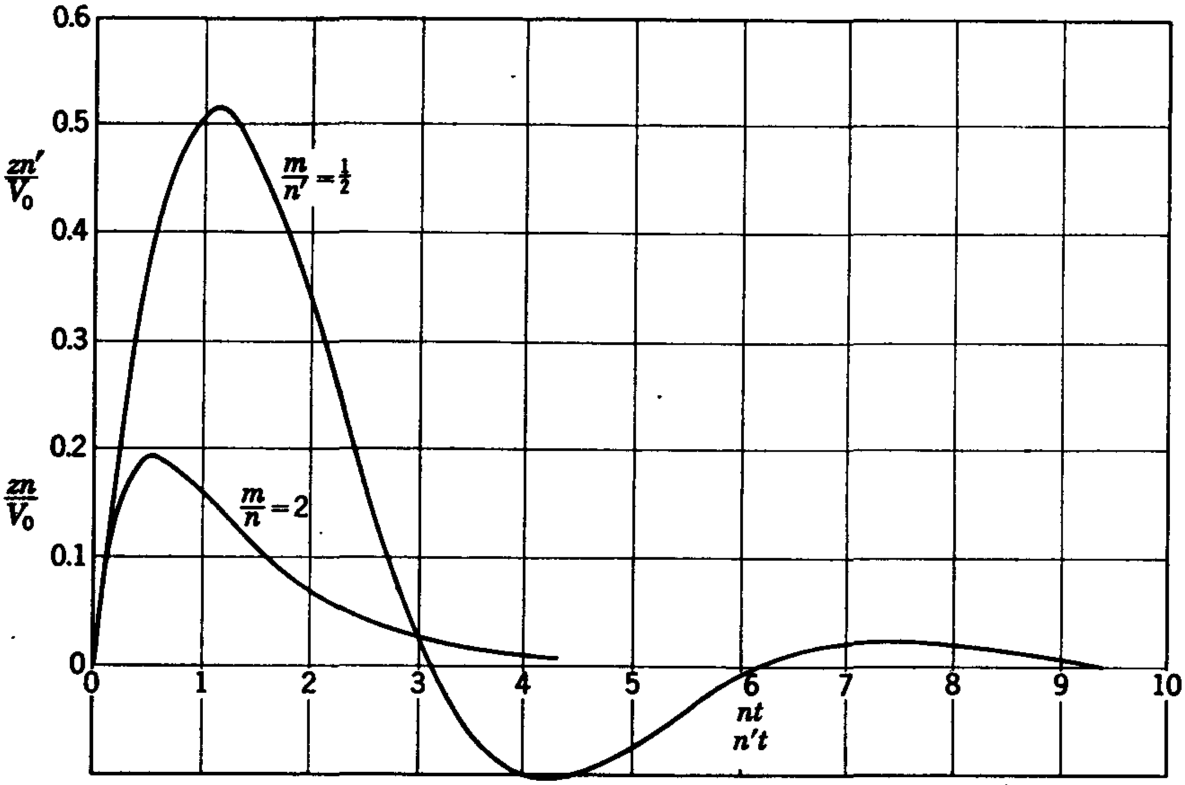


FIG. 10.14. Position of meniscus as a function of time for oscillation of liquid in a U-tube with laminar resistance.

Two principal cases[1] are to be considered. When

$$\frac{16\nu}{D^2} > \sqrt{\frac{2g}{L}}$$

n is a real number and the viscosity is so great that the motion is damped out in a partial cycle with z never becoming negative, Fig. 10.14 $\left(\frac{m}{n} = 2\right)$. The time t_0 for maximum z to occur is found by differentiating z [Eq. (10.9.5)] with respect to t and equating to zero,

$$\frac{dz}{dt} = 0 = \frac{V_0}{n} (-me^{-mt} \sinh nt + ne^{-mt} \cosh nt)$$

[1] A third case, $16\nu/D^2 = \sqrt{2g/L}$ must be treated separately, yielding $z = V_0te^{-mt}$. The resulting oscillation is for a partial cycle only and is a limiting case of $16\nu/D^2 > \sqrt{2g/L}$.

or

$$\tanh nt_0 = \frac{n}{m} \tag{10.9.6}$$

Substitution of this value of t into Eq. (10.9.5) yields the maximum displacement Z.

$$Z = \frac{V_0}{\sqrt{m^2 - n^2}} \left(\frac{m - n}{m + n}\right)^{\frac{m}{2n}} = V_0 \sqrt{\frac{L}{2g}} \left(\frac{m - n}{m + n}\right)^{\frac{m}{2n}} \tag{10.9.7}$$

The second case, when

$$\frac{16\nu}{D^2} < \sqrt{\frac{2g}{L}}$$

results in a negative term within the radical

$$n = \sqrt{-1 \left[\frac{2g}{L} - \left(\frac{16\nu}{D^2}\right)^2\right]} = i \sqrt{\frac{2g}{L} - \left(\frac{16\nu}{D^2}\right)^2} = in'$$

in which $i = \sqrt{-1}$ and n' is a real number. Replacing n by in' in Eq. (10.9.5) produces the real function

$$z = \frac{V_0}{in'} e^{-mt} \sinh in't = \frac{V_0}{n'} e^{-mt} \sin n't \tag{10.9.8}$$

since

$$\sin n't = \frac{1}{i} \sinh in't$$

The resulting motion of z is an oscillation about $z = 0$ with decreasing amplitude, as shown in Fig. 10.14 for the case $m/n' = \frac{1}{2}$. The time t_0 of maximum or minimum displacement is obtained from Eq. (10.9.8) by equating $dz/dt = 0$, producing

$$\tan n't_0 = \frac{n'}{m} \tag{10.9.9}$$

There are an indefinite number of values of t_0 satisfying this expression, corresponding with all the maximum and minimum positions of a meniscus. By substitution of t_0 into Eq. (10.9.8)

$$Z = \frac{V_0}{\sqrt{n'^2 + m^2}} e^{-(m/n') \tan^{-1} (n'/m)} = V_0 \sqrt{\frac{L}{2g}} e^{-(m/n') \tan^{-1} (n'/m)} \tag{10.9.10}$$

Example 10.13: A 1.0-in.-diameter U-tube contains oil, $\nu = 1 \times 10^{-4}$ ft²/sec, with a total column length of 120 in. Applying air pressure to one of the tubes makes the gage difference 16 in. By quickly releasing the air pressure the oil column is free to oscillate. Find the maximum velocity, the maximum Reynolds number, and the equation for position of one meniscus z, in terms of time.

The assumption is made that the flow is laminar, and Reynolds number will be computed on this basis. The constants m and n are

$$m = \frac{16\nu}{D^2} = \frac{16 \times 10^{-4}}{(1/12)^2} = 0.2302$$

$$n = \sqrt{\left(\frac{16\nu}{D^2}\right)^2 - \frac{2g}{L}} = \sqrt{(0.2302)^2 - \frac{2 \times 32.2}{10}} = \sqrt{-1}\, 2.527 = i2.527$$

or

$$n' = 2.527$$

Equations (10.9.8), (10.9.9), and (10.9.10) apply to this case, as the liquid will oscillate above and below $z = 0$. The oscillation starts from the maximum position, i.e., $Z = 0.667$ ft. By use of Eq. (10.9.10) the velocity (fictitious) when $z = 0$ at time t_0 before the maximum is determined to be

$$V_0 = Z\sqrt{\frac{2g}{L}}\, e^{(m/n')\tan^{-1}(n'/m)} = 0.667\sqrt{\frac{64.4}{10}}\, e^{(0.2302/2.527)\tan^{-1}(2.527/0.2302)} = 1.935 \text{ ft/sec}$$

and

$$\tan n't_0 = \frac{n'}{m} \qquad t_0 = \frac{1}{2.527}\tan^{-1}\frac{2.527}{0.2302} = 0.586 \text{ sec}$$

Hence by substitution into Eq. (10.9.8),

$$z = 0.766\, e^{-0.2302(t+0.586)} \sin 2.527(t + 0.586)$$

in which $z = Z$ at $t = 0$. The maximum velocity (actual) occurs for $t > 0$. Differentiating with respect to t to obtain the expression for velocity,

$$V = \frac{dz}{dt} = -0.1763\, e^{-0.2302(t+0.586)} \sin 2.527(t + 0.586) + 1.935\, e^{-0.2302(t+0.586)} \cos 2.527(t + 0.586)$$

Differentiating again with respect to t and equating to zero to obtain maximum V produces

$$\tan 2.527(t + 0.586) = -0.1837$$

The solution in the second quadrant should produce the desired maximum, $t = 0.584$ sec. Substituting this time into the expression for V produces $V = -1.48$ ft/sec. The corresponding Reynolds number is

$$\mathbf{R} = \frac{VD}{\nu} = 1.48 \times \frac{1}{12} \times 10^4 = 1234$$

hence the assumption of laminar resistance is justified.

c. Turbulent Resistance. In the majority of practical cases of oscillation, or surge, in pipe systems there is turbulent resistance. With large pipes and tunnels the Reynolds number is large except for those time periods when the velocity is very near to zero. The assumption of fluid resistance proportional to the square of the average velocity is made (constant f). It closely approximates true conditions, although it yields too small a resistance for slow motions, in which case resistance is almost negligible. The equations will be developed for f = constant for oscillation within a simple U-tube. This case will then be extended to include oscillation of flow within a pipe or tunnel between two reservoirs, taking into account the minor losses. The assumption is again made that resistance in unsteady flow is given by steady flow resistance at the same velocity. The resistance due to a column of liquid of length L is

$$\gamma A h_f = \gamma A f \frac{L}{D} \frac{V^2}{2g}$$

The equation of motion (Fig. 10.13) for z decreasing is

$$-2zA\gamma + \gamma \frac{Af}{2g} \frac{L}{D} \left(\frac{dz}{dt}\right)^2 = \frac{\gamma A L}{g} \frac{d^2z}{dt^2}$$

After simplifying,

$$\frac{d^2z}{dt^2} - \frac{f}{2D} \left(\frac{dz}{dt}\right)^2 + \frac{2g}{L} z = 0 \qquad (10.9.11)$$

The sign of the middle term becomes positive for motion in the $+z$-direction. The equation may be integrated once,[1] producing

$$\left(\frac{dz}{dt}\right)^2 = \frac{4gD^2}{f^2L} \left(1 + \frac{fz}{D}\right) + Ce^{fz/D} \qquad (10.9.12)$$

in which C is the constant of integration. To evaluate the constant, if $z = z_m$ for $dz/dt = 0$

$$C = -\frac{4gD^2}{f^2L} \left(1 + \frac{fz_m}{D}\right) e^{-fz_m/D}$$

and

$$\left(\frac{dz}{dt}\right)^2 = \frac{4gD^2}{f^2L} \left[1 + \frac{fz}{D} - \left(1 + \frac{fz_m}{D}\right) e^{f(z-z_m)/D}\right] \qquad (10.9.13)$$

Although this equation cannot be integrated again, numerical integration of particular situations yields z as a function of t. The equation, however, may be used to determine the magnitude of successive oscillations.

[1] By substitution of

$$p = \frac{dz}{dt} \qquad \frac{d^2z}{dt^2} = \frac{dp}{dt} = \frac{dp}{dz}\frac{dz}{dt} = p\frac{dp}{dz}$$

then

$$p\frac{dp}{dz} - \frac{f}{2D} p^2 + \frac{2gz}{L} = 0$$

This equation may be made exact by multiplying by the integrating factor $e^{-fz/D}$. For the detailed method see A. Cohen, "Differential Equations," p. 11, D. C. Heath & Co., Boston, 1906.

At the instants of maximum or minimum z, say z_m and z_{m+1}, respectively, $dz/dt = 0$ and Eq. (10.9.13) simplifies to

$$\left(1 + \frac{fz_m}{D}\right) e^{-fz_m/D} = \left(1 + \frac{fz_{m+1}}{D}\right) e^{-fz_{m+1}/D} \tag{10.9.14}$$

Since the original equation, Eq. (10.9.11), holds only for decreasing z, z_m must be positive and z_{m+1} negative. To find z_{m+2} the other meniscus could be considered and z_{m+1} as a positive number substituted into the left-hand side of the equation to determine a minus z_{m+2} in place of z_{m+1} on the right-hand side of the equation.

Example 10.14: A U-tube consisting of 2.0-ft-diameter pipe with $f = 0.03$ has a maximum oscillation (Fig. 10.13) of $z_m = 20$ ft. Find the minimum position of the surface and the following maximum.

From Eq. (10.9.14)

$$\left(1 + \frac{0.03 \times 20}{2}\right) e^{-0.03\times 20/2} = (1 + 0.015z_{m+1})e^{-0.015z_{m+1}}$$

or

$$(1 + 0.015z_{m+1})e^{-0.015z_{m+1}} = 0.9603$$

which is satisfied by $z_{m+1} = -16.6$ **ft.** Using $z_m = 16.6$ in Eq. (10.9.14)

$$(1 + 0.015z_{m+1})e^{-0.015z_{m+1}} = (1 + 0.015 \times 16.6)e^{-0.015\times 16.6} = 0.974$$

which is satisfied by $z_{m+1} = -14.2$ ft. Hence, the minimum water surface is $z = -16.6$ ft and the next maximum is $z = 14.2$ ft.

Equation (10.9.14) may be solved graphically: if $\phi = fz/D$, then

$$F(\phi) = (1 + \phi)e^{-\phi} \tag{10.9.15}$$

which is conveniently plotted with $F(\phi)$ as ordinate and both $-\phi$ and $+\phi$ on the same abscissa scale (Fig. 10.15). Successive values of ϕ are found as indicated by the dotted stepped line.

Although z cannot be found as a function of t from Eq. (10.9.13), V is given as a function of z, since $V = dz/dt$. The maximum value of V is found by equating $dV^2/dz = 0$ to find its position z', thus

$$\frac{dV^2}{dz} = 0 = \frac{f}{D} - \left(1 + \frac{fz_m}{D}\right) e^{f(z'-z_m)/D} \frac{f}{D}$$

After solving for z'

$$z' = z_m - \frac{D}{f} \ln\left(1 + \frac{fz_m}{D}\right)$$

and after substituting back into Eq. (10.9.13)

$$V_m{}^2 = \frac{4gD^2}{f^2L}\left[\frac{fz_m}{D} - \ln\left(1 + \frac{fz_m}{D}\right)\right] \tag{10.9.16}$$

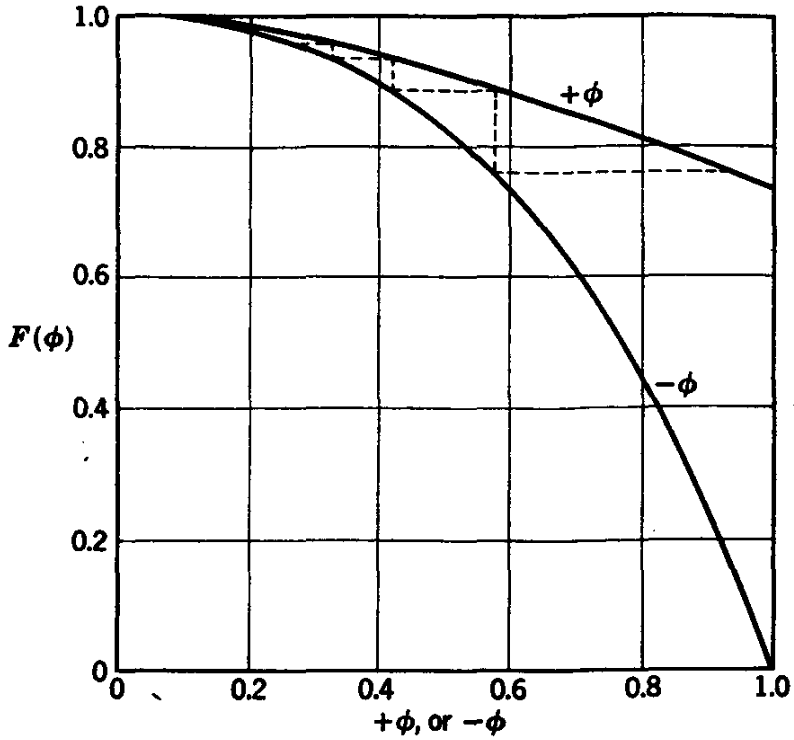


FIG. 10.15. Graphical solution of $F(\phi) = (1 + \phi)e^{-\phi}$.

Oscillation of Two Reservoirs. The equation for oscillation of two reservoirs connected by a pipeline is the same as that for oscillation of a U-tube, except for value of constant terms. If z_1 and z_2 represent displacements of the reservoir surfaces from their equilibrium positions (Fig. 10.16) and if z represents displacement of a water particle within the connecting pipe from its equilibrium position,

$$zA = z_1A_1 = z_2A_2$$

in which A_1 and A_2 are the reservoir areas, assumed to be constant in this derivation. Taking into account minor losses in the system by using the equivalent length L_e of pipe and fittings plus other minor losses, the equation of motion is

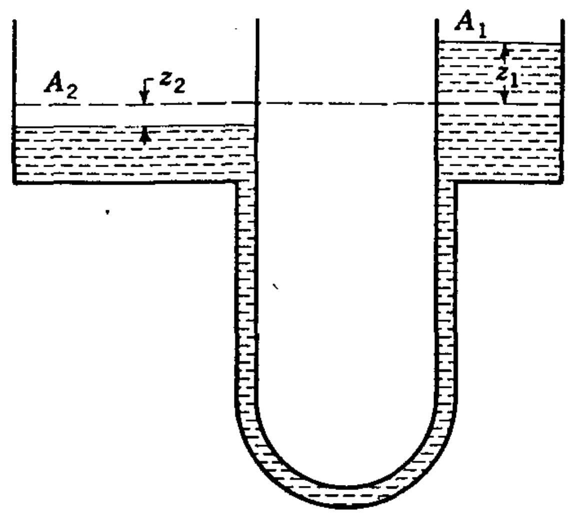


FIG. 10.16. Oscillation of two reservoirs.

$$-\gamma A(z_1 + z_2) + \frac{\gamma A f L_e}{2gD}\left(\frac{dz}{dt}\right)^2 = \frac{\gamma A L}{g}\frac{d^2z}{dt^2}$$

for z decreasing. After simplifying

$$\frac{d^2z}{dt^2} - \frac{f}{2D}\frac{L_e}{L}\left(\frac{dz}{dt}\right)^2 + \frac{gA}{L}\left(\frac{1}{A_1} + \frac{1}{A_2}\right)z = 0 \qquad (10.9.17)$$

After comparing with Eq. (10.9.11), f is replaced by fL_e/L, and $2g/L$ by $gA(1/A_1 + 1/A_2)/L$. In Eq. (10.9.15)

$$\phi = f\frac{L_e}{L}\frac{z}{D}$$

Example 10.15: In Fig. 10.16 a valve is opened suddenly in the pipeline when $z_1 = 40$ ft. $L = 2000$ ft, $A_1 = 200$ ft², $A_2 = 300$ ft², $D = 3.0$ ft, $f = 0.024$, and minor losses are $3.50V^2/2g$. Determine the subsequent maximum negative and positive surges in the reservoir A_1.

The equivalent length of minor losses is

$$\frac{KD}{f} = \frac{3.5 \times 3}{0.024} = 438 \text{ ft}$$

Then $L_e = 2000 + 438 = 2438$ and

$$z_m = \frac{z_1 A_1}{A} = \frac{40 \times 200}{2.25\pi} = 1132 \text{ ft}$$

The corresponding ϕ is

$$\phi = f\frac{L_e}{L}\frac{z_m}{D} = 0.024 \times \frac{2438}{2000} \times \frac{1132}{3} = 11.04$$

and

$$F(\phi) = (1 + \phi)e^{-\phi} = (1 + 11.04)e^{-11.04} = 0.000193$$

which is satisfied by $\phi \cong -1.0$. Then

$$F(\phi) = (1 + 1)e^{-1} = 0.736 = (1 + \phi)e^{-\phi}$$

which is satisfied by $\phi = -0.593$. The values of z_m are, for $\phi = -1$

$$z_m = \frac{\phi LD}{fL_e} = \frac{-1 \times 2000 \times 3}{0.024 \times 2438} = -102.6$$

and for $\phi = 0.593$

$$z_m = \frac{0.593 \times 2000 \times 3}{0.024 \times 2438} = 60.9$$

The corresponding values of z_1 are

$$z_1 = z_m\frac{A}{A_1} = -102.6 \times \frac{2.25\pi}{200} = -3.63 \text{ ft}$$

and

$$z_1 = 60.9 \times \frac{2.25\pi}{200} = 2.15 \text{ ft}$$

In this example it should be noted that the subsequent oscillations are almost independent of the original z_1, so long as z_1 is greater than about 20 ft.

10.10. Establishment of Flow. The problem of determination of time for flow to become established in a pipeline when a valve is suddenly opened is easily handled when friction and minor losses are taken into account. After a valve is opened (Fig. 10.17), the head H is available to accelerate the flow in the first instants, but as the velocity increases the accelerating head is reduced by friction and minor losses. If L_e is the

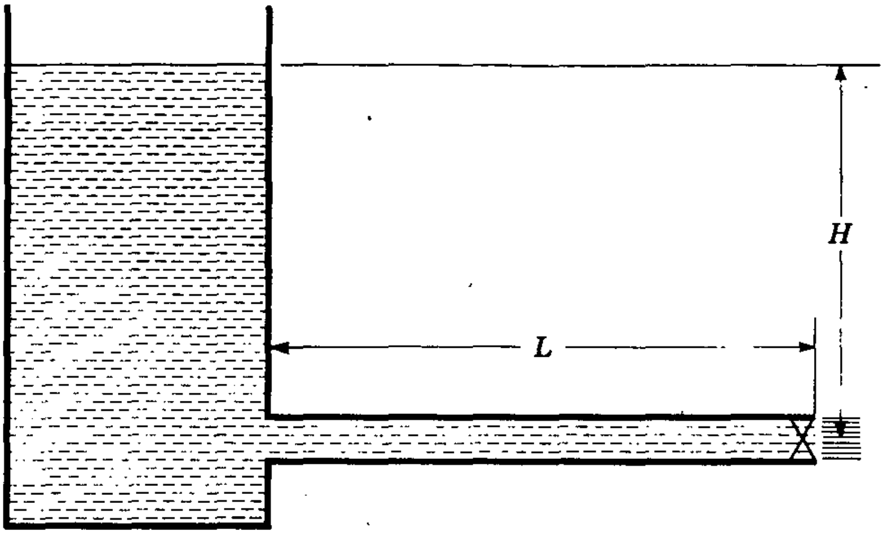


Fig. 10.17. Notation for establishment of flow.

equivalent length of the pipe system, the final velocity V_0 is given by application of the Bernoulli equation

$$H = f\frac{L_e}{D}\frac{V_0^2}{2g} \tag{10.10.1}$$

The equation of motion is

$$\gamma A\left(H - f\frac{L_e}{D}\frac{V^2}{2g}\right) = \frac{\gamma A L}{g}\frac{dV}{dt}$$

By solving for dt and rearranging, with Eq. (10.10.1),

$$\int_0^t dt = \frac{LV_0^2}{gH}\int_0^V \frac{dV}{V_0^2 - V^2}$$

After performing the integration

$$t = \frac{LV_0}{2gH}\ln\frac{V_0 + V}{V_0 - V} \tag{10.10.2}$$

The velocity V approaches V_0 asymptotically; i.e., mathematically it takes infinite time for V to attain the value V_0. Practically, for V to reach 0.99 V_0 takes

$$t = \frac{LV_0}{gH}\frac{1}{2}\ln\frac{1.99}{0.01} = 2.646\frac{LV_0}{gH}$$

V_0 must be determined by taking minor losses into account, but Eq. (10.10.2) does not contain L_e.

Example 10.16: In Fig. 10.17 the minor losses are $16V^2/2g$, $f = 0.030$, $L = 10{,}000$ ft, $D = 8.0$ ft, and $H = 60$ ft. Determine the time, after the sudden opening of a valve, for velocity to attain nine-tenths of the final velocity.

$$L_e = 10{,}000 + \frac{16 \times 8}{0.03} = 14{,}270 \text{ ft}$$

From Eq. (10.10.1)

$$V_0 = \sqrt{\frac{2gHD}{fL_e}} = \sqrt{\frac{64.4 \times 60 \times 8}{0.030 \times 14{,}270}} = 8.5 \text{ ft/sec}$$

After substituting $V = 0.9\, V_0$ into Eq. (10.10.2)

$$t = \frac{10{,}000 \times 8.5}{64.4 \times 60} \ln \frac{1.90}{0.10} = 64.8 \text{ sec}$$

10.11. Surge Control. The oscillation of flow in pipelines, when compressibility effects are not important, is referred to as *surge*. For sudden

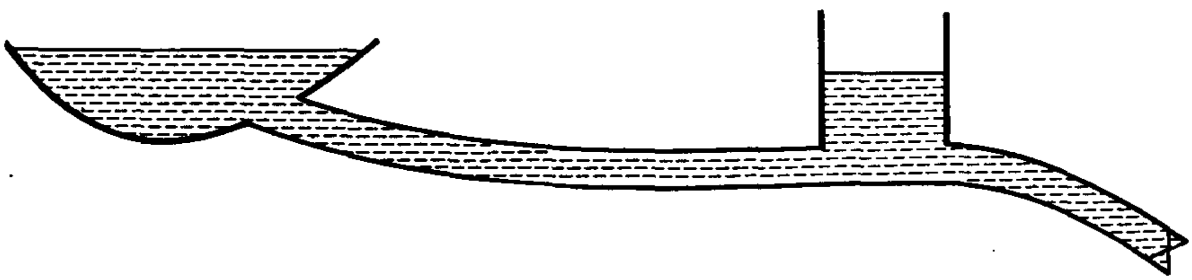


FIG. 10.18. Surge tank on a long pipeline.

deceleration of flow due to closure of the flow passage, compressibility of the liquid and elasticity of the pipe walls must be considered; this phenomenon, known as *water hammer*, is discussed in Sec. 10.12. Oscillations in a U-tube are special cases of surge. As one means of eliminating water hammer provision is made to permit the liquid to surge into a tank (Fig. 10.18). The valve at the end of a pipeline may be controlled by a turbine governor, and may rapidly stop the flow if the generator loses its load. To quickly destroy all momentum in the long pipe system would require high pressure which in turn would require a very costly pipeline. With a surge tank as near the valve as feasible, although surge will occur between the reservoir and surge tank, the developing of high pressure in this reach is prevented. It is still necessary to design the pipeline between surge tank and valve to withstand water hammer.

Surge tanks may be classified as *simple*, *orifice*, and *differential*. The simple surge tank has an unrestricted opening into it, and must be of adequate size so that it will not overflow (unless a spillway is provided) and so that it will not be emptied and air permitted to enter the pipeline.

It must also be of such size that it will not fluctuate in resonance with the governor action on the valve. The period of oscillation of a simple surge tank is relatively long.

The orifice surge tank has a restricted opening or orifice between pipeline and tank and, hence, allows more rapid pressure changes in the pipeline than the simple surge tank. The more rapid pressure change causes a more rapid adjustment of flow to the new valve setting and losses through the orifice aid in dissipating excess available energy resulting from valve closure.

The differential surge tank (Fig. 10.19) is in effect a combination of an orifice surge tank and a simple surge tank of small cross-sectional area. In case of rapid valve opening a limited amount of liquid is

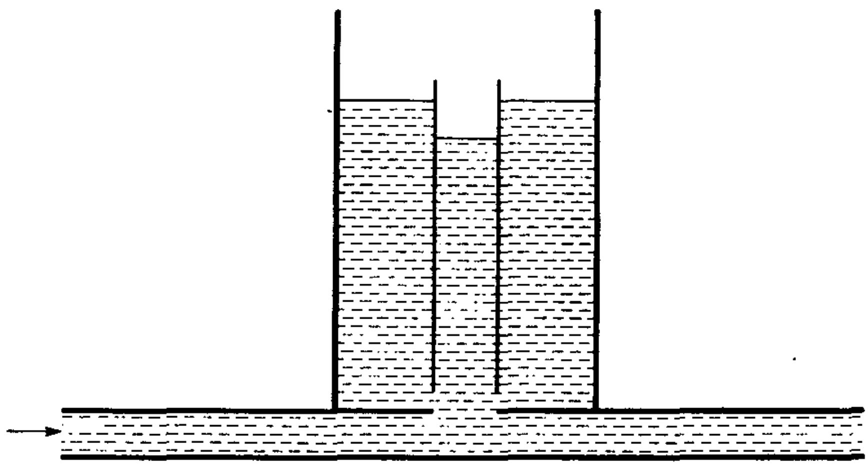


FIG. 10.19. Differential surge tank.

directly available from the central riser and flow from the large tank supplements this flow. For sudden valve closures the central riser may be designed so that it overflows into the outside tank.

Surge tanks operating under air pressure are utilized in certain circumstances, such as after a reciprocating pump. They are generally uneconomical for large pipelines.

Detailed analysis of surge tanks entails a numerical integration of the equation of motion for the liquid in the pipeline, taking into account the particular rate of valve closure, together with the continuity equation. The particular type of surge tank to be selected for a given situation is dependent upon a detailed study of the economics of the pipeline system. High-speed digital computers are most helpful in their design.

Another means of controlling surge and water hammer is to supply a quick-opening bypass valve that opens when the control valve closes. The quick-opening valve has a controlled slow closure at such a rate that excessive pressure is not developed in the line. The bypass valve wastes liquid, however, and does not provide relief from surge due to opening of the control valve or starting of a pump.

10.12. Water Hammer. Water hammer may occur either upstream or downstream from a valve in a pipeline. When sudden closure of a valve occurs, the upstream momentum must be reduced to zero very rapidly, which creates a high pressure at the valve and causes a wave of high pressure to move upstream from the valve. On the downstream side of the valve, momentum of the liquid causes it to continue downstream unless the static pressure is high enough to bring it to rest as pressure is reduced at the valve. Usually boiling (cavitation) occurs downstream. Eventually the liquid comes to rest and is then accelerated upstream toward the valve, condensing the vapor and permitting impact of the liquid column against the valve. This develops a high-pressure wave that moves downstream.

Analysis of water hammer deals with two cases: rapid valve closure and slow valve closure. In this treatment fluid friction is ignored and the assumption of perfect elasticity of liquid and pipe walls is made, as they greatly simplify the analysis. The only cases considered here are with the valve at the downstream end of a pipe.

Rapid Valve Closure. The maximum pressure rise at the valve will be shown to be the same whether the valve is closed instantaneously or in any time less than that required for the pressure wave to travel to the upstream end of the pipe and be reflected to the valve. If the speed of pressure wave is c and the pipe length L, rapid closure occurs when time of closure t_c is less than $2L/c$. The case of instantaneous closure is first considered.

With h the head rise at the valve due to closure, application of the momentum equation

$$\Sigma F_x = \rho Q(V_{x\,\text{out}} - V_{x\,\text{in}})$$

supplies one relation between head h, initial velocity V_0, and wave speed c. The only unbalanced force acting on the liquid in the axial direction is $-\gamma h A$ if friction is neglected. The term ρQ, the mass per second having its momentum changed, is $\rho A c$, as the pressure wave reduces the velocity from V_0 to 0 as it passes, and it travels a distance c in unit time. Thus

$$-\gamma h A = \rho A c(0 - V_0)$$

or

$$h = \frac{V_0 c}{g} \tag{10.12.1}$$

This equation applies to any length of pipe.

To determine the value of the speed c of the pressure wave, the principle of work and energy is applied. For a length L of pipe, the kinetic energy $\gamma A L V_0^2/2g$ before the pressure wave occurs must be converted into elastic energy in compressing the liquid and in stretching the pipe walls. The compressibility of liquid is given by $K = -\forall\, \Delta p/\Delta\forall$, in which $\forall$ is the volume of liquid subjected to pressure Δp [Eq. (1.7.1)]. Since $\Delta p = \gamma h$, the volume reduction $\Delta\forall$ is $\forall\gamma h/K$, or for length L, $AL\gamma h/K$. The volume reduction multiplied by average pressure is the work of compression

$$\frac{\gamma h}{2}\frac{AL\gamma h}{K}$$

The work done in stretching the pipe walls is the product of the average force exerted in the pipe wall and the additional strain, or extension of pipe-wall circumference. From the formula for pipe tension, $T = pr = \gamma hD/2$, in which T is the force per unit length of pipe wall and D is the pipe diameter. The unit stress is T/t' in which t' is the wall thickness. The unit strain is $T/t'E$ and the strain $\pi DT/t'E$, in which E is the modulus of elasticity of pipe-wall material. The average force in the pipe wall due to water hammer is $LT/2 = \gamma hLD/4$. Hence, the work done in expanding the pipe wall is

$$\frac{\pi D}{t'E}\cdot\frac{\gamma hD}{2}\cdot\frac{\gamma hLD}{4}$$

The expression for conversion of kinetic energy to work of compression of liquid and expansion of pipe wall is

$$\gamma AL\frac{V_0^2}{2g} = (\gamma h)^2\frac{AL}{2K} + (\gamma h)^2\frac{\pi D^2}{4}\frac{LD}{2Et'}$$

After simplifying and after solving for h,

$$h = \frac{V_0\sqrt{K/\rho}}{g\sqrt{1 + \dfrac{KD}{Et'}}} \qquad (10.12.2)$$

By comparison with Eq. (10.12.1)

$$c = \frac{\sqrt{K/\rho}}{\sqrt{1 + \dfrac{KD}{Et'}}} \qquad (10.12.3)$$

For the case of very rigid pipes, when KD/Et' is small compared with unity, $c = \sqrt{K/\rho}$, which is Eq. (6.2.3). The speed of sound in water at ordinary temperatures is about

$$c = \sqrt{300{,}000 \times 144/1.935} = 4720 \text{ ft/sec}$$

Hence, for water in a pipe

$$c = \frac{4720}{\sqrt{1 + \dfrac{KD}{Et'}}} \qquad (10.12.4)$$

in which K/E and D/t' are dimensionless. The effect of elasticity of the pipe walls is to reduce the speed of the pressure wave. As an extreme example, for rubber $E \cong 800$ psi and for $D/t = 8$ the value of c is about 86 ft/sec. For steel $E = 3 \times 10^7$ psi and for $D/t' = 100$ the value of c is about 3340 ft/sec.

Example 10.17: A valve is suddenly closed in a water main in which the velocity is 3.50 ft/sec and $c = 3800$ ft/sec. What is the pressure rise at the valve?

$$h = \frac{V_0 c}{g} = \frac{3.5 \times 3800}{32.2} = 413 \text{ ft, or } 179 \text{ psi}$$

It is important that the sequence of events taking place in a pipe after instantaneous closure is thoroughly understood. At the instant of valve closure the fluid nearest the valve is compressed, brought to rest, and the pipe wall stretched. As soon as the first layer is compressed the process is repeated for the next layer. The fluid upstream from the valve continues to move downstream with undiminished speed until successive layers have been compressed back to the source. The high pressure moves upstream as a wave, bringing the fluid to rest as it passes, compressing it, and expanding the pipe. When the wave reaches the upstream end of the pipe, all the fluid is under the extra head h, all the momentum has been lost, and all the kinetic energy has been converted into elastic energy.

There is an unbalanced condition at the upstream (reservoir) end at the instant of arrival of the pressure wave, as the reservoir pressure is unchanged. The fluid starts to flow backward, beginning at the upstream end. This flow returns the pressure to the value which was normal before closure, the pipe wall returns to normal, and the fluid has a velocity V_0 in the backward sense. This process of conversion travels downstream toward the valve at the speed of sound c in the pipe. At the instant $2L/c$ the wave arrives at the valve, pressures are back to normal along the pipe, and velocity is everywhere V_0 in the backward direction.

Since the valve is closed no fluid is available to maintain the flow at the valve and a low pressure develops ($-h$) such that the fluid is brought to rest. This low-pressure wave travels upstream at speed c and everywhere brings the fluid to rest, causes it to expand because of the lower pressure, and allows the pipe walls to contract. (If the static pressure in the pipe is not sufficiently high to sustain head, $-h$, above vapor pressure, the liquid vaporizes in part and continues to move backward over a longer period of time.)

At the instant the negative pressure wave arrives at the upstream end of the pipe, $3L/c$ sec after closure, the fluid is at rest, but uniformly at head $-h$ less than before closure. This leaves an unbalanced condition at the reservoir, and fluid flows into the pipe, acquiring a velocity V_0 forward and returning the pipe and fluid to normal conditions as the wave progresses downstream at speed c. At the instant this wave reaches the valve, conditions are exactly the same as at the instant of closure, $4L/c$ sec earlier.

This process is then repreated every $4L/c$ sec. The action of fluid friction and imperfect elasticity of fluid and pipe wall, neglected heretofore, is to damp out the vibration and eventually cause the fluid to come permanently to rest.

The sequence of events taking place in a pipe may be compared with the sudden stopping of a freight train by the engine hitting an immovable object. The car behind the engine compresses the spring in its forward coupling and stops as it exerts a force against the engine, and each car in turn keeps moving at its original speed until the preceding one suddenly comes to rest. When the caboose is at rest all the energy is stored in compressing the coupling springs (neglecting losses). The caboose has an unbalanced force exerted on it, and starts to move backward, which in turn causes an unbalanced force on the next car setting it in backward motion. This action proceeds as a wave toward the engine, causing each car to move at its original speed in a backward direction. If the engine is immovable the car next to it is stopped by a tensile force in the coupling between it and the engine, analogous to the low-pressure wave in water hammer. The process repeats itself car by car until the train is again at rest, with all couplings in tension. The caboose is then acted upon by the unbalanced tensile force in its coupling and is set into forward motion, followed in turn by the rest of the cars. When this wave reaches the engine all cars are in motion as before the original impact. Then the whole cycle is repeated again. Friction acts to reduce the energy to zero in a very few cycles.

Closure of a valve at any time before $2L/c$ sec causes the pressure at the valve to rise to the same peak V_0c/g as in the case of instantaneous closure. As the valve is being closed, conveniently analyzed as if in discrete steps with instantaneous partial closure, the pressure rise at each step is, from Eq. (10.12.1),

$$\Delta h = \frac{c\,\Delta V}{g} \tag{10.12.5}$$

At the instant of complete closure, when no reflected negative waves have had time to return to the valve,

$$\Sigma\,\Delta h = \frac{c\Sigma\,\Delta V}{g} = \frac{cV_0}{g}$$

and the full pressure rise is developed. The full head at the valve then lasts only from the instant of complete closure to time $2L/c$ from start of first closure, since a negative wave arrives at this time with successive negative waves following.

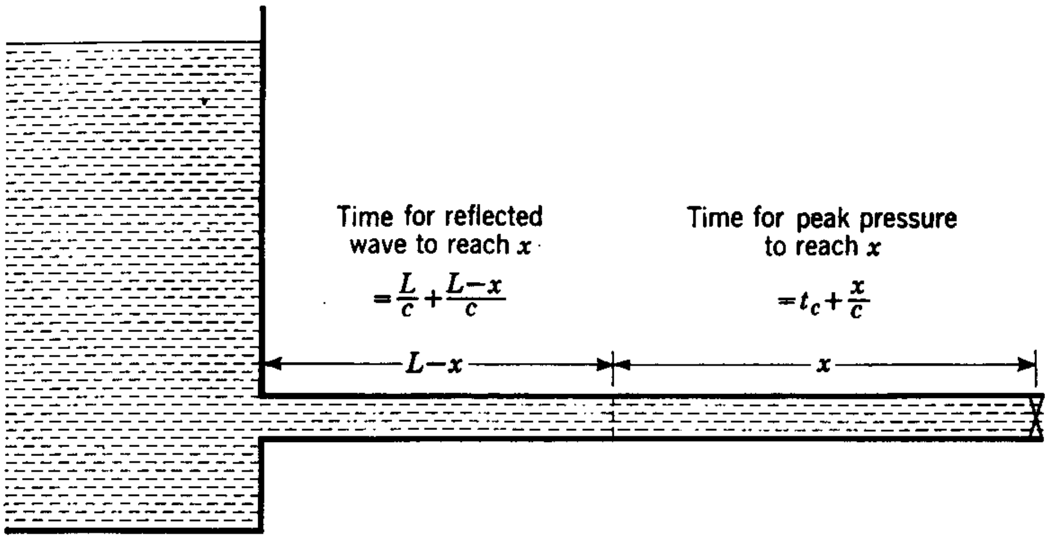


FIG. 10.20. Notation for meeting of peak pressure wave and reflected wave.

For time of closure t_c between O and $2L/c$, the length x of pipe, Fig. 10.20, over which the peak head cV_0/g acts, is found by equating the times for the waves to meet,

$$\frac{L}{c} + \frac{L - x}{c} = t_c + \frac{x}{c}$$

or

$$x = L - \frac{ct_c}{2} \qquad (10.12.6)$$

For $t_c = 0$, $x = L$, and all the pipe is subjected to peak head. For $t_c = L/c$, $x = L/2$.

To compute the pressure rise at the valve as a function of time for rapid closure a numerical process is utilized. The valve is treated as an orifice with a constant coefficient C_d and variable area, A_v,

$$VA = C_dA_v\sqrt{2gh} \qquad (10.12.7)$$

in which V is velocity in the pipe, A the pipe area, and h the head acting across the valve. With h_0 the head across the valve when $V = V_0$ and $A_v = A_{v0}$,

$$V_0A = C_dA_{v0}\sqrt{2gh_0}$$

When this equation is divided into Eq. (10.12.7),

$$\frac{V}{V_0} = \frac{A_v}{A_{v0}} \sqrt{\frac{h}{h_0}} \tag{10.12.8}$$

Equation (10.12.5) may be placed into dimensionless form,

$$\frac{\Delta h}{h_0} = \frac{cV_0}{gh_0} \frac{\Delta V}{V_0} \tag{10.12.9}$$

The fractional part of the valve area, expressed by A_v/A_{v0}, is a function of time, but will be considered as a series of sudden partial closures. For one step at time t_1,

$$\frac{V - \Delta V}{V_0} = \left(\frac{A_v}{A_{v0}}\right)_{t_1} \sqrt{\frac{h + \Delta h}{h_0}} \tag{10.12.10}$$

and may be solved simultaneously with Eq. (10.12.9) to find $\Delta h/h_0$ and $\Delta V/V_0$ for $t = t_1$. The new values of V, h, and $(A_v/A_{v0})_{t_2}$ are then inserted and Eqs. (10.12.9) and (10.12.10) solved again for $\Delta h/h_0$ and $\Delta V/V_0$.

Example 10.18: A 60-in.-diameter steel pipeline 1.0 in. thick and 3730 ft long carries water at 2 ft/sec. A valve at the downstream end of the pipe has a head of 200 ft across it initially. For valve area closure in time $t_c = 2.0$ sec, as given, find the pressure at the valve for the first 4 seconds.

$t/t_c =$	0	0.2	0.4	0.6	0.8	1.0
$A_v/A_{v0} =$	1.0	0.85	0.60	0.35	0.10	0

The speed of the pressure wave is, according to Eq. (10.12.4),

$$c = \frac{4720}{\sqrt{1 + \dfrac{3 \times 10^5 \times 60}{3 \times 10^7 \times 1}}} = 3730 \text{ ft/sec}$$

The time for the wave to be reflected is

$$\frac{2L}{c} = \frac{2 \times 3730}{3730} = 2 \text{ sec}$$

Equation (10.12.9) becomes

$$\frac{\Delta h}{h_0} = \frac{3730 \times 2}{32.2 \times 200} \frac{\Delta V}{V_0} = 1.16 \frac{\Delta V}{V_0}$$

The valve is assumed to stay open the first 0.40 sec and then suddenly to close to $A_v/A_{v0} = 0.85$.

Results of the numerical computation are conveniently tabulated. The first three columns are initially filled in, as well as the first row.

t	$\dfrac{t}{t_c}$	$\dfrac{A_v}{A_{v0}}$	$\dfrac{\Delta V}{V_0}$	$\dfrac{\Delta h}{h_0}$	$\dfrac{V}{V_0}$	$\dfrac{h}{h_0}$	p, psi
0.0	0.0	1.00	.....	......	1.00	1.00	87
0.4	0.2	0.85	0.101	0.12	0.899	1.12	97
0.8	0.4	0.60	0.202	0.23	0.697	1.35	118
1.2	0.6	0.35	0.249	0.29	0.448	1.64	143
1.6	0.8	0.10	0.307	0.36	0.141	2.00	174
2.0	1.0	0.00	0.141	0.16	0.00	2.16	188
2.4	1.2	0.00	.....	−0.23	0.00	1.93	168
2.8	1.4	0.00	.....	−0.47	0.00	1.46	127
3.2	1.6	0.00	.....	−0.58	0.00	0.88	77
3.6	1.8	0.00	.....	−0.72	0.00	0.16	14
4.0	2.0	0.00	.....	−0.32	0.00	−0.16	−14

For $t/t_c = 0.20$, from Eq. (10.12.10)

$$1 - \frac{\Delta V}{V_0} = 0.85\sqrt{1 + \frac{\Delta h}{h_0}}$$

By solving the last two equations by trial or by solving a quadratic, $\Delta V/V_0 = 0.101$, $\Delta h/h_0 = 0.12$. These values are entered into the table and V/V_0 and h/h_0 computed. For $t/t_c = 0.40$

$$0.899 - \frac{\Delta V}{V_0} = 0.60\sqrt{1.12 + \frac{\Delta h}{h_0}}$$

or

$$0.899 - \frac{1}{1.16}\frac{\Delta h}{h_0} = 0.60\sqrt{1.12 + \frac{\Delta h}{h_0}}$$

which is satisfied by $\Delta h/h_0 = 0.23$, $\Delta V/V_0 = 0.202$. The table is completed in this manner down to $t/t_c = 1.0$. At $t/t_c = 1.0$ the valve is closed completely and the head rise $\Delta h/h_0$ is that necessary to reduce the velocity to zero, or $1.16 \times 0.141 = 0.16$. At $t/t_c = 1.2$ the pressure wave generated at $t/t_c = 0.2$ returns to the valve as a reflected negative wave $2\Delta h/h_0 = -0.23$. Similarly at $t/t_c = 1.4$ the wave $2\Delta h/h_0 = -0.47$ arrives and reduces the head. These waves continue to reduce the head until $h/h_0 = -0.16$ at $t/t_c = 2.0$.

Slow Valve Closure. When the time of closure is greater than $2L/c$, reflected waves have time to arrive at the valve before it is completely closed and to reduce the pressure rise. Development of the general differential equations of water hammer is beyond the scope of this treatment, but the general principles may be comprehended by a numerical study with the valve considered as closing by sudden increments.

Equations (10.12.9) and (10.12.10) are applicable for determination of pressure rise when the reflected waves are taken into account. To assist in the numerical calculation it is convenient to consider valve-closure-time increments as a multiple or simple fraction of $2L/c$. The method of handling the reflections is best illustrated by an example.

Example 10.19: Find the maximum pressure rise in the pipeline of Example 10.18 when the time of closure is 10 sec.

t	$\dfrac{t}{t_c}$	$\dfrac{A_v}{A_{v0}}$	$\dfrac{\Delta V}{V_0}$	$\dfrac{\Delta h}{h_0}$	$\dfrac{\Sigma\Delta h}{h_0}$	$\dfrac{V}{V_0}$	$\dfrac{h}{h_0}$
0	0.0	1.00	.....	....	....	1.00	1.00
2	0.2	0.85	0.101	0.12	0.12	0.899	1.12
4	0.4	0.60	0.249	0.29	0.17	0.650	1.17
6	0.6	0.35	0.274	0.32	0.15	0.376	1.15
8	0.8	0.10	0.268	0.31	0.16	0.108	1.16
10	1.0	0.00	0.108	0.12	....	0.00	0.96

The first three columns of the table may be filled in first. The valve is considered to remain open the first 2 sec, then to close to $A_v/A_{v0} = 0.85$ instantaneously. The row for $t/t_c = 0.2$ is the same as before, as the first reflection does not arrive until $t/t_c = 0.4$. At this time

$$0.899 - \frac{\Delta V}{V_0} = 0.60\sqrt{1 - 0.12 + \frac{\Delta h}{h_0}}$$

as the head h/h_0 is reduced by the first reflected wave. After solving with

$$\frac{\Delta h}{h_0} = 1.16\,\frac{\Delta V}{V_0}$$

$\Delta V/V_0 = 0.249$ and $\Delta h/h_0 = 0.29$. Under the column heading $\Sigma\,\Delta h/h_0$ the difference of 0.29 and 0.12 is taken for value of the reflected wave at $t/t_c = 0.6$. The 0.12 value becomes positive at this time and the 0.29 is negative.

For $t/t_c = 0.6$,

$$0.650 - \frac{\Delta V}{V_0} = 0.35\sqrt{1 - 0.17 + \frac{\Delta h}{h_0}}$$

which produces $\Delta V/V_0 = 0.274$ and $\Delta h/h_0 = 0.32$, with total head $1 - 0.17 + 0.32 = 1.15$.

For $t/t_c = 0.8$,

$$0.376 - \frac{\Delta V}{V_0} = 0.10\sqrt{1 - 0.15 + \frac{\Delta h}{h_0}}$$

which determines $\Delta V/V_0 = 0.268$, $\Delta h/h_0 = 0.31$, and $h/h_0 = 1.16$.

For complete closure, $t/t_c = 1.0$, $\Delta V/V_0$ must be 0.108, and $\Delta h/h_0 = 0.108 \times 1.16 = 0.12$. The cumulative reflected head is 0.16, and the head $h/h_0 = 1 - 0.16 + 0.12 = 0.96$. The peak pressure is

$$p = 1.17 \times 200 \times 0.433 = 102 \text{ psi}$$

a reduction of about 86 psi from the case of rapid closure.

Solving the water-hammer problem may take the form of finding the rate of valve closure such that the pressure in the pipe is maintained at a fixed maximum during the closing cycle. Equations (10.12.9) and (10.12.10) are utilized, taking reflections into account. It is again convenient to consider sudden incremental closures at periods that coincide with the return to the valve of the reflected pressure waves. The value of $\Delta h/h_0$ is found for the instant the reflected wave arrives such that the head remains constant at the allowable maximum at the valve. Then $\Delta V/V_0$ is found for each $\Delta h/h_0$ from Eq. (10.12.9) and A_v/A_{v0} from Eq. (10.12.10).

Example 10.20: A pipeline 1610 ft long has an initial velocity of flow of 12 ft/sec and an initial head h_0 of 100 ft across a valve at its downstream end. $c = 3220$ ft/sec. Determine the valve closure as a function of time so that the head in the pipe does not exceed 180 ft.

$$h_0 = 100 \qquad \frac{h_m}{h_0} = 1.8$$

and

$$\frac{\Delta h}{h_0} = \frac{cV_0}{gh_0}\frac{\Delta V}{V_0} = \frac{3220 \times 12}{32.2 \times 100}\frac{\Delta V}{V_0} = 12\frac{\Delta V}{V_0}$$

The reflected wave returns in 1 sec. At $t = 1$ the valve is assumed to make its

t	$\frac{\Delta V}{V_0}$	$\frac{\Delta h}{h_0}$	$\frac{V}{V_0}$	$\frac{A_v}{A_{v0}}$	$\frac{h}{h_0}$
0	......	.....	1.0	1	1
1	0.0667	0.8	0.9333	0.696	1.8
2	0.1333	1.6	0.8000	0.597	1.8
3	0.1333	1.6	0.6777	0.505	1.8
4	0.1333	1.6	0.5444	0.406	1.8
5	0.1333	1.6	0.4111	0.307	1.8
6	0.1333	1.6	0.2778	0.207	1.8
7	0.1333	1.6	0.1445	0.108	1.8
8	0.1333	1.6	0.0112	0.0084	1.8
9	0.0112	0.134	0.0	0.0	0.334
10	0.0	0.0	0.0	0.0	1.667

first sudden partial closure such that $\Delta h/h_0 = 0.8$. Hence

$$\frac{\Delta V}{V_0} = \frac{0.8}{12} = 0.0667$$

From Eq. (10.12.10)

$$1 - 0.0667 = \frac{A_v}{A_{v0}} \sqrt{1 + 0.8}$$

and $A_v/A_{v0} = 0.696$. At $t = 2$, the first pressure wave reflects at the valve and $\Delta h/h_0$ must satisfy the following condition if the head is to remain $h/h_0 = 1.8$

$$1 - 0.8 + \frac{\Delta h}{h_0} = 1.8$$

or $\Delta h/h_0 = 1.6$, $\Delta V/V_0 = 0.1333$. Then

$$0.9333 - 0.1333 = \frac{A_v}{A_{v0}} \sqrt{1.8}$$

and $A_v/A_{v0} = 0.597$, as shown in the above table. At $t = 3$

$$1 + 0.8 - 1.6 + \frac{\Delta h}{h_0} = 1.8$$

and $\Delta h/h_0 = 1.6$ again, $\Delta V/V_0 = 0.1333$, and

$$0.8000 - 0.1333 = \frac{A_v}{A_{v0}} \sqrt{1.8}$$

with $A_v/A_{v0} = 0.505$. This process is repeated with constant reductions in valve area until $t = 9$. At this time a $\Delta h/h_0$ of 0.134 is all that is required to stop the flow. The head is maintained at 180 ft until then, but drops to $0.334 \times 100 = 33.4$ ft at $t = 9$. Neglecting friction the head at the valve would continue to oscillate between 33.4 ft and 166.6 ft. If the head is built up by $\Delta h/h_0 = 0.4$ at $t = 0.50$ and by another $\Delta h/h_0 = 0.4$ at $t = 1.0$, the calculation may be carried out on a $\frac{1}{2}$-sec basis, yielding the same answer.

With motorized valves, i.e., valves operated by electric motor or by air or hydraulic cylinders or diaphragms, any practical law of closure may be obtained with use of a *valve positioner*. The valve positioner is a control device acting on the motor to move the valve stem to any position indicated by a control signal, and to hold it there accurately regardless of flow, pipe pressure, or air or hydraulic supply pressure. A profiled cam moving at constant speed may be used to convey the desired signal to the positioner so that any reasonable stem motion is obtained.

PROBLEMS

10.1. Sketch the hydraulic and energy grade lines for Fig. 10.21. $H = 24$ ft.

10.2. Calculate the value of K for the valve of Fig. 10.21 so that the discharge of Prob. 10.1 is reduced by one-half. Sketch the hydraulic and energy grade lines.

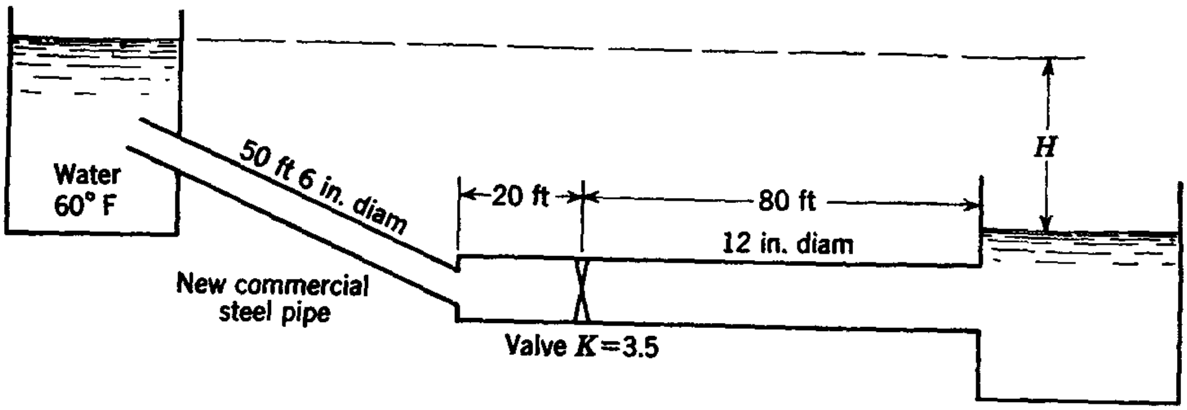


FIG. 10.21

10.3. Compute the discharge of the system in Fig. 10.22. Draw the hydraulic and energy grade lines.

10.4. What head is needed in Fig. 10.22 to produce a discharge of 10 cfs?

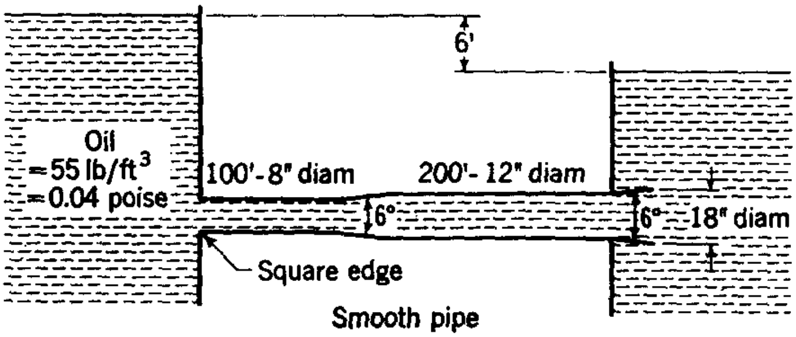


FIG. 10.22

10.5. Calculate the discharge through the siphon of Fig. 10.23 with the conical diffuser removed. $H = 4$ ft.

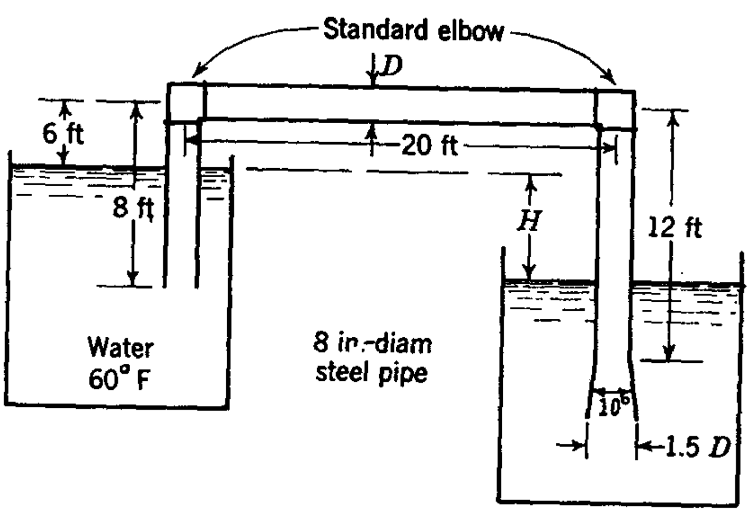


FIG. 10.23

10.6. Calculate the discharge in the siphon of Fig. 10.23 for $H = 8$ ft. What is the minimum pressure in the system?

10.7. Find the discharge through the siphon of Fig. 10.24. What is the pressure at A? Estimate the minimum pressure in the system.

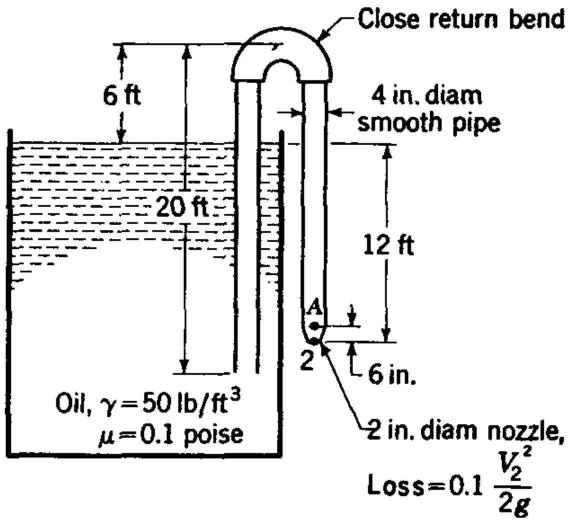


Fig. 10.24

10.8. Neglecting minor losses other than the valve, sketch the hydraulic grade line for Fig. 10.25. The globe valve has a loss coefficient $K = 4.5$.

10.9. What is the maximum height of point A (Fig. 10.25) for no cavitation? Barometer reading 29.5 in. mercury.

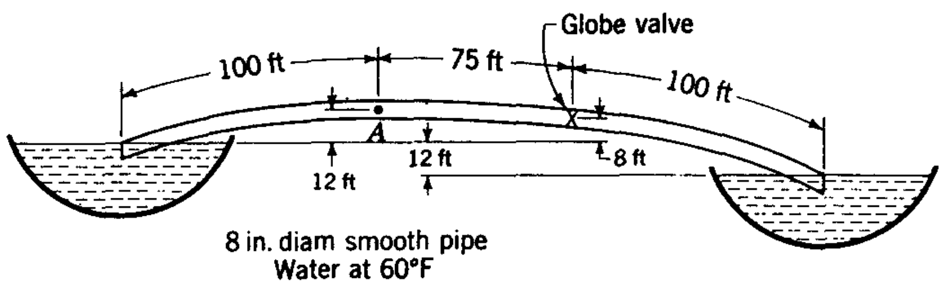


Fig. 10.25

10.10. Two reservoirs are connected by three commercial steel pipes in series, $L_1 = 1000$ ft, $D_1 = 8$ in.; $L_2 = 1200$ ft, $D_2 = 1$ ft; $L_3 = 4000$ ft, $D_3 = 18$ in. When $Q = 3$ cfs water at 70°F, determine the difference in elevation of the reservoirs.

10.11. Solve Prob. 10.10 by the method of equivalent lengths.

10.12. For a difference in elevation of 30 ft in Prob. 10.10, find the discharge by determining the friction factors.

10.13. For a difference in elevation of 40 ft in Prob. 10.10, determine the discharge by the method of equivalent lengths.

10.14. What diameter smooth pipe is required to convey 100 gpm kerosene at 90°F 500 ft with a head of 16 ft? There are a valve and other minor losses with total K of 7.6.

10.15. Air at atmospheric pressure and 60°F is carried through two horizontal pipes ($\epsilon = 0.06$) in series. The upstream pipe is 400 ft of 24 in. diameter, and the downstream pipe is 100 ft of 36 in. diameter. Estimate the equivalent length of 18-in. smooth pipe. Neglect minor losses.

10.16. What pressure drop, in inches of water, is required for flow of 6000 cfm in Prob. 10.15? Include losses due to sudden expansion.

10.17. Two pipes are connected in parallel between two reservoirs; $L_1 = 8000$ ft, $D_1 = 48$-in.-diameter old cast-iron pipe, $f_1 = 0.026$; $L_2 = 8000$ ft, $D_2 = 42$ in., $\epsilon_2 = 0.003$. For a difference in elevation of 12 ft, determine the total flow of water at 70°F.

10.18. For 160 cfs flow in the system of Prob. 10.17, determine the difference in elevation of reservoir surfaces.

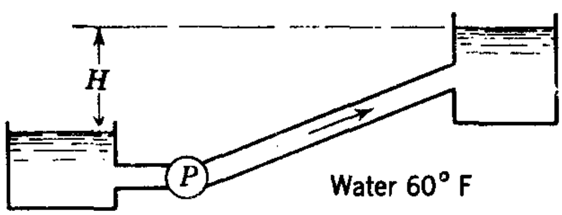
10.19. Three smooth tubes are connected in parallel: $L_1 = 40$ ft, $D_1 = \frac{1}{2}$ in.; $L_2 = 60$ ft, $D_2 = 1$ in.; $L_3 = 50$ ft, $D_3 = \frac{3}{4}$ in. For total flow of 30 gpm oil, $\gamma = 55$ lb/ft^3, $\mu = 0.65$ poise, what is the drop in hydraulic grade line between junctions?

10.20. Determine the discharge of the system of Fig. 10.26 for $L = 2000$ ft, $D = 18$ in., $\epsilon = 0.0015$, and $H = 25$ ft, with the pump A characteristics given.

10.21. Determine the discharge through the system of Fig. 10.26 for $L = 4000$ ft, $D = 24$-in. smooth pipe, $H = 40$ ft, with pump B characteristics.

10.22. Construct a head-discharge-efficiency table for pumps A and B (Fig. 10.26) connected in series.

10.23. Construct a head-discharge-efficiency table for pumps A and B (Fig. 10.26) connected in parallel.



PUMP A

H, ft	Q, cfs	e, %
70	0	0
60	2.00	59
55	2.56	70
50	3.03	76
45	3.45	78
40	3.82	76.3
35	4.11	72
30	4.38	65
25	4.59	56.5
20	4.73	42

PUMP B

H, ft	Q, cfs	e, %
80	0	0
70	2.60	54
60	3.94	70
50	4.96	80
40	5.70	73
30	6.14	60
20	6.24	40

FIG. 10.26

10.24. Find the discharge through the system of Fig. 10.26 for pumps A and B in series; 5000 ft of 12-in. clean cast-iron pipe, $H = 100$ ft.

10.25. Determine the horsepower needed to drive pumps A and B in Prob. 10.24.

10.26. Find the discharge through the system of Fig. 10.26 for pumps A and B in parallel; 5000 ft of 18-in. steel pipe, $H = 30$ ft.

10.27. Determine the horsepower needed to drive the pumps in Prob. 10.26.

10.28. For $H = 40$ ft in Fig. 10.27, find the discharge through each pipe. $\mu = 0.08$ poise; $\gamma = 60$ lb/ft^3.

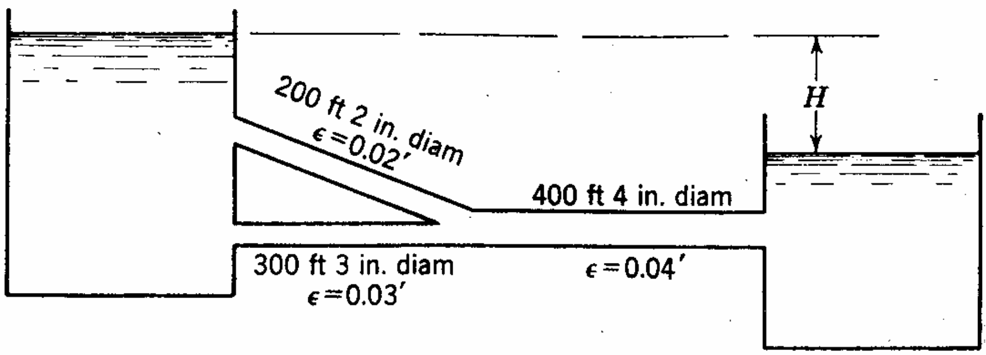


FIG. 10.27

10.29. Find H in Fig. 10.27 for 1 cfs flowing. $\mu = 0.05$ poise; $\rho = 1.8$ slugs/ft^3.

10.30. Find the equivalent length of 12-in.-diameter clean cast-iron pipe to replace the system of Fig. 10.28. For $H = 30$ ft, what is the discharge?

10.31. With velocity of 4 ft/sec in the 8-in.-diameter pipe of Fig. 10.28, calculate the flow through the system and the head H required.

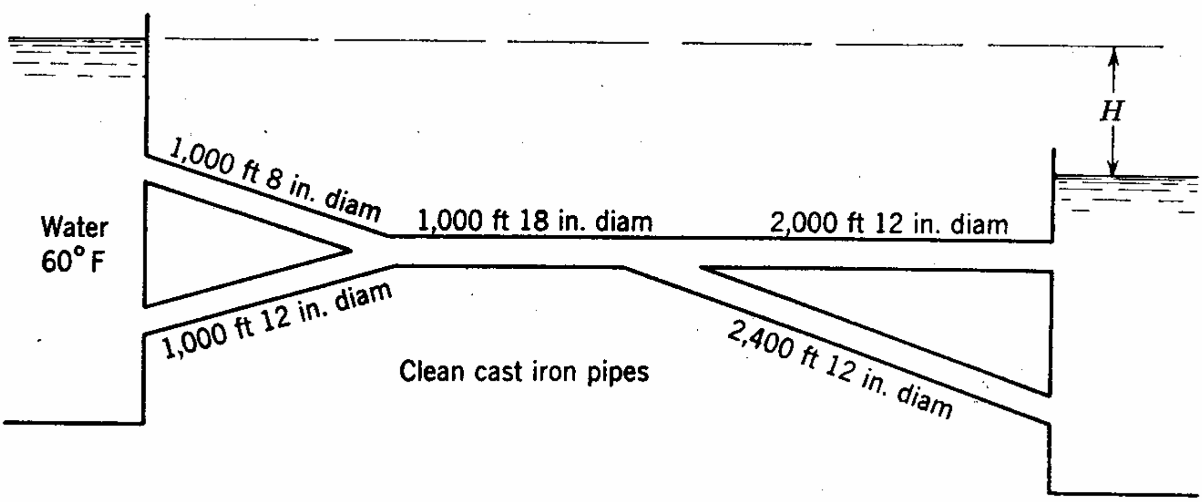


FIG. 10.28

10.32. In Fig. 10.29 find the flow through the system when the pump is removed.

10.33. If the pump of Fig. 10.29 is delivering 3 cfs toward J, find the flow into A and B and the elevation of the hydraulic grade line at J.

10.34. The pump is adding 10 fluid horsepower to the flow (toward J) in Fig. 10.29. Find Q_A and Q_B.

10.35. With pump A of Fig. 10.26 in the system of Fig. 10.29, find Q_A, Q_B, and the elevation of the hydraulic grade line at J.

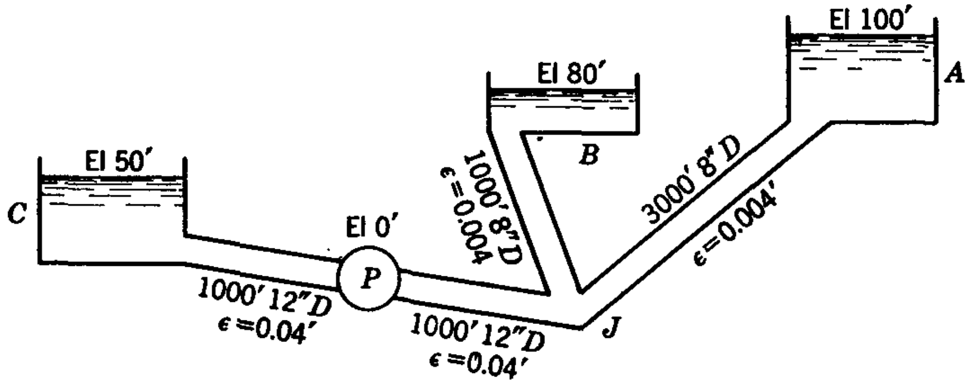


FIG. 10.29

10.36. With pump B of Fig. 10.26 in the system of Fig. 10.29, find the flow into B and the elevation of the hydraulic grade line at J.

10.37. For flow of 1 cfs into B of Fig. 10.29, what head is produced by the pump? For pump efficiency of 70 per cent, how much power is required?

10.38. Find the flow through the system of Fig. 10.30 for no pump in the system.

10.39. With pumps A and B of Fig. 10.26 in parallel in the system of Fig. 10.30, find the flow into B, C, and D and the elevation of the hydraulic grade line at J_1 and J_2.

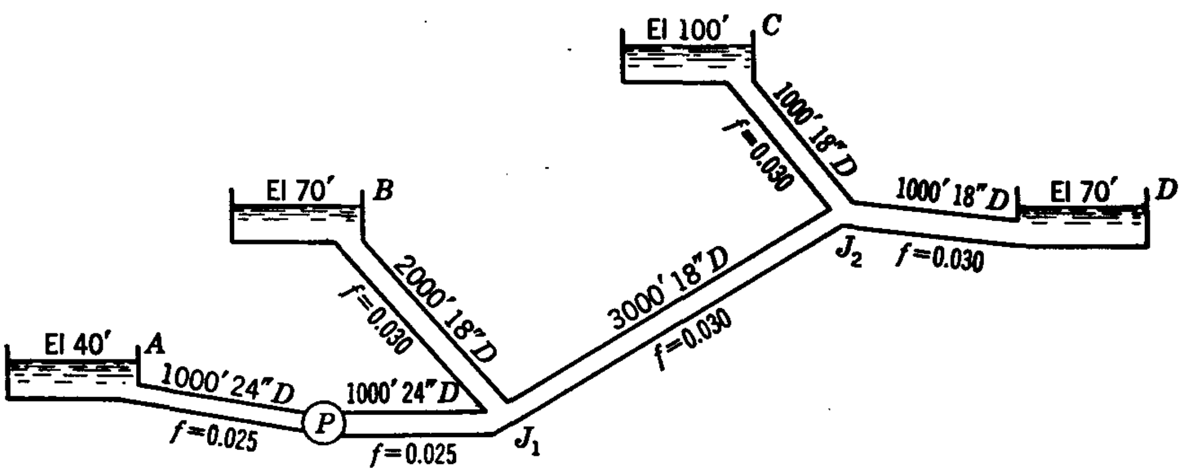


FIG. 10.30

10.40. Calculate the flow through each of the pipes of the network shown in Fig. 10.31. $n = 2$.

10.41. Determine the flow through each line of Fig. 10.32. $n = 2$.

10.42. Find the distribution through the network of Fig. 10.31 for $n = 1$.

10.43. Find the distribution through the network of Fig. 10.32 for $n = 1$.

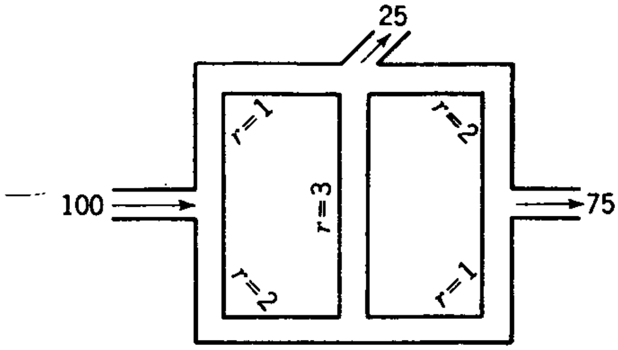


FIG. 10.31

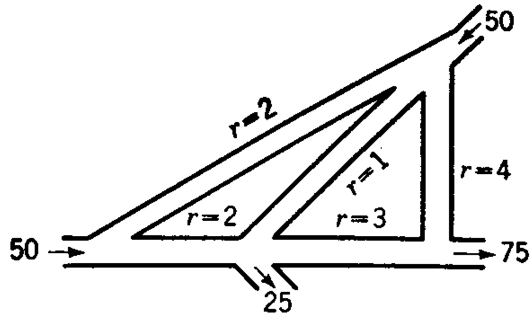


FIG. 10.32

10.44. Determine the slope of the hydraulic grade line for flow of atmospheric air at 80°F through a rectangular 18- by 6-in. galvanized-iron conduit. $V =$ 30 ft/sec.

10.45. What size square conduit is needed to convey 10 cfs water at 60°F with slope of hydraulic grade line of 0.001? $\epsilon = 0.003$.

10.46. Calculate the discharge of oil, sp gr 0.85, $\mu = 0.04$ poise, through 100 ft of 2- by 4-in. sheet-metal conduit when the head loss is 2 ft. $\epsilon = 0.0005$.

10.47. A duct, with cross section an equilateral triangle 1 ft on a side, conveys 6 cfs water at 60°F. $\epsilon = 0.003$. Calculate the slope of the hydraulic grade line.

10.48. A clean cast-iron water pipe 24 in. in diameter has its absolute roughness double in 5 years of service. Estimate the head loss per 1000 ft for a flow of 15 cfs when the pipe is 25 years old.

10.49. An 18-in.-diameter pipe has an f of 0.020 when new for 5 ft/sec water flow at 60°F. In 10 years $f = 0.029$ for $V = 3$ ft/sec. Find f for 4 ft/sec at end of 20 years.

10.50. Determine the period of oscillation of a U-tube containing one pint of water. The cross-sectional area is 0.50 in.². Neglect friction.

10.51. A U-tube containing alcohol is oscillating with maximum displacement from equilibrium position of 6.0 in. The total column length is 40 in. Determine the maximum fluid velocity and the period of oscillation. Neglect friction.

10.52. A liquid, $\nu = 0.002$ ft²/sec, is in a U-tube 0.50 in. in diameter. The total liquid column is 60 in. long. If one meniscus is 12 in. above the other meniscus when the column is at rest, determine the time for one meniscus to move to within 1.0 in. of its equilibrium position.

10.53. Develop the equations for motion of a liquid in a U-tube for laminar resistance when $16\nu/D^2 = \sqrt{2g/L}$. SUGGESTION: Try $z = e^{-mt}(c_1 + c_2t)$.

10.54. A U-tube contains liquid oscillating with a velocity 6 ft/sec at the instant the menisci are at the same elevation. Find the time to the instant the menisci are next at the same elevation, and determine the velocity then. $\nu = 1 \times 10^{-4}$ ft²/sec, $D = \frac{1}{4}$ in., $L = 30$ in.

10.55. A 10-ft-diameter horizontal tunnel has 10-ft-diameter vertical shafts spaced one mile apart. When valves are closed isolating this reach of tunnel, the water surges to a depth of 50 ft in one shaft when it is 20 ft in the other shaft. For $f = 0.022$ find the height of the next two surges.

10.56. Two standpipes 20 ft in diameter are connected by 3000 ft of 8.0-ft-diameter pipe, $f = 0.020$ and minor losses are 4.5 velocity heads. One reservoir level is 30 ft above the other one when a valve is rapidly opened in the pipeline. Find the maximum fluctuation in water level in the standpipe.

10.57. A valve is quickly opened in a pipe 4000 ft long, $D = 2.0$ ft, with a 1-ft-diameter nozzle on the downstream end. Minor losses are $4V^2/2g$, with V the velocity in the pipe, $f = 0.024$, $H = 30$ ft. Find the time to attain 95 per cent of the steady-state discharge.

10.58. A globe valve ($K = 10$) at the end of a pipe 2000 ft long is rapidly opened. $D = 3.0$ ft, $f = 0.018$, minor losses $2V^2/2g$, and $H = 75$ ft. How long does it take for the discharge to attain 80 per cent of its steady state value?

10.59. A steel pipeline is 36 in. in diameter and has a $\frac{3}{8}$-in. wall thickness. When it is carrying water, determine the speed of a pressure wave.

10.60. Benzine ($K = 150{,}000$ psi, $S = 0.88$) flows through $\frac{3}{4}$-in. ID steel tubing with $\frac{1}{8}$-in. wall thickness. Determine the speed of a pressure wave.

10.61. Determine the maximum time for rapid valve closure on the pipeline: $L = 3000$ ft, $D = 4$ ft, $t' = \frac{1}{2}$ in., steel pipe, $V_0 = 10$ ft/sec, water flowing.

10.62. A valve is closed in 5 sec at the downstream end of a 10,000-ft pipeline carrying water at 6 ft/sec. $c = 3400$ ft/sec. What is the peak pressure developed by the closure?

10.63. Determine the length of pipe in Prob. 10.62 subjected to the peak pressure.

10.64. A valve is closed at the downstream end of a pipeline in such a manner that only one-third of the line is subjected to maximum pressure. At what proportion of the time $2L/c$ is it closed?

10.65. A pipeline, $L = 6000$ ft, $c = 3000$ ft/sec, has a valve on its downstream end, $V_0 = 8$ ft/sec and $h_0 = 60$ ft. It closes in 3 increments, spaced 1 sec apart, each area reduction being one-third of the original opening. Find the pressure at the gate and at the midpoint of the pipeline at 1 sec intervals for 5 sec after initial closure.

10.66. A pipeline, $L = 2000$ ft, $c = 4000$ ft/sec, has a valve at its downstream end, $V_0 = 6$ ft/sec and $h_0 = 100$ ft. Determine the pressure at the valve for the closure:

A_v/A_{v0}	1	0.75	0.60	0.45	0.30	0.15	0
t, sec	0	0.5	1.0	1.5	2.0	2.5	3.0

10.67. In Prob. 10.66 determine the peak pressure at the valve for uniform area reduction in 3.0 sec.

10.68. Find the maximum area reduction for $\frac{1}{2}$-sec intervals for the pipeline of Prob. 10.66 when the maximum head at the valve is not to exceed 160 ft.

10.69. The hydraulic grade line is

(*a*) always above the energy grade line
(*b*) always above the closed conduit
(*c*) always sloping downward in the direction of flow
(*d*) the velocity head below the energy grade line
(*e*) upward in direction of flow when pipe is inclined downward

10.70. In solving a series-pipe problem for discharge, Bernoulli's equation is used along with the continuity equation to obtain an expression that contains a $V^2/2g$ and f_1, f_2, etc. The next step in the solution is to assume

(*a*) Q (*b*) V (*c*) **R** (*d*) $f_1, f_2, \ldots$ (*e*) none of these quantities

10.71. One pipe system is said to be equivalent to another pipe system when the following two quantities are the same:

(*a*) h, Q (*b*) L, Q (*c*) L, D (*d*) f, D (*e*) V, D

10.72. In parallel-pipe problems

(*a*) the head losses through each pipe are added to obtain the total head loss
(*b*) the discharge is the same through all the pipes
(*c*) the head loss is the same through each pipe
(*d*) a direct solution gives the flow through each pipe when the total flow is known
(*e*) a trial solution is not needed

10.73. Branching-pipe problems are solved

(*a*) analytically by using as many equations as unknowns
(*b*) by the Hardy Cross method of correcting assumed flows
(*c*) by equivalent lengths
(*d*) by assuming a distribution which satisfies continuity and computing a correction
(*e*) by assuming the elevation of hydraulic grade line at the junction point and trying to satisfy continuity

10.74. In networks of pipes

(*a*) the head loss around each elementary circuit must be zero
(*b*) the (horsepower) loss in all circuits is the same
(*c*) the elevation of hydraulic grade line is assumed for each junction
(*d*) elementary circuits are replaced by equivalent pipes
(*e*) friction factors are assumed for each pipe

10.75. The following quantities are computed by using $4R$ in place of diameter, for noncircular sections:

(*a*) velocity, relative roughness
(*b*) velocity, head loss
(*c*) Reynolds number, relative roughness, head loss
(*d*) velocity, Reynolds number, friction factor
(*e*) none of these answers

10.76. Experiments show that in the aging of pipes

(*a*) the friction factor increases linearly with time
(*b*) a pipe becomes smoother with use
(*c*) the absolute roughness increases linearly with time
(*d*) no appreciable trends can be found
(*e*) the absolute roughness decreases with time

10.77. In the analysis of unsteady-flow situations the following formulas may be utilized:

(*a*) equation of motion, Bernoulli equation, momentum equation
(*b*) equation of motion, continuity equation, momentum equation
(*c*) equation of motion, continuity equation, Bernoulli equation
(*d*) momentum equation, continuity equation, Bernoulli equation
(*e*) none of these answers

10.78. Neglecting friction, the maximum difference in elevation of the two menisci of an oscillating U-tube is 1.0 ft, $L = 3.0$ ft. The period of oscillation is, in seconds,

(*a*) 0.52 (*b*) 1.92 (*c*) 3.27 (*d*) 20.6 (*e*) none of these answers

10.79. The maximum speed of the liquid column in Prob. 10.78 is, in feet per second,

(*a*) 0.15 (*b*) 0.31 (*c*) 1.64 (*d*) 3.28 (*e*) none of these answers

10.80. In frictionless oscillation of a U-tube, $L = 4.0$ ft, $z = 0$, $V = 6$ ft/sec. The maximum value of z is, in feet,

(*a*) 0.75 (*b*) 1.50 (*c*) 6.00 (*d*) 24.0 (*e*) none of these answers

10.81. In analyzing the oscillation of a U-tube with laminar resistance, the assumption is made that the

(*a*) motion is steady
(*b*) resistance is constant
(*c*) Darcy-Weisbach equation applies
(*d*) resistance is a linear function of the displacement
(*e*) resistance is the same at any instant as if the motion were steady

10.82. When $16\nu/D^2 = 5$ and $2g/L = 12$ in oscillation of a U-tube with laminar resistance,

(*a*) the resistance is so small that it may be neglected
(*b*) the menisci oscillate about the $z = 0$ axis
(*c*) the velocity is a maximum when $z = 0$
(*d*) the velocity is zero when $z = 0$
(*e*) the speed of column is a linear function of z

10.83. In laminar resistance to oscillation in a U-tube, $m = 1$, $n = \frac{1}{2}$, $V_0 = 3$ ft/sec when $t = 0$ and $z = 0$. The time of maximum displacement of meniscus is, in seconds,

(*a*) 0.46 (*b*) 0.55 (*c*) 0.93 (*d*) 1.1 (*e*) none of these answers

10.84. In Prob. 10.83 the maximum displacement, in feet, is

(*a*) 0.53 (*b*) 1.06 (*c*) 1.16 (*d*) 6.80 (*e*) none of these answers

10.85. In analyzing the oscillation of a U-tube with turbulent resistance, the assumption is made that

(*a*) the Darcy-Weisbach equation applies
(*b*) the Hagen-Poiseuille equation applies
(*c*) the motion is steady
(*d*) the resistance is a linear function of velocity
(*e*) the resistance varies as the square of the displacement

10.86. The maximum displacement is $z_m = 20$ ft for $f = 0.020$, $D = 1.0$ ft in oscillation of a U-tube with turbulent flow. The minimum displacement, $(-z_{m+1})$ of the same fluid column is

(*a*) -13.3 (*b*) -15.7 (*c*) -16.5 (*d*) -20 (*e*) none of these answers

10.87. When a valve is suddenly opened at the downstream end of a long pipe connected at its upstream end with a water reservoir,

(*a*) the velocity attains its final value instantaneously if friction is neglected
(*b*) the time to attain nine-tenths of its final velocity is less with friction than without friction
(*c*) the value of f does not affect the time to acquire a given velocity
(*d*) the velocity increases exponentially with time
(*e*) the final velocity is attained in less than $2L/c$ sec

10.88. Surge may be differentiated from water hammer by

(*a*) the time for a pressure wave to traverse the pipe
(*b*) the presence of a reservoir at one end of the pipe
(*c*) the rate of deceleration of flow
(*d*) the relative compressibility of liquid to expansion of pipe walls
(*e*) the length-diameter ratio of pipe

10.89. Water hammer occurs only when

(*a*) $2L/c > 1$ (*b*) $V_0 > c$ (*c*) $2L/c = 1$ (*d*) $K/E < 1$
(*e*) compressibility effects are important

10.90. Valve closure is *rapid* only when

(*a*) $2L/c \geq t_c$ (*b*) $L/c \geq t_c$ (*c*) $L/2c \geq t_c$ (*d*) $t_c = 0$
(*e*) none of these answers

10.91. The head rise at a valve due to sudden closure is

(*a*) $c^2/2g$ (*b*) V_0c/g (*c*) $V_0c/2g$ (*d*) $V_0^2/2g$ (*e*) none of these answers

10.92. The speed of a pressure wave through a pipe depends upon

(*a*) the length of pipe
(*b*) the original head at the valve
(*c*) the viscosity of fluid
(*d*) the initial velocity
(*e*) none of these answers

10.93. When the velocity in a pipe is suddenly reduced from 10 ft/sec to 6 ft/sec by downstream valve closure, for $c = 3220$ ft/sec, the head rise in feet is

(*e*) 1000 (*b*) 600 (*c*) 400 (*d*) 300 (*e*) none of these answers

10.94. When $t_c = L/2c$ the proportion of pipe length subjected to maximum head is, in per cent,

(*a*) 25 (*b*) 50 (*c*) 75 (*d*) 100 (*e*) none of these answers

10.95. When the steady-state value of head at a valve is 120 ft the valve is given a sudden partial closure such that $\Delta h = 80$ ft. The head at the valve at the instant this reflected wave returns is

(*a*) -80 (*b*) 40 (*c*) 80 (*d*) 200 (*e*) none of these answers

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11

FLOW IN OPEN CHANNELS

A broad coverage of topics in open-channel flow, including both steady and unsteady flows, has been selected for this chapter. Steady uniform flow was discussed in Sec. 5.8, and application of the momentum equation to the hydraulic jump in Sec. 3.9. Weirs were introduced in Sec. 9.5. In this chapter open-channel flow is first classified and then the *shape* of optimum canal cross sections is discussed followed by a section on flow through a floodway. The hydraulic jump and its application to stilling basins is then treated, followed by a discussion of specific energy and critical depth which leads into gradually varied flow. Water surface profiles are classified and related to channel control sections. Transitions are next discussed, with one special application to the critical-depth meter. The closing section deals with unsteady flow in the form of positive and negative surge waves.

The mechanics of flow in open channels is more complicated than closed-conduit flow owing to the presence of a free surface. The hydraulic grade line coincides with the free surface, and, in general, its position is unknown.

For laminar flow to occur, the cross section must be extremely small, the velocity very small, or the kinematic viscosity extremely high. One example of laminar flow is given by a thin film of liquid flowing down an inclined or vertical plane. This case is treated by the methods developed in Chap. 5 (see Prob. 5.12). Pipe flow has a lower critical Reynolds number of 2000, and this same value may be applied to an open channel when the diameter D is replaced by $4R$. R is the hydraulic radius, which is defined as the cross-sectional area of the channel divided by the wetted perimeter. In the range of Reynolds number, based on R in place of D, $\mathbf{R} = VR/\nu < 500$ flow is laminar, $500 < \mathbf{R} < 2000$ flow is *transitional* and may be either laminar or turbulent, and $\mathbf{R} > 2000$ flow is generally turbulent.

Most open-channel flows are turbulent, usually with water as the liquid. The methods for analyzing open-channel flow are not developed to the extent of those for closed conduits. The equations in use assume complete turbulence, with the head loss proportional to the square of the velocity. Although practically all data on open-channel flow have been obtained from experiments on the flow of water, the equations should yield reasonable values for other liquids of low viscosity. The material in this chapter applies to turbulent flow only.

11.1. Classification of Flow. Open-channel flow occurs in a large variety of forms, from flow of water over the surface of a plowed field during a hard rain to the flow at constant depth through a large prismatic channel. It may be classified as steady or unsteady, uniform or non-uniform. *Steady uniform flow* occurs in very long inclined channels of constant cross section, in those regions where "terminal velocity" has been reached, i.e., where the head loss due to turbulent flow is exactly supplied by the reduction in potential energy due to the uniform decrease in elevation of the bottom of the channel. The depth for steady uniform flow is called the *normal depth.* In steady uniform flow the discharge is constant, and the depth is everywhere constant along the length of the channel. Several equations are in common use for determining the relation among the average velocity, the shape of the cross section, its size and roughness, and the slope, or inclination, of the channel bottom (Sec. 5.8).

Steady nonuniform flow occurs in any irregular channel in which the discharge does not change with the time; it also occurs in regular channels when the flow depth and, hence, the average velocity change from one cross section to another. For gradual changes in depth or section, called *gradually varied flow,* methods are available, by numerical integration or step-by-step means, for computing flow depths for known discharge, channel dimensions and roughness, and given conditions at one cross section. For those reaches of a channel where pronounced changes in velocity and depth occur in a short distance, as in a transition from one cross section to another, model studies are frequently made. The *hydraulic jump* is one example of steady nonuniform flow; it is discussed in Secs. 3.9 and 11.4.

Unsteady uniform flow rarely occurs in open-channel flow. *Unsteady nonuniform flow* is common but is extremely difficult to analyze. Wave motion is an example of this type of flow, and its analysis is complex when friction is taken into account. The positive and negative *surge wave* in a rectangular channel is analyzed, neglecting effects of friction, in Sec. 11.10.

Flow is also classified as *tranquil* or *rapid.* When flow occurs at low velocities so that a small disturbance can travel upstream and thus change upstream conditions, it is said to be tranquil flow[1] ($\mathbf{F} < 1$). Conditions

[1] See Sec. 4.4 for definition and discussion of the Froude number $\mathbf{F}$.

upstream are affected by downstream conditions, and the flow is controlled by the downstream conditions. When flow occurs at such high velocities that a small disturbance, such as an elementary wave, is swept downstream, the flow is described as *shooting* or *rapid* ($\mathbf{F} > 1$). Small changes in downstream conditions do not effect any change in upstream conditions; hence, the flow is controlled by upstream conditions. When flow is such that its velocity is just equal to the velocity of an elementary wave, the flow is said to be critical ($\mathbf{F} = 1$).

Velocity Distribution. The velocity at a solid boundary must be zero, and in open-channel flow it generally increases with distance from the boundaries. The maximum velocity does not occur at the free surface but is usually below the free surface a distance of 0.05 to 0.25 of the depth. The average velocity along a vertical line is sometimes determined by measuring the velocity at 0.6 of the depth, but a more reliable method is to take the average of the velocities at 0.2 and 0.8 of the depth, according to measurements of the U.S. Geological Survey.

11.2. Best Hydraulic Channel Cross Sections. For the cross section of channel for conveying a given discharge for given slope and roughness factor, some shapes are more efficient than others. In general, when a channel is constructed, the excavation, and possibly the lining, must be paid for. Based on the Manning formula it is shown that when the area of cross section is a minimum, the wetted perimeter is also a minimum, so both lining and excavation approach their minimum value for the same dimensions of channel. The *best hydraulic section* is one that has the least wetted perimeter, or its equivalent, the least area for the type of section. The Manning formula is

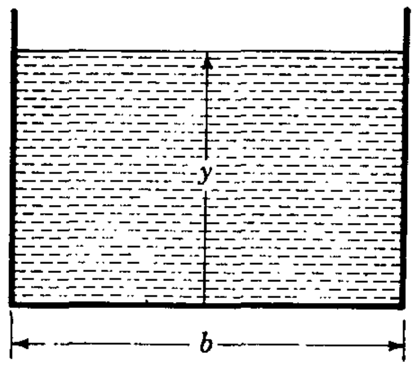

FIG. 11.1. Rectangular cross section.

$$Q = \frac{1.49}{n} AR^{\frac{2}{3}}S^{\frac{1}{2}} \tag{11.2.1}$$

in which Q is the discharge (cubic feet per second), A the cross-sectional flow area (square feet), R (area divided by wetted perimeter P) the hydraulic radius (feet), S the slope of energy grade line, and n the Manning roughness factor (Table 5.2, Sec. 5.8). With Q, n, and S known, Eq. (11.2.1) may be written

$$A = cP^{\frac{2}{5}} \tag{11.2.2}$$

in which c is known. This equation shows that P is a minimum when A is a minimum. To find the best hydraulic section for a *rectangular* channel (Fig. 11.1) $P = b + 2y$, and $A = by$. Then

$$A = (P - 2y)y = cP^{\frac{2}{5}}$$

by elimination of b. The value of y is sought for which P is a minimum. Differentiating with respect to y

$$\left(\frac{dP}{dy} - 2\right)y + P - 2y = \frac{2}{5}\,cP^{-\frac{3}{5}}\,\frac{dP}{dy}$$

After setting $dP/dy = 0$, $P = 4y$, or since $P = b + 2y$,

$$b = 2y \tag{11.2.3}$$

Therefore, the depth is one-half the bottom width, independent of the size of rectangular section.

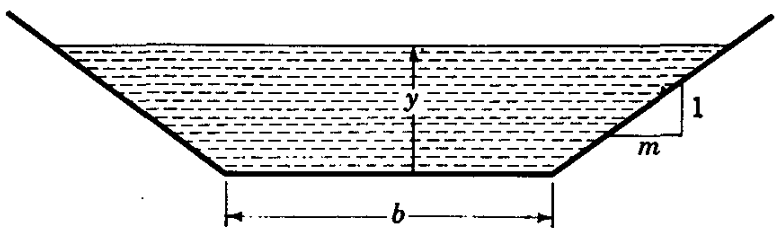

FIG. 11.2. Trapezoidal cross section.

To find the best hydraulic *trapezoidal* section (Fig. 11.2) $A = by + my^2$, $P = b + 2y\sqrt{1 + m^2}$. After eliminating b and A in these equations and Eq. (11.2.2),

$$A = by + my^2 = (P - 2y\sqrt{1 + m^2})y + my^2 = cP^{\frac{2}{5}} \tag{11.2.4}$$

By holding m constant and by differentiating with respect to y, $\partial P/\partial y$ is set equal to zero, thus

$$P = 4y\sqrt{1 + m^2} - 2my \tag{11.2.5}$$

Again, by holding y constant, Eq. (11.2.4) is differentiated with respect to m, and $\partial P/\partial m$ is set equal to zero, producing

$$\frac{2m}{\sqrt{1 + m^2}} = 1$$

After solving for m

$$m = \frac{\sqrt{3}}{3}$$

and after substituting for m in Eq. (11.2.5)

$$P = 2\sqrt{3}\,y \qquad b = 2\frac{\sqrt{3}}{3}\,y \qquad A = \sqrt{3}\,y^2 \tag{11.2.6}$$

which shows that $b = P/3$ and, hence, the sloping sides have the same length as the bottom. As $\tan^{-1} m = 30°$, the best hydraulic section is one-half a hexagon. For trapezoidal sections with m specified (maximum slope at which wet earth will stand) Eq. (11.2.5) is used to find the best bottom width-to-depth ratio.

The semicircle is the best hydraulic section of all possible open-channel cross sections.

Example 11.1: Determine the dimensions of the most economical trapezoidal brick-lined channel to carry 8000 cfs with a slope of 0.0004.

With Eq. (11.2.6),

$$R = \frac{A}{P} = \frac{y}{2}$$

and by substituting into Eq. (11.2.1)

$$8000 = \frac{1.49}{0.016} \sqrt{3}\, y^2 \left(\frac{y}{2}\right)^{\frac{2}{3}} \sqrt{0.0004}$$

or

$$y^{\frac{8}{3}} = 3930 \qquad y = 22.3 \text{ ft}$$

and from Eq. (11.2.6), $b = 25.8$ ft.

11.3. Steady Uniform Flow in a Floodway. A practical open-channel problem of importance is the computation of discharge through a floodway (Fig. 11.3). In general the floodway is much rougher than the

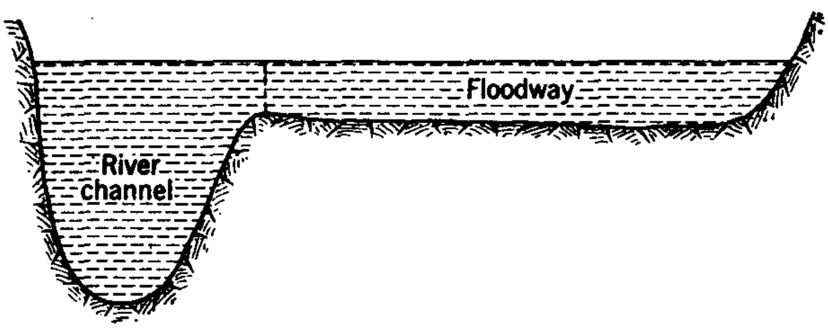

FIG. 11.3. Floodway cross section.

river channel, and its depth (and hydraulic radius) is much less. The slope of energy grade line must be the same for both portions. The discharge for each portion is determined separately, using the dotted line of Fig. 11.3 as the separation line for the two sections (but not as solid boundary), and then the discharges are added to determine the total capacity of the system.

Since both portions have the same slope, the discharge may be expressed as

$$Q_1 = K_1 \sqrt{S} \qquad Q_2 = K_2 \sqrt{S}$$

or

$$Q = (K_1 + K_2) \sqrt{S} \tag{11.3.1}$$

in which the value of K is

$$K = \frac{1.49}{n} AR^{\frac{2}{3}}$$

from Manning's formula and is a function of depth only for a given channel with fixed roughness. By computing K_1 and K_2 for different elevations of water surface, their sum may be taken and plotted against elevation. From this plot it is easy to determine the slope of energy grade line for a given depth and discharge from Eq. (11.3.1).

11.4. Hydraulic Jump. Stilling Basins. The relations among the variables V_1, y_1, V_2, y_2 for a hydraulic jump to occur in a horizontal rectangular channel are developed in Sec. 3.9. Another way of determining the conjugate depths for a given discharge is the $F + M$-method.

FIG. 11.4. Hydraulic jump in horizontal rectangular channel.

The momentum equation applied to the free body of liquid between y_1 and y_2 (Fig. 11.4) is, for unit width ($V_1y_1 = V_2y_2 = q$),

$$\frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} = \rho q(V_2 - V_1) = \rho V_2^2 y_2 - \rho V_1^2 y_1$$

By rearranging

$$\frac{\gamma y_1^2}{2} + \rho V_1^2 y_1 = \frac{\gamma y_2^2}{2} + \rho V_2^2 y_2 \tag{11.4.1}$$

or

$$F_1 + M_1 = F_2 + M_2 \tag{11.4.2}$$

in which F is the hydrostatic force at the section and M is the momentum per second passing the section. By writing $F + M$ for a given discharge q per unit width

$$F + M = \frac{\gamma y^2}{2} + \frac{\rho q^2}{y} \tag{11.4.3}$$

a plot is made of $F + M$ as abscissa against y as ordinate, Fig. 11.5, for $q = 10$ cfs/ft. Any vertical line intersecting the curve cuts it at two points having the same value of $F + M$; hence, they are conjugate depths. The value of y for minimum $F + M$ [by differentiation of Eq. (11.4.3) with respect to y and setting $d(F + M)/dy$ equal to zero], is

$$y_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}} \tag{11.4.4}$$

The jump must always occur from a depth less than this value to a depth greater than this value. This depth is the *critical* depth, which is shown in the following section to be the depth of minimum energy. Therefore, the jump always occurs from rapid flow to tranquil flow. The fact that mechanical energy is lost in the jump prevents any possibility that it could suddenly change from the higher conjugate depth to the lower conjugate depth.

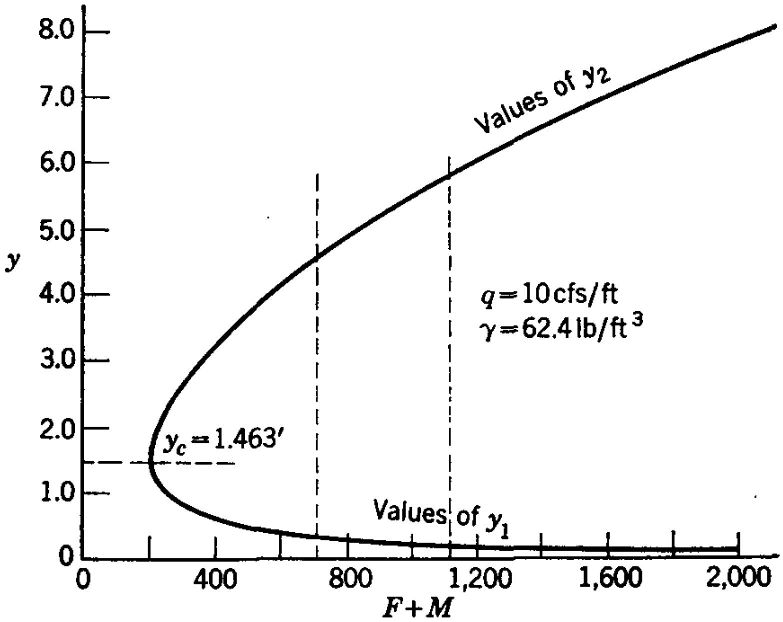

FIG. 11.5. $F + M$ curve for hydraulic jump.

The conjugate depths are directly related to the Froude numbers before and after the jump,

$$\mathbf{F}_1 = \frac{V_1^2}{gy_1} \qquad \mathbf{F}_2 = \frac{V_2^2}{gy_2} \tag{11.4.5}$$

From the continuity equation

$$V_1^2y_1^2 = g\mathbf{F}_1y_1^3 = V_2^2y_2^2 = g\mathbf{F}_2y_2^3$$

or

$$\mathbf{F}_1y_1^3 = \mathbf{F}_2y_2^3 \tag{11.4.6}$$

From Eq. (11.4.1)

$$y_1^2\left(1 + 2\frac{V_1^2}{gy_1}\right) = y_2^2\left(1 + 2\frac{V_2^2}{gy_2}\right)$$

After substituting from Eqs. (11.4.5) and (11.4.6)

$$(1 + 2\mathbf{F}_1)\mathbf{F}_1^{-\frac{2}{3}} = (1 + 2\mathbf{F}_2)\mathbf{F}_2^{-\frac{2}{3}} \tag{11.4.7}$$

The value of $\mathbf{F}_2$ in terms of $\mathbf{F}_1$ is obtained from the hydraulic jump equation [Eq. (3.9.34)]

$$y_2 = -\frac{y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2 + 2\frac{V_1^2y_1}{g}}$$

or

$$2\frac{y_2}{y_1} = -1 + \sqrt{1 + 8\frac{V_1^2}{gy_1}}$$

By using Eqs. (11.4.5) and (11.4.6)

$$\mathbf{F}_2 = \frac{8\mathbf{F}_1}{(\sqrt{1 + 8\mathbf{F}_1} - 1)^3} \qquad (11.4.8)$$

The Froude number before the jump is always greater than unity, and after the jump it is always less than unity.

Stilling Basins. A stilling basin is a structure for dissipating available energy of flow below a spillway, outlet works, chute, or canal structure. In the majority of existing installations a hydraulic jump is housed within the stilling basin and is used as the energy dissipator. This discussion is limited to rectangular basins with horizontal floors although sloping floors are used in some cases to save excavation. An authoritative and comprehensive work[1] by personnel of the Bureau of Reclamation classified the hydraulic jump as an effective energy dissipator in terms of the Froude number $\mathbf{F}_1(V_1^2/gy_1)$ entering the basin as follows:

At $\mathbf{F}_1 = 1$ to 3. Standing wave. There is only a slight difference in conjugate depths. Near $\mathbf{F}_1 = 3$ a series of small rollers develop.

At $\mathbf{F}_1 = 3$ to 6. Pre-jump. The water surface is quite smooth, the velocity is fairly uniform, and the head loss is low. No baffles required if proper length of pool is provided.

At $\mathbf{F}_1 = 6$ to 20. Transition. Oscillating action of entering jet, from bottom of basin to surface. Each oscillation produces a large wave of irregular period that can travel downstream for miles and damage earth banks and riprap. If possible, it is advantageous to avoid this range of Froude numbers in stilling-basin design.

At $\mathbf{F}_1 = 20$ to 80. Range of good jumps. The jump is well-balanced and the action is at its best. Energy absorption (irreversibilities) range from 45 to 70 per cent. Baffles and sills may be utilized to reduce length of basin.

At $\mathbf{F}_1 = 80$ upward. Effective but rough. Energy dissipation up to 85 per cent. Other types of stilling basins may be more economical.

Baffle blocks are frequently used at entrance to a basin to corrugate the flow. They are usually regularly spaced with gaps about equal to block widths. Sills, either triangular or dentated, are frequently employed at the downstream end of a basin to aid in holding the jump within the basin and to permit some shortening of the basin.

[1] *Hydraulic Laboratory Report* no. Hyd-399, Research Study on Stilling Basins, Energy Dissipators, and Associated Appurtenances, progress report II, U.S. Bur. Reclamation, Denver, June 1, 1955. In this report the Froude number was defined as $V/\sqrt{gy}$.

The basin should be paved with high quality concrete to prevent erosion and cavitation damage. No irregularities in floor or training walls should be permitted. The length of the jump, about $6y_2$, should be within the paved basin, with good riprap downstream if the material is easily eroded.

Example 11.2: A hydraulic jump occurs downstream from a 50-ft-wide sluice gate. The depth is 5.0 ft and the velocity is 60 ft/sec. Determine (*a*) the Froude number and the Froude number corresponding to the conjugate depth; (*b*) the depth and velocity after the jump; and (*c*) the horsepower dissipated by the jump.

(*a*)
$$\mathbf{F}_1 = \frac{V_1^2}{gy_1} = \frac{60^2}{32.2 \times 5} = 22.35$$

From Eq. (11.4.8)

$$\mathbf{F}_2 = \frac{8 \times 22.35}{(\sqrt{1 + 8 \times 22.35} - 1)^3} = 0.0938$$

(*b*)
$$\mathbf{F}_2 = \frac{V_2^2}{gy_2} = 0.0938, \qquad V_2 y_2 = 5 \times 60 = 300$$

then

$$V_2^3 = 300 \times 32.2 \times 0.0938$$

and $V_2 = 9.67$, $y_2 = 31.0$ ft

(*c*) From Eq. (3.9.35), the head loss in the jump, h_j, is

$$h_j = \frac{(y_2 - y_1)^3}{4y_1y_2} = \frac{(31.0 - 5)^3}{4 \times 5 \times 31.0} = 28.32 \text{ ft-lb/lb}$$

The horsepower dissipated is

$$\text{hp} = \frac{Q\gamma h_j}{550} = 50 \times 300 \times 62.4 \times \frac{28.32}{550} = 48{,}200$$

11.5. Specific Energy, Critical Depth. The energy per unit weight, E, with elevation datum taken as the bottom of the channel, is called the *specific energy*. It is a convenient quantity to use in studying open-channel flow and was introduced by Bakhmeteff in 1911. It is plotted vertically above the channel floor;

$$E = y + \frac{V^2}{2g} \qquad (11.5.1)$$

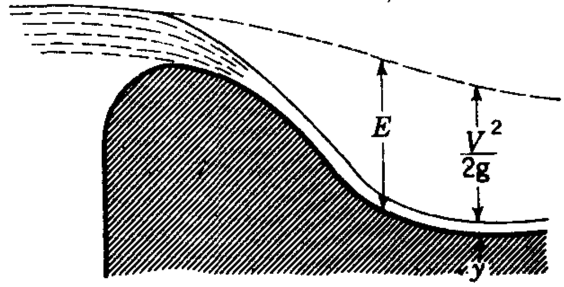

Fig. 11.6. Example of specific energy.

A plot of specific energy for a particular case is shown in Fig. 11.6. In a rectangular channel, in which q is the discharge per unit width, with $Vy = q$,

$$E = y + \frac{q^2}{2gy^2} \qquad (11.5.2)$$

It is of interest to note how the specific energy varies with the depth for a constant discharge (Fig. 11.7). For small values of y the curve goes to infinity along the E-axis, while for large values of y the velocity-head term is negligible and the curve approaches the 45° line $E = y$ asymptotically. The specific energy has a minimum value below which

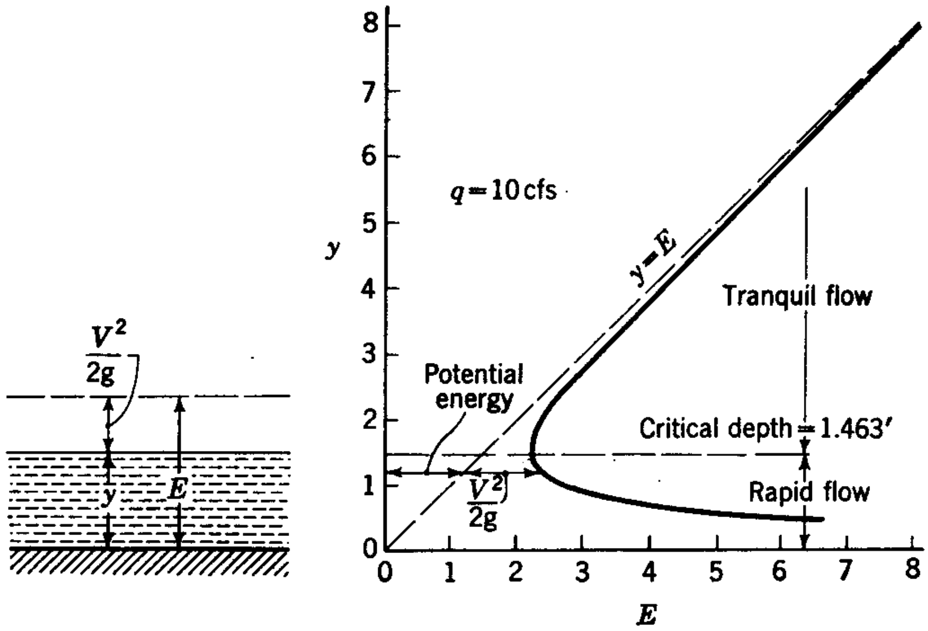

FIG. 11.7. Specific energy required for flow of a given discharge at various depths.

the given q cannot occur. The value of y for minimum E is obtained by setting dE/dy equal to zero, from Eq. (11.5.2), holding q constant,

$$\frac{dE}{dy} = 0 = 1 - \frac{q^2}{gy^3}$$

or

$$y_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}} \tag{11.5.3}$$

The depth for minimum energy y_c is called *critical depth*. By eliminating q^2 in Eqs. (11.5.2) and (11.5.3),

$$E_{\min} = \tfrac{3}{2}y_c \tag{11.5.4}$$

showing that the critical depth is two-thirds of the specific energy. By eliminating E in Eqs. (11.5.1) and (11.5.4),

$$V_c = \sqrt{gy_c} \tag{11.5.5}$$

The velocity of flow at critical condition V_c is $\sqrt{gy_c}$, which was used in Sec. 9.5 in connection with the broad-crested weir. Another method of arriving at the critical condition is to determine the maximum discharge q that could occur for a given specific energy. The resulting equations are the same as Eqs. (11.5.3) to (11.5.5).

For nonrectangular cross sections, as illustrated in Fig. 11.8, the specific-energy equation takes the form

$$E = y + \frac{Q^2}{2gA^2} \tag{11.5.6}$$

in which A is the cross-sectional area. To find the critical depth,

$$\frac{dE}{dy} = 0 = 1 - \frac{Q^2}{gA^3}\frac{dA}{dy}$$

From Fig. 11.8, the relation between dA and dy is expressed by

$$dA = T\,dy$$

in which T is the width of the cross section at the liquid surface. With this relation,

$$\frac{Q^2}{gA_c^3}T_c = 1 \tag{11.5.7}$$

The critical depth must satisfy this equation. By eliminating Q in Eqs. (11.5.6) and (11.5.7),

$$E = y_c + \frac{A_c}{2T_c} \tag{11.5.8}$$

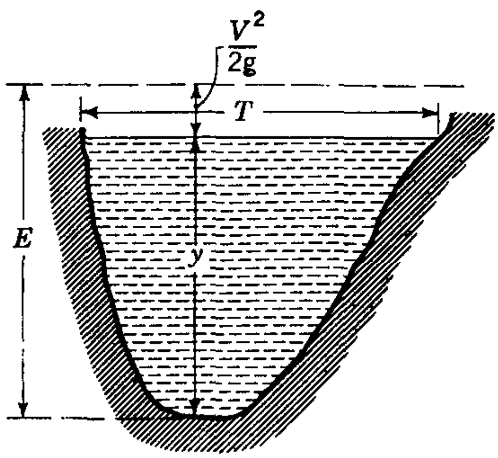

FIG. 11.8. Specific energy for a nonrectangular section.

This equation shows that the minimum energy occurs when the velocity head is one-half the average depth A/T. Equation (11.5.7) may be solved by trial for irregular sections, by plotting

$$f(y) = \frac{Q^2T}{gA^3}$$

Critical depth occurs for that value of y which makes $f(y) = 1$.

Example 11.3.: Determine the critical depth for 300 cfs flowing in a trapezoidal channel with bottom width 8 ft and side slopes one horizontal to two vertical (1 on 2).

$$A = 8y + \frac{y^2}{2} \qquad T = 8 + y$$

Hence

$$f(y) = \frac{\overline{300}^2(8 + y)}{32.2(8y + y^2/2)^3} = \frac{2795(8 + y)}{(8y + 0.5y^2)^3}$$

By trial

$y = 2$	4	3	3.2	3.24	3.26	3.30	3.28
$f(y) = 4.8$	0.52	1.33	1.08	1.03	1.02	0.975	0.997

The critical depth is 3.28 ft.

In uniform flow in an open channel, the energy grade line slopes downward parallel to the bottom of the channel, thus showing a steady decrease in available energy. The specific energy, however, remains constant along the channel, since $y + V^2/2g$ does not change. In nonuniform flow, the energy grade line always slopes downward, or the available energy is decreased. The specific energy may either increase or decrease, depending upon the slope of the channel bottom, the discharge, the depth of flow, properties of the cross section, and channel roughness. In Fig. 11.6 the specific energy increases during flow down the steep portion of the channel and decreases along the horizontal channel floor.

The specific-energy and critical-depth relationships are essential in studying gradually varied flow and in determining control sections in open-channel flow.

By comparing Figs. 11.5 and 11.7, which are both drawn for $q = 10$ cfs, it is easy to show the head loss that results from the hydraulic jump. Taking the two values of y on a vertical line from the momentum curve and plotting these points on the specific energy curve shows that the jump is always to a depth of less available energy.

11.6. Gradually Varied Flow. Gradually varied flow is steady nonuniform flow of a special class. The depth, area, roughness, bottom slope, and hydraulic radius change very slowly (if at all) along the channel. The basic assumption required is that the head-loss rate at a given section is given by the Manning formula for the same depth and discharge, regardless of trends in the depth. Solving Eq. (11.2.1) for the head loss per unit length of channel produces

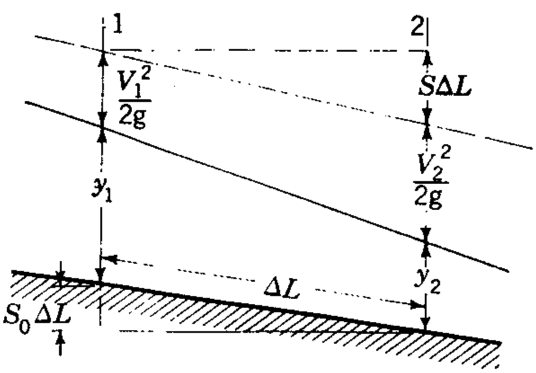

FIG. 11.9. Gradually varied flow

$$S = -\frac{\Delta E}{\Delta L} = \frac{n^2Q^2}{2.22A^2R^{\frac{4}{3}}} \qquad (11.6.1)$$

in which S is now the slope of the energy grade line, or, more specifically, the sine of the angle the energy grade line makes with the horizontal. In gradually varied flow the slopes of energy grade line, hydraulic grade line, and bottom are all different. Computations of gradually varied flow may be carried out either by the *standard step method* or by numerical *integration*. Horizontal channels of great width are treated as a special case that may be integrated.

Standard Step Method. By applying Bernoulli's equation between two sections a finite distance apart, ΔL, Fig. 11.9, including the loss term

$$\frac{V_1^2}{2g} + S_0\,\Delta L + y_1 = \frac{V_2^2}{2g} + y_2 + S\,\Delta L \qquad (11.6.2)$$

After solving for the length of reach

$$\Delta L = \frac{(V_1^2 - V_2^2)/2g + y_1 - y_2}{S - S_0} \tag{11.6.3}$$

If conditions are known at one section, e.g., section 1, and the depth y_2 is wanted a distance ΔL away, a trial solution is required. The procedure is:

a. Assume a depth y_2; then compute A_2, V_2;
b. For the assumed y_2 find an average y, P, and A for the reach [for prismatic channels $y = (y_1 + y_2)/2$ with A and R computed for this depth] and compute S;
c. Substitute in Eq. (11.6.3) to compute ΔL;
d. If ΔL is not correct, assume a new y_2 and repeat the procedure.

Example 11.4: At section 1 of a canal the cross section is trapezoidal, $b_1 = 40$ ft, $m_1 = 2$, $y_1 = 20$ ft, $V_1 = 3$ ft/sec and at section 2, 500 ft downstream, the bottom is 0.20 ft higher than at section 1, $b_2 = 50$ ft, and $m_2 = 3$. $n = 0.035$. Determine the depth of water at section 2.

$$\begin{aligned} A_1 &= b_1 y_1 + m_1 y_1^2 = 40 \times 20 + 2 \times (20)^2 = 1600 \text{ ft}^2 \\ P_1 &= b_1 + 2y_1\sqrt{m_1^2 + 1} = 40 + 2 \times 20\sqrt{2^2 + 1} = 129.6 \text{ ft} \\ S_0 &= -\frac{0.20}{500} = -0.0004 \\ Q &= V_1 A_1 = 3 \times 1600 = 4800 \text{ cfs} \end{aligned}$$

Since the bottom has an adverse slope, i.e., it is rising in the downstream direction, and since section 2 is larger than section 1, y_2 is probably less than y_1 for ΔL to be positive. Assume $y_2 = 19.8$; then

$$A_2 = 19.8 \times 50 + 3 \times (19.8)^2 = 2166 \text{ ft}^2$$

and

$$P_2 = 50 + 2 \times 19.8\sqrt{10} = 175 \text{ ft}$$

The average area $A = 1883$ and average wetted perimeter $P = 152.3$ are used to find an average hydraulic radius for the reach, $R = 12.36$. Then

$$S = \frac{n^2 Q^2}{2.22 A^2 R^{\frac{4}{3}}} = \left(\frac{0.035 \times 4800}{1883}\right)^2 \frac{1}{2.22 \times 12.36^{\frac{4}{3}}} = 0.000125$$

and

$$V_2 = \frac{4800}{2166} = 2.22$$

By substituting into Eq. (11.6.3)

$$\Delta L = \frac{(3^2 - 2.22^2)/64.4 + 20 - 19.8}{0.000125 + 0.00040} = 514 \text{ ft}$$

The value of y_2 should be slightly greater, e.g., 19.81 ft.

Numerical Integration Method. A more satisfactory procedure, particularly for flow through channels having a constant shape of cross section and constant bottom slope, is to obtain a differential equation in terms of y and L and then to perform the integration numerically. Considering ΔL an infinitesimal in Fig. 11.9, the rate of change of available energy is equal to the rate of head loss $-\Delta E/\Delta L$ given by Eq. (11.6.1), or

$$\frac{d}{dL}\left(\frac{V^2}{2g} + z_0 - S_0L + y\right) = -\frac{n^2Q^2}{2.22A^2R^{\frac{4}{3}}} \qquad (11.6.4)$$

in which $z_0 - S_0L$ is the elevation of bottom of channel at L, z_0 is the elevation of bottom at $L = 0$, and L is measured positive in the downstream direction. After performing the differentiation,

$$-\frac{V}{g}\frac{dV}{dL} + S_0 - \frac{dy}{dL} = \frac{n^2Q^2}{2.22A^2R^{\frac{4}{3}}} \qquad (11.6.5)$$

By using the continuity equation $VA = Q$ to eliminate V,

$$\frac{dV}{dL}A + V\frac{dA}{dL} = 0$$

After expressing $dA = T\,dy$, in which T is the liquid surface width of the cross section

$$\frac{dV}{dL} = -\frac{VT}{A}\frac{dy}{dL} = -\frac{QT}{A^2}\frac{dy}{dL}$$

By substituting for V in Eq. (11.6.5)

$$\frac{Q^2}{gA^3}T\frac{dy}{dL} + S_0 - \frac{dy}{dL} = \frac{n^2Q^2}{2.22A^2R^{\frac{4}{3}}}$$

and by solving for dL,

$$dL = \frac{1 - Q^2T/gA^3}{S_0 - n^2Q^2/2.22A^2R^{\frac{4}{3}}}\,dy \qquad (11.6.6)$$

After integrating,

$$L = \int_{y_1}^{y_2} \frac{1 - Q^2T/gA^3}{S_0 - n^2Q^2/2.22A^2R^{\frac{4}{3}}}\,dy \qquad (11.6.7)$$

in which L is the distance between the two sections having depths y_1 and y_2.

When the numerator of the integrand is zero, critical flow prevails; there is no change in L for a change in y (neglecting curvature of the flow and nonhydrostatic pressure distribution at this section). This is not a case of gradual change in depth, and, hence, the equations are not accurate near critical depth. When the denominator of the integrand is zero, uniform flow prevails, and there is no change in depth along the channel. The flow is at *normal depth*.

For a channel of fixed cross section, constant n and S_0, the integrand becomes a function of y only,

$$F(y) = \frac{1 - Q^2T/gA^3}{S_0 - n^2Q^2/2.22A^2R^{\frac{4}{3}}}$$

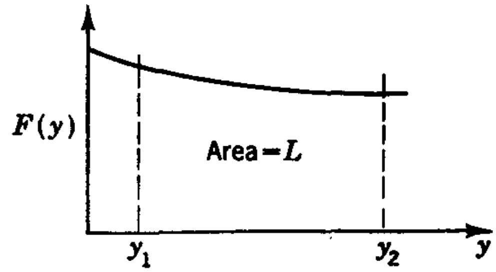

FIG. 11.10. Numerical integration of gradually varied flow equation.

and the equation may be integrated numerically by plotting $F(y)$ as ordinate against y as abscissa. The area under the curve (Fig. 11.10) between two values of y is the length L between the sections, since

$$L = \int_{y_1}^{y_2} F(y)dy$$

has exactly the same form as the area integral $\int y\,dx$.

Example 11.5: A trapezoidal channel, $b = 10$ ft, $m = 1$, $n = 0.014$, $S_0 = 0.001$ carries 1000 cfs. If the depth is 10 ft at section 1, determine the water surface profile for the next 2000 ft downstream.

To determine whether the depth increases or decreases, the slope of energy grade line at section 1 is computed, Eq. (11.6.1)

$$A = by + my^2 = 10 \times 10 + 1 \times \overline{10}^2 = 200 \text{ ft}^2$$
$$P = b + 2y\sqrt{m^2 + 1} = 38.2 \text{ ft}$$

and

$$R = \frac{200}{38.2} = 5.24 \text{ ft}$$

Then

$$S = \frac{1}{2.22}\left(\frac{0.014 \times 1000}{200}\right)^2 \frac{1}{(5.24)^{\frac{4}{3}}} = 0.000243$$

The depth is greater than critical and $S < S_0$, hence, the specific energy is increasing and this can be accomplished only by increasing the depth downstream. After substituting into Eq. (11.6.7)

$$L = \int_{10}^{y} \frac{1 - 3.105 \times 10^4T/A^3}{0.001 - 88.3/A^2R^{\frac{4}{3}}}\,dy$$

The following table evaluates the terms in the integrand:

y	A	P	R	T	Num.	$10^8 \times$ Den.	$F(y)$	L
10.0	200	38.2	5.24	30	0.8836	757	1167	0
10.5	215.2	39.8	5.41	31	0.9037	800	1129	574
11.0	232	41.1	5.64	32	0.9204	836	1101	1131
11.5	247.2	42.5	5.82	33	0.9323	862	1082	1677
12.0	264	43.9	6.01	34	0.9426	884	1067	2214

The integral $\int F(y)dy$ may be evaluated by plotting the curve and taking the area under it between $y = 10$ and the following values of y. As $F(y)$ does not vary greatly in this example, the average of $F(y)$ may be used for each reach and, when multiplied by Δy the length of reach is obtained. Between $y = 10$ and $y = 10.5$

$$\frac{1167 + 1129}{2} \times 0.50 = 574$$

Between $y = 10.5$ and $y = 11.0$

$$\frac{1129 + 1101}{2} \times 0.50 = 557$$

and so on. Five points on the water surface are known so that it may be plotted.

Horizontal Channels of Great Width. For channels of great width the hydraulic radius equals the depth; and for horizontal channel floors $S_0 = 0$; hence, Eq. (11.6.7) may be simplified. The width may be considered as unity, i.e., $T = 1$, $Q = q$ and $A = y$, $R = y$, thus

$$L = -\int_{y_1}^{y} \frac{1 - q^2/gy^3}{n^2q^2/2.22y^{\frac{10}{3}}}\, dy \tag{11.6.8}$$

or, after performing the integration,

$$L = -\frac{0.512}{n^2q^2}\left(y^{\frac{13}{3}} - y_1^{\frac{13}{3}}\right) + \frac{0.0517}{n^2}\left(y^{\frac{4}{3}} - y_1^{\frac{4}{3}}\right) \tag{11.6.9}$$

Example 11.6: After contracting below a sluice gate water flows onto a wide horizontal floor with a velocity of 40 ft/sec and a depth of 2.0 ft. Find the equation for water-surface profile, $n = 0.015$.

From Eq. (11.6.9), with x replacing L as distance from section 1, where $y_1 = 2$,

$$\begin{aligned} x &= -\frac{0.512}{(0.015 \times 80)^2}\left(y^{\frac{13}{3}} - 2^{\frac{13}{3}}\right) + \frac{0.0517}{(0.015)^2}\left(y^{\frac{4}{3}} - 2^{\frac{4}{3}}\right) \\ &= -572 - 0.3556y^{\frac{13}{3}} + 229.6y^{\frac{4}{3}} \end{aligned}$$

Critical depth occurs at Eq. (11.5.3),

$$y_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}} = \left(\frac{(80)^2}{32.2}\right)^{\frac{1}{3}} = 5.84 \text{ ft}$$

The depth must increase downstream, since the specific energy decreases, and the depth must move toward the critical value for less specific energy. The equation does not hold near the critical depth because of vertical accelerations that have been neglected in the derivation of gradually varied flow.

The various types of water-surface profile obtained in gradually varied flow are discussed in the following section.

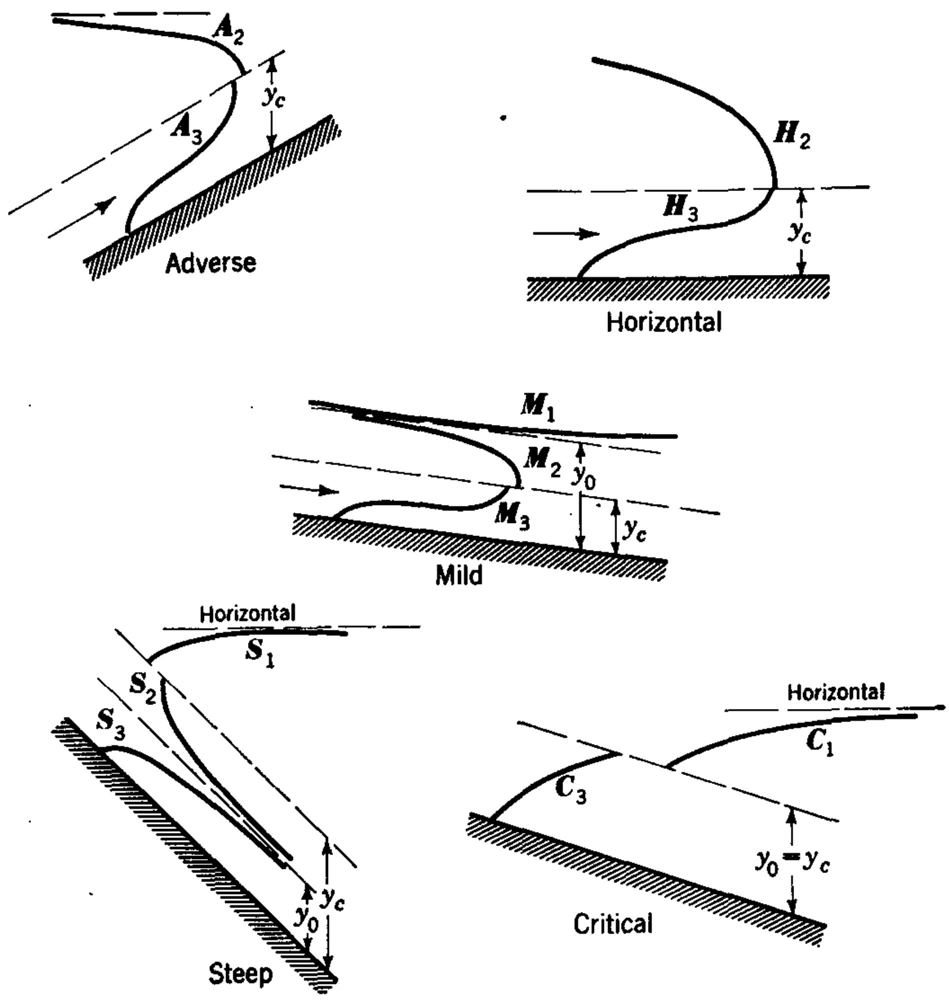

FIG. 11.11. The various typical liquid-surface profiles.

11.7. Classification of Surface Profiles A study of Eq. (11.6.7) reveals many types of surface profiles, each of which has its definite characteristics. The bottom slope is classified as *adverse, horizontal, mild, critical,* and *steep;* and, in general, the flow can be above the normal depth or below the normal depth, and it can be above critical depth or below critical depth.

The various profiles are plotted in Fig. 11.11; the procedures used are discussed for the various classifications in the following paragraphs. A very wide channel is assumed in the reduced equations which follow, with $R = y$.

Adverse-slope Profiles. When the channel bottom rises in the direction of flow, the resulting surface profiles are said to be adverse. There is no normal depth, but the flow may be either below critical depth or above critical depth. Thus, S_0 is negative. Below critical depth the numerator is negative, and Eq. (11.6.6) has the form

$$dL = \frac{1 - C_1/y^3}{S_0 - C_2/y^{\frac{10}{3}}}\, dy$$

Here, $F(y)$ is positive, and the depth increases downstream. This curve is labeled $\mathbf{A}_3$ and shown in Fig. 11.11. For depths greater than critical depth, the numerator is positive, and $F(y)$ is negative, i.e., the depth decreases in the downstream direction. For y very large, $dL/dy = 1/S_0$, which is a horizontal asymptote for the curve. At $y = y_c$, dL/dy is 0, and the curve is perpendicular to the critical-depth line. This curve is labeled $\mathbf{A}_2$.

Horizontal-slope Profiles. For a horizontal channel $S_0 = 0$, the normal depth is infinite and flow may be either below critical depth or above critical depth. The equation has the form

$$dL = -Cy^{\frac{1}{3}}(y^3 - C_1)\, dy$$

For y less than critical, dL/dy is positive, and the depth increases downstream. It is labeled $\mathbf{H}_3$. For y greater than critical ($\mathbf{H}_2$-curve) dL/dy is negative, and the depth decreases downstream. These equations are integrable analytically for very wide channels.

Mild-slope Profiles. A mild slope is one on which the normal flow is tranquil, i.e., where normal depth y_0 is greater than critical depth. Three profiles may occur, $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ for depth above normal, below normal and above critical, and below critical, respectively. For the $\mathbf{M}_1$-curve, dL/dy is positive and approaches $1/S_0$ for very large y; hence, the $\mathbf{M}_1$-curve has a horizontal asymptote downstream. As the denominator approaches zero as y approaches y_0, the normal depth is an asymptote at the upstream end of the curve. Thus, dL/dy is negative for the $\mathbf{M}_2$-curve, with the upstream asymptote the normal depth, and $dL/dy = 0$ at critical. The $\mathbf{M}_3$-curve has an increasing depth downstream, as shown.

Critical-slope Profiles. When the normal depth and the critical depth are equal, the resulting profiles are labeled $\mathbf{C}_1$ and $\mathbf{C}_3$ for depth above and below critical, respectively. The equation has the form

$$dL = \frac{1}{S_0}\frac{1 - b/y^3}{1 - b_1/y^{\frac{10}{3}}}\, dy$$

with both numerator and denominator positive for $\mathbf{C}_1$ and negative for $\mathbf{C}_3$. Therefore the depth increases downstream for both. For large y, dL/dy approaches $1/S_0$; hence, a horizontal line is an asymptote. The value of dL/dy at critical depth is $0.9/S_0$; hence, curve $\mathbf{C}_1$ is convex upward. Curve $\mathbf{C}_3$ is also convex upward, as shown.

Steep-slope Profiles. When the normal flow is rapid in a channel (normal depth less than critical depth), the resulting profiles $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ are referred to as steep profiles: $\mathbf{S}_1$ is above the normal and critical, $\mathbf{S}_2$ between critical and normal, and $\mathbf{S}_3$ below normal depth. For curve $\mathbf{S}_1$ both numerator and denominator are positive, and the depth increases downstream approaching a horizontal asymptote. For curve $\mathbf{S}_2$ the numerator is negative, and the denominator positive but approaching zero at $y = y_0$. The curve approaches the normal depth asymptotically. The $\mathbf{S}_3$-curve has a positive dL/dy as both numerator and denominator are negative. It plots as shown on Fig. 11.11.

It should be noted that a given channel may be classified as mild for one discharge, critical for another discharge, and steep for a third discharge, since normal depth and critical depth depend upon different functions of the discharge. The use of the various surface profiles is discussed in the next section.

11.8. Control Sections. A small change in downstream conditions cannot be relayed upstream when the depth is critical or less than critical; hence, downstream conditions do not control the flow. All rapid flows are controlled by upstream conditions, and computations of surface profiles must be started at the upstream end of a channel.

Tranquil flows are affected by small changes in downstream conditions and, therefore, are controlled by them. Tranquil-flow computations must start at the downstream end of a reach and be carried upstream.

Control sections occur at entrances and exits to channels and at changes in channel slopes, under certain conditions. A gate in a channel can be a control for both the upstream and downstream reaches. Three control sections are illustrated in Fig. 11.12. In *a* the flow passes through critical at the entrance to a channel, and the depth can be computed there for a given discharge. The channel is steep; therefore, computations proceed downstream. In *b* a change in channel slope from mild to steep causes the flow to pass through critical at the break in grade. Computations proceed both upstream and downstream from the control section at the break in grade. In *c* a gate in a horizontal channel provides controls both upstream and downstream from it. The various curves are labeled according to the classification in Fig. 11.11.

The hydraulic jump occurs whenever the conditions required by the momentum equation are satisfied. In Fig. 11.13, liquid issues from under a gate in rapid flow along a horizontal channel. If the channel were short enough, the flow could discharge over the end of the channel as an H_3-curve. With a longer channel, however, the jump occurs and the resulting profile consists of pieces of H_3- and H_2-curves with the jump in between. In computing these profiles for a known discharge, the H_3-curve is computed, starting at the gate (contraction coefficient must be known) and proceeding downstream until it is clear that the depth will reach critical before the end of the channel is reached. Then the

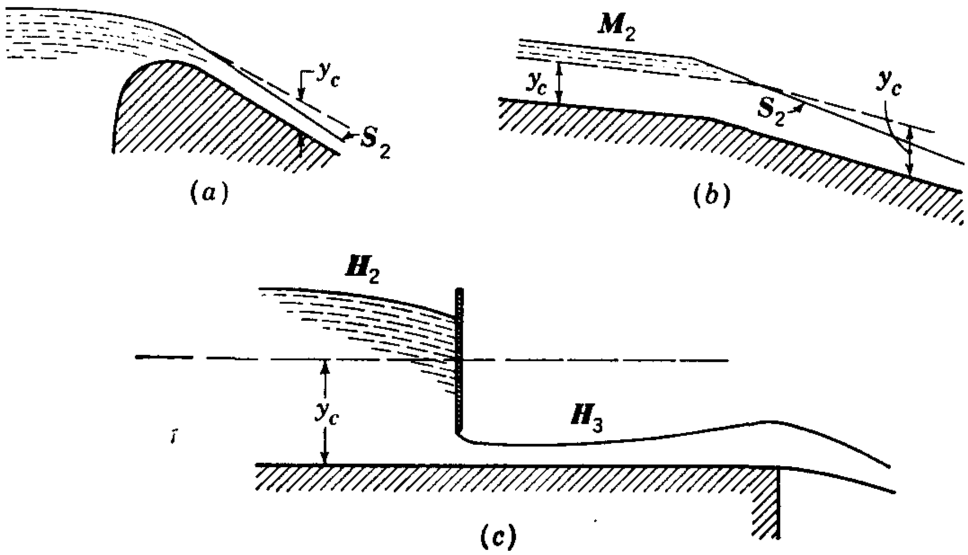

FIG. 11.12. Channel control sections.

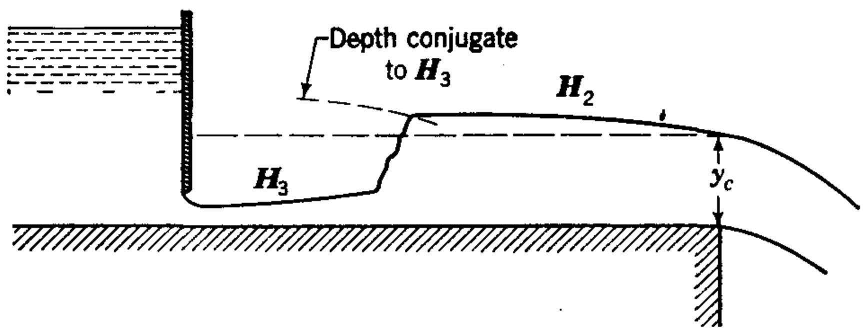

FIG. 11.13. Hydraulic jump between two control sections.

H_2-curve is computed, starting with critical depth at the end of the channel and proceeding upstream. The depths conjugate to those along H_3 are computed and plotted as shown. The intersection of the conjugate depth curve and the H_2-curve locates the position of the jump. The channel may be so long that the H_2-curve is everywhere greater than the depth conjugate to H_3. A "drowned jump" then occurs, with H_2 extending to the gate.

All sketches are drawn to a greatly exaggerated vertical scale, since usual channels have small bottom slopes.

11.9. Transitions. At entrances to channels and at changes in cross section and bottom slope, the structure that conducts the liquid from the upstream section to the new section is a *transition.* Its purpose is to change the shape of flow and surface profile in such a manner that minimum losses result. A transition for tranquil flow from a rectangular channel to a trapezoidal channel is illustrated in Fig. 11.14. By applying Bernoulli's equation from section 1 to section 2,

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + z + E_l \tag{11.9.1}$$

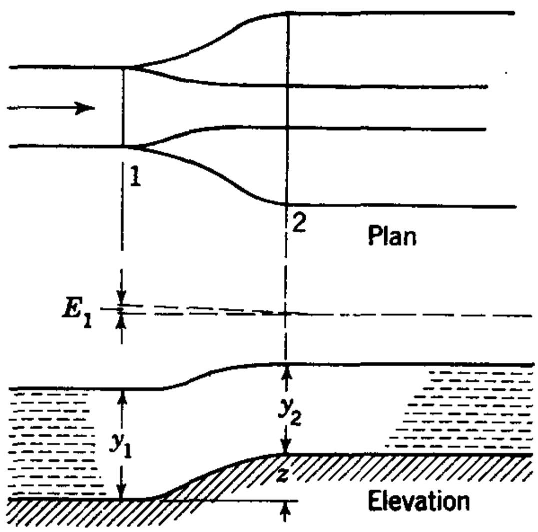

FIG. 11.14. Transition from rectangular channel to trapezoidal channel for tranquil flow.

In general, the sections and depths are determined by other considerations, and z must be determined for the expected available energy loss E_l. By good design, i.e., with slowly tapering walls and flooring with no sudden changes in cross-sectional area, the losses can be held to about one-tenth of the difference between velocity heads for accelerated flow and to about three-tenths of the difference between velocity heads for retarded flow. For rapid flow, wave mechanics is required in designing the transitions.[1]

Example 11.7: In Fig. 11.14, 400 cfs flows through the transition; the rectangular section is 8 ft wide; and $y_1 = 8$ ft. The trapezoidal section is 6 ft wide at the bottom with side slopes 1:1, and $y_2 = 7.5$ ft. Determine the rise z in the bottom through the transition

$$V_1 = \frac{400}{64} = 6.25 \qquad \frac{V_1^2}{2g} = 0.61 \qquad A_2 = 101.25 \text{ ft}^2$$

$$V_2 = \frac{400}{101.25} = 3.95 \qquad \frac{V_2^2}{2g} = 0.24 \qquad E_l = 0.3\left(\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\right) = 0.11$$

After substituting into Eq. (11.9.1)

$$z = 0.61 + 8 - 0.24 - 7.5 - 0.11 = 0.76 \text{ ft}$$

The *critical-depth meter*[2] is an excellent device for measuring discharge in an open channel. The relationships for determination of discharge are worked out for a rectangular channel of constant width, Fig. 11.15, with a raised floor over a reach of channel about $3y_c$ long. The raised floor is of such height that the restricted section becomes a control section

[1] A. T. Ippen, Channel Transitions and Controls, in "Engineering Hydraulics," ed. by H. Rouse, John Wiley & Sons, Inc., New York, 1950.

[2] H. W. King, "Handbook of Hydraulics," pp. 8-14 to 8-16, McGraw-Hill Book Company, Inc., New York, 1954.

with critical velocity occurring over it. By measuring only the upstream depth y_1, the discharge per foot of width is accurately determined. By

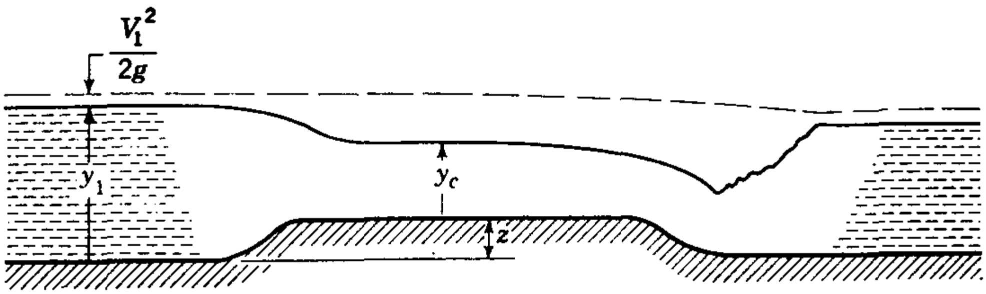

FIG. 11.15. Critical-depth meter.

applying Bernoulli's equation from section 1 to the critical section (exact location unimportant), including the transition loss term, thus:

$$\frac{V_1^2}{2g} + y_1 = z + y_c + \frac{V_c^2}{2g} + \frac{1}{10}\left(\frac{V_c^2}{2g} - \frac{V_1^2}{2g}\right)$$

Since

$$y_c + \frac{V_c^2}{2g} = E_c \qquad \frac{V_c^2}{2g} = \frac{E_c}{3}$$

in which E_c is the specific energy at critical depth

$$y_1 + 1.1\,\frac{V_1^2}{2g} = z + 1.033E_c \tag{11.9.2}$$

From Eq. (11.5.3)

$$y_c = \frac{2}{3}\,E_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}}$$

or

$$q = 3.087E_c^{\frac{3}{2}} \tag{11.9.3}$$

In Eqs. (11.9.2) and (11.9.3) E_c is eliminated and the resulting equation solved for q,

$$q = 2.94\left(y_1 - z + 1.1\,\frac{V_1^2}{2g}\right)^{\frac{3}{2}}$$

Since $q = V_1 y_1$, V_1 may be eliminated,

$$q = 2.94\left(y_1 - z + \frac{0.0171}{y_1^2}\,q^2\right)^{\frac{3}{2}} \tag{11.9.4}$$

The equation is solved by trial. As y_1 and z are known, and the right-hand term containing q is small, it may first be neglected for an approximate q. A value a little larger than the approximate q may be substituted on the right-hand side. When the two q's are the same the equation is solved. Once z and the width of channel are known, a chart or table may be prepared yielding Q for any y_1. Experiments indicate that accuracy within 2 to 3 per cent may be expected.

With tranquil flow a jump occurs downstream from the meter and with rapid flow a jump occurs upstream from the meter.

Example 11.8: In a critical-depth meter 6 ft wide with $z = 1.0$ ft the depth y_1 is measured to be 2.40 ft. Find the discharge.

In Eq. (11.9.4) as a first approximation

$$q = 2.94(1.4)^{\frac{3}{2}} = 4.87$$

As a second approximation let q be 5.00,

$$q = 2.94(1.4 + 0.00297 \times 5^2)^{\frac{3}{2}} = 5.26$$

and as a third approximation 5.32

$$q = 2.94(1.4 + 0.00297 \times \overline{5.32}^2)^{\frac{3}{2}} = 5.32$$

Then

$$Q = 6 \times 5.32 = 31.92 \text{ cfs}$$

11.10. Surge Waves. In the preceding portion of this chapter steady-flow situations have been considered. In this section an introduction to

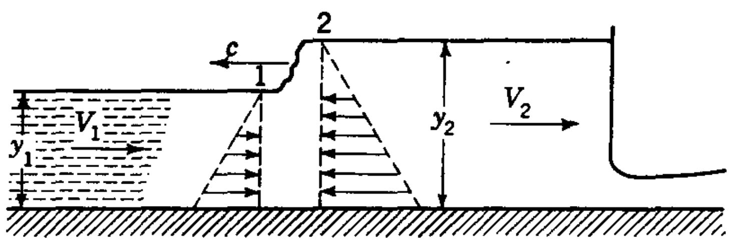

FIG. 11.16. Positive surge wave in a rectangular channel.

unsteady flow in open channels is made by studying *positive* and *negative surge waves.* When flow along a channel is decreased or increased by the closing or opening of a gate, surge waves form and travel up and down the channel. A positive surge wave results when the change causes an increase in depth, and a negative surge wave is set up by a decrease in depth. The positive surge wave may travel either upstream or downstream, depending upon the conditions, and is, in effect, a traveling hydraulic jump. The negative surge wave is unstable in that the higher portions of the wave travel more rapidly and cause a gradual decrease in depth along the channel.

Positive Surge. The equations for the positive surge wave are developed for a horizontal rectangular channel (assume unit width), neglecting the effects of friction. In Fig. 11.16, the velocity V_1 and depth y_1 have been disturbed by a closing of the gate so that a surge wave moves upstream with height $y_2 - y_1$ and velocity c. The continuity equation, stating that the flow rate into section 1 equals the flow rate out of section 2 plus the storage rate between the two sections, is

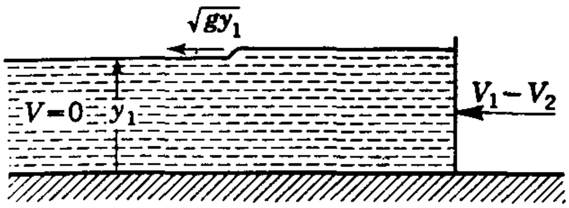

FIG. 11.17. Propagation of an elementary wave through still liquid.

$$V_1 y_1 = V_2 y_2 + c(y_2 - y_1) \qquad (11.10.1)$$

Neglecting the shear force on the bottom and sides between the two sections, the momentum equation may be applied. The mass per unit time having its momentum changed is that portion of the flow at depth y_1 that is covered by the surge wave in unit time, which is $(c + V_1)(y_1\gamma/g)$. The momentum equation is

$$\frac{\gamma y_1^2}{2} - \frac{\gamma y_2^2}{2} = (c + V_1)\frac{y_1\gamma}{g}(V_2 - V_1) \qquad (11.10.2)$$

After eliminating V_2 in the two equations,

$$\frac{2}{g}(V_1 + c)^2 = \frac{y_2}{y_1}(y_1 + y_2) \qquad (11.10.3)$$

By solving for the propagation of the wave relative to the undisturbed flow, $V_1 + c$,

$$V_1 + c = \sqrt{gy_1}\left[\frac{y_2}{2y_1}\left(1 + \frac{y_2}{y_1}\right)\right]^{\frac{1}{2}} \qquad (11.10.4)$$

The speed of an elementary wave computed with this equation by letting y_2 approach y_1, is

$$V_1 + c = \sqrt{gy} \qquad (11.10.5)$$

By imposing a velocity $-V_1$ on the flow shown by Fig. 11.16 the elementary wave speed becomes evident, as shown in Fig. 11.17, $c = \sqrt{gy}$.

By letting c equal zero in Eq. (11.10.4), the hydraulic-jump formula is obtained. The equations for the surge wave are conveniently obtained by making a steady-flow case out of the situation described in Fig. 11.16 by adding $V = c$ to each of the flows. The hydraulic-jump formula applies when V_1 is replaced by $V_1 + c$, and V_2 by $V_2 + c$.

Example 11.9: A rectangular channel 10 ft wide and 6 ft deep, discharging 600 cfs, suddenly has the discharge reduced to 400 cfs at the downstream end. Compute the height and speed of the surge wave.

$V_1 = 10$, $y_1 = 6$, $V_2 y_2 = 40$. With Eqs. (11.10.1) and (11.10.2),

$$60 = 40 + c(y_2 - 6)$$

and

$$y_2^2 - 36 = \frac{2 \times 6}{32.2}(c + 10)(10 - V_2)$$

By eliminating c and V_2,

$$y_2^2 - 36 = \frac{12}{32.2}\left(\frac{20}{y_2 - 6} + 10\right)\left(10 - \frac{40}{y_2}\right)$$

or

$$\left(\frac{y_2 - 6}{y_2 - 4}\right)^2 (y_2 + 6)y_2 = \frac{1200}{32.2} = 37.25$$

After solving for y_2 by trial, $y_2 = 8.47$ ft. Hence $V_2 = 40/8.47 = 4.72$ ft/sec. The height of surge wave is 2.47 ft, and the speed of the wave is

$$c = \frac{20}{y_2 - 6} = \frac{20}{2.47} = 8.1 \text{ ft/sec}$$

Negative Surge Wave. The negative surge wave appears as a gradual flattening and lowering of a liquid surface. It occurs, for example, in a channel downstream from a gate that is being closed, or upstream from a gate that is being opened. Its propagation is accomplished by a series of elementary negative waves superposed on the existing velocity, with each wave traveling at less speed than the one at next greater depth. Application of the momentum equation and the continuity equation to a small depth change produces simple differential expressions relating wave speed c, velocity V, and depth y. Integration of the equations yields liquid surface profile as a function of time, and velocity as a function of depth or as a function of position along the channel and time (x and t). The assumptions are made that the fluid is frictionless and that vertical accelerations are neglected.

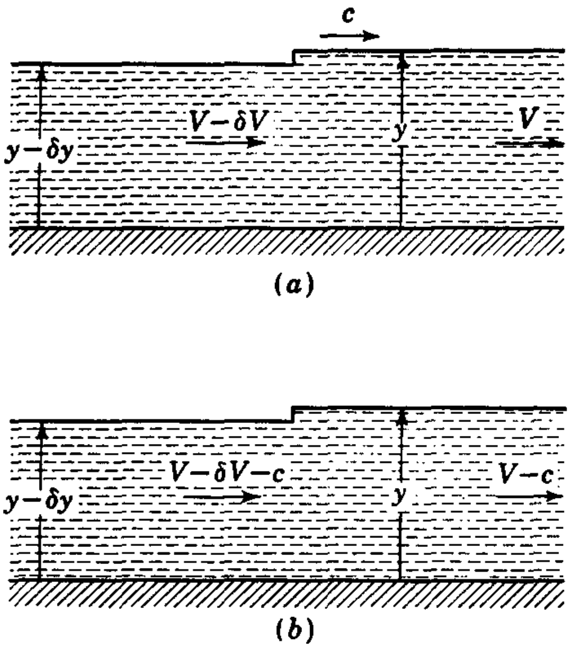

FIG. 11.18. Elementary wave.

In Fig. 11.18*a* an elementary disturbance is indicated in which the flow upstream has been slightly reduced. For application of the momentum and continuity equations it is convenient to reduce the motion to a steady one, as in Fig. 11.18*b*, by imposing a uniform velocity c to the left. The continuity equation is

$$(V - \delta V - c)(y - \delta y) = (V - c)y$$

or, by neglecting the product of small quantities,

$$(c - V)\,\delta y = y\,\delta V \tag{11.10.6}$$

The momentum equation produces

$$\frac{\gamma}{2}(y - \delta y)^2 - \frac{\gamma}{2}y^2 = \frac{\gamma}{g}(V - c)y[V - c - (V - \delta V - c)]$$

After simplifying

$$\delta y = \frac{c - V}{g}\,\delta V \tag{11.10.7}$$

By equating $\delta V/\delta y$ in Eqs. (11.10.6) and (11.10.7)

$$c - V = \pm\sqrt{gy} \tag{11.10.8}$$

or

$$c = V \pm \sqrt{gy}$$

The speed of an elementary wave in still liquid at depth y is $\sqrt{gy}$ and with flow the wave travels at the speed $\sqrt{gy}$ *relative* to the flowing liquid.

By eliminating c from Eqs. (11.10.6) and (11.10.7)

$$\frac{dV}{dy} = \sqrt{\frac{g}{y}}$$

After integrating

$$V = 2\sqrt{gy} + \text{constant}$$

For the case of a negative wave forming downstream from a gate, Fig. 11.19, after an instantaneous partial closure, $V = V_0$ when $y = y_0$, and

$$V_0 = 2\sqrt{gy_0} + \text{constant}$$

After eliminating the constant

$$V = V_0 - 2\sqrt{g}\,(\sqrt{y_0} - \sqrt{y}) \tag{11.10.9}$$

The wave travels in the $+x$-direction, so

$$c = V + \sqrt{gy} = V_0 - 2\sqrt{gy_0} + 3\sqrt{gy} \tag{11.10.10}$$

If the gate motion occurs at $t = 0$, the liquid surface position is expressed by $x = ct$, or

$$x = (V_0 - 2\sqrt{gy_0} + 3\sqrt{gy})t \tag{11.10.11}$$

By eliminating y from Eqs. (11.10.10) and (11.10.11)

$$V = \frac{V_0}{3} + \frac{2}{3}\frac{x}{t} - \frac{2}{3}\sqrt{gy_0} \tag{11.10.12}$$

which is the velocity in terms of x and t.

Example 11.10: In Fig. 11.19 find the Froude number of the undisturbed flow such that the depth y_1 at the gate is just zero when the gate is suddenly closed. For $V_0 = 20$ ft/sec, find the liquid-surface equation.

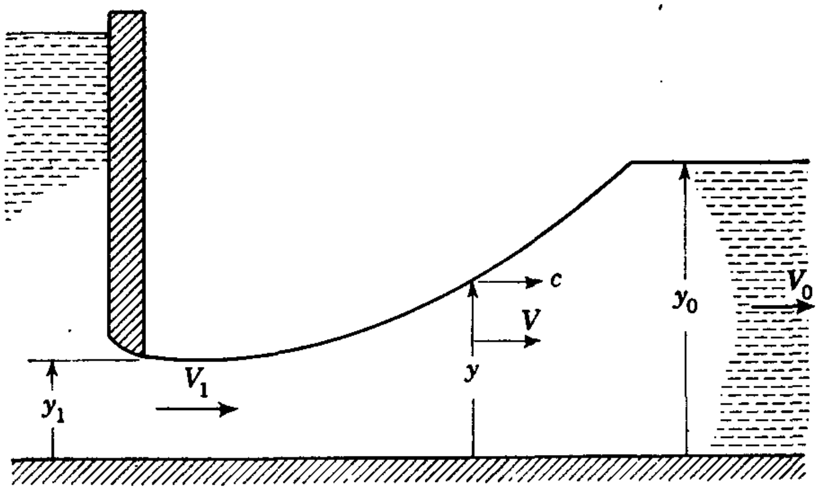

FIG. 11.19. Negative wave after gate closure.

It is required that $V_1 = 0$ when $y_1 = 0$ at $x = 0$ for any time after $t = 0$. In Eq. (11.10.9), with $V = 0$, $y = 0$

$$V_0 = 2\sqrt{gy_0}$$

or

$$\mathbf{F}_0 = \frac{V_0^2}{gy_0} = 4$$

For $V_0 = 20$,

$$y_0 = \frac{V_0^2}{4g} = \frac{20^2}{4g} = 3.11 \text{ ft}$$

By use of Eq. (11.10.11)

$$\begin{aligned} x &= (20 - 2\sqrt{32.2 \times 3.11} + 3\sqrt{32.2y})t \\ &= 17.04\sqrt{y}\, t \end{aligned}$$

The liquid surface is a parabola with vertex at the origin and surface concave upward.

Example 11.11: In Fig. 11.19 the gate is partially closed at the instant $t = 0$ so that the discharge is reduced by 50 per cent. $V_0 = 20$ ft/sec, $y_0 = 8$ ft. Find V_1, y_1 and the surface profile.

The new discharge is

$$q = \frac{20 \times 8}{2} = 80 = V_1 y_1$$

By use of Eq. (11.10.9)

$$V_1 = 20 - 2\sqrt{32.2}\,(\sqrt{8} - \sqrt{y_1})$$

Then V_1 and y_1 are found by trial from the last two equations, $V_1 = 14.5$ ft/sec, $y_1 = 5.52$ ft. The liquid-surface equation, from Eq. (11.10.11), is

$$x = (20 - 2\sqrt{8g} + 3\sqrt{gy})t$$

or

$$x = (17.04\sqrt{y} - 12.08)t$$

which holds for the range of values of y between 5.52 and 8.0.

Dam Break. An idealized dam-break water-surface profile, Fig. 11.20, may be obtained from Eqs. (11.10.9) to (11.10.12). From a frictionless, horizontal channel with depth of water y_0 on one side of a gate and no

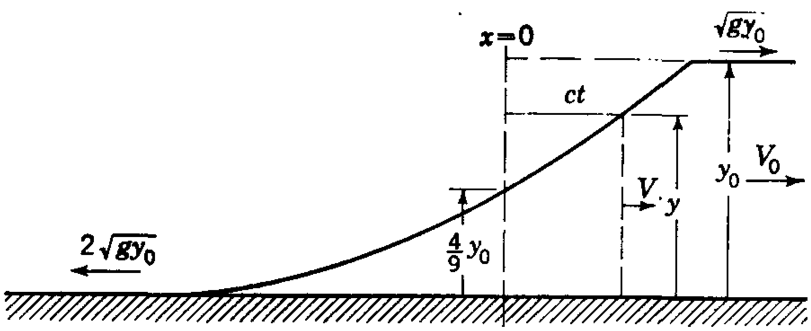

FIG. 11.20. Dam-break profile.

water on the other side of the gate, the gate is suddenly removed. Vertical accelerations are neglected. $V_0 = 0$ in the equations and y varies from y_0 to 0. The velocity at any section, Eq. (11.10.9), is

$$V = -2\sqrt{g}\,(\sqrt{y_0} - \sqrt{y}) \tag{11.10.13}$$

always in the downstream direction. The water-surface profile is, Eq. (11.10.11),

$$x = (3\sqrt{gy} - 2\sqrt{gy_0})t \tag{11.10.14}$$

At $x = 0$, $y = 4y_0/9$, the depth remains constant and the velocity past the section $x = 0$ is, from Eq. (11.10.13),

$$V = -\tfrac{2}{3}\sqrt{gy_0}$$

also independent of time. The leading edge of the wave feathers out to zero height and moves downstream at $V = c = -2\sqrt{gy_0}$. The water surface is a parabola with vertex at the leading edge, concave upward.

With an actual dam break, ground roughness causes a positive surge, or wall of water, to move downstream; i.e., the feathered edge is retarded by friction.

PROBLEMS

11.1. Show that for laminar flow to be assured down an inclined surface, the discharge per unit width cannot be greater than 500ν. (See Prob. 5.12.)

11.2. Calculate the depth of laminar flow of water at 70°F down a plane surface making an angle of 30° with the horizontal for the lower critical Reynolds number. (See Prob. 5.12.)

11.3. Calculate the depth of turbulent flow at $\mathbf{R} = VR/\nu = 500$ for flow of water at 70°F down a plane surface making an angle θ of 30° with the horizontal. Use Manning's formula. $n = 0.01$; $S = \sin\theta$.

11.4. A rectangular channel is to carry 40 cfs at a slope of 0.009. If the channel is lined with galvanized iron, $n = 0.011$, what is the minimum number of square feet of metal needed for each 100 ft of channel? Neglect freeboard.

11.5. A trapezoidal channel, with side slopes 2 on 1 (2 horizontal to 1 vertical), is to carry 600 cfs with a bottom slope of 0.0009. Determine the bottom, width, depth, and velocity for the best hydraulic section. $n = 0.025$.

11.6. A trapezoidal channel made out of brick, with bottom width 6 ft and with bottom slope 0.001, is to carry 600 cfs. What should the side slopes and depth of channel be for the least number of bricks?

11.7. What radius semicircular corrugated-metal channel is needed to convey 90 cfs 1 mile with a head loss of 7 ft? Can you find another cross section that requires less perimeter?

11.8. Determine the best hydraulic trapezoidal section to convey 3000 cfs with a bottom slope of 0.001. The lining is finished concrete.

11.9. Calculate the discharge through the channel and floodway of Fig. 11.21 for steady uniform flow, with $S = 0.0009$ and $y = 8$ ft.

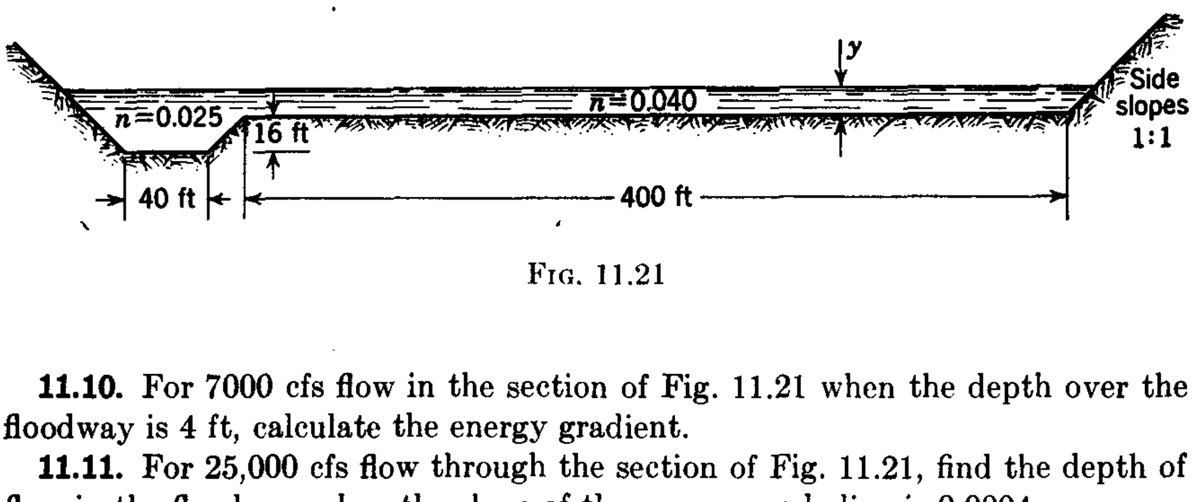

Fig. 11.21

11.10. For 7000 cfs flow in the section of Fig. 11.21 when the depth over the floodway is 4 ft, calculate the energy gradient.

11.11. For 25,000 cfs flow through the section of Fig. 11.21, find the depth of flow in the floodway when the slope of the energy grade line is 0.0004.

11.12. Draw an $F + M$-curve for 80 cfs/ft of width.

11.13. Draw the specific-energy curve for 80 cfs/ft of width on the same chart as Prob. 11.12.

11.14. Prepare a plot of Eq. (11.4.7).

11.15. With $q = 100$ cfs/ft and $\mathbf{F}_1 = 12$, determine v_1, y_1, and the conjugate depth y_2.

11.16. Determine the two depths having a specific energy of 6 ft for 30 cfs/ft.

11.17. What is the critical depth for flow of 18 cfs/ft of width?

11.18. What is the critical depth for flow of 10 cfs through the cross section of Fig. 5.48?

11.19. Determine the critical depth for flow of 300 cfs through a trapezoidal channel with a bottom width of 8 ft and side slopes of 1 on 1.

11.20. An unfinished concrete rectangular channel 12 ft wide has a slope of 0.0009. It carries 480 cfs and has a depth of 7 ft at one section. By using the step method and taking one step only, compute the depth 1000 ft downstream.

11.21. Solve Prob. 11.20 by taking two equal steps. What is the classification of this water-surface profile?

11.22. A very wide gate (Fig. 11.22) admits water to a horizontal channel. Considering the pressure distribution hydrostatic at section O, compute the depth at section O and the discharge per foot of width, when $y = 3.0$ ft.

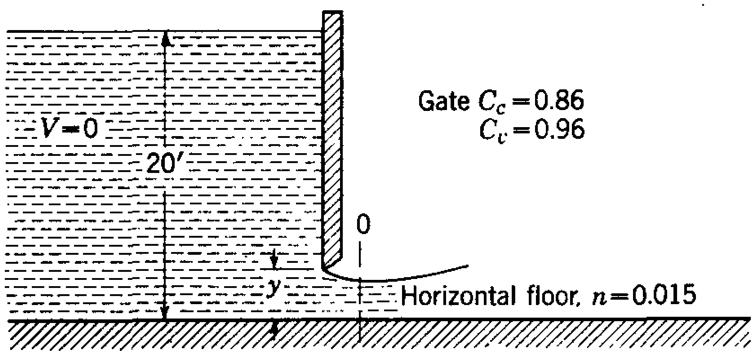

FIG. 11.22

11.23. If the depth at section O of Fig. 11.22 is 2 ft and the discharge per foot of width is 65.2 cfs, compute the water surface curve downstream from the gate.

11.24. Draw the curve of conjugate depths for the surface profile of Prob. 11.23.

11.25. If the very wide channel in Fig. 11.22 extends downstream 2000 ft and then has a sudden drop off, compute the flow profile upstream from the end of the channel for $q = 65.2$ cfs/ft by integrating the equation for gradually varied flow.

11.26. Using the results of Probs. 11.24 and 11.25, determine the position of a hydraulic jump in the channel.

11.27. In Fig. 11.23 the depth downstream from the gate is 2 ft, and the velocity is 40 ft/sec. For a very wide channel, compute the depth at the downstream end of the adverse slope.

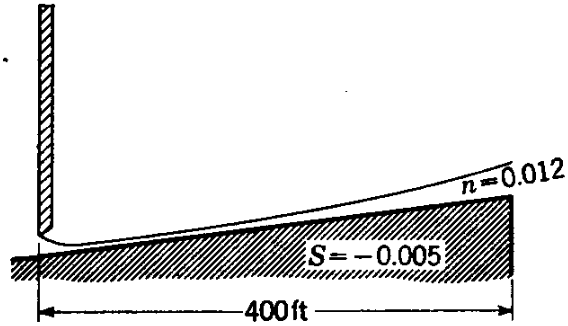

FIG. 11.23

11.28. Sketch (without computation) and label all the liquid-surface profiles that can be obtained from Fig. 11.24 by varying z_1, z_2 and the lengths of the channels, for $z_2 < z_1$ and with a steep, inclined channel.

11.29. In Fig. 11.24 determine the possible combinations of control sections for various values of z_1, z_2 and various channel lengths, for $z_1 > z_2$, and with the inclined channel always steep.

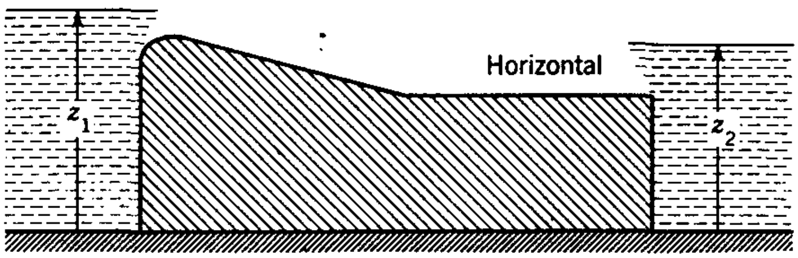

FIG. 11.24

11.30. Sketch the various liquid surface profiles and control sections for Fig. 11.24 obtained by varying channel length for $z_2 > z_1$.

11.31. Show an example of a channel that is mild for one discharge and steep for another discharge. What discharge is required for it to be critical?

11.32. Sketch the various combinations of liquid profiles obtainable from the channel profile of Fig. 11.25 for various values of z_1, z_2.

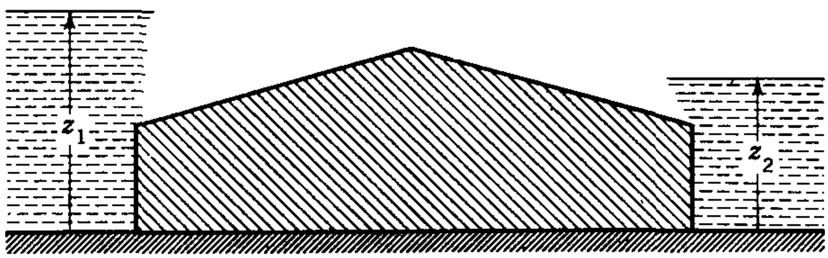

FIG. 11.25

11.33. Design a transition from a trapezoidal section, 8 ft bottom width and side slopes 1 on 1, depth 4 ft, to a rectangular section, 6 ft wide and 6 ft deep, for a flow of 250 cfs. The transition is to be 20 ft long, and the loss is one-tenth of the difference between velocity heads. Show the bottom profile, and do not make any sudden changes in cross-sectional area.

11.34. A transition from a rectangular channel, 8 ft wide and 6 ft deep, to a trapezoidal channel, bottom width 12 ft and side slopes 2 on 1, with depth 4 ft, has a loss of four-tenths of the difference between velocity heads. The discharge is 200 cfs. Determine the difference between elevations of channel bottoms.

11.35. A critical-depth meter 20 ft wide has a rise in bottom of 2.0 ft. For an upstream depth of 3.52 ft determine the flow through the meter.

11.36. With flow approaching a critical-depth meter site at 20 ft/sec and a Froude number of 10, what is the minimum amount the floor must be raised?

11.37. Derive the equations for surge waves in a rectangular channel by reducing the problem to a steady-flow case.

11.38. Derive the equation for propagation of an elementary wave through still liquid by applying the momentum and continuity equations to the case shown in Fig. 11.26.

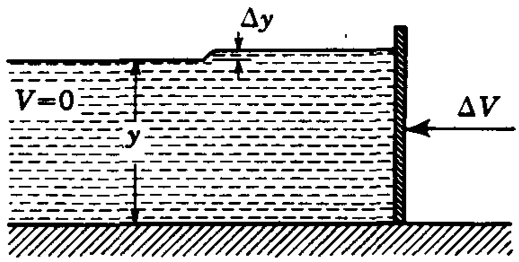

FIG. 11.26

11.39. A rectangular channel is discharging 50 cfs per foot of width at a depth of 10 ft when the discharge upstream is suddenly increased to 70 cfs/ft. Determine the speed and height of the surge wave.

11.40. In a rectangular channel with velocity 6 ft/sec flowing at a depth of 6 ft a surge wave 1.0 ft high travels upstream. What is the speed of the wave, and how much is the discharge reduced per foot of width?

11.41. A rectangular channel 10 ft wide and 6 ft deep discharges 1000 cfs when the flow is completely stopped downstream by closure of a gate. Compute the height and speed of the resulting positive surge wave.

11.42. Determine the depth downstream from the gate of Prob. 11.41 after it closes.

11.43. Find the downstream water surface of Prob. 11.41 3 sec after closure.

11.44. Determine the water surface 2 sec after an ideal dam breaks. Original depth is 100 ft.

11.45. In open-channel flow

(*a*) the hydraulic grade line is always parallel to the energy grade line
(*b*) the energy grade line coincides with the free surface
(*c*) the energy and hydraulic grade lines coincide
(*d*) the hydraulic grade line can never rise
(*e*) the hydraulic grade line and free surface coincide

11.46. Gradually varied flow is

(*a*) steady uniform flow
(*b*) steady nonuniform flow
(*c*) unsteady uniform flow
(*d*) unsteady nonuniform flow
(*e*) none of these answers

11.47. Tranquil flow must always occur

(*a*) above normal depth
(*b*) below normal depth
(*c*) above critical depth
(*d*) below critical depth
(*e*) on adverse slopes

11.48. Shooting flow can never occur

(*a*) directly after a hydraulic jump
(*b*) in a mild channel
(*c*) in an adverse channel
(*d*) in a horizontal channel
(*e*) in a steep channel

11.49. Flow at critical depth occurs when

(*a*) changes in upstream resistance alter downstream conditions
(*b*) the specific energy is a maximum for a given discharge
(*c*) any change in depth requires more specific energy
(*d*) the normal depth and critical depth coincide for a channel
(*e*) the velocity is given by $\sqrt{2gy}$

11.50. The best hydraulic rectangular cross section occurs when (b = bottom width, y = depth)

(*a*) $y = 2b$ (*b*) $y = b$ (*c*) $y = b/2$ (*d*) $y = b^2$
(*e*) $y = b/5$

11.51. The best hydraulic canal cross section is defined as

(*a*) the least expensive canal cross section
(*b*) the section with minimum roughness coefficient
(*c*) the section that has a maximum area for a given flow
(*d*) the one that has a minimum perimeter
(*e*) none of these answers

11.52. The hydraulic jump always occurs from

(*a*) an $\mathbf{M}_3$-curve to an $\mathbf{M}_1$-curve
(*b*) an $\mathbf{H}_3$-curve to an $\mathbf{H}_2$-curve
(*c*) an $\mathbf{S}_3$-curve to an $\mathbf{S}_1$-curve
(*d*) below normal depth to above normal depth
(*e*) below critical depth to above critical depth

11.53. Critical depth in a rectangular channel is expressed by

(*a*) $\sqrt{Vy}$ (*b*) $\sqrt{2gy}$ (*c*) $\sqrt{gy}$ (*d*) $\sqrt{q/g}$ (*e*) $(q^2/g)^{\frac{1}{3}}$

11.54. Critical depth in a nonrectangular channel is expressed by

(*a*) $Q^2T/gA^3 = 1$ (*b*) $QT^2/gA^2 = 1$ (*c*) $Q^2A^3/gT^2 = 1$
(*d*) $Q^2/gA^3 = 1$ (*e*) none of these answers

11.55. The specific energy for the flow expressed by $V = 8.02$ ft/sec, $y = 2$ ft is, in foot-pounds per pound,

(*a*) 3 (*b*) 4 (*c*) 6.02 (*d*) 10.02 (*e*) none of these answers

11.56. The minimum possible specific energy for a flow is 2.475 ft-lb/lb. The discharge per foot of width, in cubic feet per second, is

(*a*) 4.26 (*b*) 12.02 (*c*) 17 (*d*) 22.15 (*e*) none of these answers

11.57. The profile resulting from flow under the gate in Fig. 11.27 is classified as

(*a*) $\mathbf{H}_1$ (*b*) $\mathbf{H}_2$ (*c*) $\mathbf{H}_3$ (*d*) $\mathbf{A}_2$ (*e*) $\mathbf{A}_3$

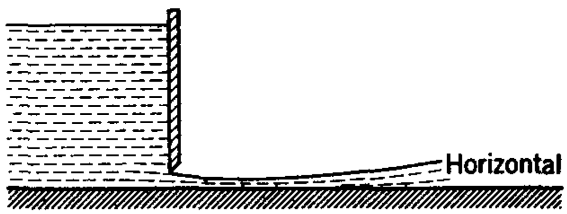

FIG. 11.27

11.58. The number of different possible surface profiles that can occur for any variations of z_1, z_2 and length of channel in Fig. 11.28 is ($z_1 \neq z_2$)

(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 5 (*e*) 6

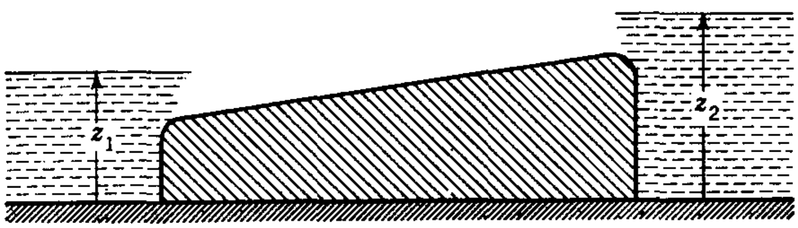

FIG. 11.28

11.59. The loss through a diverging transition is about

(*a*) $0.1\,\dfrac{(V_1 - V_2)^2}{2g}$ (*b*) $0.1\,\dfrac{(V_1^2 - V_2^2)}{2g}$ (*c*) $0.3\,\dfrac{(V_1 - V_2)^2}{2g}$

(*d*) $0.3\,\dfrac{V_1^2 - V_2^2}{2g}$ (*e*) none of these answers

11.60. A critical-depth meter

(*a*) measures the depth at the critical section
(*b*) is always preceded by a hydraulic jump
(*c*) must have tranquil flow immediately upstream
(*d*) always has a hydraulic jump downstream
(*e*) always has a hydraulic jump associated with it

11.61. An elementary wave can travel upstream in a channel, $y = 4$ ft, $V = 8$ ft/sec, with a velocity of

(*a*) 3.35 ft/sec (*b*) 11.35 ft/sec (*c*) 16.04 ft/sec (*d*) 19.35 ft/sec (*e*) none of these answers

11.62. The speed of an elementary wave in a still liquid is given by

(*a*) $(gy^2)^{1/3}$ (*b*) $2y/3$ (*c*) $\sqrt{2gy}$ (*d*) $\sqrt{gy}$ (*e*) none of these answers

11.63. A negative surge wave

(*a*) is a positive surge wave moving backwards
(*b*) is an inverted positive surge wave
(*c*) can never travel upstream
(*d*) can never travel downstream
(*e*) is none of the above

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Appendixes

A

FORCE SYSTEMS, MOMENTS, AND CENTROIDS

The material in this appendix has been assembled to aid in working with force systems. Simple force systems are briefly reviewed and first and second moments, including the product of inertia, are discussed. Centroids and centroidal axes are defined.

Simple Force Systems. A free-body diagram for an object or portion of an object shows the action of all other bodies on it. The action of the earth on the object is called a *body force* and is proportional to the mass of the object. In addition, forces and couples may act on the object by contact with its surface. When the free body is at rest or is moving in a straight line with uniform speed, it is said to be in *equilibrium*. By Newton's second law of motion, since there is no acceleration of the free body, the summation of all force components in any direction must be zero and the summation of all moments about any axis must be zero.

Two force systems are equivalent if they have the same value for summation of forces in every direction and the same value for summation of moments about every axis. The simplest equivalent force system is called the *resultant* of the force system. Equivalent force systems always cause the same motion (or lack of motion) of a free body.

In coplanar force systems the resultant is either a force or a couple. In noncoplanar parallel force systems the resultant is either a force or a couple. In general noncoplanar systems the resultant may be a force, a couple, or a force and a couple.

The action of a fluid on any surface may be replaced by the resultant force system that causes the same external motion or reaction as the distributed fluid force system. In this situation the fluid may be considered to be completely removed, the resultant acting in its place.

First and Second Moments. Centroids. The moment of an area, volume, weight, or mass may be determined in a manner analogous to that of determining the moments of a force about an axis.

First Moments. The moment of an area A about the y-axis (Fig. A.1) is expressed by

$$\int_A x\,dA$$

in which the integration is carried out over the area. To determine the moment about a parallel axis, e.g., $x = k$, the moment becomes

$$\int_A (x - k)\,dA = \int_A x\,dA - kA \tag{A.1}$$

which shows that there will always be a parallel axis $x = k = \bar{x}$, about

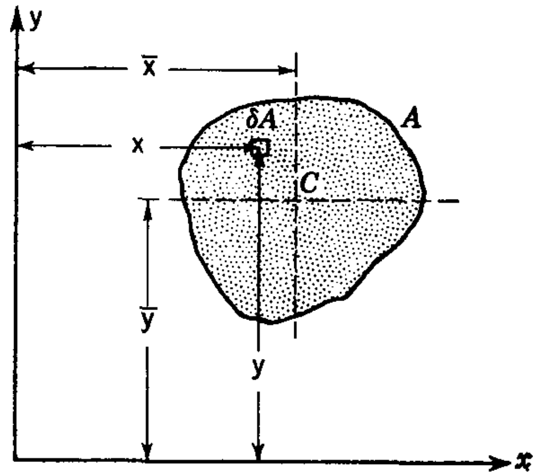

FIG. A.1. Notation for first and second moments.

which the moment is zero. This axis is called a *centroidal* axis and is obtained from Eq. (A.1) by setting it equal to zero and solving for $\bar{x}$,

$$\bar{x} = \frac{1}{A}\int_A x\,dA \tag{A.2}$$

Another centroidal axis may be determined parallel to the x-axis,

$$\bar{y} = \frac{1}{A}\int_A y\,dA \tag{A.3}$$

The point of intersection of centroidal axes is called the *centroid* of the area. It may easily be shown, by rotation of axes, that the first moment of the area is zero about any axis through the centroid. When an area has an axis of symmetry, it is a centroidal axis, because the moments of corresponding area elements on each side of the axis are equal in magnitude and opposite in sign. When location of the centroid is known, the first moment for any axis may be obtained without integration by taking the product of area and distance from centroid to the axis,

$$\int_A z\,dA = \bar{z}A \tag{A.4}$$

The centroidal axis of a triangle, parallel to one side, is one-third the altitude from that side; the centroid of a semicircle of radius a is $4a/3\pi$ from the diameter.

By taking the first moment of a volume V about a plane, say the yz-plane, the distance to its centroid is similarly determined,

$$\bar{x} = \frac{1}{V}\int_V x\,dV \tag{A.5}$$

The mass center of a body is determined by the same procedure,

$$x_m = \frac{1}{M}\int_M x\,dm \tag{A.6}$$

in which dm is an element of mass and M is the total mass of the body. For practical engineering purposes the *center of gravity* of a body is at its mass center.

Second Moments. The second moment of an area A (Fig. A.1) about the y-axis is

$$I_y = \int_A x^2\,dA \tag{A.7}$$

It is called the *moment of inertia* of the area and is always positive since dA is always considered positive. After transferring the axis to a parallel axis through the centroid C of the area,

$$I_c = \int_A (x - \bar{x})^2\,dA = \int_A x^2\,dA - 2\bar{x}\int_A x\,dA + \bar{x}^2\int_A dA$$

Since

$$\int_A x\,dA = \bar{x}A \qquad \int_A x^2\,dA = I_y \qquad \int_A dA = A$$

therefore

$$I_c = I_y - \bar{x}^2A \qquad \text{or} \qquad I_y = I_c + \bar{x}^2A \tag{A.8}$$

In words, the moment of inertia of an area about any axis is the sum of the moment of inertia about a parallel axis through the centroid and the

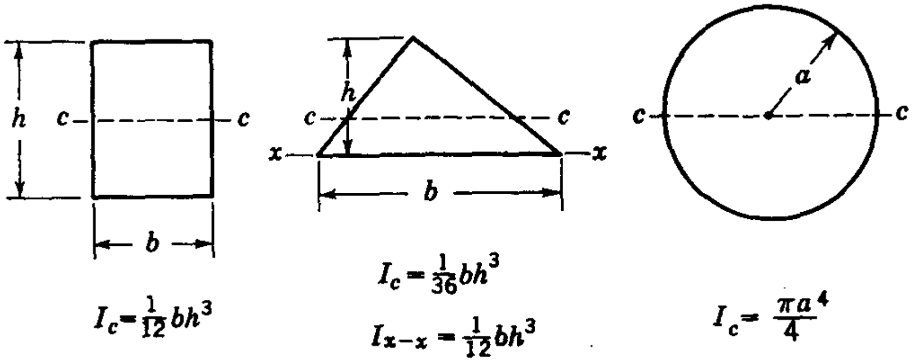

FIG. A.2. Moments of inertia of simple areas about centroidal axes.

product of the area and square of distance between axes. Figure A.2 shows moments of inertia for three simple areas.

The *product of inertia* I_{xy} of an area is expressed by

$$I_{xy} = \int_A xy\, dA \tag{A.9}$$

with the notation of Fig. A.1. It may be positive or negative. Writing the expression for product of inertia $\bar{I}_{xy}$ about centroidal axes parallel to the xy-axes produces

$$\bar{I}_{xy} = \int_A (x - \bar{x})(y - \bar{y})\, dA = \int_A xy\, dA - \bar{x} \int_A y\, dA - \bar{y} \int_A x\, dA + \bar{x}\bar{y}A$$

After simplifying, and solving for I_{xy},

$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A \tag{A.10}$$

Whenever either axis is an axis of symmetry of the area, the product of inertia is zero. The product of inertia I_{xy} of a triangle having two sides b and h along the positive coordinate axis is $b^2h^2/24$.

B

PARTIAL DERIVATIVES AND TOTAL DIFFERENTIALS

Partial Derivatives. A partial derivative is an expression of the rate of change of one variable with respect to another variable when all other variables are held constant. When one sees a partial derivative, he should determine which variables are considered constant. For example, the temperature T at any point throughout a plane might be expressed as an equation containing space coordinates and time, x, y, and t. To determine how the temperature changes at some point, e.g., x_0, y_0, with the time, the actual numbers for coordinates are substituted, and the equation becomes a relation between T and t only. The rate of change of temperature with respect to time is dT/dt, which is written as a total differential because T and t are the only two variables in the equation. When one wants an expression for rate of change of temperature with time at any point, x, y, then these are considered to be constants and the derivative of the equation with respect to t is taken. This is written $\partial T/\partial t$, to indicate that the other variables x, y have been held constant. Substitution of particular values of x, y into the expression yields $\partial T/\partial t$, in terms of t. As a specific case, if

$$T = x^2 + xyt + \sin t$$

then

$$\frac{\partial T}{\partial t} = xy + \cos t$$

For the point (1,2)

$$\frac{\partial T}{\partial t} = 2 + \cos t$$

which could have been obtained by first substituting (1,2) into the equation for T,

$$T = 1 + 2t + \sin t$$

and then by taking the total derivative

$$\frac{dT}{dt} = 2 + \cos t$$

If one wants to know the variation of temperature along any line parallel to the x-axis at a given instant of time, then $\partial T/\partial x$ is taken and the specific y-coordinates of the line and the time are substituted later; thus

$$\frac{\partial T}{\partial x} = 2x + yt$$

in which y, t have been considered constant. For the line through $y = 2$, at time $t = 4$

$$\frac{\partial T}{\partial x} = 2x + 8$$

and the rate of change of T with respect to x at this instant can be found at any point x along the particular line.

In the function

$$u = f(x,y)$$

x and y are independent variables, and u is the dependent variable. If y is held constant, u becomes a function of x alone, and its derivative may be determined as if u were a function of one variable. It is denoted by

$$\frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial u}{\partial x}$$

and is called the partial derivative of f with respect to x or the partial derivative of u with respect to x. Similarly, if x is held constant, u becomes a function of y alone and $\partial u/\partial y$ is the partial of u with respect to y. These partials are defined by

$$\frac{\partial u}{\partial x} = \frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x,y) - f(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y + \Delta y) - f(x,y)}{\Delta y}$$

Examples:

1. $u = x^3 + x^2y^3 + 3y^2$

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy^3 \qquad \frac{\partial u}{\partial y} = 3x^2y^2 + 6y$$

2. $u = \sin(ax + by^2)$

$$\frac{\partial u}{\partial x} = a\cos(ax + by^2) \qquad \frac{\partial u}{\partial y} = 2by\cos(ax + by^2)$$

3. $u = x \ln y$

$$\frac{\partial u}{\partial x} = \ln y \qquad \frac{\partial u}{\partial y} = \frac{x}{y}$$

Total Differentials. When u is a function of one variable only, $u = f(x)$,

$$\frac{du}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

and

$$\Delta u = f'(x)\Delta x + \epsilon\, \Delta x$$

in which

$$\lim_{\Delta x \to 0} \epsilon = 0$$

After applying the limiting process to Δu, then

$$du = f'(x)\, \Delta x \equiv f'(x)\, dx$$

is the differential du.

When $u = f(x,y)$, the differential du is defined in a similar manner. If x and y take on increments Δx, Δy, then

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x,y)$$

in which Δx, Δy may approach zero in any manner. If Δu approaches zero regardless of the way in which Δx and Δy approach zero, then $u = f(x,y)$ is called a *continuous* function of x and y. In the following it is assumed that $f(x,y)$ is continuous and that $\partial f/\partial x$ and $\partial f/\partial y$ are also continuous.

By adding and subtracting $f(x,y + \Delta y)$ to the expression for Δu,

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x,y)$$

Then

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \frac{\partial f(x, y + \Delta y)}{\partial x} \Delta x + \epsilon_1\, \Delta x$$

in which $\lim_{\Delta y \to 0} \epsilon_1 = 0$, because

$$\lim_{\Delta y \to 0} \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} = \frac{\partial f(x, y + \Delta y)}{\partial x}$$

Furthermore

$$\lim_{\Delta y \to 0} \frac{\partial f(x, y + \Delta y)}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

as the derivative is continuous, and

$$\frac{\partial f(x, y + \Delta y)}{\partial x} = \frac{\partial f(x,y)}{\partial x} + \epsilon_2$$

in which $\lim_{\Delta y \to 0} \epsilon_2 = 0$. Similarly

$$f(x, y + \Delta y) - f(x,y) = \frac{\partial f(x,y)}{\partial y} \Delta y + \epsilon_3\, \Delta y$$

in which $\lim\limits_{\Delta y \to 0} \epsilon_2 = 0$. By substituting into the expression for Δu,

$$\Delta u = \frac{\partial f(x,y)}{\partial x} \Delta x + \frac{\partial f(x,y)}{\partial y} \Delta y + (\epsilon_1 + \epsilon_2) \Delta x + \epsilon_3 \Delta y$$

If the limit is taken as Δx and Δy approach zero, the last two terms drop out since they are the product of two infinitesimals and, hence, are of a higher order of smallness. The total differential of u is obtained,

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

If x and y in $u = f(x,y)$ are functions of one independent variable, e.g., t, then u becomes a function of t alone and has a derivative with respect to t if the functions $x = f_1(t)$, $y = f_2(t)$ are assumed differentiable. An increment in t results in increments Δx, Δy, Δu which approach zero with Δt. By dividing the expression for Δu by Δt,

$$\frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta t} + (\epsilon_1 + \epsilon_2)\frac{\Delta x}{\Delta t} + \epsilon_3 \frac{\Delta y}{\Delta t}$$

and by taking the limit as Δt approaches zero,

$$\frac{du}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

The same general form results for additional variables, namely,

$$u = f(x,y,t)$$

in which x, y are functions of t; then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial t}$$

C

PHYSICAL PROPERTIES OF FLUIDS

TABLE C.1. PHYSICAL PROPERTIES OF WATER†

Temp °F	Specific weight γ lb/ft³	Density ρ slugs/ft³	Viscosity μ lb-sec/ft² $10^5\,\mu =$	Kinematic viscosity ν ft²/sec $10^5\,\nu =$	Surface tension σ lb/ft $100\,\sigma =$	Vapor pressure head p_v/γ ft	Bulk modulus of elasticity K lb/in² $10^{-3}\,K =$
32	62.42	1.940	3.746	1.931	0.518	0.20	293
40	62.43	1.940	3.229	1.664	0.514	0.28	294
50	62.41	1.940	2.735	1.410	0.509	0.41	305
60	62.37	1.938	2.359	1.217	0.504	0.59	311
70	62.30	1.936	2.050	1.059	0.500	0.84	320
80	62.22	1.934	1.799	0.930	0.492	1.17	322
90	62.11	1.931	1.595	0.826	0.486	1.61	323
100	62.00	1.927	1.424	0.739	0.480	2.19	327
110	61.86	1.923	1.284	0.667	0.473	2.95	331
120	61.71	1.918	1.168	0.609	0.465	3.91	333
130	61.55	1.913	1.069	0.558	0.460	5.13	334
140	61.38	1.908	0.981	0.514	0.454	6.67	330
150	61.20	1.902	0.905	0.476	0.447	8.58	328
160	61.00	1.896	0.838	0.442	0.441	10.95	326
170	60.80	1.890	0.780	0.413	0.433	13.83	322
180	60.58	1.883	0.726	0.385	0.426	17.33	318
190	60.36	1.876	0.678	0.362	0.419	21.55	313
200	60.12	1.868	0.637	0.341	0.412	26.59	308
212	59.83	1.860	0.593	0.319	0.404	33.90	300

† This table was compiled primarily from *A.S.C.E. Manual of Engineering Practice*, No. 25, Hydraulic Models, 1942.

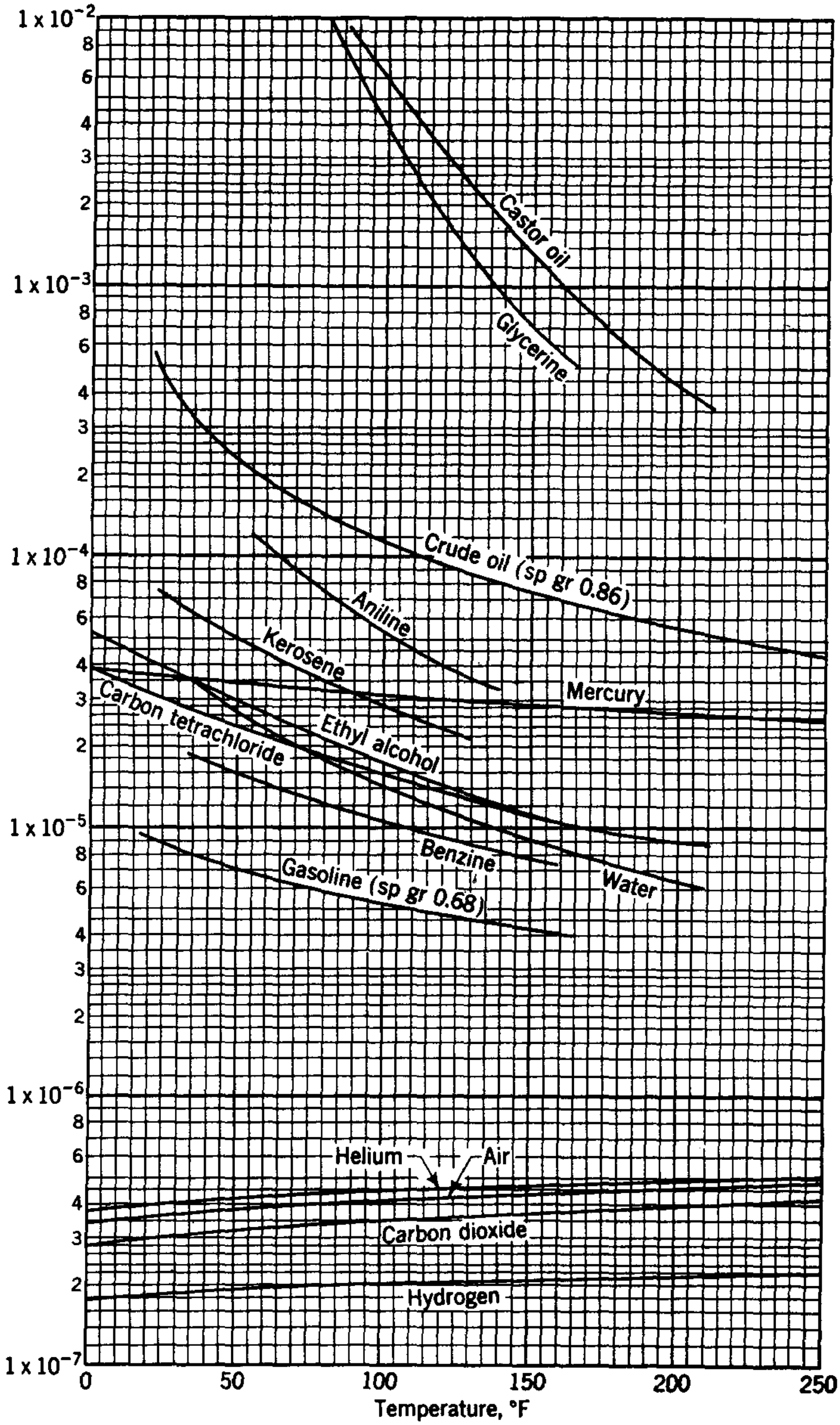

Fig. C.1. Absolute viscosities of certain gases and liquids.

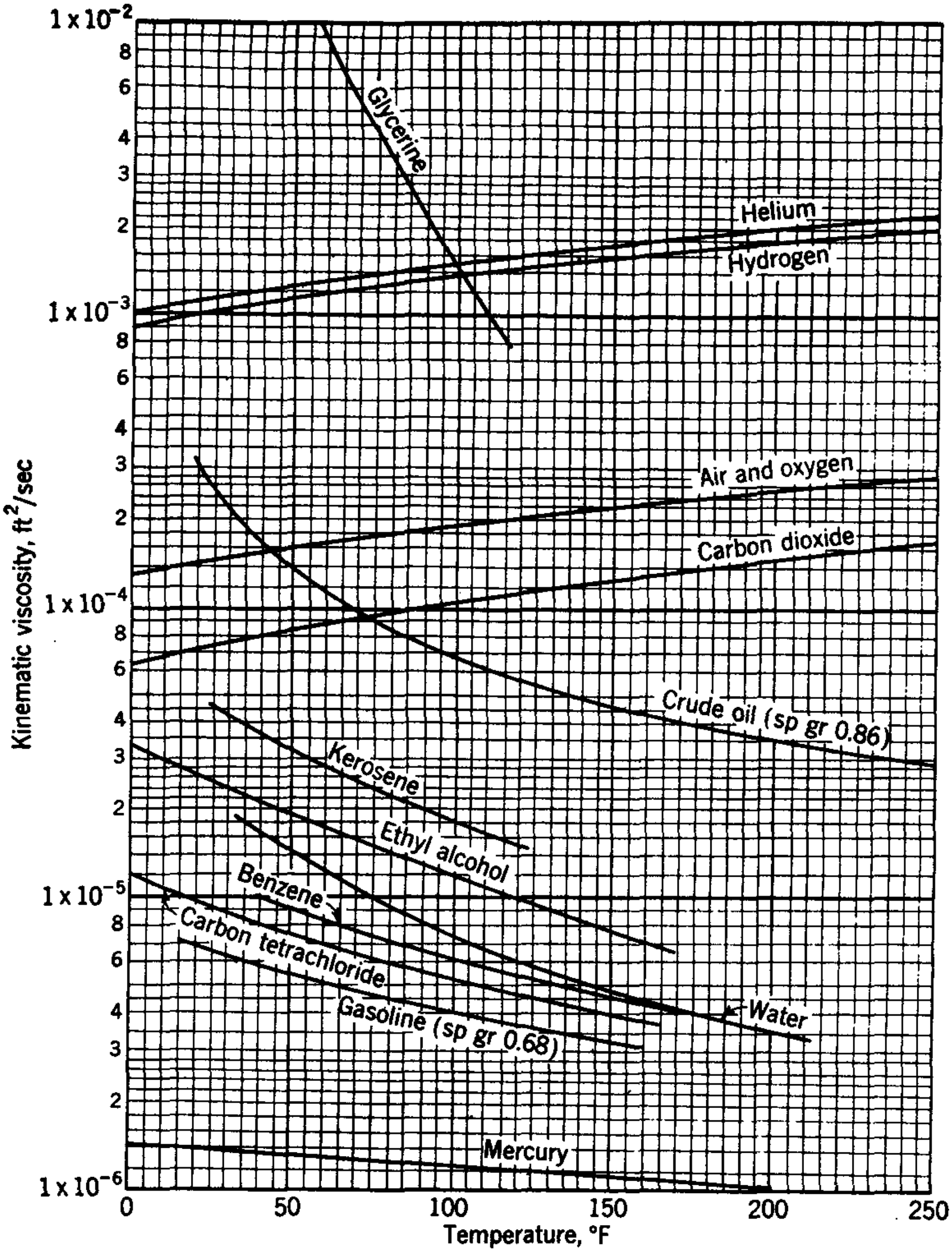

FIG. C.2. Kinematic viscosities of certain gases and liquids. The gases are at standard pressure.

TABLE C.2. PROPERTIES OF GASES AT LOW PRESSURES AND 80°F

Gas	Chemical formula	Molecular weight M	Gas constant R, ft-lb/lb$_m$ °R	Specific heat, Btu/lb$_m$ °R		Specific-heat ratio k
				c_p	c_v	
Air	...	29.0	53.3	0.240	0.171	1.40
Carbon monoxide	CO	28.0	55.2	0.249	0.178	1.40
Helium	He	4.00	386.	1.25	0.753	1.66
Hydrogen	H_2	2.02	766.	3.43	2.44	1.40
Nitrogen	N_2	28.0	55.2	0.248	0.177	1.40
Oxygen	O_2	32.0	48.3	0.219	0.157	1.40
Water vapor	H_2O	18.0	85.8	0.445	0.335	1.33

D

NOTATION

Symbol	*Quantity*	*Units (ft-lb-sec)*	*Dimensions (M,L,T)*
a	Constant		
a	Acceleration	ft/sec²	LT^{-2}
$\mathbf{a}$	Acceleration vector	ft/sec²	LT^{-2}
a^*	Velocity	ft/sec	LT^{-1}
A	Area	ft²	L^2
$\mathbf{A}$	Adverse slope	none	
b	Distance	ft	L
b	Constant		
c	Speed of surge wave	ft/sec	LT^{-1}
c	Speed of sound	ft/sec	LT^{-1}
c_p	Specific heat, constant pressure	ft-lb/slug °R	
c_v	Specific heat, constant volume	ft-lb/slug °R	
C	Concentration	No./ft³	L^{-3}
C	Coefficient	none	
C	Stress	lb/ft²	$ML^{-1}T^{-2}$
$\mathbf{C}$	Critical slope	none	
D'	Volumetric displacement	ft³	L^3
D	Diameter	ft	L
e	Efficiency	none	
E	Specific energy	ft-lb/lb	L
E	Losses per unit weight	ft-lb/lb	L
E	Modulus of elasticity	lb/ft²	$ML^{-1}T^{-2}$
	Friction factor	none	
F	Force	lb	MLT^{-2}
$\mathbf{F}$	Force vector	lb	MLT^{-2}
$\mathbf{F}$	Froude number	none	
F_B	Buoyant force	lb	MLT^{-2}
g	Acceleration of gravity	ft/sec²	LT^{-2}
g_0	Gravitational constant	lb_m-ft/lb-sec²	
G	Mass flow rate per unit area	slug/sec-ft²	$ML^{-2}T^{-1}$
h	Head, vertical distance	ft	L
h	Enthalpy per unit mass	ft-lb/slug	L^2T^{-2}
H	Head	ft	L
$\mathbf{H}$	Horizontal slope	none	
I	Moment of inertia	ft⁴	L^4

Symbol	Quantity	Units (*ft-lb-sec*)	Dimensions (M,L,T)
J	Junction point	none	
k	Specific-heat ratio	none	
K	Bulk modulus of elasticity	lb/ft^2	$ML^{-1}T^{-2}$
K	Minor loss coefficient	none	
L	Length	ft	L
L	Lift	lb	MLT^{-2}
l	Length, mixing length	ft	L
ln	Natural logarithm	none	
m	Mass	slug	M
m	Form factor, constant	none	
m	Strength of source	ft^3/sec	L^3T^{-1}
$\dot{m}$	Mass per unit time	slug/sec	MT^{-1}
M	Molecular weight		
M	Momentum per unit time	lb	MLT^{-2}
M	Mild slope	none	
M	Mach number	none	
$\overline{MG}$	Metacentric height	ft	L
n	Exponent, constant	none	
n	Normal direction	ft	L
n	Manning roughness factor		
n	Number of moles		
$\mathbf{n}_1$	Normal unit vector		
N	Rotation speed	1/sec	T^{-1}
p	Pressure	lb/ft^2	$ML^{-1}T^{-2}$
p	Force	lb	MLT^{-2}
p	Height of weir	ft	L
p	Wetted perimeter	ft	L
q	Discharge per unit width	ft^2/sec	L^2T^{-1}
q	Velocity	ft/sec	LT^{-1}
$\mathbf{q}$	Velocity vector	ft/sec	LT^{-1}
q_H	Heat transfer per unit mass	ft-lb/slug	L^2T^{-2}
Q	Discharge	ft^3/sec	L^3T^{-1}
Q_H	Heat transfer per unit time	ft-lb/sec	ML^2T^{-3}
r	Coefficient		
r	Radial distance	ft	L
$\mathbf{r}$	Position vector	ft	L
R	Hydraulic radius	ft	L
R	Gas constant	ft-lb/slug °R	
R, R'	Gage difference	ft	L
R	Reynolds number	none	
s	Distance	ft	L
s	Entropy per unit mass	ft-lb/slug °R	
s	Slip	none	
S	Entropy	ft-lb/°R	
S	Specific gravity, slope	none	
S	Stroke ratio	none	
S	Steep slope	none	
t	Time	sec	T
t, t'	Distance, thickness	ft	L

Symbol	Quantity	Units (ft-lb-sec)	Dimensions (M,L,T)
T	Temperature	°R	
T	Torque	lb-ft	ML^2T^{-2}
T	Tensile force/ft	lb/ft	MT^{-2}
T	Top width	ft	L
u	Velocity, velocity component	ft/sec	LT^{-1}
u	Peripheral speed	ft/sec	LT^{-1}
u	Internal energy	ft-lb/slug	L^2T^{-2}
u^*	Shear stress velocity	ft/sec	LT^{-1}
U	Velocity	ft/sec	LT^{-1}
v	Velocity, velocity component	ft/sec	LT^{-1}
v_s	Specific volume	ft³/slug	$M^{-1}L^3$
$\forall$	Volume	ft³	L^3
$\mathbf{V}$	Velocity vector	ft/sec	LT^{-1}
V	Velocity	ft/sec	LT^{-1}
w	Velocity component	ft/sec	LT^{-1}
w	Work per unit mass	ft-lb/slug	L^2T^{-2}
W	Work per unit time	ft-lb/sec	ML^2T^{-3}
W	Work of expansion	ft-lb	ML^2T^{-2}
W	Weight	lb	MLT^{-2}
$\mathbf{W}$	Weber number	none	
x	Distance	ft	L
x_p	Distance to pressure center	ft	L
X	Body-force component per unit mass	lb/slug	LT^{-2}
y	Distance, depth	ft	L
y_p	Distance to pressure center	ft	L
Y	Expansion factor	none	
Y	Body-force component per unit mass	lb/slug	LT^{-2}
z	Vertical distance	ft	L
Z	Vertical distance	ft	L
Z	Body-force component per unit mass	lb/slug	LT^{-2}
α	Kinetic-energy correction factor	none	
α	Angle, coefficient	none	
β	Momentum correction factor	none	
β	Blade angle	none	
Γ	Circulation	ft²/sec	L^2T^{-1}
∇	Vector operator	1/ft	L^{-1}
γ	Specific weight	lb/ft³	$ML^{-2}T^{-2}$
δ	Boundary-layer thickness	ft	L
ϵ	Kinematic eddy viscosity	ft²/sec	L^2T^{-1}
ϵ	Roughness height	ft	L
η	Eddy viscosity	lb-sec/ft²	$ML^{-1}T^{-1}$
η	Head ratio	none	
η	Efficiency		
θ	Angle	none	
κ	Universal constant	none	
λ	Scale ratio	none	
μ	Viscosity	lb-sec/ft²	$ML^{-1}T^{-1}$
μ	Constant		
ν	Kinematic viscosity	ft²/sec	L^2T^{-1}

Symbol	*Quantity*	*Units (ft-lb-sec)*	*Dimensions (M,L,T)*
ϕ	Velocity potential	ft²/sec	L^2T^{-1}
ϕ	Function		
π	Constant	none	
Π	Dimensionless parameter	none	
ρ	Density	slug/ft³	ML^{-3}
σ	Surface tension	lb/ft	MT^{-2}
σ	Cavitation index	none	
τ_0	Shear stress	lb/ft²	$ML^{-1}T^{-2}$
ψ	Stream function, two dimensions	ft²/sec	L^2T^{-1}
ψ	Stokes stream function	ft³/sec	L^3T^{-1}
ω	Angular velocity	rad/sec	T^{-1}

ANSWERS TO EVEN-NUMBERED PROBLEMS

Chapter 1

1.2. 10 ft/sec
1.4. 63.4 lb
1.6. 0.001 slug-ft/kip-sec^2
1.8. 10.72 ft/sec^2
1.10. 1.67×10^{-6} lb-sec/ft^2
1.12. 0.000475 slug/ft-sec
1.14. 0.00346 lb-sec/ft^2; 1.66 poise
1.16. 0.00249 poise
1.18. 5.88; 5.66
1.20. $v_s = g/\gamma$
1.22. $\rho = 0.0036$ slug/ft^3; 0.0144 slug
1.24. 0.000616
1.26. 4468 psia
1.28. $\rho = \rho_0 e^{(p-p_0)/K}$
1.30. 3000 psi
1.32. 15.53 psia
1.34. 0.155 in.

Chapter 2

2.2. 187.2; -62.4; -62.4; -312
2.4. -0.866; 0.866; 0.866; 3.20
2.6. $p = p_1\left(\dfrac{T_0 + \beta y}{T_0 + \beta y_1}\right)^{-g/R\beta}$; $\rho = \dfrac{p_1}{R}\dfrac{(T_0 + y_1)^{g/R\beta}}{(T_0 + y)^{1+g/R\beta}}$
2.8. 12.58 psia; 0.00205 slug/ft^3
2.10. 2920 ft
2.12. -4.62 ft water; 4.08 in. mercury suction; 12.45 psia; 28.8 ft water abs; 0.847 atmosphere; 25.42 in. mercury abs
2.14. 87.1
2.16. 1.096 psi
2.18. 32.4
2.20. 0.515
2.22. -3.34 ft
2.24. -0.26 in.
2.26. (*a*) 6.62 in.; (*b*) 6.52 in.
2.28. $0.223R$
2.30. 11.71 ft/sec^2
2.32. -0.173 psi; -0.173 psi; 0.35 psi; 0.35 psi
2.34. 0.51 psi; 2.24 psi; 0.51 psi
2.36. 32.3 ft/sec^2
2.38. $p_A = 0.52$ psi; 140 ft/sec^2
2.42. 3.27 rad/sec
2.44. $\omega = 5.67$ rad/sec
2.46. $2\sqrt{gh_0}/r_0$
2.50. Surface of sphere g/ω^2 below center
2.52. 1600 lb
2.54. -156.6 lb
2.56. 0.868 lb
2.58. 1914 lb
2.60. 0.793 below AB
2.62. 798 lb
2.64. 35,900 lb-ft
2.66. 11.58 ft
2.68. 0.856 ft
2.70. $\gamma bh^2/3$; $3h/4$
2.72. $y_p = 4.313$ ft

2.74. (a) 2.47 ft from top; (b) 2.33 ft from top
2.76. $y_p = 1.25(h - 4) + 6.67/(h - 4)$ **2.78.** $y_p = h/2$; $x_p = b/4$
2.80. $h = 0.77$ ft **2.82.** $y = 1$ ft
2.86. (a) $\bar{x} = 34.57$ ft; (b) $C_{max} = 12{,}870$ lb/ft²; $C_{min} = 1110$ lb/ft²
2.88. 3995 lb-ft **2.90.** 471 lb
2.92. 235 lb; stable
2.94. (a) $h = 6.605(\sin^2\theta \cos\theta)^{\frac{1}{3}}$; ($b$) stable, $9.49° < \theta < 54.78°$
2.96. Same steel required **2.98.** 562 lb; 1685 lb
2.102. 649 lb **2.104.** $R = 1.567$ ft
2.106. 5120 lb-ft
2.108. (a) 99.8 lb; (b) 548 lb/ft; (c) $S = 0.699$
2.110. 16.22 ft³ **2.112.** $W = 0.00666$ lb
2.114. $\frac{4}{3}$ ft **2.116.** 0.3 ft; 168.5 lb
2.118. 149.3 **2.120.** $6 < l < 15.9$
2.122. No **2.124.** Not stable

Chapter 3

3.2. 0.622 ft-lb/slug; 2.27 hp **3.4.** $x = \sqrt{y} = 2 - z$
3.6. Turbulent **3.8.** 24 ft/sec; 40 ft/sec
3.10. 542 ft/sec **3.12.** 60 per cent
3.14. Yes **3.20.** No
3.22. 20,060,000 ft-lb **3.24.** 100 ft
3.28. 1.035 **3.30.** 5.50
3.32. 1.80 ft; 12.78 ft **3.34.** 2.01 ft; 10.68 ft
3.36. 4.02 ft; 11.99 ft **3.38.** 27.8 ft/sec
3.40. $B \rightarrow A$ **3.42.** 0.63 cfs
3.44. 1.324 cfs **3.46.** 16 ft
3.48. 0.59 cfs; 6.64 psi **3.50.** $r = \dfrac{1}{4(1 + y/H)^{\frac{1}{4}}}$
3.52. 7.44 ft **3.54.** -5.2 psi
3.56. 2.43 cfs; $p_2 = -4.35$ psi; $p_3 = 0.554$ psi
3.58. 1.365 cfs; 64.5 ft **3.60.** 24.5 cfs; 26,280 lb-ft
3.62. 592 **3.64.** 719
3.66. 6.1 **3.68.** 1.559 cfs; 0.3; 0.0544 hp
3.70. $H/6$ **3.72.** 10.11 psi
3.74. 0.02 ft-lb/slug °R **3.76.** $\frac{4}{3}$
3.78. 1.183 **3.82.** 126 lb
3.84. Tension **3.86.** No change in magnitude of forces
3.88. 6898 lb; 11,330 lb **3.90.** 13.05 lb
3.92. Efficiency = 59 per cent **3.94.** 4335 lb; 86.3 per cent; 404 hp
3.96. 2903 lb **3.98.** 27.5 per cent
.100. 19 cfs **3.102.** 5420 ft/sec
104. 86.55 ft/sec; 686.1 ft **3.106.** 80 per cent
8. 2610 ft/sec **3.110.** 116,200 ft
$F_x = 228.3$ lb; $F_y = 568$ lb **3.114.** $\rho q_0(u/V_0)(V_0 + u)^2 \sin^2\theta$
$V_0/3$ **3.120.** $\theta_1 = 49° - 50.5'$; $\theta_2 = 47° - 12'$
$_x = 258$ lb; $F_y = 89.4$ lb **3.124.** $143° - 25'$
568 ft-lb/lb **3.128.** 65.6 hp
$-y_2/2 + \sqrt{(y_2/2)^2 + 2V_2{}^2y_2/g}$

3.132. $y_1 = \dfrac{y_2}{(2V_1^2/gy_2) - 1}$

3.134. 9.61 ft; 5.73 ft-lb/lb

3.136. 32.9 cfs

3.138. 0.25 in.

3.140. 74.4 lb toward left

3.142. 78.6 ft/sec; 367 ft

3.144. 537 rpm

3.146. 68.8 rpm

3.148. 0.363 lb-ft

Chapter 4

4.2. (a) $\rho V^2/\Delta p$; (b) $Fg^2/\rho V^6$; (c) $t\,\Delta p/\mu$

4.4. 86,400,000 slug

4.6. Dimensionless; T^{-1}; FLT^{-1}; FL; FL; FL

4.8. $f\left(\dfrac{\mu D}{\rho Q}, \dfrac{Q^3\rho^5 g}{\mu^5}, \dfrac{\Delta h}{l}\right) = 0$

4.12. $F_B = f(\rho g V)$

4.16. $f\left(k, \dfrac{p}{\rho V^2}\right)$

4.18. 13.6 ft/sec

4.20. $\gamma H^4 f_2\left(\dfrac{\omega H^3}{Q}, e\right)$

4.22. $F = \rho V^2 D^2 f(\mathbf{R},\mathbf{M})$

4.24. $F = \dfrac{\mu^2}{\rho_0} f(\mathbf{R})$

4.26. 85 ft/sec; 524 ft³/sec; same when expressed in velocity heads

Chapter 5

5.4. $dp/dl = 2\mu(U/a^2)$; $Q = Ua/3$

5.6. 0.1884 lb; 0.188×10^{-5} cfs

5.8. $Q = -\dfrac{a^3}{12\mu}\dfrac{dp}{dl} + \dfrac{a}{2}(U - V)$

5.10. $\frac{54}{35}$; $\frac{6}{5}$

5.14. 6.0575 lb/ft² to right

5.16. Efficiency $= \frac{1}{3}$

5.18. $\frac{4}{3}$

5.20. 0.042 lb/ft²

5.22. 0.0017 cfs; 63

5.24. 0.00136 cfs

5.26. 109.7 ft

5.28. $16000\mu^2/\rho D^2$

5.30. (a) 20 lb/ft²/ft; (b) 35.5; (c) 0.0196 lb/ft

5.32. 0.0059 cfs

5.34. 271

5.36. $l/r_0 = \frac{7}{6}k(y/r_0)$

5.38. $\epsilon/u_* r_0 = ky/r_0$

5.40. 41 ft

5.44. $\delta = 0.254x/\mathbf{R}_x^{\frac{1}{6}}$

5.46. 7.61 ft/sec

5.48. 19.7 ft

5.50. 7.59 ft

5.52. 107 lb

5.54. 2040 ft/sec; 32°

5.56. 0.0568 ft/sec

5.58. 66.72 cfs

5.60. 0.025

5.62. 217.5 cfs

5.64. 0.0285

5.66. 4.04 ft

5.68. 9.7 ft

5.70. 6.25

5.72. $V \sim y^{\frac{2}{3}}$

5.74. $y = 0.34B$

5.76. 0.0215

5.78. 321 ft

5.80. ~50 ft

5.82. 0.01 ft

5.86. 0.013

5.88. 0.352

5.90. 56.9 psi

5.92. 14.8 hp

5.94. 0.149 cfs

5.96. 7680

5.98. 7.44

5.100. 0.10 ft

5.102. Smoother plate

5.104. 7.7 ft

5.106. 2.31 ft diameter

5.108. 2.68 ft

5.110. $K = 9$; 485 ft

5.112. 26 cfs

5.114. (*a*) 7.6 ft; (*b*) 4.32 ft; (*c*) 89.4 ft
5.116. 15.75
5.122. 73.1 psi

Chapter 6

6.2. 0.331 Btu/lb_m °R
6.4. 19.88
6.6. 1.368 Btu/°R
6.8. 4020 Btu/slug
6.10. $\rho_1/\rho_2 = (T_1/T_2)^{1/(n-1)}$
6.12. 2
6.14. 20 per cent
6.18. 105°F, 44.4 psia
6.20. Same
6.22. 57.6 lb_m/sec; 0.54; 81.6 psia; 48°F
6.24. 211.2 psia; 0.914 lb_m/ft^3; 164°F
6.26. 0.331 lb/sec
6.28. 0.34 in.
6.30. 0.263 ft; 0.315 ft; 0.394 ft
6.32. 1450 ft/sec; 3.55 lb_m/sec
6.34. 0.065; 0.98
6.36. 0.636; 9.88 psia; 822°F; 1117 ft/sec
6.38. $\mathbf{M}_u = 1.55$; $\mathbf{M}_d = 0.712$; $p_d = 20.4$ psia; 312°F
6.42. 17 per cent
6.44. 0.86 ft
6.46. 0.1545 lb_m/sec
6.48. 4,110 Btu/lb_m
6.50. 0.056 slug/sec
6.52. 197.1 Btu/lb_m to the system
6.54. $q_H = (V_2^2 - V_1^2)/2$
6.56. 80 ft
6.58. 5.07; $\Delta p = 0.184$ psi
6.60. 0.108 ft

Chapter 7

7.4. $\omega_x = 0.5$; $\omega_y = -1$; $\omega_z = -0.5$
7.6. $w = -2z(x + y)$
7.8. $\frac{7}{2}(x^2 - y^2) - 4x - 3y + c$
7.12. $\psi = \theta + c$
7.14. $\phi = 26x + c$
7.16. $\left.\dfrac{\partial \phi}{\partial r}\right|_{r=a} = 0$; q_∞ finite
7.18. $u = -3.58$ ft/sec; $v = w = 0$; $u = 0.6125$ ft/sec, $v = 1.18$ ft/sec, $w = 1.18$ ft/sec
7.20. $\phi = 10(x + 1/\sqrt{x^2 + \hat{\omega}^2})$; $\psi = 10(\cos\theta + (r^2/2)\sin^2\theta)$
7.22.
$$\phi = \frac{2.73}{4\pi}\left(\frac{1}{\sqrt{(x - 1.545)^2 + \hat{\omega}^2}} - \frac{1}{\sqrt{(x + 1.545)^2 + \hat{\omega}^2}}\right) + 10x$$
$$\psi = \frac{2.73}{4\pi}\left(\frac{1}{\cos\theta_1} - \frac{1}{\cos\theta_2}\right) + \frac{10r^2}{2}\sin^2\theta$$
7.24. $p = 196.8 - 871\sin^2\theta$ lb/ft^2
7.26. Flow into a well
7.28. 200 ft^3/sec
7.30. $l^2 - 4a^2 = 16\pi\mu a/U$

Chapter 8

8.2. $Q_c = Q(n/N)$; $H_c = H(n/N)^2$, c = corrected, n = constant speed
8.4. Synchronizing causes a discrepancy
8.6. $Q = 0.125Q_1$; $H = 4H_1$
8.8. 319 rpm
8.10. 89 in.; 300 rpm
8.12. 14.05 ft
8.14. (*a*) 1.78 ft; (*b*) 1200 rpm; (*c*) 202 hp; 1.45 hp
8.16. 14.75°
8.18. $r = 3$, $V_u = 61.4$ ft/sec; $r = 1$, $V_u = 184.2$ ft/sec
8.20. 117.5 ft
8.22. 93.24 per cent
8.24. $H = 54.4 - 17.2Q$
8.26. (*a*) 506 rpm; (*b*) 13.0 ft; (*c*) 30.8 lb-ft; (*d*) 2.97 hp; (*e*) 581 lb/ft^2
8.30. 50.22
8.32. 6.8 in.
8.36. 272 lb-ft; 96.6 per cent
8.38. 11.6 ft

Chapter 9

9.2. 4.27 psi
9.4. 14.12 ft/sec
9.6. 68.1 ft/sec
9.8. 1.203
9.10. 39.4 cfs
9.12. 584.5 ft/sec; 73.5°F
9.14. 1.29 gpm
9.16. 28.35 gpm
9.18. $y = 0.017x^2$
9.20. $Y = H \cos^2 \alpha$
9.24. $C_v = 0.95$; $C_d = 0.75$; $C_c = 0.79$
9.26. 0.713 ft-lb/lb; 35.8 ft-lb/sec
9.28. 5.31 ft-lb/lb; 454.5 ft-lb/sec
9.30. 2.16 in.
9.32. 10.25 in. diameter
9.34. $r = 1.815y^{\frac{1}{4}}$
9.36. $r = 0.1515y^{\frac{3}{4}}$
9.38. 89.3 sec
9.40. 200 cfs
9.42. 0.875 psi
9.44. 0.0108 slug/sec; 746 ft/sec
9.46. 3.06 in.
9.48. 0.00787 slug/sec
9.50. 0.64 cfs
9.52. 25 cfs
9.54. 1.77 ft
9.56. (*a*) 2.32 ft; (*b*) 1.67 ft
9.58. 0.856 lb-ft
9.60. 1.79×10^{-5} slug/ft-sec

Chapter 10

10.2. 164.8
10.4. 44.8 ft
10.6. 3.975 cfs; 9.94 psia
10.10. 31.9 ft
10.12. 2.90 cfs
10.14. 0.25 ft
10.16. 2.72 in.
10.18. 38.7 ft
10.20. 4.46 cfs
10.24. 2.82 cfs
10.26. 8.48 cfs
10.28. $Q_1 = 0.0734$ cfs; $Q_2 = 0.165$ cfs; $Q_{total} = 0.238$ cfs
10.30. 9225 ft
10.32. $Q_{AJ} = 1.27$ cfs; $Q_{BJ} = 1.25$ cfs; $Q_{JC} = 2.52$ cfs
10.34. $Q_A = 0.42$ cfs; $Q_B = 2.02$ cfs
10.36. 1.86 cfs; 106.0 ft
10.38. $Q_{BJ_1} = 9.43$; $Q_{J_1A} = 16.63$; $Q_{J_2J_1} = 7.20$; $Q_{CJ_2} = 5.68$; $Q_{DJ_2} = 1.52$ cfs
10.40. 58.5, 41.5; 31; 44; 2.5
10.42. 61.5, 38.5; 31, 44; 5.5
10.44. 0.391
10.46. 0.165 cfs
10.48. 4.43 ft
10.50. 1.718 sec
10.52. 5.15 sec
10.54. 0.901 sec; 0.217 ft/sec
10.56. 27 ft
10.58. 12.9 sec
10.60. 3493 ft/sec
10.62. 274 psi
10.64. $\frac{2}{3}$

Chapter 11

11.2. 0.00219 ft
11.4. 562 ft^2 per 100 ft
11.6. $m = \sqrt{3}/3$; 7.11 ft
11.8. $m = \sqrt{3}/3$; $b = 13.45$ ft; $y = 11.65$ ft
11.10. 0.000165
11.16. 1.825 ft; 5.54 ft
11.18. 1.903 ft
11.20. 7.4 ft
11.22. 1.72 ft; 56.6 cfs/ft
11.26. 385 ft
11.34. 2.10 ft rise
11.36. 1.24 ft
11.40. 9.62 ft/sec; 9.61 cfs/ft
11.42. $y = 0.031(x/3t + 3.71)^2$
11.44. $y = 0.00345(x/2 + 644)^2$

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