ASSOSA UNIVERSITY DEPARTMENT OF MATHEMATICS

Course Module for Applied Mathematics I

6/19/2014

Compiled By Group A

1. Tamru Abera-----------------Chapter-1
2. Shimels Geletaw-----------------Chapter-2
3. Maasho Brhane-----------------Chapter-3
4. Michael Merga(Yihenew Miteku)---------Chapter-4
5. Tsegay Tesfay-----------------Chapter-5
6. Nurie Hakima-----------------Chapter-6

Contents

Chapter one

1. Vectors and Vector Spaces

Unit objective

- Define Scalars and Vectors in R^2, R^3 and R^n
- Perform Addition and scalar multiplication Vectors in R^2, R^3 and R^n
- Perform Scalar product of Vectors in R^2, R^3 and R^n
- Perform Cross product of Vectors in R^3
- Define Lines and planes
- Define the axioms of a vector space
- Define Subspaces, linear combinations and generators
- Differentiate Linear dependence and independence of vectors
- Define Bases and dimension of a vector space
- Define direct sum and direct product of subspaces

1.1. Scalars and Vectors

Definition: A physical quantity which has magnitude but not direction is called a scalar.

Example: speed, distance, temperature, etc.

Definition: A physical quantity which has both magnitude and direction is called a vector.

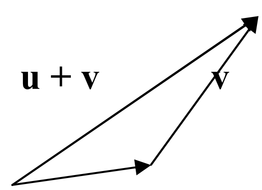
Example: velocity, acceleration, force, etc.

Notation: Vectors are mostly denoted by **bold letters** or arrow headed letters and in this chapter they are denoted by bold letters like **u, v and w.**

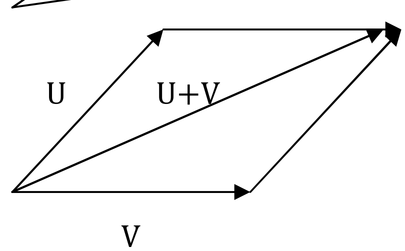
1.1.1. Addition and Scalar Multiplication of Vectors

Definition1.2.1 If **u** and **v** are vectors positioned so that the initial point of **v** is at the terminal point of **u,** then the sum **u** + **v** is the vector from the initial point of **u** to the terminal point of **v.**

Graphically:

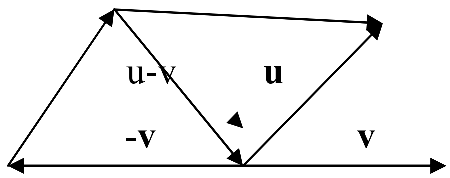


Triangular law of addition of vectors

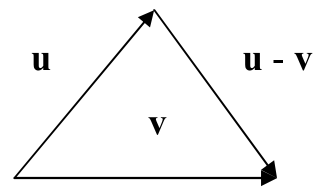


Parallelogram law of addition of vectors

The difference of two vectors **u** and **v** graphically is shown as follows:



or



Note: A zero vector is denoted by bold zero or **0.**

Definition 1.2.2 If $\boldsymbol{\alpha}$ is a scalar and **u** is a vector, then the scalar multiple $\boldsymbol{\alpha}$**u** is the vector whose length is $|\boldsymbol{\alpha}|$ times the length of **u** and having the same direction as **u** if $\boldsymbol{\alpha} > 0$ and opposite in direction to **u** if $\boldsymbol{\alpha} < 0$.

Note: If $\boldsymbol{\alpha} = 0$ or **u** = **0** then $\boldsymbol{\alpha}$ **u** = **0**

1.1.1 **Component form of vectors in** $\mathbb{R}^2$ **,** $\mathbb{R}^3$ ***and*** $\mathbb{R}^n$

1.1.2 **Component form of a vector in** $\mathbb{R}^2$**:** A vector **u** in $\mathbb{R}^2$ is given by **u** = (a_1, a_2) where the entries a_1, a_2 are called components of the vector **u**.

Definition: Let **a** = (a_1, a_2) and **b** = (b_1, b_2) be vectors in $\mathbb{R}^2$. Then:

1. **a** + **b** = $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ (**addition of vectors**)
2. **a**– **b** = $(a_1, a_2) - (b_1, b_2) = (a_1 - b_1, a_2 - b_2)$ (**subtraction of vectors**)

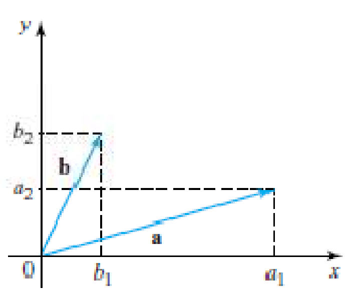


Figure (a)

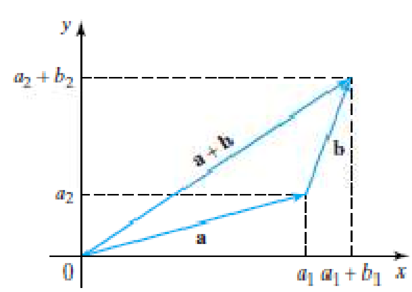


Figure (b)

3. **a=b** if and only if $a_1 = b_1$ and $a_2 = b_2$ (**equality of vectors**)
4. $\boldsymbol{\alpha}$ **a** = $\boldsymbol{\alpha}$ $(a_1, a_2) = (\boldsymbol{\alpha}\, a_1, \boldsymbol{\alpha}\, a_2)$ (**multiplying a vector a by scalar** $\boldsymbol{\alpha}$**)**

Theorem1.2.1 For any vectors **u**, **v** and **w** in $\mathbb{R}^2$ and any scalars $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the following relations hold true:

a) **u + v = u + v**
b) **(u + v) + w = u + (v + w)**

c) $\mathbf{u} + \mathbf{0} = \mathbf{u}$

d) $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u} = \beta(\alpha\mathbf{u})$

e) $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

f) $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$

g) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

h) $1\mathbf{u} = \mathbf{u}$

Example: If **u** = (1, 2) and **v** = (4, 5), then find:

a) $\mathbf{u} + \mathbf{v}$ b) $\mathbf{v} - \mathbf{u}$ c) $2\mathbf{u} + 3\mathbf{v}$

Solution:

a) $\mathbf{u} + \mathbf{v} = (1, 2) + (4, 5) = (1 + 4, 2 + 5) = (5, 7)$

b) $\mathbf{v} - \mathbf{u} = (1, 2) - (4, 5) = (1 - 4, 2 - 5) = (-3, -3)$

c) $2\mathbf{u} + 3\mathbf{v} = 2(1, 2) + 3(4, 5) = (2, 4) + (12, 15) = (14, 19)$

Component form of vectors in $\mathbb{R}^3$: A vector **u** in $\mathbb{R}^3$ is given by $\mathbf{u} = (a_1, a_2, a_3)$ where the entries a_1, a_2 and a_3 are called components

of the vector **u**.

Definition1.2.4 Let $\mathbf{u} = (a_1, a_2, a_3)$ and $\mathbf{v} = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$.

Then:

a) $\mathbf{u} + \mathbf{v} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2 + a_3 + b_3)$ (addition of vectors)

b) $\mathbf{u} - \mathbf{v} = (a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$ (subtraction of vectors)

c) $\mathbf{u} = \mathbf{v}$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$ (equality of vectors)

d) $\alpha\mathbf{u} = \alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a^3)$ (scalar multiplying a vector **u** by a scalar α)

Definition 1.2.5 The magnitude, **length** or nor m of a vector $\mathbf{u} = (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ *is* denoted by $\|\mathbf{u}\|$, and defined by:

$$\|\mathbf{u}\| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$$

It is similar for vectors in $\mathbb{R}^2$.

Example: If **u** = (3, 4, 5), then find $\|\mathbf{u}\|$ in 3-space.

Solution: $\|\mathbf{u}\| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2} = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}$

Theorem1.2.3 For every vector **u** and any scalar α the following properties hold true:

a) $\|\mathbf{u}\| \geq \mathbf{0}$,

b) $\|\mathbf{u}\| = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$

c) $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$

Proof: left as an exercise for the students.

Definition: A vector **u** is said to be a unit vector if $\|\mathbf{u}\| = \mathbf{1}$

For any vector **u,** the **unit** vector in the direction of **u** is given by:

$$\frac{1}{\|\mathbf{u}\|}\boldsymbol{u} = \frac{\boldsymbol{u}}{\|\mathbf{u}\|}.$$

Example: For a vector **u** = (2, 3, 5), find a unit vector in the direction of **u.**

Solution: The unit vector in the direction of **u** is given by $\frac{\boldsymbol{u}}{\|\mathbf{u}\|}$.

$\|\mathbf{u}\| = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$.

Hence $\frac{\boldsymbol{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{38}}(2, 3, 5) = (\frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}})$

Exercise:

1. Find the norm of **u** if **u** = $(\frac{-2}{7}, \frac{3}{7}, \frac{6}{7})$.
2. Find the unit vector in the direction of **v** if **v** = (2, 3, 6).

There are two especial unit vectors in $\mathbb{R}^2$ and three in $\mathbb{R}^3$ sometimes called standard unit vectors.

These are **i** = (1, 0), **j** = (0, 1) and **i** = (1, 0, 0) **j** = (0, 1, 0) **k** = (0, 0, 1) respectively. These unit vectors are used in simplifying the description and operations on vectors. We can write any vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ as follows:

a) **u** = $(a_1, a_2) = (a_1, 0) + (0, a_2) = a_1 (1, 0) + a_2 (0, 1) = a_1\mathbf{i} + a_2\mathbf{j}$
b) **u** = $(a_1, a_2, a_3) = (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) = a_1 (1, 0, 0) + a_2 (0, 1, 0) + a_3 (0, 0, 1)$
$= a_1\mathbf{i} + a_2\mathbf{j} + a_3 \mathbf{k}$

Note: Vectors given in component form can be expressed by using the unit vectors **i, j** and **k.**

Example: Describe the following vectors by using the appropriate unit vectors.

a) **u** = (4, 5)
b) **v** = (1, -2, 9)

Solution:

a) **u** = (4, 5)
= 4(1, 0) + 5(0, 1)
= 4**i** +5**j**

b) **v** = (1,-2,9)
=1(1, 0, 0) - 2(0, 1, 0) +9(0, 0, 1)
= **i -2j** +9**k**

The position vector of a point **P**(x_1, y_1, z_1) in $\mathbb{R}^3$ is the vector $\overrightarrow{\boldsymbol{OP}} = (x_1, y_1, z_1)$ whose initial point is the origin **O** and whose terminal point is **P.**

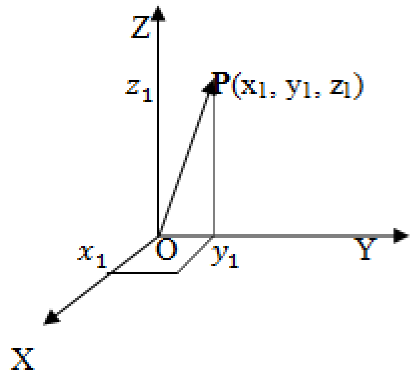


Fig 1.2.3

1.2. Dot (Scalar) Product

In this and the following section, we shall consider two kinds of products between vectors that originate in the study of mechanics, electricity and magnetism. The first of these products is known as the **dot** or **inner** or **scalar product**, which yields a scalar.

Definition: Let $\boldsymbol{u} = (a_1, a_2, \dots, a_n)$ *and* $\boldsymbol{v} = (b_1, b_2, \dots, b_n)$ be two vectors in n-space ($\mathbb{R}^n$). The dot product of **u** and **v** is given by **u·v** = $a_1b_1 + a_1b_1 + \cdots + a_nb_n$.

Example: Let **u** = (a_1, a_2, a_3) and **v** = (b_1, b_2, b_3) be two vectors in 3-space. The dot product of **u** and **v** is denoted by **u·v** and defined as:

$$\mathbf{u}\cdot\mathbf{v} = a_1b_1 + a_2b_2 + a_3b_3$$

Note: The **dot product** of a vector **u** with itself is given by:

$$\mathbf{u}\cdot\mathbf{u} = a_1a_1 + a_2a_2 + a_3a_3$$

$$= {a_1}^2 + {a_2}^2 + {a_3}^2 = \|\mathbf{u}\|^2$$

Hence $\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$

Example: For vectors **u** = (1, -2, 4) and **v** = (3, 0, 2) find **u·v.**

<u>**Solution**</u>: **u·v** = (1, -2, 4) ·(3, 0, 2) = (1) (3) + (-2) (0) + (4) (2) = 3 + 0 + 8 = 11

<u>**Properties of dot product**</u>

1. **u·v = v·u (dot product is commutative)**
2. **u·(v + w) = u·v + u·w (left distributive property of dot product over addition of vectors)**

3. **(w·u)·v = w·(u·v) = u·(w·v)(Associative property of dot product)**
4. **(u + v) ·w = u·w +v·w(right distributive property of dot product over addition of vectors)**

Definition1.3.2 Two vectors **u** and **v** are said to be :

1. parallel if there exists a scalar $\alpha \neq 0$ such that **u** = α **v**
2. in opposite direction if there exists a scalar $\alpha < 0$ such that **u** = α **v.**

Note: If two vectors **u** and **v** are parallel it is denoted by **u**$//$**v** and related by **u** = α **v** for some scalar $\alpha \neq 0$.

Example: Verify whether the following three given vectors are parallel or not.

u = (3, 2, -1) **v** = (-6, -4, 2) and **w** = $(\frac{3}{2}, 1, \frac{-1}{2})$.

<u>**Solution**</u>:1) **u** and **w** are parallel because **u**= $2(\frac{3}{2}, 1, \frac{-1}{2}) = 2$**w** or **w**= $\frac{1}{2}$**u**.

2) $\boldsymbol{u}$ and **v** are also parallel because **u**= $\frac{-1}{2}$v or **v**= -2**u** but opposite in direction.

3) **v** and **w** are also parallel because **v**= -4**w** $or\ \boldsymbol{w} = -\frac{1}{4}$**v** but opposite in direction.

<u>**Exercise**</u>: Find a vector having the same direction as **u** = (-2, 4, 2) but has magnitude 6.

Consider two vectors u and v. The square of the norm of their sum is given as follows:

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) &= \boldsymbol{u}\cdot(\boldsymbol{u}+\boldsymbol{v}) + \boldsymbol{v}\cdot(\boldsymbol{u}+\boldsymbol{v})\\ &= \boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v}\\ &= \|\boldsymbol{u}\|^2 + 2\boldsymbol{u}\cdot\boldsymbol{v} + \|\boldsymbol{v}\|^2\end{aligned}$$

And hence we see that $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$ if and only if $\boldsymbol{u}\cdot\boldsymbol{v} = \mathbf{0}$

Definition1.3.3 Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are called orthogonal (Perpendicular) to each other if and only if $\boldsymbol{u}\cdot\boldsymbol{v} = \mathbf{0}$

Example: Let **u** = (2, 2, -1) and **v** = (5, -4, 2) then show that **u·v=0**

Solution: **u·v**= $(2,2,-1)\cdot(5,-4,2) = 0$

Exercise

a. Find the value of b so that the vectors **u** = (-2, 4, 2) and **v** = (b, b^2, b) are orthogonal.
b. Find two vectors orthogonal to $\boldsymbol{u} = (1,2,3)$ and each of their components are non-zero.

Angle Between Two Vectors

Theorem1.3.1 If θ is the angle between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ then $\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\boldsymbol{cos\theta}$

Proof: Applying the law of cosine to ΔOPQ in the figure below we get:

$\|\overrightarrow{PQ}\|^2 = \|\overrightarrow{OQ}\|^2 + \|\overrightarrow{OP}\|^2 - 2\|\overrightarrow{OQ}\|\|\overrightarrow{OP}\| \cos\theta$ -------(1)

But from the figure $\overrightarrow{PQ} = \boldsymbol{u} - \boldsymbol{v}$, $\overrightarrow{OP} = \boldsymbol{v}$ and $\overrightarrow{OQ}$=**u** and hence equation (1) becomes:

$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\boldsymbol{cos\theta}$ ---------------------------------- (2)

But $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathrm{v} + \|\mathbf{v}\|^2 (verify\ this?)$ and substituting this in (2) we get: $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\boldsymbol{cos\theta}$

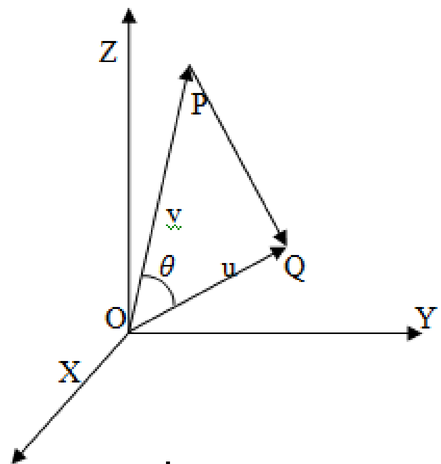


Fig 1.3.1

Corollary 1.3.1 If θ is the angle between the vectors **u** and **v** then: $\cos\theta = \frac{(\boldsymbol{u}\cdot\boldsymbol{v})}{\|\mathbf{u}\|\|\mathbf{v}\|}$

Example:

a) Find the angle between **u** = (1, -2, 2) and **v** = (-3, 6, -6)
b) If $\|\mathbf{u}\| = \mathbf{4}$, $\|\mathbf{v}\| = \mathbf{6}$ and the angle between them is $\frac{\pi}{3} = 60°$ then find **u**·**v**

Solution: a) u= (1,-2, 2) **and v=** (-3, 6,-6)

$$\|\mathbf{u}\| = \|(1,-2,2)\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

$$\|\mathbf{v}\| = \|(-3,6,-6)\| = \sqrt{(-3)^2 + (6)^2 + (-6)^2} = \sqrt{81} = 9$$

u·**v** = (1)(-3)+(-2)(6)+(2)(-6) = -3-12-12 = -27

And $\cos\theta = \frac{(\boldsymbol{u}\cdot\boldsymbol{v})}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{-27}}{\mathbf{27}} = \mathbf{-1} \Rightarrow \boldsymbol{\theta} = \boldsymbol{cos^{-1}}(\mathbf{-1}) = \mathbf{180°} = \boldsymbol{\pi}$ **in radian measure**

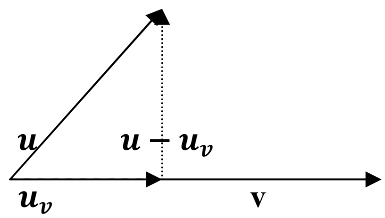
Exercise

1. Find the angle between the X- axis and **v** = (1, -2, -2)
2. If the vertices of a triangle are $P(1,-3,2)$, $Q(2,0,-4) and\ R(6,-2,-5)$ verify the type of the triangle.
3. If **u** = (2,-3,4) and **v** = (-1,2.0), and **w** = (5,-1,2) find the angle between **u**-2**v** and **u**+2**w**

Definition: Two vectors **u** and **v** are said to be ortho-normal if $\|\mathbf{u}\| \cdot \|\mathbf{v}\| = \mathbf{1}$.

1.3. Orthogonal projection

For two vectors **u** and **v** such that $\mathbf{v} \neq 0$, consider the following figure:



$\boldsymbol{u_v}$ *is* Orthogonal projection of vector **u** onto **v**

Now $\boldsymbol{u_v}$ have two properties:

1. $\boldsymbol{u_v}$ is parallel to **v** denoted by $\boldsymbol{u_v}//\boldsymbol{v}$
2. $\boldsymbol{u} - \boldsymbol{u_v}$ is perpendicular to **v** denoted by $(\boldsymbol{u} - \boldsymbol{u_v}) \perp \boldsymbol{v}$.

From (1) since $\boldsymbol{u_v}//\boldsymbol{v}$ then there exists $t \in \mathbb{R}$ such that $\boldsymbol{u_v} = t\boldsymbol{v}$ and from (2) since $(\boldsymbol{u} - \boldsymbol{u_v}) \perp \boldsymbol{v}$, then we have $(\boldsymbol{u} - \boldsymbol{u_v}) \cdot \boldsymbol{v} = 0$ that is :

$$\boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u_v} \cdot \boldsymbol{v} = 0$$

$\Rightarrow \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u_v} \cdot \boldsymbol{v}$

$\Rightarrow \boldsymbol{u} \cdot \boldsymbol{v} = (t\boldsymbol{v}) \cdot \boldsymbol{v}$

$\Rightarrow \boldsymbol{u} \cdot \boldsymbol{v} = t\|\mathbf{v}\|^2$

$\Rightarrow \ t = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\mathbf{v}\|^2}$

Then $\boldsymbol{u_v} = t\boldsymbol{v} = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\mathbf{v}\|^2}\boldsymbol{v}$ -- (1)

Definition: Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors such that $\boldsymbol{u} \neq \mathbf{0}$ and $\boldsymbol{v} \neq \mathbf{0}$ then the projections of vector **u** on to vector **v** $\boldsymbol{u_v}$, and **vector** $\boldsymbol{v}$

onto $\boldsymbol{u}$, $\boldsymbol{v_u}$ are given by:

$$\boldsymbol{u_v} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\mathbf{v}\|^2}\boldsymbol{v} \text{ and similarly } \boldsymbol{v_u} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\mathbf{u}\|^2}\boldsymbol{u}$$

Example: Let **u** = (1, 2, -1) and **v** = (3, 0, -2) then find $\boldsymbol{u_v}$ and $\boldsymbol{v_u}$

Solution: u·v = 3+0+2 = 5

$\|\boldsymbol{u}\|^2 = 6$ ***and*** $\|\mathbf{v}\|^2 = 13$

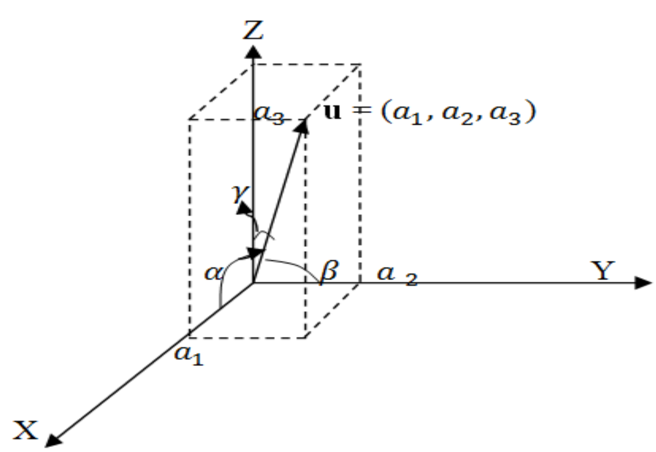
$\boldsymbol{u_v} = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\mathbf{v}\|^2}\boldsymbol{v} \quad = \frac{5}{13}(\mathbf{3}, \mathbf{0}, -\mathbf{2})$ *and* $\boldsymbol{v_u} = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\mathbf{u}\|^2}\boldsymbol{u} \quad = \frac{5}{13}(\mathbf{1}, \mathbf{2}, -\mathbf{1})$

Exercise:

1. Let $\boldsymbol{u} = (1,3,-4)$ *and* $\boldsymbol{v} = (5,-1,0)$. Find the projection of vector u onto $\boldsymbol{v}$ and $\boldsymbol{v}$ onto vector $\boldsymbol{u}$.
2. Find the angle between the following pairs of vectors:
   a. **u**=**i** + **j** + **k**, **v**=2**i** + **j** – **k**
   b. **u**= 2**i** − **j** + 3**k**, **v**= 2**i** + **j** + 3**k**
   c. **u**=3**i** − **j** + **k**, **v** = $\boldsymbol{i}$ − 2**j** + 3**k**
   d. **u**=**i** − 2**j** + **k**, **v**= 4**i** − 8**j** + 16**k**
3. Given **u** = 2**i** − 3**j** − 3**k**, **v** = **i** + **j** + 2**k**, and **w** = 3**i** −2**j** − **k**, find the angles between the following pairs of vectors:
   a. **u** + **v** and **v** – 2**w**
   b. 2**u** − **w** and, **u** + **v** – **w**
   c. v + 3**w** and **u** – 2**w**.
4. Find the component of the force **F** = 4**i** + 3**j** + 2**k** in the direction of the vector **i** + **j** + **k**.
5. Find the component of the force **F** = 2**i** + 5**j** − 3**k** in the direction of the vector 2**i** + **j** − 2**k**.
6. Given that **u** = **i** + 2**j** + 2**k** and v = 2**i** − 3**j** + **k**, find
   a. the projection of **u** onto **v**, and
   b. the projection of **v** onto **u**.
7. Given that **a** = 3**i** + 6**j** + 9**k** and **b** = **i** + 2**j** + 3**k**,
   a. Find the projection of **a** onto the line of **b** and
   b. Compare the magnitude of **a** with the result found in (a) and comment on the result.

1.4. Direction angles and Direction cosines

Definition: The direction angles of a non-zero vector **u** are the angles $\alpha,\ \beta\ \ and\ \gamma$ in the interval $[0, \pi]$ that **u** makes with the X-, Y- and Z-axes respectively as in the figure below:



The cosine of these direction angles, $cos\ \alpha$, $cos\ \beta$ $and\ cos\ \gamma$ are called direction cosines of the vector **u**.

From the figure above we have:

$$\cos\alpha = \frac{\boldsymbol{u}\cdot i}{\|\mathbf{u}\|\|\mathrm{i}\|} = \frac{a_1}{\|\mathbf{u}\|} \qquad \cos\beta = \frac{\boldsymbol{u}\cdot j}{\|\mathbf{u}\|\|\mathrm{j}\|} = \frac{a_2}{\|\mathbf{u}\|}$$

$$\cos\gamma = \frac{\boldsymbol{u}\cdot k}{\|\mathbf{u}\|\|\mathrm{k}\|} = \frac{a_3}{\|\mathbf{u}\|}$$

$$\Rightarrow \quad \begin{cases} a_1 = \|\mathbf{u}\|\cos\alpha \\ a_2 = \|\mathbf{u}\|\cos\beta \\ a_3 = \|\mathbf{u}\|\cos\gamma \end{cases}$$

Exercise:

1. Find the direction cosines and corresponding angles for the following vectors:
   a. **i** + **j** + **k**.
   b. **i** − 2**j** + 2**k**.
   c. 4**i** − 2**j** + 3**k**.

2. Find the direction cosines and corresponding angles for the following vectors:
   a. **i** − **j** − **k**.
   b. 2**i** + 2**j** − 5**k**.
   c. −4**j** − **k**

From the equations above we observe that:

$$\boldsymbol{u} = (a_1, a_2, a_3) = (\|\mathbf{u}\|\cos\alpha, \|\mathbf{u}\|\cos\beta, \|\mathbf{u}\|\cos\gamma) = \|\mathbf{u}\|(\cos\alpha, \cos\beta, \cos\gamma)$$

$\Rightarrow \frac{\boldsymbol{u}}{\|\mathbf{u}\|} = (\cos\alpha, \cos\beta, \cos\gamma)$ which indicates that the direction cosines of **u** are the components of the unit vector in the direction of **u**.

Example: Find the direction angles of the vector **u** = (6, 2, 3).

Solution: $\cos\alpha = \frac{\boldsymbol{u}\cdot i}{\|\mathbf{u}\|\|\mathrm{i}\|} = \frac{a_1}{\|\mathbf{u}\|} = \frac{6}{7}$

$$\cos\beta = \frac{\boldsymbol{u}\cdot j}{\|\mathbf{u}\|\|\mathrm{j}\|} = \frac{a_2}{\|\mathbf{u}\|} = \frac{2}{7}$$

$$\cos\gamma = \frac{\boldsymbol{u}\cdot k}{\|\mathbf{u}\|\|\mathrm{k}\|} = \frac{a_3}{\|\mathbf{u}\|} = \frac{3}{7}$$

Exercise: If a vector has direction angles $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$ then find the third direction angle γ.

1.5. Cross product of vectors

For two non-parallel vectors **u** and **v,** how can we find a non-zero vector **w** which is orthogonal to both **u** and **v**? This problem has a standard solution called the **cross product** of **u** and **v** denoted by $\boldsymbol{u}\times\boldsymbol{v}$.

Definition: If $\boldsymbol{u} = (a_1, a_2, a_3)$ and $\boldsymbol{v} = (b_1, b_2, b_3)$ are two vectors, then the cross product of **u** and **v** is defined as:

$$\boldsymbol{u}\times\boldsymbol{v} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

Remark:

1. The cross product of two vectors is a vector.
2. An easy way to remember the cross product is as follows:

   For $\boldsymbol{u} = (a_1, a_2, a_3) = a_1i + a_2j + a_3k$ and

   $\boldsymbol{v} = (b_1, b_2, b_3) = b_1i + b_2j + b_3k$

Then by repeating the first and the second column we get:

$$\begin{array}{ccccc} i & j & k & i & j \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array} = \boldsymbol{u}\times\boldsymbol{v}$$

$$\Rightarrow \ \boldsymbol{u}\times\boldsymbol{v} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$

3. In determinant form:

$$\boldsymbol{u}\times\boldsymbol{v} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

$$= (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$

Example: Let $\boldsymbol{u} = i - 2j + k$ and $\boldsymbol{v} = -i + 2j + 5k$ then find $\boldsymbol{u}\times\boldsymbol{v}$.

Solution: $\boldsymbol{u}\times\boldsymbol{v} = [(-2)(5) - (1)(2)]\boldsymbol{i} + [(1)(-1) - (1)(5)]\boldsymbol{j} + [(1)(2) - (-2)(-1)]\boldsymbol{k}$

$= -12\boldsymbol{i} - 6\boldsymbol{j}$

Theoerm1.4.1 Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in $\mathbb{R}^3$ and $t \in \mathbb{R}$, then:

1. $\boldsymbol{u}\times\boldsymbol{v} \perp \boldsymbol{u}$ ***and*** $\boldsymbol{u}\times\boldsymbol{v} \perp \boldsymbol{v}$
2. $\boldsymbol{u}\times\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{u}$
3. $\boldsymbol{u}\times\boldsymbol{u} = \boldsymbol{0}$
4. $(t\boldsymbol{u})\times\boldsymbol{v} = t(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{u}\times(t\boldsymbol{v})$
5. $\|\boldsymbol{u}\times\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - (\boldsymbol{u}\cdot\boldsymbol{v})^2$

6. $(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{w}=\boldsymbol{u}\times\boldsymbol{w}+\boldsymbol{v}\times\boldsymbol{w}$
7. $\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})=\boldsymbol{v}\cdot(\boldsymbol{w}\times\boldsymbol{u})=\boldsymbol{w}\cdot(\boldsymbol{u}\times\boldsymbol{v})$

The proof is left for the students as an exercise.

Theorem1.4.2 If θ is the angle between the vectors **u** and **v** where $0\le\theta\le\pi$ then:

$$\|\boldsymbol{u}\times\boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin(\theta)$$

Proof: From the definition of cross product and length of a vector we have:

$$\|\boldsymbol{u}\times\boldsymbol{v}\|^2 = (a_2b_3-a_3b_2)^2+\ (a_3b_1-a_1b_3)^2+\ (\,a_1b_2-a_2b_1)^2$$

$$= {a_2}^2{b_3}^2-2a_2b_3a_3b_2+{a_3}^2{b_2}^2+{a_3}^2{b_1}^2-2a_1b_1a_3b_3+{a_1}^2{b_3}^2+{a_1}^2{b_2}^2-2a_1b_1a_2b_2+{a_2}^2{b_1}^2$$

$= ({a_1}^2+\ {a_2}^2+\ {a_3}^2)\left({b_1}^2+\ {b_2}^2+\ {b_3})^2\right)-(a_1b_1+a_2b_2+a_3b_3)^2 \qquad = \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2-(\boldsymbol{u}\cdot\boldsymbol{v})^2$

$= \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2-\|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2\cos^2\theta$

$= \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2(1-\cos^2\theta)$

$= \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2\sin^2\theta \quad \Rightarrow \|\boldsymbol{u}\times\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2\sin^2\theta$

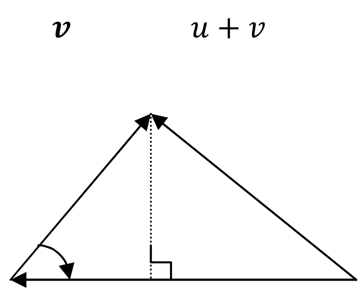
Since $\sin\theta\ge 0\ for\ 0\le\theta\le\pi$,we can take square root and hence we have $\|\boldsymbol{u}\times\boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin\theta$

Corollary: Two non-zero vectors **u** and **v** are parallel if and only if $\|\boldsymbol{u}\times\boldsymbol{v}\|=0$.

Proof: left as an exercise

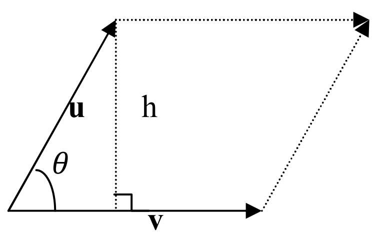
1.6. Application of cross product

1. Consider the triangle whose edges are the vectors **u** and **v** as in the figure below:



$\|\boldsymbol{v}\|$Area of $\Delta=\frac{1}{2}base\times height=\frac{1}{2}\|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin\theta$ since $height=\ \|\boldsymbol{v}\|\sin\theta \quad =\frac{1}{2}\|\boldsymbol{u}\times\boldsymbol{v}\|$

2. Let **u** and **v** be vectors in $\mathbb{R}^3$ and consider the figure below: $\sin\theta=\frac{h}{\|\boldsymbol{u}\|}\Rightarrow h=\|\boldsymbol{u}\|\sin\theta$

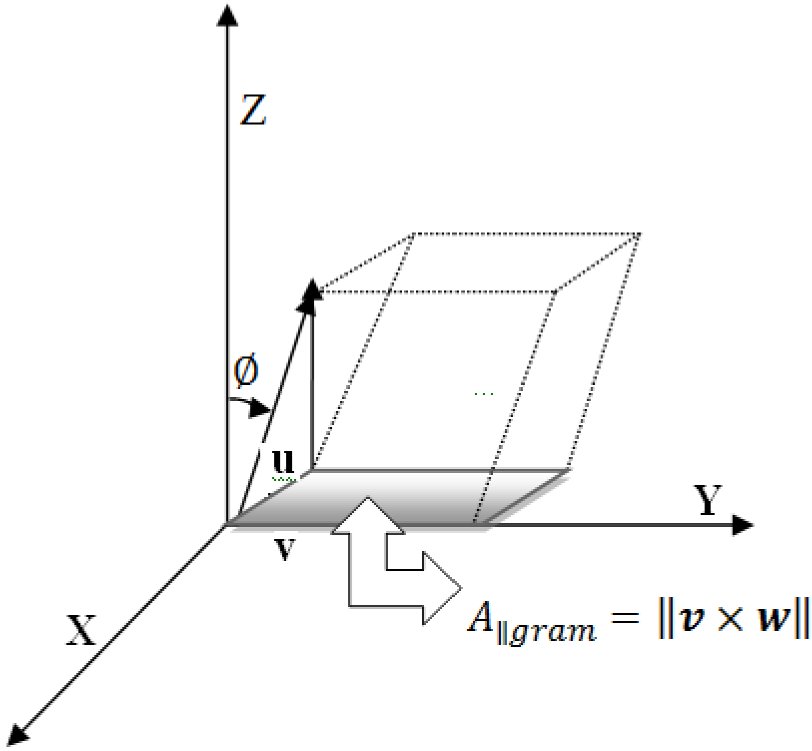


Area= $base\times height=\|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin\theta=\|\boldsymbol{u}\times\boldsymbol{v}\|$

Thus $\|\boldsymbol{u}\times\boldsymbol{v}\|$ is the area of the parallelogram spanned by the vectors **u** and **v**.

3. For vectors **u**, **v** and **w** in $\mathbb{R}^3$ as in the figure below; area of the parallelogram is given by

$$A_p = \|\boldsymbol{v} \times \boldsymbol{w}\| \qquad \text{and} \qquad \cos\emptyset = \frac{h}{\|\boldsymbol{u}\|} \Rightarrow h = \|\boldsymbol{u}\|\cos\emptyset$$



Volume $= base\ area \times height \ = \|\boldsymbol{v} \times \mathbf{w}\|\|\boldsymbol{u}\|\cos\emptyset \ = \pm\|\boldsymbol{u}\|\|\boldsymbol{v} \times \boldsymbol{w}\|\cos\emptyset \ = \pm\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$

Hence volume of the parallelepiped spanned by the vectors **u**, **v** and **w** is

$$V = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$$

<u>**Note**</u>: The expression $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is called triple scalar product of **u**, **v** and **w**.

Example: Find the volume of the parallelepiped spanned by $\boldsymbol{u} = (2, -1, -1)$, $\boldsymbol{v} = (1, 1, 3)$ and $\boldsymbol{w} = (-1, 1, 5)$

Solution: $u \times v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & -1 \\ 1 & 1 & 3 \end{vmatrix} = -2\boldsymbol{i} - 7\boldsymbol{j} + 3\boldsymbol{k}$

Therefore**, volume** $\boldsymbol{V} = |\boldsymbol{w}.(\boldsymbol{u} \times \boldsymbol{v})| = |(\mathbf{-1})(\mathbf{-2}) + \mathbf{1}(\mathbf{-7}) + \mathbf{5}(\mathbf{3})| = \mathbf{11}$ **unit cube**

Exercise:

1. Determine the area of a triangle with vertices $p_1(1,5,2)$, $p_2(-1,3,0)$and $p_3(0,1,4)$
2. For vectors **u**, **v** and **w** in $\mathbb{R}^3$ show that:

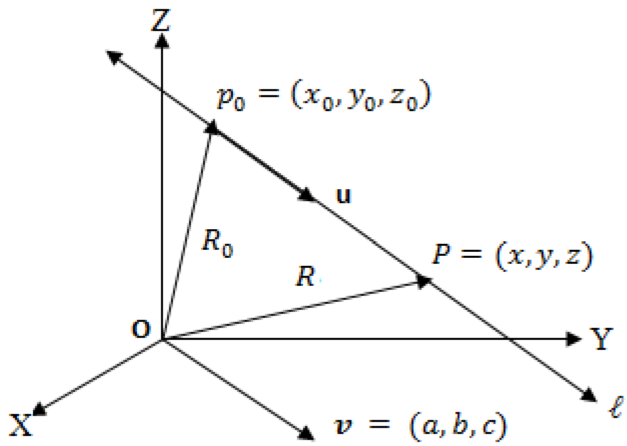
a) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$

b) $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$

1.7. Lines and planes in $\mathbb{R}^3$.

Equations of lines in space

Let ℓ be a line in $\mathbb{R}^3$ and $p_0 = (x_0, y_0, z_0)$ be a point on ℓ and **v** be a vector which is parallel to ℓ as in the figure below:



Let P be arbitrary point on ℓ so **u** is the vector P_0P, then by triangular method we have:

$$R = R_0 + \boldsymbol{u}$$

Since vector **u** is parallel to vector **v** we have $\boldsymbol{u} = t\boldsymbol{v}$ and hence the above equation becomes

$$R = R_0 + t\boldsymbol{v} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

This equation is called **vector equation** of the line ℓ.

If $\boldsymbol{v} = (a, b, c)$ then $t\boldsymbol{v} = (ta, tb, tc)$ and $R = (x, y, z)$, $R_0 = (x_0, y_0, z_0)$ and hence equation (1) becomes:

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$$

$$(x, y, z) = (x_0 + ta, y_0 + tc, z_0 + tc)$$

$\Rightarrow \begin{cases} \boldsymbol{x = x_0 + ta} \\ \boldsymbol{y = y_0 + tb} \\ \boldsymbol{z = z_0 + tc} \end{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$ is called the **parametric equation** of ℓ.

Example: Find vector equation and parametric equation of a line passing through $(5,1,3)$ and parallel to the vector $i + 4j - 2k$.

Another way of describing ℓ is to eliminate the parameter t from equation (2) above;

that is:

$$x = x_0 + ta \Rightarrow t = \frac{x - x_0}{a}, \quad a \neq 0$$
$$y = y_0 + tb \Rightarrow t = \frac{y - y_0}{b}, \quad b \neq 0$$
$$z = z_0 + tc \Rightarrow t = \frac{z - z_0}{c}, \quad c \neq 0$$

Then for $a, b, c \neq 0$ we have:

$\boldsymbol{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (3)$ is called **symmetric equation** of ℓ

Exercise:

a) Find the parametric equation of the line that passes through the point $P_0 = (2,4,-3)$ and $P_1 = (3,-1,1)$.

b) At what point does this line intersect the $XY - plane$?

Properties of a line in $\mathbb{R}^3$

Let $\ell_1: R(\alpha) = R_0 + \alpha\boldsymbol{v}$ and $\ell_2: R'(\beta) = R_0' + \beta\boldsymbol{v}'$ be two distinct lines, then:

1. $\ell_1$and ℓ_2 will intersect if and only if there are α and β in $\mathbb{R}$ so that $R(\alpha) = R'(\beta)$
2. The lines $\ell_1$and ℓ_2 are parallel iff their direction vectors are parallel. That is; $\ell_1 // \ell_2$ *iff* $\boldsymbol{v} // \boldsymbol{v}'$
3. If $\ell_1$and ℓ_2 are intersecting lines, then the angle between $\ell_1$and ℓ_2 is the angle between their direction vectors. That is:
$$\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{v}'}{\|\boldsymbol{v}\|\|\boldsymbol{v}'\|}$$
4. Non-parallel and non-intersecting lines are called skew lines.
5. $\ell_1$is perpendicular to ℓ_2 iff $\boldsymbol{v}$ is perpendicular to $\boldsymbol{v}'$. That is:
$$\ell_1 \perp \ell_2 \; iff \;\; \boldsymbol{v} \perp \boldsymbol{v}' \;\; i.e\, \boldsymbol{v} \cdot \boldsymbol{v}' = 0$$

Example: Find the point of intersections of the lines $\ell_1: R(\alpha) = (i - 6j - 1k) + \alpha(i + 2j + k)$ and

$\ell_2: R'(\beta) = (4j - 2k) + \beta(2i + 2j + 2k)$ and find the angle between them.

Solution: $R(\alpha) = R'(\beta) \Rightarrow (i - 6j - 1k) + \alpha(i + 2j + k) = (4j - 2k) + \beta(2i + 2j + 2k)$

$$\begin{cases} 1+\alpha = 2\beta \dots\dots\dots\dots.1 \\ -6+2\alpha = 4+2\beta \dots\dots.2 \\ -1+\alpha = -2+2\beta \dots\dots..3 \end{cases}$$

Multiplying equation (1) by 2 and subtracting equation (2) we get $8 = -4 + 2\beta$ which implies $\beta = 6$. And also $\alpha = 11$

Therefore, the point of intersection is (12, 16, 10).

The angle between the two lines is the same as the angle between their direction vectors $v_1 = (1,2,1)\ and\ v_2 = (2,2,2)$.

That is, let it be θ. Then $\cos\theta = \frac{(v_1 \cdot v_2)}{\|v_1\|\|v_2\|} = \frac{\mathbf{8}}{\sqrt{\mathbf{72}}} = \frac{\mathbf{8}}{\mathbf{6\sqrt{2}}} = \frac{\mathbf{4}}{\mathbf{3\sqrt{2}}}$

$$\theta = cos^{-1}(\frac{\mathbf{4}}{\mathbf{3\sqrt{2}}})$$

1.8. Distance between a point and a line

Theorem1.5.1.2.1 Let ℓ be a line with vector equation $P = P_0 + t\boldsymbol{v}$ and let Q be any point not on ℓ, then the distance between Q and ℓ is given by:

$$d = \frac{\|\overrightarrow{P_0Q} \times \boldsymbol{v}\|}{\|\boldsymbol{v}\|}$$

Example: Find the distance between the point $Q(2,-1,3)$ and the line with symmetric equation $\frac{x-1}{2} = \frac{y+1}{3} = \frac{-z}{6}$.

Solution: $P_0 = (1,-1,0)$ from the symmetric equation of the line and $v = (2,3,-6)$

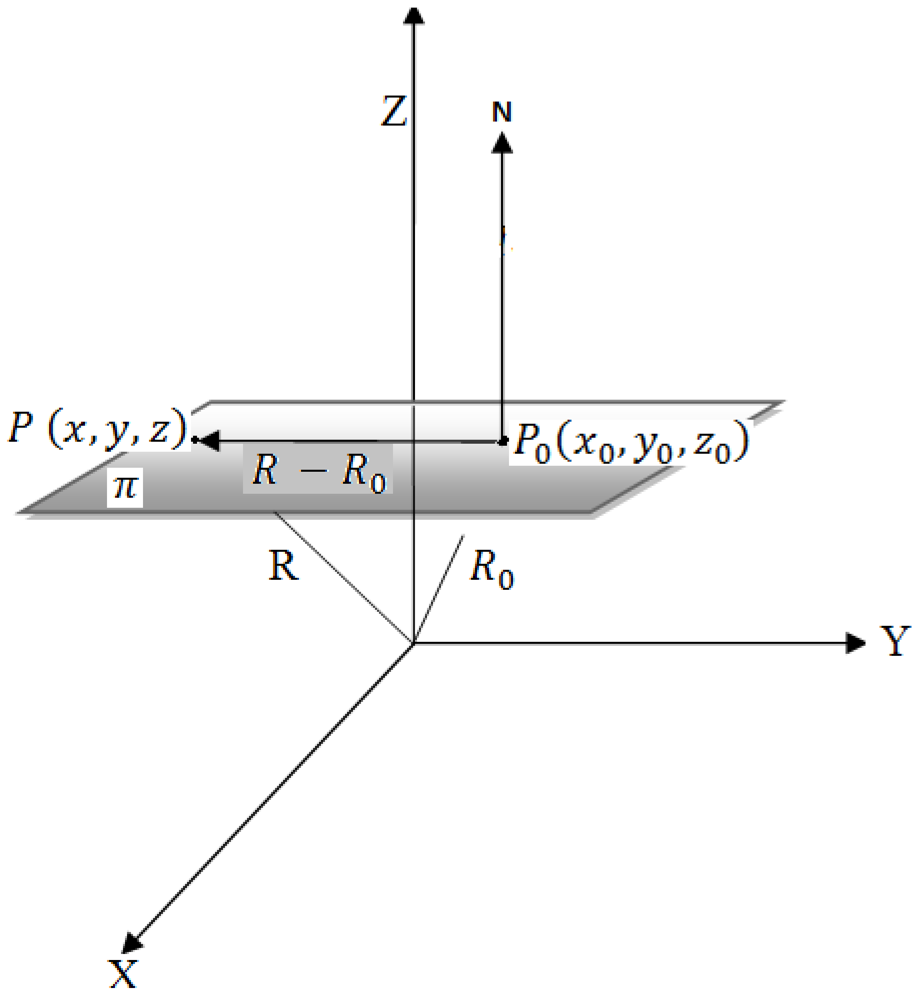
So the distance between the point Q(2,-1,3) and the line is $d = \frac{\|\overrightarrow{P_0Q} \times \boldsymbol{v}\|}{\|\boldsymbol{v}\|}$ where $\overrightarrow{P_0Q} = (1,0,3)$ and $\|\boldsymbol{v}\| = 7$

$$\mathrm{d} = \frac{\|(1,0,3)\times(\mathbf{2},\mathbf{3},-\mathbf{6})\|}{7} = 2.18\ unit.$$

Exercise: Let ℓ be a line passing through $P(1,2,1)$ and $Q(3,-1,4)$ then find the distance between ℓ and the origin.

1.9. Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ on the plane and a vector $\boldsymbol{N}$ that is orthogonal to the plane. In the figure below let $P(x,y,z)$is a point on the plane π.



Since $R - R_0$ is perpendicular to N, we have:

$$N \cdot (R - R_0) = 0.$$

This implies $N \cdot R = N.R_0$ ……………..(1) is **vector equation** of the plane π.

Remark:

1. Let $N = (a, b, c)$, $R = (x, y, z)$ and $R_0 = (x_0, y_0, z_0)$ then equation (1) becomes:

$$(a, b, c) \cdot (x, y, z) = (a, b, c) \cdot (x_0, y_0, z_0)$$

$$\Rightarrow ax + by + cz = ax_0 + by_0 + cz_0$$

$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \text{ ………………….(2)}$$

which is called the **scalar equation** of the plane π.

2. Let $d = ax_0 + by_0 + cz_0$, then equation (2) becomes:

$ax + by + cz = d$ ……………….(3)

which is called the **standard equation** of the plane π.

Example:

1. Find the equation of a plane containing the point $P(2,4,1)$ and normal vector $N(2,3,4)$.
2. Determine the line of intersection of the planes;

   $\pi_1: 2x - y + z = 4$ and $\pi_2: x + 3y - z = 2$

   Solutions:

1. Let $R(x, y, z)$ be an arbitrary point on the plane containing the point P(2, 4, 1) and normal vector N(2, 3, 4).

   Then the equation of the plane is

$$(x, y, z) \cdot (2, 3, 4) = (2, 4, 1) \cdot (2, 3, 4)$$

   $2x + 3y + 4z = 20$ is the standard equation of the plane.

2. To find the line of intersection of the planes $\pi_1: 2x - y + z = 4$ and $\pi_2: x + 3y - z = 2$

   $\begin{cases} 2x - y + z = 4 \\ x + 3y - z = 2 \end{cases}$……………..(1).

   Here we have two equations with three variables and it is impossible to solve for the three variables at same time.

   So, we give some value for one of the three variables and solve for the other two in order to get two intersection points of the two planes which can help us to find the equation of the line of intersection.

   Let $z = 0$. Hence (1) becomes $\begin{cases} 2x - y = 4 \dots\dots\dots..(a) \\ x + 3y = 2 \dots\dots\dots.(b) \end{cases}$. Multiplying equation (a) by 3 and adding the result to (b) we get $x = 2$ which implies $y = 0$. Thus, $P_1(2, 0, 0)$ is one of the intersection point of the two planes.

   Again let $z = -1$. Then equation (a) becomes $\begin{cases} 2x - y = 5 \\ x + 3y = 1 \end{cases}$. Solving these simultaneously we get $x = \frac{16}{7}$ and $y = -\frac{3}{7}$ so $P_2(\frac{16}{7}, -\frac{3}{7}, -1)$ is another point of intersection of the two planes. Therefore, the equation of the line of intersection of the two planes is given by $(x, y, z) = (2, 0, 0) + t(P_2 - P_1)$ for an arbitrary point (x, y, z) on the line of intersection of the two planes.

So, $(x, y, z) = (2 - \frac{2}{7}t, -\frac{3}{7}t, -t) \Rightarrow \begin{cases} \boldsymbol{x = 2 - \frac{2}{7}t} \\ \boldsymbol{y = -\frac{3}{7}t} \\ \boldsymbol{z = -t} \end{cases}$, $t \in \mathbb{R}$ is the parametric equation of the line of intersection of the two planes.

Exercise:

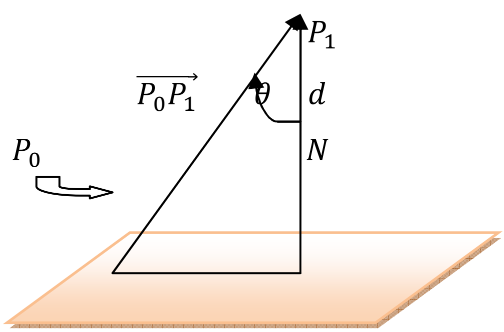
1. Find the equation of the plane that contains the points $P(1,3,2), Q(3,-1,6)$ and $R(5,2,0)$ which are not collinear but coplanar.
2. Find the point at which the line with parametric equation $x = 2 + 3t, y = -4t, and z = 5 + t$ intersects the plane

$$\pi: 4x + 5y - 2z = 18$$

3. Find the angle between the planes $\pi_1: x + y + z = 1$ and $\pi_2: x - 2y + 3z = 1$ and find the symmetric equation for the line of intersection ℓ of these planes.

1.10. Distance between a point and a plane

Consider the following figure;



From the figure we have: $\cos\theta = \frac{d}{\|\overrightarrow{P_0P_1}\|}$ $\qquad \Rightarrow d = \|\overrightarrow{P_0P_1}\| \cos\theta$

In addition to this; $N \cdot \overrightarrow{P_0P_1} = \|N\| \|\overrightarrow{P_0P_1}\| \cos\theta$ $\qquad \Rightarrow N \cdot \overrightarrow{P_0P_1} = \|N\| d$

$$\Rightarrow d = \frac{|N \cdot \overrightarrow{P_0P_1}|}{\|N\|} = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}$$

Example: Find the distance between the planes $3x + y - 4z = 2$ and the point (1,-1, 0).

Solution: $d = \frac{|N \cdot \overrightarrow{P_0P_1}|}{\|N\|} = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}}$ where $N = (3,1,-4)$ and $P_0 = (1,-1,0) = (x_0, y_0, z_0)$

1.11. Vector Space

Definition: A **vector space** is a non-empty set $\boldsymbol{V}$ of elements called vectors together with the operations of addition (+) and scalar multiplication (·) over a field F such that the following axioms (laws) holds.

For all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$and scalars $a, b \in F$ we have:

1. $\boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{V}$ **(closure property of vector addition)**
2. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ **(commutativity of vector addition)**
3. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$ (associability of vector addition)
4. $\exists \boldsymbol{0} \in \boldsymbol{V} \ni: \boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u} = \boldsymbol{0} + \boldsymbol{u}$ **(additive identity)**
5. $\exists -\boldsymbol{u} \in \boldsymbol{V} \ni: \boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0} = (-\boldsymbol{u}) + \boldsymbol{u}$ (additive inverse)
6. $a\boldsymbol{u} \in \boldsymbol{V}$ (closed under scalar multiplication)
7. $a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$ (distributive property of scalar over sum of vectors)
8. $(a + b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}$ (distributive vector over sum of scalars law)
9. $(ab)\boldsymbol{u} = a(b\boldsymbol{u})$ (associative law)
10. $1\boldsymbol{u} = \boldsymbol{u}$(monoidal law)

Remark: A field F is to mean like $\mathbb{Q}$, $\mathbb{R}$, etc…

Example:

1. Let $\mathbb{R}^{\boldsymbol{n}}$,for positive integer $\boldsymbol{n}$(n-space of real numbers) and let V be a set of n-tuples of elements of $\mathbb{R}$**.**

   We can define operations as follows:

   Addition: $(a_1, a_2, \dots \dots \dots a_n) + (b_1, b_2, \dots \dots \dots b_n) = (a_1 + b_1, a_2 + b_2, \dots \dots \dots a_n + b_n)$ and **scalar multiplication** $\alpha(a_1, a_2, \dots \dots \dots a_n) = (\alpha a_1, \alpha a_2, \dots \dots \dots \alpha a_n)$ and the zero vector is $\boldsymbol{0} = (0, 0, \dots \dots \dots 0)$then show that $\mathbb{R}^{\boldsymbol{n}}$is a vector space.

2. Let $\boldsymbol{P}_{\boldsymbol{n}}(t)$be the set of polynomials with degree less than or equal to n of the form:

   $\boldsymbol{P}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots \dots \dots + a_s t^s, s \in \mathbb{R}, s \leq n$ and we define operations as follows:

   Addition: usual addition of polynomials and scalar multiplication: multiplication of a polynomial by a number; the zero vector is **0**=0 then show that $\boldsymbol{P}_{\boldsymbol{n}}(t)$is a vector space.

Some properties of vector space

For all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$and scalar $\alpha \in F$ we have:

1. If $\boldsymbol{u} + \boldsymbol{w} = \boldsymbol{v} + \boldsymbol{w}$ then $\boldsymbol{u} = \boldsymbol{v}$

2. $\alpha\mathbf{0} = \mathbf{0}$
3. $0\boldsymbol{u} = \mathbf{0}$
4. If $\alpha \neq 0$ and $\alpha\boldsymbol{u} = \mathbf{0}$ $then$ $\boldsymbol{u} = \mathbf{0}$
5. $(-\alpha\boldsymbol{u}) = \alpha(-\boldsymbol{u})$

Proof: Exercise

1.12. Sub Spaces

Definition: A sub set $\boldsymbol{w}$ of the vector space $\boldsymbol{V}$ is called a subspace of $\boldsymbol{V}$ if $\boldsymbol{w}$ is also a vector space under the addition and scalar multiplication defined on $\boldsymbol{V}$.

Theorem: For a subset $\boldsymbol{w}$ of the vector space $\boldsymbol{V}$, $\boldsymbol{w}$ is a subspace of $\boldsymbol{V}$ iff:

1. $\mathbf{0} \in \boldsymbol{w}$ for $\mathbf{0} \in \boldsymbol{V}$
2. $\forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{w},\ \boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{w}$
3. $\forall \boldsymbol{u} \in \boldsymbol{w}$ $and \alpha \in F,\ \alpha\boldsymbol{u} \in \boldsymbol{w}$

Example: Consider the vector space $\mathbb{R}^{\mathbf{2}}$; then show that the subset $\boldsymbol{w} = \{(0, y)/\ y \in \mathbb{R}\}$is a subspace of $\mathbb{R}^{\mathbf{2}}$.

Solution:

i. $(0, 0) \in W$ so W$\neq \emptyset$

ii. $Let\ w_1 = (0, y_1)$ and $w_2 = (0, y_2)$ be in W. Then $w_1 + w_2 = (0, y_1 + y_2)$ for some $y_1, y_2 \in \mathbb{R}$ which implies $w_1 + w_2 \epsilon$ W.

iii. $Let\ \alpha \epsilon \mathbb{R}\ and\ w = (0, y) be\ in\ W.\ \alpha w = (0, \alpha y) \epsilon W$ for some $y \epsilon \mathbb{R}$.

Therefore, W is a subspace of $\mathbb{R}^{\mathbf{2}}$

Linear Combinations

Definition: Let $\boldsymbol{V}$ be a vector space. We say that a vector $\boldsymbol{v} \in \boldsymbol{V}$ is a linear combination of vectors $\boldsymbol{u_1},\ \boldsymbol{u_2}, \dots \dots \dots \boldsymbol{u_n}$if there exists $a_1,\ a_2, \dots \dots \dots a_n \in F$ such that:

$$\boldsymbol{v} = a_1\boldsymbol{u_1} + a_2\boldsymbol{u_2} + \dots \dots \dots + a_n\boldsymbol{u_n}$$

Example: Consider the vector space $\mathbb{R}^{\mathbf{2}}$.Let $\boldsymbol{v} = (8, 13), \boldsymbol{u_1} = (1, 2) and \boldsymbol{u_2} = (2, 3)$ then express $\boldsymbol{v}$ as a linear combination of $\boldsymbol{u_1} and \boldsymbol{u_2}$.

Solution: Let $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $v = \alpha_1 \boldsymbol{u_1} + \alpha_2 \boldsymbol{u_2}$

$$\begin{aligned}(8, 13) &= \alpha_1 (1, 2) + \alpha_2 (2,3)\\ &= (\alpha_1, 2\alpha_1) + (2\alpha_2, 3\alpha_2)\\ &= (\alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2)\end{aligned}$$

$$\begin{cases} \alpha_1 + 2\alpha_2 = 8 \dots\dots (1) \\ 2\alpha_1 + 3\alpha_2 = 13 \dots\dots (2) \end{cases}$$

Multiplying equation (1) by -2 and adding the result to equation (2) gives $\alpha_2 = 3\ and \alpha_1 = 2$.

Therefore, $\boldsymbol{v}$ **is the linear combination of** $\boldsymbol{u_1}$ $\boldsymbol{and}$ $\boldsymbol{u_2}$.

Example: Consider the space $P(t)$ (space of all polynomials). Let $\boldsymbol{v} = 5t^2 + 2t + 1$, and $u_1 = t^2 + t,\ u_2 = t + 1,\ u_3 = t^2 + 1$ then express $\boldsymbol{v}$ as a linear combination of $\boldsymbol{u_1},\ \boldsymbol{u_2} and \boldsymbol{u_3}$.

Solution: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$$5t^2 + 2t + 1 = \alpha_1 (t^2 + t) + \alpha_2 (t + 1) + \alpha_3 (t^2 + 1)$$

$$\begin{cases} \alpha_1 + \alpha_3 = 5 \dots\dots (1) \\ \alpha_1 + \alpha_2 = 2 \dots\dots\dots\dots (2) \\ \alpha_2 + \alpha_3 = 1 \dots\dots\dots\dots\dots (3) \end{cases}$$

Solving this system of linear equation simultaneously or by elimination method we get $\boldsymbol{\alpha_1 = 3}$, $\boldsymbol{\alpha_2 = -1\ and\ \ \alpha_3 = 2}$

Hence $\boldsymbol{v}$ **is the linear combination of** $\boldsymbol{u_1, u_2\ and\ u_3}$

Exercise :

1. Write each of the following as a linear combination of $x + 1, x^2 + x\ and\ x^2 + 2$
   a. $x^2 + 3x + 2$
   b. $2x^2 - 3x + 1$
   c. x

1.13. Linear dependence and independence

Definition: Let $\boldsymbol{V}$be a vector space over a field F, then a set of vectors $\{\boldsymbol{v_1},\ \boldsymbol{v_2}, \dots\dots\dots \boldsymbol{v_n}\}$ is called **linearly independent** if there exist $a_1,\ a_2, \dots\dots\dots a_n \in F$ such that $a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2} + \dots\dots\dots + a_n\boldsymbol{v_n} = \boldsymbol{0}\ such\ that\ \ a_1 = a_2 = \cdots = a_n = 0$.

A set of vectors that is not linearly independent is said to be linearly dependent.

Example: Show that $\{(1,0,-1), (2,1,2), (3,-2,0)\}$ is linearly independent in $\mathbb{R}^3$ over the field $\mathbb{R}$.

Solution: Let $\alpha_1\ \alpha_2\ , \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1 (1,0,-1) + \alpha_2 (2,1,2) + \alpha_3 (3,-2,0) = (0,0,0) .$$

$$\begin{cases} \alpha_1 + 2\alpha_1 + 3\alpha_1 = 0 \dots\dots\dots (1) \\ \alpha_1 - 2\alpha_1 = 0 \ \dots\dots\dots (2) \\ -\alpha_1 + 2\alpha_1 = 0 \dots\dots\dots\dots (3) \end{cases}$$

Solving this system of linear equation by elimination method we get $\alpha_1 = 0$, $\alpha_1 = 0$ *and* $\alpha_1 = 0$

Thus **{(1, 0,-1), (2, 1, 2), (3,-2, 0)} is linearly independent in** $\mathbb{R}^3$

Example: Show that $\{1 + x,\ 3x + x^2,\ 2 + x - x^2\}$ is linearly independent in $\boldsymbol{P_2}(x)$ over $\mathbb{R}$**.**

Solution: Let $\alpha_1 , \alpha_2 , \alpha_3 \boldsymbol{\epsilon} \mathbb{R}$ such that $\alpha_1 (x+1) + \alpha_2 (3x + x^2) + \alpha_3 (2 + x - x^2) = 0x^2 + 0x + 0$

$$(\alpha_2 - \alpha_3)x^2 + (\alpha_1 + 3 \alpha_2 + \alpha_3)x + (\alpha_1 + 2 \alpha_3) = 0x^2 + 0x + 0$$

$$\begin{cases} \alpha_2 - \alpha_3 = 0 \ldots\ldots\ldots\ldots\ldots (1) \\ \alpha_1 + 3 \alpha_2 + \alpha_3 = 0 \ldots\ldots (2) \\ \alpha_1 + 2\alpha_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3) \end{cases}$$

by comparing the coefficients of x^2, x *and* the constant terms at the left and right hand-sides. Solving this we get

$$\alpha_1 = 0, \alpha_2 = 0 \text{ and } \alpha_3 = 0 \boldsymbol{.}$$

Therefore**,** $\{\boldsymbol{1 + x},\ \boldsymbol{3x + x^2},\ \boldsymbol{2 + x - x^2}\}$ is linearly independent in $\boldsymbol{P_2}(x)$ over $\mathbb{R}$**.**

Exercise

1. suppose that $\{ \boldsymbol{u_1},\ \boldsymbol{u_2}\}$ is linearly independent set in a vector space $\boldsymbol{V}$.

   Show that $\{ \boldsymbol{u_1} + \boldsymbol{u_2},\ \boldsymbol{u_1} - \boldsymbol{u_2}\}$ is linearly independent.
2. Let $\{(3,-5,0), (5,0,1), (8,-5,1)\}$ be vectors in $\mathbb{R}^3$ then show that it is linearly independent.
3. Verify whether the following subsets of the vector space $V = \mathbb{R}^3$ are linearly independent or dependent.
   a. $W = \{(1,-1,0), (3,2,-1), (3,5,-2)\}$
   b. $U = \{(1,1,1), (1,-1,1), (0,0,1)\}$

1.14. Spanning sets

Definition: Let $\boldsymbol{V}$ be a vector space. The set of vectors $\{\boldsymbol{v_1},\ \boldsymbol{v_2}, \ldots\ldots\ldots \boldsymbol{v_n}\}$ is called **Spanning set** of $\boldsymbol{V}$ if every element of $\boldsymbol{V}$ is a linear combination of $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots\ldots\ldots \boldsymbol{v_n}\}$.

In this case the space $\boldsymbol{V}$ is called a **span** of the vectors and it is denoted by $\boldsymbol{V} =< \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots\ldots\ldots \boldsymbol{v_n} >$

Example: Consider the vector space $\mathbb{R}^3$. Then the vectors $\boldsymbol{v_1} = (1,0,0)$, $\boldsymbol{v_2} = (0,1,0)$and $\boldsymbol{v_3} = (0,0,1)$ form a spanning set of $\mathbb{R}^3$.

Solution: Let $v = (a,b,c) \epsilon\ \mathbb{R}^3$. Then $\boldsymbol{v} = (a,b,c) = (a,0,0) + (0,b,0) + (0,0,c)$

$$= a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$$= a\boldsymbol{v_1} + b\boldsymbol{v_2} + c\,\boldsymbol{v_3}$$

Therefore, $\{\boldsymbol{v_1} = (\mathbf{1},\mathbf{0},\mathbf{0}),\ \boldsymbol{v_2} = (\mathbf{0},\mathbf{1},\mathbf{0}), \boldsymbol{v_3} = (\mathbf{0},\mathbf{0},\mathbf{1})\ \}$ **is the spanning set of** $\mathbb{R}^3$.

Example: Consider the vector space $\mathbb{R}^3$; then the vectors $\boldsymbol{v_1} = (1,1,1)$, $\boldsymbol{v_2} = (1,1,0)$and $\boldsymbol{v_3} = (1,0,0)$ form a spanning set of $\mathbb{R}^3$.

Solution: Consider the vector space $\mathbb{R}^3$; then show that the vectors $\boldsymbol{v_1} = (1,1,1)$, $\boldsymbol{v_2} = (1,1,0)$and $\boldsymbol{v_3} = (0,1,1)$ form a spanning set of $\mathbb{R}^3$.

Definition: A set $\{\boldsymbol{v_1},\ \boldsymbol{v_2}, \dots\dots\dots \boldsymbol{v_n}\}$ of vectors in a vector space $\boldsymbol{V}$ is called a **basis** of $\boldsymbol{V}$ if it satisfies the following two conditions:

1. $\{\boldsymbol{v_1},\ \boldsymbol{v_2}, \dots\dots\dots \boldsymbol{v_n}\}$ is linearly independent
2. $\{\boldsymbol{v_1},\ \boldsymbol{v_2}, \dots\dots\dots \boldsymbol{v_n}\}$ is the spanning set of V

Example: For the vector space $\mathbb{R}^2$, the set of vectors $\boldsymbol{v_1} = (1,0) and \boldsymbol{v_2} = (0,1)$ is a **basis** of $\mathbb{R}^2$.

Example: Let $S = \{(1,1,1),\ (1,1,0),\ (1,0,0)\}$ .show that S forms a Basis for $\mathbb{R}^3$.

Solution:

i. Let $\alpha_1, \alpha_2, \alpha_3 \epsilon\mathbb{R}$. Such that $\boldsymbol{\alpha_1}\,(\mathbf{1},\mathbf{1},\mathbf{1}) + \boldsymbol{\alpha_2}\,(\mathbf{1},\mathbf{1},\mathbf{0}) + \boldsymbol{\alpha_3}\,(\mathbf{1},\mathbf{0},\mathbf{0}) = (\mathbf{0},\mathbf{0},\mathbf{0})$

   This implies $(\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1) = (0,0,0)$

   $\begin{cases} \boldsymbol{\alpha_1} + \boldsymbol{\alpha_2} + \boldsymbol{\alpha_3} = \mathbf{0} \\ \boldsymbol{\alpha_1} + \boldsymbol{\alpha_2} = \mathbf{0} \\ \boldsymbol{\alpha_1} = \mathbf{0} \end{cases}$ which gives $\boldsymbol{\alpha_1} = \mathbf{0}$, $\boldsymbol{\alpha_2} = \mathbf{0}, \boldsymbol{\alpha_3} = \mathbf{0}$. Thus $(1,1,1)$, $(1,1,0) and\ (1,0,0)\ are\ linearly\ independent.$

ii. **Let** $\boldsymbol{v} = (a,b,c)\ \boldsymbol{\epsilon}\ \mathbb{R}^3$ such that $\boldsymbol{v} = \boldsymbol{\alpha_1}\,\boldsymbol{v_1} + \boldsymbol{\alpha_2}\,\boldsymbol{v_2} + \boldsymbol{\alpha_3}\,\boldsymbol{v_3}$ for $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,0)$

   and $\alpha_1, \alpha_2, \alpha_3 \epsilon\mathbb{R}$..

   $(a,b,c) = \alpha_1\,(1,1,1) + \alpha_2\,(1,1,0) + \alpha_3\,(1,0,0) = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1)$

   $\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = a \dots..(1) \\ \alpha_1 + \alpha_2 = b \dots\dots.(2) \\ \alpha_1 = c \dots\dots\dots\dots(3) \end{cases}$ .This implies $\alpha_1 = c, \alpha_2 = b - c, \alpha_3 = b$ **.**

   So, $S = \{(1,1,1),\ (1,1,0),\ (1,0,0)\}$ is a spanning set of $\mathbb{R}^3$.

Therefore, $S = \{(1,1,1),\ (1,1,0),\ (1,0,0)\}$ is a basis of $\mathbb{R}^3$.

Example: Let $S = \{(1,0,0),\ (0,1,0) and\ \ (1,0,0)\}$, then show that S forms a Basis for $\mathbb{R}^3$.

Solution:

a. **Let** $\alpha_1, \alpha_2, \alpha_3 \epsilon \mathbb{R}$ such that $\alpha_1\,(1,0,0) + \alpha_2\,(0,1,0) + \alpha_3\,(1,0,0) = (0,0,0)$

$\Rightarrow (\alpha_1 + \alpha_3, \alpha_2, 0) = (0,0,0)$. **This implies** $\boldsymbol{\alpha_1 + \alpha_3 = 0}$ such that $\boldsymbol{\alpha_1\ and\ \ \alpha_3}$ **may have value other than zero.**

Hence no need of verifying whether it spans or not so that $S = \{(1,0,0),\ (0,1,0),\ (1,0,0)\}$ is not a basis of $\mathbb{R}^3$.

Exercises:

1. Show that $B = \{(1,-1,1), (2,1,-1), (-1,2,1)\}$ is a basis for the vector space $\mathbb{R}^3$ over a field $\mathbb{R}$

Definition: If a vector space $\boldsymbol{V}$ has a Basis of n elements; then n is called **dimension** of $\boldsymbol{V}$and written as $dim\boldsymbol{V} = n$

Example: $dim\mathbb{R}^{\mathbf{3}} = 3,\ dim\mathbb{R}^{\mathbf{2}} = 2\ and\ dim\mathbb{R}^{\boldsymbol{n}} = n, etc.$

Miscellaneous Exercises

1. Given that **a** = 2**i** + 3**j** − **k**, **b** = **i** − **j** + 2**k**, and **c** = 3**i** + 4**j** + **k**, find (a) **a** + 2**b** − **c**, (b) a vector **d** such that **a** + **b** + **c** + **d** = **0**, and (c) a vector **d** such that **a** − **b** + **c** + 3**d** = **0**.
2. Given **a** = **i** + 2**j** + 3**k**, **b** = 2**i** − 2**j** + **k**, find (a) a vector **c** such that 2**a** + **b** + 2**c** = **i** + **k**, (b) a vector **c** such that
   a − 2**b** + **c** = **i** + **j** − 2**k**
3. Given that **a** = 3**i** + 2**j** − 3**k**, **b** = 2**i** − **j** + 5**k**, and **c** = 2**i** + 5**j** + 2**k**, find (a) 2**a** + 3**b** − 3**c**, (b) a vector **d** such that
   a + 3**b** − 2**c** + 3**d** =**0**, and (c) a vector **d** such that 2**a** − 3**d** = **b** + 4**c**.
4. Given that *A* and *B* have the respective position vectors 2**i** + 3**j** − **k** and **i** + 2**j** + 4**k**, find the vector *AB* and unit vector in the direction of *AB*.
5. Given that *A* and *B* have the respective position vectors 3**i** − **j** + 4**k** and 2**i** + **j** + **k**, find the vector *AB* and the position vector **c** of the mid-point of *AB*.
6. Given that *A* and *B* have the respective position vectors **a** and **b**, find the position vector of a point *P* on the line *AB* located between *A* and *B* such that (length *AP*)/(length *PB*) = *m/n*, where *m, n* > 0 are any two real numbers.

7. Find the strength of the magnetic field vector **H** = 5**i** + 3**j** + 7**k** in the direction of 2**i** − **j** + 2**k**, where a unit vector represents one unit of magnetic flux. Ans 7
8. Find the distance of point P from the origin given that its position vector is **r** =2**i** + 4**j** − 3**k**. (b) If a general point P in space has position vector **r** = x**i** + y**j** + z**k**, describe the surface defined by _**r**_ = 3 and find its Cartesian equation.

   Ans a) $\|\boldsymbol{r}\|=\sqrt{29}$ b) $x^2+y^2+z^2=9$.

   Three points with position vectors **a**, **b**, and **c** will be **collinear** (lie on a line) if the parallelogram with adjacent sides **a** − **b** and

   a – **c** has zero geometrical area. Use this result in Exercises (9) through (11) to determine which sets of points are collinear.
9. (2, 2, 3), (6, 1, 5), (−2, 4, 3).
10. (1, 2, 4), (7, 0, 8), (−8, 5, −2).
11. (2, 3, 3), (3, 7, 5), (0,−5,−1)
12. (1, 3, 2), (4, 2, 1), (1, 0, 2).
13. The volume of a tetrahedron is one-third of the product of the area of its base and its vertical height. Show the volume V of the tetrahedron in Fig. 2.22, in which three edges formed by the vectors **a**, **b**, and **c** are directed away from a vertex, is given by

   $V = (1/6)|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$

Chapter Two

2. Matrices and Determinants

INTRODUCTION: Information in science and mathematics is often organized into rows and columns to form rectangular arrays, called "matrices" (plural of "matrix"). Matrices are often tables of numerical data that arise from physical observations, but they also occur in various mathematical contexts. For example, we shall see in this chapter that to solve a system of equations such as

$$\begin{array}{rcrcl} x & + & 2y & = & 5 \\ 3x & - & y & = & 1 \end{array}$$

all of the information required for the solution is embodied in the matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & -1 & 1 \end{bmatrix}$$

and that the solution can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs to solve systems of linear equations because computers are well suited for manipulating arrays of numerical information. However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a wide variety of applications. In this chapter we will begin the study of matrices.

2.1. Definition of Matrix

Definition:-A **matrix** is a rectangular array of numbers or variables, which we will enclose in brackets. The numbers (or variables) are called **entries** or, less commonly, *elements* of the matrix. The horizontal lines of entries are called **rows,** and the vertical lines of entries are called **columns**. A matrix with $\boldsymbol{m}$ rows and $\boldsymbol{n}$ columns has the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{Or} A = [a_{ij}], \text{ where } i = 1, 2, \ldots, m \text{ and } j = 1,2, \ldots, n$$

By an $\boldsymbol{m \times n}$**matrix** (read as"$\boldsymbol{m}$ *by* $\boldsymbol{n}$ *matrix"*) we mean a matrix with $\boldsymbol{m}$rows and $\boldsymbol{n}$columns—rows always come first! is called the **size/order/shape/dimension** of the matrix. We shall denote matrices by capital boldface letters **A**, **B**, **C**, …, or by writing the general entry in brackets; like $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$, and so on..The element $\boldsymbol{a_{ij}}$,is called the $\boldsymbol{ij}$ entry, appears in row $\boldsymbol{i}$ and column $\boldsymbol{j}$.Thus $\boldsymbol{a_{21}}$ is the entry in Row **2** and Column **1**.

Example: Consider the following matrices

$$A = \begin{bmatrix} 1 & 4 & -12 \\ 5 & -1 & 109 \\ -1 & -8 & 7\ 5 \end{bmatrix} B = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}$$

The dimension/size of matrix A is 3×4 and the dimension/size of matrix B is 3×3. The entry a_{33} in matrix A is 7 and the entry a_{24} in matrix A is 9.

NB: Matrices are important because they let us to express large amounts of data and functions in an organized and concise form.

Example: Given a matrix A = $\begin{bmatrix} 3 & 4 & 2 & -3 \\ 1 & -1 & 10 & 9 \\ -10 & -8 & 7 & 5 \end{bmatrix}$, then

a) Find the size of A

b) List columns of A

c) List rows of A

d) List elements of A

Exercise:

1. Let t be a real number. Assume that B =$[t]$. Then, determine:

   a) The size of B

   b) The rows of B

   c) the no of columns of B

   d) the elements B

2. Construct a 3x3 matrix whose a_{ij} entry is given by 2j – i

2.2. Types of Matrices

In matrix theory, there are many special kinds of matrices that are important because they possess certain properties. The following is a list of some of these matrices.

i. **Zero Matrix (Null Matrix):** Matrix that consists of all zero entries is called a **zero matrix** and is denoted by bold zero,**0 or**$\mathbf{0}_{mxn}$.

   Example: $0_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

ii. **Square Matrix:** any mxn matrix is called square matrix if $m = n$. The order of nxn square matrix is nxn or simply n.

   Example:$A_{3x3} = \begin{bmatrix} 1 & 5 & -3 \\ 2 & 0 & -2 \\ 2 & 3 & 7 \end{bmatrix}$

iii. **Rectangular Matrix:** A matrix of any size is called a **rectangular matrix**; this includes square matrices as a special case.

iv. **Row Matrix (row vector):** A $1 \times n$ matrix is called a row matrix (row vector).

$$A = [a_{11}\ a_{12}\ \dots a_{1n}\ ] = (a_1, a_2, \dots, a_n)$$

v. **Column Matrix (column vector):** A $nx1$ matrix is called a column matrix (column vector)

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{pmatrix}$$

vi. **Diagonal Matrix:** A $n \times n$ square matrix D is said to be a **diagonal matrix** if all its entries except the main diagonal entries are zeros.

$$D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

Remark: A diagonal matrix each of whose diagonal elements are equal is called a **scalar matrix**.

vii. **Identity Matrix (Unit Matrix):** is a diagonal matrix in which all diagonal elements are 1.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$ is an $n \times n$ identity matrix.

viii. **Triangular Matrix:** A square matrix in which all the entries above the main diagonal are zero is called ***lower triangular***, and a square matrix in which all the entries below the main diagonal are zero is called ***upper triangular***. A matrix that is either upper triangular or lower triangular is called ***triangular***.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

per Triangular ***Lower Triangular***

<u>**Remark**</u>**:** Observe that diagonal matrices are both upper triangular and lower triangular since they have zeros below and above the main diagonal.

Example: $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{4} & \mathbf{3} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} \end{bmatrix}$ $\qquad$ $\boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{-5} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{5} \\ \mathbf{0} & \mathbf{0} & \mathbf{-2} \end{bmatrix}$

Lower triangular matrix $\qquad$ Upper triangular matrix

2.3. Basic Operations on Matrices

Equality of Matrices

Definition: Two matrices are defined to be ***equal***, denoted by **A = B**,if they have the same size and their corresponding entries are equal.

Example: Find the value of x and y if matrix A is equal to matrix$B$.

$$A = \begin{bmatrix} x+1 & 3 \\ 0 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 3 \\ 0 & y-2 \end{bmatrix}$$

Solution: the two matrices are equal if and only if $x = 3$ and$y = 8$.

Matrix Addition and Subtraction

Definition: Let A = $[a_{ij}]$and B = $[b_{ij}]$be two matrices of the same size, say, $m \times n$ matrices. Then the ***sum of A and B, written A + B,*** is the matrix obtained by adding the entries of *B* to the corresponding entries of *A*. The ***difference A – B*** is the matrix obtained by subtracting the entries of *B* from the corresponding entries of *A*. In matrix notation, if A = $[a_{ij}]$and B = $[b_{ij}]$have the same size, then

$$A + B = [a]_{ij} + [b]_{ij} = [a_{ij} + b_{ij}] \; and \; A - B = [a_{ij}]_{m\times n} - [b_{ij}]_{m\times n} = [a_{ij} - b_{ij}]_{m\times n}$$

Example: Consider the following matrices

$$A = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 5 \\ 9 & -2 & 0 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 6 & -1 & 3 \\ 2 & 5 & 0 & -4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Then find$A + B \; and \; A - B$

Solution: $A + B = \begin{bmatrix} -1+1 & 2+6 & 3-1 & 4+3 \\ 1+2 & 0+5 & 2+0 & 5-4 \\ 9+3 & -2+4 & 0+1 & 6+2 \end{bmatrix} = \begin{bmatrix} 0 & 8 & 2 & 7 \\ 3 & 5 & 2 & 1 \\ 12 & 2 & 1 & 8 \end{bmatrix}$

$$A - B = \begin{bmatrix} -1-1 & 2-6 & 3+1 & 4-3 \\ 1-2 & 0-5 & 2-0 & 5+4 \\ 9-3 & -2-4 & 0-1 & 6-2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 4 & 1 \\ -1 & -5 & 2 & 9 \\ 6 & -6 & -1 & 4 \end{bmatrix}$$

Remark: Matrices of different sizes cannot be added or subtracted.

Scalar Multiplication of Matrix

Definition: If *A* is any matrix and *c* is any scalar; then the ***product cA*** is the matrix obtained by multiplying each entry of the matrix *A* by *c*. The matrix *cA* is said to be a ***scalar multiple*** of *A*.i.e.

$$cA = c\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix} = [ca_{ij}]_{mxn}$$

Example: Let$A = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 5 \\ 9 & -2 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 & -1 & 3 \\ 2 & 5 & 0 & -4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$. Then compute the following

a. $2A - B$

b. $\frac{1}{3}A + 2A$

Solution: a. $2A - B = \begin{bmatrix} -2 & 4 & 6 & 8 \\ 2 & 0 & 4 & 10 \\ 18 & -4 & 0 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 6 & -1 & 3 \\ 2 & 5 & 0 & -4 \\ 3 & 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 7 & 5 \\ 0 & -5 & 4 & 14 \\ 15 & -8 & -1 & 10 \end{bmatrix}$

$$\text{b.}\frac{1}{3}A + 2A = \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} & 1 & \frac{4}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & \frac{5}{3} \\ 3 & \frac{-2}{3} & 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 6 & 8 \\ 2 & 0 & 4 & 10 \\ 18 & -4 & 0 & 12 \end{bmatrix} = \begin{bmatrix} \frac{-7}{3} & \frac{14}{3} & 7 & \frac{28}{3} \\ \frac{7}{3} & 0 & \frac{14}{3} & \frac{35}{3} \\ 21 & \frac{-14}{3} & 0 & 14 \end{bmatrix}$$

Exercise:

1. Find the values of x and $y$for the following matrix equation.

$$2\begin{bmatrix} x+2 & y+3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & y \\ 6 & z \end{bmatrix}$$

2. Find matrix A if $2A = \begin{bmatrix} 2 & -1 \\ 6 & 0 \end{bmatrix}$

Remark: If $A_1, A_2, \ldots, A_n$ are matrices of the same size and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form$c_1A_1 + c_2A_2 + \cdots + c_nA_n$is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with ***coefficients***$c_1, c_2, \ldots, c_n$.

Properties of Matrix Addition and Scalar Multiplication

Suppose A, B, and C are $m \times n$matrices (having the same size) and α and β are scalars. Then

i. $A + B = B + A$(commutative law of addition)

ii. $(A + B) + C = A + (B + C)$(Associative law of addition)

iii. $A + \mathbf{0} = \mathbf{0} + A$ (Existence of additive identity)

iv. $A + (-A) = (-A) + A = \mathbf{0}$(Existence of additive inverse)

v. $\alpha(A + B) = \alpha A + \alpha B$

vi. $(\alpha + \beta)A = \alpha A + \beta A$

vii. $(\alpha(\beta A)) = \alpha(\beta A)$

viii. $1A = A$ and $0A = \mathbf{0}$

2.4. Product of Matrices

Definition: Let matrix A be an $m \times n$ matrix and B be an $n \times k$ matrix (i.e. the number of columns of A is equal to the number of rows of B). Then the product of A and B, denoted by AB, is an $m \times k$ matrix which is obtained by multiplying the corresponding elements of row i of A by column j of B and adding the product, i.e. if $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{jk}\right]_{n\times k}$, then $AB = C = [c_{ik}]_{m\times k}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$, for $i = 1,2, \dots, m$

- Consider the following matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{bmatrix}$$

Rows of A are:

$$\begin{aligned} r_1 &= [a_{11}a_{12} \dots a_{1n}] \\ r_2 &= [a_{11}a_{12} \dots a_{1n}] \\ &\vdots \\ r_m &= [a_{m1}a_{m2} \dots a_{mn}] \end{aligned}$$

Columns of B are:

$$\begin{aligned} c_1 &= [b_{11}b_{21} \dots b_{n1}] \\ c_2 &= [b_{12}b_{22} \dots b_{n2}] \\ &\vdots \\ c_k &= [b_{1k}b_{2k} \dots b_{nk}] \end{aligned}$$

Then

$$AB = \begin{bmatrix} r_1.c_1 & r_1.c_2 & \cdots & r_1.c_k \\ r_2.c_1 & r_2.c_2 & \cdots & r_2.c_k \\ \vdots & \vdots & \ddots & \vdots \\ r_m.c_1 & r_m.c_2 & \cdots & r_m.c_k \end{bmatrix} = [c_{ik}]_{m\times k}$$

Example: Find the product of the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Solution:

$$\text{AB} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Exercise: Determine the size of the product matrix AB if the sizes of A and B are 4×5 and 5×7 respectively.

Note: The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.

Properties of Matrix Multiplication

1) $AB \neq BA$, (matrix product is not commutative.)
2) $A(BC) = (AB)C$, (matrix multiplication is associative)
3) $A(B+C) = (AB + AC)\ and\ (B+C)A = BA + CA$, (multiplication of matrices is distributive with respect to addition)
4) If $AB = \mathbf{0}$, it does not mean that either A = **0** or B = **0**.

Example: For matrix A and B given by $A = \begin{bmatrix} 10 \\ 10 \end{bmatrix} B = \begin{bmatrix} 00 \\ 11 \end{bmatrix}$ we have

$AB = \begin{bmatrix} 00 \\ 00 \end{bmatrix}$ is a null matrix even though A and B are not a null matrix.

5) The relation $AB = AC\ or\ BA = CA$ does not imply that $B = C$. In other words the cancelation law doesn't hold as for real numbers.

Example: if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

We have, $AB = \begin{bmatrix} 9 & 10 & 7 \\ 6 & 7 & 6 \\ 9 & 8 & -1 \end{bmatrix} = AC$, but $B \neq C$

Exercise: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, then verify that:

a) $\mathrm{A(BC) = (AB)C}$ b) $\mathrm{A(B + C) = AB + AC}$

Transpose of a matrix

Definition: If A is an $m \times n$ matrix, then the ***transpose of A***, denoted by $\boldsymbol{A}^T$, is defined to be the matrix $n \times m$ that results from interchanging the rows and columns of A.

Remark: The transpose of a row matrix is column matrix and the transpose of a column matrix is a row matrix.

Example: The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

Properties of Matrix Transpose

1. $(A + B)^T = A^T + B^T$
2. $(A^T)^T = A$
3. $(AB)^T = B^T A^T$

Proof: Ex.

Note: The transpose of a lower triangular matrix is upper triangular and the transpose of an upper triangular matrix is lower triangular.

Orthogonal Matrix: An **orthogonal** matrix **A** is a matrix such that $AA^T = A^T A =$ **I**. A typical orthogonal matrix is:

$$A = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

Trace of Matrix

Definition: If A is a square matrix, then the ***trace of A***, denoted by $tr(A)$, is defined to be the sum of the entries on the diagonal of A. The trace of A is undefined if A is not a square matrix.

Example: The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33}$ $\qquad$ $\operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$

Polynomial of Matrix

For any $n \times n$ square matrix A and for any polynomial,

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_o$ where a_i are scalars, we define $f(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_o I$. If $f(A) = (0_{ij})$, then A is a zero (root) of the polynomial.

Examples: Let $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}$ and let $f(x) = 2x^2 + x + 3$. Then compute $f(A)$.

Exercise:

1. Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$. Then compute:

a) $f(A)\ if\ f(x) = 3x^3 - 4x^2 + 2x$ $\qquad$ b) $f(-3A)\ if\ f(x) = 3x^2 - 2x$

Symmetric and Skew-Symmetric Matrices:

Definition: Let A be a square matrix. Then $A$is said to be

- Symmetric if $A = A^T$,i.e. $a_{ij} = a_{ji}$.
- Skew-symmetric (anti-symmetric) if $A^T = -A$, i.e.$a_{ij} = -a_{ji}$.

Example: A = $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$ B = $\begin{bmatrix} 0 & 5 & -2 \\ -5 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ C = $\begin{bmatrix} 0 & 0 & 1 \\ -5 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

Symmetric Skew-Symmetric Neither

Exercise: Determine whether the following matrices are symmetric, skew symmetric or neither.

a) $A = \begin{bmatrix} 3 & b \\ b & 3 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $B = \begin{bmatrix} 0 & 5 & -4 & 3 \\ -5 & 0 & -7 & -2 \\ 4 & 7 & 0 & 1 \\ -3 & 2 & 1 & 0 \end{bmatrix}$

d) $D = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$

e) $E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.5. Elementary Row Operations and Echelon Form of Matrices

i. Elementary Row Operations

A matrix $\boldsymbol{A}$ is said to be *row equivalent* to matrix $\boldsymbol{B}$, written $\boldsymbol{A} \sim \boldsymbol{B}$ if matrix $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by a finite sequence of ***elementary row operations***. These *elementary row operations* are:

i. Interchanging the i^{th} row by the j^{th} row (i.e $R_i \leftrightarrow R_j$)

ii. Multiplying the i^{th} row by a none zero scalar (i.e$R_i \to kR_i$).

iii. Replacing the i^{th}row by k times the j^{th} row plus i^{th} row (i.e$R_i \to kR_j + R_i$)

Example: Apply all elementary row operations on the given matrix:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & -4 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 5 & 7 & -4 \\ 1 & 2 & -1 \\ 2 & 6 & 1 \end{bmatrix} R_2 \to 2R_2 \begin{bmatrix} 2 & 6 & 1 \\ 2 & 4 & -2 \\ 5 & 7 & -4 \end{bmatrix} R_2 \to 2R_3 + R_2 \begin{bmatrix} 2 & 6 & 1 \\ 11 & 16 & -9 \\ 5 & 7 & -4 \end{bmatrix} = \boldsymbol{B}$$

Exercise: Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 2 \\ -1 & -2 & 7 \end{bmatrix}$. Then find matrix $\boldsymbol{B}$ which is row equivalent to $\boldsymbol{A}$ with $\boldsymbol{a_{11}} = \boldsymbol{0}$ and$\boldsymbol{a_{32}} = \boldsymbol{0}.$

Remark: The first non-zero entry in a row is called the ***leading entry*** (***pivotal entry***) of that row.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & -1 \\ -4 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Entries $2, -2$ and -4 are leading entries of $rows$ $1, 2, 3$ respectively, and no leading entry for row 3.

ii. Echelon Form of a Matrix

Definition: An $m \times n$ matrix $A$is said to be in ***echelon form (EF)*** provided the following two conditions hold.

1. Any zero rows (if there is) are at the bottom of the non-zero rows.
2. The leading entry of all rows is at the right side of the leading entries of the above rows (i.e. all entries below the leading entry are zero).

Example: $\boldsymbol{A} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 2 & 6 & 1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix}$ $\boldsymbol{C} = \begin{bmatrix} 2 & 6 & 9 & 1 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ $\boldsymbol{D} = \begin{bmatrix} 2 & 6 & 9 \\ 0 & 1 & 7 \\ 0 & 2 & 8 \end{bmatrix}$

Matrices **A, B**and **C** are in echelon form, but matrix **D** is not.

Definition: An $m \times n$ matrix $A$is said to be in ***row-echelon form (REF)*** provided the following two conditions hold.

1. The matrix is in ***echelon form.***
2. All leading entries are equal to 1.

Example: $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 1 & 6 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix}$ $\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\boldsymbol{D} = \begin{bmatrix} 1 & 6 & 9 \\ 0 & 1 & 7 \\ 0 & 1 & 8 \end{bmatrix}$

Matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are in row echelon form **(REF)** but $\boldsymbol{D}$ is not.

Definition$_3$: An $m \times n$ matrix $A$is said to be in ***reduced row-echelon form (RREF)*** provided the following two conditions hold.

1. The matrix is in ***row echelon form.***
2. All non-leading entries in a *column*, which contains the leading entries, are equal to 0.

Example: $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\boldsymbol{C} = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\boldsymbol{D} = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Remark: Any matrix can be reduced to its echelon form by applying some elementary row operations on the given matrix.

Example: Reduce matrix $\boldsymbol{A}$ to its row echelon form by applying elementary row operations where

$$A = \begin{bmatrix} 3 & -10 & 5 \\ -1 & 12 & 2 \\ 1 & -5 & 2 \end{bmatrix}$$

Solution: Applying $R_1 \leftrightarrow R_3$ we have: $\begin{bmatrix} 1 & -5 & 2 \\ -1 & 12 & 2 \\ 3 & -10 & 5 \end{bmatrix}$

Applying $R_2 \to R_1 + R_2$ and $R_3 \to R_3 - 3R_1$ we get: $\begin{bmatrix} 1 & -5 & 2 \\ 0 & 7 & 4 \\ 0 & 5 & -1 \end{bmatrix}$

Applying $R_2 \to \frac{R_2}{7}$ we get: $\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 4/7 \\ 0 & 5 & -1 \end{bmatrix}$

Applying $R_3 \to R_3 - 5R_2$ we get: $\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 4/7 \\ 0 & 0 & -27/7 \end{bmatrix}$

Appling $R_3 \to -{}^{7}/_{27}\, R_3$ we get: $\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 4/7 \\ 0 & 0 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 - \frac{4R_3}{7}$ we get: $\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This is in row echelon form.

Rank of a Matrix

Suppose an $\boldsymbol{m} \times \boldsymbol{n}$ matrix **A**is reduced by row operations to an echelon form $\boldsymbol{E}$. Then the rank of $\boldsymbol{A}$**,** denoted by$\boldsymbol{rank}(\boldsymbol{A})$**,** is defined to be,

Rank($\boldsymbol{A}$) = number of pivots (leading entries) or

= number of nonzero rows in $\boldsymbol{E}$ or

= number of basic columns in $\boldsymbol{A}$,

where the basic columns of ***A***are defined to be those columns in ***A***which contain the ***pivotal*** positions.

Example: Let matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 6 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then the***rank (A)=3*** since matrix $\boldsymbol{A}$ is in echelon form and has three non-zero rows.

Example: Determine the rank, and identify the basic columns in the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$rank(A) = \mathbf{2}$ and

Basic Columns $= \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}, \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{4} \end{pmatrix} \right\}$

Exercise: Reduce each of the following matrices to its echelon form and determine its rank & identify the basic columns.

a. $A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{bmatrix}$

b. $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 2 & 6 & 0 \\ 1 & 2 & 5 \\ 3 & 8 & 6 \end{bmatrix}$

2.6. Inverse of a Matrix and Its Properties

Definition: If A is an $n \times n$ square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, where I the $n \times n$identity matrix, then A is said to be ***invertible (non-singular)***and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

Example: The matrix

$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 2 & -5 \\ -\mathbf{1} & 3 \end{bmatrix}$

Since

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

And

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Remark:

1. $A^{-1} \neq \frac{1}{A}$
2. Matrices which are not square matrices have no inverses
3. Not all square matrices have inverse
4. $(A^{-1})^T = (A^T)^{-1}$

Properties of Inverse Matrices

1. If B and C are both inverses of a matrix A, then B = C. that is the inverse of a matrix is unique and is denoted by A^{-1}.
2. If A and B are invertible matrices of the same size, then
   a) AB is invertible and
   b) $(AB)^{-1} = B^{-1}A^{-1}$
3. A^{-1} is invertible and $(A^{-1})^{-1} = A$
4. Let $D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & d_n \end{bmatrix}$ be an $n \times n$ diagonal matrix where all $d_i \neq 0$ (for $i = 1, 2, \dots, n$).

   Then the inverse of D, denoted by D^{-1}, is given by $D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \dots & 0 \\ 0 & \frac{1}{d_2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \frac{1}{d_n} \end{bmatrix}$

Exercise: Verify that $DD^{-1} = D^{-1}D = I$.

Finding Inverse of a Matrix by Using Elementary Row Operations (Gauss-Jordan Elimination Method)

Let $\boldsymbol{A}$be an $n \times n$matrix and let $\boldsymbol{I_n}$ be an $n \times n$ identity matrix. Then to find the inverse of $\boldsymbol{A}$.

1. Adjoin the identity $n \times n$matrix $\boldsymbol{I_n}$ to $\boldsymbol{A}$to form the augmented matrix $(\boldsymbol{A}: \boldsymbol{I_n})$.
2. Compute the reduced echelon form of $(\boldsymbol{A}: \boldsymbol{I_n})$. If the reduced echelon form is of the type $(\boldsymbol{I_n}: \boldsymbol{B})$, then B is the inverse of $\boldsymbol{A}$. If the reduced echelon form is not of the type $(\boldsymbol{I_n}: \boldsymbol{B})$, in that the first $n \times n$sub matrix is not $\boldsymbol{I_n}$, then $\boldsymbol{A}$has no inverse.

Example: Find the inverse of A if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution:$[A: I_3] = \begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 2 & 5 & 3 & \vdots & 0 & 1 & 0 \\ 1 & 0 & 8 & \vdots & 0 & 0 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ we get:

$$\begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -3 & \vdots & -2 & 1 & 0 \\ 1 & 0 & 8 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 - 2R_2$ we get:

$$\begin{bmatrix} 1 & 0 & 9 & \vdots & 5 & -2 & 0 \\ 0 & 1 & -3 & \vdots & -2 & 1 & 0 \\ 1 & 0 & 8 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_1$ we get:

$$\begin{bmatrix} 1 & 0 & 9 & \vdots & 5 & -2 & 0 \\ 0 & 1 & -3 & \vdots & -2 & 1 & 0 \\ 0 & 0 & -1 & \vdots & -5 & 2 & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 + 9R_3$; $R_2 \to R_2 - 3R_3$ and $R_3 \to -R_3$we get:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -40 & 16 & 9 \\ 0 & 1 & 0 & \vdots & -13 & -5 & -3 \\ 0 & 0 & 1 & \vdots & 5 & -2 & -1 \end{bmatrix}$$

Therefore, $B = \begin{bmatrix} -40 & 16 & 9 \\ -13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} = A^{-1}$

Example: Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ by using elementary row operations (Gauss-Jordan Method).

Solution: $[A \ \vdots\ I_3] = \begin{bmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 2 & -1 & 3 & \vdots & 0 & 1 & 0 \\ 4 & 1 & 8 & \vdots & 0 & 0 & 1 \end{bmatrix}$

Applying$R_2 \to R_2 - 2R_1$,we get:

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -1 & \vdots & -2 & 1 & 0 \\ 4 & 1 & 8 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

Applying$R_3 \to R_3 - 4R_1$we get:

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -1 & \vdots & -2 & 1 & 0 \\ 0 & 1 & 0 & \vdots & -4 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_2 + R_3$we get:

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -1 & \vdots & -2 & 1 & 0 \\ 0 & 0 & -1 & \vdots & -6 & 1 & 1 \end{bmatrix}$$

Applying$R_3 \to -R_3$ we get:

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -1 & \vdots & -2 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 6 & -1 & -1 \end{bmatrix}$$

Applying$R_1 \to R_1 - 2R_3$ and $R_2 \to R_2 + R_3$ we get:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -11 & 2 & 2 \\ 0 & -1 & 0 & \vdots & 4 & 0 & -1 \\ 0 & 0 & 1 & \vdots & 6 & -1 & -1 \end{bmatrix}$$

Applying $R_2 \to -R_2$ we get:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -11 & 2 & 2 \\ 0 & 1 & 0 & \vdots & -4 & 0 & 1 \\ 0 & 0 & 1 & \vdots & 6 & -1 & -1 \end{bmatrix}$$

$$[I_3 \vdots B] = \begin{bmatrix} 1 & 0 & 0 & \vdots & -11 & 2 & 2 \\ 0 & 1 & 0 & \vdots & -4 & 0 & 1 \\ 0 & 0 & 1 & \vdots & 6 & -1 & -1 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} = A^{-1}$$

Example: Determine the inverse of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

Solution: $$(A{:}\,\boldsymbol{I_3}) = \begin{bmatrix} 1 & -1 & -2 & \vdots & 1 & 0 & 0 \\ 2 & -3 & -5 & \vdots & 0 & 1 & 0 \\ -1 & 3 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

Appling $R_2 \to R_2 - 2R_1$& $R_3 \to R_3 + R_1$we get

$$\begin{bmatrix} 1 & -1 & -2 & \vdots & 1 & 0 & 0 \\ 0 & -1 & -1 & \vdots & -2 & 1 & 0 \\ 0 & 2 & 3 & \vdots & 1 & 0 & 1 \end{bmatrix}$$

Appling $R_2 \to -R_2$we get

$$\begin{bmatrix} 1 & -1 & -2 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 2 & -1 & 0 \\ 0 & 2 & 3 & \vdots & 1 & 0 & 1 \end{bmatrix}$$

Appling $R_1 \to R_2 + R_1$& $R_3 \to R_3 - 2R_2$we get

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 3 & -1 & 0 \\ 0 & 1 & 1 & \vdots & 2 & -1 & 0 \\ 0 & 0 & 1 & \vdots & -3 & 2 & 1 \end{bmatrix}$$

Appling $R_1 \to R_3 + R_1$& $R_2 \to R_2 - R_3$we get

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & 1 & 1 \\ 0 & 1 & 0 & \vdots & 5 & -3 & -1 \\ 0 & 0 & 1 & \vdots & -3 & 2 & 1 \end{bmatrix}$$ Thus $$\boldsymbol{A^{-1}} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

Exercise: Determine inverse of the following matrices, if it exists.

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & 1 \\ 5 & -2 & -3 \end{bmatrix} C = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} D = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

2.7. Determinant of a Matrix and Its Properties

Introduction: In this section, we shall study the "determinant function," which is a real-valued function of a matrix variable in the sense that it associates a real number $f(A)$ with a square matrix A. Our work on determinant functions will have important applications to the theory of systems of linear equations and will also lead us to an explicit formula for the inverse of an invertible matrix.

Definition: A *"determinant"* is a certain kind of function that associates a real number with a square matrix$\boldsymbol{A}$**,** and it is usually denoted by $\boldsymbol{det}(\boldsymbol{A})$ or$|\boldsymbol{A}|$. i.e. if $\boldsymbol{A}$ is an $n \times n$ square matrix, then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

a. Determinant of $\mathbf{1 \times 1}$ $\boldsymbol{M\ atrices}$

Let $A = [a_{11}]$be 1×1 matrix. Then $\det(A) = a_{11}$.

Example: Let $A = [2]$. Then $\det(A) = 2$

b. Determinant of $\mathbf{2 \times 2}$ $\boldsymbol{Matrices}$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a2×2 matrix. Then $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Example: Find $\det(A)$ if $A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$

Solution: $\det(A) = 1(0) - 2(-2) = 4$

c. Determinant of $\boldsymbol{n \times n}$ matrix

Definition-1: Let $\boldsymbol{A}$ be an $n \times n$square matrix and $\boldsymbol{M_{ij}}$ be the $(n-1) \times (n-1)$ matrix obtained from matrix $\boldsymbol{A}$ by deleting i^{th} row and j^{th} column containing the entry a_{ij}. Then $\mathbf{det}\,(\boldsymbol{M_{ij}})$ is called the *minor* of a_{ij}.

Remark: the matrix $\boldsymbol{M_{ij}}$ is called sub matrix of $\boldsymbol{A}$.

Definition-2: The cofactor of a_{ij}, denoted by C_{ij}, is defined as $C_{ij} = (-1)^{i+j}\det(M_{ij})$.

Example: Let $A = \begin{bmatrix} 2 & 5 & -4 \\ 3 & -1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$. Then find the minor and cofactor of a_{11} and a_{23}.

Solution: Minor of $a_{11} = \det(M_{11}) = \begin{vmatrix} -1 & 2 \\ 4 & 6 \end{vmatrix} = -1(6) - 2(4) = -14$

Cofactor of $a_{11} = C_{11} = (-1)^{1+1}\det(M_{11}) = -14$

Minor of $a_{23} = \det(M_{23}) = \begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} = 2(4) - 5(5) = -17$

Cofactor of $a_{23} = C_{23} = (-1)^{2+3}\det(M_{23}) = 17$

Definition-3: The determinant of an $n \times n$ matrix A is given by either of the following two formulas.

i. $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$, for fixed $i = 1,2,3,\ldots$

ii. $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$, for fixed $j = 1,2,3,\ldots$

Example: Evaluate the determinant for the matrices

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

Solution: Let us take row-1 of matrix A for i. Then

$$\mathbf{det}(\boldsymbol{A}) = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{3} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2(24) - 1(14) + 3(-13) = \mathbf{-5}$$

Let us take column-1 of matrix B for j.

$$\mathbf{det}(\boldsymbol{B}) = \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{4} a_{i1}C_{i1} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}$$

$$= 0C_{11} + 0C_{21} + 0C_{31} + 2(6) = \mathbf{12}$$

Example: Let $A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3)\ and\ C = (c_1, c_2, c_3)$ be vectors in R^3. Then show that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = A \cdot (B \times C)$

Solution*:* $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

$= (a_1, a_2, a_3) \cdot (b_2c_3 - b_3c_2,\ b_1c_3 - b_3c_1,\ b_1c_2 - b_2c_1)$

$= A \cdot (B \times C)$

Note: The *determinant* of a 3×3 square matrix $A = \begin{bmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{bmatrix}$ is given by

$$det(A) = \begin{vmatrix} a_{11} a_{12} a_{13} \\ a_{21} a_{22} a_{23} \\ a_{31} a_{32} a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} a_{23} \\ a_{32} a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} a_{23} \\ a_{31} a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} a_{22} \\ a_{31} a_{32} \end{vmatrix}$$

Properties of Determinants

Let ***A*, *B*** and **C** be $\boldsymbol{n \times n}$ square matrices and $\boldsymbol{k}$ be any scalars. Then

a. $\det(A) = \det(A^T)$

b. $\det(kA) = k^n \det(A)$, where n is the size of A.

c. $\det(A + B) \neq \det(A) + \det(B)$

d. $\det(AB) = \det(A)\det(B)$

e. $\det(A^{-1}) = \frac{1}{\det(A)}$, if A is invertible (non-singular) matrix.

f. Determinant of any $n \times n$ triangular matrix A is the product of its diagonal entries.

   i.e. $\det(A) = a_{11}a_{22} \dots a_{nn}$

g. Determinant of $n \times n$ diagonal matrix D is the product of its diagonal entries. Thus $\det(I_n) = 1$, where I_n is an $n \times n$ identity matrix.

h. If any rows (or columns) of a square matrix A are proportional to each other (i.e. one is the scalar multiple of the other), then $\det(A) = 0$.

i. If any row (or column) of a square matrix A is zero, then $\det(A) = 0$.

Exercise:

1. Evaluate the determinant for each of the following matrices.

$$A = \begin{bmatrix} 2 & 16 & 40 & 16 & 14 \\ 0 & -3 & 22 & -3 & -18 \\ 0 & 0 & 4 & 1 & 20 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ -8 & \sqrt{2} & 0 & 0 \\ 7 & 0 & -1 & 0 \\ 9 & 5 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & -3 & 4 & 5 \\ 4 & -5 & 6 & 5 & -2 \\ 7 & 8 & -8 & 6 & 20 \\ 3 & 6 & -9 & 12 & 15 \\ 0 & -2 & 3 & 4 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 6 \\ 3 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Let $\det(A) = 4$ and $\det(B) = -3$, and let A and B be a 5×5 matrices. Then find

   i. $\det(2A)$
   ii. $\det(A^T)$
   iii. $\det(AB)$
   iv. $\det(A^{-1})$

3. Let $A = \begin{bmatrix} x-1 & -2 \\ x-2 & x-1 \end{bmatrix}$ and let $\det(A) = 0$. Then find the value/s of x.

Theorem-1: Let A be an invertible matrix. Then $det(A^{-1}) = (det(A))^{-1}$.

Proof: $I = AA^{-1}$ then taking determinant on both sides we get:

$$det(I) = det(AA^{-1})$$

$$\Rightarrow 1 = det(A)det(A^{-1})$$

$$\Rightarrow det(A^{-1}) = \frac{1}{det(A)} = (det(A))^{-1}$$

Example: If $A = \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix}$, then $det(A^{-1}) = \frac{1}{det(A)} = \frac{1}{3}$.

<u>Note:</u> We can evaluate the determinant of any square matrix A by reducing it to its echelon form by keeping the following conditions.

a. If B is a matrix which results by applying the elementary row operation of multiplying any particular row of A by a non-zero constant k, then $\det(B) = kdet(A)$.

b. If B is a matrix that results when two rows or two columns of A are interchanged, then$\det(B) = -\det(A)$.

c. If B is a matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then $\det(B) = \det(A)$.

Exercise: Let$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ and let $\det(A) = -12$. The find $\det(B)$, $\det(C)$and $\det(D)$ if

$\boldsymbol{B} = \begin{bmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & k \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} a & b & c \\ g & h & k \\ d & e & f \end{bmatrix}$ and $D = \begin{bmatrix} a & b & c \\ d+2a & b+2b & f+2c \\ g & h & k \end{bmatrix}$

Adjoint and Inverse of a Matrix

Definition-1: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$be an$n \times n$square matrix and let c_{ij} be the cofactor of a_{ij}.Then the matrix $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$whose $(ij)^{th}$ entry is c_{ij} is called **matrix of cofactors** of entries of A and its transpose $C^T = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$is called the **adjoin**t **of** $\boldsymbol{A}$, denoted by $\boldsymbol{adj}(\boldsymbol{A})$.

Example: Find the matrix of cofactors and adjoint of the matrix

$$A = \begin{bmatrix} 3 & 4 & 6 \\ 3 & 9 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

Solution:

The cofactors of each elements of A are

$$C_{11} = (-1)^{1+1}\det(M_{11}) = (-1)^{1+1}\begin{vmatrix} 9 & 0 \\ 2 & -1 \end{vmatrix} = -9$$

$$C_{12} = (-1)^{1+2}det(M_{12}) = (-1)^{1+2}\begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} = 3$$

$$C_{13} = (-1)^{1+3}\det(M_{13}) = (-1)^{1+3}\begin{vmatrix} 3 & 9 \\ 1 & 2 \end{vmatrix} = -3$$

$$C_{21} = (-1)^{2+1}\det(M_{21}) = (-1)^{2+1}\begin{vmatrix} 4 & 6 \\ 2 & 0 \end{vmatrix} = 12$$

$$C_{22} = (-1)^{2+2}\det(M_{22}) = (-1)^{2+2}\begin{vmatrix} 3 & 6 \\ 1 & -1 \end{vmatrix} = -9$$

$$C_{23} = (-1)^{2+3}\det(M_{23}) = (-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2$$

$$C_{31} = (-1)^{3+1}\det(M_{31}) = (-1)^{3+1}\begin{vmatrix} 4 & 6 \\ 9 & 0 \end{vmatrix} = -54$$

$$C_{32} = (-1)^{3+2}\det(M_{32}) = (-1)^{3+2}\begin{vmatrix} 3 & 6 \\ 3 & 0 \end{vmatrix} = 18$$

$$C_{33} = (-1)^{3+3}\det(M_{33}) = (-1)^{3+3}\begin{vmatrix} 3 & 4 \\ 3 & 9 \end{vmatrix} = 15$$

Matrix of Cofactors$= C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -9 & 3 & -3 \\ 12 & -9 & -2 \\ -54 & 18 & 15 \end{bmatrix}$

$$\textbf{Adj}(\textbf{A}) = C^T = \begin{bmatrix} -9 & 12 & -54 \\ 3 & -9 & 18 \\ -3 & -2 & 15 \end{bmatrix}$$

Exercise: Find the adjoint of A if:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 8 \\ 2 & -4 & 0 \end{bmatrix}$$

Definition-2: Let A be an$n \times n$ square matrix with $\det(A) \neq 0$. Then the inverse of a matrix A, denoted by A^{-1}, is defined as $\boldsymbol{A^{-1}} = \frac{\mathbf{1}}{\mathbf{det}(A)}\boldsymbol{adj(A)}$.

Note: If $\det(A) = 0$, then the matrix A has no inverse (i.e. A is singular matrix).

Example: Find the inverse of the matrix $A = \begin{bmatrix} 3 & 4 & 6 \\ 3 & 9 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Solution: $\det(A) = -33$

$$\text{Adj(A)} = \begin{bmatrix} -9 & 12 & -54 \\ 3 & -9 & 18 \\ -3 & -2 & 15 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} adj(A) = -\frac{1}{33}\begin{bmatrix} -9 & 12 & -54 \\ 3 & -9 & 18 \\ -3 & -2 & 15 \end{bmatrix} = \begin{bmatrix} \frac{3}{11} & \frac{-4}{11} & \frac{19}{11} \\ -\frac{1}{11} & \frac{3}{11} & \frac{-6}{11} \\ \frac{1}{11} & \frac{2}{33} & \frac{-5}{11} \end{bmatrix}$$

Exercise: Verify whether each of the following matrices has an inverse, and find the inverse if it has.

$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ -4 & 5 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

2.8. System of Linear Equations

A system of m linear equations with n-unknown (variables) is given by

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{matrix}$$

The matrix form of this system of linear equations is

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbf{b}}$$

$$\boldsymbol{Ax = b}$$

Where, A is the coefficient matrix, x is the unknown matrix (vector) and b is the known matrix (vector).

The augmented matrix, $[A \vdots b]$, for the above system of linear equation is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \vdots b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} \vdots b_2 \\ \vdots & \vdots & \ddots & \vdots \ \vdots \ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \vdots b_m \end{bmatrix}$$

Remark: If a system of linear equations is the form $\boldsymbol{Ax = 0}$, i.e. all entries of b are equal to 0, then the system is called homogeneous system of linear equation, otherwise it is called non-homogeneous.

Example: write the coefficient matrix and the augmented matrix for the following system of linear equations.

$$3x_1 - x_2 + x_3 = 4$$
$$x_1 + x_2 + x_3 = 6$$
$$x_1 - x_2 - x_3 = -4$$

Solution: Coefficient Matrix = $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

Augmented Matrix= $[A \vdots b] = \begin{bmatrix} 3 & -1 & 1 & \vdots & 4 \\ 1 & 1 & 1 & \vdots & 6 \\ 1 & -1 & -1 & \vdots & -4 \end{bmatrix}$

2.9. Solving System of Linear Equations

I. Cramer's Rules

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a system of n linear equations in n unknowns such that $\det(A) \neq 0$. Then the system has a unique solution. This solution is:

$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)}$

Where A_i(for $i = 1, 2, \ldots, n$) is the matrix obtained by replacing the entries in the i^{th}column of A by the entries in the matrix $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

Example: Using Cramer's Rule, solve the following system of linear equations.

$$x_1 + x_3 = 6$$
$$-3x_1 + x_2 + x_3 = 30$$
$$-x_1 - x_2 + x_3 = 8$$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

$\det(A) = 44$, $\det(A_1) = -40$, $\det(A_2) = 72$, and $\det(A_3) = 152$

Thus $\boldsymbol{x_1} = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \boldsymbol{\frac{-10}{11}}$ $\boldsymbol{x_2} = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \boldsymbol{\frac{18}{11}}$ and $\boldsymbol{x_3} = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \boldsymbol{\frac{38}{11}}$

Exercise: Solve the following system of linear equations using Cramer's Rule:

a) $\begin{cases} 5x + 7y = 12 \\ 10x + y + 3z = 14 \\ x + 6y + 2z = 9 \end{cases}$

b) $$\begin{aligned} 2x_1 - x_2 - x_3 - x_4 &= 6 \\ x_1 - 5x_2 - 3x_3 - x_4 &= 1 \\ 5x_1 + x_2 - 7x_3 + 6x_4 &= -3 \\ -x_1 - x_2 - x_3 &= 3 \end{aligned}$$

II. Gaussian Elimination Method

Definition: The process of using elementary row operations to transfer an augmented matrix of linear system in to one whose augmented matrix is in row echelon form is called Gaussian elimination.

Let$Ax = b$ be a system of linear equations. Then, to solve the system by using Gaussian elimination method, use the following procedures.

I. Write down the augmented matrix for the system.

II. Reduce this augmented matrix to its row echelon form.

III. Use back substitution to arrive at the solution.

Example: By using the Gaussian-Elimination method, solve the following system of linear equations.

$$\begin{aligned} 2x_1 - x_2 - x_3 - x_4 &= 6 \\ x_1 - 5x_2 - 3x_3 - x_4 &= 1 \\ 5x_1 + x_2 - 7x_3 + 6x_4 &= -3 \\ -x_1 - x_2 - x_3 &= 3 \end{aligned}$$

Exercise: By using the Gaussian-Elimination method, solve the following system of linear equations.

i. $$\begin{aligned} 4x_1 + x_2 + x_3 + x_4 &= 6 \\ 3x_1 + 7x_2 - x_3 + x_4 &= 1 \\ 7x_1 + 3x_2 - 5x_3 + 8x_4 &= -3 \\ x_1 + x_2 + x_3 + 2x_4 &= 3 \end{aligned}$$

ii. $$\begin{aligned} -x_1 - 4x_2 + 2x_3 + x_4 &= -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 &= 14 \\ -x_1 + x_2 + 3x_3 + x_4 &= 4 \\ x_1 - 2x_2 + x_3 - 4x_4 &= -4 \end{aligned}$$

III. Inverse Matrix Method

Theorem: If $\boldsymbol{A}$ is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\boldsymbol{B}$**,** the system $Ax = b$ has exactly one solution namely $x = A^{-1}b = \frac{AdjA}{detA}b$

Example: Solve the following system of linear equations by using inverse method.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 3 \\ \qquad\quad x_1 + 8x_3 = 17 \end{cases}$$

In matrix form, this system can be written as:$Ax = b$ where

A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$

But $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ -13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$ and we have:

$$x = A^{-1}b = \begin{bmatrix} -40 & 16 & 9 \\ -13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Thus$x_1 = 1,\ x_2 = \ -1\ and\ \ x_3 = 2\ are\ the\ solutions$

Exercise: Solve the following system of linear equations by using inverse method.

a) $\begin{aligned} x + y &= 2 \\ 5x + 6y &= 9 \end{aligned}$

b) $\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$

Remark:-

1. A system of equations that has no solution is called ***inconsistent***; if there is at least one solution of the system, it is called ***consistent***.
2. A system of linear equations may not have solutions, or has exactly one solution (unique solution), or infinitely many solutions.
3. Every homogeneous system of linear equations ($i.e,.\boldsymbol{Ax} = \mathbf{0}$) is consistent, since all$x_i = 0$ (for $i = 1, 2, 3, \dots$) is a solution. This solution is called the ***trivial solution***; if there are other solutions, they are called ***nontrivial solutions***. If this system has nontrivial solutions, then these solutions are infinite.
4. If the number of variables is greater than the number of equations in a given system of linear equations, then the system has infinite solutions. The arbitrary values that are assigned to the free variables are often called ***parameters***.
5. Let $Ax = b$ be a system of non-homogenous linear equations and the number of variables are equal to the number of equation (i.e. A be a square matrix). And let A_i(for $i = 1, 2, \dots, n$) be a matrix obtained by replacing the entries in the i^{th}column of A by the entries in the matrix $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$. Then
   i. if $\det(A) \neq 0$, then the system has a unique solution.
   ii. if $\det(A) = 0$, then the system has
      a. infinitely many solutions if $\det(A_i) = 0$ for all $i = 1,2,3 \dots$

b. no solution if at least one of the $\det(A_i) \neq 0$ for some $i = 1,2,3 \ldots$

2.10. Eigen values and Eigenvectors

Definition: Let A be an $n \times n$ square matrix and x be a non-zero column vector. Then $x$is called the eigenvector (or right eigenvector or right characteristic vector) of A if there exists a scalar λ such that

$$Ax = \lambda x \ldots\ldots (1)$$

Then, λ is called an eigenvalue or characteristics value of A. Eigenvalue may be zero, but eigenvector cannot be zero vectors.

Example: show that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -2$ for the matrix $\begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$.

Solution: from equation (1) we have

$$\begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

From equation (1), $Ax - \lambda x = (A - \lambda I_n)x = 0$

- $|A - \lambda I_n|$ is called the characteristic polynomial of A.
- The equation $|A - \lambda I_n| = 0$ is called the characteristic equation.
- For each eigenvalue λ, the corresponding eigenvector is found by substituting λ back into the characteristic equation $|A - \lambda I_n| = 0$.

Example: Determine the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$

Solution: For this matrix $A - \lambda I = \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 5 \\ -2 & -4-\lambda \end{bmatrix}$ and hence $\det(A - \lambda I) = (3-\lambda)(-4-\lambda) - 5(-2) = \lambda^2 + \lambda - 2 = 0$. Thus, the characteristic equation of A is $\lambda^2 + \lambda - 2 = 0$ and up on solving this we get $\lambda = 1$ and $\lambda = -2$

i. The eigenvectors to $\boldsymbol{\lambda = 1}$ will be obtained by solving equation (1) above for $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. With this value of λ after substituting and rearranging, we get:

$$\left(\begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is equivalent to the set of linear equations given below:

$$2x_1 + 5x_2 = 0$$
$$-2x_1 - 5x_2 = 0$$

The solution to this system is $x_1 = \frac{-5}{2} x_2$ with x_2 arbitrary, so the eigenvectors corresponding to $\lambda = 1$ are,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-5}{2} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-5}{2} \\ 1 \end{bmatrix} \text{ with } x_2 \text{ arbitrary.}$$

ii. When $\boldsymbol{\lambda = -2}$**,** equation (1) above may be written as:

$$\left(\begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} 5 & 5 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is equivalent to the set of linear equations given below:

$$5x_1 + 5x_2 = 0$$
$$-2x_1 - 2x_2 = 0$$

The solution to this system is $x_1 = -x_2$ with x_2 arbitrary, so the eigenvectors corresponding to $\lambda = -2$ are,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ with } x_2 \text{arbitrary}$$

Exercise: Determine the eigenvalue and eigenvector of the following matrices if there exist:

$$A = \begin{bmatrix} 5 & 2 & 2 \\ 3 & 6 & 3 \\ 6 & 6 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 6 & 5 \end{bmatrix}$$

CHAPTER THREE

LIMIT AND CONTINUTY

3.1. Definition of limit

Definition: A function is a relation in which no two distinct ordered pairs have the same first element. If f is a function with domain A and range a subset of B, we write

$$f : A \to B$$

If $f : A \to B$ is given by a rule that maps x from A to y in B, then we write $y = f(x)$.

Definition: A function $f : A \to B$ is said to be

a) Odd, if and only if, for any $x \in A$, $f(-x) = -f(x)$.

b) Even, if and only if, for any $x \in A$, $f(-x) = f(x)$. The evenness or oddness of a function is called its parity.

Definition: A function $f : A \to B$ is said to be one-to-one (an injection), if and only if, each element of the range is paired with exactly one element of the domain, i.e.

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2, for\ any\ x_1\ and\ x_2 \in A.$$

Definition: A function $f : A \to B$ is onto (a surjection), if and only if, Range of f = B.

Definition: A function $f : A \to B$ is a one-to-one correspondence (a bijection), if and only if, f is one-to-one and onto.

From preparatory we have been evaluating the limit of a function by using its intuitive definition. That is we have said that limit of $f(x)$ as x approaches to a is L and write;

$$\lim_{x \to a} f(x) = L$$

If we can make $f(x)$ close enough to L by choosing x close enough to $a$but distinct from a.

Definition: (Intuitive Definition of Limit): Suppose f is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of x arbitrarily close to a(as close to L as we like) by taking x to be sufficiently close to a(on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ approach $a$as $x$approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number Las $x$gets closer and closer to the number a(from either side of a) but $x \neq a$.

Although this intuitive definition is sufficient for solving limit problems it is not prices enough. In this section we see the formal definition of limit, which we call the $\varepsilon - \delta$ definition of limit.

Definition: (Formal definition of limit): Let $f$be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $\boldsymbol{f(x)}$as x approaches a is $\boldsymbol{L}$**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$there is a number $\delta > 0$such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

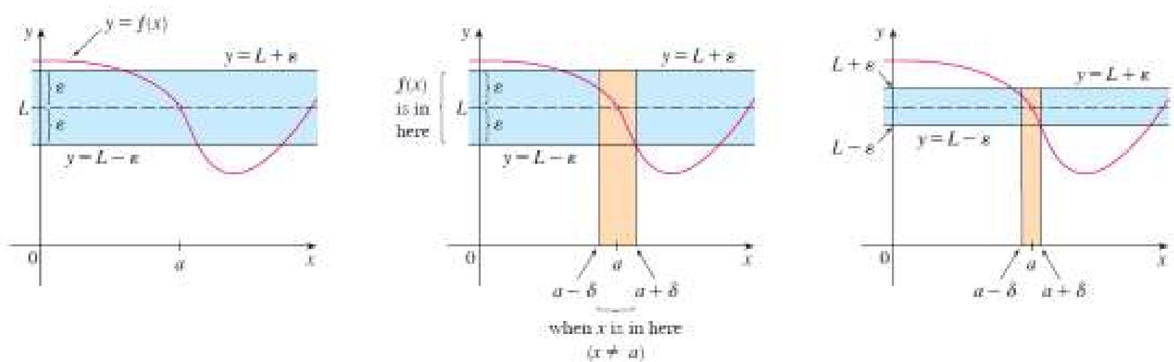
Since $|x - a|$ is the distance from $x$to a and $|f(x) - L|$is the distance from $f(x)$ to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$\lim\limits_{x \to a} f(x) = L$Means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from $x$to a sufficiently small (but not 0).

Alternatively,

$\lim\limits_{x \to a} f(x) = L$Means that the values of $f(x)$can be made as close as we please to L by taking $x$close enough to a (but not equal to a).

Diagrammatically observe the following:



Examples on limit

Even if it is very difficult to use the formal definition of limit to handle all limit problems let us see how we can use it for evaluating same important limits that may help us in developing rules by the way of which we can evaluate limits without using the formal definition.

How to Find Algebraically a $\boldsymbol{\delta}$ for a Given $\boldsymbol{f, L, a}$ and $\boldsymbol{\varepsilon} > 0$

The process of finding a$\boldsymbol{\delta} > 0$ such that for all x, $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \varepsilon$to find an open interval say (c, d) containing a on which the inequality holds for all $x \neq a$.
2. *Find a value of* $\boldsymbol{\delta} > 0$ that places the open interval $(a - \delta, a + \delta)$ centered at a inside the interval (c, d). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq a$ in this $\boldsymbol{\delta}$ **interval.**

Example: show that $\lim_{x \to 3} 4x - 5 = 7$

Solution: Based on the definition we have $f(x) = 4x - 5$, L = 7 and a = 3

We need to show for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \Rightarrow |f(x) - 7| < \varepsilon$$

Now consider $|f(x) - 7| < \varepsilon$ we have

$$|(4x - 5) - 7|$$
$$= |4x - 12|$$
$$= |4(x - 3)|$$
$$= 4|x - 3|$$

Thus

$$0 < |x - 3| < \delta \Rightarrow 4|x - 3| < \varepsilon$$
$$0 < |x - 3| < \delta \Rightarrow |x - 3| < \frac{\varepsilon}{4}$$

This suggests us we can choose $\delta = \frac{\varepsilon}{4}$

Now for checking if $0 < |x - 3| < \delta$, then:

$$0 < |x - 3| < \delta \Rightarrow 4|x - 3| < 4\delta$$
$$\Rightarrow |4(x - 3)| < 4\delta$$
$$\Rightarrow |4x - 12)| < 4\delta$$
$$\Rightarrow |(4x - 5) - 7)| < 4\delta$$

$$\Rightarrow |(4x-5)-7)| < 4\frac{\varepsilon}{4}$$

$$\Rightarrow |(4x-5)-7)| < \varepsilon$$

$$\therefore\ by\ the\ definition\ of\ limit\ \ we\ have;\ \lim_{x\to 3} 4x-5 = 7$$

THE VALUE OF δ IS NOT UNIQUE

In preparation for our next example, we note that the value of δ in the above Definition is not unique; once we have found a value of δ that fulfils the requirements of the definition, and then any *smaller* positive number δ_1 will also fulfil those requirements. I.e. if it is true that;

$|f(x)-L| < \varepsilon$ If$0 < |x-a| < \delta$it will also be true that$|f(x)-L| < \varepsilon$ if $0 < |x-a| < \delta_1$

This is because $\{x:\ 0 < |x-a| < \delta_1\}$ is a subset of $\{x:\ 0 < |x-a| < \delta\}$

Example: show that:

a) $\lim_{x\to 3} x^2 = 9$
b) $\lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$

Solution:

a) Here we have $f(x) = x^2$, L = 9 and a = 3

We need to show for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

That is $0 < |x-3| < \delta \Rightarrow |x^2-9| < \varepsilon$

Now consider $|x^2-9| < \varepsilon$

$|x^2-9| = |x+3||x-3| < \varepsilon$ We wish to bound the factor |x + 3|. This can be done by setting fixed number for δ then let $\delta \leq 1$ then we have;

$$|x-3| < 1$$

$$\Rightarrow -1 < x-3 < 1$$

$$\Rightarrow -1+6 =< x+3 < 1+6$$

$$\Rightarrow 5 < x+3 < 7$$

Consequently we can have that $|x+3| < 7$ and hence we have;

$$|x^2-9| = |x+3||x-3| < 7|x-3| < 7\delta$$

This suggests us we can choose $\delta = \min\{\frac{\varepsilon}{7}, 1\}$. This proves that$\lim_{x\to 3} x^2 = 9$.

b) Here we have $f(x) = \frac{1}{x}$, L = $\frac{1}{2}$ and a = 2

We need to a number $\delta > 0$ given that a number $\varepsilon > 0$ such that;

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

That is $0 < |x - 2| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$

Now consider $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|2-x|}{|2x|} = \frac{1}{2}\frac{|2-x|}{|x|}$$

In similar manner of the second example above we have to bound the factor |x| then let $\delta \leq 1$ and we have;

$|x - 2| < 1 \Rightarrow -1 < x - 2 < 1 \Rightarrow 1 < x < 3$ consequently we can have $|x| < 3$ and this implies $\left|\frac{1}{x}\right| > \frac{1}{3}$ hence we have:

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|2-x|}{|2x|} = \frac{|x-2|}{2|x|} > \frac{|x-2|}{6}$$

$\Rightarrow \frac{|x-2|}{6} < \frac{|x-2|}{2|x|} < \varepsilon$

$\Rightarrow \frac{|x-2|}{6} < \varepsilon \Rightarrow |x - 2| = 6\varepsilon = \delta_2$

This suggests us we can choose $\delta = \min\{1,\ 6\varepsilon\}$ and hence this proves that $\lim\limits_{x\to 2}\frac{1}{x} = \frac{1}{2}$

Exercise: show that:

a) $\lim\limits_{x\to 0} x^2 = 0$
b) $\lim\limits_{x\to 2} 3x - 5 = 1$

Uniqueness of limit

Theorem1: If the limit of a function f at a exists then, this limit is unique. Equivalently if L and M are both limits of f at a then $L = M$.

Proof: *suppose that* f *has two distinct limits* L and M $(L \neq M)$ *that is* $\lim\limits_{x\to a} f(x) = L$ *and* $\lim\limits_{x\to a} f(x) = M$

Now since $\lim\limits_{x\to a} f(x) = L$ *then given* $\frac{\varepsilon}{2} > 0$ *there exist* $\delta_1 > 0$ *such that*

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

And since $\lim\limits_{x\to a} f(x) = M$ *then given* $\frac{\varepsilon}{2} > 0$ *there exist* $\delta_2 > 0$ *such that*

$$0 < |x-a| < \delta_2 \Rightarrow |f(x) - M| < \frac{\varepsilon}{2}$$

Now take $\delta = \min(\delta_1, \delta_2)$ *and we have:*

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$ *And* $0 < |x-a| > \delta \Rightarrow |f(x) - M| < \frac{\varepsilon}{2}$

In particular choose $\varepsilon = |L - M|$ *and hence we have:*

$$\begin{aligned} |L - M| &= |L - f(x) + f(x) - M| \\ &\leq |L - f(x)| + |f(x) - M| \\ &= |f(x) - L| + |f(x) - M| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = |L - M| \end{aligned}$$

$\Rightarrow |L - M| < |L - M|$ *This is a contradiction*

Example: show that $\lim\limits_{x\to 0} f(x) = 0$ where $f(x) = \frac{x^2}{x^2+1}$ (hint $\left|\frac{x}{x^2+1}\right| < |x|$)

Solution: Given $\varepsilon > 0$ we need to find $\delta > 0$ such that;

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ That is $0 < |x-0| < \delta \Rightarrow \left|\frac{x^2}{x^2+1} - 0\right| < \varepsilon$

$$\Rightarrow\ 0 < |x| < \delta \Rightarrow \left|\frac{x^2}{x^2+1}\right| < \varepsilon$$

Now consider $\left|\frac{x^2}{x^2+1}\right|$ we have:

$$\left|\frac{x^2}{x^2+1}\right| = |x|\left|\frac{x}{x^2+1}\right| < |x||x| = |x|^2 < \varepsilon$$

$$\Rightarrow |x| < \sqrt{\varepsilon}$$

Now we can choose $\delta = \sqrt{\varepsilon}$ such that $0 < |x| < \delta \Rightarrow \left|\frac{x^2}{x^2+1}\right| < \sqrt{\varepsilon}$

3.2. Basic limit Theorems

Theorem: Suppose $\lim\limits_{x\to a} f(x) = \mathrm{L}$ and $\lim\limits_{x\to a} g(x) = \mathrm{M}$ then for any real number the following holds true.

1. $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = L \pm M$
2. $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x) = cL$
3. $\lim\limits_{x\to a} f(x) \cdot g(x) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = LM$
4. $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\mathrm{L}}{\mathrm{M}}, (for\ g(x) \neq 0, M \neq 0)$
5. $\lim\limits_{x\to a}[f(x)]^n = [\lim\limits_{x\to a} f(x)]^{\mathrm{n}}$

6. $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)}$
7. $ln\left|\lim_{x \to a} f(x)\right| = \lim_{x \to a} ln|f(x)|$ for $\lim_{x \to a} f(x) > 0$
8. $\lim_{x \to a} (f(x))^{g(x)} = e^{\lim_{x \to a} g(x) ln|f(x)|} for\ \ f(x) > 0$
9. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}\ \ for\, n\, is\, even \lim_{x \to a} f(x) > 0$

Proof: 1 since $\lim_{x \to a} f(x) =$ Land$\lim_{x \to a} f(x) =$ M then by definition for given $\varepsilon_1, \varepsilon_2 > 0$ there exists $\delta_1, \delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon_1$$
$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon_2$$

Now choose $\delta = \min\{\delta_1, \delta_2\}$ and $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$ then

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$
$$0 < |x - a| < \delta \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$$

We need to show $0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - (L + M)| < \varepsilon$

Now consider $|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)|$

$$\leq |(f(x) - L)| + |(g(x) - M)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - (L + M)| < \varepsilon$

Therefore $\lim_{x \to a}[f(x) + g(x)] = L + M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

Proof 2. Given that $\lim_{x \to a} f(x) = L$ we need to show for every $\varepsilon > 0$ $(given)$ there exist a $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |cf(x) - cL| < \varepsilon$

$$\Rightarrow |c||f(x) - L| < \varepsilon$$
$$\Rightarrow |f(x) - L| < \frac{\varepsilon}{|c|}$$

Now we can take $|c|\varepsilon > 0$ then there exist $\delta > 0$ such that;

$$0 < |x - a| < \delta \Rightarrow |cf(x) - cL| < |c|\varepsilon$$
$$\Rightarrow |c||f(x) - L| < |c|\varepsilon$$
$$|f(x) - L| < \varepsilon$$

Therefore $\lim_{x \to a} cf(x) = c\lim_{x \to a} f(x) = cL$

Proof 8. Using number 7 we can prove it informally as follows:

Let $y = \lim_{x \to a} \left(f(x)\right)^{g(x)}$

$\Rightarrow \ln y = ln\left\{\lim_{x \to a} \left(f(x)\right)^{g(x)}\right\}$

$$= \lim_{x \to a} ln\left(f(x)\right)^{g(x)} (by\ number\ 7)$$

$$= \lim_{x \to a} g(x) lnf(x)$$

$\Rightarrow\ e^{lny} = e^{\lim_{x \to a} g(x) lnf(x)}$

$\Rightarrow y = e^{\lim_{x \to a} g(x) lnf(x)}$

Therefore $\lim_{x \to a} (f(x))^{g(x)} = \mathrm{e}^{\lim_{x \to a} g(x) ln|f(x)|} for\ \ f(x) > 0$

Proof the rest as an exercise.

Activity:

1. Evaluate the following limits:
   a) $\lim_{x \to 4} \sqrt[3]{\frac{x}{-7x+1}}$
   b) $\lim_{x \to 3} \frac{x^3-27}{x-3}$
   c) $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$
   d) $\lim_{x \to 4} \frac{\sqrt{x+5}-3}{x-4}$
   e) $\lim_{x \to 2} \frac{\sqrt{1+\sqrt{2+x}}-3}{x-2}$
   f) $\lim_{x \to 2a} \frac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}}$
   g) $\lim_{x \to 3} e^{\left(\frac{x^2-9}{x-3}\right)}$

2. Show that the following limits are true.
   a) $\lim_{x \to a} \left(\frac{x^m-a^m}{x^n-a^n}\right) = \frac{m}{n} a^{m-n} \quad , m, n \in Q$
   b) $\lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1} n \in N$

Example: Find$\lim_{x \to 0} \frac{4x^2+6x+6}{\sin^2 x-1}$

Solution: Using the basic limit theorems above we can solve it as follows; since all limits of the terms exists we can write it

$$\lim_{x \to 0} \frac{4x^2 + 6x + 6}{\sin^2 x - 1} = \frac{\lim_{x \to 0} 4 + \lim_{x \to 0} 6x + \lim_{x \to 0} 6}{\lim_{x \to 0} \sin^2 x - \lim_{x \to 0} 1} = \frac{4\lim_{x \to 0} x^2 + 6\lim_{x \to 0} x + 6}{\lim_{x \to 0} \sin^2 x - 1} = \frac{0 + 0 + 6}{0 - 1} = -6$$

Theorem: (Substitution Theorem)

Suppose $\lim\limits_{x\to a} f(x) = c$ and $f(x) \neq c$ for all x in some open interval about a with the possible exception of a itself. Suppose also that $\lim\limits_{x\to a} g(x)$ exists, then $\lim\limits_{x\to a} g\big(f(x)\big) = \lim\limits_{y\to c} g(y)$

Example: Find

a) $\lim\limits_{x\to 0}\sqrt{1-\mathrm{x}^2}$
b) $\lim\limits_{x\to \frac{\pi}{12}}\sqrt{\sin 2\mathrm{x}}$

Solution:

a) let $y = 1 - x^2$ then as $x \to 0, y = 1 - x^2 \to 1$

$$\Rightarrow \lim_{x\to 0}\sqrt{1-\mathrm{x}^2} = \lim_{y\to 1}\sqrt{\mathrm{y}} = 1$$

Therefore $\lim\limits_{x\to 0}\sqrt{1-\mathrm{x}^2} = 1$

b) let $y = 2x$ then as $x \to \frac{\pi}{12}, y = 2x \to \frac{\pi}{6}$

$$\Rightarrow \lim_{x\to \frac{\pi}{12}}\sqrt{\sin 2\mathrm{x}} = \lim_{y\to \frac{\pi}{6}}\sqrt{\sin \mathrm{y}}$$

And now let $z = \sin y$ then as $y \to \frac{\pi}{6}, z = \sin y \to \frac{1}{2}$

$$\Rightarrow \lim_{y\to \frac{\pi}{6}}\sqrt{\sin \mathrm{y}} = \lim_{z\to \frac{1}{6}}\sqrt{\mathrm{z}} = \frac{\sqrt{2}}{2}$$

***Exercise**:* Find $\lim\limits_{x\to -\frac{\pi}{4}} \tan^{\frac{2}{5}} x$

Squeezing (sandwich) theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing a except possibly at a itself. Suppose also that:

$\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = L$. Then $\lim\limits_{x\to a} f(x) = L$

Example: given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ for all $x \neq 0$ then find $\lim\limits_{x\to 0} u(x)$

Solution: observe that $\lim\limits_{x\to 0}\left(1 - \frac{x^2}{4}\right) = 1 = \lim\limits_{x\to 0}(1 + \frac{x^2}{2})$ then by the sandwich theorem we have that $\lim\limits_{x\to 0} u(x) = 1$

3.3. One sided Limits

This method is applied to find the limit at $x = a$ when the function is defined differently for the cases when $x > a$, $x = a$ and $x < a$.

1. **Right hand limit**: We say that the right hand limit of $f(x)$ at $x = a$ is L, if $f(x) \to L$ as $x \to a$ through values greater than a. And we write:
   $\lim_{x \to a^+} f(x) = L$ If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a < x < a + \delta$ then $|f(x) - L| < \varepsilon$
2. **Left hand limit**: We say that the left hand limit of $f(x)$ at $x = a$ is M, if $f(x) \to M$ as $x \to a$ through values less than a. And we write:
   $\lim_{x \to a^-} f(x) = M$ If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a - \delta < x < a$ then $|f(x) - M| < \varepsilon$

Example: Show that $\lim_{x \to 0^+} \sqrt{x} = 0$

Solution: here we have $f(x) = \sqrt{x}$, $L = 0$ *and* $a = 0$ then for every number $\varepsilon > 0$ we need to find a number $\delta > 0$ such that:

$$0 < x < 0 + \delta \Rightarrow \left|\sqrt{x} - 0\right| < \varepsilon$$

$$0 < x < \delta \Rightarrow \sqrt{x} < \varepsilon$$

$$0 < x < \delta \Rightarrow x < \varepsilon^2$$

Then we can choose $\delta = \varepsilon^2$

For checking $0 < x < \delta \Rightarrow \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon \Rightarrow \left|\sqrt{x} - 0\right| < \varepsilon$

Therefore $\lim_{x \to 0^+} \sqrt{x} = 0$

Exercise: Show that $\lim_{x \to 1^-} \sqrt{1 - x^2} = 0$

Limit of a function derived from one-sided limits

Definition: We say that $\lim_{x \to a} f(x) = L$, if $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$ however, if $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$ or if any one of the limits $\lim_{x \to a^+} f(x)$ *or* $\lim_{x \to a^-} f(x)$ doesn't exist, then we say that $\lim_{x \to a} f(x)$ doesn't exist.

Example: Let $f(x) = \begin{cases} 2x + 1 \; for \; x \leq 2 \\ x + 3 \; \; for \; x > 2 \end{cases}$ then find $\lim_{x \to 2} f(x)$

Solution: using one sided limits we can solve as follows;

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x + 3 = 5$and $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2x + 1 = 5$

$\Rightarrow \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) = 5$

Exercise: Let $h(x) = \begin{cases} 4 - x^2 \; for \; x \leq 1 \\ 2 + x^2 \; \; for \; x > 1 \end{cases}$ then find $\lim_{x \to 1} h(x)$

3.4. Infinite limits and limit at infinity

Infinite limits:

Definition: Let $f(x)$ be defined for all x in some open interval containing a except possibly that $f(x)$ need not be defined at$a$. We will write:

$\lim\limits_{x\to a} f(x) = \infty$ If given any positive number M, we can find a number $\delta > 0$ such that $f(x)$ satisfies: $f(x) > M \ \ if \ \ 0 < |x-a| < \delta$ that is $0 < |x-a| < \delta \Rightarrow f(x) > M$

Definition: Let $f(x)$ be defined for all x in some open interval containing a except possibly that $f(x)$ need not be defined at$a$. We will write:

$\lim\limits_{x\to a} f(x) = -\infty$ If given any negative number M, we can find a number $\delta > 0$ such that $f(x)$ satisfies: $f(x) < M \ \ if \ \ 0 < |x-a| < \delta$ that is $0 < |x-a| < \delta \Rightarrow f(x) < M$

Example: Show that $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$

Solution: let M be given positive number, then we need to find a number $\delta > 0$ such that

$$0 < |x| < \delta \Rightarrow \frac{1}{x^2} > M$$

But $\frac{1}{x^2} > M \Leftrightarrow x^2 < \frac{1}{M} \Leftrightarrow |x| < \frac{1}{\sqrt{M}}$

Now if we choose $\delta = \frac{1}{\sqrt{M}}$ then $\frac{1}{x^2} > M$

Therefore $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$

Limit at Infinity:

Definition: Let $f(x)$ be defined for all x in some infinite open interval extending in the positive x-direction. We will write:

$\lim\limits_{x\to \infty} f(x) = L$ If given $\varepsilon > 0$, there corresponds a positive number N such that $|f(x) - L| < \varepsilon$ if $x > N$

Definition: Let $f(x)$ be defined for all x in some infinite open interval extending in the negative x-direction. We will write:

$\lim\limits_{x\to -\infty} f(x) = L$ If given $\varepsilon > 0$, there corresponds a negative number N such that $|f(x) - L| < \varepsilon$ if $x < N$

Example: Prove that $\lim\limits_{x\to \infty} \frac{1}{x} = 0$

Solution: Applying the above definition $f(x) = \frac{1}{x}$ $and\ L = 0$ we need to find a number N>0 such that

$$\left|\frac{1}{x} - 0\right| < \varepsilon \; if \; x > N$$

Since $x \to \infty$ we can assume that $x > 0$, then we can write it without absolute values as

$$\frac{1}{x} < \varepsilon \; if \; x > N$$

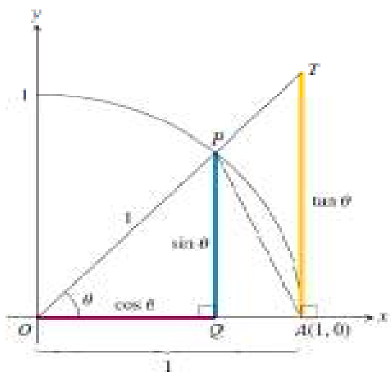
$$\Rightarrow x > \frac{1}{\varepsilon} \; if \; x > N$$

Then we can choose $N = \frac{1}{\varepsilon}$ which satisfies the required proof.

Limits involving $\frac{\sin\theta}{\theta}$

Theorem: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (θ in radians)

Proof: Consider the following diagram:



The aim is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

From the figure Area $\Delta OAP < area\ sector\ OAP < area\ \Delta OAT$. We can express these areas in terms of θas follows:

$$Area\ \Delta OAP \ = \frac{1}{2} base \ * height \ = \frac{1}{2}(1) sin\ \theta = \frac{1}{2} sin\ \theta$$

$$area\ sector\ OAP = \frac{1}{2} r^2 \theta \ = \frac{\theta}{2}$$

$$Area\ \Delta OAT = \frac{1}{2} base \ * height = \frac{1}{2} \tan\theta$$

$$Thus \frac{1}{2} sin\ \theta < \frac{\theta}{2} < \frac{1}{2} \tan\theta$$

Dividing all terms by $\frac{1}{2}\sin\theta$ ($\theta\ is\ positive\ 0 < \theta < \frac{\pi}{2}$) we get;

$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$ and by taking limit as $\theta \to 0$ through and the sandwich theorem we get $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Example: using $\lim_{x\to\infty}\frac{\sin\theta}{\theta}=1$ show that:

a) $\lim_{h\to 0}\frac{\cos h-1}{h}=0$ b) $\lim_{x\to 0}\frac{\sin 2x}{5x}=\frac{2}{5}$

Solution: a) By using the half angle formula $\cos h=1-2\sin^2(\frac{h}{2})$ then we can have; $\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\frac{-2\sin^2(\frac{h}{2})}{h}=-\lim_{h\to 0}\frac{\sin^2\left(\frac{h}{2}\right)}{\frac{h}{2}}=-\lim_{\theta\to 0}\frac{sin(\theta)}{\theta}\cdot\sin\theta$ (by letting $\theta=\frac{h}{2}$) thus we get

$$-\lim_{\theta\to 0}\frac{sin(\theta)}{\theta}\cdot\sin\theta=-1(1)(0)=0$$

b) $\lim_{x\to 0}\frac{\sin 2x}{5x}=\frac{2}{5}$ since $\lim_{x\to 0}\frac{\sin 2x}{5x}=\lim_{x\to 0}\frac{(\frac{2}{5})\sin 2x}{(\frac{2}{5})5x}=\frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}=\frac{2}{5}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ $(by\ letting\ \theta=2x)$ thus we get

$$\frac{2}{5}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{2}{5}(1)=\frac{2}{5}$$

Exercise: Find

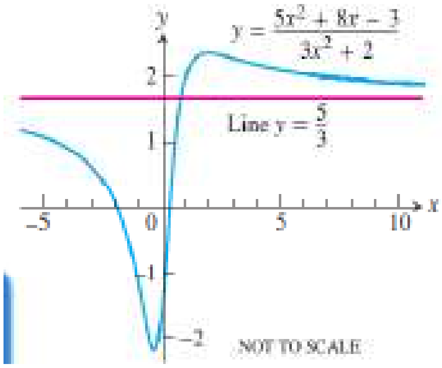
a) $\lim_{x\to 0}\frac{\tan x}{x}$
b) $\lim_{x\to 0}\frac{\sin 2x}{x}$
c) $\lim_{x\to 0}\frac{\sin 3x}{5x}$

Asymptotes

1. **Horizontal Asymptote**: a line $y=b$ is horizontal asymptote of the graph of a function $y=f(x)$ if either:
   $\lim_{x\to\infty}f(x)=b$ Or $\lim_{x\to-\infty}f(x)=b$

Example: The curve $\frac{5x^2+8x-3}{3x^2+2}$

Whose sketched is in the Figure below has the line $y=\frac{5}{3}$as a horizontal asymptote on both the right and the left because; $\lim_{x\to\infty}f(x)=\frac{5}{3}$ and $\lim_{x\to-\infty}f(x)=\frac{5}{3}$



Example: Find $\lim_{x \to \infty} \sin \frac{1}{x}$

Solution: we can solve this by substituting a new variable $t = \frac{1}{x}$ and as $x \to \infty$ $t \to 0^+$ and hence,

$$\lim_{x \to \infty} \sin \frac{1}{x} = \lim_{t \to 0^+} \sin t = 0$$

2. **Vertical Asymptote**: a line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either:

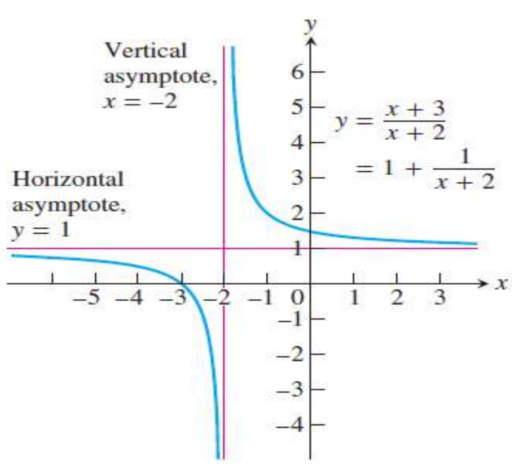
$$\lim_{x \to a^+} f(x) = \pm\infty \; or \; \lim_{x \to a^-} f(x) = \pm\infty$$

Example: Find the Horizontal and Vertical Asymptotes of the curve $f(x) = \frac{x+3}{x+2}$

Solution: We are interested in the behaviour as $x \to \pm\infty$ and as $x \to -2$ where the denominator is zero. Using long division we get;

$$y = 1 + \frac{1}{x+2}$$

We now see that the curve in question is the graph of $y = \frac{1}{x}$shifted 1 unit up and 2 units left as in the figure below. The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and$x = -2$.



3.5. Continuity:

Definition: A function f is continuous at a point a, if and only if the following three conditions are satisfied:

1. $f(a)$ is defined
2. $\lim\limits_{x \to a} f(x)$ exist
3. $\lim\limits_{x \to a} f(x) = f(a)$

Example: Determine whether the following functions are continuous or not at $x = 2$.

$$f(x) = \frac{x^2-4}{x-2}, \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & if\ x \neq 2 \\ 3 & if\ x = 2 \end{cases} \ and\ h(x) = \begin{cases} \dfrac{x^2-4}{x-2} & if\ x \neq 2 \\ 4 & if\ x = 2 \end{cases}$$

Solution: now a function f is continuous at a point a, if and only if the above three conditions are satisfied; then

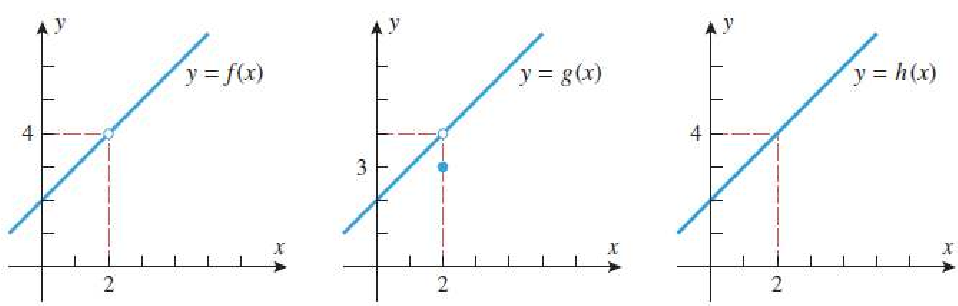
In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at $x = 2$, and hence all three have the same limit at $x = 2$, which is,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2}(x+2) = 4$$

The function *f* is undefined at $x = 2$, and hence is not continuous at $x = 2$(Figure *a*).

The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit as *x* approaches 2; hence, *g* is also not continuous at $x = 2$ (Figure *b*).

The value of the function *h* at$x = 2$is $h(2) = 4$, which is the same as the limit as *x* approaches 2; hence, *h* is continuous at $x = 2$ (Figure *c*).



(*a*) (*b*) (*c*)

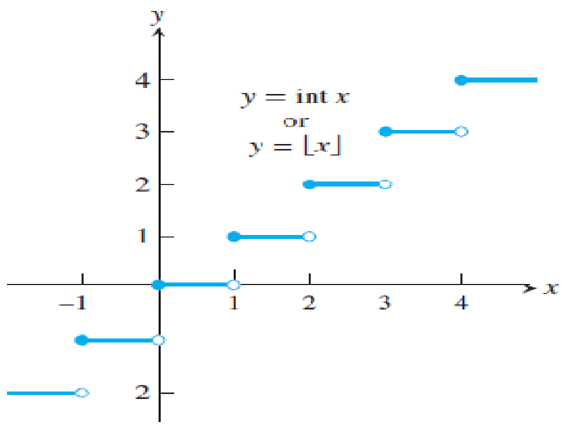
Exercise: let $f(x) = \frac{x^2-5x+4}{x^2-9}$ determine the numbers where f is continuous.

Definition: A function f is continuous from the right at a number a if $\lim\limits_{x\to a^+} f(x) = f(a)$ and f is continuous from the left at $a\ if\ \lim\limits_{x\to a^-} f(x) = f(a)$.

Example: The Greatest Integer Function$f(x) = \lfloor x\rfloor$. It is discontinuous at every integer because the limit does not exist at any integer n:

$\lim\limits_{x\to n^-} \lfloor x\rfloor = n - 1$ And $\lim\limits_{x\to n^+} \lfloor x\rfloor = n$

So the left-hand and right-hand limits are not equal as $x \to n$. Since $\lfloor n\rfloor = n$ the greatest integer function is right-continuous at every integer n (but not left-continuous).



Definition: A function f is continuous on an interval if it is continuous at every number in the interval. If f is defined only on one side of an end point of the interval, we understand continuous at the end point to mean continuous from the right or continuous from the left.

Example: Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1,1]$

Solution: we are going to check the continuity of f at any point on the interval (-1,1) and at the end points of the interval as follows;

For any $a \in (-1,1) i.e -1 < a < 1$ using the limit laws we have:

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(1 - \sqrt{1 - x^2}\right) = 1 - \lim_{x\to a}\left(\sqrt{1 - x^2}\right) = 1 - \left(\sqrt{\lim_{x\to a}(1 - x^2)}\right) = 1 - \sqrt{1 - a^2}$$

$$= f(a)$$

Thus by definition f is continuous at a if $-1 < a < 1$ now we are going to show that the limit exists at the end points, that is:

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+}\left(1 - \sqrt{1 - x^2}\right) = 1 = f(-1)$ And $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}\left(1 - \sqrt{1 - x^2}\right) = 1 = f(1)$

Hence f is continuous from the right at -1 and continuous from the left at 1.

Therefore f is continuous on $[-1,1]$

Theorem: If f and g are cont. at a and c is constant, then the following functions are also cont. at $a.f+g$

1. $f-g$
2. Cf
3. fg
4. $\frac{f}{g}$ where $g \neq 0$

Proof:

1. Since f and g are continuous at a we have:
   $\lim_{x\to a} f(x) = f(a)and\lim_{x\to a} g(x) = g(a)$
   $\Rightarrow \lim_{x\to a}(f+g)(x) = \lim_{x\to a}[f(x)+g(x)]$
   $= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ (By limit laws)
   $= f(a) + g(a)$
   $= (f+g)(a)$
   Thus $f+g$ is continuous at a

Proof the rest as an exercise

Theorem:

a) Any polynomial function is continuous everywhere. That is continuous on $\boldsymbol{R} = (-\infty, \infty)$
b) Any rational function is continuous where ever it is defined; that is continuous on its domain.

Theorem: In general the following types of functions are continuous at every number in their domains:

- polynomial function
- rational function
- root functions
- trigonometric and their inverses
- exponential functions
- logarithmic functions

Example: where is the function $f(x) = \frac{\ln x + \tan^{-1} x}{x^2-1}$ continuous?

Solution: By the above theorem $y = \ln x$ is continuous for all $x > 0$ and $y = \tan^{-1} x$ is continuous on $\boldsymbol{R}$. Thus by theorem of addition from previous theorem $y = \ln x + \tan^{-1} x$ is continuous at $(0, \infty)$. The denominator $y = x^2 - 1$ is polynomial, so it is continuous at all positive numbers x except where$x^2 - 1 = 0$. So f is continuous on the interval $(0, 1)$ and $(1, \infty)$.

Exercise: Evaluate $\lim_{x\to\pi}\frac{\sin x}{2+\cos x}$

Theorem: If f is continuous at b and $\lim_{x\to a} g(x)=b$, then $\lim_{x\to a} f(g(x))=f(b)$. In other words $\lim_{x\to a} f(g(x))=f(\lim_{x\to a} g(x))$.

Proof: Exercise

Example: Evaluate $\lim_{x\to 1}\sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$

Solution: Since inverse trigonometric functions are continuous then we can use the above theorem: thus

$$\lim_{x\to 1}\sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)=\sin^{-1}\left(\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}\right)$$

$$=\sin^{-1}\left(\lim_{x\to 1}\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right)(because\ 1-x=(1-\sqrt{x})(1-\sqrt{x}))$$

$$=\sin^{-1}\left(\lim_{x\to 1}\frac{1}{(1+\sqrt{x})}\right)$$

$$=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}=30°$$

Exercise: By using the above theorem show that: $\lim_{x\to a}\sqrt[n]{g(x)}=\sqrt[n]{\lim_{x\to a} g(x)}$

Theorem: If g is continuous at a and f is continuous at $g(a)$ then the composite function fog given by $f(g(x))$ is continuous at a.

Proof: Because g is continuous at a we have $\lim_{x\to a} g(x)=g(a)$. Now f is continuous at $g(a)$; thus, we can apply the above Theorem to the composite function fog, thereby giving us

$$\lim_{x\to a}(fog)(x)=\lim_{x\to a} f\big(g(x)\big)=f\left(\lim_{x\to a} g(x)\right)=f(g(a))$$

Therefore fog is continuous at a.

Example: Where are the following functions continuous?

a) $h(x)=\sin x^2$ and
b) $F(x)=\ln(1+\cos x)$

Solution:

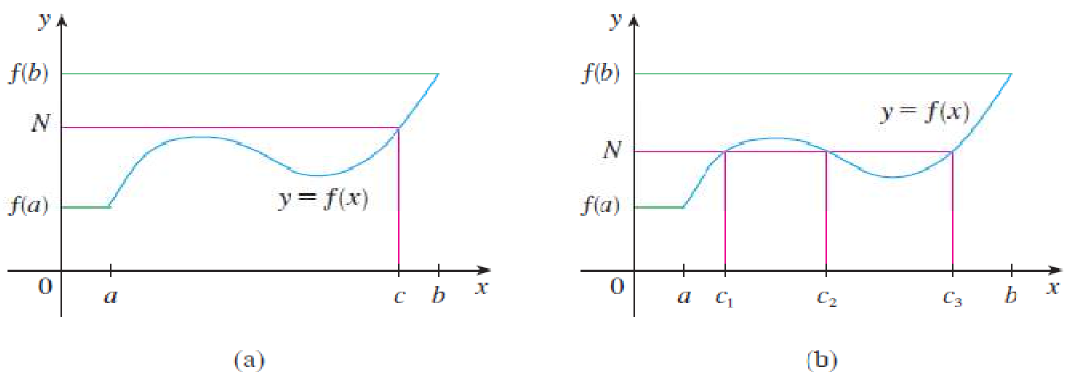
a) $h(x)=\sin x^2$, here we can write $h(x)=f(g(x))$where $g(x)=x^2\ and\ f(x)=\sin x$. Now g is continuous on $\boldsymbol{R}$, (because g is polynomial) and f is continuous everywhere. Thus $h(x)=f(g(x))$ is continuous on $\boldsymbol{R}$by the above theorem.

b) $F(x) = \ln(1 + \cos x)$, now $f(x) = \ln x$is continuous and $g(x) = 1 + \cos x$ is continuous. Then $F(x) = \ln(1 + \cos x)$ is continuous wherever it is defined. But $\ln(1 + \cos x)$ is defined when $1 + \cos x > 0$. So it is undefined when $\cos x = -1$ that is when $x = \pm\pi,\ \pm 3\pi,\ \pm 5\pi$ ... thus F is discontinuous when x is an odd multiple of π and is continuous on the intervals between these values.

3.6. The intermediate value theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number c in $[a, b]$ such that $f(c) = N$

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by the Figure below. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].



Geometrically, the Intermediate Value Theorem says that any horizontal line $y = N$crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

Example: Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2

Solution: Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore we take, $a = 1,\ b = 2$ and $N = 0$in the above Theorem. Then we have

$$f(1) = 4(1)^3 - 6(1)^2 + 3(1) - 2 = -1 < 0$$

And

$$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2 = 12 > 0$$

$\Rightarrow f(1) < 0 < f(2)$, that is $N = 0$ is a number between$f(1)$ *and* $f(2)$. Now $f$is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number $c$between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1, 2)$.

From the basic limit theorems which we have seen in the previous section we have the following special limits as a summary:

1. $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} x^n = a^n$
4. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}\ n \in Z^+$ if n is negative we assume that $a > 0$.
5. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}, n \in Z$

Exercises

1. Evaluate the following limits.
   a) $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$
   b) $\lim_{x \to 0} \frac{5x^2 - 4}{x + 1}$
   c) $\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$
   d) $\lim_{x \to \infty} \frac{2x + 5}{x^2 - 7x + 3}$
   e) $\lim_{x \to 0} \frac{|x|}{x}$
2. Find the vertical and horizontal asymptotes of the following functions.
   a) $f(x) = \frac{4x - 5}{3x + 2}$
   b) $f(x) = \frac{x^2 - 5x + 6}{x - 3}$
3. Show that whether the following functions are continuous or not at $x = 2$.
   a) $f(x) = \frac{x^2 - x - 2}{x - 2}$
   b) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & if\ x = 2 \end{cases}$
4. Show that the function $f(x) = 1 - \sqrt{4 - x^2}$ is continuous on the interval $[-2, 2]$.
5. Find the value of k, if possible, that will make the function continuous.
   $a)\ f(x) = \begin{cases} 7x - 2 & \text{if } x \leq 1 \\ kx^2 & if\ x > 1 \end{cases}$
   b) $f(x) = \begin{cases} kx^2 & \text{if } x \leq 2 \\ 2x + k & if\ x > 2 \end{cases}$

CHAPTER FOUR

DERIVATIVES AND APPLICATION OF DERIVATIVES

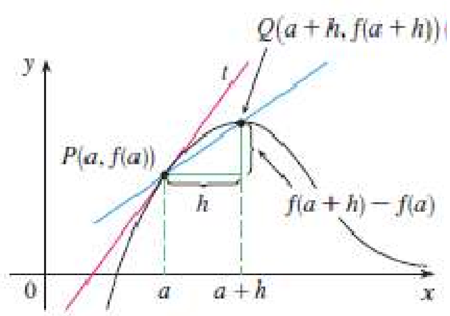
Unit out comes

- **At the end of this chapter the learners will be able to:**
  - ✓ **Determine the differentiability of a function at a point.**
  - ✓ **Find derivative of some functions**
  - ✓ **Apply sum, difference product quotient rule differentiation of function.**
  - ✓ **Find the derivative of inverse trigonometric and inverse hyperbolic functions.**
  - ✓ **Apply derivatives to solve some real life problems**
  - ✓ **Sketch graphs of some functions using derivatives.**

4.1. Definition of derivatives; Basic rules

Definition : If point $p(x_o, y_o)$ is a point on the graph of a function f then the tangent line to the graph of f at p is defined to be the line through p with slope $m_{tan} = \lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h}$

, provides this limit exist. Where $h = x - x_o$



Example : Find the slope of the tangent line to the graph of $f(x) = x^2$, at the point $(1,1)$.

Solution: Given $f(x) = x^2$ which implies that $f(x_o + h) = (x_o + h)^2$ and $f(x_o) = x_o^2$, $p(x_o, y_o) = (1,1)$

$$m_{tan} = \lim_{h\to 0} \frac{f(x_o+h)-f(x_o)}{h} = \lim_{h\to 0} \frac{(x_o+h)^2-(x_o)^2}{h} = \lim_{h\to 0} \frac{x_o{}^2+2hx_o+h^2-x_o{}^2}{h} = \lim_{h\to 0} \frac{2hx_o+h^2}{h}$$

$$= \lim_{h\to 0} 2x_o+h = 2x_o.$$

But our $x_o = 1$, so $m_{tan} = 2$

Normal Line.

Definition : The normal line to a curve at a given point is the line perpendicular to the tangent line at that point.

Definition: Let a be a number in the domain of f. If $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exist, we call this limit the derivative of f at a and we write it $f'(a)$ so that $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

Or $f'(a) = \lim_{h\to a} \frac{f(a+h)-f(a)}{h}$,if $x = a + h$

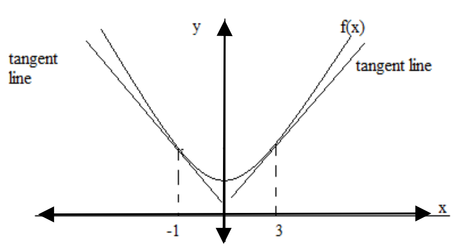
If this limit exist we say that f has a derivative at a. i.e f is differentiable at a or $f'(a)$ exist.

$f'(a)$ is the slope of the tangent to the graph of f at $(a, f(a))$.

Example : Let f(x) = $\frac{x^2}{4} + 1$. Find $f'(-1)$ and $f'(3)$ and draw the lines tangent to the graph of f at the corresponding points.

Solution: $f'(x) = \lim_{x\to -1} \frac{\left(\frac{x^2}{4}+1\right)-\left(\frac{(-1)^2}{4}+1\right)}{x-(-1)} = \lim_{x\to -1} \frac{\left(\frac{x^2}{4}+1\right)-1/4}{x+1} = -1/2.$

Similarly we obtain $f'(3) = 3/2$



Exercise

Find $f'(a)$for the given values of a.

i)$f(x) = 1/x$,a=-2 b) $f(x) = -4x + 7$ C)$f(x) = |x|$ at a=-$\sqrt{2}$

d)$f(x) = \begin{cases} x^2 \; for \; x < 2 \\ 4x - 4 \; forx \geq 2 \end{cases}$ $a = 2$

Derivative notation

The process of finding a derivative is called differentiation. When the independent variable is x, ie $y = f(x)$ the differentiation operation is often denoted by $\frac{d}{dx}(f(x)) = f'(x) = \frac{dy}{dx}$

Derivatives of constants

Theorem : The derivatives of constant function are 0. If $f(x) = c$, where c is any real number, then $f'(x) = 0$

Proof: $f(x) = c$ .From definition derivative we have

$$f'(x) = \lim_{h\to 0}\frac{f(x_o+h)-f(x_o)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0.$$

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example: Let $f(x) = x^5$, then $f'(x) = 5x^4$

Definition: A function f is said to be differentiable at a point a if $f'(a)$ exists. It is differentiable on an open interval $(a, b), [a, \infty) or (-\infty, a]$ if it is differentiable at every number in the interval.

Theorem: If f is differentiable at a then f is continuous at a.

Note. The converse of this theorem is not true.

4.2. Basic rules

If f and g are differentiable at x and c is any constant then $f+g, f-g, f.g, cf, f/g$ where $g \neq 0$ are differentiable at x

1) $(f+g)' = f' + g'$ (sum rule)
2) $(f-g)' = f' - g'$ (difference rule)
3) $(f.g)' = f'.g + f.g'$ (product rule)
4) $(cf)' = cf'$ constant multiple rule
5) $(f/g)' = \frac{f'.g - f.g'}{(g)^2}$ Provided $g \neq 0$. (Quotient rule).

Proof. (1-4) are left as exercise

Proof .5 From definition of derivative, we have

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h.g(x).g(x+h)}$$

Adding and subtracting $f(x).g(x)$ in the numerator yields

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x)}{h.g(x).g(x+h)}$$

$$= \lim_{h\to 0} \left(\frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x).g(x+h)} \right)$$

$$. \qquad = \left(\frac{\lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0} g(x) . \lim_{h\to 0} g(x+h)} \right)$$

$$= \frac{g(x).f'(x) - f(x)g'(x)}{(g(x)^2}$$

Example: If $f(x) = \sqrt{x}g(x)$ where $g(4) = 2$ *and* $g'(4) = 3$, find $f'(4)$

Solution Applying the product rule, we get

$$f'(x) = \frac{d}{dx}\left[\sqrt{x}g(x)\right] = \sqrt{x}\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}\left[\sqrt{x}\right]$$

$$=\sqrt{x}g'(x) + g(x)\frac{x^{-\frac{1}{2}}}{\frac{1}{2}} = \sqrt{x}g'(x) + \frac{g(x)}{2\sqrt{x}}$$

$$f'(4) = \sqrt{4}g'(4) + \frac{g(4)}{2\sqrt{4}} = 6.5$$

Example: Let $h(x) = \frac{x^2-1}{x^4+1}$,then find $h'(x)$

Solution. Let $f(x) = x^2-1$ and $g(x) = x^4+1$, which implies $f'(x) = 2x$ and $g'(x) = 4x^3$

$$= \frac{g(x).f'(x) - f(x)g'(x)}{(g(x))^2} = \frac{(2x)(x^4+1) - (x^2-1)(4x^3)}{(x^4+1)^2}$$

$$= \frac{-2x^5+4x^3+2x}{(x^4+1)^2}.$$

Example: Find an equation of the tangent line to the curve $y = \frac{e^x}{(1+x^2)}$ at the point $(1, 1/e)$

SOLUTION According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}$$

So the slope of the tangent line at $(1, e/2)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 0$$

This means that the tangent line at $(1, e/2)$ is horizontal and its equation is $y = e/2$.

Theorem: The chain Rule

If g is differentiable at the point x and f is differentiable at the point $g(x)$, then composition fog is differentiable at the point x and $(gof)'(x) = g'(f(x))f'(x)$.

Example: *Let* $k(x) = \sqrt{1+x^4}$ then find $k'(x)$

Solution: Let $f(x) = 1 + x^4$ and $g(x) = \sqrt{x}$ then $f'(x) = 4x^3$ and $g'(x) = \frac{1}{2\sqrt{x}}$ for $x > 0$

$$k'(x) = g'\big(f(x)\big)f'(x) = \frac{4x^3}{2\sqrt{f(x)}} = \frac{2x^3}{\sqrt{1+x^4}}$$

4.3. Derivatives of some functions

1) Derivative of trigonometric function

Let $f(x) = sinx$ from the definition of derivative we have

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h) - sinx}{h} = \lim_{h\to 0}\frac{\text{sinxcosh} + \text{cosxsinh} - sinx}{h}$$

$$= \lim_{h\to 0}\left[sinx\left(\frac{cosh-1}{h}\right) + cosx\left(\frac{sinh}{h}\right)\right]$$

$$= \lim_{h\to 0}\left[cosx\left(\frac{sinh}{h}\right) - sinx\left(\frac{1-cosh}{h}\right)\right]$$

$$= cosx\lim_{h\to 0}\frac{sinh}{h} - sinx\lim_{h\to 0}\frac{1-cosh}{h} = cosx(-1) - sinx(0) =$$

$cosx$

The derivative of $g(x) = cosx$ can be obtained in the same way and $(cos)' = -sinx$.

Note 1)$(tanx)' = sec^2x$ 2)$(secx)' = secxtanx$ 3) $(cot)' = -csc^2x$ 4)$(cscx)' = -cscxcotx$

Example : Find $f'(x) = x^2tanx$

Solution: using the product rule.

$f'(x) = x^2(tanx)' + tanx(x^2)' = x^2sec^2 + 2xtanx.$

Derivatives of exponential Functions

Let $Lt\ f(x) = e^x$

From definition of derivatives $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h}$

$$= e^x \lim_{h\to}\frac{e^h-1}{h} =$$

$e^x \ln e = e^x$

Therefore,$(e^x)' = e^x \ln e = e^x$

If $f(x) = a^x, for\ a > 0\ and\ a \neq 1$, then $\big(f(x)\big)' = a^x lna$

Example: Let $f(x) = 2^x$, then $\left(f(x)\right)' = 2^x ln2$.

Example: Differentiate $y = e^{\sin x}$

Solution: By chain rule

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{sinx}\right) = e^{sinx}\frac{d}{dx}(sinx) = e^{sinx}cosx$$

Derivatives of Logarithmic Functions

Let $f(x) = log_b\, x\,, x > 0$ and $b > 0\ \ , b \neq 1$,then from the definition of derivative we obtain

$$f'(x) = \lim_{h\to 0}\frac{\log_b(x+h) - \log_b x}{h} = \lim_{h\to 0} 1/h \log_b\left(\frac{x+h}{x}\right) = \lim_{h\to 0} 1/h \log_b\left(1 + \frac{h}{x}\right)$$

$$= \lim_{h\to 0}\frac{1}{vx}\log_b(1+v) \quad (\text{ let } v = h/x)$$

$$= 1/x \lim_{h\to 0}\log_b(1+v)^{1/v} = 1/x \log_b\left[\lim_{v\to 0}(1+v)^{1/v}\right]$$

$$= 1/x \log_b e = \frac{1}{xlnb}$$

Note. If $f(x) = lnx$ then $f'(x) = 1/x$

Example: Let $f(x) = ln(x^2+1)$, then find $f'(x)$

Solution $\boldsymbol{f'(x) = \frac{2x}{x^2+1}}$

4.4. Higher order derivatives

If f is a function then f' is the function that assigns the number $f(x)$ to each x at which f is differentiable. Since f' is a function we can carry the process at step further and define $f''(x)$ by

$f"(a) = \lim_{x\to a}\frac{f'(x) - f'(a)}{x-a}$ Whenever this limit exists we call $f''(a)$ the second derivative of f at a.

Example: Let $f(x) = sinx$, find $f''(x)$.

Solution: $f(x) = sinx \Rightarrow f'(x) = cosx \Rightarrow f''(x) = -sinx.$

For $n \geq +3$ we define the nth derivative of f at a by

$$f^{(n)}(a)=\lim_{x\to a}\frac{f^{(n-1)}(x)-f^{(n-1)}(a)}{x-a}$$

f is n times differentiable if $f^{(n)}(x)$ exists for all x in the domain.

Example: If $f(x)=3x^4-2x^3+x^2-4x+2$

$$f'(x)=12x^3-6x^2+2x \quad -4$$

$$f''(x)=36x^2-12x \quad +2$$

$$f''(x)=36x^2-12x \quad +2$$

$$f'''(x)=72x-12$$

$$f^{(4)}(x)=72$$

$$f^{(5)}(x)=0$$

$\vdots$

$$f^{(5)}(x)=0 \quad (n\geq 5)$$

Example : Let $f(x)=e^{cx}$. Find a formula for the n^{th} derivative of f.

Solution: $f(x)=e^{cx}\Rightarrow f'(x)=ce^{cx}\Rightarrow f''(x)=c^2e^{cx}\Rightarrow f(3)(x)=c^3e^{cx}$

In general for any positive integer n.

$f^{(n)}(x)=c^n e^{cx}$.

Example: Let:$f(x)=\frac{1}{1-x}$, then find a formula for the n^{th} derivative of f.

Solution: By applying the quotient rule

:$f'(x)=\frac{1}{(1-x)^2}$, $\quad f''(x)=\frac{2}{(1-x)^3}$,$f^{(3)}(x)=\frac{6}{(1-x)^4}$

Now , $1=1!$, $2=2!$ $\quad 6=3!$

So by proceeding in this way we obtain $f^{(n)}(x)=\frac{n!}{(1-x)^{n+1}}$
)

Exercise

Find a formula for nth derivative of the following function.

a)$f(x) = cosx$ b) $f(x) = sinx$ c) $f(x) = \frac{1}{x}$

4.5. Implicit differentiation

Function defined explicitly and Implicitly

Up to now, we have been concerned with differentiating functions that are expressed in the form $y = f(x)$. An equation of this form is said to be defined y explicitly as a function of x, because the variable y appears alone on one side of equation. However, sometimes function are defined by equation in which y is not alone on one side.

For example: the equation $yx + y + 1 = x$ (1)

is not of the form $y = f(x)$,however, this equation still defines y as a function of x since it can be $y = \frac{x-1}{x+1}$. Thus we say that (1) defines y implicitly as a function of x the function being $y = \frac{x-1}{x+1}$

In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the function defined implicitly.

Example: Find $\frac{dy}{dx}$ if $xy = 1$

Solution: One way to find $\frac{dy}{dx}$ is writing this equation as y=1/x$\Rightarrow \frac{dy}{dx} = -1/x^2$

However, there is another way to obtain this derivative.

i.e$\frac{d}{dx}[xy] = \frac{d}{dx}[1] \;\Rightarrow\; x\frac{d}{dx} + y = 0$

$$\Rightarrow \frac{dy}{dx} = -y/x \quad \text{but } y = 1/x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{1}{x}}{x} = -1/x^2$$

This method of obtaining derivative is called implicit differentiation.

Example: Use implicit differentiation to find $\frac{dy}{dx}$ if $5y^2 + siny = x^2$

Solution:$\frac{d}{dx}(5y^2 + siny) = \frac{d}{dx}(x^2) \quad \Rightarrow 5\frac{d}{dx}(y^2) + \frac{d}{dx}(siny) = 2x$(the chain rule)

$$\Rightarrow \; 10y\frac{dy}{dx} + cos\frac{dy}{dx} = 2x$$

Solving for $\frac{dy}{dx}$, we obtain $\frac{dy}{dx} = \frac{2x}{10y+cosy}$ (*)

Note that this formula involves both x and y .In order to obtain a formula for $\frac{dy}{dx}$, that involves x alone, we would have to solve the original equation for y in terms and then substituting in to (*). However, it is impossible to do this. So we are forced to leave the formula for $\frac{dy}{dx}$ in terms of x and y.

Example: Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$.

Solution: Differentiating both side of $4x^2 - 2y^2 = 9$ implicitly yield

$$8x - 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x}{y} \qquad (*)$$

Differentiating both side of (*) implicitly yields $\frac{d^2y}{dx^2} = \frac{(y)(2)-(2x)(dy/dx)}{y^2}$ (**)

Substituting (*) in to (**) and simplifying using the original equation we obtain

$$\frac{d^2y}{dx^2} = \frac{2y - 2x\left(\frac{2x}{y}\right)}{y^2} = \frac{2y^2 - 4x^2}{y^3} = -9/y^3$$

Example: Find the slope of the tangent line at $(2, -1)$ and $(2,1)$ to $y^2 - x + 1 = 0$.

Solution: Solving for y in terms of x and then evaluating the derivative of $y = \sqrt{x-1}$ at $(2,1)$ and the derivative of $y = -\sqrt{x-1}$ at $(2, -1)$.

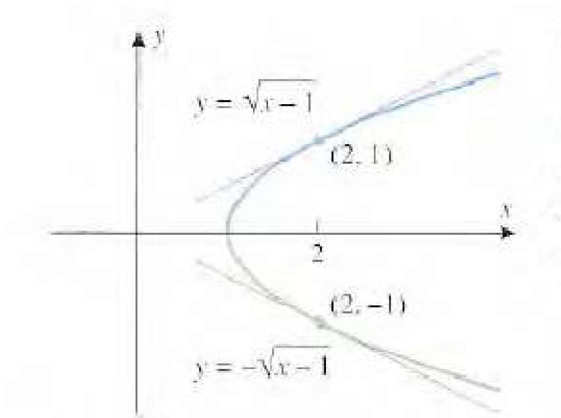
$$\frac{d}{dx}(y^2 - x + 1) = \frac{d}{dx}(0)$$

$$\Rightarrow 2y\frac{dy}{dx} - 1 = 0$$

$$\therefore \qquad \frac{dy}{dx} = 1/2y$$

At $(2,-1)$, we have $y = -1$ and at $(2,1)$ we have $y = 1$ the slope of the tangent lines at those points are

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = -1/2 \qquad and\ m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = 1/2$$



Exercise

Find $\frac{dy}{dx}$ $\ and$,find $\frac{d^2y}{dx^2}$,by implicit differentiation

1) $y = \sqrt[3]{2x-5}$
2) $sin(x^2y^2) = x$
3) $tan^3(xy + y) = x$
4) $x^2 = \frac{coty}{1+cscy}$
5) $3x^2 - 4y^2 = 7$
6) $x^3 + y = 1$
7) $y + siny = x$

4.6. Derivatives of Inverse functions

Let f be a function, then f has an inverse provided that there is a function g such that the domain of g is the range of f such that that $f(x) = y$ iff $g(y) = x$, for all x in the domain of f and y in the range of f.

Definition: Assume that the function f has an inverse and let f^{-1} be the unique function having as its domain the range of f satisfying $f(x) = y$ iff $f^{-1}(y) = x$ for all x in the domain of f and y in the range$f$. Then f^{-1} is the inverse of f.

From the above definition we can see that, if f is a function the inverse of f is f^{-1} such that for each x in the domain of f

$(f^{-1}of)(x) = x = (fof^{-1})(x)$.

Domain of f= range of f^{-1}

Range of f^{-1} =domain of f

Theorem : A function f has an inverse iff for some numbers x_1 and x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

i.e. A function f is said to be invertible iff it is one to one.

Steps to find inverse of a function

1) Write $y = f(x)$

2) Interchange x and y

3) Solve for y in terms of x in step 2

4) Write $f^{-1}(y)$ for y

Example: Let $f(x) = 3x - 2$,then find the inverse of f

Solution: Let $y = 3x - 2 \Rightarrow x = 3y - 2 \Rightarrow x + y = 3y \Rightarrow y = \dfrac{x}{3} + 2/3$

$\Rightarrow f^{-1}(x) = \frac{x}{3} + 2/3$

Theorem: Let f be continuous on an interval$I$, and let the values assigned by f to the points in I form an interval$J$. If f has an inverse, then f^{-1} is continuous on J.

Theorem: Suppose that f has an inverse and is continuous on an open interval I containing a. Assume also that$f'(a)$ exists and $f'(a) \neq 0$ and $f(a) = c$.Then$(f^{-1})'(c)$ exist and

$$(f^{-1})'(c) = \frac{1}{f'(a)}$$

Proof. Using the fact that $f^{-1}(c) = a$ and definition of derivatives we find that

$$(f^{-1})'(c) = \lim_{y\to c} \frac{f^{-1}(y)-f^{-1}(c)}{y-c} = \lim_{y\to c} \frac{f^{-1}(y)-a}{f(f^{-1}(y))-f(a)} \qquad (*)$$

First notice that f^{-1} is continuous at c. Therefore $\lim_{y\to c} f^{-1}(y) = f^{-1}(c) = a$.

So that if $x = f^{-1}(y)$ then x approaches a as y approaches c.

More over the fact that f^{-1} has an inverse and $f^{-1}(c) = a$, implies that $f^{-1}(y) \neq$ a for $y \neq c$

$$from\ (*) (f^{-1})'(c) = \lim_{y\to c} \frac{f^{-1}(y)-a}{f(f^{-1}(y))-f(a)} = \lim_{x\to a} \frac{x-a}{f(x)-f(a)} = \lim_{x\to a} \frac{\frac{1}{f(x)-f(a)}}{x-a} = \frac{1}{f'(a)}$$

Example: Let $f(x) = x^7+8x^3+4x-2$, Find $(f^{-1})'(-2)$

Solution: Let us find the value of a for which $f(a) = -2$, but $f(0) = -2$, so $a = 0$

Since $f'(x) = 7x^6+24x^2+4$. It follows that $f'(0) = 4$.

$$\Rightarrow (f^{-1})'(-2) = 1/f'(0) = 1/4.$$

We conclude this section with brief discussion of the general relation ship between the derivative of f and f^{-1}. For this purpose suppose that both function are differentiable and

let $y = f^{-1}(x) \Rightarrow x = f(x)$ (*)

and differentiating implicitly with respect to x yields $\frac{d}{dx}(x) = \frac{d}{dx}(f(y))$

$$\Rightarrow 1 = f'(y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}.$$

Thus from (*) we obtain the formula that relates the derivative of f^{-1} to the derivative of f

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

4.7. Inverse Trigonometric Functions and Their Derivatives.

.None of the six basic trigonometric functions is one to one because they all repeat periodically and do not pass the horizontal line test. Thus to define inverse trigonometric function we must first restrict the domain of trigonometric function to make them one to one.

The inverse of these restricted functions are denoted by

$\sin^{-1} x$,$\cos^{-1} x$, $\tan^{-1} x$,$\sec^{-1} x$,$\csc^{-1} x$ and $\cot^{-1} x$.(or alternatively by $arcsinx$, $arccosx$, $arctanx$,$arcsecx$,$arccscx$ and $arccotx$)

y=$\sin^{-1} x$ is equivalent to $siny = x\ if\ -1 \le x \le 1$ and$-\pi/2 \le y \le \pi/2$

$y = cos^{-1}x$ is equivalent to $cosy = x$ if$-1 \le x \le 1$ and $0 \le y \le \pi$.

$y = tan^{-1}x$ is equivalent to $tany = x$ if$-\infty < x < \infty$ and$-\pi/2 < y < \pi/2$.

$y = \cot^{-1} x$ is equivalent to $\cot y = x$ for$-\infty < x < \infty$ and $0 < y < \pi$.

$y = sec^{-1}x$ is equivalent to $secy = x$, if $|x| \ge 1$ and $0 \le y \le \pi\ and\ y \ne \pi/2$.

$y = \csc^{-1} x$ is equivalent tocsc $y = x$ if $|x| \ge 1$. $-\frac{\pi}{2} \le y \le \frac{\pi}{2} and\ y \ne 0$.

Identities for inverse trigonometric functions

1) $sin^{-1}x + cos^{-1}x = \pi/2$

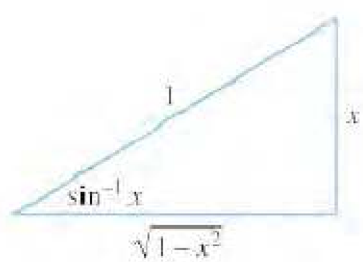
2) $\sin(\sin^{-1} x) = \sqrt{1 - x^2}$,$-1 \le x \le 1$

3) $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$,$-1 \le x \le 1$

4)$tan(\tan^{-1} x) = \frac{x}{\sqrt{1-x^2}}$,$-1 < x < 1$

5)$sec(tan^{-1}x) = \sqrt{1+x^2}$

6) $sin(sec^{-1}x) = \frac{\sqrt{x^2-1}}{x}$, $|x| \ge 1$



$\cos(\sin^{-1} x) = \sqrt{1-x^2}$

Now let us use implicit differentiation to obtain the derivative formula for $y = sin^{-1}x$

$$y = \sin^{-1} x \text{ if and only if } x = siny$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(siny)$$

$$\Rightarrow \; 1 = cosy \Rightarrow \frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}$$

Therefore, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

By applying the same procedure, we obtain the following

$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$ $\qquad$ $(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$

$(\tan^{-1} x)' = \frac{1}{1+x^2}$ $\qquad$ $(\cot^{-1} x)' = -\frac{1}{1+x^2}$

$$(c\,\mathrm{sc}^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}$$

4.8. Hyperbolic and inverse hyperbolic functions

The hyperbolic functions are special combinations of the exponential functions e^x and e^{-x} that occurs in certain applications. These functions have properties very similar to the properties of trigonometric functions. We shall define the hyperbolic functions and its inverse hyperbolic.

The two important hyperbolic functions are defined as follows

Definition: The hyperbolic sine function is defined by $inhx = \frac{e^x - e^{-x}}{2}$.

The hyperbolic cosine function is defined by $coshx = \frac{e^x - e^{-x}}{2}$.

From the definition, we conclude that the hyperbolic sine and cosine are defined for all real number x and y and differentiable

Notice that sinhx is an odd function with sinh0=0 and the coshx is an even function with cosh0=1

Since $0<e^x<1$ for $x<0$ and $e^x>1$ for $x>0$, it follows that sinhx<0 for x<0and sinhx>0 for x>0 and coshx>0 for all x.

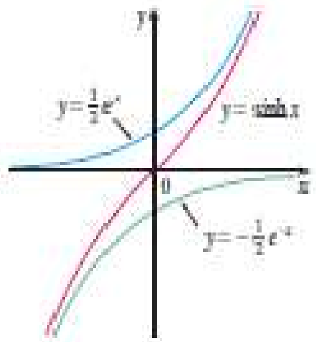


FIGURE 1
$y = \sinh x = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

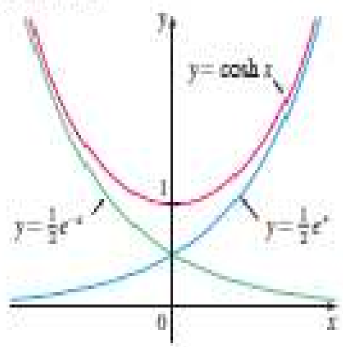


FIGURE 2
$y = \cosh x = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

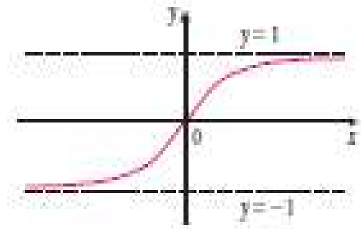


FIGURE 3
$y = \tanh x$

Direct calculation shows that

$cosh^2x - sinh^2 = 1$

This can be proved by writing.

$cosh^2x - sinh^2x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{1}{4}(e^{2x} + 2e^0 + e^{-2x}) - \frac{1}{4}\ (e^{2x} + 2e^0 + e^{-2x})$=1

We define the other four hyperbolic functions in terms of sinhx and coshx

$$tanhx = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad sechx = \frac{1}{coshx} = \frac{2}{e^x + e^{-x}}$$

$$cothx = \frac{coshx}{sinhx} = \frac{e^x+e^{-x}}{e^x-e^{-x}} \qquad cschx = \frac{1}{sinhx} = \frac{2}{e^x-e^{-x}}$$

$$\frac{d}{dx}sinhx = \frac{e^x + e^{-x}}{2} = coshx \qquad \frac{d}{dx}tanhx = sec^2hx$$

$$\frac{d}{dx}coshx = \frac{e^x - e^{-x}}{2} = sinhx$$

$$\frac{d}{dx}sechx = -sechxtanhx$$

$$\frac{d}{dx}cothx = -csc^2hx \qquad \frac{d}{dx}cschx = -cschxcothx$$

The six hyperbolic functions are related by many identities called hyperbolic identities. We list a few of them.

$$tanh^2x + sech^2h = 1 \qquad coth^2x - csch^2h = 1$$

$$\sin h(-x) = -sinhx \qquad \cosh(-x) = coshx$$

$$sinhx(x \pm y) = sinhxcoshy \pm coshxsinhy \qquad (*)$$

$$\cosh(x \pm y) = coshxcoshy \pm sinhxsinhy \qquad (**)$$

$$\tanh(x \pm y) = \frac{tanhx \pm tanhy}{1 \pm tanhxtanhy}$$

In verifying (*) we use the following relation

$coshx + sinhx = e^{x}$ $\qquad$ $coshx - sinhx = e^{-x}$

If we let $x = y$ in (*) and (**), we obtain the hyperbolic double angle formulas.

$$Sinh2x = 2sinhxcoshx$$

$$cosh2x = cosh^2x + sinh^2x$$

Using the identity $cosh^2x - sinh^2x = 1$ one can show that

$$cosh2x = 2sinh^2x + 1$$

$$cosh2x = 2cosh^2x - 1$$

Note To obtain the above hyperbolic formulae from the corresponding trigonometric formulae of circular function, replace $cosx$ by $coshx$ and $sinx$ by $sinhx$ where i=$\sqrt{-1}$.For instance we know that

$cos^2x + sin^2x = 1$.Making substitutions described above, we get

$cosh^2x + (isinhx)^2 = 1 or\ cosh^2x - sinh^2x = 1$. Other formulae can be similarly obtained

If u is a differentiable function of x then

$$\frac{d}{dx} sinhu = coshu \frac{du}{dx}$$

$$\frac{d}{dx} coshu = sihu\frac{du}{dx}$$

$$\frac{d}{dx} tanhu = sech^2u\frac{du}{dx}$$

$$\frac{d}{dx} cschu = -cschucothu\frac{du}{dx}$$

4.9. The Inverse Hyperbolic Functions

The function $f(x) = coshx$ is one to one for $x \geq 0$ and has inverse $y = cosh^{-1}x$ for all $x \geq 0$

Definition : a) $y = sinh^{-1}x \ iff \ x = sinhy$ for all x and y.

b) $y = cosh^{-1}x \ iff \ x = sinhy$ for $x \geq 1$ $and for$ $y \geq 0$

c) $y = tanh^{-1}x \ iff \ x = tanhy$ for $-1 < x < 1$ and for all y

d) $y = coth^{-1}x \ iff \ x = cothy$ for $|x|>1$ and $y\neq 0$

e) $y = sech^{-1}x \ iff \ x = sechy$ for $0 < x \leq 1$ and$y \geq 0$

f) $y = csch^{-1}x \ iff \ x = cschy$ for $x \neq 0$ and $y \neq 0$

Theorem : 1) $sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$ $(-\infty < x < \infty)$

2) $cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right)$ $(x \geq 1)$

3) $tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $(-1 < x < 1)$

4) $coth^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{x-1}\right)$ $(|x| > 1)$

5) $sech^{-1}x = \frac{1}{2}\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ $(0 < x \leq 1)$.

6) $csch^{-1}x = \frac{1}{2}\ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$ $x \neq 0$

$\boldsymbol{Proof}1$: $let \ y = sinh^{-1}x$

$\Rightarrow sinhy = \frac{e^y - e^{-y}}{2} = \frac{e^{2y}-1}{2e^y}$

$$\Rightarrow x = \frac{e^{2y}-1}{2e^y} \;\Rightarrow\; 2xe^y x = e^{2y} - 1 \; or (e^y)^2 - 2xe^y + 1 = 0$$

Our interest is to write y as a function of x so using quadratic formula

$$e^y = \frac{2x + \sqrt{4x^2+4}}{2} = x + \sqrt{x^2+1} \;\; \Rightarrow y = ln\left(x + \sqrt{x^2+1}\right) \;\; for\ all\ x$$

Proof (2-6) left as exercise

Derivative of inverse Hyperbolic.

$$\frac{d}{dx} sinh^{-1}x = \frac{1 + \frac{2x}{2\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

Theorem

$$\frac{d}{dx} sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx} cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$$

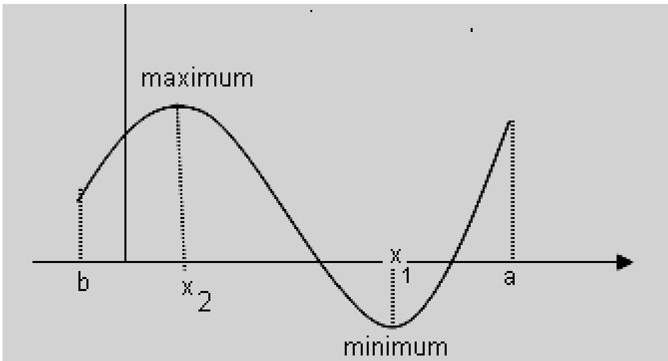
$$\frac{d}{dx} tanh^{-1}x = \frac{1}{1-x^2}$$

4.10. Application of derivative

Extreme of a function.

Definition

a. A function f has a maximum value (absolute maximum) on a set I if there is a number d in I such that $f\ (x) \le f(d)$ for all x in I (figure below). We call $f(d)$ the maximum value of f on I.
b. A function f has a minimum value(absolute minimum.) on a set I if there is a number c in I such that $f(x) \ge f(c)$ for all x in I (see figure below). We call the minimum value of f on I.
c. A value of f that is either a maximum value or minimum value on I is called an extreme value of f on I.



If the set I is the complete domain of f, and if f has a maximum value on I, then this maximum value is called the maximum value or (sometimes global maximum value) of f similarly, when it exists, the minimum value of f on its domain is the minimum value (or some times the global minimum value) of f.

A function f may or may not have extreme value on a set I depending on f and on$I$.

Example:

1. If $f(x) = x$, then on $[0,1]$ the function $f$has the maximum value of 1 and minimum value of 0 (as shown in the figure a below)
2. If $f(x) = \tan x$, then on $(-\frac{\pi}{2}, \frac{\pi}{2})$, the function f has neither a maximum value nor a minimum value (as shown in figure b below)
3. If $f(x) = |x|$, for $-1 \leq x < 0$ and $0 < x \leq 1$and $f(0) = 1$, then on [-1,1] the function has the maximum value of 1 but has no minimum value because f does not assume the value 0(as shown in figure c below).
4. If $f(x) = x$ for $-\infty < x < \infty$, then $(-\infty, \infty)$ the function has neither a maximum value nor a minimum value.

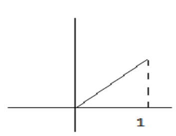


fig a

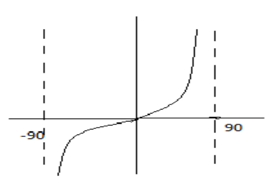


fig .b

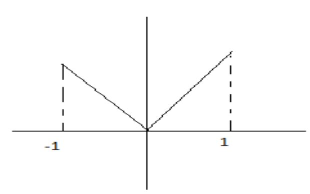


fig.c

Theorem (Extreme –value theorem)

If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$

Example: The function $f(x) = 2x + 1$, is continuous and hence has both an absolute maximum and absolute minimum on every closed interval and in particular on the interval [0,3] i.e $f(0) = 1$

(Minimum value) and $f(3) = 7$ (maximum value)

Theorem : suppose c is an interior point of an interval I and $f(c)$ is an extreme value of f on I and$f(c)$ is an extreme value of f on I .

If $f'(c)$ exists then $f'(c) = 0$.

Proof: The statement of the theorem is equivalent to the assertion that if c is any number interior to I such that $f'(c)$ exists and is not equal to 0, and then $f(c)$ is not an extreme value of f on I. therefore we assume that $f'(c) \neq 0$

Consider the following cases

Case 1 $f'(c) > 0$

Since $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$

$$\frac{f(x) - f(c)}{x - c} > 0$$

For all $x \neq c$ in some open interval about c.

For such x

If $x > c$, then $f(x) - f(c) = (x - c)\frac{f(x)-f(c)}{x-c} > 0$

Because $(x - c) > 0$

Therefore $f(x) > f(c)$, so that f does not have a maximum value at c. in the same way

If $x < c$, then $f(x) - f(c) = (x - c)\left(\frac{f(x)-f(c)}{x-c}\right) < 0$

Thus $f(x) < f(c)$, so that f does not have a minimum value at c hence if $f'(c) > 0$ then f has neither a maximum value nor minimum value at c.

The case $f'(c) < 0$ is treated in the same way.

The points at which either $f'(x) = 0$ or f is not differentiable are called the critical points of f.

Example: Find the critical number for $f(x) = 4x - x^2$

Solution: To find critical number c first find $f'(x)$

$f(x) = 4x - x^2$

$f'(x) = 4 - 2x$

$f'(x) = 0$

$\Rightarrow 4 - 2x = 0 \Rightarrow x = 2($ critical number)

(2,4) is critical point.

Finding extreme values on [$\boldsymbol{a, b}$]

1. Compute the values of f at all critical numbers in (a, b)
2. Compute the values of f at the end points $a \; and \; b$.
3. The largest of those value in step 1 and step2 is the maximum value of f on $[a, b]$ and the smallest of those values is the minimum value of f on $[a, b]$
   Example: Let$f(x) = x - x^3$, find the extreme values of f on [0, 1] and determine at which number in [0,1] they occur.
   Solution: since f is continuous on [0,1] it has extreme value on [0,1]
   First let us find the critical number.
   $f'(x) = 0$ this implies that $1 - 3x^2 = 0$,$\Rightarrow x = -\frac{\sqrt{3}}{3}$ or $x = \frac{\sqrt{3}}{3}$ but $-\frac{\sqrt{3}}{3} \notin [0,1]$
   Thus the extreme value of f on [0,1] can occur only at one of the end points 0 and 1 or at the critical numbers $\frac{\sqrt{3}}{3}$ in (0,1) $f(0) = 0, f(\frac{\sqrt{3}}{3}) == \frac{\sqrt{3}}{3} - \left(\frac{\sqrt{3}}{3}\right)^3 = \frac{2\sqrt{3}}{9}$ and $f(1) = 0$, so the maximum value of f on [0,1] is $\frac{2\sqrt{3}}{9}$ occurs at$\frac{2\sqrt{3}}{9}$ and its minimum value is 0. Which occurs at 0 and 1.

Exercise

1) A sheet of card board 12 in square is used to make an open box by cutting a square of equal size from the four corners and folding up the sides. what size squares should be cut to obtain a box with largest possible volume?

2) Find all critical number (if any) of the given functions

a. $f(x) = x + \frac{1}{x}$ c. $g(x) = \frac{1}{\sqrt{x^2+1}}$

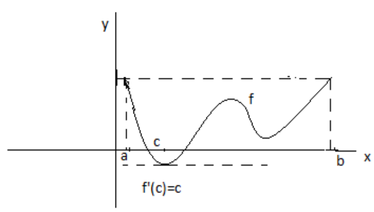
b. $f(x) = \sin x$ d. $f(x) = |x - 2|$ e. $f(x) = x^2 e^x$

2) Find all extreme value (if any) of the given function on the given interval. Determine at which numbers in the interval their value occur.
   a. $f(x) = x^2 - x$, [0.2]
   b. $g(x) = \sqrt{1 + x^2}$, [-2,3]
   c. $f(x)= \cos \pi x$,$(\frac{1}{3}, 1)$
3) A mass connected to spring moves along the x-axis so that its x co-ordinate at a time t is given by: $x(t) = 2 \sin 2t + \sqrt{3cos2t}$
   What is the maximum distance of the mass from the origin?

4.11. Mean value theorem

Theorem (Rolle's theorem)
Let f be continuous on [a,b] and differentiable on [a,b]. if $f(a) = f(b)$, then there is a number c in (a,b) such that $f'(c) = 0$.

Proof. If f is constant, then its derivative is 0, so that $f'(c) = 0$, for each c in (a,b). if f is not constant, then its maximum and minimum values (which exist by the maximum-minimum theorem) are distinct. Since $f(a) = f(b)$, either the maximum or the minimum must occur at a number c in (a,b). by hypothesis,f is differentiable atc so that $f'(c) = 0$



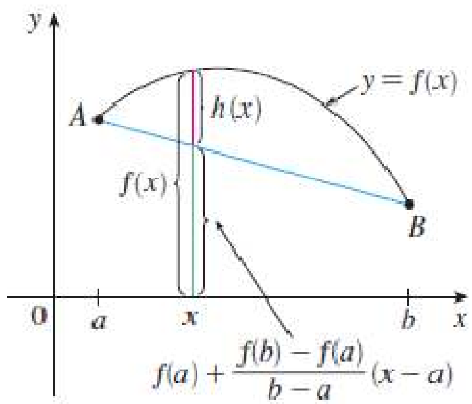
Theorem: (Mean value theorem)

Let f be continuous on [a,b] and differentiable on (a,b). then there is a number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof:

We introduce an auxilar function h that allows us to simplify the proof by using Rolle's theorem

(see adjacent fig.)



h is continuous on [a, b] and differentiable on (a, b), $h(a) = h(b) = 0$, so by Rolle's theorem there is a number c in (a, b) such that $h'(c) = 0$

$\Rightarrow h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, for $a < x < b$

And thus

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Example: Let $f(x) = \frac{1}{3}x^3 + 2x$, the show that f satisfies the mean value theorem on the interval [0, 2]

Solution: $f'(c) = \frac{f(3)-f(0)}{3-0} = 5$

We find a number c in (0, 3) such that $f'(c) = 5$

$$f'(x) = x^2 + 2 \Rightarrow f'(c) = c^2 + 2$$
$$\Rightarrow 5 = c^2 + 2$$
$$\Rightarrow c^2 = 3 \;\Rightarrow c = \pm\sqrt{3}$$

Since c must be in (0, 3), so $c = \sqrt{3}$

Exercise

Find all number c in the interval (a, b) for which the mean value theorem satisfied.

a. $f(x) = x^2 - 6x; a = 0, b = 4$

b. $f(x) = x^3 - 6x, a = -2, b = 0$

c. $f(x) = -3 + \sqrt{x}\,, a = 0, b = 1$

d. $f(x) = 3\left(x + \frac{1}{x}\right), a = \frac{1}{3}, b = 3$

Increasing and decreasing function

Let f be defined on an interval I then

a. f is increasing on I if $f(x_1) < f(x_2)$, where ever x_1 and $x_2$are in I and $x_1<x_2$.

b. $f$is decreasing on I if $f(x_1) > f(x_2)$, where ever x_1 and $x_2$are in I and $x_1<x_2$.
Graphically, a function is increasing on I if its graph slopes upward to the right and it is decreasing on I if its graph slopes down ward to the right.

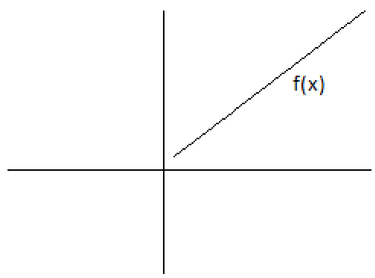


fig a f is increasing

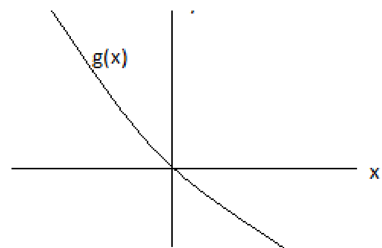


fig b g is decreasing

Theorem : Let f be continuous on an interval I and differentiable at each interior point of I

a. If $f'(x) > 0$ at each interior point of I then f is increasing on I more over f is increasing on I if $f'(x) > 0$ except for a finite number of points x in I

b. If $f'(x) < 0$ at each interior point of I, then f is decreasing on I moreover, f is decreasing on I if $f'(x) < 0$ except for a finite number of points x in I

Proof .Left as exercise

Example: Let $f(x) = \frac{1}{3}x^3 - x^2 + x - 5$, show that f is increasing on $(-\infty, \infty)$

Solution: $f'(x) = x^2 - 2x + 1 = (x-1)^2$

Since $f'(x) > 0$ for all x except x=1, where $f'(x) = 0$

Hence f is increasing on $(-\infty, \infty)$

Example: Find where the Function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where decreasing.

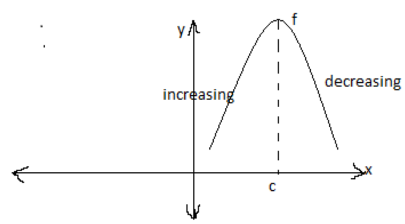
Solution: $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$

We divide the real line in to intervals whose end points are the critical numbers -1,0 and 2 and arrange in a chart.

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x<-1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty,-1)$
$1<x<0$			$\vert$		increasing on $(\ \ 1, 0)$
$0<x<2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x>2$	$+$	$+$	$+$	$-$	increasing on $(2,\infty)$

4.5.3.The first and second derivative tests.

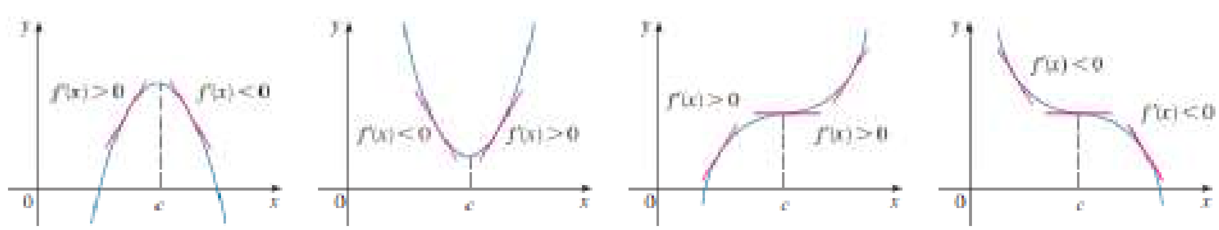
Definition: A function f has a relative maximum value (respectively a relative minimum value) at c if $f(c)$ is the maximum value (respectively, the minimum value) of f on an open interval containing c. A value that is either a relative maximum value or relative minimum value is called a relative extreme value.



Theorem: Let f be differentiable on an open interval about the number c except possibly at c, where f is continuous.

a. Iff' changes sign from positive to negative at c, then f has a relative maximum value at c.

b. If f' changes sign from negative to positive at c then f has a relative minimum value at c.



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

Example: Let $(x) = \frac{1}{4}x^3 - 3x$, find the relative extreme value of f

Solution: $f(x) = \frac{1}{4}x^3 - 3x$

$$\Rightarrow f'(x) = \frac{3}{4}x^2 - 3 = \frac{3}{4}(x^2 - 4) = \frac{3}{4}(x+2)(x-2)$$

				-2			2				
$x+2$		-	-	0		+	+		+		
$x-2$	-	-	-	-		-	0		+	+	
$f(x) = \frac{3}{4}(x+2)(x-2)$	+	+		0	-	-	0	+	+	+	

$f'(x)$ Changes sign from positive to negative at -2, so $f(-2) = 4$ is relative maximum value.

$f'(x)$ Changes sign from negative to positive at 2, so $f(2) = -4$ is relative minimum value of f.

Example: Let $f \quad (x) = (x-1)^2(x-3)^2$ .Determine the relative extreme of f

Solution: $\boldsymbol{f'(x)} = 2(x-1) \quad (x-3)^2 + 2(x-1)^2(x-3)$

$$= 2(x-1) \quad (x-3) \quad [(x-3) + (x-1)]$$

$$= 4(x-1) \quad (x-2)(x-3)$$

1 2 3

$\boldsymbol{x-1} - - - \boldsymbol{0} + + + + + +$

$\boldsymbol{x-2} - - - - - - - - - \boldsymbol{0} + + + + +$

$\boldsymbol{x-3} - - - - - - - - - - - - - \boldsymbol{0} + + +$

$\boldsymbol{f'(x)} == 4(x-1) \quad (x-2)(x-3) \quad - - - - 0 + + + 0 - - 0 + + + +$

From the sign chart $f'(x)$ Changes sign from positive to negative at 2, so $f(2) = 1$ is relative maximum value.

$f'(x)$ Changes sign from negative to positive at 1 and 3 so $f(1) = 0$ and $f(3) = 0$ are relative minimum value of f.

Theorem of second derivative test

Assume that $f'(c) = 0$

a. If $f''(c) < 0$ then $f(c)$ is a relative maximum value of f
b. If $f''(c) > 0$ then $f(c)$ is a relative minimum value of f
c. If $f''(c) = 0$ then we cannot draw any conclusion about a relative extreme value of f at c.

Proof: a. by hypothesis $f''(c) = \lim_{x \to c} \frac{f'(x)-f'(c)}{x-c} < 0$

Since $f'(c) = 0$ by hypothesis, it follows that for all $x \neq c$ in some open interval $(c - \sigma, c + \sigma)$

$\frac{f'(x)}{x-c} = \frac{f'(x)-f'(c)}{x-c} < 0$

If $c - \sigma < x < c$ then $x - c < 0$, so that $f'(x) > 0$. If $c < x < c + \sigma$ then $x - c > 0$ so that $f'(x) < 0$. This means that f' changes sign from positive to negative at c.

So by first derivative test f a. relative maximum value at c.

The proof (b) is analogous to the proof of (a)

Example: Let $f(x) = x^3 - 3x - 2$. Using the second derivative test. Find the relative extreme value of f.

Solution $f(x) = x^3 - 3x - 2$

$\Rightarrow f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$ and $f''(x) = 6x$

$f''(x) = 0$, when $x = -1 \; or \; x = 1$

Since $f''(-1) = -6 < 0 \; and \; f''(1) = 6 > 0$

$f(-1) = 0$ is relative maximum value of f where $f(-1) = -4$ is a relative minimum value of f these are the only relative extreme of f.

Exercise

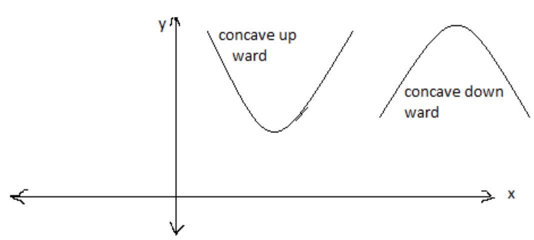
1. Determine the value of c at which
   a. $f(x) = x^2 + 6x - 11$
   b. $f(t) = \frac{t^2-t+1}{t^2+t+1}$
   c. $f(t) = \sin t + \frac{1}{2}t$
2. Use the first derivative test to determine the relative extreme values (if any) of the following.
   a. $f(x) = 4x^2 - \frac{1}{x}$
   b. $f(x) = x\sqrt{1-x^2}$
3. Use the second derivative test to determine the relative extreme value (if any) of the function.
   a. $f(x) = -4x^2 + 3x - 1$
   b. $f(x) = x^3 - 3x^2 - 24x + 1$
   c. $f(t) = e^t - e^{-t}$

Concavity and inflection point

Definition: If f is differentiable on open interval I, then f is said to be concave up on I if f' is increasing on I and f is said to be concave down if f' is decreasing on I.

Theorem : Let f be twice differentiable on an open interval I

a) If $f''(x) > 0$ on I then f is concave up ward on I
b) If $f''(x) < 0$ on I then f is concave downward on on I.



Example: Find open intervals on which on which the function $f(x) = x^3 - 3x^2 + 1$ is concave up and concave down.

Solution: $f(x) = x^3 - 3x^2 + 1 \Rightarrow f'(x) = 3x^2 - 6x.$

$\Rightarrow f''(x) = 6x - 6 = 6(x-1)$

$\Rightarrow f''(x) < 0$ if $x < 1$ and $f''(x) > 0$ if $x < 1$

$\Rightarrow$ f is concave up on $(1, \infty)$ and concave down ward on $(-\infty, 1)$

Inflection points

Definition: If f is continuous on an open interval I containing xo and if f changes the direction of concavity at that point then we say that f has an inflection point at xo and we call the point $(x_o, f(x_o))$ on the graph of f an inflection point of f.

How to find inflection point.

1) Find the value of c for which $f''(x) = 0$
2) For each value of c found in step 1 determine whether $f''(x)$ changes sign at c.
3) If f changes sign at c, the point $(c, f(c))$ is inflection point of f.

Example : Let $f(x) = x^4 - 6x^2 + 8x + 10$. Find the inflection point of the graph of f

Solution: $f(x) = x^4 - 6x^2 + 8x + 10$

$\Rightarrow f'(x) = 4x^3 - 12x + 8 \Rightarrow f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x-1)(x+1)$

		-1		1	
$x+1$	- - -	0	+ + +		+ + +
$x-1$	- - - - -	-	- - - - - - - - -	0	+ +
$f''(x) = 12(x-1)(x+1)$	+ + +	0	- - - - - - - -	0	+ ++

From the sign chart

f changes sign at -1 and 1

$(1, f(1)) = (1,13)$ and $(1, f(x)) = (1, -3)$ are inflection points of f

Exercise

Find the inflection point of the function $f(x) = xe^{-x}$

4.12. Curve sketching

A knowledge of derivative helps greatly in sketching the graph of function.

Example : Let $f(x) = \frac{2}{1+x^2}$, sketch the graph of f by noting all the relevant properties.

Solution. y –intercept: $f(0) = 2$.

No x-intercept because $f(x) > 0$ for all x

$$f'(x) = \frac{-4x}{(1+x^2)^2}$$

Since $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$, it follows that f is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$ and $f(0) = 2$ is the maximum value of f.

$$f''(x) = \frac{4(3x^2 - 1)}{(1 + x^2)^3} = \frac{4(\sqrt{3}x - 1)(\sqrt{3}x + 1)}{(1 + x^2)^3}$$

$-\frac{\sqrt{3}}{3}$ $\qquad\qquad$ $\frac{\sqrt{3}}{3}$

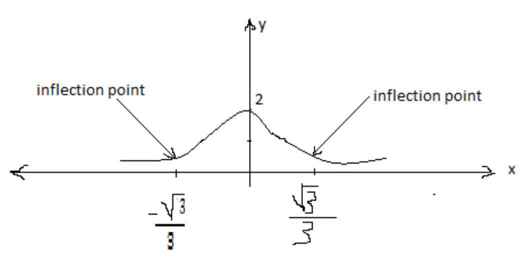
$\sqrt{3}x + 1$	$- - -\ 0 + + + + + + + + + + + + + + +$
$\sqrt{3}x - 1$	$- - - - - - - - - - - - - - - - - - 0 + + +$
$f'(x) = \dfrac{4(3x^2 - 1)}{(1 + x^2)^3}$	$+ + + + + + + 0 - - - - - - - - - - - - 0 + + + +$

From the sign chart

f is concave up ward on $(-\infty, -\frac{\sqrt{3}}{3})$ and $(\sqrt{3}/3, \infty)$ and concave downward on$(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ with inflection point at $(\frac{\sqrt{3}}{3}, 3/2)$ and $(-\frac{\sqrt{3}}{3}, 3/2)$

And $\lim_{x \to -\infty} \frac{2}{1+x^2} = \lim_{x \to \infty} \frac{2}{1+x^2} = 0$

This implies that x-axis is the horizontal asymptote of f.



Exercise

Sketch the graph of a)$f(x) = x^3 - 8x^2 + 16x - 3$.

b) $g(x) = \frac{x+2}{x-3}$

4.13. Related rate

When spherical balloon is inflated, the radius r and the volume v of the balloon are function of time t.

$V = \frac{4}{3}\pi r^3.$

Using the chain rule to differentiate v with respect to t

$\frac{dv}{dt} = \frac{dv}{dt}\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$

The rates $dv/dt$and dr/dt are related

Example: Suppose a spherical balloon is inflated at the rate of 10 cubic centimeter per minute

How fast the radius of the radius of the balloon increasing when the radius is 5cm?

Solution:$\frac{dv}{dt}$=$4\pi r^2\frac{dr}{dt}$ But $\frac{dv}{dt} = 10$,$r = 5cm.$

$\Rightarrow$ 10=$4\pi(5)^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt}$=$\frac{1}{10\pi}$.

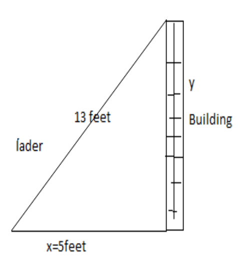
Therefore, when the radius is $5cm$, the radius is increasing at the rate $1/10\pi$ per minute.

Example: One end of a 13 feet ladder is on the floor and the other end rests on a vertical wall. If the bottom end is drawn away from the wall at 3 feet per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 5 feet from the wall?

Solution: Let y be the height of the top of the ladder above the floor and let x be the distance beween the base of the wall and the bottom of the ladder.

$dx/dt = 3$, we required to find dy/dt.

From Pythagoras we have



$x^2 + y^2 = 13^2$

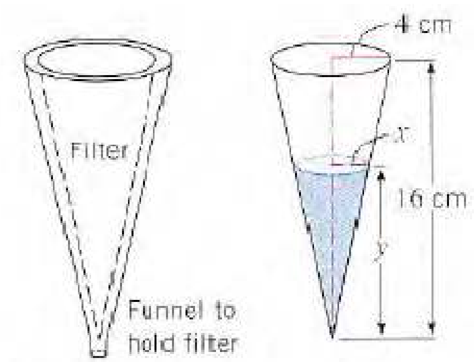
$$\Rightarrow 2x dx/dt + 2y dy/dt = 0$$

$\Rightarrow dy/dt = -x/y \; dx/dt$ .at the instant $x = 5 \Rightarrow y = 12$

$\Rightarrow dy/dt = -5/12(3) = -5/4$

When the bottom of the ladder is 5 feet from the wall, the top is sliding down at the rate 5/4 feet per second.

Example: Suppose that liquid is to be cleared of sediment by pouring it through a conical filter that is 16cm high and has a radius of 4cm at the top(fig. below). Suppose that the liquid flows out of the cone at constant rate of $2cm^2/min$. at what rate is the depth of the liquid changing when the level is 8cm?



Solution: Let

t = time elapsed from the initial observation (min)
V = volume of liquid in the cone at time t (cm^3)
y = depth of the liquid in the cone at time t (cm)
x = radius of the liquid surface at time t (cm)

From the formula for the volume of a cone, the volume V, the radius x, and the depth y are related by

$$V = \frac{1}{3}\pi x^2 y \qquad (1)$$

By using similarity of triangles we have

$$\frac{x}{y} = \frac{4}{16} \quad \text{or} \quad x = \frac{1}{4}y$$

Substituting this in to (1) gives

$$V = \frac{\pi}{48} y^3 \qquad (2)$$

Differentiating (2) with respect to t gives

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2 \frac{dy}{dt}\right)$$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2}$$

The minus sign indicates y is decreasing with time

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min}$$

Exercise.

A ladder 15 feet long leans against a vertical wall .Suppose that when the bottom of the ladder is x feet from the wall the bottom is being pushed toward the wall at the rate of $x/2$ feet per second. How fast is the top of the ladder rising at the moment the bottom is 5 feet from the wall.

4.14. L'Hopital's rule

Suppose that $\lim_{x\to *} f(x)$ and $\lim_{x\to *} g(x)$ are both 0 assume $g'(x) \neq 0$ for x near $*$

Then $\lim_{x\to *} \frac{f(x)}{g(x)} = \lim_{x\to *} \frac{f'(x)}{g'(x)}$.

The indeterminate form 0/0

If $\lim_{x\to *} f(x) = 0 = \lim_{x\to *} g(x)$, we say that $\lim_{x\to *} \frac{f(x)}{g(x)}$ has indeterminate form 0/0

Example: Find $\lim_{x\to 0} \frac{sin4x}{sin3x}$

Solution:$\lim_{x\to 0} sin4x = \lim_{x\to 0} sin3x = 0$.

By applying L'Hopital's Rule

$\lim_{x \to 0} \frac{sin4x}{sin3x} = \lim_{x \to 0} \frac{4cos4x}{3co3x} = 4/3.$

The indeterminate form ∞/∞

Suppose $\lim_{x \to *} f(x)$=∞ $or - \infty$ and $\lim_{x \to *} g(x) = \infty$ $or - \infty$ then we say that $\lim_{x \to *} \frac{f(x)}{g(x)}$

has indeterminate form ∞/∞, L' Hopital's rule is valid in this case.

$\lim_{x \to *} \frac{f(x)}{g(x)} = \lim_{x \to *} \frac{f'(x)}{g'(x)}$ (provided that the later limit exist as a number ∞ $or - \infty$).

Example: In each part confirm that the limit is an indeterminate form of the type ∞/∞

And apply L'Hopital's rule

a)$\lim_{x \to \infty} \frac{x}{e^x}$ b) $\lim_{x \to 0^+} \frac{x}{cscx}$

solution:a) The numerator and the denominator have a limit of ∞ so we have indeterminate form of the type ∞/∞.

Applying the L'Hopital's rule

$$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$$

c) The numerator has a limit of $-\infty$ and the denominator has alimit of $+\infty$ so we have indeterminate form of the type ∞/∞. Applying L'Hopital's rule we get.

$$\lim_{x \to 0^+} \frac{x}{cscx} = \lim_{x \to 0^+} \frac{1/x}{cscxcotx} = \lim_{x \to 0^+} \frac{sinx}{x} tanx = \lim_{x \to 0^+} \frac{sinx}{x} \lim_{x \to 0^+} tanx = (-1)0 = 0$$

Other Indeterminate Forms

Indeterminate form	**Example**
1) $0.\infty$	$\lim_{x \to 0} xlnx$
2) 0^0	$\lim_{x \to 0} x^x$
3) 1^∞	$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$
4) ∞^0	$\lim_{x \to \infty} x^{1/x}$
5) $\infty - \infty$	$\lim_{x \to o^+}(cscx - cotx)$

When we find the limit of the indeterminate forms of the type listed above, we have to rewrite the given limit in a way that enable us to L' Hopital's rule

Example: Find $\lim_{x\to o^+} x\ln x$

Solution: $\lim_{x\to 0^+} x=0$ and $\lim_{x\to 0^+}\ln x = -\infty$

The limit is of the form

$0.\infty$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\left(\frac{\ln x}{1/x}\right) = \lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right) = \lim_{x\to 0^+}(-x)$$

$\therefore \lim_{x\to o^+} x\ln x = 0$

4. Review Questions

1) a) Find the slope of tangent line to the curve $y = x - x^3$ at the point (1,0). ans -2

b) Find the equation of the tangent line passing through the origin and tangent to the graph of the function $y = \ln x$. ans $y = \frac{1}{e}x$

c) Find equation of the line tangent to curve $y = 1 + x^3$,which is parallel to the line $12x - y = 1$ ans $y = 12x - 15$.

2) Find $f'(a)$ if $f(x) = xe^x, a = 1$ ans $2e$

3) Find $f'(x)$ if $i) f(x) = e^x\sin x$ ans $e^x\sin x + e^x\cos x$

ii) $f(x) = \cos(\sin x)$ ans $-[\sin(\sin x)]\cos x$

$iii)$ $f(x) = \ln(\ln x)$ ans $\frac{1}{x\ln x}$

4) Use implicit differentiation to find $\frac{dy}{dx}$.

a) $\frac{\sin y}{y^2+1} = 3x$ ans $\frac{3(y^2+1)}{(y^2+1)\cos y - 2y\sin y}$

ii) $xe^y = y + x^2$ ans $\frac{2x - e^y}{xe^y - 1}$

iii) $x^2 + y^2 = \frac{y^2}{x^2}$ ans $\frac{x^4+y^2}{xy - x^3y}$

5)Find the dimension of a rectangle with perimeter 100m whose area is as large as possible . ans 25m by 25m.

6) Differentiate the following.

a) $f(x) = sech\sqrt{x}$ ans $-\frac{1}{2\sqrt{x}} sech\sqrt{x}\, tanh\sqrt{x}$

b) $f(x) = sinh^2\sqrt{1-x^2}$ ans $-\frac{2x}{\sqrt{1-x^2}} sinh\sqrt{1-x^2}\ cosh\sqrt{1-x^2}$

c)) $f(x) = sinh^{-1}(-3x^2)$ ans $\frac{-6x}{\sqrt{9x^2+1}}$

7) If $1200cm^2$ of material is available to make a box with square base and an open top find the largest possible volume of the box. Ans $4000cm^3$

8) Find the point on the line $y = 4x + 7$ that is closest to the origin. Ans $(\frac{-28}{17}, \frac{7}{17})$

9)Find the the interval on which f is increasing or decreasing

a) $f(x) = x^3 - 12x + 1$. ans increasing on $(-\infty, -2)$,$(2, \infty)$ and decreasing on $(-2,2)$

b) $f(x) = xe^x$. ans increasing on $(-1, \infty)$, and decreasing on (-$\infty, -1)$

10)Find the local maximum and local minimum values of the function

$f(x) = x + \sqrt{1-x}$. Ans Local max. $f\left(\frac{3}{4}\right) = 5/4$

Chapter Five

Integration

Unit out comes

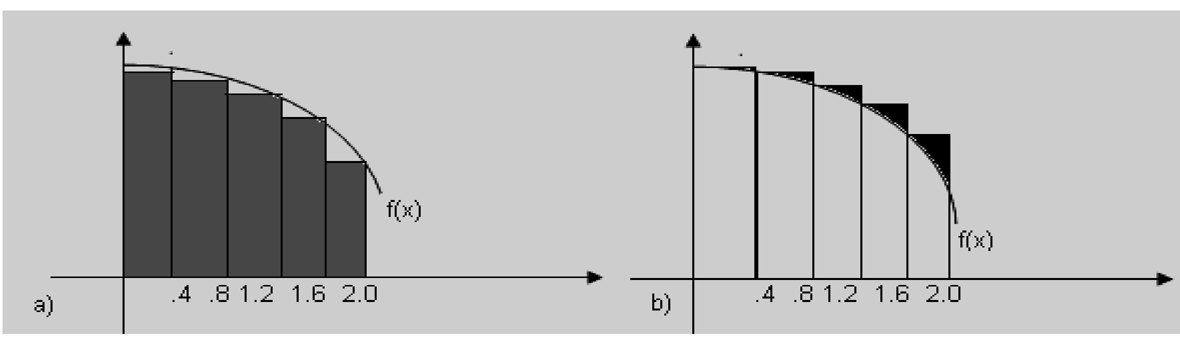
- **At the end of this chapter the learners will be able to:**
  - **Estimating area with Finite Sums**
  - **Find The Definite Integral as a Limit of Riemann Sums**
  - **Determining the Properties of the Definite Integral**
  - **Defining the Fundamental Theorem of Calculus**
  - **Evaluate Indefinite Integrals and the Substitution Rule**
  - **Integrate different functions by using Techniques of integration**
  - **Define and Evaluate Improper integral.**

Introduction

One of the great achievements of classical geometry was to obtain formulas for the areas and volumes of triangles, spheres, and cones. In chapter five and chapter six we will study methods of finding formulas and calculate the areas and volumes of these and other more general shapes. The method we develop, called integration. The integral has many applications in statistics, economics, the sciences, and engineering.

5.1. Estimating Area with Finite Sums

Consider the following plane region under the curve y=f(x)=$-x^2+5$.Use the five rectangles as in the figure below for the function f(x) =-x^2 +5 and the x-axis between the graph of x=0 and x=2



Solution: The right end points of the five interval are $\frac{2}{5}i$ where i=1,2,3,4,5 the width of each rectangle is $\frac{2}{5}$ since area of a rectangle is $\boldsymbol{area = hight * width}$ then the area of the five rectangles on the intervals $\left[0,\frac{2}{5}\right],\left[\frac{2}{5},\frac{4}{5}\right],\left[\frac{4}{5},\frac{6}{5}\right]\left[\frac{6}{5},\frac{8}{5}\right],\left[\frac{8}{5},\frac{10}{5}\right]$ is

i. Evaluate f at the right end point from fig (a)

$$\text{Area (a)} = \sum_{i=1}^{5} f\left(\frac{2i}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i}{5}\right)^2+5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48$$

ii. Evaluate f at the left end point from fig (b)

$$\text{Area (b)}=\sum_{i=l}^{5} f\left(\frac{2i-2}{5}\right)\left(\frac{2}{5}\right) \;=\sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^{2}+5\right]\left(\frac{2}{5}\right) \;=\frac{202}{25}=8.08$$

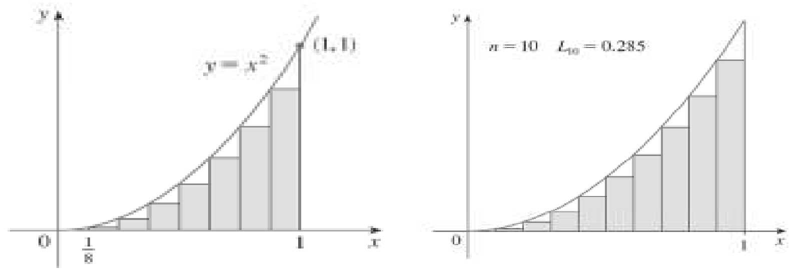
By combining the results in parts (a) and (b), you can conclude that

$$6.48 \le (\text{area of the region under f}) \le 8.08$$

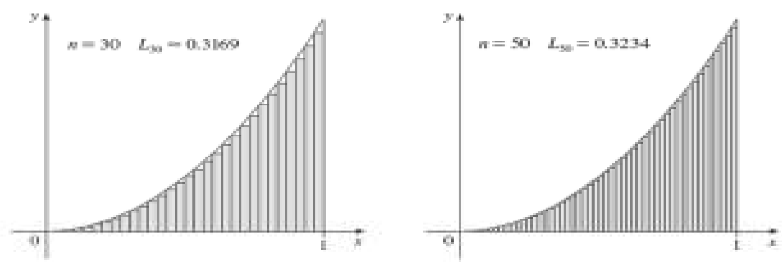
We get an upper estimate of the area of R in fig (b) by using five rectangles containing R is 8.08. We get lower estimate of the area of R in fig (a) by using five rectangles containing R is 6.08. And we know that the area of the region under the curve is between 6.08 and 8.08.

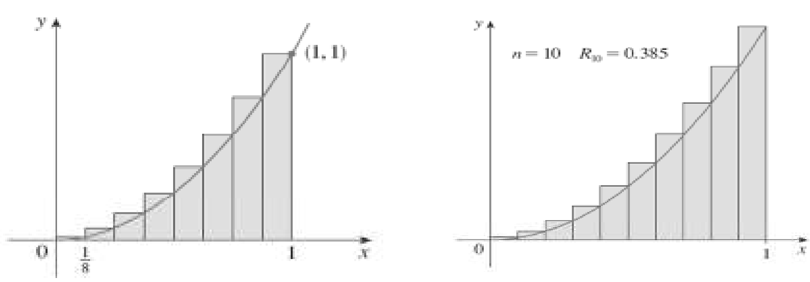
Note: As the number of rectangles increase the upper sum decreases but lower sum increases. At some point the two sums will overlap as the number of rectangles tends to infinity.

Example:. Consider the function$f(x) = x^2$.

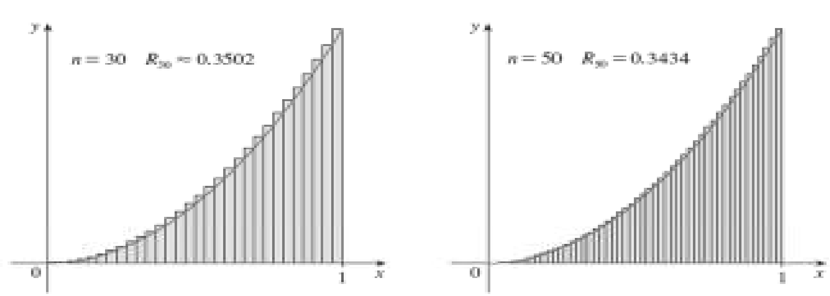


(a) Using left endpoints

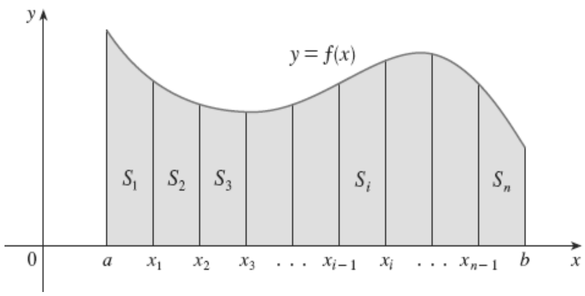


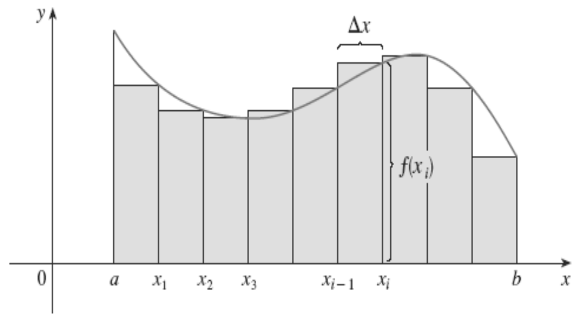


(b) Using right endpoints



Definition: Let f(x) be a nonnegative and continuous function on an interval $[a,b]$ and let S be the region bounded by the graph of f and the vertical line x=a, x=b and the x-axis as show in figure below, Then, the area A(s) is given by





$A(S) = \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i)\,\Delta x_i$ where $z_i = [x_{i-1}, x_i]$ $i = 1,2,3,\dots,n$ and $\Delta x = \dfrac{b-a}{n}$

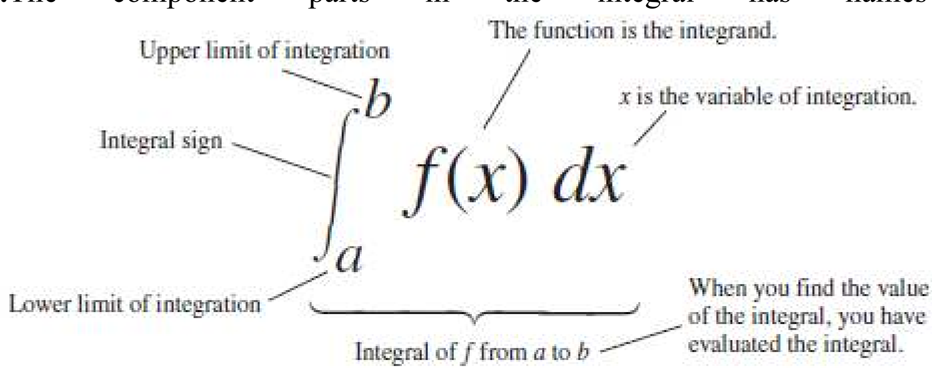
5.2. The Definite Integral

The Definite Integral as a Limit of Riemann Sums

If f is continuous function defined for $a \le x \le b$,we divide the interval [a,b] in to n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be the endpoints of these subinterval and we let $z_i = [x_{i-1}, x_i]$ $i = 1,2,3,\dots,n$ be any sample points in these subintervals, where $z_1 = [x_0, x_1]$,$z_2 = [x_1, x_2]$ and so on. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i)\,\Delta x$$

Note-1. The component parts in the integral has names as follows



Note-2. The sum $\sum_{i=1}^{n} f(z_i)\,\Delta x$ is called a Riemann sum after the German Mathematician Bernhard Riemann (1826-1866). If f(x) is positive then the Riemann Sum can be interpreted as the sum areas of the approximating rectangles.

Note-3. The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable.

If we decide to use t or u instead of x we simply write the integral as

$$\int_a^b f(t)dt \text{ or } \int_a^b f(u)du \text{ instead of } \int_a^b f(x)dx$$

THEOREM 1
The Existence of Definite Integrals
A continuous function is integrable. That is, if a function f is continuous on an interval [a, b], then its definite integral over [a, b] exists.

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — Also a Definition
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any Number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

Example: Suppose that f and g are integrable and

$$\int_1^2 f(x)dx = -4 \quad , \int_1^5 f(x)dx = 6 \quad \text{and} \int_1^5 g(x)dx = 4$$

Then, find

a. $\int_2^2 g(x)dx$ b. $\int_5^1 f(x)dx$ c. $\int_2^5 f(x)dx$ d. $\int_1^5 (4f(x) - g(x))dx$

Solution.

a. $\int_{2}^{2} g(x)dx = 0$ By rule two (the zero width interval)

b. $\int_{5}^{1} g(x)dx = -\int_{1}^{5} g(x)dx = -(4) = -4$ By rule 1

c. $\int_{1}^{5} f(x)dx = \int_{1}^{2} f(x)dx + \int_{2}^{5} f(x)dx$ from rule 5

$$\int_{2}^{5} f(x)dx = \int_{1}^{5} f(x)dx - \int_{1}^{2} f(x)dx = 6 - (-4) = 10$$

d. $\int_{1}^{5} (4f(x) - g(x))dx = 4\int_{1}^{5} f(x)dx - \int_{1}^{5} g(x)dx$ combination of rule 3 and 4

$$4(6) - 4 = 20$$

Example: show that the value of

$$\int_{0}^{1} \sqrt{1 + cosx}dx < 1.5$$

Solution: The Max-Min Inequality for definite integrals (Rule 6) says that minf. $(b - a)$ is the lowerbound for the value $\int_{a}^{b} f(x)dx$ and maxf. $(b - a)$ is the upper bound

minf=1 and maxf=$\sqrt{1 + 1} = \sqrt{2} \approx 1.414$

∴ The upper bound for the integral $\int_{0}^{1} \sqrt{1 + \cos x}dx \leq \sqrt{2}(1 - 0) = \sqrt{2}$

Since the integral $\int_{0}^{1} \sqrt{1 + \cos x}dx$ is bounded above by $\sqrt{2}$, then it is less than 1.5.

Exercise:

1. Suppose that f and g are integrable and

$$\int_{1}^{2} f(x)dx = 2 \quad , \int_{1}^{3} f(x)dx = 4 \quad \text{and} \int_{1}^{3} g(x)dx = 1$$

Then, find

a. $\int_{1}^{1} g(x)dx$ b. $\int_{3}^{1} f(x)dx$ c. $\int_{2}^{3} f(x)dx$ d. $\int_{1}^{5} (2f(x) + 5g(x))dx$

2. Use the min-max Inequality to find the upper and lower bounds for the values of

a. $\int_{0}^{1} \frac{1}{1+x^2} dx$ b. $\int_{0}^{0.5} \frac{1}{1+x^2} dx$

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: *differential calculus* and *integral calculus*.

The Fundamental Theorem of Calculus Part 1

If f is continuous on [a, b] then$F(x) = \int_a^x f(t)dt$ is continuous on [a,b] and differentiable on (a,b) and its derivative is f(x);

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\, dt = f(x)$$

Example:

Use the Fundamental Theorem to find

a. $\frac{d}{dx}\int_{a}^{x} \text{costdt}$ b. $\frac{d}{dx}\int_{0}^{x} \frac{1}{1+t^2} dt$

Solution:

a. $\frac{d}{dx}\int_{a}^{x} \text{cosxdt} = cosx$

b. $\frac{d}{dx}\int_{0}^{x} \frac{1}{1+t^2} dt = \frac{1}{1+x^2}$

Fundamental Theorem, Part 2 (The Evaluation Theorem)

If f is continuous at every point of [a, b] and F is any antiderivative of f on [a, b], then

$\int_a^b f(x)dx = F(b) - F(a)$ *, where* $F(x)$ *is the antiderivative of* $f(x)$*, that is,* $F'(x) = f(x)$.

The theorem says that to calculate the definite integral of f over [a, b] all we need todo is:

a. Find an antiderivative F of f, and
b. Calculate the number

$$\int_a^b f(x)dx = F(b) - F(a)$$

The usual notation $F(b) - F(a)$ *is*

$F(x)]_a^b$ or $[F(x)]_a^b$ depending on whether F has one or more terms.

Example

Use the FTC (Fundamental Theorem of Calculus) to evaluate the following.

$$a. \int_1^2 x^2\,dx$$

$$b. \int_0^\pi cosx\,dx$$

$$c. \int_0^3 f(x)\,dx \quad where\ f(x) = \begin{cases} 9-x, & x<1 \\ x^2+7, & x \geq 1 \end{cases}$$

Solution:

a. Since the antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$ then

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

b. Since the antiderivative of $f(x) = cosx$ is $F(x) = sinx$ then

$$\int_0^\pi cosx\,dx = sinx]_0^\pi = sin\pi - sin0 = 0$$

c. By using the property-5 we can write as

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx$$

$$= \int_0^3 f(x)\,dx$$

$$= \int_0^1 (9-x)\,dx + \int_1^3 (x^2+7)dx$$

$$= \ 9x - \frac{x^2}{2}\Big]_0^1 + \frac{x^3}{3} + 7x\Big]_1^3$$

$$= \left(9 - \tfrac{1}{2}\right) + \left(\tfrac{27}{3} + 21\right) - \left(\tfrac{1}{3} + 1\right) = 33.1667$$

5.3. Indefinite Integrals and the Substitution Rule

A definite integral is a number defined by taking the limit of Riemann sums associated with partitions of a finite closed interval whose norms go to zero. The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed easily if we can find an antiderivative of the function. Antiderivatives generally turn out to be more difficult to find than derivatives. However, it is well worth the effort to learn techniques for computing them. The set of all antiderivatives of the function f is called the **Indefinite integral** of f with respect to x, and is symbolized by

$$\int f(x)dx$$

When we to find the indefinite integral of a function f, remember that it always includes an arbitrary constant C.

Note: A definite integral $\int_a^b f(x)dx$ is a number. But an indefinite integral $\int f(x)dx$ is a function plus an arbitrary constant C.
Here are some examples of derivative formulas and their equivalent integration formulas:

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2\,dx = x^3 + C$
$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}dx = \sqrt{x} + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t\,dt = \tan t + C$
$\frac{d}{du}[u^{3/2}] = \frac{3}{2}u^{1/2}$	$\int \frac{3}{2}u^{1/2}\,du = u^{3/2} + C$

Properties of the indefinite integral

If f and g are continuous, and if k is any constant, then

a) $\int kf(x)dx = k\int f(x)dx$

b) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Some of the most important are given in Table in the table below

Integration Formulas

Differentiation Formula	Integration Formula
$\frac{d}{dx}[x] = 1$	$\int dx = x + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r (r \neq -1)$	$\int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
$\frac{d}{dx}[sinx] = cosx$	$\int cosxdx = sinx + C$
$\frac{d}{dx}[cosx] = -sinx$	$\int (-sinx)dx = cosx + C$
$\frac{d}{dx}[tanx] = sec^2x$	$\int (sec^2x)dx = tanx + C$
$\frac{d}{dx}[-cotx] = csc^2x$	$\int (csc^2x)dx = -cotx + C$

$\frac{d}{dx}[secx] = secxtanx$	$\int (secxtanx)dx = secx + C$
$\frac{d}{dx}[-cscx] = cscxcotx$	$\int (cscxcotx)dx = -cscx + C$
$\frac{d}{dx}[e^x] = e^x$	$\int (e^x)dx = e^x + C$
$\frac{d}{dx}[ln\|x\|] = \frac{1}{x}$	$\int \left(\frac{1}{x}\right) dx = ln\|x\| + C$

Example: Describe the anti-derivative of 3x

Solution We have k=3 and f(x)=x

$$\int kf(x)dx = k\int f(x)dx \Longrightarrow \int 3xdx = 3\int x^1 dx = 3\frac{x^2}{2} + c$$

Example: Fill the following table using the basic rules of integration

Original integral	Rewrite	Integrate	simplify
$\int \frac{1}{x^3}dx$	$\int x^{-3}dx$	$\frac{x^{-3+1}}{-3+1}$+C	$\frac{x^{-2}}{-2} + C = \frac{-1}{2x^2}$+C
$\int \sqrt{x}dx$	$\int x^{\frac{1}{2}}dx$	$\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$+C	$\frac{2x^{\frac{3}{2}}}{3} + C$
$\int 2\sin xdx$	$2\int \sin xdx$	2(-cosx)+C	-2cosx+C
$\int 2e^x dx$	____________	____________	____________
$\int \left(\frac{\cos x}{\sin^2 x}\right)dx$	$\int \left(\frac{\cos x}{\sin x \sin x}\right)dx$	$\int \cot x \csc xdx$	$\csc x + C$

Example: Evaluate

$a. \quad \int (2x + 3\cos x)dx \qquad b. \int (x + x^3)dx$

Solution: a.

Using the above properties

$$\int (2x + 3\cos x)dx = \int 2xdx + 3\int \cos xdx$$

$$= 2\int xdx + 3\int \cos xdx$$

$$= 2\left(\frac{x^2}{2}\right) + C_1 + 3\sin x + C_2$$

$$= x^2 + 3\sin x + C \text{ where } C=C_1+C_2$$

Solution: ***b.***

$$\int (x + x^3)\,dx = \int xdx + \int x^3 dx = \frac{x^2}{2} + C_1 + \frac{x^3}{3} + C_2$$

$$= \frac{x^2}{2} + \frac{x^3}{3} + C \;\; where\; C = C_1 + C_2$$

The above property (b) can be extended to more than two functions, which can be formulated as follows:

$$\int [C_1 f_1(x) + C_2 f_2(x) + \cdots + C_n f_n(x)] = C_1 \int f_1(x)dx + C_2 \int f_2(x)dx + \cdots + C_n \int f_n(x)dx$$

Example: Evaluate

$$\int \frac{2x^2 + x^2\sqrt[3]{x} - 1}{x^2} dx$$

Solution: First we need to write the integrand in a simpler form

$$\int \frac{2x^2 + x^2\sqrt[3]{x} - 1}{x^2} dx = \int \left(2 + x^{\frac{1}{2}} - x^{-2}\right) dx$$

$$= \int \left(2 + x^{\frac{1}{2}} - x^{-2}\right) dx$$

$$= 2\int dx + \int x^{\frac{1}{2}} dx - \int x^{-2} dx$$

$$= 2x + x^{\frac{3}{2}} - x^{-1} + C$$

$$= 2x + x^{\frac{3}{2}} - \frac{1}{x} + C$$

Exercise: Evaluate the following integrals

1. (a) $\int x^8\,dx$ (b) $\int x^{5/7}\,dx$ (c) $\int x^3\sqrt{x}\,dx$
2. (a) $\int \sqrt[3]{x^2}\,dx$ (b) $\int \frac{1}{x^6}\,dx$ (c) $\int x^{-7/8}\,dx$
3. $\int \left[5x + \frac{2}{3x^5}\right] dx$
4. $\int [x^{-3} - 3x^{1/4} + 8x^2]\,dx$
5. $\int \left[\frac{10}{y^{3/4}} - \sqrt[3]{y} + \frac{4}{\sqrt{y}}\right] dy$

5.4. Techniques of integration

Integration by Substitution

Theorem: Let f(x) and g(x) be functions with both fog and g' continuous on an interval I. If F is an antiderivative (indefinite integral) of f on I, then

(i) $\int f(g(x))g'(x)dx = F(g(x)) + C$

Example: Find $\int (x+6)^{21}\,dx$

Solution To solve this we use integration by substitution

Let g(x)=x+6 implies $g'(x)=1$ and $f(g(x))=(g(x))^{21}$

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

$$\int \left(g(x)^{21}\right)g^{/}(x)dx = \frac{(g(x))^{21+1}}{21+1} + C = \frac{(g(x))^{22}}{22} + C = \frac{(x+6)^{22}}{22} + C$$

Equivalently we can let

$$u = x + 6 \; implies \; du = dx$$

$$\int (x+6)^{21}dx = \int (u)^{21}du = \frac{u^{22}}{22} + C = \frac{(x+6)^{22}}{22} + C$$

Example: consider the following patterns

Original integral	**g(x)=u**	g' **(x)=du**	$f(g(x))g^{/}(x)dx$	
$\int 2x(x^2+1)^4 dx$	$u=x^2+1$	du=2xdx	$\int u^4 du$	___________
$\int 3x^2\left(\sqrt{x^3+1}\right)dx$	$u=x^3+1$	$du=3x^2dx$	$\int \sqrt{u}du$	___________
$\int 2\sin 6x dx$	u=6x	du=6dx	$\frac{1}{3}\int \sin u du$	___________
$\int 2e^{3x}dx$	u=3x	du=3dx	$\frac{2}{3}\int e^{u}du$	___________

The logarithm as an integral

Definition: The natural logarithm is the function defined on the integral $(0\,\infty)$ by lnx $= \int_1^x \frac{1}{t}dt$

Corollary $\frac{d}{dx}\left(\int_1^x \frac{1}{t}dt\right) = \frac{1}{x} \Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$

Example: Evaluate $\int_2^6 \frac{1}{x}dx$ in terms of logarithm

Solution: $\int_2^6 \frac{1}{x}dx = \ln x\Big|_2^6 = \ln 6 - \ln 2 = \ln\frac{6}{2} = \ln 3$

Note: $\int \frac{1}{x}dx = \ln|x| + c$

Example: Find $\int \frac{x^4}{x^5+1}dx$

Solution: let $u = x^5+1$ so that $du = 5x^4dx$

$$\int \frac{x4}{x^5+1}dx = \int \frac{1}{x^5+1}\, x^4dx = \frac{1}{5}\int \frac{1}{u}dx$$

$$= \frac{1}{5}\ln|u| + c = \frac{1}{5}\ln|x^5+1| + c$$

Exercise:

1. Show that $\int \tan x dx = -\ln|\cos x| + c$
2. Evaluate the integrals

i) $\int \frac{1}{x-1}dx$ ii) $\int \frac{x}{x^2+4}dx$ iii) $\int \frac{2}{1-4x}dx$ iv) $\int \frac{x^2}{1-x^2}dx$ v) $\int_0^1 \frac{x+2}{x^2-4x-1}dx$

Integration by parts

Theorem Let F and G be differentiable on $[a,b]$, and assume that F' and G' are continuous on $[a,b]$ then.

$\int f(x)G'(x)dx = F(x)G(x) - \int F'(x)G(x)dx.$

and $\int_a^b F(x)G'(x)dx = F(x).G(x) - \int_a^b F'(x)G(x)dx$

$\int udv = uv - \int vdu$

Example: Find $\int x\cos xdx$

Solution: the integral $x\cos x$ can naturally be split in to the two parts x and cosx

Let $u = x$ implies $du = dx$ $\quad$ $dv = \cos xdx$, $v = \sin x$

$$uv = \int vdu + \int udv$$

$$x\sin x = \int \sin xdx + \int x\cos xdx$$

$$x\sin x + \cos x + c = \int x\cos xdx$$

Example: Find $\int 2xe^{3x}dx$

Since 2x is easily differentiable and e^{3x} is easily integrable we can

Let $u = 2x$ and $dv = e^{3x}dx$

$du = 2dx$ and $v = \frac{1}{3}e^{3x}$

$$\frac{1}{3}2xe^{3x} = \frac{2}{3}\int e^{3x}dx + \int 2xe^{3x}dx = \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + c$$

Exercise: Find $\int_0^1 x^5e^{-x}dx$

Reduction Formula

Note: 1. $\int \sin^n xdx = \frac{-1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}xdx$

2. $\int \cos^n xdx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}xdx$

Example: Find $\int \cos^5 x dx$

Let n=5 in the reduction formula

$$\int \cos^5 x dx = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\int \cos^3 x dx$$

A second application

$$\int \cos^3 x dx = \int \frac{1}{3}\cos^2 x \sin x + \frac{3}{2}\int \cos x dx$$

$$= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + c,$$

Consequently

$$\int \cos^5 x dx = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left(\frac{2}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + c\right)$$

$$= \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + c$$

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)dx$ in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficultly, are natural conditions for integration by parts. However, if many repeat are required the calculation that saves a great deal of work. It is called tabular integration and is illustrated in the following examples.

Example: Find $\int x^2 e^x dx$ by tabular integration

Solution: with $f(x) = x^2$ and $g(x) = e^x$, we list

$f(x)$ and its derivative		$g(x)$ and its integrals
x^2		e^x
$2x$	(+)	e^x
2	(-)	e^x
0	(+)	e^x

We add the products of the functions connected by the arouse, with the middle sign changed, to obtain

$$\int x^2 e^2\, dx = \int x^2 e\, dx = x^2 e^x - 2x\, e^x + 2e^x + c$$

Example: Find $\int x^3 \sin x dx$ by tabular integration

Solution :With $f(x) = x^3$ and $g(x) =$ sinx, we list,

$f(x)$ and its Derivates		$g(x)$ and its integrals
x^3	(+)	$\sin\ x$
		$\cos x$
$3x^2$	(-)	
		$\sin x$
$6x$	(+)	
		$\cos x$
6	(-)	
		sinx
0		

We add the products of the functions connected by the arouse, with every other sign changed,

$$\int x^5 \sin xdx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$$

Exercise

1. Evaluate the integrals by tabular Method.

a) $\int x^3 e^x dx$ b) $\int (x^2 - 5x) e^x dx$ c) $\int p^4 e^{-p} dp$

2. Use integration by parts to evaluate the following integralsEvaluate $\int_0^{\frac{7}{6}} \cos^5 xdx.$

a) $\int x \ln x^2$

b) $\int xe^{-1} dx$

c) $\int_0^{\frac{\Pi}{2}} (x + x \sin x) dx$

d) $\int \frac{\sin x}{e^x} dx$

Summary of common integrals using integration by parts

1. for integrating of the form

$\int x^n e^{ax} dx, \int x^n \sin kdx,$ or $\int x^n \cos axdx$, Let $u = x^n$ and let $dv = e^{ax} dx, \sin axdx, \ \cos axdx.$

2) For integrals of the form

$\int x^n \ln xdx,$ Arcs sin ax, or arctanax Let $dv = x^n dx$

3) For integrals of the form $\int e^{ax} \sin bxdx$ Or $\int e^{ax} \cos bxdx$

Let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax} dx$

Note

Choose u in this order: LIPET

Logs, **I**nverse trig, **P**olynomial, **E**xponential, **T**rig

Example:

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln x\,dx$$

LIPET

logarithmic factor $\Rightarrow$ $u = \ln x$ $\quad dv = dx$

$$u\,v - \int v\,du$$

$$du = \frac{1}{x}dx \qquad v = x$$

$$x\ln x - x + C$$

Partial fraction

Consider the following rational functions

1) $\dfrac{2x^3}{x^2+3}$ We can divide the function by the long division as follows

$$\begin{array}{r|l} & 2x \\ \hline x^2+2 & 2x^3 \\ & -\left(2x^3+6x\right) \\ \hline & -6x \end{array}$$

$$\frac{2x^3}{x^2+3} = 2x + \frac{-6x}{x^2+3}$$

Then $\displaystyle\int \frac{2x^3}{x^2+3}dx = \int 2x\,dx - \int \frac{6x}{x^2+3}dx$

2) $\displaystyle\frac{2x+4}{x^2+3x+2} = \frac{2x+4}{(x+2)(x+1)} = \frac{2(x+2)}{(x+2)(x+1)} = \frac{2}{x+1}$

$$\int \frac{2x+4}{x^2+3x+2}dx = \int \frac{2}{x+1}dx$$

Examples of Integration by partial fraction:

Example: Evaluate $\int \frac{2x+3}{x^3+2x^2+x}dx$

Solution:

$$\frac{2x+3}{x^3+2x^2+x} = \frac{2\left(x+\frac{3}{2}\right)}{x(x+1)^2} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

Determine A,B and C

$A(x+1)^2+B(x(x+1))+2x+3$

$A(x^2+2x+1)+B(x^2+x)+cx=2x+3$

$Ax^2+2Ax+A+Bx^2+Bx+Cx=2x+3$

$A=3$

$A+B=0$
$B=-A$
$B=-3$

$2A+B+C=2$

$C=-1$

$$\frac{2x+3}{x(x+1)^2}=\frac{3}{x}-\frac{3}{x+1}-\frac{1}{(x+1)^2}$$

$$\int \frac{(2x+3)}{x(x+1)^2}dx=\int \frac{3}{x}dx-3\int \frac{1}{x+1}dx-\int \frac{1}{(x+1)^2}dx$$

$$=\ln|x|-3\ln|x+3|+\frac{1}{x+1}+C_1$$

$$=3lin\left|\frac{x}{x+1}\right|+\frac{1}{x+1}+C_1$$

Example: Evaluate $\int \frac{x^2+2x+7}{x^3+x^2-2}dx$

Solution: $\frac{x^2+2x+7}{x^5+x^2-2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+C}$

Find A and B and C

$A=2,$ $B=-1$ $C=-3$

$$\frac{x^2+2x+7}{x^3+x^2-2} = \frac{2}{x-1} - \frac{x+3}{x^2+2x+2}$$

Thus $\int \frac{x^2+2x+7}{x^3+x^2+2}dx=\int \frac{2}{x-1}dx+\int \frac{x+3}{x^2+2x+2}dx$

$$=\int \frac{2}{x-1}dx + \int \frac{x+3}{(x+1)^2+1}dx$$

Let $u=x+1$ $x=u-1\Rightarrow x+3=u+2$

$$= \int \frac{2}{x-1}dx + \int \frac{u+2}{u^2+1}du$$

$$= \int \frac{2}{x-1}dx + \int \frac{u}{u^2+1}du + \int \frac{2}{u^2+1}du$$

$$= 2\ln|x-1| + \frac{1}{2}\ln\left((x+1)^2+1\ \right)^2 + 2\arctan(x+1) + c$$

Summary for determining the constants by partial fraction is given as follows.

1. $\frac{numerrator}{(x+p)(x+q)} = \frac{A}{(x+p)} + \frac{B}{(x+q)}$
2. $\frac{numeantor}{(x+p)^2} = \frac{A}{(x+p)} + \frac{B}{(x+p)^2}$
3. $\frac{numerator}{(x^2+p)(x+q)^2} = \frac{Ax+B}{x^2+p} + \frac{C}{x+q} + \frac{D}{(x+q)^2}$

The Heaviside "cover-up" method for linear factor

When the degree of the polynomial f(x) is less than the degree of g(x), and $g(x) = (x-r_1)(x-r_2)..........(x-r_n)$ is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fraction.

Example: find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

Solution: If we multiply both sides of the above expression by $(x-1)$ to get

$$\frac{x^2+1}{(x-1)(x-3)} = A + \frac{B(x-1)}{(x-2)} + \frac{C(x-1)}{(x-3)}$$

And let x=1

$$\frac{1^2+1}{(-1)(-2)} = \frac{2}{2} = A + 0 + 0 \Rightarrow A = 1$$

These the value of A is the number we would have obtained if we covered the factor $(x-1)$ in the denominator of the original function.

$\frac{x^2+1}{(x-1)(x-2)(x-3)}$ and evaluate the rest at x=1

A= $\frac{(1)^2+1}{(x-1)(1-2)(1-3)} = \frac{2}{(-1)(-2)} = -1$

B= $\frac{2^2+1}{(2-1)(x-2)(2-3)} = \frac{5}{(1)(-1)} = -5$

Finally C is found covering the $(x-3)$ in above equation and evaluating at x=3

$$C = \frac{3^2+1}{(3-1)(3-2)(x-3)} = \frac{10}{2\times 1} = 5$$

Example: Evaluate

$$\int \frac{x+4}{x^3+3x^2-10x}dx$$

Solution: the degree of $f(x) = x+4$ is less than the degree of $g(x) = x^3+3x^2-10x$, and with $g(x)$ factored.

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x+5)}$$

$$A = \frac{0+4}{x(0-2)(0+5)} = \frac{4}{(-2)(5)} = \frac{-2}{5}$$

cover up

$$B = \frac{2+4}{2(x-2)(2+5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

cover up

$$C = \frac{-5+4}{(-5)(-5-2)x+5} = \frac{-1}{(-5)(-7)} = \frac{-1}{35}$$

cover up

Therefore: $\dfrac{x+4}{x(x-2)(x+5)} = \dfrac{-2}{5x} + \dfrac{3}{7(x-2)} - \dfrac{1}{35(x+5)}$

and $\displaystyle\int \frac{x+4}{x(x-2)(x+5)}dx \quad = \quad \frac{-2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x = 5| + C$

Home Task

Evaluating the following indefinite integral

a) $\displaystyle\int \frac{x^2}{(x-1)(x^2+2x+1)}dx$

b) $\displaystyle\int \frac{z}{z^3-z^2-6z}dz$

Exercise

1. a) $\int \frac{x}{x+1}dx$ b) $\int \frac{x^2}{x^2+1}$ c) $\int \frac{3x}{(x-2)^2}dx$

2. Evaluate the following integrals

   a. $\int \frac{e^x}{1-e^{3x}}dx$

   b) $\int \frac{\sin^2 x\cos x}{\sin^2 x+1}dx$

   c) $\int e^x\left(x^2+1\right)dx$

3. Verify that $\int_a^{a+k\pi} \sin^2 xdx = \int_a^{a+k\pi} \cos^2 xdx = \frac{k\pi}{2}$ For k is any integer

5.5. Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines,but it is often simpler to work with other functions, as in the integral

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^{m} x \cos^{n} xdx$$

where m and n are non-negative

➢ When m is odd $(m = 2k + 1)$	Substitute $sin^2x = 1 - cos^2x$
➢ If m is even and n is odd	Substitute $cos^2x = 1 - sin^2x$
➢ If both m and n are even	Substitute $cos^2x = \frac{1+cos2x}{2}$ and Substitute $sin^2x = \frac{1 - cos2x}{2}$

Example:

Evaluate the following indefinite integrals

$$a. \int sin^3xcos^2xdx$$

$$b. \int cos^3xdx$$

$$c. \int sin^2xcos^2xdx$$

Solution a. Since m is odd then we Substitute $sin^2x = 1 - cos^2x$

$$\int sin^3xcos^2xdx = \int (1 - \cos^2 x)sinxos^2xdx = \int (1 - \cos^2 x)cos^2xsinxdx$$

Let $u = cosx \implies du = -sinxdx\ then$

$$\int (1 - u^2)u^2du = -\int (u^2 - u^4)du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{cos^5x}{5} - \frac{cos^3x}{3} + C$$

Solution b. Since m=0 is Even and n=3 is Odd then we substitute $cos^2x = 1 - sin^2x$

$$\int cos^3xdx = \int cos^2xcosxdx = \int (1 - sin^2x)cosxdx$$

Let $u = sinx \implies du = cosxdx\ then$

$$\int (1 - u^2)du = u - \frac{u^3}{3} + C = sinx\ - \frac{sin^3x}{3} + C$$

Solution c.m=2 and n=2 are both even then we substitute$cos^2x = \frac{1+cos2x}{2}$ and

$$sin^2x = \frac{1 - cos2x}{2}$$

$$\int sin^2xcos^2xdx = \int \left(\frac{1 - cos2x}{2}\right)\left(\frac{1 + cos2x}{2}\right)dx = \frac{1}{4}\int (1 - cos^22x)dx$$

Let $u = 2x \implies du = 2dx\ then$

$$\frac{1}{2}\int (1 - cos^2u)du = \frac{1}{2}\left(\int 1du - \int cos^2udu\right)$$

$$= \frac{1}{2}\left(u - \int cos^2udu\right)$$

$$= \frac{1}{2}\left(u - \int \frac{1 + cos2u}{2}du\right)$$

$$= \frac{1}{2}\left(u - \frac{1}{2}u - \frac{1}{4}sin2u\right) + C$$

$$= \frac{1}{2}\left(2x - x - \frac{1}{4}sin4x\right) + C$$

$$= \frac{1}{2}\left(x - \frac{1}{4}sin4x\right) + C$$

Exercise:

Evaluate the following indefinite integrals

$$a. \int sin^5xdx$$

$$b. \int 4tan^4xdx$$

$$c. \int sin^4xcos^2xdx$$

Products of Sines and Cosines

Integrals of the form

$$\int \sin mx \sin nx\, dx\,, \int \sin mx \cos nx\, dx \;\;, and \int \cos mx \cos nx\, dx$$

Can be solved by using the following identities

i. $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$

ii. $\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$

iii. $\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$

Example:

Evaluate the following indefinite integrals

$a. \int \sin 2x \cos 3x\, dx$

$b. \int \cos 4x \cos 3x\, dx$

$c. \int \sin 2x \sin 3x\, dx$

Solution a. By using property (i)with m=2 is Even and n=3

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\sin 2x \sin 3x = \frac{1}{2}[\cos(2-3)x - \cos(2+3)x]$$

$$\sin 2x \sin 3x = \frac{1}{2}[\cos(-x) - \cos(5x)]\text{then}$$

$$\int \sin 2x \cos 3x\, dx = \frac{1}{2}\int [\cos(-x) - \cos(5x)]dx$$

$$= \frac{1}{2}\int [\cos(-x) - \cos(5x)]dx$$

$$= \frac{1}{2}\left\{\int cosxdx - \int cos5xdx\right\}$$

$$since\ cosine\ is\ an\ even\ function \cos(-x) = cosx$$

$$=\frac{1}{2}\left(sinx - \frac{1}{5}sin5x\right) + C$$

Solution b. By using property (iii)with m=4 is Even and n=3

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(4-3)x + \cos(4+3)x]$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(x) + \cos(7x)]\text{then}$$

$$\int \cos 4x \cos 3x\, dx = \frac{1}{2}\int [\cos(x) + \cos(7x)]dx$$

$$=\frac{1}{2}\left(\int \cos(x)dx+\int \cos(7x)dx\right)$$
$$=\frac{1}{2}\left(sinx+\frac{1}{7}sin7x\right)+C$$

Solution c. Exercise

Exercise: Evaluate the following definite integrals

a. $\int_{-\pi}^{\pi} \sin 3x \sin 3x dx$

b. $\int_{0}^{\pi} \cos 3x \cos 4x dx$

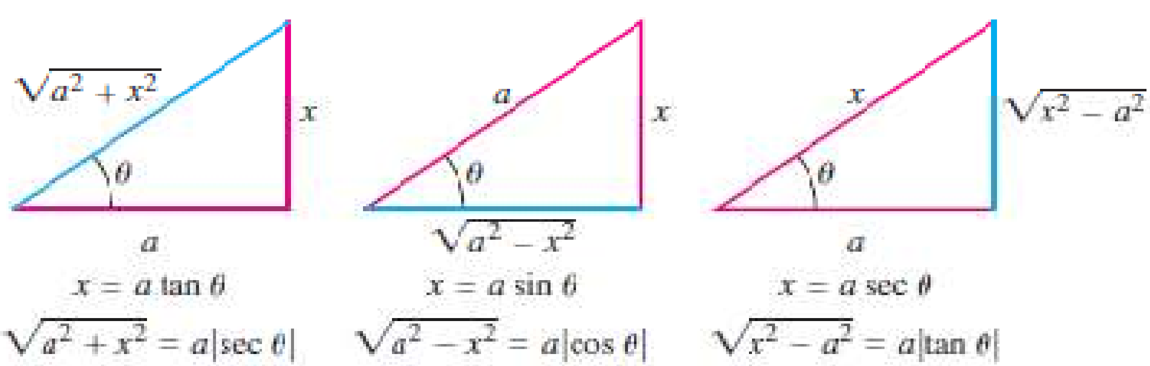
c. $\int_{0}^{\frac{\pi}{2}} \cos 3x \sin 4x dx$

5.6. Trigonometric Substitutions

Trigonometric substitutions can be effective in transforming integrals involving$\sqrt{a^2-x^2}\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$into integrals we can evaluate directly we use the table below.

Expressions in integrand	Trigonometric substitution	Interval(s)
$\sqrt{a^2-x^2}$	$x=asin\theta$	$\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$,
$\sqrt{a^2+x^2}$	$x=atan\theta$	$\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$,
$\sqrt{x^2-a^2}$	$x=asec\theta$	$0\le\theta\le\frac{\pi}{2}$,$or\ \pi\le\theta\le\frac{3\pi}{2}$

The substitutions come from the reference of right triangles as shown below



Reference triangles for the three basic substitutions identifying the sides labeled x and $a$for each substitution.

Example:

Evaluate the following indefinite integrals

$$a.\int \frac{x^2}{\sqrt{9-x^2}}dx \qquad b.\int \frac{3}{\sqrt{1+9x^2}}dx \qquad c.\int \frac{x^2}{(x^2-1)^{\frac{3}{2}}}dx$$

Solution: a. We set $x = 3sin\theta \Rightarrow dx = 3cos\theta d\theta$

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2 x)} = 3cosx \text{ for} \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\int \frac{x^2}{\sqrt{9-x^2}}dx = \int \frac{(3sin\theta)^2}{3cos\theta}3cos\theta d\theta = 9\int sin^2\theta d\theta$$

From the above we substitute $sin^2\theta$ by $\frac{1-cos2\theta}{2}$ hence we get

$$9\int sin^2\theta d\theta = 9\int \left(\frac{1-cos2\theta}{2}\right)d\theta = \frac{9}{2}\left(\theta - \frac{sin2\theta}{2}\right) + C$$

From $x = 3sin\theta$ we get $\theta = sin^{-1}\left(\frac{x}{3}\right)$ and $\sin 2\theta = 2\sin\theta\cos\theta$

$$\frac{9}{2}\left(\theta - \frac{sin2\theta}{2}\right) + C = \frac{9}{2}\left(sin^{-1}\left(\frac{x}{3}\right) - \sin\theta\cos\theta\right) + C$$

$$= \frac{9}{2}\left(sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\left(sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{9}\right) + C = \frac{9}{2}sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C$$

Solution: b. we set $u = 3x \Rightarrow du = 3dx$ and $u^2 = 9x^2$ then from this

$$\int \frac{3}{\sqrt{1+9x^2}}dx = \int \frac{1}{\sqrt{1+u^2}}du$$

From the above table we can set $u = tan\theta \Rightarrow du = \sec^2\theta d\theta$ and from the trigonometric identities we have $1 + \tan^2\theta = \sec^2\theta$ then

$$\int \frac{1}{\sqrt{1+u^2}}du = \int \frac{1}{\sqrt{1+tan^2\theta}}sec^2\theta d\theta = \int \frac{sec^2\theta d\theta}{\sqrt{\sec^2\theta}} = \int \frac{sec^2\theta}{|\sec\theta|}d\theta$$

For $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then we can have

$$\int \frac{sec^2\theta}{|\sec\theta|}d\theta = \int (1)\sec\theta\, d\theta$$

$$= \int \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right)\sec\theta d\theta = \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}d\theta$$

Let $v = \sec\theta + \tan\theta \Rightarrow dv = (\sec^2\theta + \sec\theta\tan\theta)d\theta$

$$\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}d\theta = \int \frac{1}{v}dv = ln|v| + C$$

$= ln|\sec\theta + \tan\theta|$+C but $\theta = \tan^{-1}u = \tan^{-1}3x$

$=ln|\sec(\tan^{-1}3x) + 3x| + C$

Solution c. Exercise

Exercise: Evaluate the following indefinite integrals

$$a. \int \frac{1}{x\sqrt{4-ln^2x}}dx$$
$$b. \int \frac{1}{\sqrt{e^{2y}-1}}dy$$
$$c. \int \frac{x^2}{x^2+1}dx$$

5.7. Improper Integrals

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are calledimproper integrals of Type I.

1. If f(x) is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)dx$$
2. If f(x) is continuous on $[-\infty, a)$, then
$$\int_{-\infty}^a f(x)\,dx = \lim_{b\to-\infty}\int_b^a f(x)dx$$
3. If f(x) is continuous on $[-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

Where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example:

Evaluate the following improper integrals

$$a. \int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$$
$$b. \int_{-\infty}^{0}\theta e^{\theta}d\theta$$

Solution: a. According to the definition in part 3above we have

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$$
$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_{0}^{\infty}\frac{1}{1+x^2}dx$$
$$= \lim_{b\to-\infty}\int_b^0\frac{1}{1+x^2}dx \quad + \lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}dx$$

$$= \lim_{b\to-\infty} \tan^{-1} x]_b^0 + \lim_{b\to\infty} \tan^{-1} x]_0^b = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$therefore \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi$$

Solution b. Exercise

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

1. If f(x) is continuous on (a, b] and is discontinuous at a then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)dx$$
2. If f(x) is continuous on [a, b) and is discontinuous at b then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)dx$$
3. If f(x) is discontinuous at c, where$a < c < b$,and continuous on $[a, c) \cup (c, b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

In each case, if the limit is finite we say the improper integral **converges** and thatthe limit is the value of the improper integral. If the limit does not exist, the integral **diverges**.

Example:

Show weather the following improper integrals converge or diverge.

$$a. \int_0^1 \frac{1}{1-x} dx$$

$$b. \int_0^3 \frac{1}{(x-1)^{\frac{3}{2}}} dx$$

Solution: a. The function $\frac{1}{1-x}$ is discontinuous at x=1 the as defined in 2 above we can have

$$\int_0^1 \frac{1}{1-x} dx = \lim_{c\to 1^-} \int_0^c \frac{1}{1-x} dx \quad = \lim_{c\to 1^-} ln(1-x)]_0^c = \lim_{c\to 1^-} [\ln(1-c) - 0] = \infty$$

The limit is infinite, so the integral diverges.

Solution: b.

The integrand has a vertical asymptote at x=1 and is continuous on [0, 1)and (1, 3]

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b$$

$$= \lim_{b\to 1^-} [3(b-1)^{1/3} + 3] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3$$

$$= \lim_{c\to 1^+} \left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$

Exercise:

Show weather the following improper integrals converge or diverge.

$a. \int_0^3 \frac{1}{1-x}dx$

$b. \int_0^1 (-lnx)dx$

Determine whether the integral converges or diverges, and if it converges, find its value.

1. $\int_0^{\infty} \frac{x}{1+x^2} dx$

2. $\int_{-\infty}^{0} \frac{1}{(x+3)^2} dx$

3. $\int_1^{\infty} \frac{1}{\sqrt{x^2-1}} dx$

4. $\int_{-\infty}^{\infty} x e^{-x^2} dx$

5. $\int_0^{9} \frac{1}{\sqrt{x}} dx$

6. $\int_0^{\pi/2} \sec^2 x dx$

7. $\int_{-2}^{0} \frac{1}{\sqrt{4-x^2}} dx$

8. $\int_0^{\pi} \sec x dx$

9. $\int_{-2}^{7} \frac{1}{(x+1)^{2/3}} dx$

10. $\int_0^{1} \frac{3x^2-1}{x^3-x} dx$

Miscellaneous Exercise

Evaluate the following integrals

1. $\int_1^2 (x^4 - 3x^2 + 4x - 2) dx$
2. $\int_1^4 \left(\sqrt{x} + \frac{1}{2\sqrt{x}} \right) dx$
3. $\int (\theta^2 + \sec^2 \theta) d\theta$
4. $\int_{\pi/4}^{\pi/2} (-7\sin x + 3\cos x) dx$
5. $\int_0^{\pi/2} f(x) dx$, where

$$f(x) = \begin{cases} \sec^2 \theta & for \quad 0 \le x \le \frac{\pi}{4} \\ \csc^2 \theta & for \quad \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$$

6. $\int \left(x^5 \sqrt{x^2 - 1} \right) dx$
7. $\int (1+4x)\sqrt{1+2x+4x^2}\, dx$
8. $\int \sec x \tan x \sqrt{3 + \sec x}\, dx$; u=3+secx
9. $\int \frac{\sqrt[3]{x}}{\left(\sqrt[3]{x}+1\right)^5} dx$
10. Find the indefinite integrals
    a) $\int x e^{-x} dx$

b) $\int x^3 \cos x dx$

c) $\int x^2 4^x dx$

12. Find the integral

a) $\int \frac{x}{x+1} dx$

b) $\int \frac{2}{x^2 - x - 6} dx$

c) $\int_{-1}^{0} \left(\frac{x^2 + x + 1}{x+1} \right) dx$

13. By making the substitution u=1-x show that

$\int_0^1 x^n (1-x)^m dx = \int_0^1 x^m (1-x)^n dx$ for any non negative integers m and n.

14. Evaluate the integrals below

$$a. \int_0^{\frac{\pi}{8}} sin^5 2x \; cos2xdx$$

$$b. \int_0^{\frac{\pi}{8}} (2x-5)(x-3)^9 \, dx$$

15. Evaluate the integrals below

$$a. \int_0^{\pi} sin4xcos3xdx \qquad b. \int_0^{\pi/2} sin^4 xcos^3 xdx$$

$$c. \int \left(\frac{cosx - sin2x}{sinx} \right) dx \qquad d. \int_0^{\pi/2} (2 - sinx)^2 dx$$

$$e. \int sin5xsinxdx \qquad f. \int_0^{\pi/4} \sqrt{1 - cos4x} dx$$

16. Evaluate the improper integrals below

CHAPTER SIX

APPLICATION OF THE INTEGRAL

The mathematical application will include area, volume, length of a curve, and surface area and the other work, force and momentum and centre of mass so on are for physics, engineering...

6.1. AREA

Suppose we want to find the area of a region bounded above by the curve $y = f(x)$ and, below by the curve $= g(x)$and on the left and right by lines $x = a \; and$ $x = b$ $(Fig\ 6.1)$ $if\ f\ and\ g\ are\ continuous$.

We first approximate the region with n vertical rectangles based on a partition P = $\{x_0 , x_1 , \dots , x_n\}$ $of\ [a , b]$ (Fig 6.2) the area of the k^{th} rectangle (Fig 6.3) is

$$\Delta A_k = haight \times width = [f(c_k) - g(c_k)]\Delta x_k$$

We then approximate the area of the region by adding the areas of the n rectangles

A$\approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n}[f(c_k) - g(c_k)]\Delta x_k$ which is called Rieman sum

As $\|p\| \to 0$ the area of the right approach the limit $\int_a^b [f(x) - g(x)]dx$ because $f\ and\ g$ are continuous. We take the area of the region to be the value of this integral. That is,

$$A \approx \lim_{|P| \to 0} \sum_{K=1}^{n} [f(c_k) - g(c_k)]\Delta x_k = \int_a^b [f(x) - g(x)]dx$$

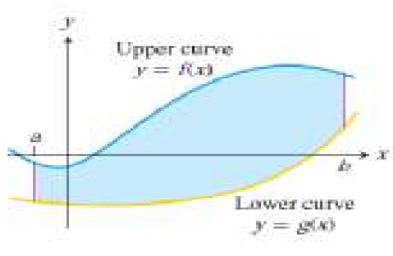


Fig 6.1

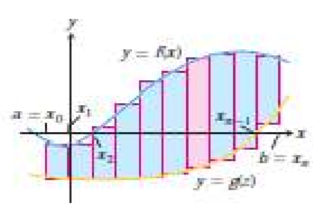


Fig 6.2

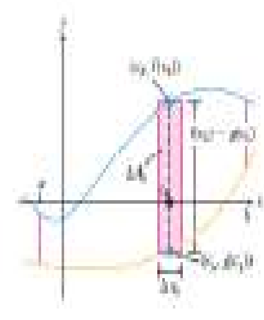


Fig 6.3

Area between curves

Definition; If $f\ and\ g$ are continuous with $f(x) \geq g(x)$ throughout $[a, b]$ then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from $a\ to\ b$ is the integral of $A = \int_a^b (f - g)dA = \int_a^b (f(x) - g(x))dx$

EXAMPLE:

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

solution First we sketch the two curves

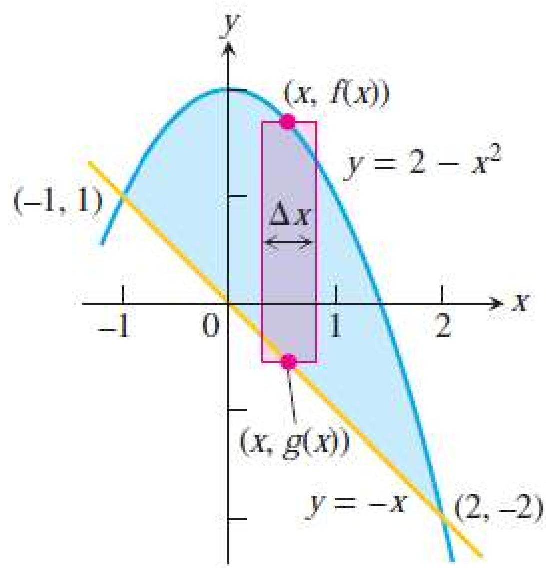


Fig 6.4

The limits of integration are found

By solving $y = 2 - x^2$ and $y = -x$ simultaneously for x

$$2 - x^2 = -x \quad equate\ f(x) and\ g(x)$$

$$2 - x^2 - x = 0 \quad rewrite$$

$$(x+1)(x-2) = 0\ factor$$

$$x = -1\ and\ x = 2$$

The region runs from $x = -1\ and\ x = -2$ is the limit of integration from

$a = -1\ and\ b = 2$ then the area between the curves is

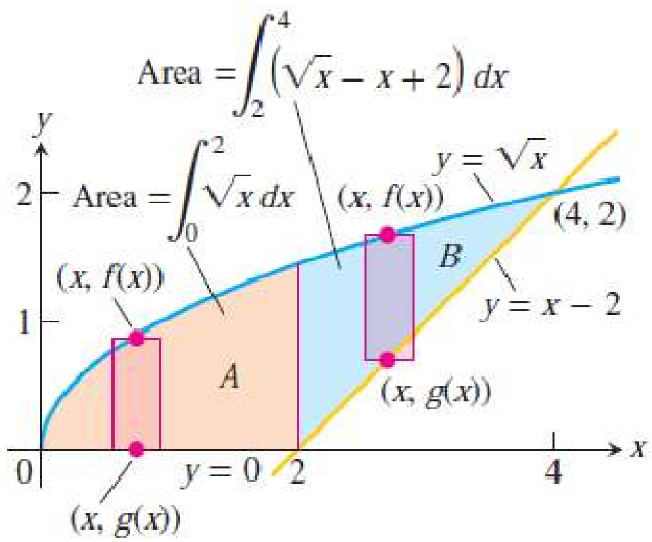
$\int_a^b (f(x) - g(x))dx = \int_{-1}^{2}[(2 - x^2) - (-x)]dx = \frac{9}{2}$

Example:

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution: Shows that the region upper boundary is the graph of

$f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \leq x \leq 2$ to $g(x) = x - 2$ for $2 \leq x \leq 4$ (there is agreement at $x = 2$). We subdivide the region at $x = 2$ into subre-
region from A and B



Figur 6.5

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\begin{aligned} \sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x). \\ x &= (x-2)^2 = x^2 - 4x + 4 && \text{Square both sides.} \\ x^2 - 5x + 4 &= 0 && \text{Rewrite.} \\ (x-1)(x-4) &= 0 && \text{Factor.} \\ x &= 1, \qquad x = 4. && \text{Solve.} \end{aligned}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$\begin{aligned} &\text{For } 0 \le x \le 2: && f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x} \\ &\text{For } 2 \le x \le 4: && f(x) - g(x) = \sqrt{x} - (x-2) = \sqrt{x} - x + 2 \end{aligned}$$

We add the area of subregions A and B to find the total area:

$$\begin{aligned} \text{Total area} &= \underbrace{\int_0^2 \sqrt{x}\, dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\, dx}_{\text{area of } B} \\ &= \left[\frac{2}{3} x^{3/2}\right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4 \\ &= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right) \\ &= \frac{2}{3}(8) - 2 = \frac{10}{3}. \end{aligned}$$

Exercises

1. Let $f(x) = 5x \; and \; g(x) = x^2$, and let R be the region between the graphs of $f \; and \; g \;\; on \; [0,3]$ find the area of the region

6.2. VOLUMES

If f is a function continuous on an interval $[a\,, b]$ then $\int_a^b f(x)dx$ is the limit of the Riemann sums for $f \; on \; [a, b]$ as the lengths of the subintervals derived from the partition of $[a, b]$ approach 0.

We will describe briefly here our procedure for introducing the other application of the integral. For each application our goal will be to find a formula for a quantity I (such as the volume of solid region) we will proceed in the following way

In each case it will be reasonable to expect that $\sum_{k=1}^{n} f(t_k)\,\Delta x_k$ should approach I as the norm of the partition P tends to 0 this idea is expressed by writing $I = \sum_{k=1}^{n} f(t_k)\,\Delta x_k$

We will conclude that I = $\int_a^b f(x)dx$

Volume can be found by different methods

a. The cross-sectional method
b. The disc method
c. The washer method
d. The shell method

a. The Cross-Sectional Method

Consider the cross-sectional area A of the region D is a function that is continuous on $[a\,, b]$ let P= $(x_0, x_1, x_2 ..., x_n)$ be a portion of $[a. b]$ for each k between 1 and n ,let t_k be an arbitrary number in the sub intervals $[x_{k-1}, x_k]$, if Δx_k is small ,the volume Δv_k of the part of D between x_{k-1}, x_k is approximately equal to the product of the cross-sectional area A(t_k) and the length Δx_k

Thus $\Delta V \approx$ (cross-sectional area)×(length)

$$= A\,(t_k)\Delta x_k$$

Since the volume V of D is the sum of $\Delta v_1, \Delta v_2, \ldots, \Delta v_n$. It follows that V should be approximately $\sum_{k=1}^{n} A(t_k)\,\Delta x_k$

The volume of a solid of known integral cross-sectional area A(x) from $x = a \ to \ x = b$ is the integral of A from $a \ to \ b$ $\quad V = \int_a^b A(x)dx$

Example: Find the volume V of the solid D in which cross-section at x is semicircular region with radius r is x^2 for $0 \leq x \leq 1$

Solution: The area of a semicircular region with radius r is $\frac{1}{2}\pi r^2$.

Thus the cross-sectional area A(x) of x is given by A(x) $= \frac{1}{2}\pi(x^2)^2$.

There for we conclude that

V$=\int_0^1 A(x)dx$

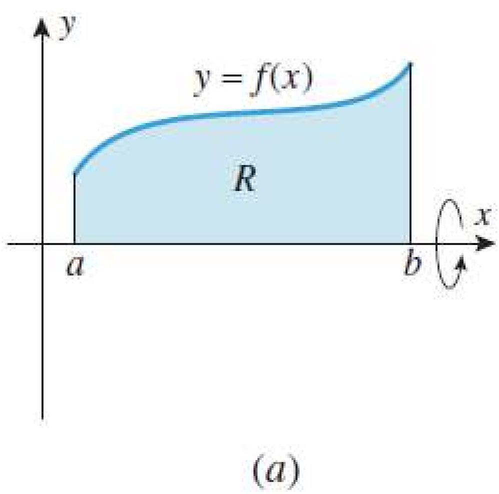
$=\int_0^1 \frac{1}{2}\pi x^4$

$=\frac{\pi}{10}$

b. The Disc Method

When the graph of a continuous non-negative function f on an interval $[a, b]$ is revolved about the x axis, it generates a solid region having circular cross-sections, that is cross-sections are circular disc (Fig 6.6) ,since the radius of the cross-section at $x \ is \ f(x)$ it follows that

$A(x) = \pi[f(x)]^2$ (1)



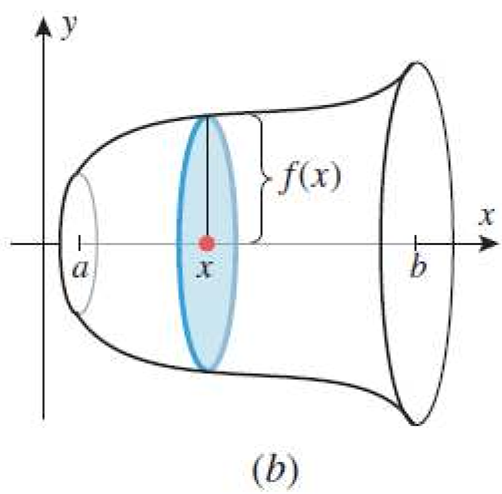


Figure 6.6 Figure 6.7

Thus we obtain a formula for the volume V of the solid that is generated

$V = \int_a^b \pi[f(x)]^2 \, dx$

Because the cross-sections are disc, it is called the disc method

Example: Find the volume V of a sphere of radius r

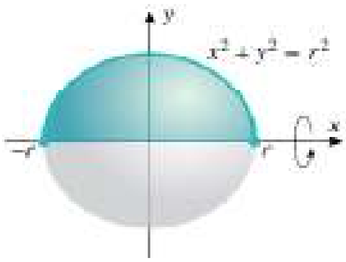


Figure 6.8

Solution: In Figure6.8 A sphere is generated by revolving a semicircle about its diameter ,if we let

$f(x) = \sqrt{r^2 - x^2}$ for $-r \leq x \leq r$ .Then

V=$\int_{-r}^{r}(\sqrt{r^2 - x^2})^2 dx = \frac{4}{3}\pi r^3$

Example: Find the volume of the solid that is obtained when the region under the Curve $y = \sqrt{x}$ over the interval [1,4] is revolved about the x-axis

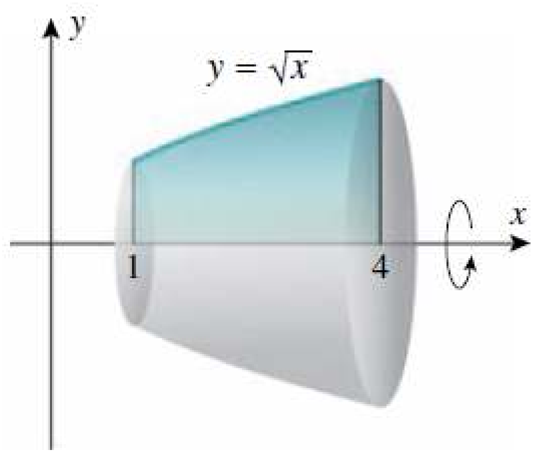


Figure 6.9

Solution: the volume is

$$V = \int_a^b \pi[f(x)]^2 \, dx = \int_1^4 \pi x \, dx = \frac{\pi x^2}{2}\Big]_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$$

Exercises

1. Let R be the region between the graph of the function and the x axis on the given interval. Find the volume v of the solid obtained by revolving R about x axis
   $a.\ f(x) = x^2$;[0,3] $b.\ f(x) = x^2$; [1,2] $c.\ f(x) = \sqrt{xsinx}$; $[0, \pi]$

c. The Washer Method

We present a formula for the volume of the solid region generated by revolving a more general plane region about the x-axis .let f and g be function such that

$0 \leq g(x) \leq f(x) \quad for a \leq x \leq b$

Then the plane region between the graph of f and the x-axis on $[a, b]$ is composed of the region between the graph of g on $[a, b]$ and the region between the graph of g and the x-axis on $[a, b]$

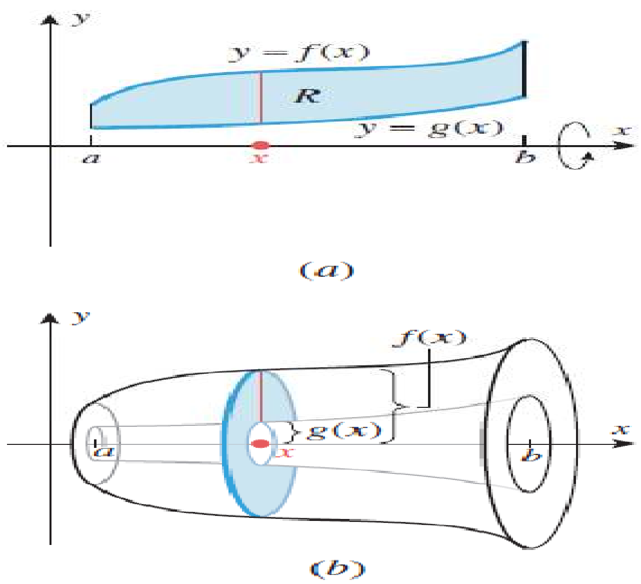


Figure 6.10

The volume V of the solid generated by revolving the region between the graph of f and g on $[a, b]$ is given by V=$\int_a^b \pi[f(x)^2 - g(x)^2]dx$

Example: Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval [0, 2] is revolved about the x-axis.

Solution first sketches the region (Figure); then imagine revolving it about the x-axis. From

(6.10) the volume is

$$V=\int_a^b \pi([f(x)]^2-[g(x)]^2)\,dx=\int_0^2 \pi\left([\tfrac{1}{2}+x^2]^2-x^2\right)dx$$

$$=\int_0^2 \pi\left(\frac{1}{4}+x^4\right)dx=\pi\left[\frac{x}{4}+\frac{x^5}{5}\right]_0^2=\frac{69\pi}{10}$$

Exercise

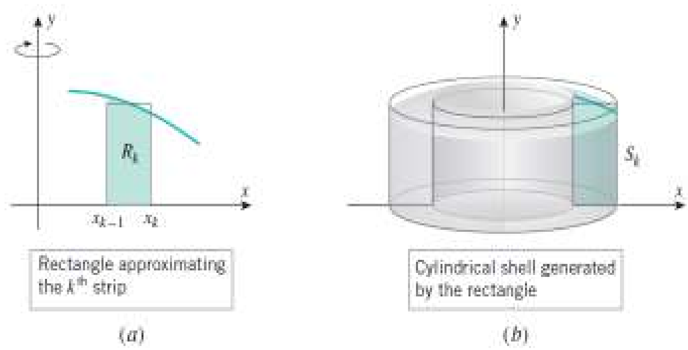
1. Let R be the region between the graph of f *and* g on the given interval. Find the volume v of the solid obtained by revolving R about x axis

   $a.\, f(x)=\sqrt{x+1}, g(x)=\sqrt{x-1}$;[1,3] $b.\, f(x)=x+1, g(x)=x-1$;[1,4]

The Shell Method

We obtain a formula for the volume of the solid generated by revolving about the x-axis the region between the graphs of f *and* g *on* $[a,b]$.we can also revolve such a region about the y *axis* and find a corresponding formula for the volume of the solid so generated.

To begin, let us determine the volume V of a cylindrical shell obtained by revolving a rectangle about the y-axis



(Fig6.11)

Suppose the rectangle is bounded by the x-axis, the line y=c, the lines x=a and x=b, where b$\geq a \geq 0$ *and* $c \geq 0$ then since the volume of the cylindrical shell is the difference of the volume of the outer and the inner cylinders ,it follows that

V = volume of outer cylinder – volume of inner cylinder

$$=\pi b^2 c-\pi a^2 c=\pi c(b^2-a^2)$$

If we replace a *by* x_{k-1} *and* b *by* x_k then we obtain

$$V=\pi c({x_k}^2-{x_{k-1}}^2)=\pi c(x_k+x_{k-1})(x_k-x_{k-1}) \text{ ....(*)}$$

Now let f be a continuous nonnegative function on $[a, b]$,with $a \geq 0$. We wish to define the volume V of the solid region in Figure above ,obtained by revolving about the y-axis the region R between the graph of f and the x axis on [a,b]. Let P=$\{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of [a,b]. For each k between 1 and n, let t_k be the midpoint $^{(x_k + x_{k-1})}/_2$ of the subinterval $[x_{k-1}, x_k]$ .if Δx_k is small, the volume ΔV_k of the portion of the solid between the revolve lines$x = x_{k-1}$ and $x = x_k$ is approximately equal to the volume of the corresponding cylindrical shell with height $f(x)$ by (*), with $f(t_k)$ replacing c, this means that $\Delta V \approx \pi f(t_k)(x_k + x_{k-1})(x_k - x_{k-1}) = 2\pi t_k f(t_k)\Delta x_k$

Therefore the volume V of the solid, which is the sum of $\Delta v_1, \Delta v_2, \ldots, \Delta v_n$

Should be approximately $\sum_{k=1}^{n} 2\pi t_k f(t_k)\Delta x_k$ which is a Riemann sum for $2\pi x f$ on $[a.b]$ .as a result $V = \lim_{\|p\| \to 0} \sum_{k=1}^{n} 2\pi t_k f(t_k)\Delta x_k = \int_a^b 2\pi x f(x)dx$

Thus we are led to the following formula for volume

$$V = \int_a^b 2\pi f(x)dx$$

Let f and g be continuous on $[a, b], with\ for\ a \geq 0$,and suppose that

$$g(x) \leq f(x)\ for\ a \leq x \leq b$$

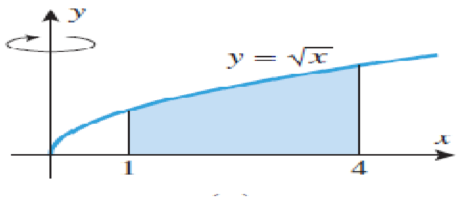
Then let R be the region between the graphs of f and g on $[a, b]$ Figure.

The volume V of the solid obtain by revolving about the y $axis$ is given by

$$V = \int_a^b 2\pi x[f(x) - g(x)]dx$$

► **Example 1** Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis is revolved about the y-axis.

Solution: First sketch the region (Figure6.12);then imagine revolving about y-axis



Figur 6.12

Since $f(x) = \sqrt{x}\,, a = 1\,, and\ b = 4$ the volume is given by

$$V = \int_1^4 2\pi x\sqrt{x}\,dx = 2\pi\int_1^4 x^{3/2}\,dx = \left[2\pi \cdot \frac{2}{5}x^{5/2}\right]_1^4 = \frac{4\pi}{5}[32-1] = \frac{124\pi}{5}$$

Exercises

1. Let R be the region between the graph of the function and x axis on the given interval. Find the volume v of the solid generated by revolving R about the y-axis
   $a. f(x) = \sqrt{x^2+1}; [0,\sqrt{3}]$ $b. f(x) = e^{2x+1}; [0,1]$ $c. f(x) = lnx\,; [1,3]$

6.3. LENGTH OF A CURVE

Consider the graph of a function f with a continuous derivative on closed interval [a,b]. If f is linear, that is ,if the graph of f is a line segment ,then the length L of the graph is the distance between $(a, f(a))\ and\ (b, f(b))$ so that

$$L = \sqrt{(b-a)^2 + (f(b) - f(a))^2}$$

When f is not necessarily linear, we let P = $\{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of [a,b] and approximate the graph of f by a polygonal line l whose vertices are $(x_0, f(x_0)$,($x_1, f(x_1)$),....,($x_n, f(x_n)$) figure. Let ΔL_k be the length of the portion of the graph of f joining $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. If Δx_k is small, $L_k$is approximately equal to the length of the line segment joining $(x_{k-1}, f(x_{k-1})$ and $(x_k, f(x_k))$. In other words

$$\Delta L_k = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} \qquad (1)$$

The Mean value Theorem, applied to f on the interval $[x_{k-1}, x_k]$, implies that

$f(x_k) - f(x_{k-1} = f'(t_k)(x_k - x_{k-1})$ for some $t_k\ in\ (x_{k-1}, x_k)$. Therefore (1) can be rewritten $\Delta L_k = \sqrt{(x_k - x_{k-1})^2 + (f'(t_k)(x_k - x_{k-1}))^2} \quad = \sqrt{1 + (f'(t_k))^2}(x_k - x_{k-1})$ Therefore the total length L of the graph of f ,which is the sum of the length $L_1, L_2, \dots, L_n$ should approximately $\sum_{k=1}^{n} \sqrt{1 + [f'(t_k)]^2} \quad \Delta x_k$ is a Riemann sum for $\sqrt{1 + [f']^2}$ on $[a, b]$. Therefore it seems that

$$L = \lim_{\|p\| \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(t_k)]^{+2}} \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Figure 6.15

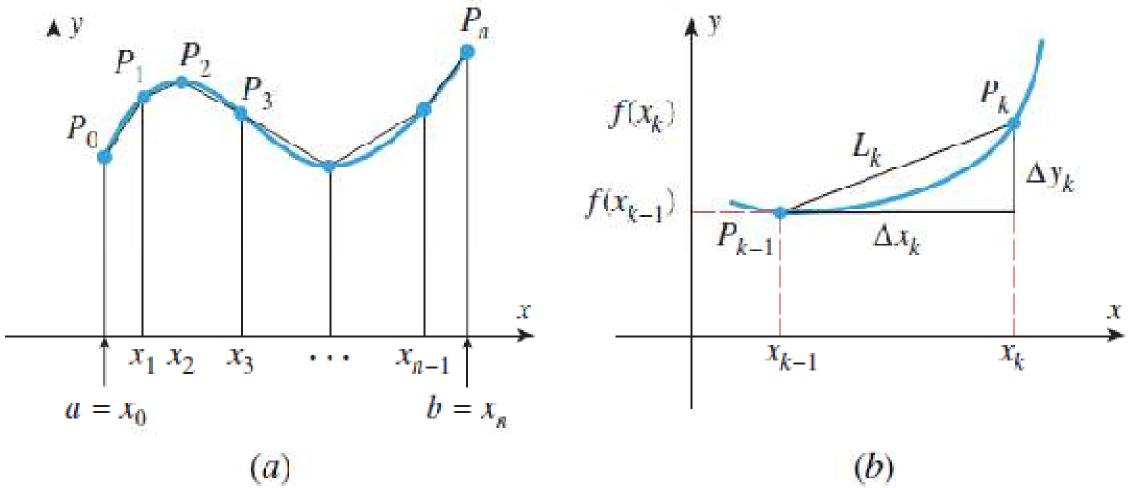


Figure 6.13

This leads us to make the following definition

DEFINITION If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

▶ **Example 1** Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$

Solution

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$ and since the curve extends from $x = 1$ to $x = 2$, it follows

$$L = \int_1^2 \sqrt{1 + \left(\tfrac{3}{2}x^{1/2}\right)^2}\, dx = \int_1^2 \sqrt{1 + \tfrac{9}{4}x}\, dx$$

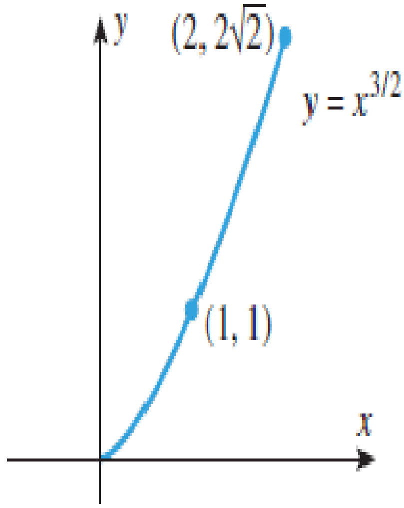


Figure 6.14

To evaluate this integral we make the u-substitution

$$u = 1 + \frac{9}{4}x, \quad du = \frac{9}{4}dx$$

and then change the x-limits of integration ($x = 1, x = 2$) to the corresponding u-limits ($u = \frac{13}{4}, u = \frac{22}{4}$):

$$L = \frac{4}{9}\int_{13/4}^{22/4} u^{1/2}\,du = \frac{8}{27}u^{3/2}\bigg]_{13/4}^{22/4} = \frac{8}{27}\left[\left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$= \frac{22\sqrt{22} - 13\sqrt{13}}{27} \approx 2.09$$

In Exercises1-3 find the length of the graph of the given function

1. $f(x) = 2x + 3 \; for \; 1 \le x \le 5$
2. $g(x) = x^3 + \frac{1}{2x} \; for \; 1 \le x \le 3$
3. $g(x) = x^4 + \frac{1}{32x^2} \;\; for \; 1 \le x \le 2$

6.4. Area of a Surface

A higher-dimensional version of the length of a curve is the area of a surface. It has been known that the surface area S of a cube of sides s is given by S $=6s^2$ and the surface area S of a cylinder of radius r and height h is given by S=2πrh. However , our analysis of the areas of other surfaces will be based on the surface area of a frustum of a cone. If the frustum has slant height l and radii $r_1$and r_2 ,then the surface area S is given by

$$S = 2\pi\left(\frac{r_1 + r_2}{2}\right)l = \pi(r_1 + r_2)l \qquad (1)$$

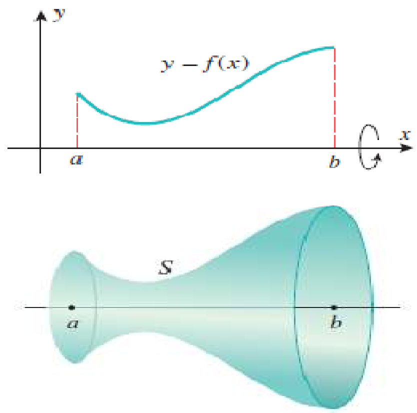


Fig 6.15

More generally, suppose f is defined on$[a, b]$, and$f(x) \geq 0 \; for \; a \leq x \leq b$.we will derive a formula for the S of the surface obtained by revolving the graph of f about x axis Figure. In order to be able to use the length of the graph of f on $[a, b]$in our calculations of surface area, we will assume that f is continuous differentiable on $[a, b]$. Now let P=$\{x_0, x_1, \dots, x_n\}$ be a partition on $[a, b]$,and let ΔS be the area of the portion of the surface between the revolved lines $x = x_{k-1} and \; x = x_k . if \; \Delta x_k$ is small, the surface area s_k is approximately equal to the surface area of the frustum of the corresponding cone. That is the frustum whose slant height is equal to the length of the line between $(x_{k-1}, f(x_{k-1}))$ and $x_k, f(x_k)$ and the radii of whose ends are $f(x_{k-1}) and \; f(x_k)$ from (1) this means that

$$\Delta s_k \approx \pi[f(x_{k-1}) + f(x_k)]\sqrt{(x_k - x_{k-1})^2 + [f(x_k) - f(x_{k-1})]^2}$$

The Mean Value theorem can be applied to show that for some $t_k \; in \; [x_{k-1}, x_k]$,

$$\sqrt{(x_k, x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} = \sqrt{1 + [f'(t_k)]^2}\,\Delta x_k \qquad 3$$

Since Δx_k is assumed to be small $x_{k-1} and \; x_k$are close together, with t_k between them. Since $f' and \; f$ are continuous on $[x_k, x_{k-1}]$ it follows that $f(x_{k-1}) + f(x_k)$ should be approximately $f(t_k) + f(t_k)$ $, that \; is \; 2f(t_k)$ .so from (3) and (2) we deduce that

$$\Delta S_k \approx \pi[2f(t_k)]\sqrt{1 + [f'(t_k)]^2}$$

Consequently the area S of the complete surface, which equal the sum of $\Delta S_1, \Delta S_2, \dots, \Delta S_n$ should approximately $\sum_{k=1}^{n} 2\pi f(t_k)\sqrt{1 + [f'(t_k)]^2}\,\Delta x_k$

Which is a Riemann sum for $2\pi f(t_k)\sqrt{1+[f'(t_k)]^2}$ on $[a,b]$. Therefore we define the surface area S by

$$S = \lim_{\|p\|\to 0} \sum_{k=1}^{n} 2\pi f(t_k)\sqrt{1+[f'(t_k)]^2} \ \ \Delta x_k = \int_a^b 2\pi f(x)\sqrt{1+[f'(t_k)]^2}dx$$

DEFINITION Let f be nonnegative and continuously differentiable on [a,b]. The area of the surface obtained by revolving the graph of f about the x axis is defined by

$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(t_k)]^2}dx$

DEFINITION If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

► **Example 1** Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

Solution. First sketch the curve; then imagine revolving it about the x axis (Figure

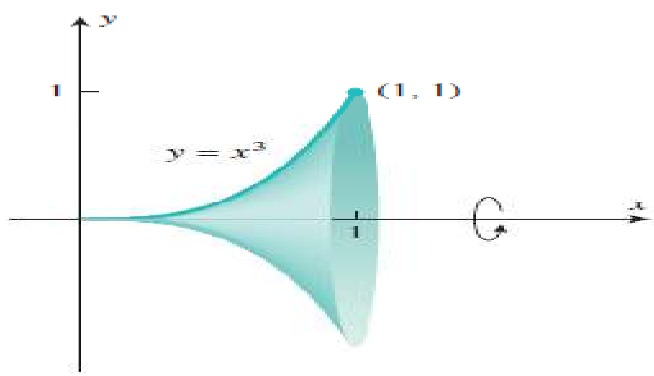


Figure 6.16

Since $y = x^3$, we have $dy/dx = 3x^2$, and hence from (4) the surface area S is

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2}\, dx$$

$$= 2\pi \int_0^1 x^3 (1 + 9x^4)^{1/2}\, dx$$

$$= \frac{2\pi}{36} \int_1^{10} u^{1/2}\, du \qquad \boxed{\begin{array}{c} u = 1 + 9x^4 \\ du = 36x^3\, dx \end{array}}$$

$$= \frac{2\pi}{36} \cdot \frac{2}{3} u^{3/2} \Big]_{u=1}^{10} = \frac{\pi}{27}(10^{3/2} - 1) \approx 3.56$$

Exercises:

Find the area S of the surface generated by revolving about the x axis the graph of f on the given interval

a. $f(x) = \sqrt{4 - x^2}$; [1/2,3/2]

b. $f(x) = \frac{1}{3}x^3$;$[0,\sqrt{2}]$

Exercises

1. Sketch and find the area of the region to the parabola $x = 2y^2$ to the right of the y-axis , and between $y = 1\ and\ y = 3$
2. Find the area bounded by the curve $y = 1 - x^{-2}$ and the lines $y = 1 , x = 1\ and\ x = 4$

$1 - 5$ Find the volume the solid that results when the region enclosed by the given curves is revolved about x-axis

1. $y = 9 - x^2\ y = 0$
2. $y = \sqrt{25 - x^2}\ y = 3$
3. $y = sinx , y = cosx\ x = 0 , x = \frac{\pi}{4}$
4. $y = \frac{e^{3x}}{\sqrt{1+e^{6x}}} x = 0 , x = 1, y = 0$
5. $y = \frac{1}{\sqrt{4+x^2}}\ x = -2, x = 2\ y = 0$

$6-10$ Find the volume the solid that results when the region enclosed by the given curves is revolved about y-axis

6. $x = cscy\; y = \frac{\pi}{4}\; y = \frac{3\pi}{4}, x = 0$
7. $y = x^2\,,\; x = y^2$
8. $y = \sqrt{\frac{1-x^2}{x^2}}\quad (x > 0)$
9. $y = lnx\;\; , x = 0\,, y = 0\,, y = 1$
10. $x = y^2\,, x = y + 2$
11. Let V be the volume of the solid that results when the region enclosed by $y = \frac{1}{x}\,, y = 0, y = 0\,, x = 2\; and\; x = b(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which V=3
12. Find the length of the arc of $24xy = x^2 + 48$ from $x = 2\; to\; x = 4$
13. Find the length of the arc of $y^3 = 8x^2\; from\; x = 1\; to\; x = 8$